University of Alberta

ON PRICING CONTINGENT CLAIMS AND EXPECTED UTILITY MAXIMIZATION IN TWO INTEREST RATES FINANCIAL MARKETS VIA COMPLETION

by

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Your file Votre référence
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To my family

ABSTRACT

In this thesis, we consider the problems of hedging contingent claims and utility maximization in the framework of two interest rates financial markets. The upper and lower hedging prices are derived for European options by means of auxiliary completions of the initial market.

The well known financial markets consider a unique interest rates for both credit and deposit purposes, see for instance the Black –Scholes, Merton and Cox–Ingersoll–Ross models. The latter assumption allows to derive a unique price called *the fair price* on which the buyer and seller of the claim agree. Instead of a unique price, the hedging problem in the two interest rates financial markets admits an interval of initial prices on which buyer and seller can agree.

Using a similar technique of market completions, we consider the problem of an investor searching to maximize the expected utility of his terminal wealth, in the two interest rates financial markets. We show that under suitable conditions the latter problem can be reduced to a standard investment problem. This methodology is then adapted on the problem of shortfall risk minimization.

We finally consider the problem of pricing equity linked–life insurance contracts with guarantee in the two interest rates financial markets considered. To address the problem, we adapt a technique used by Melnikov [37] to price equity linked-life insurance contract, in one interest rate jump–diffusion financial market.

On one hand, the motivation of these problems lies on the realities of financial

markets where, the credit rate is always higher than the deposit rate. Taking into account such a constraint brings new difficulties in the problems of pricing contingent claims and utility maximization (investment).

On the other hand, the previous works in this topic were mostly made in a Black–Scholes and Cox–Ross–Rubinstein (binomial) frameworks. Yet, it is common knowledge in the area of finance that stock prices involve jumps, in order to take into account new extreme events. Therefore, besides a Black–Scholes model, we have considered a pure Merton (or pure jump) model and a two factors jump–diffusion model.

The thesis is organized as follows. In chapter 2, we consider a Black–Scholes and Merton model. We solve the hedging and utility maximization problems on a two interest rates financial market. Further, we give the solution of the shortfall risk minimization problem. In chapter 3, the same problems are considered but in a more general setting of two factors jump–diffusion model. In chapter 4, we consider the problem of pricing pure endowment life insurance contract with guarantee on the different interest rates financial markets. We give an approximation of the interval of survival probabilities and the corresponding policy–holder interval of ages. Finally, chapter 5 allows us to compare our hedging results on a jump–diffusion basis to those obtained by Bergman on a Black–Scholes model.

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Chapter 1

Introduction

In what follows we give the relevant literature on two interest rates financial markets.

Early works on two interest rates financial markets include Barron and Jensen [5], the latter authors use a utility based technique to solve the problem of pricing. Later, Karatzas and Cvitanic [19],[18] see also [30] consider a more general setting of constrained market, solve the problem of pricing contingent claims and utility maximization, using a technique of duality theory on a Black–Scholes (with deterministic coefficient) setting. In particular, they consider the case where the drift of the wealth process is concave as function of the portfolio, which includes the case of a two interest rates model. Then, they adapt the convex duality method to derive the upper hedging price of the claim by solving a Hamilton–Jacobi–Bellman equation (they called such a price the seller price in opposite to Korn [31], who called it the buyer price). They also solve the utility maximization from terminal wealth problem in the two interest financial market using the same convex duality method. Korn [31] adapts Karatzas and Cvitanic methodology to derive both upper (buyer 's price) and lower (seller's price) hedging prices of European call and put options, in a Black–Scholes setting. He then extends the obtained results to a special set of contingent claims. Bergman [7], considers the same pricing problem and derives the buyer and seller prices by using duplicating strategies and partial differential equation with switching interest rates. He argues that in the case of a call for instance, the long position involves only borrowing therefore the Black–Scholes pricing with borrowing rate is applicable.

Bart [6] considers the problem in a Cox–Ross–Rubinstein model and gives a description of the complete structure of optimal hedging. His methodology is adapted from Volkov and Kramkov [48]. An elegant exposition of markets with structural constraints (transaction costs, short selling restrictions, different credit and deposit rates) can be found in Melnikov et al. [39]. The authors provide as in Bart the structure of an optimal hedge but besides the Cox–Ingersoll–Ross model, the Black–Scholes, Merton and jump–diffusion models are considered.

On the expected shortfall risk minimization side, Föllmer and Leukert [26], [27] consider a one interest rate financial market, find the optimal hedge of an investor whose initial capital is less than the fair price (Black–Scholes) of the option . The criteria to choose the optimal strategy being maximizing the probability of successful hedge for the quantile hedge and, minimizing the expected value of the shortfall (that amount the portfolio is short of covering the claim: $(f_T - X_T)^+$), weighted by a loss function. Nakano [45] follows a methodology suggested in Föllmer and Leukert [27] (convex duality method) to derive the optimal strategy of the problem in a two factor jump–diffusion model. He treats the problem as a utility maximization one, and adapts Karatzas–

Cvitanic methodology. In parallel, Melnikov–Krutchenko [33] solve the same problem via efficient hedging which is based on the Neyman–Pearson lemma. In our application on expected shortfall risk minimization on a two interest rates financial market, we have used the Nakano results. We provide conditions under which the two interest rates problem of shortfall risk minimization can be reduced to the similar problem on an unconstrained market.

We now turn to equity linked–life insurance contracts. Since Brennan–Schwartz [13], equity–linked life insurance contracts have been widely studied in finance and insurance see for instance Brennan–Schwartz [14], Moller [43] [44], Boyle–Schwartz [11]. Common to those articles is: they were focused on how to price those instruments on complete or incomplete markets. More recently, Melnikov [36],[37], [38] used imperfect hedge (quantile and efficient) to price pure endowment life insurance contract with guarantee. We use the latter results to derive the survival probabilities and age interval of a policy–holder of the contract, on two interest rates financial markets.

In this thesis the problems will be considered in a Black–Scholes, pure Merton, and a Jump-diffusion setting. Jump–diffusion are a special case of Lévy processes. More details in this topic are provided for instance in Corcuera, Nualart, Schoutens [21] and Bertoin [8].
The first paper on jump–diffusion in option pricing was introduced by Merton [34],[42] on a (B, S)–market. In his setting, two types of uncertainties arise, one from the diffusion the other from the jumps, which made the pricing by using no–arbitrage argument alone impossible. In order to price the European option, Merton assumed the jumps diversifiable. However, the latter assumption induces the existence of a market portfolio containing the asset without

showing the jumps characteristics (see for instance Björk and Naslund [10]). The advantage of the jump–diffusion model used here is based on the completion of the market. The latter characteristic allows to price options by the sole use of no–arbitrage argument (see for instance Runggaldier [46]) for a survey. Other papers related to jump–diffusion and their estimations include Honore [28], Kou [32], Runggaldier [46], Mercurio–Runggaldier [41] and Ball and Torous [3] who also show the impact of jumps on option pricing. In contrast to Ball and Torous who find that the difference between the Merton (original model) and Black–Scholes model is highest for out-of-the-money calls and lowest for in-the-money calls, in the model used here the largest difference occurs at-the-money.

Chapter 2

The pure Merton and Black–Scholes models

2.1 Introduction

In this chapter we begin by recalling some standard results on pricing derivatives, investment and also on shortfall risk minimization problems. We also provide the corresponding results in two interest rates financial markets (Black–Scholes and pure Merton).

2.2 Preliminaries

Let $(\Omega, \mathcal{F}, F = (\mathcal{F}_t), P)$ be a standard stochastic basis on which we consider a financial market with 2 assets: a bank account B_t and a stock S_t evolving according to the following stochastic differential equations

$$\begin{aligned} dB_t &= rB_t dt; & (2.2.1) \\ dS_t &= S_{t^-}\Big(\mu dt - \nu d\Pi_t\Big); & (2.2.2) \\ or & & \\ dS_t &= S_t\Big(\mu dt + \sigma dW_t\Big), & (2.2.3) \end{aligned}$$

We assume $\mu \in R$, $\sigma > 0$, $\nu < 1$, $S_{t^-} = \lim_{u \uparrow t} S_u$, and $\Pi = \{\Pi_t, \mathcal{F}_t\}_{t \geq 0}$ is a P–Poisson process with intensity $\lambda > 0$, W_t is a Brownian motion. The time horizon T is finite.

Denote the above financial markets by (B, S). The model will be said a pure jump (or Merton) model if equations (2.2.1) and (2.2.2) are verified, otherwise if equations (2.2.1) and (2.2.3) hold a Black–Scholes model.

Definition 2.1. A *portfolio* π is an $\mathcal{F}_t$–predictable process.

In the (B, S)–market, at any time t, the position of an investor is given by a portfolio $\pi = (\beta_t, \gamma_t)$ where β and γ are respectively the number of units invested in the bank account and stock. The balance equation of the portfolio process π at any time t is given by

$$V_t^\pi = \beta_t B_t + \gamma_t S_t \qquad a.s.$$

Definition 2.2. A portfolio π is said *self–financing* and denoted (SF) if

$$dV_t^\pi = \beta_t dB_t + \gamma_t dS_t$$

alternatively

$$\widetilde{V_t^\pi} = x + \int_0^t \gamma_u d\widetilde{S_u} \qquad a.s. \qquad with \qquad \widetilde{V_t} = \frac{V_t}{B_t},\ \widetilde{S_u} = \frac{S_u}{B_u}.$$

Such a portfolio will be said *admissible* if

$$V_t^\pi > 0 \;\; a.s, \qquad \forall\, t > 0\,. \tag{2.2.4}$$

In the sequel we will denote the set of admissible strategies with initial capital x by $\mathcal{A}(x)$.

The (SF) condition means that there will be no inflow or outflow of capital during the hedging period. One might think of it in terms of variations. The variations of V_t come solely from the variations of the underlying assets and not the number of assets involved.

Definition 2.3. A process X_t will be said a *wealth process* if generated by a self–financing and admissible strategy ($X_t \geq 0, \quad \forall \quad t \geq 0$) and, a debt process Y_t if $-Y_t$ is generated by a self financing and admissible strategy ($Y_t \leq 0, \quad \forall \quad t \geq 0$).

As in Korn ([31]), we call buyer (resp. seller), the agent purchasing the claim at time $t = 0$ (resp. selling f_T). Consequently, the buyer of a claim f_T can be seen as an agent whose initial investment x grows to X_t at time $t \leq T$ ($X_T = f_T$), while the seller represents an agent whose initial debt y grows to Y_T, hence will have to reimburse $-Y_T = f_T$ at time T. The process Y gives the position of the seller.

Definition 2.4. A *contingent claim* f_T is a non–negative $\mathcal{F}_T$–measurable random variable.

Definition 2.5. The portfolio π is called a hedge for the buyer of the claim f_T if the wealth process verifies

$$X_T^\pi(x) \geq f_T \qquad a.s.,$$

such a hedge will be said *minimal* and denoted π^* if further for any other hedge π,

$$X_t^{\pi} \geq X_t^{\pi^*} \quad a.s. \quad \forall \quad 0 \leq t \leq T$$

In parallel for the seller of the same claim, we say that a portfolio π with initial debt y is a hedge if

$$Y_T^{\pi} \leq -f_T \qquad a.s.$$

Such a hedge will be said minimal and denoted by π^* if further for any other hedge π,

$$Y_t^{\pi} \leq Y_t^{\pi^*} \quad a.s. \quad \forall \quad 0 \leq t \leq T$$

2.3 The complete (B, S) market results

We first give a fundamental result in the (B, S)–market

Statement 2.6. The (B, S)–market is complete.

1) In the pure Merton model, the density Z_t of the martingale measure P^* is given by the following equality

$$Z_t = \frac{dP^*}{dP}\Big|\mathcal{F}_t = \mathcal{E}_t(N_.) = \exp\Big\{(\lambda - \lambda^*)t + (\ln\lambda^* - \ln\lambda)\Pi_t\Big\}, \quad (2.3.1)$$

where $N_t = \psi(\Pi_t - \lambda_t)$, $\psi = \frac{\lambda^*}{\lambda} - 1$ with $\lambda^* = \frac{\mu - r}{\nu}$, $\nu \neq 0$.

2) In the Black–Scholes model, the density Z_t is provided by the formula

$$Z_t = \mathcal{E}_t(N_.) = \exp\Big\{\phi W_t - \frac{1}{2}\phi^2 t\Big\}, \quad (2.3.2)$$

with $N_t = \phi W_t$ and $\phi = -\frac{\mu - r}{\sigma}$.

3) Under the new measure P^*, the given Poisson process Π has intensity λ^* and, $W_t^* = W_t - \phi t$ is a Wiener process.

The proof of this statement can be found in Melnikov et al. [39].
From the theory of complete market (*One law price*), at any time t, the position of a buyer X_t of a claim f_T is given by the opposite position of a seller $-Y_t$ of the same claim. The dynamics of such positions are provided below

$$\frac{dX_t}{X_{t-}} = \frac{dY_t}{Y_{t-}} = (1-\alpha_t)r\ dt + \alpha_t \frac{dS_t}{S_{t-}}, \tag{2.3.3}$$

where $\alpha_t = \frac{\gamma_t S_{t-}}{X_{t-}}$ (resp. $\alpha_t = \frac{\gamma_t S_{t-}}{Y_{t-}}$) denotes the proportion of capital invested on stocks. The derivation of the formula can be found on the Appendix. We next give some standard hedging results.

2.3.1 Hedging

Consider the buyer case. The notion of hedge defined above is perfect since it occurs without any risk in the probabilistic sense, namely $P(X_T^\pi(x) < f_T) = 0$. Now, if there is a risk involved ($P(X_T^\pi(x) < f_T) \neq 0$) then, hedging in this context will be said imperfect. One example of such an imperfect hedge is quantile hedging. It consists in finding the minimal initial capital required, given the probability of hedge. Alternatively, one can find the maximal probability of a hedge, given the initial capital of the hedge i.e.

$$Max_{\{x \leq x_0 < C_r\}} P\Big(X_T^\pi(x) - f_T > 0 \Big),$$

where C_r is the fair price of the claim f_T. Solving this problem requires the use of the Neyman–Pearson Lemma (see Melnikov [39], Föllmer and Leukert

[26]).

One more example of imperfect hedge we will need later is given by the efficient hedge methodology. In this case, one minimizes the expected shortfall weighted by a loss function, given a constraint on the initial capital of the hedge i.e.

$$Min_{\{x<C_r\}} E\left[l_p(f_T - X_T^{\pi}(x))^+\right], \qquad p > 1.$$

Quantile hedge can be retrieved from efficient hedge by taking the indicator function as the loss function.

Perfect hedge results

We give the fair price of the Merton and Black–Scholes models in their corresponding settings.

Lemma 2.1. *At any time t, the fair price of a European call option $f_T = (S_T^1 - K)^+$ in the (B, S)–market is given by*

$$\begin{aligned} C_r(t) &= E^*\left[e^{-r(T-t)}(S_T - K)^+ \,|\mathcal{F}_t\right] \\ &= \begin{cases} S_t\Psi(n_0, \lambda^*(1-\nu)(T-t)) - Ke^{(-r(T-t))}\Psi(n_0, \lambda^*(T-t)), & \textit{if } \nu < 0; \\ \\ S_t\Phi(n_0, \lambda^*(1-\nu)(T-t)) - Ke^{(-r(T-t))}\Phi(n_0, \lambda^*(T-t)), & \textit{if } 0 < \nu < 1; \end{cases} \qquad (2.3.4) \\ &= V_0 \qquad (2.3.5) \end{aligned}$$

Where $V_0 = \inf\{x \geq 0; \exists\ \theta \in \mathcal{A}(x);\ V_T^{\theta,x} \geq F_T\ a.s.\}$ *represents the initial capital of the minimal hedge,* $\Psi(k,y) = \sum_{n=k}^{\infty} \frac{(y^n)}{n!} e^{(-y)}$*, and* $\Phi(k,y) = \sum_{n=0}^{k} \frac{(y^n)}{n!} e^{(-y)}$*,*

$$n_0 = \begin{cases} \left[\left[\frac{\ln[K/S_t] - \mu(T-t)}{ln[1-\nu]}\right]\right] + 1 & \textit{if } \nu < 0 \\ \left[\left[\frac{\ln[K/S_t] - \mu(T-t)}{ln[1-\nu]}\right]\right] & \textit{if } 0 < \nu < 1. \end{cases} \qquad (2.3.6)$$

with $[[a]]$ *s.t.* $[[a]] \leq a < [[a]] + 1$.

For the Black–Scholes model the following holds

$$\begin{aligned} C_r(t) &= E^*\left[e^{-r(T-t)}(S_T - K)^+ \,|\mathcal{F}_t\right], \\ &= S_t\Phi(d_1(S_t, K, \sigma, r, T, t)) - Ke^{-r(T-t)}\Phi(d_2(S_t, K, \sigma, r, T, t)), \quad (2.3.7) \\ &= V_0, \quad (2.3.8) \end{aligned}$$

with

$$\begin{aligned} d_1(S_t, K, \sigma, r, T, t) &= \frac{\ln(\frac{S_t}{K}) + (T-t)(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}, \\ d_2(S_t, K, \sigma, r, T, t) &= d_1(S_t, K, \sigma, r, T, t) - \sigma\sqrt{T-t}, \quad (2.3.9) \\ \Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt. \end{aligned}$$

The corresponding prices in the different models for European puts are derived from the Put–Call parity relation:

$$P_r(t) = C_r(t) - S_t + Ke^{-r(T-t)}.$$

We only give the proof of equality (2.3.4) in the pure Merton model, the proof of equality (2.3.5) is similar to the one developed in Karatzas–Shreve [30] for a Black–Scholes setting. We omit the proof of the Black–Scholes case since it can be found in Melnikov et al [39], Elliott–Kopp [24], and Karatzas–Shreve [30] among others.

Proof. From the stochastic exponent relation (6.2.2) in the Appendix

$$\begin{aligned} S_T &= S_t e^{\{\mu(T-t)+(\Pi_T-\Pi_t)\ln(1-\nu)\}} \quad (2.3.10) \\ &= S_t e^{\{r(T-t)+\nu\lambda^*(T-t)+(\Pi_T-\Pi_t)\ln(1-\nu)\}} \quad (2.3.11) \end{aligned}$$

For the computation of $E^*[e^{-r(T-t)}(S_T - K)^+ \,|\mathcal{F}_t]$, it is more convenient to use the second formulation of S_T

$$E^*[e^{-r(T-t)}(S_T - K)^+ \,|\mathcal{F}_t] = E^*[(S_t e^{\{\nu\lambda^*(T-t)+(\Pi_T-\Pi_t)\ln(1-\nu)\}} - Ke^{-r(T-t)})^+ \,|\mathcal{F}_t]$$

Keeping in mind that Π_t is a P^* Poisson process with intensity λ^* and hence $(\Pi_t - \lambda^* t)$ is a P^*–martingale, we derive the following

$$\begin{aligned} E^*[e^{-r(T-t)}(S_T-K)^+|\mathcal{F}_t] &= S_t e^{\{\nu\lambda^*(T-t)\}} E^*\left[e^{((\Pi_T-\Pi_t)\ln(1-\nu))} I_{\{S_T>K\}}|\mathcal{F}_t\right] \\ &\quad - K e^{-r(T-t)} E^*\left[I_{\{S_T>K\}}|\mathcal{F}_t\right]. \end{aligned}$$

We rewrite the set $\{S_T > K\}$ as

$$\{S_T > K\} = \left\{(\Pi_T - \Pi_t)\ln(1-\nu) > \ln\left(\frac{Ke^{-r(T-t)}}{S_t}\right) - \nu\lambda^*(T-t)\right\}$$

Depending on the sign of $\ln(1-\nu)$ we derive

$$\{S_T > K\} = \begin{cases} \{(\Pi_T - \Pi_t) > \frac{\ln(\frac{Ke^{-r(T-t)}}{S_t}) - \nu\lambda^*(T-t)}{\ln(1-\nu)}\}, & \text{if } \ln(1-\nu) > 0 \ (\nu < 0) \\ \{(\Pi_T - \Pi_t) < \frac{\ln(\frac{Ke^{-r(T-t)}}{S_t}) - \nu\lambda^*(T-t)}{\ln(1-\nu)}\}, & \text{if } \ln(1-\nu) < 0 \ (0 < \nu < 1). \end{cases}$$

Assume $\nu < 0$, then we obtain

$$\begin{aligned} C_r(t) &= S_t e^{\{\nu\lambda^*(T-t)\}} E^*\left[e^{((\Pi_T-\Pi_t)\ln(1-\nu))} I_{\left\{(\Pi_T-\Pi_t) > \frac{\ln(\frac{K}{S_t}) - \mu(T-t)}{\ln(1-\nu)}\right\}} |\mathcal{F}_t\right] \\ &\quad - K e^{-r(T-t)} E^*\left[I_{\left\{(\Pi_T-\Pi_t) > \frac{\ln(\frac{K}{S_t}) - \mu(T-t)}{\ln(1-\nu)}\right\}} |\mathcal{F}_t\right]. \end{aligned}$$

From the Poisson process properties $\Pi_T - \Pi_t$ is independent of $\mathcal{F}_t$, hence

$$\begin{aligned} C_r(t) &= S_t e^{\{\nu\lambda^*(T-t)\}} E^*\left[e^{((\Pi_T-\Pi_t)\ln(1-\nu))} I_{\left\{(\Pi_T-\Pi_t) > \frac{\ln(\frac{K}{S_t}) - \mu(T-t)}{\ln(1-\nu)}\right\}}\right] \\ &\quad - K e^{-r(T-t)} E^*\left[I_{\left\{(\Pi_T-\Pi_t) > \frac{\ln(\frac{K}{S_t}) - \mu(T-t)}{\ln(1-\nu)}\right\}}\right] \end{aligned}$$

$$\begin{aligned} C_r(t) &= S_t \sum_{j=n_0}^{\infty} \frac{(\lambda^*(1-\nu)(T-t))^j}{j!} e^{-\lambda^*(1-\nu)(T-t)} \\ &\quad - K e^{-r(T-t)} \sum_{j=n_0}^{\infty} \frac{(\lambda^*(T-t))^j}{j!} e^{-\lambda^*(T-t)}. \end{aligned}$$

Where

$$n_0 = \inf\Big\{ j, \quad S_t \frac{(\lambda^*(1-\nu)(T-t))^j}{j!} e^{-\lambda^*(1-\nu)(T-t)} - K e^{-r(T-t)} \frac{(\lambda^*(T-t))^j}{j!} e^{-\lambda^*(T-t)} > 0 \Big\} \tag{2.3.13}$$

and, from above if $\ln(1-\nu) > 0$ (*i.e.* $\nu < 0$), we have

$$n_0 = \inf\Big\{ n \ s.t. \quad n > \frac{\ln(\frac{Ke^{-r(T-t)}}{S_t}) - \nu\lambda^*(T-t)}{\ln(1-\nu)} \Big\}.$$

We finally arrive to

$$C_r(t) = S_t \Psi(n_0, \lambda^*(1-\nu)(T-t)) - Ke^{(-r(T-t))}\Psi(n_0, \lambda^*(T-t)).$$

Applying a same methodology for $0 < \nu < 1$ yields to

$$C_r(t) = S_t \Phi(n_0, \lambda^*(1-\nu)(T-t)) - Ke^{(-r(T-t))}\Phi(n_0, \lambda^*(T-t)).$$

□

We next establish a special property of the fair prices given by lemma 2.1 in what follows.

Theorem 2.7. *The price $C_r(t)$, of a European Call option is a non–decreasing function of r and the price of a European Put option $P_r(t)$ is a non–increasing function of r.*

Proof. We derive the pure Merton model results. For convenience purposes we will consider separately the cases $\nu \in (-\infty, 0)$ and $\nu \in (0,1)$. We first turn to the case $\nu < 0$.

The case of a European Call option with $\nu < 0$.

Let us note that in our context, $\lambda^* = \frac{\mu - r}{\nu}$, hence $\frac{\partial \lambda^*}{\partial r} = \frac{-1}{\nu}$ and,

$$\frac{\partial C_r(0)}{\partial r} = S_0 \frac{\partial \Psi(n_0, \lambda^* T(1-\nu))}{\partial r} + KTe^{-rT}\Psi(n_0, \lambda^* T) - Ke^{-rT}\frac{\partial \Psi(n_0, \lambda^* T)}{\partial r}$$

The computation of $\frac{\partial \Psi(n_0, \lambda^* T)}{\partial r}$ yields

$$\frac{\partial \Psi(n_0, \lambda^* T)}{\partial r} = -\frac{T}{\nu} e^{[-\lambda^* T]} \frac{[\lambda^* T]^{n_0 - 1}}{(n_0 - 1)!}$$

Identically,

$$\frac{\partial \Psi(n_0, \lambda^* T(1-\nu))}{\partial r} = -\frac{T(1-\nu)}{\nu} e^{[-\lambda^* T(1-\nu)]} \frac{[\lambda^* T(1-\nu)]^{n_0 - 1}}{(n_0 - 1)!}$$

The following relation hold

$$\frac{\partial \Psi(n_0, \lambda^* T(1-\nu))}{\partial r} = (1-\nu)^{n_0} e^{[\nu \lambda * T]} \frac{\partial \Psi(n_0, \lambda^* T)}{\partial r}$$

Therefore the previous computations allow us to rewrite $\frac{\partial C_r(0)}{\partial r}$

$$\frac{\partial C_r(0)}{\partial r} = \frac{\partial \Psi(n_0, \lambda^* T)}{\partial r}\left(S_0(1-\nu)^{n_0} e^{\nu \ \lambda * T} - Ke^{-rT}\right) + KTe^{-rT}\Psi(n_0, \lambda^* T)$$

By definition of n_0, $S_0(1-\nu)^{n_0} e^{\nu \ \lambda * T} - Ke^{-rT}$ is positive or null, and for ν negative, the expression $\frac{\partial \Psi(n_0, \lambda^* T)}{\partial r}$ is positive consequently

$$\frac{\partial C_r(0)}{\partial r} \geq 0$$

The case of a European Put option with $\nu < 0$.

The parity Put–call relationship $P_t = C_t - S_t + Ke^{-r(T-t)}$ taken at time $t = 0$

yields to

$$
\begin{aligned}
\frac{\partial P_r(0)}{\partial r} &= \frac{\partial \Psi(n_0,\lambda^* T)}{\partial r}\Big(S_0(1-\nu)^{n_0} e^{\nu\ \lambda * T} - Ke^{-rT}\Big) \\
&\quad + KTe^{-rT}\Psi(n_0,\lambda^* T) - TKe^{-rT} \\
&= \frac{\partial \Psi(n_0,\lambda^* T)}{\partial r}\Big(\big[S_0(1-\nu)^{n_0} - S_0(1-\nu)^{n_0-1} + S_0(1-\nu)^{n_0-1}\big] \\
&\qquad e^{\nu\ \lambda * T} - Ke^{-rT}\Big) - TKe^{-rT}\sum_0^{n_0-1} e^{(-\lambda^* T)}\frac{(\lambda^* T)^k}{k!} \\
&= S_0(1-\nu)^{n_0-1} e^{[\nu\lambda * T]}\frac{\partial \Psi(n_0,\lambda^* T)}{\partial r}\Big(1-\nu-1\Big) + \frac{\partial \Psi(n_0,\lambda^* T)}{\partial r} \\
&\quad \Big(e^{[\nu\lambda * T]} S_0(1-\nu)^{n_0-1} - Ke^{-rT}\Big) - TKe^{-rT}\sum_0^{n_0-1} e^{(-\lambda^* T)}\frac{(\lambda^* T)^k}{k!} \\
&= -\nu S_0(1-\nu)^{n_0-1} e^{[\nu\lambda * T]}\frac{\partial \Psi(n_0,\lambda^* T)}{\partial r} + \\
&\quad \frac{\partial \Psi(n_0,\lambda^* T)}{\partial r}\Big(e^{[\nu\lambda * T]} S_0(1-\nu)^{n_0-1} - Ke^{-rT}\Big) \\
&\quad - TKe^{-rT} e^{(-\lambda^* T)}\frac{(\lambda^* T)^{n_0-1}}{n_0-1!} - TKe^{-rT}\sum_0^{n_0-2} e^{(-\lambda^* T)}\frac{(\lambda^* T)^k}{k!}.
\end{aligned}
$$

Note $-\nu\frac{\partial \Psi(n_0,\lambda^* T)}{\partial r} = Te^{(-\lambda^* T)}\frac{(\lambda^* T)^{n_0-1}}{n_0-1!}$ then, rearranging the terms yields to

$$
\begin{aligned}
\frac{\partial P_r(0)}{\partial r} &= [S_0(1-\nu)^{n_0-1} e^{[\nu\lambda * T]} - Ke^{-rT}]\frac{\partial \Psi(n_0,\lambda^* T)}{\partial r}(1-\nu) \\
&- TKe^{-rT}\sum_0^{n_0-2} e^{(-\lambda^* T)}\frac{(\lambda^* T)^k}{k!}.
\end{aligned}
$$

As $\nu < 0$, by definition of n_0, the first term of the right hand side of the equality is negative, the second term also. Hence the following holds

$$
\frac{\partial P_r(0)}{\partial r} \leq 0.
$$

We now consider the case where $0 < \nu < 1$.

The case of European Call option with $0 < \nu < 1$.

$$\begin{aligned} C_r(0) &= S_0\Phi(n_0, \lambda^*T(1-\nu)) - Ke^{(-rT)}\Phi(n_0, \lambda^*T), \text{ with} \\ \Phi(k, y) &= \sum_{n=0}^{k} \frac{(y^n)}{n!} e^{(-y)} \end{aligned}$$

$$\begin{aligned} \frac{\partial C_r(0)}{\partial r} &= S_0 \frac{\partial[\Phi(n_0, \lambda^*T(1-\nu))]}{\partial r} + KTe^{-rT}\Phi(n_0, \lambda^*T) \\ &- Ke^{-rT}\frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} \\ \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} &= \frac{T}{\nu} e^{-\lambda^*T} \frac{(\lambda^*T)^{n_0}}{n_0!} \\ \frac{\partial \Phi(n_0, \lambda^*T(1-\nu))}{\partial r} &= (1-\nu)^{n_0+1} e^{\nu\lambda^*T} \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} \end{aligned}$$

$$\begin{aligned} \frac{\partial C_r(0)}{\partial r} &= S_0(1-\nu)^{n_0+1} e^{-\lambda^*T\nu} \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} - Ke^{(-rT)} \frac{\partial \Phi(n_0, \lambda^*T(1-\nu))}{\partial r} \\ &-S_0(1-\nu)^{n_0} e^{\nu\lambda^*T} \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} + S_0(1-\nu)^{n_0} e^{\nu\lambda^*T} \frac{\Phi(n_0, \lambda^*T)}{\partial r} \\ &+KTe^{-rT}\Phi(n_0, \lambda^*T) \\ &= -\nu \frac{\Phi(n_0, \lambda^*T)}{\partial r} S_0(1-\nu)^{n_0} + \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} (S_0(1-\nu)^{n_0} e^{\nu\lambda^*T} \\ &-Ke^{-rT}) + KTe^{-rT}\Phi(n_0, \lambda^*T) \end{aligned}$$

Now writing $\Phi(n_0, \lambda^*T)$ as function of $\Phi(n_0-1, \lambda^*T)$ and simplifying we obtain

$$\begin{aligned} \frac{\partial C_r(0)}{\partial r} &= \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} \left[S_0(1-\nu)^{n_0} e^{\nu\lambda^*T} - Ke^{-rT}\right] [1-\nu] \\ &+ KTe^{[-rT]}\Phi(n_0-1, \lambda^*T). \end{aligned}$$

From there, it's clear that

$$\frac{\partial C_r(0)}{\partial r} \geq 0$$

The case of a European Put option with $0 < \nu < 1$.

Let us first rewrite $\frac{\partial P_r(0)}{\partial r}$.

$$\begin{aligned} \frac{\partial P_r(0)}{\partial r} &= \left[S_0(1-\nu)^{n_0+1} e^{-\lambda^*T\nu} - Ke^{-rT}\right] \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r} \\ &+KTe^{-rT}[\Phi(n_0, \lambda^*T) - 1]. \end{aligned}$$

We shall note that by definition of n_0, $\frac{\partial \Phi(n_0,\lambda^*T)}{\partial r}\left[S_0(1-\nu)^{n_0+1}e^{-\lambda^*T\nu} - Ke^{-rT}\right]$ is negative, and $KTe^{-rT}\left[\Phi(n_0,\lambda^*T)-1\right]$ is also negative. Hence we obtain

$$\frac{\partial P_r(0)}{\partial r} \leq 0$$

Let us now turn to the Black–Scholes case. It is well known that

$$\begin{aligned}
\frac{\partial C_r(0)}{\partial r} &= KTe^{-rT}\Phi(d_2(S_0,K,\sigma,r,T,0)) \\
\frac{\partial P_r(0)}{\partial r} &= KTe^{-rT}\left[\Phi(d_2(S_0,K,\sigma,r,T,0))-1\right]
\end{aligned}$$

Hence the theorem is also proved for the Black–Scholes model □

We now will provide the results of the investment problem in the auxiliary market.

2.3.2 Investing

Consider an investor with a given initial capital x. The investment problem consists in maximizing the expected utility of the investor terminal wealth, namely we will find: the price function

$$u(x) = \sup_{\pi\in SF} E[U(X_T^{\pi}(x))]\,, \tag{2.3.14}$$

the optimal terminal wealth $X_T^{\pi^*}$ and the optimal strategy α^*. We give the solution of the problem in a complete market setting, then we consider its application on the Merton and Black–Scholes models (see Melnikov [39], Karatzas–Shreve [30], Cvitanic [16],[17], among others).

Consider a utility function $U: R_+ \longrightarrow R$, concave, non–decreasing, continuously differentiable, and

$$\left|\left|\left| \begin{array}{l} \lim_{x\to\infty} U'(x) = 0\,, \\ \lim_{x\to 0} U'(x) = \infty\,. \end{array} \right.\right.\right. \tag{2.3.15}$$

Let $u(x)$ be the price function in the (B, S^1, S^2)-market. The investment problem (see, for instance Karatzas and Shreve [30], Melnikov et al [39]) consists in finding

$$u(x) \;=\; \sup_{\pi\in SF} E\left(U(X_T^\pi(x))\right) = E\left(U(X_T^{\pi^*}(x))\right)\,, \tag{2.3.16}$$

The above problem can be transformed as

$$u(x) \;=\; \sup_{y\in\chi} E\left(U(Y_T(x))\right) = E\left(U(Y_T^*(x))\right)$$

where $\chi = \{Y positive:\ Y_t(x) = x + \int_0^t \gamma_u d\tilde{X}_s\}$ with γ a predictable process and $\tilde{X}$ is a P^* local martingale. To solve the problem, we need the next theorem:

Theorem 2.8. *Let $V(y)$ the conjugate function of $U(x)$, whose relations to the latter is given by what follows:*

$$V(y) = \sup_{x>0} U(x) - xy \qquad y > 0\,, \tag{2.3.17}$$

$$U(x) = \inf_{y>0} V(y) + xy \qquad x > 0\,. \tag{2.3.18}$$

We denote $I(x) = ((U')^{-1}(x)) = -V'(x)$, $y_0 = \inf\{y:\ v(y) < \infty\}$ and $x_0 = \lim_{y\to y_0}(-v'(y))$. The functions $v(y)$ and $u(x)$ verifies what follows.

1) The function $u(x) < \infty$ is continuously differentiable for all $x > 0$, stricly concave on $(0, x_0)$, and the function $v(x) < \infty$ is continuously *differentiable for all $y > 0$, strictly convex on (y_0, ∞).*

$$v(y) = \sup_{x>0} u(x) - xy \qquad y > 0\,, \tag{2.3.19}$$

$$u(x) = \inf_{y>0} v(y) + xy \qquad x > 0. \tag{2.3.20}$$

2) If $y = u'(x)$ where $x < x_0$ and $y < y_0$ then the optimal solution of (3.2.19) is

$$Y_T^*(x) = I(yZ_T). \tag{2.3.21}$$

We next provide the solution of Problem (2.3.14) in the setting of the pure Merton model. Consider the case $U(x) = \ln(x)$ in the (B, S) financial market. We substitute $U(x)$ in (2.3.17), derive $V(y) = -\ln(y) - 1$ and,

$$\begin{aligned} v(y) &= E[V(yZ_T)] = -E[\ln(yZ_T)] - 1 = -\ln(y) - 1 - E[\ln(Z_T)] \\ &= -\ln(y) - 1 - \Big(\ln(\lambda^*) - \ln(\lambda)\Big)\lambda T + (\lambda^* - \lambda)T. \end{aligned}$$

We substitute the above expression of $v(y)$ into equality (2.3.20), and find the price function

$$u(x) = \ln(x) - \Big(\ln(\lambda^*) - \ln(\lambda)\Big)\lambda T + (\lambda^* - \lambda)T. \tag{2.3.22}$$

Next, we derive the optimal proportions invested on the different assets involved. From (2.3.21), we know that

$$\begin{aligned} Y_T^*(x) &= \frac{X_T^{\pi^*}}{B_T} \\ &= I(yZ_T) = \frac{1}{yZ_T} = \frac{x}{Z_T} \\ &= x \cdot \exp\left\{-\Pi_T \ln\Big(\frac{\lambda^*}{\lambda}\Big) + (\lambda^* - \lambda)T\right\} \end{aligned} \tag{2.3.23}$$

and solving (2.3.3) for $\alpha_t := \alpha$ yields

$$\frac{X_T^\pi}{B_T} = x \cdot \exp\left\{\alpha(\mu - r) + ln(1 - \alpha\nu)\Pi_T\right\}. \tag{2.3.24}$$

We identify expression (2.3.23) and (2.3.24) and obtain the value of α

$$\alpha_t = \alpha = \frac{\mu - r - \nu\mu}{\nu(\mu - r)}, \tag{2.3.25}$$

Let us now turn to the Black–Scholes model setting, where as above the problem (2.3.14) is considered. We follow the same methodology as in the pure Merton model case. Let $U(x) = \ln(x)$, following the same procedure we derive

$$\begin{aligned} V(y) &= -\ln(y) - 1, \\ v(y) &= E[V(yZ_T)] = -\ln(y) - 1 - E[\ln(Z_T)] = -\ln(y) - 1 + \frac{\phi^2}{2}T, \end{aligned}$$

The cost function is given by

$$u(x) = \ln(x) + \frac{\phi^2}{2}T, \tag{2.3.26}$$

and the discounted optimal terminal wealth follows

$$Y_T^*(x) = x \cdot \exp\left\{-\phi W_T + \frac{\phi^2}{2}T\right\},$$

From relation (2.3.21) we obtain

$$\begin{aligned} Y_T^*(x) &= \frac{X_T^{\pi^*}}{B_T}, \\ &= I(yZ_T) = \frac{1}{yZ_T} = \frac{x}{Z_T} \\ &= x \cdot \exp\left\{-\phi W_T + \frac{\phi^2}{2}T.\right\} \end{aligned}$$

Solving equation (2.3.3) for $\alpha_t = \alpha$ yields to

$$\frac{X_T^\pi}{B_T} = x \cdot \exp\left\{\left(\alpha(\mu - r) - \frac{(\alpha\sigma)^2}{2}\right)T + \alpha\sigma W_T\right\}. \tag{2.3.27}$$

We identify expression (2.3.27) and (2.3.27) and obtain the value for α

$$\alpha_t := \alpha = \frac{\mu - r}{\sigma^2}. \tag{2.3.28}$$

The later ratio is called the Merton point.

2.3.3 Minimizing the shortfall risk

We previously introduced the notion of imperfect hedge (quantile and efficient hedge). When minimizing the expected shortfall, the investor has an initial capital less than the required Black–Scholes fair price. In this situation where the inequality $X_T^\pi(x) < f_T$ also holds, the investor wants to find the optimal strategy that minimizes the expected value of his shortfall $(f_T - X_T^\pi(x))^+$ weighted by a loss function say $l_p(x) = \frac{X^p}{p}$ with $p > 1$ (see Föllmer and Leukert [26], [27]).

The problem of shortfall risk minimization can be seen as an investment problem (see Nakano [45]) or an efficient hedge problem (see Melnikov and Kruchenko [33]). One can see the relation between efficient hedge and shortfall risk minimization in Föllmer and Leukert [27]. We will follow the Nakano methodology.

Nakano adapts the methodology used in Cvitanic [17], [16], Karatzas–Shreve [30] for the maximization of the expected value of the utility function of an investor to solve the problem of minimization of an expected shortfall in a case of a one interest rate jump–diffusion market. The procedure was first suggested in Föllmer and Leukert [27]. The problem is given by

$$u(x) = \inf_{\begin{cases} \pi \in \mathcal{A}, \\ x < E^*[f(S_T^1)e^{-rT}] \end{cases}} E[l_p((f_T - X_T^\pi(x))^+)] \,. \tag{2.3.29}$$

where the set $\mathcal{A} = \{\pi \ s.t. \ E\left[\sup_{0\le t\le T} |X_t^\pi(0)|^p\right] < \infty\}$, $f_T \in L^{p+\epsilon}(\Omega, \mathcal{F}_T, P)$, for some $\epsilon > 0$ and, E^* is the expected value taken under the unique martingale measure. From Nakano [45], the solution of the problem (3.2.24) in the setting of a (B, S)–market consists in finding the perfect hedge of the claim f_T (see Section 2.1.1) and the optimal strategy for an expected utility maximization problem (Theorem 2.8 helps in solving this part, see also Karatzas–Shreve [30], Nakano [45]).

Theorem 2.9. *The optimal hedge $\pi^* = \pi_{f_T} - \pi_0$, where π_{f_T} is the perfect hedge (in the Nakano sense, see [45]) for f_T and, π_0 is the optimal portfolio for*

$$J(z) := \inf_{\pi\in\mathcal{A}_0(z)} E[l_p(X_T^\pi(z))] \tag{2.3.30}$$

with $z = x_{f_T} - x$. We denote

$$\mathcal{A}_0(z) = \{\pi \in \mathcal{A} \ and \ X_t^\pi \ge 0 \qquad t \in [0, T] \qquad a.s.\}.$$

In the pure Merton model a solution is characterized by what follows.

a) Let $\alpha_0(t)$ be the optimal portfolio proportions associated to π_0 solution of (2.3.30), then $\alpha_0(t) := \alpha_0 = (\alpha_0^1, \alpha_0^2)$ of $J(z)$ is given by

$$\alpha_0 = -\frac{1}{\nu}\left(\frac{\lambda^*}{\lambda}\right)^{q-1}, \tag{2.3.31}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

b) The price function

$$u(x) = l_p(x_{f_T} - x)\exp\left(-(p-1)aT\right), \tag{2.3.32}$$

with $a = -qr - \lambda\left((q-1) - q\left(\frac{\lambda^*}{\lambda}\right) + \left(\frac{\lambda^*}{\lambda}\right)^q\right)$.

c) The optimal terminal wealth is given by

$$X_T^{\pi_{f_T}-\pi_0}(x) = f_T - (x_{f_T} - x)(Z_T)^{q-1}\exp\left(-\left(a + \frac{r}{p-1}\right)T\right).$$

In the Black–Scholes setting we derive

a) Let $\alpha_0(t)$ be the optimal portfolio proportions associated to π_0 solution of (2.3.30), then $\alpha_0(t) := \alpha_0 = (\alpha_0^1, \alpha_0^2)$ of $J(z)$ is given by

$$\alpha_0 = \frac{1}{\sigma}\frac{\phi}{p-1}, \tag{2.3.33}$$

b) The price function

$$u(x) = l_p(x_{f_T} - x)\exp\left(-(p-1)aT\right), \tag{2.3.34}$$

with $a = -qr + \frac{1}{2}q(q-1)\phi^2$

c) The optimal terminal wealth is given by

$$X_T^{\pi_{f_T}-\pi_0}(x) = f_T - (x_{f_T} - x)(Z_T)^{q-1}\exp\left(-\left(a + \frac{r}{p-1}\right)T\right).$$

Let us now turn to a constrained market case.

2.4 The constrained market results

We previously provided the solution to the problems of pricing a contingent claim, investing, and expected shortfall minimization in a context of a unique interest rate (unconstrained) pure jump and pure diffusion financial markets. We now consider the previous problems in the context of a financial market where the risk free borrowing and lending rates differ (B^1, B^2, S), where B^1 is the lending account, B^2 the borrowing account, we assume the corresponding interest rates are both constants and $r_1 < r_2$. We also consider that investors will not borrow and lend money simultaneously. The dynamics of the different assets involved this market are given by equations (2.2.2) and what follows

$$dB_t^1 = B_t^1 r_1 dt\,, \tag{2.4.1}$$

$$dB_t^2 = B_t^2 r_2 dt\,, \tag{2.4.2}$$

in a Merton model. In a Black–Scholes model, the stock price verifies (2.2.3) and the lending and borrowing accounts follows respectively (2.4.1) and (2.4.2). We denote the above financial markets by (B^1, B^2, S).

A portfolio π is provided by $(\beta^1, \beta^2, \gamma)$ (see Definition 2.1), where β^i denotes the number of units of the $i^t h$ bond, and γ the number of units of stock in the portfolio.

At any time t, the value of such a portfolio is given by

$$\begin{aligned} V_t &= \beta_t^1 B_t^1 + \beta_t^2 B_t^2 + \gamma_t S_t \qquad a.s.\,, \\ &\textit{with} \qquad \beta^1 \geq 0 \qquad \beta^2 \leq 0\,. \end{aligned} \tag{2.4.3}$$

We assume that the definitions of a wealth and debt processes (see 2.3–2.5) still hold. The standard methods for pricing and investing do not work in this setting. One way to solve the problems of pricing claims, investment, and risk minimization is to introduce suitable auxiliary markets $(B^d, S)_{d\in[0,r^2-r^1]}$ where the bank account B^d admits an interest rate $r^d = r^1 + d$. Each such an auxiliary market is complete (see Statement 2.6). From relation (2.1) or (2.3.7) the fair price C_{r^d} of the the claim f_T is known. The initial prices of a buyer and seller in the (B^1, B^2, S)–market are given by the initial cost of the minimal hedge provided the latter exists.

To solve the investment and shortfall risk minimization problems of the (B^1, B^2, S)–market, we first consider the problems in a standard unconstrained market (B^d, S), then derive the corresponding results in the two interest rates model (B^1, B^2, S).

2.4.1 Hedging

We consider a contingent claim f_T. Hedging this claim requires the construction of a wealth process X_t and debt process Y_t for respectively the buyer and seller of the claim. In the constrained (B^1, B^2, S)–market the buyer and seller positions do not coincide, it is therefore necessary to distinguish the two processes.

Under the assumptions made in the previous section, the wealth and debt processes will have the following dynamics

$$dX_t = X_{t^-}\left[(1-\alpha_t)^+ r^1 dt - (1-\alpha_t)^- r^2 dt + \alpha_t \frac{dS_t}{S_{t^-}}\right], \tag{2.4.4}$$

$$dY_t = Y_{t^-}\left[(1-\alpha_t)^+ r^2 dt - (1-\alpha_t)^- r^1 dt + \alpha_t \frac{dS_t}{S_{t^-}}\right] \tag{2.4.5}$$

where, $\alpha_t = \frac{\gamma_t S_{t-}}{X_{t-}}$ (resp. $\alpha_t = \frac{\gamma_t S_{t-}}{Y_{t-}}$) is the proportion of cash invested in the stock for a buyer (resp. seller), and

$$\begin{aligned} a^+ &= \max(a, 0) \\ a^- &= -\min(a, 0) \,. \end{aligned}$$

The derivation of the above formulas will be found on the Appendix. Let us turn to the buyer case. The objective of the buying agent is to pay (or invest) the minimal initial amount "possible". Further, the terminal wealth $X_T^{\pi}(x)$ of the buyer should at least match the value of the claim (i.e. $X_T^{\pi}(x) \geq f_T$). Hence, the initial price required for a buying agent is

$$V_0 = \inf\{x \geq 0 : \; \exists \, \pi \in \mathcal{A}(x), \; s.t. \; X_T^{x,\pi} \geq f_T\} \,. \tag{2.4.6}$$

We will sometimes refer to the above as the upper hedging price of the two interest rates financial market. We give next a statement relating the (B^d, S)–market to the (B^1, B^2, S)–market for both the Merton and Black–Scholes models.

Statement 2.10. Let $d = (d_t) \in [0, r^2 - r^1]$ a predictable process and assume that α_d the optimal hedging strategy against f_T in the (B^d, S) market verifies

$$(r^2 - r^1 - d_t)(1 - \alpha_t)^- + d_t(1 - \alpha_t)^+ = 0 \,, \tag{2.4.7}$$

Then, C_{r^d} the initial of the minimal hedge against f_T in (B^d, S) is equal to C_+ the initial price of the minimal hedging strategy in (B^1, B^2, S).

This statement was proved in Korn [31] on a Black–Scholes setting. We give the proof in a Merton type model.

Proof. Let α be the optimal hedging strategy against f_T in the (B^d, S)–market and assume it verifies (2.4.7), then α is a hedge against the same claim in the (B^1, B^2, S)–market.

To prove this claim let C_{r^d} be the initial capital associated to α in (B^d, S), and denote the wealth processes in (B^d, S) and (B^1, B^2, S) respectively by $V^{\alpha,d}$ and V^α, then

$$\begin{aligned}\frac{dV_t^{\alpha,d}}{V_t^{\alpha,d}} &= (1-\alpha_t)r^d dt + \alpha_t \frac{dS_t}{S_{t-}}, \\ &= r^1(1-\alpha_t)dt + d(1-\alpha_t)^+ dt - d(1-\alpha_t)^- dt + \frac{dS_t}{S_{t-}}.\end{aligned}$$

Now, from equation (2.4.7), we derive

$$d(1-\alpha_t)^+ = -(r^2 - r^1 - d)(1-\alpha_t)^-$$

and, by substitution

$$\begin{aligned}\frac{dV_t^{\alpha,d}}{V_t^{\alpha,d}} &= r^1(1-\alpha_t)dt - (r^2 - r^1 - d)(1-\alpha_t)^- dt - d(1-\alpha_t)^- dt + \frac{dS_t}{S_{t-}} \\ &= r^1(1-\alpha_t)^+ dt - r^2(1-\alpha_t)^- dt + \frac{dS_t}{S_{t-}} \\ &= \frac{dV_t^\alpha}{V_t^\alpha}.\end{aligned}$$

Consequently for a same initial capital C_{r^d}, the next equalities hold

$$\begin{aligned}V_t^\alpha(C_{r^d}) &= V_t^{\alpha,d}(C_{r^d}) \\ V_T^\alpha(C_{r^d}) &= V_T^{\alpha,d}(C_{r^d}) = f_T.\end{aligned}$$

Now we have proved the claim made at the beginning of the proof, let us show that such a strategy α is optimal in the (B^1, B^2, S)–market. It suffices to show

$$E^{d,*}[f(S_T^1)e^{-r^d T}] \leq x,$$

where x is the initial capital of (α_a) an arbitrary hedge against f_T in the (B^1, B^2, S)–market and, $E^{d,*}$ is the expected value under the risk neutral measure $P^{*,d}$ (for simplification we shall write P^*).

If $V_t^{\alpha_a}$ is the wealth process associated to α_a then

$$E^{d,*}[V_T^{\alpha_a} e^{-r^d T}] \leq x\,. \tag{2.4.8}$$

To show relation (2.4.8) let us denote $X_t := V_t e^{-r^d t}$ and using Kolmogorov–Ito formula we derive for a Black–Scholes model we derive

$$dX_t \;=\; V_{t-}^{\alpha_a} e^{-r^d t}\left(\left[(1-\alpha_t^a)^-(r^1-r^2) - d(1-\alpha_t^a)\right] dt + \alpha_t^a \sigma dW_t^{d,*}\right), \tag{2.4.9}$$

and the dynamic of the discounted wealth process in the pure Merton model follows

$$dX_t \;=\; V_{t-}^{\alpha_a} e^{-r^d t}\left(\left[(1-\alpha_t^a)^-(r^1-r^2) - d(1-\alpha_t^a)\right] dt - \alpha_t^a \nu \; d(\Pi_t - \lambda^* t)\right). \tag{2.4.10}$$

Since $r^1 < r^2$, we note

$$(1-\alpha_t^a)^-(r^1-r^2) - d(1-\alpha_t^a) \leq 0$$

hence,

$$\int_0^T dX_t \leq \int_0^T -V_t^{\alpha_a} e^{-r^d t} \alpha_t^a \nu d(\Pi_t - \lambda^* t)\,.$$

Since $-V_t e^{-r^d t}\alpha_t^a \nu \; d(\Pi_t - \lambda^* t)$ is a P^* local martingale and any local martingale bounded from below is a super–martingale (the process X_t is non–negative). Upon taking the P^* expectation, we obtain, for all t in $[0,T]$,

$$E^{d,*}\left[\int_0^T dX_t\right] \leq E^{d,*}\left[\int_0^T -V_t^{\alpha_a} e^{-r^d t}\alpha_t^a \nu d(\Pi_t - \lambda^* t)\right].$$

$$E^{d,*}[X_t] = E^{d,*}[V_t^{\alpha_a} e^{-r^d t}] \leq x. \qquad (2.4.11)$$

The initial capital of α is less than the initial capital of any arbitrary strategy in the (B^1, B^2, S) market. Hence $C_{r^d} = C_+$ by definition of C_+.

A similar proof for the put yields to $P_{r^d} = P_+$. □

We next give an example of the European put and call prices for a buyer in the (B^1, B^2, S)–market.

Example 2.11. Let us begin with the Merton model

A) From the Merton call price (2.3.4), and the self–financing property of the portfolio (Definition 2.2), the amount of cash invested at any time t in any (B^d, S)–market is negative hence $(1-\alpha)^+ = 0$ and solving for (2.4.7) yields to $d = r^2 - r^1$ and $C_+ = C_{r^2}$. The European put option admits a positive amount of cash invested hence $(1-\alpha)^- = 0$ and solving for equation (2.4.7) yields to $d = 0$ and $P_+ = P_{r^1}$.

B) The Black–Scholes call price (2.3.7) shows a negative amount of cash invested (i.e. $(1-\alpha)^+ = 0$), hence $d = r^2 - r^1$ and $C_+ = C_{r^2}$. A similar method for the European put option yields $P_+ = P_{r^1}$.

We now turn to the seller case, we consider the latter has a debt process of initial amount y. In the (B^1, B^2, S)–market denote the lowest amount initial that allows the seller to repay his debt f_T by C_- and let $Y_t(y)$ be the debt process giving the position of the seller then, $C_- = |y|$. Consequently the minimal initial debt in the latter market follows

$$-C_- = \sup\{y \leq 0 \;/\; \exists\, \alpha = (\alpha^1, \alpha^2) \in \mathcal{A}'(x) s.t., \; Y_T^\alpha \leq -f_T\} \qquad (2.4.12)$$

and $\mathcal{A}'(x)$ represents the set of self financing strategies α such that the generated debt process $Y_t^\alpha \leq 0, \; \forall t \geq 0$.

Statement 2.12. Let $d = (d_t)$ be a predictable process in $[0, r^2 - r^1]$, and let α_d, the minimal hedging strategy against f_T in (B^d, S^1, S^2) verify the equation

$$(r^2 - r^1 - d_t)(1 - \alpha_t)^+ + d_t(1 - \alpha_t)^- = 0. \qquad (2.4.13)$$

Then,

1. the strategy α_d is a hedge against $-f_T$ in (B^1, B^2, S).
2. if further, C_{r^d} (resp. P_{r^d}) the fair price of the claim in (B^d, S^1, S^2) verifies $C_{r^d} = \inf_{k \in [0, r^2 - r^1]} C_{r^k}$ (resp. $P_{r^d} = \inf_{k \in [0, r^2 - r^1]} P_{r^k}$) then,

   $$C_{r^d} = C_-(\text{resp. } P_{r^d} = P_-),$$

   where C_- (resp. P_-) is the initial debt of the minimal hedge (see 2.4.12), also called the seller price.

We next enounce a Lemma that helps us to proof Statement 2.12.

Lemma 2.2. *The minimal hedging strategy against f_T in (B^d, S) is the minimal hedging strategy against $-f_T$ in the same market.*

Proof. In the unconstrained (B^d, S)–market, the stochastic differential equations of the debt and wealth processes coincide, then if α_d is a hedge against

f_T in (B^d, S) we have $V_T^{\alpha_d,x} = f_T$. Now, taking $y = -x$ as initial price for the debt process yields $Y_T = -V_T^{\alpha_d,x} = -f_T$. Henceforth, α_d is a hedge against $-f_T$ in (B^d, S).

$\square$

Proof. (of Statement 2.12)

Let (α_d, C_{r^d}) be the pair hedge, initial price against f_T in the (B^d, S)–market. Further assume α_d verify the relation (2.4.13) then, from the previous Lemma α_d is a hedge against $-f_T$ with initial price $-C_{r^d}$. From equality (2.4.13) the pair $(\alpha_d, -C_{r^d})$ hedges against $-f_T$ in the (B^1, B^1, S)–market. It remains to show that α_d is the optimal hedge in the latter market.

Let us assume that $C_{r^d} = \inf_{k\in[0,r^2-r^1]} C_{r^k}$, and let y be the initial value of the debt process generated by α, an arbitrary strategy in (B^1, B^2, S).
We shall show that $y \leq \sup_{k\in[0,r^2-r^1]}(-C_{r^k}) := -C_{r^d}$. Henceforth, any hedging strategy against $-f_T$ has an initial value less than $-C_{r^d}$, but $-C_{r^d}$ is itself an initial debt of a hedge (α_d) against $-f_T$ in (B^1, B^2, S). Consequently $-C_{r^d}$ gives the lowest initial debt in (B^1, B^2, S).
Let us show that $y \leq \sup_{k\in[0,r^2-r^1]}(-C_{r^k}) := -C_{r^d}$.
Any hedging strategy against $-f_T$ in (B^1, B^2, S) is a hedging strategy against the same claim in $(B^{\tilde{d}}, S)$ where

$$\tilde{d} = \begin{cases} r^2 - r^1 & \text{if } 1 - \alpha_t^1 - \alpha_t^2 \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.14}$$

But, $-C_{r^{\tilde{d}}} \leq -C_{r^d}$ by definition. Therefore $y \leq -C_{r^d}$.
The proof holds for both Put and Call options. It follows that $C_- = P_{r^d}$ (resp. $P_- = P_{r^d}$).

$\square$

Example 2.13. We give the prices of the seller for the European put and call options.

A) For the Merton case, the price of a European call in any (B^d, S)–market and at any time t is provided by formula (2.3.4). From the self–financing property (see Definition 2.2), the amount of cash in the portfolio is always negative. Hence $(1-\alpha)^+ = 0$ and taking this into account in relation (2.4.13) yields to $d = 0$ and $C_- = C_{r^1}(0)$. Similarly for the European put option the amount in the bank account is always positive hence $(1-\alpha)^- = 0$, $d = r^2 - r^1$ and $P_- = P_{r^2}(0)$.

B) Regarding the Black–Scholes model, the price of a European contingent claim is given by formula (2.3.7) and the amount of cash invested, at any time t, in any market (B^d, S) is negative. Hence $(1-\alpha)^+ = 0$ and solving equation (2.4.13) yields to $d = 0$ and $C_- = C_{r^1}(0)$. A similar procedure for the European put yields $P_- = P_{r^2}(0)$.

A more intuitive way to derive the possible option prices in a case of a different interest rates market would consist in approximating the arbitrage–free region by taking the interval $[\inf_{d\in[0,r^2-r^1]} C_{r^d}, \quad \sup_{d\in[0,r^2-r^1]} C_{r^d}]$ where C_{r^d} is the price of the option in the (B^d, S)–market. From Theorem (2.7), the arbitrage–free region in the case of a European put option is $[C_{r^1}, C_{r^2}]$ and for a call option we derive $[P_{r^2}, P_{r^1}]$. Consequently, the completion methodology confirm the intuition in these particular cases.

2.4.2 Investing

We consider the investment problem described in (2.3.14) on a setting of a two interest–rates (B^1, B^2, S)–market. Hence the next theorem follows

Theorem 2.14. *Consider a logarithmic utility function $U(x) = \ln(x)$ in the (B^1, B^2, S)–market. Denote the wealth processes generated by a portfolio π in (B^d, S) and (B^1, B^2, S) markets by $X_t^{\pi,d}$ and X_t^{π} respectively, and let them verify equations (2.3.3) and (2.4.4) respectively. Now, assume α^* the optimal strategy on the stock in the (B^d, S)–market verifies (2.4.7) then,*

1) *the cost function $u(x)$ is given by (2.3.22) for the pure Merton model and (2.3.26) for the Black–Scholes one.*

2) *The optimal proportions invested in the (B^1, B^2, S)–market are α^* on stocks, $(1-\alpha^*)^+$ on the deposit account, and $(1-\alpha^*)^+$ on the credit account.*

3) *The optimal terminal wealth are given by (2.3.24) and (2.3.27) in resp. the Merton and Black–Scholes models.*

Proof. Let α^* be the optimal proportion in the (B^d, S)–market and assume it verifies equation (2.4.7) then α^* is optimal in the (B^1, B^2, S) financial market. For any strategy π the relation $X_t^{\pi}(x) \leq X_t^{\pi,d}(x)$ holds since there are better investing conditions in the (B^d, S)–market than in the (B^1, B^2, S) one. Thus

$$\sup_{\pi \in SF} E\left[U(X_T^{\pi}(x))\right] \leq \sup_{\pi \in SF} E\left[U(X_T^{\pi,d}(x))\right] = E\left[U(X_T^{\pi^*,d}(x))\right] \overset{(2.4.7)}{=} E\left[U(X_T^{\pi^*}(x))\right] ,$$

and

$$\sup_{\pi \in SF} E\left[U(X_T^{\pi}(x))\right] = E\left[U(X_T^{\pi^*}(x))\right] = E\left[U(X_T^{\pi^*,d}(x))\right] = u(x) . \quad (2.4.15)$$

Regarding the optimal proportions in the (B^1, B^2, S)-market, From (2.4.7) and (2.4.15), we derive

$$X_T^{\pi^*}(x) = X_T^{\pi^*,d}(x) = Y_T^{*,d}(x)e^{r^d T}. \tag{2.4.16}$$

It remains to derive the optimal proportions in the (B^1, B^2, S)-market. We already know that α^* is optimal in the (B^1, B^2, S), and it is given respectively in the Merton and in Black scholes model by (2.3.25) and (2.3.28).

Thus, one invests α^* in stocks and the rest in the bank accounts, the positive part $(1-\alpha^*)^+$ in the deposit account and $(1-\alpha^*)^-$ in the credit account. □

In the next example we give some analytical solutions.

Example 2.15. The optimal proportion found is constant (since the optimization problem is solved on the set of $\{\alpha_t := \alpha\}$). Therefore $(1-\alpha^*)$ is either positive or negative. Assume $(1-\alpha^*)$ is positive (i.e. $\alpha^* < 1$), then by solving equation (2.4.7) we obtain $d = 0$ hence $r^d = r^1$. Thus we derive the optimal proportions invested in stocks on respectively the Merton and Black–Scholes models

$$\alpha^* = \frac{\mu - r^1 - \nu\lambda}{\nu(\mu - r^1)}, \quad \text{if} \quad \frac{\mu - r^1 - \nu\lambda}{\nu(\mu - r^1)} < 1, \tag{2.4.17}$$

$$\alpha^* = \frac{\mu - r^1}{\sigma^2}, \quad \text{if} \quad \frac{\mu - r^1}{\sigma^2} < 1. \tag{2.4.18}$$

We consider now $(1-\alpha^*)$ is negative (i.e. $\alpha^* > 1$), then by solving equation (2.4.7) we obtain $d = r^2 - r^1$ and therefore $r^d = r^2$. From there the optimal proportions invested on the stocks in resp.the Merton and Black–Scholes models are

$$\alpha^* = \frac{\mu - r^2 - \nu\lambda}{\nu(\mu - r^2)}, \quad \text{if} \quad \frac{\mu - r^2 - \nu\lambda}{\nu(\mu - r^2)} > 1,$$
$$\alpha^* = \frac{\mu - r^2}{\sigma^2}, \quad \text{if} \quad \frac{\mu - r^2}{\sigma^2} > 1.$$

2.4.3 Minimizing the Shortfall Risk

We consider the problem similar to (3.2.24) in the two interest rates financial market (B^1, B^2, S). Such a problem is described as follows.

$$u(x) = \inf_{\begin{cases} \pi \in \mathcal{A}, \\ x < C_- := \inf_d E^{d,*}[f(S_T)e^{-r^d T}] \end{cases}} E[l_p((f_T - X_T^{\pi,d}(x))^+)] \quad (2.4.19)$$

The solution of the problem (2.4.19) is provided by the next Theorem

Theorem 2.16. *Denote the wealth processes with initial capital x in respectively the (B^d, S) and (B^1, B^2, S)–market by $X_t^{\pi,d}(x)$ and $X_t^{\pi}(x)$. Suppose that $X_t^{\pi,d}(x)$ and $X_t^{\pi}(x)$ verify resp. (2.3.3) and (2.4.4). Assume that α the optimal proportion of problem 3.2.24 in the (B^d, S) market verify (2.4.7) and, assume also that α_{f_T} the optimal hedge of f_T in the (B^d, S)–market fulfills the conditions of Statement 2.12 then,*

1) the cost function $u(x)$ is given by (2.3.32)

2) and, the optimal proportions invested are

$$\alpha_t = \frac{\alpha_f X_{t-}^{\pi_{f_T}}(x_{f_T}) - \alpha_0 X_{t-}^{\pi_0}(x_{f_T} - x)}{X_{t-}^{\pi_{f_T} - \pi_0}(x)} \qquad \text{on } S,$$

$(1-\alpha)^-$ on the credit account and $(1-\alpha)^+$ on the deposit account.

The proof of this Theorem is identical to the one given in the jump–diffusion case.

Chapter 3

The jump–diffusion model

3.1 Introduction

In this chapter we first give the standard results of the problem of pricing, investing and minimizing the shortfall risk in an unconstrained jump–diffusion financial market. Then, as in the previous chapter we extend those results to a two interest rates financial market. We also exploit a particular characteristic of the European put and call options to approximate the arbitrage free interval of prices.

3.2 The complete market results

Let $\{\Omega, \mathcal{F}, \mathrm{F} = (\mathcal{F}_t)_{t\geq 0}, P\}$ be a standard stochastic basis. Assume there are two risky assets S^i, $i = 1, 2$, whose prices are described by the equations

$$dS_t^i = S_{t-}^i\Big(\mu^i dt + \sigma^i dW_t - \nu^i d\Pi_t\Big), \qquad i = 1, 2. \tag{3.2.1}$$

Here W is a standard Wiener process and Π is a Poisson process with positive intensity λ. The filtration F is generated by the independent processes W and

Π, $\mu^i \in \mathrm{R}$, $\sigma^i > 0$, $\nu^i < 1$.

We also assume that there is a bank account B verifying

$$dB_t = B_t r dt. \tag{3.2.2}$$

Denote (B, S^1, S^2) the market described by the above assets, and consider a contingent claim f_T , and a portfolio $\pi = (\beta, \gamma^1, \gamma^2)$ (see Definitions 2.4 and 2.1), where we denote respectively by β and γ^i the number of units of bond and i^{th} stock in the wealth. The value of such a portfolio π is given in this model by

$$V_t = \beta_t B_t + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2, \qquad a.s. \tag{3.2.3}$$

Definition 3.1. A portfolio π is said *self-financing* (SF) if it verifies the following property

$$dV_t = \beta_t dB_t + \gamma_t^1 dS_t^1 + \gamma_t^2 dS_t^2, \ a.s. \tag{3.2.4}$$

Such a portfolio will be said *admissible* if

$$V_t \geq 0 \ \ a.s \, \forall \, t \geq 0.$$

The set of admissible portfolios with initial capital x is denoted by $\mathcal{A}(x)$. The buyer and the seller positions can be identified with the *wealth process* X_t and the *debt process* Y_t respectively (see Definition 2.3).

Let $\sigma^1\nu^2 \neq \sigma^2\nu^1$, and the parameters

$$\begin{aligned} \phi &= -\frac{(\mu^1 - r)\nu^2 - (\mu^2 - r)\nu^1}{\sigma^1\nu^2 - \sigma^2\nu^1}, \\ \psi &= \frac{(\mu^1 - r)\sigma^2 - (\mu^2 - r)\sigma^1}{\sigma^2\nu^1 - \sigma^1\nu^2}\lambda^{-1} - 1, \end{aligned} \tag{3.2.5}$$

then the next Statement holds (see Melnikov et al [39]).

Statement 3.2. The (B, S^1, S^2)–market is complete, and the density of the martingale measure $Z_t = \frac{dP^*}{dP}$ is provided below

$$Z_t = \mathcal{E}_t(N) = \exp\Big\{\phi W_t - \frac{\phi^2}{2}t + (\lambda - \lambda^*)t + (\ln \lambda^* - \ln \lambda)\Pi_t\Big\}, \quad (3.2.6)$$

where $N_t = \phi W_t + \psi(\Pi_t - \lambda t)$. Under the martingale measure, the given Poisson process Π has intensity $\lambda^* = \lambda(1 + \psi)$ and $W_t^* = W_t - \phi t$ is a Wiener process.

From the above Statement, the (B, S^1, S^2)–market is complete, thus according to the *One Law Price*, $Y_t(x)$ the position of the seller of the Claim f_T is obtained by taking $-X_t(x)$, where X_t is the buyer position. The dynamics of the wealth and debt processes follow

$$\frac{dV_t}{V_{t-}} = \frac{dY_t}{Y_{t-}} = \left[(1 - \alpha_t^1 - \alpha_t^2)r\,dt + \alpha_t^1 \frac{dS_t^1}{S_{t-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t-}^2}\right]. \quad (3.2.7)$$

We turn to standard hedging results in the complete market.

3.2.1 Hedging results in an auxiliary market

As in the previous chapter we are considering a perfect hedge. In this section, the contingent claims will be of the form $f_T = f(S_T^1)$. The initial price of the derivative is provided next.

Lemma 3.1. *The fair price of the call option $(S_T^1 - K)^+$ in this (B, S^1, S^2)–market is given by the formula*

$$\begin{aligned} E^* e^{-rT}(S_T^1 - K)^+ &= \sum_{n=0}^{\infty}\left(e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!} C_{BS}[S_0^1(1 - \nu^1)^n e^{\nu^1 \lambda^* T}, r, \sigma^1, K]\right) \\ &= C_r(0) = V_0, \end{aligned} \quad (3.2.8)$$

where $V_0 = \inf\{x \geq 0 : \ \exists\, \pi \in \mathcal{A}(x), \ s.t. \ X_T^\pi(x) \geq f(S_T^1)\}$,

$$C_{BS}(x, r, \sigma, K) = x\Phi(d_1) - Ke^{-rT}\Phi(d_2)\,,$$
$$d_1 = \frac{\ln(\frac{x}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}}, \ d_2 = d_1 - \sigma\sqrt{T}\,,$$
$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\,.$$

We derive the well known price of a Put option from the Call–Put parity.

For more details on the Lemma 3.1 one can Aase [1], Bardhan and Chao [4], Colwell and Elliott [15], Mercurio and Runggaldier [41], Melnikov et al. [39].

The next Lemma studies the monotonic properties of C_r (see formula (3.2.8)) as a function of r.

Lemma 3.2. *If the following inequalities are fulfilled*

3.1) $\frac{\partial \lambda^*}{\partial r} \geq 0$, *or*

3.2) $\frac{\partial \lambda^*}{\partial r} \leq 0$ *and* $\nu^1 \geq 0$, *or*

3.3) $\frac{\partial \lambda^*}{\partial r} \leq 0$, $\nu^1 \leq 0$ *and* $\frac{\nu^1 \frac{\partial \lambda^*}{\partial r}}{1+\nu^1 \frac{\partial \lambda^*}{\partial r}} \leq \Phi(d_2(0))$

then, $\rho_C := \frac{\partial C}{\partial r}$ *is positive.*

If the next inequalities are verified

3.1') $\frac{\partial \lambda^*}{\partial r} \leq 0$, *or*

3.2') $\frac{\partial \lambda^*}{\partial r} \geq 0$ *and* $\nu^1 \leq 0$, *or*

3.3') $\frac{\partial \lambda^*}{\partial r} \geq 0$, $\nu^1 \geq 0$ *and* $\Phi(d_2(0)) \leq \frac{1}{1+\nu^1 \frac{\partial \lambda^*}{\partial r}}$

then, $\rho_P := \frac{\partial P}{\partial r}$ *is negative.*

Proof. We start the proof with the case of the call option.

a) **The case of a call option.**

For convenience purposes we shall choose to use a different representation of $\frac{\partial C}{\partial r}$ whether $\frac{\partial \lambda^*}{\partial r}$ is positive or negative. Differentiating (3.2.8) yields to

$$\begin{aligned}\frac{\partial C}{\partial r} &= T\frac{\partial \lambda^*}{\partial r}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}A(n) + KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}\Phi(d_2(n)),\\ &\text{if} \quad \frac{\partial \lambda^*}{\partial r} \geq 0. \end{aligned} \tag{3.2.9}$$

$$\begin{aligned}\frac{\partial C}{\partial r} &= T\frac{\partial \lambda^*}{\partial r}(1-\nu^1)\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}B(n)\\ &- \nu T\frac{\partial \lambda^*}{\partial r}Ke^{-rT}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}\Big(\Phi(d_2(n+1)) - \Phi(d_2(n))\Big)\\ &+ KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}\Phi(d_2(n)),\\ &\text{if} \quad \frac{\partial \lambda^*}{\partial r} \leq 0. \end{aligned} \tag{3.2.10}$$

Where $A(n)$ and $B(n)$ have the following expressions

$$\begin{aligned}A(n) &= S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}\Big(\Phi(d_1(n+1)) - \Phi(d_1(n))\Big)\\ &\quad -Ke^{-rT}\Big(\Phi(d_2(n+1)) - \Phi(d_2(n))\Big),\\ B(n) &= S_0(1-\nu^1)^{n}e^{\nu^1\lambda^* T}\Big(\Phi(d_1(n+1)) - \Phi(d_1(n))\Big)\\ &\quad -Ke^{-rT}\Big(\Phi(d_2(n+1)) - \Phi(d_2(n))\Big),\end{aligned}$$

we denote σ^1 by σ, and

$$\begin{aligned}d_2(n) &= \frac{\ln[S/K] + n\ln(1-\nu^1) + \nu^1\lambda^* T + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},\\ d_1(n) &= d_2(n) + \sigma\sqrt{T}.\end{aligned}$$

One can easily show that $A(n) \geq 0$ and $B(n) \leq 0$.

We only give the proof that $A(n) \geq 0$ since a similar method is used to show that $B(n) \leq 0$.

$$A(n) = S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}\Big(\Phi(d_1(n+1)) - \Phi(d_1(n))\Big) - Ke^{-rT}\Big(\Phi(d_2(n+1)) - \Phi(d_2(n))\Big),$$

where $d_i(n+1) = d_i(n) + \frac{\ln(1-\nu)}{\sigma\sqrt{T}}, \qquad i = 1, 2.$

$$\begin{aligned}
A(n) &= \frac{1}{\sqrt{2\pi}}\left(S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}\int_{d_1(n)}^{d_1(n+1)} e^{-\frac{x^2}{2}}\,dx - Ke^{-rT}\int_{d_2(n)}^{d_2(n+1)} e^{-\frac{x^2}{2}}\,dx\right)\\
&= \frac{1}{\sqrt{2\pi}}\Bigg(S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}\int_0^{\frac{ln(1-\nu^1)}{\sigma\sqrt{T}}} e^{-\frac{(x+d_1(n))^2}{2}}\,dx\\
&\qquad - Ke^{-rT}\int_0^{\frac{ln(1-\nu^1)}{\sigma\sqrt{T}}} e^{-\frac{(x+d_2(n))^2}{2}}\,dx\Bigg)\\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\frac{ln(1-\nu^1)}{\sigma\sqrt{T}}}\left(S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}e^{-\frac{(x+d_1(n))^2}{2}} - Ke^{-rT}e^{-\frac{(x+d_2(n))^2}{2}}\right)dx\\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\frac{ln(1-\nu^1)}{\sigma\sqrt{T}}} S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^* T}e^{-\frac{(x+d_1(n))^2}{2}}\left(1 - \frac{e^{x\sigma\sqrt{T}}}{(1-\nu^1)}\right)dx
\end{aligned}$$

If $\ln(1-\nu^1)$ is negative then $(1 - \frac{e^{x\sigma\sqrt{T}}}{(1-\nu^1)})$ is also negative. Hence, $A(n)$ is positive. Identically, if $\ln(1-\nu^1)$ is positive then $(1 - \frac{e^{x\sigma\sqrt{T}}}{(1-\nu^1)})$ is also positive. Hence, $A(n)$ is always positive. Consequently

1) For $\frac{\partial\lambda^*}{\partial r} \geq 0$, from the sign of $A(n)$ and equation (3.2.9) we obtain $\frac{\partial C}{\partial r} > 0$.

2) Similarly for $\frac{\partial\lambda^*}{\partial r} \leq 0$, since $B(n) \leq 0$ and from equation (3.2.10), we only need to find the sign of

$$-\nu^1 T\frac{\partial\lambda^*}{\partial r}Ke^{-rT}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}\Big(\Phi(d_2(n+1)) - \Phi(d_2(n))\Big) + KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^* T)^n}{n!}e^{-\lambda^* T}\Phi(d_2(n)). \tag{3.2.11}$$

Expression (3.2.11) can be transformed as

$$KTe^{-rt}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}\left(\Phi(d_2(n))(1+\nu^1\frac{\partial\lambda^*}{\partial r})-\nu^1\frac{\partial\lambda^*}{\partial r}\Phi(d_2(n+1))\right). \tag{3.2.12}$$

To guarantee the positivity of the above expression, it is sufficient to prove that

$$X=\left(\Phi(d_2(n))(1+\nu^1\frac{\partial\lambda^*}{\partial r})-\nu^1\frac{\partial\lambda^*}{\partial r}\Phi(d_2(n+1))\right).$$

is positive.

Let us now consider two cases $\nu^1 \geq 0$ or $\nu^1 \leq 0$ and note that from the expression of $d_2(n)$ the following always holds

$$\nu^1\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\leq 0.$$

(a) If $\nu^1 \leq 0$, then from the previous relation, Φ is a non decreasing function of n and $\left(1+\nu^1\frac{\partial\lambda^*}{\partial r}\right)>0$, therefore

$$\left(\Phi(d_2(0))(1+\nu^1\frac{\partial\lambda^*}{\partial r})-\nu^1\frac{\partial\lambda^*}{\partial r}\right)<X.$$

Hence, if $\Phi(d_2(0))\geq\frac{\nu^1\frac{\partial\lambda^*}{\partial r}}{1+\nu^1\frac{\partial\lambda^*}{\partial r}}$ then $X>0$ and $\frac{\partial C}{\partial r}>0$.

(b) If $\nu^1\geq 0$ then Φ is a non increasing function of n i.e. $\Phi(d_2(n+1))\leq\Phi(d_2(n))$, and $\Phi(d_2(n+1))<X$.

X is therefore non negative and $\frac{\partial C}{\partial r}>0$.

b) **The case of a put option**

The ρ of the put option is given by the following

$$\begin{aligned}\frac{\partial P}{\partial r} &= T\frac{\partial \lambda^*}{\partial r}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}\Big(S_0(1-\nu^1)^{n+1}e^{\nu^1\lambda^*T}\Big(\Phi(d_1(n+1))-\Phi(d_1(n))\Big)\\ &\quad -Ke^{-rT}\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\Big)\\ &\quad -KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}(1-\Phi(d_2(n))), \qquad \text{if } \frac{\partial \lambda^*}{\partial r}\leq 0. \end{aligned} \tag{3.2.13}$$

And

$$\begin{aligned}\frac{\partial P}{\partial r} &= T\frac{\partial \lambda^*}{\partial r}(1-\nu^1)\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}\Big(S_0(1-\nu^1)^{n}e^{\nu^1\lambda^*T}\Big(\Phi(d_1(n+1))-\Phi(d_1(n))\Big)\\ &- Ke^{-rT}\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\Big)\\ &- \nu^1T\frac{\partial \lambda^*}{\partial r}Ke^{-rT}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\\ &- KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}(1-\Phi(d_2(n))), \qquad \text{if } \frac{\partial \lambda^*}{\partial r}\geq 0. \end{aligned} \tag{3.2.14}$$

1. If $\frac{\partial \lambda^*}{\partial r}\leq 0$, from the sign of $A(n)$ we get $\frac{\partial P}{\partial r}<0$.

2. Now if $\frac{\partial \lambda^*}{\partial r}>0$, since $B(n)\leq 0$ and $(1-\nu^1)>0$, the first term of $\frac{\partial P}{\partial r}$ is negative. We only need to determine the sign of

$$\begin{aligned}&-\nu^1T\frac{\partial \lambda^*}{\partial r}Ke^{-rT}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\\ &\qquad -KTe^{-rT}\sum_{n\geq 0}\frac{(\lambda^*T)^n}{n!}e^{-\lambda^*T}\Big(1-\Phi(d_2(n))\Big). \end{aligned} \tag{3.2.15}$$

As in the Call case, let us note that $\nu^1\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\leq 0$, we shall consider again two cases $\nu^1\geq 0$ and $\nu^1\leq 0$. Again we consider the problem of finding the sign of

$$\begin{aligned}&-\nu^1\frac{\partial \lambda^*}{\partial r}\Big(\Phi(d_2(n+1))-\Phi(d_2(n))\Big)\\ &\qquad -\Big(1-\Phi(d_2(n))\Big)\end{aligned} \tag{3.2.16}$$

(a) For $\nu^1 \geq 0$, we rewrite the expression (3.2.16) as

$$\begin{aligned} Y &= \left(-\nu^1 \frac{\partial \lambda^*}{\partial r}\left(\Psi(d_2(n)) - \Psi(d_2(n+1))\right) - (\Psi(d_2(n)))\right), \\ &= \left(\Psi(d_2(n))\left(-1 - \nu^1 \frac{\partial \lambda^*}{\partial r}\right) + \nu^1 \frac{\partial \lambda^*}{\partial r}\Psi(d_2(n+1))\right) \end{aligned} \tag{3.2.17}$$

where $\Phi = 1 - \Psi$.

The above is negative if an upper bound of Y is negative. But, for the same reason as in the call case (here Ψ is an increasing function of n), so

$$Y < \left(\Psi(d_2(0))(-1 - \nu^1 \frac{\partial \lambda^*}{\partial r}) + \nu^1 \frac{\partial \lambda^*}{\partial r}\right).$$

Hence for

$$\Psi(d_2(0)) = 1 - \Phi(d_2(0))) > \frac{\nu^1 \frac{\partial \lambda^*}{\partial r}}{1 + \nu^1 \frac{\partial \lambda^*}{\partial r}}$$

or

$$\Phi(d_2(0)) < \frac{1}{1 + \nu^1 \frac{\partial \lambda^*}{\partial r}}$$

$\frac{\partial P}{\partial r}$ is negative.

(b) For $\nu^1 \leq 0$ we note that $\Phi(d_2(n))$ (resp. $\Psi(d_2(n))$) is a non decreasing (resp. non increasing) function of n and

$$Y \leq -\Psi(d_2(n)).$$

Hence Y is negative and so is $\frac{\partial P}{\partial r}$. □

Let us turn to the investment problem

3.2.2 Investing in an auxiliary market

In parallel to the hedging agent who wants to find the optimal strategy to hedge his claim f_T (see previous sections), an investing agent is to find the optimal strategy that allows him to maximize the expected utility of his terminal wealth. Assume a given utility function $U : R_+ \longrightarrow R$, concave, non–decreasing, continuously differentiable, and

$$\left\| \begin{array}{l} \lim_{x\to\infty} U'(x) = 0\,, \\ \lim_{x\to 0} U'(x) = \infty\,. \end{array} \right. \tag{3.2.18}$$

Let $u(x)$ be the cost function in the (B^d, S^1, S^2)-market. The investment problem (see, for instance Karatzas and Shreve [30], Melnikov et al [39]) consists in finding

$$u(x) \;=\; \sup_{\pi\in SF} E\left(U(X_T^{\pi,d}(x))\right) = E\left(U(X_T^{\pi^*,d}(x))\right)\,, \tag{3.2.19}$$

The same transformation as in the previous chapter is applicable

$$u(x) \;=\; \sup_{y\in\chi} E\left(U(Y_T(x))\right) = E\left(U(Y_T^*(x))\right)$$

where $\chi = \{Y positive:\; Y_t(x) = x + \int_0^t \gamma_u d\tilde{X}_s\}$ with γ a predictable process and $\tilde{X}$ is a P^* local martingale. To solve the problem, we use Theorem 2.8. We consider the case $U(x) = \ln(x)$ in the (B^d, S^1, S^2) financial market. Substituting $U(x)$ in (2.3.17), we derive $V(y) = -\ln(y) - 1$ and,

$$\begin{aligned} v(y) \;&=\; E[V(yZ_T)] = -E[\ln(yZ_T)] - 1 = -\ln(y) - 1 - E[\ln(Z_T)] \\ &=\; -\ln(y) - 1 + \frac{\phi^2}{2}T - \left(\ln(\lambda^*) - \ln(\lambda)\right)\lambda T + (\lambda^* - \lambda)T. \end{aligned}$$

We substitute the above expression of $v(y)$ into equality (2.3.20), and find the cost function

$$u(x) = \ln(x) + \frac{\phi^2}{2}T - \Big(\ln(\lambda^*) - \ln(\lambda)\Big)\lambda T + (\lambda^* - \lambda)T. \tag{3.2.20}$$

Next, we derive the optimal proportions invested on the different assets involved. From (2.3.21), we know that

$$\begin{aligned} Y_T^*(x) &= \frac{X_T^{\pi^*,d}}{B_T} = I(yZ_T) = \frac{1}{yZ_T} = \frac{x}{Z_T} \\ &= x\exp\Big\{-\phi W_T + \frac{\phi^2}{2}T - \Pi_T \ln\Big(\frac{\lambda^*}{\lambda}\Big) + (\lambda^* - \lambda)T\Big\} \end{aligned} \tag{3.2.21}$$

and solving (3.2.7) for $\alpha_t = \alpha$ yields

$$\begin{aligned} \frac{X_T^{\pi,d}}{B_T^d} &= x\exp\Big\{(\alpha^1(\mu^1 - r_d) + \alpha^2(\mu^2 - r_d) - \frac{(\alpha^1\sigma^1 + \alpha^2\sigma^2)^2}{2})T \\ &\qquad + (\alpha^1\sigma^1 + \alpha^2\sigma^2)W_T + \ln(1 - \alpha^1\nu^1 - \alpha^2\nu^2)\Pi_T\Big\}. \end{aligned} \tag{3.2.22}$$

We identify expression (3.2.21) and (3.2.22) and obtain the values for α^1 and α^2

$$\alpha^1 = \frac{\phi\nu^2 - \sigma^2\left(\frac{\lambda}{\lambda^*} - 1\right)}{\nu^1\sigma^2 - \nu^2\sigma^1}, \qquad \alpha^2 = \frac{\phi\nu^1 - \sigma^1\left(\frac{\lambda}{\lambda^*} - 1\right)}{\nu^2\sigma^1 - \nu^1\sigma^2}. \tag{3.2.23}$$

In the (B^d, S^1, S^2) market, the optimal proportions invested are α^1 on the first stock, α^2 on the second stock and the rest $(1 - \alpha^1 - \alpha^2)$ on the bank account.

3.2.3 The shortfall risk minimization problem in an auxiliary market

In the (B^d, S^1, S^2)–market, we previously found the optimal strategy and initial capital required to hedge perfectly a claim $f(S^1_T)$ (see Section 1 of the current chapter). We also derived the optimal investment strategy and terminal wealth of an expected utility maximization problem.

Consider now an investor whose initial capital x is less than the required $E^{d,*}[e^{-r^dT}f_T]$. In such a case a perfect hedge is no longer possible but one can minimize the risk of shortfall given the initial cost's constraint

$$u(x) = \inf_{\begin{cases} \pi \in \mathcal{A}, \\ x < E^{d,*}[f(S^1_T)e^{-r^dT}] \end{cases}} E[l_p((f_T - X^{\pi,d}_T(x))^+)]\,. \tag{3.2.24}$$

The loss function $l_p(x) = \frac{x^p}{p}$ with $p > 1$, $\mathcal{A} = \{\pi \ s.t. \ E\left[\sup_{0\le t\le T}|X^\pi_t(0)|\right] < \infty\}$ and $f_T \in L^{p+\epsilon}(\Omega, \mathcal{F}_T, P)$, for some $\epsilon > 0$.

The Theorem (2.8) from Chapter 2. gives the optimal solution of the problem. The latter is characterized in the (B^d, S^1, S^2)–market by what follows.

a) Let $\alpha_0(t)$ be the optimal portfolio proportions associated to π_0 solution of the problem (2.3.30), then $\alpha_0(t) := \alpha_0 = (\alpha_0^1, \alpha_0^2)$ of $J(z)$ is given by

$$\begin{aligned} \sigma_1\alpha_0^1 + \sigma_2\alpha_0^2 &= \frac{\phi}{p-1}, \\ \nu_1\alpha_0^1 + \nu_2\alpha_0^2 &= -\left(\frac{\lambda^*}{\lambda}\right)^{q-1}, \end{aligned} \tag{3.2.25}$$

Hence

$$\alpha_0^1 = \frac{\frac{\phi\nu^2}{p-1} + \sigma^2\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\nu^2\sigma^1 - \nu^1\sigma^2}, \qquad \alpha_0^2 = \frac{\frac{\phi\nu^1}{p-1} + \sigma^1\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\sigma^2\nu^1 - \sigma^1\nu^2}, \tag{3.2.26}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

b) The cost function

$$u(x) = l_p(x_{f_T} - x)e^{(-(p-1)aT)}, \tag{3.2.27}$$

with $a = -qr^d + \frac{1}{2}q(q-1)\phi^2 - \lambda\left((q-1) - q\left(\frac{\lambda^*}{\lambda}\right) + \left(\frac{\lambda^*}{\lambda}\right)^q\right)$.

c) The optimal terminal wealth is given by

$$X_T^{\pi_{f_T} - \pi_0, d}(x) = f_T - (x_{f_T} - x)(Z_T)^{q-1}e^{-\left(a + \frac{r^d}{p-1}\right)T}.$$

3.3 The constrained market results

We consider the same standard stochastic basis. Assume there are two risky assets S^i, $i = 1, 2$, whose prices are given by the equations (3.2.1).

We also assume that there are a deposit account B^1 and a credit account B^2 satisfying to

$$dB_t^i = B_t^i r^i dt, \qquad i = 1, 2. \tag{3.3.1}$$

Denote (B^1, B^2, S^1, S^2) the market described by the above assets, and any non-negative $\mathcal{F}_T$–measurable random variable f_T is called a *contingent claim* with the maturity time T. In the (B^1, B^2, S^1, S^2)–market, a *portfolio* $\pi = (\beta^1, \beta^2, \gamma^1, \gamma^2)$ is an $\mathcal{F}_t$–predictable process, where we denote respectively by β^i and γ^i the number of units of the i^{th} bond and i^{th} stock in the wealth. The value of the portfolio π is given by

$$V_t = \beta_t^1 B_t^1 + \beta_t^2 B_t^2 + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2, \ a.s. \tag{3.3.2}$$

A portfolio π is said *self–financing* (SF) if it verifies the following property

$$dV_t = \beta_t^1 dB_t^1 + \beta_t^2 dB_t^2 + \gamma_t^1 dS_t^1 + \gamma_t^2 dS_t^2, \ a.s. \tag{3.3.3}$$

we now turn to the hedging in the two interest rates (B^1, B^2, S^1, S^2)–market.

3.3.1 Hedging in a two interest rates market

Let us now turn to the hedging in the (B^1, B^2, S^1, S^2)–market. We first define suitable auxiliaries markets $(B^d, S^1, S^2)_{d\in[0,r^2-r^1]}$, where a bank account $B = B^d$ is defined by the interest rates $r^d = r^1 + d$. From (3.2.8), the minimal initial hedging cost C_{r^d} of the claim f_T in the complete (B^d, S^1, S^2) market is known. In the (B^1, B^2, S^1, S^2)–market the buyer or seller price of the claim f_T will be given by the initial cost of the *minimal hedge* provided the latter exists.

We finally study the optimal investment problem in the given jump–diffusion financial market given by relations (3.2.1) and (3.3.1). Considering the problem in a standard unconstrained financial market, we then derive the corresponding results in a two interest rates model.

As in Chapter 2 under the assumptions made, The Wealth and Debt processes have the following dynamics:

$$dV_t = V_{t^-}\left[(1-\alpha_t^1-\alpha_t^2)^+ r^1 dt - (1-\alpha_t^1-\alpha_t^2)^- r^2 dt + \alpha_t^1 \frac{dS_t^1}{S_{t^-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t^-}^2}\right], \tag{3.3.4}$$

$$dY_t = Y_{t^-}\left[(1-\alpha_t^1-\alpha_t^2)^+ r^2 dt - (1-\alpha_t^1-\alpha_t^2)^- r^1 dt + \alpha_t^1 \frac{dS_t^1}{S_{t^-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t^-}^2}\right]. \tag{3.3.5}$$

Where α_t represents again the proportion invested in stocks. The derivation of the above formulas are provided in the Appendix.

Consider first the position of a buyer. From a buyer's viewpoint, the investor wishes to invest the minimal initial amount and, at the same time generate a terminal wealth matching at least f_T. Consequently the buyer's price is defined as follows:

$\inf\{x \geq 0 \ \exists\ \alpha \in \mathcal{A}(x)$, s.t. $V_T^{x,\alpha} \geq f_T \ a.s.\}$, and represents the initial capital of the minimal hedge (if it exists) against the claim f_T. The upper hedging price (or buyer price) in the jump–diffusion (B^1, B^2, S^1, S^2)–market will be given by what follows (see Korn [31] for the Black–Scholes model).

Statement 3.1. Let $d = (d_t) \in [0, r^2 - r^1]$ be a predictable process and let $\alpha := (\alpha^1, \alpha^2)$ the optimal hedge against f_T in the (B^d, S^1, S^2)–market verify

$$(r^2 - r^1 - d_t)(1 - \alpha_t^1 - \alpha_t^2)^- + d_t(1 - \alpha_t^1 - \alpha_t^2)^+ = 0, \tag{3.3.6}$$

$C_{r^d}(0)$ (resp. $P_{r^d}(0)$) the initial price of the minimal hedge in (B^d, S^1, S^2) against f_T is equal to C_+ (resp. P_+) the initial price of the minimal hedging strategy in (B^1, B^2, S^1, S^2). Namely

$$C_{r^d}(0) = C_+(\text{ resp. } P_{r^d}(0) = P_+).$$

Proof. Let us first show that under relation (3.3.6), the minimal hedging strategy (α) in (B^d, S^1, S^2) is a hedging strategy in (B^1, B^2, S^1, S^2). Let C_{r^d} be the initial capital associated to that hedge in (B^d, S^1, S^2). If α verifies (3.3.6), then the stochastic differential equations of the wealth processes $V_t^{\alpha,d}$ and V_t^α of respectively (B^d, S^1, S^2) and (B^1, B^2, S^1, S^2) coincide. Taking C_{r^d} as the initial price in both markets yields an equality between the two processes at any time $t \in [0, T]$. Consequently $V_T^{\alpha,d} = V_T^\alpha = f(S_T^1)$.

Now, let us show that under the assumption of the Statement 3.1. the strategy α is minimal among the hedges against $f(S_T^1)$ in the (B^1, B^2, S^1, S^2) market.

For that purpose, it suffices to establish the following

$$E^{d,*}[f(S^1_T)e^{-r^dT}] \leq x,$$

where x represents the initial capital of α_a an arbitrary strategy in (B^1, B^2, S^1, S^2). Let $V_t^{\alpha_a}$ be the wealth process generated by α_a in the (B^1, B^2, S^1, S^2) market. We will prove that $E^{d,*}[V_T^{\alpha_a}e^{-r^dT}] \leq x$.

Consider the discounted wealth process $X_t := V_t^{\alpha_a}e^{-r^dt}$, then by using Ito–Kolmogorov's formula we obtain

$$\begin{aligned} dX_t &= V_{t-}^{\alpha_a}e^{-r^dt}\Big(\big[(1-\alpha_t^{1,a}-\alpha_t^{2,a})^-(r^1-r^2) - d(1-\alpha_t^{1,a}-\alpha_t^{2,a})\big]\,dt \\ &+ (\alpha_t^{1,a}\sigma^1+\alpha_t^{2,a}\sigma^2)dW_t^* - (\alpha_t^{1,a}\nu^1+\alpha_t^{2,a}\nu^2)d(\Pi_t-\lambda^*t)\Big). \qquad (3.3.7) \end{aligned}$$

Now, we note that

$$(1-\alpha_t^{1,a}-\alpha_t^{2,a})^-(r^1-r^2) - d(1-\alpha_t^{1,a}-\alpha_t^{2,a}) \leq 0$$

and,

$$(\alpha_t^{1,a}\sigma^1+\alpha_t^{2,a}\sigma^2)dW_t^* - (\alpha_t^{1,a}\nu^1+\alpha_t^{2,a}\nu^2)d(\Pi_t-\lambda^*t)$$

is a P^* local martingale. Whence upon first integrating the relation (3.3.7) then taking the P^* expectation, we obtain, for all t in $[0,T]$,

$$E^{d,*}[X_t] = E^{d,*}[V_t^{\alpha_a}e^{-r^dt}] \leq x. \qquad (3.3.8)$$

α_a is a hedge for f_T, yields $V_T^{\alpha_a}e^{-r^dT} = X_T \geq f_Te^{-r^dT}$ henceforth

$$E^{d,*}[f_Te^{-r^dT}] \leq E^{d,*}[X_T] = E^{d,*}[V_T^{\alpha_a}e^{-r^dT}] \leq x.$$

From there, $C_{r^d} = E^{d,*}[f_Te^{-r^dT}] \leq x$ where x is the initial capital of an arbitrary hedge for f_T in (B^1, B^2, S^1, S^2). Further, provided relation (3.3.6) fulfilled, C_{r^d} is an initial price of a hedge for f_T in the latter market. Therefore

$$C_{r^d} = C_+,$$

where C_+ is the initial capital of the minimal hedge in (B^1, B^2, S^1, S^2).

The proof is similar for the Put case hence $P_{r^d} = P_+$. □

Secondly we study the position of a seller. From a seller's perspective, one can imagine the seller contracting an initial debt of value $|y|$, where y is the initial value of a debt process Y. The seller's objective is to find a strategy α with the lowest initial debt $C_- = |y|$, that allows him to "honor his contract" ($Y_T \leq -f_T$). The minimal initial debt possible given by C_- is described by what follows,

$$-C_- = \sup\{y \leq 0 \;/\; \exists\; \alpha = (\alpha^1, \alpha^2) \in \mathcal{A}'(x) s.t.,\; Y_T^\alpha \leq -f_T\}$$

and $\mathcal{A}'(x)$ represents the set of self financing strategies α such that the generated debt process $Y_t^\alpha \leq 0,\ \forall t \geq 0$.

Statement 3.2. Let $d = (d_t)$ be a predictable process in $[0, r^2 - r^1]$, and let α_d, the minimal hedging strategy against f_T in (B^d, S^1, S^2) verify the equation

$$(r^2 - r^1 - d_t)(1 - \alpha_t^1 - \alpha_t^2)^+ + d_t(1 - \alpha_t^1 - \alpha_t^2)^- = 0. \tag{3.3.9}$$

Then,

1. The proportion α_d is a hedge against $-f_T$ in (B^1, B^2, S^1, S^2).

2. If further, C_{r^d} (resp. P_{r^d}) the fair price of the claim in (B^d, S^1, S^2) verifies $C_{r^d} = \inf_{k \in [0, r^2 - r^1]} C_{r^k}$ (resp. $P_{r^d} = \inf_{k \in [0, r^2 - r^1]} P_{r^k}$) then,

$$C_{r^d} = C_- (\text{resp. } P_{r^d} = P_-),$$

   where C_- (resp. P_-) is the initial debt of the minimal hedge (i.e., the seller price). Namely $-C_-$ (resp. $-P_-$) $= \sup\{y \leq 0 \;/\; \exists\; \alpha \in \mathcal{A}'(x) s.t.,\; Y_T \leq -f_T\}$.

In order to proof Statement 3.2 let us first state the following

Lemma 3.3. *The minimal hedging strategy against f_T in (B^d, S^1, S^2) (for a buyer) is the minimal hedging strategy against $-f_T$ (for a seller) in the same market.*

Proof. In the unconstrained (B^d, S^1, S^2)–market, the stochastic differential equations of the debt and wealth processes coincide, then if α_d is a hedge against f_T in (B^d, S^1, S^2) we have $V_T^{\alpha_d, x} = f_T$. Now, taking $y = -x$ as initial price for the debt process yields $Y_T = -V_T^{\alpha_d, x} = -f_T$. Henceforth, α_d is a hedge against $-f_T$ in (B^d, S^1, S^2).

□

Proof. (of Statement 3.2) Provided relation (3.3.9) is verified, α_d is a hedge in (B^1, B^2, S^1, S^2) against $-f_T$, with initial price $-C_{r^d}$. Only, we need to find the minimal hedge in the latter market. Let us assume that $C_{r^d} = \inf_{k \in [0, r^2 - r^1]} C_{r^k}$, and let y be the initial value of the debt process generated by α, an arbitrary strategy in (B^1, B^2, S^1, S^2).

We shall show that $y \leq \sup_{k \in [0, r^2 - r^1]}(-C_{r^k}) := -C_{r^d}$. Henceforth, any hedging strategy against $-f_T$ has an initial value less than $-C_{r^d}$, but $-C_{r^d}$ is itself an initial debt of a hedge (α_d) against $-f_T$ in (B^1, B^2, S^1, S^2). Consequently $-C_{r^d}$ gives the lowest initial debt in (B^1, B^2, S^1, S^2).

Let us show that $y \leq \sup_{k \in [0, r^2 - r^1]}(-C_{r^k}) := -C_{r^d}$.

Any hedging strategy against $-f_T$ in (B^1, B^2, S^1, S^2) is a hedging strategy against the same claim in $(B^{\tilde{d}}, S^1, S^2)$ where

$$\tilde{d} = \begin{cases} r^2 - r^1 & \text{if } 1 - \alpha_t^1 - \alpha_t^2 \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.10}$$

But, $-C_{r\bar{d}} \leq -C_{r^d}$ by definition. Therefore $y \leq -C_{r^d}$.

The proof holds for both Put and Call options. □

Let us give an approximation of the arbitrage free prices of the claim $f_T = (S_T^1 - K)^+$. The key ingredient of the method relies on the following. Taking the supremum (resp. infimum) over the auxiliary markets of the actual prices we find some natural approximations for the upper and lower hedging prices of the claim, (the buyer and seller prices of the claim f_T), and hence we approximate the arbitrage–free interval of prices by taking

$$\left[\inf_{d \in [0, r^2 - r^1]} C_{r^d}, \sup_{d \in [0, r^2 - r^1]} C_{r^d}\right].$$

Exploiting the Call–Put parity a similar method is used for $f_T = (K - S_T^1)^+$. Let us realize this in the following.

The next Theorem contains our main pricing formulas.

Let us first introduce the following conditions:

(I) $\nu^1 \frac{\partial \lambda^*}{\partial r} \leq 0$,

(II) $\nu^1 \geq 0$ and $\frac{\partial \lambda^*}{\partial r} \geq 0$ and $\Phi(d_2(0)) \leq \frac{1}{1+\nu^1 \frac{\partial \lambda^*}{\partial r}}$,

(III) $\nu^1 \leq 0$ and $\frac{\partial \lambda^*}{\partial r} \leq 0$ and $\frac{\nu^1 \frac{\partial \lambda^*}{\partial r}}{1+\nu^1 \frac{\partial \lambda^*}{\partial r}} \leq \Phi(d_2(0))$.

Under these conditions combining Lemma 3.1, Statement 3.1 and Statement 3.2 we arrive to the main result.

Theorem 3.3. *If condition (I) or (II) or (III) are fulfilled, then the following pricing formulas hold.*

$$\begin{aligned} \sup_{d \in [0, r^2 - r^1]} C_{r^d} &\geq C_{r^2}, & \sup_{d \in [0, r^2 - r^1]} P_{r^d} &\geq P_{r^1} \\ \inf_{d \in [0, r^2 - r^1]} C_{r^d} &\leq C_{r^1}, & \inf_{d \in [0, r^2 - r^1]} P_{r^d} &\leq P_{r^2}. \end{aligned} \tag{3.3.11}$$

Corollary 3.1. *(See Korn [31]) The Black–Scholes model verifies (3.3.11) since conditions (I), (II) and (III) are fulfilled:* $\nu^1 = 0$, $\nu^1 \frac{\partial \lambda^*}{\partial r} = 0$ *and* $0 \leq \Phi(d_2) \leq 1$.

Corollary 3.2. *The pure jump case (the Merton model) verifies (3.3.11) since condition (I) holds:* $\nu^1 \frac{\partial \lambda^*}{\partial r} = -1$.

3.3.2 Investing in a two interest rates financial market

In the (B^d, S^1, S^2) market, the optimal proportions invested are α^1 on the first stock, α^2 on the second stock and the rest $(1 - \alpha^1 - \alpha^2)$ on the bank account. Let us turn to the two interest rate financial market (B^1, B^2, S^1, S^2). The previous results lead to the following theorem

Theorem 3.4. *Let the wealth processes in the* (B^d, S^1, S^2) *and* (B^1, B^2, S^1, S^2) *financial markets* $X_t^{\pi,d}(x)$ *and* $X_t^{\pi}(x)$ *verify (3.3.4) and (3.2.7) respectively, and let (3.3.6) hold for* α_t *the optimal proportions in the* (B^d, S^1, S^2)*–market. Then considering a logarithmic utility function, in the* (B^1, B^2, S^1, S^2)*–market*

1) *the cost function* $u(x)$ *is given by (3.2.20) and,*

2) *the optimal proportions invested on the different assets are* α^1 *on* S^1, α^2 *on* S^2, $(1 - \alpha^1 - \alpha^2)^+$ *on the deposit account (if* $(1 - \alpha^1 - \alpha^2) > 0$*) and* $(1 - \alpha^1 - \alpha^2)^-$ *on the credit account (if* $(1 - \alpha^1 - \alpha^2) < 0$*).*

Proof. Let α^* the optimal proportions in the (B^d, S^1, S^2)–market verify (3.3.6) then, α^* is optimal for the (B^1, B^2, S^1, S^2)–market. For any hedge π, we have $X_t^{\pi}(x) \leq X_t^{\pi,d}(x)$ and,

$$\sup_{\pi \in SF} E\left[U(X_T^{\pi}(x))\right] \leq \sup_{\pi \in SF} E\left[U(X_T^{\pi,d}(x))\right] = E\left[U(X_T^{\pi^*,d}(x))\right] \overset{(3.3.6)}{=} E\left[U(X_T^{\pi^*}(x))\right].$$

Hence

$$\sup_{\pi \in SF} E\left[U(X_T^\pi(x))\right] = E\left[U(X_T^{\pi^*}(x))\right] = E\left[U(X_T^{\pi^*,d}(x))\right] = u(x)\,. \quad (3.3.12)$$

Regarding the optimal proportions in the (B^1, B^2, S^1, S^2)-market, From (3.3.6) and (3.3.12), we derive

$$X_T^{\pi^*}(x) = X_T^{\pi^*,d}(x) = Y_T^{*,d}(x)e^{r^d T}. \quad (3.3.13)$$

Solving equation (3.3.4) for $X_T^{\pi^*}(x)$ and identifying the latter with $Y_T^{*,d}e^{r^d T}$ the optimal proportions invested in the (B^1, B^2, S^1, S^2)–market are α^1 on S^1, α^2 on S^2.

Since α^1, α^2 are constants then if $(1-\alpha^1-\alpha^2) > 0$ we invest $(1-\alpha^1-\alpha^2)^+$ on B^1 and if $(1-\alpha^1-\alpha^2) < 0$ we invest $-(1-\alpha^1-\alpha^2)^-$ on B^2. □

Consider a particular case $\sigma^1 = 0$, $\nu^2 = 0$ and assume $(1-\alpha^1-\alpha^2) \geq 0$ (only the lending rate r^1 is applicable). Then, the market (B^d, S^1, S^2) is defined by

$$\begin{cases} dB_t = r^d B_t dt, & B_0 > 0\,, \\ dS_t^1 = S_t^1(\mu^1 dt - \nu^1 d\Pi_t), & S_0^1 > 0\,, \\ dS_t^2 = S_t^2(\mu^2 dt - \sigma_2 dW_t), & S_0^2 > 0\,. \end{cases}$$

From relation (3.2.5) we obtain that

$$\phi = -\frac{\mu^2 - r^1}{\sigma_2} \qquad \lambda^* = \frac{\mu^1 - r^1}{\nu^1}\,. \quad (3.3.14)$$

Exploiting (3.3.14) and (3.2.23) we derive

$$\alpha^1 = \frac{\mu^1 - r^1 - \lambda\nu^1}{\nu^1(\mu^1 - r)}\,, \qquad \alpha^2 = \frac{\mu^2 - r^1}{(\sigma_2)^2}\,. \quad (3.3.15)$$

Note that α_1 and α_2 are the Merton point in a pure jump and pure diffusion model respectively. In the setting of a two interest rates financial market with the above assumptions the optimal proportions invested are defined by (3.3.15) for S^1 and S^2 respectively, and $(1-\alpha^1-\alpha^2)$ in B^1 and 0 in B^2.

3.3.3 Shortfall risk minimization problem in a two interest rates market

Consider now the same problem in a two interest–rates financial market. We give in what follows the solution of problem (3.2.24) in such a setting.

Theorem 3.5. *Let $X_t^{\pi,d}(x)$ and $X_t^{\pi}(x)$ the wealth processes in the (B^d, S^1, S^2) and (B^1, B^2, S^1, S^2) with initial capital x verify respectively (3.3.4) and (3.2.7). Further, assume α_t the optimal proportion (for Problem 3.2.24) in the (B^d, S^1, S^2)-market verifies (3.3.6), assume also that α_{f_T} the optimal strategy hedging f_T in the (B^d, S^1, S^2) fulfills the conditions provided in Statement 3.2. Then, in the (B^1, B^2, S^1, S^2)-market*

1) the cost function (3.2.24) is given by (3.2.27).

2) the optimal proportions invested are

$$\alpha_t^1 = \frac{\alpha_f^1 X_{t-}^{\pi_{f_T}}(x_{f_T}) - \alpha_0^1 X_{t-}^{\pi_0}(x_{f_T} - x)}{X_{t-}^{\pi_{f_T}-\pi_0}(x)} \qquad on\ S^1,$$
$$\alpha_t^2 = \frac{\alpha_f^2 X_{t-}^{\pi_{f_T}}(x_{f_T}) - \alpha_0^2 X_{t-}^{\pi_0}(x_{f_T} - x)}{X_{t-}^{\pi_{f_T}-\pi_0}(x)} \qquad on\ S^2,$$

and $(1 - \alpha_t^1 - \alpha_t^2)^+$ on the deposit account and $(1 - \alpha_t^1 - \alpha_t^2)^-$ on the credit account.

Proof. Let α_t^*, the optimal proportions invested for the problem (3.2.24) in the (B^d, S^1, S^2)–financial market verify (3.3.6). Then, for all $0 \le t \le T$, $X_t^{\pi^*,d}(x) = X_t^{\pi^*}(x)$ and α^* is optimal for the same problem in the (B^1, B^2, S^1, S^2)–financial market (since the (B^1, B^2, S^1, S^2)–market admits higher deposit rate and lower lending rate than the (B^d, S^1, S^2)–market. Therefore for any hedge π, $X_t^{\pi}(x) \le X(x)_t^{\pi,d}$ and

$$\inf_{\pi\in\mathcal{A},\{x<E^{d,*}[e^{-r^dT}f_T]\}} E\left[l_p\left((f_T - X_T^{\pi,d}(x))^+\right)\right] \leq \inf_{\pi\in\mathcal{A},\{x<E^{d,*}[e^{-r^dT}f_T]\}} E\left[l_p\left((f_T - X_T^{\pi}(x))^+\right)\right]. \tag{3.3.16}$$

Now if α^* is optimal for the left hand side of inequality (3.3.16) and verifies equality (3.3.6) then it is also optimal for the right hand side.

The cost function $u(x)$ is given by

$$\begin{aligned}
u(x) &= \inf_{\pi\in\mathcal{A},\{x<E^{d,*}[e^{-r^dT}f_T]\}} E\left[l_p\left((f_T - X_T^{\pi,d}(x))^+\right)\right] \\
&= E\left[l_p\left((f_T - X_T^{\pi^*,d}(x))^+\right)\right] \\
&= E\left[l_p\left((f_T - X_T^{\pi^*}(x))^+\right)\right] \\
&= \inf_{\pi\in\mathcal{A},\{x<E^{d,*}[e^{-r^dT}f_T]\}} E\left[l_p\left((f_T - X_T^{\pi}(x))^+\right)\right] \\
&= \inf_{\{\pi\in\mathcal{A},\{x<C_-=\inf_d E^{d,*}[e^{-r^dT}f_T]\}\}} E\left[l_p\left((f_T - X_T^{\pi}(x))^+\right)\right].
\end{aligned}$$

Note that the last equality is obtained from

$$\inf_{\{\pi\in\mathcal{A},\{x<E^{d,*}[e^{-r^dT}f_T]\}\}} E\left[l_p\left((f_T - X_T^{\pi}(x))^+\right)\right] \leq \inf_{\{\pi\in\mathcal{A},\{x<C_-=\inf_d E^{d,*}[e^{-r^dT}f_T]\}\}} E\left[l_p\left((f_T - X_T^{\pi}(x))^+\right)\right]$$

and since α_{f_T} the optimal proportions hedging f_T is assumed to verify the conditions of Statement 3.2. (i.e. $C_- = C_{r^d}$). The optimal terminal wealth is $X_T^{\pi}(x) = X_T^{\pi,d}(x) = X_T^{\pi_f}(x_f) - X_T^{\pi_0}(x - x_f)$.

The optimal proportions invested are given by

$$
\begin{aligned}
\pi &= \pi_f - \pi_0 \\
\gamma_t S &= \gamma_{t,f} S - \gamma_{t,0} S \\
\alpha_t X_{t^-} &= \alpha_{t,f} X^f_{t^-} - \alpha_{t,0} X^0_{t^-}
\end{aligned}
\tag{3.3.17}
$$

□

Chapter 4

Application on life insurance contract with guarantee

4.1 Introduction

In this section we begin by giving some standard results on quantile hedging in the Merton, Black–Scholes, and Jump–diffusion models. We introduce a new type of instrument with both finance and insurance risk called *unit–linked* or *pure endowment* insurance contract with guarantee (see Brennan and Schwartz [13],[14], Moller [43], Aase and Persson [2]). Melnikov [36] proposes efficient and quantile hedging to price these instruments on a framework of one interest rates financial market. We exploit the latter results to approximate the interval of survival probabilities for the holder of unit–linked insurance contract with guarantee, in the (B^1, B^2, S) and (B^1, B^2, S^1, S^2) markets. At the end of the chapter, we provide 2 numerical examples on which the interval of ages required for a policy–holder of the latter contract is derived on a two interest rates Black–Scholes model.

4.2 Standard results on quantile hedging

Since Föllmer and Leukert [26], [27], we know that an investor unwilling to pay the fair price of a claim f_T (see formula 3.2.8), could pay less than the required fair price but at a higher risk. The quantile hedging problem can be modelized as follows

$$Max_{\{x \leq x_0 < C_r\}} P(X_T^{\pi}(x) \geq f_T)\,, \qquad (4.2.1)$$

where π is in the set of Self–financing strategies.

Consider an investor whose initial capital x_0 is less than C_r the fair price of the contingent claim f_T. The quantile problem consists in finding the strategy that maximizes the probability of successful hedge for such an investor. The set of successful hedge for a strategy π with an initial capital x and a contingent claim f_T is given by $\mathcal{A}(x, \pi, f_T) = \{\omega, X_T^{\pi}(\omega)(x) \geq f_T\}$. The solution of the problem (4.2.1) is provided by the next lemma.

Lemma 4.1. *An optimal strategy for the Problem (4.2.1) is given by the perfect hedge of the modified option $fI_{\tilde{A}}$ where $\tilde{A}$ has the form*

$$\tilde{A} = \left\{\omega \ s.t. \qquad \frac{dP}{dP^*} > const \cdot \ f_T\right\}$$

More details on the proof of Lemma 4.1 can be found in Föllmer and Leukert [26] or Melnikov et al [39]. Alternatively to the problem (4.2.1) (primal problem), we could have formulated the dual Problem: given the probability of the maximal set of successful hedge, derive the minimal initial capital required to hedge the claim f_T.

We next provide the analytical results of respectively the pure Merton, Black–Scholes and jump–diffusion models (see Melnikov et al [39], Melnikov [36]).

Statement 4.1. Let x_0 be as previously the initial capital held by the investor and assume

$$\alpha_M = -\frac{\ln(\lambda^*) - \ln(\lambda)}{\ln(1-\nu)} \tag{4.2.2}$$

then, two cases arise.

1a) If $\alpha_M \leq 1$ then $\tilde{A}$, the set of successful hedge is given by $\{\Pi_T < d\} = \{S_T < b\}$, where $d = \frac{1}{\ln(1-\nu)} * \left(\ln(\frac{b}{S_0}) - \mu T\right)$ and b is the solution of

$$\begin{aligned} x_0 &= S_0\Big[\Phi(n_K, \lambda^*(1-\nu)T) - \Phi(n_b, \lambda^*(1-\nu)T)\Big] \\ &\qquad -Ke^{-rT}\Big[\Phi(n_K, \lambda^*T) - \Phi(n_b, \lambda^*T)\Big], \end{aligned} \tag{4.2.3}$$

with $n_X = \left[\left[\frac{\ln\frac{X}{S_0} - \mu T}{\ln(1-\nu)}\right]\right] + 1.$

2a) Else, if $\alpha_M > 1$, then

$$\tilde{A} = \{S_T < b_1\} \cup \{S_T > b_2\} = \{\Pi_T < d_1\} \cup \{\Pi_T > d_2\},$$

where b_1 and b_2 are solutions of

$$\begin{aligned} x_0 &= S_0\Big[\Phi(n_K, \lambda^*(1-\nu)T) - \Phi(n_{b1}, \lambda^*(1-\nu)T) + \Phi(n_{b2}, \lambda^*(1-\nu)T)\Big] \\ &\quad -Ke^{-rT}\Big[\Phi(n_K, \lambda^*T) - \Phi(n_{b1}, \lambda^*T) + \Phi(n_{b2}, \lambda^*T)\Big]. \end{aligned} \tag{4.2.4}$$

We now provide the quantile hedge results in a Black–Scholes framework.

Statement 4.2. Let x_0 be the initial capital held by the investor and Denote α_{BS} by

$$\alpha_{BS} := \frac{\mu - r}{\sigma^2} = -\frac{\phi}{\sigma}. \tag{4.2.5}$$

Then, we derive what follows

1b) For $\alpha_{BS} \leq 1$, the maximal set of successful hedge

$$\tilde{A} = \{S_T < B\} = \{W_T^* < b\} = \left\{W_T < b - \left(\frac{\mu - r}{\sigma}\right)T\right\}$$

and, the solution of the problem (4.2.1) is given by

$$P(\tilde{A}) = \Phi\left(\frac{b - \frac{\mu - r}{\sigma}T}{\sqrt{T}}\right), \tag{4.2.6}$$

where b is derived from the initial capital constraint

$$x_0 = S_0\left[\Phi(d_1) - \Phi\left(\sigma\sqrt{T} - \frac{b}{\sqrt{T}}\right)\right] - Ke^{-rT}\left[\Phi(d_2) - \Phi\left(-\frac{b}{\sqrt{T}}\right)\right], \tag{4.2.7}$$

and d_1 and d_2 are provided by formula (2.3.9).

2b) For $\alpha_{BS} > 1$, the maximal set of successful hedge

$$\tilde{A} = \{S_T < B_1\} \cup \{S_T > B_2\} = \left\{W_T < b_1 - \left(\frac{\mu - r}{\sigma}\right)T\right\} \cup \left\{W_T > b_2 - \left(\frac{\mu - r}{\sigma}\right)T\right\},$$

hence the solution of problem (4.2.1) is given by

$$P(\tilde{A}) = \Phi\left(\frac{b_1 - \frac{\mu - r}{\sigma}T}{\sqrt{T}}\right) + \Phi\left(\frac{b_2 - \frac{\mu - r}{\sigma}T}{\sqrt{T}}\right),$$

with b_1, b_2 verifying

$$x_0 = S_0\left[\Phi(d_1) - \Phi\left(\sigma\sqrt{T} - \frac{b_1}{\sqrt{T}}\right) + \Phi\left(\sigma\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)\right] \tag{4.2.8}$$

$$-Ke^{-rT}\left[\Phi(d_2) - \Phi\left(-\frac{b_1}{\sqrt{T}}\right) + \Phi\left(-\frac{b_2}{\sqrt{T}}\right)\right]. \tag{4.2.9}$$

We finally arrive to the jump–diffusion model.

Statement 4.3. Let ϕ be as in formula (3.2.5) and, x_0 the initial capital held by the investor, then $\tilde{A}$ the set of successful hedge is such that

1) If $\frac{-\phi}{\sigma} \leq 1$ then, the conditioned maximal set of successful hedge

$$\tilde{A}|_{\{\Pi_T=n\}} = \{S_T < B(n)\} = \{W_T^* < b(n)\} = \{W_T < b(n) + \phi T\},$$

where $\forall n$, $B(n)$ is the unique solution of

$$x^{\frac{-\phi}{\sigma 1}} = h^n \text{const}(x-K)^+ \text{ and }, \qquad h = \frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\phi}{\sigma 1}}}.$$

Further, the initial capital of the quantile hedge is given by

$$\begin{aligned} x_0 &= \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \cdot \Big[S(1-\nu)^n e^{\nu\lambda^* T} \Big(\Phi(d_+(n)) - \Phi(b_+(n)) \Big) \\ &\qquad - K e^{-rT} \Big(\Phi(d_-(n)) - \Phi(b_-(n)) \Big) \Big]. \end{aligned} \tag{4.2.10}$$

2) Else if $\frac{-\phi}{\sigma} > 1$ then, the conditioned set of successful hedge $\tilde{A}|_{\{\Pi_T=n\}}$ is provided by

$$\begin{aligned} \tilde{A}|_{\{\Pi_T=n\}} &= \{S_T < B_1(n)\} \cup \{S_T > B_2(n)\} \\ &= \{W_T < b_1(n) + \phi T\} \cup \{W_T > b_2(n) + \phi T\}, \end{aligned}$$

with $B_1(n)$ and $B_2(n)$ solutions of the equation (4.2.10). The latter admits two solutions when $\frac{-\phi}{\sigma} > 1$. The initial capital of the quantile hedge follows

$$\begin{aligned} x_0 &= \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \cdot \Big[S(1-\nu)^n e^{\nu\lambda^* T} \Big(\Phi(d_+(n)) - \Phi(b1_+(n)) + \Phi(b2_+(n)) \Big) \\ &\qquad - K e^{-rT} \Big(\Phi(d_-(n)) - \Phi(b1_-(n)) + \Phi(b2_-(n)) \Big) \Big], \end{aligned}$$

(4.2.11)

where we denote

$$
\begin{aligned}
d_-(n) &= \frac{\ln(\frac{S(1-\nu)^n e^{\nu\lambda^* T}}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, & d_+(n) &= d_-(n) + \sigma\sqrt{T}, \\
b_-(n) &= \frac{\ln(\frac{S(1-\nu)^n e^{\nu\lambda^* T}}{B(n)}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, & b_+(n) &= b_-(n) + \sigma\sqrt{T}, \\
&= \frac{-b(n)}{\sqrt{T}}, & b_+(n) &= \sigma\sqrt{T} - \frac{b(n)}{\sqrt{T}}, \\
bi_-(n) &= \frac{\ln(\frac{S(1-\nu)^n e^{\nu\lambda^* T}}{B_i(n)}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, & bi_+(n) &= bi_-(n) + \sigma\sqrt{T}, \\
&= \frac{-bi(n)}{\sqrt{T}} & bi_+(n) &= \sigma\sqrt{T} - \frac{bi(n)}{\sqrt{T}},
\end{aligned}
$$

$i = 1, 2$.

We have chosen to express the quantile hedging initial price as a function of $b_\pm(n)$, $bi_\pm(n)$, for convenience purposes on the proof. We have provided the formulas necessary to obtain the quantile hedging initial price as a function of $b(n)$ and $bi(n)$ (as it was expressed in the pure Merton and Black–Scholes models). Also to avoid cumbersome notations we have respectively denoted the parameters μ^1, σ^1, ν^1 by μ, σ, ν.

4.3 Pricing unit–linked insurance contract using quantile hedge

A unit–linked insurance contract holds both finance risk (from the stock market evolution in $(\Omega, \mathcal{F}, P)$) and insurance risk (from the mortality of the insured in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$). To take into account theses features a probability space

$(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, P \times \tilde{P})$ is considered. Unit–linked insurance contract are paid upon survival of the insured at the maturity of the contract. Let the random variable $T(x) \in (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be the survival time of an insured whose current age is x, and K the guarantee of the contract then, the terminal payoff of this insurance claim is described by

$$\max(S_T, K) I_{\{T(x)>T\}}$$

To price this contract, we consider the probability measure $\hat{P} = P^* \times \tilde{P}$, where P^* is the new measure defined by Statement (2.6) and denote the premium of the contract in the auxiliary market (B, S) (resp. (B, S^1, S^2)) by ${}_TU_x$ and let ${}_Tp_x = \tilde{P}(T(x) > T)$ then,

$$\begin{aligned} {}_TU_x &= \hat{E}[e^{-rT} \max(S_T, K) I_{\{T(x)>T\}}] \\ &= \hat{E}[e^{-rT}(K + (S_T - K)^+) I_{\{T(x)>T\}}] \\ &= {}_Tp_x K e^{-rT} + {}_Tp_x C_r \,. \end{aligned}$$

Let us rewrite the above equation as follows

$${}_TU_x - {}_Tp_x K e^{-rT} = {}_Tp_x C_r \,.$$

Since ${}_Tp_x C_r < C_r$ the quantile hedge of f_T with initial capital constraint ${}_Tp_x C_r$ is considered (see Melnikov [36]). From the solution of the quantile hedge the next equation holds

$${}_Tp_x C_r = x_0 \,,$$

where x_0 verifies relation (4.2.3) or (4.2.4) in the Merton model, (4.2.7) or (4.2.8) in the Black–Scholes model and (4.2.10) or (4.2.11) in the Jump–diffusion model. We summarize next the survival probabilities obtained in the different models of unconstrained markets.

1) The pure Merton model

If $\alpha_M \leq 1$ the survival probability is given by the next equality

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_r} \cdot \Big[S_0 \Big(\Phi(n_0, \lambda^* T(1-\nu)) - \Phi(n_1, \lambda^* T(1-\nu)) \Big) \\
&\qquad - K e^{-rT} \Big(\Phi(n_0, \lambda^* T) - \Phi(n_1, \lambda^* T) \Big) \Big] .
\end{aligned}
$$

Else if $\alpha_M > 1$, it follows that

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_r} \cdot \Big[S_0 \Big(\Phi(n_0, \lambda^* T(1-\nu)) - \Phi(n_1, \lambda^* T(1-\nu)) + \Phi(n_2, \lambda^* T(1-\nu)) \Big) \\
&\qquad - K e^{-rT} \Big(\Phi(n_0, \lambda^* T) - \Phi(n_1, \lambda^* T) + \Phi(n_2, \lambda^* T) \Big) \Big] ,
\end{aligned}
$$

Here, the initial price of the pure Merton model with strike price K is denoted by C_r.

2) The Black–Scholes model.

From equation (4.2.8) we derive the survival probabilities

a) for the case where $\frac{-\phi}{\sigma} \leq 1$,

$$
{}_T p_x = \frac{1}{C_r} \cdot S_0 \left[\Phi(d_1) - \Phi\Big(\sigma\sqrt{T} - \frac{b}{\sqrt{T}}\Big) \right] - K e^{-rT} \left[\Phi(d_2) - \Phi\Big(-\frac{b}{\sqrt{T}}\Big) \right] ,
$$

b) and for the case where $\frac{-\phi}{\sigma} > 1$ the next holds

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_r} \cdot S_0 \left[\Phi(d_1) - \Phi\Big(\sigma\sqrt{T} - \frac{b_1}{\sqrt{T}}\Big) + \Phi\Big(\sigma\sqrt{T} - \frac{b_2}{\sqrt{T}}\Big) \right] \\
&\qquad - K e^{-rT} \left[\Phi(d_2) - \Phi\Big(-\frac{b_1}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2}{\sqrt{T}}\Big) \right] .
\end{aligned}
$$

We have denoted the Black–Scholes initial price with strike price K by C_r.

3) The Jump–diffusion model

a) For $\frac{-\phi}{\sigma_1} \leq 1$, the survival probability is given by

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_r^K} \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \cdot \Big[S(1-\nu)^n e^{\nu \lambda^* T} \Big(\Phi(d_+(n)) - \Phi(b_+(n)) \Big) \\
&\qquad - K e^{-rT} \Big(\Phi(d_-(n)) - \Phi(b_-(n)) \Big) \Big] .
\end{aligned}
$$

b) For $\frac{-\phi}{\sigma_1} > 1$, we obtain

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_r^K} \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \cdot \Big[S(1-\nu)^n e^{\nu \lambda^* T} \Big(\Phi(d_+(n)) - \Phi(b1_+(n)) + \Phi(b2_+(n)) \Big) \\
&\qquad - K e^{-rT} \Big(\Phi(d_-(n)) - \Phi(b1_-(n)) + \Phi(b2_-(n)) \Big) \Big] ,
\end{aligned}
$$

where we have denoted C_r^K the jump–diffusion initial price with strike price K.

4.4 The constrained market

In the previous section we have considered the problem of pricing unit linked life insurance contracts and derived the survival probability of the insured (purchaser of the claim) in a (B, S) (resp. (B, S^1, S^2))–market. In this section we consider the same problem in a two interest rates financial ((B^1, B^2, S) and (B^1, B^2, S^1, S^2)–markets). The solution of the problem follows from the previous chapters. We consider the auxiliary (B^d, S)–market for the pure Merton and Black–Scholes models and (B^d, S^1, S^2) for the jump–diffusion model, where the applicable interest rate $r^d := r^1 + d$ is such that $r^1 \leq r^d \leq r^2$. We derive monotonic properties of the survival probability in the different models. This monotonicity allows us to approximate the interval of survival probabilities in the two interest rates (B^1, B^2, S)–market (resp. (B^1, B^2, S^1, S^2)–

market). From there we can retrieve the interval of possible ages of the insured, using the Table of Bowers et al. [12].

1) The structure of the survival probability in the (B^1, B^2, S)–market depends on the parameters of the initial model. We provide in what follows the interval of the survival probabilities on a two interest rates Merton model.

Theorem 4.4. *Let* ${}_Tp_x^{r^d}$ *be the survival probability in the* (B^d, S) *market, and let* $I := [\inf_{d\in\ [0,r^2-r^1]}\ {}_Tp_x^{r^d}, \sup_{d\in\ [0,r^2-r^1]}\ {}_Tp_x^{r^d}]$ *be an approximation of the interval of survival probabilities in the* (B^1, B^2, S) *market.*

a) If $\alpha_M \leq 1 \quad \forall\ d \in [0, r^2 - r^1]$ *then,* $I = [{}_Tp_x^{r^2},\ {}_Tp_x^{r^1}]$.

b) Else if $\alpha_M > 1, \quad \forall\ d \in [0, r^2 - r^1]$*, we distinguish two cases according to* $\nu < 0$ *or* $0 < \nu < 1$.

a)

The case where the parameter $\nu < 0$.

If $n_2 < n^*$ *then,* $I = [{}_Tp_x^{r^1}, {}_Tp_x^{r^2}]$.

Else if $n^* < n_1$ *then,* $I = [{}_Tp_x^{r^2}, {}_Tp_x^{r^1}]$

b)

The case where $0 < \nu < 1$.

If $n_2 < n^*$ *then,* $I = [{}_Tp_x^{r^2}, {}_Tp_x^{r^1}]$.

If $n^* < n_1$ *then,* $I = [{}_Tp_x^{r^1}, {}_Tp_x^{r^2}]$.

where, we denote

$$n_0 = \frac{ln(\frac{K}{S}) - \mu T}{\ln(1-\nu)}, \qquad n_1 = \frac{ln(\frac{C}{S}) - \mu T}{\ln(1-\nu)}, \qquad n_2 = \frac{ln(\frac{C_2}{S}) - \mu T}{\ln(1-\nu)}$$

$$n^* = \begin{cases} \dfrac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^* T}}, & \textit{for } 0 < \nu < 1 \\ \dfrac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^* T}} + 1, & \textit{for } \nu < 0. \end{cases}$$

2) For the Black–Scholes model the following theorem shall hold

Theorem 4.5. *Let ${}_Tp_x^{r^d}$ be the survival probability in the (B^d, S) market,*

1) The case where $\frac{\mu - r^d}{\sigma^2} \leq 1 \; \forall d \in [0, r^2 - r^1]$.

The interval

$$I = [{}_Tp_x^{r^2}, {}_Tp_x^{r^1}]$$

2) The case where $\frac{\mu - r^d}{\sigma^2} > 1 \; \forall d \in [0, r^2 - r^1]$,

[a)] if $\frac{-b_2}{\sqrt{T}} < x_I$ and $\frac{-b_1}{\sqrt{T}} < x_I$ where x_I is given by

$$x_I = -\frac{\sigma\sqrt{T} S \Phi(d_+)}{C_{r^d}}$$

then,

$$I = [{}_Tp_x^{r^2}, {}_Tp_x^{r^1}].$$

[b)] Else if $\frac{-b_2}{\sqrt{T}} > x_I$ and $\frac{-b_1}{\sqrt{T}} > x_I$ then,

$$I = [{}_Tp_x^{r^1}, {}_Tp_x^{r^2}].$$

3) For the Jump–diffusion model

Theorem 4.6. *Let ${}_Tp_x^{r^d}$ be the survival probability in the (B^d, S^1, S^2)–market, let ϕ verify formula (3.2.5) and, denote*

$$I := [\inf_{d \in [0, r^2 - r^1]} {}_Tp_x^{r^d}, \sup_{d \in [0, r^2 - r^1]} {}_Tp_x^{r^d}]$$

then, the following cases hold

***A) the case where** $-\frac{\phi}{\sigma} \leq 1$,* *a1) if* $\nu\frac{\partial\lambda^*}{\partial r^d} > -1$ *then,* $I = \left[{}_T p_x^{r^2},\ {}_T p_x^{r^1}\right]$

a2) if $\nu\frac{\partial\lambda^*}{\partial r^d} = -1$ *then, we are in the pure jump case (see previous results).*

a3) if $\nu\frac{\partial\lambda^*}{\partial r^d} < -1$ *then,* $I = \left[{}_T p_x^{r^1},\ {}_T p_x^{r^2}\right]$

B) For the case where $-\frac{\phi}{\sigma} > 1$, *three possibilities arise.*

b1) If $\nu\frac{\partial\lambda^*}{\partial r^d} > -1$,

and if $\forall\ n,\ b1_-(n) < b^*_-(n)$ *then, we obtain* $I = \left[{}_T p_x^{r^2},\ {}_T p_x^{r^1}\right]$

else if $\forall\ n,\ b^*_-(n) < b2_-(n)$ *then,* $I = \left[{}_T p_x^{r^1},\ {}_T p_x^{r^2}\right]$.

b2) The case $\nu\frac{\partial\lambda^*}{\partial r^d} = -1$ *leads again to the pure jump case.*

b3) If $\nu\frac{\partial\lambda^*}{\partial r^d} < -1$,

and if $\forall\ n,\ b1_-(n) < b^*_-(n)$ *then,* $I = \left[{}_T p_x^{r^1},\ {}_T p_x^{r^2}\right]$,

else if $\forall\ n,\ b^*_-(n) < b2_-(n)$ *then,* $I = \left[{}_T p_x^{r^2},\ {}_T p_x^{r^1}\right]$.

Where, we denote

$$b^*_-(n) = \frac{-\frac{\partial C^K_{r^d}}{\partial r^d} + C^K_{r^d} T\frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right) - TC^K_{r^d}\left(\nu\frac{\partial\lambda^*}{\partial r^d} + 1\right)}{TC^K_{r^d}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}}}.$$

The next remark is a direct consequence of the above theorem and the previous results from Chapter 3.

Remark 1. If the inequality $-\frac{\phi}{\sigma} \leq 1$ is verified and $-1 < \nu\frac{\partial\lambda^*}{\partial r^d} < 0$ then the following equalities are true.

1) The interval of survival probabilities $I = [{}_Tp_x^{r^2},\ {}_Tp_x^{r^1}]$.

2) The arbitrage free interval of the Call option is given by

$$[\inf_d C_{r^d},\ \sup C_{r^d}] = [C_{r^1},\ C_{r^2}].$$

3) Identically, the arbitrage free region of a put option follows $[\inf_d P_{r^d},\ \sup P_{r^d}] = [P_{r^2},\ P_{r^1}]$.

We now will provide the proof of the above Theorems. Throughout all the proof we will denote the interval of survival probabilities by

$$I := [\inf_{d\in[0,r^2-r^1]} {}_Tp_x^{r^d},\ \sup_{d\in[0,r^2-r^1]} {}_Tp_x^{r^1}].$$

Let us begin by the proof of the Theorem 4.4 .

Proof. The Pure Merton Model

1. Assume $\alpha_M = \alpha^* \leq 1$. Then two cases will be considered:

   The case where $\nu < 0$.

   Denote $C^C_{r^d} = S(\Phi(n_1, \lambda^* T(1-\nu))) - Ke^{-rT}\Phi(n_1, \lambda^* T)$, we recall that

$$C_{r^d} = S(\Phi(n_0, \lambda^* T(1-\nu))) - Ke^{-rT}\Phi(n_0, \lambda^* T)$$

   where n_0 is given by formula (2.3.6) and $n_1 = \frac{\ln(\frac{C}{S})-\mu T}{\ln(1-\nu)}$, and we recall that since $C > K$ we have $n_0 < n_1$. We derive the following formulas

$$\begin{aligned}\frac{\partial\Phi(n,\lambda^*T)}{\partial r^d} &= -\frac{T}{\nu}\frac{(\lambda^*T)^{n-1}}{n-1!}e^{-\lambda^*T}\\ \frac{\partial\Phi(n,\lambda^*T(1-\nu))}{\partial r^d} &= (1-\nu)^{n-1}e^{\nu\lambda^*T}\frac{\partial\Phi(n,\lambda^*T)}{\partial r^d}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial C_{r^d}}{\partial r^d} &= \frac{\partial \Phi(n_0, \lambda^* T)}{\partial r^d}\left[S(1-\nu)^{n_0} e^{\nu\lambda^* T} - Ke^{-r^d T}\right] + KTe^{-r^d T}\Phi(n_0, \lambda^* T). \\
\frac{\partial C^C_{r^d}}{\partial r^d} &= \frac{\partial \Phi(n_1, \lambda^* T)}{\partial r^d}\left[S(1-\nu)^{n_1} e^{\nu\lambda^* T} - Ke^{-r^d T}\right] + KTe^{-r^d T}\Phi(n_1, \lambda^* T)
\end{aligned}
$$

Note that ${}_Tp_x = \frac{C_{r^d} - C^C_{r^d}}{C_{r^d}} = 1 - \frac{C^C_{r^d}}{C_{r^d}}$ hence

$$
\begin{aligned}
\frac{\partial {}_Tp_x^{r^d}}{\partial r^d} &= -\frac{\partial\left(\frac{C^C_{r^d}}{C_{r^d}}\right)}{\partial r}, \\
(C_{r^d})^2 \frac{\partial {}_Tp_x^{r^d}}{\partial r^d} &= C^C_{r^d}\frac{\partial C_{r^d}}{\partial r^d} - C_{r^d}\frac{\partial C^C_{r^d}}{\partial r^d},
\end{aligned}
$$

and we derive

$$
\begin{aligned}
(C_{r^d})^2 \frac{\partial {}_Tp_x^{r^d}}{\partial r^d} &= \left(S(1-\nu)^{n_1} e^{\nu\lambda^* T} - Ke^{-r^d T}\Phi(n_1, \lambda^* T)\right)\frac{\partial C_{r^d}}{\partial r^d} \\
&- C_{r^d}\left(\left[S(1-\nu)^{n_1} e^{\nu\lambda^* T} - Ke^{-r^d T}\right]\frac{\partial \Phi(n_1, \lambda^* T)}{\partial r^d} + KTe^{-r^d T}\Phi(n_1, \lambda^* T\right) \\
&= S\left(\Phi(n_1, \lambda^* T(1-\nu))\frac{\partial C_{r^d}}{\partial r^d} - C_{r^d}(1-\nu)^{n_1} e^{\nu\lambda^* T}\frac{\partial \Phi(n_1, \lambda^* T)}{\partial r^d}\right) \\
&- Ke^{-r^d T}\left(\Phi(n_1, \lambda^* T)\frac{\partial C_{r^d}}{\partial r^d} - C_{r^d}\frac{\partial \Phi(n_1, \lambda^* T)}{\partial r^d} + C_{r^d} T\Phi(n_1, \lambda^* T)\right)
\end{aligned}
$$

Let

$$
(C_{r^d})^2 \frac{\partial (Tp_x(n_1))}{\partial r^d} = f(n_1).
$$

In order to find the sign of the above we shall compute $\Delta f_{n_1} = f(n_1) - f(n_1 - 1)$. Using the following equalities

$$
\begin{aligned}
\Phi(n_1, \lambda^* T) - \Phi((n_1 - 1), \lambda^* T) &= -\frac{(\lambda^* T)^{n_1 - 1}}{(n_1 - 1)!}e^{-\lambda^* T}, \\
\Phi(n_1, \lambda^* T(1-\nu)) - \Phi((n_1 - 1), \lambda^* T(1-\nu)) &= -\frac{(\lambda^* T(1-\nu))^{n_1 - 1}}{(n_1 - 1)!}e^{-\lambda^* T(1-\nu)},
\end{aligned}
$$

$$\frac{\partial\Phi(n_1,\lambda^*T)}{\partial r^d} - \frac{\partial\Phi((n_1-1),\lambda^*T)}{\partial r^d} = \frac{T}{\nu}\frac{(\lambda^*T)^{n_1-2}}{(n_1-2)!}e^{-\lambda^*T}\Big[1-\frac{\lambda^*T}{n_1-1}\Big],$$

$$\frac{\partial\Phi(n_1,\lambda^*T(1-\nu))}{\partial r^d} - \frac{\partial\Phi((n_1-1),\lambda^*T(1-\nu))}{\partial r^d} = \frac{T}{\nu}\frac{(\lambda^*T(1-\nu))^{n_1-2}}{(n_1-2)!}e^{-\lambda^*T(1-\nu)} \Big[1-\frac{\lambda^*T(1-\nu)}{n_1-1}\Big],$$

we then derive

$$\Delta f_{n_1} = \frac{(\lambda^*T)^{n_1-1}}{(n_1-1)!}e^{-\lambda^*T}\big[S(1-\nu)^{n_1-1}e^{\nu\lambda^*T} - Ke^{-r^dT}\big] \left(-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu} - C_{r^d}\frac{T}{\nu}\frac{n_1-1}{\lambda^*T}\right).$$

When $\nu < 0$, n_0 represents $\inf\{n \ s.t. \ S(1-\nu)^n e^{\nu\lambda^*T} - Ke^{-r^dT} > 0\}$. Now from the choice of the maximal set of successful hedging $C > K$. That yields to $n_1 > n_0$ and by definition of n_0 we obtain $S(1-\nu)^{n_1-1}e^{\nu\lambda^*T} - Ke^{-r^dT} > 0$ Δf_{n_1} is therefore positive (i.e. f_{n_1} increasing) if the expression

$$-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu} - C_{r^d}\frac{T}{\nu}\frac{n_1-1}{\lambda^*T} > 0$$

i.e

$$n_1 - 1 > \frac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^*T}}$$

$$n_1 > \left[\left[\frac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^*T}} + 1\right]\right] + 1.$$

Otherwise Δf_{n_1} is negative and hence f_{n_1} decreasing. Combining the above results with $f(n_0) = 0$ and $\lim_{n_1\to\infty} f(n_1) = 0$ implies $f(n_1) < 0$ always.

Case where $0 < \nu < 1$

Note that in this case the definition of $\Phi(n,\lambda^*T)$ is given by $\Phi(n,\lambda^*T) = \sum_{k=0}^{n}\frac{(\lambda^*T)^k}{k!}e^{-\lambda^*T}$ and is different from the previous case where $\Phi(n,\lambda^*T) =$

$\sum_{k=n}^{\infty} \frac{(\lambda^*T)^k}{k!} e^{-\lambda^*T}$, the following holds

$$
\begin{aligned}
\frac{\partial \Phi(n, \lambda^*T)}{\partial r^d} &= \frac{T}{\nu}\frac{(\lambda^*T)^n}{n!} e^{-\lambda^*T} \\
\frac{\partial \Phi(n, \lambda^*T(1-\nu))}{\partial r^d} &= (1-\nu)^{n+1} e^{\nu\lambda^*T} \frac{\partial \Phi(n, \lambda^*T)}{\partial r^d}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_{r^d}}{\partial r^d} &= \frac{\partial \Phi(n_0, \lambda^*T)}{\partial r^d}\Big(S(1-\nu)^{n_0} e^{\nu\lambda^*T} - Ke^{-r^dT}\Big)(1-\nu) + KTe^{-r^dT}\Phi(n_0 - 1, \lambda^*T)\,, \\
\frac{\partial C^C_{r^d}}{\partial r^d} &= \frac{\partial \Phi(n_1, \lambda^*T)}{\partial r^d}\Big(S(1-\nu)^{n_1} e^{\nu\lambda^*T} - Ke^{-r^dT}\Big)(1-\nu) + KTe^{-r^dT}\Phi(n_1 - 1, \lambda^*T)\,.
\end{aligned}
$$

As $(C_{r^d})^2 \frac{\partial_T p_x^{r^d}}{\partial r^d} = C^C_{r^d}\frac{\partial C_{r^d}}{\partial r^d} - C_{r^d}\frac{\partial C^C_{r^d}}{\partial r^d}$

$$
\begin{aligned}
(C_{r^d})^2 \frac{\partial_T p_x^{r^d}}{\partial r^d} &= \frac{\partial C_{r^d}}{\partial r^d}\Big(S\Phi(n_1, \lambda^*T(1-\nu)) - Ke^{-r^dT}\Phi(n_1, \lambda^*T)\Big) \\
&\quad - C_{r^d} \cdot \Bigg(\Big[S(1-\nu)^{n_1} e^{\nu\lambda^*T} - Ke^{-r^dT}\Big]\frac{\partial \Phi(n_1, \lambda^*T)}{\partial r^d}(1-\nu) \\
&\quad + KTe^{-r^dT}\Phi(n_1 - 1, \lambda^*T)\Bigg)
\end{aligned}
$$

Denote the above equality by $f(n_1)$ and as previously we shall compute $\Delta f_{n_1} = f(n_1) - f(n_1 - 1)$ then,

$$
\begin{aligned}
f(n_1) &= S\left(\frac{\partial C_{r^d}}{\partial r^d}\Phi(n_1, \lambda^*T(1-\nu)) - C_{r^d}(1-\nu)^{n_1+1} e^{\nu\lambda^*T}\partial_{r^d}\Phi(n_1, \lambda^*T)\right) \\
&- Ke^{-r^dT}\left(\frac{\partial C_{r^d}}{\partial r^d}\Phi(n_1, \lambda^*T) - C_{r^d}(1-\nu)\frac{\partial \Phi(n_1, \lambda^*T)}{\partial r^d} + C_{r^d}T\Phi(n_1 - 1, \lambda^*T)\right).
\end{aligned}
$$

Hence

$$
\Delta f_{n_1} = \frac{(\lambda^*T)^{n_1}}{n_1!} e^{-\lambda^*T}\Big[S(1-\nu)^{n_1} e^{\nu\lambda^*T} - Ke^{-r^dT}\Big]\left(\frac{\partial C_{r^d}}{\partial r^d} - C_{r^d}\frac{T(1-\nu)}{\nu} + C_{r^d}\frac{T}{\nu}\frac{n_1}{\lambda^*T}\right)
$$

As above the following shall hold for $0 < \nu < 1$.

In this case $n_0 = \sup\Big\{n \quad s.t. \quad S(1-\nu)^{n_1} e^{\nu\lambda^*T} - Ke^{-r^dT} > 0\Big\}$. From the choice of the set of successful hedging $C > K$ and the sign of $\ln(1-\nu)$, we derive $n_1 < n_0$ and $S(1-\nu)^{n_1} e^{\nu\lambda^* T} - Ke^{-r^d T} > 0$. Consequently, Δf_{n_1} is positive if

$$n_1 > \frac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^* T}},$$

and negative otherwise. i.e. f is increasing for

$$n_1 \geq \left[\left[\frac{-\frac{\partial C_{r^d}}{\partial r^d} + C_{r^d}\frac{T(1-\nu)}{\nu}}{C_{r^d}\frac{T}{\nu\lambda^* T}}\right]\right] + 1.$$

The expression of $f(n_1)$ implies $f(0) < 0$ and, $f(n_0) = 0$. Consequently $f(n_1) < 0$ always.

2. If $\alpha_M > 1$ then, from the expression of ${}_Tp_x^{r^d}$ we derive

$$_Tp_x = \frac{C_{r^d} - C_{r^d}^{C_1}(n_1) + C_{r^d}^{C_2}(n_2)}{C_{r^d}} = 1 - \frac{C_{r^d}^{C_1}(n_1)}{C_{r^d}} + \frac{C_{r^d}^{C_2}(n_2)}{C_{r^d}}$$

we obtain next

$$(C_{r^d})^2 \frac{\partial {}_Tp_x^{r^d}}{\partial r^d} = -(C_{r^d})^2 \frac{\partial\left(\frac{C_{r^d}^{C_1}(n_1)}{C_{r^d}}\right)}{\partial r^d} + (C_{r^d})^2 \frac{\partial\left(\frac{C_{r^d}^{C_2}(n_2)}{C_{r^d}}\right)}{\partial r^d}$$

where we denote

$$\begin{aligned} C_{r^d}^{C_1} &= S\Phi(n_1, \lambda^* T(1-\nu)) - Ke^{-r^d T}\Phi(n_1, \lambda^* T), \\ C_{r^d}^{C_2} &= S\Phi(n_2, \lambda^* T(1-\nu)) - Ke^{-r^d T}\Phi(n_2, \lambda^* T). \end{aligned}$$

The above expression is $f(n_1) - f(n_2)$ where

$$f(n_1) = -(C_{r^d})^2 \frac{\partial\left(\frac{C_{r^d}^{C_1}(n_1)}{C_{r^d}}\right)}{\partial r^d}.$$

Now, depending on $\nu < 0$ or $0 < \nu < 1$ we have derived the expression of $f(n_1)$ in the previous part.

Case where $\nu < 0$

We have $n_0 < n_1 < n_2$, and the function f is decreasing $[n_0, x_I]$ and, increasing in $[x_I,\ \infty)$. Hence, we consider two cases

1) for $n_1, n_2\ \in [n_0,\ x_I]$, we derive

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} = f(n_1) - f(n_2) > 0,$$

the latter yields to

$$I = [_T p_x^{r^1},\ _T p_x^{r^1}].$$

2) for $n_1, n_2\ \in [x_I,\ \infty]$ the survival probability

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} = f(n_1) - f(n_2) < 0,$$

hence

$$I = [_T p_x^{r^2},\ _T p_x^{r^1}].$$

Case where $0 < \nu < 1$

We have $n_2 < n_1 < n_0$, and the function f is decreasing in $[0, x_I]$ and increasing in $[x_I, n_0]$. Therefore we consider two possibilities:

1) for $n_1, n_2\ \in [0,\ x_I]$,

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} = f(n_1) - f(n_2) < 0$$

consequently we derive

$$I = [_T p_x^{r^2},\ _T p_x^{r^1}].$$

2) and for $n_1, n_2\ \in [x_I,\ n_0]$,

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} = f(n_1) - f(n_2) > 0$$

which yields to

$$I = [_T p_x^{r^1},\ _T p_x^{r^2}].$$

□

We turn now to the proof of Black–Scholes model.

Proof of the Theorem (4.5)

The case where $\alpha_{BS} = \frac{\mu - r^d}{\sigma^2} \leq 1 \quad \forall d \in [0, r^2 - r^1]$.

The survival probability is given by what follows

$$_Tp_x^d = \frac{S_0\left(\Phi(d_+) - \Phi\left(\frac{-b}{\sqrt{T}} + \sigma\sqrt{T}\right)\right) - Ke^{-r^dT}\left(\Phi(d_-) - \Phi\left(\frac{-b}{\sqrt{T}}\right)\right)}{C_{r^d}}$$

Hence differentiating the above yields to the following relation

$$(C_{r^d})^2 \frac{\partial_T p_x^d}{\partial r^d} = -\frac{\sqrt{T}}{\sigma\sqrt{2\pi}}\left(Se^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}+\sigma\sqrt{T}\right)^2} - Ke^{-r^dT}e^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}\right)^2}\right)C_{r^d}$$
$$+ SKTe^{-r^dT}\left(\Phi(d_-)\Phi\left(\frac{-b}{\sqrt{T}} + \sigma\sqrt{T}\right) - \Phi(d_+)\Phi\left(\frac{-b}{\sqrt{T}}\right)\right)$$

Next we prove that $\frac{\partial_T p_x^{r^d}}{\partial r^d}$ is negative.

In order to derive the sign of the survival probability's derivative, let us consider the function f given by

$$f\left(-\frac{b}{\sqrt{T}}\right) = (C_{r^d})^2 \frac{\partial_T p_x^d}{\partial r^d}.$$

Note that $\frac{-b}{\sqrt{T}} \leq d_-$ (since C, the endpoint of the set of successful hedging is chosen such that $C \geq K$).

Now $f(d_-) = 0$, $\lim_{\frac{-b}{\sqrt{T}} \to -\infty} f\left(\frac{-b}{\sqrt{T}}\right) = 0$ and f is a continuous function in $(-\infty, d_-]$.

We next show that f decreases in $(-\infty, x_I)$ then increases in (x_I, d_-) where

$x_I = \frac{\sigma\sqrt{T}S\Phi(d_+)}{C_{r^d}}$

$$\begin{aligned} f'\Big(\frac{-b}{\sqrt{T}}\Big) &= Se^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}+\sigma\sqrt{T}\right)^2}\left[\Big(\frac{-b}{\sqrt{T}}+\sigma\sqrt{T}\Big)C_{r^d}+\frac{\sigma T}{\sqrt{T}}Ke^{-r^dT}\Phi(d_-)\right] \\ &- Ke^{-r^dT}e^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}\right)^2}\left[\Big(\frac{-b}{\sqrt{T}}\Big)C_{r^d}+\frac{\sigma T}{\sqrt{T}}S\Phi(d_+)\right] \\ &= \left(Se^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}+\sigma\sqrt{T}\right)^2} - Ke^{-r^dT}e^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}\right)^2}\right)\left[\Big(\frac{-b}{\sqrt{T}}\Big)C_{r^d}+\sigma\sqrt{T}S\Phi(d_+)\right] \end{aligned}$$

As $-\frac{b}{\sqrt{T}} < d_- \implies \left(Se^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}+\sigma\sqrt{T}\right)^2} - Ke^{-r^dT}e^{\frac{-1}{2}\left(\frac{-b}{\sqrt{T}}\right)^2}\right) > 0$, it follows that the sign of f' is determined by the sign of $\left[\Big(\frac{-b}{\sqrt{T}}\Big)C_{r^d}+\sigma\sqrt{T}S\Phi(d_+)\right]$.

The case $\alpha_{BS} > 1\ \forall d \in [0, r^2 - r^1]$.

We proceed in the same way as in the pure Merton model. Consequently, the survival probability is given by

$$_Tp_x = \frac{C_{r^d} - C^{C_1}_{r^d}\left(\frac{-b_1}{\sqrt{T}}\right) + C^{C_2}_{r^d}\left(\frac{-b_2}{\sqrt{T}}\right)}{C_{r^d}} = 1 - \frac{C^{C_1}_{r^d}\left(\frac{-b_1}{\sqrt{T}}\right)}{C_{r^d}} + \frac{C^{C_2}_{r^d}\left(\frac{-b_2}{\sqrt{T}}\right)}{C_{r^d}}$$

we obtain next

$$\begin{aligned} (C_{r^d})^2\frac{\partial _Tp_x^{r^d}}{\partial r^d} &= -(C_{r^d})^2\frac{\partial\left(\frac{C^{C_1}_{r^d}\left(\frac{-b_1}{\sqrt{T}}\right)}{C_{r^d}}\right)}{\partial r^d} + (C_{r^d})^2\frac{\partial\left(\frac{C^{C_2}_{r^d}\left(\frac{-b_2}{\sqrt{T}}\right)}{C_{r^d}}\right)}{\partial r^d} \\ &= -f\Big(\frac{-b_1}{\sqrt{T}}\Big) + f\Big(\frac{-b_2}{\sqrt{T}}\Big) \end{aligned}$$

¿From the properties of the function f, and given $C_2 > C_1 > K$, we have $\frac{-b_2}{\sqrt{T}} < \frac{-b_1}{\sqrt{T}} < d_-$ and hence

for $\frac{-b_1}{\sqrt{T}} < x_I$ and $\frac{-b_2}{\sqrt{T}} < x_I$ the next relation holds

$$\begin{aligned} (C_{r^d})^2\frac{\partial _Tp_x^{r^d}}{\partial r^d} &= -f\Big(\frac{-b_1}{\sqrt{T}}\Big) + f\Big(\frac{-b_2}{\sqrt{T}}\Big) \geq 0, \\ I &= \left[{}_Tp_x^{r^1}, {}_Tp_x^{r^2}\right] \end{aligned}$$

and for $\frac{-b_1}{\sqrt{T}} > x_I$ and $\frac{-b_2}{\sqrt{T}} > x_I$ we derive

$$
\begin{aligned}
(C_{r^d})^2 \frac{\partial _T p_x^{r^d}}{\partial r^d} &= -f\Big(\frac{-b_1}{\sqrt{T}}\Big) + f\Big(\frac{-b_2}{\sqrt{T}}\Big) \leq 0\,, \\
I &= \Big[{}_T p_x^{r^2}, {}_T p_x^{r^1} \Big]
\end{aligned}
$$

where we denote

$$
\begin{aligned}
C_{r^d}^{C_1} &= S\Phi\Big(\frac{-b_1}{\sqrt{T}} + \sigma\sqrt{T}\Big) - Ke^{-r^d T}\Phi\Big(\frac{-b_1}{\sqrt{T}}\Big)\,, \\
C_{r^d}^{C_2} &= S\Phi\Big(\frac{-b_2}{\sqrt{T}} + \sigma\sqrt{T}\Big) - Ke^{-r^d T}\Phi\Big(\frac{-b_2}{\sqrt{T}}\Big)\,.
\end{aligned}
$$

Let us turn to the jump–diffusion case

Proof. of the jump–diffusion case see Theorem 4.6

A) If $-\frac{\phi}{\sigma_1} \leq 1 \quad \forall d \in [0,\ r^2 - r^1]$

The corresponding survival probability is provided next:

$$
\begin{aligned}
{}_T p_x &= \frac{1}{C_{r^d}^K} \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \Big[S(1-\nu)^n e^{\nu\lambda^* T} \left(\Phi(d_+(n)) - \Phi(b_+(n))\right) \\
&\qquad - Ke^{-rT} \left(\Phi(d_-(n)) - \Phi(b_-(n))\right)\Big]\,.
\end{aligned}
$$

Rewriting the above yields to ${}_T p_x = 1 - \frac{C_{r^d}^B}{C_{r^d}^K}$ where, the following holds

$$
\begin{aligned}
C_{r^d}^K &= \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \Big[S(1-\nu)^n e^{\nu\lambda^* T} \Phi(d_+(n)) - Ke^{-rT}\Phi(d_-(n))\Big] \\
C_{r^d}^B &= \sum_{n=0}^{\infty} \frac{(\lambda^* T)^n e^{-\lambda^* T}}{n!} \Big[S(1-\nu)^n e^{\nu\lambda^* T} \Phi(b_+(n)) - Ke^{-rT}\Phi(b_-(n))\Big]
\end{aligned}
$$

Henceforth

$$
(C_{r^d}^K)^2 \frac{\partial _T p_x^{r^d}}{\partial r^d} = C_{r^d}^K \frac{\partial C_{r^d}^B}{\partial r^d} - C_{r^d}^B \frac{\partial C_{r^d}^K}{\partial r^d}.
$$

As

$$\frac{\partial C^B_{r^d}}{\partial r^d} = T\frac{\partial\lambda^*}{\partial r^d}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\left(\frac{n}{\lambda^*T}-1+\nu\right)\left[S(1-\nu)^n e^{\nu\lambda^*T}\Phi(b_+(n)) - Ke^{-rT}\Phi(b_-(n))\right]$$
$$+\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)KTe^{-r^dT}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\Phi(b_-(n))$$
$$+\frac{T\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\left(S(1-\nu)^n e^{\nu\lambda^*T}e^{\frac{-1}{2}(d_+(n))^2} - Ke^{-rT}e^{\frac{-1}{2}(d_-(n))^2}\right),$$

we derive

$$(C^K_{r^d})^2\frac{\partial_T p^{r^d}_x}{\partial r^d} = C^K_{r^d}\Bigg(T\frac{\partial\lambda^*}{\partial r^d}$$
$$\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\left(\frac{n}{\lambda^*T}-1+\nu\right)\left[S(1-\nu)^n e^{\nu\lambda^*T}\Phi(b_+(n)) - Ke^{-rT}\Phi(b_-(n))\right]$$
$$+\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)KTe^{-r^dT}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\Phi(b_-(n))$$
$$+ \frac{T\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\left[S(1-\nu)^n e^{\nu\lambda^*T}e^{\frac{-1}{2}(d_+(n))^2} - Ke^{-rT}e^{\frac{-1}{2}(d_-(n))^2}\right]\Bigg)$$
$$-\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\left[S(1-\nu)^n e^{\nu\lambda^*T}\Phi(b_+(n)) - Ke^{-rT}\Phi(b_-(n))\right]\frac{\partial C^K_{r^d}}{\partial r^d}$$
$$:= \sum_{n=0}^{\infty}A(n,b_-(n))\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!} \tag{4.4.1}$$

where,

$$
\begin{aligned}
A(n,b_-(n)) &= S(1-\nu)^n e^{\nu\lambda^* T} \\
&\left(\frac{\partial C^K_{r^d}}{\partial r^d}\Phi(b_+(n)) - C^K_{r^d}\Phi(b_+(n))T\frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right)\right. \\
&\left.-TC^K_{r^d}e^{-\frac{1}{2}(b_+(n))^2}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}\sqrt{2\pi}}\right) \\
&-Ke^{-r^d T} \\
&\left(\frac{\partial C^K_{r^d}}{\partial r^d}\Phi(b_-(n)) - C^K_{r^d}\Phi(b_-(n))T\frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right)\right. \\
&\left.-TC^K_{r^d}e^{-\frac{1}{2}(b_-(n))^2}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}\sqrt{2\pi}} + TC^K_{r^d}\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)\Phi(b_-(n))\right)
\end{aligned}
$$

Let us fix n and consider $A(n,b_-(n))$ as a function of $b_-(n)$, $A(n,b_-(n)) = f(b_-(n))$, differentiating f w.r.t. $b_-(n)$ yields to what follows.

$$
\begin{aligned}
\sqrt{(2\pi)}\frac{\partial[A(n,b_-(n))]}{\partial b_-(n)} &= S(1-\nu)^n e^{\nu\lambda^* T}e^{-\frac{1}{2}(b_+(n))^2} \\
&\left(\frac{\partial C^K_{r^d}}{\partial r^d} - C^K_{r^d}T\frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right)\right. \\
&\left.+b_+(n)TC^K_{r^d}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}}\right) \\
&-Ke^{-r^d T}e^{-\frac{1}{2}(b_-(n))^2}\left(\frac{\partial C^K_{r^d}}{\partial r^d} - C^K_{r^d}T\frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right)\right. \\
&\left.-TC^K_{r^d}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}} + b_-(n)TC^K_{r^d}\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)\right)
\end{aligned}
$$

$$\sqrt{(2\pi)}\frac{\partial[A(n,b_-(n))]}{\partial b_-(n)} = \left(S(1-\nu)^n e^{\nu\lambda^* T} e^{-\frac{1}{2}(b_+(n))^2} - Ke^{-r^d T} e^{-\frac{1}{2}(b_-(n))^2}\right)$$
$$\left(\frac{\partial C^K_{r^d}}{\partial r^d} - C^K_{r^d} T \frac{\partial \lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right)\right.$$
$$\left. + b_-(n) T C^K_{r^d} \frac{\left(\nu \frac{\partial \lambda^*}{\partial r^d} + 1\right)}{\sigma\sqrt{T}} + TC^K_{r^d}\left(\nu\frac{\partial \lambda^*}{\partial r^d} + 1\right)\right).$$

As $b_n < d_- \ \forall n > 0$,

$$\left(S(1-\nu)^n e^{\nu\lambda^* T} e^{-\frac{1}{2}(b_+(n))^2} - Ke^{-r^d T} e^{-\frac{1}{2}(b_-(n))^2}\right) \geq 0.$$

A1) If $\left(\nu\frac{\partial\lambda^*}{\partial r^d} + 1\right) > 0$ then $A(b_-(n))$ decreases from $(-\infty, b^*_-(n))$ then, increases from $(b^*_-(n), d_-(n))$ where $b^*_-(n)$ is given by what follows

$$b^*_-(n) = \frac{-\frac{\partial C^K_{r^d}}{\partial r^d} + C^K_{r^d} T \frac{\partial\lambda^*}{\partial r^d}\left(\frac{n}{\lambda^* T} - 1 + \nu\right) - TC^K_{r^d}\left(\nu\frac{\partial\lambda^*}{\partial r^d} + 1\right)}{TC^K_{r^d}\frac{\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)}{\sigma\sqrt{T}}}$$

Further $\lim_{b_-(n)\to -\infty} A(b_-(n)) = 0$, and $A(d_-(n)) = 0$. Henceforth $A(b_-(n)) < 0$. Now lets us show that $A(d_-(n)) = 0$ holds.

As

$$(C^K_{r^d})^2 \frac{\partial_T p^{r^d}_x}{\partial r^d} = C^K_{r^d}\frac{\partial C^B_{r^d}}{\partial r^d} - C^B_{r^d}\frac{\partial C^K_{r^d}}{\partial r^d} =$$
$$\sum_{n=0}^{\infty} e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}\sum_{j=0}^{j=n} e^{-\lambda^* T}\frac{n!}{j!(n-j)!}\left[C^K_{r^d}(n-j)\partial_{r^d}[C^B_{r^d}(j)] - C^B_{r^d}(n-j)\partial_{r^d}[C^K_{r^d}(j)]\right]$$

identifying the above expression to equation(4.4.1) yields

$$A(n, b_-(n)) = \sum_{j=0}^{j=n} e^{-\lambda^* T}\frac{n!}{j!(n-j)!}\left[C^K_{r^d}(n-j)\partial_{r^d}[C^B_{r^d}(j)] - C^B_{r^d}(n-j)\partial_{r^d}[C^K_{r^d}(j)]\right].$$

Henceforth

$$A(n, d_-(n)) = 0$$

A2) Using a similar method we obtain $\left(\nu\frac{\partial\lambda^*}{\partial r^d} + 1\right) < 0 \implies A(b_-(n)) > 0$

A3) If $\left(\nu\frac{\partial\lambda^*}{\partial r^d}+1\right)=0$ then $\frac{\partial A(b_-(n))}{\partial b_-(n)}=0$, and $b(n)=0$ lead to the pure Merton case (see previous part).

A4) Fixing $\nu=\frac{\partial\lambda^*}{\partial r}=0$, and $b_-(n)=b_-(0)$ lead to the Black-Scholes case

We turn now to the alternative case.

B) If $-\frac{\phi}{\sigma}>1\ \forall d\in[0,\ r^2-r^1]$ we shall consider two cases.

B1) The case where $\nu\frac{\partial\lambda^*}{\partial r^d}+1>0$

The survival probability is provided by

$$
\begin{aligned}
{}_Tp_x^{r^d} &= \frac{1}{C^K_{r^d}}\sum_{n=0}^{\infty}\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}\cdot\Big[S(1-\nu)^n e^{\nu\lambda^*T}\left(\Phi(d_+(n))-\Phi(b1_+(n)+\Phi(b2_+(n))\right)\\
&\qquad + \Phi(b2_-(n))\Big]\\
&= 1-\frac{C^{B1}_{r^d}}{C^K_{r^d}}+\frac{C^{B2}_{r^d}}{C^K_{r^d}}
\end{aligned}
$$

differentiating the above expression and using the previous part yield to
differentiating the above expression and using the previous part yield to

$$
\begin{aligned}
\frac{\partial_T p_x^{r^d}}{\partial r^d} &= \frac{\partial\left(-\frac{C^{B1}_{r^d}}{C^K_{r^d}}\right)}{\partial r^d}+\frac{\partial\left(\frac{C^{B2}_{r^d}}{C^K_{r^d}}\right)}{\partial r^d} \qquad \text{and,}\\
(C^K_{r^d})^2\frac{\partial_T p_x^{r^d}}{\partial r^d} &= \sum_{n=0}^{\infty}\Big[A(n,b1_-(n))-A(n,b2_-(n))\Big]\frac{(\lambda^*T)^n e^{-\lambda^*T}}{n!}
\end{aligned}
$$

Let us fix n, from the previous part $A(n,b_-(n))$ is decreasing from $(-\infty,b^*_-(n)]$ and, increasing in $[b^*_-(n),d_-(n)]$. As $b2_-(n)<b1_-(n)<d_-(n)$, it follows from the variations of $A(b(n))$ that

1) If $\forall n,\ b2_-(n),\ b1_-(n)\in(-\infty,\ b^*_-(n)]$ then,

$$A(n,b1_-(n))-A(n,b2_-(n))<0\,,$$

and hence

$$\frac{\partial_T p_x^{r^d}}{\partial r^d}<0\,,\qquad \text{and}\quad I=[{}_Tp_x^{r^2},\ {}_Tp_x^{r^1}].$$

2) If $\forall n,\ \ b2_-(n),\ b1_-(n) \in [b^*_-(n)\ d_-(n]$ then,

$$A(n, b1_-(n)) - A(n, b2_-(n)) > 0\,,$$

hence,

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} > 0 \qquad \text{and} \quad I = [{}_T p_x r^1,\ {}_T p_x r^2].$$

B2) Case $\nu\frac{\partial \lambda^*}{\partial r^d} + 1 < 0$

The expression $A(n, b_-(n))$ is increasing from $(-\infty, b^*_-(n)]$ then, decreasing in $[b^*_-(n), d_-(n)]$.

As $b2_-(n) < b1_-(n) < d_-(n)$, it follows that

1) If $\forall n,\ \ b2_-(n),\ b1_-(n) \in (-\infty,\ b^*_-(n)]$ then,

$$A(n, b1_-(n)) - A(n, b2_-(n)) > 0$$

consequently,

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} > 0 \qquad \text{and} \qquad I = [{}_T p_x r^1,\ {}_T p_x r^2].$$

2) If $\forall n,\ \ b2_-(n),\ b1_-(n) \in [b^*_-(n)\ d_-(n)]$ then,

$$A(n, b1_-(n)) - A(n, b2_-(n)) < 0$$

and hence

$$\frac{\partial_T p_x^{r^d}}{\partial r^d} < 0, \qquad \text{and} \qquad I = [{}_T p_x^{r^2},\ {}_T p_x^{r^1}].$$

□

4.5 Numerical results on a Black–Scholes setting

For simplicity we have used the Black–Scholes model with the parameters of the $S\&P500$ provided on Table 5.2. Hence a volatility $\sigma = 0.195$, a mean

$\mu = 0.109$ are considered. Further, we assume the interest rates $r^1 = 0.14$ and $r^2 = 0.19$ (since for very small and or two close interest rates, there is no much difference between the upper and lower hedging prices), the strike price $K = \$1200$ and the stock price $S_0 = \$1205$.

For a given risk level ϵ, we derive $b = \sqrt{T}\Phi^{-1}(1-\epsilon) + \frac{\mu - r}{\sigma}T$ from formula (4.2.6). We compute the initial quantile hedge price and the survival probability associated to the given risk level. Tables 4.1, 4.2 and 4.3 provide the interval of survival probabilities and the corresponding policy holder interval of ages for different maturities and level of risk.

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	170.438	152.248	137.631
${}_Tp_x(r^1)$	0.9275	0.828	0.7489
$x_0(r^2)$	199.074	171.659	151.382
${}_Tp_x(r^2)$	0.8932	0.7702	0.6792
$I = [{}_Tp_x(r^2),\ {}_Tp_x(r^1)]$	[0.8932, 0.9275]	[0.7702, 0.828]	[0.6792, 0.7489]
Age Interval	[80.37, 84.75]	[90.98, 95.02]	[96.52, 103.39,]

Table 4.1: The next table gives the required interval of ages for a policy holder of the pure endowment insurance contract in a two interest rates financial market . It is obtained for a maturity $T = 1$ year. The Black–Scholes lower and upper hedging prices are provided respectively by $C_{r^1} = \$183.756$ and $C_{r^2} = \$222.869$. The value $x_0(r)$ is the initial price of the quantile hedge with r being the applicable interest rates.

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	389.35	345.376	311.728
${}_Tp_x(r^1)$	0.9152	0.8118	0.7327
$x_0(r^2)$	438.598	361.953	311.222
${}_Tp_x(r^2)$	0.8363	0.6902	0.5934
$I = [{}_Tp_x(r^2),\ {}_Tp_x(r^1)]$	[0.8363, 0.9152]	[0.6902, 0.8118]	[0.5934, 0.7327]
Age Interval	[68.19, 76.99]	[78.79, 85.03]	[83.05, 89.05]

Table 4.2: This table provides the required interval of ages of policy holders in the two interest rates market. The maturity $T = 3$ years, the Black–Scholes lower and upper hedging price are provided by $C_{r^1} = \$425.409$ and $C_{r^2} = \$524.409$.

Remark 2. The results provided on Table 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 are obtained by using the Bowers et al. Table. Further, we have used some special smoothing technique to derive the appropriate interval of ages x.

We next consider different parameters and derive as above, the corresponding survival probability tables (see Tables 4.4, 4.5, 4.6). The stock price $S = \$150$, the strike $K = \$145$, $\mu = 0.2$, $\sigma = 0.9$,

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	549.824	482.57	433.075
$_Tp_x(r^1)$	0.9011	0.7909	0.7098
$x_0(r^2)$	571.706	451.625	378.434
$_Tp_x(r^2)$	0.7771	0.6138	0.5143
$I = [_Tp_x(r^2),\ _Tp_x(r^1)]$	[0.7771, .9011]	[0.6138, 0.7909]	[0.5143, 0.7098]
Age Interval	[70.11, 80.89]	[80.14, 88.23]	[84.16, 91.98]

Table 4.3: The table provides for a maturity $T = 5$ years, the survival probabilities, initial quantile prices and the age of policy–holder in the two interest rates financial market. The lower and upper Black–Scholes hedging prices are provided by $C_{r^1} = \$610.136$ and $C_{r^2} = \$735.683$

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	51.63	41.86	34.94
$_Tp_x(r^1)$	0.85	0.68	0.57
$x_0(r^2)$	53.12	42.66	35.38
$_Tp_x(r^2)$	0.83	0.67	0.55
$I = [_Tp_x(r^2),\ _Tp_x(r^1)]$	[0.83, 0.85]	[0.67, 0.68]	[0.55, 0.57]
Age Interval	[89.2, 90]	[102.2, 104]	[,]

Table 4.4: The table provides for a maturity $T = 1$ year, the survival probabilities, initial quantile prices and the age of a policy–holder in the two interest rates financial market. The lower and upper Black–Scholes hedging prices are provided by $C_{r^1} = \$60.73$ and $C_{r^2} = \$63.25$

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	78.28	51.43	39.59
${}_T p_x(r^1)$	0.72	0.52	0.40
$x_0(r^2)$	71.67	50.63	38.48
${}_T p_x(r^2)$	0.69	0.49	0.37
$I = [{}_T p_x(r^2),\ {}_T p_x(r^1)]$	[0.69, 0.72]	[0.49, 0.52]	[0.37, 0.40]
Age Interval	[83.7, 84.9]	[91.8, 93]	[96.8, 98]

Table 4.5: The table provides for a maturity $T = 3$ years, the survival probabilities, initial quantile prices and the age of policy–holder in the two interest rates financial market. The lower and upper Black–Scholes hedging prices are provided by $C_{r^1} = \$98.88$ and $C_{r^2} = \$103.25$

Risk level (ϵ)	0.01	0.03	0.05
$x_0(r^1)$	70.68	46.02	33.10
${}_T p_x(r^1)$	0.59	0.38	0.27
$x_0(r^2)$	68.33	43.13	30.46
${}_T p_x(r^2)$	0.55	0.35	0.24
$I = [{}_T p_x(r^2),\ {}_T p_x(r^1)]$	[0.55, 0.59]	[0.35, 0.38]	[0.24, 0.27]
Age Interval	[82.1, 83.3]	[89, 90.4]	[93.3, 94.7]

Table 4.6: The table provides for a maturity $T = 5$ years, the survival probabilities, initial quantile prices and the age of policy–holder in the two interest rates financial market. The lower and upper Black–Scholes hedging prices are provided by $C_{r^1} = \$118.45$ and $C_{r^2} = \$122.92$

Chapter 5

Graphs and numerical results

5.1 Introduction

In this section we begin by giving 2 examples that show how adding a jumps of different sizes affect the arbitrage free interval of prices. We compare some of our results to the Bergman's [7]. We provide a selection of graphs comparing the Black–Scholes's option prices to those obtained from the jump–diffusion model and also graphs showing the difference between the upper and lower hedging prices.

5.2 Numerical example

We give two examples to illustrate the results obtained and to allow a comparison with previous results from Bergman (1981). As in Bergman let us consider the pricing of a Bear Spread of calls. A Bear Spread can consist in buying a Call with strike price K_1 and selling a Call with strike price K_2 where $K_2 < K_1$. The price of this instrument is provided by $C(K_2) - C(K_1)$ in a one interest rates financial market. As we have seen in previous chapters, for the 2 interest rates market, the buyer and seller price will differ. Their value are respectively provided by $C_-(K_2) - C_+(K_1)$ and $C_-(K_1) - C_+(K_2)$.

In the first example, we consider the following parameters for the Black–Scholes case $S = S^1 = 30$, $\sigma = 0.3$, $K_2 = 30$, $K_1 = \$20$, $r_1 = 14\%$, $r_2 = 19\%$. Regarding the jump–diffusion case the previous parameters will represent the parameters of the first stock and will be completed by $\mu_1 = 0.05$, $\nu^1 = -0.2$. The second stock's parameters are $\mu_2 = 0.09$, $\sigma_2 = 0.15$, $\nu_2 = -0.35$. It is well known from empirical results in finance that the log normal return of stocks have an excess of Kurtosis which is not taken into account in the Black–Scholes pricing. Jump–diffusion models such as the Merton [42] capture better this excess of kurtosis see (Kou [32], Honore [28]). Hence, 5.1 (Table 1, 2), 5.2 and 5.3 show the importance of adding a jump component in the jump–diffusion model used here.

In the first part of Figure 5.1 (Table 1), we consider the Calls separately and derive the upper, lower hedging prices for the Black–Scholes case (Note that for $\nu^1 = 0$, The jump–diffusion and Black–Scholes prices coincide). We also derive the spread between the upper and lower hedging prices. We obtain slightly different results than Bergman. For instance for the upper hedging price of the $C(K_1)$, Bergman gets 13.26 while we obtain 13.519. This is probably due to the software used or he used a method of duplication. One observes that with larger jump sizes of S^1 (in absolute value) some interesting differences arise on the interval of possible prices between the Black–Scholes model and the jump–diffusion one. The farther we move away from the Black–Scholes model the larger the spread of the option prices becomes.

However this characteristic will not be verified in the second example.

The second part of Figure 5.1 (Table 2) gives the interval of prices of a Spread of Calls in a two interest rates market. Again, the same conclusion can be drawn regarding the jump size. Let us note in the case of the Spread of Calls it has been shown by Bergman in a Black–Scholes setting that such interval of prices ($[C_-, C_+]$) can be narrowed by solving a Partial Differential Equation with switching function. This was possible because that the latter method offsets borrowing and lending while in the method used here, one simultaneously borrows and lends for the case of the Spread of Calls. Such problem would also occur in the case considered in Korn [31] for the sum of a Put and Call.

In our second example we consider different parameters than the previously in the previous example: $\mu^1 = 0.25$, $\mu^2 = 0.2$, $\sigma^1 = 0.3$, $\sigma^2 = 0.5$, $\nu^1 = 0.7$, $\nu^2 = 0.6$ and, we derive Figure 5.2 and Figure 5.3.

Figures 5.4 and 5.5 confirm the results found in Figure 5.1

5.3 Graphs and comparison between the Jump–diffusion and Black–Scholes models

The following graphs are mostly using the data of the indices $S\&P500$, $S\&P600$ and the DJI (Dow Jones Industrial Average). Daily quotations of the indices from January / September 1995 to January 2004 have been used. To estimate the parameters of the jump–diffusion and Black–Scholes models, we have used the method of moments. Let us denote the log return of the stock by X, then from equation (2.2.3) for the Black–Scholes model, we derive the following discretization.

$$\begin{aligned} X_i &= \ln\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\,, \\ &= (\mu - \frac{\sigma^2}{2})\Delta t + \sigma \Delta W_t \end{aligned} \tag{5.3.1}$$

and from equation (3.2.1) for the jump–diffusion model we obtain

$$X_i = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma \Delta W_t + \ln(1-\nu)\Delta \Pi_t\,, \tag{5.3.2}$$

where we denote

$$\Delta t = t_i - t_{i-1}, \quad \Delta W_t = W_{t_i} - W_{t_{i-1}} = Z\sqrt{t_{i-1} - t_i}\,, \quad \text{and } \Delta \Pi_t = \Pi_{t_i} - \Pi_{t_{i-1}} \tag{5.3.3}$$

with $Z \sim N(0,1)$ and, $\Delta \Pi_t \sim B(\lambda t)$ where B is a Bernoulli random variable with intensity λ.

For the jump–diffusion (resp. Black–Scholes) model, matching the 4 (resp. 2) first moments of the random variable X_i to their empirical estimations provided by what follows

$$\frac{1}{n}\sum_{i=1}^{n}(x_i)^j, \qquad x_i = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right), \quad j = 1..4 \quad (\text{resp. } j = 1, 2),$$

leads to the parameters estimates for the Jump–diffusion model in Table 5.1. Similarly the parameters estimates of the Black–Scholes model are provided

Index	μ	σ	ν	λ
$S\&P500$	.109320	.195255	.269618	.154027e-2
$S\&P600$	.897931e-1	.216732	.172591	.833322e-2
OIX	.548259e-1	.236278	.557323	.358894e-4
DJT	.304172e-1	.267501	.284170	.112879e-1
DJI	.691585e-1	.207432	.184409	.130687e-1
TXX	.196158	.414473	-.276029	.508951e-1
RUT	.103595	.201855	.114260	.667166e-1
$Nasdaq$	.197425	.393108	-.631982	.417905e-2

Table 5.1: Jump–diffusion parameters

in Table 5.2.

Let us note that the parameters obtained on the Table 5.1 and Table 5.2 are in general very close. For the Dow Jones Transportation Index, the Black–Scholes model does not give any result. It appears that in our estimations the parameter λ are fairly small. For instance the rate of jumps for the $S\&P500$ in a year is around 0.015. However the jump diffusion option price does not depend on λ in our model, hence this should not affect much our results.

The parameters are all expressed in annual terms since we have chosen ($\Delta t = \frac{1}{261}$). We consider $r^1 = 0.14$ and $r^2 = 0.19$. For such interest rates, the

Index	μ	σ
$S\&P500$	.108912	.195644
$S\&P600$	.883639e-1	.217421
OIX	.548086e-1	.236328
DJT	NA	NA
DJI	.667660e-1	.208737
TXX	.210076	.418105
RUT	.959917e-1	.204274
$Nasdaq$	.199973	.394381

Table 5.2: Black Scholes parameters

parameter λ^* is positive for the pairs ($S\&P500$, $S\&P600$), ($S\&P500$, DJI). Hence the choice of pricing the options on the index $S\&P500$ using as second stock either the $S\&P600$ or the DJI. Note that we could have as well used an option on the $S\&P500$ as the second stock (see Runggaldier [46], Melnikov et al. [39])

We provide next a selection of graphs comparing the jump–diffusion option prices to the Black–Scholes ones and also the upper and lower hedging prices. The graphs shows the following important features.

1) The spread (difference of the Jump–diffusion model to the Black–Scholes one) reaches its maximum when the initial value of the stock is around the strike price for shorter maturities (see Figure 5.7), however this characteristic does not seem to hold for longer maturities (see Figure 5.8). In contrast, Ball and Torous [3], compare the original Merton model to the Black–Scholes and shows that the highest difference occurs for out-of-the-money calls and lowest

for in-the-money calls.

2) Most of the changes on the spread of the options prices with different interest rates occurs when the stock price is close to the strike price value.

Table 1

		$\nu^1=-0.6$	$\nu^1=-0.4$	$\nu^1=-0.2$	$\nu^1=0$	$\nu^1=0.2$	$\nu^1=0.4$	$\nu^1=0.6$	$\nu^1=0.8$	$\nu^1=0.9$	$\nu^1=0.99$
					Separate	Calls					
$K_1=20$	C_+	13.843	13.552	13.525	13.519	13.529	13.581	13.718	13.913	13.999	14.068
	C_-	12.745	12.712	12.706	12.705	12.707	12.718	12.744	12.776	12.790	12.801
	$C_+ - C_-$	1.097	0.840	0.818	0.813	0.821	0.862	0.974	1.136	1.209	1.267
$K_2=30$	C_+	8.220	6.787	6.545	6.490	6.538	6.670	6.839	6.984	7.044	7.092
	C_-	5.982	5.714	5.667	5.657	5.665	5.688	5.715	5.736	5.745	5.752
	$C_+ - C_-$	2.238	1.072	0.877	0.833	0.872	0.982	1.124	1.247	1.298	1.34

Table 2

		$\nu^1=-0.6$	$\nu^1=-0.4$	$\nu^1=-0.2$	$\nu^1=0$	$\nu^1=0.2$	$\nu^1=0.4$	$\nu^1=0.6$	$\nu^1=0.8$	$\nu^1=0.9$	$\nu^1=0.99$
					Spread of Calls						
Ask	$C_+^{K_1} - C_-^{K_2}$	7.861	7.838	7.857	7.862	7.863	7.892	8.003	8.176	8.254	8.316
Bid	$C_-^{K_1} - C_+^{K_2}$	4.524	5.924	6.161	6.215	6.168	6.647	5.905	5.792	5.746	5.708
	$Ask - Bid$	3.336	1.913	1.696	1.647	1.694	1.845	2.098	2.384	2.508	2.607

Figure 5.1: The tables show the behavior of the jump–diffusion option prices as the jump size changes. A comparison with the Black–Scholes model (which corresponds to $\nu^1 = 0$) is provided.

Table 1

		$\nu^1 = 0.64$	$\nu^1 = 0.72$	$\nu^1 = 0.76$	$\nu^1 = 0.8$	$\nu^1 = 0.88$	$\nu^1 = 0.92$	$\nu^1 = 0.96$	$\nu^1 = 0.97$	$\nu^1 = 0.98$	$\nu^1 = 0.99$
					Separate	Calls					
$K_1 = 20$	C_+	14.42	14.55	14.61	14.66	14.74	14.77	14.80	14.80	14.81	14.81
	C_-	14.01	14.19	14.27	14.33	14.43	14.47	14.51	14.52	14.52	14.53
	$C_+ - C_-$	0.40	0.36	0.34	0.32	0.30	0.29	0.28	0.28	0.28	0.28
$K_2 = 30$	C_+	8.07	8.03	8.02	8	7.98	7.97	7.96	7.96	7.96	7.96
	C_-	7.80	7.74	7.72	7.70	7.66	7.65	7.64	7.63	7.63	7.63
	$C_+ - C_-$	0.27	0.29	0.29	0.30	0.31	0.32	0.32	0.32	0.32	0.33

Table 2

		$\nu^1 = 0.64$	$\nu^1 = 0.72$	$\nu^1 = 0.76$	$\nu^1 = 0.8$	$\nu^1 = 0.88$	$\nu^1 = 0.92$	$\nu^1 = 0.96$	$\nu^1 = 0.97$	$\nu^1 = 0.98$	$\nu^1 = 0.99$
					Spread of Calls						
Ask	$C_+^{K_1} - C_-^{K_2}$	6.61	6.81	6.89	6.96	7.07	7.12	7.16	7.17	7.17	7.18
Bid	$C_-^{K_1} - C_+^{K_2}$	5.93	6.15	6.25	6.33	6.43	6.50	6.54	6.55	6.56	6.57
	$Ask - Bid$	0.67	0.65	0.64	0.63	0.62	0.61	0.61	0.61	0.61	0.61

Figure 5.2: The tables show the behavior of the jump–diffusion option prices as the jump size changes.

Table 1

		$\nu^1 = 0.64$	$\nu^1 = 0.72$	$\nu^1 = 0.76$	$\nu^1 = 0.8$	$\nu^1 = 0.88$	$\nu^1 = 0.92$	$\nu^1 = 0.96$	$\nu^1 = 0.97$	$\nu^1 = 0.98$	$\nu^1 = 0.99$
					Separate Calls						
$K_1 = 20$	$\frac{C_+ - C_{BS}}{C_{BS}}$	0.066	0.076	0.081	0.084	0.090	0.092	0.094	0.095	0.095	0.096
	$\frac{C_- - C_{BS}}{C_{BS}}$	0.103	0.117	0.123	0.128	0.136	0.139	0.142	0.142	0.143	0.144
$K_2 = 30$	$\frac{C_+ - C_{BS}}{C_{BS}}$	0.244	0.283	0.235	0.233	0.230	0.228	0.227	0.227	0.227	0.226
	$\frac{C_- - C_{BS}}{C_{BS}}$	0.379	0.368	0.364	0.361	0.355	0.352	0.350	0.349	0.349	0.348

Figure 5.3: The table shows the behavior of the value added by the jump–diffusion option prices as the jump size changes relatively to the Black Scholes model. The largest difference occurs at the money

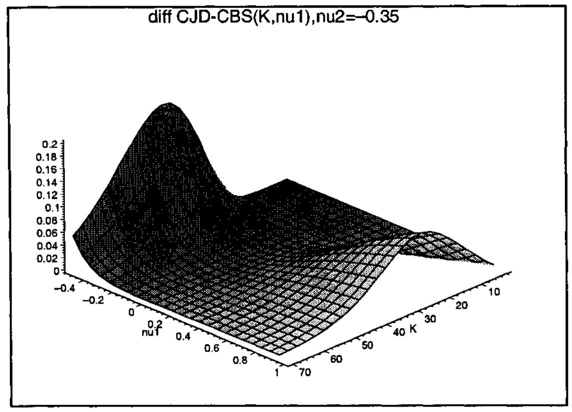

Figure 5.4: The graph provides the difference between the option prices of the jump–diffusion and Black–Scholes models, while using the Bergman's parameters (see Section 5.2). The graph confirms the results found on Figure 5.1. The farther the parameter ν^1 is from zero the larger is the spread between the two models.

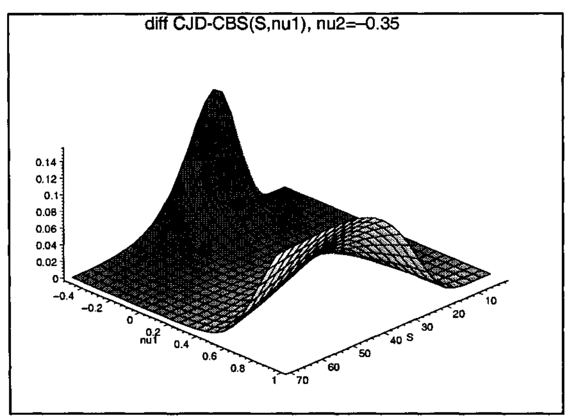

Figure 5.5: This graph shows the difference between the option prices between the jump–diffusion and Black–Scholes models plotted in Figure 5.4. The graph is a function of the maturity and the stock price. The same conclusions as those made in Figure 5.4 and 5.1 are drawn here.

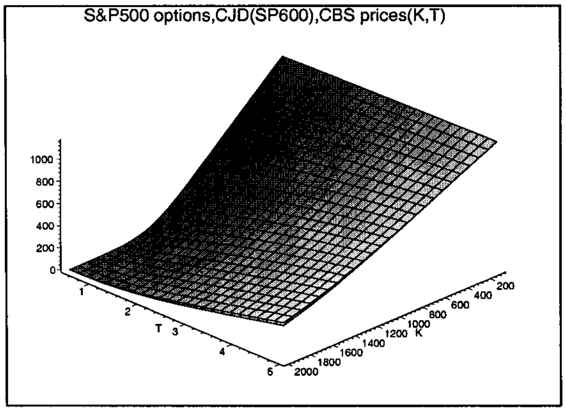

Figure 5.6: Graphs of the option prices derived from the Jump–diffusion and Black–Scholes models for the $S\&P500$. The prices are given as functions of the strike price K and the maturity date T. The parameters of S^1 ($S\&P500$) are $\mu_1 = 0.109$, $\sigma_1 = 0.195$, $\nu_1 = 0.269$. The second asset S^2 is the $S\&P600$, whose parameters are $\mu_2 = 0.089$, $\sigma_2 = 0.216$, $\nu_2 = 0.172$ (see Table 5.1). The quotation of the $S\&P500$ at the closure of the market, the $11th$ of February was \$1205. We considered the interest–rate $r^1 = 0.14$.

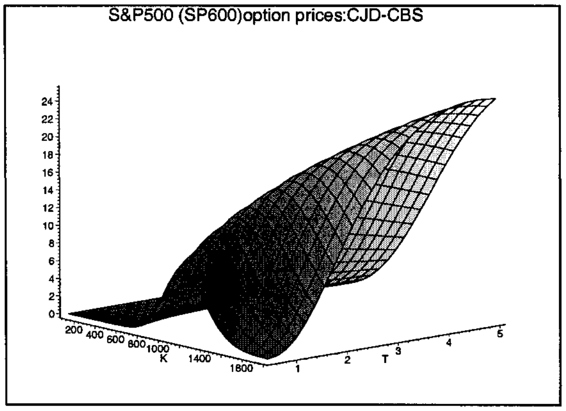

Figure 5.7: The spread between the jump–diffusion model to the Black–Scholes model ($C^{JD} - C^{BS}$) as a function of the strike price K and the maturity T. The initial graphs are provided on Figure 5.6. For shorter maturities the spread reaches its maximum around the initial value of the stock price $S^1 = \$1205$.

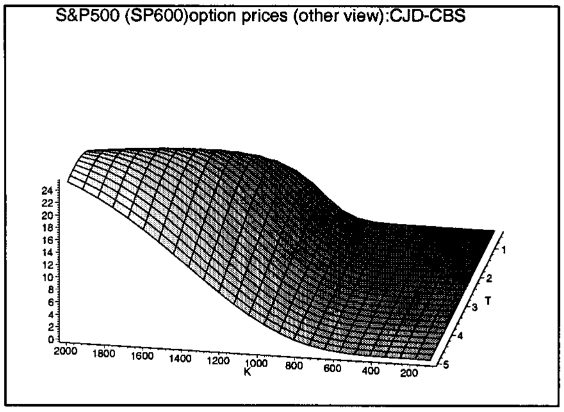

Figure 5.8: The graph shows the difference of the option prices $C^{JD} - C^{BS}$ as a function of the maturity T and strike price K. The view focuses on longer term maturities. The spread is an increasing function of the strike price.

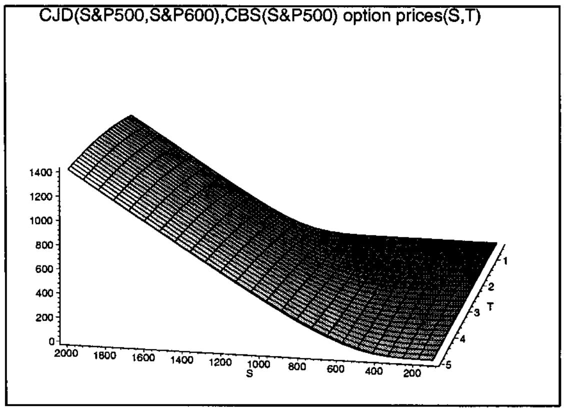

Figure 5.9: These graphs give the jump–diffusion and Black–Scholes option prices on the $S\&P500$. The graph is a function of the maturity and the stock price. The strike price is set equal to \$1200. As in Figure 5.6, the option prices are very close.

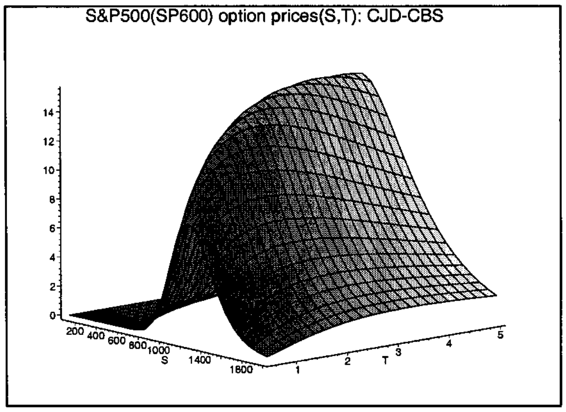

Figure 5.10: This graph provides the difference between the option prices on the $S\&P500$ $(C^{JD} - C^{BS})$ as function of the maturity T and the stock price S^1. The strike price K is set equal to \$1200. The maximum of the spread occurs around the strike price $S^1 = K = \$1200$ and increases with the maturity. The initial graphs are provided in Figure 5.5.

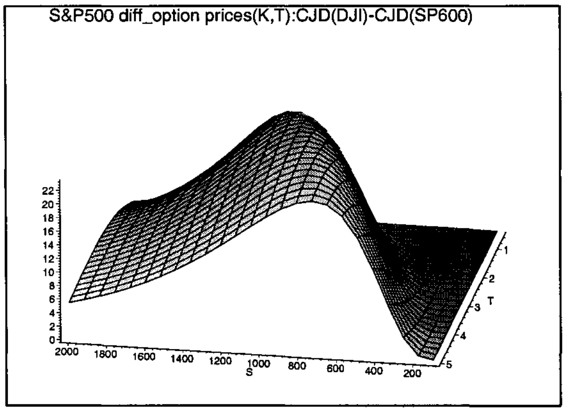

Figure 5.11: The graph shows the difference between the options prices obtained from two jump–diffusion models on the $S\&P500$. One of which uses the DJI as second stock, while the second option uses the $S\&P600$ as second stock: $C^{JD}(S\&P500, DJI) - C^{JD}(S\&P500, S\&P600)$. The graphs are function of (S^1, T). We can compare this graph with Figure 5.10.

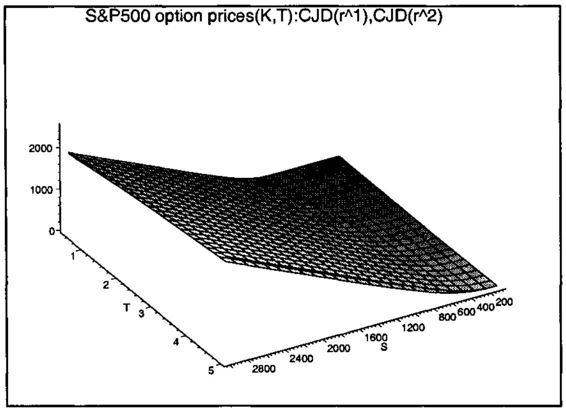

Figure 5.12: Graphs of the option prices on the $S\&P500$ for the jump–diffusion model. The second stock S^2 is the $S\&P600$. The upper graph is the option price for $r^2 = 0.19$ and the lower one is the option price for $r^1 = 0.14$. The parameter K is set equal to \$1200.

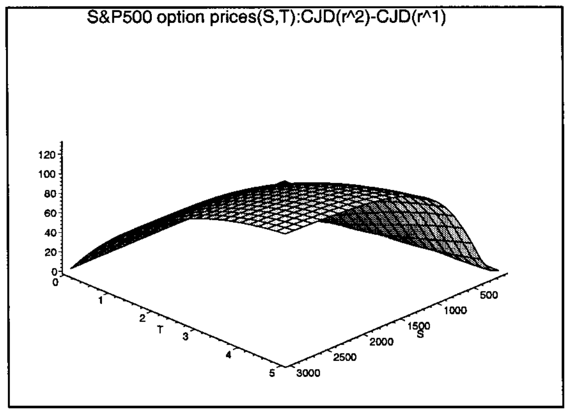

Figure 5.13: Difference between the jump–diffusion option prices $(C^{JD}(r^2) - C^{JD}(r^1))$ as a function of the stock price ($S\&P500$) and the maturity T. The initial graphs are shown in Figure 5.12. The parameters of this model verify the conditions $\nu^1 > 0$, $\frac{\partial \lambda^*}{\partial r} = \frac{\sigma^2 - \sigma^1}{\sigma^2 \nu^1 - \sigma^1 \nu^2} = -0.594 < 0$, hence it was expected to have $C^{JD}(r^2)$ as the upper hedging price and $C^{JD}(r^1)$ a lower hedging price. We note that in some cases when the conditions of Lemma 3.2 was not verified, the values $C(r^2)$ (resp. $C(r^1)$) did not provide the upper (resp. lower) hedging price.

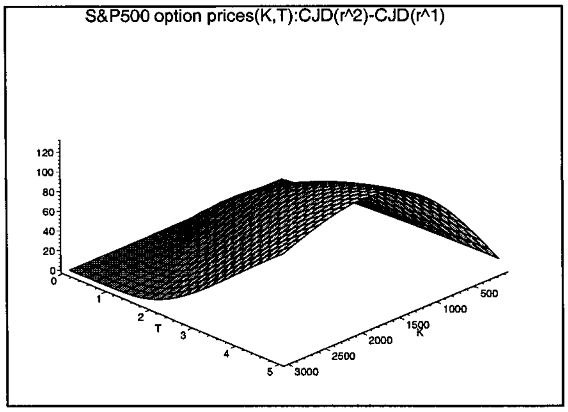

Figure 5.14: This graph provides the spread $(C^{JD}(r^2) - C^{JD}(r^1))$ on the $S\&P500$ as a function of the strike price and maturity. From the parameters of the model, the graph is consistent with the results provided by Lemma 3.2. The maximum of the spread occurs around $K = \$1200$, very close to the initial value of the stock price.

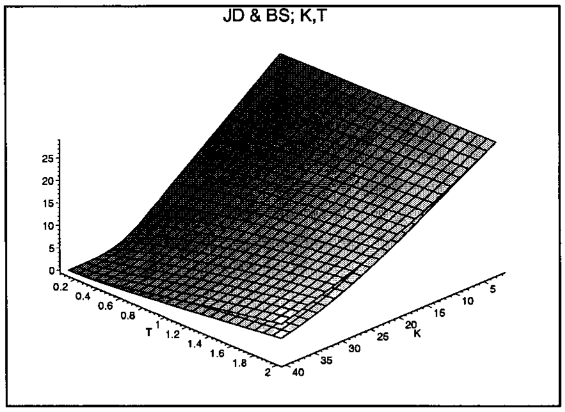

Figure 5.15: These graphs compare the Black-Scholes model to the Jump–diffusion one for the parameters $\mu^1 = 0.25,\ \mu^2 = 0.2,\ \sigma^1 = 0.3, \sigma^2 = 0.5, \nu^1 = 0.7,\ \nu^2 = 0.6$ and $S = \$30$.

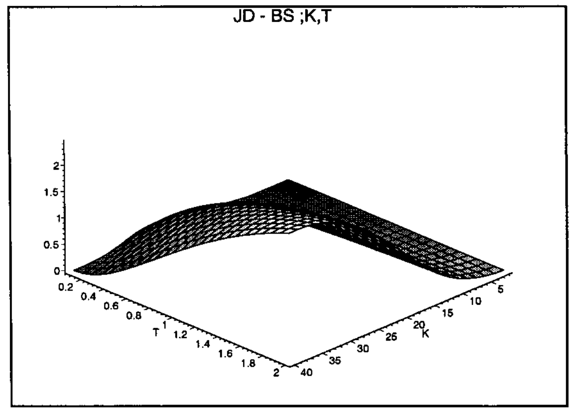

Figure 5.16: The graph provides the difference between the jump–diffusion and the Black–Scholes models for the parameters (of the second example) $\mu^1 = 0.25,\ \mu^2 = 0.2,\ \sigma^1 = 0.3, \sigma^2 = 0.5, \nu^1 = 0.7,\ \nu^2 = 0.6$.

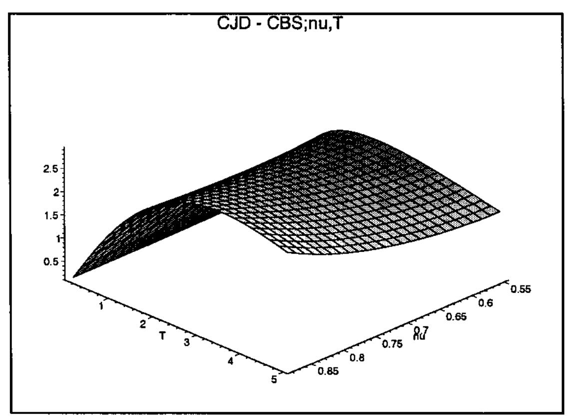

Figure 5.17: The graph gives the difference between the jump–diffusion and the Black-Scholes models. The parameters chosen are $\mu^1 = 0.25,\ \mu^2 = 0.2,\ \sigma^1 = 0.3, \sigma^2 = 0.5,\ \nu^2 = 0.6$. Here $S = K = \$30$. The graph is plotted as a function of the jumps ν^1 and the maturity T.

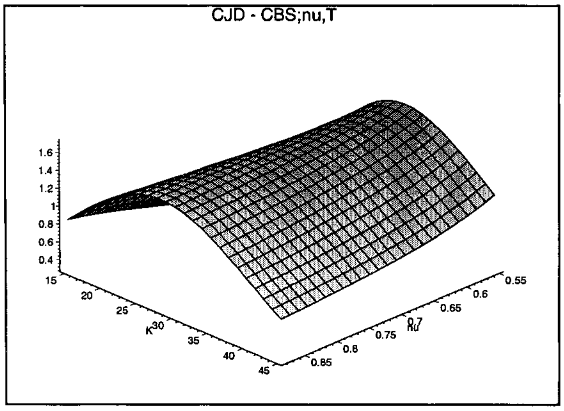

Figure 5.18: The graph gives the difference between the jump–diffusion and the Black-Scholes models. The parameters chosen are $\mu^1 = 0.25,\ \mu^2 = 0.2,\ \sigma^1 = 0.3, \sigma^2 = 0.5,\ \nu^2 = 0.6$. The graph is plotted as a function of the jump ν^1 and the exercise price K. The value of the stock is $S = \$30$. Most of the difference occurs at-the-money.

Bibliography

[1] Aase, K. K. Contingent claim valuation when the security price is a combination of an Ito process and a random point process. Stoch. Process. Appl. **1988**, *28*, 185–220.

[2] Aase, K. K.; Persson, S. Pricing of unit linked insurance policies Scandinavian Actuarial Journal **1994** 1: 26–52

[3] Ball C.A.; Torous, W.N. On jumps in common stock prices and their impact on call option pricing. Journal of Finance 40 **1985***1* 154–173.

[4] Bardhan, J.; Chao, X. Pricing options on securities with discontinuous returns. Stoch. Process. Appl. **1993**, *48*, 123–137.

[5] Barron E.N., Jensen R. A stochastic control approach to the pricing of options. Mathematics of Operation Research **1990**, *15*, 49–79.

[6] Bart, Y. Option hedging in the binomial model with differing interest rates. Uspekhi Math. Nauk 53 **1998**, no 5, 227–228; English trans., Russian Math. Surveys **1998**, 53, 1084–1085.

[7] Bergman, Y. Option pricing with different interest rates for borrowing and for lending. Working Paper University of California Berlekey **1981**, *109.*

[8] Bertoin, J.; Levy processes. Cambridge tracts in mathematics *1996* 12 Cambridge University Press, Cambridge.

[9] Bjork, T.; Kabanov, Y.; Runggaldier W. Bond Market structure in the presence of marked Poisson Processes, Math. Finance, **1997**, *7*, 211–239.

[10] Bjork, T.; Naslund, B. Diversified Portfolios in continuous time European Financial Review , **1998** *1*, 361–387.

[11] Boyle, P.; Schwartz E. Equilibrium prices of guarantees under equity linked contracts. Journal of Risk and Insurance *44* **1977**, 639–680.

[12] Bowers, N. L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.I. 1997. Actuarial Mathematics. Society of Actuaries, Schaumburg, Illinois, **1994**.

[13] Brennan, M. J., Schwartz, E.S. The pricing of equity–linked life insurance policies with an asset value Guarantee. Journal of Financial Economics **1976**, *3*: 195–213.

[14] Brennan, M.J.; Schwartz, E.S. Alternative investment strategies for the issuers of equity–linked life insurance with an asset value guarantee. J. of Financial Economics *3* 195–213 **1979**.

[15] Colwell, D.; Elliott, R. Discontinuous asset prices and non–attainable contingent claims and corporate policy. Math. Finance **1993**, *3*, 295–318.

[16] Cvitanic, J. Theory of portfolio optimization in markets with frictions; In *Handbooks in Math. Finance: Option Pricing, Interest Rates and Risk Management*; E. Jouini and M. Musiela, Eds.; Cambridge University Press: 2001.

[17] Cvitanic, J. Optimal trading under constraints. In *Lectures Notes in Mathematics*. Springer–Verlag: Berlin, 1997; 1656, 123–190.

[18] Cvitanic, J.; Karatzas, I. Convex duality in constrained portfolio optimization. The Annals of Applied Probability **1992**, *(2)*, 767–818.

[19] Cvitanic, J.; Karatzas, I. Hedging contingent claims with constrained portfolio. The Annals of Applied Probability **1993**, *3(3)*, 652–681.

[20] Cvitanic, J.; Pham, H.; Touzi, N. Super–replication in Stochastic Volatility Models under Portfolio Constraints. J. of Appl. Probability, **1999**, *36*, 523–545.

[21] Corcuera, J. M.; Nualart, D.; and Schoutens W.: Completion of a Lévy Market by Power–Jump Assets, *preprint.*

[22] Corcuera, J. M.; Guerra, J.; Nualart,D.; Schoutens W. : Optimal investment in a Lévy Market, *preprint.*

[23] Delbaen, F. Equity linked policies, Bulletin de l'association Royale des Actuaires Belges,**1986**, 33–52.

[24] Elliott R.; Kopp P. E. *Mathematics of financial markets*; Springer–Verlag: Berlin, 1998.

[25] Föllmer, H.; Kramkov, D. O. Optional decompositions under constraints. Probability Theory and Related Fields **1997**, *109*, 1–25.

[26] Föllmer, H.; Leukert, P. Quantile hedging , Finance and Stochastics, **1999**, 3, 251–273.

[27] Föllmer, H.; Leukert, P. Efficient hedging: Cost versus Shortfall Risk, Finance and Stochastics, **2000**, 3, 251–273.

[28] Honore, P. Pitfalls in estimating jump–diffusion models. Working paper CAF, **1998**, *18*.

[29] Kane, S.; Melnikov, A. On pricing contingent claims on a two interest rates financial market via completions, *Submitted to the Journal of Stochastic Analysis and Applications.*

[30] Karatzas, I.; Shreve, S. *Methods of mathematical finance*; Springer–Verlag: New York, 1998.

[31] Korn, R. Contingent claim valuation in a market with different interest rates. Mathematical Methods of Operations Research **1995**, *42*, 255–274.

[32] Kou, S. G. A jump–diffusion model for option pricing. Management Science, **2002**, *48*, 1086–1101.

[33] Krutchenko, R.; Melnikov, A. V. Quantile hedging for a jump-diffusion financial market. Trends in Mathematics, Ed M. Kohlmann, Birkhäuser Verlag Basel /Switzerland, **2001**, 215–229.

[34] Merton, R.C. Option pricing when underlying stock returns are discontinuous. Journal of financial economics, **1976**.

[35] Merton, R.C. Continuous time finance. Blackwell, Oxford **1990**.

[36] Melnikov, A. V. Risk analysis in finance and insurance Chapman & Hall/CRC **2003**.

[37] Melnikov, A. V. Quantile hedging of equity linked life insurance policies Doklady Mathematics (Proceedings of the Russian Acad. of Sciences) **2004**.

[38] Melnikov, A. V. Efficient hedging of equity linked life insurance policies Doklady Mathematics (Proceedings of the Russian Acad. of Sciences) **2004**.

[39] Melnikov, A. V.; Nechaev, M.; Volkov, S. *Mathematics of financial obligations*; AMS Providence 2002; 194 pp.

[40] Melnikov, A. V. Financial markets: Stochastic analysis and the pricing of derivative securities, TVP, Moscow 1997; English transl., Amer. Math. Soc. Providence, RI, 1999.

[41] Mercurio, F.; W. Runggaldier, Option pricing for jump–diffusions: approximations and their interpretation. Math. Finance, **1993**, *3*, 191–200.

[42] Merton, R. C. *Continuous–time finance*; Basil–Blackwell: Oxford, 1990.

[43] Moller, T. Risk minimizing hedging strategies for unit linked life insurance contracts. Astin Bulletin **1998** 28: 17–47.

[44] Moller, T. Hedging equity linked life insurance contracts North American Actuarial Journal **2002**.

[45] Nakano, Y. ; Minimization of Shortfall risk in a Jump–diffusion model. Statistics & Probability Letters **2004**,*67, 87–95.*

[46] Runggaldier, W. J.; Jump–Diffusion Models, Handbook of heavy tailed Distributions in Finance, Edited by S.T. Rachev, Elsvier Science B.V. **2003**, *5, 170–207.*

[47] Soner, H.; Touzi, N. Super–replication under gamma constraints. Journal on Control and Optimization **2000**, *39*, 73–96.

[48] Volkov S.N, Kramkov, D.O. On the methodology of hedging options, Obzrenie Prikl. i prom. Mathem. 4:1 **1997** 18–65.

Chapter 6

Appendix

In this chapter we give some properties on stochastic processes that are relevant to the previous chapters of the thesis

6.1 Some properties about Poisson processes

Definition 6.1. A Poisson process X_t with intensity $\lambda > 0$ is defined as follows:

1) The initial value of the process $X_0 = 0$ a.s.;

2) The increment $X_t - X_s$ is independant of $\mathcal{F}_s \quad \forall\, s \leq t$;

3) Further $X_t - X_s$ has a Poisson distribution with parameter $\lambda(t-s)$.

Statement 6.2. If X_t is a Poisson process in $(\Omega, F, P, \mathcal{F}_t)$ with intensity λ, then

1) The process $X_t - \lambda t$ is a P–martingale.

2) The process $X_t - \lambda^* t$ is a P^*–martingale where λ^* verifies (3.2.5) and P^* is defined as in Statement 3.2.

6.2 Preliminaries in stochastic exponential

Statement 6.3. A process M is said local martingale if there is a sequence of stopping times $\tau_n \uparrow \infty$ as $n \to \infty$ such that $M^{\tau^n} := M_{t\tau_n}$ is a martingale for all n

Statement 6.4. A semi–martingale is a process of the form

$$X_t = X_0 + A_t + M_t \,,$$

where $A \in \mathcal{V}$, $M \in \mathcal{M}_{loc}$ and X_0 is $\mathcal{F}_0$–measurable.

The set of finite variation in each finite interval is denoted by $\mathcal{V}$. The set $\mathcal{M}_{loc}$ represents the set of local martingales.

Statement 6.5. Let X_t and Y_t be semi–martingales, then

a) the unique solution of the equation

$$Y_t = 1 + \int_0^t Y_{s-} dX_s \qquad \left(\frac{dY_t}{Y_{t-}} = dX_s \right) \tag{6.2.1}$$

is given by the Dolean exponent (stochastic exponent)

$$Y_t = Y_0 \mathcal{E}_t(X) = Y_0 e^{(X_t - X_0 - \frac{1}{2}\langle X^c_\cdot, X^c_\cdot \rangle_t)} \Pi_{\{s \le t\}} (1 + \Delta X_s) \exp\left(-\Delta X_s\right) \tag{6.2.2}$$

where $\langle X^c, X^c \rangle$ represents the quadratic characteristic of a continuous martingale part of $X :\ X^c$

b) the stochastic multiplication (Yor's formula)

$$\mathcal{E}_t(X_\cdot) * \mathcal{E}_t(Y_\cdot) = \mathcal{E}_t(X_\cdot + Y_\cdot + [X, Y]) \,, \tag{6.2.3}$$

where $[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum \Delta X_s \Delta Y_s$.

Multivariate Kolmogorov–Ito for a jump–diffusion Process (Dolean–Meyer).

Statement 6.6. Let $f : R^n \to R^1$ be a twice differentiable function and X_t be a semimartingale with values in R^n, then

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \sum_{i=1}^{i=n} \frac{\partial f(X_{s-})}{\partial x_i} dX_s^i + \frac{1}{2}\int_0^t \sum_{i=1}^{i=n} \frac{\partial^2 f(X_s^i)}{\partial x_i \partial x_j} d\left\langle (X_\cdot^i)^c, (X_\cdot^j)^c \right\rangle_s \\ &+ \sum_{s\le t} \Big[f(X_s) - f(X_{s-}) - \sum_{i=1}^{n} \frac{\partial f(X_{s-})}{\partial x_i}(X_s^i - X_{s-}^i) \Big]. \end{aligned} \tag{6.2.4}$$

6.3 Dynamics of the wealth and debt processes

The derivation of the dynamic of the wealth process generated by the portfolio π with initial capital x in the (B, S)-market

$$X_t = \beta_t B_t + \gamma_t S_t \tag{6.3.1}$$

The self financing property (2.2) yields to

$$\begin{aligned} dX_t &= \beta_t dB_t + \gamma_t dS_t\,, \\ \frac{dX_t}{X_{t-}} &= \frac{\beta_t}{X_{t-}} dB_t + \frac{\gamma_t S_{t-}}{X_{t-}} \frac{dS_t}{S_{t-}} \end{aligned} \tag{6.3.2}$$

Writing $\alpha_t = \frac{\gamma_t S_{t-}}{X_{t-}}$ and substituing into the relation

$$1 = \beta_t \frac{B_t}{X_{t-}} + \gamma_t \frac{S_{t-}}{X_{t-}}\,,$$

yield to $\beta_t = \frac{1-\alpha_t}{B_t}$, consequently

$$\frac{dX_t}{X_{t-}} = (1-\alpha_t) r dt + \alpha_t \frac{dS_t}{S_{t-}}.$$

We identically show that

$$\frac{dY_t}{Y_{t-}} = (1-\alpha_t) r dt + \alpha_t \frac{dS_t}{S_{t-}}.$$

Hence (2.3.3) holds.

Derivation of the stochastic differential equations of the buyer and seller in the two interest rates (B, S) financial market. We have from the self financing property

$$\begin{aligned} dX_t &= \beta_t^1 dB_t^1 + \beta_t^2 dB_t^2 + \gamma_t dS_t \qquad \text{where } \beta^1 > 0,\ \beta^2 < 0 \\ \frac{dX_t}{X_{t-}} &= \frac{\beta_t^1 dB_t^1}{X_{t-}} + \frac{\beta_t^2 dB_t^2}{X_{t-}} + \frac{\gamma_t dS_t}{X_{t-}} \end{aligned}$$

Using the previous notation of α_t, the signs of β^1 and β^2 we derive

$$\frac{dX_t}{X_{t-}} = (1-\alpha_t)^+ r^1 dt - (1-\alpha_t)^- r^2 dt + \alpha_t \frac{dS_t}{S_{t-}}$$

identically taking into account that the process Y_t is negative, the dynamic of a seller in the two interest rates market follows

$$\frac{dY_t}{Y_{t-}} = (1-\alpha_t)^+ r^2 dt - (1-\alpha_t)^- r^1 dt + \alpha_t \frac{dS_t}{S_{t-}}.$$

We now turn to the derivation of the dynamic of the wealth and debt processes in the (B^1, B^2, S^1, S^2)–market. Let us begin with the wealth process, $X_t \geq 0$ and from the self financing condition

$$\begin{aligned} dX_t &= \beta_t^1 dB_t^1 + \beta_t^2 dB_t^2 + \gamma_t^1 dS_t^1 + \gamma_t^2 dS_t^2 \qquad \text{where } \beta^1 > 0,\ \beta^2 < 0 \\ \frac{dX_t}{X_{t-}} &= \frac{\beta_t^1 B_t^1}{X_{t-}} \frac{dB_t^1}{B_t^1} + \frac{\beta_t^2 B_t^2}{X_{t-}} \frac{dB_t^2}{B_t^2} + \frac{\gamma_t^1 S_{t-}^1}{X_{t-}} \frac{dS_t^1}{S_{t-}^1} + \frac{\gamma_t^2 S_{t-}^2}{X_{t-}} \frac{dS_t^2}{S_{t-}^2} \end{aligned}$$

Denote $\alpha^1 = \frac{\gamma_t^1 S_{t-}^1}{X_{t-}}$ and $\alpha^2 = \frac{\gamma_t^2 S_{t-}^2}{X_{t-}}$, then we obtain,

$$\frac{dX_t}{X_{t-}} = (1-\alpha_t^1-\alpha_t^2)^+ r^1 dt - (1-\alpha_t^1-\alpha_t^2)^- r^2 dt + \alpha_t^1 \frac{dS_t^1}{S_{t-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t-}^2}.$$

An identical procedure yields to the stochastic differential equation of the seller

.

$$\frac{dY_t}{Y_{t-}} = (1-\alpha_t^1-\alpha_t^2)^+ r^2 dt - (1-\alpha_t^1-\alpha_t^2)^- r^1 dt + \alpha_t^1 \frac{dS_t^1}{S_{t-}^1} + \alpha_t^2 \frac{dS_t^2}{S_{t-}^2}.$$