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$\mathcal{H}_\infty$ Performance of Mechanical and Power Networks

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Abstract: This paper investigates the robustness of two well-known applications of second-order consensus dynamics, namely mechanical and power networks. For uniform subsystem parameters, we derive expressions for the $\mathcal{H}_\infty$ norms of mechanical and power networks, from external disturbances to body displacements and to generator phase angles, respectively. The closed-form expressions are in terms of the physical parameters (damping coefficients and inertias) of the dynamics and in terms of the spectrum of the grounded Laplacian matrix associated with the network. We then analyze the dependence of the $\mathcal{H}_\infty$ norm of each network on both the network structure and the physical parameters. For a fixed network topology, we find that each system norm can be minimized by choosing the damping coefficient within a specified range. Theoretical contributions are verified via two illustrative examples for mechanical and power networks, in which we show that the network structure, number of the reference nodes and their location in the network can have considerable effects on the system $\mathcal{H}_\infty$ norm.

Keywords: Network robustness, Mechanical networks, Power networks, Second order consensus, Grounded Laplacian matrix.

1. INTRODUCTION

Connecting system-theoretic notions to graph theory has attracted much attention in the past decade due to growing interest in networked control systems. Algebraic and spectral graph theory have become key tools in the study of systems on graphs, and have been used to study stability Olfati-Saber et al. (2007), robustness Bamieh et al. (2012); Patterson and Bamieh (2010); Pirani et al. (2017a), controllability Olshevsky (2014); Pequito et al. (2016); Rahmani et al. (2009) and observability Sundaram and Hadjicostis (2011). A foundational system-theoretic concept is robustness to uncertainties or external disturbances. Among the robustness metrics proposed in systems and control theory, the $\mathcal{H}_\infty$ norm plays a substantial role in robust control synthesis and uncertainty modelling. Recently, extensive research has been done characterizing the $\mathcal{H}_2$ norm in terms of graph-theoretic notions Bamieh et al. (2012); Fitch and Leonard (2013); Patterson et al. (2015). Graph-theoretic characterizations of the $\mathcal{H}_\infty$ norm have received less attention however. In this paper we investigate some graph-theoretic approaches to the $\mathcal{H}_\infty$ norm of mechanical and power networks, which are two well-known examples of second-order consensus dynamics.

Second-order consensus dynamics have been heavily investigated in recent years Ren and R. Beard (2007), due to their diverse applications. Stability, performance and robustness of such systems have also been analyzed Lin et al. (2008a,b); Ren and Beard (2008). However, in order to investigate the dependence of system robustness on the physical model parameters, it is desirable to understand the precise relationship between the robustness metric and those parameters. The model of mechanical networks presented in this paper has diverse applications Acosta et al. (2005); Sandberg and Murray (2008, 2009). One of these applications is cooperative adaptive cruise control in vehicle platooning problems, in which vehicles are controlled to track reference inter-vehicular distances as well as a desired velocity, targeting safety issues and fuel consumption considerations Barooah et al. (2009); Hao and Barooah (2013); Jovanovic and Bamieh (2005); Peters et al. (2014); Sebastian et al. (2015). The other example of a second-order consensus systems considered in this paper is a network of power generators. During past years, there have been great efforts on revisiting power network dynamics from control-theoretic and cyber-physical perspectives Dorfler and Bullo (2010); Schiffer et al. (2014); Simpson-Porco et al. (2013). In this direction, some research has been done to analyze the effect of the network structure as well as physical control parameters on certain robustness and performance metrics Poolla et al. (2016); Tegling et al. (2015a,b); Teixeira et al. (2015).

The contributions of this paper are as follows:

- For homogeneous damping and mass/inertia constants, we derive closed-form expressions for the $\mathcal{H}_\infty$ norms of mechanical and power networks. The norms are in terms of physical parameters of the subsystems and in terms of the spectrum of the grounded Laplacian matrix.
- We investigate the dependence of these $\mathcal{H}_\infty$ norms on physical parameters (in particular, on the homoge-

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neous damping coefficient c and mass/inertia m), and on the spectrum of the grounded Laplacian matrix. We demonstrate this analysis via two examples, one for each type of network.

The paper is organized as follows. In Section 2 some notations and definitions used in this paper are introduced. Section 3 presents the models for mechanical and power networks, and characterizes the spectral properties of the associated dynamic matrices in terms of the grounded Laplacian matrix. In Section 4, we derive expressions for the $\mathcal{H}_\infty$ norm for mechanical and power networks and describe its dependence on the physical parameters of the system. The effect of network structure on the $\mathcal{H}_\infty$ norm for mechanical and power networks is discussed in Section 5. Section 6 concludes and proposes future research.

2. NOTATIONS AND DEFINITIONS

We denote an undirected graph (network) by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is a set of nodes (or vertices) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Neighbors of node $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The unweighted adjacency matrix of the graph is given by a symmetric and binary $n \times n$ matrix A, where element $A_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and zero otherwise. The degree of node v_i is denoted by $d_i \triangleq \sum_{j=1}^{n} A_{ij}$. For a given set of nodes $X \subset \mathcal{V}$, the *edge-boundary* (or just boundary) of the set is given by $\partial X \triangleq \{(v_i, v_j) \in \mathcal{E} \mid v_i \in X, v_j \in \mathcal{V} \setminus X\}$. The Laplacian matrix of the graph is given by $L \triangleq D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$. For a given subset $\mathcal{S} \subset \mathcal{V}$ of nodes (which we term *grounded nodes*), the *grounded Laplacian* induced by $\mathcal{S}$ is denoted by $L_{g\mathcal{S}}$ or simply L_g, and is obtained by removing the rows and columns of L corresponding to the nodes in $\mathcal{S}$. For the case where the underlying network is connected and there exists at least one grounded node, the grounded Laplacian matrix L_g is a positive-definite matrix Pirani and Sundaram (2016). For a given set $\mathcal{I}$, the number of members (cardinality) of the set is denoted by $|\mathcal{I}|$.

3. NETWORK MODELS

In this section, we discuss two applications of second-order consensus dynamics: mechanical mass-spring-damper networks, and networks of power generators.

3.1 Mechanical Network

Consider a simple system comprised of two bodies connected to each other via a spring and a viscous damper, as shown in Fig. 1(a). Suppose that the position of one of the bodies (called the reference or leader) is fixed [1] and the other acts as the follower. This is the simplest form of a network control system of masses, springs and dampers which will be addressed in this paper. The dynamics of the reference body is $\dot{x}_1(t) = 0$. From Newton's law, the dynamics of the follower body are

$$m_2 \ddot{x}_2 = k\,(x_1 - x_2) + c\,(\dot{x}_1 - \dot{x}_2) + w_2(t), \qquad (1)$$

[1] We can easily extend this to the case where the reference body tracks an input signal, i.e. $\ddot{x}_1(t) = u(t)$.

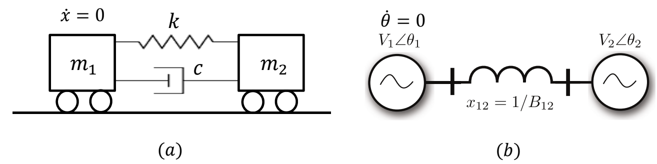

Fig. 1. A simple schematic figure of (a) mechanical interactions with grounded body m_1, (b) Connected generators with resistance x_{12} and slack generator with $\dot{\theta} = 0$.

where m, c, k are mass, damping and stiffness constants, respectively. The follower is subject to an external forcing disturbance, which we denote by $w_2(t)$.

Now we extend the above system to a network of connected masses with arbitrary topology. Consider a connected network of n masses $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Each mass $v_i \in \mathcal{V}$ is either a follower $v_i \in \mathcal{F}$ or a reference mass $v_i \in \mathcal{R}$. The position and the velocity of each mass v_i is denoted by scalars x_i and $\dot{x}_i(t)$, respectively. The equation of motion of each follower is

$$m_i \ddot{x}_i = \sum_{j \in \mathcal{N}_i} k_{ij}\,(x_j - x_i) + c_{ij}\,(\dot{x}_j - \dot{x}_i) + w_i(t), \quad v_i \in \mathcal{F}, \qquad (2)$$

where $k_{ij}, c_{ij} > 0$ are spring stiffness and damping constant between m_i and m_j and $w_i(t)$ models the disturbance and uncertainties in dynamics of the i-th body. The dynamics of the reference body is

$$\dot{x}_i = 0, \quad v_i \in \mathcal{R}. \qquad (3)$$

Here we consider homogeneous mass, stiffness and damping constants, i.e. $m_i = m$, $k_i = k$ and $c_i = c$, $\forall i = 1, 2, ..., n$. The objective of this study is to investigate the robustness of the networked control system to external disturbances. Thus, we only consider the dynamics of the followers, since they are affected by the disturbances. In vector notation, the dynamics of the followers are

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{|\mathcal{F}|} & I_{|\mathcal{F}|} \\ -\frac{k}{m} L_g & -\frac{c}{m} L_g \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{|\mathcal{F}|} \\ \frac{1}{m} I_{|\mathcal{F}|} \end{bmatrix} \mathbf{w}(t),$$
$$\mathbf{y} = \mathbf{x}, \qquad (4)$$

where L_g is the grounded Laplacian matrix formed by removing rows and columns corresponding to the reference bodies and $\mathbf{y}(t)$ is the output of interest, in this case the positions of the follower bodies.

3.2 Power Network

Consider a power network of n buses $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ and a set of edges $\mathcal{E}$, as shown in Fig. 1(b). Here we assume a Kron-reduced transmission network model, where all buses are modeled as generators and branch resistances are neglected. At each node $i = 1, 2, ..., n$, there is a generator with inertia m_i, damping c_i, and voltage phase angle θ_i. The dynamics of the i-th generator is described by the *swing equation*

$$m_i \ddot{\theta}_i + c_i \dot{\theta}_i = P_{m,i} - P_{e,i} + w_i(t), \qquad (5)$$

where $P_{m,i}$ is the constant mechanical power input from turbine and $w_i(t)$ models disturbances arising from changing generation or changing local load. The term $P_{e,i}$ is the real electrical power injected from i-th generator to the network, and is given by

$$P_{e,i} = \sum_{j\in\mathcal{N}_i} V_i V_j B_{ij} \sin(\theta_i - \theta_j), \tag{6}$$

where V_i is the nodal voltage magnitude and $-B_{ij} < 0$ is the susceptance associated with edge $(v_i, v_j) \in \mathcal{E}$. We further approximate this using the so-called DC Power Flow, where $V_i \simeq V_j \simeq 1$ and $|\theta_i - \theta_j| \ll 1$, leading to the linear model

$$P_{e,i} \approx \sum_{j\in\mathcal{N}_i} B_{ij}(\theta_i - \theta_j). \tag{7}$$

Substituting (7) into (5) yields the LTI system

$$m_i\ddot{\theta}_i + c_i\dot{\theta}_i \approx -\sum_{j\in\mathcal{N}_i} B_{ij}(\theta_i - \theta_j) + P_{m,i} + w_i(t), \tag{8}$$

We consider a reference (slack) generator, $i \in \mathcal{R}$, which has a constant phase

$$\dot{\theta}_i = 0, \quad v_i \in \mathcal{R}. \tag{9}$$

Similar to the case of mechanical network, here we assume that $m_i = m$, $B_{ij} = b$ and $c_i = c$ for all $i = 1, 2, ..., |\mathcal{F}|$ Tegling et al. (2015b). After shifting the equilibrium point of (8) to the origin, the term proportional to $P_{m,i}$ may be removed and the dynamics of the followers can be written in vector form as

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{|\mathcal{F}|} & I_{|\mathcal{F}|}\\ -\frac{b}{m}L_g & -\frac{c}{m}I_{|\mathcal{F}|}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{|\mathcal{F}|}\\ \frac{1}{m}I_{|\mathcal{F}|}\end{bmatrix}\mathbf{w}(t),$$
$$\mathbf{y} = \theta, \tag{10}$$

where we have taken $\mathbf{y} = \theta$ as the output of interest; this is the phase difference between each bus angle and the (zero) [2] angle of the reference bus. Other kinds of performance outputs (including frequency) are considered in Pirani et al. (2017b).

In Section 4 we derive expressions for the $\mathcal{H}_\infty$ norms of the systems (4) and (10), respectively. Since the smallest eigenvalue of L_g plays a crucial role in the robustness of (4) and (10) to external disturbances, the following theorem provides some graph-theoretic bounds on λ_1. [3]

Theorem 1. (Pirani and Sundaram (2016)). Consider a connected network $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with a set of reference nodes $\mathcal{R} \subset \mathcal{V}$. Let L_g be the grounded Laplacian matrix for $\mathcal{G}$. Let $\beta_i = |\mathcal{N}_i \cap \mathcal{R}|$ be the number of reference nodes in follower v_i's neighborhood. Then

$$\max\left\{\frac{|\partial\mathcal{R}|u_{\min}}{|\mathcal{F}|}, \min_{i\in\mathcal{F}}\{\beta_i\}\right\} \leq \lambda_1 \leq \frac{|\partial\mathcal{R}|}{|\mathcal{F}|} \leq \max_{i\in\mathcal{F}}\{\beta_i\} \leq |\mathcal{R}|, \tag{11}$$

where $u_{\min}$ is the smallest component of the eigenvector $\mathbf{u}$ corresponding to λ_1. Here we normalized the elements of $\mathbf{x}$ such that $||\mathbf{u}||_\infty = 1$.

4. $\mathcal{H}_\infty$ PERFORMANCE OF MECHANICAL AND POWER NETWORKS

We now present closed-form expressions for the $\mathcal{H}_\infty$ system norm of mechanical and power networks, from external disturbances to the mass displacements and the phase angles, respectively. We then discuss how each system norm depends on the physical parameters of the subsystems.

Theorem 2. Consider the mechanical network described by (4) and the power network described by (10).

(i) **Mechanical Network:** For the mechanical network (4), the $\mathcal{H}_\infty$ norm from disturbances to position errors is

$$||G||_\infty^{\text{Mech}} = \begin{cases}\dfrac{2m}{c\lambda_1^{\frac{3}{2}}\sqrt{4mk - c^2\lambda_1}}, & \text{if } \dfrac{\lambda_1 c^2}{2km} \leq 1,\\ \dfrac{1}{k\lambda_1} & \text{otherwise.}\end{cases} \tag{12}$$

(ii) **Power Network:** For the power network (10), the $\mathcal{H}_\infty$ norm from disturbances to phase angle errors is

$$||G||_\infty^{\text{Power}} = \begin{cases}\dfrac{2m}{c\sqrt{4bm\lambda_1 - c^2}}, & \text{if } \dfrac{c^2}{2bm\lambda_1} \leq 1,\\ \dfrac{1}{b\lambda_1} & \text{otherwise.}\end{cases} \tag{13}$$

Proof. We prove the result for the mechanical network; the proof for the power network is similar. Computing the transfer function of (4) gives

$$G(s) = \left(ms^2 I + (cs+k)L_g\right)^{-1}$$
$$= U\underbrace{\left(ms^2 I + (cs+k)\Lambda\right)^{-1}}_{\text{diag}(G_i(s))}U^T, \tag{14}$$

where $U = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{|\mathcal{F}|}]$ is a matrix formed by eigenvectors of L_g and $\text{diag}(G_i(s))$ is a diagonal matrix with diagonal elements $G_i(s) = \frac{1}{ms^2 + c\lambda_i s + k\lambda_i}$. Then we have

$$|G_i(j\omega)|^2 = \frac{1}{\underbrace{(m\omega^2 - k\lambda_i)^2 + c^2\lambda_i^2\omega^2}_{f(\omega)}}. \tag{15}$$

Maximizing $|G_i(j\omega)|^2$ with respect to ω is equivalent to minimizing $f(\omega)$. By setting $\frac{df(\omega)}{d\omega} = 0$, we get $\bar{\omega}_1 = 0$ and $\bar{\omega}_2 = (\frac{k\lambda_i}{m} - \frac{c^2\lambda_i^2}{2m^2})^{\frac{1}{2}}$ as critical points. Here $\bar{\omega}_2$ is the global minimizer of $f(\omega)$, unless $\frac{\lambda_i c^2}{2km} > 1$. Hence, we have

$$\bar{\omega} = \begin{cases}\left(\dfrac{k\lambda_i}{m} - \dfrac{c^2\lambda_i^2}{2m^2}\right)^{\frac{1}{2}}, & \text{if } \dfrac{\lambda_i c^2}{2km} \leq 1,\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

substituting the values in (16) into $|G_i(j\omega)|$ yields

$$\sup_\omega |G_i(j\omega)| = \begin{cases}\dfrac{2m}{c\lambda_i^{\frac{3}{2}}\sqrt{4mk - c^2\lambda_i}}, & \text{if } \dfrac{\lambda_i c^2}{2km} \leq 1,\\ \dfrac{1}{k\lambda_i} & \text{otherwise.}\end{cases} \tag{17}$$

Both terms in (17) are monotonic decreasing functions of λ_i, which is clear from the second term, $\frac{1}{k\lambda_i}$. For the first term, by differentiating with respect to λ_i we find that $\arg\min_{\lambda_i}\left(\frac{2m}{c\lambda_i^{\frac{3}{2}}\sqrt{4mk - c^2\lambda_i}}\right) = \frac{3km}{c^2}$, which is out of the range of $\lambda_i \leq \frac{2km}{c^2}$. Hence, for system $\mathcal{H}_\infty$ norm we have

$$||G||_\infty^{\text{Mech}} = \sup_\omega \max_i ||G_i(j\omega)||,$$

[2] Without loss of generality we assume that the states of reference nodes are zero, i.e., $\mathbf{x} = \mathbf{0}$ and $\theta = \mathbf{0}$.

[3] For simplicity, λ_i refers to $\lambda_i(L_g)$, unless indicated.

which yields the result. □

According to Theorem 2, the $\mathcal{H}_\infty$ norms for both mechanical and power networks are monotonic decreasing functions of λ_1. Hence, increasing the smallest eigenvalue of the grounded Laplacian matrix (by changing the network topology, and managing the number and location of the reference nodes) will always decrease the $\mathcal{H}_\infty$ norm and consequently provide a more robust network control system. We will return to the effect of network structure on the norm in Section 5. However, due to the fact that in most cases the network structure and the number and location of reference nodes are fixed, it is more convenient to use local feedback controllers to increase the robustness of the overall system. To this end, in the following proposition we analyze the behaviour of the damping coefficient c and inertia m on system $\mathcal{H}_\infty$ norms for both mechanical and power networks.

Proposition 3. Consider the mechanical network described by (4) and the power network described by (10). The following statements hold:

(i) **Mechanical Network:** The $\mathcal{H}_\infty$ norm of a mechanical network (12) is a non-increasing function of the damping coefficient c and is bounded from below as

$$||G||_\infty^{\text{Mech}} \geq \frac{1}{k\lambda_1}, \tag{18}$$

with equality sign for all $c \geq \sqrt{\frac{2mk}{\lambda_1}}$. Moreover, the norm is a non-decreasing function of the mass m, and is bounded from below by (18), which holds with equality sign for all $m \leq \frac{c^2\lambda_1}{2k}$.

(ii) **Power Network:** The $\mathcal{H}_\infty$ norm of the power network (13) is a non-increasing function of the damping coefficient c, and it is bounded from below as

$$||G||_\infty^{\text{Power}} \geq \frac{1}{b\lambda_1}, \tag{19}$$

with equality sign for all $c \geq \sqrt{2bm\lambda_1}$.Moreover, the norm is a non-decreasing function of the inertia m, and is bounded from below by (19), which holds with equality sign for all $m \leq \frac{c^2}{2b\lambda_1}$.

Proof. The proof for both the mechanical and power networks is similar; here we consider the power network case. First note that $||G||_\infty^{\text{Power}}$ is a continuous function of c and m (it can be verified by checking the values in boundaries between two sub-functions in (13)). Let $f(c)$ denote $||G||_\infty^{\text{Power}}$ as a function of c. For the first sub-function in (13) we have $f(c) = \frac{2m}{c\sqrt{4bm\lambda_1 - c^2}}$. Differentiating with respect to c yields

$$f'(c) = \frac{2m}{(4bm\lambda_1 - c^2)^{\frac{3}{2}}} - \frac{2m}{c^2(4bm\lambda_1 - c^2)^{\frac{1}{2}}}.$$

By solving $f'(c) = 0$, we reach to the critical point $\bar{c} = \sqrt{2bm\lambda_1}$, which is exactly the boundary of the range of this sub-function in (13). For $c < \bar{c}$ we have $f'(c) < 0$ and for $c > \bar{c}$, based on the second sub-function in (13) we have $\frac{df(c)}{dc} = 0$ (since it is independent of c). Hence the minimum value of $f(c)$ is attained for all $c \geq \sqrt{2bm\lambda_1}$.

Now consider $||G||_\infty^{\text{Power}}$ as a function of inertia, and denote this function by $g(m)$. Differentiating $g(m)$ for the first sub-function in (13) with respect to m yields

$$g'(m) = \frac{2}{c(4bm\lambda_1 - c^2)^{\frac{1}{2}}} - \frac{4mb\lambda_1}{c(4bm\lambda_1 - c^2)^{\frac{3}{2}}}, \tag{20}$$

where $\bar{m} = \frac{c^2}{2b\lambda_1}$ is its critical point, which is the boundary of the range of this sub-function in (13). Considering the fact that for all $m < \bar{m}$ (based on the second sub-function), we have $g(m) = \frac{1}{b\lambda_1}$ and for $m > \bar{m}$, we have $g'(m) > 0$, the result is obtained. □

Fig. 2 shows the behavior of the power network $\mathcal{H}_\infty$ norm as a function of both damping and inertia, for several values of λ_1. We make two comments. First, while in general the $\mathcal{H}_\infty$ norm is non-differentiable, both expressions from Theorem 2 are continuously differentiable functions of c and m. Second, given a fixed network structure and one of the values of c or m, the other parameter can be adjusted to maximize system robustness. For example, if m is fixed, one should choose $c = \sqrt{\frac{2mk}{\lambda_1}}$ in mechanical and $c = \sqrt{2mb\lambda_1}$ in power network to yield the minimum system $\mathcal{H}_\infty$ norm.

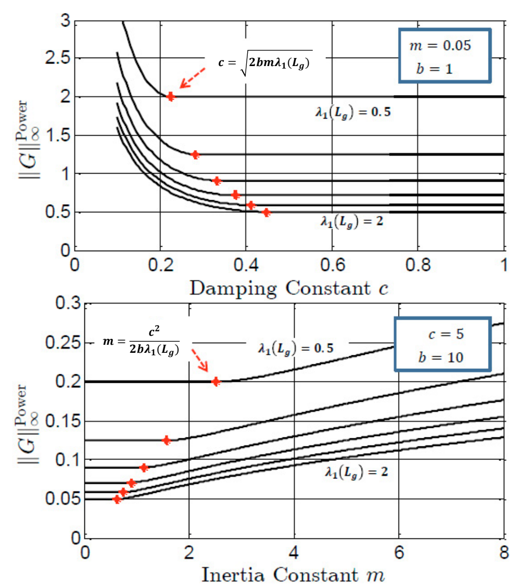

Fig. 2. $||G||_\infty^{\text{Power}}$ as a function of damping constant c and inertia constant m for different values of λ_1.

5. THE EFFECT OF NETWORK ON $\mathcal{H}_\infty$ PERFORMANCE

In this section, we present examples which illustrate how the network structure and the number and location of reference nodes can affect the $\mathcal{H}_\infty$ performance of mechanical and power networks.

5.1 Mechanical Network

Consider a network of masses, springs and dampers with $m_i = m = 1$, $k_i = k = 100$, $c_i = c = 10$ for all $i = 1, 2, ..., 5$. There are two reference and five follower masses, as shown in Fig. 3. The goal of this example is to show that interconnections between followers and the reference masses provide more robust network than those within the follower masses.

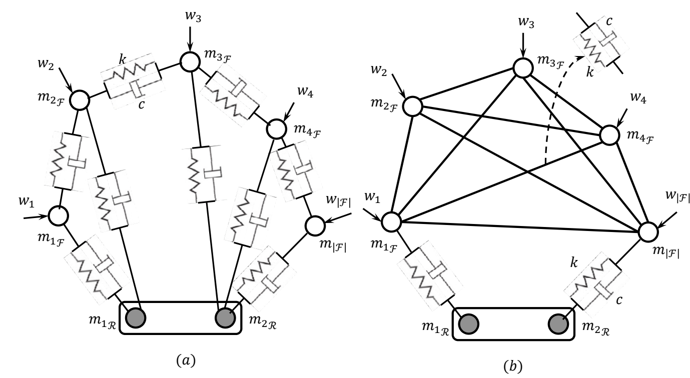

Fig. 3. Mechanical network with (a) line graph between followers and (b) complete graph between followers.

In Fig. 3 (a), since each follower is exactly connected to one reference node, we have $\min_{i\in\mathcal{F}}\{\beta_i\} = \max_{i\in\mathcal{F}}\{\beta_i\} = 1$ in (11), which gives $\lambda_1 = 1$. By substituting into (12) we have $||G||_\infty^{\text{Mech}} = 0.01$. In Fig. 3 (b) the follower set $\mathcal{F}$ forms a complete graph and two of them are connected to the reference set $\mathcal{R}$. In this case we have $\lambda_1 \leq \frac{|\partial\mathcal{R}|}{|\mathcal{F}|} = \frac{2}{5}$. Substituting this value into (12), and noting that here $c^2 < \frac{2km}{\lambda_1}$, we get $||G||_\infty \geq 0.04$. Therefore, the interconnections from followers to reference nodes are more important than the interconnections between followers for robustness, and the graph-theoretic bounds (11) can be used as design criteria to improve system robustness.

5.2 Power Network

In power networks, there exists a limited flexibility in changing the network structure. However, in order to analyze such systems a reference node (with reference phase angle) should be chosen first. Since the $\mathcal{H}_\infty$ norm is a monotonic decreasing function of λ_1, given a power network with fixed m, b and c, maximizing the smallest eigenvalue of the grounded Laplacian matrix and minimizing $||G||_\infty^{\text{Power}}$ are equivalent. Hence, the choice of a reference node has a considerable effect on the $\mathcal{H}_\infty$ performance. From the reference (leader) selection point of view, in order to minimize the $||G||_\infty^{\text{Power}}$ norm, a reference should be chosen which maximizes λ_1. This is discussed in the following definition.

Definition 1. Consider a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. The *grounding centrality* of each vertex $v_s \in \mathcal{V}$, denoted by $I(s)$, is $I(s) = \lambda_1(L_{gs})$. The set of *grounding central* vertices in the graph $\mathcal{G}$ is given by $IC(\mathcal{G}) = \text{argmax}_{v_s\in\mathcal{V}}\,\lambda_1(L_{gs})$.

According to the above definition, a grounding central vertex $v_s \in IC(\mathcal{G})$ is a vertex that maximizes λ_1, if chosen as a reference, over all possible choices of single reference nodes Pirani et al. (2016).

Fig. 4 shows a network called a *broom tree*, $B_{n,\Delta}$, comprised of a star S_Δ with Δ leaf vertices and a path of length $n-\Delta-1$ attached to the center of the star Stevanovic and Ilic (2010).

Consider Fig. 4 representing a power network with $m = \frac{20}{2\pi f}$ and $c = \frac{10}{2\pi f}$, based on Sauer and Pai (1999), with $f = 60$Hz and $b = 1$. Fig. 5 plots the value of the $\mathcal{H}_\infty$ norm as a function of the node which is chosen as a reference (the slack bus choice). As shown in this figure, the reference node which minimizes the system $\mathcal{H}_\infty$ norm is node number 8. This node does not have the highest degree ("degree central" node), nor is it the closest node to the other nodes (the "closeness central" node). In the graph shown in Fig. 4, node 7 is both the degree and closeness central node. By increasing Δ and consequently n in the broom tree, the grounding central node becomes farther from the degree and closeness central nodes in the network.

6. SUMMARY AND CONCLUSIONS

In this paper we investigated the robustness of mechanical and power network dynamical systems in the sense of their system $\mathcal{H}_\infty$ norm. For homogeneous subsystem parameters, we derived-closed form expressions for the $\mathcal{H}_\infty$ norm in terms of physical properties of each dynamics (mass/inertia, coupling weights and damping coefficients) and in terms of the network structure, encapsulated in the smallest eigenvalue of the grounded Laplacian matrix. An interesting and important avenue for extending the results presented in this paper is to derive the system norms when the physical subsystem parameters are heterogeneous. Moreover, it would be informative to compare the $\mathcal{H}_\infty$ norm to the $\mathcal{H}_2$ norm in terms of their dependency on physical parameters and network structure.

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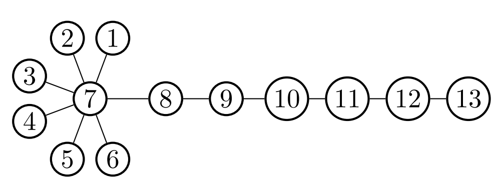

Fig. 4. Broom tree with $\Delta = 6$, $n = 13$.

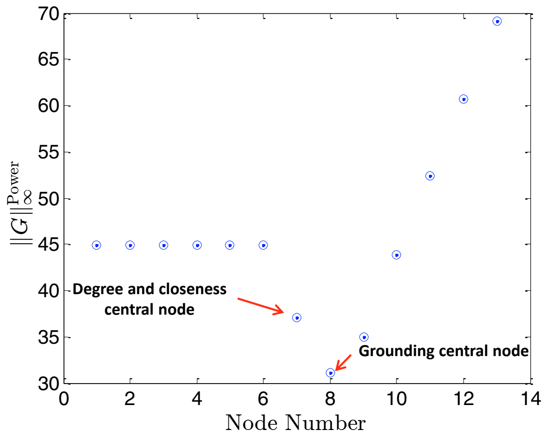

Fig. 5. $\mathcal{H}_\infty$ norm (13) versus the node label in Broom tree $B_{13,6}$.

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