THE UNIVERSITY OF ALBERTA

WEAK RADIATIVE DECAY OF BARYONS

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PRAVEER ASTHANA

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to Dada

whose strength helps us rest today

and to Dadi

whose compassion knew of no bounds in nursing others

to Nana

whose brilliance helps us shine today

and to Nani

whose generosity knew to no bounds in giving others

ABSTRACT

The purpose of this thesis is to do a quark-level calculation of weak radiative decay of baryons which is complete and also consistent with quantum field-theoretic principles.

First, the problem of weak radiative decay of baryons is formulated within the rigorous Bethe-Salpeter formalism. With suitable choices of interpolating fields for the initial and final baryons $\mathcal{B}$ and $\mathcal{B}'$, a reduction formula for the transition amplitude $\mathcal{B} \to \mathcal{B}'\gamma$ is derived in a manner parallel to the standard LSZ reduction procedure. One discovers that the Feynman rules for bound quarks are very different from those for free quarks. In this fashion, one recovers the methodology of analyzing composite particle reactions pioneered by Nishijima and Mandelstam in a way which is easily amenable to generalizations. A suitable parametrization for the Bethe-Salpeter amplitude of a $\frac{1}{2}^+$ baryon is suggested for calculational purposes. Various subtleties associated with the normalization of this amplitude are discussed.

Second, specializing this reduction formula to the case of $\Sigma^+ \to p\gamma$, a detailed analysis of this decay is carried out. Its asymmetry parameter is calculated. Differences with usual quark-model calculations are highlighted and commented upon.

ACKNOWLEDGEMENT

> *" A cold coming we had of it*
> *Just the worst time of the year*
> *For a journey, and such a long journey:*
> *The ways deep and the weather sharp,*
> *The very dead of winter "*
>
> *T. S. Eliot*

All my teachers since childhood have complained that I write a lot. And, I will not use the opportunity of writing this acknowledgement to prove them wrong.

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TABLE OF CONTENTS

CHAPTER | PAGE

1. Introduction ... 1
   - 1.1 Experimental Status of these Decays ... 2
   - 1.2 Theoretical Explanations ... 3
     - 1.2.1 Baryon-Level Calculations ... 4
     - 1.2.2 Quark-Level Analysis ... 11

2. Hyperon radiative decays in the Bethe-Salpeter Formalism ... 21
   - 2.1 Kinematics of a three-quark system ... 21
   - 2.2 The Bethe-Salpeter amplitude for a $\frac{1}{2}$ + baryon ... 23
   - 2.3 Relationship between the BS amplitude and the 6-quark Green function ... 26
   - 2.4 A consistent parametrization of the BS amplitude for $\frac{1}{2}$ Baryons ... 27
   - 2.5 Question of Normalization of the Parametrized BS Amplitude. ... 33
   - 2.6 The Nishijima-Mandelstam Formula for Hyperon Radiative Decays ... 35

3. Analysis of $\Sigma^+ \to p\gamma$ Decay ... 51
   - 3.1 Simplified Form of the Nishijima-Mandelstam formula for $\Sigma^+ \to p\gamma$ Decay ... 51
   - 3.2 Calculation of the Required Amputated Green Function ... 57
   - 3.3 Calculation of the unnormalized $\Sigma^+ \to p\gamma$ Amplitude ... 83

Chapter		Page
4.	Conclusion	113
Bibliography		120
Appendix A.	Flavour Structure of the Octet Baryons	124
Appendix B.	An Integral Relation between χ and $\overline{\chi}$	128
Appendix C.	Relevant Part of the Glashow-Salam-Weinberg Lagrangian	133
Appendix D.	Evaluation of the Feynman Parameter Integrals.	135
Appendix E.	Evaluation of the Momentum Integrals	149

LIST OF TABLES

Table	Description	Page
1.1	Experimental Status of $\frac{1}{2}^{+}$-Weak Radiative Decays	2
3.1	Values of Δ_i's	101 103
3.2	Values of Δ_i's	104 106
3.3	Values of Δ_i's	107 109
3.4	Values of Δ_i's	110 112

LIST OF FIGURES

Figure	Description	Page
1.1	The Decay $\mathcal{B} \to \mathcal{B}'\gamma$	3
1.2	Typical baryon pole diagrams for $\Sigma^+ \to p\gamma$	6
1.3a	1-quark transition	11
1.3b	A typical example of 1-quark transition permitted by GSW theory	11
1.4a	2-quark transition	12
1.4b	A typical example of 2-quark transition permitted by GSW theory	12
1.5a	3-quark transition	12
1.5b	A typical example of 3-quark transition permitted by GSW theory	12
1.6	Amputation of the 7-point Green function to obtain σ	18

Figure	Description	Page
2.1	Graphical depiction of the irreducible kernel V.	29
2.2	Graphical representation of the amputation procedure	46
2.3	Graphical representations of σ, $\bar{\chi}$ and χ in momentum space	49
3.1	Momentum-space Green functions required for the calculation of σ	59-60
3.2	$G_1^{(1)},\ldots,G_{28}^{(1)}$ in the Feynman-t'Hooft gauge	63-65
3.3	Permuted diagrams for G_1	65
3.4	$G_1^{(1)}$ in detail in configuration space	67
3.5	$G_1^{(2)}$, $G_1^{(3)}$ and $G_1^{(4)}$ in detail in configuration space	71
3.6	Feynman diagram for writing down $\sigma_1(p_1,p_2,p_3;p_4;p_1,p_2,p_3)_{\widetilde{\delta uD},\widetilde{\kappa uE},\widetilde{\sigma dF};\tilde{\mu}';\widetilde{\alpha uA},\widetilde{\beta uB},\widetilde{\gamma sC}}$	76
3.7	Feynman diagrams for various σ_i's	78-79
4.1	The self-energy diagrams	114

Figure	Description	Page
4.2	Quark propagator responsible for the imaginary part of the amplitude —— an example	115
E.1	Contour for the z-integral	152

CHAPTER 1

INTRODUCTION

> " *We shall not cease from exploration*
> *And the end of all our exploring*
> *Will be to arrive where we started*
> *And know the place for the first time* "
>
> *T. S. Eliot*

The quest for an understanding of weak interactions has revolutionized modern particle physics. This revolution has been twofold - first, weak interaction with its subtle violations of various invariances has unravelled some of the mysteries of symmetry principles and their modes of implementation in Nature; second, since it affects hadrons and leptons alike it has provided important information regarding the nature of strong and electromagnetic interactions too. The history of these developments is too well known to be traced once again. With its striking success at presently accessible energies, the Glashow-Salam-Weinberg (GSW) theory has acquired a canonical flavour. This solves the puzzle of electroweak interaction for quarks and leptons. We can now use it to discover some of the subtleties of strong interactions. Analysis and subsequent understanding of hadronic electroweak decays will enable us to do precisely this.

Weak radiative decay of baryons (e.g., $\Sigma^+ \to p\gamma$) are cleaner to analyze as compared to other hadronic decay modes of these particles.

Therefore, it is not a surprise that, despite being at the level of a few tenths of a percent in branching ratio, these decays have undergone such intense investigation since their discovery about 30 years ago.

Restricting ourselves to the $\frac{1}{2}+$ baryon octet, there are 6 possible decay modes of this kind [1,2]

$$\begin{array}{l} \Sigma^+ \to p\gamma \\ \Xi^- \to \Sigma^-\gamma \\ \Xi^\circ \to \Sigma^\circ\gamma \\ \Xi^\circ \to \Lambda\gamma \\ \Lambda \to n\gamma \\ \Sigma^\circ \to n\gamma \end{array} \qquad (1.1)$$

1.1 Experimental Status of these decays

The experimental status of these six decays is summarized in Table 1.1.

TABLE 1.1 Experimental Status of $\frac{1}{2}+$ Weak Radiative Decays

Decay Mode	Branching Fraction %	Asymmetry Parameter	REFERENCE
$\Sigma^+ \to p\gamma$	$(1.22 \pm 0.10)\times 10^{-3}$	-0.72 ± 0.29	[3,66]
$\Xi^- \to \Sigma^-\gamma$	$<1.2 \times 10^{-3}$		[3]
$\Xi^\circ \to \Sigma^\circ\gamma$	$<7 \times 10^{-3}$		[3]
$\Xi^\circ \to \Lambda\gamma$	$(1.1 \pm 0.2)10^{-3}$		[4]
$\Lambda \to n\gamma$	$(1.02 \pm 0.33)10^{-3}$		[5]
$\Sigma^\circ \to n\gamma$			

1.2. Theoretical Explanations

At the baryon level, the decay $\mathcal{B} \to \mathcal{B}'\gamma$ can be represented by the Feynman diagram in Fig. 1.1.

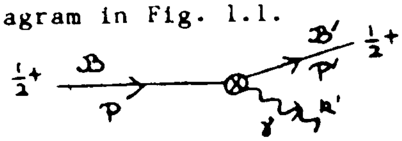


Fig. 1.1 .
The Decay $\mathcal{B} \to \mathcal{B}'\gamma$

It is easy to see that the most general matrix element for this process will have the following Lorentz structure [6-8]

$$\mathcal{M}_{fi} \equiv \epsilon^*_{\mu'}\Gamma^{\mu'} \equiv G_F e \quad \bar{u}^{S'}_{\mathcal{B}'}(P')\,\epsilon^*_{\mu'}[\gamma^{\mu'}(A + B\gamma_5) + i\sigma^{\mu'\nu'}k'_{\nu'}(C + D\gamma_5) + k'^{\mu'}(E + F\gamma_5)]u^S_{\mathcal{B}}(P) \qquad (1.2)$$

The Lorentz scalar quantities $A,\ldots,F$ are sometimes called the "**form factors**" for these decays. The gauge invariance condition $k'_{\mu'}\Gamma^{\mu'} = 0$ relates A, B, E and F and this relationship can be derived if desired [6,63]. However, in processes like ours, the photon is on its mass-shell (i.e., it is a real photon). For real photons, the form factors E and F drop out due to the transversality condition $\epsilon^*.k' = 0$. Gauge invariance condition $k'_{\mu'}\Gamma^{\mu'} = 0$ requires that A and B should be absent too. So, eqn. (1.2) becomes

$$\mathcal{M}_{fi} \equiv \epsilon^*_{\mu'}\Gamma^{\mu'} = -G_F e\,\bar{u}^{S'}_{\mathcal{B}'}(P')(C+D\gamma_5)\not\epsilon^*\not k' u^S_{\mathcal{B}}(P) \qquad (1.3)$$

Starting with eqn. (1.3), the decay rate can be obtained. It is given by

$$\Gamma = \frac{G_F^2\,e^2}{2\pi}\left(|C|^2 + |D|^2\right)k'^3 \ . \qquad (1.4)$$

The asymmetry parameter is given by

$$\alpha = \frac{2\mathrm{Re}(CD^*)}{|C|^2 + |D|^2} \tag{1.5}$$

It is obvious that the asymmetry parameter is independent of the normalization of the transition amplitude in eqn. (1.3).

The challenge for theorists is to calculate the form factors C and D from the underlying dynamics. Various attempts so far fall broadly into the following two classes.

1.2.1. Baryon-Level Calculations

In this class, the quark-substructures of $\mathcal{B}$ and $\mathcal{B}'$ are completely ignored. These baryon-level calculations can be further classified as follows.

1.2.1(1) Pure symmetry calculations.

The matrix element in eqn. (1.3) corresponds to an effective weak interaction Hamiltonian of the form

$$\mathcal{H}_{eff} = -\frac{1}{2} G_F e\, \bar{\psi}_{\mathcal{B}'}(x)(C + D\gamma_5)\sigma_{\mu'\nu'}\, \psi_{\mathcal{B}}(x)\, F^{\mu'\nu'} + \mathrm{h.c.} \tag{1.6}$$

where $\psi_{\mathcal{B}}(x)$ and $\psi_{\mathcal{B}'}(x)$ are the Dirac field operators for the initial and final baryon respectively. $F^{\mu'\nu'}$ is the electromagnetic field tensor. Even before embarking on detailed dynamics to calculate C and D, one can derive many meaningful relationships between various decay modes in eqn. (1.1) just based on the internal symmetry properties of $\psi_{\mathcal{B}}$ and $\bar{\psi}_{\mathcal{B}'}$. A great deal of work has been done in this area [2,9-13]. For the purpose of illustration, let us consider a simple case [2].

The $\frac{1}{2}^+$ baryons transform as an octet under SU(3). Let us denote the octet baryon tensor by $\mathcal{B}^i_j$. Obviously from eqn. (1.6) then, the effective Hamiltonian belongs to the direct product representation formed by combining $\mathcal{B}$ and $\bar{\mathcal{B}}$. When we combine two octets, we get a host of irreducible representations [14]. The trick is to assign $\mathcal{H}_{eff}$ to a sensible representation. To this end, SU(3) selection rules operating in weak hadronic decays and CP invariance come to our rescue. One assumes, as a simple and natural choice, that both the p.c. and p.v. parts of hadronic weak hamiltonian transform either as a symmetric octet (i.e. like $T^3_2 + T^2_3$) or an antisymmetric octet (i.e. $T^3_2 - T^2_3$) under SU(3). $T^{ij....}_{k\ell....}$ stands for a generic SU(3) tensor. Electromagnetic interaction transforms as T^1_1 under SU(3). Combining these two, the effective hamiltonian, $\mathcal{H}_{eff}$, in eqn. (1.6) should transform as $(T^{31}_{21} + T^{21}_{31})$ or as $(T^{31}_{21} - T^{21}_{31})$. This dictates the following SU(3) structure for $\mathcal{H}_{eff}$.

$$\begin{aligned}\mathcal{H}_{eff}(p.c) = {} & a_1(\bar{\mathcal{B}}^3_2\mathcal{B}^1_1 \pm \bar{\mathcal{B}}^2_3\mathcal{B}^1_1 + h.c) \\ & + a_2(\bar{\mathcal{B}}^1_2\mathcal{B}^3_1 - \frac{1}{3}\bar{\mathcal{B}}^i_2\mathcal{B}^3_i + h.c) \\ & + a_3(\bar{\mathcal{B}}^3_1\mathcal{B}^1_2 - \frac{1}{3}\bar{\mathcal{B}}^3_i\mathcal{B}^i_2 + h.c)\end{aligned} \tag{1.7}$$

where the $\pm$ sign corresponds to the form $(T^{31}_{21} \pm T^{21}_{31})$. $\mathcal{H}_{eff}(p.v.)$ has a similar SU(3) structure. Invoking the explicit forms for $\mathcal{B}$ and $\bar{\mathcal{B}}$ in terms of their particle content, one can immediately obtain the following expressions for the p.c. amplitude $C(\mathcal{B} \to \mathcal{B}'\gamma)$.

$$C(\Sigma^+ \to p) = \frac{2}{3} a_3$$

$$C(\Xi^- \to \Sigma^-) = \frac{2}{3} a_2$$

$$C(\Xi^\circ \to \Sigma^\circ) = \pm \frac{a_1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} a_2$$

$$C(\Xi^\circ \to \Lambda) = \pm \frac{a_1}{\sqrt{6}} - \frac{1}{3\sqrt{6}} a_2 + \frac{2}{3\sqrt{6}} a_3 \qquad (1.8)$$

$$C(\Lambda \to n) = \frac{a_1}{\sqrt{6}} + \frac{2}{3\sqrt{6}} a_2 - \frac{1}{3\sqrt{6}} a_3$$

$$C(\Sigma^\circ \to n) = \frac{a_1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} a_3$$

Similar expressions can be found for the p.v. amplitude D . From these, certain sum rules can be deduced [2]. At present, however, the available experimental information is not enough to check these sum rules.

1.2.1(2) Pole Model Calculations.

The pole model goes a step further and tries to calculate C and D . There are many variations of this approach [1,15,16]. We pick a simple case [1,2] as an example. $\mathcal{B} \to \mathcal{B}'\gamma$ is thought to proceed according to the two tree diagrams drawn in Fig. 1.2.

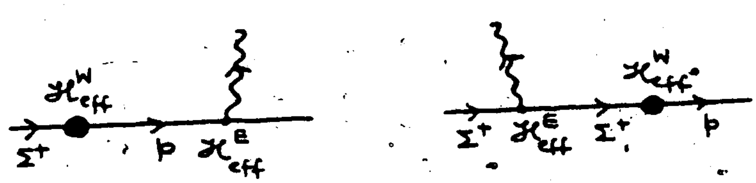


Fig 1.2. Typical baryon pole diagrams for $\Sigma^+ \to p\gamma$.

Obviously, the effective interaction Hamiltonian is of the form

$$\mathcal{H}^{int}_{eff} = \mathcal{H}^{W}_{eff} + \mathcal{H}^{E}_{eff} \tag{1.9}$$

Assigning $\mathcal{H}^{W}_{eff}$ again to the SU(3) octet to uphold $\Delta I = \frac{1}{2}$ rule, we get the following expression for it:

$$\begin{aligned}\mathcal{H}^{W}_{eff} &= a(\bar{\mathcal{B}}^{c}_{2}\mathcal{B}^{3}_{c} + \bar{\mathcal{B}}^{c}_{3}\mathcal{B}^{2}_{c}) + b(\bar{\mathcal{B}}^{3}_{c}\mathcal{B}^{c}_{2} + \bar{\mathcal{B}}^{2}_{c}\mathcal{B}^{c}_{3}) \\ &+ a'(\bar{\mathcal{B}}^{c}_{2}\gamma_5\mathcal{B}^{3}_{c} - \bar{\mathcal{B}}^{c}_{3}\gamma_5\mathcal{B}^{2}_{2}) \\ &+ b'(\bar{\mathcal{B}}^{3}_{c}\gamma_5\mathcal{B}^{c}_{2} - \bar{\mathcal{B}}^{2}_{c}\gamma_5\mathcal{B}^{c}_{3})\end{aligned} \tag{1.10}$$

The constants a,b,a' and b' are determined from a similar pole-diagram analysis of $\mathcal{B} \to \mathcal{B}' + \pi$ decays. In one such model calculation [1]

$$\begin{aligned} a &= 3.5 \times 10^{-4} \text{ MeV} \\ b &= -8.4 \times 10^{-4} \text{ MeV} \\ a' &= -9.0 \times 10^{-5} \text{ MeV} \\ b' &= -2.5 \times 10^{-5} \text{ MeV}\end{aligned} \tag{1.11}$$

$\mathcal{H}^{E}_{eff}$ has the following from

$$\mathcal{H}^{E}_{eff} = -\left[ieq_{\mathcal{B}}\ \bar{\psi}_{\mathcal{B}}\ \gamma_{\mu'}\ \psi_{\mathcal{B}}\ A^{\mu'} + \frac{1}{2}e\ \bar{\psi}_{\mathcal{B}}\ \frac{\mu_{\mathcal{B}}}{2m_{\mathcal{B}}}\ \sigma_{\mu'\nu'}\ \psi_{\mathcal{B}}\ F^{\mu'\nu'}\right] \tag{1.12}$$

where $q_{\mathcal{B}}$ is the charge of the baryon and $\mu_{\mathcal{B}}$ is its magnetic moment. Evaluating the tree diagrams in Fig. 1.2 with eqns. (1.9)-(1.12), we obtain (e.g. for $\Sigma^{+} \to p\gamma$)

$$C(\Sigma^+ \to p\gamma) = e\left(\frac{\mu_p}{2m_N} - \frac{\mu_{\Sigma^+}}{2m_\Sigma}\right)\frac{b}{m_\Sigma - m_N}$$

$$D(\Sigma^+ \to p\gamma) = -e\left(\frac{\mu_{\Sigma^+}}{2m_\Sigma} + \frac{\mu_p}{2m_N}\right)\frac{b'}{m_\Sigma + m_N} \qquad (1.13)$$

$m_{\mathcal{B}}$ stands for the baryon mass. Using the SU(3) value $\mu_{\Sigma^+} = \mu_p = 1.79$ we obtain

$$\frac{\Gamma(\Sigma^+ \to p\gamma)}{\Gamma(\Sigma^+ \to p\pi^\circ)} = 0.28\% \qquad (1.14)$$

which agrees well with the experimental value [1-3]. However, in the above scheme the parity conserving part is the dominant contribution to the decay. This fails to reproduce the asymmetry parameter [1,3].

Some modifications of the pole model include other resonances as intermediate states [17-19]. These calculations more or less, reproduce the experimental results for $\Sigma^+ \to p\gamma$ and $\Lambda^\circ \to n\gamma$ [5]. There has been an attempt to include meson poles also [20].

1.2.1(3) Current Algebra Calculations

Techniques of current algebra have also been applied to these decays [21]. In this approach, the $\mathcal{B} \to \mathcal{B}'\gamma$ amplitude is first related to $\mathcal{B} \to \mathcal{B}'\pi\gamma$ amplitude by a low-energy theorem. One then expands the $\mathcal{B} \to \mathcal{B}'\pi\gamma$ amplitude in powers of the photon momentum k' using Low's procedure [22]. By keeping the k'^{-1} and k'° order terms only, one can express $\mathcal{B} \to \mathcal{B}'\gamma$ amplitude in terms of $\mathcal{B} \to \mathcal{B}'\pi$ amplitude. Such an analysis leads to a sizable parity violating contribution to $\mathcal{B} \to \mathcal{B}'\gamma$ amplitude. Results are in reasonable agreement with experiment [3,21].

1.2.1(4) Unitarity Calculations

In another class of calculations, unitarity has been used to get lower bounds on these decay rates. The trick is to realize that the real and the imaginary parts of C and D do not interfere in the decay rate. So, if one can determine at least the imaginary part, one can easily get a lower bound on the decay rate. Unitarity allows us to compute the imaginary part of the amplitude. By virtue of unitarity the imaginary part of the amplitude can be determined in terms of a sum over physical, experimentally measurable amplitudes for intermediate states. In one such claculation [23,24] only $N\pi$ intermediate states were considered. One can write

$$\mathrm{Im}\, \mathcal{M}_{fi} = \frac{1}{2} (2\pi)^4 \int \frac{d^3p'}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \delta^4(P-p'-q) \frac{m}{p_0'} \frac{1}{2q_0} \mathcal{M}^{\dagger}_{fn} \mathcal{M}_{ni} \quad . \tag{1.15}$$

P, p' and q are the initial baryon, intermediate nucleon and pion momenta respectively; m is the intermediate nucleon mass. n, the intermediate state corresponds to $N\pi$ state. So, the imaginary part gets related to the pionic decay and photoproduction amplitudes, both of which are known. Borrowing those one gets the following unitarity bounds

$$\frac{\Gamma(\Sigma^+ \to p\gamma)}{\Gamma(\Sigma^+ \to \text{all})} \gtrsim 6.9 \times 10^{-6}$$

$$\frac{\Gamma(\Lambda \to n\gamma)}{\Gamma(\Lambda \to \text{all})} \gtrsim 8.5 \times 10^{-4} \qquad (1.16)$$

There have been still bolder attempts [23,24] along these lines to even estimate the real parts of these amplitudes using dispersion techniques supplemented with current-algebra and PCAC. The results are far from being satisfactory.

1.2.1(5) Critique of the baryon-level calculations.

There baryon-level calculations have employed a variety of techniques to guess the structure of the hadronic weak Hamiltonian from the observed data. The list of techniques is impressive and different models have succeeded, to varied degrees, in putting forward convincing mechanisms for these decays. At the present stage in the history of particle physics, however, these baryon-level calculations can hardly be called satisfactory explanations for hyperon weak radiative decays. We know that the hyperons are made up of quarks and the GSW theory gives us very definite mechanisms for such flavour-changing weak radiative transitions. Baryon-level calculations, by the very nature of the analyses involved, tell us very little as to how quark dynamics gives rise to the observed decays. They fail to give us a microsocpic picture; they are in the pre-GSW spirit. Now that the GSW theory is so well established, one must try to explain these decays at the quark level. This has precisely been the recent line of investigation.

1.2.2. Quark-level Analyses

§1.2.2(1) The quark-level analyses of these decays start with Gilman and Wise in 1979 [25]. They assumed that these transitions are dominated by a single s quark decaying into a d quark and a photon, while the other two quarks remain 'spectators' (Fig. 1.3a). GSW theory promises such a mechanism (Fig. 1.3b). The quark-level amplitude is related to the baryon-level amplitude following the usual nonrelativistic quark model (NRQM) methods [26,27] . $\Sigma^+ \to p\gamma$ decay rate was used as an input parameter in this calculation to predict the other decay rates. It was realized that this 1-quark transition

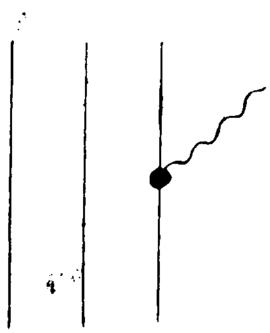


Fig. 1.3a 1-quark transition

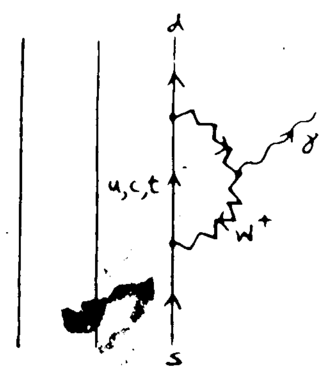


Fig. 1.3b A typical example of 1-quark transition permitted by GSW theory

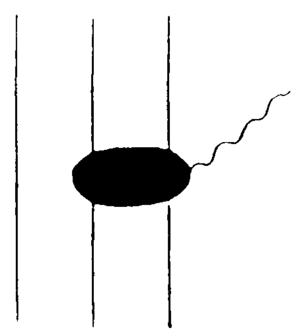


Fig. 1.4a 2-quark transition

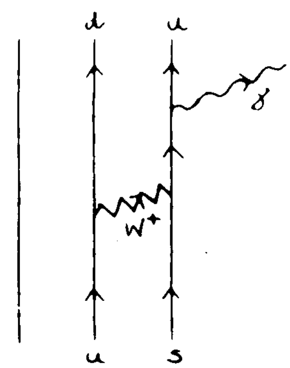


Fig. 1.4b A typical example of 2-quark transition permitted by GSW theory

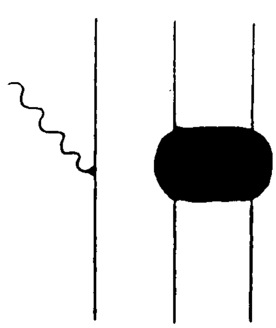


Fig. 1.5a 3-quark transition

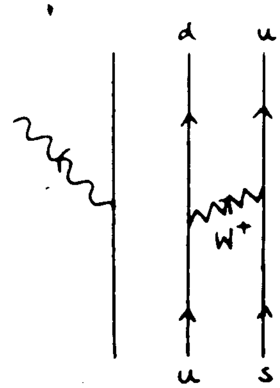


Fig. 1.5b A typical example of 3-quark transition permitted by GSW theory

was incapable of reproducing the experimental data [5,25].

In addition to the 1-quark transition, 2-quark and 3-quark transitions also contribute to these decays in the same order (Figs. 1.4 and 1.5). Kamal and Verma [28] looked at the contributions of these additional mechanisms. Following the NRQM methods again, they combined the 2-quark amplitude with the 1-quark one. Their model had three free parameters which were fixed by the $\Sigma^+ \to p\gamma$ decay rate, its asymmetry parameter and the bound on $\Xi^- \to \Sigma^-\gamma$ decay rate. Their predictions for $\Lambda \to n\gamma$ and $\Xi^0 \to \Lambda\gamma$ are in agreement with the recent experimental data [4,5]. However, they had to invoke an additional dimensionful parameter to combine the 2-quark amplitude with the 1-quark one despite the fact that both these contributions are of the same order in the coupling constants.

Almost at the same time, Lo [29] came up with a very detailed analysis of hyperon radiative decays within the MIT Bag Model [26,30-31]. He included the 3-quark contributions as well. He also took account of the short-range gluonic corrections. The salient features of his calculation are:

(i) This calcuation has only one parameter, viz., the overall normalization, which is fixed by the $\Sigma^+ \to p\gamma$ rate.

(ii) The calculated value of $\Sigma^+ \to p\gamma$ asymmetry parameter is $\alpha = -0.154$. Though the sign is correct, the magnitude is a bit too small.

(iii) The 3-quark amplitude identically vanishes in his calculation.

Some other quark-level attempts have also appeared in the literature [32] but the cases mentioned above, more or less, summarize the main thrust of various quark-level investigations.

The scenario that emerges out of all these efforts can be summed up as follows.

(i) There is little hope of solving the riddle of these decays unless one combines all the permitted quark processes.

(ii) All these calculations take $\Sigma^+ \to p\gamma$ rate and asymmetry parameter as inputs. Experimentally this is the best measured process but there is still no ab initio calculation of these parameters.

(iii) All these analyses either do not calculate or fail to reproduce the large negative asymmetry parameter of $\Sigma^+ \to p\gamma$ decay. Being independent of the normalization, this should be a more reliable piece of prediction. All sorts of conjectures have been put forward in the literature to explain this discrepancy [10,29,33].

1.2.2(2) Critique of the quark-level calculations

These calculations have all been done following the NRQM or MIT bag model (MITBM) methodologies. Though the NRQM and the MITBM have been very successful in explaining many facets of hadronic phenomenology, one will have to be a bit prudent in their application to hyperon radiative decays. These processes are complicated because both spectator and non-spectator processes compete in the same order. So before we seek alternative quark-level mechanisms (i.e. beyond GSW theory) to explain these decays, it is important to ask whether the calculations above have been rigorous enought. It turns out, and as will be explained in detail in the sequel, that there are some serious field-theoretic consistency problems in the above analyses. To appreciate this criticism, let us scrutinize the NRQM method of obtaining a baryon-level amplitude from the quark-level one.

We take up the $\Sigma^+ \to p\gamma$ case as an illustration. This is a two-body decay process. One goes to the rest-frame of Σ^+ and seeks a quark-level expression for the amplitude

$$\langle p(E',+k'\hat{z};\downarrow),\gamma(k',-k'\hat{z};\lambda=-1)\ \text{out}|\Sigma^+(M,0;\uparrow)\text{in}\rangle$$

where $\uparrow$ or $\downarrow$ give us the spin $\frac{1}{2}$ components in the positive z-direction. One uses the SU(6) wave functions for the baryons and assumes that

$$\begin{aligned}
&\langle p(E',+k'\hat{z};\downarrow),\gamma(k',-k'\hat{z};\lambda=-1)\ \text{out}|\Sigma^+(M,0;\uparrow)\text{in}\rangle \\
&\propto \int d^3\vec{p}_1{}' d^3\vec{p}_2{}' d^3\vec{p}_3{}' d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p}_3\ \psi_p^*(\vec{p}_1{}',\vec{p}_2{}',\vec{p}_3{}')\psi_{\Sigma^+}(\vec{p}_1,\vec{p}_2,\vec{p}_3) \\
&\langle [u(\vec{p}_1{}',\uparrow)u(\vec{p}_2{}',\downarrow)d(\vec{p}_3{}',\downarrow) + u(\vec{p}_1{}',\downarrow)u(\vec{p}_2{}',\uparrow)d(\vec{p}_3{}',\downarrow) \\
&-2u(\vec{p}_1{}',\downarrow)u(\vec{p}_2{}',\downarrow)d(\vec{p}_3{}',\uparrow) + u(\vec{p}_1{}',\uparrow)d(\vec{p}_2{}',\downarrow)u(\vec{p}_3{}',\downarrow) \\
&-2u(\vec{p}_1{}',\downarrow)d(\vec{p}_2{}',\uparrow)u(\vec{p}_3{}',\downarrow) + u(\vec{p}_1{}',\downarrow)d(\vec{p}_2{}',\downarrow)u(\vec{p}_3{}',\uparrow) \\
&-2d(\vec{p}_1{}',\uparrow)u(\vec{p}_2{}',\downarrow)u(\vec{p}_3{}',\downarrow) + d(\vec{p}_1{}',\downarrow)u(\vec{p}_2{}',\uparrow)u(\vec{p}_3{}',\downarrow) \\
&+d(\vec{p}_1{}',\downarrow)u(\vec{p}_2{}',\downarrow)u(\vec{p}_3{}',\uparrow)]\gamma\ \text{out}| \\
&[2u(\vec{p}_1,\uparrow)s(\vec{p}_2,\downarrow)u(\vec{p}_3,\uparrow) + 2s(\vec{p}_1,\downarrow)u(\vec{p}_2,\uparrow)u(\vec{p}_3,\uparrow) \\
&+2u(\vec{p}_1,\uparrow)u(\vec{p}_2,\uparrow)s(\vec{p}_3,\downarrow) - u(\vec{p}_1,\downarrow)s(\vec{p}_2,\uparrow)u(\vec{p}_3,\uparrow) \\
&-s(\vec{p}_1,\uparrow)u(\vec{p}_2,\downarrow)u(\vec{p}_3,\uparrow) - u(\vec{p}_1,\uparrow)u(\vec{p}_2,\downarrow)s(\vec{p}_3,\uparrow) \\
&-u(\vec{p}_1,\downarrow)u(\vec{p}_2,\uparrow)s(\vec{p}_3,\uparrow) - u(\vec{p}_1,\uparrow)s(\vec{p}_2,\uparrow)u(\vec{p}_3,\downarrow) \\
&-s(\vec{p}_1,\uparrow)u(\vec{p}_2,\uparrow)u(\vec{p}_3,\downarrow)]\text{in}\rangle
\end{aligned} \tag{1.17}$$

$\psi_p^*(\vec{p}_1{}',\vec{p}_2{}',\vec{p}_3{}')$ and $\psi_{\Sigma^+}(\vec{p}_1,\vec{p}_2\ \vec{p}_3)$ are some normalizable 'shape factors' usually obtained from a potential model. The overall normalization of the amplitude is fixed by a known decay process. In order to obtain each of the quark matrix elements in eqn. (1.17), one does a quark-diagram analysis by treating the quarks as free. One then performs a nonrelativistic reduction of those amplitudes and feeds them back into eqn. (1.17). It is claimed that the right-hand side of eqn. (1.17) is, then, proportional to the left hand side. From a strict field-theoretic point of view, simple integration of a free-quark matrix element over some shape factors does not yield the matrix element between the corresponding bound states.

In fact the correct method to analyze bound-state reactions within the rigorous Bethe-Salpeter (BS) formalism was put forward by Nishijima and Mandelstam around 1955 [34-38]. As will be shown later, an application of that method to hyperon radiative decays yields the following formula for the $\Sigma^+ \to p\gamma$ decay amplitude

$$\begin{aligned}
&\langle p(P',S'),\gamma(k',\lambda')\ \mathrm{out}|\Sigma^+(P,S)\mathrm{in}\rangle \\
&= Z_3^{1/2}\,\frac{1}{2P^0}\,\frac{1}{2P'^0}\int d^4y_1 d^4y_2 d^4y_3 d^4y_4 d^4x_1 d^4x_2 d^4x_3 \\
&\quad f^*_{k',\lambda'}(y_4)^{\mu'}\ \bar{\chi}(y_1,y_2,y_3)^{p:P',S'}_{\delta dD,\kappa eE,\sigma fF} \\
&\quad \sigma(y_1,y_2,y_3;y_4;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC} \\
&\quad \chi(x_1,x_2,x_3)^{\Sigma^+:P,S}_{\alpha aA,\beta bB,\gamma cC}
\end{aligned} \tag{1.18}$$

where Z_3 is the photon renormalization constant; $(\delta dD,\kappa eE,\sigma fF)$ and $(\alpha aA,\beta bB,\gamma cC)$ are the (Dirac, flavour, colour) indices of the final and initial quarks, (y_1,y_2,y_3) and (x_1,x_2,x_3) are the spacetime labels of the final and initial quarks, y_4 is the spacetime label of the photon. All repeated indices are summed over. $\bar{\chi}^p$ and χ^{Σ^+} are the

BS amplitudes for p and Σ^+ respectively. The quantity σ in eqn. (1.18) is the 'amputated 7-point Green function' given by

$$G(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$$
$$\equiv \langle 0|T\psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)A^{\mu'}(y_4)\overline{\psi}_{\alpha aA}(x_1)\overline{\psi}_{\beta bB}(x_2)\overline{\psi}_{\gamma cC}(x_3)|0\rangle$$
$$= \int d^4y_1'd^4y_2'd^4y_3'd^4y_4'd^4x_1'd^4x_2'd^4x_3'$$
$$G(y_1,y_2,y_3;y_1',y_2',y_3')_{\delta dD,\kappa eE,\sigma fF;\tilde{\delta}\tilde{d}\tilde{D},\tilde{\kappa}\tilde{e}\tilde{E},\tilde{\sigma}\tilde{f}\tilde{F}}\ G(y_4,y_4')^{\mu'\tilde{\mu}'}$$
$$\sigma(y_1',y_2',y_3';y_4';x_1',x_2',x_3')_{\tilde{\delta}\tilde{d}\tilde{D},\tilde{\kappa}\tilde{e}\tilde{E},\tilde{\sigma}\tilde{f}\tilde{F};\tilde{\mu}';\tilde{\alpha}\tilde{a}\tilde{A},\tilde{\beta}\tilde{b}\tilde{B},\tilde{\gamma}\tilde{c}\tilde{C}}$$
$$G(x_1',x_2',x_3';x_1,x_2,x_3)_{\tilde{\alpha}\tilde{a}\tilde{A},\tilde{\beta}\tilde{b}\tilde{B},\tilde{\gamma}\tilde{c}\tilde{C};\alpha aA,\beta bB,\gamma cC} \tag{1.19}$$

where

$$G(y_1,y_2,y_3;y_1',y_2',y_3')_{\delta dD,\kappa eE,\sigma fF;\tilde{\alpha}\tilde{a}\tilde{A},\tilde{\beta}\tilde{b}\tilde{B},\tilde{\gamma}\tilde{c}\tilde{C}}$$
$$\equiv -\langle 0|T\psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)\overline{\psi}_{\tilde{\delta}\tilde{d}\tilde{D}}(y_1')\overline{\psi}_{\tilde{\kappa}\tilde{e}\tilde{E}}(y_2')\overline{\psi}_{\tilde{\sigma}\tilde{f}\tilde{F}}(y_3')|0\rangle \tag{1.20}$$

$$G(y_4,y_4')^{\mu'\tilde{\mu}'} \equiv \langle 0|TA^{\mu'}(y_4)A^{\tilde{\mu}'}(y'_4)|0\rangle \tag{1.21}$$

and

$$G(x_1',x_2',x_3';x_1,x_2,x_3)_{\tilde{\alpha}\tilde{a}\tilde{A},\tilde{\beta}\tilde{b}\tilde{B},\tilde{\gamma}\tilde{c}\tilde{C};\alpha aA,\beta bB,\gamma cC}$$
$$\equiv -\langle 0|T\psi_{\tilde{\alpha}\tilde{a}\tilde{A}}(x_1')\psi_{\tilde{\beta}\tilde{b}\tilde{B}}(x_2')\psi_{\tilde{\gamma}\tilde{c}\tilde{C}}(x_3')\overline{\psi}_{\alpha aA}(x_1)\overline{\psi}_{\beta bB}(x_2)\overline{\psi}_{\gamma cC}(x_3)|0\rangle \tag{1.22}$$

All field operators in the above expressions are Heisenberg operators. Diagramatically, eqn. (1.19) implies that from the 7-point Green function, full 6-point quark Green functions are amputated in the initial and final state and full photon leg is amputated in the final state. Whatever is left of the 7-point Green function gives us σ. (Fig. 1.6). The procedure to do this order-by-order

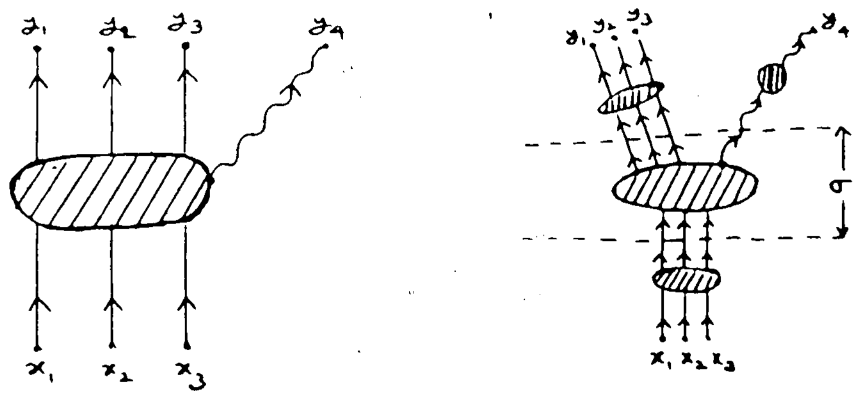


Fig. 1.6. Amputation of the 7-point Green function to obtain σ.

was first elucidated by Mandelstam [36].

The amputation required by the 'Nishijima-Mandelstam formula' i.e. eqn. (1.18) is, therefore, very different. We recall that if the final and initial quarks were treated as free, one will amputate the seven legs separately to all orders [35,39,40]. Such an amputation will never give rise to bound states as the baryon-pole comes in the 6-quark Green function. Hence, the usual quark model philosophy has serious consistency problems. It turns out that the correct amputation gives very different Feynman rules for bound quarks [36-38]. In particular, one finds that the spectator lines contribute inverse propagators. So, in the usual quark model calculations one is not doing the perturbation theory for bound quarks correctly. It is for this reason that Kamal and Verma [28] had to introduce a new parameter to add quark diagrams in the same order. In processes like ours, where spectator and nonspectator processes compete in the same order, it becomes all the more important to do the perturbation theory for bound quarks correctly. It is not obvious that these consistency requirements are met in the MITBM calculations either.

To sum up, we realize that the method of relating the quark-level amplitude to the baryon-level amplitude in the NRQM and the MITBM is far from being rigorous. A rigorous BS calculation introduces many nontrivial features in the analysis. It is, therefore, important to investigate these effects in detail before one looks for alternative mechanisms at the quark level to explain these decays.

It is such a rigorous and detailed analysis of these decays that is taken up in this thesis. We calculate the amplitude according to eqn. (1.18). GSW theory then exactly tells us how to calculate σ. For $\overline{\chi}^{P}$ and χ^{Σ^+}, we assume a covariant and normalizable Gaussian in relative momentum variables. The invariance properties of the baryons are built into them. Their parameters are fixed by the quark and baryon masses and the radii of the baryons. This scheme, then, enables us to predict the asymmetry parameter for these decays without any ambiguity. As we shall see in Chapter 2, the normalization condition for BS amplitude needs the strong interaction dynamics as an input. In our scheme —— where we have used phenomenological insight to parameterize the BS amplitude without actually deriving it from some strong interaction dynamics —— it is not possible to rigorously normalize it. The overall normalization of the decay amplitude, therefore, remains undetermined. As a result, this scheme does not permit us to predict the rate for an individual decay process.

In Chapter 2 of this thesis, we formulate the problem of weak radiative decay of baryons within the precincts of the BS formalism. We also point out the approximation and assumptions involved in the choice of $\overline{\chi}^{p}$ and $\chi^{\Sigma^{+}}$.

In Chapter 3, we derive Feynman rules for bound quarks and then use them to calculate the $\Sigma^{+} \rightarrow p\gamma$ asymmetry parameter.

In Chapter 4, we end the thesis by commenting on our results. Specifically the differences with the usual quark model calculations are highlighted.

CHAPTER 2

HYPERON RADIATIVE DECAYS IN THE BETHE-SALPETER FORMALISM

> " *If you came this way,*
> *Taking the route you would be likely to take*
> *From the place where you would be likely to come from* "
>
> *T. S. Eliot*

In this chapter, we first discuss how $\frac{1}{2}+$ baryons are described in the Bethe-Salpeter formalism [30,35,37,38,40-44]. Then the S-matrix element for hyperon radiative decay will be derived following the method of Nishijima and Mandelstam [35-38]. All field operators in this chapter, unless explicitly stated, are Heisenberg operators.

2.1. Kinematics of a three-quark system

A baryon $\mathcal{B}$ consists of three interacting quarks described by the field operators $\psi_{\alpha aA}(x_1)$, $\psi_{\beta bB}(x_2)$ and $\psi_{\gamma cC}(x_3)$. (αaA, βbB, γcC) are the (Dirac, Flavour, Colour) indices of the three quarks respectively and (x_1,x_2,x_3) are their spacetime locations. Instead of (x_1,x_2,x_3), it is sometimes useful to work in **baryonic coordinates** (X,ξ,η) given by

$$X = f_1 x_1 + f_2 x_2 + f_3 x_3$$
$$\xi = x_1 - x_2 \qquad (2.1)$$
$$\eta = \frac{1}{2}(x_1 + x_2 - 2x_3)$$

where the coefficients f_1, f_2, f_3 satisfy

$$f_1 + f_2 + f_3 = 1 \qquad (2.2)$$

In our case, where the three quarks have non-zero masses, we can choose

$$f_i = \frac{m_i}{m_1 + m_2 + m_3} \quad ; \; i = 1,2,3 \qquad (2.3)$$

The discussion in this chapter, however, is independent of any specific choice of f_1, f_2, f_3 so long as they satisfy eqn. (2.2). It can be easily shown that the Jacobian of the above transformation is equal to one.

The inverse transformation is given by

$$x_1 = X + f_3\eta - (f_1 + \frac{f_3}{2} - 1)\xi \equiv X + F_1(\xi,\eta)$$
$$x_2 = X + f_3\eta - (f_1 + \frac{f_3}{2})\xi \equiv X + F_2(\xi,\eta) \qquad (2.4)$$
$$x_3 = X + (f_3 - 1)\eta - (f_1 + \frac{f_3}{2} - \frac{1}{2})\xi \equiv X + F_3(\xi,\eta)$$

Let p_1, p_2, p_3 be the 4-momenta of the quarks at x_1, x_2, x_3 respectively. Again, instead of (p_1, p_2, p_3) we can work in terms of a transformed set of momentum variables ($, p_\xi, p_\eta$). These two sets are related by

$$\mathcal{P} = p_1 + p_2 + p_3$$
$$p_\xi = \frac{1}{2}(2f_2 + f_3)p_1 - \frac{1}{2}(2f_1 + f_3)p_2 - \frac{1}{2}(f_1 - f_2)p_3 \qquad (2.5)$$
$$p_\eta = f_3 p_1 + f_3 p_2 - (f_1 + f_2)p_3$$

The Jacobian of this transformation is also equal to one. The inverse transformation is given by

$$p_1 = f_1\mathcal{P} + p_\xi + \frac{1}{2} p_\eta$$
$$p_2 = f_2\mathcal{P} - p_\xi + \frac{1}{2} p_\eta \qquad (2.6)$$
$$p_3 = f_3\mathcal{P} - p_\eta$$

It is straightforward to obtain the following useful result

$$\mathcal{P} X + p_\xi \xi + p_\eta \eta = p_1 x_1 + p_2 x_2 + p_3 x_3 \qquad (2.7)$$

2.2 The Bethe-Salpeter amplitude for a $\frac{1}{2}^+$ baryon

The bound-state properties of a $\frac{1}{2}^+$ baryon are contained in the **BS amplitude** [30,35,37,38,40-44]

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \equiv \langle 0|T\ \psi_{\alpha aA}(x_1)\ \psi_{\beta bB}(x_2)\ \psi_{\gamma cC}(x_3)|\mathcal{B}:P,S\rangle \qquad (2.8)$$

where P,S are the 4-momentum and of spin of the baryon respectively. Other flavour and colour quantum numbers of the baryon are obvious from $\mathcal{B}$. For instance, if $\mathcal{B}$ is a proton, then we know that it belongs to the octet of flavour-SU(3) and singlet of colour-SU(3). Thus, without risking any loss of information, we have suppressed the flavour and colour quantum numbers of $\mathcal{B}$. (αaA, βbB, γcC) remain the same as before. The baryon $\mathcal{B}$ is on mass-shell, i.e.

$$P^2 = M^2 \tag{2.9}$$

where M is the baryon mass.

Based on spacetime translation invariance properties of the field operators in eqn. (2.8), we can immediately write

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{1}{(2\pi)^{3/2}} e^{-iP.X} \chi[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{2.10}$$

where the **relative BS amplitude** $\chi[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$ is defined by

$$\chi[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$\equiv (2\pi)^{3/2} \langle 0|T\ \psi_{\alpha aA}(F_1(\xi,\eta))\ \psi_{\beta bB}(F_2(\xi,\eta))\ \psi_{\gamma cC}(F_3(\xi,\eta))|\mathcal{B}:P,S\rangle \tag{2.11}$$

We shall always write the arguments of the relative BS amplitude inside square brackets.

Corresponding to χ, we have a **conjugate BS amplitude** defined by

$$\bar{\chi}(y_1,y_2,y_3)^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \equiv \langle \mathcal{B}:P,S|T\ \bar{\psi}_{\delta dD}(y_1)\ \bar{\psi}_{\kappa eE}(y_2)\ \bar{\psi}_{\sigma fF}(y_3)|0\rangle \tag{2.12}$$

The notation used in eqn. (2.12) must be self-evident by now. We use a different set of coordinates and indices for future convenience. Spacetime translation invariance again implies

$$\bar{\chi}(y_1,y_2,y_3)^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} = \frac{1}{(2\pi)^{3/2}} e^{iP.Y} \bar{\chi}[\xi',\eta']^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \tag{2.13}$$

(Y,ξ',η') are related to (y_1,y_2,y_3) through three constants f_1', f_2', f_3' in exactly the same way as (X,ξ,η) are related to (x_1,x_2,x_3)

through the constants (t_1, t_2, t_3). The **conjugate relative BS amplitude** is defined by

$$\bar{\chi}[\xi', \eta']^{\mathcal{B}:P,S}_{\delta dD, \kappa eE, \sigma fF}$$

$$= (2\pi)^{3/2} \langle \mathcal{B}:P,S | T\, \bar{\psi}_{\delta dD}(F_1'(\xi',\eta'))\, \bar{\psi}_{\kappa eE}(F_2'(\xi',\eta'))\, \bar{\psi}_{\sigma fF}(F_3'(\xi',\eta')) | 0 \rangle \tag{2.14}$$

where $F_1'(\xi',\eta')$, $F_2'(\xi',\eta')$ and $F_3'(\xi',\eta')$ are obviously related to $(t_1', t_2', t_3', \xi', \eta')$ in exactly the same way as $F_1(\xi,\eta)$, $F_2(\xi,\eta)$ and $F_3(\xi,\eta)$ are related to $(t_1, t_2, t_3, \xi, \eta)$.

In the interest of continuity, it is wise to introduce the momentum space representations of χ and $\bar{\chi}$ at this stage. We define the **BS amplitude in momentum space** by the following Fourier transform

$$\chi(p_1, p_2, p_3)^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC}$$

$$= \int d^4x_1 d^4x_2 d^4x_3\; e^{ip_1x_1 + ip_2x_2 + ip_3x_3}\; \chi(x_1, x_2, x_3)^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC} \tag{2.15}$$

Using eqns. (2.10) and (2.7) in eqn. (2.15) we obtain

$$\chi(p_1, p_2, p_3)^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC} = \frac{(2\pi)^4}{(2\pi)^{3/2}}\, \delta^4(P - \mathcal{P})\; \dot{\chi}[p_\xi, p_\eta]^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC} \tag{2.16}$$

where

$$\chi[p_\xi, p_\eta]^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC} = \int d^4\xi\, d^4\eta\; e^{ip_\xi \xi + ip_\eta \eta}\; \chi[\xi, \eta]^{\mathcal{B}:P,S}_{\alpha aA, \beta bB, \gamma cC} \tag{2.17}$$

Similarly, **the conjugate BS amplitude in momentum space** is defined by

$$\bar{\chi}(p_1',p_2',p_3')^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF}$$

$$= \int d^4y_1 d^4y_2 d^4y_3 \; e^{-ip_1'y_1 - ip_2'y_2 - ip_3'y_3} \; \bar{\chi}(y_1,y_2,y_3)^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \qquad (2.18)$$

Again using eqn. (2.13) in eqn. (2.18) we obtain

$$\bar{\chi}(p_1',p_2',p_3')^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} = \frac{(2\pi)^4}{(2\pi)^{3/2}} \delta^4(P-\mathcal{P}') \; \bar{\chi}[p_\xi',p_\eta']^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \qquad (2.19)$$

where

$$\bar{\chi}[p_\xi',p_\eta']^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF}$$

$$= \int d^4\xi' d^4\eta' \; e^{-ip_\xi'\xi' - ip_\eta'\eta'} \; \bar{\chi}[\xi',\eta']^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \qquad (2.20)$$

Once again $(\mathcal{P}',p_\xi',p_\eta')$ are related to (p_1',p_2',p_3') through (f_1',f_2',f_3') in exactly the same fashion as $(\mathcal{P},p_\xi,p_\eta)$ are related to (p_1,p_2,p_3) through (f_1,f_2,f_3).

2.3 Relationship between the BS amplitude and the 6-quark Green function

The BS amplitude χ and its conjugate $\bar{\chi}$ are intimately related to the 6-quark Green function defined by

$$G(x_1,x_2,x_3;y_1,y_2,y_3)_{\alpha aA,\beta bB,\gamma cC;\delta dD,\kappa eE,\sigma fF}$$

$$\equiv -\langle 0|T\; \psi_{\alpha aA}(x_1)\; \psi_{\beta bB}(x_2)\; \psi_{\gamma cC}(x_3)\; \bar{\psi}_{\delta dD}(y_1)\; \bar{\psi}_{\kappa eE}(y_2)\; \bar{\psi}_{\sigma fF}(y_3)|0\rangle \qquad (2.21)$$

The minus sign in front is a matter of convention [30,37,38,43]. Again, exploiting the spacetime translation invariance of the field operators, we can define the Fourier transform of G as follows:

$$G(x_1,x_2,x_3;y_1,y_2,y_3)_{\alpha aA,\beta bB,\gamma cC;\delta dD,\kappa eE,\sigma fF}$$
$$= \frac{1}{(2\pi)^{16}} \int d^4P d^4p_\xi d^4p_\eta d^4p'_\xi d^4p'_\eta \; e^{-iP(X-Y)}$$
$$e^{-i(p_\xi\xi+p_\eta\eta)} \; e^{i(p'_\xi\xi'+p'_\eta\eta')}$$
$$G(P;p_\xi,p_\eta;p'_\xi,p'_\eta)_{\alpha aA,\beta bB,\gamma cC;\delta dD,\kappa eE,\sigma fF} \tag{2.22}$$

It can be rigorously shown [30,37,43,45] that, in momentum space G has a pole at the bound-state energy and the residue at the pole is related to χ and $\bar{\chi}$. The exact relationship is as follows:

$$G(P;p_\xi,p_\eta;p'_\xi,p'_\eta)_{\alpha aA,\beta bB,\gamma cC;\delta dD,\kappa eE,\sigma fF}$$
$$= \sum_{\mathcal{B}:S} \left(\frac{-i}{2\pi}\right) \frac{1}{(2\pi)^3} \frac{1}{2P^0} \frac{1}{P^0-E_p+i\epsilon}$$
$$\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \; \bar{\chi}[p'_\xi,p'_\eta]^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF}$$
$$+ \text{(terms regular at } P^0 = E_p) \tag{2.23}$$

where

$$P^0 = E_p = \sqrt{\vec{P}^2 + M^2} \tag{2.24}$$

This relation will play a pivotal role in our discussions.

2.4 A Consistent Parametrization of the BS Amplitude for $\frac{1}{2}^+$ Baryons .

The BS amplitude χ satisfies the **Bethe-Salpeter equation**

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = -\int d^4x'_1 d^4x'_2 d^4x'_3 d^4y'_1 d^4y'_2 d^4y'_3$$

$$\{[iS'(x_1-x_1')]_{\alpha\tilde{\alpha}}\,\delta_{a\tilde{a}}\,\delta_{A\tilde{A}}[iS'(x_2-x_2')]_{\beta\tilde{\beta}}\,\delta_{b\tilde{b}}\,\delta_{B\tilde{B}}$$

$$[iS'(x_3-x_3')]_{\gamma\tilde{\gamma}}\,\delta_{c\tilde{c}}\,\delta_{C\tilde{C}} + \text{permutation over } x_1,x_2,x_3 \text{ legs}\}$$

$$V(x_1',x_2',x_3';y_1',y_2',y_3')_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\delta dD,\kappa eE,\sigma fF}$$

$$\chi(y_1',y_2',y_3')^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF} \tag{2.25}$$

The function V is called the **irreducible kernel** [37,40,41,43] for the 6-quark Green function G. It is related to G as follows

$$G(x_1,x_2,x_3;y_1,y_2,y_3)_{\alpha aA,\beta bB,\gamma cC;\delta dD,\kappa eE,\sigma fF}$$

$$= \{[iS'(x_1-y_1)]_{\alpha\delta}\,\delta_{ad}\,\delta_{AD}[iS'(x_2-y_2)]_{\beta\kappa}\,\delta_{be}\,\delta_{BE}[iS'(x_3-y_3)]_{\gamma\sigma}\,\delta_{cf}\,\delta_{CF}$$

$$+ \text{permutation over } x_1,x_2,x_3 \text{ legs}\}$$

$$-\int d^4x_1'd^4x_2'd^4x_3'd^4y_1'd^4y_2'd^4y_3'$$

$$\{[iS'(x_1-x_1')]_{\alpha\tilde{\alpha}}\,\delta_{a\tilde{a}}\,\delta_{A\tilde{A}}[iS'(x_2-x_2')]_{\beta\tilde{\beta}}\,\delta_{b\tilde{b}}\,\delta_{B\tilde{B}}[iS'(x_3-x_3')]_{\gamma\tilde{\gamma}}\,\delta_{c\tilde{c}}\,\delta_{C\tilde{C}}$$

$$+ \text{permutation over } x_1,x_2,x_3 \text{ legs}\}$$

$$V(x_1',x_2',x_3';y_1',y_2',y_3')_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}}$$

$$\mathcal{G}(y_1',y_2',y_3';y_1,y_2,y_3)_{\widetilde{\delta dD},\widetilde{\kappa eF},\widetilde{\sigma fF};\delta dD,\kappa eE,\sigma fF} \qquad (2.26)$$

Graphically, the relationship between V and G has been shown in Figure. 2.1

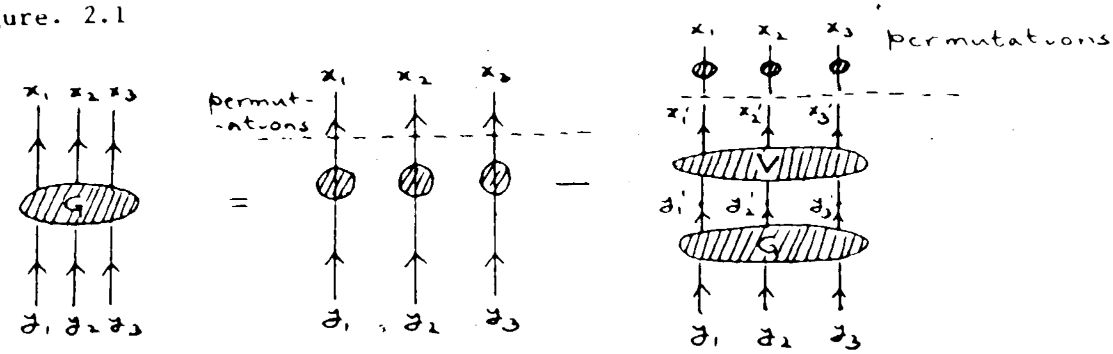


Figure 2.1 Graphical depiction of the irreducible kernel V.

It is obvious that we need to solve the BS equation to get an explicit form for χ. There are two major difficulties in doing this. First, we need the explicit form of V in order to solve eqn. (2.25). This amounts to knowing the long-distance strong interaction dynamics responsible for binding of quarks in the baryon. This information is not available at the present time. Secondly, even if V were known explicitly the solution of eqn. (2.25) is fraught with formidable difficulties. It is almost impossible to obtain a convenient closed form solution [30,35,37,40,41,46,47]. There have been attempts in the literature to solve eqn. (2.25) with suitable kernels under various approximations [30,40-42,47,49-51,53,54]. In our calculations, we adopt the more pragmatic approach of Tomozawa and Esteve et al. [38,43]. We **parametrize** χ straightaway in such a manner that it has the correct spin and internal quantum numbers and has a reasonable momentum dependence which is amenable to analytic computation of hadronic S-matrix elements. This parametrization of χ can be thought of as **a low-energy structure factor** for the baryon whose credibility is to be established phenomenologically. Following Tomozawa [43 we **assume** that the BS amplitude for $\frac{1}{2}^+$ baryons is given by

$$\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= N\,\frac{(2\pi)^4}{(2\pi)^{3/2}}\,\delta^4(P-\mathcal{P})[(D_\rho)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\rho}_{abc}$$

$$+\ (D_\lambda)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\lambda}_{abc}]\quad \chi[p_\xi,p_\eta]^P \tag{2.27}$$

where

$$(D_\lambda)^{P,S}_{\alpha\beta\gamma} = \left(\frac{-1}{\sqrt{2}}\right)\left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\alpha\beta} u^S_\gamma(P) \tag{2.28}$$

$$(D_\rho)^{P,S}_{\alpha\beta\gamma} = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right)\left\{\left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\beta\gamma} u^S_\alpha(P) - \left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\gamma\alpha} u^S_\beta(P)\right\} \tag{2.29}$$

$\mathcal{F}^{\mathcal{B}:\rho}_{abc}$ and $\mathcal{F}^{\mathcal{B}:\lambda}_{abc}$ are the ρ and λ type baryon octet tensors [26,27,30,42,55]. In order to avoid confusion between different conventions in the literature, their components for various octet baryons are listed in Appendix A. ε_{ABC} is the colour singlet tensor. $\chi[p_\xi,p_\eta]^P$ on the right-hand side in eqn. (2.27) does not have indices anymore. It is a Lorentz invariant function of p_ξ, p_η and P. N is the normalization constant. It can be shown that the BS amplitude in eqn. (2.27) satisfies all the invariance properties required for the baryon octet [43-56] . For the momentum-dependent part, we assume

$$\chi[p_\xi,p_\eta]^P = \exp\{-\frac{1}{2\alpha^2}[8(\frac{P\cdot p_\xi}{M})^2 - 4p_\xi^2 + 6(\frac{P\cdot p_\eta}{M})^2 - 3p_\eta^2]\} . \qquad (2.30)$$

It can be shown that this expression for $\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$ satisfies the required antisymmetry properties under the interchange of any two quarks demanded by the definition of the BS amplitude. It must be reiterated that eqn. (2.27) is **an ansatz** for the BS amplitude. Its utility will be established phenomenologically.

Next question that arises is: given $\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$ by eqn. (2.27), how can one determine $\bar{\chi}(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$? This innocent sounding question is, in fact, a highly nontrivial one because of the presence of T-product in definitions of χ and $\bar{\chi}$ [36,40,43,45,50,57]. For two-fermion bound states, an integral relation between χ and $\bar{\chi}$ was derived by Lurie, McFarlane and Takahashi [45] using microscopic causality. As shown in Appendix B, this can be generalized to 3-fermion bound states using canonical commutation relations. The integral relation is

$$\int dp_\xi^\circ dp_\eta^\circ\, \bar{\chi}[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\tilde{\alpha}aA,\tilde{\beta}bB,\tilde{\gamma}cC} = -\int dp_\xi^\circ dp_\eta^\circ\, \chi[p_\xi,p_\eta]^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC}\, \gamma^\circ_{\alpha\tilde{\alpha}}\, \gamma^\circ_{\beta\tilde{\beta}}\, \gamma^\circ_{\gamma\tilde{\gamma}} . \qquad (2.31)$$

Eqn. (2.31) is a rigorous equality. **A consistent** way in which eqn. (2.31) can be satisfied is obviously [43]

$$\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\tilde{\alpha}aA,\tilde{\beta}bB,\tilde{\gamma}cC} = -\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC}\,\gamma^\circ_{\alpha\tilde{\alpha}}\,\gamma^\circ_{\beta\tilde{\beta}}\,\gamma^\circ_{\gamma\tilde{\gamma}} \tag{2.32}$$

In configuration space, this relation reads as follows

$$\bar{\chi}[\xi,\eta]^{\mathcal{B}:P,S}_{\tilde{\alpha}aA,\tilde{\beta}bB,\tilde{\gamma}cC} = -\chi[\xi,\eta]^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC}\,\gamma^\circ_{\alpha\tilde{\alpha}}\,\gamma^\circ_{\beta\tilde{\beta}}\,\gamma^\circ_{\gamma\tilde{\gamma}} \tag{2.33}$$

We do not propose to go into the uniqueness of this choice. This is **a consistent** choice and we will use this relation to obtain $\bar{\chi}$ from χ.

For the derivation of the Nishijima-Mandelstam formula, we need another integral relation. From eqns. (2.27) and (2.16), it is obvious that the relative BS amplitude is

$$\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = N[(D_\rho)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\rho}_{abc} + (D_\lambda)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\lambda}_{abc}]$$

$$\chi[p_\xi,p_\eta]^{P} \tag{2.34}$$

From eqns (2.34) and (2.17) we obtain

$$\chi[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = N[(D_\rho)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\rho}_{abc} + (D_\lambda)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\lambda}_{abc}]$$

$$\chi[\xi,\eta]^{P} \tag{2.35}$$

It is not important for us to know the functional form of $\chi[\xi,\eta]^P$. It is sufficient to know that it is a function of ξ, η and P. From eqns. (2.35) and (2.33), we obtain

$$\bar{\chi}[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = N^*[(\bar{D}_\rho)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\rho}_{abc} + (\bar{D}_\lambda)^{P,S}_{\alpha\beta\gamma}\,\frac{1}{\sqrt{6}}\,\varepsilon_{ABC}\,\mathcal{F}^{\mathcal{B}:\lambda}_{abc}]$$

$$\chi[\xi,\eta]^{P*} \tag{2.36}$$

where

$$(\bar{D}_\rho)^{P,S}_{\alpha\beta\gamma} = \left(\frac{1}{\sqrt{2}}\right)\left[\gamma^\circ\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\gamma^\circ\right]^*_{\alpha\beta}\bar{u}^S_\gamma(P) \tag{2.37}$$

$$(\bar{D}_\lambda)^{P,S}_{\alpha\beta\gamma} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right)\Big\{\left[\gamma^\circ\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\gamma^\circ\right]^*_{\beta\gamma}\bar{u}^S_\alpha(P) - \left[\gamma^\circ\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\gamma^\circ\right]^*_{\gamma\alpha}\bar{u}^S_\beta(P)\Big\} \tag{2.38}$$

From eqns. (2.35) and (2.36), it is not very difficult to prove the following integral relation

$$\int d^4\xi d^4\eta\ \bar{\chi}[\xi,\eta]^{\mathcal{B}':P',S'}_{\alpha aA,\beta bB,\gamma cC}\ \chi[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = C(P)\delta_{\mathcal{B}'\mathcal{B}}\ \delta_{S'S} \tag{2.39}$$

where $\mathcal{B}$ and $\mathcal{B}'$ are two different octet baryons. $\delta_{\mathcal{B}'\mathcal{B}}$ implies a Kronecker delta over all internal quantum numbers of the baryons. The detailed form of $C(P)$ is irrelevant. Eqn. (2.39) is obviously an orthogonality relation. This **must not** be confused with the normalization condition for χ and $\bar{\chi}$.

2.5. Question of Normalization of the Parametrized BS Amplitude

The BS amplitude satisfies the normalization condition [37,40-43]

$$\frac{-i}{(2\pi)^4}\int d^4p'_\xi d^4p'_\eta d^4p_\xi d^4p_\eta\ \bar{\chi}[p'_\xi,p'_\eta]^{\mathcal{B}:P,S}_{\delta dD,\kappa eE,\sigma fF}$$

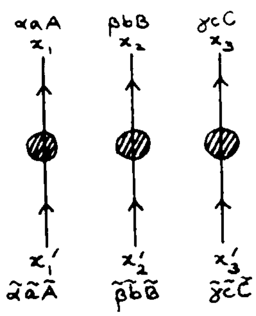
$$\Big\{\frac{\partial}{\partial P^0}\big[I(P;p'_\xi,p'_\eta;p_\xi,p_\eta)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$$

$$+ V(P;p'_\xi,p'_\eta;p_\xi,p_\eta)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}\big]\Big\}$$

$$\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = 2P^\circ \tag{2.40a}$$

where $I(P;p'_\xi,p'_\eta;p_\xi,p_\eta)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$ is the Fourier transform of $I(y_1,y_2,y_3;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$ defined in the same way as shown in eqn. (2.22). And, $I(y_1,y_2,y_3;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$ is given by the following relation

$$\int d^4x_1 d^4x_2 d^4x_3 I(y_1,y_2,y_3;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$$



$+$ permutation over x_1,x_2,x_3

$$= \delta^4(y_1-x'_1)\delta_{\delta\tilde{\alpha}}\delta_{d\tilde{a}}\delta_{D\tilde{A}}\delta^4(y_2-x'_2)\delta_{\kappa\tilde{\beta}}\delta_{e\tilde{b}}\delta_{E\tilde{B}}\delta^4(y_3-x'_3)\delta_{\sigma\tilde{\gamma}}\delta_{f\tilde{c}}\delta_{F\tilde{C}} \tag{2.40b}$$

$V(P;p'_\xi,p'_\eta;p_\xi,p_\eta)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$ is, once again, the Fourier transform of $V(y_1,y_2,y_3;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$ defined in the same manner as in eqn. (2.22).

It is obvious that we need to know V, i.e. the long-range strong interaction dynamics, even to normalize the BS amplitude. As mentioned earlier, this information is not available at the present time. Does it mean that our plan to work in the BS formalism is a mistake to start with? Not quite! We shall see that the normalization constant N affects only the overall normalization of our S-matrix element. So, quantities like the asymmetry parameter, which are independent of the overall normalization, can still be reliably calculated.

In order to obtain the decay rate, it is true, we need the normalization constant. The scheme of using a parametrized BS amplitude for calculating decay amplitudes is therefore, incapable of predicting the decay rate for a specific process.

Right now, we only want to emphasize that we plan to use such a BS model to calculate hyperon radiative decays. We realize that it is quite formidable to solve the BS equation exactly. However, as we shall see, one needs the BS amplitude to do the perturbation theory for bound quarks correctly. We are trying to find out a pragmatic solution to this impasse [38,43]. Our goal is not to solve the bound-state problem for baryons, but to establish a form of the BS amplitude, with a normalization prescription for it, which is theoretically consistent and phenomenologically reliable.

2.6. The Nishijima-Mandelstam Formula for Hyperon Radiative Decays

Nishijima and Mandelstam had developed a method of formally writing down the S-matrix elements for composite particle reactions in terms of the BS amplitudes for bound states and suitably amputated Green functions. Though their method is outlined at quite a few places in the literature [35-38,58], we again derive such a formula for hyperon radiative decays. The reason is that our method is very close to the standard LSZ reduction formalism [37,39,40,57] and, to our knowledge, this parallel has not been brought to attention in the literature so far. Let us now proceed with the derivation. We are looking for the S-matrix element

$$S_{fi} = \langle \mathcal{B}'(P',S'),\gamma(k',\lambda')\text{out}|\mathcal{B}(P,S)\text{in}\rangle \tag{2.42}$$

for the reaction $\mathcal{B} \to \mathcal{B}'\gamma$. (P,S) are the momentum and spin of the initial baryon; (P',S') are the momentum and spin of the final baryon and (k',λ') are the momentum and the helicity of the photon.

Step 1. Reduction of the Final Photon

Photon is an elementary field. Following the standard LSZ reduction procedure [37,39,40,57], we can express eqn. (2.42) as follows

$$S_{fi} = \left(\frac{-i}{\sqrt{Z_3}}\right) \int d^4y_4\, f^*_{k',\lambda'}(y_4)_{\mu'}\, \overrightarrow{\Box}_{y_4} \langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle \tag{2.43}$$

where

$$f_{k',\lambda'}(y_4)_{\mu'} = \frac{1}{\sqrt{(2\pi)^3 2k'_o}}\, e^{-ik'\cdot y_4}\, \epsilon_{\mu'}(k',\lambda') \tag{2.44}$$

Z_3 is the renormalization constant for photons and $\epsilon_{\mu'}(k',\lambda')$ is the photon polarization. $A^{\mu'}(y_4)$ is the Heisenberg field for photons.

Step 2. Reduction of the Initial $\frac{1}{2}^+$ Baryon

Since the initial $\frac{1}{2}^+$ baryon is an observed particle, we must have a corresponding asymptotic field to create (or annihilate) it. It will satisfy the free Dirac equation. Following the notations of Bjorken and Drell [39], we denote it by

$$\psi^{\mathcal{B}}_{\substack{in\\out}}(X) = \int d^3P \sum_{\pm S} \left[b^{\mathcal{B}}_{\substack{in\\out}}(P,S)\, U_{PS}(X) + d^{\mathcal{B}\dagger}_{\substack{in\\out}}(P,S)\, V_{PS}(X) \right] \tag{2.45}$$

where X is the position of the initial baryon. Obviously, this corresponds to its centre of mass. P, S are its momentum and spin. Also [39]

$$U_{ps}(x) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}}\, u(p,s)\, e^{-ipx} \tag{2.46}$$

$$V_{ps}(x) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}}\, v(p,s)\, e^{ipx} \tag{2.47}$$

for a generic Dirac particle of mass 'm' with momentum 'p' and spin 's'. u(p,s) and v(p,s) are the Dirac spinors in Bjorken and Drell notation [39]. $b_{\substack{in\\out}}$, $b^{\dagger}_{\substack{in\\out}}$, $d_{\substack{in\\out}}$ and $d^{\dagger}_{\substack{in\\out}}$ can be expressed in terms of $\psi_{\substack{in\\out}}$ [39]. We will need only two of them, viz.

$$b_{\substack{in\\out}}(p,s) = \int d^3x\, U^{\dagger}_{ps}(x)\, \psi_{\substack{in\\out}}(x) \tag{2.48}$$

$$b^{\dagger}_{\substack{in\\out}}(p,s) = \int d^3x\, \psi^{\dagger}_{\substack{in\\out}}(x)\, U_{ps}(x) \tag{2.49}$$

Let us consider now

$$\langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle$$
$$= \langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)b_{in}^{\mathcal{B}\dagger}(P,S)|0\rangle$$
$$= \langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)b_{in}^{\mathcal{B}\dagger}(P,S) - b_{out}^{\mathcal{B}\dagger}(P,S)A^{\mu'}(y_4)|0\rangle \qquad (2.50)$$

The last step in eqn. (2.50) is not surprising because $b^{\dagger}_{out}(P,S)$ acting on the final bra state gives zero. Using eqn. (2.49) in eqn. (2.50), we obtain

$$\langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle$$
$$= \langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)\int d^3X\, \psi_{in}^{\mathcal{B}\dagger}(X)\, U_{PS}(X)$$
$$- \int d^3X\, \psi_{out}^{\mathcal{B}\dagger}(X)\, U_{PS}(X)\, A^{\mu'}(y_4)|0\rangle \qquad (2.51)$$

It can be shown that

$$\partial_{X^\circ} \int d^3X\, \psi^{\mathcal{B}\dagger}_{\substack{in\\out}}(X)\, U_{PS}(X) = 0 \qquad (2.52)$$

Hence, we can take the limit $X^\circ \to \mp\infty$ in eqn. (2.51).

$$\langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle$$
$$= \lim_{X^\circ\to-\infty} \langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)\int d^3X\, \psi_{in}^{\mathcal{B}\dagger}(X)\, U_{PS}(X)|0\rangle$$
$$- \lim_{X^\circ\to+\infty} \langle \mathcal{B}'(P',S')\text{out}|\int d^3X\, \psi_{out}^{\mathcal{B}\dagger}(X)\, U_{PS}(X) A^{\mu'}(y_4)|0\rangle \qquad (2.53)$$

We have to now define a suitable interpolating field for the baryon in order to proceed further. There is a considerable freedom in this choice [37]. We will invent one suitable interpolating field by invoking the BS amplitude for the bound state $\mathcal{B}$. This is the heart of Nishijima-Mandelstam technique. Consider the field

$$\tilde{\psi}^{\mathcal{B}}(X) = \frac{1}{C(P)} \sqrt{\frac{M}{E_P}} \int d^4\xi\, d^4\eta \,[\Gamma\, \psi_{\alpha aA}(x_1)\, \psi_{\beta bB}(x_2)\, \psi_{\gamma cC}(x_3) \sum_{\pm S''} u^\dagger(P,S'')\chi[\xi,\eta]^{\mathcal{B}:P,S''}_{\alpha aA,\beta bB,\gamma cC} \qquad (2.54)$$

where $C(P)$ is the function defined in eqn. (2.39). The rest of the notation should be obvious by now. Using eqns. (2.54), (2.12), (2.13), (2.39) and orthogonality for Dirac spinors [39], it is not difficult to show that

$$\langle \mathcal{B}(P,S)|\tilde{\psi}^{\mathcal{B}}(X)|0\rangle = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{M}{E_P}}\, u^\dagger(P,S)\, e^{iP\cdot X} \qquad (2.55)$$

Since $\tilde{\psi}^{\mathcal{B}}(X)$ satisfies the appropriate normalization for the asymptotic field, it is a consistent choice for the interpolating field [37]. Therefore, we can use the asymptotic limit

$$\tilde{\psi}^{\mathcal{B}}(X) \xrightarrow[X^\circ \to \mp\infty]{} \psi^{\mathcal{B}\,\dagger}_{\substack{in\\out}}(X) \qquad (2.56)$$

in the sense of a weak operator relation. Using this asymptotic condition in eqn. (2.53), we obtain

$$\begin{aligned}
&\langle \mathcal{B}'(P',S')out|A^{\mu'}(y_4)|\mathcal{B}(P,S)in\rangle \\
&= -\Big(\lim_{X^\circ\to+\infty} - \lim_{X^\circ\to-\infty}\Big) \int d^3X\, d^4\xi\, d^4\eta \\
&\quad \langle \mathcal{B}'(P',S')out|T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)|0\rangle \\
&\quad \frac{1}{C(P)} \sqrt{\frac{M}{E_P}} \sum_{\pm S''} u^\dagger(P,S)\, U_{PS}(X)\, \chi[\xi,\eta]^{\mathcal{B}:P,S''}_{\alpha aA,\beta bB,\gamma cC}
\end{aligned} \qquad (2.57)$$

Using eqn. (2.46) and the orthogonality relation for Dirac spinors [39] in eqn. (2.57), we obtain

$$\langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle$$

$$= -\left(\lim_{X^\circ\to+\infty} - \lim_{X^\circ\to-\infty}\right)\int d^3X\, d^4\xi\, d^4\eta$$

$$\langle \mathcal{B}'(P',S')\text{out}|T\, A^{\mu'}(y_4)\, \bar\psi_{\alpha aA}(x_1)\, \bar\psi_{\beta bB}(x_2)\, \bar\psi_{\gamma cC}(x_3)|0\rangle$$

$$\frac{1}{(2\pi)^{3/2}}\;\frac{1}{C(P)}\;\chi[\xi,\eta]^{\mathcal{B}\,:P,S}_{\alpha aA,\beta bB,\gamma cC}\; e^{-iP\cdot X} \qquad (2.58)$$

We can convert the infinite limits in eqn. (2.58) into an integral in the following manner

$$\langle \mathcal{B}'(P',S')\text{out}|A^{\mu'}(y_4)|\mathcal{B}(P,S)\text{in}\rangle$$

$$= -\int_{-\infty}^{\infty} dX^\circ \frac{\partial}{\partial X^\circ}\left\{\int d^3X\, d^4\xi\, d^4\eta\right.$$

$$\langle \mathcal{B}'(P',S')\text{out}|TA^{\mu'}(y_4)\, \bar\psi_{\alpha aA}(x_1)\, \bar\psi_{\beta bB}(x_2)\, \bar\psi_{\gamma cC}(x_3)|0\rangle$$

$$\left.\frac{1}{(2\pi)^{3/2}}\;\frac{1}{C(P)}\;\chi[\xi,\eta]^{\mathcal{B}\,:P,S}_{\alpha aA,\beta bB,\gamma cC}\; e^{-iP\cdot X}\right\}$$

$$= -\int d^4X d^4\xi d^4\eta\, \frac{\partial}{\partial X^\circ}\left\{\langle \mathcal{B}'(P',S')\text{out}|T\, A^{\mu'}(y_4)\, \bar\psi_{\alpha aA}(x_1)\, \bar\psi_{\beta bB}(x_2)\right.$$

$$\bar\psi_{\gamma cC}(x_3)|0\rangle$$

$$\left.\frac{1}{(2\pi)^{3/2}}\;\frac{1}{C(P)}\;\chi[\xi,\eta]^{\mathcal{B}\,:P,S}_{\alpha aA,\beta bB,\gamma cC}\; e^{-iP\cdot X}\right\}$$

$$= -\int d^4Xd^4\xi d^4\eta\left\{\langle \mathcal{B}'(P',S')\text{out}|T\, A^{\mu'}(y_4)\, \bar\psi_{\alpha aA}(x_1)\, \bar\psi_{\beta bB}(x_2)\bar\psi_{\gamma cC}(x_3)|0\rangle\right.$$

$$\left.\left[\overleftarrow{\frac{\partial}{\partial X^\circ} - iE_P}\right]\frac{1}{(2\pi)^{3/2}}\;\frac{1}{C(P)}\;\chi[\xi,\eta]^{\mathcal{B}\,:P,S}_{\alpha aA,\beta bB,\gamma cC}\; e^{-iP\cdot X}\right\} \qquad (2.59)$$

Putting eqn. (2.59) in eqn. (2.43), our reduction formula looks like

$$S_{fi} = \left(\frac{-i}{\sqrt{Z_3}}\right)(-1) \int d^4y_4 d^4X d^4\xi d^4\eta \; f^{*}_{k',\lambda'}(y_4)_{\mu'} \overleftrightarrow{\partial}_{y_4}$$

$$\langle \mathcal{B}'(P',S')\text{out}|T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)|0\rangle$$

$$\overleftarrow{\left[\frac{\partial}{\partial X^{o}} - iE_p\right]} \frac{1}{(2\pi)^{3/2}} \frac{1}{C(P)} \chi[\xi,\eta]^{\mathcal{B}\,:\,P,S}_{\alpha aA,\beta bB,\gamma cC}\; e^{-iP\cdot X} \qquad (2.60)$$

Step 3. Reduction of the Final $\frac{1}{2}^{+}$ Baryon

We now proceed with the reduction of the final $\frac{1}{2}^{+}$ baryon. We must have an asymptotic Dirac field to create (or annihilate) it. Following the notations of Bjorken and Drell [39], we denote it by

$$\psi^{\mathcal{B}'}_{\substack{in\\out}}(Y) = \int d^3P' \sum_{\pm S'} \left[b^{\mathcal{B}'}_{\substack{in\\out}}(P',S')\, U_{P'S'}(Y) + d^{\mathcal{B}'\dagger}_{\substack{in\\out}}(P',S')V_{P'S'}(Y)\right] \qquad (2.61)$$

where P' is the final baryon momentum; S' is its spin and Y' is its position. Consider now

$$\begin{aligned}
&\langle \mathcal{B}'(P',S')\text{out}|T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)|0\rangle \\
&= \langle 0|b^{\mathcal{B}'}_{out}(P',S')T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)|0\rangle \\
&= \langle 0|b^{\mathcal{B}'}_{out}(P',S')[T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)] \\
&- (-1)^{3\times 3}\, [T\, A^{\mu'}(y_4)\, \bar{\psi}_{\alpha aA}(x_1)\, \bar{\psi}_{\beta bB}(x_2)\, \bar{\psi}_{\gamma cC}(x_3)]\, b^{\mathcal{B}'}_{in}(P'S')|0\rangle \qquad (2.62)
\end{aligned}$$

Using eqn. (2.48) in eqn. (2.62), we obtain

$$\langle \mathcal{B}'(P',S')\text{out}|T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)|0\rangle$$
$$= \langle 0|\int d^3Y\ U^\dagger_{P'S'}(Y)\ \psi^{\mathcal{B}'}_{out}(Y)[T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)]$$
$$- (-1)^{3\times 3}[T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)]\int d^3Y\ U^\dagger_{P'S'}(Y)\psi^{\mathcal{B}'}_{out}(Y)|0\rangle \tag{2.63}$$

Again, since for the Dirac field, in general,

$$\frac{\partial}{\partial Y^\circ}\int d^3Y\ U^\dagger_{P'S'}(Y)\ \psi^{\mathcal{B}'}_{\substack{in\\out}}(Y) = 0 \tag{2.64}$$

we can take the asymptotic limit in eqn. (2.63).

$$\langle \mathcal{B}'(P',S')\text{out}|T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)|0\rangle$$
$$= \lim_{Y^\circ\to+\infty} \langle 0|\int d^3Y\ U^\dagger_{P'S'}(Y)\psi^{\mathcal{B}'}_{out}(Y)$$
$$[T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)]|0\rangle$$
$$- \lim_{Y^\circ\to-\infty} \langle 0|(-1)^{3\times 3}[T\ A^{\mu'}(y_4)\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)]$$
$$\int d^3Y\ U^\dagger_{P'S'}(Y)\ \psi^{\mathcal{B}'}_{in}(Y)|0\rangle \tag{2.65}$$

We need to define, once again, a suitable interpolating field now.

Consider the construct

$$\tilde{\psi}^{\mathcal{B}'}(Y) = \frac{1}{C(P')}\sqrt{\frac{M'}{E_{P'}}}\int d^4\xi' d^4\eta' \sum_{\pm S''} u(P',S'')\ \bar{\chi}\ [\xi',\eta']^{\mathcal{B}':P',S''}_{\delta dD,\kappa eE,\sigma fF}$$
$$[T\ \psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)] \tag{2.66}$$

where $C(P')$ is the function defined in eqn. (2.39). The notation in eqn. (2.66) should, otherwise, be clear by now. Again, with the help of eqns. (2.66), (2.8), (2.10), (2.39) and orthogonality for Dirac spinors, we can show

$$\langle 0|\tilde{\psi}^{\mathcal{B}'}(Y)|\mathcal{B}'(P',S')\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{M'}{E_{P'}}}\; u(P',S')\; e^{-iP'\cdot Y} \qquad (2.67)$$

Since it satisfies the appropriate normalization for the asymptotic field, it is **a good choice** for the corresponding interpolating field [37]. Therefore,

$$\tilde{\psi}^{\mathcal{B}'}(Y) \xrightarrow[Y^{\circ}\to\mp\infty]{} \psi^{\mathcal{B}'}_{\substack{in\\out}}(Y) \qquad (2.68)$$

in the sense of a weak operator relation. Now, using eqns. (2.68) and (2.66) in eqn. (2.65) and following steps similar to those outlined in step 2, we obtain

$$\langle\mathcal{B}'(P',S')out|T\, A^{\mu'}(y_4)\,\bar{\psi}_{\alpha aA}(x_1)\,\bar{\psi}_{\beta bB}(x_2)\,\bar{\psi}_{\gamma cC}(x_3)|0\rangle$$
$$= \int d^4Yd^4\xi' d^4\eta' \frac{1}{(2\pi)^{3/2}}\frac{1}{C(P')}\, e^{iP'Y}\,\bar{\chi}\,[\xi',\eta']^{\mathcal{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF}$$
$$[iE_{P'} + \overrightarrow{\frac{\partial}{\partial Y^{\circ}}}]\langle 0|T\,\psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)A^{\mu'}(y_4)$$
$$\bar{\psi}_{\alpha aA}(x_1)\,\bar{\psi}_{\beta bB}(x_2)\,\bar{\psi}_{\gamma cC}(x_3)|0\rangle \qquad (2.69)$$

Putting eqn. (2.69) in eqn. (2.60), we obtain the final reduction formula

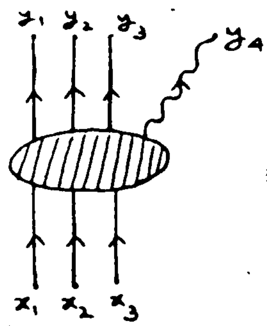
$$S_{fi} = \left(\frac{-i}{\sqrt{Z_3}}\right)(-1)(+1)\int d^4Yd^4\xi' d^4\eta' d^4y_4 d^4X d^4\xi d^4\eta\; f^{*}_{k',\lambda'}(y_4)_{\mu'}$$
$$\frac{1}{(2\pi)^{3/2}}\frac{1}{C(P')}\, e^{iP'Y}\,\bar{\chi}\,[\xi',\eta']^{\mathcal{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF}\;\overrightarrow{\square}_{y_4}\,[iE_{P'} + \overrightarrow{\frac{\partial}{\partial Y^{\circ}}}]$$
$$\langle 0|T\,\psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)\,A^{\mu'}(y_4)\,\bar{\psi}_{\alpha aA}(x_1)\,\bar{\psi}_{\beta bB}(x_2)\,\bar{\psi}_{\gamma cC}(x_3)|0\rangle$$
$$[\overleftarrow{\frac{\partial}{\partial X^{\circ}}} - iE_P]\,\frac{1}{(2\pi)^{3/2}}\frac{1}{C(P)}\,\chi\,[\xi,\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}\, e^{-iP\cdot X} \qquad (2.70)$$

As in any S-matrix calculation, the next step is to amputate the

Green function in eqn. (2.70) to get rid of the differential operators. Since the initial and final baryons are not elementary but composite particles, the amputation procedure is slightly different in the present case [35-38,58]. This is the next trick in the Nishijima-Mandelstam method.

Step 4. Amputating the 7-point Function

Let us amputate the 7-point function in eqn. (2.70) in the following manner.



$$\equiv G(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta dD,\kappa eE,\sigma fF;\alpha aA,\beta bB,\gamma cC}$$

$$\equiv \langle 0|T\, \psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)\, A^{\mu'}(y_4)\, \bar\psi_{\alpha aA}(x_1)\, \bar\psi_{\beta bB}(x_2)\, \bar\psi_{\gamma cC}(x_3)|0\rangle$$

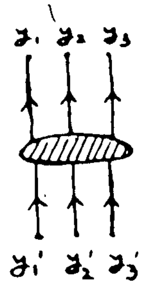
$$= \int d^4y_1' d^4y_2' d^4y_3' d^4y_4' d^4x_1' d^4x_2' d^4x_3'$$

$$G(y_1,y_2,y_3;y_1',y_2',y_3')_{\delta dD,\kappa eE,\sigma fF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}} \quad G(y_4,y_4')^{\mu'\tilde{\mu}'}$$

$$\sigma(y_1',y_2',y_3';y_4';x_1',x_2',x_3')_{\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF};\tilde{\mu}';\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC}}$$

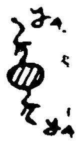
$$G(x_1',x_2',x_3';x_1,x_2,x_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha aA,\beta bB,\gamma cC} \tag{2.71}$$

where

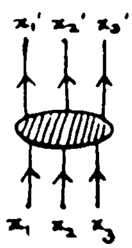


$$\equiv G(y_1, y_2, y_3; y_1', y_2', y_3')_{\delta dD, \kappa eE, \sigma fF; \widetilde{\delta dD}, \widetilde{\kappa eE}, \widetilde{\sigma fF}}$$

$$\equiv \langle 0|T\ \psi_{\delta dD}(y_1)\psi_{\kappa eE}(y_2)\psi_{\sigma fF}(y_3)\ \bar{\psi}_{\widetilde{\delta dD}}(y_1')\ \bar{\psi}_{\widetilde{\kappa eE}}(y_2')\ \bar{\psi}_{\widetilde{\sigma fF}}(y_3')|0\rangle \quad (2.72)$$



$$\equiv G(y_4, y_4')^{\mu'\tilde{\mu}'} \equiv \langle 0|T\ A^{\mu'}(y_4)A^{\tilde{\mu}'}(y_4')|0\rangle \quad (2.73)$$



$$\equiv G(x_1', x_2', x_3'; x_1, x_2, x_3)_{\widetilde{\alpha aA}, \widetilde{\beta bB}, \widetilde{\gamma cC}; \alpha aA, \beta bB, \gamma cC}$$

$$\equiv -\langle 0|T\psi_{\widetilde{\alpha aA}}(x_1')\psi_{\widetilde{\beta bB}}(x_2')\psi_{\widetilde{\gamma cC}}(x_3')\ \bar{\psi}_{\alpha aA}(x_1)\ \bar{\psi}_{\beta bB}(x_2)\ \bar{\psi}_{\gamma cC}(x_3)|0\rangle \quad (2.74)$$

It is obvious from eqn. (2.71) that one removes from the 7-point function, full 6-quark Green functions in the initial and final states. Amputation for the photon leg is as usual. Whatever is left of the F-point function is σ, **the amputated Green function.** Diagrammatically, this is illustrated in Fig. 2.2.

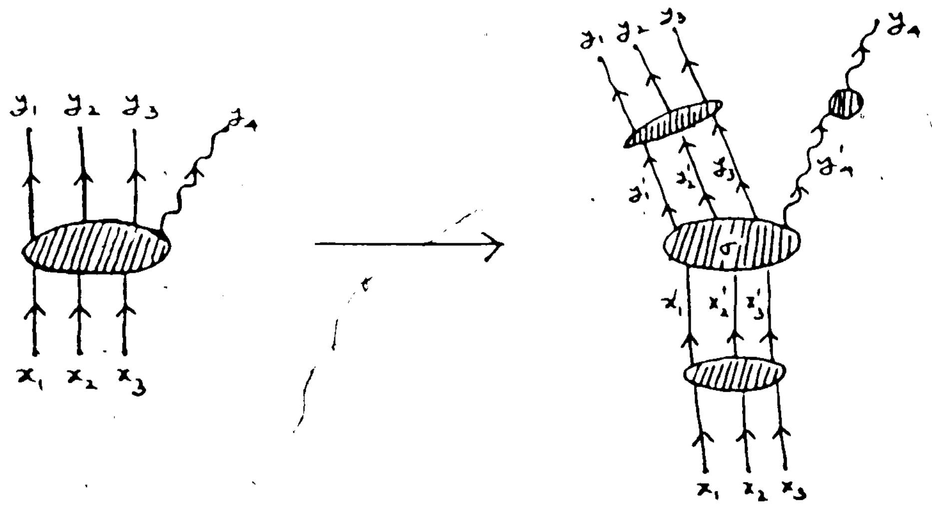


Fig. 2.2 Graphical representation of the amputation procedure.

Putting eqn. (2.71) in eqn. (2.70), we obtain

$$
\begin{aligned}
S_{fi} = {} & \left(\frac{-i}{\sqrt{Z_3}}\right)(-1)(+1) \int d^4y'_1 d^4y'_2 d^4y'_3 d^4y'_4 d^4x'_1 d^4x'_2 d^4x'_3 \\
& d^4Y d^4\xi' d^4\eta' d^4y_4 d^4X d^4\xi d^4\eta \; f^*_{k',\lambda'}(y_4)_{\mu'} \\
& \frac{1}{(2\pi)^{3/2}} \frac{1}{C(P')} e^{iP'Y} \bar{\chi}[\xi',\eta']^{\mathfrak{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF} \{\Box_{y_4} G(y_4,y'_4)^{\mu'\tilde{\mu}'}\} \\
& \{[iE_{P'} + \overrightarrow{\frac{\partial}{\partial Y^\circ}}] \, G(y_1,y_2,y_3;y'_1,y'_2,y'_3)_{\sigma dD,\kappa eE,\sigma fF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}} \\
& \sigma(y'_1,y'_2,y'_3;y'_4;x'_1,x'_2,x'_3)_{\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF};\tilde{\mu}';\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC}} \\
& \{G(x'_1,x'_2,x'_3;x_1,x_2,x_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha aA,\beta bB,\gamma cC} \; [\overleftarrow{\frac{\partial}{\partial X^\circ}} - iE_P]\} \\
& \frac{1}{(2\pi)^{3/2}} \frac{1}{C(P)} \chi[\xi,\eta]^{\mathfrak{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \; e^{-iPX}
\end{aligned}
\tag{2.75}
$$

We know that on-the-shell [59]

$$
G(y_4,y'_4)^{\mu'\tilde{\mu}'} = -g^{\mu'\tilde{\mu}'} Z_3 \, i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(y_4-y'_4)}}{k^2+i\epsilon} \tag{2.76}
$$

Also, from eqns. (2.22), (2.23) and (2.24) we can write

$$G(y_1,y_2,y_3;y_1',y_2',y_3')_{\delta dD,\kappa eE,\sigma fF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}}$$

$$= \frac{1}{(2\pi)^{16}} \int d^4K'\, d^4p_\xi'\, d^4p_\eta'\, d^4p_\xi''\, d^4p_\eta''\; e^{-iK'(Y-Y')}\; e^{-i(p_\xi'\xi' + p_\eta'\eta')}\; e^{i(p_\xi''\xi'' + p_\eta''\eta'')}$$

$$\Big\{ \sum_{\mathcal{B}'':S''} \left(\frac{-i}{2\pi}\right) \frac{1}{(2\pi)^3}\, \frac{1}{2K'^0}\, \frac{1}{K'^0 - E_{K'} + i\epsilon}\; \chi\,[p_\xi', p_\eta']^{\mathcal{B}'':K',S''}_{\delta dD,\kappa eE,\sigma fF}$$

$$\bar{\chi}\,[p_\xi'', p_\eta'']^{\mathcal{B}'':P',S''}_{\delta dD,\kappa eE,\sigma fF} + \left(\begin{matrix}\text{terms regular}\\ \text{at } K'^0 = E_{K'}\end{matrix}\right) \Big\} \tag{2.77}$$

The notation is a little confusing here. (Y',ξ'',η'') are the baryonic coordinates corresponding to (y_1',y_2',y_3') and (Y,ξ',η') are the baryonic coordinates corresponding to (y_1,y_2,y_3).

Similarly,

$$G(x_1',x_2',x_3';x_1,x_2,x_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha aA,\beta bB,\gamma cC}$$

$$= \frac{1}{(2\pi)^{16}} \int d^4K\, d^4p_\xi'''\, d^4p_\eta'''\, d^4p_\xi\, d^4p_\eta\; e^{-iK(X'-X)}\; e^{-i(p_\xi'''\xi''' + p_\eta'''\eta''')}\; e^{i(p_\xi\xi + p_\eta\eta)}$$

$$\Big\{ \sum_{\mathcal{B}''':S'''} \left(\frac{-i}{2\pi}\right) \frac{1}{(2\pi)^3}\, \frac{1}{2K^\circ}\, \frac{1}{K^\circ - E_{K'} + i\epsilon}\; \chi\,[p_\xi''', p_\eta''']^{\mathcal{B}''':K''',S'''}_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC}}$$

$$\bar{\chi}\,[p_\xi, p_\eta]^{\mathcal{B}''':K,S'''}_{\alpha aA,\beta bB,\gamma cC} + \left(\begin{matrix}\text{terms regular}\\ \text{at } K^\circ = E_K\end{matrix}\right) \Big\} \tag{2.78}$$

Once again (X',ξ''',η''') and (X,ξ,η) are the baryonic coordinates corresponding to (x_1',x_2',x_3') and (x_1,x_2,x_3) respectively. We put eqns. (2.76)-(2.78) in eqn. (2.75). Manipulations after this stage are messy but straightforward. We skip the details. It involves the following steps: (i) we let the differential operators in eqn. (2.75) act on their respective Green functions; (ii) we do the y_4,X,Y,K and

K' integrals; (iii) we use the orthogonality relation, viz. eqn. (2.39), for the relative BS amplitudes. After these steps we obtain the much awaited **Nishijima-Mandelstam Formula for hyperon radiative decays in configuration space**

$$S_{fi} = Z_3^{1/2} \frac{1}{2P^\circ} \frac{1}{2P'^\circ} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 d^4x_1 d^4x_2 d^4x_3$$
$$f^*_{k',\lambda'}(y_4)^{\mu'} \bar{\chi}(y_1,y_2,y_3)^{\mathfrak{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF}$$
$$\sigma(y_1,y_2,y_3;y_4;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC}$$
$$\chi(x_1,x_2,x_3)^{\mathfrak{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \qquad (2.79)$$

We shall now rewrite eqn. (2.79) in momentum space because it is much more convenient to do practical calculations in momentum space. We define required Fourier transforms in the following manner.

$$\sigma(y_1,y_2,y_3;y_4;x_1,x_2,x_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC}$$
$$= \int \frac{d^4p_1'}{(2\pi)^4} \frac{d^4p_2'}{(2\pi)^4} \frac{d^4p_3'}{(2\pi)^4} \frac{d^4p_4'}{(2\pi)^4} \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4}$$
$$e^{-ip_1'y_1 - ip_2'y_2 - ip_3'y_3 - ip_4'y_4 + ip_1x_1 + ip_2x_2 + ip_3x_3}$$
$$\sigma(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC} \qquad (2.80)$$

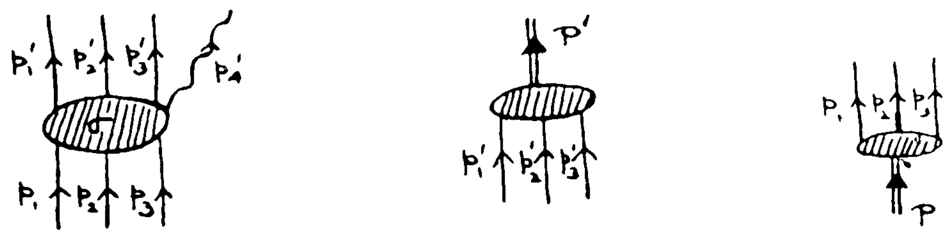
The Fourier transform of $\bar{\chi}$ is defined by

$$\bar{\chi}(y_1,y_2,y_3)^{\mathfrak{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF}$$
$$= \int \frac{d^4p_1'}{(2\pi)^4} \frac{d^4p_2'}{(2\pi)^4} \frac{d^4p_3'}{(2\pi)^4} e^{ip_1'y_1 + ip_2'y_2 + ip_3'y_3} \bar{\chi}(p_1',p_2',p_3')^{\mathfrak{B}':P',S''}_{\delta dD,\kappa eE,\sigma fF} \qquad (2.81)$$

and that of χ is defined by

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\, e^{-ip_1x_1-ip_2x_2-ip_3x_3}\, \chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{2.82}$$

Pictorally, these Fourier transforms are drawn in Fig. 2.3.



a. $\sigma(p_1',p_2',p_3';p_4';p_1,p_2,p_3)$ b. $\bar{\chi}(p_1',p_2',p_3')$ c. $\chi(p_1,p_2,p_3)$

Fig. 2.3 Graphical representations of $\sigma, \bar{\chi}$ & χ in momentum space. Indices have been suppressed.

We have adopted the convention that all momenta coming to a vertex are positive whereas all those leaving a vertex are negative. In this sense, BS amplitudes can also be looked at as quark-quark-baryon vertices.

Putting eqns. (2.80)-(2.82) in eqn. (2.79) and carrying out the spacetime integrations, we obtain

$$S_{fi} = \int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\frac{d^4p_3'}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}$$

$$\left[\frac{Z_3^{1/2}}{\sqrt{(2\pi)^3 2k_0'}}\ \epsilon^{\mu'*}(k',\lambda')\right]\left[\frac{1}{2P'^{\circ}}\ \bar{\chi}\ (p_1',p_2',p_3')^{\mathcal{B}':P',S'}_{\delta dD,\kappa eE,\sigma fF}\right]$$

$$\sigma(p_1',p_2',p_3';k';p_1,p_2,p_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC}$$

$$\left[\frac{1}{2P^{\circ}}\ \chi\ (p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}\right] \qquad (2.83)$$

Eqn. (2.83) is the **Nishijima-Mandelstam Formula for hyperon radiative decays in momentum space.**

CHAPTER 3

ANALYSIS of $\Sigma^+ \to p\gamma$ DECAY

" *Here is no water but only rock*
Rock and no water and the sandy road
The road winding above among the mountains
Which are mountains of rock without water "

T. S. Eliot

In Chapter 2, we developed a method for analyzing hyperon radiative decays within the Bethe-Salpeter formalism. We shall now use that method to calculate the asymmetry parameter and decay rate for $\Sigma^+ \to p\gamma$.

3.1. Simplified Form of the Nishijima-Mandelstam formula for $\Sigma^+ \to p\gamma$ Decay.

The starting point of our calculation is the Nishijima-Mandelstam formula, i.e. eqn. (2.83). For the specific case of $\Sigma^+ \to p\gamma$, it reads as follows

$$S_{\Sigma^+ \to p\gamma} = \int [\Pi \frac{d^4p}{(2\pi)^4}] \; [\frac{Z_3^{1/2}}{\sqrt{(2\pi^3) 2k'_0}} \epsilon^{\mu' *}(k',\lambda')]$$

$$[\frac{1}{2P'_0} \bar{\chi}(p_1', p_2', p_3')^{P:P',S'}_{\delta dD, \kappa eE, \sigma fF}]$$

$$\sigma(p_1', p_2', p_3'; k'; p_1, p_2, p_3)_{\delta dD, \kappa eE, \sigma fF; \mu'; \alpha aA, \beta bB, \gamma cC}$$

$$[\frac{1}{2P^0}\,\chi(p_1,p_2,p_3)^{\Sigma^+:P,S}_{\alpha aA,\beta bB,\gamma cC}] \tag{3.1}$$

where

$$[\Pi\,\frac{d^4p}{(2\pi)^4}] \equiv \frac{d^4p_1'}{(2\pi)^4}\,\frac{d^4p_2'}{(2\pi)^4}\,\frac{d^4p_3'}{(2\pi)^4}\,\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{d^4p_3}{(2\pi)^4} \tag{3.2}$$

In order to define the baryonic coordinates for Σ^+, we choose the three required constants to be

$$f_{1,2,3} = \frac{m_{a,b,c}}{m_a+m_b+m_c} \equiv f_{a,b,c} \tag{3.3}$$

Obviously,

$$f_a + f_b + f_c = 1 \tag{3.4}$$

The baryonic coordinates for Σ^+ are, then,

$$\begin{aligned} \mathcal{P}(p_1,p_2,p_3) &= p_1 + p_2 + p_3 \\ p_\xi(f_ap_1;f_bp_2;f_cp_3) &= \frac{1}{2}\,(2f_b + f_c)p_1 - \frac{1}{2}\,(2f_a+f_c)p_2 - \frac{1}{2}\,(f_a-f_b)p_3 \\ p_\eta(f_ap_1;f_bp_2;f_cp_3) &= f_cp_1 + f_cp_2 - (f_a+f_b)p_3 \end{aligned} \tag{3.5}$$

where we have explicitly written down the arguments of $\mathcal{P}$, p_ξ, p_η for reasons that will become obvious later. From eqns. (2.27)-(2.30) and (3.3)-(3.5), we can write

$$\chi(p_1,p_2,p_3)^{\Sigma^+:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= N_{\Sigma^+}\,\frac{(2\pi)^4}{(2\pi)^{3/2}}\,\delta^4[P-\mathcal{P}(p_1,p_2,p_3)]$$

$$[(D_\rho)^{P,S}_{\alpha\beta\gamma} \frac{1}{\sqrt{6}} \epsilon_{ABC} \mathcal{F}^{\Sigma^{\dagger}:\rho}_{abc} + (D_\lambda)^{P,S}_{\alpha\beta\gamma} \frac{1}{\sqrt{6}} \epsilon_{ABC} \mathcal{F}^{\Sigma^{\dagger}:\lambda}_{abc}]$$

$$\times \chi[p_\xi(f_a p_1; f_b p_2; f_c p_3), p_\eta(f_a p_1; f_b p_2; f_c p_3)]^P \tag{3.6}$$

where

$$(D_\rho)^{P,S}_{\alpha\beta\gamma} = \left(\frac{-1}{\sqrt{2}}\right)\left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\alpha\beta} u^S_\gamma(P) \tag{3.7}$$

$$(D_\lambda)^{P,S}_{\alpha\beta\gamma} = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{3}}\right)\left\{\left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\beta\gamma} u^S_\alpha(P) - \left[\left(\frac{\not{P}+M}{2M}\right)\gamma^5 C\right]_{\gamma\alpha} u^S_\beta(P)\right\} \tag{3.8}$$

and

$$\chi[p_\xi(f_a p_1; f_b p_2; f_c p_3), p_\eta(f_a p_1; f_b p_2; f_c p_3)]^P$$

$$= \exp\left\{ -\frac{1}{2\alpha^2} \left(8\left[\frac{P\cdot p_\xi(f_a p_1; f_b p_2; f_c p_3)}{M}\right]^2 - 4[p_\xi(f_a p_1; f_b p_2; f_c p_3)]^2 + 6\left[\frac{P\cdot p_\eta(f_a p_1; f_b p_2; f_c p_3)}{M}\right]^2 - 3[p_\eta(f_a p_1; f_b p_2; f_c p_3)]^2 \right)\right\} \tag{3.9}$$

This function χ has the interesting property that it is symmetric under any permutation of the pairs $(f_a p_1; f_b p_2; f_c p_3)$.

Similarly, for the proton the baryonic coordinates are defined through the three constants

$$f'_{1,2,3} = \frac{m_{d,e,f}}{m_d + m_e + m_f} \equiv f'_{d,e,f} \tag{3.10}$$

They also satisfy

$$f'_d + f'_e + f'_f = 1 \tag{3.11}$$

The baryonic coordinates for the proton are given by

$$\mathcal{P}'(p'_1, p'_2, p'_3) = p'_1 + p'_2 + p'_3$$

$$p'_\xi(f'_d p'_1; f'_e p'_2; f'_f p'_3) = \frac{1}{2}(2f'_e + f'_f)p'_1 - \frac{1}{2}(2f'_d + f'_f)p'_2 - \frac{1}{2}(f'_d - f'_e)p'_3$$

$$p'_\eta(f'_d p'_1; f'_e p'_2; f'_f p'_3) = f'_f p'_1 + f'_f p'_2 - (f'_d + f'_e) p'_3 \tag{3.12}$$

From eqns. (2.30), (2.36)-(2.38) and (3.10)-(3.12), we can write the conjugate BS amplitude for the proton as follows

$$\bar{\chi}(p'_1, p'_2, p'_3)^{p:P',S'}_{\delta dD,\kappa eE,\sigma fF}$$
$$= N^*_p \frac{(2\pi)^4}{(2\pi)^{3/2}} \delta^4[P' - \mathcal{P}'(p'_1, p'_2, p'_3)]$$
$$[(\bar{D}_\rho)^{P',S'}_{\delta\kappa\sigma} \frac{1}{\sqrt{6}} \epsilon_{DEF} \Upsilon^{p:\rho}_{def} + (\bar{D}_\lambda)^{P',S'}_{\delta\kappa\sigma} \frac{1}{\sqrt{6}} \epsilon_{DEF} \Upsilon^{p:\lambda}_{def}]$$
$$\chi[p'_\xi(f'_d p'_1; f'_e p'_2; f'_f p'_3), p'_\eta(f'_d p'_1; f'_e p'_2; f'_f p'_3)]^{P'*} \tag{3.13}$$

where

$$(\bar{D}_\rho)^{P',S'}_{\delta\kappa\sigma} = (\frac{1}{\sqrt{2}})[\gamma^\circ(\frac{\not{P}'+M'}{2M'})\gamma^5 C\gamma^\circ]^*_{\delta\kappa} \bar{u}_\sigma^{S'}(P')$$
$$= (\frac{1}{\sqrt{2}})[(\frac{\not{P}'^T+M'}{2M'})\gamma^5 C]_{\delta\kappa} \bar{u}_\sigma^{S'}(P') \tag{3.14}$$

$$(\bar{D}_\lambda)^{P',S'}_{\delta\kappa\sigma} = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{3}})\{[\gamma^\circ(\frac{\not{P}'+M'}{2M'})\gamma^5 C\gamma^\circ]^*_{\kappa\sigma} \bar{u}_\delta^{S'}(P')$$
$$- [\gamma^\circ(\frac{\not{P}'+M'}{2M'})\gamma^5 C\gamma^\circ]^*_{\sigma\delta} \bar{u}_\kappa^{S'}(P')\}$$
$$= (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{3}})\{[(\frac{\not{P}'^T+M'}{2M'})\gamma^5 C]_{\kappa\sigma} \bar{u}_\delta^{S'}(P')$$
$$- [(\frac{\not{P}'^T+M'}{2M'})\gamma^5 C]_{\sigma\delta} \bar{u}_\kappa^{S'}(P')\} \tag{3.15}$$

and

$$\chi[p'_\xi(f'_d p'_1;f'_e p'_2;f'_f p'_3),\ p'_\eta(f'_d p'_1;f'_e p'_2;f'_f p'_3)]^{P'}$$
$$= \exp\Big\{-\frac{1}{2\alpha'^2}\Big(8\Big[\frac{P'\cdot p'_\xi(f'_d p'_1;f'_e p'_2;f'_f p'_3)}{M'}\Big]^2 - 4[p'_\xi(f'_d p'_1;f'_e p'_2;f'_f p'_3)]^{2}$$
$$+ 6\Big[-\frac{P'\cdot p'_\eta(f'_d p'_1;f'_e p'_2;f'_f p'_3)}{M'}\Big]^2 - 3[p'_\eta(f'_d p'_1;f'_e p'_2;f'_f p'_3)]^2\Big)\Big\} \qquad (3.16)$$

Just like χ in eqn. (3.9), χ in the equation above is also symmetric under any permutation of the pairs $(f'_d p'_1;f'_e p'_2;f'_f p'_3)$ . .

From the definition of the amputated Green function, one can also show that

$$\sigma(p'_1,p'_2,p'_3;k';p_1,p_2,p_3)_{\delta dD,\kappa eE,\sigma fF;\mu';\alpha aA,\beta bB,\gamma cC}$$
$$= (-1)^{P_1}(-1)^{P_2}\ \sigma(P_1(p'_1,p'_2,p'_3);k';\ P_2(p_1,p_2,p_3)\ )_{P_1(\delta dD,\kappa eE,\sigma fF);\mu';\ P_2(\alpha aA,\beta bB,\gamma cC)} \qquad (3.17)$$

where $P_1(\ldots.)$ and $P_2(\ldots.)$ stand for permutation of quantities inside their arguments.

From Appendix A, we borrow the non-zero components of the flavour baryon tensors.

$$\Upsilon^{p:\rho}_{udu} = \frac{1}{\sqrt{2}} \quad ; \quad \Upsilon^{p:\rho}_{duu} = \frac{-1}{\sqrt{2}} \qquad (3.18)$$

$$\Upsilon^{p:\lambda}_{uud} = \frac{2}{\sqrt{6}} \quad ; \quad \Upsilon^{p:\lambda}_{udu} = \Upsilon^{p:\lambda}_{duu} = \frac{-1}{\sqrt{6}} \qquad (3.19)$$

$$\Upsilon^{\Sigma^+:\rho}_{usu} = \frac{1}{\sqrt{2}} \quad ; \quad \Upsilon^{\Sigma^+:\rho}_{suu} = \frac{-1}{\sqrt{2}} \qquad (3.20)$$

$$\Upsilon^{\Sigma^+:\lambda}_{uus} = \frac{2}{\sqrt{6}} \quad ; \quad \Upsilon^{\Sigma^+:\lambda}_{usu} = \Upsilon^{\Sigma^+:\lambda}_{suu} = \frac{-1}{\sqrt{6}} \qquad (3.21)$$

We now simplify eqn. (3.1) in the following manner:

(i) We first put eqns. (3.6) and (3.13) in eqn. (3.1).

(ii) We carry out the summation over the flavour indices by using eqns. (3.18)-(3.21).

(iii) We rename various dummy variables and then use the symmetry properties of χ's, σ, ϵ and the following two relations

$$[(\frac{\not{P}+M}{2M})\gamma^5 C]_{\alpha\beta} = -[(\frac{\not{P}+M}{2M})\gamma^5 C]_{\beta\alpha} \qquad (3.22)$$

and

$$[(\frac{\not{P}'^T+M'}{2M'})\gamma^5 C]_{\delta\kappa} = -[(\frac{\not{P}'^T+M'}{2M'})\gamma^5 C]_{\kappa\delta} \qquad (3.23)$$

After a very tedious calculation, one gets the following simple form for $S_{\Sigma^+\to p\gamma}$.

$$S_{\Sigma^+\to p\gamma} = [\frac{Z_3^{1/2}}{\sqrt{(2\pi)^3 2k_0'}}]\ [N_p^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$
$$[(\frac{-1}{6})\epsilon_{DEF}\ \epsilon_{ABC}]\ D_{\delta\kappa\sigma\alpha\beta\gamma}\ \epsilon^{\mu'*}(k',\lambda')$$
$$W(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} \qquad (3.24)$$

where

$$W(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC}$$
$$= \int[\Pi \frac{d^4p}{(2\pi)^4}]\ \delta^4[P'-\mathcal{P}'(p_1',p_2',p_3')]\ \delta^4[P-\mathcal{P}(p_1,p_2,p_3)]$$
$$\chi[p_\xi'(f_u'p_1';f_u'p_2';f_d'p_3'),\ p_\eta'(f_u p_1';f_u'p_2';f_d'p_3')]^P$$
$$\sigma(p_1',p_2',p_3';k';p_1,p_2,p_3)_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC}$$
$$\chi[p_\xi(f_u'p_1';f_u'p_2';f_s'p_3'),\ p_\eta(f_u p_1';f_u'p_2';f_s'p_3')]^P \qquad (3.25)$$

It is obvious that there are no dummy flavour indices in W. We have already carried out that sum. Appearance of (uud) and (uus) in W indicates that it depends on various properties of these quarks such as their masses, charges etc.

$D_{\delta\kappa\sigma\alpha\beta\gamma}$ in eqn. (3.25) is given by

$$\begin{aligned} D_{\delta\kappa\sigma\alpha\beta\gamma} = [\,&(\mathcal{M}_1)_{\delta\sigma}\,\bar{u}_\kappa^{S'}(P') \quad (\mathcal{M}_2)_{\alpha\gamma}\,u_\beta^{S}(P) \\ -\,&(\mathcal{M}_1)_{\delta\sigma}\,\bar{u}_\kappa^{S'}(P') \quad (\mathcal{M}_2)_{\gamma\beta}\,u_\alpha^{S}(P) \\ -\,&(\mathcal{M}_1)_{\sigma\kappa}\,\bar{u}_\delta^{S'}(P') \quad (\mathcal{M}_2)_{\alpha\gamma}\,u_\beta^{S}(P) \\ +\,&(\mathcal{M}_1)_{\sigma\kappa}\,\bar{u}_\delta^{S'}(P') \quad (\mathcal{M}_2)_{\gamma\beta}\,u_\alpha^{S}(P)\,] \end{aligned} \tag{3.26}$$

where the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are given by

$$\mathcal{M}_1 = \left(\frac{\not{P}'^{T} + M'}{2M'}\right)\gamma^5 C \tag{3.27}$$

and

$$\mathcal{M}_2 = \left(\frac{\not{P} + M}{2M}\right)\gamma^5 C \tag{3.28}$$

It is obvious that if we know W then we can put it in eqn. (3.24) and carry out the summations over colour and Dirac indices to calculate the desired S-matrix element. In order to do that, we need to know the amputated Green function σ.

3.2. Calculation of the required Amputated Green Function

From eqns. (2.71) and (2.80), it is clear that the amputated Green function $\sigma(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC}$ is given by the following equation.

$$G(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$
$$= \int \frac{d^4p_1'}{(2\pi)^4}\,\frac{d^4p_2'}{(2\pi)^4}\,\frac{d^4p_3'}{(2\pi)^4}\,\frac{d^4p_4'}{(2\pi)^4}\,\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{d^4p_3}{(2\pi)^4}$$

$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}} \quad G(q_4',p_4')^{\mu'\tilde{\mu}'}$$

$$\sigma(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF};\tilde{\mu}';\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC}}$$

$$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha uA,\beta uB,\gamma sC} \tag{3.29}$$

The momentum-space Green functions in eqn. (3.29) are defined by the following Fourier transforms of their configuration-space counterparts in eqns. (2.71)-(2.74).

$$G(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$\equiv \langle 0|T\psi_{\delta uD}(y_1)\psi_{\kappa uE}(y_2)\psi_{\sigma dF}(y_3)A^{\mu'}(y_4)\bar{\psi}_{\alpha uA}(x_1)\bar{\psi}_{\beta uB}(x_2)\bar{\psi}_{\gamma sC}(x_3)|0\rangle$$

$$= \int \frac{d^4q_1'}{(2\pi)^4}\frac{d^4q_2'}{(2\pi)^4}\frac{d^4q_3'}{(2\pi)^4}\frac{d^4q_4'}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4} e^{-iq_1'y_1-iq_2'y_2-iq_3'y_3-iq_4'y_4+iq_1x_1+iq_2x_2+iq_3x_3}$$

$$G(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC} \tag{3.30}$$

$$G(y_1,y_2,y_3;y_1',y_2',y_3')_{\delta uD,\kappa uE,\sigma dF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}}$$

$$\equiv -\langle 0|T\psi_{\delta uD}(y_1)\psi_{\kappa uE}(y_2)\psi_{\sigma dF}(y_3)\bar{\psi}_{\widetilde{\delta dD}}(y_1')\bar{\psi}_{\widetilde{\kappa eE}}(y_2')\bar{\psi}_{\widetilde{\sigma fF}}(y_3')|0\rangle$$

$$= \int \frac{d^4q_1'}{(2\pi)^4}\frac{d^4q_2'}{(2\pi)^4}\frac{d^4q_3'}{(2\pi)^4}\frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\frac{d^4p_3'}{(2\pi)^4} e^{-iq_1'y_1-iq_2'y_2-iq_3'y_3+ip_1'y_1'+ip_2'y_2'+ip_3'y_3'}$$

$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}} \tag{3.31}$$

$$G(y_4,y_4')^{\mu'\tilde{\mu}'}$$

$$\equiv \langle 0|TA^{\mu'}(y_4)\,A^{\tilde{\mu}'}(y_4')|0\rangle$$

$$= \int \frac{d^4 q_4'}{(2\pi)^4} \frac{d^4 p_4'}{(2\pi)^4} e^{-iq_4' y_4 + ip_4' y_4'} G(q_4', p_4')^{\mu' \tilde{\mu}'} \tag{3.32}$$

and, finally,

$$G(x_1', x_2', x_3'; x_1, x_2, x_3)_{\widetilde{\alpha aA}, \widetilde{\beta bB}, \widetilde{\gamma cC}; \alpha uA, \beta uB, \gamma sC}$$

$$\equiv -\langle 0 | T\psi_{\widetilde{\alpha aA}}(x_1')\psi_{\widetilde{\beta bB}}(x_2')\psi_{\widetilde{\gamma cC}}(x_3')\bar{\psi}_{\alpha uA}(x_1)\bar{\psi}_{\beta uB}(x_2)\bar{\psi}_{\gamma sC}(x_3) | 0 \rangle$$

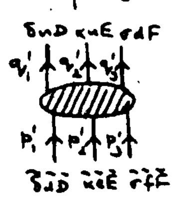
$$= \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{d^4 q_3}{(2\pi)^4} e^{-ip_1 x_1' - ip_2 x_2' - i p_3 x_3' + iq_1 x_1 + iq_2 x_2 + iq_3 x_3}$$

$$G(p_1, p_2, p_3; q_1, q_2, q_3)_{\widetilde{\alpha aA}, \widetilde{\beta bB}, \widetilde{\gamma cC}; \alpha uA, \beta uB, \gamma sC} \tag{3.33}$$

These momentum-space Green functions have been drawn in Fig. 3.1.

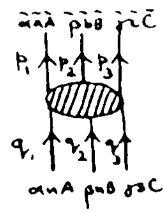
For the evaluation of these Green functions, we shall employ the GSW theory [60-62] and standard perturbative techniques [35,37,39,40,46,57,59]. Relevant part of the GSW Lagrangian is given in Appendix C.

$$G(q_1', q_2', q_3'; q_4'; q_1, q_2, q_3)^{\mu'}_{\delta uD, \kappa uE, \sigma dF; \alpha uA, \beta uB, \gamma sC}$$



$$G(q_1', q_2', q_3'; p_1', p_2', p_3')_{\delta uD, \kappa uE, \sigma dF; \widetilde{\delta dD}, \widetilde{\kappa eE}, \widetilde{\sigma fF}}$$

$$G(q_4', p_4')^{\mu' \tilde{\mu}'}$$

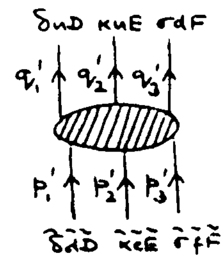


$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha uA,\beta uB,\gamma sC}$

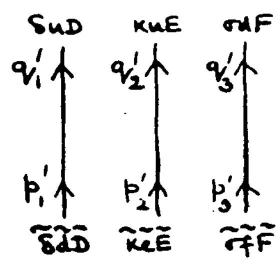
Fig. 3.1. Momentum-space Green functions required for the calculation of σ.

As for the 6-particle Green functions in eqn. (3.29), we shall work only in the lowest order, i.e.

$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\widetilde{\delta dD},\widetilde{\kappa eE},\widetilde{\sigma fF}} \equiv$$



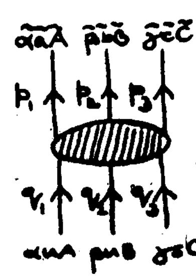
$=$



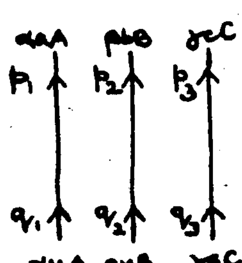
$+$ permutations (3.34)

Similarly,

$$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\widetilde{\alpha aA},\widetilde{\beta bB},\widetilde{\gamma cC};\alpha uA,\beta uB,\gamma sC} \equiv$$



$=$



$+$ permutations (3.35)

It is, therefore, obvious that

$G(q_1', q_2', q_3'; p_1', p_2', p_3')_{\delta uD, \kappa uE, \sigma dF; \widetilde{\delta d D}, \widetilde{\kappa e E}, \widetilde{\sigma f F}}$ will be nonvanishing only

for $(\tilde{d}, \tilde{e}, \tilde{f}) = (u,u,d)$ or any of its permutations. Similarly,

$G(p_1, p_2, p_3; q_1, q_2, q_3)_{\widetilde{\alpha a A}, \widetilde{\beta b B}, \widetilde{\gamma c C}; \alpha uA, \beta uB, \gamma sC}$ will be nonvanishing only

for $(\tilde{a}, \tilde{b}, \tilde{c}) = (u,u,s)$ or any of its permutations.

We now do the sum over $(\tilde{d}, \tilde{e}, \tilde{f})$ and $(\tilde{a}, \tilde{b}, \tilde{c})$ in eqn. (3.29). It is clear that we shall get nine terms. Interchanging some of the dummy variables and indices and using the antisymmetry properties of G's and σ, we obtain

$$
\begin{aligned}
& G(q_1', q_2', q_3'; q_4'; q_1, q_2, q_3)^{\mu'}_{\delta uD, \kappa uE, \sigma dF; \alpha uA, \beta uB, \gamma sC} \\
&= 9 \int \frac{d^4 p_1'}{(2\pi)^4} \frac{d^4 p_2'}{(2\pi)^4} \frac{d^4 p_3'}{(2\pi)^4} \frac{d^4 p_4'}{(2\pi)^4} \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \; G(q_4', p_4')^{\mu' \tilde{\mu}'} \\
& \quad G(q_1', q_2', q_3'; p_1', p_2', p_3')_{\delta uD, \kappa uE, \sigma dF; \tilde{\delta} u\tilde{D}, \tilde{\kappa} u\tilde{E}, \tilde{\sigma} d\tilde{F}} \\
& \quad \sigma(p_1', p_2', p_3'; p_4'; p_1, p_2, p_3)_{\tilde{\delta} u\tilde{D}, \tilde{\kappa} u\tilde{E}, \tilde{\sigma} d\tilde{F}; \tilde{\mu}'; \tilde{\alpha} u\tilde{A}, \tilde{\beta} u\tilde{B}, \tilde{\gamma} s\tilde{C}} \\
& \quad G(p_1, p_2, p_3; q_1, q_2, q_3)_{\tilde{\alpha} u\tilde{A}, \tilde{\beta} u\tilde{B}, \tilde{\gamma} s\tilde{C}; \alpha uA, \beta uB, \gamma sC}
\end{aligned}
\tag{3.36}
$$

The factor of 9 arises because two quarks are identical in the initial as well as the final state. In order to calculate σ, we shall compute the 7-point function on the left-hand side in the lowest possible order in the GSW theory and cast it in the form of eqn. (3.36). We can, then, read off σ from that expression. It is, however, convenient to do this job in pieces as explained below.

The lowest order contributions to

$$G(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

are of third order in the electroweak coupling, viz., of the order g^2e. In fact, in the Feynman-t'Hooft gauge,

$$G(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

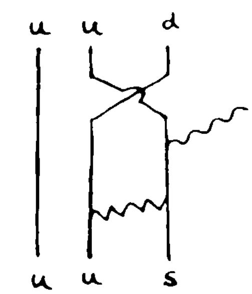
is a sum of (28×4) terms. Suppressing the argument and indices

$$G = \left(G_1^{(1)}+\ldots.+G_1^{(4)}\right) +\ldots.+ \left(G_{28}^{(1)}+\ldots.+ G_{28}^{(4)}\right) \qquad (3.37)$$

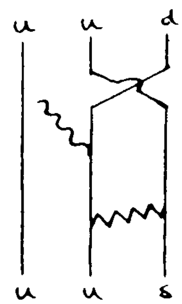
We have drawn $G_1^{(1)}, G_2^{(1)},\ldots,G_{28}^{(1)}$ in Fig. 3.2. The other three diagrams in each case come from permutation of two identical quarks both in the initial and final state. This is illustrated by drawing $G_1^{(2)},\ldots,G_1^{(4)}$ in detail in Fig. 3.3.

We shall cast each of these 28 contributions separately in the form of eqn. (3.36) as follows

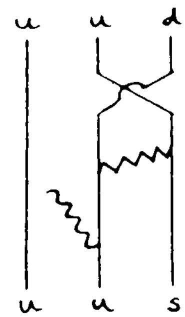
$$G_i^{(1)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC} + \ldots.$$
$$+ G_i^{(4)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$
$$= 9\int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\frac{d^4p_3'}{(2\pi)^4}\frac{d^4p_4'}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\; G(q_4',p_4')^{\mu'\tilde{\mu}'}$$
$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F}}$$
$$\sigma_i(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F};\mu';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}}$$
$$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C};\alpha uA,\beta uB,\gamma sC} \qquad (3.38)$$



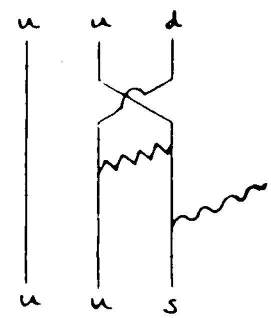
$G_1^{(1)}$



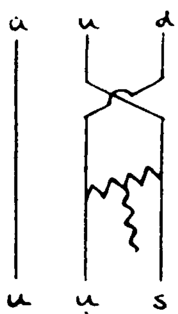
$G_2^{(1)}$



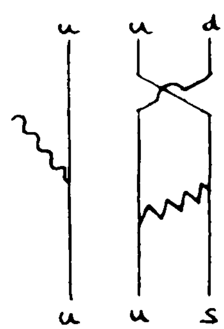
$G_3^{(1)}$



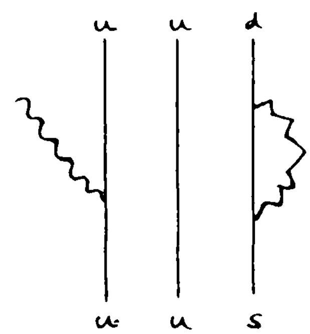
$G_4^{(1)}$



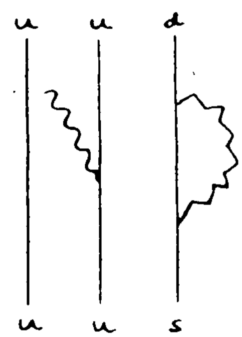
$G_5^{(1)}$



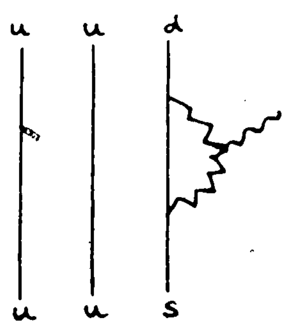
$G_6^{(1)}$



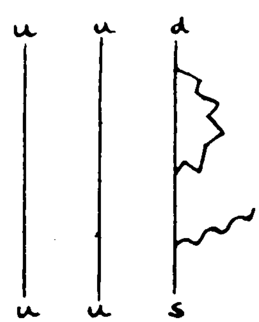
$G_7^{(1)}$



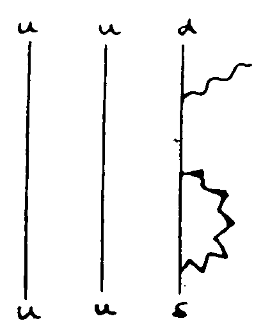
$G_8^{(1)}$



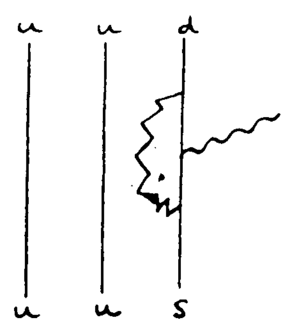
$G_9^{(1)}$



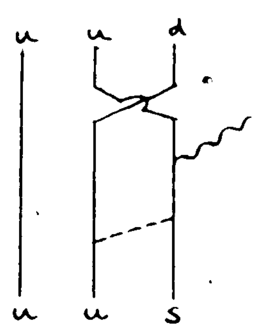
$G_{10}^{(1)}$



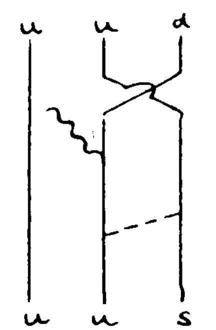
$G_{11}^{(1)}$



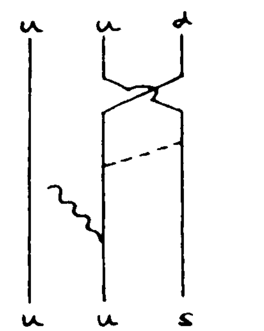
$G_{12}^{(1)}$



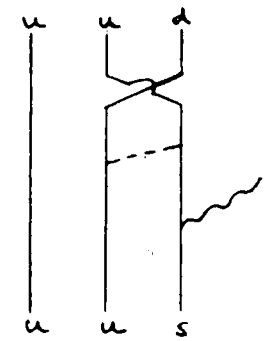
$G^{(1)}_{13}$



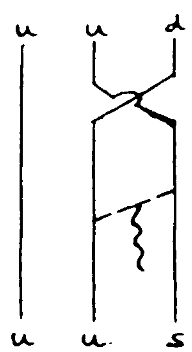
$G^{(1)}_{14}$



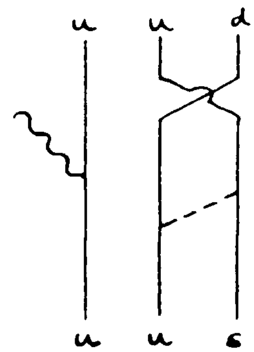
$G^{(1)}_{15}$



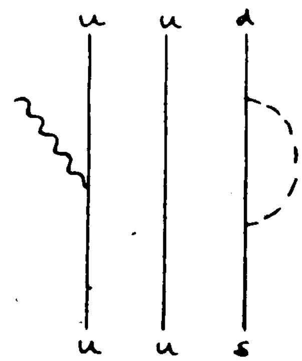
$G^{(1)}_{16}$



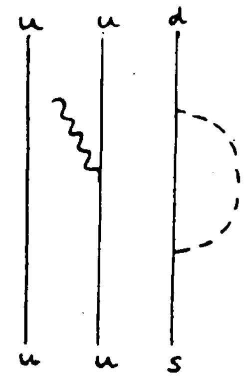
$G^{(1)}_{17}$



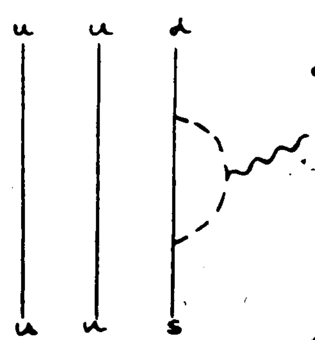
$G^{(1)}_{18}$



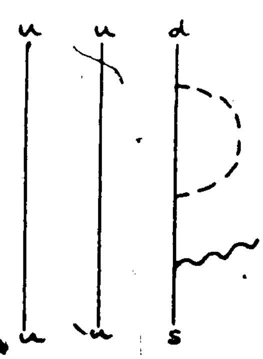
$G^{(1)}_{19}$



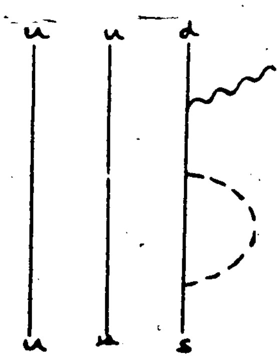
$G^{(1)}_{20}$



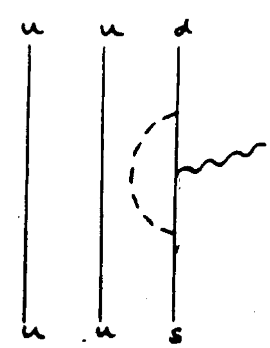
$G^{(1)}_{21}$



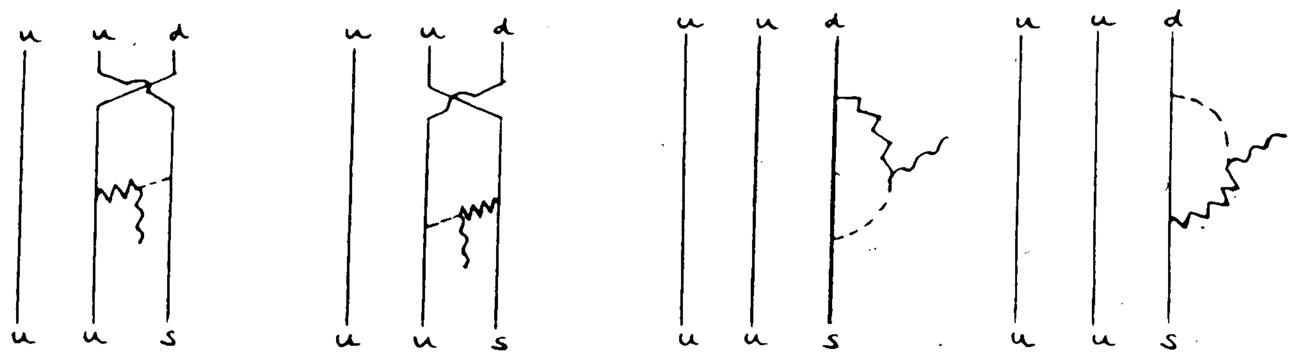
$G^{(1)}_{22}$



$G^{(1)}_{23}$

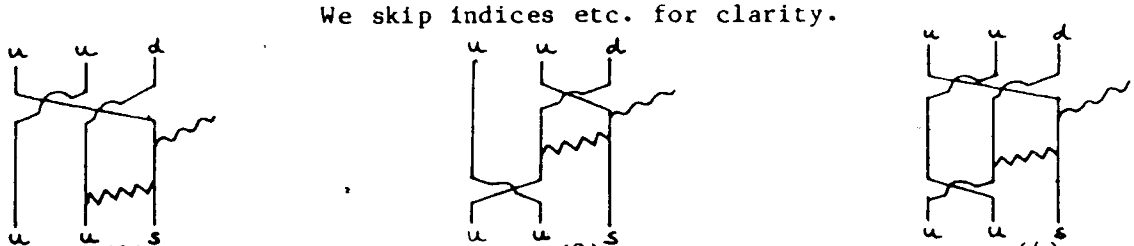


$G^{(1)}_{24}$



$G^{(1)}_{25}$ $G^{(1)}_{26}$ $G^{(1)}_{27}$ $G^{(1)}_{28}$

Fig. 3.2. $G^{(1)}_1, \ldots, G^{(1)}_{28}$ in the Feynman-t'Hooft gauge. Sawtooth lines are W-lines; wavy lines are photon lines and dashed lines are would-be Goldstones. We skip indices etc. for clarity.



$G^{(2)}_1$ $G^{(3)}_1$ $G^{(4)}_1$

Fig. 3.3. Permuted diagrams for G_1 .

From eqns. (3.37) and (3.38)

$$G(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma uA;\alpha uA,\beta uB,\gamma sC}$$

$$= 9 \int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\frac{d^4p_3'}{(2\pi)^4}\frac{d^4p_4'}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\; G(q_4',p_4')^{\mu'\tilde{\mu}'}$$

$$\left\{ G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F}} \right.$$

$$\left[\sum_{j=1}^{28} \sigma_j(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F};\tilde{\mu}';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}} \right]$$

$$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C};\alpha uA,\beta uB,\gamma sC} \tag{3.39}$$

From eqns. (3.36) and (3.39), we get

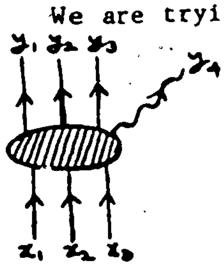
$$\sigma(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}s\tilde{F};\tilde{\mu}';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}}$$

$$= \sum_{j=1}^{28} \sigma_j(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F};\tilde{\mu}';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}} \tag{3.40}$$

where the σ_j's are defined by eqn. (3.38).

As an illustration, let us try to calculate σ_1 in detail. This will give us straightforward Feynman rules and detailed calculation for other σ_j's will, then, be unnecessary.

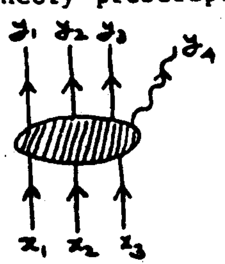
We are trying to calculate the 7-point function given by



$$= G(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$\equiv \langle 0|T\psi_{\delta uD}(y_1)\psi_{\kappa uE}(y_2)\psi_{\sigma dF}(y_3)A^{\mu'}(y_4)\bar{\psi}_{\alpha uA}(x_1)\bar{\psi}_{\beta uB}(x_2)\bar{\psi}_{\gamma sC}(x_3)|0\rangle \tag{3.41}$$

The lowest order contribution to this Green function is of 3rd order in the GSW Lagrangian. Therefore, following the standard perturbation theory prescription [35,37,39,40,46,57,59]



$$= G(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (i)^3 \int d^4z_1\, d^4z_2\, d^4z_3$$

$$\langle 0|T\, \psi_{\delta uD}(y_1)_{in}\, \psi_{kuE}(y_2)_{in}\, \psi_{\sigma dF}(y_3)_{in}\, A^{\mu'}(y_4)_{in}$$

$$\bar{\psi}_{\alpha a A}(x_1)_{in}\ \psi_{\beta b B}(x_2)_{in}\ \psi_{\gamma s C}(x_3)_{in}$$
$$\mathcal{L}_{int}(z_1)\,\mathcal{L}_{int}(z_2)\,\mathcal{L}_{int}(z_3)|0\rangle^C \qquad (3.41)$$

where different pieces of the GSW Lagrangian will give rise to different diagrams or terms contributing to G. The supersript C on $\langle 0|\ldots|0\rangle$ in (3.41) implies that we should only keep the connected diagrams in the expansion. Out of this expansion, we want to pick up $G_1^{(1)}$ given by the following Feynman diagram

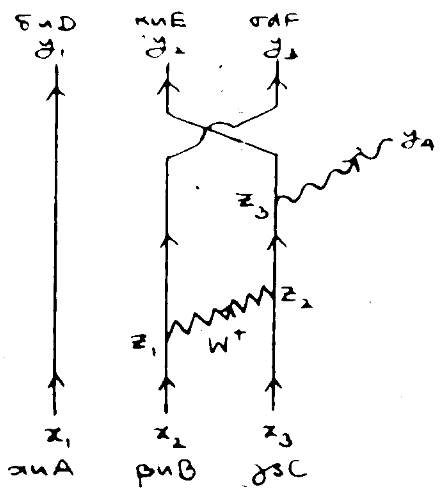


Fig. 3.4. $G_1^{(1)}$ in detail in configuration space

It is obvious that the following pieces of the GSW Lagrangian will give rise to the diagram above after contractions in eqn. (3.41).

$$\mathcal{L}^{duW^-}(z_1) = C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta}\ \bar{\psi}_{\eta dR}(z_1)_{in}\,\psi_{\zeta uR}(z_1)_{in}W^-_{\lambda'}(z_1)_{in} \qquad (3.42)$$

$$\mathcal{L}^{usW^+}(z_2) = C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\ \bar{\psi}_{\xi uT}(z_2)_{in}\ \psi_{\tau sT}(z_2)_{in}W^-_{\nu'}(z_2)_{in} \qquad (3.43)$$

$$\mathcal{L}^{nnA}(z_3) = C(nnA)_{\mu\nu\beta'}\ \bar{\psi}_{\mu nL}(z_3)_{in}\ \psi_{\nu nL}(z_3)_{in}\ A^{\beta'}(z_3)_{in} \qquad (3.44)$$

n stands for the flavour index and R,T,L for colour indices. The vertex factors, $C(duW^-)^{\lambda'}_{\nu\omega;\omega\zeta}$, $C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}$ and $C(nnA)_{\mu\nu\beta'}$, are listed in Appendix C.

Putting eqns. (3.42)-(3.44) in eqn. (3.41) and doing the contractions following Wick's [35,37,39,40,46,57,59] theorem which gives rise to Fig. 3.4, one obtains

$$G_1^{(1)}(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (-1)^{18}\,\frac{(i)^3}{3!}\,\delta_{DA}\delta_{EC}\delta_{FB}\int_{\Omega} d^4z_1 d^4z_2 d^4z_3\; C(duW^-)^{\lambda'}_{\eta\omega,\omega\zeta}$$
$$C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\; C(uuA)_{\mu\nu\beta'}\; [iS^u(y_1-x_1)]_{\delta\alpha}\; [iS^u(y_2-z_3)]_{\kappa\mu}$$
$$[iS^d(y_3-z_1)]_{\sigma\eta}\; [iS^u(z_1-x_2)]_{\zeta\beta}\; [iS^s(z_2-x_3)]_{\tau\gamma}$$
$$[iS^u(z_3-z_2)]_{\nu\xi}\; [i\Delta^W(z_2-z_1)]_{\nu'\lambda'}\; [iD(y_4-z_3)]^{\mu'\beta'} \qquad (3.45)$$

where S, Δ and D are the quark, W and photon propagators respectively. In the course of this derivation, one realizes that the sign of this term, viz. $(-1)^{18}$, is solely determined by the contractions between the fermions in eqn. (3.41). In other words, one has to just consider the fermions in $\langle 0|\cdots|0\rangle$ in eqn. (3.41), i.e. in

$$\langle 0|T\;\psi(y_1)_{in}\;\psi(y_2)_{in}\;\psi(y_3)_{in}\;\bar\psi(x_1)_{in}\;\bar\psi(x_2)_{in}\;\bar\psi(x_3)_{in}$$
$$\bar\psi(z_1)_{in}\;\psi(z_1)_{in}\;\bar\psi(z_2)_{in}\;\psi(z_2)_{in}\;\bar\psi(z_3)_{in}\;\psi(z_3)_{in}\;\cdots|0\rangle \qquad (3.46)$$

and contract them in a manner which gives rise to the particular diagram in Fig. 3.4. The sign picked up in doing these contractions determines the sign of this particular contribution to G.

Introducing the momentum-space representations of various propagators in eqn. (3.45), we obtain

$$G_1^{(1)}(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (i)^3 \, \delta_{DA}\delta_{EC}\delta_{FB} \int \frac{d^4q_1'}{(2\pi)^4} \frac{d^4q_2'}{(2\pi)^4} \frac{d^4q_3'}{(2\pi)^4} \frac{d^4q_4'}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4}$$

$$C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta} \; C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau} \; C(uuA)_{\mu\nu\beta'}$$

$$e^{-iq_1'y_1 - iq_2'y_2 - iq_3'y_3 - iq_4'y_4 + iq_1'x_1 + iq_2x_2 + iq_3x_3}$$

$$(2\pi)^4\delta^4(q_2'+q_3'+q_4'-q_2-q_3)$$

$$\frac{[i(\not q_1'+m_u)]_{\delta\tilde{\delta}}}{q_1'^2-m_u^2+i\epsilon} \; \delta_{\tilde{\delta}\alpha} \; \frac{[i(\not q_2'+m_u)]_{\kappa\mu}}{q_2'^2-m_u^2+i\epsilon} \; \frac{[i(\not q_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\epsilon}$$

$$\frac{-ig^{\mu'\beta'}}{q_4'^2+i\epsilon} \; \frac{[i(\not q_2+m_u)]_{\zeta\beta}}{q_2^2-m_u^2+i\epsilon} \; \frac{[i(\not q_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\epsilon}$$

$$\frac{[i(\not q_2+\not q_3-\not q_3'+m_u)]_{\nu\xi}}{(q_2+q_3-q_3')^2-m_u^2+i\epsilon} \; \frac{-ig_{\nu'\lambda'}}{(q_2-q_3')^2 - m_W^2 + i\epsilon} \qquad (3.47)$$

For purposes of amputation later, we introduce the inverse propagator for the spectator quark by the definition

$$\frac{[-i(\not q_1'-m_u)]_{\tilde{\delta}\tilde{\alpha}}[i(\not q_1'+m_u)]_{\tilde{\alpha}\alpha}}{q_1'^2 - m_u^2+i\epsilon} = \delta_{\tilde{\delta}\alpha} \qquad (3.48)$$

Also, introducing an integral over q_1 in a trivial fashion, eqn. (3.47) assumes the following form

$$G_1^{(1)}(y_1,y_2,y_3;y_4;x_1,x_2,x_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (i)^3 \, \delta_{DA}\delta_{EC}\delta_{FB} \int \frac{d^4q_1'}{(2\pi)^4} \frac{d^4q_2'}{(2\pi)^4} \frac{d^4q_3'}{(2\pi)^4} \frac{d^4q_4'}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4}$$

$$C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta} \; C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau} \; C(uuA)_{\mu\nu\beta'}$$

$$e^{-iq_1'y_1 - iq_2'y_2 - iq_3'y_3 - iq_4'y_4 + iq_1x_1 + iq_2x_2 + iq_3x_3}$$

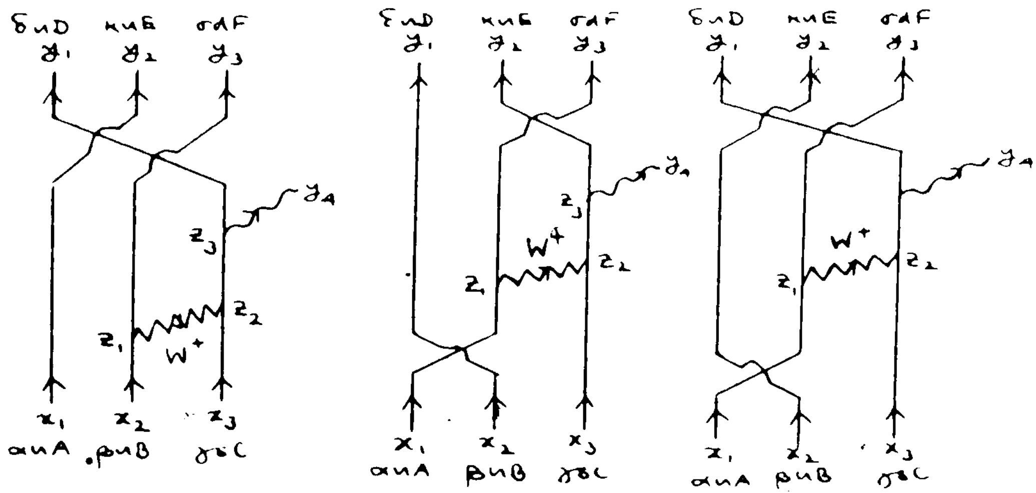
$$(2\pi)^4\delta^4(q_1'-q_1) \; (2\pi)^4\delta^4(q_2'+q_3'+q_4'-q_2-q_3)$$

$$\frac{[i(\not{q}_1'+m_u)]_{\delta\tilde{\delta}}}{q_1'^2-m_u^2+i\epsilon}\frac{[i(\not{q}_2'+m_u)]_{\kappa\mu}}{q_2'^2-m_u^2+i\epsilon}\frac{[i(\not{q}_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\epsilon}\frac{-ig^{\mu'\beta'}}{q_4'^2+i\epsilon}$$
$$[-i(\not{q}_1'-m_u)]_{\tilde{\delta}\tilde{\alpha}}\frac{[i(\not{q}_2+\not{q}_3-\not{q}_3'+m_u)]_{\nu\xi}}{(q_2+q_3-q_3')^2-m_u^2+i\epsilon}\frac{-ig_{\nu'\lambda'}}{(q_2-q_3')^2-m_w^2+i\epsilon}$$
$$\frac{[i(\not{q}_1+m_u)]_{\tilde{\alpha}\alpha}}{q_1^2-m_u^2+i\epsilon}\frac{[i(\not{q}_2+m_u)]_{\zeta\beta}}{q_2^2-m_u^2+i\epsilon}\frac{[i(\not{q}_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\epsilon} \qquad (3.49)$$

Comparing eqn (3.49) with the definition of G in momentum space, viz. eqn (3.30), we immediately obtain $G_1^{(1)}$ in momentum space. It is given by

$$G_1^{(1)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$
$$= (i)^3\ \delta_{DA}\delta_{EC}\delta_{FB}\ C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta}C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}C(uuA)_{\mu\nu\beta'}$$
$$(2\pi)^4\delta^4(q_1'-q_1)\ (2\pi)^4\delta^4(q_2'+q_3'+q_4'-q_2-q_3)$$
$$\frac{[i(\not{q}_1'+m_u)]_{\delta\tilde{\delta}}}{q_1'^2-m_u^2+i\epsilon}\frac{[i(\not{q}_2'+m_u)]_{\kappa\mu}}{q_2'^2-m_u^2+i\epsilon}\frac{[i(\not{q}_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\epsilon}\frac{-ig^{\mu'\beta'}}{q_4'^2+i\epsilon}$$
$$[-i(\not{q}_1'-m_u)]_{\tilde{\delta}\tilde{\alpha}}\frac{[i(\not{q}_2+\not{q}_3-\not{q}_3'+m_u)]_{\nu\xi}}{(q_2+q_3-q_3')^2-m_u^2+i\epsilon}\frac{-ig_{\nu'\lambda'}}{(q_2-q_3')^2-m_w^2+i\epsilon}$$
$$\frac{[i(\not{q}_1+m_u)]_{\tilde{\alpha}\alpha}}{q_1^2-m_u^2+i\epsilon}\frac{[i(\not{q}_2+m_u)]_{\zeta\beta}}{q_2^2-m_u^2+i\epsilon}\frac{[i(\not{q}_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\epsilon} \qquad (3.50)$$

In exactly the same fashion one can calculate $G_1^{(2)}$, $G_1^{(3)}$ and $G_1^{(4)}$ which are given in detail by the Feynman diagrams in Fig. 3.5.



$G_1^{(2)}$ $G_1^{(3)}$ $G_1^{(4)}$

Fig. 3.5 $G_1^{(2)}$, $G_1^{(3)}$ and $G_1^{(4)}$ in detail in configuration space.

They are given by the following expression

$$G_1^{(2)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (-1)\,(i)^3\,\delta_{DC}\delta_{EA}\delta_{FB}\;C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta}\;C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\;C(uuA)_{\mu\nu\beta'}$$

$$(2\pi)^4\delta^4(q_2'-q_1)\;(2\pi)^4\delta^4(q_1'+q_3'+q_4'-q_2-q_3)$$

$$\frac{[i(\not q_1'+m_u)]_{\delta\mu}}{q_1'^2-m_u^2+i\epsilon}\;\frac{[i(\not q_2'+m_u)]_{\kappa\tilde{\delta}}}{q_2'^2-m_u^2+i\epsilon}\;\frac{[i(\not q_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\epsilon}\;\frac{-ig^{\mu'\beta'}}{q_4'^2+i\epsilon}$$

$$[-i(\not q_2'-m_u)]_{\tilde{\delta}\tilde{\alpha}}\;\frac{[i(\not q_2-\not q_3'+\not q_3+m_u)]_{\nu\xi}}{(q_2-q_3'+q_3)^2-m_u^2+i\epsilon}\;\frac{-ig_{\nu'\lambda'}}{(q_2-q_3')^2-m_W^2+i\epsilon}$$

$$\frac{[i(\not q_1+m_u)]_{\tilde{\alpha}\alpha}}{q_1^2-m_u^2+i\epsilon}\;\frac{[i(\not q_2+m_u)]_{\zeta\beta}}{q_2^2-m_u^2+i\epsilon}\;\frac{[i(\not q_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\epsilon} \qquad (3.51)$$

$$G_1^{(3)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= (-1)\ (i)^3\ \delta_{DB}\delta_{EC}\delta_{FA}\ C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta}\ C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\ C(uuA)_{\mu\nu\beta'}$$
$$(2\pi)^4\delta^4(q_1'-q_2)(2\pi)^4\delta^4(q_2'+q_3'+q_4'-q_1-q_3)$$

$$\frac{[i(\not q_1'+m_u)]_{\delta\tilde\delta}}{q_1'^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_2'+m_u)]_{\kappa\mu}}{q_2'^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\varepsilon}\ \frac{-ig^{\mu'\beta'}}{q_4'^2+i\varepsilon}$$
$$[-i(\not q_1'-m_u)]_{\tilde\delta\tilde\alpha}\ \frac{-ig_{\nu'\lambda'}}{(q_1-q_3')^2-m_w^2+i\varepsilon}\ \frac{[i(\not q_2'+\not q_4'+m_u)]_{\nu\xi}}{(q_2'+q_4')^2-m_u^2+i\varepsilon}$$
$$\frac{[i(\not q_1+m_u)]_{\zeta\alpha}}{q_1^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_2+m_u)]_{\tilde\alpha\beta}}{q_2^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\varepsilon} \qquad (3.52)$$

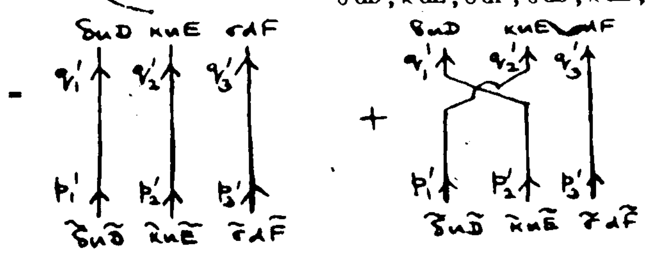
$$G_1^{(4)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$
$$= (i)^3\ \delta_{DC}\delta_{EB}\delta_{FA}\ C(duW^-)^{\lambda'}_{\eta\omega;\omega\zeta}\ C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\ C(uuA)_{\mu\nu\beta'}$$
$$(2\pi)^4\delta^4(q_2'-q_2)\ (2\pi)^4\delta^4(q_1'+q_3'+q_4'-q_1-q_3)$$
$$\frac{[i(\not q_1'+m_u)]_{\delta\mu}}{q_1'^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_2'+m_u)]_{\kappa\tilde\delta}}{q_2'^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_3'+m_d)]_{\sigma\eta}}{q_3'^2-m_d^2+i\varepsilon}\ \frac{-ig^{\mu'\beta'}}{q_4'^2+i\varepsilon}$$
$$[-i(\not q_2'-m_u)]_{\tilde\delta\tilde\alpha}\ \frac{[i(\not q_1'+\not q_4'+m_u)]_{\nu\xi}}{(q_1'+q_4')^2-m_u^2+i\varepsilon}\ \frac{-ig_{\nu'\lambda'}}{(q_1-q_3')^2-m_w^2+i\varepsilon}$$
$$\frac{[i(\not q_1+m_u)]_{\zeta\alpha}}{q_1^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_2+m_u)]_{\tilde\alpha\beta}}{q_2^2-m_u^2+i\varepsilon}\ \frac{[i(\not q_3+m_s)]_{\tau\gamma}}{q_3^2-m_s^2+i\varepsilon} \qquad (3.53)$$

We have now completed the calculation of $G_1^{(1)},\dots,G_1^{(4)}$ in momentum space. We now calculate the final and initial 6-point functions which have to be amputated. As mentioned earlier, we amputate the 6-point functions only in the lowest order. Following steps similar to those which we have followed above, we can calculate

$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\tilde\delta u\tilde D,\tilde\kappa u\tilde E,\tilde\sigma d\tilde F}$$

In the lowest order the expression is

$$-G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\tilde\delta u\tilde D,\tilde\kappa u\tilde E,\tilde\sigma d\tilde F}$$



$$= (-1)\,(2\pi)^4\delta^4(q_1'-p_1')\,\delta_{D\tilde D}\,(2\pi)^4\delta^4(q_2'-p_2')\,\delta_{E\tilde E}\,(2\pi)^4\delta^4(q_3'-p_3')\,\delta_{F\tilde F}$$

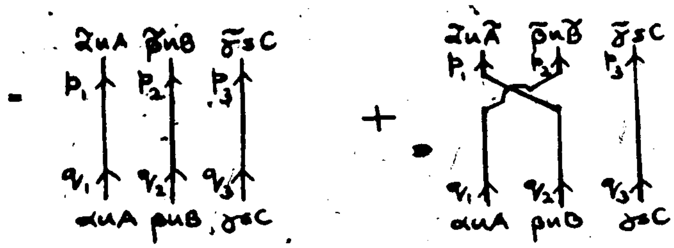
$$\frac{[i(\not p_1'+m_u)]_{\delta\tilde\delta}}{p_1'^2-m_u^2+i\epsilon}\;\frac{[i(\not p_2'+m_u)]_{\kappa\tilde\kappa}}{p_2'^2-m_u^2+i\epsilon}\;\frac{[i(\not p_3'+m_d)]_{\sigma\tilde\sigma}}{p_3'^2-m_d^2+i\epsilon}$$

$$+ (+1)\,(2\pi)^4\delta^4(q_1'-p_2')\,\delta_{D\tilde E}\,(2\pi)^4\delta^4(q_2'-p_1')\,\delta_{E\tilde D}\,(2\pi)^4\delta^4(q_3'-p_3')\,\delta_{F\tilde F}$$

$$\frac{[i(\not p_2'+m_u)]_{\delta\tilde\kappa}}{p_2'^2-m_u^2+i\epsilon}\;\frac{[i(\not p_1'+m_u)]_{\kappa\tilde\delta}}{p_1'^2-m_u^2+i\epsilon}\;\frac{[i(\not p_3'+m_d)]_{\sigma\tilde\sigma}}{p_3'^2-m_d^2+i\epsilon} \qquad (3.54)$$

Similarly, the initial 6-point function is given by

$$-G(p_1,p_2,p_3;q_1,q_2,q_3)_{\tilde\alpha u\tilde A,\tilde\beta u\tilde B,\tilde\gamma s\tilde C;\alpha uA,\beta uB,\gamma sC}$$



$$= (-1)\ (2\pi)^4\delta^4(p_1-q_1)\ \delta_{\tilde{A}A}\ (2\pi)^4\delta^4(p_2-q_2)\ \delta_{\tilde{B}B}\ (2\pi)^4\delta^4(p_3-q_3)\ \delta_{\tilde{C}C}$$

$$\frac{[i(\not p_1+m_u)]_{\tilde{\alpha}\alpha}}{p_1^2-m_u^2+i\epsilon}\ \frac{[i(\not p_2+m_u)]_{\tilde{\beta}\beta}}{p_2^2-m_u^2+i\epsilon}\ \frac{[i(\not p_3+m_s)]_{\tilde{\gamma}\gamma}}{p_3^2-m_s^2+i\epsilon}$$

$$+ (+1)\ (2\pi)^4\delta^4(p_1-q_2)\ \delta_{\tilde{A}B}\ (2\pi)^4\delta^4(p_2-q_1)\ \delta_{\tilde{B}A}\ (2\pi)^4\delta^4(p_3-q_3)\ \delta_{\tilde{C}C}$$

$$\frac{[i(\not p_1+m_u)]_{\tilde{\alpha}\beta}}{p_1^2-m_u^2+i\epsilon}\ \frac{[i(\not p_2+m_u)]_{\tilde{\beta}\alpha}}{p_2^2-m_u^2+i\epsilon}\ \frac{[i(\not p_3+m_s)]_{\tilde{\gamma}\gamma}}{p_3^2-m_s^2+i\epsilon} \qquad (3.55)$$

Also in the lowest order, from eqn. (3.32)

$$G(q_4',p_4')^{\mu'\tilde{\mu}'} = (2\pi)^4\delta^4(q_4'-p_4')\ \frac{-ig^{\mu'\tilde{\mu}'}}{p_4'^2+i\epsilon} \qquad (3.56)$$

From eqns. (3.50)-(3.56) after a tedious but straightforward calculation, it is easy to show that

$$G_1^{(1)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$+ \ldots + G_1^{(4)}(q_1',q_2',q_3';q_4';q_1,q_2,q_3)^{\mu'}_{\delta uD,\kappa uE,\sigma dF;\alpha uA,\beta uB,\gamma sC}$$

$$= 9\int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\frac{d^4p_3'}{(2\pi)^4}\frac{d^4p_4'}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\ G(q_4',p_4')^{\mu'\tilde{\mu}'}$$

$$G(q_1',q_2',q_3';p_1',p_2',p_3')_{\delta uD,\kappa uE,\sigma dF;\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F}}$$

$$\{\ \frac{1}{9}(i)^3\ \delta_{\widetilde{DA}}\ \delta_{\widetilde{EC}}\ \delta_{\widetilde{FB}}\ \ C(du\ W^-)^{\lambda'}_{\eta\omega;\omega\zeta}\ \ C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tau}\ \ C(uuA)_{\mu\nu\beta'}$$

$$(2\pi)^4\delta^4(p_1'-p_1)\ (2\pi)^4\delta^4(p_2'+p_3'+p_4'-p_2-p_3)\ \delta_{\tilde{\kappa}\mu}\ \delta_{\tilde{\sigma}\eta}\ \delta^{\beta'}_{\tilde{\mu}'}$$

$$[-i(\not{p}_1'-m_u)]_{\tilde{\delta}\tilde{\alpha}} \quad \frac{[i(\not{p}_2'+\not{p}_4'+m_u)]_{\nu\xi}}{(p_2'+p_4')^2-m_u^2+i\varepsilon} \quad \frac{-ig_{\nu'\lambda'}}{(p_2-p_3')^2-m_w^2+i\varepsilon} \quad \delta_{\xi\tilde{\beta}} \quad \delta_{\tau\tilde{\gamma}} \Big\}$$

$$G(p_1,p_2,p_3;q_1,q_2,q_3)_{\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C};\alpha uA,\beta uB,\gamma sC} \tag{3.57}$$

And therefore, from equation (3.38) and (3.57), by definition

$$\sigma_1(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F};\tilde{\mu}';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}}$$

$$= \frac{1}{9}\,(i)^3\,\delta_{\tilde{D}\tilde{A}}\,\delta_{\tilde{E}\tilde{C}}\,\delta_{\tilde{F}\tilde{B}}\quad C(duW^-)^{\lambda'}_{\tilde{\sigma}\omega;\omega\tilde{\beta}}\quad C(usW^+)^{\nu'}_{\xi\lambda;\lambda\tilde{\gamma}}\quad C(uuA)_{\tilde{\kappa}\nu\tilde{\mu}'}$$

$$(2\pi)^4\delta^4(p_1'-p_1)\;(2\pi)^4\delta^4(p_2'+p_3'+p_4'-p_2-p_3)$$

$$[-i(\not{p}_1'-m_u)]_{\tilde{\delta}\tilde{\alpha}} \quad \frac{[i(\not{p}_2'+\not{p}_4'+m_u)]_{\nu\xi}}{(p_2'+p_4')^2-m_u^2+i\varepsilon} \quad \frac{-ig_{\nu'\lambda'}}{(p_2-p_3')^2-m_w^2+i\varepsilon} \tag{3.58}$$

The form of eqn. (3.58) immediately suggests that we can write this expression down by drawing a Feynman diagram for σ_1 and following a definite set of Feynman rules. This is what we point out next. This will make detailed calculations for $\sigma_2, \ldots, \sigma_{28}$ unnecessary.

Eqn. (3.58) corresponds to the Feynman diagram in Fig. 3.6.

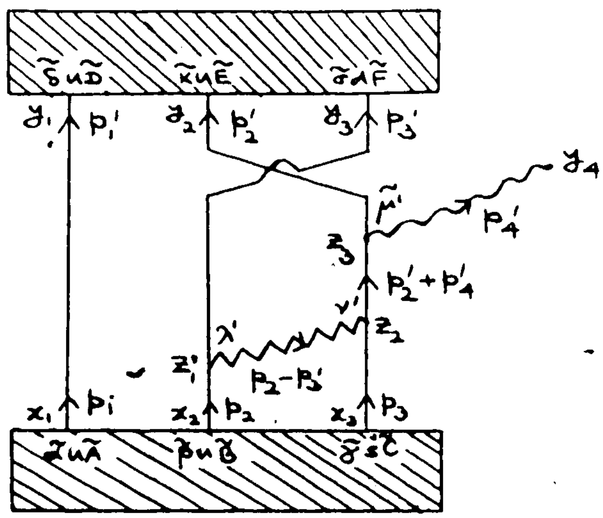


Fig. 3.6 Feynman diagram for writing down

$$\sigma_1(p_1',p_2',p_3';p_4';p_1,p_2,p_3)_{\tilde{\delta}u\tilde{D},\tilde{\kappa}u\tilde{E},\tilde{\sigma}d\tilde{F};\tilde{\mu}';\tilde{\alpha}u\tilde{A},\tilde{\beta}u\tilde{B},\tilde{\gamma}s\tilde{C}}$$

The Feynman rules are the usual ones except for the following changes:

(1) a symmetry factor of $\frac{1}{9}$ is to be included due to the identity of two initial and two final quarks;

(2) The hatched boxes in Fig. 3.6 denote the bound states or BS amplitudes. A spectator line (i.e. the one which joins the two BS amplitudes and is not connected to any vertex) contributes the factor

$$(2\pi)^4\delta^4(p-q)[-i(\not{p}-m)]_{\delta\alpha}$$

where p and q are the initial and final momenta, m is the mass of the spectator quark and δ, α are the initial and final Dirac spinor indices;

(3) lines connecting the vertices with bound states do not receive any contribution;

(4) no permutation of identical initial or final quark legs is to be taken into account;

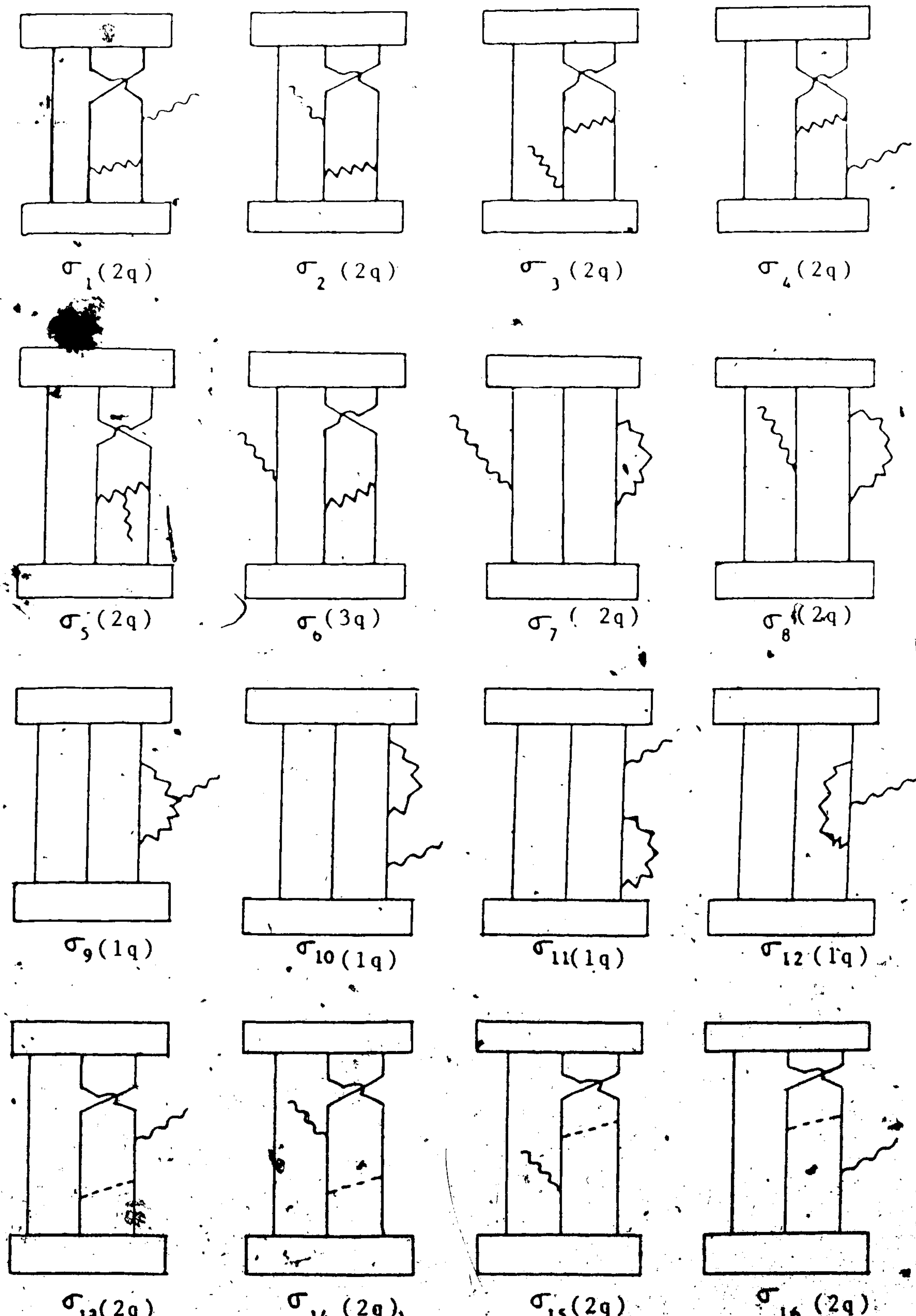
(5) for deciding the sign of the contribution, we observe the following. Three quarks are being annihilated at y_1, y_2, y_3 ; three quarks are being created at x_1, x_2, x_3 ; one quark each is being annihilated and created at z_1, z_2, z_3 . This diagram comes from the Wick's contraction of the following vacuum expectation value (VEV)

$$\langle 0|T\ \psi(y_1)_{in}\ \psi(y_2)_{in}\ \psi(y_3)_{in}$$
$$\bar{\psi}(x_1)_{in}\ \bar{\psi}(x_2)_{in}\ \bar{\psi}(x_3)_{in}\ \bar{\psi}(z_1)_{in}\ \psi(z_1)_{in}\ \bar{\psi}(z_2)_{in}\ \psi(z_2)_{in}\ \bar{\psi}(z_3)_{in}\ \psi(z_3)_{in}$$
$$\ldots|0\rangle \qquad (3.59)$$

One takes this VEV, and does the required contractions to obtain the diagram in Fig. 3.6. Whatever sign one picks up in this process is the sign of the diagram in Fig. 3.6. Similar procedure will have to be followed for other diagrams. Method should be obvious by now.

We now only need to draw the Feynman diagrams corresponding to each of the 28 σ_i's and then follow the Feynman rules explained above to obtain the required expression.

The diagrams corresponding to different σ_i's are drawn in Fig. 3.7. In order to be tidy, we do not label the diagrams in Fig. 3.7. It is not difficult to label various momenta, spacetime points, indicies, etc. by looking at the example in Fig. 3.6. Also, in brackets, we have indicated the nature of quark diagram i.e., whether it is a 1-quark, 2-quark or 3-quark diagram.



σ_1 (2q)
σ_2 (2q)
σ_3 (2q)
σ_4 (2q)
σ_5 (2q)
σ_6 (3q)
σ_7 (2q)
σ_8 (2q)
σ_9 (1q)
σ_{10} (1q)
σ_{11} (1q)
σ_{12} (1q)
σ_{13} (2q)
σ_{14} (2q)
σ_{15} (2q)
σ_{16} (2q)

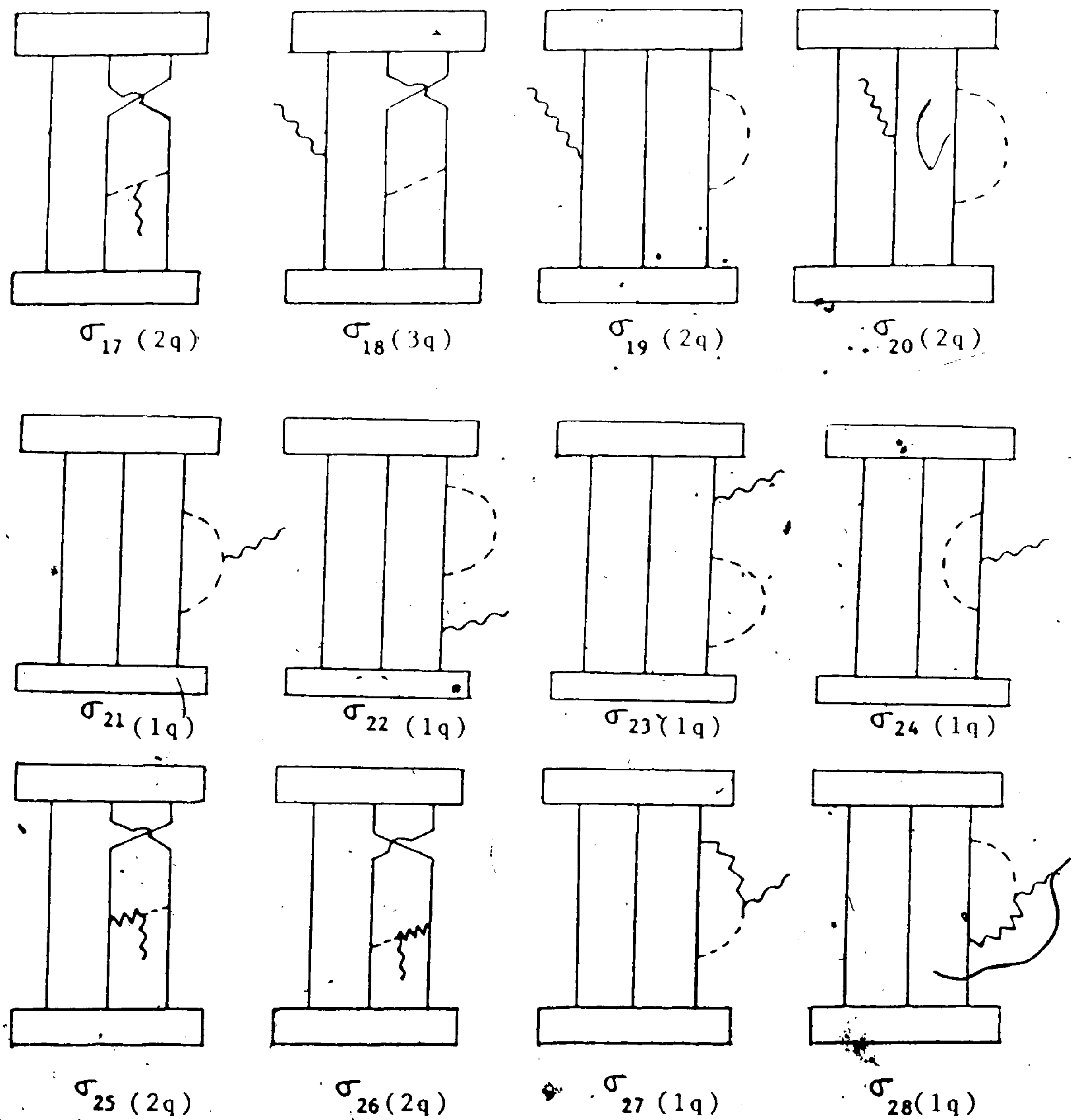


Fig. 3.7 Feynman diagrams for various σ_i's. As before, solid lines are quark lines; sawtooth lines are W lines; wavy lines are photon lines; dashed lines are the would-be Goldstones.

Following our Feynman rules, we can write down the expressions for various σ_i's. However, just by looking at the diagrams and vertex factors in Appendix C, we can immediately conclude that some of the diagrams are much smaller as compared to others. We shall neglect those in our calculations. Let us weed out those diagrams. For example, we observe that the momentum structures of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are exactly the same as those of $\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}$. The quark-quark-Goldstone vertex in $\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}$ are suppressed by $(\sim m_q^2/m_W^2)$ as compared to quark-quark-W vertex in $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ where m_q is a generic quark mass. So, $\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}$ can be neglected in comparison to $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Similarly, σ_{18} can be neglected in comparison to σ_6 for the same reason. Also, diagrams $\sigma_5, \sigma_{17}, \sigma_{25}$ and σ_{26} have two W-propagators in comparison to only one in other diagrams without loops. So, they are also suppressed by $(\sim 1/m_W^2)$. Expressions for other σ_i's are listed below. We suppress the argument and indices of σ_i's.

$$\sigma_1 = \frac{1}{9}(i)^3\, \delta_{DA}\delta_{EC}\delta_{FB}(i)[C(uuA)_{\kappa\nu\mu}, C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma} C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta}(-g_{\nu'\lambda'})]$$
$$(2\pi)^4\delta^4(p_1'-p_1)\ (2\pi)^4\delta^4(p_2'+p_3'+k'-p_2-p_3)$$
$$[-i(\not{p}_1'-m_u)]_{\delta\alpha}\ \frac{[i(\not{p}_2'+\not{k}'+m_u)]_{\nu\xi}}{(p_2'+k')^2-m_u^2+i\epsilon}\ \frac{1}{(p_2'-p_3')^2-m_W^2+i\epsilon} \qquad (3.60)$$

$$\sigma_2 = \frac{1}{9}(i)^3\, \delta_{DA}\delta_{EC}\delta_{FB}(i)[C(usW^+)^{\nu'}_{\kappa\eta;\eta\gamma} C(ddA)_{\sigma\xi\mu}, C(duW^-)^{\lambda'}_{\nu\omega;\omega\beta}(-g_{\nu'\lambda'})]$$
$$(2\pi)^4\delta^4(p_1'-p_1)\ (2\pi)^4\delta^4(p_2'+p_3'+k'-p_2-p_3)$$
$$[-i(\not{p}_1'-m_u)]_{\delta\alpha}\ \frac{[i(\not{p}_3'+\not{k}'+m_d)]_{\xi\nu}}{(p_3'+k')^2-m_d^2+i\epsilon}\ \frac{1}{(p_2'-p_3')^2-m_W^2+i\epsilon} \qquad (3.61)$$

$$\sigma_3 = \frac{1}{9}(i)^3 \delta_{DA}\delta_{EC}\delta_{FB}(i)[C(usW^+)^{\nu'}_{\kappa\eta;\eta\gamma}C(duW^-)^{\lambda'}_{\sigma\omega;\omega\xi}C(uuA)_{\nu\beta\mu'}(-g_{\nu'\lambda'})]$$
$$(2\pi)^4\delta^4(p_1'-p_1)\ (2\pi)^4\delta^4(p_2'+p_3'+k'-p_2-p_3)$$
$$[-i(\not p_1'-m_u)]_{\delta\alpha}\ \frac{[i(\not p_2-\not k'+m_u)]_{\xi\nu}}{(p_2-k')^2-m_u^2+i\epsilon}\ \frac{1}{(p_2'-p_3)^2-m_w^2+i\epsilon} \tag{3.62}$$

$$\sigma_4 = \frac{1}{9}(i)^3 \delta_{DA}\delta_{EC}\delta_{FB}(i)[C(usW^+)^{\nu'}_{\kappa\eta;\eta\nu}C(ssA)_{\xi\gamma\mu'}C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta}(-g_{\nu'\lambda'})]$$
$$(2\pi)^4\delta^4(p_1'-p_1)\ (2\pi)^4\delta^4(p_2'+p_3'+k'-p_2-p_3)$$
$$[-i(\not p_1'-m_u)]_{\delta\alpha}\ \frac{[i(\not p_3'-\not k'+m_s)]_{\nu\xi}}{(p_3-k')^2-m_s^2+i\epsilon}\ \frac{1}{(p_2-p_3')^2-m_w^2+i\epsilon} \tag{3.63}$$

$$\sigma_6 = \frac{1}{9}(i)^3 \delta_{DA}\delta_{EC}\delta_{FB}C(uuA)_{\delta\alpha\mu'}[C(usW^+)^{\nu'}_{\kappa\omega;\omega\gamma}C(duW^-)^{\lambda'}_{\sigma\eta;\eta\beta}(-ig_{\nu'\lambda'})]$$
$$(2\pi)^4\delta^4(p_1-p_1'-k')\ (2\pi)^4\delta^4(p_2+p_3-p_2'-p_3')$$
$$\frac{1}{(p_2-p_3')^2-m_w^2+i\epsilon} \tag{3.64}$$

We can combine σ_7 with σ_{19} and σ_8 with σ_{20} to expedite the calculations

$$\sigma_7+\sigma_{19} = -\frac{1}{9}\delta_{DA}\delta_{EB}\delta_{FC}U_{d\ell}U^\dagger_{\ell s}(2\pi)^4\delta^4(p_1'+k'-p_1)$$
$$C(uuA)_{\delta\alpha\mu'}(2\pi)^4\delta^4(p_2'-p_2)(\not p_2-m_u)_{\kappa\beta}\ (2\pi)^4\delta^4(p_3'-p_3)$$
$$\{[E_1(p_3)(\gamma_{\beta'})_{\sigma\zeta}(L)_{\zeta\gamma}+E_2(p_3)(\gamma_{\beta'})_{\sigma\zeta}(R)_{\zeta\gamma}]p_3^{\beta'}+(E_3(p_3)L+E_4(p_3)R)_{\sigma\gamma}\} \tag{3.65}$$

$$\sigma_8+\sigma_{20} = -\frac{1}{9}\delta_{DA}\delta_{EB}\delta_{FC}U_{d\ell}U^\dagger_{\ell s}(2\pi)^4\delta^4(p_1'-p_1)(\not p_1-m_u)_{\delta\alpha}$$
$$(2\pi)^4\delta^4(p_2'+k'-p_2)C(uuA)_{\kappa\beta\mu'}(2\pi)^4\delta^4(p_3'-p_3)$$
$$\{[E_1(p_3)(\gamma_{\beta'})_{\sigma\zeta}(L)_{\zeta\gamma}+E_2(p_3)(\gamma_{\beta'})_{\sigma\zeta}(R)_{\zeta\gamma}]p_3^{\beta'}+(E_3(p_3)L+E_4(p_3)R)_{\sigma\gamma}\} \tag{3.66}$$

Both in eqns. (3.65) and (3.66), there is a sum over 'ℓ', viz. the quark flavour in the self-energy loop. $E_1(p_3),\ldots,E_4(p_3)$ come from the renormalized self-energy calculations for flavour-changing single quark transitions. We follow the calculations of Deshpande and Eilam [63]. The correspondence with Deshpande and Eilam is as follows:

$$\begin{aligned} E_1(p_3) &= (f+C) \text{ in Deshpande-Eilam paper} \\ E_2(p_3) &= (h+D) \text{ in Deshpande-Eilam paper} \\ E_3(p_3) &= (\psi+A) \text{ in Deshpande-Eilam paper} \\ E_4(p_3) &= (\phi+B) \text{ in Deshpande-Eilam paper} \end{aligned} \tag{3.67}$$

We have changed the symbols to avoid confusion. Also, R and L stand for

$$R = \left(\frac{1+\gamma_5}{2}\right) \qquad L = \left(\frac{1-\gamma_5}{2}\right) \tag{3.68}$$

It is to be noted that $E_i(p_3)$'s depend on 'ℓ' through the properties of the internal quark in the loop.

We can combine all ℓ-quark contributions into one

$$\sigma_9 + \cdots + \sigma_{12} + \sigma_{21} + \cdots + \sigma_{24} + \sigma_{27} + \sigma_{28} = -\frac{1}{9}\delta_{DA}\delta_{EB}\delta_{FC}U_{d\ell}U^{\dagger}_{\ell s}(2\pi)^4\delta^4(p_1'-p_1)[-i(\not p_1-m_u)]_{\delta\alpha}$$

$$(2\pi)^4\delta^4(p_2'-p_2)[-i(\not p_2-m_u)]_{\kappa\beta}(2\pi)^4\delta^4(p_3'+k'-p_3)V^{\mu'}(p_3)_{\sigma\gamma} \tag{3.69}$$

$V^{\mu'}(p_3)$ in eqn. (3.69) is the renormalized flavour-changing radiative vertex calculated by Deshpande and Eilam [63] and by Ma and Pramudita

[64], we follow the work of Deshpande and Eilam. They list the vertex in terms of integrals over Feynman parameters. Calculation of those Feynman-parameter integrals has been outlined in Appendix D. $V^{\mu'}(p_3)$ is also dependent on 'ℓ' through the properties of the internal quark in the loop.

This completes the calculation of the required amputated Green functions.

3.3. Calculation of the unnormalized $\Sigma^+ \to p\gamma$ Amplitude

Like σ, we should also calculate the S-matrix element for $\Sigma^+ \to p\gamma$ in pieces. Put eqn. (3.40) into eqn. (3.25) to obtain

$$W(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} = \sum_{j=1}^{28} W_j(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} \tag{3.70}$$

where

$$\begin{aligned} W_j(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} &\equiv \int[\Pi \frac{d^4p}{(2\pi)^4}]\, \delta^4[P'-\mathcal{P}'(p_1',p_2',p_3')]\delta^4[P-\mathcal{P}(p_1,p_2,p_3)] \\ &\times[p_\xi'(f_u'p_1';f_u'p_2';f_d'p_3'),p_\eta'(f_u'p_1';f_u'p_2';f_d'p_3')]^{P'} \\ &\sigma_j(p_1',p_2',p_3';k';p_1,p_2,p_3)_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} \\ &\times[p_\xi(f_up_1;f_up_2;f_sp_3),p_\eta(f_up_1;f_up_2;f_sp_3)]^{P} \end{aligned} \tag{3.71}$$

Putting eqn. (3.71) in eqn. (3.24), we obtain

$$S_{\Sigma^+\to p\gamma} = \sum_{j=1}^{28} \mathcal{A}_j(P,P',k') \tag{3.72}$$

where

$$\mathcal{A}_j(P,P',k') \equiv \left[\frac{Z_3^{1/2}}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$
$$\left[\left(\frac{-1}{6}\right) \epsilon_{DEF}\epsilon_{ABC}\right] D_{\delta\kappa\sigma\alpha\beta\gamma}\ \epsilon^{\mu'*}(k',\lambda')$$
$$W_j(P,P',k')_{\delta uD,\kappa uE,\sigma dF;\mu';\alpha uA,\beta uB,\gamma sC} \qquad (3.73)$$

$D_{\delta\kappa\sigma\alpha\beta\gamma}$ in eqn. (3.73) has already been defined in eqn. (3.26). Our next job is to calculate $\mathcal{A}_j$'s. As before, we shall show the details of calculation in one particular case and then quote the result for the others. Let us consider the calculation of $\mathcal{A}_1$ in detail.

From eqn. (3.71)

$$W_1 = \int\left[\Pi \frac{d^4p}{(2\pi)^4}\right] \delta^4[P' - \mathcal{P}(p_1',p_2',p_3')]\ \delta^4[P - \mathcal{P}(p_1,p_2,p_3)]$$
$$\chi[p_\xi'(f_u'p_1';f_u'p_2';f_d'p_3'),p_\eta'(f_u'p_1';f_u'p_2';f_d'p_3')]^{P'}$$
$$(\sigma_1)$$
$$\chi[p_\xi(f_up_1;f_up_2;f_sp_3),p_\eta(f_up_1;f_up_2;f_sp_3)]^{P} \qquad (3.73a)$$

From eqns. (3.16) and (3.9), the explicit functional forms of χ's in eqn. (3.73a) are

$$\chi[p_\xi'(f_u'p_1';f_u'p_2';f_d'p_3'),p_\eta'(f_u'p_1';f_u'p_2';f_d'p_3')]^{P'}$$

$$= \exp\left\{-\frac{1}{2\alpha'^2}\left[8\left(\frac{P'\cdot p_\xi'}{M'}\right)^2 - 4p_\xi'^2 + 6\left(\frac{P'\cdot p_\eta'}{M'}\right)^2 - 3p_\eta'^2\right]\right\} \qquad (3.73b)$$

and

$$\chi[p_\xi(f_up_1;f_up_2;f_sp_3),p_\eta(f_up_1;f_up_2;f_sp_3)]^{P}$$

$$= \exp\{-\frac{1}{2\alpha^2}\left[8\left(\frac{P\cdot p_\xi}{M}\right)^2 - 4p_\xi^2 + 6\left(\frac{P\cdot p_\eta}{M}\right)^2 - 3p_\eta^2\right]\} \tag{3.74}$$

From eqns. (3.10)-(3.12), $(\mathcal{P}', p'_\xi, p'_\eta)$ are related to (p'_1, p'_2, p'_3) by

$$\begin{aligned} p'_1 &= f'_u \mathcal{P}' + p'_\xi + \frac{1}{2} p'_\eta \\ p'_2 &= f'_u \mathcal{P}' - p'_\xi + \frac{1}{2} p'_\eta \\ p'_3 &= f'_d \mathcal{P}' - p'_\eta \end{aligned} \tag{3.75}$$

where

$$f'_u = \frac{m_u}{m_u + m_u + m_d} \quad ; \quad f'_d = \frac{m_d}{m_u + m_u + m_d} \tag{3.76}$$

and

$$f'_u + f'_u + f'_d = 2f'_u + f'_d = 1 \tag{3.77}$$

Similarly, from eqns. (3.3)-(3.5), $(\mathcal{P}, p_\xi, p_\eta)$ are related to (p_1, p_2, p_3) by

$$\begin{aligned} p_1 &= f_u \mathcal{P} + p_\xi + \frac{1}{2} p_\eta \\ p_2 &= f_u \mathcal{P} - p_\xi + \frac{1}{2} p_\eta \\ p_3 &= f_s \mathcal{P} - p_\eta \end{aligned} \tag{3.78}$$

where

$$f_u = \frac{m_u}{m_u + m_u + m_s} \quad ; \quad f_s = \frac{m_s}{m_u + m_u + m_s} \tag{3.79}$$

and

$$f_u + f_u + f_s = 2f_u + f_s = 1 \tag{3.80}$$

Put the expression for σ_1 from eqn. (3.60) and $\chi[\ldots]^{P'}$ and $\chi[\ldots]^{P}$ from eqns. (3.73) and (3.74) in eqn. (3.72).

$$W_1 = \frac{1}{9}\ (i)^3\ \delta_{DA}\delta_{EC}\delta_{FB}(i)$$

$$[C(uuA)_{\kappa\nu\mu}, \; C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma} \; C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta} \; (-g_{\nu'\lambda'})]$$

$$\int \frac{d^4p_1'}{(2\pi)^4} \frac{d^4p_2'}{(2\pi)^4} \frac{d^4p_3'}{(2\pi)^4} \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \cdots$$

$$\delta^4[P' - \mathcal{P}'(p_1', p_2', p_3')] \; \delta^4[P - \mathcal{P}(p_1, p_2, p_3)]$$

$$(2\pi)^4 \delta^4(p_1' - p_1) \; (2\pi)^4 \delta^4(p_2' + p_3' + k' - p_2 - p_3)$$

$$\exp\{-\frac{1}{2\alpha'^2}[8(\frac{P'\cdot p_\xi'}{M'})^2 - 4p_\xi'^2 + 6(\frac{P'\cdot p_\eta'}{M'})^2 - 3p_\eta'^2]\}$$

$$[-i(\not{p}_1' - m_u)]_{\delta\alpha} \; \frac{[i(\not{p}_2' + \not{k}' + m_u)]_{\nu\xi}}{(p_2' + k')^2 - m_u^2 + i\epsilon} \qquad \frac{1}{(p_2 - p_3')^2 - m_W^2 + i\epsilon}$$

$$\exp\{-\frac{1}{2\alpha^2}[8(\frac{P\cdot p_\xi}{M})^2 - 4p_\xi^2 + 6(\frac{P\cdot p_\eta}{M})^2 - 3p_\eta^2]\} \qquad (3.81)$$

We now manipulate the expression in eqn. (3.81) in the following manner.

(i) We change the integration variables from (p_1', p_2', p_3') to $(\mathcal{P}', p_\xi', p_\eta')$ and from (p_1, p_2, p_3) to $(\mathcal{P}, p_\xi, p_\eta)$.

(ii) We do the $\mathcal{P}$, $\mathcal{P}'$ and p_η integrals using three of the delta functions. We are then left with the overall delta function $\delta^4(P' + k' - P)$.

(iii) We are left with integrals over p_ξ', p_η' and p_ξ. We then transform (p_ξ', p_η') to (p, q) according to the following equations

$$p = f_u'P' + p_\xi' + \frac{1}{2} p_\eta' + k'$$
$$q = f_u'P' - p_\xi' + \frac{1}{2} p_\eta' + k' \qquad (3.82)$$

The Jacobian of this tranformation is equal to one. p_ξ is left unchanged by this transformation.

After these tedious manipulations, eqn. (3.81) assumes the following form.

$$W_1 = \frac{1}{9} (i)^3 \frac{1}{(2\pi)^{20}} \delta_{DA}\delta_{EC}\delta_{FB} (2)^4 (i)$$

$$[C(uuA)_{\kappa\nu\mu}, C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma} C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta} (-g_{\nu'\lambda'})]$$

$$(2\pi)^4\delta^4(P'+k'-P)\int d^4p d^4q d^4p_\xi$$

$$\exp\{-\frac{1}{2\alpha'^2} (8[\frac{P'\cdot\frac{1}{2}(p-q)}{M'}]^2 - 4[\frac{1}{2}(p-q)]^2 + 6[\frac{P'\cdot(p+q-2f_u'P'-2k')}{M'}]^2$$
$$- 3(p+q-2f_u'P'-2k')^2)\}$$

$$[-i(\not p-\not k'-m_u)]_{\delta\alpha} \frac{[i(\not q+m_u)]_{\nu\xi}}{q^2-m_u^2+i\epsilon}$$

$$\frac{1}{[(f_u'-f_d')P'+\frac{1}{2}(p-q)+\frac{3}{2}(p+q-2f_u'P'-2k')-2p_\xi]^2-m_W^2+i\epsilon}$$

$$\exp\{-\frac{1}{2\alpha^2}(8[\frac{P\cdot p_\xi}{M}]^2 - 4p_\xi^2 + 24[\frac{P\cdot(p-k'-f_u P-p_\xi)}{M}]^2 - 12(p-k'-f_u P-p_\xi)^2)\}. \tag{3.83}$$

We realize that m_w is much-much bigger than P, P' or any of the relative momenta. So, in eqn. (3.83), we can very safely approximate

$$\frac{1}{[(f_u'-f_d')P'+\frac{1}{2}(p-q)+\frac{3}{2}(p+q-2f_u'P'-2k')-2p_\xi]^2-m_w^2+i\epsilon} \simeq (-\frac{1}{m_w^2}) \tag{3.84}$$

Such a thing, however, cannot be done for the u-quark propagator because the relative momentum q meets the singularity well within its cutoff value provided by the BS amplitudes. We use the Schwinger representation [38-40]

$$\frac{1}{q^2-m_u^2+i\epsilon} = -i\int_0^\infty dz\; e^{iz(q^2-m_u^2+i\epsilon)} , \tag{3.85}$$

for the u-quark propagator. This helps us to do the momentum integrals exactly. The z-integral is finally managed numerically as outlined in Appendix E.

Making these two substitutions in eqn. (3.83), we obtain

$$W_1 = \frac{1}{9}(i)^3 \frac{1}{(2\pi)^{20}} \delta_{DA}\delta_{EC}\delta_{FB} (2)^4$$

$$[C(uuA)_{\kappa\nu\mu}, C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma} C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta} (-g_{\nu'\lambda'})]$$

$$(2\pi)^4\delta^4(P'+k'-P)\,(-\frac{1}{m_w^2})$$

$$[(\gamma_{\alpha'})_{\delta\alpha}(\gamma_{\beta'})_{\nu\xi} I_1^{\alpha'\beta'} + (\gamma_{\alpha'})_{\delta\alpha}(m_u)_{\nu\xi} I_2^{\alpha'}$$

$$- (k'+m_u)_{\delta\alpha} (\gamma_{\alpha'})_{\nu\xi} I_3^{\alpha'} - (k'+m_u)_{\delta\alpha} (m_u)_{\nu\xi} T_1] \qquad (3.86)$$

where

$$I_1^{\alpha'\beta'} = \int_0^\infty dz\, e^{iz(-m_u^2+i\varepsilon)} \int d^4p\, d^4q\, d^4p_\xi \; p^{\alpha'} q^{\beta'} e^{\mathcal{E}_1} \qquad (3.87)$$

$$I_2^{\alpha'} = \int_0^\infty dz\, e^{iz(-m_u^2+i\varepsilon)} \int d^4p\, d^4q\, d^4p_\xi \; p^{\alpha'} e^{\mathcal{E}_1} \qquad (3.88)$$

$$I_3^{\alpha'} = \int_0^\infty dz\, e^{iz(-m_u^2+i\varepsilon)} \int d^4p\, d^4q\, d^4p_\xi \; q^{\alpha'} e^{\mathcal{E}_1} \qquad (3.89)$$

$$T_1 = \int_0^\infty dz\, e^{iz(-m_u^2+i\varepsilon)} \int d^4p\, d^4q\, d^4p_\xi \; e^{\mathcal{E}_1} \qquad (3.90)$$

$\mathcal{E}_1$ in eqns. (3.87)-(3.90) stands for the following quantity.

$$\mathcal{E}_1 \equiv -\frac{1}{2\alpha'^2}\left(2\left[\frac{P'\cdot(p-q)}{M'}\right]^2 - (p-q)^2 + 6\left[\frac{P'\cdot(p+q-2f_u'P'-2k')}{M'}\right]^2 - 3(p+q-2f_u'P'-2k')^2\right)$$

$$+izq^2$$

$$-\frac{1}{2\alpha^2}\left(8\left[\frac{P\cdot p_\xi}{M}\right]^2 - 4p_\xi^2 + 24\left[\frac{P\cdot(p-k'-f_uP-p_\xi)}{M}\right]^2 - 12(p-k'-f_uP-p_\xi)^2\right) \qquad (3.91)$$

As discussed in Appendix E, the integrals in eqns. (3.87)-(3.90) have the following Lorentz sturcture.

$$T_1 = \text{a Lorentz scalar} \qquad (3.92)$$

$$I_3^{\alpha'} = T_2P^{\alpha'} + T_3k'^{\alpha'} \qquad (3.93)$$

$$I_2^{\alpha'} = T_4P^{\alpha'} + T_5k'^{\alpha'} \qquad (3.94)$$

$$I_1^{\alpha'\beta'} = T_6g^{\alpha'\beta'} + T_7P^{\alpha'}P^{\beta'} + T_8k'^{\alpha'}k'^{\beta'}$$

$$+ T_9(P^{\alpha'}k'^{\beta'}+P^{\beta'}k'^{\alpha'}) + T_{10}(P^{\alpha'}k'^{\beta'}-P^{\beta'}k'^{\alpha'})$$
$$+ T_{11}\epsilon^{\alpha'\beta'\gamma'\delta'}(P_{\kappa'}k'_{\sigma'}-P_{\sigma'}k'_{\kappa'}) \qquad (3.95)$$

The scalar coefficients $T_1,\ldots,T_{11}$ can be calculated following the procedure outlined in Appendix E. The expression for W_1 is, then, known.

Having calculated W_1, we proceed to calculate $\mathcal{A}_1$. From eqn. (3.73),

$$\mathcal{A}_1 \equiv \left[\frac{Z_3^{1/2}}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^{*}N_\Sigma + \frac{1}{2P'^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\frac{1}{2P^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$
$$\cdot\left[\left(\frac{-1}{6}\right)\epsilon_{DEF}\,\epsilon_{ABC}\right]D_{\delta\kappa\sigma\alpha\beta\gamma}\,\epsilon^{\mu'*}(k',\lambda')W_1 \qquad (3.96)$$

where we have suppressed the arguments and indices of $\mathcal{A}_1$ and W_1 for simplicity in writing. We proceed in a manner outlined by the following steps.

(i) We put the expression for W_1 from eqn. (3.86) in eqn. (3.96).

(ii) We put $Z_3 = 1$ because no renormalization corrections to the outgoing photon leg were considered.

(iii) The colour factors in $\mathcal{A}_1$ are $\left[\left(\frac{-1}{6}\right)\epsilon_{DEF}\epsilon_{ABC}\right]\delta_{DA}\delta_{EC}\delta_{FB}$. We can easily carry out this colour sum.

$$\left[\left(\frac{-1}{6}\right)\epsilon_{DEF}\epsilon_{ABC}\right]\delta_{DA}\delta_{EC}\delta_{FB} = \frac{1}{6}\epsilon_{ABC}\epsilon_{ABC} = \frac{1}{6}(3!) = 1 \qquad (3.97)$$

The expression for $\mathcal{A}_1$ at this stage looks like

$$\mathcal{A}_1 = i(2\pi)^4\delta^4(P'+k'-P)$$

$$[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}][N_p {}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$

$$\frac{1}{9} \frac{1}{(2\pi)^{20}} (2)^4 (\frac{1}{m_W^2})$$

$$\{D_{\delta\kappa\sigma\alpha\beta\gamma} \varepsilon^{\mu'*}(k',\lambda')$$

$$C(uuA)_{\kappa\nu\mu'} C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma} C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta}(-g_{\nu'\lambda'})$$

$$[(\gamma_{\alpha'})_{\delta\alpha}(\gamma_{\beta'})_{\nu\xi} I_1^{\alpha'\beta'} + (\gamma_{\alpha'})_{\delta\alpha}(m_u)_{\nu\xi} I_2^{\alpha'}$$

$$- (\not{k}'+m_u)_{\delta\alpha}(\gamma_{\alpha'})_{\nu\xi} I_3^{\alpha'} - (\not{k}'+m_u)_{\delta\alpha}(m_u)_{\nu\xi} T_1] \qquad (3.98)$$

After this stage, we do the following manipulations.

(i) We put the expression for $D_{\delta\kappa\sigma\alpha\beta\gamma}$ from eqn. (3.26) in eqn. (3.98).

(ii) We put in the expressions for $C(uuA)_{\kappa\nu\mu'}$, $C(usW^+)^{\nu'}_{\xi\lambda;\lambda\gamma}$ and $C(duW^-)^{\lambda'}_{\sigma\omega;\omega\beta}$ from Appendix C.

(iii) We match the Dirac indices so that the $\bar{u}^{S'}(P')$ coming from $D_{\delta\kappa\sigma\alpha\beta\gamma}$ stands on the left and $u^S(P)$ stands on the extreme right. It is obvious that we shall have a product of gamma matrices in between to make the expression, a scalar. At this stage $\mathcal{A}_1$ looks like

$$\mathcal{A}_1 = i(2\pi)^4\delta^4(P'+k'-P)$$

$$[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}][N_p {}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$

$$\frac{1}{9} \frac{1}{(2\pi)^{20}} (2)^4 (\frac{1}{m_W^2}) \varepsilon^{\mu'*}(k',\lambda')[I^{(1)} + I^{(2)} - I^{(3)} - I^{(4)}] \qquad (3.99)$$

where

$$I^{(1)} = (eQ_u)\left(\frac{g}{\sqrt{2}}\right)^2 U^{\dagger}_{us} U_{du}$$

$$[- \bar{u}^{S'}(P')\gamma_{\mu'}\gamma_{\beta'}\gamma_{\nu'}\left(\frac{1-\gamma_5}{2}\right) (\mathcal{M}_2)^T \gamma^T_{\alpha'} (\mathcal{M}_1)\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right)u^S(P)$$

$$+ \bar{u}^{S'}(P')\gamma_{\mu'}\gamma_{\beta'}\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right) (\mathcal{M}_2)\left(\frac{1-\gamma_5}{2}\right)\gamma^T_{\nu'}(\mathcal{M}_1)^T\gamma_{\alpha'}u^S(P)$$

$$+ \bar{u}^{S'}(P')\gamma_{\alpha'}(\mathcal{M}_2)\left(\frac{1-\gamma_5}{2}\right) (\gamma^{\nu'})^T\gamma^T_{\beta'}\gamma^T_{\mu'}(\mathcal{M}_1)^T\gamma_{\nu'}\left(\frac{1-\gamma_5}{2}\right)u^S(P)$$

$$- \bar{u}^{S'}(P')\gamma_{\alpha'}u^S(P)\mathrm{Tr}\{\gamma_{\mu'}\gamma_{\beta'}\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right) (\mathcal{M}_2)\left(\frac{1-\gamma_5}{2}\right)\gamma^T_{\nu'}(\mathcal{M}_1)\}]I_1^{\alpha'\beta'} \tag{3.100}$$

$I^{(2)}$, $I^{(3)}$ and $I^{(4)}$ have very similar expressions. We shall not convey anything more by writing those cumbersome expressions. $\mathcal{M}_1$ and $\mathcal{M}_2$ in eqn. (3.100) are given by eqns. (3.27) and (3.28).

It is a bit disturbing to find charge conjugation matrices C and transposes in these expressions. But it happens that we always have just enough number of C's and trasposes so that they undo the effect of each other. After using the algebraic properties of C in eqn. (3.100) and in $I^{(2)}$, $I^{(3)}$ and $I^{(4)}$, $\mathcal{A}_1$ assumes the following form

$$\mathcal{A}_1 = i(2\pi)^4\delta^4(P'+k'-P)$$

$$\left[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}\right]\left[N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\frac{1}{2P^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$

$$\frac{1}{9}\frac{1}{(2\pi)^{20}}(2)^4\left(\frac{1}{m_W^2}\right)\epsilon^{\mu'*}(k',\lambda')[I^{(1)} + I^{(2)} - I^{(3)} - I^{(4)}] \tag{3.101}$$

where $I^{(1)}$ is given by

$$I^{(1)} = (eQ_u)\left(\frac{g}{\sqrt{2}}\right)^2 U^{\dagger}_{us} U_{du}$$

$$\bar{u}^{S'}(P')[\gamma_{\mu'}\gamma_{\beta'}\gamma_{\nu'}\left(\frac{1-\gamma_5}{2}\right)\gamma^5\left(\frac{-\not{P}+M}{2M}\right)\gamma_{\alpha'}\left(\frac{-\not{P}'+M'}{2M'}\right)\gamma_5\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right)$$

$$-\gamma_{\mu'}\gamma_{\beta'}\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right)\left(\frac{\not{P}+M}{2M}\right)\gamma^5\left(\frac{1-\gamma_5}{2}\right)\gamma_{\nu'}\gamma^5\left(\frac{\not{P}'+M'}{2M'}\right)\gamma_{\alpha'}$$

$$-\gamma_{\alpha'}\left(\frac{\not{P}+M}{2M}\right)\gamma^5\left(\frac{1-\gamma_5}{2}\right)\gamma^{\nu'}\gamma_{\beta'}\gamma_{\mu'}\gamma^5\left(\frac{\not{P}'+M'}{2M'}\right)\gamma_{\nu'}\left(\frac{1-\gamma_5}{2}\right)$$

$$-\gamma_{\alpha'}Tr\{\gamma_{\mu'}\gamma_{\beta'}\gamma^{\nu'}\left(\frac{1-\gamma_5}{2}\right)\left(\frac{\not{P}+M}{2M}\right)\gamma^5\left(\frac{1-\gamma_5}{2}\right)\gamma_{\nu'}\left(\frac{-\not{P}'+M'}{2M'}\right)\gamma^5\}]I_1^{\alpha'\beta'}u^S(P) \quad (3.102)$$

We again skip writing similar expressions for $I^{(2)}$, $I^{(3)}$ and $I^{(4)}$.

At this stage, we contract various Lorentz tensors and multiply the Dirac matrices (and take traces wherever required) between $\bar{u}^{S'}(P')$ and $u^S(P)$. We always impose the following constraints

$$P^2 = M^2$$
$$P' = P-k'. \quad (3.103)$$
$$\varepsilon^*\cdot k' = 0$$

And, finally we use the equations of motion

$$\not{P}u^S(P) = M\,u^S(P)$$
$$\bar{u}^{S'}(P')\not{P}' = M'\,\bar{u}^{S'}(P'). \quad (3.104)$$

This part of the calculation was handled on computer using the REDUCE2 symbolic manipulation package [65]. After this, $\mathcal{A}_1$ looks like

$$\mathcal{A}_1 = i(2\pi)^4\delta^4(P'+k'-P)$$

$$\left[\frac{1}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p^* N_\Sigma + \frac{1}{2P'^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\frac{1}{2P^0}\frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$

$$\frac{1}{9}\frac{1}{(2\pi)^{20}}(2)^4\left(\frac{1}{m_W^2}\right)(eQ_u)\left(\frac{B}{\sqrt{2}}\right)^2 U^{\dagger}_{us}U_{du}$$

$$\bar{u}^{S'}(P')[\Delta_{281}\not\epsilon^{*}+\Delta_{282}\gamma^5\not\epsilon^{*}+\Delta_{283}\not\epsilon^{*}\not k'+\Delta_{284}\gamma^5\not\epsilon^{*}\not k']u^{S}(P)$$

(3.105)

Various Δ_i's are functions of M, M', $P\cdot k'$, α, x', m_u, m_d, m_s explicitly and also implicity through the T_i's. Just for the sake of illustration, we present the FORTRAN expression for Δ_{281} in the footnote below.

```
 CS(281) = (-6.D0*PKPR*M**2*CT(7) + 6.D0*PKPR*M*CT(2)*MU - 10.D0*
1PKPR*M*CT(2)*MPR + 10.D0*PKPR*M*MU*CT(5) - 10.D0*PKPR*M*MU*CT(1) +
2 10.D0*PKPR*M*MPR*CT(9) - 10.D0*PKPR*M*MPR*CT(10) - 2.D0*PKPR*CT(
32)*MPR**2 + 2.D0*PKPR*MU*CT(5)*MPR - 2.D0*PKPR*MU*CT(1)*MPR + 2.
4D0*PKPR*MPR**2*CT(9) - 2.D0*PKPR*MPR**2*CT(10) + 3.D0*M**4*CT(7) -
5 3.D0*M**3*CT(2)*MU - 4.D0*M**3*CT(2)*MPR + 4.D0*M**3*MU*CT(5) +
69.D0*M**3*MU*CT(4) - 4.D0*M**3*MU*CT(1) + 9.D0*M**3*CT(7)*MPR + 4.
7D0*M**3*MPR*CT(9) - 4.D0*M**3*MPR*CT(10) - 9.D0*M**2*CT(2)*MU*MPR
8+ M**2*CT(2)*MPR**2 - 9.D0*M**2*MU**2*CT(1) - M**2*MU*CT(5)*MPR +
99.D0*M**2*MU*CT(4)*MPR + M**2*MU*CT(1)*MPR + 6.D0*M**2*CT(7)*MPR**
*2 - 6.D0*M**2*CT(6) - M**2*MPR**2*CT(9) + M**2*MPR**2*CT(10) - 6.
1D0*M*CT(2)*MU*MPR**2 + 4.D0*M*CT(2)*MPR**3 - 9.D0*M*MU**2*CT(1)*
2MPR - 4.D0*M*MU*CT(5)*MPR**2 + 4.D0*M*MU*CT(1)*MPR**2 + 24.D0*M*
3CT(6)*MPR - 4.D0*M*MPR**3*CT(9) + 4.D0*M*MPR**3*CT(10) - CT(2)*
4MPR**4 + MU*CT(5)*MPR**3 - MU*CT(1)*MPR**3 + MPR**4*CT(9) - MPR**
54*CT(10)) / (4.D0*M*MPR)
```

In the FORTRAN expression above

CS(281) ⟷ Δ_{281}

CT(I) ⟷ T_i

MPR ⟷ M'

MU ⟺ m_u

PKPR ⟷ P.k'

Values of $\Delta_{281},\ldots,\Delta_{284}$ are given in Table 3.1 - 3.4

Rearranging eqn. (3.105) a little bit, we obtain

$$\mathcal{A}_1 = i(2\pi)^4 \delta^4(P'+k'-P)$$
$$\left[\frac{1}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^*N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$
$$\frac{1}{9} \frac{1}{(2\pi)^{16}} U_{us}^{\dagger} U_{du} \frac{g^2 e}{32\pi^2 m_w^2} \left(\frac{16Q_u}{\pi^2}\right)$$
$$\bar{u}^{S'}(P')[\Delta_{281} \epsilon^* + \Delta_{282} \gamma^5 \epsilon^* + \Delta_{283} \epsilon^* k' + \Delta_{284} \gamma^5 \epsilon^* k']u^S(P) \tag{3.106}$$

Having explained the evaluation of $\mathcal{A}_1$ in detail, we now list the result for other $\mathcal{A}_i$'s

$$\mathcal{A}_2 = i(2\pi)^4 \delta^4(P'+k'-P)$$
$$\left[\frac{1}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^*N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$
$$\frac{1}{9} \frac{1}{(2\pi)^{16}} U_{us}^{\dagger} U_{du} \frac{g^2 e}{32\pi^2 m_w^2} \left(-\frac{16Q_d}{\pi^2}\right)$$
$$\bar{u}^{S'}(P')[\Delta_{419} \epsilon^* + \Delta_{420} \gamma^5 \epsilon^* + \Delta_{421} \epsilon^* k' + \Delta_{422} \gamma^5 \epsilon^* k']u^S(P) \tag{3.107}$$

$$\mathcal{A}_3 = i(2\pi)^4 \delta^4(P'+k'-P)$$
$$\left[\frac{1}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^*N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$

$$\frac{1}{9}\ \frac{1}{(2\pi)^{16}}\ U^{\dagger}_{us}U_{du}\ \frac{g^2 e}{32\pi^2 m_w^2}\left(-\frac{16Q_u}{\pi^2}\right)$$

$$\bar{u}^{S'}(P')[\mathcal{S}_{511}\ \not{\epsilon}^{*}+\mathcal{S}_{512}\ \gamma^5\not{\epsilon}^{*}+\mathcal{S}_{513}\not{\epsilon}^{*}\not{K}' + \mathcal{S}_{514}\gamma^5\not{\epsilon}^{*}\not{K}']u^S(P) \tag{3.108}$$

$$\mathcal{A}_4 = i(2\pi)^4\delta^4(P'+k'-P)$$

$$\left[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}\right]\left[N_p{}^{*}N_{\Sigma^+}\ \frac{1}{2P'^0}\ \frac{(2\pi)^4}{(2\pi)^{3/2}}\ \frac{1}{2P^0}\ \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$

$$\frac{1}{9}\ \frac{1}{(2\pi)^{16}}\ U^{\dagger}_{us}U_{du}\ \frac{g^2 e}{32\pi^2 m_w^2}\left(-\frac{16Q_s}{\pi^2}\right)$$

$$\bar{u}^{S'}(P')[\mathcal{S}_{665}\ \not{\epsilon}^{*}+\mathcal{S}_{666}\ \gamma^5\not{\epsilon}^{*}+\mathcal{S}_{667}\not{\epsilon}^{*}\not{K}' + \mathcal{S}_{668}\gamma^5\not{\epsilon}^{*}\not{K}']u^S(P) \tag{3.109}$$

$$\mathcal{A}_9 + \ldots + \mathcal{A}_{12} + \mathcal{A}_{21} + \ldots + \mathcal{A}_{24} + \mathcal{A}_{27} + \mathcal{A}_{28}$$

$$= i(2\pi)^4\delta^4(P'+k'-P)$$

$$\left[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}\right]\left[N_p{}^{*}N_{\Sigma^+}\ \frac{1}{2P'^0}\ \frac{(2\pi)^4}{(2\pi)^{3/2}}\ \frac{1}{2P^0}\ \frac{(2\pi)^4}{(2\pi)^{3/2}}\right]$$

$$\frac{1}{9}\ \frac{1}{(2\pi)^{16}}\ U^{\dagger}_{d\ell}U^{\dagger}_{\ell s}\ \frac{g^2 e}{32\pi^2 m_w^2}(-1)$$

$$\bar{u}^{S'}(P')[\mathcal{S}^{\ell}_{1513}\ \not{\epsilon}^{*}+\mathcal{S}^{\ell}_{1514}\ \gamma^5\not{\epsilon}^{*}+\mathcal{S}^{\ell}_{1515}\not{\epsilon}^{*}\not{K}'$$

$$+\mathcal{S}^{\ell}_{1516}\gamma^5\not{\epsilon}^{*}\not{K}']u^S(P) \tag{3.110}$$

$$\mathcal{A}_7 + \mathcal{A}_{19} = i(2\pi)^4\delta^4(P'+k'-P)$$

$$[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}][N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$

$$\frac{1}{9} \frac{1}{(2\pi)^{16}} U^\dagger_{d\ell} U^\dagger_{\ell s} \frac{g^2 e}{32\pi^2 m_w^2} (Q_u)$$

$$\bar{u}^{S'}(P')[\mathcal{S}^\ell_{1565} \epsilon^* + \mathcal{S}^\ell_{1566} \gamma^5 \epsilon^* + \mathcal{S}^\ell_{1567} \epsilon^* k' + \mathcal{S}^\ell_{1568} \gamma^5 \epsilon^* k'] u^S(P) \qquad (3.111)$$

$$\mathcal{A}_8 + \mathcal{A}_{20} = i(2\pi)^4 \delta^4(P'+k'-P)$$

$$[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}][N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$

$$\frac{1}{9} \frac{1}{(2\pi)^{16}} U^\dagger_{d\ell} U^\dagger_{\ell s} \frac{g^2 e}{32\pi^2 m_w^2} (Q_u)$$

$$\bar{u}^{S'}(P')[\mathcal{S}^\ell_{1617} \epsilon^* + \mathcal{S}^\ell_{1618} \gamma^5 \epsilon^* + \mathcal{S}^\ell_{1619} \epsilon^* k' + \mathcal{S}^\ell_{1620} \gamma^5 \epsilon^* k'] u^S(P) \qquad (3.112)$$

And, finally,

$$\mathcal{A}_6 = i(2\pi)^4 \delta^4(P'+k'-P)$$

$$[\frac{1}{\sqrt{(2\pi)^3 2k'_0}}][N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}]$$

$$\frac{1}{9} \frac{1}{(2\pi)^{16}} U^\dagger_{us} U_{du} \frac{g^2 e}{32\pi^2 m_w^2} \left(-\frac{16 i Q_u}{\pi^2}\right)$$

$$\bar{u}^{S'}(P')[\mathcal{S}_{1621} \epsilon^* + \mathcal{S}_{1622} \gamma^5 \epsilon^* + \mathcal{S}_{1623} \epsilon^* k' + \mathcal{S}_{1624} \gamma^5 \epsilon^* k'] u^S(P) \qquad (3.113)$$

Adding all the contributions above, we get the total transition amplitude for $\Sigma^+ \to p\gamma$.

$$S_{\Sigma^+ \to p\gamma} = i(2\pi)^4 \delta^4(P'+k'-P)$$

$$\left[\frac{1}{\sqrt{(2\pi)^3 2k_0'}}\right]\left[N_p{}^* N_{\Sigma^+} \frac{1}{2P'^0} \frac{(2\pi)^4}{(2\pi)^{3/2}} \frac{1}{2P^0} \frac{(2\pi)^4}{(2\pi)^{3/2}}\right] .$$

$$\frac{1}{9} \frac{1}{(2\pi)^{16}} \frac{g^2 e}{32\pi^2 m_W^2}$$

$$\bar{u}^{S'}(P')[\tilde{A}\not{\epsilon}^* + \tilde{B}\gamma^5\not{\epsilon}^* + \tilde{C}\,\not{\epsilon}^*\not{k}' + \tilde{D}\gamma^5\not{\epsilon}^*\not{k}']u^S(P) \qquad (3.114)$$

Before we quote the values of the 'unnormalized form factors', $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$, we should point out the subtlety involved in choosing the input quark masses and baryon sizes (α and α').

Since we parametrized our BS amplitude from the very outset, we do not know what strong interaction dynamics gives rise to the Gaussian form that we have used. We, therefore, do not have a dynamical constraint on the quark masses and size parameters (α and α'). In view of the formidable character of the BS equation, this appraoch needs no apologies. The question is: what quark masses and size parameters do we use in such a situation? Looking into the literature, we find some concrete evidence of the performance of such a Gaussian BS amplitude in describing hadronic processes. Tomozawa [43] used such a BS amplitude for proton to calculate the decay rate for $p \to e^+\pi^0$. He used constituent quark masses and $\alpha = 632$MeV. These are close to the values used in all quark model calculations except that $\alpha = 632$ MeV is a bit too high. So, as the first choice for the input parameters, we tried constituent quark masses and $\alpha = \alpha' = 500$ MeV. The second piece of evidence comes from the calculation of

Esteve et al [38]. They used a similar Gaussian BS amplitude for mesons to calculate $K^+ \to \pi^+\pi^0$. The input quark masses were chosen to be the current quark masses and $\alpha \simeq 316$ MeV. This value of α very closely corresponds to a radius of 0.6 fm and that is the charge radius of the pion. Following this prescription, we calculated $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ with current quark masses and $\alpha = \alpha' = 246.660725$ MeV (this corresponds to a radius of 0.8 fm which is the charge radius of the proton). We quote the results separately for each of these two sets of input parameters. In each case, we only report the numbers with m_t = 40 GeV. It is clear from Tables 3.1-3.4 that the results with m_t = 80 GeV differ very slightly.

<u>Results with constituent quark masses and $\alpha = \alpha' = 500$ MeV.</u>

	Real Part	Imaginary Part	
$\tilde{A}(\mathrm{Mev}^{14})$ =	-0.102×10^{32}	0.784×10^{32}	(3.115)
$\tilde{B}(\mathrm{Mev}^{14})$ =	-0.338×10^{31}	0.906×10^{30}	(3.116)
$\tilde{C}(\mathrm{Mev}^{13})$ =	-0.462×10^{28}	-0.346×10^{28}	(3.117)
$\tilde{D}(\mathrm{Mev}^{13})$ =	-0.368×10^{28}	-0.825×10^{28}	(3.118)

Since the asymmetry parameter is independent of the overall normalization, we rightaway get the following value for it from eqns. (1.5), (3.117) and (3.118)

$$\alpha_{\Sigma^+} = +0.792 \qquad (3.119)$$

Results with current quark masses and $\alpha = \alpha' = 246.660725$ MeV

	Real Part	Imaginary Part	
$\tilde{A}(\mathrm{Mev}^{14})$	$= -0.108\times10^{28}$	-0.197×10^{28}	(3.120)
$\tilde{B}(\mathrm{Mev}^{14})$	$= -0.354\times10^{27}$	0.785×10^{27}	(3.121)
$\tilde{C}(\mathrm{Mev}^{13})$	$= 0.938\times10^{25}$	0.321×10^{25}	(3.122)
$\tilde{D}(\mathrm{Mev}^{13})$	$= -0.278\times10^{25}$	0.455×10^{25}	(3.123)

From eqns. (1.5), (3.122) and (3.123),

$$\alpha_{\Sigma^+} = -0.181 \tag{3.124}$$

We see, a rather significant, difference between the two resutls quoted above. We comment upon these results in Chapter 4.

TABLE 3.1 Values of Δ_i's

Input Parameters : $m_u = 350$ MeV ; $m_d = 350$ MeV ; $m_s = 550$ MeV
$m_c = 1.5$ GeV ; $m_t = 40$ GeV
$\alpha = \alpha' = 500$ MeV ; M = 1189.36 MeV
M' = 938.2796 MeV

Coefficients of ϕ^* and $\gamma^5\phi$ have the dimensions $(\text{MeV})^{14}$; Those of ϕ^*k' and $\gamma^5\phi^*k'$ have the dimensions $(\text{MeV})^{13}$.

Δ_i's	Real Part	Imaginary Part
Δ_{281}	-0.34892E+32	-0.67618E+32
Δ_{282}	-0.17697E+31	-0.15950E+31
Δ_{283}	0.72057E+28	0.93888E+28
Δ_{284}	-0.16411E+29	-0.18145E+29
Δ_{419}	0.30309E+32	0.69609E+32
Δ_{420}	-0.17697E+31	-0.15950E+31
Δ_{421}	-0.47587E+29	-0.44616E+29
Δ_{422}	-0.16411E+29	-0.18145E+29
Δ_{511}	0.64858E+32	-0.46304E+32
Δ_{512}	0.92775E+31	-0.49701E+31
Δ_{513}	-0.23666E+29	0.22579E+29
Δ_{514}	-0.45269E+28	0.51357E+28
Δ_{665}	0.80150E+32	-0.48897E+32
Δ_{666}	-0.51366E+31	0.26153E+31
Δ_{667}	-0.52419E+29	0.42575E+29
Δ_{668}	0.86352E+28	-0.61937E+28
Δ^u_{1513}	-0.11653E+32	-0.20819E+31
Δ^u_{1514}	-0.17989E+31	-0.22401E+31
Δ^u_{1515}	0.12794E+29	0.10710E+29

Δ^{u}_{1516}	0.17826E+28	-0.32853E+28
Δ^{c}_{1513}	-0.11637E+32	-0.20784E+31
Δ^{c}_{1514}	-0.17989E+31	-0.22396E+31
Δ^{c}_{1515}	0.12775E+29	0.10692E+29
Δ^{c}_{1516}	0.17821E+28	-0.32846E+28
Δ^{t}_{1513}	-0.50555E+31	-0.64646E+30
Δ^{t}_{1514}	-0.18809E+31	-0.21777E+31
Δ^{t}_{1515}	0.49942E+28	0.33257E+28
Δ^{t}_{1516}	0.15997E+28	-0.31938E+28
Δ^{u}_{1565}	0.56459E+31	
Δ^{u}_{1566}	-0.16707E+30	
Δ^{u}_{1567}	-0.75980E+27	
Δ^{u}_{1568}	0.15481E+27	
Δ^{c}_{1565}	0.56391E+31	
Δ^{c}_{1566}	-0.16709E+30	
Δ^{c}_{1567}	-0.75887E+27	
Δ^{c}_{1568}	0.15516E+27	
Δ^{t}_{1565}	0.29789E+31	
Δ^{t}_{1566}	-0.18437E+30	
Δ^{t}_{1567}	-0.39493E+27	
Δ^{t}_{1568}	0.31791E+27	
Δ^{u}_{1617}	0.56459E+31	
Δ^{u}_{1618}	-0.16707E+30	
Δ^{u}_{1619}	-0.75980E+27	
Δ^{u}_{1620}	0.15481E+27	
Δ^{c}_{1617}	0.56391E+31	
Δ^{c}_{1618}	-0.16709E+30	
Δ^{c}_{1619}	-0.75887E+27	
Δ^{c}_{1620}	0.15516E+27	
Δ^{t}_{1617}	0.29789E+31	

Δ^{t}_{1618}	-0.18437E+30
Δ^{t}_{1619}	-0.39493E+27
Δ^{t}_{1620}	0.31791E+27
Δ_{1621}	-0.34787E+33
Δ_{1622}	0.0
Δ_{1623}	0.0
Δ_{1624}	0.0

TABLE 3.2 Values of Δ_i's

Input Parameters : $m_u = 350$ MeV ; $m_d = 350$ MeV ; $m_s = 550$ MeV
$m_c = 1.5$ GeV ; $m_t = 80$ GeV
$\alpha = \alpha' = 500$ MeV ; M = 1189.36 MeV
M' = 938.2796 MeV

Coefficients of ϕ^* and $\gamma^5\phi$ have the dimensions $(MeV)^{14}$; Those of ϕ^*k' and $\gamma^5\phi^*k'$ have the dimensions $(MeV)^{13}$.

Δ_i's	Real Part	Imaginary Part
Δ_{281}	-0.34892E+32	-0.67618E+32
Δ_{282}	-0.17697E+31	-0.15950E+31
Δ_{283}	0.72057E+28	0.93888E+28
Δ_{284}	-0.16411E+29	-0.18145E+29
Δ_{419}	0.30309E+32	0.69609E+32
Δ_{420}	-0.17697E+31	-0.15950E+31
Δ_{421}	-0.47587E+29	-0.44616E+29
Δ_{422}	-0.16411E+29	-0.18145E+29
Δ_{511}	0.64858E+32	-0.46304E+32
Δ_{512}	0.92775E+31	-0.49701E+31
Δ_{513}	-0.23666E+29	0.22579E+29
Δ_{514}	-0.45269E+28	0.51357E+28
Δ_{665}	0.80150E+32	-0.48897E+32
Δ_{666}	-0.51366E+31	0.26153E+31
Δ_{667}	-0.52419E+29	0.42575E+29
Δ_{668}	0.86352E+28	-0.61937E+28
Δ^u_{1513}	-0.11653E+32	-0.20819E+31
Δ^u_{1514}	-0.17989E+31	-0.22401E+31
Δ^u_{1515}	0.12794E+29	0.10710E+29

Δ^{u}_{1516}	0.17826E+28	-0.32853E+28
Δ^{c}_{1513}	-0.11637E+32	-0.20784E+31
Δ^{c}_{1514}	-0.17989E+31	-0.22396E+31
Δ^{c}_{1515}	0.12775E+29	0.10692E+29
Δ^{c}_{1516}	0.17821E+28	-0.32846E+28
Δ^{t}_{1513}	0.25818E+31	0.10514E+31
Δ^{t}_{1514}	-0.21539E+31	-0.23110E+31
Δ^{t}_{1515}	-0.41224E+28	-0.54087E+28
Δ^{t}_{1516}	0.13892E+28	-0.33893E+28
Δ^{u}_{1565}	0.56459E+31	
Δ^{u}_{1566}	-0.16707E+30	
Δ^{u}_{1567}	-0.75980E+27	
Δ^{u}_{1568}	0.15481E+27	
Δ^{c}_{1565}	0.56391E+31	
Δ^{c}_{1566}	-0.16709E+30	
Δ^{c}_{1567}	-0.75887E+27	
Δ^{c}_{1568}	0.15516E+27	
Δ^{t}_{1565}	-0.42598E+28	
Δ^{t}_{1566}	-0.22335E+30	
Δ^{t}_{1567}	0.14402E+26	
Δ^{t}_{1568}	0.54852E+27	
Δ^{u}_{1617}	0.56459E+31	
Δ^{u}_{1618}	-0.16707E+30	
Δ^{u}_{1619}	-0.75980E+27	
Δ^{u}_{1620}	0.15481E+27	
Δ^{c}_{1617}	0.56391E+31	
Δ^{c}_{1618}	-0.16709E+30	
Δ^{c}_{1619}	-0.75887E+27	
Δ^{c}_{1620}	0.15516E+27	
Δ^{t}_{1617}	-0.42598E+28	

Δ^{t}_{1618}	-0.22335E+30
Δ^{t}_{1619}	0.14402E+26
Δ^{t}_{1620}	0.54852E+27
Δ_{1621}	-0.34787E+33
Δ_{1622}	0.0
Δ_{1623}	0.0
Δ_{1624}	0.0

TABLE 3.3 Values of Δ_i's

Input Parameters : $m_u = 5.1$ MeV ; $m_d = 8.9$ MeV ; $m_s = 175$ MeV
$m_c = 1.35$ GeV ; $m_t = 40$ GeV
$\alpha = \alpha' = 500$ MeV ; M = 1189.36 MeV
M' = 938.2796 MeV

Coefficients of $\not{\epsilon}^*$ and $\gamma^5\not{\epsilon}$ have the dimensions $(\text{MeV})^{14}$; Those of $\not{\epsilon}^* k'$ and $\gamma^5\not{\epsilon}^* k'$ have the dimensions $(\text{MeV})^{13}$.

Δ_i's	Real Part	Imaginary Part
Δ_{281}	0.20467E+28	0.11866E+29
Δ_{282}	0.52164E+27	0.37474E+28
Δ_{283}	0.11553E+26	0.65063E+25
Δ_{284}	0.11950E+26	0.99442E+25
Δ_{419}	-0.28892E+27	-0.92386E+28
Δ_{420}	0.58593E+26	0.28518E+28
Δ_{421}	0.19292E+25	0.11975E+26
Δ_{422}	0.18531E+25	0.67669E+25
Δ_{511}	0.60431E+28	0.11081E+29
Δ_{512}	0.21805E+28	0.37416E+28
Δ_{513}	-0.27281E+26	0.50207E+25
Δ_{514}	0.24606E+26	-0.79395E+25
Δ_{665}	-0.98065E+27	-0.16455E+29
Δ_{666}	0.22825E+27	0.38826E+28
Δ_{667}	0.98591E+24	0.12600E+26
Δ_{668}	-0.44722E+24	-0.34122E+25
Δ^u_{1513}	-0.43330E+28	0.79917E+16
Δ^u_{1514}	0.16835E+28	0.76326E+15
Δ^u_{1515}	0.26898E+25	0.12487E+15

Δ^{u}_{1516}	0.15220E+24	-0.10916E+15
Δ^{c}_{1513}	0.43330E+28	0.79917E+16
Δ^{c}_{1514}	0.16835E+28	0.76326E+15
Δ^{c}_{1515}	0.26898E+25	0.12487E+15
Δ^{c}_{1516}	0.15220E+24	-0.10916E+15
Δ^{t}_{1513}	0.44667E+28	-0.27958E+17
Δ^{t}_{1514}	0.16288E+28	0.36492E+16
Δ^{t}_{1515}	0.27846E+25	-0.43685E+15
Δ^{t}_{1516}	0.16728E+24	-0.52192E+15
Δ^{u}_{1565}	0.53776E+27	
Δ^{u}_{1566}	0.91271E+25	
Δ^{u}_{1567}	-0.84400E+23	
Δ^{u}_{1568}	-0.34701E+23	
Δ^{c}_{1565}	0.53776E+27	
Δ^{c}_{1566}	0.91271E+25	
Δ^{c}_{1567}	-0.84400E+23	
Δ^{c}_{1568}	-0.34701E+23	
Δ^{t}_{1565}	0.43606E+27	
Δ^{t}_{1566}	0.89436E+25	
Δ^{t}_{1567}	-0.68447E+23	
Δ^{t}_{1568}	-0.34196E+23	
Δ^{u}_{1617}	0.53776E+27	
Δ^{u}_{1618}	0.91271E+25	
Δ^{u}_{1619}	-0.84400E+23	
Δ^{u}_{1620}	-0.34701E+23	
Δ^{c}_{1617}	0.53776E+27	
Δ^{c}_{1618}	0.91271E+25	
Δ^{c}_{1619}	-0.84400E+23	
Δ^{c}_{1620}	-0.34701E+23	
Δ^{t}_{1617}	0.43606E+27	

Δ^{t}_{1618}	0.89436E+25
Δ^{t}_{1619}	-0.68447E+23
Δ^{t}_{1620}	-0.34196E+23
Δ_{1621}	-0.35965E+28
Δ_{1622}	0.0
Δ_{1623}	0.0
Δ_{1624}	0.0

TABLE 3.4 Values of Δ_i's

Input Parameters : $m_u = 5.1$ MeV ; $m_d = 8.9$ MeV ; $m_s = 175$ MeV
$m_c = 1.35$ GeV ; $m_t = 80$ GeV
$\alpha = \alpha' = 500$ MeV ; $M = 1189.36$ MeV
$M' = 938.2796$ MeV

Coefficients of ψ^* and $\gamma^5\psi$ have the dimensions $(\mathrm{MeV})^{14}$; Those of $\psi^* k'$ and $\gamma^5\psi^* k'$ have the dimensions $(\mathrm{MeV})^{13}$.

Δ_i's	Real Part	Imaginary Part
Δ_{281}	0.20467E+28	0.11866E+29
Δ_{282}	0.52164E+27	0.37474E+28
Δ_{283}	0.11553E+26	0.65063E+25
Δ_{284}	0.11950E+26	0.99442E+25
Δ_{419}	-0.28892E+27	-0.92386E+28
Δ_{420}	0.58593E+26	0.28518E+28
Δ_{421}	0.19292E+25	0.11975E+26
Δ_{422}	0.18531E+25	0.67669E+25
Δ_{511}	0.60431E+28	0.11081E+29
Δ_{512}	0.21805E+28	0.37416E+28
Δ_{513}	-0.27281E+26	0.50207E+25
Δ_{514}	0.24606E+26	-0.79395E+25
Δ_{665}	-0.98065E+27	-0.16455E+29
Δ_{666}	0.22825E+27	0.38826E+28
Δ_{667}	0.98591E+24	0.12600E+26
Δ_{668}	-0.44722E+24	-0.34122E+25
Δ^{u}_{1513}	-0.43330E+28	0.79917E+16
Δ^{u}_{1514}	0.16835E+28	0.76326E+15
Δ^{u}_{1515}	0.26898E+25	0.12487E+15

Δ^{u}_{1516}	0.15220E+24	-0.10916E+15
Δ^{c}_{1513}	-0.43330E+28	0.79917E+16
Δ^{c}_{1514}	0.16835E+28	0.76326E+15
Δ^{c}_{1515}	0.26898E+25	0.12487E+15
Δ^{c}_{1516}	0.15220E+24	-0.10916E+15
Δ^{t}_{1513}	-0.38155E+28	-0.74725E+17
Δ^{t}_{1514}	0.13118E+28	0.75388E+16
Δ^{t}_{1515}	0.23866E+25	-0.11676E+16
Δ^{t}_{1516}	0.15179E+24	-0.10782E+16
Δ^{u}_{1565}	0.53776E+27	
Δ^{u}_{1566}	0.91271E+25	
Δ^{u}_{1567}	-0.84400E+23	
Δ^{u}_{1568}	-0.34701E+23	
Δ^{c}_{1565}	0.53776E+27	
Δ^{c}_{1566}	0.91271E+25	
Δ^{c}_{1567}	-0.84400E+23	
Δ^{c}_{1568}	-0.34701E+23	
Δ^{t}_{1565}	0.35339E+27	
Δ^{t}_{1566}	0.95804E+25	
Δ^{t}_{1567}	-0.55484E+23	
Δ^{t}_{1568}	-0.36872E+23	
Δ^{u}_{1617}	0.53776E+27	
Δ^{u}_{1618}	0.91271E+25	
Δ^{u}_{1619}	-0.84400E+23	
Δ^{u}_{1620}	-0.34701E+23	
Δ^{c}_{1617}	0.53776E+27	
Δ^{c}_{1618}	0.91271E+25	
Δ^{c}_{1619}	-0.84400E+23	
Δ^{c}_{1620}	-0.34701E+23	
Δ^{t}_{1617}	0.35339E+27	

s^{t}_{1618}	0.95804E+25
s^{t}_{1619}	-0.55484E+23
s^{t}_{1620}	-0.36872E+23
s_{1621}	-0.35965E+28
s_{1622}	0.0
s_{1623}	0.0
s_{1624}	0.0

CHAPTER 4

CONCLUSION

> " *What we call the beginning is often the end*
> *And to make an end is to make a beginning.*
> *The end is where we start from.* "
>
> *T. S. Eliot*

As pointed out in the Introduction, our major undertaking in this thesis was to formulate the problem of weak radiative decay of $\frac{1}{2}^+$ baryons within the rigorous Bethe-Salpeter framework of quantum field theory. This task was fulfilled in Chapter 2. We derived a reduction formula for $\rightarrow \gamma$ in a manner which closely resembles the standard LSZ Reduction Formalism. This method of deriving the 'Nishijima-Mandelstam Formula' does not seem to be available in the literature. Apart from its pedagogical — and, to some extent, aesthetic — value, this method can be very easily generalized to derive reduction formulas for other composite particle reactions.

The next motive was to see how the consistency requirments of BS formalism affect our understanding of weak radiative decay of baryons. A detailed calculation of $\Sigma^+ \rightarrow p\gamma$ decay was carried out to observe such effects. Let us discuss those observations one by one.

As we saw in Chapter 3, we faced no problem in adding 1-quark processes with the 2-quark and 3-quark ones. The Nishijima-Mandelstam amputation procedure unambiguously told us what to do. So, unlike Kamal and Verma [28], we did not have to invoke any additional parameter to add these contributions. In the quark model calculations, quarks are put on their mass-shell. Then these different classes of diagrams correspond to disconnected pieces in the LSZ formula. In our opinion, that may be the origin of the ambiguity experienced in adding these diagrams. We faced no such difficulty.

The quarks in our calculation are not on their mass-shell — baryons, obviously, are. This feature brings in some changes. Most notably, we needed to use full off-shell $s \to d\gamma$ vertex as calculated by Deshpande and Eilam [63] instead of only the on-shell vertex. Also, the contributions of the diagrams in Fig. 4.1 would have been zero if the quarks were put on-shell due to the on-shell

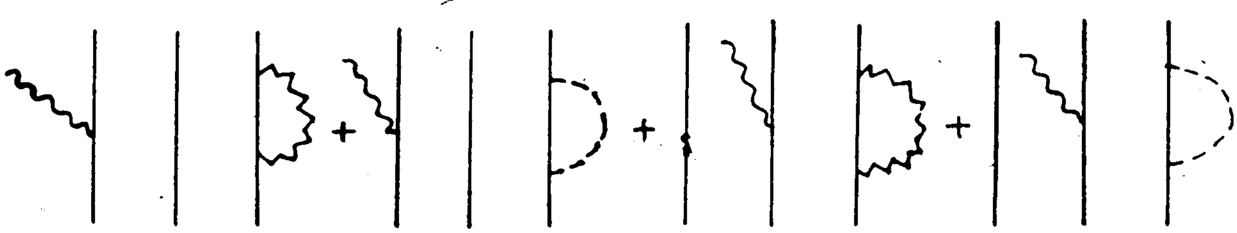


Fig. 4.1. The self-energy diagrams

renormalization condition used by Deshpande and Eilam.

If we look at the contributions of the exchange and the 1-quark processes, we see that they have both real and imaginary parts. The imaginary parts come from a proper integration of the quark propagators in these diagrams. As an example, we show the occurrence of such propagator in one of the exchange diagrams. If we look

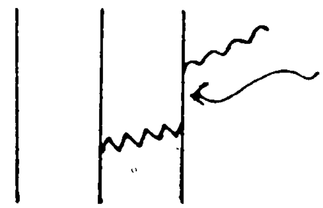


Fig. 4.2. Quark propagator responsible for the imaginary part of the amplitude ----an example.

at the numbers carefully in Tables 3.1 and 3.2, we realize that the imaginary part is comparable to the real part. So, any calculation where a (p/m)-expansion of the quark propagator is carried out will miss this contribution. However, this contribution is very much present and can play an important role in determining the sign of the asymmetry parameter.

The most important conclusion of our calculations is arrived at by comparing the contributions of 2-quark and 1-quark processes. THEY ARE COMPARABLE. The relative strength of these amplitudes has been a matter of debate [25,32] in the past. In fact, Lo's calculation [29] in the MITBM explicitly shows that the 1-quark contribution is smaller than the 2-quark one. So is the conclusion of Eckert and Morel [32] in a quark model calculation. Our calculation, unambiguously, disputes these observations. If nothing more, it certainly points out that such conjectures are highly dependent on the hadron model one is using. Our calculation — which is fully covariant, where the quarks inside the baryons are off-shell, as they should be, and where the quark propagators are properly integrated out — clearly show that the

2-quark and 1-quark contributions are of the same order in magnitude.

We, now, come to the value of the asymmetry parameter. In order to calculate this piece of experimental information theoretically, one needs the quark masses and the size parameters, α and α', of the baryons as input. Since we have parametrized the BS amplitude using phenomenological insight, we do not have a bound-state dynamics which can tell us what values of these parameters are to be used. Keeping in view the formidable difficulties encountered in solving the BS equation, this approach does not require any further defence. In such a situation, one has to rely on the evidence in literature regarding the performance of such a Gaussian BS amplitude in describing hadronic decay processes. One finds two pieces of evidence. Tomozawa [43] used such a Gaussian BS amplitude for proton to calculate $p \to e^+\pi^0$. He chose the constituent masses for quarks and $\alpha = 632$ MeV. These values are very similar to the usual quark model values except for the difference that α is a bit too large. So, we calculated α_{Σ^+} with constituent quark masses and $\alpha = \alpha' = 500$ MeV. The result was $\alpha_{\Sigma^+} = +0.792$. On the other hand, Esteve et al [38] calculated the meson decay $K^+ \to \pi^+\pi^0$ using a similar Gaussian for mesons. They used current masses for the quarks and $\alpha = 316$ Mev. This value of α corresponds very closely to the charge radius of pion. Picking up this thread, we calculated α_{Σ^+} again with current quark masses and $\alpha = \alpha' = 246$ MeV (this corresponds to the charge radius of the proton). The result for $\alpha_{\Sigma^+} = -0.181$. Thus, we conclude that α_{Σ^+} is extermely sensitive to the values of these input parameters. Only future calculations of other weak radiative decays will fortify the claim of one or the other set of values. Till then, we cannot make any more conclusive a statement. It must be mentioned, however, that it is the fully relativistic nature of our analysis that has permitted us to investigate the effect of these two very different sets of inputs clearly.

At this stage, we ask ourselves the following question: what have we learnt from the calculations presented in this thesis? We can point out the following things:

(i) The Bethe-Salpeter formalism brings in some non-trivial changes in the method of analysis;

(ii) the off-shell and quark-propagator effects are substantial;

(iii) one must consider all quark processes in analyzing these decays.

Keeping in view the lack of quark-level understanding of these processes, these conclusions can be of valuable help.

We now consider those two aspects of weak radiative decay of baryons that have not been considered in this thesis. The first one is the question of normalization of the decay amplitude. As we realize, this requires us to know the normalization constants for the BS amplitudes. A correct evaluation of these normalization constants requires the knowledge of long-range strong interaction dynamics. In the absence of this information, we can only hope to find a 'normalization prescription' for the decay amplitude. Various possibilities come to mind:

(i) Normalize, for example, the $\Sigma^+ \to p\gamma$ amplitude by the amplitude for another process like $\Sigma^- \to ne^-\bar{\nu}$.

$\Sigma^- \to ne^-\bar{\nu}$ is a simpler process to analyze and the experimental rate is available.

(ii) Calculate the normalization constants N_p and N_{Σ^+} for the proton and Σ^+ BS amplitudes respectively from the BS normalization condition by neglecting the irreducible kernel V. Then use these values of N_p and N_{Σ^+} to normalize the $\Sigma^+ \to p\gamma$ amplitude.

The phenomenological validity of any of these prescriptions can only be checked by explicit calculations. Such an investigation was not carried out in this thesis. Without this, we cannot judge the performance of the present calculation in accounting for the observed rates.

The second aspect that we have not considered is the role of hard-gluonic corrections in these decays. Within the MITBM and the quark model, such effects have been considered [29,32]. In these investigations, one considers QCD-corrected $s \to d\gamma$ vertex and QCD-corrected two-quark $\to$ two-quark strangeness-changing transition. The quarks are always put on their mass-shells. The results of these investigations have been summarized by Bergstrom and Singer [32]. Even the QCD-corrected $s \to d\gamma$ vertex cannot reproduce the $\Sigma^+ \to p\gamma$ rate. It is smaller by a factor of a thousand. When one combines the QCD-corrected $s \to d\gamma$ in Lo's calculation (which, we remember, already has QCD-corrections in 2-quark and 3-quark processes) one finds that α_{Σ^+} changes to -0.22 from -0.154 [29,32]. In Lo's calculation, one cannot find the decay rate because the overall normalization of $\Sigma^+ \to p\gamma$ is, once again, undetermined. It seems, therefore, that when the quarks are put on their mass-shell the hard-gluonic corrections do not improve the situation in case of $\Sigma^+ \to p\gamma$. This does not rule out their importance in other decay modes [32]. The question is: how much difference will they make in our results? To answer this question, one will have to consider gluonic corrections in their full glory with the quarks off-shell.

BIBLIOGRAPHY

1. R.H. GRAHAM and S. PAKVASA, Phys. Rev. 140(1965), B1144.

2. R.E. MARSHAK, RIAZUDDIN AND C.P. RYAN, "Theory of weak Interactions in Particle Physics", Wiley Interscience, 1969.

3. REVIEW OF PARTICLE PROPERTIES, Phys. Lett. B170(1986).

4. Reported at International High Energy Physics Conference, Berkeley, 1986.

5. S.F. BIAGI et al, Z. Phys C30(1986), 201.

6. LO CHONG-HUAH, Phys. Rev. D26(1982), 199.

7. R.E. Behrends, Phys. Rev. 111(1958), 1691.

8. B.W. LEE AND R. SHROCK, Phys. Rev. D16(1977), 1444.

9. K.C. TRIPATHY, S.N. BISWAS AND R.P. SAXENA, Phys. Rev. 144(1966), 1163.

10. N. VASANTI, Phys. Rev. D13(1976), 1889.

11. KUSUM SHARMA, RAMESH C. VERMA AND M.P. KHANNA, J. Phys. G. 5(1979), 1519.

12. P.N. PANDIT, RAMESH C. VERMA AND M.P. KHANNA, Prog. Theor. Phys. 64(1980) 2158.

13. A. COHEN, Phys. Lett. B160(1985), 177.

14. P. CARRUTHERS, "Introduction to Unitary Symmetry", Interscience Publishers, 1966.

15. S.P. ROSEN AND S. PAKVASA, in "Advances in Particle Physics", Interscience Publishers, 1968.

16. M.K. GAILLARD, Nuovo Cimento 6A(1971), 559.

17. K.G. RAUH, Z. Phys. C10(1981), 81.

18. M.B. GAVELA, A. Le YAOUANC, L. OLIVER, O. PENE, J.C. RAYNAL AND T.N. PHAM, Phys. Lett. B101(1981), 417.

19. MICHAEL D. SCADRON AND MIHAI VISINESCU, Phys. Rev. D28(1983), 117.

20. K. GAVROGLU AND H.P.W. GOTTLIEB, Nucl. Phys. B (1974), 168.

21. M.A. AHMED, Nuovo Cimento LVIII(1968), 728.

22. F. LOW, Phys. Rev. 110(1958) 974.

23. GLENNYS R. FARRAR, Phys. Rev. D4(1971), 212.

24. YA. I. KOGAN AND M.A. SHIFMAN, ITEP preprint, 34, 1983.

25. FREDERICK J. GILMAN AND MARK B. WISE, Phys. Rev. D19(1979), 976.

26. F.E. CLOSE, "An Introduction to Quarks and Partons", Academic Press, 1979.

27. J.J.J. KOKKEDEE, "The Quark Model", W.A. Benjamin, Inc., 1969.

28. A.N. KAMAL AND RAMESH C. VERMA, Phys. Rev. D26(1982), 190.

29. LO CHONG-HUAH, Phys. Rev. D26(1982), 199.

30. D. FLAMM AND F. SCHOBERL, "Introduction to the Quark Model of Elementary Particles, Vol. 1," Gordon and Breach Science Publishers, 1982.

31. A.CHODOS, R.L. JAFFE, K. JOHNSON, C.B. THORN AND V.F. WEISSKOPF, Phys. Rev. D9(1974), 3471; A. CHODOS, R.L. JAFFE, K. JOHNSON AND C.B. THORN, Phys. Rev. D10(1974), 2599; T. DeGRAND, R.L. JAFFE, K. JOHNSON AND J. KISKIS, Phys. Rev. D12(1975) 2060; K. JOHNSON, Acta Physica Polonica, B6(1975) 865; A. CHODOS AND C.B. THORN, Phys. Rev. D12(1975) 2733; A.J.G. HEY, in "Topics in Quantum Field Theory and Gause Theories", Springer-Verlag 1978; P.H. HASENFRATZ AND J. KUTI, Phys. Rep. C40(1978), 75; Quantum Flavordynamics, Quantum Chromodynamics and Unified Theories", Plenum Press, 1980.

32. YU. I. SKOVPEN', Sov. J. Nul. Phys. 34(1981) 773; P. ECKERT AND B. MOREL, Universite de Geneve preprint, UGVA-DPT 1982/03-340, 1982; SHOEB ABDULLAH AND S.K. SINGH, Z. Phys. C18(1983) 31; AVINASH SHARMA AND RAMESH C. VERMA, J. Phys. G12(1986), 1329; LARS BERGSTROM AND PAUL SINGER, Phys. Lett. B169(1986), 297; NORIHITO TOYOTA AND KANJI FUJI, Prog. Theor. Phys. 75(1986), 340.

33. M.A. AHMED AND G.G. ROSS, Phys. Lett. B59(1975), 293; MARY K. GAILLARD, XUE QIAN LI AND SERGE RUDAZ, Phys. Lett. B158(1985), 158.

34. KAZUHIKO NISHIJIMA, Prog. Theor. Phys. 10(1953), 549; ibid 12(1954) 279; ibid. 13(1955) 305; Phys Rev. 111(1958), 995.

35. KAZUHIKO NISHIJIMA, "Fields and Particles", W.A. Benjamin, Inc., 1969.

36. S. MANDELSTAM, Proc. Royal Society A233(1955), 248.

37. D. LURIE, "Particles and Fields", Interscience Publishers, 1968.

38. J.G. ESTEVE, A. MORALES AND R. NUNEZ-LAGOS, J. Phys. G9(1983), 357; J. ABAD, J.G. ESTEVE AND A.F. PACHECO, Phys. Rev. D30(1984), 1488.

39. J.D. BJORKEN AND SIDNEY D. DRELL, "Relativistic Quantum Mechanics" and "Relativistic Quantum Fields", McGraw-Hill Inc., 1965.

40. CLAUDE ITZYKSON AND JEAN-BERNARD ZUBER, "Quantum Field Theory", McGraw Hill Inc., 1980.

41. NOBORU NAKANISHI, Supp. Prog. Theor. Phys. 43(1969), 1.

42. M. BOHM AND R.F. MEYER, Ann. Phys. 120(1979), 360; M. BOHM AND T. GUDEHUS, Nuovo Cimento LVIIA(1968) 587.

43. YUKIO TOMOZAWA, J. Math. Phys. 24(1983), 369; Phys. Rev. Lett 46(1981), 463.

44. A. LeYAOUANC, L. OLIVER, O. PENE AND J.C. RAYNAL, Orsay Preprint LPTHE-LPTPE Orsay 83/3, 1983.

45. D. LURIE, A.J. MACFARLANE AND Y. TAKAHASKI, Phys. Rev. 140(1965), B1091.

46. PAUL ROMAN, "Introduction to Quantum Field Theory", John Wiley and Sons, Inc., 1969.

47. JACK S. GOLDSTEIN, Phys. Rev. 91(1953), 1516.

48. E.E. SALPETER AND H.A. BETHE, Phys. Rev. 84(1951),1232; E.E. SALPETER, Phys. Rev. 87(1952) 328.

49. H.S. GREEN AND S.N. BISWAS, Prog. Theor. Phys. 18(1957), 121.

50. C.H. LLEWELLYN SMITH, Ann. Phys. 53(1969), 521; Nuovo Cimento LXA(1969), 348.

51. M. SUNDARESAN AND P. WATSON, Ann. Phys. 59(1970), 375.

52. R.F. MEYER, Nucl. Phys. B71(1974), 226.

53. A.N. MITRA, Z. Phys. C8(1981), 25; A.N. MITRA AND I. SANTHANAM, Z. Phys. C8(1981), 33; D.S. KULSHRESHTHA, A.N. MITRA AND I. SANTHANAM, Phys. Rev. D26(1982) 3131; D.S. KULSHRESHTHA AND A.N. MITRA, Phys. Rev. D28(1983), 587; A.N. MITRA AND ANURADHA MITTAL, Phys. Rev. D29(1984), 1399; N. NIMAI SINGH, Y.K. MATHUR AND A.N. MITRA, Few Body Systems 1(1986), 47.

54. LI YI-MING AND ZHANG JIAN-ZU, Nuovo Cimento 97A(1987), 281.

55. D.B. LICHTENBERG, "Unitary Symmetry and Elementary Particles", Academic Press, 1978.

56. RICHARD G.LIPES, Phys. Rev. D5(1972), 2849

57. SILVAN S. SCHWEBER, "An Introduction to Relativistic Quantum Field Theory", Row, Peterson and Co., 1961.

58. STANLEY J. BRODSKY AND JOEL R. PRIMACK, Ann. Phys. 52(1969), 315.

59. C. NASH, "Relativistic Quantum Fields", Academic Press, 1978.

60. E.S. ABERS AND B.W. LEE, Phys. Rep. C9(1973), 1.

61. K. AOKI, Z. HIOKI, R. KAWABE, M. KONUMA AND T. MUTA, Supp. Prog. Theor. Phys. 73(1982), 1.

62. CHRIS QUIGG, "Guage Theories of the Strong, Weak and Electromagnetic Interactions", The Benjamin/Cummings Publishing Company, Inc., 1983.

63. NILENDRA G. DESHPANDE AND GAD EILAM, Phys. Rev. D26(1982) 2463; NILENDRA G. DESHPANDE AND M. NAZERIMONFARED, Nucl. Phys. B213(1983), 390.

64. E. MA AND A. PRAMUDITA, Phys. Rev. D24(1981), 1410.

65. ANTHONY C. HEARN, "Reduce 2 User's Manual", 1974.

66. M. KOBAYASHI et al, Phys. Rev. Abstracts 18(1987), 45.

67. I.S. GRADSHTEYN AND I.M. RYZHIK, "Tables of Integrals, Series and Products", Academic Press, 1980.

68. R.E. BELLMAN, "Introduction to Matrix Analysis", McGraw Hill Inc., 1960.

APPENDIX A

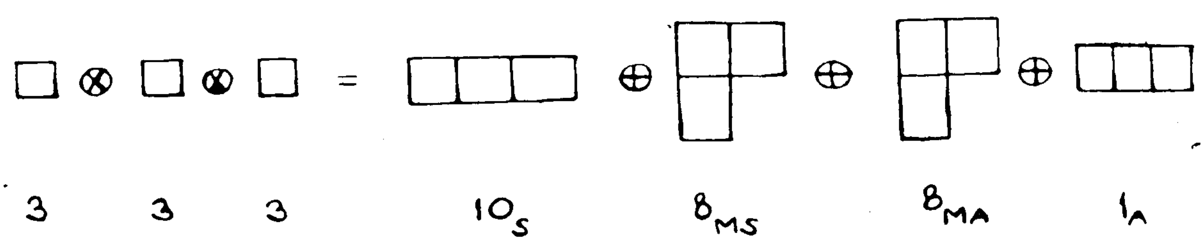
FLAVOUR STRUCTURE OF THE OCTET BARYONS

A baryon consists of three quarks. So, a baryon state can be written as

$$|\mathcal{B}\rangle = \mathcal{F}^{\mathcal{B}}_{abc}\ |\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle \qquad \text{(A.1)}$$

$\mathcal{B}$ represents the baryon and we have suppressed its flavour quantum numbers, e.g. isospin etc. a, b and c are the flavour indices which run over all quark flavours. $|\psi_a\rangle$, $|\psi_b\rangle$ and $|\psi_c\rangle$ are the three quark states and $\mathcal{F}^{\mathcal{B}}_{abc}$ are the various Clebsch-Gordan coefficients which we call the **baryon flavour tensor.**

Restricting to the strange vector only, each of a, b, and c runs over (u,d,s). Each of $|\psi_a\rangle$, $|\psi_b\rangle$ and $|\psi_c\rangle$ transforms as the fundamental 3 representation of SU(3). As a result $|\mathcal{B}\rangle$ transforms under the product representation $3 \otimes 3 \otimes 3$. This product representation decomposes into four irreducible families to which the Physical baryons belong. The Clebsch-Gordan series is



(A.2)

10_S is symmetric under the flavour indices of all three quarks; 1_A is antisymmetric under the three flavour indices. 8_{MS} is symmetric under the flavour indices of the first two quarks whereas 8_{MA} is antisymmetric under the flavour indices of the first two quarks.

According to the SU(3) spectroscopy of baryons, the $\frac{1}{2}^+$ baryons belong to the two octets. From eqn. (A.1), therefore,

$$|\mathcal{B} \in 8_{MS}\rangle = \Upsilon_{abc}^{\mathcal{B}\in 8_{MS}} |\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle \equiv \Upsilon_{abc}^{\mathcal{B}:\lambda} |\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle \tag{A.3}$$

and

$$|\mathcal{B} \in 8_{MA}\rangle = \Upsilon_{abc}^{\mathcal{B}\in 8_{MA}} |\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle \equiv \Upsilon_{abc}^{\mathcal{B}:\rho} |\psi_a\rangle \otimes |\psi_b\rangle \otimes |\psi_c\rangle \tag{A.4}$$

We now list $\Upsilon_{abc}^{\mathcal{B}:\lambda}$ and $\Upsilon_{abc}^{\mathcal{B}:\rho}$ for various $\frac{1}{2}^+$ baryons [55].

Various $\Upsilon_{abc}^{\mathcal{B}:\lambda}$

$$\Upsilon_{uud}^{p:\lambda} = \frac{2}{\sqrt{6}} \; ; \quad \Upsilon_{udu}^{p:\lambda} = \Upsilon_{duu}^{p:\lambda} = \frac{-1}{\sqrt{6}} \tag{A.5}$$

$$\Upsilon_{udd}^{n:\lambda} = \Upsilon_{dud}^{n:\lambda} = \frac{1}{\sqrt{6}} \; ; \quad \Upsilon_{ddu}^{n:\lambda} = \frac{-2}{\sqrt{6}} \tag{A.6}$$

$$\Upsilon_{usd}^{\Lambda^0:\lambda} = \Upsilon_{sud}^{\Lambda^0:\lambda} = \frac{1}{2} \; ; \quad \Upsilon_{dsu}^{\Lambda^0} = \Upsilon_{sdu}^{\Lambda^0} = \frac{-1}{2} \tag{A.7}$$

$$\Upsilon_{uus}^{\Sigma^+:\lambda} = \frac{2}{\sqrt{6}} \; ; \quad \Upsilon_{usu}^{\Sigma^+:\lambda} = \Upsilon_{suu}^{\Sigma^+:\lambda} = \frac{-1}{\sqrt{6}} \tag{A.8}$$

$$\Upsilon^{\Sigma^0:\lambda}_{uds} = \Upsilon^{\Sigma^0:\lambda}_{dus} = \frac{2}{\sqrt{12}} \; ; \; \Upsilon^{\Sigma^0:\lambda}_{usd} = \Upsilon^{\Sigma^0:\lambda}_{dsu} = \Upsilon^{\Sigma^0:\lambda}_{sud} = \Upsilon^{\Sigma^0:\lambda}_{sdu} = \frac{-1}{\sqrt{2}} \tag{A.9}$$

$$\Upsilon^{\Sigma^-:\lambda}_{dds} = \frac{2}{\sqrt{6}} \; ; \; \Upsilon^{\Sigma^-:\lambda}_{dsd} = \Upsilon^{\Sigma^-:\lambda}_{sdd} = \frac{-1}{\sqrt{6}} \tag{A.10}$$

$$\Upsilon^{\Xi^0:\lambda}_{uss} = \Upsilon^{\Xi^0:\lambda}_{sus} = \frac{1}{\sqrt{6}} \; ; \; \Upsilon^{\Xi^0:\lambda}_{ssu} = \frac{-2}{\sqrt{6}} \tag{A.11}$$

$$\Upsilon^{\Xi^-:\lambda}_{dss} = \Upsilon^{\Xi^-:\lambda}_{sds} = \frac{1}{\sqrt{6}} \; ; \; \Upsilon^{\Xi^-:\lambda}_{ssd} = \frac{-2}{\sqrt{6}} \tag{A.12}$$

Various $\Upsilon^{\mathfrak{B}:\rho}_{abc}$

$$\Upsilon^{p:\rho}_{udu} = \frac{1}{\sqrt{2}} \; ; \; \Upsilon^{p:\rho}_{duu} = \frac{-1}{\sqrt{2}} \tag{A.13}$$

$$\Upsilon^{n:\rho}_{udd} = \frac{1}{\sqrt{2}} \; ; \; \Upsilon^{n:\rho}_{dud} = \frac{-1}{\sqrt{2}} \tag{A.14}$$

$$\psi^{\Lambda^0:\rho}_{uds} = \frac{2}{\sqrt{12}}\ ; \quad \psi^{\Lambda^0:\rho}_{dus} = \frac{-2}{\sqrt{12}}\ ; \quad \psi^{\Lambda^0:\rho}_{sdu} = \psi^{\Lambda^0:\rho}_{usd} = \frac{1}{\sqrt{12}}\ ;$$
$$\psi^{\Lambda^0:\rho}_{dsu} = \psi^{\Lambda^0:\rho}_{sud} = \frac{-1}{\sqrt{12}} \qquad \text{(A.15)}$$

$$\psi^{\Sigma^+:\rho}_{usu} = \frac{1}{\sqrt{2}}\ ; \quad \psi^{\Sigma^+:\rho}_{suu} = \frac{-1}{\sqrt{2}} \qquad \text{(A.16)}$$

$$\psi^{\Sigma^0:\rho}_{usd} = \psi^{\Sigma^0:\rho}_{dsu} = \frac{1}{2}\ ; \quad \psi^{\Sigma^0:\rho}_{sud} = \psi^{\Sigma^0:\rho}_{sdu} = \frac{-1}{2} \qquad \text{(A.17)}$$

$$\psi^{\Sigma^-:\rho}_{dsd} = \frac{1}{\sqrt{2}}\ ; \quad \psi^{\Sigma^-:\rho}_{sdd} = \frac{-1}{\sqrt{2}} \qquad \text{(A.18)}$$

$$\psi^{\Xi^0:\rho}_{uss} = \frac{1}{\sqrt{2}}\ ; \quad \psi^{\Xi^0:\rho}_{sus} = \frac{-1}{\sqrt{2}} \qquad \text{(A.19)}$$

$$\psi^{\Xi^-:\rho}_{dss} = \frac{1}{\sqrt{2}}\ ; \quad \psi^{\Xi^-:\rho}_{sds} = \frac{-1}{\sqrt{2}} \qquad \text{(A.20)}$$

APPENDIX B

AN INTEGRAL RELATION BETWEEN χ and $\bar{\chi}$

Before we derive the desired integral relation, let us prove a lemma which will be needed in the course of that derivation.

Lemma. If

$$\int d^3p\,d^3q\,e^{-i\vec{p}\cdot\vec{x}-i\vec{q}\cdot\vec{y}}\,f(\vec{p},\vec{q}) = \int d^3p\,d^3q\,e^{-i\vec{p}\cdot\vec{x}-i\vec{q}\cdot\vec{y}}\,g(\vec{p},\vec{q})$$

then

$$f(\vec{p},\vec{q}) = g(\vec{p},\vec{q}) \tag{B.1}$$

Proof of the Lemma. Since $\vec{p}$ and $\vec{q}$ are integration variables, the first equality can be rewritten as

$$\int d^3p'd^3q'e^{-i\vec{p}'\cdot\vec{x}-i\vec{q}'\cdot\vec{y}}\,f(\vec{p}',\vec{q}') = \int d^3p'd^3q'e^{-i\vec{p}'\cdot\vec{x}-i\vec{q}'\cdot\vec{y}}\,g(\vec{p}',\vec{q}') \tag{B.2}$$

Multiply both sides by $e^{i\vec{p}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}$ and integrate over $d^3x\,d^3y$.

$$(2\pi)^6\int d^3p'd^3q'\delta^3(\vec{p}-\vec{p}')\delta^3(\vec{q}-\vec{q}')f(\vec{p}',\vec{q}')$$
$$= (2\pi)^6\int d^3p'd^3q'\delta^3(\vec{p}-\vec{p}')\delta^3(\vec{q}-\vec{q}')g(\vec{p}',\vec{q}')$$

i.e.

$$f(\vec{p},\vec{q}) = g(\vec{p},\vec{q}) \tag{B.3}$$

Q.E.D.

We now proceed to derive the desired integral relation. We start with the equal-time BS amplitude. From eqn. (2.8),

$$\chi(\vec{x}_1t,\vec{x}_2t,\vec{x}_3t)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \equiv \langle 0|\psi_{\alpha aA}(\vec{x}_1t)\psi_{\beta bB}(\vec{x}_2t)\psi_{\gamma cC}(\vec{x}_3t)|\mathcal{B}:P,S\rangle \tag{B.4}$$

Similarly, from eqn. (2.12)

$$\bar{\chi}(\vec{x}_1 t, \vec{x}_2 t, \vec{x}_3 t)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \equiv \langle \mathcal{B}:P,S|\bar{\psi}_{\alpha aA}(\vec{x}_1 t)\bar{\psi}_{\beta bB}(\vec{x}_2 t)\bar{\psi}_{\gamma cC}(\vec{x}_3 t)|0\rangle \tag{B.5}$$

The T-product drops out at equal times. Starting from these two definitions, it is not very difficult to prove that

$$\chi(\vec{x}_1 t, \vec{x}_2 t, \vec{x}_3 t)^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC} = -\bar{\chi}(\vec{x}_1 t, \vec{x}_2 t, \vec{x}_3 t)^{\mathcal{B}:P,S}_{\alpha' aA,\beta' bB,\gamma' cC}\gamma^0_{\alpha'\alpha}\gamma^0_{\beta'\beta}\gamma^0_{\gamma'\gamma} \tag{B.6}$$

Now, from eqn. (2.15)

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{1}{(2\pi)^{12}}\int d^4p_1 d^4p_2 d^4p_3\, e^{-ip_1x_1-ip_2x_2-ip_3x_3}\,\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.7}$$

Also from eqn. (2.18)

$$\bar{\chi}(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{1}{(2\pi)^{12}}\int d^4p_1 d^4p_2 d^4p_3\, e^{ip_1x+ip_2x_2+ip_3x_3}\,\bar{\chi}(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.8}$$

From spacetime translation invariance, we can write

$$\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{(2\pi)^4}{(2\pi)^{3/2}}\,\delta^4(P-\mathcal{P})\,\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.9}$$

and, also

$$\bar{\chi}(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{(2\pi)^4}{(2\pi)^{3/2}}\,\delta^4(P-\mathcal{P})\,\bar{\chi}[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.10}$$

In eqns. (B.9) and (B.10) we must remember that the right-hand sides are still functions of (p_1,p_2,p_3). They only occur in the specific combinations $(\mathcal{P},p_\xi,p_\eta)$ due to space time translation invariance. Let us now substitute for $\chi(p_1,p_2,p_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$ from eqn. (B.9) in eqn. (B.7)

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{1}{(2\pi)^{12}} \int d^4p_1 d^4p_2 d^4p_3\, e^{-ip_1x_1-ip_2x_2-ip_3x_3}$$

$$\frac{(2\pi)^4}{(2\pi)^{3/2}}\, \delta^4(P-\mathcal{P})\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.11}$$

Let us now change the integration variables from (p_1,p_2,p_3) to $(\mathcal{P},p_\xi,p_\eta)$.

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} = \frac{1}{(2\pi)^8(2\pi)^{3/2}} \int d^4\mathcal{P}\, d^4p_\xi d^4p_\eta\, e^{-i\mathcal{P}X-ip_\xi\xi-ip_\eta\eta}$$

$$\delta^4(P-\mathcal{P})\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.12}$$

After carrying out the $\mathcal{P}$ integration, we obtain

$$\chi(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= \frac{1}{(2\pi)^8(2\pi)^{3/2}}\, e^{-iPX} \int d^4p_\xi d^4p_\eta\, e^{-ip_\xi\xi-ip_\eta\eta}\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.13}$$

Following similar steps, we can prove from eqns. (B.8) and (B.10),

$$\bar{\chi}(x_1,x_2,x_3)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= \frac{1}{(2\pi)^8(2\pi)^{3/2}}\, e^{iPX} \int d^4p_\xi d^4p_\eta\, e^{ip_\xi\xi+ip_\eta\eta}\bar{\chi}[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC} \tag{B.14}$$

We have now all the ingredients for deriving the desired relation.

From eqn. (B.13), we can obtain at equal times

$$\chi(\vec{x}_1t,\vec{x}_2t,\vec{x}_3t)^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= \frac{1}{(2\pi)^8(2\pi)^{3/2}}\, e^{-iP^0t+i\vec{P}\cdot\vec{X}} \int d^4p_\xi d^4p_\eta\, e^{i\vec{p}_\xi\vec{\xi}+i\vec{p}_\eta\vec{\eta}}\chi[p_\xi,p_\eta]^{\mathcal{B}:P,S}_{\alpha aA,\beta bB,\gamma cC}$$

$$= \frac{1}{(2\pi)^8 (2\pi)^{3/2}} e^{-iP^0 t + i\vec{P}\cdot\vec{X}} \int d^3 p_\xi d^3 p_\eta e^{i\vec{p}_\xi \vec{\xi} + i\vec{p}_\eta \vec{\eta}}$$

$$(\int dp_\xi^0 dp_\eta^0 \ \chi[p_\xi, p_\eta]^{\mathfrak{B}:P,S}_{\alpha aA, \beta bB, \gamma cC}) \quad \text{(B.15)}$$

Similarly, from eqn. (B.14)

$$\bar{\chi}(\vec{x}_1 t, \vec{x}_2 t, \vec{x}_3 t)^{\mathfrak{B}:P,S}_{\alpha aA, \beta bB, \gamma cC}$$

$$= \frac{1}{(2\pi)^8 (2\pi)^{3/2}} e^{iP^0 t - i\vec{P}\cdot\vec{X}} \int d^3 p_\xi d^3 p_\eta e^{-i\vec{p}_\xi \vec{\xi} - i\vec{p}_\eta \vec{\eta}}$$

$$(\int dp_\xi^0 dp_\eta^0 \ \bar{\chi}[p_\xi, p_\eta]^{\mathfrak{B}:P,S}_{\alpha aA, \beta bB, \gamma cC}) \quad \text{(B.16)}$$

Putting eqns. (B.15) and (B.16) in eqn. (B.6), we get

$$\frac{1}{(2\pi)^8 (2\pi)^{3/2}} e^{iP^0 t - i\vec{P}\cdot\vec{X}} \int d^3 p_\xi d^3 p_\eta e^{-i\vec{p}_\xi \vec{\xi} - i\vec{p}_\eta \vec{\eta}}$$

$$(\int dp_\xi^0 dp_\eta^0 \ \chi[p_\xi, p_\eta]^{\mathfrak{B}:P,S*}_{\alpha aA, \beta bB, \gamma cC})$$

$$= - \frac{1}{(2\pi)^8 (2\pi)^{3/2}} e^{iP^0 t - i\vec{P}\cdot\vec{X}} \int d^3 p_\xi d^3 p_\eta e^{-i\vec{p}_\xi \vec{\xi} - i\vec{p}_\eta \vec{\eta}}$$

$$(\int dp_\xi^0 dp_\eta^0 \ \bar{\chi}[p_\xi, p_\eta]^{\mathfrak{B}:P,S}_{\alpha' aA, \beta' bB, \gamma' cC}) \ \gamma^0_{\alpha'\alpha} \gamma^0_{\beta'\beta} \gamma^0_{\gamma'\gamma}$$

i.e. $\int d^3 p_\xi d^3 p_\eta e^{-i\vec{p}_\xi \vec{\xi} - i\vec{p}_\eta \vec{\eta}}$

$$(\int dp_\xi^0 dp_\eta^0 \ \chi[p_\xi, p_\eta]^{\mathfrak{B}:P,S*}_{\alpha aA, \beta bB, \gamma cC})$$

$$= - \int d^3 p_\xi d^3 p_\eta e^{-i\vec{p}_\xi \vec{\xi} - i\vec{p}_\eta \vec{\eta}}$$

$$(\int dp_\xi^0 dp_\eta^0 \ \bar{\chi}[p_\xi, p_\eta]^{\mathfrak{B}:P,S}_{\alpha' aA, \beta' bB, \gamma' cC}) \ \gamma^0_{\alpha'\alpha} \gamma^0_{\beta'\beta} \gamma^0_{\gamma'\gamma} \quad \text{(B.17)}$$

Invoking in eqn (B.17), the Lemma we proved above, we get

$$\int dp_\xi^0 dp_\eta^0 \ \chi[p_\xi, p_\eta]^{\mathfrak{B}:P,S*}_{\alpha aA, \beta bB, \gamma cC}$$

$$= - \int dp_\xi^0 dp_\eta^0 \ \bar{\chi}[p_\xi, p_\eta]^{\mathfrak{B}:P,S}_{\alpha' aA, \beta' bB, \gamma' cC} \ \gamma^0_{\alpha'\alpha} \gamma^0_{\beta'\beta} \gamma^0_{\gamma'\gamma} \quad \text{(B.18)}$$

Multiplying both the sides by $\gamma^0_{\alpha\tilde{\alpha}}\gamma^0_{\beta\tilde{\beta}}\gamma^0_{\gamma\tilde{\gamma}}$ we obtain

$$\int dp^0_\xi dp^0_\eta\, \chi[p_\xi, p_\eta]^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC}\, \gamma^0_{\alpha\tilde{\alpha}}\gamma^0_{\beta\tilde{\beta}}\gamma^0_{\gamma\tilde{\gamma}} = -\int dp^0_\xi dp^0_\eta\, \bar{\chi}[p_\xi, p_\eta]^{\mathcal{B}:P,S}_{\tilde{\alpha} aA,\tilde{\beta} bB,\tilde{\gamma} cC} \qquad (B.19)$$

Changing $\tilde{\alpha}$ to α' and multiplying both sides by (-1),

$$-\int dp^0_\xi dp^0_\eta\, \chi[p_\xi, p_\eta]^{\mathcal{B}:P,S*}_{\alpha aA,\beta bB,\gamma cC}\, \gamma^0_{\alpha\alpha'}\gamma^0_{\beta\beta'}\gamma^0_{\gamma\gamma'} = \int dp^0_\xi dp^0_\eta\, \bar{\chi}[p_\xi, p_\eta]^{\mathcal{B}:P,S}_{\alpha' aA,\beta' bB,\gamma' cC} \qquad (B.20)$$

This the desired result.

APPENDIX C

RELEVANT PART OF THE GLASHOW-SALAM-WEINBERG LAGRANGIAN

Following is the interaction piece of the Glashow-Salam-Weinberg Lagrangian which induces the flavour-changing radiative transitions we are concerned with. As before, we denote Lorentz indices by primed Greek indices and Dirac matrix indices by unprimed Greek indices. We are following Aoki et al. [61]

$$\mathcal{L}_{int} = C(W^+W^-A)^{\alpha'\beta'\gamma'\delta'}[(\partial_{\alpha'}W^+_{\beta'})W^-_{\gamma'}A_{\delta'}+(\partial_{\alpha'}W^-_{\beta'})A_{\gamma'}W^+_{\delta'}+(\partial_{\alpha'}A_{\beta'})W^+_{\gamma'}W^-_{\delta'}]$$
$$+ C(IiW^+)^{\mu'}_{\alpha\beta;\beta\gamma}\,\bar{\psi}^I_\alpha\psi^i_\gamma W^+_{\mu'} + C(iIW^-)^{\mu'}_{\alpha\beta;\beta\gamma}\,\bar{\psi}^i_\alpha\psi^I_\gamma W^-_{\mu'}$$
$$+ C(nnA)^{\mu'}_{\alpha\beta}\,\bar{\psi}^n_\alpha\psi^n_\beta A_{\mu'} + C(AW^+\chi^-)^{\mu'\nu'}A_{\mu'}W^+_{\nu'}\chi^- + C(AW^-\chi^+)^{\mu'\nu'}A_{\mu'}W^-_{\nu'}\chi^+$$
$$+ C(Ii\chi^+)_{\alpha\beta}\,\bar{\psi}^I_\alpha\psi^i_\beta\chi^+ + C(iI\chi^-)_{\alpha\beta}\bar{\psi}^i_\alpha\psi^I_\beta\chi^-$$
$$+ C(A\chi^-\chi^+)A_{\mu'}(\chi^-\overleftrightarrow{\partial}^{\mu'}\chi^+) \qquad (C.1)$$

where the vertex factors, viz. the C's, are as follows.

$$C(W^+W^-A)^{\alpha'\beta'\gamma'\delta'} = ie\,(g^{\alpha'\gamma'}g^{\beta'\delta'}-g^{\alpha'\delta'}g^{\beta'\gamma'}) \qquad (C.2)$$

$$C(IiW^+)^{\mu'}_{\alpha\beta;\beta\gamma} = \frac{g}{\sqrt{2}}\,U^\dagger_{Ii}\,\gamma^{\mu'}_{\alpha\beta}\left(\frac{1-\gamma_5}{2}\right)_{\beta\gamma} \qquad (C.3)$$

$$C(iIW^-)^{\mu'}_{\alpha\beta;\beta\gamma} = \frac{g}{\sqrt{2}}\,U_{iI}\,\gamma^{\mu'}_{\alpha\beta}\left(\frac{1-\gamma_5}{2}\right)_{\beta\gamma} \qquad (C.4)$$

$$C(nnA)^{\mu'}_{\alpha\beta} = eQ_n\gamma^{\mu'}_{\alpha\beta} \qquad (C.5)$$

$$C(AW^+\chi^-)^{\mu'\nu'} = -ie\,m_W g^{\mu'\nu'} \qquad (C.6)$$

$$C(AW^-\chi^+)^{\mu'\nu'} = ie\,m_W g^{\mu'\nu'} \qquad (C.7)$$

$$C(Ii\chi^+)_{\alpha\beta} = \frac{ig}{m_W\sqrt{2}}\,U^\dagger_{Ii}\left[m_I\left(\frac{1-\gamma_5}{2}\right)_{\alpha\beta} - m_i\left(\frac{1+\gamma_5}{2}\right)_{\alpha\beta}\right] \qquad (C.8)$$

$$C(iI\chi^{-})_{\alpha\beta} = \frac{ig}{m_W\sqrt{2}} U_{iI}[m_i(\frac{1-\gamma_5}{2})_{\alpha\beta} - m_I(\frac{1+\gamma_5}{2})_{\alpha\beta}] \quad (C.9)$$

$$C(A\chi^{-}\chi^{+}) = ie \quad (C.10)$$

In the expressions above, ψ_i refers to d, s or b quark; ψ_I refers to u, c, or t quark; and ψ_n refers to any of the six quarks. Q_n refers to the charge of ψ_n. $W^{\pm}$ are the charged intermediate vector bosons. $\chi^{\pm}$ are the charged would-be Goldstones which enter the calculation in the Feynman -t' Hooft gauge [63]. m_I and m_i are the masses of ψ_I and ψ_i respectively and m_W is the W-mass. e is the electronic charge. g is related to the Fermi coupling constant G_F by [40]

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} \quad (C.11)$$

U is the mixing matrix. Our U is, in fact, $V^{\dagger}$ where V is the usual Kobayashi-Maskawa matrix in the literature [61]. Also, the differential operator $\overleftrightarrow{\partial}^{\mu'}$ means the following

$$A\overleftrightarrow{\partial}^{\mu'}B \equiv A.\partial^{\mu'}B - \partial^{\mu'}A.B \quad (C.12)$$

APPENDIX D

EVALUATION OF THE FEYNMAN PARAMETER INTEGRALS

The Feynman parameter integrals involved in the calculation of the 1-quark vertex $s \to d\gamma$ and the flavour-changing self-energy $s \to d$ can be expressed in terms of some generic integrals. We dicuss the evaluation of those generic integrals in this appendix.

D.1. 1st Integral

The simplest Feynman parameter integral that occurs is the following

$$\mathcal{J}_{sel} = \int_0^1 d\alpha\ \alpha\ \ell n\tilde{\Delta} \tag{D.1}$$

where

$$\tilde{\Delta} = \tilde{m}_W^2\alpha + \tilde{m}_\ell^2(1-\alpha) - \tilde{p}_3^2\alpha(1-\alpha) \tag{D.2}$$

and

$$\tilde{m}_W \equiv \frac{m_W}{\sqrt{4\pi\mu^2}}\ ;\quad \tilde{m}_\ell \equiv \frac{m_\ell}{\sqrt{4\pi\mu^2}}\ ;\quad \tilde{p}_3 \equiv \frac{p_3}{\sqrt{4\pi\mu^2}} \tag{D.3}$$

m_W is the W-mass, m_ℓ is the internal quark mass in the loop and μ is the scale introcuded in the course of dimensional regularization. The physical results are independent of μ. p_3 in the s-quark momentum.

For the sake of some convenience in comparison with other integrals, we do a change of variables in eqn. (D.1) given by

$$\alpha' = 1 - \alpha \tag{D.4}$$

Eqn. (D.1) then becomes

$$\vartheta_{sel} = \int_0^1 d\alpha'(1-\alpha')\ell n[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha' - \tilde{p}_3^2\alpha'(1-\alpha')] \qquad (D.5)$$

An exact integration in eqn. (D.5) will involve complicated functions of p_3 [63]. This is not particularly convenient because p_3 is still to be integrated with the BS amplitudes. In fact, this is also unnecessary. The finite size of baryons offers a strong cutoff for quark momenta beyond 500 MeV. This is reflected in the $-\frac{1}{2\alpha^2}$ factor in the exponent of BS amplitude with $\alpha \simeq 500$ MeV. So, despite the fact that p_3 is integrated from $-\infty$ to $+\infty$, the integrand hardly gets any contribution from $p_3 >> 500$ MeV. With p_3 around 500 Mev, $(p_3^2/m_W^2) << 1$. We can carry out expansion in powers of these quantities. Let us consider the logarithm in eqn. (D.5)

$$\ell n[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha' - \tilde{p}_3^2\alpha'(1-\alpha')]$$

$$= \ell n[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha'] + \ell n\left[1 - \frac{\tilde{p}_3^2\alpha'(1-\alpha')}{\{\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha'\}}\right] \qquad (D.6)$$

We realize that

$$\frac{\tilde{p}_3^2\alpha'(1-\alpha')}{\{\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha'\}} = \frac{p_3^2}{m_W^2}\,\frac{\alpha'(1-\alpha')}{\left\{(1-\alpha') + \frac{m_\ell^2}{m_W^2}\alpha'\right\}} << 1 \qquad (D.7)$$

So, following Aoki et al [61], we expand the second lograithm in eqn. (D.6).

$$\ell n[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha' - \tilde{p}_3^2\alpha'(1-\alpha')]$$

$$= \ell n[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha'] - \frac{\tilde{p}_3^2\alpha'(1-\alpha')}{[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha']} - \dots \qquad (D.8)$$

Put eqn. (D.8) in eqn. (D.5)

$$\vartheta_{se1} = \int_0^1 d\alpha'(1-\alpha')\ln[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha'] - \int_0^1 d\alpha'(1-\alpha')\frac{\tilde{p}_3^2\alpha'(1-\alpha')}{[\tilde{m}_W^2(1-\alpha') + \tilde{m}_\ell^2\alpha']} - \cdots \tag{D.9}$$

Both of these integrals exist in tables [67] and can be done easily. The result looks like

$$\vartheta_{se1} = F_1(m_\ell^2, m_W^2, \mu^2) + \frac{p_3^2}{m_W^2} F_2(m_\ell^2, m_W^2) \tag{D.10}$$

where

$$F_1(m_\ell^2, m_W^2, \mu^2) \equiv \frac{1}{2}\ln\tilde{m}_W^2 - \frac{1}{4} + \frac{1}{2}\frac{m_\ell^2}{m_W^2 - m_\ell^2} + \frac{1}{2}\left(\frac{m_\ell^2}{m_W^2 - m_\ell^2}\right)^2 \ln\left(\frac{m_\ell^2}{m_W^2}\right) \tag{D.11}$$

and

$$F_2(m_\ell^2, m_W^2) \equiv -\frac{1}{6}\frac{1}{\left(1 - \frac{m_\ell^2}{m_W^2}\right)} + \frac{1}{2}\frac{m_\ell^2}{m_W^2}\frac{1}{\left(1 - \frac{m_\ell^2}{m_W^2}\right)^2} + \frac{m_\ell^4}{4 m_W^4}\frac{1}{\left(1 - \frac{m_\ell^2}{m_W^2}\right)^3}\left[1 + \ln\left(\frac{m_\ell^2}{m_W^2}\right)\right] + \frac{m_\ell^6}{6 m_W^6}\frac{1}{\left(1 - \frac{m_\ell^2}{m_W^2}\right)^4}\ln\left(\frac{m_\ell^2}{m_W^2}\right) \tag{D.12}$$

D.2. The 2nd Integral

The second integral is of the form

$$\vartheta_{se2} = \int_0^1 d\alpha \, \ln\tilde{\Delta} \tag{D.13}$$

Obviously, we can follow exactly the same steps as in D.1. The final result is

$$\vartheta_{se2} = F_3(m_\ell^2, m_W^2, \mu^2) + \frac{p_3^2}{m_W^2} F_4(m_\ell^2, m_W^2) \tag{D.14}$$

where

$$F_3(m_\ell^2, m_W^2, \mu^2) = \ell n\ m_W^2 - 1 - \left(\frac{m_\ell^2}{m_W^2 - m_\ell^2}\right) \ell n \left(\frac{m_\ell^2}{m_W^2}\right) \tag{D.15}$$

and

$$F_4(m_\ell^2, m_W^2) = \frac{1}{\left(1 - \frac{m_\ell^2}{m_W^2}\right)} \left[-\frac{1}{2} - \frac{m_\ell^2}{m_W^2\left(1 - \frac{m_\ell^2}{m_W^2}\right)} \left\{1 + \ell n\left(\frac{m_\ell^2}{m_W^2}\right)\right\} - \frac{m_\ell^4}{m_W^4\left(1 - \frac{m_\ell^2}{m_W^2}\right)^2} \ell n\left(\frac{m_\ell^2}{m_W^2}\right)\right] \tag{D.16}$$

D.3. The 3rd integral

The third generic integral is of the kind

$$\mathcal{I}_{34} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{1}{X} \tag{D.17}$$

where

$$X = m_W^2(1-\alpha_p) + m_\ell^2\alpha_p - p_3^2\alpha_p(1-\alpha_p) + 2p_3\cdot k'\alpha_p\alpha_k \tag{D.18}$$

In eqn. (D.18), k' is the photon momentum. Other symbols have already been explained in Sec. D.1

The α_k-integral in eqn. (D.17) exists in tables [67] and can be performed easily. After α_k integration, eqn. (D.17) assumes the following form

$$\mathcal{I}_{34} = \int_0^1 d\alpha \; \frac{1}{p\; 2p_3\cdot k'\alpha_p} \ln\{1 + \frac{2p_3\cdot k'\alpha_p(1-\alpha_p)}{[m_W^2(1-\alpha_p)+m_\ell^2\alpha_p - p_3^2\alpha_p(1-\alpha_p)]}\} \tag{D.19}$$

For reasons similar to those mentioned in Sec. D.1,

$$\frac{2p_3\cdot k'\alpha_p(1-\alpha_p)}{[m_W^2(1-\alpha_p) + m_i^2\alpha_p - p_3^2\alpha_p(1-\alpha_p)]} = \frac{2p_3\cdot k'}{m_W^2} \; \frac{\alpha_p(1-\alpha_p)}{[(1-\alpha_p) + \frac{m_\ell^2}{m_W^2}\alpha_p - \frac{p^2}{m_W^2}\alpha_p(1-\alpha_p)]}$$

$$\ll 1 \tag{D.20}$$

This is true in the entire domain of α_p. So, we can carry out an expansion of the logarithm in eqn. (D.19). As a result, we obtain

$$\mathcal{I}_{34} = \int_0^1 d\alpha_p\{\frac{(1-\alpha_p)}{[m_W^2(1-\alpha_p) + m_\ell^2\alpha_p - p_3^2\alpha_p(1-\alpha_p)]}$$
$$- \frac{1}{2}\frac{2(p_3\cdot k')\alpha_p(1-\alpha_p)^2}{[m_W^2(1-\alpha_p) + m_\ell^2\alpha_p - p_3^2\alpha_p(1-\alpha_p)]^2} + O(\frac{p^4}{m_W^6})\} \tag{D.21}$$

It is obvious from eqn. (D.7) that

$$\frac{p_3^2\alpha_p(1-\alpha_p)}{[m_W^2(1-\alpha_p) + m_\ell^2\alpha_p]} \ll 1 \qquad (D.22)$$

in the entire domain of α_p. So, we can expand the denominators in eqn. (D.21) leading to the following expression.

$$\mathcal{J}_{34} = \int_0^1 d\alpha_p \frac{(1-\alpha_p)}{[m_W^2(1-\alpha_p) + m_\ell^2\alpha_p]} + \int_0^1 d\alpha_p \frac{(p_3^2 - p_3\cdot k')\alpha_p(1-\alpha_p)^2}{[m_W^2(1-\alpha_p) + m_\ell^2\alpha_p]^2} + O(\frac{p_3^4}{m_W^6}) \qquad (D.23)$$

Making a change of variable

$$\alpha = 1 - \alpha_p \qquad (D.24)$$

eqn. (D.23) assumes the following form

$$\mathcal{J}_{34} = \int_0^1 d\alpha \frac{\alpha}{[m_\ell^2 + (m_W^2 - m_\ell^2)\alpha]} + \int_0^1 d\alpha_p \frac{(p_3^2 - p_3\cdot k')\alpha_p(1-\alpha)\alpha^2}{[m_\ell^2 + (m_W^2 - m_\ell^2)\alpha]^2} + O(\frac{p_3^4}{m_W^6}) \qquad (D.25)$$

These integrals exist in tables [67]. Borrowing from there

$$\mathcal{J}_{34} = \frac{1}{m_W^2} F_5(m_\ell^2, m_W^2) + \frac{1}{m_W^2}(\frac{p_3^2}{m_W^2}) F_{21}(m_\ell^2, m_W^2) - \frac{1}{m_W^2}(\frac{p_3\cdot k'}{m_W^2}) F_{21}(m_\ell^2, m_W^2) \qquad (D.26)$$

where

$$F_5(m_\ell^2, m_W^2) \equiv \frac{1}{(1-\frac{m_\ell^2}{m_W^2})} + \frac{m_\ell^2}{m_W^2}\frac{1}{(1-\frac{m_\ell^2}{m_W^2})^2} \ln(\frac{m_\ell^2}{m_W^2}) \qquad (D.27)$$

$$F_{19}(m_\ell^2,m_W^2) \equiv \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} + \left(\frac{m_\ell^2}{m_W^2}\right) \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} + 2\left(\frac{m_\ell^2}{m_W^2}\right) \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^3} \ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad (D.28)$$

$$F_{20}(m_\ell^2,m_W^2) \equiv \frac{1}{2\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} - 3\left(\frac{m_\ell^2}{m_W^2}\right) \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)} \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^3}$$

$$- 3\left(\frac{m_\ell^4}{m_W^4}\right) \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^4} \ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad (D.29)$$

$$F_{21}(m_\ell^2,m_W^2) \equiv F_{19}(m_\ell^2,m_W^2) - F_{20}(m_\ell^2,m_W^2) \qquad (D.30)$$

All the integrals in the sequel will be evaluated by following a similar expansion procedure. So, we shall only quote the final result in each case.

D.4. The 4th Integral

The 4th integral is of the form

$$\vartheta_{35} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \frac{\alpha_p}{X} \qquad (D.31)$$

X is given by eqn. (D.18). The reuslt of this integration is

$$\vartheta_{35} = \frac{1}{m_W^2} F_7(m_\ell^2,m_W^2) \qquad (D.32)$$

$F_7(m_\ell^2,m_W^2)$ is given by

$$F_7(m_\ell^2,m_W^2) \equiv F_5(m_\ell^2,m_W^2) - F_6(m_\ell^2,m_W^2) \qquad (D.33)$$

$F_5(m_\ell^2, m_W^2)$ is given by eqn. (D.27) and

$$F_6(m_\ell^2,m_W^2) \equiv \frac{1}{2\left(1-\frac{m_\ell^2}{m_W^2}\right)} - \left(\frac{m_\ell^2}{m_W^2}\right)\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} - \frac{m_\ell^4}{m_W^4}\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^3}\ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad \text{(D.34)}$$

D.5. The 5th Integral

Next, we consider

$$\vartheta_{36} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} \frac{\alpha_p^2}{X}\, d\alpha_k \qquad \text{(D.35)}$$

This integral is finally given by

$$\vartheta_{36} = \frac{1}{m_W^2} F_9(m_\ell^2,m_W^2) \qquad \text{(D.36)}$$

$F_9(m_\ell^2,m_W^2)$ is given by

$$F_9(m_\ell^2,m_W^2) \equiv F_5(m_\ell^2,m_W^2) - 2F_6(m_\ell^2,m_W^2) + F_8(m_\ell^2,m_W^2) \qquad \text{(D.37)}$$

We have already listed $F_5(m_\ell^2,m_W^2)$ and $F_6(m_\ell^2,m_W^2)$ in eqns. (D.27) and (D.34). $F_8(m_\ell^2,m_W^2)$ is given by

$$F_8(m_\ell^2,m_W^2) \equiv \frac{1}{3\left(1-\frac{m_\ell^2}{m_W^2}\right)} - \frac{1}{2}\frac{m_\ell^2}{m_W^2}\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} + \frac{m_\ell^4}{m_W^4}\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^3}$$

$$+ \frac{m_\ell^6}{m_W^6}\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^4}\ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad \text{(D.38)}$$

D.6. The 6th Integral

$$\vartheta_{37} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \frac{\alpha_k}{X} \qquad (D.39)$$

Carrying out the integration,

$$\vartheta_{37} = \frac{1}{m_W^2} F_{10}(m_\ell^2, m_W^2) \qquad (D.40)$$

where

$$F_{10}(m_\ell^2, m_W^2) \equiv \frac{1}{2} F_6(m_\ell^2, m_W^2) \qquad (D.41)$$

and $F_6(m_\ell^2, m_W^2)$ is given by eqn. (D.34).

D.7. The 7th Integral

$$\vartheta_{38} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \frac{\alpha_k^2}{X} \qquad (D.42)$$

This integral is given by

$$\vartheta_{38} = \frac{1}{m_W^2} F_{11}(m_\ell^2, m_W^2) \qquad (D.43)$$

where

$$F_{11}(m_\ell^2, m_W^2) = \frac{1}{3} F_8(m_\ell^2, m_W^2) \qquad (D.44)$$

and $F_8(m_\ell^2, m_W^2)$ is given by (D.38)

D.8. The 8th Integral

$$\vartheta_{39} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_p \alpha_k}{X} \qquad (D.45)$$

We get, after integration,

$$\vartheta_{39} = \frac{1}{m_W^2} F_{12}(m_\ell^2, m_W^2) \qquad (D.46)$$

where

$$F_{12}(m_\ell^2, m_W^2) \equiv \frac{1}{2} [F_6(m_\ell^2, m_W^2) - F_8(m_\ell^2, m_W^2)] \qquad (D.47)$$

$F_6(m_\ell^2, m_W^2)$ and $F_8(m_\ell^2, m_W^2)$ are listed above.

D.9. The 9th Integral

The 9th integral has the following form

$$\vartheta_{40} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{1}{Y} . \qquad (D.48)$$

Y is given by the following expression

$$Y = m_W^2 \alpha_p + m_\ell^2 (1-\alpha_p) - p_3^2 \alpha_p (1-\alpha_p) + 2 p_3 \cdot k' \alpha_p \alpha_k \qquad (D.49)$$

After the integrations, we obtain

$$\vartheta_{40} = \frac{1}{m_W^2} F_{14}(m_\ell^2, m_W^2) + \frac{1}{m_W^2} \left(\frac{p_3^2}{m_W^2}\right) F_{23}(m_\ell^2, m_W^2)$$
$$- \frac{1}{m_W^2} \left(\frac{p_3 \cdot k'}{m_W^2}\right) F_{23}(m_\ell^2, m_W^2) \qquad (D.50)$$

where

$$F_{14}(m_\ell^2, m_W^2) \equiv F_{13}(m_\ell^2, m_W^2) - F_5(m_\ell^2, m_W^2) \qquad (D.51)$$

$F_5(m_\ell^2, m_W^2)$, we have already listed above. $F_{13}(m_\ell^2, m_W^2)$ is given by

$$F_{13}(m_\ell^2,m_W^2) \equiv -\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)} \ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad \text{(D.52)}$$

Also, $F_{23}(m_\ell^2,m_W^2)$ is given by

$$F_{23}(m_\ell^2,m_W^2) \equiv F_{22}(m_\ell^2,m_W^2) - 2F_{19}(m_\ell^2,m_W^2) + F_{20}(m_\ell^2,m_W^2) \qquad \text{(D.53)}$$

$F_{19}(m_\ell^2,m_W^2)$ and $F_{20}(m_\ell^2,m_W^2)$ are given by eqns. (D.28) and (D.29). $F_{22}(m_\ell^2,m_W^2)$ is given by the following expression

$$F_{22}(m_\ell^2,m_W^2) \equiv -\frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)} - \frac{1}{\left(1-\frac{m_\ell^2}{m_W^2}\right)^2} \ln\left(\frac{m_\ell^2}{m_W^2}\right) \qquad \text{(D.54)}$$

D.10. The 10th Integral

$$\vartheta_{41} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_p}{Y} \qquad \text{(D.55)}$$

After the integrations, we obtain

$$\vartheta_{41} = \frac{1}{m_W^2} F_7(m_\ell^2,m_W^2) \qquad \text{(D.56)}$$

where $F_7(m_\ell^2,m_W^2)$ is given by eqn. (D.33).

D.11. The 11th Integral

$$\vartheta_{42} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_p^2}{Y} \qquad \text{(D.57)}$$

This integral is given by

$$\vartheta_{42} = \frac{1}{m_W^2} F_{15}(m_\ell^2, m_W^2) \qquad (D.58)$$

where

$$F_{15}(m_\ell^2, m_W^2) \equiv 2F_{12}(m_\ell^2, m_W^2) \qquad (D.59)$$

$F_{12}(m_\ell^2, m_W^2)$ is given by eqn. (D.47)

D.12. The 12th Integral

$$\vartheta_{43} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_k}{Y} \qquad (D.60)$$

After integrations, we obtain

$$\vartheta_{43} = \frac{1}{m_W^2} F_{16}(m_\ell^2, m_W^2) \qquad (D.61)$$

where

$$F_{16}(m_\ell^2, m_W^2) \equiv \frac{1}{2} \left\{ F_{13}(m_\ell^2, m_W^2) - 2F_5(m_\ell^2, m_W^2) + F_6(m_\ell^2, m_W^2) \right\} \qquad (D.62)$$

We have already listed $F_5(m_\ell^2, m_W^2)$, $F_6(m_\ell^2, m_W^2)$ and $F_{13}(m_\ell^2, m_W^2)$.

D.13. The 13th Integral

$$\vartheta_{44} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_k^2}{Y} \qquad (D.63)$$

We obtain, after integrations,

$$\vartheta_{44} = \frac{1}{m_W^2} F_{17}(m_\ell^2, m_W^2) \qquad (D.64)$$

where

$$F_{17}(m_\ell^2,m_W^2) \equiv \frac{1}{3}\{F_{13}(m_\ell^2,m_W^2) - 3F_5(m_\ell^2,m_W^2) + 3F_6(m_\ell^2,m_W^2) - F_8(m_\ell^2,m_W^2)\} \tag{D.65}$$

$F_5(m_\ell^2,m_W^2)$, $F_6(m_\ell^2,m_W^2)$, $F_8(m_\ell^2,m_W^2)$ and $F_{13}(m_\ell^2,m_W^2)$ have already been listed above.

D.14. The 14th Integral

$$\mathcal{J}_{45} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \frac{\alpha_p\alpha_k}{Y} \tag{D.66}$$

After performing the integration, we find

$$\mathcal{J}_{45} = \frac{1}{m_W^2} F_{18}(m_\ell^2,m_W^2) \tag{D.67}$$

where

$$F_{18}(m_\ell^2,m_W^2) \equiv \frac{1}{2} F_9(m_\ell^2,m_W^2) \tag{D.68}$$

and $F_9(m_\ell^2,m_W^2)$ is given by eqn. (D.37).

D.15. The 15th Integral

$$\mathcal{J}_{48} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \; \ln \frac{X}{X_0} \tag{D.69}$$

where

$$X_0 \equiv m_W^2(1-\alpha_p) + m_\ell^2\alpha_p - m_s^2\alpha_p(1-\alpha_p) + (m_s^2-m_d^2)\alpha_p\alpha_k \tag{D.70}$$

$\mathcal{J}_{48}$ is given by

$$\mathcal{J}_{48} = \left(\frac{p_3\cdot k'}{m_W^2}\right) F_{15}(m_\ell^2, m_W^2) - \left(\frac{p_3^2}{m_W^2}\right) F_{15}(m_\ell^2, m_W^2)$$

$$- \left[\frac{1}{2}\,\frac{(m_s^2 - m_\alpha^2)}{m_W^2} - \frac{m_s^2}{m_W^2}\right] F_{15}(m_\ell^2, m_W^2) \qquad \text{(D.71)}$$

$F_{15}(m_\ell^2, m_W^2)$ has already been listed above.

D.16. The 16th Integral

$$\mathcal{J}_{49} = \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_k \;\; \ell n \; \frac{Y}{Y_0} \qquad \text{(D.72)}$$

where

$$Y_0 \equiv m_W^2\alpha_p + m_\ell^2(1-\alpha_p) - m_s^2\alpha_p(1-\alpha_p) + (m_s^2 - m_d^2)\alpha_p\alpha_k \qquad \text{(D.73)}$$

After the integrations, we obtain

$$\mathcal{J}_{49} = \left(\frac{p_3\cdot k'}{m_W^2}\right) F_9(m_\ell^2, m_W^2) - \left(\frac{p_3^2}{m_W^2}\right) F_9(m_\ell^2, m_W^2)$$

$$- \left[\frac{1}{2}\,\frac{(m_s^2 - m_d^2)}{m_W^2} - \frac{m_s^2}{m_W^2}\right] F_9(m_\ell^2, m_W^2) \qquad \text{(D.74)}$$

We have already listed $F_9(m_\ell^2, m_W^2)$ above.

These generic integrals were sufficient to do all the other integrals in the 1-quark vertex $s \to d\gamma$ as well as in the flavour-changing self-energy $s \to d$.

APPENDIX E

EVALUATION OF THE MOMENTUM INTEGRALS

All the 4-momentum integrals that we encountered can be conveniently classified into eight generic types if we study their behaviour under Lorentz transformations. It is, therefore, enough to discuss the method of evaluating those eight generic types.

E.1 Intgerals of type 1.

This class consists of the following two integrals

$$I = \int d^4p d^4q \cdots e^{f(p^{\kappa'}, q^{\sigma'}, \ldots, P^{\mu'}, k'^{\nu'}, M, M', \alpha, \alpha', m_u, m_d, m_s)} \qquad (E.1)$$

$$\tilde{I} = \int_0^\infty dz\, e^{iz(-m_q^2+i\varepsilon)} \int d^4p d^4q \cdots e^{\tilde{f}(p^{\kappa'}, q^{\sigma'}, \ldots, P^{\mu'}, k'^{\nu'}, M, M', \alpha, \alpha', m_u, m_d, m_s, z)} \qquad (E.2)$$

Let us first explain the notation. In eqn. (E.1), $p^{\kappa'}$, $q^{\sigma'}$, ... are the 4-momenta to be integrated; $P^{\mu'}$, $k'^{\nu'}$ are the initial baryon and final photon momenta respectively; α and α' are the radii of the initial and final baryons; m_u, m_d and m_s are the quark masses. f is a real Lorentz invariant function of its arguments. It is then obvious that I will be real too. In eqn. (E.2), we have an additional member in the argument list of $\tilde{f}$, viz. z. This is the Schwinger parameter already familiar to us. $\tilde{f}$ is a complex Lorentz-invariant function of its arguments. We have one more z-integral after the 4-momentum integrals. m_q^2 stands generically for any of m_u^2, m_d^2 or m_s^2. It is obvious that $\tilde{I}$ is going to be complex. We should mention that f and $\tilde{f}$ stand for any function of their arguments and we use these symbols in all the integrals in a generic sense. Obviously, they are not necessarily the same function in case of all the integrals.

Having explained the notation, let us now study the properties of these integrals under Lorentz transformations. Since f and $\tilde{f}$ are Lorentz invariant functions and the measure $d^4p\,d^4q \cdots$ (also dz in the second case) are also Lorentz invariant, it is obvious that I and $\tilde{I}$ will also be Lorentz scalars. After the integrations, we are only left with the quantities $P^{\mu'}, k'^{\nu'}$, M, M', α, α', m_u, m_d, m_s. So, I and $\tilde{I}$ will be Lorentz scal[illegible] formed out of these quantities. The complex nature of $\tilde{I}$ does not interfere with this argument. So, I and $\tilde{I}$ can depend only on $P\cdot k', M, M', \alpha, \alpha', m_u, m_d, m_s$. We denote it symbolically as follows

$$I = T_1(P\cdot k', M, M', \alpha, \alpha', m_u, m_d, m_s) \tag{E.3}$$

$$\tilde{I} = T_2(P\cdot k', M, M', \alpha, \alpha', m_u, m_d, m_s) \tag{E.4}$$

T_1 is a real scalar and T_2, a complex scalar. We have to determine T_1 and T_2.

The fact that T_1 and T_2 are Lorentz scalars greatly simplifies their actual evaluation. Since they are scalars, we can evaluate them in any frame. In particular, we can go to the rest frame of the initial baryon

$$P = (M,0,0,0) \tag{E.5}$$

Our process being a two-body decay further simplifies the situation. Since T_1 and T_2 can only depend on $P\cdot k'$ (apart from other scalars which do not matter in this argument), it means that in the rest frame it will only depend on Mk'^0 which for the photon is $M|\vec{k}'|$. Since it only depends on the magnitude of the photon momentum in the rest frame, we can also choose it to be in the negative z-direction i.e.

$$k' = (k'^0, 0, 0, -k'^0) \quad \text{with} \quad k'^0 = |\vec{k}'| \tag{E.6}$$

The trick is to expand f and $\tilde{f}$ in terms of its Cartesian components keeping eqns. (E.5) and (E.6) in mind. In terms of the integration variables $(p^0, p^1, p^2, p^3, q^0, q^1, q^2, q^3, \cdots)$ the momentum integral then becomes a **generalized Gaussian integral**. In fact, due to our simple kinematics, in all cases it turned out to be a product of two 4-dimensional Gaussian integrals of the type discussed in Section E.10 at the end of this appendix. They were evaluated using the formulas given in Section E.10. That took care of the momentum integrations. For T_1, this completes the story.

In case of T_2, we have yet another scalar integral to do, viz., over z. After the momentum integrations, T_2 looks like

$$T_2 = \int_0^\infty dz\, e^{iz(-m_q^2+i\varepsilon)}\, \tilde{f}'(P\cdot k', M, M', \alpha, \alpha', m_u, m_d, m_s, z) \tag{E.7}$$

where $\tilde{f}'$ is another complex function of its arguments. This integration was performed numerically. The exponential factor $e^{iz(-m_q^2+i\varepsilon)}$, however, presents difficulty in doing the numerical integration because it is an oscillating function of z. The analytic behaviour of $\tilde{f}'$ in complex z plane comes to our rescue. It turns out that all the singularities of $\tilde{f}'$ lie on (or very close to) the positive and negative imaginary axes excluding the origin. It also dies off at infinity.

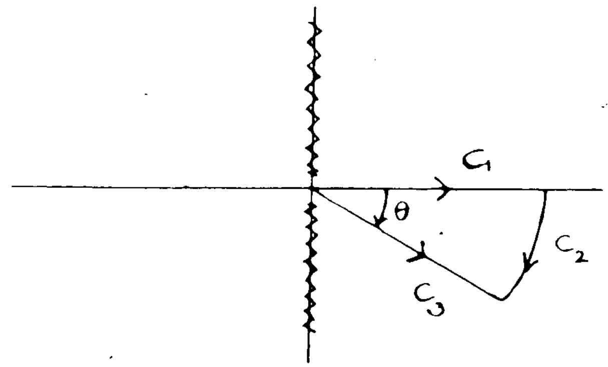


Fig. E.1 Contour for the z-integral

So, our integrand is analytic on and inside the contour shown in Fig. E.1. From Cauchy's theorem, therefore,

$$T_2 = \int_{C_3} dz\ e^{-izm_q^2}\ \tilde{f}'(P\cdot k',M,M',\alpha,\alpha',m_u,m_d,m_s,z) \tag{E.8}$$

where we have now dropped the $i\epsilon$ term. A parameteric representation of contour C_3 is

$$z = -i\left(\frac{1+i\ \cot\ \theta}{m_q^2}\right)\ y \quad ; \quad 0 \leq y < \infty \tag{E.9}$$

where θ is the angle by which the axis has been rotated in the fourth quadrant. From eqn. (E.9)

$$dz = -i\left(\frac{1+i\ \cot\ \theta}{m_q^2}\right)\ dy \tag{E.10}$$

Hence, the integral in eqn. (E.8) looks like

$$T_2 = -i\left(\frac{1+i\ \cot\ \theta}{m_q^2}\right)\int_0^\infty dy\ e^{-(1+i\ \cot\ \theta)y}\ \tilde{f}''(P\cdot k',M,M',\alpha,\alpha',m_u,m_d,m_s,y) \tag{E.11}$$

where $\tilde{f}''$ is yet another complex function of its arguments. Now the integrand has acquired a damped part and the numerical integration can be easily performed. The value of the integral is independent of θ. It is varied in the numerical computation to achieve a fast enough convergence of the integration subroutines. The integral was stabilized to 10 digits in the numerical computation. This completes the evaluation of T_2.

E.2 Integrals of type 2.

The integrals in this class are

$$I^{\alpha'} = \int d^4p d^4q \cdots (p^{\alpha'};q^{\alpha'}; \quad \cdots)e^{f(p^{\kappa'},q^{\sigma'},\ldots,p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \tag{E.12}$$

$$\tilde{I}^{\alpha'} = \int_0^{\infty} dz\; e^{iz(-m_q^2+i\varepsilon)} \int d^4p d^4q \cdots (p^{\alpha'};q^{\alpha'}; \quad \cdots)e^{\tilde{f}(p^{\kappa'},q^{\sigma'},\ldots,p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s,z)} \tag{E.13}$$

$(p^{\alpha'};q^{\alpha'};\ldots)$ in the integrand implies that we have any 'one' of these tensors multiplying the exponential in the integrand. We shall consistently use this symbol for the same purpose in the sequel. Rest of the notation should be clear from Section E.1.

Following our Lorentz invariance arguments, we see that due to the presence of $(p^{\alpha'};q^{\alpha'};\ldots)$ $I^{\alpha'}$ and $\tilde{I}^{\alpha'}$ are now first-rank Lorentz tensors. After the integrations, they can only be functions of $p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s$. We can, therefore, very generally write

$$\vartheta^{\alpha'} = T_1 p^{\alpha'} + T_2 k'^{\alpha'} \tag{E.14}$$

Once again, these T_1 and T_2 should not be confused with T_1 and T_2 in Section E.1 or in the sequel. They are being used in a generic sense. $\vartheta^{\alpha'}$ stand for either $I^{\alpha'}$ or $\tilde{I}^{\alpha'}$. The tensor structure is the same for both of them. Values of T_1 and T_2 will be different in each case. These will be real in case of $I^{\alpha'}$ and complex in case of $\tilde{I}^{\alpha'}$. Both T_1 and T_2 are Lorentz scalars built out of $P^{\mu'}, k'^{\nu'}, M, M', \alpha, \alpha', m_u, m_d, m_s$. So, they are functions of $P\cdot k', M, M', \alpha, \alpha', m_u, m_d, m_s$. We suppress their arguments for the sake of clarity. If we can calculate the values of T_1 and T_2, then our integrals are known.

Towards this end, let us first invert eqn. (E.14).

$$T_i = \tilde{\Lambda}^i_{\alpha'} \mathcal{J}^{\alpha'} \quad ; \quad i = 1,2 \tag{E.15}$$

where

$$\tilde{\Lambda}^1_{\alpha'} = \frac{1}{(P\cdot k')} k'_{\alpha'} \tag{E.16}$$

$$\tilde{\Lambda}^2_{\alpha'} = \frac{1}{(P\cdot k')} P_{\alpha'} - \frac{M^2}{(P\cdot k')^2} k'_{\alpha'} \tag{E.17}$$

We call $\tilde{\Lambda}^i_{\alpha'}$, **the projectors** . Since T_1 and T_2 are Lorentz scalars, we can evaluate them in the special frame given by eqns. (E.5) and (E.6). This leads to considerable simplification. In this frame, only certain components of the projectors are nonzero. To be specific, only $\tilde{\Lambda}^i_0$ and $\tilde{\Lambda}^i_3$ are nonzero. Their values can be calculated from eqns. (E.16) and (E.17). So, we need to calculate only $\mathcal{J}^0$ and $\mathcal{J}^3$ in the special frame to know T_1 and T_2 completely from eqn. (E.15). These components of I^{α} and $\tilde{I}^{\alpha'}$ can be calculated in exactly the same manner as outlined in Section E.1. To repeat, the components of $I^{\alpha'}$ reduce to generalized Gaussian integrals to be discussed at the end of this appendix. For the components of $\tilde{I}^{\alpha'}$, the momentum integrals are once again generalized Gaussian integrals. The z-integral can be performed numerically as discussed in the last section. Once we have the values of $\mathcal{I}^0$ and $\mathcal{I}^3$, we can carry out the sum in eqn. (E.15). That will give us T_1 and T_2 and hence $\mathcal{I}^{\alpha'}$.

E.3. Integrals of type 3

These integrals come in the following form

$$I^{(\alpha'\beta')} = \int d^4p d^4q \cdots (p^{\alpha'}p^{\beta'};q^{\alpha'}q^{\beta'}; \cdots)e^{f(p^{\kappa'},q^{\sigma'},\ldots,p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \qquad \text{(E.18)}$$

$$\tilde{I}^{(\alpha'\beta')} = \int_0^\infty dz\; e^{iz(-m_q^2+i\varepsilon)} \int d^4p d^4q \cdots (p^{\alpha'}p^{\beta'};q^{\alpha'}q^{\beta'}; \cdots)e^{\tilde{f}(p^{\kappa'},q^{\sigma'},\ldots,p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s,z)} \qquad \text{(E.19)}$$

The notation needs no further explanation. These integrals differ from integral of type 2 in a more complicated Lorentz structure only. So, it is sufficient for us to write their Lorentz structure and projectors. Rest of the procedure must be obvious to us by now.

Their Lorentz structure is given by the following equation

$$\mathcal{I}^{(\alpha'\beta')} = T_1 g^{\alpha'\beta'} + T_2 p^{\alpha'}p^{\beta'} + T_3 k'^{\alpha'}k'^{\beta'} + T_4(p^{\alpha'}k'^{\beta'} + p^{\beta'}k'^{\alpha'}) \qquad \text{(E.20)}$$

Once again, $\mathcal{I}^{(\alpha'\beta')}$ stands generically for either $I^{(\alpha'\beta')}$ or $\tilde{I}^{(\alpha'\beta')}$. It is obvious that T_i's will, in general, be different in the two cases. Also in the previous case they will be real and complex in the latter. We do not repeat writing the expressions separately for each of them in the interest of clarity. We shall consistently follow this mode of presentation in the sequel.

The inverse relations are given by

$$T_i = \tilde{\Lambda}^i_{\alpha'\beta'} \vartheta^{(\alpha'\beta')} \quad ; \quad i = 1, \ldots, 4 \tag{E.21}$$

where the projectors

$$\tilde{\Lambda}^1_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^2}\left[M^2 ♃^3_{\alpha'\beta'} + (P\cdot k')^2 ♃^1_{\alpha'\beta'} - (P\cdot k') ♃^4_{\alpha'\beta'}\right] \tag{E.22}$$

$$\tilde{\Lambda}^2_{\alpha'\beta'} = \frac{1}{(P\cdot k')^2} ♃^3_{\alpha'\beta'} \tag{E.23}$$

$$\tilde{\Lambda}^3_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^4}\left[3M^4 ♃^3_{\alpha'\beta'} + M^2(P\cdot k')^2 ♃^1_{\alpha'\beta'} - 3M^2(P\cdot k') ♃^4_{\alpha'\beta'} + 2(P\cdot k')^2 ♃^2_{\alpha'\beta'}\right] \tag{E.24}$$

$$\tilde{\Lambda}^4_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^3}\left[-3M^2 ♃^3_{\alpha'\beta'} - (P\cdot k')^2 ♃^1_{\alpha'\beta'} + 2(P\cdot k') ♃^4_{\alpha'\beta'}\right] \tag{E.25}$$

and the tensors $♃^i_{\alpha'\beta'}$ are given by

$$♃^1_{\alpha'\beta'} = g_{\alpha'\beta'} \tag{E.26}$$

$$♃^2_{\alpha'\beta'} = P_{\alpha'} P_{\beta'} \tag{E.27}$$

$$♃^3_{\alpha'\beta'} = k'_{\alpha'} k'_{\beta'} \tag{E.28}$$

$$♃^4_{\alpha'\beta'} = P_{\alpha'} k'_{\beta'} + P_{\beta'} k'_{\alpha'} \tag{E.29}$$

We first calculate the non-zero components of $\tilde{\Lambda}^i_{\alpha'\beta'}$ in our special frame. Then the corresponding components of $\vartheta^{(\alpha'\beta')}$ are calculated as per the procedure mentioned earlier. Carrying out the sum in (E.21) we get T_i's.

E.4 Integrals of type 4.

The integrals next in order of complexity are

$$I^{\alpha'\beta'} = \int d^4p\,d^4q \cdots (p^{\alpha'}q^{\beta'};\ldots)\, e^{f(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \tag{E.30}$$

$$\tilde{I}^{\alpha'\beta'} = \int_0^\infty dz\; e^{iz(-m_q^2+i\varepsilon)} \int d^4p\,d^4q \cdots (p^{\alpha'}q^{\beta'};\;\cdots)\, e^{\tilde{f}(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s,z)} \tag{E.31}$$

These integrals have the following Lorentz structure

$$\vartheta^{\alpha'\beta'} = T_1 g^{\alpha'\beta'} + T_2 P^{\alpha'}P^{\beta'} + T_3 k'^{\alpha'}k'^{\beta'} + T_4(P^{\alpha'}k'^{\beta'}+P^{\beta'}k'^{\alpha'}) + T_5(P^{\alpha'}k'^{\beta'}-P^{\beta'}k'^{\alpha'}) + T_6\varepsilon^{\alpha'\beta'\kappa'\sigma'}(P_{\kappa'}k'_{\sigma'}-P_{\sigma'}k'_{\kappa'}) \tag{E.32}$$

The inverse relations are

$$T_i = \tilde{\Lambda}^i_{\alpha'\beta'}\vartheta^{\alpha'\beta'} \quad ; \; i = 1,\ldots,6 \tag{E.33}$$

where the projectors are given by

$$\tilde{\Lambda}^1_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^2}\left[M^2 Ŧ^3_{\alpha'\beta'} + (P\cdot k')^2 Ŧ^1_{\alpha'\beta'} - (P\cdot k') Ŧ^4_{\alpha'\beta'}\right] \tag{E.34}$$

$$\tilde{\Lambda}^2_{\alpha'\beta'} = \frac{1}{(P\cdot k')^2} Ŧ^3_{\alpha'\beta'} \tag{E.35}$$

$$\tilde{\Lambda}^3_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^4}\left[3M^4 Ŧ^3_{\alpha'\beta'} + M^2(P\cdot k')^2 Ŧ^1_{\alpha'\beta'} - 3M^2(P\cdot k') Ŧ^4_{\alpha'\beta'} + 2(P\cdot k')^2 Ŧ^2_{\alpha'\beta'}\right] \tag{E.36}$$

$$\tilde{\Lambda}^4_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^3}\left[-3M^2 Ŧ^3_{\alpha'\beta'} - (P\cdot k')^2 Ŧ^1_{\alpha'\beta'} + 2(P\cdot k') Ŧ^4_{\alpha'\beta'}\right] \tag{E.37}$$

$$\tilde{\Lambda}^5_{\alpha'\beta'} = \frac{1}{2(P\cdot k')^2} Ŧ^5_{\alpha'\beta'} \tag{E.38}$$

$$\tilde{\Lambda}^{6}_{\alpha'\beta'} = \frac{1}{8(P\cdot k')^2} \mathcal{T}^{6}_{\alpha'\beta'} \qquad \text{(E.39)}$$

$\mathcal{T}^{i}_{\alpha'\beta'}$ are given by

$$\mathcal{T}^{1}_{\alpha'\beta'} = g_{\alpha'\beta'} \qquad \text{(E.38)}$$

$$\mathcal{T}^{2}_{\alpha'\beta'} = P_{\alpha'}P_{\beta'} \qquad \text{(E.39)}$$

$$\mathcal{T}^{3}_{\alpha'\beta'} = k'_{\alpha'}k'_{\beta'} \qquad \text{(E.40)}$$

$$\mathcal{T}^{4}_{\alpha'\beta'} = P_{\alpha'}k'_{\beta'} + P_{\beta'}k'_{\alpha'} \qquad \text{(E.41)}$$

$$\mathcal{T}^{5}_{\alpha'\beta'} = P_{\alpha'}k'_{\beta'} - P_{\beta'}k'_{\alpha'} \qquad \text{(E.42)}$$

$$\mathcal{T}^{6}_{\alpha'\beta'} = \varepsilon_{\alpha'\beta'\gamma'\delta'}(P^{\gamma'}k'^{\delta'} - P^{\gamma'}k'^{\gamma'}) \qquad \text{(E.43)}$$

E.5. Integrals of type 5

The simplest of the third rank integrals are

$$I^{(\alpha'\beta'\gamma')} = \int d^4p d^4q \cdots \quad (p^{\alpha'}p^{\beta'}p^{\gamma'};q^{\alpha'}q^{\beta'}q^{\gamma'};\cdots)$$
$$e^{f(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \qquad \text{(E.44)}$$

$$\tilde{I}^{(\alpha'\beta'\gamma')} = \int_0^\infty dz\, e^{iz(-m_q^2+i\varepsilon)} \int d^4p d^4q \cdots \quad (p^{\alpha'}p^{\beta'}p^{\gamma'};q^{\alpha'}q^{\beta'}q^{\gamma'};\cdots)$$
$$e^{\tilde{f}(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s,z)} \qquad \text{(E.45)}$$

These integrals have the following Lorentz structure

$$\begin{aligned}\mathcal{J}^{(\alpha'\beta'\gamma')} = {} & T_1(g^{\alpha'\beta'}P^{\gamma'} + g^{\beta'\gamma'}P^{\alpha'} + g^{\gamma'\alpha'}P^{\beta'}) \\ & + T_2(g^{\alpha'\beta'}k'^{\gamma'} + g^{\beta'\gamma'}k'^{\alpha'} + g^{\gamma'\alpha'}k'^{\beta'}) \\ & + T_3P^{\alpha'}P^{\beta'}P^{\gamma'} + T_4k'^{\alpha'}k'^{\beta'}k'^{\gamma'} \\ & + T_5(P^{\alpha'}P^{\beta'}k'^{\gamma'} + P^{\beta'}P^{\gamma'}k'^{\alpha'} + P^{\gamma'}P^{\alpha'}k'^{\beta'})\end{aligned}$$

$$+ T_6(P^{\alpha'}k'^{\beta'}k'^{\gamma'} + P^{\beta'}k'^{\gamma'}k'^{\alpha'} + P^{\gamma'}k'^{\alpha'}k'^{\beta'}) \tag{E.46}$$

The inverse relations are given by

$$T_i = \tilde{\Lambda}^i_{\alpha'\beta'\gamma'}\mathcal{J}^{(\alpha'\beta'\gamma')} \quad ; \quad i = 1,\ldots,6 \tag{E.47}$$

where the projectors are given by

$$\tilde{\Lambda}^1_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^3}\left[M^2 \mathcal{F}^4_{\alpha'\beta'\gamma'} + (P\cdot k')^2 \mathcal{F}^2_{\alpha'\beta'\gamma'} - 2(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'}\right] \tag{E.48}$$

$$\tilde{\Lambda}^2_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^4}\left[-M^4 \mathcal{F}^4_{\alpha'\beta'\gamma'} - M^2(P\cdot k')^2 \mathcal{F}^2_{\alpha'\beta'\gamma'}\right.$$

$$\left. + 3M^2(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'} + (P\cdot k')^3 \mathcal{F}^1_{\alpha'\beta'\gamma'} - 2(P\cdot k')^2 \mathcal{F}^5_{\alpha'\beta'\gamma'}\right] \tag{E.49}$$

$$\tilde{\Lambda}^3_{\alpha'\beta'\gamma'} = \frac{1}{(P\cdot k')^3}\mathcal{F}^4_{\alpha'\beta'\gamma'} \tag{E.50}$$

$$\tilde{\Lambda}^4_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^6}\left[-5M^6 \mathcal{F}^4_{\alpha'\beta'\gamma'} - 3M^4(P\cdot k')^2 \mathcal{F}^2_{\alpha'\beta'\gamma'}\right.$$

$$+ 15M^4(P\cdot k')^4 \mathcal{F}^6_{\alpha'\beta'\gamma'} + 3M^2(P\cdot k')^3 \mathcal{F}^1_{\alpha'\beta'\gamma'}$$

$$\left. - 12M^2(P\cdot k')^2 \mathcal{F}^5_{\alpha'\beta'\gamma'} + 2(P\cdot k')^3 \mathcal{F}^3_{\alpha'\beta'\gamma'}\right] \tag{E.51}$$

$$\tilde{\Lambda}^5_{\alpha'\beta'\gamma'} = \frac{1}{(P\cdot k')^4}\left[-2M^2 \mathcal{F}^4_{\alpha'\beta'\gamma'} - (P\cdot k')^2 \mathcal{F}^2_{\alpha'\beta'\gamma'} + 3(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'}\right] \tag{E.52}$$

$$\tilde{\Lambda}^6_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^5}\left[5M^4 \mathcal{F}^4_{\alpha'\beta'\gamma'} + 3M^2(P\cdot k')^2 \mathcal{F}^2_{\alpha'\beta'\gamma'}\right.$$

$$\left. - 12M^2(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'} - 2(P\cdot k')^3 \mathcal{F}^1_{\alpha'\beta'\gamma'} + 6(P\cdot k')^2 \mathcal{F}^5_{\alpha'\beta'\gamma'}\right] \tag{E.53}$$

and the $\mathcal{F}^i_{\alpha'\beta'\gamma'}$ are given by

$$\mathcal{F}^1_{\alpha'\beta'\gamma'} = g_{\alpha'\beta'}P_{\gamma'} \tag{E.54}$$

$$\mathcal{F}^2_{\alpha'\beta'\gamma'} = g_{\alpha'\beta'}k'_{\gamma'} \tag{E.55}$$

$$\mathcal{F}^3_{\alpha'\beta'\gamma'} = P_{\alpha'}P_{\beta'}P_{\gamma'} \tag{E.56}$$

$$\mathcal{F}^{4}_{\alpha'\beta'\gamma'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'} \qquad (E.57)$$

$$\mathcal{F}^{5}_{\alpha'\beta'\gamma'} = P_{\alpha'}P_{\beta'}k'_{\gamma'} \qquad (E.58)$$

$$\mathcal{F}^{6}_{\alpha'\beta'\gamma'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'} \qquad (E.59)$$

E.6. Integrals of type 6.

The tougher third rank integrals that we encountered were of the following form.

$$I^{(\alpha'\beta')\gamma'} = \int d^4p d^4q \cdots \ (p^{\alpha'}p^{\beta'}q^{\gamma'};q^{\alpha'}q^{\beta'}p^{\gamma'};\cdots) \ e^{f(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \qquad (E.60)$$

$$\tilde{I}^{(\alpha'\beta')\gamma'} = \int_0^{\infty} dz\ e^{iz(-m_q^2+i\varepsilon)} \int d^4p d^4q \cdots \ (p^{\alpha'}p^{\beta'}q^{\gamma'};q^{\alpha'}q^{\beta'}p^{\gamma'};\cdots) \ e^{\tilde{f}(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s,z)} \qquad (E.61)$$

The Lorentz structure of these integrals is given by

$$\begin{aligned}
\mathcal{J}^{(\alpha'\beta')\gamma'} = {} & T_1 P^{\alpha'}P^{\beta'}P^{\gamma'} + T_2 P^{\alpha'}P^{\beta'}k'^{\gamma'} + T_3(P^{\alpha'}k'^{\beta'}+P^{\beta'}k'^{\alpha'})k'^{\gamma'} \\
& + T_4 k'^{\alpha'}k'^{\beta'}k'^{\gamma'} + T_5(P^{\alpha'}k'^{\beta'}+P^{\beta'}k'^{\alpha'})P^{\gamma'} + T_6 P^{\gamma'}k'^{\alpha'}k'^{\beta'} \\
& + T_7(P^{\alpha'}g^{\beta'\gamma'}+P^{\beta'}g^{\alpha'\gamma'}) + T_8(k'^{\alpha'}g^{\beta'\gamma'}+k'^{\beta'}g^{\alpha'\gamma'}) \\
& + T_9 P^{\gamma'}g^{\alpha'\beta'} + T_{10}k'^{\gamma'}g^{\alpha'\beta'} \\
& + T_{11}(P^{\alpha'}\varepsilon^{\beta'\gamma'\delta'\sigma'}P_{\delta'}k'_{\sigma'} + P^{\beta'}\varepsilon^{\alpha'\gamma'\delta'\sigma'}P_{\delta'}k'_{\sigma'}) \\
& + T_{12}(k'^{\alpha'}\varepsilon^{\beta'\gamma'\delta'\sigma'}P_{\delta'}k'_{\sigma'} + k'^{\beta'}\varepsilon^{\alpha'\gamma'\delta'\sigma'}P_{\delta'}k'_{\sigma'})
\end{aligned} \qquad (E.62)$$

The inverse relations are

$$T_i = \tilde{\Lambda}^{i}_{\alpha'\beta'\gamma'}\mathcal{J}^{(\alpha'\beta')\gamma'} \qquad i = 1,\ldots,12 \qquad (E.63)$$

where the projectors are given by

$$\tilde{\Lambda}^{1}_{\alpha'\beta'\gamma'} = \frac{1}{(P\cdot k')^3}\,\mathcal{F}^{4}_{\alpha'\beta'\gamma'} \tag{E.64}$$

$$\tilde{\Lambda}^{2}_{\alpha'\beta'\gamma'} = \frac{1}{(P\cdot k')^4}\Big[-2M^2\mathcal{F}^{4}_{\alpha'\beta'\gamma'} - (P\cdot k')^2\mathcal{F}^{8}_{\alpha'\beta'\gamma'} + (P\cdot k')\mathcal{F}^{3}_{\alpha'\beta'\gamma'} + 2(P\cdot k')\mathcal{F}^{6}_{\alpha'\beta'\gamma'}\Big] \tag{E.65}$$

$$\begin{aligned}\tilde{\Lambda}^{3}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^5}\Big[&5M^4\mathcal{F}^{4}_{\alpha'\beta'\gamma'} + M^2(P\cdot k')^2\mathcal{F}^{10}_{\alpha'\beta'\gamma'}\\ &+ 2M^2(P\cdot k')^2\mathcal{F}^{8}_{\alpha'\beta'\gamma'} - 7M^2(P\cdot k')\mathcal{F}^{3}_{\alpha'\beta'\gamma'} - 5M^2(P\cdot k')\mathcal{F}^{6}_{\alpha'\beta'\gamma'}\\ &- (P\cdot k')^3\mathcal{F}^{9}_{\alpha'\beta'\gamma'} - (P\cdot k')^3\mathcal{F}^{7}_{\alpha'\beta'\gamma'} + (P\cdot k')^2\mathcal{F}^{2}_{\alpha'\beta'\gamma'}\\ &+ 5(P\cdot k')^2\mathcal{F}^{5}_{\alpha'\beta'\gamma'}\Big]\end{aligned} \tag{E.66}$$

$$\begin{aligned}\tilde{\Lambda}^{4}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^6}\Big[&-5M^6\mathcal{F}^{4}_{\alpha'\beta'\gamma'} - M^4(P\cdot k')^2\mathcal{F}^{10}_{\alpha'\beta'\gamma'}\\ &- 2M^4(P\cdot k')^2\mathcal{F}^{8}_{\alpha'\beta'\gamma'} + 10M^4(P\cdot k')\mathcal{F}^{3}_{\alpha'\beta'\gamma'} + 5M^4(P\cdot k')\mathcal{F}^{6}_{\alpha'\beta'\gamma'}\\ &+ M^2(P\cdot k')^3\mathcal{F}^{9}_{\alpha'\beta'\gamma'} + 2M^2(P\cdot k)^3\mathcal{F}^{7}_{\alpha'\beta'\gamma'} - 4M^2(P\cdot k')^2\mathcal{F}^{2}_{\alpha'\beta'\gamma'}\\ &- 8M^2(P\cdot k')^2\mathcal{F}^{5}_{\alpha'\beta'\gamma'} + 2(P\cdot k')^3\mathcal{F}^{1}_{\alpha'\beta'\gamma'}\Big]\end{aligned} \tag{E.67}$$

$$\begin{aligned}\tilde{\Lambda}^{5}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^4}\Big[&-4M^2\mathcal{F}^{4}_{\alpha'\beta'\gamma'} - (P\cdot k')^2\mathcal{F}^{10}_{\alpha'\beta'\gamma'} - (P\cdot k')^2\mathcal{F}^{8}_{\alpha'\beta'\gamma'}\\ &+ 5(P\cdot k')\mathcal{F}^{3}_{\alpha'\beta'\gamma'} + (P\cdot k')\mathcal{F}^{6}_{\alpha'\beta'\gamma'}\Big]\end{aligned} \tag{E.68}$$

$$\begin{aligned}\tilde{\Lambda}^{6}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^5}\Big[&5M^4\mathcal{F}^{4}_{\alpha'\beta'\gamma'} + M^2(P\cdot k')^2\mathcal{F}^{10}_{\alpha'\beta'\gamma'}\\ &+ 2M^2(P\cdot k')^2\mathcal{F}^{8}_{\alpha'\beta'\gamma'} - 10M^2(P\cdot k')\mathcal{F}^{3}_{\alpha'\beta'\gamma'} - 2M^2(P\cdot k')\mathcal{F}^{6}_{\alpha'\beta'\gamma'}\end{aligned}$$

$$- 2(P\cdot k')^3 \mathcal{T}^7_{\alpha'\beta'\gamma'} + 4(P\cdot k)^2 \mathcal{T}^2_{\alpha'\beta'\gamma'} + 2(P\cdot k')^2 \mathcal{T}^5_{\alpha'\beta'\gamma'}] \tag{E.69}$$

$$\tilde{\Lambda}^7_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^3} [M^2 \mathcal{T}^4_{\alpha'\beta'\gamma'} + (P\cdot k')^2 \mathcal{T}^8_{\alpha'\beta'\gamma'} - (P\cdot k')\mathcal{T}^3_{\alpha'\beta'\gamma'} - (P\cdot k')\mathcal{T}^6_{\alpha'\beta'\gamma'}] \tag{E.70}$$

$$\tilde{\Lambda}^8_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^4} [-M^4 \mathcal{T}^4_{\alpha'\beta'\gamma'} - M^2 (P\cdot k')^2 \mathcal{T}^8_{\alpha'\beta'\gamma'} + 2M^2 (P\cdot k')\mathcal{T}^3_{\alpha'\beta'\gamma'} + M^2 (P\cdot k')\mathcal{T}^6_{\alpha'\beta'\gamma'} + (P\cdot k')^3 \mathcal{T}^7_{\alpha'\beta'\gamma'} - (P\cdot k')^2 \mathcal{T}^2_{\alpha'\beta'\gamma'} - (P\cdot k)^2 \mathcal{T}^5_{\alpha'\beta'\gamma'}] \tag{E.71}$$

$$\tilde{\Lambda}^9_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^3} [M^2 \mathcal{T}^4_{\alpha'\beta'\gamma'} + (P\cdot k')^2 \mathcal{T}^{10}_{\alpha'\beta'\gamma'} - 2(P\cdot k')\mathcal{T}^3_{\alpha'\beta'\gamma'}] \tag{E.72}$$

$$\tilde{\Lambda}^{10}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^4} [-M^4 \mathcal{T}^4_{\alpha'\beta'\gamma'} - M^2 (P\cdot k')^2 \mathcal{T}^{10}_{\alpha'\beta'\gamma'} + 2M^2 (P\cdot k')\mathcal{T}^3_{\alpha'\beta'\gamma'} + M^2 (P\cdot k')\mathcal{T}^6_{\alpha'\beta'\gamma'} + (P\cdot k')^3 \mathcal{T}^9_{\alpha'\beta'\gamma'} - 2(P\cdot k')^2 \mathcal{T}^5_{\alpha'\beta'\gamma'}] \tag{E.73}$$

$$\tilde{\Lambda}^{11}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^3} \mathcal{T}^{12}_{\alpha'\beta'\gamma'} \tag{E.74}$$

$$\tilde{\Lambda}^{12}_{\alpha'\beta'\gamma'} = \frac{1}{2(P\cdot k')^4} [-M^2 \mathcal{T}^{12}_{\alpha'\beta'\gamma'} + (P\cdot k')\mathcal{T}^{11}_{\alpha'\beta'\gamma'}] \tag{E.75}$$

The tensors $\mathcal{T}^i_{\alpha'\beta'\gamma'}$ are given by

$$\mathcal{T}^1_{\alpha'\beta'\gamma'} = P_{\alpha'} P_{\beta'} P_{\gamma'} \tag{E.76}$$

$$\mp^{2}_{\alpha'\beta'\gamma'} = P_{\alpha'}P_{\beta'}k'_{\gamma'} \quad \text{(E.77)}$$

$$\mp^{3}_{\alpha'\beta'\gamma'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'} \quad \text{(E.78)}$$

$$\mp^{4}_{\alpha'\beta'\gamma'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'} \quad \text{(E.79)}$$

$$\mp^{5}_{\alpha'\beta'\gamma'} = P_{\alpha'}k'_{\beta'}P_{\gamma'} \quad \text{(E.80)}$$

$$\mp^{6}_{\alpha'\beta'\gamma'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'} \quad \text{(E.81)}$$

$$\mp^{7}_{\alpha'\beta'\gamma'} = P_{\alpha'}g_{\beta'\gamma'} \quad \text{(E.82)}$$

$$\mp^{8}_{\alpha'\beta'\gamma'} = k'_{\alpha'}g_{\beta'\gamma'} \quad \text{(E.83)}$$

$$\mp^{9}_{\alpha'\beta'\gamma'} = P_{\gamma'}g_{\alpha'\beta'} \quad \text{(E.84)}$$

$$\mp^{10}_{\alpha'\beta'\gamma'} = k'_{\gamma'}g_{\alpha'\beta'} \quad \text{(E.85)}$$

$$\mp^{11}_{\alpha'\beta'\gamma'} = P_{\alpha'}\varepsilon_{\beta'\gamma'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.86)}$$

$$\mp^{12}_{\alpha'\beta'\gamma'} = k'_{\alpha'}\varepsilon_{\beta'\gamma'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.87)}$$

E.7. Integrals of type 7

In this class we only encountered real integrals of the kind

$$I^{(\alpha'\beta'\gamma'\delta')} = \int d^4p\,d^4q \ \ldots \ (p^{\alpha'}p^{\beta'}p^{\gamma'}p^{\delta'};q^{\alpha'}q^{\beta'}q^{\gamma'}q^{\delta'};\ldots)\; e^{f(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \quad \text{(E.88)}$$

The Lorentz structure of this integral is given by

$$\begin{aligned}
I^{(\alpha'\beta'\gamma'\delta')} &= T_1 P^{\alpha'}P^{\beta'}P^{\gamma'}P^{\delta'} \\
&+ T_2(P^{\alpha'}P^{\beta'}P^{\gamma'}k'^{\delta'} + P^{\alpha'}P^{\beta'}P^{\delta'}k'^{\gamma'} + P^{\alpha'}P^{\gamma'}P^{\delta'}k'^{\beta'} + P^{\beta'}P^{\gamma'}P^{\delta'}k'^{\alpha'}) \\
&+ T_3(P^{\alpha'}P^{\beta'}k'^{\gamma'}k'^{\delta'} + P^{\alpha'}P^{\gamma'}k'^{\delta'}k'^{\beta'} + P^{\alpha'}P^{\delta'}k'^{\beta'}k'^{\gamma'} + P^{\beta'}P^{\gamma'}k'^{\delta'}k'^{\alpha'} \\
&\quad + P^{\beta'}P^{\delta'}k'^{\alpha'}k'^{\gamma'} + P^{\gamma'}P^{\delta'}k'^{\alpha'}k'^{\beta'})
\end{aligned}$$

$$
\begin{aligned}
&+ T_4 (P^{\alpha'} k'^{\beta'} k'^{\gamma'} k'^{\delta'} + P^{\beta'} k'^{\gamma'} k'^{\delta'} k'^{\alpha'} + P^{\gamma'} k'^{\delta'} k'^{\alpha'} k'^{\beta'} \\
&\quad + P^{\delta'} P^{\alpha'} k'^{\beta'} k'^{\gamma'}) \\
&+ T_5 k'^{\alpha'} k'^{\beta'} k'^{\gamma'} k'^{\delta'} \\
&+ T_6 (P^{\alpha'} P^{\beta'} g^{\gamma'\delta'} + P^{\alpha'} P^{\gamma'} g^{\delta'\beta'} + P^{\alpha'} P^{\delta'} g^{\beta'\gamma'} + P^{\beta'} P^{\gamma'} g^{\delta'\alpha'} \\
&\qquad + P^{\beta'} P^{\delta'} g^{\alpha'\gamma'} + P^{\gamma'} P^{\delta'} g^{\alpha'\beta'}) \\
&+ T_7 (P^{\alpha'} k'^{\beta'} g^{\gamma'\delta'} + P^{\beta'} k'^{\alpha'} g^{\gamma'\delta'} + P^{\alpha'} k'^{\gamma'} g^{\delta'\beta'} + P^{\gamma'} k'^{\alpha'} g^{\delta'\beta'} \\
&\qquad + P^{\alpha'} k'^{\delta'} g^{\beta'\gamma'} + P^{\delta'} k'^{\alpha'} g^{\beta'\gamma'} + P^{\beta'} k'^{\gamma'} g^{\delta'\alpha'} + P^{\gamma'} k'^{\beta'} g^{\delta'\alpha'} \\
&\qquad + P^{\beta'} k'^{\delta'} g^{\alpha'\gamma'} + P^{\delta'} k'^{\beta'} g^{\gamma'\alpha'} + P^{\gamma'} k'^{\delta'} g^{\alpha'\beta'} + P^{\delta'} k'^{\gamma'} g^{\alpha'\beta'}) \\
&+ T_8 (k'^{\alpha'} k'^{\beta'} g^{\gamma'\delta'} + k'^{\alpha'} k'^{\gamma'} g^{\delta'\beta'} + k'^{\alpha'} k'^{\delta'} g^{\beta'\gamma'} + k'^{\beta'} k'^{\gamma'} g^{\delta'\alpha'} \\
&\qquad + k'^{\beta'} k'^{\delta'} g^{\alpha'\gamma'} + k'^{\gamma'} k'^{\delta'} g^{\alpha'\beta'}) \\
&+ T_9 (g^{\alpha'\beta'} g^{\gamma'\delta'} + g^{\alpha'\gamma'} g^{\delta'\beta'} + g^{\alpha'\delta'} g^{\beta'\gamma'})
\end{aligned}
\tag{E.89}
$$

The inverse relations are

$$T_i = \tilde{\Lambda}^i_{\alpha'\beta'\gamma'\delta'} I^{(\alpha'\beta'\gamma'\delta')} \qquad i = 1,\ldots,9 \tag{E.90}$$

where the projectors are given by

$$\tilde{\Lambda}^1_{\alpha'\beta'\gamma'\delta'} = \frac{1}{(P\cdot k')^4} \mathfrak{T}^5_{\alpha'\beta'\gamma'\delta'} \tag{E.91}$$

$$
\begin{aligned}
\tilde{\Lambda}^2_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^5} \Big[&-5M^2 \mathfrak{T}^5_{\alpha'\beta'\gamma'\delta'} - 3(P\cdot k')^2 \mathfrak{T}^8_{\alpha'\beta'\gamma'\delta'} \\
&+ 8(P\cdot k') \mathfrak{T}^4_{\alpha'\beta'\gamma'\delta'} \Big]
\end{aligned}
\tag{E.92}
$$

$$
\begin{aligned}
\tilde{\Lambda}^3_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6} \Big[&15M^4 \mathfrak{T}^5_{\alpha'\beta'\gamma'\delta'} + 12M^2 (P\cdot k')^2 \mathfrak{T}^8_{\alpha'\beta'\gamma'\delta'} \\
- 40M^2 (P\cdot k') \mathfrak{T}^4_{\alpha'\beta'\gamma'\delta'} &+ (P\cdot k')^4 \mathfrak{T}^9_{\alpha'\beta'\gamma'\delta'} - 12(P\cdot k')^3 \mathfrak{T}^7_{\alpha'\beta'\gamma'\delta'}
\end{aligned}
$$

$$+ 24(P\cdot k')^2 \mathcal{F}^3_{\alpha'\beta'\gamma'\delta'}] \qquad (E.93)$$

$$\tilde{\Lambda}^4_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^7}\Big[-35M^6\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'} - 30M^4(P\cdot k')^2\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'}$$
$$+ 120M^4(P\cdot k')\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'} - 3M^2(P\cdot k')^4\mathcal{F}^9_{\alpha'\beta'\gamma'\delta'} + 48M^2(P\cdot k')^3\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$- 120M^2(P\cdot k')^2\mathcal{F}^3_{\alpha'\beta'\gamma'\delta'} - 12(P\cdot k)^4\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'}$$
$$+ 32(P\cdot k')^3\mathcal{F}^2_{\alpha'\beta'\gamma'\delta'}\Big] \qquad (E.94)$$

$$\tilde{\Lambda}^5_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^8}\Big[35M^8\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'} + 30M^6(P\cdot k')^2\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'}$$
$$- 140M^6(P\cdot k')\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'} + 3M^4(P\cdot k')^4\mathcal{F}^9_{\alpha'\beta'\gamma'\delta'} - 60M^4(P\cdot k')^3\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$+ 180M^4(P\cdot k')^2\mathcal{F}^3_{\alpha'\beta'\gamma'\delta'} + 24M^2(P\cdot k')^4\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'}$$
$$- 80M^2(P\cdot k')^3\mathcal{F}^2_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^4\mathcal{F}^1_{\alpha'\beta'\gamma'\delta'}\Big] \qquad (E.95)$$

$$\tilde{\Lambda}^{6}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^{4}}\left[M^{2}\Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^{2}\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}\right] \tag{E.96}$$

$$\tilde{\Lambda}^{7}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^{5}}\left[-5M^{4}\Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} - 6M^{2}(P\cdot k')^{2}\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} + 16M^{2}(P\cdot k')\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^{4}\Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^{3}\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} - 12(P\cdot k')^{2}\Upsilon^{3}_{\alpha'\beta'\gamma'\delta'}\right] \tag{E.97}$$

$$\tilde{\Lambda}^{8}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^{6}}\left[5M^{6}\Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} + 6M^{4}(P\cdot k')^{2}\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} - 20M^{4}(P\cdot k')\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} + M^{2}(P\cdot k')^{4}\Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 12M^{2}(P\cdot k')^{3}\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} + 24M^{2}(P\cdot k')^{2}\Upsilon^{3}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^{4}\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k')^{3}\Upsilon^{2}_{\alpha'\beta'\gamma'\delta'}\right] \tag{E.98}$$

$$\tilde{\Lambda}^{9}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^{4}}\left[M^{4}\Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} + 2M^{2}(P\cdot k')^{2}\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} - 4M^{2}(P\cdot k')\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^{4}\Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^{3}\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^{2}\Upsilon^{3}_{\alpha'\beta'\gamma'\delta'}\right] \tag{E.99}$$

The tensors $\Upsilon^{i}_{\alpha'\beta'\gamma'\delta'}$ are given by

$$\Upsilon^{1}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}P_{\delta'} \tag{E.100}$$

$$\Upsilon^{2}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}k'_{\delta'} \tag{E.101}$$

$$\Upsilon^{3}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}k'_{\gamma'}k'_{\delta'} \tag{E.102}$$

$$\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \tag{E.103}$$

$$\mathcal{T}^{5}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \qquad (E.104)$$

$$\mathcal{T}^{6}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}g_{\gamma'\delta'} \qquad (E.105)$$

$$\mathcal{T}^{7}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \qquad (E.106)$$

$$\mathcal{T}^{8}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \qquad (E.107)$$

$$\mathcal{T}^{9}_{\alpha'\beta'\gamma'\delta'} = g_{\alpha'\beta'}g_{\gamma'\delta'} \qquad (E.108)$$

E.8. Integrals of type 8.

Once again, we only encountered real integrals of the following kind in this class.

$$I^{(\alpha'\beta'\gamma')\delta'} = \int d^4p\,d^4q \ \ldots \ (p^{\alpha'}p^{\beta'}p^{\gamma'}q^{\delta'};q^{\alpha'}q^{\beta'}q^{\gamma'}p^{\delta'};\ldots)$$
$$e^{f(p^{\kappa'},q^{\sigma'},\ldots,p^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \qquad (E.109)$$

The Lorentz structure of this integral is given by

$$I^{(\alpha'\beta'\gamma')\delta'} = T_1 P^{\alpha'}P^{\beta'}P^{\gamma'}P^{\delta'} + T_2 P^{\alpha'}P^{\beta'}P^{\gamma'}k'^{\delta'}$$
$$+ T_3 (P^{\alpha'}P^{\beta'}k'^{\gamma'} + P^{\beta'}P^{\gamma'}k'^{\alpha'} + P^{\gamma'}P^{\alpha'}k'^{\beta'})P^{\delta'}$$
$$+ T_4 (P^{\alpha'}P^{\beta'}k'^{\gamma'} + P^{\beta'}P^{\gamma'}k'^{\alpha'} + P^{\gamma'}P^{\alpha'}k'^{\beta'})k'^{\delta'}$$
$$+ T_5 (P^{\alpha'}k'^{\beta'}k'^{\gamma'} + P^{\beta'}k'^{\gamma'}k'^{\alpha'} + P^{\gamma'}k'^{\alpha'}k'^{\beta'})P^{\delta'}$$
$$+ T_6 (P^{\alpha'}k'^{\beta'}k'^{\gamma'} + P^{\beta'}k'^{\gamma'}k'^{\alpha'} + P^{\gamma'}k'^{\alpha'}k'^{\beta'})k'^{\delta'}$$
$$+ T_7 k'^{\alpha'}k'^{\beta'}k'^{\gamma'}P^{\delta'} + T_8 k'^{\alpha'}k'^{\beta'}k'^{\gamma'}k'^{\delta'}$$
$$+ T_9 (P^{\alpha'}P^{\beta'}g^{\gamma'\delta'} + P^{\beta'}P^{\gamma'}g^{\alpha'\delta'} + P^{\gamma'}P^{\alpha'}g^{\beta'\delta'})$$
$$+ T_{10} (P^{\alpha'}k'^{\beta'}g^{\gamma'\delta'} + P^{\beta'}k'^{\alpha'}g^{\gamma'\delta'} + P^{\alpha'}k'^{\gamma'}g^{\beta'\delta'} + P^{\gamma'}k'^{\alpha'}g^{\beta'\delta'}$$
$$+ P^{\beta'}k'^{\gamma'}g^{\alpha'\delta'} + P^{\gamma'}k'^{\beta'}g^{\alpha'\delta'})$$

$$+ T_{11}(k'^{\alpha'}k'^{\beta'}g^{\gamma'\delta'} + k'^{\beta'}k'^{\gamma'}g^{\alpha'\delta'} + k'^{\gamma'}k'^{\alpha'}g^{\beta'\delta'})$$

$$+ T_{12}(P^{\alpha'}g^{\beta'\gamma'} + P^{\beta'}g^{\gamma'\alpha'} + P^{\gamma'}g^{\alpha'\beta'})P^{\delta'}$$

$$+ T_{13}(P^{\alpha'}g^{\beta'\gamma'} + P^{\beta'}g^{\gamma'\alpha'} + P^{\gamma'}g^{\alpha'\beta'})k'^{\delta'}$$

$$+ T_{14}(k'^{\alpha'}g^{\beta'\gamma'} + k'^{\beta'}g^{\gamma'\alpha'} + k'^{\gamma'}g^{\alpha'\beta'})P^{\delta'}$$

$$+ T_{15}(k'^{\alpha'}g^{\beta'\gamma'} + k'^{\beta'}g^{\gamma'\alpha'} + k'^{\gamma'}g^{\alpha'\beta'})k'^{\delta'}$$

$$+ T_{16}(P^{\alpha'}P^{\beta'}\varepsilon^{\gamma'\delta'\eta'\sigma'} + P^{\beta'}P^{\gamma'}\varepsilon^{\alpha'\delta'\eta'\sigma'} + P^{\gamma'}P^{\alpha'}\varepsilon^{\beta'\delta'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$

$$+ T_{17}(P^{\alpha'}k'^{\beta'}\varepsilon^{\gamma'\delta'\eta'\sigma'} + P^{\beta'}k'^{\alpha'}\varepsilon^{\gamma'\delta'\eta'\sigma'} + P^{\beta'}k'^{\gamma'}\varepsilon^{\alpha'\delta'\eta'\sigma'}$$

$$+ P^{\gamma'}k'^{\beta'}\varepsilon^{\alpha'\delta'\eta'\sigma'} + P^{\gamma'}k'^{\alpha'}\varepsilon^{\beta'\delta'\eta'\sigma'} + P^{\alpha'}k'^{\gamma'}\varepsilon^{\beta'\delta'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$

$$+ T_{18}(k'^{\alpha'}k'^{\beta'}\varepsilon^{\gamma'\delta'\eta'\sigma'} + k'^{\beta'}k'^{\gamma'}\varepsilon^{\alpha'\delta'\eta'\sigma'}$$

$$+ k'^{\gamma'}k'^{\alpha'}\varepsilon^{\beta'\delta'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$

$$+ T_{19}(g^{\alpha'\beta'}\varepsilon^{\gamma'\delta'\eta'\sigma'} + g^{\beta'\gamma'}\varepsilon^{\alpha'\delta'\eta'\sigma'} + g^{\gamma'\alpha'}\varepsilon^{\beta'\delta'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$

$$+ T_{20}(g^{\alpha'\beta'}g^{\gamma'\delta'} + g^{\beta'\gamma'}g^{\alpha'\delta'} + g^{\gamma'\alpha'}g^{\beta'\delta'}) \qquad \text{(E.110)}$$

The inverse relations are given by

$$T_i = \tilde{\Lambda}^i_{\alpha'\beta'\gamma'\delta'} I^{(\alpha'\beta'\gamma')\delta'} \quad ; \quad i = 1,\ldots,20 \qquad \text{(E.111)}$$

where the projectors are given by

$$\tilde{\Lambda}^1_{\alpha'\beta'\gamma'\delta'} = \frac{1}{(P\cdot k')^4} \digamma^8_{\alpha'\beta'\gamma'\delta'} \qquad \text{(E.112)}$$

$$\tilde{\Lambda}^2_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^5}\left[-5M^2\digamma^8_{\alpha'\beta'\gamma'\delta'} - 3(P\cdot k')^2\digamma^{11}_{\alpha'\beta'\gamma'\delta'}\right.$$

$$\left. + 3(P\cdot k')\digamma^6_{\alpha'\beta'\gamma'\delta'} + 5(P\cdot k')\digamma^7_{\alpha'\beta'\gamma'\delta'}\right] \qquad \text{(E.113)}$$

$$\tilde{\Lambda}^3_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^5}\left[-5M^2\digamma^8_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2\digamma^{15}_{\alpha'\beta'\gamma'\delta'}\right.$$

$$\left. - (P\cdot k')^2\digamma^{11}_{\alpha'\beta'\gamma'\delta'} + 7(P\cdot k')\digamma^6_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')\digamma^7_{\alpha'\beta'\gamma'\delta'}\right] \qquad \text{(E.114)}$$

$$\tilde{\Lambda}^4_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6}\Big[15M^4\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'} + 5M^2(P\cdot k')^2\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 7M^2(P\cdot k')^2\mathcal{F}^{11}_{\alpha'\beta'\gamma'\delta'} - 25M^2(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'} - 15M^2(P\cdot k')\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$+ (P\cdot k')^4\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} - 6(P\cdot k')^3\mathcal{F}^{10}_{\alpha'\beta'\gamma'\delta'} - 5(P\cdot k')^3\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$- (P\cdot k')^3\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} + 18(P\cdot k')^2\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k')^2\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'}\Big] \tag{E.115}$$

$$\tilde{\Lambda}^5_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6}\Big[15M^4\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'} + 7M^2(P\cdot k')^2\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 5M^2(P\cdot k')^2\mathcal{F}^{11}_{\alpha'\beta'\gamma'\delta'} - 35M^2(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'} - 5M^2(P\cdot k')\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$+ (P\cdot k')^4\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} - 6(P\cdot k')^3\mathcal{F}^{10}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^3\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$- 5(P\cdot k')^3\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k')^2\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'} + 18(P\cdot k')^2\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'}\Big] \tag{E.116}$$

$$\tilde{\Lambda}^6_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^7}\Big[-35M^6\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'} - 15M^4(P\cdot k')^2\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$- 15M^4(P\cdot k')^2\mathcal{F}^{11}_{\alpha'\beta'\gamma'\delta'} + 85M^4(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'} + 35M^4(P\cdot k')\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$- 3M^2(P\cdot k')^4\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} + 22M^2(P\cdot k')^3\mathcal{F}^{10}_{\alpha'\beta'\gamma'\delta'} + 15M^2(P\cdot k')^3\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 11M^2(P\cdot k')^3\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} - 70M^2(P\cdot k')^2\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'}$$
$$- 50M^2(P\cdot k')^2\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k')^4\mathcal{F}^{12}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^4\mathcal{F}^9_{\alpha'\beta'\gamma'\delta'}$$
$$+ 28(P\cdot k')^3\mathcal{F}^3_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3\mathcal{F}^2_{\alpha'\beta'\gamma'\delta'}\Big] \tag{E.117}$$

$$\tilde{\Lambda}^7_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^7}\Big[-35M^6\mathcal{F}^8_{\alpha'\beta'\gamma'\delta'} - 15M^4(P\cdot k')^2\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$- 15M^4(P\cdot k')^2\mathcal{F}^{11}_{\alpha'\beta'\gamma'\delta'} + 105M^4(P\cdot k')\mathcal{F}^6_{\alpha'\beta'\gamma'\delta'} + 15M^4(P\cdot k')\mathcal{F}^7_{\alpha'\beta'\gamma'\delta'}$$
$$- 3M^2(P\cdot k')^4\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} + 30M^2(P\cdot k')^3\mathcal{F}^{10}_{\alpha'\beta'\gamma'\delta'} + 3M^2(P\cdot k')^3\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 15M^2(P\cdot k')^3\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} - 30M^2(P\cdot k')^2\mathcal{F}^5_{\alpha'\beta'\gamma'\delta'} - 90M^2(P\cdot k')^2\mathcal{F}^4_{\alpha'\beta'\gamma'\delta'}$$

$$- 12(P\cdot k')^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 12(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 20(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.118)}$$

$$\tilde{\Lambda}^{8}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^8}[35M^8 \mp^{8}_{\alpha'\beta'\gamma'\delta'} + 15M^6(P\cdot k')^2 \mp^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 15M^6(P\cdot k')^2 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 105M^6(P\cdot k') \mp^{6}_{\alpha'\beta'\gamma'\delta'} - 35M^6(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 3M^4(P\cdot k')^4 \mp^{20}_{\alpha'\beta'\gamma'\delta'} - 30M^4(P\cdot k')^3 \mp^{10}_{\alpha'\beta'\gamma'\delta'} - 15M^4(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$- 15M^4(P\cdot k')^3 \mp^{13}_{\alpha'\beta'\gamma'\delta'} + 90M^4(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 90M^4(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 12M^2(P\cdot k')^4 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 12M^2(P\cdot k')^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 60M^2(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'}$$
$$- 20M^2(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^4 \mp^{1}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.119)}$$

$$\tilde{\Lambda}^{9}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4}[M^2 \mp^{8}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \mp^{11}_{\alpha'\beta'\gamma'\delta'}$$
$$- (P\cdot k') \mp^{6}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.120)}$$

$$\tilde{\Lambda}^{10}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5}[-5M^4 \mp^{8}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^2 \mp^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$- 5M^2(P\cdot k')^2 \mp^{11}_{\alpha'\beta'\gamma'\delta'} + 11M^2(P\cdot k') \mp^{6}_{\alpha'\beta'\gamma'\delta'} + 5M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}$$
$$- (P\cdot k')^4 \mp^{20}_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k')^3 \mp^{10}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$+ (P\cdot k')^3 \mp^{13}_{\alpha'\beta'\gamma'\delta'} - 6(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} - 6(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.121)}$$

$$\tilde{\Lambda}^{11}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^6}[5M^6 \mp^{8}_{\alpha'\beta'\gamma'\delta'} + M^4(P\cdot k')^2 \mp^{15}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 5M^4(P\cdot k')^2 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 15M^4(P\cdot k') \mp^{6}_{\alpha'\beta'\gamma'\delta'} - 5M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}$$
$$+ M^2(P\cdot k')^4 \mp^{20}_{\alpha'\beta'\gamma'\delta'} - 10M^2(P\cdot k')^3 \mp^{10}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$- M^2(P\cdot k')^3 \mp^{13}_{\alpha'\beta'\gamma'\delta'} + 10M^2(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 14M^2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 4(P\cdot k')^4 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \Upsilon^{3}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \Upsilon^{2}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.122)$$

$$\tilde{\Lambda}^{12}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4}[M^2 \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.123)$$

$$\tilde{\Lambda}^{13}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5}[-5M^4 \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} - 5M^2(P\cdot k')^2 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'}$$

$$- M^2(P\cdot k')^2 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} + 11M^2(P\cdot k')\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} + 5M^2(P\cdot k')\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'}$$

$$- (P\cdot k')^4 \Upsilon^{20}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^3 \Upsilon^{10}_{\alpha'\beta'\gamma'\delta'} + 5(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ (P\cdot k')^3 \Upsilon^{13}_{\alpha'\beta'\gamma'\delta'} - 10(P\cdot k')^2 \Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2 \Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.124)$$

$$\tilde{\Lambda}^{14}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5}[-5M^4 \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} - 5M^2(P\cdot k')^2 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'}$$

$$- M^2(P\cdot k')^2 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} + 15M^2(P\cdot k')\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'}$$

$$- (P\cdot k')^4 \Upsilon^{20}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^3 \Upsilon^{10}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 5(P\cdot k')^3 \Upsilon^{13}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2 \Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} - 10(P\cdot k')^2 \Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.125)$$

$$\tilde{\Lambda}^{15}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^6}[5M^6 \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} + 5M^4(P\cdot k')^2 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^4(P\cdot k')^2 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} - 15M^4(P\cdot k')\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} - 5M^4(P\cdot k')\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^2(P\cdot k')^4 \Upsilon^{20}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^3 \Upsilon^{10}_{\alpha'\beta'\gamma'\delta'} - 5M^2(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 5M^2(P\cdot k')^3 \Upsilon^{13}_{\alpha'\beta'\gamma'\delta'} + 14M^2(P\cdot k')^2 \Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} + 10M^2(P\cdot k')^2 \Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 4(P\cdot k')^4 \Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k')^3 \Upsilon^{3}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.126)$$

$$\tilde{\Lambda}^{16}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4} \Upsilon^{18}_{\alpha'\beta'\gamma'\delta'} \qquad (E.127)$$

$$\tilde{\Lambda}^{17}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5}\left[-5M^2 \digamma^{18}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^2 \digamma^{19}_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k')\digamma^{17}_{\alpha'\beta'\gamma'\delta'}\right] \quad (E.128)$$

$$\tilde{\Lambda}^{18}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^6}\left[5M^4 \digamma^{18}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \digamma^{19}_{\alpha'\beta'\gamma'\delta'} - 10M^2(P\cdot k')\digamma^{17}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^2\digamma^{16}_{\alpha'\beta'\gamma'\delta'}\right] \quad (E.129)$$

$$\tilde{\Lambda}^{19}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^4}\left[M^2 \digamma^{18}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \digamma^{19}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')\digamma^{17}_{\alpha'\beta'\gamma'\delta'}\right] \quad (E.130)$$

$$\tilde{\Lambda}^{20}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^4}\left[M^4 \digamma^{8}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \digamma^{15}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \digamma^{11}_{\alpha'\beta'\gamma'\delta'} - 3M^2(P\cdot k')\digamma^{6}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')\digamma^{7}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^4\digamma^{20}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^3\digamma^{10}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^3\digamma^{14}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^3\digamma^{13}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2\digamma^{5}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2\digamma^{4}_{\alpha'\beta'\gamma'\delta'}\right] \quad (E.131)$$

The tensors $\digamma^{i}_{\alpha'\beta'\gamma'\delta'}$ are given by

$$\digamma^{1}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}P_{\delta'} \quad (E.132)$$

$$\digamma^{2}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}k'_{\delta'} \quad (E.133)$$

$$\digamma^{3}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}k'_{\gamma'}P_{\delta'} \quad (E.134)$$

$$\digamma^{4}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}k'_{\gamma'}k'_{\delta'} \quad (E.135)$$

$$\digamma^{5}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'}P_{\delta'} \quad (E.136)$$

$$\digamma^{6}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \quad (E.137)$$

$$\digamma^{7}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'}P_{\delta'} \quad (E.138)$$

$$\digamma^{8}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \quad (E.139)$$

$$\mathcal{F}^{9}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}g_{\gamma'\delta'} \tag{E.140}$$

$$\mathcal{F}^{10}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \tag{E.141}$$

$$\mathcal{F}^{11}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \tag{E.142}$$

$$\mathcal{F}^{12}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\delta'}g_{\beta'\gamma'} \tag{E.143}$$

$$\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\delta'}g_{\beta'\gamma'} \tag{E.144}$$

$$\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}P_{\delta'}g_{\beta'\gamma'} \tag{E.145}$$

$$\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\delta'}g_{\beta'\gamma'} \tag{E.146}$$

$$\mathcal{F}^{16}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}\epsilon_{\gamma'\delta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \tag{E.147}$$

$$\mathcal{F}^{17}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}\epsilon_{\gamma'\delta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \tag{E.148}$$

$$\mathcal{F}^{18}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}\epsilon_{\gamma'\delta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \tag{E.149}$$

$$\mathcal{F}^{19}_{\alpha'\beta'\gamma'\delta'} = g_{\alpha'\beta'}\epsilon_{\gamma'\delta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \tag{E.150}$$

$$\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} = g_{\alpha'\beta'}g_{\gamma'\delta} \tag{E.151}$$

E.9. Integrals of type 9

All the integrals in this class were also real. They were of the form

$$I^{(\alpha'\beta')(\gamma'\delta')} = \int d^4p\,d^4q \;\ldots\; (p^{\alpha'}p^{\beta'}q^{\gamma'}q^{\delta'};\ldots)\; e^{f(p^{\kappa'},q^{\sigma'},\ldots,P^{\mu'},k'^{\nu'},M,M',\alpha,\alpha',m_u,m_d,m_s)} \tag{E.152}$$

This integral has the following Lorentz structure.

$$I^{(\alpha'\beta')(\gamma'\delta')} = T_1 P^{\alpha'}P^{\beta'}P^{\gamma'}P^{\delta'} + T_2 P^{\alpha'}P^{\beta'}(P^{\gamma'}k'^{\delta'} + P^{\delta'}k'^{\gamma'})$$

$$+ T_3(P^{\alpha'}k'^{\beta'} + P^{\beta'}k'^{\alpha'})P^{\gamma'}P^{\delta'} + T_4 P^{\alpha'}P^{\beta'}k'^{\gamma'}k'^{\delta'}$$
$$+ T_5(P^{\alpha'}k'^{\beta'} + P^{\beta'}k'^{\alpha'})(P^{\gamma'}k'^{\delta'} + P^{\delta'}k'^{\gamma'}) + T_6 P^{\gamma'}P^{\delta'}k'^{\alpha'}k'^{\beta'}$$
$$+ T_7(P^{\alpha'}k'^{\beta'} + P^{\beta'}k'^{\alpha'})k'^{\gamma'}k'^{\delta'} + T_8(P^{\gamma'}k'^{\delta'} + P^{\delta'}k'^{\gamma'})k'^{\alpha'}k'^{\beta'}$$
$$+ T_9 k'^{\alpha'}k'^{\beta'}k'^{\gamma'}k'^{\delta'} + T_{10}P^{\alpha'}P^{\beta'}g^{\gamma'\delta'}$$
$$+ T_{11}(P^{\alpha'}k'^{\beta'} + P^{\beta'}k'^{\alpha'})g^{\gamma'\delta'} + T_{12}k'^{\alpha'}k'^{\beta'}g^{\gamma'\delta'}$$
$$+ T_{13}(P^{\alpha'}P^{\gamma'}g^{\delta'\beta'} + P^{\alpha'}P^{\delta'}g^{\gamma'\beta'} + P^{\beta'}P^{\gamma'}g^{\delta'\alpha'} + P^{\beta'}P^{\delta'}g^{\gamma'\alpha'})$$
$$+ T_{14}(P^{\alpha'}k'^{\gamma'}g^{\delta'\beta'} + P^{\alpha'}k'^{\delta'}g^{\gamma'\beta'} + P^{\beta'}k'^{\gamma'}g^{\delta'\alpha'} + P^{\beta'}k'^{\delta'}g^{\gamma'\alpha'})$$
$$+ T_{15}(P^{\gamma'}k'^{\alpha'}g^{\delta'\beta'} + P^{\delta'}k'^{\alpha'}g^{\gamma'\beta'} + P^{\gamma'}k'^{\beta'}g^{\delta'\alpha'} + P^{\delta'}k'^{\beta'}g^{\gamma'\alpha'})$$
$$+ T_{16}(k'^{\alpha'}k'^{\gamma'}g^{\delta'\beta'} + k'^{\alpha'}k'^{\delta'}g^{\gamma'\beta'} + k'^{\beta'}k'^{\gamma'}g^{\delta'\alpha'} + k'^{\beta'}k'^{\delta'}g^{\gamma'\alpha'})$$
$$+ T_{17}P^{\gamma'}P^{\delta'}g^{\alpha'\beta'} + T_{18}(P^{\gamma'}k'^{\delta'} + P^{\delta'}k'^{\gamma'})g^{\alpha'\beta'} + T_{19}k'^{\gamma'}k'^{\delta'}g^{\alpha'\beta'}$$
$$+ T_{20}(P^{\alpha'}P^{\gamma'}\varepsilon^{\delta'\beta'\eta'\sigma'} + P^{\alpha'}P^{\delta'}\varepsilon^{\gamma'\beta'\eta'\sigma'} + P^{\beta'}P^{\gamma'}\varepsilon^{\delta'\alpha'\eta'\sigma'}$$
$$+ P^{\beta'}P^{\delta'}\varepsilon^{\gamma'\alpha'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$
$$+ T_{21}(P^{\alpha'}k'^{\gamma'}\varepsilon^{\delta'\beta'\eta'\sigma'} + P^{\alpha'}k'^{\delta'}\varepsilon^{\gamma'\beta'\eta'\sigma'} + P^{\beta'}k'^{\gamma'}\varepsilon^{\delta'\alpha'\eta'\sigma'}$$
$$+ P^{\beta'}k'^{\delta'}\varepsilon^{\gamma'\alpha'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$
$$+ T_{22}(P^{\gamma'}k'^{\alpha'}\varepsilon^{\delta'\beta'\eta'\sigma'} + P^{\delta'}k'^{\alpha'}\varepsilon^{\gamma'\beta'\eta'\sigma'} + P^{\gamma'}k'^{\beta'}\varepsilon^{\delta'\alpha'\eta'\sigma'}$$
$$+ P^{\delta'}k'^{\beta'}\varepsilon^{\gamma'\alpha'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$
$$+ T_{23}(k'^{\alpha'}k'^{\gamma'}\varepsilon^{\delta'\beta'\eta'\sigma'} + k'^{\alpha'}k'^{\delta'}\varepsilon^{\gamma'\beta'\eta'\sigma'} + k'^{\beta'}k'^{\gamma'}\varepsilon^{\delta'\alpha'\eta'\sigma'}$$
$$+ k'^{\beta'}k'^{\delta'}\varepsilon^{\gamma'\alpha'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$
$$+ T_{24}(g^{\alpha'\gamma'}\varepsilon^{\delta'\beta'\eta'\sigma'} + \gamma^{\alpha'\delta'}\varepsilon^{\gamma'\beta'\eta'\sigma'} + g^{\beta'\gamma'}\varepsilon^{\delta'\alpha'\eta'\sigma'}$$
$$+ k'^{\delta'}k'^{\beta'}\varepsilon^{\gamma'\alpha'\eta'\sigma'})P_{\eta'}k'_{\sigma'}$$

$$+ T_{25} g^{\alpha'\beta'} g^{\gamma'\delta'} + T_{26}(g^{\beta'\gamma'} g^{\delta'\alpha'} + g^{\beta'\delta'} g^{\gamma'\alpha'}) \tag{E.153}$$

The inverse relations are given by

$$T_i = \tilde{\Lambda}^{i}_{\alpha'\beta'\gamma'\delta'} I^{(\alpha'\beta')(\gamma'\delta')} \quad ; \quad i = 1,\ldots,26 \tag{E.154}$$

where the projectors are given by,

$$\tilde{\Lambda}^{1}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{(P\cdot k')^4} \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} \tag{E.155}$$

$$\tilde{\Lambda}^{2}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^5} [-5M^2 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2 \Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$
$$- (P\cdot k')^2 \Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k') \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k') \Upsilon^{7}_{\alpha'\beta'\gamma'\delta'}]$$

$$\tilde{\Lambda}^{3}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^5} [-5M^2 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2 \Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$
$$- (P\cdot k')^2 \Upsilon^{19}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k') \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} + 6(P\cdot k') \Upsilon^{7}_{\alpha'\beta'\gamma'\delta'}] \tag{E.156}$$

$$\tilde{\Lambda}^{4}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6} [15M^4 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} + 12M^2 (P\cdot k')^2 \Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$
$$- M^2 (P\cdot k')^2 \Upsilon^{19}_{\alpha'\beta'\gamma'\delta'} + M^2 (P\cdot k')^2 \Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} - 30M^2 (P\cdot k') \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'}$$
$$- 10M^2 (P\cdot k') \Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^4 \Upsilon^{26}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^4 \Upsilon^{25}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^3 \Upsilon^{18}_{\alpha'\beta'\gamma'\delta'} - 12(P\cdot k')^3 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^3 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^2 \Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} + 14(P\cdot k')^2 \Upsilon^{6}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^2 \Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}] \tag{E.157}$$

$$\tilde{\Lambda}^{5}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6} [15M^4 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} + 6M^2 (P\cdot k')^2 \Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 3M^2 (P\cdot k')^2 \Upsilon^{19}_{\alpha'\beta'\gamma'\delta'} + 3M^2 (P\cdot k')^2 \Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} - 20M^2 (P\cdot k') \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'}$$
$$- 20M^2 (P\cdot k') \Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^4 \Upsilon^{25}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \Upsilon^{18}_{\alpha'\beta'\gamma'\delta'}$$
$$- 2(P\cdot k')^3 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 20(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.158)}$$

$$\tilde{\Lambda}^{6}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{4(P\cdot k')^6} [15M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 12M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^2(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} - 10M^2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$- 30M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 12(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 14(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.159)}$$

$$\tilde{\Lambda}^{7}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^7} [-35M^6 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 20M^4(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- 5M^4(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - 5M^4(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 70M^4(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 50M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10M^2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 20M^2(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 12M^2(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'} + 6M^2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 80M^2(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'}$$

$$- 30M^2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'} - 10M^2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k')^4 \mp^{13}_{\alpha'\beta'\gamma'\delta'}$$

$$- 4(P\cdot k')^4 \mp^{17}_{\alpha'\beta'\gamma'\delta'} + 24(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 8(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.160)}$$

$$\tilde{\Lambda}^{8}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^7} [-35M^6 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 20M^4(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- 5M^4(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - 5M^4(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 50M^4(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 70M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6M^2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 12M^2(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} + 20M^2(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10M^2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 80M^2(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} - 10M^2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$- 30M^2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^4 \mp^{10}_{\alpha'\beta'\gamma'\delta'}$$

$$- 8(P\cdot k')^4 \mp^{13}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 24(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.161)}$$

$$\tilde{\Lambda}^9_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^8}[35M^8 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 20M^6(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 5M^6(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} + 5M^6(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} - 70M^6(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$- 70M^6(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2M^4(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} + M^4(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 10M^4(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} - 20M^4(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 20M^4(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 10M^4(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} + 120M^4(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 30M^4(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'} + 30M^4(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^4 \mp^{10}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 16M^2(P\cdot k')^4 \mp^{13}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^4 \mp^{17}_{\alpha'\beta'\gamma'\delta'}$$

$$- 40M^2(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'} - 40M^2(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^4 \mp^{1}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.162)}$$

$$\tilde{\Lambda}^{10}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4}[M^2 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.163)}$$

$$\tilde{\Lambda}^{11}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5}[-5M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- 3M^2(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - 7M^2(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 10M^2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - 3(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 16(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$-22(P\cdot k')^2\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.178)$$

$$\tilde{\Lambda}^{26}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^4}[M^4\Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^2\Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$
$$- M^2(P\cdot k')^2\Upsilon^{19}_{\alpha'\beta'\gamma'\delta'} - M^2(P\cdot k')^2\Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'}$$
$$- 2M^2(P\cdot k')\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^4\Upsilon^{26}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^4\Upsilon^{25}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^3\Upsilon^{18}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3\Upsilon^{15}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3\Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^3\Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'}$$
$$+ 2(P\cdot k')^2\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.179)$$

The tensor $\Upsilon^{i}_{\alpha'\beta'\gamma'\delta'}$ are given by

$$\Upsilon^{1}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}P_{\delta'} \qquad (E.180)$$

$$\Upsilon^{2}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}P_{\gamma'}k'_{\delta'} \qquad (E.181)$$

$$\Upsilon^{3}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}P_{\gamma'}P_{\delta'} \qquad (E.182)$$

$$\Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}k'_{\gamma'}k'_{\delta'} \qquad (E.183)$$

$$\Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}P_{\gamma'}k'_{\delta'} \qquad (E.184)$$

$$\Upsilon^{6}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}P_{\delta'}k'_{\alpha'}k'_{\beta'} \qquad (E.185)$$

$$\Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \qquad (E.186)$$

$$\Upsilon^{8}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}k'_{\delta'}k'_{\alpha'}k'_{\beta'} \qquad (E.187)$$

$$\Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}k'_{\gamma'}k'_{\delta'} \qquad (E.188)$$

$$\Upsilon^{10}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\beta'}g_{\gamma'\delta'} \qquad (E.189)$$

$$\Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \qquad (E.190)$$

$$\Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\beta'}g_{\gamma'\delta'} \qquad (E.191)$$

$$+ 2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.164)$$

$$\tilde{\Lambda}^{12}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^6} [5M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 4M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 3M^4(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} + 7M^4(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} - 10M^4(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$- 10M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2M^2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - 3M^2(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 6M^2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 14M^2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} + 24M^2(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2M^2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k)^4 \mp^{10}_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k)^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'}]$$

$$(E.165)$$

$$\tilde{\Lambda}^{13}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4} [M^2 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- (P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.166)$$

$$\tilde{\Lambda}^{14}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5} [-5M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 8M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^2(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 10M^2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} - 6(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.167)$$

$$\tilde{\Lambda}^{15}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5} [-5M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 8M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^2(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 6M^2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 8(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$- 6(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.168)}$$

$$\tilde{\Lambda}^{16}_{\alpha'\beta'\gamma'\delta'} = - \frac{1}{8(P\cdot k')^6} [-5M^6 \mp^{9}_{\alpha'\beta'\gamma'\delta'} - 8M^4(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ M^4(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} + M^4(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 10M^4(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$- 10M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2M^2(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} + 8M^2(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} + 8M^2(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2M^2(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 12M^2(P\cdot k')^2 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 6M^2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$- 6M^2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^4 \mp^{13}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3 \mp^{3}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 4(P\cdot k')^3 \mp^{2}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.169)}$$

$$\tilde{\Lambda}^{17}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4} [M^2 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.170)}$$

$$\tilde{\Lambda}^{18}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5} [-5M^4 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 4M^2(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- 7M^2(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - 3M^2(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 6M^2(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10M^2(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - 3(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10(P\cdot k')^3 \mp^{18}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \mp^{15}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^3 \mp^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6(P\cdot k')^3 \mp^{11}_{\alpha'\beta'\gamma'\delta'} - 16(P\cdot k')^2 \mp^{5}_{\alpha'\beta'\gamma'\delta'} + 2(P\cdot k')^2 \mp^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2(P\cdot k')^2 \mp^{4}_{\alpha'\beta'\gamma'\delta'}] \qquad \text{(E.171)}$$

$$\tilde{\Lambda}^{19}_{\alpha'\beta'\gamma'\delta'} = - \frac{1}{8(P\cdot k')^6} [-5M^6 \mp^{9}_{\alpha'\beta'\gamma'\delta'} + 4M^4(P\cdot k')^2 \mp^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$- 7M^4(P\cdot k')^2 \mp^{19}_{\alpha'\beta'\gamma'\delta'} - 3M^4(P\cdot k')^2 \mp^{12}_{\alpha'\beta'\gamma'\delta'} + 10M^4(P\cdot k') \mp^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 10M^4(P\cdot k') \mp^{7}_{\alpha'\beta'\gamma'\delta'} + 2M^2(P\cdot k')^4 \mp^{26}_{\alpha'\beta'\gamma'\delta'} - 3M^2(P\cdot k')^4 \mp^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 14M^2(P\cdot k')^3 \Upsilon^{18}_{\alpha'\beta'\gamma'\delta'} - 4M^2(P\cdot k')^3 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'} - 4M^2(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 6M^2(P\cdot k')^3 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} - 24M^2(P\cdot k')^2 \Upsilon^{5}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k')^2 \Upsilon^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 2M^2(P\cdot k')^2 \Upsilon^{4}_{\alpha'\beta'\gamma'\delta'} - 4(P\cdot k')^4 \Upsilon^{17}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 8(P\cdot k')^3 \Upsilon^{3}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.172)$$

$$\tilde{\Lambda}^{20}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{2(P\cdot k')^4} \Upsilon^{23}_{\alpha'\beta'\gamma'\delta'} \qquad (E.173)$$

$$\tilde{\Lambda}^{21}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5} [-5M^2 \Upsilon^{23}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^2 \Upsilon^{24}_{\alpha'\beta'\gamma'\delta'}$$

$$+ (P\cdot k') \Upsilon^{21}_{\alpha'\beta'\gamma'\delta'} + 5(P\cdot k') \Upsilon^{22}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.174)$$

$$\tilde{\Lambda}^{22}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^5} [-5M^2 \Upsilon^{23}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k')^2 \Upsilon^{24}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 5(P\cdot k') \Upsilon^{21}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k') \Upsilon^{22}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.175)$$

$$\tilde{\Lambda}^{23}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^6} [5M^4 \Upsilon^{23}_{\alpha'\beta'\gamma'\delta'} + M^2(P\cdot k')^2 \Upsilon^{24}_{\alpha'\beta'\gamma'\delta'}$$

$$- 5M^2(P\cdot k') \Upsilon^{21}_{\alpha'\beta'\gamma'\delta'} - 5M^2(P\cdot k') \Upsilon^{22}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 4(P\cdot k')^2 \Upsilon^{20}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.176)$$

$$\tilde{\Lambda}^{24}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^4} [M^2 \Upsilon^{23}_{\alpha'\beta'\gamma'\delta'} + (P\cdot k')^2 \Upsilon^{24}_{\alpha'\beta'\gamma'\delta'}$$

$$- (P\cdot k') \Upsilon^{21}_{\alpha'\beta'\gamma'\delta'} - (P\cdot k') \Upsilon^{22}_{\alpha'\beta'\gamma'\delta'}] \qquad (E.177)$$

$$\tilde{\Lambda}^{25}_{\alpha'\beta'\gamma'\delta'} = \frac{1}{8(P\cdot k')^4} [M^4 \Upsilon^{9}_{\alpha'\beta'\gamma'\delta'} - 4M^2(P\cdot k')^2 \Upsilon^{16}_{\alpha'\beta'\gamma'\delta'}$$

$$+ 3M^2(P\cdot k')^2 \Upsilon^{19}_{\alpha'\beta'\gamma'\delta'} + 3M^2(P\cdot k')^2 \Upsilon^{12}_{\alpha'\beta'\gamma'\delta'} - 2M^2(P\cdot k') \Upsilon^{8}_{\alpha'\beta'\gamma'\delta'}$$

$$- 2M^2(P\cdot k') \Upsilon^{7}_{\alpha'\beta'\gamma'\delta'} - 2(P\cdot k')^4 \Upsilon^{26}_{\alpha'\beta'\gamma'\delta'} + 3(P\cdot k')^4 \Upsilon^{25}_{\alpha'\beta'\gamma'\delta'}$$

$$- 6(P\cdot k')^3 \Upsilon^{18}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3 \Upsilon^{15}_{\alpha'\beta'\gamma'\delta'} + 4(P\cdot k')^3 \Upsilon^{14}_{\alpha'\beta'\gamma'\delta'}$$

$$- 6(P\cdot k')^3 \Upsilon^{11}_{\alpha'\beta'\gamma'\delta'} + 8(P\cdot k')^2 \Upsilon^{6}_{\alpha'\beta'\gamma'\delta'}$$

$$\mathcal{F}^{13}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\gamma'}g_{\delta'\beta'} \quad \text{(E.192)}$$

$$\mathcal{F}^{14}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\gamma'}g_{\delta'\beta'} \quad \text{(E.193)}$$

$$\mathcal{F}^{15}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}k'_{\alpha'}g_{\delta'\beta'} \quad \text{(E.194)}$$

$$\mathcal{F}^{16}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\gamma'}g_{\delta'\beta'} \quad \text{(E.195)}$$

$$\mathcal{F}^{17}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}P_{\delta'}g_{\alpha'\beta'} \quad \text{(E.196)}$$

$$\mathcal{F}^{18}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}k'_{\delta'}g_{\alpha'\beta'} \quad \text{(E.197)}$$

$$\mathcal{F}^{19}_{\alpha'\beta'\gamma'\delta'} = k'_{\gamma'}k'_{\delta'}g_{\alpha'\beta'} \quad \text{(E.198)}$$

$$\mathcal{F}^{20}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}P_{\gamma'}\epsilon_{\delta'\beta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.199)}$$

$$\mathcal{F}^{21}_{\alpha'\beta'\gamma'\delta'} = P_{\alpha'}k'_{\gamma'}\epsilon_{\delta'\beta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.200)}$$

$$\mathcal{F}^{22}_{\alpha'\beta'\gamma'\delta'} = P_{\gamma'}k'_{\alpha'}\epsilon_{\delta'\beta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.201)}$$

$$\mathcal{F}^{23}_{\alpha'\beta'\gamma'\delta'} = k'_{\alpha'}k'_{\gamma'}\epsilon_{\delta'\beta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.202)}$$

$$\mathcal{F}^{24}_{\alpha'\beta'\gamma'\delta'} = g_{\alpha'\gamma'}\epsilon_{\delta'\beta'\zeta'\xi'}P^{\zeta'}k'^{\xi'} \quad \text{(E.203)}$$

$$\mathcal{F}^{25}_{\alpha'\beta'\gamma'\delta'} = g_{\alpha'\beta'}g_{\gamma'\delta'} \quad \text{(E.204)}$$

$$\mathcal{F}^{26}_{\alpha'\beta'\gamma'\delta'} = g_{\beta'\gamma'}g_{\delta'\alpha'} \quad \text{(E.205)}$$

E.10. The Generalized Gaussian Integrals

All the generalized Gaussian integrals needed in the evaluation of the momentum integrals above can be derived from the four-dimensional Gaussian integral

$$I_1 = \int dx_1 dx_2 dx_3 dx_4 \; e^{-x^T C x - s^T x} \quad \text{(E.206)}$$

In eqn. (E.206), x and s are column vectors given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad ; \quad s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} . \qquad \text{(E.207)}$$

and C is the following 4×4 symmetric matrix

$$C = \begin{bmatrix} C_{11} & \frac{1}{2} C_{12} & \frac{1}{2} C_{13} & \frac{1}{2} C_{14} \\ & C_{22} & \frac{1}{2} C_{23} & \frac{1}{2} C_{24} \\ & & C_{33} & \frac{1}{2} C_{34} \\ \text{Symmetric} & & & C_{44} \end{bmatrix} \qquad \text{(E.208)}$$

s_i's and C_{ij}'s can, in general, be complex numbers.

This Gaussian integral can be done by successively completing squares in the exponential and then using the standard Gaussian integral formula. This method is standard and is dealt with in detail at various places [68]. So, we skip the details of this part of the calculation.

The final result is

$$I_1 = \frac{\sqrt{\pi}}{\sqrt{D_1}} \frac{\sqrt{\pi}}{\sqrt{D_2}} \frac{\sqrt{\pi}}{\sqrt{D_3}} \frac{\sqrt{\pi}}{\sqrt{D_4}} \exp[D_1 a_1^2 + D_2 a_2^2 + D_3 a_3^2 + D_4 a_4^2] \qquad \text{(E.209)}$$

where

$$D_1 = C_{11} \qquad \text{(E.210)}$$

$$D_2 = C_{22} - D_1 b_{12}^2 \qquad \text{(E.211)}$$

$$D_3 = C_{33} - D_1 b_{13}^2 - D_2 b_{23}^2 \qquad \text{(E.212)}$$

$$D_4 = C_{44} - D_1 b_{14}^2 - D_2 b_{24}^2 - D_3 b_{34}^2 \qquad \text{(E.213)}$$

b_{ij}'s are given by

$$b_{1k} = \frac{1}{2D_1} C_{1k} \quad ; \quad k = 2,3,4 \qquad (E.214)$$

$$b_{2k} = \frac{1}{2D_2} [C_{2k} - (2D_1 b_{12}) b_{1k}] \quad ; \quad k = 3,4 \qquad (E.215)$$

$$b_{34} = \frac{1}{2D_3} [C_{34} - (2D_1 b_{13}) b_{14} - (2D_2 b_{23}) b_{24}] \qquad (E.216)$$

and the a_i's are given by

$$a_1 = \frac{1}{2D_1} s_1 \qquad (E.217)$$

$$a_2 = \frac{1}{2D_2} [s_2 - 2D_1 a_1 b_{12}] \qquad (E.218)$$

$$a_3 = \frac{1}{2D_3} [s_3 - 2D_1 a_1 b_{13} - 2D_2 a_2 b_{23}] \qquad (E.219)$$

$$a_4 = \frac{1}{2D_4} [s_4 - 2D_1 a_1 b_{14} - 2D_2 a_2 b_{24} - 2D_3 a_3 b_{34}] \qquad (E.220)$$

There are four conditions for the existence of this integral. They are

$$\text{Re } D_1 > 0 \qquad (E.221)$$

$$\text{Re } D_2 > 0 \qquad (E.222)$$

$$\text{Re } D_3 > 0 \qquad (E.223)$$

$$\text{Re } D_4 > 0 \qquad (E.224)$$

We must also mention that if any of the D_i's is complex, then by its square root in eqn. (E.209) we always mean the principal value of the square root [68].

The other generalized Gaussian integrals that show up are all of the form

$$\int dx_1 dx_2 dx_3 dx_4 [\text{Polynomial in } x_1, x_2, x_3, x_4] e^{-x^T C x - s^T x}$$

As mentioned above, they can all be obtained by differentiating I_1 with one or more of the s_i's.