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Sufficient Aggregation of Performance Measures

by

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Abstract

Aggregating performance measures considerably reduces the complexity of a performance evaluation system. This dissertation analyzes the nature and characteristics of the aggregation of performance measures in institutional settings of multiple tasks and multiple periods.

In multi-task settings, the number of tasks restricts the feasibility of a statistically sufficient aggregation and the nature of an economically sufficient aggregation. Statistical sufficiency of aggregation can be achieved only by multi-dimensional aggregate measures, whose minimum dimensionality is given by the number of tasks. If the number of aggregate measures is less than the number of tasks, an economically sufficient aggregation incurs loss of information even if there is no loss of information through the likelihood ratio. The results support the use of multi-dimensional aggregate measures to preserve the information content of performance measures in multi-task agencies.

In a multi-period setting, the inter-temporal correlations among performance measures restrict the feasibility of a statistically sufficient aggregation and the nature of an economically sufficient aggregation. When performance measures are inter-temporally correlated, there is no statistically sufficient aggregation and an economically sufficient aggregation of the basic measures depends on the effort level to induce. The optimal aggregation is characterized by the agent's characteristics and the economic situation of the agency as well as the statistical properties of performance measures.

In a long-term contract with multiple tasks, the inter-temporal covariance risk has a monotonic impact on the endogenous allocation of effort through the optimal relative incentive rate. The inter-temporal covariance risk, as well as the within-period risk premium, prevents the first best allocation of effort from being endogenously achieved even if the first best allocation is feasible.

Contents

Chapter 1

Introduction

This dissertation analyzes the aggregation of performance measures in settings with multiple periods and multiple tasks. Each period, an accounting system generates performance measures to measure and report the managers' activities for the period. The performance measures give imperfect information on the managers' unobservable activities for the period. In practice, multiple performance measures are often used on an activity such that those performance measures are complementary in providing information on the unobservable activity. Also, performance measures are often inter-temporally correlated such that some information on the future period performance measures is available from the realized performance measures.

Aggregation of performance measures is an integral part of accounting. Most accounting performance measures are aggregated to generate performance measures at a higher level. For example, segment earnings are aggregated into firm earnings, and branch production quality measures are aggregated to generate the firm-wide production quality measure.

By aggregating performance measures, it is possible to considerably reduce the complexity of a performance evaluation system. A management accountant may construct aggregated performance measures out of numerous basic performance measures. In designing a performance evaluation system for managers of a firm, a management accountant is endowed with the discretion to decide how to aggregate performance measures. This dissertation raises two questions on the aggregation of performance measures:

1. Is it possible to construct aggregated performance measures such that no information on the unobservable activity is lost in the aggregation process?

2. Is it possible to construct aggregated performance measures, which are efficient and optimal for every manager in all firms? Or should a performance evaluation system be individually designed for each manager such that different managers are evaluated on different performance measures?

This dissertation employs institutional settings of multiple periods and multiple tasks in answering the above questions. In a single-period setting with a single-task, Amershi, Banker, and Datar (1990) show that a "universal" aggregate measure, which is good enough to substitute for the basic measures for inducing all effort levels from every manager in all firms, is rarely feasible because a statistical condition ("all a or no a"condition[1]) is not generally satisfied. In settings of multiple periods and multiple tasks, this dissertation also finds a "universal"aggregate measure generally infeasible, but the reasons for the infeasibility are qualitatively different from that of Amershi, Banker, and Datar (1990).

In the chapter on multiple tasks, this dissertation shows that it is not only a statistical condition ("all a or no a"condition) but also the number of aggregated performance measures and the number of tasks that together decide the feasibility of a "universal"aggregate measure. In a multi-task setting, the number of tasks restricts the feasibility of a statistically sufficient aggregation, which is characterized by no loss of information, and the nature of an economically sufficient aggregation, which is characterized by no economic loss to the principal. Statistical sufficiency of aggregation can be achieved only by multi-dimensional aggregate measures, whose minimum dimensionality is given by the number of tasks. If the number of aggregate measures is less than the number of tasks, there is no aggregation with no loss of information. Also, if the number of aggregate measures is less than the number of tasks, a performance evaluation system should be "tailored"for each manager such that different managers are evaluated on different performance measures.

In the chapter on multiple periods, this dissertation shows that it is not only a statistical condition ("all a or no a"condition) but also the existence of the inter-temporal correlations among performance measures that prevents a "universal"aggregate measure. In a multi-period setting, the inter-temporal correlations among performance measures restrict the feasibility of a statistically sufficient aggregation and the nature of an economically sufficient aggregation. As long as performance measures are inter-temporally correlated, there is no aggregation with no loss of information in the aggregation process and an optimal aggregate measure depends on the characteristics of a manager and the economic situation of a

[1] The "all a or no a"condition was discussed in Holmstrom (1979) and Amershi, Banker, and Datar (1990), and implies no loss of information through the likelihood ratio. The technical detail is presented with Definition 8.

firm. Thus, in a multi-period setting, a performance evaluation system should be "tailored" for each manager such that different managers are evaluated on different performance measures.

Finally, in the chapter on relative incentive rates, this dissertation shows that the inter-temporal covariance risk of performance measures, as well as the within-period risk premium, prevents the first best allocation of effort from being achieved even if the first best allocation is feasible. In the presence of inter-temporal correlations among performance measures, the first best allocation of effort is not optimal to the principal. The endogenous allocation of effort is examined through the optimal relative incentive rate. As the analysis is extended to a multi-period and multi-task setting, the optimal relative incentive rate is no longer explained by the relative "signal-to-noise" ratio of performance measures. In particular, the compensation risk regarding the inter-temporal covariance of performance measures affects the relative incentive rate such that a performance measure with bigger inter-temporal covariance risk is assigned a weaker relative incentive rate.

Chapter 2

Multiple Tasks

2.1 Introduction

Aggregating performance measures can considerably reduce the complexity of performance measurement and evaluation systems. However, if a significant loss of information is incurred in an aggregation process, the aggregation of performance measures might not be desirable.

Since proposed by Kaplan and Norton (1992, 1993), the balanced scorecard has been widely practiced in businesses. Although there have been many studies and a rich literature on the balanced scorecard, they are largely silent on the aggregation side of the balanced scorecard as addressed in Datar, Kulp, and Lambert (2001).

This study shows in a multi-task setting that aggregating performance measures into a single aggregate measure generally incurs loss of information and that the use of multiple aggregate measures is necessary to preserve the information content of performance measures.

When a management accountant designs a performance evaluation system for providing incentive compensation to managers, the management accountant may aggregate the basic performance measures into a smaller number of aggregate performance measures. In designing the performance evaluation system, an important question is whether the aggregate performance measures can carry all the information from the basic performance measures without losing any information in the aggregation process. Another question is whether the aggregate performance measures can be commonly used for all managers or should be individually "tailored"for each manager such that different managers are evaluated on different performance measures.

These questions are more clearly answered in a multi-task setting, in which the information content of performance measures is richer than that in a single-task setting. Typically, managers undertake multiple tasks. For example, a manager may work as a cost-center and also be responsible for quality control. Motivating managers for multiple tasks is qualitatively different from a single-task setting as the allocation of effort among tasks is considered (Holmstrom and Milgrom 1991;

Feltham and Xie 1994; Datar, Kulp, and Lambert 2001). This study shows that the number of aggregate performance measures and the number of tasks are together critical in answering the questions raised above. If the number of aggregate performance measures is no less than the number of tasks, it is possible to have a universal evaluation system, in which the aggregate performance measures do not lose any information in the aggregation process and also the same aggregate performance measures are used for all managers, for inducing all effort levels, and even for all firms.

Otherwise, if the number of aggregate performance measures is less than the number of tasks, then it is no longer possible to have a universal evaluation system. In this case, aggregate performance measures incur loss of information in the aggregation process, and the optimality of a certain set of aggregate performance measures is limited to a specific manager, for inducing a specific level of effort, and in a specific firm. Different aggregate performance measures should be used for different managers in different settings.

In answering the aforementioned questions, this study employs a principal-agent model and examines the following questions. In multi-task settings, this study asks whether it is possible to construct a one-dimensional aggregate measure, that is good enough to substitute for the basic measures in all agencies. If not, this study questions the minimum dimensionality of multiple aggregate measures, which are good enough to substitute for the basic measures in all agencies. For an aggregation to be "universally"good for all agencies, the aggregation should incur no loss of information regardless of the characteristics of the agency (utility functions of the participants) and regardless of the economic condition of the agency (the agent's effort to induce). Such an aggregation is statistically sufficient and the aggregate measures are sufficient statistics, which are determined only by the statistical properties of performance measures.

The next question in multi-task settings is on the nature of aggregate measures, which are good enough to substitute, with no economic loss to the principal, for the basic measures in the contract for inducing a single effort level. Such an aggregation is economically sufficient and the aggregate measures can depend on the characteristics of the agency (utility functions of the participants) and the economic condition of the agency (the agent's effort to induce). The final question in multi-task settings is whether an economically sufficient aggregation necessarily incurs loss of information.

This study introduces a hierarchy of sufficiency concepts of aggregation of performance measures, and examines the feasibility and the nature of sufficient aggregation of performance measures in multi-task settings. This study extends the analysis on the sufficient aggregation of Amershi, Banker, and Datar (1990) not only to multi-task settings but also for the implementation of arbitrary effort levels as opposed to only the optimal effort level.

For statistical sufficiency of aggregation, this study shows that in general, no one-dimensional sufficient statistic exists when the analysis is extended to a multi-task setting. In addition, this study also shows that in general, no one-dimensional sufficient implementation statistic, which requires statistical sufficiency for a single effort level instead of all effort levels, exists in a multi-task setting. Instead, statistical sufficiency of aggregation can be achieved only by multi-dimensional (aggregate) measures. This study shows in a multi-task setting that the minimum dimensionality of statistically sufficient aggregate measures is the number of tasks. In general, the number of aggregate measures should be no less than the number of tasks in order to preserve the information content of performance measures in the aggregation process.

For economic sufficiency of aggregation, this study analyzes the efficient aggregation of performance measures, that is the aggregation in the minimum cost contract for inducing an exogenously fixed arbitrary effort level. This study shows in a multi-task setting that if the number of aggregate measures is less than the number of tasks, an economically sufficient aggregation incurs loss of information and conveys strictly less information than the basic measures. This result implies that if the dimension of aggregation is smaller than that of the effort level to induce, an economically sufficient aggregation depends on the characteristics of the agency and takes varying forms for each manager and each firm.

This study contributes to the literature by first discussing the sufficient aggregation of performance measures in multi-task settings. The informativeness condition in Holmstrom (1979) proposes that additional performance measures are informative on the agent's unobservable effort unless the existing measure is a sufficient statistic. The result of Holmstrom (1979) implies that the concept of sufficient statistics characterizes the optimal aggregation of performance measures. On the other hand, Amershi, Banker, and Datar (1990) show that sufficient statistics do not generally characterize the optimal aggregation of performance measures, but the optimal aggregation is generally economically sufficient and agency-specific. The study shows that because a statistical condition ("all a or no a" condition) is generally not satisfied, the optimal aggregate measures are agency-specific and not sufficient statistics. The analyses in Holmstrom (1979) and Amershi, Banker, and Datar (1990) are restricted to a single-task and single-period setting. Recently, Christensen, Şabac, and Tian (2010) analyze the efficient contract in multi-task agencies and discuss the role of the likelihood ratio and the variance of the likelihood ratio in ranking performance measures.

In a multi-task setting, this study shows that the concept of sufficient statistics is too restrictive to characterize the optimal aggregation of performance measures. Although the restrictiveness of the concept of sufficient statistics found in this study is similar to that of Amershi, Banker, and Datar (1990) in a single-task setting, the reason for the restrictiveness in a multi-task setting is qualitatively different from that of a single-task setting. This study shows that it is not only a statistical condition ("all a or no a" condition) but also a multi-task setting that causes the optimal aggregate measures to be agency-specific and not sufficient statistics. In a multi-task setting, the number of tasks, in addition to a statistical condition ("all a or no a" condition), is a constraining factor that causes the optimal aggregation to be agency-specific. In particular, this study shows that, even when the "all a or no a" condition is satisfied, the optimal aggregation of performance measures is not statistically sufficient if the number of aggregate measures is less than the number of tasks.

Taking the two-stage optimization approach of Grossman and Hart (1983), this study analyzes the efficient aggregation as opposed to the optimal aggregation of performance measures. The optimal aggregation is regarded as a special case of the efficient aggregation. Most previous studies, including Banker and Datar (1989) and Amershi, Banker, and Datar (1990), focus on the optimal aggregation, that is the aggregation in equilibrium to induce the endogenously determined optimal effort level. This study also contributes to the literature by extending the analysis of aggregation from the optimal aggregation to the efficient aggregation to induce an exogenously fixed effort level, which is not necessarily optimal.

The rest of this study is organized as follows: Section 2 discusses the sufficiency concepts for aggregation of performance measures. Section 3 analyzes the sufficient statistic in multi-task settings. Section 4 analyzes the sufficient implementation statistic in multi-task settings. Section 5 analyzes the economically sufficient aggregation in multi-task settings. Section 6 concludes the study.

2.2 Sufficiency concepts for aggregation of performance measures

Before reviewing the sufficiency concepts for aggregation, it may be helpful to look at an agency, in which a risk neutral principal induces productive effort from a risk averse agent by offering a contract. The general setting in this study, unless specified otherwise, involves multiple tasks:

$$\vec{a} = (a_1, a_2, \cdots, a_p) \in \mathbb{R}^p \tag{2.1}$$

and multiple performance measures:

$$\vec{y} = (y_1, y_2, \cdots, y_q) \in \mathbb{R}^q. \tag{2.2}$$

Assume that there are at least as many performance measures $(y_1, y_2, \cdots, y_q)$ as the number of tasks $(a_1, a_2, \cdots, a_p)$: $p \leq q$.

Following the two-stage optimization approach of Grossman and Hart (1983), this study analyzes the efficient contract, which is to induce an exogenously fixed arbitrary effort level at the minimum cost. The optimal contract, which is to induce the endogenously determined optimal effort level, is regarded as a special case of the efficient contract. Assume that the agent's utility function has an additively separable action cost :

$$U^a\,(\,C(\vec{y}),\,\vec{a}\,) = U\,(\,C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)\,) - v\,(\,a_1,\,a_2,\,\cdots\,,\,a_p\,)\,, \tag{2.3}$$

where $C\,(\cdot)$ denotes the contract, $U\,(\cdot)$ is the agent's utility of consumption function, and $v\,(\cdot)$ is the agent's cost of action. The agent's expected utility is as follows :

$$\begin{aligned} E\,[\,U^a\,(\,C(\vec{y}),\,\vec{a}\,)\,] = \int \cdots \int & U\,(\,C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)\,)\,d\,(\vec{y}\,|\,\vec{a})\,dy_1 \cdots dy_q \\ & - v\,(\,a_1,\,a_2,\,\cdots\,,\,a_p\,)\,, \end{aligned} \tag{2.4}$$

where

$$d\,(\vec{y}\,|\,\vec{a}) = d\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,|\,a_1,\,a_2,\,\cdots\,,\,a_p\,) \tag{2.5}$$

is the joint density function of the basic measures $(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)$. Without loss of generality, the participation constraint is :

$$E\,[\,U^a\,(\,C(\vec{y}),\,\vec{a}\,)\,] \geq 0\,, \tag{2.6}$$

and the incentive compatibility constraint, taking the first order approach, is :

$$\begin{aligned} & \nabla_a E\,[\,U^a\,(\,C(\vec{y}),\,\vec{a}\,)\,] \\ & = \textstyle\int \cdots \int U\,(\,C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)\,)\,\nabla_a d\,(\vec{y}\,|\,\vec{a})\,dy_1 \cdots dy_q - \nabla_a v\,(\,a_1,\,a_2,\,\cdots\,,\,a_p\,) \\ & = 0\,, \end{aligned}$$

$$(2.7)$$

where ∇_a denotes the gradient vector of partial derivatives $(\partial/\partial a_1, \partial/\partial a_2, \cdots, \partial/\partial a_p)$.[1] The efficient contract for inducing an exogenously fixed effort level $(a_1, a_2, \cdots, a_p)$ is obtained by minimizing the expected compensation:

$$E[C(y_1, y_2, \cdots, y_q)] = \int \cdots \int C(y_1, y_2, \cdots, y_q)\, d(\vec{y}\,|\,\vec{a})\, dy_1 \cdots dy_q\,, \quad (2.8)$$

subject to the participation constraint (2.6) and the incentive compatibility constraint (2.7).

$$\max_{C,\lambda,\vec{\mu}} \mathcal{L} = -E[C(y_1, y_2, \cdots, y_q)] + \lambda E[U^a(C(\vec{y}), \vec{a})] + \vec{\mu} \cdot \nabla_a E[U^a(C(\vec{y}), \vec{a})]\,, \quad (2.9)$$

where λ is a Kuhn-Tucker multiplier and $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_p)$ is a vector of Lagrange multipliers. The first order condition with respect to the contract $C(y_1, y_2, \cdots, y_q)$ is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C} = \; & -\int \cdots \int d(\vec{y}\,|\,\vec{a})\, dy_1 \cdots dy_q \\ & +\lambda \int \cdots \int U'(C(y_1, y_2, \cdots, y_q))\, d(\vec{y}\,|\,\vec{a})\, dy_1 \cdots dy_q \\ & +\vec{\mu} \cdot \int \cdots \int U'(C(y_1, y_2, \cdots, y_q))\, \nabla_a d(\vec{y}\,|\,\vec{a})\, dy_1 \cdots dy_q \; = \; 0\,. \end{aligned} \quad (2.10)$$

Therefore, the efficient contract $C(y_1, y_2, \cdots, y_q)$ in inducing an exogenously fixed effort level $(a_1, a_2, \cdots, a_p)$ is characterized by:

$$\frac{1}{U'(C(y_1, y_2, \cdots, y_q))} = \lambda + \vec{\mu} \cdot L(\vec{y}\,|\,\vec{a})\,, \quad (2.11)$$

where

$$L(\vec{y}\,|\,\vec{a}) = \frac{\nabla_a d(\vec{y}\,|\,\vec{a})}{d(\vec{y}\,|\,\vec{a})} \quad (2.12)$$

[1] The first order approach is justified if local incentive compatibility implies global incentive compatibility, which is not always the case (Christensen and Feltham 2005, Page 66).

is the likelihood ratio of the joint density function $d\,(\vec{y}\,|\,\vec{a})$ (2.5). From (2.11), it is apparent that the likelihood ratio $L\,(\vec{y}\,|\,\vec{a})$ (2.12) represents the statistical properties of the basic performance measures $\vec{y} = (\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ for incentive contracting purposes.

The efficient contract $C\,(\vec{y})$ in (2.11) depends on the characteristics of the agency participants (the agent's utility function $U\,(\cdot)$) and the economic condition of the agency (the effort level to induce $\vec{a} = (\,a_1,\, a_2,\, \cdots\,,\, a_p\,)$). The denominator of the left hand side of (2.11) depends on the agent's utility function $U\,(\cdot)$ and the numerator results from the risk neutrality of the principal. The right hand side of (2.11) depends on the effort level to induce $\vec{a}$, which in turn depends on the economic condition of the agency, through the likelihood ratio $L\,(\vec{y}\,|\,\vec{a})$ and the vector of Lagrange multipliers $\vec{\mu}$. The vector of Lagrange multipliers $\vec{\mu}$ in (2.11) is endogenous and generally cannot be explicitly solved out. As the left hand side of (2.11) depends on the agent's utility function $U\,(\cdot)$, the vector of Lagrange multipliers $\vec{\mu}$ generally depends on the agent's utility function $U\,(\cdot)$ as well as the effort level to induce $\vec{a}$.

2.2.1 Hierarchy of sufficiency concepts for aggregation

The literature has discussed several sufficiency concepts for aggregation of performance measures: statistical sufficiency for all effort levels ([1]), statistical sufficiency for a single effort level ([2]), economic sufficiency for an arbitrary effort level ([3]), and economic sufficiency for the optimal effort level ([4]). The four sufficiency concepts for aggregation of performance measures are in the order of diminishing requirement ([1] $\Rightarrow$ [2] $\Rightarrow$ [3] $\Rightarrow$ [4]). Amershi, Banker, and Datar (1990) discuss statistical sufficiency for all effort levels ([1]) and economic sufficiency for the optimal effort level ([4]) in the basic setting of a single-task and single-period agency, but statistical sufficiency for a single effort level ([2]) and economic sufficiency for an arbitrary effort level ([3]) are not discussed in Amershi, Banker, and Datar (1990).

Statistical sufficiency for a single effort level ([2]) and economic sufficiency for an arbitrary effort level ([3]) involve an exogenously fixed (single) effort level to induce. As Şabac (2009) discusses, the dual roles, imposed on performance measures, of the optimal decision making (the optimal effort level to induce) and the efficient use of information (the sufficient aggregation of performance measures) become more difficult to disentangle, as institutionally richer settings such as multi-task settings are employed. Using an exogenously fixed (single) effort level ([2] and [3]) makes it possible to separate the dual roles of performance measures and helps one to concentrate on the efficient use of information. Moreover, using an exogenously fixed (single) effort level ([2] and [3]) is relevant to the analysis in decentralized agencies. For example, when the headquarters decide the production level of a division and the division figures out the optimal way of attaining the production level, an exogenously fixed effort level is relevant to the decision problem of the division.

While the problem of inducing the agent's effort generally depends on the agent's utility function $U^a(\cdot)$ and the agent's effort level to induce $\vec{a}$, the first question on the aggregation of performance measures is whether it is possible to construct an aggregate measure with no loss of information regardless of the characteristics of the agency participants (the agent's utility function $U^a(\cdot)$) *and* the economic condition of the agency (the effort level to induce $\vec{a} = (a_1, a_2, \cdots, a_p)$). The concept of statistical sufficiency represents no loss of information in the aggregation process for all utility functions $U^a(\cdot)$ and for all effort levels to induce $\vec{a}$. In fact, the concept of statistical sufficiency is not restricted to agencies and a statistically sufficient aggregation incurs no loss of information for *all* kinds of decision problems, including agencies.

An analogy of the principal inferring the agent's unobservable effort level to a statistician inferring the unknown parameter values (in this case the effort $\vec{a}$) is helpful. If a set of aggregate measures exists that contains all the information from the basic measures, the statistician can work with a simpler set of signals and incur no loss of information in drawing inferences concerning the unknown parameters. Such a set of aggregate measures is statistically sufficient for the basic measures with respect to the parameters. Provided with a set of statistically sufficient aggregate measures, which is by nature "utility-independent" and "effort-independent,"the principal incurs no loss of information on the agent's unobservable effort level in the aggregation process.

Statistical sufficiency for all effort levels ([1]) is defined in terms of aggregate measures being sufficient statistics for the basic measures $\vec{y} = (\, y_1,\, y_2,\, \cdots\, ,\, y_q\,)$ with respect to the agent's unobservable effort $\vec{a} = (\, a_1, a_2, \cdots , a_p\,) \in \mathbb{R}^p$. Amershi and Hughes (1989) discuss the role of a one-dimensional sufficient statistic for inducing the agent's effort. In this study, the concept of statistical sufficiency is not limited to a one-dimensional aggregate measure but extended to multi-dimensional aggregate measures.

Definition 1 *(Statistical sufficiency for all effort levels* [1] *)*

An aggregation is statistically sufficient for all effort levels if, and only if, there are non-trivial jointly sufficient statistics $\overrightarrow{T\,(\vec{y})}$ *of the basic measures* $\vec{y} = (\, y_1,\, y_2,\, \cdots\, ,\, y_q\,)$ *:*

$$\overrightarrow{T\,(\vec{y})} = (\, T_1\,(\vec{y})\, ,\, T_2\,(\vec{y})\, ,\, \cdots\, ,\, T_j\,(\vec{y})\,)\, , \;\; j \leq q\, . \tag{2.13}$$

For $j = 1$*,* $T\,(\vec{y})$ *is a one-dimensional sufficient statistic of the basic measures.*

When an aggregation is statistically sufficient for all effort levels, the aggregate measures are determined only by the statistical properties of the basic measures. That is, the *same* aggregation $\overrightarrow{T\,(\vec{y})} = (\, T_1\,(\vec{y})\, ,\, T_2\,(\vec{y})\, ,\, \cdots\, ,\, T_j\,(\vec{y})\,)$ is good enough to substitute for the basic measures $\vec{y} = (\, y_1,\, y_2,\, \cdots\, ,\, y_q\,)$ for *all* utility functions $U^a(\cdot)$ and for *all* effort levels to induce $\vec{a}$. The existence of sufficient statistics $\overrightarrow{T\,(\vec{y})}$ is shown by the factorization criterion (DeGroot 1986, Page 364 ; 1970, Page 156).

Definition 2 *(Sufficient statistic ; factorization criterion)*

The statistics $\overrightarrow{T\,(\vec{y})} = (\, T_1\,(\vec{y})\, ,\, T_2\,(\vec{y})\, ,\, \cdots\, ,\, T_j\,(\vec{y})\,)$ *are jointly sufficient for the basic measures* $\vec{y} = (\, y_1,\, y_2,\, \cdots\, ,\, y_q\,)$ *if, and only if, the joint density function* $d\,(\vec{y}\,|\,\vec{a})$ *(2.5) is factorized for all effort levels* $\vec{a} = (\, a_1, a_2, \cdots , a_p\,) \in \mathbb{R}^p$ *:*

$$d\,(\vec{y}\,|\,\vec{a}) = u\,(\, y_1,\, y_2,\, \cdots\, ,\, y_q\,)\; v\,(\, T_1\,(\vec{y})\, ,\, T_2\,(\vec{y})\, ,\, \cdots\, ,\, T_j\,(\vec{y})\, ;\, a_1,\, a_2,\, \cdots\, ,\, a_p\,)\, , \tag{2.14}$$

where $u(\cdot)$ and $v(\cdot)$ are non-negative functions. The function $v(\cdot)$ depends on the basic measures $(y_1, y_2, \cdots, y_q)$ only through $\overrightarrow{T(\vec{y})} = (T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$.

In particular, a one-dimensional statistic $T(\vec{y})$ is sufficient for the basic measures $\vec{y} = (y_1, y_2, \cdots, y_q)$ if, and only if, the factorization criterion (2.14) is satisfied with $T(\vec{y})$ for all effort levels $\vec{a} = (a_1, a_2, \cdots, a_p) \in \mathbb{R}^p$ such that the function $v(\cdot)$ depends on the basic measures $(y_1, y_2, \cdots, y_q)$ only through $T(\vec{y})$.

The basic measures $(y_1, y_2, \cdots, y_q)$ are trivial jointly sufficient statistics as the basic measures themselves trivially satisfy the factorization criterion (2.14). The discussion in the sequel excludes the trivial jointly sufficient statistics since they do not help in reducing the complexity of an information system.

A statistically sufficient aggregation is characterized by no loss of information in the aggregation process for all utility functions $U^a(\cdot)$ and for all effort levels to induce $\vec{a}$, and actually for all kinds of decision problems. The aggregate measures $(T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$ are jointly sufficient statistics for the basic measures $(y_1, y_2, \cdots, y_q)$ with respect to the agent's unobservable effort $(a_1, a_2, \cdots, a_p)$ if observing the aggregate measures without observing the basic measures is as useful as observing the basic measures in estimating *all* levels of the agent's effort. All the information relevant to estimating the agent's unobservable effort $(a_1, a_2, \cdots, a_p)$ is preserved and passed onto the sufficient statistics $(T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$ from the basic measures $(y_1, y_2, \cdots, y_q)$.

While preserving all the relevant information, sufficient statistics enable an information system to work with a simpler set of measures than the basic measures. As the value of sufficient statistics comes from simplicity, the simplest sufficient statistics are of interest. The minimal sufficient statistics are the simplest set of jointly sufficient statistics and defined as follows (DeGroot 1986, Page 368).

Definition 3 *(Minimal statistical sufficiency)*

A set of sufficient statistics is minimal if it can be represented as a function of any other sufficient statistics.

Obviously, when a one-dimensional sufficient statistic exists, the minimal sufficient statistic is one-dimensional.

Although statistical sufficiency of aggregation is clearly defined in terms of sufficient statistics, the requirement is excessively strong for the effort-inducement purpose in an agency context. Gjesdal (1981) shows that the incentive (agency) objective and the decision-making objective do not give rise to identical rankings of information systems. For the purpose of controlling the agent's effort, the required information may be different from that for decision-making purposes.

Whereas sufficient statistics incur no loss of information for all kinds of decision problems as well as for all kinds of the agent's utility functions $U^a(\cdot)$ and all effort levels to induce $\vec{a}$, the problem of aggregating performance measures often arises in an agency setting. In addition, the principal may aim at inducing a certain level of effort rather than all effort levels from the agent.

Thus, it is natural to ask whether it is possible to construct an aggregate measure with no loss of information in inducing a single effort level, rather than all effort levels, in an agency context. In particular, the question is whether a set of aggregate measures, which contains all the information from the basic measures regardless of the characteristics of the agency participants (the agent's utility function $U^a(\cdot)$), exists when the agent's induced effort is fixed at a single level. Such a set of aggregate measures is statistically sufficient for a single effort level as opposed to all effort levels. With a set of "utility-independent" aggregate measures, the principal incurs no loss of information in the aggregation process for inducing a single effort level. Statistical sufficiency for a single effort level ([2]) is attained by a set of aggregate measures, which is statistically sufficient for a single effort level.

Definition 4 *(Statistical sufficiency for a single effort level* [2] *)*

For inducing a single effort level $\vec{a} = (\, a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$, an aggregation is statistically sufficient if there is a set of jointly sufficient implementation statistics $\overrightarrow{\psi\,(\vec{y};\,\vec{a})}$ of the basic measures $\vec{y} = (\, y_1,\, y_2,\, \cdots,\, y_q\,)$:

$$\overrightarrow{\psi\,(\vec{y};\,\vec{a})} = (\, \psi_1\,(\vec{y};\,\vec{a})\,,\, \psi_2\,(\vec{y};\,\vec{a})\,,\, \cdots,\, \psi_j\,(\vec{y};\,\vec{a})\,)\,,\;\; j \le q\,. \tag{2.15}$$

For $j = 1$, $\psi(\vec{y};\, \vec{a})$ is a one-dimensional sufficient implementation statistic of the basic measures for inducing a single effort level.

Following Christensen and Feltham (2005, Page 101), jointly sufficient implementation statistics are defined in terms of the likelihood ratio $L(\vec{y}\,|\,\vec{a})$ (2.12) as follows:

Definition 5 *(Sufficient implementation statistic)*

For inducing a single effort level $\vec{a} = (\, a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$, the statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})} = (\, \psi_1(\vec{y};\, \vec{a}),\, \psi_2(\vec{y};\, \vec{a}),\, \cdots,\, \psi_j(\vec{y};\, \vec{a})\,)$ are jointly sufficient implementation statistics if for all $\psi \in \Psi \subset \mathbb{R}^j$ it holds that

$$L(\vec{y}'\,|\,\vec{a}) = L(\vec{y}''\,|\,\vec{a}),\ \forall\, (\vec{y}',\, \vec{y}'') \in \vec{Y}(\psi) \equiv \{\vec{y} \in \vec{Y}\,|\,\overrightarrow{\psi(\vec{y};\, \vec{a})} = \psi\}\,. \tag{2.16}$$

Definition 5 is satisfied if, and only if, the likelihood ratio $L(\vec{y}\,|\,\vec{a})$ can be written as a function $q(\cdot)$, which depends on the basic measures $\vec{y} = (\, y_1,\, y_2,\, \cdots,\, y_q\,)$ only through a set of jointly sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})} = (\, \psi_1(\vec{y};\, \vec{a}),\, \psi_2(\vec{y};\, \vec{a}),\, \cdots,\, \psi_j(\vec{y};\, \vec{a})\,)$:

$$L(\vec{y}\,|\,\vec{a}) = \frac{\nabla_a d(\vec{y}\,|\,\vec{a})}{d(\vec{y}\,|\,\vec{a})} = q\,(\, \psi_1(\vec{y};\, \vec{a}),\, \psi_2(\vec{y};\, \vec{a}),\, \cdots,\, \psi_j(\vec{y};\, \vec{a})\,)\,. \tag{2.17}$$

In particular, for inducing a single effort level $\vec{a} = (\, a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$, if condition (2.17) is satisfied with a one-dimensional statistic $\overrightarrow{\psi(\vec{y};\, \vec{a})} = \psi(\vec{y};\, \vec{a})$, then $\psi(\vec{y};\, \vec{a})$ is a one-dimensional sufficient implementation statistic.

The key qualification for a set of jointly sufficient implementation statistics is that it should generate the same likelihood ratio as the basic measures in inducing a single fixed effort level; the likelihood ratio can be written as a function of a set of jointly sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})}$ as in (2.17). A set of sufficient implementation statistics can induce a single effort level across agencies regardless of the agent's utility function $U^a(\cdot)$. In the characterization of the efficient contract (2.11), the vector of Lagrange multipliers $\vec{\mu}$ generally depends on the agent's utility function $U(\cdot)$. When the likelihood ratio $L(\vec{y}\,|\,\vec{a})$ is represented as a function of a set of jointly sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})}$ as in (2.17), the set of jointly sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})}$ substitutes for the basic measures $\vec{y}$ regardless of the agent's utility function $U(\cdot)$ in (2.11).

Statistical sufficiency of aggregation for all effort levels implies statistical sufficiency for a single effort level ([1] $\Rightarrow$ [2]), but the converse is not true. A set of jointly sufficient statistics $\overrightarrow{T(\vec{y})} = (\, T_1(\vec{y})\,,\, T_2(\vec{y})\,,\, \cdots\,,\, T_j(\vec{y})\,)$ is always a set of jointly sufficient implementation statistics.

Proposition 1 ([1] $\Rightarrow$ [2])

If an aggregation of performance measures is statistically sufficient for all effort levels, then the aggregation is statistically sufficient for any single fixed effort level.

If there is no loss of information from the aggregation process in inducing all effort levels, then there is no loss of information in inducing any single effort level.

While statistical sufficiency of aggregation requires no loss of information in the aggregation process, economic sufficiency of aggregation requires no economic loss to the principal although there can be some loss of information in the aggregation process. Now, the question on the aggregation of performance measures is whether it is possible to construct an aggregate measure in inducing a single effort level with no economic loss to the principal as opposed to no loss of information. Such an aggregate measure is economically sufficient and can depend both on the characteristics of the agency participants (the agent's utility function $U^a(\cdot)$) and on the economic condition of the agency (the effort level to induce $\vec{a}$). Economic sufficiency of aggregation ([3] and [4]) is attained by a set of aggregate measures, which is economically sufficient for a single effort level.

Definition 6 *(Economic sufficiency for an arbitrary effort level* [3] *)*

An aggregation of performance measures is economically sufficient for a single effort level if there exist a set of aggregate measures :

$$\overrightarrow{\zeta(\vec{y}\,;\, U^a(\cdot), \vec{a})} = \Big(\, \zeta_1(\vec{y}\,;\, U^a(\cdot), \vec{a})\,,\, \zeta_2(\vec{y}\,;\, U^a(\cdot), \vec{a})\,,\, \cdots\,,\, \zeta_j(\vec{y}\,;\, U^a(\cdot), \vec{a})\,\Big) \quad (2.18)$$

and a function $l(\cdot)$ such that the set of aggregate measures $\overrightarrow{\zeta(\vec{y}\,;\, U^a(\cdot), \vec{a})}$ can be *substituted for the basic measures* $\vec{y} = (\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ *in the efficient contract to induce a single effort level* $(\,a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$ *:*

$$C\,(\,\vec{y}\,) = l\left(\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}\right), \tag{2.19}$$

where the function $l\,(\cdot)$ *depends on the basic measures* $\vec{y} = (\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ *only through the aggregate measures* $\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}$*.*

A set of economically sufficient aggregate measures $\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}$ can depend on the agent's utility function $U^a(\cdot)$ and the effort level to induce $\vec{a}$. In the characterization of the efficient contract (2.11), the term $\vec{\mu} \cdot L\,(\vec{y}\,|\,\vec{a})$ depends on the agent's utility function $U\,(\cdot)$ as well as the effort level to induce $\vec{a}$ because the vector of Lagrange multipliers $\vec{\mu}$ generally depends on the agent's utility function $U\,(\cdot)$ and the effort level to induce $\vec{a}$.

For inducing a single effort level, the principal is as well off with a set of economically sufficient aggregate measures $\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}$ as with the basic measures $\vec{y} = (\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$. When an aggregation is economically sufficient, the aggregate measure of the basic measures can vary depending on the agent's utility function $U^a(\cdot)$ in inducing a single effort level. That is, *some* aggregation $\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}$ may be good enough to substitute for the basic measures $\vec{y} = (\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ in the efficient contract for inducing a single effort level $\vec{a} = (\,a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$, but the aggregation can depend on the agent's utility function $U^a(\cdot)$, which is an agency-specific factor.

Whereas statistically sufficient aggregate measures (sufficient statistics $\overrightarrow{T\,(\vec{y})}$ and sufficient implementation statistics $\overrightarrow{\psi\,(\vec{y};\, \vec{a})}$) are independent of the agent's utility function $U^a(\cdot)$ and thus agency-independent, relaxing the requirement of no information loss may allow economically sufficient aggregate measures $\overrightarrow{\zeta\,(\vec{y}\,;\, U^a(\cdot), \vec{a})}$ to depend on the agent's utility function $U^a(\cdot)$. As the agent's utility function $U^a(\cdot)$ is agency-specific, an economically sufficient aggregation is, by nature, specific to an agency as opposed to the agency-independence of a statistically sufficient aggregation.

Statistical sufficiency of aggregation for a single effort level implies economic sufficiency of aggregation for the single effort level ([2] $\Rightarrow$ [3]), but the converse is not true.

Proposition 2 ([2] $\Rightarrow$ [3])

Assume that the agent's utility function has a separable action cost. For inducing an exogenously fixed single effort level, if an aggregation of performance measures is statistically sufficient, then the aggregation is economically sufficient.

Proposition 2 shows that if the agent's utility function has a separable action cost (additively or multiplicatively), a set of jointly sufficient implementation statistics $\overrightarrow{\psi\,(\vec{y};\,\vec{a})} = (\,\psi_1\,(\vec{y};\,\vec{a})\,,\,\psi_2\,(\vec{y};\,\vec{a})\,,\,\cdots\,,\,\psi_j\,(\vec{y};\,\vec{a})\,)$ for a single effort level $\vec{a} = (\,a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$ is always economically sufficient for the single effort level. If there is no loss of information from an aggregation process in inducing a single effort level, then the principal can, with no economic loss, replace the basic measures with the aggregate measures in an efficient contract for inducing the single effort level.

Finally, the aggregation of performance measures $\overrightarrow{\zeta\,(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger)}$ in inducing the second-best effort level $\vec{a}^\dagger = \left(a_1^\dagger, a_2^\dagger, \cdots, a_p^\dagger\right) \in \mathbb{R}^p$ is a special case of the economically sufficient aggregation $\overrightarrow{\zeta\,(\vec{y}\,;\,U^a(\cdot),\vec{a})}$ in inducing an arbitrary effort level $\vec{a} = (\,a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$.

Definition 7 *(Economic sufficiency for the optimal effort level* [4] *)*

For inducing the endogenously determined optimal effort level $\vec{a}^\dagger = \left(a_1^\dagger, a_2^\dagger, \cdots, a_p^\dagger\right) \in \mathbb{R}^p$*, an aggregation of performance measures is economically sufficient if there exist a set of aggregate measures :*

$$\overrightarrow{\zeta\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)} = \left(\,\zeta_1\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right),\,\zeta_2\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right),\,\cdots\,,\,\zeta_j\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)\,\right) \tag{2.20}$$

and a function $l^\dagger\,(\cdot)$ *such that the set of aggregate measures* $\overrightarrow{\zeta\,(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger)}$ *can be* *substituted for the basic measures* $\vec{y} = (\, y_1,\, y_2, \cdots, \, y_q \,)$ *in the optimal contract* $C^\dagger$ *:*

$$C^\dagger\,(\,\vec{y}\,) = l^\dagger \left(\overrightarrow{\zeta\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)} \right), \tag{2.21}$$

where the function $l^\dagger\,(\cdot)$ *depends on the basic measures* $\vec{y} = (\, y_1,\, y_2, \cdots, \, y_q \,)$ *only through the aggregate measures* $\overrightarrow{\zeta\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)}$*.*

As the optimal contract $C^\dagger\,(\,\vec{y}\,)$, which implements the second-best effort level $\vec{a}^\dagger$, is a special case of the efficient contract $C\,(\,\vec{y}\,)$, an economically sufficient aggregation $\overrightarrow{\zeta\,(\vec{y}\,;\,U^a(\cdot),\vec{a})}$ for an arbitrary single effort level always implies an economically sufficient aggregation $\overrightarrow{\zeta\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)}$ for the optimal effort level ([3] $\Rightarrow$ [4]). As the optimal effort level to induce $\vec{a}^\dagger$ depends on the economic condition of the agency and the agent's utility function $U^a(\cdot)$, an economically sufficient aggregation $\overrightarrow{\zeta\left(\vec{y}\,;\,U^a(\cdot),\vec{a}^\dagger\right)}$ for the optimal effort level is agency-specific.

2.2.2 Joint distributions of performance measures

This section introduces the joint distributions of performance measures for the analysis in the sequel. If the sufficient aggregation is linear in the basic measures, not only the analysis is more tractable but also the relative incentive rate on the basic measures is defined. Banker and Datar (1989) use a family of joint distributions, under which the economically sufficient aggregation for the optimal effort level ([4]) is linear in the basic measures. This study introduces a subfamily of joint distributions used in Banker and Datar (1989) and Amershi, Banker, and Datar (1990), under which every sufficient aggregation ([1], [2], [3], [4]) is linear in the basic measures.

With q-dimensional basic measures $\vec{y}_q = (\, y_1,\, y_2, \cdots, \, y_q \,) \in \mathbb{R}^q$ and p-dimensional tasks $\vec{a}_p = (\, a_1,\, a_2, \cdots, \, a_p \,) \in \mathbb{R}^p$ $(\, p \leq q \,)$, the exponential family of joint density functions (DeGroot 1970, Page 161) can be written as follows:

$$\rho(\vec{y}_q;\, \vec{a}_p) = exp\Big[\, \sum_{i=1}^{k} g_i(\vec{a}_p)\, h_i(\vec{y}_q) + \alpha(\vec{a}_p) + \beta(\vec{y}_q) \,\Big], \tag{2.22}$$

where $g_i(\cdot)$, $h_i(\cdot)$, $\alpha(\cdot)$, and $\beta(\cdot)$ are functions. If the likelihood ratio (2.12) is linear in the basic measures $\vec{y}_q = (\, y_1,\, y_2, \cdots ,\, y_q\,)$, the analysis becomes more tractable. If $h_i(\vec{y}_q) = y_i$ for each i in (2.22), then the likelihood ratio becomes linear in the basic measures. With $k = q$ and $h_i(\vec{y}_q) = y_i$, (2.22) can be written as follows:

$$\rho(\vec{y}_q;\, \vec{a}_p) = exp\Big[\, \sum_{i=1}^{q} g_i(\vec{a}_p)\, y_i + \alpha(\vec{a}_p) + \beta(\vec{y}_q)\,\Big], \tag{2.23}$$

and the likelihood ratio is linear in the basic measures $\vec{y}_q = (\, y_1,\, y_2, \cdots ,\, y_q\,)$ when the joint distribution of the basic measures belongs to the family (2.23).

A special case of the family (2.23), with two performance measures y and z and a single task $a \in \mathbb{R}$, is used in Banker and Datar (1989) and Amershi, Banker, and Datar (1990):

$$\phi(y, z;\, a) = exp\,[\, p(a)\, y + q(a)\, z - r(a) + s(y) + t(z - \gamma\, y)\,]\,, \tag{2.24}$$

where $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, $s(\cdot)$, and $t(\cdot)$ are arbitrary functions and γ is a scalar parameter. The family of joint density functions $\phi(y, z;\, a)$ (2.24) includes many common distributions such as (truncated) normal, exponential, gamma, chi-square, and inverse Gaussian. Banker and Datar (1989) show that when the joint distribution of the basic measures y and z belongs to the family (2.24), the economically sufficient aggregation for the optimal effort level ([4]) can be restricted, without loss of generality, to the linear aggregation of the basic measures y and z.

To characterize minimal sufficient statistics, the analysis may be restricted to a subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ of $\rho(\vec{y}_q;\, \vec{a}_p)$ (2.23):

$$\varphi(\vec{y}_q;\, \vec{a}_p) = exp\left[\, \sum_{i=1}^{q} g_i(\vec{a}_p)\, y_i + \alpha(\vec{a}_p) + \beta(\vec{y}_q)\,\right], \tag{2.25}$$

in which the effort-dependent coefficient $g_i(\vec{a}_p)$ is linear in the tasks

$\vec{a}_p = (\, a_1,\, a_2, \cdots,\, a_p\,)$ such that:

$$\begin{bmatrix} g_1(\vec{a}_p) \\ g_2(\vec{a}_p) \\ g_3(\vec{a}_p) \\ \vdots \\ g_q(\vec{a}_p) \end{bmatrix} = \Omega_{qp}\, \vec{a}_p = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1p} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2p} \\ \omega_{31} & \omega_{32} & \cdots & \omega_{3p} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{q1} & \omega_{q2} & \cdots & \omega_{qp} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}. \tag{2.26}$$

The analysis in later sections will show that when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), every sufficient aggregation ([1], [2], [3], [4]) is linear in the basic measures.

The joint normal density function $f\,(\vec{y}_q;\, \vec{a}_p)$ belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25),[2] if the agent's action affects the expected value but not the variance-covariance of the basic measures such that:

$$\vec{y}_q = M_{qp}\, \vec{a}_p + \vec{\varepsilon}_q\,, \tag{2.29}$$

where M_{qp} is the sensitivity matrix of the basic measures $(\, y_1,\, y_2, \cdots,\, y_q\,)$ with respect to the effort $(\, a_1,\, a_2, \cdots,\, a_p\,)$, and $\vec{\varepsilon}_q$ is the vector of error terms. If necessary for a more explicit analysis, the following joint normal density function with two basic measures y and z for two tasks (a_1, a_2) can be employed:

$$f(y, z;\, \vec{a}) = \frac{1}{2\pi\sqrt{1-\rho^2_{\varepsilon\delta}}\,\sigma_1\,\sigma_2}\, exp\left[-\frac{1}{2(1-\rho^2_{\varepsilon\delta})}\,\Xi\right], \tag{2.30}$$

[2] The joint normal density function is as follows:

$$f\,(\, y_1,\, y_2, \cdots,\, y_q;\, a_1,\, a_2, \cdots,\, a_p\,) = \frac{1}{(2\,\pi)^{\frac{q}{2}}\,\sqrt{det(\Sigma_{qq})}}\, exp\left[-\frac{1}{2}\,\vec{x}_q^T\,\Sigma_{qq}^{-1}\,\vec{x}_q\right], \tag{2.27}$$

where $\vec{x}_q = \vec{y}_q - M_{qp}\,\vec{a}_p$, Σ_{qq} is the variance-covariance matrix of the basic measures $(\, y_1,\, y_2, \cdots,\, y_q\,)$, the superscript T denotes the transpose of the vector, and $det(\cdot)$ denotes the determinant of a matrix. As the input value of the exponential function in (2.27) is decomposed as follows:

$$\vec{x}_q^T\,\Sigma_{qq}^{-1}\,\vec{x}_q = \vec{y}_q^T\,\Sigma_{qq}^{-1}\,\vec{y}_q + (M_{qp}\,\vec{a}_p)^T\,\Sigma_{qq}^{-1}\,(M_{qp}\,\vec{a}_p) - 2\,\vec{y}_q^T\left(\Sigma_{qq}^{-1}\,M_{qp}\right)\vec{a}_p\,, \tag{2.28}$$

the term $\vec{y}_q^T\left(\Sigma_{qq}^{-1}\,M_{qp}\right)\vec{a}_p$ in (2.28) satisfies the linearity condition (2.26).

where $\Xi = (y - \vec{m} \cdot \vec{a})^2 / \sigma_1^2 - 2\,\rho_{\varepsilon\delta}\,(y - \vec{m} \cdot \vec{a})\,(z - \vec{k} \cdot \vec{a}) / (\sigma_1\,\sigma_2) + (z - \vec{k} \cdot \vec{a})^2 / \sigma_2^2$. The basic measures y and z are given by the agent's two-dimensional effort $\vec{a} = (a_1, a_2)$ together with the sensitivity vectors $\vec{m} = (m_1, m_2)$ and $\vec{k} = (k_1, k_2)$ as follows:

$$y = m_1\,a_1 + m_2\,a_2 + \varepsilon\,, \tag{2.31}$$

$$z = k_1\,a_1 + k_2\,a_2 + \delta\,, \tag{2.32}$$

where $\varepsilon \sim N(0, \sigma_1^2)$, $\delta \sim N(0, \sigma_2^2)$, and $Cov(y, z) = \rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2$.

2.3 Sufficient statistics in multi-task settings

Using a richer institutional setting of a multi-task agency puts more restrictions on the aggregation of performance measures than in a single-task case. Whereas Amershi, Banker, and Datar (1990) show in a single-task setting that it is possible to construct a one-dimensional "utility-independent" and "effort-independent" aggregate measure, the analysis of this section shows in a multi-task setting that a one-dimensional "utility-independent" and "effort-independent" aggregation of performance measures is infeasible in general. Instead, this section shows that statistical sufficiency of aggregation can be achieved only by multiple aggregate measures, whose minimum dimensionality is given by the number of tasks. In general, the dimension of statistically sufficient aggregate measures should be no less than the number of tasks.

The analysis begins with a setting of two performance measures $(\,y,\,z\,)$ and two tasks $\vec{a} = (\,a_1,\,a_2\,) \in \mathbb{R}^2$. With Definition 1 and Definition 2, *one-dimensional* statistical sufficiency for all effort levels in a two-task setting would be achieved by a one-dimensional sufficient statistic $T\,(y, z)$ if, and only if, the joint density function $d\,(y, z;\,a_1, a_2)$ can be factorized for all effort levels $(a_1, a_2) \in \mathbb{R}^2$:

$$d\,(y, z;\,a_1, a_2) = g\,(y, z)\;h\,(T\,(y, z)\,;\,a_1, a_2)\,, \tag{2.33}$$

where $g\,(\cdot)$ and $h\,(\cdot)$ are non-negative functions. If the joint distribution of the basic measures y and z is restricted to the family of joint density functions (2.24):

$$\phi(y, z;\, a_1, a_2) = exp\left[\, p(a_1, a_2)\, y + q(a_1, a_2)\, z - r(a_1, a_2) + s(y) + t(z - \gamma\, y)\,\right], \tag{2.34}$$

then the feasibility of the factorization (2.33) depends on whether the aggregate measure $p(a_1, a_2)\, y + q(a_1, a_2)\, z$ is factorized to be a function of a one-dimensional sufficient statistic $T\,(y, z)$ for all effort levels $(a_1, a_2) \in \mathbb{R}^2$:

$$p(a_1, a_2)\, y + q(a_1, a_2)\, z = h\,(T\,(y, z)\,;\, a_1, a_2)\,, \tag{2.35}$$

where $h\,(\cdot)$ is a non-negative function. For (2.35) to hold, the effort-dependent coefficients $p(a_1, a_2)$ and $q(a_1, a_2)$ should be factored so that the function $h\,(\cdot)$ depends on y and z only through a one-dimensional statistic $T\,(y, z)$, which is independent of the effort level $(a_1, a_2) \in \mathbb{R}^2$. Generally, (2.35) does not hold for the family of joint density functions (2.34). This simple example shows that in general, a one-dimensional sufficient statistic does not exist for multi-task agencies.

To see why a one-dimensional sufficient statistic does not exist in a multi-task setting, consider a particular case, in which the joint distribution of the basic measures y and z belongs to the subfamily $\varphi(y, z;\, a_1, a_2)$ in (2.25) such that the effort-dependent coefficients $p(a_1, a_2)$ and $q(a_1, a_2)$ are linear in the effort levels a_1 and a_2:

$$\begin{aligned} \varphi(y, z;\, a_1, a_2) &= exp\begin{bmatrix} p(a_1, a_2)\, y + q(a_1, a_2)\, z \\ \quad -r(a_1, a_2) + s(y) + t(z - \gamma\, y) \end{bmatrix}, && (2.36) \\ p(a_1, a_2) &= c_1\, a_1 + c_2\, a_2\,, && (2.37) \\ q(a_1, a_2) &= d_1\, a_1 + d_2\, a_2\,, && (2.38) \end{aligned}$$

where c_1, c_2, d_1, d_2 are arbitrary constants. The subfamily $\varphi(y, z;\, a_1, a_2)$ (2.36) of joint density functions can be factorized:

$$\begin{aligned}\varphi(y,z;\,a_1,a_2) =& exp\,[\,s(y)+t(z-\gamma\,y)\,]\\ &\times exp\,[\,a_1\,T_1\,(y,z)+a_2\,T_2\,(y,z)\,]\;exp\,[\,-r(a_1,a_2)\,]\,,\end{aligned} \tag{2.39}$$

which satisfies the factorization criterion (Definition 2) with the following two-dimensional jointly sufficient statistics:

$$T_1\,(y,z) \;=\; c_1\,y+d_1\,z\,, \tag{2.40}$$

$$T_2\,(y,z) \;=\; c_2\,y+d_2\,z\,. \tag{2.41}$$

Even if the joint distribution of performance measures y and z is restricted to the subfamily $\varphi(y,z;\,a_1,a_2)$ (2.36), a one-dimensional sufficient statistic is generally infeasible in a two-task setting. With the effort-dependent coefficients $p(a_1,a_2)$ and $q(a_1,a_2)$ linear in the effort levels a_1 and a_2 as in (2.37) and (2.38), the following shows that condition (2.35) for the existence of a one-dimensional sufficient statistic $T\,(y,z)$ is not generally satisfied:

$$p(a_1,a_2)\,y+q(a_1,a_2)\,z = a_1\,T_1\,(y,z)+a_2\,T_2\,(y,z)\,, \tag{2.42}$$

where $T_1\,(y,z)$ and $T_2\,(y,z)$ are as in (2.40) and (2.41).

The subsequent analysis demonstrates that a one-dimensional sufficient statistic $T\,(y,z)$ is feasible only in a single-task setting or single-performance-measure setting, but generally a one-dimensional sufficient statistic $T\,(y,z)$ is infeasible in a multi-task and multi-performance-measure setting. A special case for the existence of a one-dimensional sufficient statistic $T\,(y,z)$ is when the implementable action space is restricted to a one-dimensional linear subspace. For example, consider an agency, in which the agent is a firm producing two kinds of space-science items for the government. The two items have an identical cost structure and there is no other market for those items. For a technical reason, the government demands the two items in a certain relative proportion, say $w_1 \in \mathbb{R}$. Then, the induced efforts of the firm for producing the two items will be proportional according to the optimal relative proportion:

$$\frac{a_1}{a_2} = w_1,\ \ w_1 \in \mathbb{R}\,. \tag{2.43}$$

In this case, the principal needs to estimate only one parameter and the agency is effectively a single-task setting. Under condition (2.43), a one-dimensional sufficient statistic is well defined from (2.42):

$$T\,(y, z) = w_1\,T_1\,(y, z) + T_2\,(y, z)\,. \tag{2.44}$$

Now, assume that the implementable action space is $\mathbb{R}^2$, and thus each element of two-dimensional tasks $(\,a_1,\,a_2\,)$ is a parameter to estimate. Because a statistically sufficient aggregation requires every aggregate measure to be independent of all parameters to estimate and also because the two-dimensional sufficient statistics $T_1\,(y, z)$ (2.40) and $T_2\,(y, z)$ (2.41) are both linear in the basic measures y and z, (2.42) can be factored with a one-dimensional sufficient statistic if, and only if, an element of the two-dimensional sufficient statistics is a multiple of the other: without loss of generality,

$$T_1\,(y, z) = w_2\,T_2\,(y, z)\ \ ,\ \ w_2 \in \mathbb{R}\,, \tag{2.45}$$

which implies $c_1 = w_2\,c_2$ and $d_1 = w_2\,d_2$. Then, the effort-dependent coefficients $p(a_1, a_2)$ and $q(a_1, a_2)$ are proportional regardless of the effort level $(a_1, a_2) \in \mathbb{R}^2$:

$$\frac{p(a_1, a_2)}{q(a_1, a_2)} = \frac{c_2}{d_2} \in \mathbb{R}. \tag{2.46}$$

Under condition (2.46), the factorization requirement (2.35) for the existence of a one-dimensional sufficient statistic is always satisfied with $T\,(y, z) = (\,c_2\,/\,d_2\,)\ y + z$, and the agency effectively has a single performance measure $T\,(y, z) = (\,c_2\,/\,d_2\,)\ y + z$ with respect to the agent's effort $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$ in the joint density function

$\varphi(y, z;\, a_1, a_2)$ (2.39). Therefore, when the joint distribution of performance measures is restricted to the family $\varphi(y, z;\, a_1, a_2)$ (2.39), a one-dimensional sufficient statistic is generally infeasible in a setting of multiple tasks and multiple performance measures.

In order to prove the impossibility of a one-dimensional sufficient statistic, the joint distribution of the basic measures y and z is further restricted to the joint normal density function $f(y, z;\, \vec{a})$ (2.30) with two tasks $\vec{a} = (a_1, a_2)$. By the factorization criterion (Definition 2), the following two-dimensional jointly sufficient statistics are obtained from the joint normal density function $f(y, z;\, \vec{a})$ (2.30) :

$$T_1(y, z) = \frac{(m_1 - \phi_1\, k_1)}{\sigma_1^2}\, y + \frac{(k_1 - \phi_2\, m_1)}{\sigma_2^2}\, z\,, \tag{2.47}$$

$$T_2(y, z) = \frac{(m_2 - \phi_1\, k_2)}{\sigma_1^2}\, y + \frac{(k_2 - \phi_2\, m_2)}{\sigma_2^2}\, z\,, \tag{2.48}$$

where $\phi_1 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_2^2$ and $\phi_2 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_1^2$. Then, the question is whether the jointly sufficient statistics $(T_1(y, z),\, T_2(y, z))$ ((2.47) and (2.48)) are minimal such that no one-dimensional sufficient statistic exists. The following lemma shows that the jointly sufficient statistics (2.47) and (2.48) are minimal if the two performance measures y and z are not perfectly aligned.

Lemma 1

Assume that the basic measures y and z follow the joint normal density function $f(y, z;\, \vec{a})$ (2.30) with two tasks $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$, and that the basic measures y and z are not perfectly aligned. Then, the jointly sufficient statistics $T_1(y, z)$ (2.47) and $T_2(y, z)$ (2.48) are minimal.

Given Lemma 1, the following proposition shows that a one-dimensional sufficient statistic is infeasible in a two-task setting, unless the two performance measures y and z are perfectly aligned.[3]

[3] It is assumed in Proposition 3 that the basic measures y and z are not perfectly aligned. What if they are? In that case, a one-dimensional sufficient statistic exists because the linear dependence of the basic measures y and z ($\vec{m} = w\, \vec{k},\; w \in \mathbb{R}$) is equivalent to the special case in (2.46), which is effectively in a single-performance-measure setting. The informativeness condition of an additional performance measure in Feltham and Xie (1994) is a special case with $\vec{k} = 0$. A one-dimensional sufficient statistic $T(y, z) = (m \,/\, \sigma_1^2)\, y + (-\phi_2\, m \,/\, \sigma_2^2)\, z$, $\phi_2 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_1^2$ is well defined unless

Proposition 3 *(No one-dimensional sufficient statistic in a two-task setting)*

With the same assumptions in Lemma 1, a one-dimensional sufficient statistic of the basic measures y and z is infeasible.

Proposition 3 shows that in general, it is infeasible to achieve statistical sufficiency for all effort levels ([1]) with a one-dimensional aggregate measure if the agent's effort is on two tasks. Amershi, Banker, and Datar (1990) show in a single-task setting that when the joint distribution of the basic measures belongs to the family $\phi(y, z; a)$ (2.24), a one-dimensional statistically sufficient aggregation can be achieved. Proposition 3 shows that the result of Amershi, Banker, and Datar (1990) does not generally hold in a multi-task setting.

Now, the question is on the minimum number of aggregate measures necessary to achieve statistical sufficiency for all effort levels ([1]). The following proposition answers this question, and generalizes the result of Proposition 3 not only to a general setting of q-dimensional basic measures $(y_1, y_2, \cdots, y_q)$ and p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$ from a two-performance-measure and two-task setting, but also to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25) of joint distributions from the joint normal distribution $f(y, z; \vec{a})$ (2.30) of the basic measures. While Proposition 3 provides a special case that proves the impossibility of a one-dimensional sufficient statistic in a multi-task setting, the following proposition provides the general minimum requirement for jointly sufficient statistics in a multi-task setting.

In a general setting of q-dimensional basic measures $(y_1, y_2, \cdots, y_q)$ and p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$, the principal's statistical problem is to estimate each of the p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$. The constant matrix Ω_{qp} in (2.26) is assumed to be full-rank ($rank\,(\Omega_{qp}) = p,\ p \leq q$) such that all the columns of the constant matrix Ω_{qp} are linearly independent.[4] The following lemma shows that when the basic measures follow a joint distribution of the sub-

$\rho_{\varepsilon\delta} = 0$ and the relative weight on the basic measures y and z is $-\sigma_2\,/\,(\rho_{\varepsilon\delta}\,\sigma_1)$ as in Feltham and Xie (1994).

[4] If the constant matrix Ω_{qp} is not full-rank, then the principal estimates less than p-dimensional parameters $\vec{a} = (a_1, a_2, \cdots, a_p)$. The concept of statistical sufficiency does not require an agency, and thus the only imposition on the estimation of the p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$ regarding the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25) is the maximum likelihood. The first order condition from

family $\varphi(\vec{y}_q;\,\vec{a}_p)$ (2.25), there exists a set of p-dimensional jointly sufficient statistics $\overrightarrow{T\,(\vec{y}_q)} = (\,T_1\,(\vec{y}_q)\,,\,T_2\,(\vec{y}_q)\,,\,\cdots\,,\,T_p\,(\vec{y}_q)\,)$, which characterizes the likelihood ratio of the subfamily $\varphi(\vec{y}_q;\,\vec{a}_p)$.

Lemma 2

Assume that q-dimensional basic measures $(\,y_1,\,y_2,\cdots\,,\,y_q\,)$ *follow a joint distribution of the subfamily* $\varphi(\vec{y}_q;\,\vec{a}_p)$ *(2.25), the agent's effort is on p-dimensional tasks* $(\,a_1,\,a_2,\cdots\,,\,a_p\,)$*, the principal estimates each of the p-dimensional tasks* $\vec{a} = (a_1, a_2, \cdots, a_p)$*, and the constant matrix* Ω_{qp} *in (2.26) is full-rank (*$rank\,(\Omega_{qp}) = p$*,* $p \le q$ *). Then, there exists a set of p-dimensional jointly sufficient statistics* $\overrightarrow{T\,(\vec{y}_q)} = (\,T_1\,(\vec{y}_q)\,,\,T_2\,(\vec{y}_q)\,,\,\cdots\,,\,T_p\,(\vec{y}_q)\,)$ *given by:*

$$
\begin{aligned}
T_1\,(\vec{y}_q) &= \omega_{11}\,y_1 + \omega_{21}\,y_2 + \cdots + \omega_{q1}\,y_q\,,\\
T_2\,(\vec{y}_q) &= \omega_{12}\,y_1 + \omega_{22}\,y_2 + \cdots + \omega_{q2}\,y_q\,,\\
&\cdots\cdots\cdots\\
T_p\,(\vec{y}_q) &= \omega_{1p}\,y_1 + \omega_{2p}\,y_2 + \cdots + \omega_{qp}\,y_q\,.
\end{aligned}
\tag{2.53}
$$

(2.25) is as follows:

$$\nabla_a\varphi(\vec{y}_q;\,\vec{a}_p) = \Omega_{qp}^T\,\vec{y}_q + \nabla_a\alpha(\vec{a}_p) = 0\,, \tag{2.49}$$

where the superscript T denotes the transpose of the matrix. The first order condition (2.49) is a system of the following p equations:

$$
\begin{aligned}
&{Col_1}^T\,\vec{y}_q + \alpha_1(\vec{a}_p) = 0\,,\\
&{Col_2}^T\,\vec{y}_q + \alpha_2(\vec{a}_p) = 0\,,\\
&\quad\cdots\cdots\cdots\\
&Col_{p-1}^T\,\vec{y}_q + \alpha_{p-1}(\vec{a}_p) = 0\,,\\
&{Col_p}^T\,\vec{y}_q + \alpha_p(\vec{a}_p) = 0\,,
\end{aligned}
\tag{2.50}
$$

where Col_i denotes the column i of the constant matrix Ω_{qp} in (2.26) and $\alpha_i(\cdot)$ denotes the partial derivative $\partial\alpha/\partial a_i$. Now suppose, without loss of generality, that the column p of Ω_{qp} is a linear combination of the other columns $1, 2, \cdots, (p-1)$:

$$Col_p = \kappa_1\,Col_1 + \kappa_2\,Col_2 + \cdots + \kappa_{p-1}\,Col_{p-1}\,, \tag{2.51}$$

where $\kappa_1, \kappa_2, \cdots, \kappa_{p-1}$ are constants and not all of them are zero. Substituting (2.51) into the last equation in (2.50), multiplying respectively the rest $(p-1)$ equations with $-\kappa_1, -\kappa_2, \cdots, -\kappa_{p-1}$, and summing up all the p equations result in:

$$\alpha_p(\vec{a}_p) = \kappa_1\,\alpha_1(\vec{a}_p) + \kappa_2\,\alpha_2(\vec{a}_p) + \cdots + \kappa_{p-1}\,\alpha_{p-1}(\vec{a}_p)\,. \tag{2.52}$$

By (2.51) and (2.52), the last equation in (2.50) turns out to be a linear combination of the rest $(p-1)$ equations with coefficients $\kappa_1, \kappa_2, \cdots, \kappa_{p-1}$. Thus, the last equation in (2.50) is redundant, and the first order condition (2.49) has p unknowns $(\,a_1,\,a_2,\cdots\,,\,a_p\,)$ and $p-1$ equations. Therefore, the maximum likelihood estimator $\left(\,a_1^*,\,a_2^*,\cdots\,,\,a_p^*\,\right)$ lies in a proper subspace of $\mathbb{R}^p$ and the principal does not need to estimate all the p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$.

Furthermore, the likelihood ratio of the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25) is given by :

$$L(\vec{y}_q;\, \vec{a}_p)_{p\times 1} = \frac{\nabla_a \varphi(\vec{y}_q;\, \vec{a}_p)}{\varphi(\vec{y}_q;\, \vec{a}_p)} = \begin{bmatrix} T_1\left(\vec{y}_q\right) \\ T_2\left(\vec{y}_q\right) \\ \vdots \\ T_p\left(\vec{y}_q\right) \end{bmatrix} + \nabla_a \alpha(\vec{a}_p)\,, \tag{2.54}$$

where the p-dimensional jointly sufficient statistics $\left(\, T_1\left(\vec{y}_q\right),\, T_2\left(\vec{y}_q\right),\, \cdots,\, T_p\left(\vec{y}_q\right)\right)$ are as in (2.53).

With the result of Lemma 2, the following proposition shows that when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), the dimension of minimal sufficient statistics is the number of tasks. The intuition for the following proposition is that the dimension of minimal sufficient statistics equals the number of "independent"tasks that have to be estimated and controlled.

Proposition 4 *(Dimension of minimal sufficient statistics)*

With the same assumptions as in Lemma 2, the p-dimensional jointly sufficient statistics $\overrightarrow{T\left(\vec{y}_q\right)} = \left(\, T_1\left(\vec{y}_q\right),\, T_2\left(\vec{y}_q\right),\, \cdots,\, T_p\left(\vec{y}_q\right)\right)$ (2.53) are minimal.

Proposition 4 shows that statistical sufficiency for all effort levels ([1]) in aggregating q-dimensional basic measures $(\, y_1,\, y_2,\, \cdots,\, y_q\,)$ with respect to the agent's p-dimensional effort $(\, a_1,\, a_2, \cdots,\, a_p\,)$ can be achieved by *no less than p* (the number of tasks) aggregate measures. Multiple aggregate measures, at least as many as the number of tasks estimated and controlled by the principal, are necessary to avoid any loss of information in the aggregation process. The result of Proposition 4 confirms and generalizes the result of Proposition 3 that it is impossible to aggregate the basic measures with no loss of information into a one-dimensional aggregate measure when the agent's effort is on multiple tasks. Proposition 4 implies that as the number of tasks increases, a higher number of aggregate measures are necessary to preserve the information content of the basic measures in the aggregation process.

2.4 Sufficient implementation statistics in multi-task settings

Whereas sufficient statistics require statistical sufficiency for all effort levels, sufficient implementation statistics require statistical sufficiency only for a single (fixed) effort level. The key qualification for a set of sufficient implementation statistics is that it should result in the same likelihood ratio as the basic measures in inducing a single effort level.

While sufficient statistics incur no loss of information under all kinds of decision-making contexts (including agency settings), sufficient implementation statistics require no loss of information under an agency setting, particularly when an effort level to induce is exogenously specified. Thus, under an agency setting, sufficient implementation statistics are better benchmark of lack of information loss than sufficient statistics.

This section shows that, when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), sufficient implementation statistics of the basic measures are equivalent to sufficient statistics ([1] = [2]) as the "all a or no a" condition is satisfied. That is, if the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), sufficient implementation statistics are independent of the effort level to induce $\vec{a}$ as well as the agent's utility function $U^a(\cdot)$. Thus, it follows from the results in the previous section that when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), the dimension of minimal sufficient implementation statistics is also given by the number of tasks.

For showing the dimension of minimal sufficient implementation statistics, it is useful to discuss the "all a or no a" condition as a sufficient condition for the existence of sufficient implementation statistics. In a single-task setting, Amershi, Banker, and Datar (1990) show that if the "all a or no a" condition is satisfied, economic sufficiency for the optimal effort level implies statistical sufficiency for all effort levels ([4] $\Rightarrow$ [1]) such that the four sufficiency concepts of aggregation are equivalent

([1] = [2] = [3] = [4]). Whereas the definition of Amershi, Banker, and Datar (1990, Page 129) requires a one-dimensional sufficient statistic, the following definition generalizes the definition of Amershi, Banker, and Datar (1990) and considers multi-dimensional jointly sufficient statistics:

Definition 8 *("All a or no a" condition)*

The "all a or no a" condition is satisfied if the likelihood ratio $L(\vec{y}\,|\,\vec{a})$ *(2.12) can be represented as a function of the same jointly sufficient statistics* $\overrightarrow{T(\vec{y})} = (T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$ *of the basic measures for all effort levels such that the likelihood ratio* $L(\vec{y}\,|\,\vec{a})$ *depends on the basic measures only through the jointly sufficient statistics:*

$$L(\vec{y}\,|\,\vec{a}) = \frac{\nabla_a d(\vec{y}\,|\,\vec{a})}{d(\vec{y}\,|\,\vec{a})} = q'(T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y})), \tag{2.55}$$

where $q'(\cdot)$ *is a function.*

The "all a or no a" condition implies that the likelihood ratio incurs no loss of information in inducing all effort levels since sufficient statistics do not lose any information of the basic measures in the aggregation process.

The "all a or no a" condition (2.55) satisfies, for all effort levels, condition (2.17) for jointly sufficient implementation statistics. When the likelihood ratio $L(\vec{y}\,|\,\vec{a})$ is a function of the same jointly sufficient statistics $\overrightarrow{T(\vec{y})} = (T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$ for all effort levels $\vec{a} \in \mathbb{R}^p$, the jointly sufficient statistics $\overrightarrow{T(\vec{y})}$ are sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\,\vec{a})}$ for inducing any single effort level $\vec{a} \in \mathbb{R}^p$ because the jointly sufficient statistics $\overrightarrow{T(\vec{y})}$ generate the same likelihood ratio $L(\vec{y}\,|\,\vec{a})$ as the basic measures $\vec{y} = (y_1, y_2, \cdots, y_q)$ (condition (2.17)) for any single effort level $\vec{a} \in \mathbb{R}^p$. Thus, if the "all a or no a" condition (Definition 8) is satisfied, then the same set of sufficient implementation statistics, which is a set of jointly sufficient statistics $\overrightarrow{T(\vec{y})}$, exists for all effort levels $\vec{a} \in \mathbb{R}^p$, and the same sufficient implementation statistics are independent of the effort level to induce $\vec{a}$ as well as the agent's utility function $U^a(\cdot)$.

If the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), the "all a or no a"condition (Definition 8) is satisfied. In (2.54) of Lemma 2, the likelihood ratio $L(\vec{y}_q;\, \vec{a}_p)_{p\times 1}$ is a function of the same p-dimensional jointly sufficient statistics $(\, T_1\,(\vec{y}_q)\,,\, T_2\,(\vec{y}_q)\,,\, \cdots\,,\, T_p\,(\vec{y}_q)\,)$ for all effort levels $\vec{a} = (a_1, a_2, \cdots, a_p) \in \mathbb{R}^p$. Given the results of Lemma 2 and Proposition 4, the following proposition has been proved.

Proposition 5 *(Dimension of minimal sufficient implementation statistics)*

With the assumptions in Lemma 2, the dimension of minimal sufficient implementation statistics is the number of tasks. In particular, statistical sufficiency for a single effort level ([2]) in aggregating the basic measures $(\, y_1,\, y_2,\, \cdots\,,\, y_q\,)$ *with respect to the agent's p-dimensional effort* $(\, a_1,\, a_2, \cdots\,,\, a_p\,)$ *is achieved by the p-dimensional jointly sufficient statistics* $\overrightarrow{T\,(\vec{y}_q)} = (\, T_1\,(\vec{y}_q)\,,\, T_2\,(\vec{y}_q)\,,\, \cdots\,,\, T_p\,(\vec{y}_q)\,)$ *(2.53), which are independent of the effort level to induce* $\vec{a}_p$ *as well as the agent's utility function* $U^a(\cdot)$*.*

From Proposition 5, it follows that, when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), there is no statistically sufficient aggregation of the basic measures with the dimensionality less than the number of tasks, even for inducing a single effort level.

2.5 Economic sufficiency : Efficient aggregation in multi-task settings

This section extends the discussion on the aggregation of performance measures in Amershi, Banker, and Datar (1990) not only to a multi-task setting but also to arbitrary effort levels. This section shows that if the number of aggregate measures is less than the number of tasks, the efficient aggregation depends on the effort level to induce $\vec{a} \in \mathbb{R}^p$ and incurs loss of information even if the "all a or no a"condition (Definition 8) is satisfied. On the other hand, the efficient aggregation in a single-task setting is characterized by sufficient statistics if the "all a or no a"condition is satisfied. As the previous section shows, when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25), the "all a or no a"condition (Definition 8) is satisfied such that the sufficient implementation statistics do not depend on the effort level to induce $\vec{a}$ as well as the agent's utility function $U^a(\cdot)$. Thus, the dependence of an efficient aggregation on the effort level to induce $\vec{a}$ indicates the lack of statistical sufficiency and loss of information in the aggregation process.

In the characterization of the efficient contract (2.11), an economically sufficient aggregation generally depends on the effort level to induce $\vec{a}$ and the agent's utility function $U(\cdot)$ through the term $\vec{\mu} \cdot L\,(\vec{y}\,|\,\vec{a})$. For explicitly observing an economically sufficient aggregation, the vector of Lagrange multipliers $\vec{\mu}$ should be solved out. There are at least two ways of solving the vector of Lagrange multipliers $\vec{\mu}$. One way is to employ a square-root utility function of the agent and the other way is to use a LEN (Linear contract, Exponential utility of the agent, Normally distributed performance measure) model. However, both ways fix a type of utility function of the agent, and the dependence of an economically sufficient aggregation on the agent's utility function $U(\cdot)$ cannot be observed.

In this section, it is assumed that the agent has a square-root utility function with an additively-separable action cost :

$$U^a\,(\,C(\,\vec{y}\,),\,\vec{a}\,) = 2\sqrt{C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)} - v\,(\,a_1,\,a_2,\,\cdots\,,\,a_p\,)\,. \tag{2.56}$$

The subsequent discussion on the efficient aggregation of performance measures follows from Christensen, Şabac, and Tian (2010). In a multi-task setting, an economically sufficient aggregation for a single effort level is characterized not only by the likelihood ratio, but also by the variance-covariance matrix of the likelihood ratio and the agent's marginal action cost. The following proposition shows an economically sufficient aggregation for a single effort level when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.25).

Proposition 6 *(Economic sufficiency for an arbitrary effort level* [3] *(Definition*

6) with a square-root utility function)

Together with the assumptions in Lemma 2, assume that the agent has a square-root utility function with an additively-separable action cost as in (2.56). For inducing a single effort level $\vec{a} = (a_1, a_2, \cdots, a_p) \in \mathbb{R}^p$*, the economically sufficient aggregation of the basic measures* $\vec{y} = (\, y_1,\, y_2,\, \cdots,\, y_q\,)$ *is characterized by:*

$$[\,\Sigma L\,(\vec{y})\,]^{-1}\,\nabla_a v\,(\,a_1,\,a_2,\,\cdots,\,a_p\,)\cdot\begin{bmatrix} T_1\,(\vec{y}_q) \\ T_2\,(\vec{y}_q) \\ \vdots \\ T_p\,(\vec{y}_q) \end{bmatrix}, \tag{2.57}$$

where $\Sigma L\,(\vec{y})$ *denotes the variance-covariance matrix of the likelihood ratio* $L\,(\vec{y}\,|\,\vec{a})$ *and the* p*-dimensional jointly sufficient statistics* $(\,T_1\,(\vec{y}_q)\,,\,T_2\,(\vec{y}_q)\,,\,\cdots\,,\,T_p\,(\vec{y}_q)\,)$ *are given by (2.53).*

When the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q;\,\vec{a}_p)$ (2.25), the "all a or no a" condition (Definition 8) is satisfied such that sufficient implementation statistics are equivalent to sufficient statistics ([1] = [2]). Proposition 6 shows how the number of aggregate measures restricts sufficiency of aggregation of performance measures in a multi-task setting. In an efficient contract, the principal can substitute multi-dimensional aggregate measures $\overrightarrow{\zeta\,(\vec{y}\,;\,U^a(\cdot),\vec{a})}$ for the basic measures (Definition 6). If multiple p (the number of tasks) aggregate measures $(\,T_1\,(\vec{y}_q)\,,\,T_2\,(\vec{y}_q)\,,\,\cdots\,,\,T_p\,(\vec{y}_q)\,)$ are used, then the aggregate measures are both statistically sufficient and economically sufficient. If less than p (the number of tasks) aggregate measures are used, then there is an economically sufficient aggregation but not a statistically sufficient aggregation.

If the principal designs the performance evaluation system such that the number of aggregate measures for the contract is greater than or equal to the number of tasks (p), then the p-dimensional jointly sufficient statistics $(\,T_1\,(\vec{y}_q)\,,\,T_2\,(\vec{y}_q)\,,\,\cdots\,,\,T_p\,(\vec{y}_q)\,)$ can be used in their entirety for the efficient contract and the information content from the basic measures is preserved. Thus, if the number of aggregate measures is no less than the number of tasks, the statistically sufficient aggregate measures $(T_1(\vec{y}_q), T_2(\vec{y}_q), \cdots, T_p(\vec{y}_q))$ can be used for inducing any single effort level with no loss of information.

On the other hand, if the principal designs the performance evaluation system such that the number of aggregate measures for the contract is less than the number of tasks (p), then the p-dimensional jointly sufficient statistics $(T_1(\vec{y}_q), T_2(\vec{y}_q), \cdots, T_p(\vec{y}_q))$ cannot be used any more in their entirety for the efficient contract and the aggregation lacks statistical sufficiency and incurs loss of information. In this case, the economically sufficient aggregate measures in (2.57) will depend on the effort level to induce $\vec{a}$ because the term $[\Sigma L(\vec{y})]^{-1} \nabla_a v(a_1, a_2, \cdots, a_p)$ involves in the aggregation in an inseparable way. Thus, if the number of aggregate measures is less than the number of tasks, the economically sufficient aggregation is not statistically sufficient and fails to preserve the information content from the basic measures $(y_1, y_2, \cdots, y_q)$. The following corollary stems from Proposition 6.

Corollary 1

With the assumptions in Proposition 6, for inducing a single effort level $\vec{a} \in \mathbb{R}^p$, an economically sufficient aggregation of the basic measures $\vec{y} = (y_1, y_2, \cdots, y_q)$ ([3]) can be statistically sufficient ([2]) only if the number of aggregate measures is no less than the number of tasks. Otherwise, if the number of aggregate measures is less than the number of tasks, then an economically sufficient aggregation necessarily incurs loss of information.

Corollary 1 shows that although the "all a or no a" condition (Definition 8) is satisfied, an economically sufficient aggregation is not statistically sufficient and incurs loss of information in a multi-task setting if the performance evaluation system is designed such that the number of aggregate measures is less than the number of tasks. The result of Corollary 1 is a multi-task phenomenon and cannot be found in a single-task setting. Corollary 1 introduces the number of tasks as a constraining factor, in addition to the "all a or no a" condition, that hinders aggregate performance measures from being sufficient statistics.

On the other hand, as Amershi, Banker, and Datar (1990) show, the "all a or no a" condition is the only constraining factor, that hinders aggregate performance measures from being sufficient statistics, in a single-task setting. If the agent's effort is on a single task ($p = 1$) in (2.57), the economically sufficient aggregation for a single effort level $a \in \mathbb{R}$ is determined only by a one-dimensional sufficient statistic $T\,(\vec{y}_q)$. It suffices for the proof of the following corollary to observe that the variance of the likelihood ratio $\Sigma L\,(\vec{y})$ and the agent's marginal action cost $\nabla_a v\,(\,a_1,\, a_2,\, \cdots,\, a_p\,)$ are scalars in (2.57) of Proposition 6 with a single task.

Corollary 2

With the assumptions in Proposition 6, assume a single task ($p = 1$)*. Then, for inducing a single effort level* $a \in \mathbb{R}$*, the economically sufficient aggregation (* [3] *) of the basic measures* $\vec{y} = (\,y_1,\, y_2,\, \cdots,\, y_q\,)$ *is always statistically sufficient (* [2] *).*

Corollary 2 is a particular case of the result in Amershi, Banker, and Datar (1990). Corollary 2 shows that if the "all a or no a" condition (Definition 8) is satisfied in a single-task setting, an economically sufficient aggregation also attains statistical sufficiency and incurs no loss of information in inducing *any* single effort level, as Amershi, Banker, and Datar (1990) show for the optimal effort level.

Now, the joint distribution of the basic measures is restricted to the joint normal density function $f(y, z; \vec{a})$ (2.30) with two performance measures ($y,\, z$) and two tasks $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$. In addition, the agent's action cost is assumed to be quadratic $v(\vec{a}) = \frac{1}{2}\,(\,\vec{a} \cdot \vec{a}\,) = \frac{1}{2}\left(\,a_1^2 + a_2^2\,\right)$ such that:

$$\nabla_a v(\vec{a}) = \vec{a}\,. \tag{2.58}$$

The following corollary explicitly shows that the economically sufficient aggregation (2.57) depends on the effort level to induce $\vec{a}$ when the number of aggregate measures is less than the number of tasks.

Corollary 3 *(Economic sufficiency for an arbitrary effort level* [3] *: Normal distribution)*

Assume that the agent has a square-root utility function with an additively-separable action cost as in (2.56), the basic measures y and z follow the joint normal density function $f(y, z; \vec{a})$ (2.30) with two tasks $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$, and the agent's marginal action cost is given by (2.58). Then, for inducing a single effort level $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$, the economically sufficient aggregation (2.57) is characterized by the relative weight on the basic measures y and z :

$$\frac{\beta^1}{\beta^2} = \frac{a_2\, k_1 - a_1\, k_2}{a_1\, m_2 - a_2\, m_1}\,. \tag{2.59}$$

The result (2.59) in Corollary 3 explicitly shows that when the number of aggregate measures is restricted to be less than the number of tasks, the economically sufficient aggregation incurs loss of information, which is indicated by the dependence of the aggregation on the effort level $\vec{a} = (a_1, a_2)$. Note that if the implementable action space is restricted to a one-dimensional linear subspace $a_1/a_2 = w_2 \in \mathbb{R}$ (effectively single-task), the economically sufficient aggregation (2.59) becomes independent of the effort level to induce $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$. Although the joint normal density function $f(y, z; \vec{a})$ (2.30) satisfies the "all a or no a" condition (Definition 8), the multi-task setting causes an economically sufficient aggregation to fail in attaining statistical sufficiency and incur loss of information if the number of aggregate measures is less than the number of tasks. Because there is no sufficient statistic of dimension less than the number of tasks (p) when the joint distribution of the basic measures belongs to the subfamily $\varphi(\vec{y}_q; \vec{a}_p)$ (2.25) (Proposition 4), any aggregation (including any economically sufficient aggregation) of dimension less than the number of tasks (p) will fail to be a sufficient statistic.

2.6 Conclusion

In an extended multi-task setting, the first question is whether there is a one-dimensional "universal" aggregation, which is "utility-independent" (regardless of the characteristics of the agency participants) and "effort-independent" (regardless of the economic condition of the agency), with no loss of information in the aggregation process. If the same one-dimensional aggregate measure is good enough to substitute for the basic measures in all firms for every manager, then the complexity involved in motivating and evaluating managers' activities could be reduced considerably. This study shows in multi-task settings that such a one-dimensional "universal" aggregation does not exist. Whereas Amershi, Banker, and Datar (1990) show the feasibility and the condition for the existence of a one-dimensional statistically sufficient aggregation in a single-task setting, a one-dimensional statistically sufficient aggregation is infeasible, even for a single effort level, in a multi-task setting.

This study shows that statistical sufficiency can be achieved only by multiple aggregate measures in a multi-task setting. In particular, the minimum dimension required for a statistically sufficient aggregation is given by the number of tasks. To avoid any loss of information in an aggregation process, the basic measures should be aggregated into at least as many aggregate measures as the number of tasks. This result implies that relying on a single aggregate measure in a multi-task agency necessarily incurs loss of information, and the more information loss can be incurred as the number of tasks increases.

Another question in an extended multi-task setting is on the nature of aggregation of performance measures. While it is infeasible to construct a one-dimensional aggregate measure with no loss of information in a multi-task setting, it is still possible to construct a one-dimensional aggregate measure with no economic loss to the principal. However, in contrast with the result of Amershi, Banker, and Datar (1990) in a single-task setting, a one-dimensional economically sufficient aggregate measure in a multi-task setting necessarily incurs loss of information. In general, when the number of aggregate measures is less than the number of tasks, any aggregation (including any economically sufficient aggregation) incurs loss of information. This result implies that when the number of aggregate measures is less than the number of tasks, an efficient performance evaluation system is "utility-dependent." With a smaller number of aggregate measures than tasks, a performance evaluation system should be "tailored" and individually designed for each manager and each firm.

In a multi-task setting, generally the optimal aggregate measures are not sufficient statistics. Although this result is similar to the result of Amershi, Banker, and Datar (1990) in a single-task setting, the reason in a multi-task setting is qualitatively different from that of a single-task setting. Whereas Amershi, Banker, and Datar (1990) show in a single-task setting that in general, the optimal aggregate measures are agency-specific and not sufficient statistics because a statistical condition ("all a or no a" condition) is generally not satisfied, this study shows that it is not only a statistical condition ("all a or no a" condition) but also a multi-task setting that causes the optimal aggregate measures to be agency-specific and not to be sufficient statistics. Particularly, even if the "all a or no a" condition is satisfied, the optimal aggregate measures are not sufficient statistics if the performance evaluation system is designed such that the number of aggregate measures is less than the number of tasks. As the analysis moves toward an institutionally richer multi-task setting, the optimal aggregate measures are even less represented by sufficient statistics.

2.7 Appendix

2.7.1 Proof of Proposition 1

By Definition 1, if an aggregation of performance measures is statistically sufficient for all effort levels ([1]), a set of jointly sufficient statistics $\overrightarrow{T(\vec{y})} = (T_1(\vec{y}), T_2(\vec{y}), \cdots, T_j(\vec{y}))$ exists for the basic measures $\vec{y} = (y_1, y_2, \cdots, y_q)$. By Definition 2, the joint density function of the basic measures is factorized as in (2.14) for all effort levels $\vec{a} = (a_1, a_2, \cdots, a_p) \in \mathbb{R}^p$. It follows that the likelihood ratio is a function of a set of jointly sufficient statistics

$\overrightarrow{T(\vec{y})} = (\, T_1(\vec{y})\,,\, T_2(\vec{y})\,,\, \cdots\,,\, T_j(\vec{y})\,)$ for all effort levels $\vec{a} = (\, a_1, a_2, \cdots, a_p\,) \in \mathbb{R}^p$:

$$
\begin{aligned}
L(\vec{y}\,|\,\vec{a}) &= \frac{\nabla_a d(\vec{y}\,|\,\vec{a})}{d(\vec{y}\,|\,\vec{a})} \\
&= \frac{\nabla_a v(\, T_1(\vec{y})\,,\, T_2(\vec{y})\,,\, \cdots\,,\, T_j(\vec{y})\,;\, a_1,\, a_2,\, \cdots\,,\, a_p\,)}{v(\, T_1(\vec{y})\,,\, T_2(\vec{y})\,,\, \cdots\,,\, T_j(\vec{y})\,;\, a_1,\, a_2,\, \cdots\,,\, a_p\,)} \\
&= q(\, T_1(\vec{y})\,,\, T_2(\vec{y})\,,\, \cdots\,,\, T_j(\vec{y})\,;\, a_1,\, a_2,\, \cdots\,,\, a_p\,)\,,
\end{aligned}
\tag{2.60}
$$

which satisfies condition (2.17) for any single fixed effort level.

2.7.2 Proof of Proposition 2

If the agent's utility function has an additively separable action cost as in (2.3), then (2.11) and (2.12) characterize the efficient contract $C(\, y_1,\, y_2,\, \cdots\,,\, y_q\,)$. The Kuhn-Tucker multiplier λ is a constant and thus the aggregation of performance measures in the efficient contract is determined by the second term $\vec{\mu} \cdot L(\vec{y}\,|\,\vec{a})$ in (2.11). By condition (2.17) for sufficient implementation statistics $\overrightarrow{\psi(\vec{y};\, \vec{a})} = (\, \psi_1(\vec{y};\, \vec{a})\,,\, \psi_2(\vec{y};\, \vec{a})\,,\, \cdots\,,\, \psi_j(\vec{y};\, \vec{a})\,)$, the characterization of the efficient contract (2.11) is equivalent to the following:

$$
\frac{1}{U'(\, C(\, y_1,\, y_2,\, \cdots\,,\, y_q\,)\,)} = \lambda + \vec{\mu} \cdot q(\, \psi_1(\vec{y};\, \vec{a})\,,\, \psi_2(\vec{y};\, \vec{a})\,,\, \cdots\,,\, \psi_j(\vec{y};\, \vec{a})\,)\,. \tag{2.61}
$$

The efficient contract is given by:

$$
C(\,\vec{y}\,) = U'^{\,-1}\left[\frac{1}{\lambda + \vec{\mu} \cdot q(\, \psi_1(\vec{y};\, \vec{a})\,,\, \psi_2(\vec{y};\, \vec{a})\,,\, \cdots\,,\, \psi_j(\vec{y};\, \vec{a})\,)} \right], \tag{2.62}
$$

and Definition 6 of economic sufficiency of aggregation for a single effort level is satisfied by (2.62) with the jointly sufficient implementation statistics $(\, \psi_1(\vec{y};\, \vec{a})\,,\, \psi_2(\vec{y};\, \vec{a})\,,\, \cdots\,,\, \psi_j(\vec{y};\, \vec{a})\,)$.

For the agent's utility function with a multiplicatively separable action cost:

$$
U^a(\, C(\,\vec{y}\,),\, \vec{a}\,) = k(\vec{a})\, U(\, C(\, y_1,\, y_2,\, \cdots\,,\, y_q\,)\,)\,,\;\; k(\vec{a}) > 0\,, \tag{2.63}
$$

it can be shown that following the same procedures, that were applied to the utility function (2.3) with an additively separable action cost, results in the efficient contract $C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)$ characterized by:

$$\frac{1}{U'\,(\,C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)\,)}=\lambda\,k(\vec{a})+\vec{\mu}\cdot\nabla_a k(\vec{a})+k(a)\,\vec{\mu}\cdot L\,(\vec{y}\,|\,\vec{a})\,, \tag{2.64}$$

which is equivalent to the following by condition (2.17) for sufficient implementation statistics $\overrightarrow{\psi\,(\vec{y};\,\vec{a})}=(\,\psi_1\,(\vec{y};\,\vec{a})\,,\,\psi_2\,(\vec{y};\,\vec{a})\,,\,\cdots\,,\,\psi_j\,(\vec{y};\,\vec{a})\,)$:

$$\begin{aligned}\frac{1}{U'\,(\,C\,(\,y_1,\,y_2,\,\cdots\,,\,y_q\,)\,)}&=\lambda\,k(\vec{a})+\vec{\mu}\cdot\nabla_a k(\vec{a})\\&\quad+k(a)\,\vec{\mu}\cdot q\,(\,\psi_1\,(\vec{y};\,\vec{a})\,,\,\psi_2\,(\vec{y};\,\vec{a})\,,\,\cdots\,,\,\psi_j\,(\vec{y};\,\vec{a})\,)\,.\end{aligned} \tag{2.65}$$

The efficient contract is given by:

$$C\,(\,\vec{y}\,)=U'^{\,-1}\left[\frac{1}{\lambda\,k(\vec{a})+\vec{\mu}\cdot\nabla_a k(\vec{a})+k(a)\,\vec{\mu}\cdot q\,(\,\psi_1\,(\vec{y};\,\vec{a})\,,\,\psi_2\,(\vec{y};\,\vec{a})\,,\,\cdots\,,\,\psi_j\,(\vec{y};\,\vec{a})\,)}\right], \tag{2.66}$$

and Definition 6 of economic sufficiency of aggregation for a single effort level is satisfied by (2.66) with the jointly sufficient implementation statistics $(\,\psi_1\,(\vec{y};\,\vec{a})\,,\,\psi_2\,(\vec{y};\,\vec{a})\,,\,\cdots\,,\,\psi_j\,(\vec{y};\,\vec{a})\,)$.

2.7.3 Proof of Lemma 1

A one-to-one relation between the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) and the maximum likelihood estimators (a_1^*, a_2^*) for the agent's effort $\vec{a}=(a_1,a_2)\in\mathbb{R}^2$ is established. From the one-to-one relation, minimal sufficiency of the maximum likelihood estimators (a_1^*, a_2^*) follows, and then minimal sufficiency of the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) is derived. In the proof, the following two points are exploited:

1. If a set of maximum likelihood estimators are jointly sufficient statistics, then the maximum likelihood estimators are minimal jointly sufficient statistics (DeGroot 1986, Page 368),

2. A one-to-one function of a set of jointly sufficient statistic is also sufficient.

The maximum likelihood estimators (a_1^*, a_2^*) for the agent's effort $\vec{a} = (a_1, a_2)$ are obtained by solving the following maximization problem:

$$\max_{a_1,\, a_2} \; f(y, z;\, \vec{a})\,, \tag{2.67}$$

where $f(y, z;\, \vec{a})$ is the joint normal density function (2.30) with $\vec{a} = (a_1, a_2)$. For notational convenience, the following covariance-adjusted sensitivities, which are assumed to be non-negative, are used:

$M1 = m_1 - \phi_1\, k_1,\; K1 = k_1 - \phi_2\, m_1,$

$M2 = m_2 - \phi_1\, k_2,\; K2 = k_2 - \phi_2\, m_2,$

where $\phi_1 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_2^2,\; \phi_2 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_1^2.$

The second order conditions are satisfied as the covariance-adjusted sensitivities are non-negative: $M1 \geq 0,\; K1 \geq 0,\; M2 \geq 0,\; K2 \geq 0$. From the first order conditions, the maximum likelihood estimators (a_1^*, a_2^*) for the agent's effort $\vec{a} = (a_1, a_2)$ are given as follows:

$$\begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix} = A^{-1} \begin{bmatrix} T_1(y, z) \\ T_2(y, z) \end{bmatrix}, \tag{2.68}$$

where $T_1(y, z)$ and $T_2(y, z)$ are as in (2.47) and (2.48), and A is a 2 by 2 matrix:

$$A_{2\times 2} = \begin{bmatrix} \frac{M1}{\sigma_1^2}\, m_1 + \frac{K1}{\sigma_2^2}\, k_1 & \frac{M1}{\sigma_1^2}\, m_2 + \frac{K1}{\sigma_2^2}\, k_2 \\ \frac{M2}{\sigma_1^2}\, m_1 + \frac{K2}{\sigma_2^2}\, k_1 & \frac{M2}{\sigma_1^2}\, m_2 + \frac{K2}{\sigma_2^2}\, k_2 \end{bmatrix}, \tag{2.69}$$

whose determinant is $(m_1 k_2 - m_2 k_1)^2\, (1 - \rho_{\varepsilon\delta}^2) \,/\, (\sigma_1^2\, \sigma_2^2)$.

If the two performance measures y and z are not perfectly aligned $(\, m_1 k_2 - m_2 k_1 \neq 0\,)$, the maximum likelihood estimators (a_1^*, a_2^*) are one-to-one to the jointly sufficient statistics $(T_1(y, z),\, T_2(y, z))$. Thus, the maximum likelihood estimators (a_1^*, a_2^*) are sufficient by Point 2. As the maximum likelihood estimators are now sufficient, the maximum likelihood estimators (a_1^*, a_2^*) are minimal jointly sufficient statistics by Point 1.

As there is a one-to-one relation (2.68) between the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) and the maximum likelihood estimators (a_1^*, a_2^*), there exist two (linear) functions $\alpha\,(\cdot)$ and $\alpha^{-1}\,(\cdot)$ such that :

$$(\,a_1^*,\, a_2^*\,) \;=\; \alpha\,(\,T_1(y,z),\, T_2(y,z)\,)\,, \tag{2.70}$$

$$(\,T_1(y,z),\, T_2(y,z)\,) \;=\; \alpha^{-1}\,(\,a_1^*,\, a_2^*\,)\,. \tag{2.71}$$

As the maximum likelihood estimators (a_1^*, a_2^*) are minimal jointly sufficient statistics, the function $\alpha\,(\cdot)$ in (2.70) is consistent with Definition 3. Because the maximum likelihood estimators (a_1^*, a_2^*) can be represented as a function of any other sufficient statistics, the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) can also be represented as a function of any other sufficient statistics by the function $\alpha^{-1}\,(\cdot)$ in (2.71). Therefore, the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) are minimal by Definition 3.

2.7.4 Proof of Proposition 3

The proof is by contradiction. Suppose that there exists a one-dimensional sufficient statistic $\tau\,(y,z)$. Because the joint normal density function $f(y,z;\,\vec{a})$ (2.30) belongs to the subfamily $\varphi(\vec{y}_q;\,\vec{a}_p)$ (2.25), in which sufficient statistics are linear in the basic measures (see Lemma 2), the one-dimensional sufficient statistic can be restricted, without loss of generality, to the class linear in the basic measures y and z :

$$\tau\,(y,z) = r_1\,y + r_2\,z\,, \tag{2.72}$$

where r_1 and r_2 are non-zero constants. Given that the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) are minimal (Lemma 1), the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) must be, by Definition 3, represented as a function of the one-dimensional sufficient statistic $\tau\,(y,z)$ (2.72). That is, there exist functions $l_1(\cdot)$ and $l_2(\cdot)$ such that:

$$\begin{aligned} T_1(y,z) &= l_1\,(\,r_1\,y + r_2\,z\,)\,, \qquad (2.73)\\ T_2(y,z) &= l_2\,(\,r_1\,y + r_2\,z\,)\,. \qquad (2.74)\end{aligned}$$

As the jointly sufficient statistics $T_1(y,z)$ (2.47) and $T_2(y,z)$ (2.48) and the one-dimensional sufficient statistic $\tau\,(y,z)$ (2.72) are all linear in the basic measures y and z, (2.73) and (2.74) can be, without loss of generality, rewritten for some non-zero constants c_1 and c_2 as follows:

$$\begin{aligned} \frac{(m_1 - \phi_1\,k_1)}{\sigma_1^2}\,y + \frac{(k_1 - \phi_2\,m_1)}{\sigma_2^2}\,z &= c_1\,(\,r_1\,y + r_2\,z\,)\,,\;\; c_1 \in \mathbb{R}\,, \qquad (2.75)\\ \frac{(m_2 - \phi_1\,k_2)}{\sigma_1^2}\,y + \frac{(k_2 - \phi_2\,m_2)}{\sigma_2^2}\,z &= c_2\,(\,r_1\,y + r_2\,z\,)\,,\;\; c_2 \in \mathbb{R}\,, \qquad (2.76)\end{aligned}$$

where $\phi_1 = \rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2\,/\,\sigma_2^2$ and $\phi_2 = \rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2\,/\,\sigma_1^2$. It follows from (2.75) and (2.76):

$$\frac{m_1 - \phi_1\,k_1}{m_2 - \phi_1\,k_2} = \frac{k_1 - \phi_2\,m_1}{k_2 - \phi_2\,m_2}\,, \qquad (2.77)$$

which leads to a contradiction:

$$(m_1\,k_2 - m_2\,k_1)\left(1 - \rho_{\varepsilon\delta}^2\right) = 0\,. \qquad (2.78)$$

Therefore, no one-dimensional sufficient statistic exists if the basic measures y and z are not perfectly aligned $(m_1 k_2 - m_2 k_1) \neq 0$.

2.7.5 Proof of Lemma 2

The joint density function of the subfamily $\varphi(\vec{y}_q;\,\vec{a}_p)$ (2.25) can be rewritten with (2.53) as follows:

$$\varphi(\vec{y}_q;\,\vec{a}_p) = exp\left[\,\sum_{i=1}^{p} T_i\,(\vec{y}_q)\;a_i + \alpha(\vec{a}_p) + \beta(\vec{y}_q)\,\right], \qquad (2.79)$$

which satisfies the factorization criterion (Definition 2) with the p-dimensional jointly sufficient statistics $\overrightarrow{T(\vec{y}_q)} = (T_1(\vec{y}_q), T_2(\vec{y}_q), \cdots, T_p(\vec{y}_q))$. By the definition in (2.12), the likelihood ratio (2.54) follows from (2.79).

2.7.6 Proof of Proposition 4

The proof is by contradiction. The constant matrix Ω_{qp} in (2.26) is full-rank ($rank(\Omega_{qp}) = p$) by assumption. Suppose that a set of $(p-1)$-dimensional jointly sufficient statistics exists. By the factorization criterion (Definition 2), the density function $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.79) should be represented as a function of a set of $(p-1)$-dimensional jointly sufficient statistics. Because the principal estimates each of the p-dimensional tasks $\vec{a} = (a_1, a_2, \cdots, a_p)$ and a sufficient statistic $T_i(\vec{y}_q)$ should be independent of all parameters to estimate, the density function $\varphi(\vec{y}_q;\, \vec{a}_p)$ (2.79) can be represented as a function of a set of $(p-1)$-dimensional jointly sufficient statistics if, and only if, an element of the p-dimensional jointly sufficient statistics $\overrightarrow{T(\vec{y}_q)} = (T_1(\vec{y}_q), T_2(\vec{y}_q), \cdots, T_p(\vec{y}_q))$ (2.53) is a function of the other $(p-1)$ elements: as an example without loss of generality,

$$T_p(\vec{y}_q) = \gamma\big(T_1(\vec{y}_q), T_2(\vec{y}_q), \cdots, T_{p-1}(\vec{y}_q)\big), \tag{2.80}$$

where $\gamma(\cdot)$ is a function. As the p-dimensional jointly sufficient statistics $\overrightarrow{T(\vec{y}_q)}$ (2.53) are all linear in the basic measures ($y_1, y_2, \cdots, y_q$), (2.80) is equivalent to an element being a linear combination of the other $(p-1)$ elements of the p-dimensional sufficient statistics:

$$T_p(\vec{y}_q) = r_1 T_1(\vec{y}_q) + r_2 T_2(\vec{y}_q) + \cdots + r_{p-1} T_{p-1}(\vec{y}_q), \tag{2.81}$$

with some constants $r_1, r_2, \cdots, r_{p-1} \in \mathbb{R}$ such that not all $r_1, r_2, \cdots, r_{p-1} = 0$. That is:

$$\begin{bmatrix} \omega_{1p} \\ \omega_{2p} \\ \vdots \\ \omega_{qp} \end{bmatrix} = r_1 \begin{bmatrix} \omega_{11} \\ \omega_{21} \\ \vdots \\ \omega_{q1} \end{bmatrix} + r_2 \begin{bmatrix} \omega_{12} \\ \omega_{22} \\ \vdots \\ \omega_{q2} \end{bmatrix} + \cdots + r_{p-1} \begin{bmatrix} \omega_{1(p-1)} \\ \omega_{2(p-1)} \\ \vdots \\ \omega_{q(p-1)} \end{bmatrix}. \tag{2.82}$$

However, this is a contradiction to the assumption that all columns of the constant matrix Ω_{qp} in (2.26) are linearly independent. Thus, no $(p-1)$-dimensional jointly sufficient statistics exist when the agent's effort is on p-dimensional tasks. Similarly, it can be shown that no jointly sufficient statistics with dimensionality $1, 2, \cdots, (p-2)$ exist.

2.7.7 Proof of Proposition 6

The proof follows Proposition 3 in Christensen, Şabac, and Tian (2010). With a square-root utility function (2.56) (that is, $U(C(y_1, y_2, \cdots, y_q)) = 2\sqrt{C(y_1, y_2, \cdots, y_q)}$), the characterization of the efficient contract (2.11) is equivalent to:

$$\sqrt{C(y_1, y_2, \cdots, y_q)} = \lambda + \vec{\mu} \cdot L(\vec{y} \,|\, \vec{a}). \tag{2.83}$$

Substituting (2.83) into the incentive compatibility constraint (2.7) with

$$\int \cdots \int \nabla_a d(\vec{y} \,|\, \vec{a})\, dy_1 \cdots dy_q = 0 \tag{2.84}$$

results in:

$$\int \cdots \int 2\, \vec{\mu} \cdot L(\vec{y} \,|\, \vec{a})\, \nabla_a d(\vec{y} \,|\, \vec{a})\, dy_1 \cdots dy_q - \nabla_a v(a_1, a_2, \cdots, a_p) = 0. \tag{2.85}$$

Using the definition $L(\vec{y}\,|\,\vec{a}) = \nabla_a d(\vec{y}\,|\,\vec{a}) \,/\, d(\vec{y}\,|\,\vec{a})$ and $E[L(\vec{y}\,|\,\vec{a})] = 0$, and rearranging the term $\vec{\mu} \cdot L(\vec{y}\,|\,\vec{a})\, L(\vec{y}\,|\,\vec{a}) = L(\vec{y}\,|\,\vec{a})\, L(\vec{y}\,|\,\vec{a})^T\, \vec{\mu}$, (2.85) becomes:

$$2\,\Sigma L(\vec{y})\;\vec{\mu} - \nabla_a v\,(\,a_1,\, a_2,\, \cdots\,,\, a_p\,) = 0\,, \tag{2.86}$$

where the variance-covariance matrix of the likelihood ratio is denoted by $\Sigma L(\vec{y})$:

$$\Sigma L(\vec{y}) = Var\,(\,L(\vec{y}\,|\,\vec{a})\,) = E\left[(\,L(\vec{y}\,|\,\vec{a}) - E[L(\vec{y}\,|\,\vec{a})]\,)\,(\,L(\vec{y}\,|\,\vec{a}) - E[L(\vec{y}\,|\,\vec{a})]\,)^T\right]. \tag{2.87}$$

The vector of Lagrange multipliers $\vec{\mu} = (\,\mu_1,\, \mu_2,\, \cdots\,,\, \mu_p\,)$ is explicitly calculated from (2.86):

$$\vec{\mu} = \frac{1}{2}\,[\,\Sigma L(\vec{y})\,]^{-1}\,\nabla_a v\,(\,a_1,\, a_2,\, \cdots\,,\, a_p\,)\,. \tag{2.88}$$

From (2.83) and (2.88), the efficient aggregation of the basic measures $(\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ is characterized by:

$$[\,\Sigma L(\vec{y})\,]^{-1}\,\nabla_a v\,(\,a_1,\, a_2,\, \cdots\,,\, a_p\,) \cdot L(\vec{y}\,|\,\vec{a})\,. \tag{2.89}$$

Finally, by applying (2.54) to (2.89), the efficient aggregation of the basic measures $(\,y_1,\, y_2,\, \cdots\,,\, y_q\,)$ is characterized by (2.57).

2.7.8 Proof of Corollary 3

The two-dimensional jointly sufficient statistics $T_1(y,z)$ and $T_2(y,z)$ from the joint normal density function $f(y,z;\,\vec{a})$ (2.30) are as in (2.47) and (2.48). The likelihood ratio $L(y,z\,;\,\vec{a})$ of the joint normal density function $f(y,z;\,\vec{a})$ (2.30) is as follows:

$$L(y,z\,;\,\vec{a})_{2\times 1} = \frac{\nabla_a f(y,z;\,\vec{a})}{f(y,z;\,\vec{a})} = \frac{1}{(1-\rho^2_{\varepsilon\delta})}\left[\begin{array}{c} \frac{M1}{\sigma_1^2}(y - \vec{m}\cdot\vec{a}) + \frac{K1}{\sigma_2^2}(z - \vec{k}\cdot\vec{a}) \\ \frac{M2}{\sigma_1^2}(y - \vec{m}\cdot\vec{a}) + \frac{K2}{\sigma_2^2}(z - \vec{k}\cdot\vec{a}) \end{array}\right], \tag{2.90}$$

where $M1 = m_1 - \phi_1 k_1$, $M2 = m_2 - \phi_1 k_2$, $K1 = k_1 - \phi_2 m_1$, $K2 = k_2 - \phi_2 m_2$, and $\phi_1 = \rho_{\varepsilon\delta} \sigma_1 \sigma_2 / \sigma_2^2$, $\phi_2 = \rho_{\varepsilon\delta} \sigma_1 \sigma_2 / \sigma_1^2$. The variance-covariance matrix of the likelihood ratio $L(y, z\,; \vec{a})$ (2.90) is explicitly obtained as follows:

$$\Sigma L(y,z)_{2\times 2} = \frac{1}{(1-\rho_{\varepsilon\delta}^2)^2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.91}$$

where $A_{11} = (M1)^2 / \sigma_1^2 + (K1)^2 / \sigma_2^2 + 2\,\rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2 \left(M1/\sigma_1^2\right) \left(K1/\sigma_2^2\right)$,

$A_{22} = (M2)^2 / \sigma_1^2 + (K2)^2 / \sigma_2^2 + 2\,\rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2 \left(M2/\sigma_1^2\right) \left(K2/\sigma_2^2\right)$,

$A_{12} = A_{21} = (M1\,M2) / \sigma_1^2 + (K1\,K2) / \sigma_2^2 + \rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2\,(M1\,K2 + M2\,K1) / (\sigma_1^2\,\sigma_2^2)$.

It turns out that the variance-covariance matrix $\Sigma L(y,z)$ (2.91) is invertible unless the basic measures y and z are perfectly aligned $(m_1 k_2 - m_2 k_1) \neq 0$. Substituting the jointly sufficient statistics $T_1(y,z)$ and $T_2(y,z)$ ((2.47) and (2.48)), the agent's marginal action cost $\nabla_a v(\vec{a})$ (2.58), and the variance-covariance matrix $\Sigma L(y,z)$ (2.91) into (2.57) of Proposition 6 results in the efficient linear aggregation, which is characterized by the following relative weight on the basic measures y and z:

$$\frac{[\Sigma L(y,z)]^{-1}\,\vec{a} \cdot (M1, M2) / \sigma_1^2}{[\Sigma L(y,z)]^{-1}\,\vec{a} \cdot (K1, K2) / \sigma_2^2} = \frac{a_2\,k_1 - a_1\,k_2}{a_1\,m_2 - a_2\,m_1}\,. \tag{2.92}$$

Chapter 3

Multiple Periods

3.1 Introduction

Inter-temporally correlated performance measures, such as earnings, share prices, and items in balanced-scorecards, are ubiquitous in business. Although inter-temporally correlated performance measures are widely used for performance evaluation purposes, no previous research has dealt with the aggregation of performance measures and relative incentive rate in a multi-period setting.

When performance measures are inter-temporally correlated, information on future performance measures can be extracted from the observation on realized performance measures. Thus, the sufficient aggregation of performance measures is affected by the inter-temporal correlations among performance measures, and the sufficient aggregation of the basic measures in a multi-period setting is qualitatively different from that of a single-period setting. This study examines the nature and characteristics of sufficient aggregations of inter-temporally correlated performance measures.

In designing a performance evaluation system for providing incentive compensation to managers, a management accountant may aggregate the basic performance measures to a smaller number of aggregate performance measures. If the management accountant is in a multi-period setting, in which performance measures are inter-temporally correlated, what would be the nature and characteristics of feasible sufficient aggregations of performance measures?

A question in a multi-period setting is whether the aggregate performance measures can carry all the information from the basic performance measures without losing any information in the aggregation process. In a multi-period setting, this study asks whether it is possible to construct a statistically sufficient aggregate measure, that is determined only by the statistical properties of performance measures and good enough to substitute for the basic measures in all agencies. A statistically sufficient aggregation is characterized by sufficient statistics of the basic measures and incurs no loss of information regardless of the characteristics of the agent and the economic situation of the agency. This study shows that in a multi-period setting, there is no statistically sufficient aggregation of performance measures even if the "all a or no a" condition is satisfied such that there is no loss of information through the likelihood ratio.

Another question in a multi-period setting is whether the aggregate performance measures can be commonly used for all managers or should be individually "tailored" for each manager such that different managers are evaluated on different performance measures. This study explicitly shows that in a multi-period setting, the optimal aggregation of performance measures depends on the agent's characteristics and the economic situation of the agency as well as the statistical properties of performance measures. In a multi-period setting, an efficient aggregation, which incurs no economic loss to the principal and substitutes for the basic measures in the minimum cost contract for inducing a single effort level, depends on the effort level to induce even if there is no information loss through the likelihood ratio. Because the effort level in turn depends on the agent's risk aversion and the first best effort level, the optimal aggregation depends on the characteristics of the agent and the economic situation of the agency.

If performance measures are inter-temporally correlated, it is no longer possible to have a universal evaluation system and the aggregate performance measure incurs loss of information in the aggregation process. The optimality of a certain aggregate performance measure is limited to a specific manger in a specific firm. In a multi-period setting, different aggregate performance measures should be used for different managers in different firms.

In a multi-period setting, this study analyzes the feasibility of a statistically sufficient aggregation and the characteristics of an economically sufficient aggregation. For tractability, this study employs a LEN model (Linear contract, Exponential utility of the agent, Normal distribution of performance measure) and the aggregation of performance measures is restricted to linear aggregation.

This study shows that there is no statistically sufficient aggregation of performance measures in a multi-period setting. Even when it is possible to construct a "myopic" aggregate measure of the basic measures which is equivalent to a sufficient statistic in a single-period setting, the “myopic”aggregate measure is not optimal or efficient in a multi-period setting.

A “myopic”aggregate measure fails to consider the “inter-temporal”effect resulting from the inter-temporal correlations of performance measures. The inter-temporal correlations among performance measures reduce the conditional variances and affect the future period incentive rates, which are ex-post efficient. As the future period incentive rates come into the principal's problem through the compensation risk regarding the inter-temporal covariances among performance measures, the inter-temporal effect should be expected and considered in equilibrium. The inter-temporal effect is rendered by the conditional density function of the future period performance measures, which is not considered in a “myopic”aggregation. As long as performance measures are inter-temporally correlated, a “myopic”aggregation is not optimal or efficient in a multi-period setting.

For economic sufficiency of aggregation, this study shows that in a multi-period setting, an efficient aggregation, which is to induce an exogenously fixed effort level, depends on the effort level to induce even if the “all a or no a”condition is satisfied such that there is no loss of information through the likelihood ratio. In particular, the relative weight on the basic measures in the efficient aggregation depends on the effort level in a multi-period setting, even when it is independent of the effort level in a single-period setting. This study also shows that in a multi-period setting, the optimal aggregation, which is to induce the endogenously determined effort level and is a special case of the efficient aggregation, depends on the agent's characteristics and the economic situation of the agency, which are respectively represented by the agent's risk aversion and the first best effort level.

By employing the two-stage optimization approach of Grossman and Hart (1983), this study analyzes how the agent's characteristics and the economic situation of the agency affect the optimal aggregation. In a multi-period setting, an efficient aggregation depends on the effort level and the statistical properties of performance measures, but not on the agent's characteristics and the economic situation of the agency. It is the optimal effort level that brings the agent's characteristics and the economic situation of the agency into the optimal aggregation.

This study contributes to the literature by first discussing the aggregation of performance measures in a multi-period agency. Holmstrom (1979) proposes the informativeness condition such that additional performance measures are informative on the agent's unobservable effort unless the existing measure is a sufficient statistic, and the result implies that the optimal aggregation of performance measures is characterized by sufficient statistics. On the other hand, Amershi, Banker, and Datar (1990) show that the optimal aggregation of performance measures is not generally characterized by sufficient statistics, but the optimal aggregation is generally economically sufficient and agency-specific because a statistical condition ("all a or no a" condition) is generally not satisfied. Christensen, Şabac, and Tian (2010) analyze the efficient contract in multi-task agencies and discuss the role of the likelihood ratio and the variance of the likelihood ratio in ranking performance measures. All of these studies are restricted to a single-period setting.

In this study, a two-period contract with renegotiation is analyzed, not only because renegotiation is an important institutional feature but also because, stripping away wealth effects and income smoothing, a long-term contract on a single-task with full commitment is equivalent to a single-period contract on multiple tasks, which was analyzed in the previous chapter. Provided with full commitment, a two-period contract on a single-task is equivalent to a single-period contract on two tasks, and the principal can use both periods measures in controlling both periods effort. On the other hand, when a two-period contract is renegotiated, the principal is no longer able to use the second period measures in controlling the first period effort. With renegotiation, the "global" aggregation of both periods measures with respect to both periods effort is inadequate. Thus, this study analyzes only the "period-by-period" aggregation of performance measures with respect to each period effort.

The analysis of sufficient aggregation of performance measures in a multi-period setting stems from the literature on the long-term contract with renegotiation. With the renegotiation-proof contract, Christensen, Feltham, and Şabac (2003, 2005) introduce the inter-temporal covariance risk and the insurance motive of the principal. Şabac (2007, 2008) studies the effect of risk externality and effort externality on the incentive dynamics in an N-period setting. Recently, Şabac (2009) analyzes the sufficiency condition and the conditional controllability in multi-period agencies. For tractability, all of these studies on the long-term contract with renegotiation employ a LEN model.

In a multi-period setting, this study shows that the concept of sufficient statistics is too restrictive to characterize the optimal aggregation of performance measures. The resulting restrictiveness of the concept of sufficient statistics is similar to that of Amershi, Banker, and Datar (1990) in a single-period setting. However, the reason for the restrictiveness in a multi-period setting is qualitatively different from that of a single-period setting. While Amershi, Banker, and Datar (1990) show in a single-period setting that in general, the optimal aggregate measures are agency-specific and not sufficient statistics because a statistical condition ("all a or no a" condition) is generally not satisfied, this study shows that it is not only a statistical condition ("all a or no a" condition) but also the existence of inter-temporal correlations among performance measures that causes the optimal aggregation in a multi-period setting to be agency-specific and not to be sufficient statistics. Particularly, this study shows that even if the "all a or no a" condition is satisfied, the optimal aggregation of performance measures is not statistically sufficient if performance measures are inter-temporally correlated.

The rest of this study is organized as follows: Section 2 discusses the modeling features. Section 3 analyzes the sufficient aggregation of performance measures in a single-period setting. Section 4 analyzes the sufficient aggregation of performance measures in a multi-period setting. Section 5 confirms the result by taking the likelihood ratio approach of Şabac (2009). Section 6 concludes the study.

3.2 Modeling features

For an economically sufficient aggregation of performance measures, this study employs a LEN model in a two-period setting with two performance measures on a single-task. The two-period contract is characterized as a renegotiation-proof contract.

3.2.1 LEN model

A risk neutral principal owns a production technology, which requires a single effort (a_1, a_2) in each of two periods $t = 1, 2$ from a risk and effort averse agent. The economic outcome from the agency is not contractible and the principal and the agent write a contract, which is linear on a sequence of two contractible performance measures $(y_t, z_t),\ t = 1, 2$.

The principal receives an economic benefit from the agent's effort $B(a_1, a_2) = b_1\, a_1 + b_2\, a_2$ and his utility is represented by $U^P = B(a_1, a_2) - C(y_1, z_1, y_2, z_2)$, where $C(y_1, z_1, y_2, z_2)$ is the compensation from the contract. The agent has a multiplicatively separable exponential utility $U^a = -exp\Big[-r\big\{C(y_1, z_1, y_2, z_2) - K(a_1, a_2)\big\}\Big]$, where r is the agent's absolute risk aversion, and $K(a_1, a_2)$ is the agent's personal action cost $K(a_1, a_2) = \frac{1}{2}\Big(a_1^2 + a_2^2\Big)$. The agent has a single consumption date and no time value of money is assumed for simplicity. Thus, only the total amount of compensation matters to the agent. A similar result in a multiple consumption date setting could be obtained if the agent is allowed access to unlimited borrowing and lending opportunity with the same interest rate as the principal's (Dutta and Reichelstein 1999). As the contract proceeds, performance measures (y_t, z_t) are realized and the agent's compensation accrues (savings account) such that future performance does not affect already accrued results.

The performance measures y_t and z_t are joint normally distributed with normally distributed residual terms:

$$y_t = m_t\,a_t + \varepsilon_t \tag{3.1}$$

$$z_t = k_t\,a_t + \delta_t, \quad t = 1,\,2\,, \tag{3.2}$$

Each performance measure y_t and z_t has sensitivity m_t and k_t to the agent's effort a_t, $t = 1,\,2$. The performance measures y_t and z_t are also affected by uncontrollable random factors ε_t and δ_t which are zero-mean normally distributed. It is assumed that the residual terms ε_t and δ_t are correlated. Both time-series and cross-sectional correlations among performance measures are allowed, without restriction on the correlation structure. For simplicity, no long-term action is assumed. The agent's effort a_t affects only the current period performance measures (y_t, z_t) and not the future period performance measures.

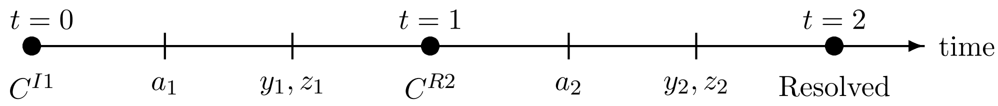
Since the performance measures y_t and z_t are normally distributed, the agent's certainty equivalent at the start of each period $t = 1,\,2$ is represented by the expected compensation minus the risk premium and action cost:

$$ACE_{t-1} = E_{t-1}\Big[C(y_1, z_1, y_2, z_2)\Big] - \frac{1}{2}\,r\,Var_{t-1}\Big[C(y_1, z_1, y_2, z_2)\Big] - K(a_1, a_2)\,, \tag{3.3}$$

where the expectation $E_{t-1}[\,\cdot\,]$ and variance $Var_{t-1}[\,\cdot\,]$ are conditional on the realized performance measures y_1 and z_1 for the second period ($t = 2$), and unconditional for the first period ($t = 1$).

To make sure that the agent participates in the contract, the principal should guarantee a reservation certainty equivalent, which is alternatively available to the agent in the labor market. Since the amount of a reservation certainty equivalent is independent of the analysis, a zero reservation certainty equivalent is assumed without loss of generality.

3.2.2 Renegotiation-proof contract



(Figure 1)

Renegotiation is an important institutional feature since it is not feasible to prevent the contract parties from mutually beneficial ex-post efficient renegotiation. A renegotiation-proof contract is a contract such that once it is agreed upon at the initial point, there is no contract at any later renegotiation point which is weakly preferred to the initial contract by both parties and strictly preferred by at least one party. That is, no ex-post Pareto improvement is made by a renegotiation offer. It is assumed that both the principal and the agent are committed to the employment relation for the full duration of the contract, but the principal can change the terms of contract if the agent agrees.

In the sequel, the superscript I is used for the initial contract at the renegotiation date ($t = 1$, see Figure 1) and R is used for renegotiation offers. At the start of the first period, $t = 0$, the principal offers to the agent an initial contract C^{I1}, which is denoted by a sequence of incentive rates :

$$C^{I1} = \left\{\alpha_0, \left(\beta_1^1, \beta_1^2\right), \left(\beta_2^1, \beta_2^2\right) \right\}, \tag{3.4}$$

where α_0 is a fixed payment, and the superscript to the incentive rate β indicates the performance measure such that 1 is for y and 2 is for z while the subscript to the incentive rate β indicates the period. The initial contract C^{I1} is the contract in effect unless replaced by a renegotiation offer. The agent either accepts or rejects it. Once the agent accepts the initial contract offer, the agent provides the first period effort a_1. Before the end of the first period, the principal and the agent observe two contractible performance measures y_1 and z_1.

At $t = 1$, the principal makes a take-it-or-leave-it renegotiation offer C^{R2} :

$$C^{R2} = \{\alpha_1, (\beta_2^1, \beta_2^2)\}, \tag{3.5}$$

where the fixed payment α_1 is a function of the realized performance measures y_1 and z_1. If the renegotiation offer is rejected, C^{I2} is the contract in effect for the second period. If accepted, C^{R2} becomes the contract in effect. The agent provides the second period effort a_2 and the principal and the agent observe two contractible performance measures y_2 and z_2 before the end of the second period. At the terminal date, $t = 2$, the agent receives the compensation based on the realized values of performance measures and the contract is resolved.

In general, when no restriction is imposed on the contract form, there is no loss of generality in restricting the analysis to renegotiation-proof contracts. As Christensen, Feltham, and Şabac $(2003, 2005)$ and Şabac $(2007, 2008)$ have proved the renegotiation-proofness principle for the LEN model, the analysis of a linear optimal contract can be, without loss of generality, restricted to the linear renegotiation-proof contract :

$$C(y_1, z_1, y_2, z_2) = \alpha_0 + \beta_1^1\, y_1 + \beta_1^2\, z_1 + \beta_2^1\, y_2 + \beta_2^2\, z_2\,. \tag{3.6}$$

At the decision point of the first period incentive rates β_1^1 and β_1^2, the second period incentive rates β_2^1 and β_2^2 are restricted to be ex-post efficient and rationally expected by the principal and the agent due to the renegotiation-proofness requirement.

The contracting and renegotiation can be summarized by the principal's problems at the start of each period $t = 1,\, 2$ as follows :

$$\max_{\beta_t^1, \beta_t^2} E_{t-1}\Big[B(a_1, a_2) - C^{Rt}\Big], \quad \Big(E\Big[B(a_1, a_2) - C^{I1}\Big] \text{ for the first period}\Big) \tag{3.7}$$

subject to the renegotiation-proofness constraint

$$\left(\beta_2^1, \beta_2^2\right) \text{ are optimal at the start of the second period}, \tag{3.8}$$

the incentive compatibility constraint

$$a_t \in \arg\max\ ACE_{t-1}(C^{Rt})\,, \quad (\,a_1 \in \arg\max\ ACE(C^{I1}) \text{ for the first period}\,) \quad (3.9)$$

and the participation constraint

$$ACE_{t-1}(C^{Rt}) \geq ACE_{t-1}(C^{It})\,. \quad (\,ACE(C^{I1}) \geq 0 \text{ for the first period}\,) \quad (3.10)$$

3.2.3 Joint distributions of performance measures and likelihood ratios

Whereas Amershi, Banker, and Datar (1990) work with optimal contracts in a single-period setting, this study employs a LEN model in a two-period setting and the aggregation of performance measures is restricted to linear aggregation. Generally, a LEN contract can be expressed as a linear function of likelihood ratios (Christensen, Şabac, and Tian 2010; Şabac 2009).

For the analysis in the sequel, this section introduces the joint normal distributions of performance measures and the likelihood ratios. At the start of the first period, the basic measures (y_1, z_1, y_2, z_2) given by (3.1) and (3.2) follow the joint normal density function:

$$f(y_1, z_1, y_2, z_2;\, a_1, a_2) = \frac{1}{(2\,\pi)^2\,\sqrt{det(\Sigma)}}\, exp\left[-\frac{1}{2}\,\vec{x}^T\Sigma^{-1}\vec{x}\right], \qquad (3.11)$$

where $\vec{x} = (y_1 - m_1\,a_1,\; z_1 - k_1\,a_1,\; y_2 - m_2\,a_2,\; z_2 - k_2\,a_2)$, Σ is the variance-covariance matrix of the basic measures (y_1, z_1, y_2, z_2), and $det(\cdot)$ denotes the determinant of matrix. The likelihood ratio L_1 of the joint density function $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$

(3.11) with respect to the first period effort a_1 is defined as follows[1]:

$$L_1 = \frac{\partial\, f(y_1, z_1, y_2, z_2;\; a_1, a_2)}{\partial\, a_1} \;/\; f(y_1, z_1, y_2, z_2;\; a_1, a_2)\,. \tag{3.13}$$

From the joint density function $f(y_1, z_1, y_2, z_2;\; a_1, a_2)$ (3.11), the marginal distribution of the first period measures y_1 and z_1 is given by the marginal density function $\overline{f_1}(y_1, z_1;\; a_1)$ of y_1 and z_1 (see the single period joint density function (2.30)):

$$\begin{aligned}\overline{f_1}(y_1, z_1;\; a_1) &= \iint f(y_1, z_1, y_2, z_2;\; a_1, a_2)\, dy_2\, dz_2 \\ &= \frac{1}{2\pi\sqrt{1-\rho_{\varepsilon\delta}^2}\,\sigma_1\,\sigma_2}\; exp\left[-\frac{1}{2(1-\rho_{\varepsilon\delta}^2)}\,\Xi_1\right],\end{aligned} \tag{3.14}$$

where

$\Xi_1 = (y_1 - m_1\, a_1)^2 \,/\, \sigma_1^2 - 2\,\rho_{\varepsilon\delta}\;(y_1 - m_1\, a_1)\,(z_1 - k_1\, a_1)\,/\,(\sigma_1\,\sigma_2) + (z_1 - k_1\, a_1)^2\,/\,\sigma_2^2$, $Var(y_1) = \sigma_1^2$, $Var(z_1) = \sigma_2^2$, and $Cov(y_1, z_1) = \rho_{\varepsilon\delta}\,\sigma_1\,\sigma_2$.

The likelihood ratio $\overline{L}_1$ with respect to the first period effort a_1 is defined and explicitly presented from the marginal density function $\overline{f_1}(y_1, z_1;\; a_1)$ (3.14):

$$\begin{aligned}\overline{L}_1 &= \frac{\partial\, \overline{f_1}(y_1, z_1;\; a_1)}{\partial\, a_1} \;/\; \overline{f_1}(y_1, z_1;\; a_1) \\ &= (1-\rho_{\varepsilon\delta}^2)\left[\frac{(m_1 - \phi_{11}\, k_1)}{Var(y_1)}\;(y_1 - m_1\, a_1) + \frac{(k_1 - \phi_{21}\, m_1)}{Var(z_1)}\;(z_1 - k_1\, a_1)\right],\end{aligned} \tag{3.15}$$

where $\phi_{11} = Cov(y_1, z_1)\,/\,Var(z_1)$ and $\phi_{21} = Cov(y_1, z_1)\,/\,Var(y_1)$. The likelihood ratio $\overline{L}_1$ (3.15) satisfies the "all a or no a" condition (Definition 8) in the original form of Amershi, Banker, and Datar (1990). The likelihood ratio $\overline{L}_1$ (3.15) can be expressed as a function of the same sufficient statistic $T_1(y_1, z_1)$ of the basic measures

[1] With $\vec{x}$ as in (3.11), the likelihood ratio L_1 is explicitly given as follows:

$$\begin{aligned}L_1 &= \vec{x}\cdot\big(\,m_1\, Row_1(\Sigma^{-1}) + k_1\, Row_2(\Sigma^{-1})\,\big) \\ &= (\,y_1,\, z_1,\, y_2,\, z_2\,)\cdot\big(\,m_1\, Row_1(\Sigma^{-1}) + k_1\, Row_2(\Sigma^{-1})\,\big) \\ &\quad - (\,m_1\, a_1,\, k_1\, a_1,\, m_2\, a_2,\, k_2\, a_2\,)\cdot\big(\,m_1\, Row_1(\Sigma^{-1}) + k_1\, Row_2(\Sigma^{-1})\,\big)\,,\end{aligned} \tag{3.12}$$

where $Row_1(\Sigma^{-1})$ and $Row_2(\Sigma^{-1})$ refer to the first and second rows in the inverse matrix of the variance-covariance matrix, Σ^{-1}. From (3.12), note that the likelihood ratio L_1 satisfies the "all a or no a" condition (Definition 8).

y_1 and z_1 for all effort levels $a_1 \in \mathbb{R}$ such that the function depends on the basic measures only through the same sufficient statistic:

$$\overline{L}_1 = q_1\left(T_1(y_1, z_1)\right) \text{ for all } a_1 \in \mathbb{R}, \tag{3.16}$$

where

$$T_1(y_1, z_1) = \frac{(m_1 - \phi_{11}\, k_1)}{Var(y_1)}\, y_1 + \frac{(k_1 - \phi_{21}\, m_1)}{Var(z_1)}\, z_1\,, \tag{3.17}$$

$\phi_{11} = Cov(y_1, z_1)\,/\,Var(z_1)$, $\phi_{21} = Cov(y_1, z_1)\,/\,Var(y_1)$, and $q_1(\cdot)$ is a function.

The joint distribution of the second period measures y_2 and z_2, conditional on the first period realized measures y_1 and z_1, is characterized by the conditional density function:

$$f_2(y_2, z_2|\, y_1, z_1, a_2) = \frac{f(y_1, z_1, y_2, z_2;\, a_1, a_2)}{\overline{f_1}(y_1, z_1;\, a_1)}\,, \tag{3.18}$$

where $\overline{f_1}(y_1, z_1;\, a_1)$ is as in (3.14) and $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$ is as in (3.11). The likelihood ratio L_2 of the conditional density $f_2(y_2, z_2|\, y_1, z_1, a_2)$ (3.18) with respect to the second period effort a_2 can be calculated from the joint density function $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$ (3.11) because $\partial\, \overline{f_1}(y_1, z_1;\, a_1)\,/\,\partial\, a_2 = 0$:

$$\begin{aligned} L_2 &= \frac{\partial\, f_2(y_2, z_2|\, y_1, z_1, a_2)}{\partial\, a_2}\, /\, f_2(y_2, z_2|\, y_1, z_1, a_2) \\ &= \frac{\partial\, f(y_1, z_1, y_2, z_2;\, a_1, a_2)}{\partial\, a_2}\, /\, f(y_1, z_1, y_2, z_2;\, a_1, a_2) \\ &= -\frac{1}{2}\, \frac{\partial\left(\vec{x}^T \Sigma^{-1} \vec{x}\right)}{\partial\, a_2} = -\frac{1}{2}\, \frac{\partial\, \vec{x}}{\partial\, a_2}\, \frac{\partial\left(\vec{x}^T \Sigma^{-1} \vec{x}\right)}{\partial\, \vec{x}} \\ &= (0,\, 0,\, m_2,\, k_2)\; \Sigma^{-1}\, \vec{x}\,, \end{aligned} \tag{3.19}$$

where the chain rule is applied in the second last step. If the inverse matrix Σ^{-1} of the variance-covariance matrix of performance measures $(y_1,\, z_1,\, y_2,\, z_2)$ is explicitly written:

$$\Sigma^{-1} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}, \tag{3.20}$$

then the likelihood ratio L_2 (3.19) is equivalent to:

$$\begin{aligned} L_2 = m_2 \Big[& s_{31}\,(y_1 - m_1\,a_1) + s_{32}\,(z_1 - k_1\,a_1) + s_{33}\,(y_2 - m_2\,a_2) + s_{34}\,(z_2 - k_2\,a_2) \Big] \\ + k_2 \Big[& s_{41}\,(y_1 - m_1\,a_1) + s_{42}\,(z_1 - k_1\,a_1) + s_{43}\,(y_2 - m_2\,a_2) + s_{44}\,(z_2 - k_2\,a_2) \Big]. \end{aligned} \tag{3.21}$$

The likelihood ratio L_2 (3.21) also satisfies the "all a or no a" condition (Definition 8) in the original form of Amershi, Banker, and Datar (1990). The likelihood ratio L_2 (3.21) can be expressed as a function of the same sufficient statistic $T_{12}(y_1, z_1, y_2, z_2)$ of the basic measures (y_1, z_1, y_2, z_2) for all effort levels $(a_1, a_2) \in \mathbb{R}^2$ such that the function depends on the basic measures only through the same sufficient statistic:

$$L_2 = q_{12}\,(\,T_{12}(y_1, z_1, y_2, z_2)\,) \;\text{ for all }\; (a_1, a_2) \in \mathbb{R}^2, \tag{3.22}$$

where

$$\begin{aligned} T_{12}(y_1, z_1, y_2, z_2) = &\;(\,m_2\,s_{31} + k_2\,s_{41}\,)\;y_1 + (\,m_2\,s_{32} + k_2\,s_{42}\,)\;z_1 \\ &+ (\,m_2\,s_{33} + k_2\,s_{43}\,)\;y_2 + (\,m_2\,s_{34} + k_2\,s_{44}\,)\;z_2\,, \end{aligned} \tag{3.23}$$

and $q_{12}(\cdot)$ is a function. The two likelihood ratios $\overline{L}_1$ (3.15) and L_2 (3.21), defined and explicitly presented above, are later used when the likelihood ratio approach of Şabac (2009) is applied.

3.3 Sufficient aggregation in a single-period setting with a single-task

Amershi, Banker, and Datar (1990) show in a single-period setting with a single-task that when the "all a or no a"condition is satisfied, it is possible to construct a "utility-independent" and "effort-independent" aggregate measure, which is determined only by the statistical properties of performance measures. The analysis of this section is in accordance with Amershi, Banker, and Datar (1990). In a single-period and single-task setting in which the "all a or no a"condition is satisfied, a statistically sufficient aggregation of performance measures for all effort levels ([1], Definition 1) exists. The same one-dimensional sufficient statistic is good enough to substitute for the basic measures in inducing all effort levels for every manager in all firms.

3.3.1 Signal-to-noise ratio

The aggregation of performance measures in a single-period setting with a single-task is determined by the relative "signal-to-noise"ratio of performance measures. Banker and Datar (1989) discuss the optimal aggregation and relative incentive rate when the joint distribution of performance measures y and z belongs to the family $\phi(y, z;\, a)$ (2.24) given in the previous chapter. Their study shows that the optimal relative incentive rate is equivalent to the relative signal-to-noise ratio of performance measures:

$$\frac{(\,\mu_{1a} - \phi_1\,\mu_{2a}\,)\,/\,Var(y)}{(\,\mu_{2a} - \phi_2\,\mu_{1a}\,)\,/\,Var(z)}\,, \tag{3.24}$$

where $\mu_{1a} = \partial\, E[\,y\,]\,/\,\partial\, a$, $\mu_{2a} = \partial\, E[\,z\,]\,/\,\partial\, a$,

and $\phi_1 = Cov(y, z)/\,Var(z)$, $\phi_2 = Cov(y, z)/\,Var(y)$.

In a single-period setting, the joint normal distribution of performance measures y and z is equivalent to the marginal density function of y_1 and z_1 (3.14). With the variance and covariance terms in the joint normal density function (3.14) and the constant sensitivity of performance measures $\mu_{1a} = m$ and $\mu_{2a} = k$ from (3.1) and (3.2), the relative signal-to-noise ratio (3.24) of performance measures is equivalent to:

$$\frac{(\, m - \phi_1\, k\,) \,/\, \sigma_1^2}{(\, k - \phi_2\, m\,) \,/\, \sigma_2^2}\,, \tag{3.25}$$

where $\phi_1 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_2^2$ and $\phi_2 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 \,/\, \sigma_1^2$.

The marginal density function of y_1 and z_1 (3.14) belongs to the family $\phi(y, z;\, a)$ (2.24) and also satisfies the "all a or no a" condition (see (3.16)). In a single-period setting with a single-task, Amershi, Banker, and Datar (1990) show that if the "all a or no a" condition is satisfied, economic sufficiency of aggregation for the optimal effort level ([4], Definition 7) implies statistical sufficiency of aggregation for all effort levels ([1], Definition 1) ([4] $\Rightarrow$ [1]) such that the four sufficiency concepts of aggregation are equivalent ([1] = [2] = [3] = [4]). This study examines the feasibility and relations of the four sufficiency concepts of aggregation in a multi-period setting. This study will show that as the analysis is extended to a multi-period setting, even if the "all a or no a" condition is satisfied, economic sufficiency of aggregation for the optimal effort level does not imply statistical sufficiency of aggregation for all effort levels ([4] $\not\Rightarrow$ [1]), and the only available sufficiency of aggregation is economic sufficiency ([3], [4]), which depends on the agent's effort level to induce.

3.3.2 Statistical sufficiency of aggregation of performance measures

In a single-period setting, it is possible to construct a statistically sufficient aggregate measure for all effort levels $a \in \mathbb{R}$ because there exists a one-dimensional sufficient statistic for the basic measures y and z with respect to the agent's effort. The existence of a one-dimensional sufficient statistic is directly shown by the factorization criterion (Definition 2). Suppressing the first period subscripts of performance measures, effort, and sensitivity, the joint density function (3.14) is factored out for all effort levels $a \in \mathbb{R}$:

$$\overline{f_1}(y, z;\, a) = g\,(y, z)\; h\,(T\,(y, z)\,;\, a)\,, \tag{3.26}$$

where $g\,(\cdot)$ and $h\,(\cdot)$ are some non-negative functions. $T(y, z)$ is a one-dimensional sufficient statistic with respect to the agent's effort $a \in \mathbb{R}$ as follows:

$$T(y, z) = \frac{(m - \phi_1\, k)}{\sigma_1^2}\, y + \frac{(k - \phi_2\, m)}{\sigma_2^2}\, z\,, \tag{3.27}$$

where $\phi_1 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 / \sigma_2^2$ and $\phi_2 = \rho_{\varepsilon\delta}\, \sigma_1\, \sigma_2 / \sigma_1^2$. It is crucial that the function $h\,(T\,(y, z)\,;\, a)$ depends on the basic measures y and z only through the sufficient statistic $T(y, z)$. Note that the relative weight on the basic measures y and z in the one-dimensional sufficient statistic $T(y, z)$ (3.27) is equal to the relative signal-to-noise ratio (3.25). Therefore, the aggregation of performance measures in a single-period setting with a single-task is determined by the relative signal-to-noise ratio of performance measures, as the "all a or no a" condition is satisfied.

The one-dimensional sufficient statistic $T\,(y, z)$ (3.27) is independent of the agent's effort level $a \in \mathbb{R}$. The one-dimensional sufficient statistic $T\,(y, z)$ can substitute for the basic measures y and z, without loss of information in the aggregation process, in inducing all effort levels for every manager in all agencies. The relative weight on the basic measures y and z in the aggregation is characterized only by the statistical properties of performance measures (signal-to-noise ratio), regardless of the agent's characteristics and the economic situation of the agency. Furthemore, the one-dimensional sufficient statistic $T\,(y, z)$ can substitute for the basic measures y and z in inducing an arbitrary effort level (off the equilibrium path) as well as the optimal effort level (on the equilibrium path).

3.4 Sufficient aggregation in a multi-period setting with a single-task

Periodic performance evaluations are widely practiced, and thus period-specific aggregate performance measures are required and generated. In this section, "period-by-period" aggregation of performance measures is analyzed. With renegotiation and inter-temporally correlated performance measures in a two-period setting, the agency problem in the first period is in a multi-period setting while the agency problem in the second period is effectively in a single-period setting. Thus, the main analysis in this section will be on the aggregation of the first period measures y_1 and z_1.

Little is known about the aggregation of performance measures in a multi-period setting. Amershi, Banker, and Datar (1990) conclude in a single-period setting with a single-task that it is the "all a or no a"condition that determines the feasibility of a "utility-independent"and "effort-independent"aggregation of performance measures.

Moving toward a richer institutional setting of multi-period puts more restrictions on the aggregation of performance measures. In contrast with the conclusion of Amershi, Banker, and Datar (1990), the following analysis shows that efficient aggregations and optimal aggregations in a two-period setting are not statistically sufficient even if the "all a or no a"condition is satisfied. Even when it is feasible to construct a "myopic"aggregate measure, which does not consider the inter-temporal correlations of performance measures and is analogous to a one-dimensional sufficient statistic of the basic measures, the principal does not use the "myopic"aggregate measure because it is not optimal or efficient in a multi-period setting.

There is no statistically sufficient "period-by-period"aggregation of performance measures in a multi-period setting. The efficient aggregation of the first period measures y_1 and z_1 depends on the sequence of efforts (a_1, a_2), while the optimal aggregation of the first period measures y_1 and z_1 depends on the agent's characteristics and the economic situation of the agency as well as the statistical properties of performance measures.

3.4.1 Full commitment benchmark

If a two-period contract is characterized with full commitment precluding renegotiation, the full commitment contract on two periods with a single-task is equivalent to a contract in a single-period setting with two tasks, which was analyzed in the previous chapter. In this case, a one-dimensional sufficient statistic is not available, but only two-dimensional jointly sufficient statistics are available. By the factorization criterion (Definition 2), the two-dimensional jointly sufficient statistics of the basic measures (y_1, z_1, y_2, z_2) with respect to the sequence of efforts (a_1, a_2) can be obtained from the joint density function $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$ (3.11). The joint density function $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$ (3.11) is factorized out for all effort levels of a_1 and a_2 :

$$f(y_1, z_1, y_2, z_2;\, a_1, a_2) = g\,(y_1, z_1, y_2, z_2)\; h\,(T_1\,(y_1, z_1, y_2, z_2)\,, T_2\,(y_1, z_1, y_2, z_2)\,;\, a_1, a_2)\,, \tag{3.28}$$

where $g\,(\cdot)$ and $h\,(\cdot)$ are some non-negative functions, and the two-dimensional jointly sufficient statistics $T_1(y_1, z_1, y_2, z_2)$ and $T_2(y_1, z_1, y_2, z_2)$ are given by :

$$T_1(y_1, z_1, y_2, z_2) \;=\; (\,y_1, z_1, y_2, z_2\,) \cdot \left(\,m_1\, Col_1(\Sigma^{-1}) + k_1\, Col_2(\Sigma^{-1})\,\right), \tag{3.29}$$
$$T_2(y_1, z_1, y_2, z_2) \;=\; (\,y_1, z_1, y_2, z_2\,) \cdot \left(\,m_2\, Col_3(\Sigma^{-1}) + k_2\, Col_4(\Sigma^{-1})\,\right), \tag{3.30}$$

where $Col_i(\Sigma^{-1})$ refers to i^{th} column in the inverse matrix Σ^{-1} of the variance-covariance matrix of the basic measures (y_1, z_1, y_2, z_2).

With full commitment, a one-dimensional efficient aggregation of the basic measures for inducing a single effort level also follows the results in the previous chapter. When the two-period linear contract $C(y_1, z_1, y_2, z_2)$ (3.6) is used with full commitment, the basic measures (y_1, z_1, y_2, z_2) are linearly aggregated into a one-dimensional aggregate measure with the incentive rates $(\beta_1^1, \beta_1^2, \beta_2^1, \beta_2^2)$ as relative weights. As shown in the previous chapter, because the number of aggregate measures (a single-dimension aggregate measure) is less than the number of tasks (effectively two-tasks in a single-period), an efficient aggregation with full commitment for inducing an exogenously fixed sequence of efforts (a_1, a_2) cannot be statistically sufficient and depends on the sequence of efforts to induce (a_1, a_2). The incentive compatibility constraints are as follows:

$$a_1 = \beta_1^1\, m_1 + \beta_1^2\, k_1\,, \tag{3.31}$$

$$a_2 = \beta_2^1\, m_2 + \beta_2^2\, k_2\,. \tag{3.32}$$

The incentive rates on the basic measures (y_1, z_1, y_2, z_2) in the efficient aggregation are given by:

$$\begin{bmatrix} \beta_1^1 \\ \beta_1^2 \\ \beta_2^1 \\ \beta_2^2 \end{bmatrix} = \frac{1}{r}\, \Sigma^{-1} \begin{bmatrix} \mu_1\, m_1 \\ \mu_1\, k_1 \\ \mu_2\, m_2 \\ \mu_2\, k_2 \end{bmatrix}, \tag{3.33}$$

where Σ is the variance-covariance matrix of (y_1, z_1, y_2, z_2) and μ_1 and μ_2 are the Lagrange multipliers to the incentive compatibility constraints (3.31) and (3.32). Substituting the incentive rates from (3.33) in the incentive compatibility constraints (3.31) and (3.32) and solving for the Lagrange multipliers μ_1 and μ_2 show that the Lagrange multipliers μ_1 and μ_2 depends on the sequence of efforts to induce:

$$\mu_1 = h_1(a_1, a_2)\,, \tag{3.34}$$

$$\mu_2 = h_2(a_1, a_2)\,, \tag{3.35}$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are some functions. Thus, the incentive rates $(\beta_1^1, \beta_1^2, \beta_2^1, \beta_2^2)$ in (3.33) depend on the sequence of efforts to induce (a_1, a_2) and the efficient aggregation into a single aggregate measure is effort-dependent.

In the two-period contract $C(y_1, z_1, y_2, z_2)$ (3.6) with full commitment, the principal can use the performance measures (y_1, z_1, y_2, z_2) in both periods to control effort (a_1, a_2). This can be seen in that each incentive rate $(\beta_1^1, \beta_1^2, \beta_2^1, \beta_2^2)$ on the basic measures (y_1, z_1, y_2, z_2) in (3.33) is affected by the sensitivities (m_1, k_1, m_2, k_2) and the variance-covariance matrix Σ of the performance measures (y_1, z_1, y_2, z_2). Thus, effort (a_1, a_2) in (3.31) and (3.32) are controlled by the performance measures (y_1, z_1, y_2, z_2) in both periods.

When the two-period contract $C(y_1, z_1, y_2, z_2)$ (3.6) is characterized with full commitment, both the "global" aggregation and the "period-by-period" aggregation of performance measures are available. The "global" aggregation of performance measures is to aggregate all of both periods basic measures (y_1, z_1, y_2, z_2) with respect to both periods effort (a_1, a_2). The two-dimensional jointly sufficient statistics $T_1(y_1, z_1, y_2, z_2)$ (3.29) and $T_2(y_1, z_1, y_2, z_2)$ (3.30) are "global" aggregate measures. On the other hand, the "period-by-period" aggregation is to aggregate the basic measures of each period. In the "period-by-period" aggregation, the first period aggregation is to aggregate the first period basic measures y_1 and z_1 with respect to the first period effort a_1 and the second period aggregation is to aggregate the second period basic measures y_2 and z_2 with respect to the second period effort a_2. The "period-by-period" aggregate measures are period-specific and time-dependent.

3.4.2 "Myopic" statistically sufficient aggregation

As discussed in the previous section, a two-period contract with full commitment on a single-task each period is equivalent to a single-period setting with two tasks. Because multi-task settings in a single-period were analyzed in the previous chapter, the subsequent analysis focuses on a two-period contract with renegotiation.

The key difference between the full commitment and renegotiation cases lies in the controllability of the first period effort a_1. With full commitment, the principal can use both periods performance measures (y_1, z_1, y_2, z_2) to control the agent's first period effort a_1, as shown in the previous section. With renegotiation, the principal cannot use the second period measures y_2 and z_2 to control the first period effort a_1. At the renegotiation point, $t = 1$, the first period effort a_1 is sunk and the second period performance measures y_2 and z_2 have not been reported. Any risky incentive compensation related to the sunk effort a_1 will not be offered at the renegotiation point, $t = 1$, because such an incentive is ex-post inefficient. Thus, with renegotiation, the second period measures y_2 and z_2 are no longer controlling tools of the first period effort a_1. As a result, when the two-period contract is renegotiated, the "global" aggregation of both periods basic measures (y_1, z_1, y_2, z_2) with respect to both periods effort (a_1, a_2) is inadequate. With renegotiation, only the "period-by-period" aggregation of performance measures is adequate.

In the sequel, the "period-by-period" aggregation of the first period measures y_1 and z_1 with respect to the first period effort a_1 will be analyzed. With renegotiation, the first period effort a_1 is sunk in the second period, and the realized values of the first period measures y_1 and z_1 are also available in the second period. Because of ex-post efficiency imposed by renegotiation, the agency problem in the second period is effectively in a single-period setting and the agency problem in the first period is of interest.

In the first period, the principal rationally expects the second period induced effort. For an exogenously given sequence of efforts (a_1, a_2), the second period effort level a_2 is fixed.[2] For the sequence of optimal efforts $(a_1^\dagger, a_2^\dagger)$, the principal can rationally expect the second period induced effort $a_2^\dagger$ due to the renegotiation-proofness condition. With the rational expectation on the second period induced effort, the principal induces the first period effort. Then, the principal uses the information obtained from the first period realized performance measures for solving the second period agency problem.

In this process, the first period measures y_1 and z_1 affect the expected compensation in two ways. The first way is a "direct" effect as the first period measures y_1 and z_1 are used in the contract $C(y_1, z_1, y_2, z_2)$ (3.6). In the first period, given

[2] With a single performance measure in the second period, there would be no room for renegotiation, at $t = 1$, on the second period incentive rate. With a single performance measure, the binding incentive compatibility constraint together with a fixed effort level would result in a fixed incentive rate. On the other hand, with the two performance measures y_2 and z_2 in the second period, an effective renegotiation takes place, at $t = 1$, on the second period incentive rates. Even with the binding incentive compatibility constraint and a fixed effort level, "under-determination" gives many alternative combinations of incentive rates on y_2 and z_2 to induce a fixed effort level.

the rational expectation on the second period induced effort, the principal induces the first period effort with the incentive rates on the first period measures y_1 and z_1 as if the agency were in a single-period setting. Note that the expected values of the first period measures y_1 and z_1 are affected only by the first period effort ((3.1) and (3.2)). Thus, the direct effect is rendered by the marginal joint distribution $\overline{f_1}(y_1, z_1; a_1)$ (3.14) of the first period measures y_1 and z_1, in which only the first period effort is relevant.

The other way is an "inter-temporal" effect as the first period measures y_1 and z_1 give information on the second period measures y_2 and z_2 through the inter-temporal correlations. The inter-temporal effect results from the inter-temporal correlations of performance measures in multi-period agencies. The inter-temporal correlations of performance measures reduce the conditional variances of the second period measures y_2 and z_2 and affect the second period incentive rates (β_2^1, β_2^2), which are ex-post efficient. Because the second period incentive rates (β_2^1, β_2^2) and the first period incentive rates (β_1^1, β_1^2) together – through the inter-temporal covariance risk $\beta_1^1\, ICR1 + \beta_1^2\, ICR2$ below explained (also see (3.67)) – affect the risk premium necessary for inducing effort from the risk averse agent, the expected compensation is affected by the first period measures y_1 and z_1. The inter-temporal effect is rendered by the joint distribution $f_2(y_2, z_2 | \, y_1, z_1, a_2)$ (3.18) of the second period measures y_2 and z_2, conditional on the first period measures y_1 and z_1.

The direct effect and the inter-temporal effect adjust each other in an inseparable way and fine-tune the compensation. The incentive rates (β_1^1, β_1^2) on the first period measures y_1 and z_1 are constituents of the inter-temporal covariance risk and affect the inter-temporal effect. In turn, the inter-temporal effect is expected in the first period and properly adjusts the direct effect. The renegotiation-proof contract reflects both the direct effect and the inter-temporal effect from the first period measures y_1 and z_1.

The inter-temporal effect comes into the principal's first period problem by the inter-temporal covariance risk factors (ICR) :

Definition 9 *(Inter-temporal covariance risk factors)*

$ICR1$ and $ICR2$ characterize the inter-temporal covariance risk with respect to the performance measures y_1 and z_1, respectively :

$$ICR1 = r\,Cov\left(\, y_1,\ \beta_2^1\, y_2 + \beta_2^2\, z_2 \,\right), \tag{3.36}$$

$$ICR2 = r\,Cov\left(\, z_1,\ \beta_2^2\, z_2 + \beta_2^1\, y_2 \,\right). \tag{3.37}$$

As a multi-period agency problem, the first period agency problem is featured with the inter-temporal covariance risk ($\beta_1^1\, ICR1 + \beta_1^2\, ICR2$) of the first period measures y_1 and z_1 with respect to the second period measures y_2 and z_2. The inter-temporal covariance risk results from the risk externality (Şabac 2008) when there exist covariances among current period and future period performance measures. The risk externality arises because the principal cannot commit to future incentive rates and "cooperate with himself" in determining incentive rates at different points in time. When positive (negative) covariance exists between current period and future period performance measures, it imposes too much (too little) compensation risk to the agent and the principal lowers (raises) current period incentive rates in order to reduce (increase) the current period induced effort. As the inter-temporal covariance risk results from the existence of inter-temporal correlations among performance measures, the inter-temporal covariance risk should be considered in inducing an exogenously fixed sequence of efforts (a_1, a_2) as well as the sequence of optimal efforts $(a_1^\dagger, a_2^\dagger)$.

As the agency problem in the first period is a multi-period agency problem with the inter-temporal effect that is characterized by the inter-temporal covariance risk factors ($ICR1$ and $ICR2$), an aggregation of the first period measures y_1 and z_1 will be "myopic" if only the direct effect is taken into account without the inter-temporal effect. In fact, it is only the first period effort a_1 (as in (3.1) and (3.2)) that affects the distribution of the first period basic measures y_1 and z_1. Thus, using only the marginal joint distribution $\overline{f_1}(y_1, z_1;\, a_1)$ (3.14), it is possible to construct a one-dimensional "myopic" aggregate measure of the first period measures y_1 and z_1, that is equivalent to a one-dimensional sufficient statistic in a single-period setting (see $T(y, z)$ (3.27)). By the factorization criterion (Definition 2) the following one-dimensional "myopic" aggregate measure of the first period period measures y_1 and z_1 is derived from the joint density function $\overline{f_1}(y_1, z_1;\, a_1)$ (3.14) :

$$T^{\,m}(y_1, z_1) = \frac{(m_1 - \phi_{11}\, k_1)}{Var(y_1)}\, y_1 + \frac{(k_1 - \phi_{21}\, m_1)}{Var(z_1)}\, z_1\,, \tag{3.38}$$

where $\phi_{11} = Cov(y_1, z_1) \,/\, Var(z_1)$ and $\phi_{21} = Cov(y_1, z_1) \,/\, Var(y_1)$.

However, the following analysis will show that in a multi-period setting, neither an efficient aggregation nor the optimal aggregation of the first period measures y_1 and z_1 is characterized by the one-dimensional "myopic" statistic $T^{\,m}(y_1, z_1)$ (3.38). In a multi-period setting with the inter-temporal effect, the principal does not use the one-dimensional "myopic" statistic $T^{\,m}(y_1, z_1)$ of the first period measures y_1 and z_1. As long as the inter-temporal covariance risk (represented by $ICR1$ and $ICR2$) exists, there is no statistically sufficient aggregation.

3.4.3 Economically sufficient aggregation : Efficient aggregation

The two-stage optimization approach of Grossman and Hart (1983) allows one to explicitly observe how the first period basic measures y_1 and z_1 are aggregated for inducing an exogenously fixed effort level in a multi-period setting. In the first stage, a sequence of arbitrary efforts (a_1, a_2), which is not necessarily the sequence of optimal efforts $(a_1^{\dagger}, a_2^{\dagger})$, is fixed and then the principal's problem is to solve for the minimum cost contract. The following proposition completes the first stage of optimization.

Proposition 7 *(Incentive rates in the minimum cost contract)*

In a two-period and single-task setting, the first period and second period ($t =$ 1, 2) *incentive rates in the minimum cost contract for inducing an exogenously fixed* *sequence of efforts* (a_1, a_2) *are as follows:*

$$\beta_t^1 = \frac{\Big(\mu_t\, m_t - ICR1\Big)\, Var_{t-1}(z_t) - \Big(\mu_t\, k_t - ICR2\Big)\, Cov_{t-1}(y_t, z_t)}{r\Big[\, Var_{t-1}(y_t)\, Var_{t-1}(z_t) - \{Cov_{t-1}(y_t, z_t)\}^2\, \Big]}\,, \tag{3.39}$$

$$\beta_t^2 = \frac{\Big(\mu_t\, k_t - ICR2\Big)\, Var_{t-1}(y_t) - \Big(\mu_t\, m_t - ICR1\Big)\, Cov_{t-1}(y_t, z_t)}{r\Big[\, Var_{t-1}(y_t)\, Var_{t-1}(z_t) - \{Cov_{t-1}(y_t, z_t)\}^2\, \Big]}\,, \tag{3.40}$$

where

$$\mu_t = \frac{a_t\, r\Big(1 - \phi_{1t}\, \phi_{2t}\Big) + ICR1\Big[\, M_t \,/\, Var_{t-1}(y_t)\, \Big] + ICR2\Big[\, K_t \,/\, Var_{t-1}(z_t)\, \Big]}{m_t\Big[\, M_t \,/\, Var_{t-1}(y_t)\, \Big] + k_t\Big[\, K_t \,/\, Var_{t-1}(z_t)\, \Big]}\,, \tag{3.41}$$

$$M_t = m_t - \phi_{1t}\, k_t,\; K_t = k_t - \phi_{2t}\, m_t,$$

$$\phi_{1t} = Cov_{t-1}(y_t, z_t) \,/\, Var_{t-1}(z_t),\; \phi_{2t} = Cov_{t-1}(y_t, z_t) \,/\, Var_{t-1}(y_t)\,.$$

$ICR1$ *and* $ICR2$ *are as in (3.36) and (3.37). For the second period, it is sufficient to set no inter-temporal covariance risk* $ICR1 = ICR2 = 0$*. The variance* $Var_{t-1}[\,\cdot\,]$ *and covariance* $Cov_{t-1}[\,\cdot\,]$ *are conditional on the realized performance measures* y_1 *and* z_1 *for the second period (t=2), and unconditional for the first period (t=1).*

As the aggregation of performance measures is restricted to linear aggregation in this study, the relative incentive rate on the basic measures characterizes the aggregation of performance measures. Banker and Datar (1989) and Datar, Kulp, and Lambert (2001) analyze the relative incentive rate, but their studies are restricted to a single-period setting. From Proposition 7, it follows that the efficient aggregation for the first period in the minimum cost contract for inducing an exogenously fixed sequence of efforts (a_1, a_2) is characterized by the following relative weight on the first period basic measures y_1 and z_1:

$$\frac{\beta_1^1}{\beta_1^2} = \frac{\Big[\, \Big(\mu_1\, m_1 - ICR1\Big) - \phi_{11}\Big(\mu_1\, k_1 - ICR2\Big)\, \Big] \,/\, Var(y_1)}{\Big[\, \Big(\mu_1\, k_1 - ICR2\Big) - \phi_{21}\Big(\mu_1\, m_1 - ICR1\Big)\, \Big] \,/\, Var(z_1)}\,. \tag{3.42}$$

Because the second period agency problem is a single-period problem, it is an expected result that the efficient aggregation of the second period basic measures y_2 and z_2 is characterized by the following one-dimensional sufficient statistic $T(y_2, z_2)$ with the posterior beliefs:

$$T(y_2, z_2) = \frac{(m_2 - \phi_{12}\, k_2)}{Var_1(y_2)}\, y_2 + \frac{(k_2 - \phi_{22}\, m_2)}{Var_1(z_2)}\, z_2\,, \tag{3.43}$$

where $\phi_{12} = Cov_1(y_2, z_2) \,/\, Var_1(z_2)$ and $\phi_{22} = Cov_1(y_2, z_2) \,/\, Var_1(y_2)$.

Although the efficient aggregation of the second period measures y_2 and z_2 is independent of the second period effort level a_2, the second period incentive rates β_2^1 and β_2^2 ((3.39) and (3.40)) depend on the second period effort level a_2 through the optimal Lagrange multiplier μ_2 (3.41) on the incentive compatibility constraint. Thus, the inter-temporal covariance risk factors $ICR1$ and $ICR2$ ((3.36) and (3.37)) in the first period depend on the second period effort level a_2 through the second period incentive rates β_2^1 and β_2^2.

The relative weight (3.42) on the first period measures y_1 and z_1 explicitly shows that an efficient aggregation in a multi-period setting depends on the sequence of efforts (a_1, a_2). The relative weight (3.42) depends on the first period effort level a_1 through the Lagrange multiplier μ_1 (3.41) and on the second period effort level a_2 through the inter-temporal covariance risk factors $ICR1$ and $ICR2$.

Note that without the inter-temporal covariance risk $ICR1 = ICR2 = 0$, the efficient aggregation (3.42) is equivalent to the one-dimensional "myopic"statistic $\beta_1^1\, y_1 + \beta_1^2\, z_1 \; = T^m(y_1, z_1)$ (3.38), which is independent of the sequence of efforts (a_1, a_2). Thus, it is evident that the existence of inter-temporal covariance risk (represented by $ICR1$ and $ICR2$) causes an efficient aggregation in a multi-period setting to depend on the sequence of efforts (a_1, a_2).

3.4.4 Economically sufficient aggregation : Optimal aggregation

The optimal incentive rates are necessary to complete the second stage of optimization. The following proposition gives the first period optimal incentive rates on the basic measures y_1 and z_1 for inducing the sequence of endogenously determined optimal efforts $(a_1^\dagger, a_2^\dagger)$.

Proposition 8 *(Incentive rates in the optimal contract)*

In a two-period setting with a single-task, the optimal incentive rates $\beta_1^{1\dagger}$ *and* $\beta_1^{2\dagger}$ *on the first period basic measures* y_1 *and* z_1 *are as follows:*

$$\beta_1^{1\dagger} = \frac{\Big(b_1\, m_1 - ICR1\Big)\Big\{k_1^2 + r\,Var(z_1)\Big\} - \Big(b_1\, k_1 - ICR2\Big)\Big\{m_1\, k_1 + r\,Cov(y_1, z_1)\Big\}}{D_1}, \tag{3.44}$$

$$\beta_1^{2\dagger} = \frac{\Big(b_1\, k_1 - ICR2\Big)\Big\{m_1^2 + r\,Var(y_1)\Big\} - \Big(b_1\, m_1 - ICR1\Big)\Big\{m_1\, k_1 + r\,Cov(y_1, z_1)\Big\}}{D_1}, \tag{3.45}$$

where

$$\begin{aligned} D_1 &= m_1^2\, r\,Var(z_1) + k_1^2\, r\,Var(y_1) \\ &\quad - 2\, m_1\, k_1\, r\,Cov(y_1, z_1) + r^2\Big[Var(y_1)Var(z_1) - \{Cov(y_1, z_1)\}^2\Big]. \end{aligned} \tag{3.46}$$

From Proposition 8, it follows that the optimal aggregation in the first period is characterized by the following relative incentive rate on the first period basic measures y_1 and z_1:

$$\frac{\beta_1^{1\dagger}}{\beta_1^{2\dagger}} = \frac{\Big[\Big(b_1\, m_1 - ICR1\Big) - \phi_{11}\Big(b_1\, k_1 - ICR2\Big)\Big] \,/\, \Big(m_1^2 + r\,Var(y_1)\Big)}{\Big[\Big(b_1\, k_1 - ICR2\Big) - \phi_{21}\Big(b_1\, m_1 - ICR1\Big)\Big] \,/\, \Big(k_1^2 + r\,Var(z_1)\Big)}, \tag{3.47}$$

where $\phi_{11} = \Big(m_1\, k_1 + r\,Cov(y_1, z_1)\Big) \,/\, \Big(k_1^2 + r\,Var(z_1)\Big)$,

$\phi_{21} = \Big(m_1\, k_1 + r\,Cov(y_1, z_1)\Big) \,/\, \Big(m_1^2 + r\,Var(y_1)\Big)$.

In the case of independent periods, $ICR1 = ICR2 = 0$, the optimal aggregation (3.47) is equivalent to the one-dimensional "myopic"statistic $T^m(y_1, z_1)$ (3.38). However, as long as the performance measures are inter-temporally correlated and the inter-temporal covariance risk (represented by $ICR1$ and $ICR2$) exists, the optimal aggregation (3.47) is not equivalent to the one-dimensional "myopic"statistic $T^m(y_1, z_1)$ (3.38) and the principal does not use the one-dimensional "myopic" statistic $T^m(y_1, z_1)$ (3.38). Thus, statistical sufficiency is not the point to the optimal aggregation in a multi-period setting.

In (3.47), the optimal aggregation in a multi-period setting depends on the agent's characteristics and the economic situation of the agency, which are respectively represented by the agent's risk aversion r and the sequence of the first best effort levels (b_1, b_2). The first best effort level of the second period b_2 affects the optimal aggregation (3.47) through the inter-temporal covariance risk factors $ICR1$ and $ICR2$. Note that the incentive rates β_1^1 and β_1^2 ((3.39) and (3.40)) in the minimum cost contract are independent of the agent's risk aversion r, and thus the efficient aggregation (3.42) is independent of not only the sequence of the first best effort levels (b_1, b_2) but also the agent's risk aversion r. Since the optimal aggregation is to induce the endogenously determined optimal effort level, it is the optimal effort level $(a_1^\dagger, a_2^\dagger)$ that brings the agent's characteristics and the economic situation of the agency into the optimal aggregation (3.47).

In the second stage of optimization, it is verified that as the sequence of optimal efforts $(a_1^\dagger, a_2^\dagger)$ is substituted, the efficient aggregation (3.42) is equivalent to the optimal aggregation (3.47). The sequence of optimal efforts $(a_1^\dagger, a_2^\dagger)$ is given by the sequence of optimal incentive rates $\Big\{ (\beta_1^{1\dagger}, \beta_1^{2\dagger}),\, (\beta_2^{1\dagger}, \beta_2^{2\dagger}) \Big\}$. The first period optimal effort level $a_1^\dagger$ can be derived using the incentive compatibility constraint (from (3.80)):

$$a_1^\dagger = \beta_1^{1\dagger}\, m_1 + \beta_1^{2\dagger}\, k_1\,, \tag{3.48}$$

where the first period optimal incentive rates $\beta_1^{1\dagger}$ and $\beta_1^{2\dagger}$ are given by (3.44) and (3.45). It is observed in (3.41) and (3.42) that the first period optimal incentive rates $\beta_1^{1\dagger}$ and $\beta_1^{2\dagger}$ affect the efficient aggregation (3.42) through the optimal effort level $a_1^\dagger$, and the second period optimal incentive rates $\beta_2^{1\dagger}$ and $\beta_2^{2\dagger}$ affect the efficient aggregation through the inter-temporal covariance risk factors $ICR1$ and $ICR2$. It can be shown that substituting the optimal effort level $a_1^\dagger$ (3.48) into (3.41) and then into (3.42) results in the optimal aggregation (3.47).

The optimal aggregation (3.47) is a special case of the efficient aggregation (3.42). Thus, the optimal aggregation (3.47) depends on the sequence of optimal effort levels $(a_1^\dagger, a_2^\dagger)$. In a two-period setting with a single-task, the two relevant joint distributions of performance measures $\overline{f_1}(y_1, z_1;\, a_1)$ (3.14) and $f_2(y_2, z_2 |\, y_1, z_1, a_2)$ (3.18) satisfy the "all a or no a" condition (Definition 8). Therefore, as the analysis is extended to a multi-period setting, the "all a or no a" condition no longer implies that an economically sufficient aggregation for the optimal effort level is also a statistically sufficient aggregation for all effort levels ([4] $\not\Rightarrow$ [1]).

3.5 Likelihood ratio approach : Efficient aggregation

In this section, the results obtained in the previous section are confirmed by taking the likelihood ratio approach of Şabac (2009). While using the LEN model, he characterizes the renegotiation-proof contract with the statistical properties and likelihood ratios of performance measures. Using the results in Lemma 2 and Lemma 3 of Şabac (2009) and applying the setting of two-performance-measures and single-task, the two-period renegotiation-proof contract $C(y_1, z_1, y_2, z_2)$, that is the minimum cost contract for inducing an exogenously fixed sequence of efforts (a_1, a_2), is presented as follows :

$$C(y_1, z_1, y_2, z_2) = C_1(y_1, z_1) + \Sigma L_2^{-1}\, a_2\, L_2\,, \tag{3.49}$$

where ΣL_2 denotes the variance of the likelihood ratio L_2 (3.21). The first part $C_1(y_1, z_1)$ in (3.49) depends only on the first period measures y_1 and z_1 and is given as follows :

$$\begin{aligned} C_1(y_1, z_1) =& ACE_0 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 + \frac{1}{2}\, r\, \Sigma\overline{L}_1^{\,-1}\big(\, a_1 - \Sigma L_{12}\, \Sigma L_2^{-1}\, a_2\, \big)^2 \\ &+ \frac{1}{2}\, r\, \Sigma L_2^{-1}\, a_2^2 + \Sigma\overline{L}_1^{\,-1}\big(\, a_1 - \Sigma L_{12}\, \Sigma L_2^{-1}\, a_2\, \big)\, \overline{L}_1\,, \end{aligned} \tag{3.50}$$

where ACE_0 denotes the agent's certainty equivalent at the start of the first period, $\Sigma\overline{L}_1$ denotes the variance of the likelihood ratio $\overline{L}_1$ (3.15), and ΣL_{12} denotes the covariance $Cov(L_1, L_2)$ in which the likelihood L_1 is as in (3.13). Whereas the likelihood ratio $\overline{L}_1$ (3.15) depends only on the first period measures y_1 and z_1, the likelihood ratio L_2 (3.21) depends on both the first and the second period measures (y_1, z_1, y_2, z_2). In particular, the coefficient of y_1 is $(\, m_2\, s_{31} + k_2\, s_{41}\,)$ and the coefficient of z_1 is $(\, m_2\, s_{32} + k_2\, s_{42}\,)$ in the likelihood ratio L_2 (3.21).

The following proposition verifies the results from Proposition 7 in the previous section.

Proposition 9 *(Likelihood ratio approach : Efficient aggregation in a two-period setting with a single-task)*

In a two-period setting with a single-task, the economically sufficient aggregation ([3]*, Definition 6) of the first period basic measures y_1 and z_1 for inducing an exogenously fixed sequence of efforts (a_1, a_2) depends on the sequence of efforts (a_1, a_2). In particular, the efficient aggregation of the first period basic measures y_1 and z_1 is characterized as follows :*

$$\Sigma\overline{L}_1^{-1} \big(\, a_1 - \Sigma L_{12}\, \Sigma L_2^{-1}\, a_2 \,\big)\, \overline{L}_1 + \Sigma L_2^{-1}\, a_2\, L_2\,. \tag{3.51}$$

If the performance measures are inter-temporally independent, the efficient aggregation (3.51) is also statistically sufficient for all effort levels ([1]*, Definition 1) and characterized by the one-dimensional "myopic"statistic $T^m(y_1, z_1)$ (3.38).*

Using the likelihood ratios, Proposition 9 restates the results from Proposition 7 by showing that in a multi-period setting, an efficient aggregation of performance measures depends on the sequence of efforts (a_1, a_2) and that an efficient aggregate measure is equivalent to the one-dimensional "myopic"statistic $T^m(y_1, z_1)$ (3.38) if the performance measures are inter-temporally independent. Thus, Proposition 9 confirms that it is the existence of inter-temporal correlations and inter-temporal covariance risk (represented by $ICR1$ and $ICR2$) of performance measures that causes an efficient aggregation in a multi-period setting to depend on the sequence of efforts (a_1, a_2).

3.6 Conclusion

The aggregation of performance measures can help to reduce the complexity of an agency problem. More often than not, an agency problem takes place in a multi-period setting, in which performance measures are inter-temporally correlated. This study examines the nature and characteristics of the aggregation of performance measures in an extended multi-period setting.

In a multi-period setting, the first question is whether there is a universal aggregation, that is determined only by the statistical properties of performance measures with no loss of information in the aggregation process. If the same aggregation is good enough to substitute for the basic measures for every manager in all firms, then the complexity involved in motivating and evaluating managers' activity could be reduced considerably.

The results obtained in this study suggest that such a universal aggregation is unlikely to exist. In a multi-period setting, even when a "myopic"aggregate measure, which is equivalent to a one-dimensional sufficient statistic in a single-period setting, is available, the "myopic"aggregate measure is neither optimal nor efficient in inducing the agent's effort. The economically sufficient aggregate measure does not attain statistical sufficiency by depending on the agent's effort level to induce even if the likelihood ratios satisfy the "all a or no a"condition. Thus, statistical sufficiency is not the point to the aggregation of performance measures in a multi-period setting. This result is consistent with the results of Şabac (2009), where he shows that renegotiation may call for another qualitatively different sufficiency concept other than statistical sufficiency by introducing a "renegotiation"sufficient statistic, which neither implies nor is implied by a sufficient statistic.

The next question in a multi-period setting is whether a performance evaluation system can be commonly applied to all managers or should be "tailored"and individually designed for each manager and each firm such that different managers are evaluated on different performance measures. The results in this study suggest that in a multi-period setting, an optimal performance evaluation system should be "tailored" for each manager in each firm. This study shows that in a multi-period setting, an efficient aggregation of performance measures depends on the effort level to induce. Also, this study explicitly shows that in a multi-period setting, the optimal aggregation of performance measures depends on the agent's characteristics and the economic situation of the agency as well as the statistical properties of performance measures.

In a multi-period setting, the optimal aggregate measures are not sufficient statistics. This result is similar to the result of Amershi, Banker, and Datar (1990) in a single-period setting, but the reason in a multi-period setting is qualitatively different from that of a single-period setting. In a multi-period setting, it is not only a statistical condition ("all a or no a" condition) but also the existence of inter-temporal correlations of performance measures and inter-temporal covariance risk in compensation that causes the optimal aggregate measures to be agency-specific and not to be sufficient statistics. In particular, the optimal aggregate measure is not a sufficient statistic of the basic measures in a multi-period setting even if the "all a or no a" condition is satisfied. Sufficient statistics do not represent the optimal aggregation of performance measure as the analysis moves toward an institutionally richer setting of multi-period.

3.7 Appendix

3.7.1 Proof of Proposition 7

This proof is composed of two parts. The first part employs the compensation C as the decision variable and shows that the Kuhn-Tucker multiplier λ_i, $t = 1, 2$ equals 1 in both periods. The second part explicitly solves for the minimum cost contract by employing the first period incentive rates β_1^1 and β_1^2 as the decision variables in order to handle the inter-temporal covariance risk.

The following shows that the first period Kuhn-Tucker multiplier λ_1 equals 1 with the joint normal density function $\overline{f_1}(y_1, z_1;\, a_1)$ (3.14). The same procedure can show that the second period Kuhn-Tucker multiplier λ_2 also equals 1 with the conditional joint normal density function $f_2(y_2, z_2|\, y_1, z_1, a_2)$ (3.18).

The renegotiation-proof contract $C(y_1, z_1, y_2, z_2)$ (3.6) can be split into two parts:

$$C(y_1, z_1, y_2, z_2) = C_1(y_1, z_1) + Z_2(y_2, z_2)\,, \tag{3.52}$$

where $C_1(y_1, z_1) = \alpha_0 + \beta_1^1\, y_1 + \beta_1^2\, z_1$ depends on the first period measures y_1 and z_1, and $Z_2(y_2, z_2) = \beta_2^1\, y_2 + \beta_2^2\, z_2$ depends on the second period measures y_2 and z_2. The expected value of the contract $C(y_1, z_1, y_2, z_2)$ (3.6) is as follows:

$$\begin{aligned} E\Big[C(y_1, z_1, y_2, z_2);\, a_1, a_2\Big] &= E\Big[C_1(y_1, z_1);\, a_1\Big] + E\Big[Z_2(y_2, z_2)\Big] \\ &= \iint C_1(y_1, z_1)\, \overline{f_1}(y_1, z_1;\, a_1)\, dy_1\, dz_1 + E\Big[Z_2(y_2, z_2)\Big]\,. \end{aligned} \tag{3.53}$$

Due to the renegotiation-proofness condition, $E\Big[Z_2(y_2, z_2)\Big]$ is taken effectively as fixed in the first period.

Given the normality of compensation, the agent's certainty equivalent at the start of the first period is:

$$\begin{aligned} ACE(a_1) &= E\Big[C(y_1, z_1, y_2, z_2);\, a_1, a_2\Big] - \frac{1}{2}\Big(a_1^2 + a_2^2\Big) - \frac{1}{2}\, r\, Var\Big[C(y_1, z_1, y_2, z_2)\Big] \\ &= \iint C_1(y_1, z_1)\, \overline{f_1}(y_1, z_1;\, a_1)\, dy_1\, dz_1 + E\Big[Z_2(y_2, z_2)\Big] - \frac{1}{2}\Big(a_1^2 + a_2^2\Big) \\ &\quad - \frac{1}{2}\, r \iiiint \Big\{C(y_1, z_1, y_2, z_2) - E\Big[C(y_1, z_1, y_2, z_2);\, a_1, a_2\Big]\Big\}^2 \\ &\qquad f(y_1, z_1, y_2, z_2;\, a_1, a_2)\, dy_1\, dz_1\, dy_2\, dz_2\,, \end{aligned} \tag{3.54}$$

where $f(y_1, z_1, y_2, z_2;\, a_1, a_2)$ is the joint density function (3.11). With (3.52) and (3.53), the derivative of the agent's certainty equivalent $ACE(a_1)$ with respect to the contract $C_1(y_1, z_1)$ equals 1:

$$\frac{\partial\, ACE(a_1)}{\partial\, C_1(y_1, z_1)} = \iint \overline{f_1}(y_1, z_1;\, a_1)\, dy_1\, dz_1 = 1\,. \tag{3.55}$$

The incentive compatibility constraint for the first period is:

$$(IC) \quad \iint C_1(y_1, z_1)\, \frac{\partial\, \overline{f_1}(y_1, z_1;\, a_1)}{\partial\, a_1}\, dy_1\, dz_1 - a_1 = 0\,. \tag{3.56}$$

A zero reservation certainty equivalent is set without loss of generality, and the participation constraint for the first period is:

$$(PC) \;\; ACE(a_1) \;\geq\; 0\,. \tag{3.57}$$

The first period problem of solving for the minimum cost contract for a fixed sequence of efforts (a_1, a_2) is written as the following maximization problem:

$$\begin{aligned} \max_{C_1(y_1,z_1),\, \lambda_1,\, \mu_1} \; \mathcal{L} = - \, E\Big[C(y_1, z_1, y_2, z_2);\, a_1, a_2\Big] & \\ + \lambda_1\, ACE(a_1) & \\ + \mu_1 \left(\iint C_1(y_1, z_1)\, \frac{\partial\, \overline{f_1}(y_1, z_1;\, a_1)}{\partial\, a_1}\, dy_1\, dz_1 - a_1 \right), & \end{aligned} \tag{3.58}$$

where λ_1 is a Kuhn-Tucker multiplier and μ_1 is a Lagrange multiplier. The first order conditions are as follows:

$$\begin{aligned} \frac{\partial\, \mathcal{L}}{\partial\, C_1(y_1, z_1)} = & - \iint \overline{f_1}(y_1, z_1;\, a_1)\, dy_1\, dz_1 + \lambda_1 \iint \overline{f_1}(y_1, z_1;\, a_1)\, dy_1\, dz_1 \\ & + \mu_1 \iint \frac{\partial\, \overline{f_1}(y_1, z_1;\, a_1)}{\partial\, a_1}\, dy_1\, dz_1 \\ = & \; 0\,, \end{aligned} \tag{3.59}$$

$$\frac{\partial\, \mathcal{L}}{\partial\, \lambda_1} = ACE(a_1) \geq 0,\;\; \lambda_1 \geq 0,\;\; \lambda_1\, ACE(a_1) = 0\,, \tag{3.60}$$

$$\frac{\partial\, \mathcal{L}}{\partial\, \mu_1} = \iint C_1(y_1, z_1)\, \frac{\partial\, \overline{f_1}(y_1, z_1;\, a_1)}{\partial\, a_1}\, dy_1\, dz_1 - a_1 = 0\,. \tag{3.61}$$

Because $\iint \frac{\partial \overline{f_1}(y_1, z_1; a_1)}{\partial a_1} dy_1 \, dz_1 = 0$, the Kuhn-Tucker multiplier λ_1 equals 1 from (3.59):

$$\lambda_1 = 1\,. \tag{3.62}$$

Thus, the participation constraint (3.57) is binding by the Kuhn-Tucker condition (3.60).

Now, the first period incentive rates β_1^1 and β_1^2 are used as decision variables in solving for the minimum cost contract. The optimization follows backward induction. However, the proof skips the second period problem and advances to the first period, because it is sufficient for the second period to set no inter-temporal covariance risk $ICR1 = ICR2 = 0$ and to use the variance $Var_1[\,\cdot\,]$ and covariance $Cov_1[\,\cdot\,]$, which are conditional on the realized first period measures y_1 and z_1.

Taking expectation to (3.6) and using (3.1) and (3.2), the first period incentive compatibility constraint can be written as:

$$(IC') \quad \beta_1^1 \, m_1 + \beta_1^2 \, k_1 - a_1 = 0\,. \tag{3.63}$$

The principal's problem at the start of the first period can be represented by the following maximization problem:

$$\max_{\alpha_0, \beta_1^1, \beta_1^2, \lambda_1, \mu_1} \mathcal{L} = -E\Big[C(y_1, z_1, y_2, z_2);\, a_1, a_2\Big] + \lambda_1 \, ACE(a_1) + \mu_1 \Big(\beta_1^1 \, m_1 + \beta_1^2 \, k_1 - a_1\Big), \tag{3.64}$$

which is reduced to maximizing the following because $\lambda_1 = 1$:

$$-\frac{1}{2}\left(a_1^2 + a_2^2\right) - \frac{1}{2}\, r\, Var\Big[C(y_1, z_1, y_2, z_2)\Big] + \mu_1 \Big(\beta_1^1 \, m_1 + \beta_1^2 \, k_1 - a_1\Big). \tag{3.65}$$

The agent's action cost is dropped from (3.65) because the sequence of efforts (a_1, a_2) is fixed. As the fixed payment α_0 can be set to satisfy the binding participation constraint (3.57), the decision of the fixed payment α_0 can be omitted. Also, the risk premium can be reduced to the relevant portion $\frac{1}{2}\, r\, Var'\Big[C(y_1, z_1, y_2, z_2)\Big]$ regarding the decision variables β_1^1 and β_1^2. Then, the principal's problem is equivalent to:

$$\max_{\beta_1^1,\, \beta_1^2,\, \mu_1} \quad \mathcal{L}' = -\frac{1}{2}\, r\, Var'\Big[C(y_1, z_1, y_2, z_2)\Big] + \mu_1 \Big(\beta_1^1\, m_1 + \beta_1^2\, k_1 - a_1\Big)\,, \tag{3.66}$$

in which by Definition 9 ((3.36) and (3.37)):

$$\begin{aligned} \frac{1}{2}\, r\, Var'\Big[C(y_1, z_1, y_2, z_2)\Big] =& \frac{1}{2}\, r\Big[\, \big(\beta_1^1\big)^2\, Var(y_1) + \big(\beta_1^2\big)^2\, Var(z_1) + 2\, \beta_1^1\, \beta_1^2\, Cov(y_1, z_1)\Big] \\ &+ \beta_1^1\, ICR1 + \beta_1^2\, ICR2\,. \end{aligned} \tag{3.67}$$

The first order conditions are as follows:

$$\frac{\partial\, \mathcal{L}'}{\partial\, \beta_1^1} = -r\Big[\, \beta_1^1\, Var(y_1) + \beta_1^2\, Cov(y_1, z_1)\,\Big] - ICR1 + \mu_1\, m_1 = 0\,, \tag{3.68}$$

$$\frac{\partial\, \mathcal{L}'}{\partial\, \beta_1^2} = -r\Big[\, \beta_1^2\, Var(z_1) + \beta_1^1\, Cov(y_1, z_1)\,\Big] - ICR2 + \mu_1\, k_1 = 0\,, \tag{3.69}$$

$$\frac{\partial\, \mathcal{L}'}{\partial\, \mu_1} = \beta_1^1\, m_1 + \beta_1^2\, k_1 - a_1 = 0\,. \tag{3.70}$$

Solving (3.68) and (3.69) for β_1^1 and β_1^2 gives:

$$\beta_1^1 = \frac{\Big(\mu_1\, m_1 - ICR1\Big)\, Var(z_1) - \Big(\mu_1\, k_1 - ICR2\Big)\, Cov(y_1, z_1)}{r\Big[\, Var(y_1)\, Var(z_1) - \{Cov(y_1, z_1)\}^2\,\Big]}\,, \tag{3.71}$$

$$\beta_1^2 = \frac{\Big(\mu_1\, k_1 - ICR2\Big)\, Var(y_1) - \Big(\mu_1\, m_1 - ICR1\Big)\, Cov(y_1, z_1)}{r\Big[\, Var(y_1)\, Var(z_1) - \{Cov(y_1, z_1)\}^2\,\Big]}\,. \tag{3.72}$$

Finally, substituting (3.71) and (3.72) into the incentive compatibility constraint (3.70) yields:

$$\mu_1 = \frac{a_1\, r\, \Big(1 - \phi_{11}\, \phi_{21}\Big) + ICR1\Big[\, M_1\, /\, Var(y_1)\,\Big] + ICR2\Big[\, K_1\, /\, Var(z_1)\,\Big]}{m_1\Big[\, M_1\, /\, Var(y_1)\,\Big] + k_1\Big[\, K_1\, /\, Var(z_1)\,\Big]}\,, \tag{3.73}$$

where $M_1 = m_1 - \phi_{11}\, k_1$, $K_1 = k_1 - \phi_{21}\, m_1$,

$\phi_{11} = Cov(y_1, z_1) \,/\, Var(z_1),\ \phi_{21} = Cov(y_1, z_1) \,/\, Var(y_1).$

3.7.2 Proof of Proposition 8

This proof is by backward induction. The optimal incentive rates $\beta_2^{1\dagger}$ and $\beta_2^{2\dagger}$ on the second period basic measures y_2 and z_2 are obtained by solving the principal's problem at the start of the second period. The principal maximizes his expected utility with the decision variables β_2^1 and β_2^2:

$$\max_{\beta_2^1, \beta_2^2} \ U_2^p = b_2\, a_2 - E\big[C|\, a_2\big]\,. \tag{3.74}$$

The agent's rational action choice is:

$$a_2 = \beta_2^1\, m_2 + \beta_2^2\, k_2\,. \tag{3.75}$$

Using the binding participation and incentive compatibility constraints, the second period optimal incentive rates $\beta_2^{1\dagger}$ and $\beta_2^{2\dagger}$ are as follows:

$$\beta_2^{1\dagger} = b_2\, r\, \frac{m_2\, Var_1(z_2) - k_2\, Cov_1(y_2, z_2)}{D_2}\,, \tag{3.76}$$

$$\beta_2^{2\dagger} = b_2\, r\, \frac{k_2\, Var_1(y_2) - m_2\, Cov_1(y_2, z_2)}{D_2}\,, \tag{3.77}$$

where

$$\begin{aligned} D_2 = {} & m_2^2\, r\, Var_1(z_2) + k_2^2\, r\, Var_1(y_2) \\ & - 2\, m_2\, k_2\, r\, Cov_1(y_2, z_2) + r^2 \Big[Var_1(y_2) Var_1(z_2) - \{Cov_1(y_2, z_2)\}^2\Big]\,. \end{aligned} \tag{3.78}$$

Given the second period optimal incentive rates $\beta_2^{1\dagger}$ and $\beta_2^{2\dagger}$, backward induction allows one to calculate the first period optimal incentive rates $\beta_1^{1\dagger}$ and $\beta_1^{2\dagger}$. At the start of the first period, the principal maximizes his expected utility with the decision variables β_1^1 and β_1^2:

$$\max_{\beta_1^1, \beta_1^2} \ U_1^p = b_1\, a_1 + b_2\, a_2 - E\big[C|\, a_1, a_2\big]\,, \tag{3.79}$$

in which the second period benefit $b_2\, a_2$ can be dropped because the second period optimal incentive rates $\beta_2^{1\dagger}$ and $\beta_2^{2\dagger}$ and effort level $a_2^{\dagger}$ are taken effectively as fixed in the first period due to the renegotiation-proofness requirement. The agent's rational action choice in the first period is:

$$a_1 = \beta_1^1\, m_1 + \beta_1^2\, k_1\,. \tag{3.80}$$

With the binding participation and incentive compatibility constraints, the principal's expected utility maximization is solved by:

$$\max_{\beta_1^1,\,\beta_1^2} \quad b_1 \left(\beta_1^1\, m_1 + \beta_1^2\, k_1\right) - \frac{1}{2}\left(\beta_1^1\, m_1 + \beta_1^2\, k_1\right)^2 - \frac{1}{2}\, r\, Var\big[C|\, a_1, a_2\big]. \tag{3.81}$$

In addition, the risk premium $\frac{1}{2}\, r\, Var\big[C|\, a_1, a_2\big]$ is reduced to a relevant risk premium $\frac{1}{2}\, r\, Var'\big[C|\, a_1, a_2\big]$ due to the renegotiation-proofness requirement:

$$\begin{aligned}\frac{1}{2}\, r\, Var'\big[C|\, a_1, a_2\big] = \frac{1}{2}\, r\, \Big[\, \big(\beta_1^1\big)^2\, Var(y_1) + \big(\beta_1^2\big)^2\, Var(z_1) + 2\, \beta_1^1\, \beta_1^2\, Cov(y_1, z_1)\Big] \\ + \beta_1^1\, ICR1 + \beta_1^2\, ICR2\,,\end{aligned} \tag{3.82}$$

where $ICR1$ and $ICR2$ are as in (3.36) and (3.37). Solving the maximization problem gives the first period optimal incentive rates $\beta_1^{1\dagger}$ and $\beta_1^{2\dagger}$:

$$\beta_1^{1\dagger} = \frac{\Big(b_1\, m_1 - ICR1\Big)\Big\{k_1^2 + r\, Var(z_1)\Big\} - \Big(b_1\, k_1 - ICR2\Big)\Big\{m_1\, k_1 + r\, Cov(y_1, z_1)\Big\}}{D_1}\,, \tag{3.83}$$

$$\beta_1^{2\dagger} = \frac{\Big(b_1\, k_1 - ICR2\Big)\Big\{m_1^2 + r\, Var(y_1)\Big\} - \Big(b_1\, m_1 - ICR1\Big)\Big\{m_1\, k_1 + r\, Cov(y_1, z_1)\Big\}}{D_1}\,, \tag{3.84}$$

where

$$\begin{aligned} D_1 = {} & m_1^2\, r\, Var(z_1) + k_1^2\, r\, Var(y_1) \\ & - 2\, m_1\, k_1\, r\, Cov(y_1, z_1) + r^2 \Big[Var(y_1) Var(z_1) - \{Cov(y_1, z_1)\}^2 \Big] \,. \end{aligned} \tag{3.85}$$

3.7.3 Proof of Proposition 9

Given the renegotiation-proof contract $C(y_1, z_1, y_2, z_2)$ (3.49) and the first part $C_1(y_1, z_1)$ (3.50), the aggregate measure of the first period basic measures y_1 and z_1 in (3.51) is economically sufficient for a single effort level by Definition 6. When the performance measures are inter-temporally independent:

$$Cov(y_1, y_2) = Cov(y_1, z_2) = Cov(z_1, z_2) = Cov(z_1, y_2) = 0\,, \tag{3.86}$$

both y_1 and z_1 vanish in the likelihood ratio L_2 (3.21):

$$(\, m_2\, s_{31} + k_2\, s_{41}\,) \;=\; 0\,, \tag{3.87}$$

$$(\, m_2\, s_{32} + k_2\, s_{42}\,) \;=\; 0\,. \tag{3.88}$$

Thus, if the performance measures are inter-temporally independent, the efficient aggregation of the first period measures y_1 and z_1 in (3.51) depends only on the likelihood ratio $\overline{L}_1$ (3.15). In this case, the efficient aggregation of the first period measures y_1 and z_1 in (3.51) is also statistically sufficient for all effort levels as the likelihood ratio $\overline{L}_1$ (3.15) is characterized by the one-dimensional sufficient statistic $T_1(y_1, z_1)$ (3.17), that is equivalent to the one-dimensional “myopic”statistic $T^m(y_1, z_1)$ (3.38).

Chapter 4

Relative Incentive Rate

4.1 Introduction

The previous two chapters analyze the sufficient aggregation of performance measures either in a multi-task setting or in a multi-period setting. In reality, the aggregation of performance measures and the design of a performance evaluation system, more often than not, take place in a multi-period setting with multiple tasks. This study examines what would be the relative incentive rate of performance measures in an institutionally richer setting of multi-period and multi-task.

This study shows that the inter-temporal covariance risk of performance measures is a factor determining the endogenous allocation of effort. This study analyzes the aggregation of performance measures in a general N-period setting with two tasks, and explicitly presents the relative incentive rate on the basic measures. The endogenous allocation of effort across multiple tasks is examined through the relative incentive rate.

The modeling setting is endogenous throughout the analysis. In particular, the endogenous allocation of effort is preserved. For tractability, this study employs a LEN model (Linear contract, Exponential utility of the agent, Normal distribution of performance measure) and the aggregation of performance measures is restricted to linear aggregation.

This study contributes to the literature by first analyzing the endogenous allocation of effort in a multi-period setting. In addition, this study first provides the explicit relative incentive rate in a multi-period and multi-task setting. Banker and Datar (1989) and Amershi, Banker, and Datar (1990) discuss the nature and characteristics of optimal aggregation of performance measures. In particular, Banker and Datar (1989) show that the optimal relative incentive rate in a single-period setting with a single-task is determined only by the signal-to-noise ratio of performance measures.

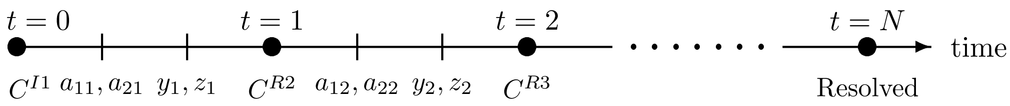
Using a LEN model, Holmstrom and Milgrom (1991) introduce the allocation of effort across multiple tasks. The allocation of effort in their study depends on whether multiple tasks are complements or substitutes to each other in terms of the agent's personal action cost. Using a LEN model in a single-period setting with multiple tasks, Datar, Kulp, and Lambert (2001) analyze the trade-off between the congruity of performance measures and the risk premium by demonstrating that an endogenously determined optimal allocation of effort may not be the first best allocation even if the first best allocation is feasible. Datar, Kulp, and Lambert (2001) show that as the endogenous allocation of effort comes into the principal's problem, the optimal relative incentive rate is no longer equivalent to the signal-to-noise ratio of performance measures.

The results of Banker and Datar (1989) and Datar, Kulp, and Lambert (2001) are restricted to a single-period setting. This study not only encompasses the results of Banker and Datar (1989) and Datar, Kulp, and Lambert (2001), but also shows how the inter-temporal covariance risk of performance measures, in addition to the within-period risk premium, interacts with the congruity of performance measures in determining the optimal endogenous allocation of effort. The rest of this study is organized as follows: Section 2 explains the modeling features. Section 3 analyzes the optimal relative incentive rate. Section 4 concludes the study.

4.2 Modeling features

This study employs a LEN model in an N-period setting with two tasks. An N-period contract is characterized with renegotiation. Because a long-term contract with full commitment is equivalent to a single-period contract on multiple tasks which was analyzed in a previous chapter, a renegotiation-proof contract is used in this study.

This study is different from the previous chapters in that this study mainly analyzes the economically sufficient aggregation for inducing the optimal effort level, rather than the statistically sufficient aggregation and the economically sufficient aggregation for an arbitrary effort level. Thus, the analysis focuses on the equilibrium path.



(Figure 2)

In each of N periods $t = 1, 2, \cdots, N$, the performance measures y_t and z_t are joint normally distributed with normally distributed residual terms:

$$\begin{aligned} y_t &= \vec{m}_t \cdot \vec{a}_t + \varepsilon_t & (4.1) \\ z_t &= \vec{k}_t \cdot \vec{a}_t + \delta_t, \quad t = 1, \cdots, N\,. & (4.2) \end{aligned}$$

Each performance measure y_t and z_t has sensitivity $\vec{m}_t = (m_{1t}, m_{2t})$ and $\vec{k}_t = (k_{1t}, k_{2t})$ to the agent's effort $\vec{a}_t = (a_{1t}, a_{2t})$, where the first subscript indicates the task and the second subscript indicates the period. Since the performance measures y_t and z_t are normally distributed, the agent's certainty equivalent is represented by the expected compensation minus the risk premium and action cost:

$$ACE_{t-1} = E_{t-1}\Big[C(\vec{y}, \vec{z})\Big] - \frac{1}{2}\, r\, Var_{t-1}\Big[C(\vec{y}, \vec{z})\Big] - K(\vec{a})\,, \qquad (4.3)$$

where the following notations for conditional expectation and variance are used:

$$\begin{aligned} E_{t-1}\Big[\,\cdot\,\Big] &= E\Big[\,\cdot\,|\, y_1, z_1, \cdots, y_{t-1}, z_{t-1}\Big] & (4.4) \\ Var_{t-1}\Big[\,\cdot\,\Big] &= Var\Big[\,\cdot\,|\, y_1, z_1, \cdots, y_{t-1}, z_{t-1}\Big]. & (4.5) \end{aligned}$$

A contract offer at $t-1$ (the start of period t, see Figure 2) is denoted by a sequence of incentive rates:

$$C^t = \Big\{\alpha_{t-1}, \big(\beta_t^1, \beta_t^2\big), \cdots, \big(\beta_N^1, \beta_N^2\big)\,\Big\}, \qquad (4.6)$$

where the fixed payment α_{t-1} is a function of the history of realized performance measures $\alpha_{t-1} = h\,(y_1, z_1, \cdots, y_{t-1}, z_{t-1})$ (α_0 some constant), and the superscript to the incentive rate β indicates the performance measure such that 1 is for y and 2 is for z while the subscript to the incentive rate β indicates the period. The initial contract $C^1 = \Big\{\alpha_0, (\beta_1^1, \beta_1^2), \cdots, (\beta_N^1, \beta_N^2)\Big\}$ is the contract in effect unless replaced by a subsequent renegotiation offer. In the sequel, the superscript I is used for the initial contract at the renegotiation time and R is used for renegotiation offers.

At the start of the first period, $t = 0$, the principal offers to the agent an initial contract C^{I1}. The agent either accepts or rejects it. Once the agent accepts the initial contract offer, the agent provides the period 1 effort $\vec{a}_1 = (a_{11}, a_{21})$. Before the end of period 1, the principal and the agent observe two contractible performance measures y_1 and z_1. At $t = 1$, the principal makes a take-it-or-leave-it renegotiation offer C^{R2}. If the renegotiation offer is rejected, C^{I2} is the contract in effect for period 2. If accepted, C^{R2} becomes the contract in effect. The agent provides the period 2 effort $\vec{a}_2 = (a_{12}, a_{22})$ and the principal and the agent observe two contractible performance measures y_2 and z_2 before the end of period 2. At the start of each period $t = 3, \cdots, N$, the same renegotiation procedure occurs with a renegotiation offer C^{Rt}. At the terminal date, $t = N$, the agent receives the compensation based on the realized values of performance measures and the contract is resolved.

Due to the renegotiation-proofness principle for the LEN model (Şabac 2007), the analysis of linear optimal contract can, without loss of generality, be restricted to a linear renegotiation-proof contract:

$$C = \alpha + \sum_{t=1}^{N} \beta_t^1 \, y_t + \sum_{t=1}^{N} \beta_t^2 \, z_t \, . \tag{4.7}$$

At the decision point of the period t incentive rates β_t^1 and β_t^2, the incentive rates in the remaining contract periods β_{t+i}^1 and β_{t+i}^2, $i = 1, \cdots, N-t$ are restricted to be ex-post efficient and rationally expected by the principal and the agent due to the renegotiation-proofness requirement.

The contracting and renegotiation can be summarized by the principal's problem at the start of period t as follows:

$$\max_{\beta_t^1, \beta_t^2} E_{t-1}\Big[B(\vec{a}) - C^{Rt}\Big], \quad \Big(E\Big[B(\vec{a}) - C^{I1}\Big] \text{ for the first period}\Big) \tag{4.8}$$

subject to the renegotiation-proofness constraint

$$\Big\{\big(\beta^1_{t+1},\beta^2_{t+1}\big),\cdots,\big(\beta^1_N,\beta^2_N\big)\Big\} \text{ are optimal at the start of periods } t+1,\cdots,N\,, \tag{4.9}$$

the incentive compatibility constraint

$$\vec{a}_t,\cdots,\vec{a}_N \in \arg\max\, ACE_{t-1}\big(C^{Rt}\big)\,, \tag{4.10}$$

and the participation constraint

$$ACE_{t-1}\big(C^{Rt}\big) \geq ACE_{t-1}\big(C^{It}\big)\,. \quad \big(ACE\big(C^{I1}\big) \geq 0 \text{ for the first period}\big) \tag{4.11}$$

4.3 Optimal relative incentive rate

This section provides the optimal relative incentive rate in a multi-period and multi-task setting. The results in this section not only encompass what have been found in Banker and Datar (1989) and Datar, Kulp, and Lambert (2001), but also explicitly show the role of inter-temporal covariance risk in determining the optimal relative incentive rate. The endogenous allocation of effort is examined through the optimal relative incentive rate on the basic measures.

In a single-period setting, Datar, Kulp, and Lambert (2001) show that there is a trade-off between maximizing the congruity of performance measures and minimizing the risk premium. The optimal relative incentive rate is affected by the within-period risk premium as well as the congruity of performance measures. In the optimal relative incentive rate, a performance measure with a bigger variance is assigned a less weight because the performance measure causes a bigger within-period risk premium. Thus, as long as a risk premium exists, the first best allocation of effort is not endogenously achieved even if it is feasible.

In a multi-period setting, this section shows that it is not only the within-period risk premium but also the inter-temporal covariance risk of performance measures that trades off with the congruity of performance measures. The optimal relative incentive rate is affected by the inter-temporal covariance risk of performance measures as well as the within-period risk premium and the congruity of performance measures. In the optimal relative incentive rate, a performance measure with a bigger inter-temporal covariance risk is assigned a less weight because the performance measure causes a bigger risk premium. As a result, the first best allocation of effort is not endogenously achieved, even if it is feasible, in a multi-period setting in which inter-temporal correlations and inter-temporal covariance risk of performance measures exist.

4.3.1 Optimal relative incentive rate

In maximizing the expected utility (4.8), the principal wants to minimize the expected compensation, which leads to minimizing the sum of the risk premium and action cost as the agent certainty equivalent (4.3) is constant by the binding participation constraint. When a residual term of a performance measure, say ε_t, is correlated with its future residual term ε_{t+i} or the other performance measure's future residual term δ_{t+i}, the future incentive rates come into the principal's problem through the inter-temporal covariance risk terms:

$$\sum_{i=1}^{N-t} \beta_t^1\, \beta_{t+i}^1\, Cov_{t-1}\big(y_t, y_{t+i}\big) + \sum_{i=1}^{N-t} \beta_t^1\, \beta_{t+i}^2\, Cov_{t-1}\big(y_t, z_{t+i}\big) \quad \text{and} \tag{4.12}$$

$$\sum_{i=1}^{N-t} \beta_t^2\, \beta_{t+i}^2\, Cov_{t-1}\big(z_t, z_{t+i}\big) + \sum_{i=1}^{N-t} \beta_t^2\, \beta_{t+i}^1\, Cov_{t-1}\big(z_t, y_{t+i}\big)\,. \tag{4.13}$$

In a multi-period setting, the principal develops an incentive to minimize the inter-temporal covariance risk regarding the incentive rates in the yet-to-come periods. In analyzing this problem, the inter-temporal covariance risk factors (ICR) are defined as follows.

Definition 10 *(N-period : Inter-temporal covariance risk factors)*

ICR_t^1 and ICR_t^2 characterize the inter-temporal covariance risk of the perfor-

mance measures y_t and z_t, respectively:

$$ICR_t^1 = r\,Cov_{t-1}\left(y_t,\ \sum_{i=1}^{N-t} \left(\beta_{t+i}^1\, y_{t+i} + \beta_{t+i}^2\, z_{t+i} \right) \right), \tag{4.14}$$

$$ICR_t^2 = r\,Cov_{t-1}\left(z_t,\ \sum_{i=1}^{N-t} \left(\beta_{t+i}^2\, z_{t+i} + \beta_{t+i}^1\, y_{t+i} \right) \right). \tag{4.15}$$

Using the inter-temporal covariance risk factors ICR_t^1 (4.14) and ICR_t^2 (4.15), the following proposition presents the optimal incentive rates on the two performance measures y_t and z_t in an N-period contract with the agent's effort on two tasks.

Proposition 10 *(Optimal incentive rates)*

The period t optimal incentive rates β_t^1 and β_t^2 are as follows:

$$\beta_t^1 = \frac{\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]\left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}(z_t)\right\} - \left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right]\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}(y_t, z_t)\right\}}{D_t}, \tag{4.16}$$

$$\beta_t^2 = \frac{\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right]\left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}(y_t)\right\} - \left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}(y_t, z_t)\right\}}{D_t}, \tag{4.17}$$

where

$$\begin{aligned} D_t =& \left(m_{1t}k_{2t} - m_{2t}k_{1t}\right)^2 + \left(\vec{m}_t \cdot \vec{m}_t\right) r\,Var_{t-1}(z_t) + \left(\vec{k}_t \cdot \vec{k}_t\right) r\,Var_{t-1}(y_t) \\ & - 2\left(\vec{m}_t \cdot \vec{k}_t\right) r\,Cov_{t-1}(y_t, z_t) \\ & + r^2 \left[Var_{t-1}(y_t)\, Var_{t-1}(z_t) - \left\{ Cov_{t-1}(y_t, z_t) \right\}^2 \right] > 0\,. \end{aligned} \tag{4.18}$$

The optimal incentive rates for the last period are obtained by substituting $ICR_N^1 = ICR_N^2 = 0$.

From Proposition 10, it follows that the optimal relative incentive rate on the two performance measures y_t and z_t is:

$$\frac{\beta_t^1}{\beta_t^2} = \frac{\left\{\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right] - \phi_{1t}\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right]\right\} \,/\, \left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}(y_t)\right\}}{\left\{\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right] - \phi_{2t}\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]\right\} \,/\, \left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}(z_t)\right\}}, \tag{4.19}$$

$$\begin{aligned} \text{where} \quad \phi_{1t} &= \left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}(y_t, z_t)\right\} / \left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}(z_t)\right\} \\ \text{and} \quad \phi_{2t} &= \left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}(y_t, z_t)\right\} / \left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}(y_t)\right\}. \end{aligned}$$

In the previous chapter (see (3.25)), it was shown in a single-period setting with a single-task that the relative signal-to-noise ratio of performance measures (Banker and Datar 1989) with the LEN model is:

$$\frac{\left(m - \phi_1\, k\right) / Var(y)}{\left(k - \phi_2\, m\right) / Var(z)}, \tag{4.20}$$

where $\phi_1 = Cov(y,z) / Var(z)$ and $\phi_2 = Cov(y,z) / Var(y)$. With a single-task in each period $\vec{m}_t = m_t$, $\vec{k}_t = k_t$, $\vec{b}_t = b_t$, and independent periods $ICR_t^1 = ICR_t^2 = 0$ (effectively a single-period setting), the optimal relative incentive rate (4.19) in each period is:

$$\frac{\beta_t^1}{\beta_t^2} = \frac{\left(m_t - \phi_{1t}\, k_t\right) / Var_{t-1}(y_t)}{\left(k_t - \phi_{2t}\, m_t\right) / Var_{t-1}(z_t)}, \tag{4.21}$$

where $\phi_{1t} = Cov_{t-1}(y_t, z_t)/Var_{t-1}(z_t)$ and $\phi_{2t} = Cov_{t-1}(y_t, z_t)/Var_{t-1}(y_t)$. The relative incentive rate (4.21) is equivalent to the relative signal-to-noise ratio (4.20) with the posterior beliefs. Thus, a multi-period agency problem with independent periods $ICR_t^1 = ICR_t^2 = 0$ and a single-task is equivalent to a repeated single-period agency problem with a single-task, and the optimal relative incentive rate in such a setting is explained by the relative signal-to-noise ratio (4.21) in each period.

In addition to the relative signal-to-noise ratio, the optimal relative incentive rate (4.19) reflects the risk externality from inter-temporally correlated performance measures and the endogenous allocation of effort across multiple tasks. The following sections discuss the risk externality and the endogenous allocation of effort.

4.3.2 Risk externality

The last period optimal relative incentive rate β_N^1/β_N^2 does not contain the inter-temporal covariance risk factors ICR_N^1 and ICR_N^2, because there is no further contract period after N and thus no inter-temporal covariance risk regarding the last period incentive rates β_N^1 and β_N^2:

$$\frac{\beta_N^1}{\beta_N^2} = \frac{\left\{\left(\vec{b}_N \cdot \vec{m}_N\right) - \phi_{1N}\left(\vec{b}_N \cdot \vec{k}_N\right)\right\} / \left\{\vec{m}_N \cdot \vec{m}_N + r\,Var_{N-1}\big(y_N\big)\right\}}{\left\{\left(\vec{b}_N \cdot \vec{k}_N\right) - \phi_{2N}\left(\vec{b}_N \cdot \vec{m}_N\right)\right\} / \left\{\vec{k}_N \cdot \vec{k}_N + r\,Var_{N-1}\big(z_N\big)\right\}}\,, \quad (4.22)$$

where $\phi_{1N} = \left\{\vec{m}_N \cdot \vec{k}_N + r\,Cov_{N-1}\big(y_N, z_N\big)\right\} / \left\{\vec{k}_N \cdot \vec{k}_N + r\,Var_{N-1}\big(z_N\big)\right\}$

and $\phi_{2N} = \left\{\vec{m}_N \cdot \vec{k}_N + r\,Cov_{N-1}\big(y_N, z_N\big)\right\} / \left\{\vec{m}_N \cdot \vec{m}_N + r\,Var_{N-1}\big(y_N\big)\right\}.$

The principal's problem at $t = N - 1$ is a single-period problem with myopic incentives in the sense that the last period incentive rates β_N^1 and β_N^2 are designed only to induce the optimal effort $\vec{a}_N$ in the last period.

On the other hand, all the incentive rates before the last period consist of two components: the myopic incentive and the risk externality adjustment. In the optimal incentive rates (4.16) and (4.17), the myopic incentive is the component which is based on the principal's benefit $\vec{b}_t$ from the current period effort $\vec{a}_t$, the sensitivity of the current period measures $\vec{m}_t$ and $\vec{k}_t$, and the posterior variances of the current period measures $Var_{t-1}\big(y_t\big)$, $Var_{t-1}\big(z_t\big)$, $Cov_{t-1}\big(y_t, z_t\big)$. That is, the myopic incentive is the component analogous to the last period optimal incentive rate, respectively:

$$\left[\left(\vec{b}_t \cdot \vec{m}_t\right)\left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}\big(z_t\big)\right\} - \left(\vec{b}_t \cdot \vec{k}_t\right)\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\big(y_t, z_t\big)\right\}\right] / D_t\,, \quad (4.23)$$

$$\left[\left(\vec{b}_t \cdot \vec{k}_t\right)\left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\left(y_t\right)\right\} - \left(\vec{b}_t \cdot \vec{m}_t\right)\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\left(y_t, z_t\right)\right\}\right] / D_t \,. \quad (4.24)$$

In the optimal incentive rates (4.16) and (4.17), the risk externality adjustment is the component which is characterized by the inter-temporal covariance risk factors ICR_t^1 and ICR_t^2. The risk externality adjustment in the optimal incentive rates (4.16) and (4.17) is given as follows, respectively :

$$\left[-ICR_t^1\left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}\left(z_t\right)\right\} + ICR_t^2\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\left(y_t, z_t\right)\right\}\right] / D_t \,, \quad (4.25)$$

$$\left[-ICR_t^2\left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\left(y_t\right)\right\} + ICR_t^1\left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\left(y_t, z_t\right)\right\}\right] / D_t \,. \quad (4.26)$$

The risk externality (Şabac 2008) results from the principal's lack of commitment to future incentive rates. The principal cannot "cooperate with himself" in determining incentive rates at different points in time. The risk externality arises when positive (negative) covariance between current and future performance measures imposes too much (too little) compensation risk to the agent so that the principal lowers (raises) current incentive rates in order to reduce (increase) the current period induced effort.

In studies with a single-period setting such as Banker and Datar (1989) and Datar, Kulp, and Lambert (2001), the optimal incentive rate consists only of the myopic incentive, because there is no inter-temporal consideration. As the analysis is extended to a multi-period setting, the risk externality adjustment becomes a factor of the optimal incentive rate, because the principal takes into account the impact of incentive rates on the inter-temporal covariance risk as well as effort inducement. Thus, when performance measures are inter-temporally correlated, the optimal relative incentive rate is also affected by the inter-temporal covariance of performance measures. With the optimal relative incentive rate (4.19), the subsequent section will show that as long as the inter-temporal covariance of performance measures exists in a multi-period setting, the first best allocation of effort is not endogenously achieved, even if it is feasible.

The risk externality adjustment (4.25) and (4.26) can be explained by two special cases: the pure-insurance and the window-dressing (Christensen, Feltham, and Şabac 2005). The pure-insurance is a special case when the principal induces null current period effort $\vec{a}_t = 0$ because the current period performance measures y_t and z_t (but not the performance measures for future periods $t+1, \cdots, N$) have no sensitivity to the agent's effort $\vec{m}_t = \vec{k}_t = 0$. The incentive rates β_t^1 and β_t^2 are used purely to minimize the risk premium. If $\vec{m}_t = \vec{k}_t = 0$ is substituted in the optimal incentive rates (4.16) and (4.17), then the pure-insurance adjustment is obtained:

$$\Big[- ICR_t^1 \, r \, Var_{t-1}\big(z_t\big) + ICR_t^2 \, r \, Cov_{t-1}\big(y_t, z_t\big) \Big] \, / \, D_t \, , \tag{4.27}$$

$$\Big[- ICR_t^2 \, r \, Var_{t-1}\big(y_t\big) + ICR_t^1 \, r \, Cov_{t-1}\big(y_t, z_t\big) \Big] \, / \, D_t \, , \tag{4.28}$$

which are respectively included in the risk externality adjustments (4.25) and (4.26).

The window-dressing is a special case when the agent's effort generates no economic benefit $\vec{b}_t = 0$ (only in the current period and not in the future periods $t+1, \cdots, N$) but the current period performance measures y_t and z_t have non-zero sensitivity to effort $\vec{m}_t \neq 0$ or $\vec{k}_t \neq 0$. In this case, non-zero window-dressing effort is induced $\vec{a}_t \neq 0$ and the principal should compensate for it. If $\vec{b}_t = 0$ is substituted in the optimal incentive rates (4.16) and (4.17), the risk externality adjustments (4.25) and (4.26) are obtained. Therefore, the risk externality adjustment is equivalent to the optimal incentive rate in the case that no productive effort is induced from the agent, but the principal cannot avoid paying for the agent's window-dressing effort in equilibrium.

The principal respectively includes the risk externality adjustment (4.25) and (4.26) in the optimal incentive rates (4.16) and (4.17) because the myopic incentive (4.23) and (4.24) is too strong or too weak in the presence of covariance among the current and future performance measures. Suppose y_t has positive covariances with future performance measures: $Cov_{t-1}\big(y_t, y_{t+i}\big) > 0$, $Cov_{t-1}\big(y_t, z_{t+i}\big) > 0$, and $ICR_t^1 > 0$ in (4.14). Then setting the incentive rate β_t^1 as the myopic incentive (4.23) is too expensive because a stronger incentive rate β_t^1 results in a bigger inter-temporal covariance risk in (4.12). Therefore, the principal reduces β_t^1 by

$-ICR_t^1 \Big\{ \vec{k}_t \cdot \vec{k}_t + r\, Var_{t-1}\big(z_t\big) \Big\} \,/\, D_t$ in (4.25) because y_t is relatively expensive in inducing effort $\vec{a}_t$ due to the positive inter-temporal covariance risk. At the same time, the principal raises β_t^2 by $ICR_t^1 \Big\{ \vec{m}_t \cdot \vec{k}_t + r\, Cov_{t-1}\big(y_t, z_t\big) \Big\} \,/\, D_t$ in (4.26) because z_t is relatively inexpensive in inducing effort $\vec{a}_t$.

Now suppose that y_t has negative covariances with future performance measures: $Cov_{t-1}\big(y_t, y_{t+i}\big) < 0$, $Cov_{t-1}\big(y_t, z_{t+i}\big) < 0$, and $ICR_t^1 < 0$ in (4.14). Then the myopic incentive (4.23) is too weak for β_t^1 because a stronger incentive rate β_t^1 results in a smaller inter-temporal covariance risk in (4.12). Thus, the principal raises β_t^1 by $-ICR_t^1 \Big\{ \vec{k}_t \cdot \vec{k}_t + r\, Var_{t-1}\big(z_t\big) \Big\} \,/\, D_t$ in (4.25) because y_t is relatively inexpensive in inducing effort $\vec{a}_t$ due to the negative inter-temporal covariance risk. The principal reduces β_t^2 by $ICR_t^1 \Big\{ \vec{m}_t \cdot \vec{k}_t + r\, Cov_{t-1}\big(y_t, z_t\big) \Big\} \,/\, D_t$ in (4.26) because z_t is relatively expensive in inducing effort $\vec{a}_t$. The impact of the risk externality from the inter-temporal covariances $Cov_{t-1}\big(z_t, z_{t+i}\big)$ and $Cov_{t-1}\big(z_t, y_{t+i}\big)$ on the optimal incentive rates is symmetric.

4.3.3 Endogenous effort allocation

In a single-period setting with multiple tasks, Datar, Kulp, and Lambert (2001, Page 82) show that the principal's utility maximization problem is equivalent to minimizing the sum of the incongruity of performance measures and the risk premium (also see the proof of Proposition 10 of this study). As a benchmark in discussing the endogenous allocation of effort, below presented is a case with a risk neutral agent, in which both the within-period risk premium and the inter-temporal covariance risk are null.

If the agent is risk neutral $r = 0$, then the principal's problem is reduced to minimizing the incongruity of performance measures, which is geometrically explained by the squared distance from the first best effort and the induced effort $||\vec{b}_t - \vec{a}_t||^2$. That is, the principal uses the incentive rates in inducing a second best effort $\vec{a}_t = \beta_t^1\, \vec{m}_t + \beta_t^2\, \vec{k}_t$ as close as possible to the first best effort $\vec{b}_t$ without concerns for the risk premium. When the agent is risk neutral $r = 0$, the optimal relative incentive rate (4.19) is equivalent to the following, which actually achieves perfect congruity $\vec{b}_t = \vec{a}_t$:

$$\frac{\left(\vec{b}_t \cdot \vec{m}_t\right)\left(\vec{k}_t \cdot \vec{k}_t\right) - \left(\vec{b}_t \cdot \vec{k}_t\right)\left(\vec{m}_t \cdot \vec{k}_t\right)}{\left(\vec{b}_t \cdot \vec{k}_t\right)\left(\vec{m}_t \cdot \vec{m}_t\right) - \left(\vec{b}_t \cdot \vec{m}_t\right)\left(\vec{m}_t \cdot \vec{k}_t\right)} = \frac{\left(b_{1t}\, k_{2t} - b_{2t}\, k_{1t}\right)\left(m_{1t}\, k_{2t} - m_{2t}\, k_{1t}\right)}{\left(b_{2t}\, m_{1t} - b_{1t}\, m_{2t}\right)\left(m_{1t}\, k_{2t} - m_{2t}\, k_{1t}\right)}. \tag{4.29}$$

The endogenous allocation of effort among multiple tasks, which is reflected in the optimal relative incentive rate (4.19) through the numerator and denominator of (4.29), geometrically minimizes the angle between the induced effort $\vec{a}_t$ and the first best effort $\vec{b}_t$ in the case of no risk premium. With a fixed first best effort $\vec{b}_t$, the squared distance $||\vec{b}_t - \vec{a}_t||^2$ is determined by the length (intensity) of the induced effort $||\vec{a}_t||$ and the angle (allocation) between the induced effort $\vec{a}_t$ and the first best effort $\vec{b}_t$. Since the length (intensity) of the induced effort is costless due to the agent's risk neutrality $r = 0$, the principal's only problem is to decide the angle (allocation) between the induced effort $\vec{a}_t$ and the first best effort $\vec{b}_t$ in minimizing the squared distance $||\vec{b}_t - \vec{a}_t||^2$. The optimal (minimal) angle is solved by the the numerator and denominator of (4.29).

When the two performance measures y_t and z_t are perfectly aligned $\left(m_{1t}\, k_{2t} - m_{2t}\, k_{1t}\right) = 0$ (effectively a single-performance-measure setting) or in a single-task setting $\vec{m}_t = m_t$, $\vec{k}_t = k_t$, and $\vec{b}_t = b_t$, both the numerator and denominator of (4.29) vanish in the optimal relative incentive rate (4.19). In a single-performance-measure setting such as Feltham and Xie (1994), the induced effort in equilibrium is constrained to a one-dimensional linear subspace and the principal cannot affect the allocation of effort. In a single-task setting such as Banker and Datar (1989), there is no effort allocation problem and thus the allocation of effort is not relevant to the principal. Therefore, either in a single-performance-measure setting or in a single-task setting, the allocation of effort is not a relevant problem to the principal. In accordance, the optimal relative incentive rate (4.19) does not contain the numerator and denominator of (4.29) in a single-performance-measure setting or a single-task setting.

In a general setting with a risk averse agent, the endogenous allocation of effort can be discussed in terms of the geometric analysis in Demski, Fellingham, and Lin (2007). Their study presents the induced effort $\vec{a}$ as a projection of the first best effort $\vec{b}$ on the implementable action space $M(\vec{a}) = Span\left(\vec{m}, \vec{k}\right)$, which is the span of the performance sensitivities to effort. The presence of a risk premium in the agency problem makes the projection non-orthogonal, and the non-orthogonal projection $\vec{a} = Proj_{M(\vec{a})}\,\vec{b}$ is viewed as first being orthogonally projected on the implementable action space $\widehat{Proj}_{M(\vec{a})}\,\vec{b}$ and then adjusted in the implementable action space. The orthogonal projection of the first best effort on the implementable action space $\widehat{Proj}_{M(\vec{a})}\,\vec{b}$ represents what would be induced without concerns for the risk premium $r = 0$.

It has been shown by (4.29) that if there is no concern for the risk premium $r = 0$, the two-task and two-performance-measure setting makes it possible to achieve perfect congruity $\vec{b}_t = \vec{a}_t$. Thus, the first best effort resides in the implementable action space $\vec{b}_t \in M(\vec{a}_t)$ in this study. If the number of tasks exceeds the number of performance measures, perfect congruity is not achievable even without concerns for the risk premium $r = 0$ because the first best effort does not reside in the implementable action space $\vec{b}_t \notin M(\vec{a}_t)$.

In general, the presence of a risk premium in the agency problem prevents the first best effort from being induced even if the first best effort resides in the implementable action space $\vec{b}_t \in M(\vec{a}_t)$. In a single-period setting, Datar, Kulp, and Lambert (2001) show that the principal may not achieve perfect congruity in order to reduce the within-period risk premium even if perfect congruity is feasible. To apply the single-period setting with multiple tasks of Datar, Kulp, and Lambert (2001), assume the current period measure z_t (but not y_t and not the performance measures for future periods $t+1, \cdots, N$) has no variance $Var\left(z_t\right) = 0$. Then, it follows that both the cross-sectional covariance and the inter-temporal covariance risk factor of z_t vanish $Cov_{t-1}\left(y_t, z_t\right) = ICR_t^2 = 0$. In addition, assume that the current period measure y_t is inter-temporally independent with the future period measures such that the inter-temporal covariance risk factor of y_t vanishes $ICR_t^1 = 0$. In this case, the optimal relative incentive rate (4.19) is equivalent to the following:

$$\frac{\left(\vec{b}_t \cdot \vec{m}_t\right)\left(\vec{k}_t \cdot \vec{k}_t\right) - \left(\vec{b}_t \cdot \vec{k}_t\right)\left(\vec{m}_t \cdot \vec{k}_t\right)}{\left(\vec{b}_t \cdot \vec{k}_t\right)\left[\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\left(y_t\right)\right] - \left(\vec{b}_t \cdot \vec{m}_t\right)\left(\vec{m}_t \cdot \vec{k}_t\right)}\,. \tag{4.30}$$

The relative incentive rate (4.30) reflects an endogenous allocation of effort which is not the first best. In accordance with the result of Datar, Kulp, and Lambert (2001), the within-period risk premium $r\,Var_{t-1}\left(y_t\right)$ in the denominator causes the relative incentive rate (4.30) to deviate from the case of perfect congruity in (4.29).

Now, while keeping the assumptions in the previous case $Var\left(z_t\right) = Cov_{t-1}\left(y_t, z_t\right) = ICR_t^2 = 0$, assume the existence of an inter-temporal covariance risk from y_t, $ICR_t^1 \neq 0$, for analyzing the allocation of effort in a multi-period setting. Note that the within-period risk premium from y_t, $r\,Var_{t-1}\left(y_t\right) \neq 0$, is minimal in the sense that without the variance of y_t, $Var_{t-1}\left(y_t\right) = 0$, the inter-temporal covariance risk from y_t also vanishes, $ICR_t^1 = 0$, and there is no risk premium in the principal's problem. With the inter-temporal covariance risk of y_t, which is represented by ICR_t^1, the optimal relative incentive rate (4.19) is now equivalent to the following:

$$\frac{\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]\left(\vec{k}_t \cdot \vec{k}_t\right) - \left(\vec{b}_t \cdot \vec{k}_t\right)\left(\vec{m}_t \cdot \vec{k}_t\right)}{\left(\vec{b}_t \cdot \vec{k}_t\right)\left[\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\left(y_t\right)\right] - \left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]\left(\vec{m}_t \cdot \vec{k}_t\right)}\,. \tag{4.31}$$

The relative incentive rate (4.31) reflects the endogenous allocation of effort when there exist the inter-temporal covariance risk as well as the within-period risk premium from y_t. The inter-temporal covariance risk factor ICR_t^1 in both the numerator and the denominator, as well as the within-period risk premium $r\,Var_{t-1}\left(y_t\right)$ in the denominator, causes the relative incentive rate (4.31) to deviate from the case of perfect congruity (4.29).

The inter-temporal covariance risk can be regarded as an "expense"for using a performance measure in inducing effort from the risk averse agent in a multi-period setting. While there is only one "expense" (the within-period risk premium) in a single-period setting, a multi-period setting incurs an additional "expense" (the inter-temporal covariance risk) for using performance measures to maximize the congruity. In a multi-period setting, the existence of the "expenses" (the inter-temporal covariance risk and the within-period risk premium) adjusts the maximization of congruity and results in the endogenous allocation of effort in (4.31).

As a result, in the presence of the inter-temporal covariance risk, the endogenous allocation of effort is not the first best even if it is feasible. As long as the inter-temporal covariance risk of performance measures exists in a multi-period setting, the endogenously determined optimal allocation of effort is not the first best allocation. In a multi-period setting, the congruity of performance measures trades off with the inter-temporal covariance risk as well as the within-period risk premium.

4.3.4 Effective signal-to-noise ratio

When the analysis is extended from a single-period and single-task setting to a single-period setting with multiple tasks, the endogenous allocation of effort becomes relevant to the principal. In accordance, the numerator and the denominator of (4.29) come into the optimal incentive rates, and the resulting optimal relative incentive rate is equivalent to (4.22). When the optimal relative incentive rate (4.22) in a single-period setting with multiple tasks is compared with the relative signal-to-noise ratio (4.21) in a single-period and single-task setting, it is observed that the terms $\left(\vec{b}_t \cdot \vec{m}_t\right)$ and $\left(\vec{b}_t \cdot \vec{k}_t\right)$ play the role of "effective sensitivity" and the terms $\left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\big(y_t\big)\right\}$ and $\left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}\big(z_t\big)\right\}$ play the role of "effective noise" of the performance measures y_t and z_t, respectively.

When the analysis is extended further to a multi-period setting with multiple tasks, the inter-temporal covariance risk of performance measures becomes relevant to the principal. In accordance, the inter-temporal covariance risk factors ICR_t^1 (4.14) and ICR_t^2 (4.15) come into the optimal incentive rates, and the resulting relative incentive rate is (4.19). From the optimal relative incentive rate (4.19) in a multi-period setting with multiple tasks, it is observed that the terms $\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]$ and $\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right]$ play the role of "effective sensitivity"and the terms $\left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\big(y_t\big)\right\}$ and $\left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}\big(z_t\big)\right\}$ play the role of "effective noise"of the performance measures y_t and z_t, respectively.

The effective noise is obviously positive. How about the effective sensitivity? The effective sensitivity is also positive as long as the two performance measures y_t and z_t have a non-negative cross-sectional correlation such that y_t and z_t are competing alternatives to the principal in inducing the agent's effort. In particular, from the optimal incentive rates (4.16) and (4.17), it can be shown that the following identities hold on the equilibrium path :

$$\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right] \equiv \beta_t^1 \left\{\vec{m}_t \cdot \vec{m}_t + r\,Var_{t-1}\big(y_t\big)\right\} + \beta_t^2 \left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\big(y_t, z_t\big)\right\}, \quad (4.32)$$

$$\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right] \equiv \beta_t^2 \left\{\vec{k}_t \cdot \vec{k}_t + r\,Var_{t-1}\big(z_t\big)\right\} + \beta_t^1 \left\{\vec{m}_t \cdot \vec{k}_t + r\,Cov_{t-1}\big(y_t, z_t\big)\right\}, \quad (4.33)$$

where no inter-temporal covariance risk exists in the last period, $ICR_N^1 = ICR_N^2 = 0$.

If the two performance measures y_t and z_t have a non-negative cross-sectional correlation, it is clear from (4.19) that the optimal relative incentive rate $\beta_t^1 \,/\, \beta_t^2$ on the two performance measures y_t and z_t is strictly decreasing in the inter-temporal covariance risk factor of y_t (ICR_t^1) and strictly increasing in the inter-temporal covariance risk factor of z_t (ICR_t^2). Therefore, a performance measure with bigger inter-temporal covariance risk has a smaller effective sensitivity and thus is assigned a weaker relative incentive rate.

Now, with the positive effective sensitivities $\left[\vec{b}_t \cdot \vec{m}_t - ICR_t^1\right]$ and $\left[\vec{b}_t \cdot \vec{k}_t - ICR_t^2\right]$, the relative incentive rates discussed in the previous section ((4.29), (4.30), (4.31)) decrease monotonically : (4.29) > (4.30) > (4.31). The existence of the within-period risk premium from y_t causes the relative incentive rate (4.30) to be less than (4.29). The existence of the inter-temporal covariance risk from y_t causes the relative incentive rate (4.31) to be further reduced from (4.30).

When the two performance measures y_t and z_t have a non-negative cross-sectional correlation, the effective sensitivities are positive. Thus, the inter-temporal covariance risk of a performance measure has a monotonic impact on the endogenous allocation of effort through the relative incentive rate. Given the binding incentive compatibility constraints ((4.46) and (4.47)), the induced allocation of effort $a_{1t} \,/\, a_{2t}$ is monotonically affected by the relative incentive rate $\beta_t^1 \,/\, \beta_t^2$ with the direction depending on the sensitivities of the performance measures y_t and z_t. It can be shown that the sign of the derivative :

$$\frac{d\,(\,a_{1t} \,/\, a_{2t}\,)}{d\,(\,\beta_t^1 \,/\, \beta_t^2\,)} \tag{4.34}$$

is decided by the term :

$$(\,m_{1t}\,k_{2t} - m_{2t}\,k_{1t}\,)\,. \tag{4.35}$$

As the relative incentive rate is monotonically affected by the inter-temporal covariance risk of a performance measure, the resulting impact on the induced allocation of effort is also monotonic, with the direction depending on the sensitivities of performance measures. Therefore, in a multi-period setting, the inter-temporal covariance risk of a performance measure has a monotonic impact on the endogenous allocation of effort through the relative incentive rate.

4.4 Conclusion

In a multi-period setting with multiple tasks, the design of a performance evaluation system should take into consideration managers' allocation of effort across multiple tasks while minimizing the imposed compensation risk to managers. This study analyzes the endogenous allocation of effort with the explicit optimal relative incentive rate in a multi-period and multi-task setting.

The endogenous allocation of effort is examined through the optimal relative incentive rate on the basic measures. In a multi-period setting, the inter-temporal covariance risk of performance measures becomes a part of an agency problem and thus relevant to the principal. In particular, the inter-temporal covariance risk weakens the effective sensitivity of a performance measure, and a performance measure with bigger inter-temporal covariance risk is assigned a weaker relative incentive rate. In a multi-period and multi-task setting, the optimal relative incentive rate is no longer explained only by the relative signal-to-noise ratio of performance measures, but also by the inter-temporal covariance risk and the allocation of effort.

This study shows that in determining the optimal endogenous allocation of effort, the congruity of performance measures trades off not only with the within-period risk premium but also with the inter-temporal covariance risk of performance measures. In a multi-period setting, the existence of the inter-temporal covariance risk of performance measures adjusts the maximization of congruity and results in the endogenous allocation of effort. As a result, in a multi-period setting, the endogenous allocation of effort is not the first best allocation even if it is feasible.

4.5 Appendix

4.5.1 Proof of Proposition 10

The proof is by backward induction. The optimal incentive rates β_N^1 and β_N^2 in the last period are obtained by solving the principal's problem at $t = N-1$. The principal maximizes his expected utility with the decision variables β_N^1 and β_N^2:

$$\max_{\beta_N^1, \beta_N^2} \; U_N^p = \left(\vec{b}_N \cdot \vec{a}_N\right) - E_{N-1}\big[C|\, \vec{a}_N\big]\,. \tag{4.36}$$

For the last period the agent's rational action choice is as follows:

$$a_{1N} \;\; = \;\; \beta_N^1\, m_{1N} + \beta_N^2\, k_{1N}\,, \tag{4.37}$$

$$a_{2N} \;\; = \;\; \beta_N^1\, m_{2N} + \beta_N^2\, k_{2N}\,. \tag{4.38}$$

Given the binding participation and incentive compatibility constraints, the principal's expected utility maximization (4.36) becomes:

$$\max_{\beta_N^1,\,\beta_N^2} \quad \Big(b_{1N},\, b_{2N}\Big)\cdot\Big(\beta_N^1\, m_{1N} + \beta_N^2\, k_{1N},\; \beta_N^1\, m_{2N} + \beta_N^2\, k_{2N}\Big)$$
$$-\tfrac{1}{2}\left[\Big(\beta_N^1\, m_{1N} + \beta_N^2\, k_{1N}\Big)^2 + \Big(\beta_N^1\, m_{2N} + \beta_N^2\, k_{2N}\Big)^2\right] - \tfrac{1}{2}\, r\, Var_{N-1}\big[C|\,\vec{a}_N\big], \tag{4.39}$$

which is equivalent to an agency loss minimization problem:

$$\min_{\beta_N^1,\,\beta_N^2} L_N = \frac{1}{2}\left[\Big(\beta_N^1\, m_{1N} + \beta_N^2\, k_{1N} - b_{1N}\Big)^2 + \Big(\beta_N^1\, m_{2N} + \beta_N^2\, k_{2N} - b_{2N}\Big)^2\right]$$
$$+\frac{1}{2}\, r\left[\Big(\beta_N^1\Big)^2 Var_{N-1}\big(y_N\big) + \Big(\beta_N^2\Big)^2 Var_{N-1}\big(z_N\big) + 2\,\beta_N^1\,\beta_N^2\, Cov_{N-1}\big(y_N, z_N\big)\right]. \tag{4.40}$$

Minimizing L_N gives the last period optimal incentive rates β_N^1 and β_N^2:

$$\beta_N^1 = \frac{\Big(\vec{b}_N\cdot\vec{m}_N\Big)\Big\{\vec{k}_N\cdot\vec{k}_N + r\, Var_{N-1}(z_N)\Big\} - \Big(\vec{b}_N\cdot\vec{k}_N\Big)\Big\{\vec{m}_N\cdot\vec{k}_N + r\, Cov_{N-1}(y_N, z_N)\Big\}}{D_N}, \tag{4.41}$$

$$\beta_N^2 = \frac{\Big(\vec{b}_N\cdot\vec{k}_N\Big)\Big\{\vec{m}_N\cdot\vec{m}_N + r\, Var_{N-1}(y_N)\Big\} - \Big(\vec{b}_N\cdot\vec{m}_N\Big)\Big\{\vec{m}_N\cdot\vec{k}_N + r\, Cov_{N-1}(y_N, z_N)\Big\}}{D_N}, \tag{4.42}$$

where

$$D_N = \Big(m_{1N}\, k_{2N} - m_{2N}\, k_{1N}\Big)^2 + \Big(\vec{m}_N\cdot\vec{m}_N\Big)\, r\, Var_{N-1}\big(z_N\big) + \Big(\vec{k}_N\cdot\vec{k}_N\Big)\, r\, Var_{N-1}\big(y_N\big)$$
$$-\,2\Big(\vec{m}_N\cdot\vec{k}_N\Big)\, r\, Cov_{N-1}\big(y_N, z_N\big)$$
$$+\,r^2\left[Var_{N-1}\big(y_N\big)\, Var_{N-1}\big(z_N\big) - \Big\{Cov_{N-1}\big(y_N, z_N\big)\Big\}^2\right]. \tag{4.43}$$

D_N is positive since

$$
\begin{aligned}
D_N \geq\ & \Big(m_{1N}\, k_{2N} - m_{2N}\, k_{1N}\Big)^2 + r^2 \left(1 - \rho_{\varepsilon\delta}^2\right) Var_{N-1}(y_N)\, Var_{N-1}(z_N) \\
& + r \left(m_{1N} \sqrt{Var_{N-1}(z_N)} - k_{1N} \sqrt{Var_{N-1}(y_N)}\right)^2 \\
& + r \left(m_{2N} \sqrt{Var_{N-1}(z_N)} - k_{2N} \sqrt{Var_{N-1}(y_N)}\right)^2
\end{aligned} \tag{4.44}
$$

where $\rho_{\varepsilon\delta}$ is the correlation between y_N and z_N.

Given the last period optimal incentive rates $\left(\beta_N^1, \beta_N^2\right)$, backward induction allows one to calculate the optimal incentive rates of all periods before the last period $\left(\beta_{N-1}^1, \beta_{N-1}^2\right), \cdots, \left(\beta_t^1, \beta_t^2\right), \cdots,\ \left(\beta_1^1, \beta_1^2\right)$. The optimal period t $(1 \leq t \leq N-1)$ incentive rates β_t^1 and β_t^2 are calculated by solving the principal's problem at $t-1$. The principal maximizes his expected utility with the decision variables β_t^1 and β_t^2:

$$
\max_{\beta_t^1, \beta_t^2} \; U_t^p = \sum_{i=t}^{N} \left(\vec{b}_i \cdot \vec{a}_i\right) - E_{t-1}\left[C \middle| \, \vec{a}_t, \cdots, \vec{a}_N\right]. \tag{4.45}
$$

The agent's rational action choice for period t is as follows:

$$
a_{1t} = \beta_t^1\, m_{1t} + \beta_t^2\, k_{1t}\,, \tag{4.46}
$$

$$
a_{2t} = \beta_t^1\, m_{2t} + \beta_t^2\, k_{2t}\,. \tag{4.47}
$$

With the binding participation and incentive compatibility constraints, the principal's expected utility maximization (4.45) is:

$$
\max_{\beta_t^1, \beta_t^2} \sum_{i=t}^{N} \left[\begin{array}{c} \Big(b_{1i},\, b_{2i}\Big) \cdot \Big(\beta_i^1\, m_{1i} + \beta_i^2\, k_{1i},\, \beta_i^1\, m_{2i} + \beta_i^2\, k_{2i}\Big) \\ -\frac{1}{2} \left[\Big(\beta_i^1\, m_{1i} + \beta_i^2\, k_{1i}\Big)^2 + \Big(\beta_i^1\, m_{2i} + \beta_i^2\, k_{2i}\Big)^2\right] \end{array} \right] - \frac{1}{2}\, r\, Var_{t-1}\left[C \middle| \, \vec{a}_t, \cdots, \vec{a}_N\right]. \tag{4.48}
$$

As the optimal incentive rates for the period $t+1, \cdots, N$ are taken as effectively fixed at $t-1$ due to the renegotiation-proofness requirement, (4.48) is reduced to:

$$
\begin{aligned}
\max_{\beta_t^1, \beta_t^2} \quad & \Big(b_{1t},\, b_{2t}\Big) \cdot \Big(\beta_t^1\, m_{1t} + \beta_t^2\, k_{1t},\, \beta_t^1\, m_{2t} + \beta_t^2\, k_{2t}\Big) \\
& -\tfrac{1}{2} \left[\Big(\beta_t^1\, m_{1t} + \beta_t^2\, k_{1t}\Big)^2 + \Big(\beta_t^1\, m_{2t} + \beta_t^2\, k_{2t}\Big)^2\right] - \tfrac{1}{2}\, r\, Var_{t-1}\left[C \middle| \, \vec{a}_t, \cdots, \vec{a}_N\right].
\end{aligned}
$$

$$(4.49)$$

In addition, the risk premium $\frac{1}{2}\, r\, Var_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big]$ is reduced to a relevant risk premium $\frac{1}{2}\, r\, Var^{'}_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big]$ due to the renegotiation-proofness requirement:

$$Var_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big] = Var_{t-1}\left(\sum_{i=t}^{N} \beta_i^1 y_i + \sum_{i=t}^{N} \beta_i^2 z_i\right) \tag{4.50}$$

$$\begin{aligned} \sim \quad & Var^{'}_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big] \\ = & \big(\beta_t^1\big)^2 Var_{t-1}(y_t) + \big(\beta_t^2\big)^2 Var_{t-1}(z_t) + 2\,\beta_t^1\,\beta_t^2\, Cov_{t-1}(y_t, z_t) \\ & + \textstyle\sum_{i=t+1}^{N} 2\,\beta_t^1\,\beta_i^1\, Cov_{t-1}(y_t, y_i) + \sum_{i=t+1}^{N} 2\,\beta_t^1\,\beta_i^2\, Cov_{t-1}(y_t, z_i) \\ & + \textstyle\sum_{i=t+1}^{N} 2\,\beta_t^2\,\beta_i^2\, Cov_{t-1}(z_t, z_i) + \sum_{i=t+1}^{N} 2\,\beta_t^2\,\beta_i^1\, Cov_{t-1}(z_t, y_i)\,. \end{aligned} \tag{4.51}$$

$Var^{'}_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big]$ is substituted for $Var_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big]$ in (4.49) and the principal's expected utility maximization at $t-1$ is equivalent to an agency loss minimization:

$$\begin{aligned} \min_{\beta_t^1,\,\beta_t^2}\; L_t = & \frac{1}{2}\left[\Big(\beta_t^1\, m_{1t} + \beta_t^2\, k_{1t} - b_{1t}\Big)^2 + \Big(\beta_t^1\, m_{2t} + \beta_t^2\, k_{2t} - b_{2t}\Big)^2\right] \\ & + \frac{1}{2}\, r\, Var^{'}_{t-1}\big[C|\,\vec{a}_t, \cdots, \vec{a}_N\big]. \end{aligned} \tag{4.52}$$

Finally, minimizing L_t gives the period t optimal incentive rates β_t^1 and β_t^2:

$$\beta_t^1 = \frac{\Big[\vec{b}_t\cdot\vec{m}_t - ICR_t^1\Big]\Big\{\vec{k}_t\cdot\vec{k}_t + r\, Var_{t-1}\big(z_t\big)\Big\} - \Big[\vec{b}_t\cdot\vec{k}_t - ICR_t^2\Big]\Big\{\vec{m}_t\cdot\vec{k}_t + r\, Cov_{t-1}\big(y_t, z_t\big)\Big\}}{D_t}, \tag{4.53}$$

$$\beta_t^2 = \frac{\Big[\vec{b}_t\cdot\vec{k}_t - ICR_t^2\Big]\Big\{\vec{m}_t\cdot\vec{m}_t + r\, Var_{t-1}\big(y_t\big)\Big\} - \Big[\vec{b}_t\cdot\vec{m}_t - ICR_t^1\Big]\Big\{\vec{m}_t\cdot\vec{k}_t + r\, Cov_{t-1}\big(y_t, z_t\big)\Big\}}{D_t}, \tag{4.54}$$

where

$$\begin{aligned} D_t = & \Big(m_{1t}\, k_{2t} - m_{2t}\, k_{1t}\Big)^2 + \Big(\vec{m}_t\cdot\vec{m}_t\Big)\, r\, Var_{t-1}\big(z_t\big) + \Big(\vec{k}_t\cdot\vec{k}_t\Big)\, r\, Var_{t-1}\big(y_t\big) \\ & - 2\,\Big(\vec{m}_t\cdot\vec{k}_t\Big)\, r\, Cov_{t-1}\big(y_t, z_t\big) + r^2\,\Big[Var_{t-1}\big(y_t\big)\, Var_{t-1}\big(z_t\big) - \Big\{Cov_{t-1}\big(y_t, z_t\big)\Big\}^2\Big]. \end{aligned}$$

$$(4.55)$$

It can be shown that D_t is positive as in (4.44).

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