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SOME REPRESENTATION AND DISTRIBUTION PROBLEMS FOR GENERALIZED r-FREE INTEGERS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME REPRESENTATION AND DISTRIBUTION PROBLEMS FOR GENERALIZED r-FREE INTEGERS", submitted by YAN KWANG FENG, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis, we study some representation and distribution problems for the class of generalized r-free integers, called (k,r)-integers. These are defined as integers whose k-free part is also r-free (where k and r are fixed integers with $0 < r < k$). A special (limiting) case of this is the r-free integers $(k = \infty)$. Many known results concerning the r-free integers follow as corollaries of our results for the (k,r)-integers. Our results include those concerning the Schnirelmann and asymptotic density for the (k,r)-integers. We study the distribution of the (k,r)-integers in a given arithmetic progression a (mod h), also the number of representations of an integer as the sum of two (k,r) type integers, sum of a prime and a (k,r) integer, etc.

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I wish to express my gratitude to Professor M.V. Subbarao for suggesting this topic. I deeply appreciate the guidance and friendly encouragement which he has provided throughout the preparation of this thesis.

CHAPTER I

Introduction

For a given natural number r, we recall that an integer n is said to be r-free whenever it is not divisible by the r-th power of any prime. In the extreme cases, unity is the only 1-free integer, and every integer is ∞-free. There is a vast literature concerning r-free integers and, in particular, square-free (or quadratfrei) number. Various results obtained in earlier years by several authors (e.g. S.S. Pillai, K. Rogers, F. Mertens, etc.) involving square-free integers, (e.g. density, representation and distribution problems) were later, in turn generalized by others (e.g. L. Mirsky, E. Cohen, T. Estermann, etc.) to r-free integers, where r is any integer ≥ 2.

In a paper published in 1960, E. Cohen [13] introduced a class of integers, called integers of the type (b,k), which are in fact a generalization of the k-free integers. For $b \geq 0$ and $k \geq 1$, an integer of the type (b,k) is defined to be an integer whose greatest k-th power divisor is also a bk-th power divisor.

Here we wish to study in some detail a class of integers $Q_{k,r}$, which we call the (k,r)-integers (defined below) of which the r-free integers and Cohen's (b,k)-type integers arise as special cases.

Let k and r be fixed integers such that $0 < r < k$. We recall that any integer $n > 1$ has the unique representation $n = a^k b$ where b is k-free. We shall call a^k the k-th power part of n and b the k-free part. If b is r-free we shall call n a generalized r-free integer or a (k,r)-integer. Notice that in the limiting case when

$k \to \infty$, a (k,r)-integer becomes a r-free integer. We may also observe that a (k,1)-integer is the same as a k-th power integer. The (k,r)-integers were first introduced in 1966 in a paper by M.V. Subbarao and V.C. Harris [1] in connection with generalization of the well known Ramanujan trigonometric sum $C(n,r)$.

The purpose of this dissertation is to study some density, representation and distribution problems involving (k,r)-integers. The results we obtain are analogous to the corresponding known ones for r-free integers, and in fact they reduce to the known results for r-free numbers, on letting $k \to \infty$. While the methods used for obtaining such parallel results are generally analogous to those for the r-free integers, there are also many significant differences of detail in many cases to warrant a separate study.

Let $Q_{k,r}(x)$ denote the number of (k,r)-integers not exceeding x . F. Gegenbauer [1] in 1885 established the classical result that

$$Q_r(x) = \frac{x}{\zeta(r)} + O(x^{\frac{1}{r}}) , \tag{1.1}$$

where $Q_r(x)$ denotes the number of r-free integers not exceeding x .

Later in 1931, this result was improved by Evelyn and Linfoot [1] who proved that

$$Q_r(x) = \frac{x}{\zeta(r)} + O(x^{\frac{1}{r}} e^{-b(\log x \log\log x)^{\frac{1}{2}}}) , \tag{1.2}$$

where $b = ar^{-3/2}$ and a is a positive constant. In Chapter II, we establish results analogous to that of (1.1) and (1.2) for (k,r)-integers. That is

$$(1.3) \qquad Q_{k,r}(x) = \begin{cases} \frac{x\zeta(k)}{\zeta(r)} + O(x^{\frac{1}{r}}) , & \text{for } r > 1 \text{ , uniformly in } k \text{ ;} \\ O(x^{\frac{1}{k}}) , & \text{for } r = 1 \text{ ,} \end{cases}$$

and

$$(1.4) \qquad Q_{k,r}(x) = \frac{x\zeta(k)}{\zeta(r)} + O(x^{\frac{1}{r}}B(x;k,r)) \quad \text{for } r > 1 \text{ ,}$$

where $B(x;k,r) = \sum_{j=1}^{[x^{\frac{1}{k}}]} \frac{e^{-b(\log(\frac{x}{j^k}) \log\log (\frac{x}{j^k}))^{\frac{1}{2}}}}{j^{k/r}}$, $b = ar^{-3/2}$, a being an absolute constant. We also estimate the partial sum of the generating series for (k,r)-integers, and discuss some properties of the error function $E(x)$ for $Q_{k,r}(x)$.

The asymptotic density $\delta(Q_{k,r})$ of the (k,r)-integers is defined to be

$$(1.5) \qquad \delta(Q_{k,r}) = \lim_{n\to\infty} \frac{Q_{k,r}(n)}{n} = \frac{\zeta(k)}{\zeta(r)} \text{ ,}$$

and their Schnirelmann density $D(Q_{k,r})$ is, by definition,

$$(1.6) \qquad D(Q_{k,r}) = \inf_{n} \frac{Q_{k,r}(n)}{n} \text{ .}$$

In Chapter II, we also establish the following result ,

$$(1.7) \qquad D(Q_{k,r}) \geq \zeta(k)(1 - \sum_{p} p^{-r}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} \text{ ,}$$

which is a generalization of Duncan's [2] result;

$$(1.8) \qquad D(Q_{k,r}) < \delta(Q_{k,r}) \text{ ;}$$

our proof is based on the occurrence of infinitely many change of sign of the error function $E(x)$ for $Q_{k,r}(x)$.

In Chapter III, we study the distribution of (k,r)-integers in residue classes (mod h) . We obtain the main result as follows:

(1.9) Let k, r, and q be integers with $k > 1$ and $0 < q < r$. Let a and h be positive integers with $d = (a,h)$.

If $d \in Q_{kq}$ and $p^{\alpha} || h$, then we have

$$Q_{kr,kq}(x;a,h) = \frac{h^{kq-1}}{\phi_{kq}(h)} \frac{x\zeta(kr)}{\zeta(kq)} \prod_{p|(a,h)_*} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha} \nmid d}} (1 - p^{-kr}) + O(dx^{\frac{1}{k}}) ;$$

the estimate is uniform in a, h, r, and q . As a special case of the above result, by letting $q = 1$ and $r \to \infty$, we obtain

$$Q_{\infty,k}(x;a,h) = Q_k(x;a,h) = \frac{h^{k-1}}{\phi_k(h)} \frac{\phi^*(H')}{H'} \frac{x}{\zeta(k)} + O(dx^{\frac{1}{k}}) ,$$

the estimate is uniform in a, h, r, and q . This is equivalent to the Cohen and Robinson result [1]. (For notation, see Chapter III, sec. 3, p.28 or p.55)

The problem of the representations of an integer > 1 as the sum of two k-free integers, was studied by Linfoot and Evelyn. In 1931, A. Page [1] generalized their result, and obtained the number of representations of any integer $n > 1$ as the sum of a k-free and a ℓ-free integers. In Chapter IV, we establish the result as follow:

(1.10) Let $T(n)$ denote the number of representations of the integer n as the sum of a (k_1r_1,k_1q_1)-integer and a (k_2r_2,k_2q_2)-integer, where $k_1 > 1$; $0 < q_1 < r_1$, $k_2 > 1$; $0 < q_2 < r_2$, and $k_2q_2 \le k_1$.
Then

$$T(n) \sim nH(n) ,$$

where

$$H(n) = \frac{\zeta(k_2 r_2) \prod_p [(1 - p^{-k_2 q_2}) + \frac{1 - p^{-k_2 r_2}}{1 - p^{-k_1 r_1}} (p^{-k_1 r_1} - p^{-k_1 q_1})]}{\prod_{p^{k_2 q_2} | n} \left[1 + \left(\frac{1 - p^{-k_2 r_2}}{1 - p^{-k_2 r_1}}\right)\left(\frac{p^{-k_1 r_1} - p^{-k_1 q_1}}{1 - p^{-k_2 q_2}}\right)\right]} .$$

In Chapter V, we study the problem of the number of representations of an integer as the sum of a prime and a (k,r)-integer, and obtain an asymptotic formula for the number $T(k,r;n)$ of such representations in the form:

(1.11)
$$T(k,r;n) = \prod_{p \nmid n} \{1 + (1 - p^{-1})^{-1}(1 - p^{-k})^{-1}(p^{-k} - p^{-r})\} \text{ Li } n + F(n) + O(n\, e^{-\frac{A}{2}\sqrt{\log n}}),$$

where $\text{Li } n = \int_2^n \frac{du}{\log u}$ and

$$F(n) = \text{Li } n \sum_{\substack{(a,n) = 1 \\ a > e^{\frac{A}{2}\sqrt{\log n}}}} \frac{\lambda(a)}{\phi(a)} + \sum_{\substack{ab+p = n \\ a > e^{\frac{A}{2}\sqrt{\log n}}}} \lambda(a)$$

and the O-constant depends at most on A only and is uniform in k and r. For the case when $(k,r) > 1$, we are able to obtain an elegant estimate; the corresponding theorem for this case is given below.

(1.12) Let k, r, q be integers with $k > 1$, and $1 \le q < r$. Then

$$T(kr,kq;n) = \text{Li } n \prod_{p \nmid n} \{1 + (1 - p^{-1})^{-1}(1 - p^{-kr})^{-1}(p^{-kr} - p^{-kq})\} + O(n\, e^{-\frac{A}{2}\sqrt{\log n}}) ,$$

where $\text{Li } n = \int_2^n \frac{du}{\log u}$

and the O-constant depends at most on A only and is uniform in k, r, and q.

(1.13) Remarks on further generalization

The notion of an r-free integer can be generalized in a very wide manner (to include our (k;r) integers) as follows.

$$\text{Let } \tilde{r} = (r_1, r_2, r_3, \dots)$$

denote a sequence of arbitrary positive integers. Then by an $\tilde{r}$-free integer we mean an integer n such that

$$p_j^{r_j} \nmid n \quad (j = 1,2,3,\dots) ,$$

where p_j denotes the j-th prime. This notion appears in one of the applications in the paper of Carlitz [2] . One may add the hypothesis that $r_j \geq 2$ for all but a finite number of j's .

More generally, given two sequences

$$\tilde{r} = (r_1, r_2, r_3, \dots)$$

$$\tilde{k} = (k_1, k_2, k_3, \dots)$$

such that

$$1 \leq r_j < k_j \quad (j = 1,2,3,\dots) ,$$

define $Q_{\tilde{k},\tilde{r}}$ as the set of integers n such that, if

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots \quad (e_j \geq 0)$$

is the canonical factorization of n , then either

$$e_j < r_j \quad \text{or} \quad e_j \geq k_j \quad (j = 1,2,3,\dots) .$$

Again one may add the hypothesis that $r_j \geq 2$ for all but a finite number of j's . With this added hypothesis one should presumably be able to extend most of the results of the thesis. The possibility of such an extension has kindly been pointed out by Professor L. Carlitz.

CHAPTER II

The (k,r)-integers and their density problem

(2.1) Notation. Throughout what follows r denotes a natural number, and k represents a natural number greater than r, or as a limiting case, $k = \infty$. We use the words "integer" or "number" to mean a "natural number". Unless otherwise stated, all the definitions and results that follow involving k are valid for both these cases: (i) when k is a finite number $> r$ and (ii) in the limiting case when $k = \infty$.

We recall that, a number n is said to be r-free whenever it is not divisible by the r-th power of any prime. Now, any integer n can be written uniquely in the form $n = a^r b$, where b is r-free. We shall call a^r the "r-th power part" and b the "r-free part" of n.

(2.2) Definition. By a (k,r)-integer we mean an integer of the form $a^k b$ where a,b are natural numbers and b is r-free.

(2.2.1) Remark. The (∞,r) numbers are the same as the r-free numbers.

(2.2.2) Remark. Eckford Cohen [13] studied a class of generalized r-free numbers which he called "integers of type (b,r), $b \geq 0$". These are, by definition, integers whose greatest r-th power divisor is also a br-th power divisor. It is clear that an integer n is of type (b,r) in the sense of E. Cohen iff it is a (br,r)-integer according to our definition.

(2.2.3) Remark. When $r = 1$, the (k,r) numbers are just the k-th power integers.

(2.3) Notation. Let $Q_{k,r}$ denote the set of all (k,r) numbers; and

$\psi = \psi_{k,r}$ the characteristic function of $Q_{k,r}$.

(2.4) Definition. Let $\lambda = \lambda_{k,r}$ be the multiplicative function defined for powers of an arbitrary prime p as follows:

$$\lambda(p^a) = \begin{cases} 1, & a \equiv 0 \pmod{k} , \\ -1, & a \equiv r \pmod{k} , \\ 0, & \text{otherwise}. \end{cases}$$

This function, was first studied by M.V. Subbarao and V.C. Harris [1] who showed that

(2.4.1) $$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(ks)}{\zeta(rs)} \quad (s > 1) .$$

Further we have

(2.5) Lemma. $\sum_{d|n} \lambda(d) = \psi(n)$.

For completeness, we shall give here a short proof.

Proof. The result being trivial for $n = 1$, we shall assume that $n > 1$. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ be the prime factorization of n , then we have

$$\sum_{d|n} \lambda(d) = \prod_{p_i} \{\lambda(1) + \lambda(p_i) + \lambda(p_i^2) + \cdots + \lambda(p_i^{a_i})\} .$$

From the definition (2.4), we see that $\lambda(1) + \lambda(p_i) + \lambda(p_i^2) + \cdots + \lambda(p_i^{a_i}) = 0$, unless $a_i \equiv 0,1,2,\ldots,(r-1) \pmod{k}$, in which case the sum becomes unity, from which we have

$$\psi(n) = \sum_{d|n} \lambda(d) = \begin{cases} 1, & n \in Q_{k,r} ; \\ 0, & n \notin Q_{k,r} . \end{cases}$$

Results (2.4.1) and (2.5) are of fundamental importance in our work.

In passing, also note the useful result that for $\lambda(a) \neq 0$, the number a must be of the form $a = b^k c^r$, where c is square-free including 1, and $\lambda(a) = \mu(c)$.

(2.6) Some general results. Suppose S is any non-empty set of positive integers and Z the set of all positive integers. Let $\psi_s(n)$ denote the characteristic function of S and let $\mu_s(n)$ be defined by the relation

$$(2.6.1) \qquad \sum_{d|n} \mu_s(d) = \psi_s(n) \quad (n = 1,2,3,\ldots) .$$

It is easy to see that this relation defines $\mu_s(n)$ uniquely, by recursion. Now, define for any arithmetic function $f(n)$,

$$(2.6.2) \qquad F_s(n) = \sum_{\substack{d|n \\ \frac{n}{d}\in S}} f(d) , \; F(n) = F_Z(n) .$$

Also define

$$(2.6.3) \qquad G_s(n) = \sum_{d|n} f(d)\mu_s(\tfrac{n}{d}) , \; G(n) = G_1(n) ,$$

where in $G_1(n)$ the suffix 1 denotes the set consisting of the integer 1 alone.

We then have the following three formulas:

$$(2.6.4) \qquad \mu_s(n) = \sum_{\substack{d\ell=n \\ \ell\in S}} \mu(d) .$$

$$(2.6.5) \qquad F_s(n) = \sum_{d|n} \mu_s(d)F(\tfrac{n}{d}) .$$

$$(2.6.6) \qquad G_s(n) = \sum_{\substack{d|n \\ d\in S}} G(\tfrac{n}{d}) .$$

These results are not difficult to prove. For a formal proof we refer to E. Cohen [13]. We could, in particular, apply these formulas for the case when $S = Q_{k,r}$.

For example, the result (2.5) can be obtained in this way, but the direct proof is easier.

(2.7) Notation. $Q_{k,r}(x)$ denotes the number of (k,r) numbers not exceeding x, so that

$$Q_{k,r}(x) = \sum_{n \le x} \psi(n) .$$

We write also

$$Q_r(x) = Q_{\infty,r}(x) = \text{the number of } r\text{-free numbers} \le x .$$

The result that

$$(2.8) \qquad Q_r(x) = \frac{x}{\zeta(r)} + O(x^{\frac{1}{r}})$$

is classical, and the proof of this by F. Gegenbauer [1] in 1885 was probably the earliest one. This result was not improved until 1931 when C.J.A. Evelyn and E.H. Linfoot [1] proved that

$$(2.9) \qquad Q_r(x) = \frac{x}{\zeta(r)} + O(x^{\frac{1}{r}} e^{-b(\log x \log\log x)^{\frac{1}{2}}}) ,$$

where $b = ar^{-\frac{3}{2}}$ and a is a positive constant. Now we turn to the case of $Q_{k,r}(x)$. In the following results the O-terms correspond to $x \to \infty$.

(2.10) Theorem.

$$Q_{k,r}(x) = \begin{cases} \frac{x\zeta(k)}{\zeta(r)} + O(x^{\frac{1}{r}}) , & \text{for } r > 1 \text{, uniformly in } k ; \\ O(x^{\frac{1}{k}}) , & \text{for } r = 1 . \end{cases}$$

Proof. Since $Q_{k,1}$ is the set of k-th power integers, the above result for $r = 1$ is immediate.

To prove the result for $r > 1$. We can use the generating function in (2.4.1) and some standard arguments. However we shall use another method based on the following useful and interesting device.

(2.11) Lemma. $Q_{k,r}(x) = Q_r(\frac{x}{1^k}) + Q_r(\frac{x}{2^k}) + \cdots\cdots$

i.e. $$Q_{k,r}(x) = \sum_{n=1}^{\infty} Q_r(\frac{x}{n^k}) .$$

This follows from the observation that if $m = a^k b$ is a (k,r)-integer, then b is r-free and a is one of the integers 1, 2, 3, ...

Actually the series above is finite, and ends with the term $Q_r(\frac{x}{N^k})$ where N is the largest integer such that

$$\frac{x}{N^k} \geq 1 \text{ , that is, } N = [x^{\frac{1}{k}}] .$$

Hence

$$(2.12) \qquad Q_{k,r}(x) = Q_r(\frac{x}{1^k}) + Q_r(\frac{x}{2^k}) + \cdots + Q_r(\frac{x}{N^k}) .$$

On using (2.8) this gives

$$Q_{k,r}(x) = \frac{x}{\zeta(r)} (\frac{1}{1^k} + \cdots + \frac{1}{N^k}) + 0((\frac{x}{1^k})^{\frac{1}{r}} + \cdots + (\frac{x}{N^k})^{\frac{1}{r}})$$

$$= \frac{x}{\zeta(r)} (\zeta(k) - \sum_{j=N+1}^{\infty} \frac{1}{j^k}) + 0[x^{\frac{1}{r}}(\zeta(\frac{k}{r}) - \sum_{j=N+1}^{\infty} \frac{1}{j^{k/r}})] ,$$

where the constants involved in the 0-terms depend only on r . Now we use the fact that $k \geq r + 1$. We get

$$Q_{k,r}(x) = \frac{\zeta(k)}{\zeta(r)} x + \frac{x}{\zeta(r)} 0(\sum_{j=N+1}^{\infty} \frac{1}{j^k}) + 0(\zeta(\frac{r+1}{r})x^{\frac{1}{r}}) + 0(x^{\frac{1}{r}} \sum_{j=N+1}^{\infty} \frac{1}{j^{\frac{r+1}{r}}}) .$$

The last two 0-terms are clearly $0(x^{\frac{1}{r}})$, uniformly in k . Now the first 0-term

$$\frac{x}{\zeta(r)} 0(\sum_{j=N+1}^{\infty} \frac{1}{j^k}) = \frac{x}{\zeta(r)} 0(\int_{x^{\frac{1}{k}}}^{\infty} t^{-k}dt) = 0(\frac{x^{\frac{1}{k}}}{(k-1)\zeta(r)}) = 0(x^{\frac{1}{r}}) ,$$

uniformly in k , because $\frac{1}{k-1} \leq \frac{1}{r}$. This proves the theorem (2.10) .

If we use the result (2.9) of Evelyn and Linfoot instead of (2.8) and proceed as above, we obtain the following improvement of theorem (2.10) .

(2.13) <u>Theorem</u>. $Q_{k,r}(x) = \frac{x\zeta(k)}{\zeta(r)} + O(x^{\frac{1}{r}}B(x;k,r))$ for $r > 1$, where

$$B(x;k,r) = \sum_{j=1}^{[x^{\frac{1}{k}}]} \frac{e^{-b(\log (\frac{x}{j^k}) \log\log (\frac{x}{j^k}))^{\frac{1}{2}}}}{j^{k/r}} , \quad b = ar^{-3/2} , \text{ a being}$$

an absolute constant.

<u>Proof</u>. $Q_{k,r}(x) = Q_r(\frac{x}{1^k}) + \ldots + Q_r(\frac{x}{N^k})$, $N = [x^{\frac{1}{k}}]$.

$$Q_{k,r}(x) = \frac{x}{\zeta(r)} \left(\sum_{j=1}^{\infty} \frac{1}{j^k} - \sum_{j=[x^{\frac{1}{k}}]+1}^{\infty} \frac{1}{j^k}\right) + O\left(\sum_{j=1}^{[x^{\frac{1}{k}}]} (\frac{x}{j^k})^{\frac{1}{r}} e^{-b\sqrt{\log(\frac{x}{j^k})\log\log(\frac{x}{j^k})}}\right)$$

$$= \frac{\zeta(k)}{\zeta(r)} x + O(x^{\frac{1}{k}}) + O(x^{\frac{1}{r}}B(x;k,r))$$

$$= \frac{\zeta(k)}{\zeta(r)} x + O(x^{\frac{1}{r}}B(x;k,r)) , \text{ since } k > r .$$

(2.14) <u>An unsolved problem</u>. The theorem (2.10) raises the following questions. Suppose we write

$$Q_r(x) = \frac{x}{\zeta(r)} + R_r(x)$$

and

$$Q_{k,r}(x) = \frac{\zeta(k)}{\zeta(r)} x + R_{k,r}(x) .$$

What are the true orders $R_r(x)$, and $R_{k,r}(x)$? If $R_{k,r}(x) = O(x^{\beta})$ uniformly in k , what is the best possible value of β ? These are deep problems, and are not likely to be solved in the near future. Even in the case of $R_r(x)$, Evelyn and Linfoot [1] showed that

$$R_r(x) \neq O(x^{\frac{1}{2r} - \delta}) \text{ for every } \delta > 0 ;$$

and, in fact, that

(2.14.1) $$R_r(x) \neq o(x^{\frac{1}{2r}}) .$$

On the other hand, Axer [1] proved that on the basis of the Riemann hypothesis,

(2.14.2) $$R_r(x) = O(x^{(2+\varepsilon)/(2r+1)}) .$$

Thus even with the assumption that the Riemann hypothesis holds, the true order of $R_r(x)$ is still open, since the gap between the results (2.14.1) and (2.14.2) is still wide.

In view of the above mentioned results of Evelyn and Linfoot it is clear that if $R_{k,r}(x) = O(x^{\beta})$ uniformly in k, then $\beta \geq \frac{1}{2r}$.

The following result is of some interest in this connection.

(2.15) Theorem. (An estimate for the partial sum of the generating series for (k,r)-integers.)

Suppose β is the number which satisfies the relation

$$Q_{k,r}(x) = \frac{\zeta(k)}{\zeta(r)} x + O(x^{\beta}) ,$$

the O-term being uniform in k.

Then

$$\sum_{n=1}^{m} \frac{\psi(n)}{n^s} = \frac{\zeta(k)}{\zeta(r)} \sum_{n=1}^{m} \frac{1}{n^s} + s \sum_{n=1}^{m} O(\frac{1}{n^{1+\sigma-\beta}}) + O(m^{\beta-\sigma}) ,$$

where $\sigma = \mathrm{Re}(s)$.

We first prove the following Lemma.

(2.16) Lemma. $$\sum_{n=1}^{m} n(\frac{1}{n^s} - \frac{1}{(n+1)^s}) + \frac{m}{(m+1)^s} = \sum_{n=1}^{m} \frac{1}{n^s} .$$

__Proof__. $\sum_{n=1}^{m} n(\frac{1}{n^s} - \frac{1}{(n+1)^s}) + \frac{m}{(m+1)^s}$

$$= \sum_{n=1}^{m} \frac{1}{n^{s-1}} - \sum_{n=1}^{m} \frac{n}{(n+1)^s} + \frac{m}{(m+1)^s}$$

$$= \sum_{n=1}^{m} \frac{1}{n^{s-1}} - \sum_{n=1}^{m-1} \frac{n}{(n+1)^s}$$

$$= \sum_{n=1}^{m} \frac{1}{n^{s-1}} - \sum_{n=1}^{m-1} \frac{(n+1)-1}{(n+1)^s}$$

$$= \sum_{n=1}^{m} \frac{1}{n^{s-1}} - \sum_{n=1}^{m-1} \frac{1}{(n+1)^{s-1}} + \sum_{n=1}^{m-1} \frac{1}{(n+1)^s}$$

$$= \sum_{n=1}^{m} \frac{1}{n^{s-1}} - \sum_{n=2}^{m} \frac{1}{n^{s-1}} + \sum_{n=2}^{m} \frac{1}{n^s}$$

$$= \sum_{n=1}^{m} \frac{1}{n^s}$$

(2.17) __Proof of the theorem (2.15)__

$$\sum_{n=1}^{m} \frac{\psi(n)}{n^s} = \sum_{n=1}^{m} \frac{Q_{k,r}(n) - Q_{k,r}(n-1)}{n^s}$$

$$= \sum_{n=1}^{m} Q_{k,r}(n)(\frac{1}{n^s} - \frac{1}{(n+1)^s}) + \frac{1}{(m+1)^s} Q_{k,r}(m)$$

$$= \sum_{n=1}^{m} \frac{n\zeta(k)}{\zeta(r)} (\frac{1}{n^s} - \frac{1}{(n+1)^s}) + \frac{1}{(m+1)^s} \frac{m\zeta(k)}{\zeta(r)}$$

$$+ \sum_{n=1}^{m} (\frac{1}{n^s} - \frac{1}{(n+1)^s}) O(n^\beta) + \frac{1}{(m+1)^s} O(m^\beta) .$$

We now use the Lemma (2.16) .

Thus we get

$$\sum_{n=1}^{m} \frac{\psi(n)}{n^s} = \frac{\zeta(k)}{\zeta(r)} \sum_{n=1}^{m} \frac{1}{n^s} + s \sum_{n=1}^{m} O(n^\beta) \int_n^{n+1} \frac{dt}{t^{s+1}} + \frac{1}{(m+1)^s} O(m^\beta)$$

$$= \frac{\zeta(k)}{\zeta(r)} \sum_{n=1}^{m} \frac{1}{n^s} + s \sum_{n=1}^{m} O(\frac{1}{n^{1+\sigma-\beta}}) + O(m^{\beta-\sigma}) ,$$

this proving the theorem.

Evelyn and Linfoot use the special case of this result together with some properties about the zeros of the Zeta function to show that $Q_r(x) - \frac{x}{\zeta(r)} \neq 0(x^{\frac{1}{2r} - \delta})$ for any $\delta > 0$. We can similarly use the above result to show that

$$Q_{k,r}(x) - \frac{x\zeta(k)}{\zeta(r)} \neq 0(x^{\frac{1}{2r} - \delta}) \text{, for any } \delta > 0 .$$

(2.18) <u>The density of the (k,r)-integers</u>

In view of result (2.9), the asymptotic density $\delta(Q_r)$ of the r-free integers

(2.18.1) $$\delta(Q_r) = \lim_{n\to\infty} \frac{Q_r(n)}{n} = \frac{1}{\zeta(r)} .$$

Their Schnirelmann density $D(Q_r)$ is, by definition,

(2.18.2) $$D(Q_r) = \inf_n \frac{Q_r(n)}{n} .$$

As R.L. Duncan [2] remarked, we have

(2.18.3) $$D(Q_2) \leq \delta(Q_2) < D(Q_3) \leq \delta(Q_3) < \dots\dots$$
$$< D(Q_k) \leq \delta(Q_k) < D(Q_{k+1}) \leq \delta(Q_{k+1}) < \dots\dots$$

Since $\frac{Q_r(n)}{n}$ is initially greater than $\delta(Q_r)$, the question arises if $D(Q_r) = \delta(Q_r)$, K. Rogers [1] showed that this is not the case for $r = 2$. In fact

(2.18.4) $$D(Q_2) = \frac{53}{88} < \frac{6}{\pi^2} = \delta(Q_2) .$$

Recently H.M. Stark [1] proved, more generally, that

(2.18.5) $$D(Q_r) < \delta(Q_r) \quad \text{for all} \quad r > 1 .$$

We shall now briefly examine the case of $Q_{k,r}$ integers. Their asymptotic density $\delta(Q_{k,r})$ is given by

(2.18.6) $$\delta(Q_{k,r}) = \lim_{n\to\infty} \frac{Q_{k,r}(n)}{n} = \frac{\zeta(k)}{\zeta(r)} .$$

Let $D(Q_{k,r})$ be their Schnirelmann density. Since

(2.18.7) $$Q_{k,1} \subset Q_{k,2} \subset Q_{k,3} \subset \dots \subset Q_{k,k-1} ,$$

it follows that

(2.18.8) $$D(Q_{k,1}) \le D(Q_{k,2}) \le D(Q_{k,3}) \le \dots \le D(Q_{k,k-1}) \le 1 .$$

We shall now prove the

(2.19) Lemma.

$$D(Q_{k,r}) \ge \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} .$$

Proof. case 1 $r \ge 2$.

It is clear that

$$Q_r(n) \ge n - \sum_p [\frac{n}{p^r}] \quad \text{and} \quad \frac{Q_r(n)}{n} > 1 - \sum_p p^{-r}$$

Since

$$Q_{k,r}(n) \ge \sum_{a=1}^{\infty} ([\frac{n}{a^k}] - \sum_p [\frac{\frac{n}{a^k}}{p^r}])$$

$$> \sum_{a=1}^{\infty} (\frac{n}{a^k} - \sum_p \frac{n}{a^k p^r}) - (n^{\frac{1}{k}} - 1) ,$$

We have

$$\frac{Q_{k,r}(n)}{n} > \sum_{a=1}^{\infty} (\frac{1}{a^k} - \sum_p \frac{1}{a^k p^r}) - \frac{n^{\frac{1}{k}} - 1}{n} = \zeta(k)(1 - \sum_p p^{-r}) + \frac{1 - n^{\frac{1}{k}}}{n} .$$

Let

$$f(x) = \frac{1 - x^{\frac{1}{k}}}{x} = \frac{1}{x} - x^{\frac{1}{k} - 1} ;$$

then

$$f'(x) = -\frac{1}{x^2} - (\frac{1}{k} - 1)x^{\frac{1}{k} - 2} ,$$

so that

$$f'(x) > 0 \quad \text{if} \quad (1 - \frac{1}{k})x^{\frac{1}{k} - 2} > \frac{1}{x^2} , \text{ i.e. } (1 - \frac{1}{k})x^{\frac{1}{k}} > 1 .$$

Thus

$$f'(x) \begin{cases} > 0 & \text{when } x > \dfrac{1}{(1 - \frac{1}{k})^k} , \\ < 0 & \text{when } x < \dfrac{1}{(1 - \frac{1}{k})^k} . \end{cases}$$

When $x = (1 - \frac{1}{k})^{-k}$ we get the minimum value of f, which is equal to

$$f((1 - \frac{1}{k})^{-k}) = \frac{1 - (1 - \frac{1}{k})^{-1}}{(1 - \frac{1}{k})^{-k}} = -\frac{1}{k}(1 - \frac{1}{k})^{k-1} .$$

Hence

$$\frac{Q_{k,r}(n)}{n} > \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} ,$$

and

$$D(Q_{k,r}) \geq \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} .$$

<u>case 2</u> $r = 1$.

In this case, $Q_{k,1}(n) = [n^{\frac{1}{k}}]$

$$\delta(Q_{k,1}) = \lim_{n\to\infty} \frac{[n^{\frac{1}{k}}]}{n} = 0 , \text{ since } k \geq 2 .$$

$$D(Q_{k,1}) = \inf_n \frac{[n^{\frac{1}{k}}]}{n} = 0 .$$

So the lemma result still holds in this case.

(2.20) <u>Remark</u>. The above proof is easily seen to hold even when $k = \infty$. The corresponding result, namely,

$$D(Q_r) > 1 - \sum_p p^{-r} ,$$

is due to R.L. Duncan [2] .

We will also be able to establish a result analogous to that of (2.18.5) for (k,r)-integers.

(2.21) Theorem.

$$D(Q_{k,r}) < \delta(Q_{k,r}) , \text{ for } 1 < r < k .$$

We need the following results, before proving the theorem.

(2.22) Lemma. If $a(n) \geq 0$ and $f(s)$ is the function determined by the series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, then $f(s)$ has a singularity at $s = \alpha$, where α is the abscissa of convergence.

(see Ayoub, R. [1], p. 17).

(2.23) Lemma.

$$|n^{-s} + (n+1)^{-s} - s \cdot n^{-s-1}| \leq |s||s+1|n^{-\sigma-2} .$$

(See Landau, E. [3], p. 700).

Now we let the error function be

$$E(x) = Q_{k,r}(x) - \frac{x\zeta(k)}{\zeta(r)} . \tag{2.24}$$

(2.25) Theorem. For any $\varepsilon > 0$, we have

(i) $E(n) > n^{\frac{1}{2r} - \varepsilon}$, for infinitely many integers n ,

(ii) $E(n) < -n^{\frac{1}{2r} - \varepsilon}$, for infinitely many integers n .

<u>Proof</u>. Let $\sum (\psi(n) - \frac{\zeta(k)}{\zeta(r)})n^{-s} = R_1(s)$, since

$$\sum (\psi(n) - \frac{\zeta(k)}{\zeta(r)})n^{-s} = \frac{\zeta(ks)\zeta(s)}{\zeta(rs)} - \frac{\zeta(k)\zeta(s)}{\zeta(r)} ,$$

and

$$\begin{aligned} R_1(s) &= \sum (\psi(n) - \frac{\zeta(k)}{\zeta(r)})n^{-s} \\ &= \sum (E(n) - E(n-1))n^{-s} \\ &= \sum E(n)(n^{-s} - (n+1)^{-s}) . \end{aligned}$$

Also, let

$$s \sum E(n)n^{-s-1} = R_2(s) ,$$

$$\sum E(n)n^{-s-1} = R_3(s) ,$$

$$\sum n^{\frac{1}{2r} - \varepsilon} \cdot n^{-s-1} = R_4(s) ,$$

and

$$\sum (n^{\frac{1}{2r} - \varepsilon} - E(n))n^{-s-1} = R_5(s) ,$$

$$\sum (n^{\frac{1}{2r} - \varepsilon} + E(n))n^{-s-1} = R_6(s) .$$

(i) Now suppose that for all $n \geq n_o$, $E(n) \leq n^{\frac{1}{2r} - \varepsilon}$. Then the series $R_5(s)$ converges for $\sigma > \frac{1}{2r} - \varepsilon$, and all but a finite number of coefficients of $R_5(s)$ are non-negative. Hence by lemma (2.22), the abscissa of convergence of $R_5(s)$ must be less than or equal to $\frac{1}{2r} - \varepsilon$. Let α be its abscissa of convergence, that is $\alpha \leq \frac{1}{2r} - \varepsilon$. By Lemma (2.23) implies $R_1(s)$ also converges for $\sigma > \alpha$. But this is false because $R_1(s)$ has singularities on $\sigma = \frac{1}{2r}$. Thus we must have $E(n) > n^{\frac{1}{2r} - \varepsilon}$ for infinitely many integers n .

(ii) Suppose that for all $n \geq n_o$, $E(n) \geq -n^{\frac{1}{2r} - \varepsilon}$, then we consider the series $R_6(s)$, proceed as in (i) and arrive at the same contradiction.

(2.26) Proof of the theorem (2.21)

By theorem (2.25), there are infinitely many integers n for which $E(n) < 0$. For such n,

$$\frac{Q_{k,r}(n)}{n} = \frac{\zeta(k)}{\zeta(r)} + \frac{E(n)}{n} < \frac{\zeta(k)}{\zeta(r)} ,$$

which proves the theorem.

(2.27) Theorem. For infinitely many integers n,

$$0 \le E(n) < 1 .$$

Proof. Let q_ℓ be the ℓ-th (k,r)-integer. Then $E(x)$ as a function of the real variable x is continuous in every interval $q_\ell \le x < q_{\ell+1}$, $\ell = 1,2,3,\ldots$. If a change of sign of $E(x)$ occurs within such an interval, then by continuity $E(x)$ has at least one zero in that interval. If $E(x) = 0$ and if $[x] = n$, so that $0 \le x - n < 1$, then

$$Q_{k,r}(n) = Q_{k,r}(x) = \frac{x\zeta(k)}{\zeta(r)} = \frac{n\zeta(k)}{\zeta(r)} + \frac{(x-n)\zeta(k)}{\zeta(r)} .$$

Hence $E(n) = Q_{k,r}(n) - \frac{n\zeta(k)}{\zeta(r)} = \frac{(x-n)\zeta(k)}{\zeta(r)}$. Thus $0 \le E(n) < \frac{\zeta(k)}{\zeta(r)} < 1$.

If a change of sign occurs at a (k,r)-integer n, then we prove that $0 < E(n) < 1$ as follows.

n is a (k,r)-integer and $E(n-\varepsilon)E(n) < 0$ for small enough $\varepsilon > 0$. Therefore

$$\begin{aligned} E(n-\varepsilon) &= Q_{k,r}(n-\varepsilon) - \frac{(n-\varepsilon)\zeta(k)}{\zeta(r)} \\ &= Q_{k,r}(n)-1 - \frac{n\zeta(k)}{\zeta(r)} + \frac{\varepsilon\zeta(k)}{\zeta(r)} \\ &= E(n) - (1 - \frac{\varepsilon\zeta(k)}{\zeta(r)}) \\ &< E(n) \text{ , for small } \varepsilon . \end{aligned}$$

Thus we must have $E(n-\varepsilon) < 0 < E(n)$. So that

$$E(n) - (1 - \frac{\varepsilon\zeta(k)}{\zeta(r)}) < 0 ,$$

i.e. $E(n) < 1 - \frac{\varepsilon\zeta(k)}{\zeta(r)} < 1$. Therefore $0 < E(n) < 1$.

Let us now consider the case when k is fixed and r varies; then we have the following theorem.

(2.28) Theorem.

$$\delta(Q_{k,r-1}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} < D(Q_{k,r}) < \delta(Q_{k,r}) .$$

Proof. The second inequality follows from theorem (2.21) . By Lemma (2.19), the first inequality will be true if

$$\frac{\zeta(k)}{\zeta(r-1)} + \zeta(k) \sum_p p^{-r} \le \zeta(k) .$$

i.e. if

$$\frac{1}{\zeta(r-1)} + \sum_p p^{-r} \le 1 .$$

Since

$$\sum_p p^{-r} < \zeta(r) - 1 ,$$

it will be sufficient to show that

$$\frac{1}{\zeta(r-1)} + \zeta(r) \le 2 .$$

Let $g(s) = \frac{1}{\zeta(s-1)} + \zeta(s)$.

Then, since

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=2}^{\infty} \Lambda(n)n^{-s} ,$$

we have

$$g'(s) = \zeta'(s) - \frac{\zeta'(s-1)}{\zeta^2(s-1)} = \sum_{n=2}^{\infty} \{\frac{n}{\zeta(s-1)} - \zeta(s)\}\Lambda(n)n^{-s} .$$

Thus $g'(s) > 0$ if $\zeta(s-1)\zeta(s) < 2$.

But $\zeta(s)\zeta(s+1)$ is a decreasing function and $\zeta(2)\zeta(3) < 2$.

Hence $g'(s) > 0$ and $g(s)$ is an increasing function for $s \geq 3$.

The desired result now follows since $\lim_{s\to\infty} g(s) = 2$.

(2.29) Corollary. $D(Q_{k,r-1}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} < D(Q_{k,r})$.

(2.30) Corollary. $\lim_{r\to\infty} D(Q_{k,r}) = 1$.

Since $\delta(Q_{k,r}) = \frac{\zeta(k)}{\zeta(r)}$, it is clear that

(2.31) $\frac{1}{\zeta(r)} \leq \ldots < \delta(Q_{k+2,r}) < \delta(Q_{k+1,r}) < \delta(Q_{k,r})$.

Now we have the following

(2.32) Theorem. $\frac{\zeta(r)}{\zeta(r-1)}\, \delta(Q_{k,r}) - \frac{1}{r} < D(Q_{k,r}) < \delta(Q_{k,r})$.

Proof. The second inequality follows from the theorem (2.21). By Lemma (2.19), the first inequality will be true if

$$\frac{\zeta(r)}{\zeta(r-1)}\,\frac{\zeta(k)}{\zeta(r)} - \frac{1}{r} < \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}(1 - \frac{1}{k})^{k-1} .$$

That is, if

$$\frac{\zeta(k)}{\zeta(r-1)} < \zeta(k)(1 - \sum_p p^{-r}) + \frac{1}{r} - \frac{1}{k}(1 - \frac{1}{k})^{k-1} .$$

But in Theorem (2.28), we have proved that

$$\frac{\zeta(k)}{\zeta(r-1)} \leq \zeta(k)(1 - \sum_p p^{-r}) \quad \text{for} \quad r \geq 2 .$$

Since $\frac{1}{r} - \frac{1}{k}(1 - \frac{1}{k})^{k-1} > 0$ for $r \geq 2$, the desired result follows immediately.

(2.33) Corollary. $\frac{\zeta(r)}{\zeta(r-1)}\, D(Q_{k+1,r}) - \frac{1}{r} < D(Q_{k,r})$.

Proof. This follows because

$$\frac{\zeta(r)}{\zeta(r-1)} D(Q_{k+1,r}) - \frac{1}{r} < \frac{\zeta(r)}{\zeta(r-1)} \delta(Q_{k+1,r}) - \frac{1}{r}$$

$$< \frac{\zeta(r)}{\zeta(r-1)} \delta(Q_{k,r}) - \frac{1}{r}$$

$$< D(Q_{k,r}) \ .$$

CHAPTER III

The distribution of (k,r)-integers in residue classes (mod h).

(3.1) Introduction. In this chapter, a and h represent integers with $h \geq 1$, and $C_{a,h}$ represents the residue class consisting of all integers which are $\equiv a \pmod h$. Also let $Q(x) = Q_{k,r}(x;a,h)$ denote the number of (k,r)-integers which do not exceed x and which belong to the residue class $C_{a,h}$. Our interest in this chapter is to obtain an estimate for $Q_{k,r}(x;a,h)$. Such an estimate was obtained for k-free integers by E. Cohen and R.L. Robinson [1]. It must be noted that while their result is a special case of ours (theorem (3.2) below), our theorem cannot be deduced with the help of their result; lemma (2.11) is not applicable for this purpose. Various special cases of this problem already exist in the literature. We shall only mention that Landau [1] considered the distribution of square-free numbers in residue classes (Cohen and Robinson [1] pointed out that Landau's error term in this connection was actually a uniform O-estimate); and that Ostmann ([1], p. 23) gave a wrong result for r-free numbers (see again Cohen and Robinson [1], p. 283). Our result for (k,r) numbers given in theorem (3.2) below is evidently new.

To simplify our analysis, we shall confine ourselves to the case $Q_{kr,kq}(x;a,h)$, where k is any integer > 1 and $0 < q < r$.

Recalling the notation that $p^a || h$ means that p^a is a unitary divisor of h, our theorem may be stated thus:

(3.2) Theorem. Let k, r, and q be integers with $k > 1$ and $0 < q < r$.

Let a and h be positive integers with $d = (a,h)$.
If $d \in Q_{kq}$ and $p^{\alpha}||h$, then we have

$$Q_{kr,kq}(x;a,h) = \frac{h^{kq-1}}{\phi_{kq}(h)} \frac{x\zeta(kr)}{\zeta(kq)} \prod_{p|(a,h)_*}(1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha}\nmid d}} (1 - p^{-kr}) + O(dx^{\frac{1}{k}}) ;$$

the estimate is uniform in a, h, r, and q.

Before proving the theorem, we have to note some lemmas. First we begin by defining some arithmetic functions.

(3.3) <u>The functions $\gamma(n)$, $\omega_s(n)$, and $\phi^*(n)$</u>.

The function $\gamma(n)$, called the 'core' of n, is defined to be the product of the distinct prime divisors of n. It is thus the largest square-free divisor of n. We define $\gamma(1) = 1$.

We also write $\omega_s(n) = \frac{\gamma^s(n)}{n}$; and $n' = \omega_s(n)$ if $n \in Q_s$.

Let $d||n$ mean that d is a unitary divisor of n, i.e. $d|n$ and $(d,\frac{n}{d}) = 1$.

As usual, let $\delta = (a,n)_*$ be the largest unitary divisor of n which divide of a. If $\delta = 1$ we say a is unitarily prime to n.

The function $\phi^*(n)$ is defined to be the number of integers a (mod n) such that $(a,n)_* = 1$. $\phi^*(n)$ is the unitary analogue of $\phi(n)$ and is given by

(3.3.1) $$\phi^*(n) = \prod_{p^e||n} (p^e - 1),$$

Let $\rho_h(a)$ denote the product of the distinct prime divisors of d which do not divide $\frac{h}{d}$.

It is clear that $\rho_h(a) = \gamma((a,h)_*)$.

We also let

$$(3.3.2) \qquad \theta_k(a,h) = \prod_{p|\rho_h(a)} \left(1 - \frac{(p^k,h)}{p^k}\right) .$$

(3.4) <u>Lemma</u>. (E. Cohen and R.L. Robinson [1]) .

if $H = (a,h)_* \in Q_k$, then

$$\theta_k(a,h) = \frac{\phi^*(H')}{H'} .$$

We supply a short proof for completeness. From (3.3.2) and the relation $\rho_h(a) = \gamma((a,h)_*)$,

$$(3.4.1) \qquad \theta_k(a,h) = \prod_{p|\gamma((a,h)_*)} \left(1 - \frac{(p^k,h)}{p^k}\right) = \prod_{p|(a,h)_*} \left(1 - \frac{(p^k,h)}{p^k}\right).$$

Since $H = (a,h)_* \in Q_k$, so (3.4.1) becomes

$$(3.4.2) \qquad \theta_k(a,h) = \prod_{p^e||(a,h)_*} \left(1 - \frac{p^e}{p^k}\right) = \prod_{p^e||H} \left(1 - \frac{1}{p^{k-e}}\right) .$$

But $p^e||H$ if and only if $p^{k-e}||H'$.

So from (3.4.2) and (3.3.1), we get

$$\theta_k(a,h) = \prod_{p^f||H'} \left(1 - \frac{1}{p^f}\right) = \frac{\phi^*(H')}{H'} .$$

(3.5) <u>Lemma</u>. Let $\psi_{kr,kq}(n)$ be the characteristic function of (kr,kq)-integers, then we have

$$\psi_{kr,kq}(n) = \sum_{m^k|n} \lambda(m) .$$

In future we shall write $\lambda_{r,q}(n)$ as $\lambda(n)$ for brevity.

(3.6) Lemma. Let $S(x;\alpha,\beta,\gamma)$ denote the number of solutions $\le x$ of the congruence

$$\alpha n \equiv \beta \pmod{\gamma}$$

where α and γ are integers ≥ 1. Then

$$S(x;\alpha,\beta,\gamma) = \frac{x}{\gamma}(\alpha,\gamma) + O(1) .$$

(3.7) Lemma. Let $d = (a,h)$ and $d \in Q_{kq}$ and let

$$f(m) = \begin{cases} \frac{\lambda(m)}{m^k}(m^k,h) , & (m^k,h)\,|\,d , \\ 0 , & \text{otherwise.} \end{cases}$$

Then

$$\sum_{m=1}^{\infty} f(m) = \frac{\zeta(kr)}{\zeta(kq)} \frac{h^{kq}}{\phi_{kq}(h)} \prod_{p|(a,h)_*} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha}\nmid d}} (1 - p^{-kr})$$

Proof. The function f is multiplicative in m, and it is easy to check that $\sum_{m=1}^{\infty} f(m)$ is absolutely convergent.
Hence we may express this series as a product in the form

$$\sum_{m=1}^{\infty} f(m) = \prod_p (1 + f(p) + f(p^2) + \dots) .$$

Hence

$$\sum_{m=1}^{\infty} f(m) = \prod_{\substack{p \\ p^{kq}\nmid h}} \Big\{\Big[1 + \frac{(p^{kr},h)}{p^{kr}} + \frac{(p^{2kr},h)}{p^{2kr}} + \dots\Big]$$

$$- \Big[\frac{(p^{kq},h)}{p^{kq}} + \frac{(p^{kr+kq},h)}{p^{kr+kq}} + \dots\Big]\Big\} ,$$

because if $p^{kq}|h$ then $(p^{kq},h) = p^{kq}$, and hence $p^{kq}|d$, which contradicts our hypothesis that $d \in Q_{kq}$, the set of all kq-free integers.

Let $p^{\alpha}||h$, then $(p^{kq},h) = p^{\alpha}$, $\alpha < kq$. If $p|h$ and $p^{\alpha} \nmid d$, then the only non vanishing term in $(1 + f(p) + f(p^2) + \ldots)$ is the first one, namely unity.

Hence

$$\sum_{m=1}^{\infty} f(m) = \prod_{\substack{p \\ p \nmid h}} \left(\frac{1 - p^{-kq}}{1 - p^{-kr}}\right) \prod_{\substack{p \\ p|h \\ p^{\alpha}|d}} \frac{1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}}{1 - p^{-kr}}$$

$$= \prod_{p} \left(\frac{1 - p^{-kq}}{1 - p^{-kr}}\right) \prod_{\substack{p \\ p|h}} \left(\frac{1 - p^{-kq}}{1 - p^{-kr}}\right)^{-1} \prod_{\substack{p \\ p|h \\ p^{\alpha}|d}} \frac{1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}}{1 - p^{-kr}}$$

$$= \frac{\zeta(kr)}{\zeta(kq)} \prod_{\substack{p \\ p|h}} \left(\frac{1 - p^{-kr}}{1 - p^{-kq}}\right) \prod_{\substack{p \\ p|h \\ p^{\alpha}|d}} \frac{1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}}{1 - p^{-kr}}$$

$$= \frac{\zeta(kr)}{\zeta(kq)} \prod_{\substack{p \\ p|h \\ p^{\alpha}|d}} \frac{1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}}{1 - p^{-kq}} \prod_{\substack{p \\ p|h \\ p^{\alpha} \nmid d}} \left(\frac{1 - p^{-kr}}{1 - p^{-kq}}\right)$$

$$= \frac{\zeta(kr)}{\zeta(kq)} \frac{h^{kq}}{\phi_{kq}(h)} \prod_{\substack{p \\ p|h \\ p^{\alpha}|d}} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha} \nmid d}} (1 - p^{-kr}) .$$

The conditions $p|h$ and $p^{\alpha}|d$ are equivalent to conditions: $p|d$ and $p \nmid \frac{h}{d}$, i.e. $p|\rho_h(a)$. Also $\rho_h(a) = \gamma((a,h)_*)$.

Thus we get

$$\sum_{m=1}^{\infty} f(m) = \frac{\zeta(kr)}{\zeta(kq)} \cdot \frac{h^{kq}}{\phi_{kq}(h)} \prod_{p|(a,h)_*} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha}\nmid d}} (1 - p^{-kr}) .$$

(3.8) Proof of theorem (3.2)

$$Q_{kr,kq}(x;a,h) = \sum_{\substack{z\le x \\ z\equiv a \pmod h}} \psi_{kr,kq}(z)$$

$$= \sum_{\substack{z\le x \\ z\equiv a \pmod h}} \sum_{m^k|z} \lambda_{r,q}(m) .$$

In future we shall write $\lambda_{r,q}(n)$ as $\lambda(n)$ for brevity.

So we have

(3.8.1) $$Q_{kr,kq}(x;a,h) = \sum_{\substack{m,n \\ m^k n \le x \\ m^k n \equiv a \pmod h}} \lambda(m) = \sum_{m\le x^{\frac{1}{k}}} \lambda(m) \sum_{\substack{n\le \frac{x}{m^k} \\ m^k n \equiv a \pmod h}} 1$$

$$= \sum_{\substack{m \le x^{\frac{1}{k}} \\ (m^k,h)|a}} \lambda(m) S(\frac{x}{m^k} ; m^k, a, h) = \sum_{\substack{m \le x^{\frac{1}{k}} \\ (m^k,h)|d}} \lambda(m) S(\frac{x}{m^k} ; m^k, a, h)$$

$$= \sum_{\substack{m \le x^{\frac{1}{k}} \\ (m^k,h)|d}} \lambda(m)\left\{\frac{x}{m^k h} (m^k,h) + O(1)\right\}$$ On using Lemma 5.

$$= \frac{x}{h} \sum_{\substack{m=1 \\ (m^k,h)|d}}^{\infty} \frac{\lambda(m)}{m^k} (m^k,h) - \frac{x}{h} \sum_{\substack{m > x^{\frac{1}{k}} \\ (m^k,h)|d}} \frac{\lambda(m)}{m^k} (m^k,h) + O(x^{\frac{1}{k}})$$

(3.8.2) $$\left| \sum_{\substack{m > x^{\frac{1}{k}} \\ (m^k,h)|d}} \frac{\lambda(m)}{m^k} (m^k,h)\right| < d \sum_{m > x^{\frac{1}{k}}} \frac{1}{m^k} = O\left(\frac{d}{x^{1 - \frac{1}{k}}}\right)$$

On combining (3.8.1), (3.8.2), and Lemma (3.7) and observing that the constants involved in the O-estimates $O(dx^{\frac{1}{k}})$ are independent of a, h, r, and q, we have

$$Q_{kr,kq}(x;a,h) = \frac{h^{kq-1}}{\phi_{kq}(h)} \frac{x\zeta(kr)}{\zeta(kq)} \prod_{p|(a,h)_*} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \prod_{\substack{p \\ p|h \\ p^{\alpha}\nmid d}} (1 - p^{-kr}) + O(dx^{\frac{1}{k}}) .$$

Hence the theorem is proved.

(3.9) <u>Corollary</u>. Let $q = 1$ and $r \to \infty$, then

$$Q_{\infty,k}(x;a,h) = Q_k(x;a,h) = \frac{h^{k-1}}{\phi_k(h)} \frac{\phi^*(H')}{H'} \frac{x}{\zeta(k)} + O(dx^{\frac{1}{k}}) ,$$

the estimate is uniform in a, h, r, and q .

This follows on observing that as $r \to \infty$

$$\zeta(kr) \to 1 , \quad \prod_{\substack{p \\ p|h \\ p^{\alpha}\nmid d}} (1 - p^{-kr}) \to 1 , \text{ and}$$

$$\prod_{p|(a,h)_*} (1 - p^{-kr} + p^{\alpha-kr} - p^{\alpha-kq}) \to \frac{\phi^*(H')}{H'} .$$

This result in an equivalent form, is due to E. Cohen and R.L. Robinson [1] .

CHAPTER IV

Representation of an integer as sum of two (r,q) type integers

(4.1) Introduction. The problem of the representations of an integer > 1 as the sum of two k-free integers, was studied by Linfoot and Evelyn [3], who published their proof in the Journal für Mathematik. Their result is the following:

Let n be any integer > 1 and let $T_k(n)$ denote the number of representations of n as the sum of two k-free integers, where k is any integer greater than 1. If $C_k = \prod_p (1-2p^{-k})$, then for any postive ε, we have

$$(4.1.1) \qquad T_k(n) = C_k n \prod_{p^k|n} \frac{p^k-1}{p^k-2} + O(n^{\frac{2}{k+1}+\varepsilon}) .$$

In 1930, T. Estermann [2] gave a short and elementary proof of the above result. In 1931, A. Page [1] generalized their result, and obtained the number of representations of any integer $n > 1$ as the sum of a k-free and a ℓ-free integer. Page established an asymptotic formula for such representations. Page's result is the following:

Let $T_{k,\ell}(n)$ denote the number of representations of n as the sum of a k-free and a ℓ-free integer. If k and ℓ be any integers such that $k \geq \ell \geq 2$, and $C = \prod_p (1-p^{-\ell}-p^{-k})$, then for any positive ε, we have

$$(4.1.2) \qquad T_{k,\ell}(n) = Cn \prod_{p^\ell|n} \left(1 + \frac{1}{p^k-p^{k-\ell}-1}\right) + O(n^{\frac{\ell+k-2}{\ell k-1}+\varepsilon}).$$

Now in this chapter, we are going to study the problem of representations of any integer $n > 1$ as the sum of a (k_1r_1, k_1q_1)-integer and a (k_2r_2, k_2q_2)-integer, where $k_1 > 1$; $0 < q_1 < r_1$, $k_2 > 1$; $0 < q_2 < r_2$, and $k_2q_2 \leq k_1$.

(4.2) Notation. Let $T(n)$ denote the number of representations of the integer n as the sum of a (k_1r_1, k_1q_1)-integer and a (k_2r_2, k_2q_2)-integer, where $k_1 > 1$; $0 < q_1 < r_1$, $k_2 > 1$; $0 < q_2 < r_2$, and $k_2q_2 \leq k_1$.

(4.3) Theorem. $T(n) \sim n\, H(n)$,

where

$$H(n) = \frac{\zeta(k_2r_2) \prod_p \left[(1-p^{-k_2q_2}) + \frac{1-p^{-k_2r_2}}{1-p^{-k_1r_1}}(p^{-k_1r_1} - p^{-k_1q_1})\right]}{\prod_{p^{k_2q_2}|n} \left[1 + \left(\frac{1-p^{-k_2r_2}}{1-p^{-k_1r_1}}\right)\left(\frac{p^{-k_1r_1} - p^{-k_1q_1}}{1-p^{-k_2q_2}}\right)\right]} .$$

We first note for later use an alternate form of theorem (3.2).

(4.4) Theorem. Let $Q^*_{r,q}(x;a,h)$ denote the number of (r,q)-integers strictly less than x and $\equiv a \pmod h$. If $kq \leq j$ and if $(m^j, a) \in Q_{kq}$, then

$$Q^*_{kr,kq}(x;a,m^j) = \left(\frac{m^{kq-j}}{\phi_{kq}(m)}\right)\left(\frac{\zeta(kr)}{\zeta(kq)}\right)x \prod_{p|m^j} (1-p^{-kr}) + O(dx^{\frac{1}{k}}) ;$$

the estimate is uniform in a, m^j, r, and q .

This follows from the theorem (3.2) on using the fact that

$$\phi_{kq}(m^j) = m^{(j-1)kq}\ \phi_{kq}(m) .$$

(4.5) Proof of theorem (4.3). Let $\psi_1(n)$ denote the characteristic **function** of (k_1r_1,k_1q_1)-integers, and $\psi_2(n)$ denote the characteristic function of the (k_2r_2,k_2q_2)-integers.

Then

$$T(n) = \sum_{a+b=n} \psi_1(a)\psi_2(b) = \sum_{a+b=n} \Big(\sum_{m^{k_1}|a} \lambda(m)\Big)\ \psi_2(b)$$

$$= \sum_{\substack{m,\ell,b \\ m^{k_1}\ell+b=n}} \lambda(m)\psi_2(b) = \sum_{m < n^{\frac{1}{k_1}}} \lambda(m) \sum_{\substack{b<n \\ b\equiv n \pmod{m^{k_1}}}} \psi_2(b)$$

$$= \sum_{m<n^{\frac{1}{k_1}}} \lambda(m)\ Q^*_{(k_2r_2,k_2q_2)}(n;n,m^{k_1}) .$$

Now let ε denote an arbitrary number such that $0 < \varepsilon < \frac{1}{k_1}$. Then, considering separately those values of $m \le n^{\frac{1}{k_1}-\varepsilon}$ and those $> n^{\frac{1}{k_1}-\varepsilon}$, we get

$$T(n) = \sum_{m\le n^{\frac{1}{k_1}-\varepsilon}} \lambda(m)\ Q^*_{(k_2r_2,k_2q_2)}(n;n,m^{k_1})$$
$$+ O\Big(\sum_{n^{\frac{1}{k_1}-\varepsilon}<m} Q^*_{(k_2r_2,k_2q_2)}(n;n,m^{k_1})\Big) .$$

Since $Q^*_{(k_2r_2,k_2q_2)}(n;a,h) = O(\frac{n}{h})$ uniformly in a, we have, by the theorem (4.4),

$$T(n) = \frac{n\zeta(k_2r_2)}{\zeta(k_2q_2)} \sum_{\substack{m \le n^{\frac{1}{k_1}-\varepsilon} \\ (m^{k_1},n)\varepsilon Q_{k_2q_2}}} \frac{\lambda(m)}{m^{k_1-k_2q_2}\phi_{k_2q_2}(m)} \prod_{p|m^{k_1}} (1-p^{-k_2r_2})$$

$$+ O(n^{\frac{1}{k_2}+\frac{1}{k_1}-\varepsilon}) + O(n \sum_{m>n^{\frac{1}{k_1}-\varepsilon}} m^{\frac{1}{k_1}}) .$$

Since $\frac{1}{k_2} + \frac{1}{k_1} \le 1$, the first O-term is $o(n)$. Also, the second O-term is $O(n^{\frac{1}{k_1}+\varepsilon(k_1-1)})$ and is therefore also $o(n)$. Hence

$$\lim_{n \to \infty} \frac{T(n)}{n} = \frac{\zeta(k_2r_2)}{\zeta(k_2q_2)} \sum_{\substack{m \le n^{\frac{1}{k_1}-\varepsilon} \\ (m^{k_1},n)\varepsilon Q_{k_2q_2}}} \frac{\lambda(m)}{m^{k_1-k_2q_2}\phi_{k_2q_2}(m)} \prod_{p|m^{k_1}} (1-p^{-k_2r_2}) .$$

The series on the right side of the above equation, summed for $m = 1$ to ∞ is absolutely convergent because it is less than

$$\sum_{m=1}^{\infty} \frac{m^{k_2q_2-k_1}}{\phi_{k_2q_2}(m)} = \sum_{m=1}^{\infty} m^{-k_1} \prod_{p|m} (1 - p^{-k_2q_2})^{-1}$$

$$< \sum_{m=1}^{\infty} m^{-k_1} \prod_{p} (1 - p^{-k_2q_2})^{-1} = \zeta(k_1)\zeta(k_2q_2) .$$

Now consider the series

$$\frac{\zeta(k_2r_2)}{\zeta(k_2q_2)} \sum_{m=1}^{\infty} g(m)$$

where

$$g(m) = \begin{cases} \dfrac{\lambda(m)}{m^{k_1-k_2q_2}\phi_{k_2q_2}(m)} \prod_{p|m^{k_1}} (1-p^{-k_2r_2}) \, , & \text{if } (m^{k_1},n) \in Q_{k_2q_2} \, , \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

The function g is multiplicative in m, and so we have

$$\sum_{m=1}^{\infty} g(m) = \prod_p \{1 + g(p) + g(p^2) + \ldots\}$$

$$= \frac{\prod_p \{1 + g(p^{r_1}) + g(p^{2r_1}) + \ldots - g(p^{q_1}) - g(p^{q_1+r_1}) - \ldots\}}{\prod_{p^{k_2q_2}|n} \{1 + g(p^{r_1}) + g(p^{2r_1}) + \ldots - g(p^{q_1}) - g(p^{q_1+r_1}) - \ldots\}}$$

$$= \frac{\prod_p \left\{1 + \left(\dfrac{1-p^{-k_2r_2}}{1-p^{-k_2q_2}}\right)\left(\dfrac{p^{-k_1r_1} - p^{-k_1q_1}}{1-p^{-k_1r_1}}\right)\right\}}{\prod_{p^{k_2q_2}|n} \left\{1 + \left(\dfrac{1-p^{-k_2r_2}}{1-p^{-k_2q_2}}\right)\left(\dfrac{p^{-k_1r_1} - p^{-k_1q_1}}{1-p^{-k_1r_1}}\right)\right\}}$$

Also note that the denominator of the above expression is a bounded function of n, since a routine calculation shows that each factor in the product of the denominator lies between 0 and 1.

It follows that

$$\frac{\zeta(k_2 r_2)}{\zeta(k_2 q_2)} \sum_{m=1}^{\infty} g(m) = H(n) \quad .$$

The theorem now follows.

(4.6) Corollary. Let $q_1 = q_2 = 1$ and $r_1 \to \infty$, $r_2 \to \infty$ then we have

$$T_{k_1,k_2}(n) \sim n\, H_{k_1 k_2}(n) \qquad \text{as} \quad n \to \infty$$

where $T_{k_1,k_2}(n)$ denotes the number of representations of the integer n as the sum of a k_1-free and a k_2-free integer, $1 < k_2 \le k_1$, and

$$H_{k_1,k_2}(n) = \prod_p (1-p^{-k_1}-p^{-k_2}) \sum_{p^{k_2}|n} \left(1 + \frac{1}{p^{k_1}-p^{k_1-k_2}-1}\right) \quad .$$

This follows by letting $q_1 = q_2 = 1$ and $r_1 \to \infty$, $r_2 \to \infty$ then $T(n) \to T_{k_1,k_2}(n)$, and $H(n) \to H_{k_1,k_2}(n)$. This result is due to E. Cohen and R.L. Robinson [1]. Another version of the same is due to A. Page [1].

(4.7) Remarks. It is perhaps possible to apply the results of Carlitz [2] concerning the asymptotic formula obtained for a expression of the form

$$\sum_{n_1+\cdots+n_s=n} \alpha_1(n_1) \cdots \alpha_s(n_s)$$

(where $\alpha_1,\ldots,\alpha_s$ are arithmetic functions subject to certain restrictions) to obtain a formula for the number of solutions of

$$n = n_1 + n_2 + \dots + n_s \quad ,$$

where

$$n_j \in Q_{k_j, r_j} \qquad (j = 1, 2, \dots, s) \quad ,$$

(more generally, $n_j \in Q_{\tilde{k}_j, \tilde{r}_j}$).

CHAPTER V

The number of representations of an integer as the sum of a prime and a (k,r)-integer.

(5.1) Introduction

Generalizing an earlier result of T. Estermann [3], L. Mirsky [3] showed that every sufficiently large integer n can be represented as the sum of a prime and a k-free integer $(k > 1)$; also for $n \to \infty$, the number $T(k;n)$ of such representation is given by

$$(5.1.1) \qquad \prod_{p \nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right) ;$$

where H is any positive number, $\mathrm{Li}\, n = \int_2^n \frac{du}{\log u}$, and the O-constant depends at most on k and H.

Since the set of r-free integers is a subset of the set of (k,r)-integers, it is trivial, after Mirsky's result that every sufficiently large integer n can be represented as a sum of a prime and a (k,r)-integer. In this chapter we shall obtain an expression for the number $T(k,r;n)$ of such representations. Note that such an expression cannot be deduced for Mirsky's result (5.1.1) above.

Our result (theorem 5.6 below) for $T(k,r;n)$ includes Mirsky's result (5.1.1) as a special case, and is in fact an improvement of his result.

Our improvement is obtained by using the following estimate namely (see Page, A. [2]), if $(d,q) = 1$, then

$$(5.1.2) \qquad \sum_{\substack{p<n, \\ p\equiv d \pmod q}} 1 = \frac{1}{\phi(q)} \mathrm{Li}\, n + O(n\, e^{-A\sqrt{\log n}}) ,$$

where A is a positive constant, in the place of the following estimate used by Mirsky: for $(d,q) = 1$

$$(5.1.3) \qquad \sum_{p<n,\, p\equiv d \pmod q} 1 = \frac{1}{\phi(q)} \mathrm{Li}\, n + O\left(\frac{n}{\log^{2H} n}\right),$$

where H is any positive number.

We will first prove our improved version of Mirsky's theorem (5.1.1).

(5.2) <u>Theorem.</u> Let r be any integer greater than 1, and A is a positive constant. Then every sufficiently large integer n can be represented as the sum of a prime and a r-free integer, and, for $n \to \infty$, the number $T(r;n) = T(n)$ of such representations is given by

$$(5.2.1) \qquad T(n) = \prod_{p \nmid n} \left(1 - \frac{1}{p^{r-1}(p-1)}\right) \mathrm{Li}\, n + O\left(n\, e^{-\frac{A}{2}\sqrt{\log n}}\right),$$

where $\mathrm{Li}\, n = \int_2^n \frac{du}{\log u}$,

and the O-constant depends at most on r and A.

<u>Proof</u>. Since $\mu_r(m) = \sum_{a^r b = m} \mu(a)$, we have

$$T(n) = \sum_{m+p=n} \mu_r(m) = \sum_{a^r b + p = n} \mu(a)$$

$$= \sum_{\substack{a^r b+p=n \\ a\le x}} \mu(a) + \sum_{\substack{a^r b+p=n \\ a>x}} \mu(a) = \sum\nolimits_1 + \sum\nolimits_2 \quad \text{(say)},$$

where the value of x will be fixed later. To evaluate $\sum_1$ we use the estimate (5.1.2). We have

$$\sum_1 = \sum_{a\le x} \mu(a) \sum_{\substack{p<n \\ p\equiv n \pmod{a^r}}} 1$$

$$= \sum_{\substack{a\le x \\ (a,n)=1}} \mu(a)\left\{\frac{1}{\phi(a^r)} \operatorname{Li} n + O(n\, e^{-A\sqrt{\log n}})\right\} + O(x)$$

$$= \operatorname{Li} n \sum_{(a,n)=1} \frac{\mu(a)}{\phi(a^r)} + O\left(\frac{n}{\log n} \sum_{a>x} \frac{1}{\phi(a^r)}\right) + O\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) + O(x)$$

$$= \prod_{p\nmid n} \left(1 - \frac{1}{\phi(p^r)}\right) \operatorname{Li} n + O\left(\frac{n}{\log n} \sum_{a>x} \frac{\log\log a}{a^r}\right) + O\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) + O(x)$$

$$= \prod_{p\nmid n} \left(1 - \frac{1}{p^{r-1}(p-1)}\right) \operatorname{Li} n + O\left(\frac{n \log\log x}{x \log n}\right) + O\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) + O(x) ,$$

where in obtaining the first O-term above, we use the fact that $r > 1$.

Furthermore

$$\left|\sum_2\right| \le \sum_{\substack{a^r b+p=n \\ a>x}} 1 \le \sum_{\substack{a^r b<n \\ a>x}} 1 = O\left(\frac{n}{x^{r-1}}\right) = O\left(\frac{n}{x}\right) .$$

Thus we have

$$T(n) = \prod_{p\nmid n} \left(1 - \frac{1}{p^{r-1}(p-1)}\right) \operatorname{Li} n + O\left(\frac{n \log\log x}{x \log n}\right)$$

$$+ O\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) + O(x) + O\left(\frac{n}{x}\right) .$$

The theorem follows immediately by putting $x = e^{\frac{A}{2}\sqrt{\log n}}$.

(5.3) <u>r-free numbers of the form $p + \ell$</u>.

Let ℓ be a given positive integer. T. Estermann [3] obtained an asymptotic formula for the number of square-free integers not exceeding n and having the form $a^2 + \ell$. L. Mirsky [3] generalized this result and obtained the following theorem.

(5.3.1) Theorem. Let ℓ be a fixed positive integer, r any integer greater than 1, and H any positive number. Then the number of r-free integers $m \le n$, having the form $m = p + \ell$, is

$$(5.3.2) \qquad U(n) = \prod_{p \nmid \ell} \left(1 - \frac{1}{p^{r-1}(p-1)}\right) \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

as $n \to \infty$, where the O-constant depends at most on r, ℓ, and H. We can improve this result also by using the estimate (5.1.2). Now we obtain the following theorem whose proof is very similar to that of theorem (5.2), and is therefore omitted.

(5.4) Theorem. Let ℓ be a fixed positive integer, r any integer greater than 1, and A is a positive constant. Let $U(n)$ denote the number of r-free integers $m \le n$, having the form $m = p + \ell$. Then, as $n \to \infty$,

$$(5.4.1) \qquad U(n) = \prod_{p \nmid \ell} \left(1 - \frac{1}{p^{r-1}(p-1)}\right) \mathrm{Li}\, n + O\left(n\, e^{-\frac{A}{2}\sqrt{\log n}}\right),$$

where the O-constant depends at most on r, ℓ, and A.

(5.5) Representation as a sum of a prime and a (k,r)-integer.

We will now establish an asymptotic formula for the number $T(k,r;n)$ of representations of an integer as the sum of a prime and a (k,r)-integer.

We note for later use a well-known result, namely

(5.5.1) Lemma. There exist a constant $C > 0$ and an integer N such that

$$\frac{C}{\phi(n)} \le \frac{\log\log n}{n} \quad \text{for} \quad n \ge N .$$

(see, for example, G.H. Hardy and E.M. Wright [1] Theorem 328, p. 267).

(5.5.2) Lemma. Let $f(a) = \frac{\lambda(a)}{\phi(a)}$, then $f(a)$ is multiplicative, and $\sum_{a=1}^{\infty} f(a)$ is absolutely convergent for $r > 1$.

Proof. Clearly $f(1) = \frac{\lambda(1)}{\phi(1)} = 1$. Also $f(a)$ is multiplicative, since $\lambda(n)$ and $\phi(n)$ are multiplicative functions.

To show that $\sum_{a=1}^{\infty} |f(a)| = \sum_{a=1}^{\infty} |\frac{\lambda(a)}{\phi(a)}|$ is convergent, by lemma (5.5.1), there exist a positive constant C and an integer N_1 such that

$$\frac{c}{\phi(a)} \le \frac{\log\log a}{a} \quad \text{for} \quad a \ge N_1 .$$

Hence

$$\left|\frac{\lambda(a)}{\phi(a)}\right| \le \frac{1}{\phi(a)} \le \frac{1}{c}\,\frac{\log\log a}{a} \quad \text{for} \quad a \ge N_1 .$$

Put $a = n^k m^r$. Then we have

$$\frac{1}{\phi(n^k m^r)} \le \frac{1}{c}\,\frac{\log\log n^k m^r}{n^k m^r} \quad (n^k m^r \ge N_1) .$$

Also, for a given positive ε with $\varepsilon < 1 - \frac{1}{r}$, there exists an integer $N_2(\varepsilon)$ such that

$$\frac{\log\log n^k m^r}{n^k m^r} \le \frac{1}{(n^k m^r)^{1-\varepsilon}} \quad (n^k m^r \ge N_2) .$$

Now let $N = \max (N_1, N_2)$ then we have

$$\frac{1}{\phi(n^k m^r)} \le \frac{1}{c}\,\frac{\log\log n^k m^r}{n^k m^r} \le \frac{1}{c}\,\frac{1}{(n^k m^r)^{1-\varepsilon}} \quad (n^k m^r > N) .$$

Hence, recalling that $\lambda(a) = 0$ unless a is of the form $a = n^k m^r$, and that then $|\lambda(a)| = 1$, we have

$$\sum_{a=1}^{\infty} |f(a)| = \sum_{\substack{n,m \\ 1 \le n^k m^r \le N}} \frac{1}{\phi(n^k m^r)} + \sum_{\substack{n,m \\ n^k m^r > N}} \frac{1}{\phi(n^k m^r)}$$

$$< N + \sum_{n,m=1}^{\infty} \frac{1}{c} \frac{1}{(n^k m^r)^{1-\varepsilon}} < \infty .$$

Hence the lemma is proved.

(5.6) Theorem. Let k,r be integers with $k > r > 1$, and A a positive constant. Then for $n \to \infty$, the number $T(k,r;n)$ of representations of n as the sum of a prime and a (k,r)-integer is given by

(5.6.1) $$T(k,r;n) = \prod_{p \nmid n} \{1 + (1 - p^{-1})^{-1}(1 - p^{-k})^{-1}(p^{-k} - p^{-r})\} \text{ Li } n + F(n) + O(n\, e^{-\frac{A}{2}\sqrt{\log n}}),$$

where $\text{Li } n = \int_2^n \frac{du}{\log u}$ and

$$F(n) = \text{Li } n \sum_{\substack{(a,n)=1 \\ a > e^{\frac{A}{2}\sqrt{\log n}}}} \frac{\lambda(a)}{\phi(a)} + \sum_{\substack{ab+p=n \\ a > e^{\frac{A}{2}\sqrt{\log n}}}} \lambda(a)$$

and the O-constant depends at most on A only and is uniform in k and r.

Proof. Let x denote a certain function of n (to be fixed later), which tends to infinity with n. As usual, p denotes a prime, and the O-notation refers to the passage $n \to \infty$. We have

$$T(k,r;n) = \sum_{\substack{p \\ m+p=n}} \psi(m) = \sum_{\substack{p \\ m+p=n}} \sum_{a|m} \lambda(a)$$

$$= \sum_{\substack{p \\ m+p=n}} \sum_{ab=m} \lambda(a) = \sum_{ab+p=n} \lambda(a)$$

$$= \sum_{\substack{ab+p=n \\ a\le x}} \lambda(a) + \sum_{\substack{ab+p=n \\ a>x}} \lambda(a) = \sum_1 + \sum_2 \text{, say .}$$

We use (5.1.2) to evaluate $\sum_1$.

$$\sum_1 = \sum_{\substack{ab+p=n \\ a\le x}} \lambda(a) = \sum_{a\le x} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1$$

$$= \sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1 + \sum_{\substack{a\le x \\ (a,n)>1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1 \quad .$$

Let us now consider $\sum_{\substack{a\le x \\ (a,n)>1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1$.

Since $(a,n) > 1$ and $n = ab + p$, we have $(a,n) = p$. It is obvious that the number of a's $\le x$, with $(a,n) = p$ for some prime p , is certainly $\le x$.

Hence
$$\sum_{\substack{a\le x \\ (a,n)>1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1 = \sum_{\substack{ab+p=n \\ a\le x \\ (a,n)=p}} \lambda(a) = 0(x) \ .$$

Next,

$$\sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod a}} 1 = \sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) \left\{ \frac{1}{\phi(a\ )} \text{ Li } n + 0\left(\frac{n}{e^{A\sqrt{\log n}}}\right) \right\}$$

$$= \text{Li } n \left(\sum_{\substack{a=1 \\ (a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a)} \right) - \text{Li } n \left(\sum_{\substack{(a,n)=1 \\ a>x}} \frac{\lambda(a)}{\phi(a)} \right) + 0\left(\frac{n}{e^{A\sqrt{\log n}}}\right) \sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) \ .$$

But $0\left(\frac{n}{e^{A\sqrt{\log n}}}\right) \sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) = 0\left(\frac{nx}{e^{A\sqrt{\log n}}}\right)$.

By lemma (5.5.2), we have

$$\sum_{\substack{a=1 \\ (a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a)} = \prod_{\substack{p \\ p\nmid n}} \{1 + \frac{\lambda(p)}{\phi(p)} + \frac{\lambda(p^2)}{\phi(p^2)} + \dots\} .$$

so that

$$\text{Li } n \left(\sum_{\substack{a=1 \\ (a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a)} \right) = \text{Li } n \prod_{\substack{p \\ p\nmid n}} \{1 + \frac{\lambda(p)}{\phi(p)} + \frac{\lambda(p^2)}{\phi(p^2)} + \dots\}$$

$$= \text{Li } n \prod_{\substack{p \\ p\nmid n}} \{1 + \frac{1}{\phi(p^k)} + \frac{1}{\phi(p^{2k})} + \frac{1}{\phi(p^{3k})} + \dots$$

$$- \frac{1}{\phi(p^r)} - \frac{1}{\phi(p^{r+k})} - \frac{1}{\phi(p^{r+2k})} - \dots\}$$

$$= \text{Li } n \prod_{\substack{p \\ p\nmid n}} \{1 + \frac{1}{p^k(1-\frac{1}{p})} + \frac{1}{p^{2k}(1-\frac{1}{p})} + \frac{1}{p^{3k}(1-\frac{1}{p})} + \dots$$

$$- [\frac{1}{p^r(1-\frac{1}{p})} + \frac{1}{p^{r+k}(1-\frac{1}{p})} + \frac{1}{p^{r+2k}(1-\frac{1}{p})} + \dots]\}$$

$$= \text{Li } n \prod_{\substack{p \\ p\nmid n}} \{1 + \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{p^k} \cdot \frac{1}{1-p^{-k}} - \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{p^r} \cdot \frac{1}{1-p^{-k}}\}$$

$$= \text{Li } n \prod_{\substack{p \\ p\nmid n}} \{1 + (1-p^{-1})^{-1}(1-p^{-k})^{-1}(p^{-k} - p^{-r})\} .$$

Now we let

$$F_1(n,x) = \text{Li } n \sum_{\substack{(a,n)=1 \\ a>x}} \frac{\lambda(a)}{\phi(a)} + \sum_{\substack{ab+p=n \\ a>x}} \lambda(a) .$$

Thus we have

$$T(k,r;n) = \text{Li}\, n \prod_{\substack{p \\ p\nmid n}} \{1 + (1 - p^{-1})^{-1}(1 - p^{-k})^{-1}(p^{-k} - p^{-r})\}$$

$$+ F_1(n,x) + O(x) + O(\frac{nx}{e^{A\sqrt{\log n}}}) .$$

Put $x = e^{\frac{A}{2}\sqrt{\log n}}$, then

$$T(k,r;n) = \text{Li}\, n \prod_{\substack{p \\ p\nmid n}} \{1 + (1 - p^{-1})^{-1}(1 - p^{-k})^{-1}(p^{-k} - p^{-r})\}$$

$$+ F(n) + O(n\, e^{-\frac{A}{2}\sqrt{\log n}}) .$$

(5.7) <u>Remarks</u>. For general values of k and r , the function $F_1(n,e^{\frac{A}{2}\sqrt{\log n}})$ does not seem to be amenable for an elegant estimate. We therefore prefer to leave it as it is for this general case when k and r are any integers with $1 < r < k$. However, in the case when $(k,r) > 1$, an elegant estimate for $F_1(n,e^{\frac{A}{2}\sqrt{\log n}})$ is possible. The corresponding theorem for this case is given below.

It should be noted that Mirsky's theorem stated in (5.1.1) is a special case of our theorem to follow, with an improved error term.

(5.8) <u>Theorem</u>. Let k, r, q be integers with $k > 1$, and $1 \leq q < r$. Then

$$T(kr,kq;n) = \text{Li}\, n \prod_{p\nmid n} \{1 + (1 - p^{-1})^{-1}(1 - p^{-kr})^{-1}(p^{-kr} - p^{-kq})\}$$

$$+ O(n\, e^{-\frac{A}{2}\sqrt{\log n}}) ,$$

where $\mathrm{Li}\ n = \int_2^n \frac{du}{\log u}$

and the O-constant depends at most on A and k, and is uniform in r and q .

Proof. The proof proceeds as in that for theorem (5.6) except that in the later part we are now able to estimate the error more precisely. For completeness, we shall give the details below,

$$T(kr,kq;n) = \sum_{m+p=n} \psi_{kr,kq}(m) = \sum_{m+p=n} \sum_{a^k|m} \lambda_{r,q}(a)$$

$$= \sum_{a^kb+p=n} \lambda(a) \qquad (\lambda(a) = \lambda_{r,q}(a))$$

$$= \sum_{\substack{a^kb+p=n \\ a\le x}} \lambda(a) + \sum_{\substack{a^kb+p=n \\ a>x}} \lambda(a) = \sum_1 + \sum_2 , \quad \text{say .}$$

We use (5.1.2) to evaluate $\sum_1$.

$$\sum_1 = \sum_{\substack{a^kb+p=n \\ a\le x}} \lambda(a) = \sum_{a\le x} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod{a^k}}} 1$$

$$= \sum_{\substack{a\le x \\ (a,n)=1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod{a^k}}} 1 + \sum_{\substack{a\le x \\ (a,n)>1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod{a^k}}} 1 .$$

Let us now consider

$$\sum_{\substack{a\le x \\ (a,n)>1}} \lambda(a) \sum_{\substack{p<n \\ p\equiv n \pmod{a^k}}} 1 .$$

Since $(a,n) > 1$ and $n = a^kb + p$, we have $(a,n) = p$.

It is obvious that the number of a's $\le x$, with $(a,n) = p$ for some prime p , is certainly $\leq x$.

Hence $$\sum_{\substack{a\leq x\\(a,n)>1}} \lambda(a) \sum_{\substack{p<n\\p\equiv n \pmod{a^k}}} 1 = \sum_{\substack{a^k b+p=n\\a\leq x\\(a,n)=p}} \lambda(a) = 0(x) .$$

Next,

$$\sum_{\substack{a\leq x\\(a,n)=1}} \lambda(a) \sum_{\substack{p<n\\p\equiv n \pmod{a^k}}} 1 = \sum_{\substack{a\leq x\\(a,n)=1}} \lambda(a)\left\{\frac{1}{\phi(a^k)} \operatorname{Li} n + 0\left(\frac{n}{e^{A\sqrt{\log n}}}\right)\right\}$$

$$= \operatorname{Li} n \left(\sum_{\substack{a=1\\(a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a^k)}\right) - \operatorname{Li} n \left(\sum_{\substack{(a,n)=1\\a>x}} \frac{\lambda(a)}{\phi(a^k)}\right) + 0\left(\frac{n}{e^{A\sqrt{\log n}}}\right) \sum_{\substack{a\leq x\\(a,n)=1}} \lambda(a) .$$

But $$0\left(\frac{n}{e^{A\sqrt{\log n}}}\right) \sum_{\substack{a\leq x\\(a,n)=1}} \lambda(a) = 0\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) .$$

By lemma (5.5.2) , we have

$$\sum_{\substack{a=1\\(a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a^k)} = \prod_{\substack{p\\p\nmid n}} \left\{1 + \frac{\lambda(p)}{\phi(p^k)} + \frac{\lambda(p^2)}{\phi(p^{2k})} + \dots\right\} .$$

so that

$$\operatorname{Li} n \left(\sum_{\substack{a=1\\(a,n)=1}}^{\infty} \frac{\lambda(a)}{\phi(a^k)}\right) = \operatorname{Li} n \prod_{\substack{p\\p\nmid n}} \left\{1 + \frac{\lambda(p)}{\phi(p^k)} + \frac{\lambda(p^2)}{\phi(p^{2k})} + \dots\right\}$$

$$= \operatorname{Li} n \prod_{\substack{p\\p\nmid n}} \left\{1 + \frac{1}{\phi(p^{kr})} + \frac{1}{\phi(p^{2kr})} + \frac{1}{\phi(p^{3kr})} + \dots \right.$$

$$\left. - \frac{1}{\phi(p^{kq})} - \frac{1}{\phi(p^{k(q+r)})} - \frac{1}{\phi(p^{k(q+2r)})} - \dots\right\}$$

$$= \text{Li } n \prod_{\substack{p \\ p \nmid n}} \{1 + \frac{1}{p^{kr}(1 - \frac{1}{p})} + \frac{1}{p^{2kr}(1 - \frac{1}{p})} + \frac{1}{p^{3kr}(1 - \frac{1}{p})} + \cdots$$

$$- [\frac{1}{p^{kq}(1 - \frac{1}{p})} + \frac{1}{p^{k(q+r)}(1 - \frac{1}{p})} + \frac{1}{p^{k(q+2r)}(1 - \frac{1}{p})} + \cdots]\}$$

$$= \text{Li } n \prod_{\substack{p \\ p \nmid n}} \{1 + \frac{1}{1 - \frac{1}{p}} \cdot \frac{1}{p^{kr}} \cdot \frac{1}{1 - p^{-kr}} - \frac{1}{1 - \frac{1}{p}} \cdot \frac{1}{p^{kq}} \cdot \frac{1}{1 - p^{-kr}}\}$$

$$= \text{Li } n \prod_{\substack{p \\ p \nmid n}} \{1 + (1 - p^{-1})^{-1}(1 - p^{-kr})^{-1}(p^{-kr} - p^{-kq})\} \text{ .}$$

Now consider

$$\text{Li } n \sum_{\substack{(a,n)=1 \\ a>x}} \frac{\lambda(a)}{\phi(a^k)}$$

$$= 0\left(\frac{n}{\log n} \sum_{a>x} \frac{1}{\phi(a^k)}\right) = 0\left(\frac{n}{\log n} \sum_{a>x} \frac{\log \log a}{a^k}\right)$$

$$= 0\left(\frac{n \log \log x}{x \log n}\right) \text{ .}$$

Furthermore

$$|\sum_2| = |\sum_{\substack{a^k b+p=n \\ a>x}} \lambda(a)| \le \sum_{\substack{a^k b+p=n \\ a>x}} 1$$

$$\le \sum_{\substack{a^k b<n \\ a>x}} 1 = 0\left(\frac{n}{x^{k-1}}\right) = 0\left(\frac{n}{x}\right) \text{ .}$$

Thus we have

$$T(kr,kq;n) = \mathrm{Li}\, n \prod_{\substack{p \\ p\nmid n}} \{1 + (1 - p^{-1})^{-1}(1 - p^{-kr})^{-1}(p^{-kr} - p^{-kq})\}$$

$$+ O\left(\frac{n \log\log x}{x \log n}\right) + O\left(\frac{nx}{e^{A\sqrt{\log n}}}\right) + O(x) + O\left(\frac{n}{x}\right) .$$

The theorem follows immediately by putting $x = e^{\frac{A}{2}\sqrt{\log n}}$.

(5.9) Remark. Letting $q = 1$, $r \to \infty$, theorem (5.8) yields an improved version Mirsky's result (5.1.1) for $T(k;n)$.

(5.10) Further extensions. The results of this chapter naturally suggest the analogous problem of the number of representations of a square integer and a (k;r)-integer; and more generally, the number of representations of a natural number n in the form

$$n = a_1^2 + \ldots + a_s^2 + n_1 + \ldots + n_t ,$$

where $a_1,\ldots,a_s,n_1,\ldots,n_t$ are natural numbers and $n_i \in Q_{k,r}$ $(i = 1,\ldots,t)$. Instead of squares of a_i's, one can consider also higher powers. We hope to examine these problems in detail on a future occasion.

Remarks on Notation and some definitions

(1) $b|a$ means that a is divisible by b . We use $b\nmid a$ to express the contrary of $b|a$.

(2) The symbol $[x]$ denotes the largest integer which does not exceed x.

(3) (a,b) denotes the greatest common divisor of a,b .

(4) A function $f(n)$ is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$, whenever $(m,n) = 1$.

(5)
$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 , \\ (-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes,} \\ 0 & \text{if } n \text{ contains a square divisor } > 1 . \end{cases}$$

$\mu(n)$ is called the Möbius function.

(6) The Riemann zeta function $\zeta(s)$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ , } (\mathrm{Re}(s) > 1) .$$

(7) Euler's function $\phi(n)$ denotes the number of positive integers not greater than and relatively prime to n ; i.e the number of positive integers m such that $0 < m \le n$ and $(m,n) = 1$.

(8) Let n be an integral variable and x a continuous variable, which tends to infinity; $g(n)$(or $g(x)$) a positive function of n(or x) , and $f(n)$(or $f(x)$) any other function of n(or x) . Then

(i) $f = O(g)$ means that $|f| < Ag$, where A is independent of n or x ,

(ii) $f = o(g)$ means that $\frac{f}{g} \to 0$,

(iii) $f \sim g$ means that $\frac{f}{g} \to 1$.

(9) $\phi_k(n)$ (Jordan's generalization of Euler's ϕ-function) denotes the number of different sets of k (equal or distinct) positive integers

$\leq n$, whose g.c.d. is relatively prime to n . $\phi_k(n) = n^k(1 - \frac{1}{p_1^k}) \ldots (1 - \frac{1}{p_q^k})$, if $p_1, \ldots, p_q$ are the distinct prime factors of n .

(10) The function $\Lambda(n)$ is defined by

$$\Lambda(n) = \begin{cases} \log p \ , & n = p^m \ , \\ 0 \ , & n \neq p^m \ . \end{cases}$$

(11) $d||n$ means that d is a unitary divisor of n , i.e. $d|n$ and $(d, \frac{n}{d}) = 1$.

(12) The function $\gamma(n)$, called the 'core' of n , is defined to be the product of the distinct prime divisors of n . We define $\gamma(1) = 1$.

(13) $H = (a,h)_*$ is the largest unitary divisor of h which divides of a . It is called the unitary g.c.d. of a with h .

(14) $H' = \omega_s(H) = \frac{\gamma^s(H)}{H}$ if $H \in Q_s$, where Q_s denotes the set of all s-free integers.

(15) The function $\phi^*(n)$ is defined to be the number of integers a (mod n) such that $(a,n)_* = 1$. $\phi^*(n)$ is the unitary analogue of $\phi(n)$ and is given by

$$\phi^*(n) = \prod_{p^e || n} (p^e - 1) \ .$$

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