University of Alberta

The effect of cosmological scalar fields on CMB anisotropies

by

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Abstract

This thesis considers the effect of perturbations in a dynamical scalar field quintessence on the large scale anisotropy of the cosmological microwave background (CMB). It is observed that evolving quintessence inhomogeneities suppress low-l multipoles of the CMB power spectrum. The description of cosmological perturbations is carried out in the conformal Newtonian (zero shear) gauge, and assuming an inverse power law form for the quintessence field potential. The CMB power spectra are calculated using the exact form of the Sachs-Wolfe effect, as opposed to widely used approximations. It is established that long wavelength perturbations of the scalar field remain significant till the present time, whereas short wavelength perturbations decay and become insignificant during the matter dominated epoch. Therefore the existence of inhomogeneities in the quintessence field affects the gravitational potential only at long wavelengths. Correspondingly, only the low-l multipoles of the CMB are affected. These results may be used in conjunction with modern CMB observations to constrain quintessence models.

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Chapter 1

Introduction

1.1 History of quintessence in cosmology

For a general isotropic and homogeneous universe, with a metric chosen to be of the Robertson-Walker form

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \tag{1.1}$$

the Einstein equations of general relativity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.2}$$

predict a dynamical cosmology. The prevalent view at the time when Einstein formulated General Relativity was, however, that the universe is static and unchanging. Therefore, in 1917, Einstein introduced into his equations a new fundamental constant which has since been variously called the cosmological constant or the 'Lambda term'. The modified equations then read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{1.3}$$

The resulting equations indeed admit static solution, but it is unstable. Consequently if subjected to an infinitesimal contraction or expansion, the universe must go on contracting or expanding. The discovery by Edwin Hubble in the 1920s of an expanding universe, proved the static Einstein universe to be unrealistic. Subsequently, Einstein abandoned the cosmological constant, calling it "his greatest blunder".

From a theoretical standpoint, the cosmological constant remained a distinct possibility and continued to be studied if only out of academic curiosity. Lemaitre (1927) used it to construct an expanding universe with a quasi-static origin in the past, rather than an initial singularity seen in expanding Friedman-Robertson-Walker (FRW) models. De Sitter invoked the Λ term in the absence of matter to get dynamic and static solutions. The de Sitter model proved important for the steady state cosmology, as well as for inflationary models of the early universe. In inflationary models [15], proposed in the early 1980s, matter (in the form of a false vacuum, as vacuum polarization or as a minimally coupled scalar field) behaved precisely like a weakly time dependent Λ term. Ya. B. Zeldovich (1968) gave the Λ term a physical basis when he observed that one loop quantum vacuum fluctuations give rise to an energy momentum tensor which upon being suitably regularized has exactly the same form as a cosmological constant. This introduced the modern view according to which the Λ term is not a fundamental constant in Einstein equations but is mimicked by some kind of matter, often called dark energy. The Λ term in the form of dark energy came to the forefront of cosmology in the last decade with detailed observations of type Ia high redshift supernovae (Riess et al [31], Perlmutter et al [26], for the latest results see [32, 27] and references listed therein) which indicated that the expansion of the universe is accelerating, which requires dominant matter component to have negative pressure, behaving similarly to the Λ term.

The nature of the dark energy, required to explain the acceleration, is still to be established. In particular, its exact equation of state is still unknown. The first attempts rescued the cosmological constant as an ad hoc explanation, although a naive interpretation of the constant in terms of a vacuum energy is inconsistent by 124 orders of magnitude with respect to the required value.

The idea that a scalar field with a suitable potential of self-interaction can play the role of dark energy was introduced by Wetterich [37], Caldwell et al [8], Ratra and Peebles [30] to alleviate the extreme fine tuning needed to allow a cosmological constant to be significant only at recent epochs. Such a scalar field is popularly called a quintessence field. The most popular potentials are inverse power and exponential laws [29] although other possibilities have been considered [28]. The main property of quintessence potentials is the existence of "tracking" behaviour due to which the motion of the field converges to a unique solution for a broad range of different initial conditions.

The dynamic nature of the dark energy in the form of the quintessence field opens many new possibilities. For example, since quintessence has been inspired by observations of an accelerating universe, it is worthwhile asking, what if the acceleration is merely transient, and not permanent as has been assumed heretofore. In this context Blais [6] proposes a double quintessence model in which the dark energy sector consists of two coupled scalar fields. In both such models, it is shown [6] that if acceleration occurs, it is necessarily transient. The possibility of transient acceleration in two one field models, the Albrecht-Skordis model and the pure exponential has also been established. Using separate conservative constraints (on the effective equation of state w_{eff}, the relative density of the dark energy Ω_Q and the present age of the universe) scenarios with a transient acceleration that has already ended at the present time or no acceleration may be constructed. A less conservative analysis using the cosmic microwave background (CMB) data rules out the last possibility. The scenario with a transient acceleration that ended by today may be implemented if at the present time the ordinary matter density $\Omega_M \geq 0.35$ and the Hubble constant $H_0 \leq 68$km/s/Mpc.

Another interesting angle is a possible link between the running of the fine structure constant α and a time evolving scalar dark energy field [12]. Provided no symmetry cancels it, there may be a term in the effective Lagrangian weakly coupling baryonic matter to the scalar field. If this field evolves over cosmological times, such a coupling would lead to a time dependence of the coupling "constants" of baryonic matter. Dirac had introduced the notion that the fundamental constants of nature may vary. In a realistic GUT scenario, the variation of different couplings is interconnected. This interdependence might be ignored thus making only the fine structure constant "variable" with all others fixed. Indeed, bounds on the time variation of these "constants" restrict the evolution of the scalar field and the strength of this coupling. Under the assumption that the change in α is of the first order given by the evolution of the quintessence field, it can be shown using current Oklo nuclear reactor, quasi-stellar object (QSO) and equivalence principle observations [11, 10, 24, 36], that the model parameters are restricted considerably stronger than observations of the CMB, large scale structure and type Ia Supernovae combined.

Quintessence fields also arise in Supergravity and M/string theory. In [13] the implications of seven popular models of quintessence based on these theories for the transition from a decelerating to an accelerating universe are explored. All seven candidates can mimic the Λ cold dark matter (Λ-CDM) model at low redshifts $0 < z < 5$. For a natural range of initial values of the quintessence field, the SUGRA and Polonyi potentials predict a redshift z_t of transition to Λ-dominated epoch to be $z_t \approx 0.5$ for $\Omega_{\Lambda 0} \approx 0.7$ in agreement with the observational value of $z_t \approx 0.46$. For reference, the Λ-CDM model with constant Λ-term has $z_t = 0.67$.

1.2 Alternatives to quintessence

A competing candidate to quintessence for dark energy has been k-essence (or kinetic essence), originally proposed in [2]. k-essence cosmologies, unlike quintessence ones, are derived from Lagrangians with non-canonical kinematic terms. In standard quintessence models we may devise situations with scalar field potentials that go to zero asymptotically. These can have cosmologically interesting properties, including "tracker" behaviour, that makes the current energy density largely independent of the initial conditions. Unfortunately, the era in which the scalar field begins to dominate can only be set by fine tuning the parameters in the theory. This may be remedied by considering a dissipative matter component interacting with dark energy. However in k-essence models [1], the solution seems not to require the consideration of dissipation. Even for potentials which are not shallow, the nonlinear kinetic terms lead to dynamical attractor behaviour that permits the avoidance of the cosmic coincidence problem.

A surprisingly simple alternative to quintessence has been to consider a cosmological model comprising only two fluids, baryons (modelled as dust) and dark matter with a van der Waals equation of state. In [9] it is shown that accelerated expansion may be obtained by suitably choosing the model parameters. Data from type Ia Supernovae and distant radio galaxies may be used to constrain the parameters of this form of quintessence.

Amongst astrophysical effects, the quintessence scalar field may enhance the abundance of dark matter relic particles [33]. The integrated Sachs-Wolfe (ISW) effect on the CMB, as measured through its correlation with galaxies, may be used [17] to study the dynamics of the dark energy through its large scale clustering properties. A canonical single scalar field or quintessence model predicts that these clustering effects will appear on the horizon scale with a strength that reflects the evolution of dark energy density.

Importantly, the dynamic nature of quintessence field results in inhomogeneities in the field being developed in the course of gravitational clustering, in contrast to the pure Lambda term which affects only homogeneous background. The goal of this thesis is to study the role of the perturbed quintessence degrees of freedom on the prediction of the observed large-scale anisotropy of the CMB.

1.3 Organization of the thesis

In this thesis we presume the quintessence model of Ratra and Peebles [30] and Wetterich [37], whereby a scalar field with a negative equation of state, constitutes the dark energy required to explain the anomalous acceleration of the Universe. Such a scalar field has a "tracking" behaviour, which is to say it tracks the dominant component of the energy content of the universe, and comes into prominence at a relatively recent cosmic era. This thesis is primarily concerned with the effect of such a dynamical quintessence on observables such as the anisotropies of the CMB. In particular, our goal is to compare the predictions for the large-scale anisotropy in the CMB in the model with dynamical quintessence field when its perturbations are taken into account versus, as is frequently done, the case of neglecting the quintessence perturbations and, as a benchmark, versus the model with the cosmological constant.

This thesis is organized as follows. All equations necessary to describe the evolution of a homogeneous, isotropic background and cosmological perturbations upon it, are derived and collected in Chapter 2. Chapter 3 contains analytical solutions to these equations in certain limiting cases. Chapter 4 contains an introduction to the cosmic microwave background (CMB) and the description of the main mechanism by which cosmological perturbations cause large scale CMB temperature anisotropies (called the Sachs-Wolfe effect after its discoverers [34]). Later in Chapter 4 we turn to the main result of the thesis, a computation of how a realistic quintessence with inhomogeneities affects these anisotropies as opposed to dynamical but homogeneous quintessence, and the static Λ term postulated originally by Einstein. Chapter 5 summarizes the results, and suggests possible future developments.

Throughout the thesis we set $c = \hbar = 1$. We do not set $G = 1$. Thus we are not using a system of Planck units. Instead, in section 2.1.3 we further completely specify the system of units by setting the present day Hubble constant $H_0 = 1$. In such units, the time is measured in $H_0^{-1} \approx 14\,Gyr$ and the length in $c/H_0 = 4283\,Mpc$ if $H_0 = 70\,\mathrm{km/s/Mpc}$. The energy (mass) density scale is then $H_0^2 G^{-1} = H_0^2 M_{pl}^2$.

This choice is more suitable for cosmological calculations. The value of the scalar field is, on the other hand, measured in inverse Planck length l_{pl}^{-1}.

Everywhere, the dot $\dot{()}$ denotes a partial derivative with respect to proper time t of the metric (1.1), while the prime $()'$ denotes a derivative with respect to conformal time $\eta = \int dt/a(t)$. We use comma $()_{,i}$ for the derivative with respect to a spatial coordinate x^i and $()_{,\varphi}$ for the derivative with respect to the scalar field value φ. The semicolon $()_{;\nu}$ is used for the covariant derivative with respect to x^ν. H as a function of time denotes the Hubble parameter $H = \dot{a}/a$. The Hubble parameter is also somewhat loosely used to denote the closely related quantity $\mathcal{H} = a'/a$. After introducing dimensionless quantitities in section 2.1.3 we retain the same symbols for them as for their dimensional counterparts, which, hopefully, does not lead to confusion.

Chapter 2

Theoretical background

In this chapter we turn to the systematic development of all necessary equations which govern the evolution of cosmological perturbations. Perturbations can only be studied in the context of the background manifold. Hence in this chapter we first outline the properties of the background spacetime, such as its metric, its assumed material constituents and derive the equations governing them.

Then, in section 2.2 we review the perturbations on such a background, and derive the perturbed Einstein equations. These equations provide a set of constraint equations giving the metric perturbation (Φ) in terms of the quantities that specify the perturbed stress energy tensor. The conservation condition and the Euler equation that govern the evolution of quantities in the perturbed stress-energy tensor in the limited case of scalar perturbations, are derived. Finally we end this chapter with the description of inhomogeneities in the quintessence field.

2.1 The background spacetime

Our approach is based on the premise that on the large scales the observed universe deviates by only a very small amount from a homogeneous, isotropic space time. Though originally a simplifying assumption, this has been verified to a remarkable degree of precision by recent observations. This makes it convenient to decompose the metric into a background metric, representing the homogeneous, isotropic ideal, and a perturbation upon it. The background metric in such a case is called the Friedman-Robertson-Walker (FRW) spacetime.

The background line element in a FRW spacetime in conformal time and cartesian coordinates is given by

$$ds^2 = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j)\,, \tag{2.1}$$

where space-like coordinates and tensor components are labelled with Latin letters $i, j = 1, 2, 3$. We reserve Greek letters to denote four-dimensional coordinates and tensor indices which vary from zero to four with zero being the time component. We assume Einstein summation convention over upper-lower pairs of repeated indices.

The background metric in these coordinates then becomes

$$g_{\mu\nu} = a^2(\eta)\begin{bmatrix} 1 & 0 \\ 0 & -\gamma_{ij} \end{bmatrix}, \tag{2.2}$$

where

$$\gamma_{ij} = F(r)\delta_{ij} \tag{2.3}$$

and

$$F(r) = [1 + \frac{K}{4}r^2]^{-2}\,. \tag{2.4}$$

For spacetimes with flat, closed and open spacelike hypersurfaces, $K = 0, -1$ and 1 respectively. There is reasonable observational evidence that $K = 0$ in which case

$$g_{\mu\nu} = a^2(\eta)\begin{bmatrix} 1 & 0 \\ 0 & -\delta_{ij} \end{bmatrix}. \tag{2.5}$$

But in the following treatment we keep the formalism as general as possible. As a prelude to computing the perturbation to the Einstein or Ricci tensors, we must first compute the affine connections. For the metric given above, the nonvanishing affine connections are as follows,

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H}\,, \\
\Gamma^0_{ij} &= \mathcal{H}\gamma_{ij}\,, \\
\Gamma^i_{0j} &= \mathcal{H}\delta^i_j\,, \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j\,, \\
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk}\,,
\end{aligned} \tag{2.6}$$

where $\mathcal{H}$ is related to the derivative a' of the scale factor with respect to conformal time η, $\mathcal{H} \equiv a'/a$, and $\tilde{\Gamma}^i_{jk}$ is the affine connection on the spatial metric γ_{ij}. For spatially flat spacetimes, γ_{ij} reduces to the Euclidean metric and the corresponding affine connections $\tilde{\Gamma}^i_{jk}$ reduce to zero. The details of these computations are given in Appendix A.1. For convenience we define $\Gamma_\mu = \Gamma^\lambda_{\mu\lambda}$. In spatially flat spacetime, it has the components

$$\begin{aligned} \Gamma_0 &= 4\mathcal{H}\,, \\ \Gamma_i &= 0\,. \end{aligned} \tag{2.7}$$

2.1.1 Einstein equations

The Ricci tensor $R_{\mu\nu}$ then has the elements

$$\begin{aligned} R_{00} &= 3\mathcal{H}'\,, \\ R_{0i} &= 0\,, \\ R_{ij} &= -[\mathcal{H}' + 2\mathcal{H}^2 + 2K]\gamma_{ij}\,. \end{aligned} \tag{2.8}$$

The Ricci curvature scalar is given by $R = -6[\frac{\mathcal{H}'+\mathcal{H}^2+K}{a^2}]$. The reader may refer to Appendix A.2, for the details of these computations. The Einstein tensor, defined as

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}Rg^\mu_\nu\,, \tag{2.9}$$

then has the components

$$\begin{aligned} G^0_0 &= \frac{3(\mathcal{H}^2+K)}{a^2}\,, \\ G^0_i &= 0\,, \\ G^i_j &= \frac{2\mathcal{H}' + \mathcal{H}^2 + K}{a^2}\delta^i_j\,. \end{aligned} \tag{2.10}$$

We consider the universe with material constituents being the radiation (describing both relic photons and relativistic neutrinos), pressureless matter (for our purposes baryons can be included in this category together with dark matter) and a scalar field which does not interact with other components, except gravitationally.

Hence, it is convenient to decompose the total stress-energy tensor $T^{\mu\nu}$ as follows,

$$T^{\mu\nu} = (T_{rm})^{\mu\nu} + (T_{sf})^{\mu\nu}, \tag{2.11}$$

where $(T_{rm})^{\mu\nu}$ and $(T_{sf})^{\mu\nu}$ refer to the stress-energy tensors of radiation-matter and the coherent scalar field respectively. Radiation and matter may be treated as a perfect fluid with no anisotropic stress and no dissipation. The stress-energy tensor is then described in terms of only three functions, the energy density ρ, pressure p and the fluid four-velocity U^μ as follows,

$$(T_{rm})^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu}. \tag{2.12}$$

The energy density in the radiation-matter fluid ρ is the sum of the energy densities of the two components. So,

$$\rho = \rho_r + \rho_m, \tag{2.13}$$

where ρ_r and ρ_m are the energy densities of radiation and matter respectively. As the energy of a photon redshifts as $1/a$, we may infer ρ_r drops as $1/a^4$. For ordinary baryonic matter, or dust, as well as for pressureless dark matter, ρ_m drops as $1/a^3$. We may therefore introduce the following substitutions,

$$\begin{aligned} \rho_r &= \frac{\rho_{r0}}{a^4}, \\ \rho_m &= \frac{\rho_{m0}}{a^3}, \end{aligned} \tag{2.14}$$

where ρ_{r0} and ρ_{m0} are the energy densities at a scale factor of unity. As dust is pressureless, the pressure of the fluid p equals the pressure due to radiation, which is inferred from the relativistic limit of the energy-momentum relations, to be,

$$p = p_r = \frac{1}{3}\rho_r. \tag{2.15}$$

For a scalar field interacting only with itself via the potential $V(\varphi)$, and minimally coupled to gravity, the corresponding energy-momentum tensor is given by

$$(T_{sf})^{\mu\nu} = \varphi^{;\mu}\varphi^{;\nu} - \left[\frac{1}{2}\varphi^{;\alpha}\varphi_{;\alpha} - V(\varphi)\right] g^{\mu\nu}, \tag{2.16}$$

where

$$\varphi^{;\mu} = g^{\mu\nu}\varphi_{;\nu}\,. \tag{2.17}$$

In a highly homogeneous universe, the scalar field shall also, to a large degree, be homogeneous. We denote $\varphi(\eta)$ to be this homogeneous part of the scalar field which drives the background isotropic model.

From the stress-energy tensor of the scalar field we may infer that the energy ρ_{sf} and pressure p_{sf} of the background scalar field are given by

$$\begin{aligned} \rho_{sf} &= \frac{1}{2a^2}\varphi'^2 + V(\varphi)\,, \\ p_{sf} &= \frac{1}{2a^2}\varphi'^2 - V(\varphi)\,. \end{aligned} \tag{2.18}$$

The combination of terms for the pressure p_{sf} is not positive definite. In fact when the potential term dominates over the usual kinetic term, the pressure becomes negative. Under such conditions, the scalar field acts as if with a negative equation of state. This property is used in the models of quintessence scalar field to explain the accelerated expansion of the universe.

Another unusual property of the scalar field, which makes it a good candidate for the role of dark energy, relates to the evolution of its energy density, as compared to the energy densities of radiation and matter. Unlike these other constituents, the energy density of the scalar field does not drop as a simple, fixed power law of the the scale factor $a(\eta)$ as in (2.14). To the contrary, for several types of potential, including the inverse power law and exponential ones, the energy density of the subdominant scalar field ρ_{sf} "tracks" the dominant component of the energy at various cosmological epochs. The exact law of the energy change quintessence is determined by the potential $V(\varphi)$ and importantly, by the equation of state of the dominant energy component. This property shall be explained analytically later in Chapter 3. Such "tracking" behaviour ensures that quintessence which dominates the present day energy balance in the universe, was not negligibly small relative to other components in the early universe. This alleviates the fine-tuning required to have the present-day value of the cosmological constant of the same order as other matter components.

Summing up the various contributions to the total energy momentum tensor T^{μ}_{ν}, we observe

$$
\begin{aligned}
T_0^0 &= \rho_r + \rho_m + \rho_{sf} \, , \\
T_i^0 &= 0 \, , \\
T_j^i &= p_r + p_{sf} \, .
\end{aligned} \tag{2.19}
$$

With the above substituted into the Einstein equations, the time-time component gives us

$$\frac{a'^2}{a^4} + \frac{K}{a^2} = 8\pi G T_t^t = 8\pi G(\rho_r + \rho_m + \rho_{sf}) \, , \tag{2.20}$$

and the trace gives us

$$\frac{a''}{a^3} + \frac{K}{a^2} = 8\pi G T_\mu^\mu \, . \tag{2.21}$$

The equations (2.20) and (2.21) are commonly known as the Friedman equations, and the corresponding cosmological models are known as the Friedman models. Incidentally, the second of these equations (2.21) follow from the first (2.20) and from the equation of energy conservation, which may be stated in conformal time as

$$p' a^3 = \left[a^3 (\rho + p) \right]' \, , \tag{2.22}$$

where ρ and p refer to the total energy density and total pressure due to all constituents of the universe. Hence, in our numerical calculations of the scale factor $a(\eta)$ we restrict our attention exclusively to the first Friedman equation (2.20). For a spatially flat universe, where $K = 0$, substituting (2.14) for the scale dependence of the energy densities of radiation and matter, and (2.18) for the energy density of the scalar field, the first FRW equation becomes

$$a'^2 = \frac{8\pi G}{3} \left(\rho_{mo} a + \rho_{ro} + \frac{1}{2} a^2 \varphi'^2 + a^4 V(\varphi) \right) \, . \tag{2.23}$$

The third term is the kinetic term associated with the scalar field and $V(\varphi)$ represents the potential that governs the quintessence like scalar field.

2.1.2 Scalar field dynamics

We have derived the equation which governs the evolution of the scale factor a. The evolution of the energy densities radiation and matter are trivial, as they are given by simple power laws in (2.14). In order to complete this treatment however, we need the energy density of the scalar field which in turn depends on the scalar field value and its time derivative. Therefore, we need the equation for the evolution of the unperturbed scalar field. This is given by the Klein-Gordon equation, which we derive shortly. The action for a free self-interacting scalar field is given by

$$S = \int \left(\frac{1}{2}\varphi^{;\mu}\varphi_{;\mu} - V(\varphi)\right)\sqrt{-g}d^4x\,. \tag{2.24}$$

On varying the action with respect to (for compactness - wrt) the scalar field φ, for convenience the variation may be decomposed into two terms, $\delta S = \delta S_1 + \delta S_2$, where δS_1 and δS_2 are given by

$$\begin{aligned} \delta S_1 &= \int \left[\frac{1}{2}(\delta\varphi^{;\mu}\varphi_{;\mu} + \varphi^{;\mu}\delta\varphi_{;\mu}) - V_{,\varphi}\delta\varphi\right]\sqrt{-g}d^4x\,, \\ \delta S_2 &= \int \left[\frac{1}{2}\varphi^{;\mu}\varphi_{;\mu} - V(\varphi)\right]\delta\sqrt{-g}d^4x\,. \end{aligned} \tag{2.25}$$

The first term may be simplified as follows,

$$\begin{aligned} \delta S_1 &= \int \left[\frac{1}{2}(\delta\varphi^{;\mu}\varphi_{;\mu} + \varphi^{;\mu}\delta\varphi_{;\mu}) - V_{,\varphi}\delta\varphi\right]\sqrt{-g}d^4x\,, \\ &= \int \left[\frac{1}{2}\delta g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} + \varphi^{;\mu}\delta\varphi_{;\mu} - V_{,\varphi}\delta\varphi\right]\sqrt{-g}d^4x\,, \\ &= \int \left[\frac{1}{2}\delta g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} + (\varphi^{;\mu}\delta\varphi)_{;\mu} - \varphi^{;\mu}{}_{;\mu}\delta\varphi - V_{,\varphi}\delta\varphi\right]\sqrt{-g}d^4x\,. \end{aligned} \tag{2.26}$$

The integral over the second term which is a full divergence vanishes, leaving us with

$$\delta S_1 = \int \left[\frac{1}{2}\delta g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - \left(\varphi^{;\mu}{}_{;\mu} + V_{,\varphi}\right)\delta\varphi\right]\sqrt{-g}d^4x\,, \tag{2.27}$$

To compute the variation of the determinant of the metric in (2.25) we use the following linear algebra relations for a matrix M

$$\delta\ln[Det(M)] = \text{Trace}\left[M^{-1}\delta M\right]\,. \tag{2.28}$$

For the metric tensor g this gives us

$$\frac{\delta(-g)}{-g} = g^{\mu\nu}\delta g_{\mu\nu}\,, \tag{2.29}$$

or,

$$\begin{aligned} \delta\sqrt{-g} &= \frac{\sqrt{-g}}{2} g^{\mu\nu}\delta g_{\mu\nu} \\ &= -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu}\,. \end{aligned} \tag{2.30}$$

Substituting the above into (2.25) we get

$$\delta S_2 = -\int \left[\frac{1}{2}\varphi^{;\lambda}\varphi_{;\lambda} - V(\varphi)\right] \frac{g_{\mu\nu}\delta g^{\mu\nu}}{2}\sqrt{-g}d^4x\,. \tag{2.31}$$

Summing up the two terms,

$$\delta S = \int \left[\frac{1}{2}\delta g^{\mu\nu}\left\{\varphi_{;\mu}\varphi_{;\nu} - \left(\frac{1}{2}\varphi^{;\mu}\varphi_{;\mu} - V(\varphi)\right)g_{\mu\nu}\right\} + \left\{\varphi^{;\mu}{}_{;\mu} + V_{,\varphi}\right\}\delta\varphi\right]\sqrt{-g}d^4x\,. \tag{2.32}$$

Assuming the variation wrt the scalar field vanishes, we get the Klein-Gordon equation:

$$\varphi^{;\mu}{}_{;\mu} + V_{,\varphi} = 0\,. \tag{2.33}$$

Since the background scalar field φ is by assumption homogeneous, all spatial derivatives vanish, and the Klein-Gordon equation is simplified to

$$\varphi'' + 2\mathcal{H}\varphi' + a^2V_{,\varphi} = 0\,. \tag{2.34}$$

2.1.3 The background evolution of the cosmological model. Tracking behaviour of the scalar field.

The evolution of the background is governed by the coupled system of equations (2.23) and (2.34). Let us introduce convenient dimensionless variables.

- We normalize the scale factor so that at the present moment $a(now) = 1$.

- At the present moment the Hubble parameter is equal to the observed value $\mathcal{H} = \frac{a'}{a} = H_0$. Henceforth, we shall use the time units in which $H_0 = 1$. This condition together with the already used $c = \hbar = 1$ completely specifies our system of units. In these units the present time is defined by $a'(now) = 1$.

- With the present day critical energy density $\rho_{crit} = 3H_0^2/(8\pi G)$ we use fractional densities $\Omega_r = \rho_r/\rho_{crit}$, $\Omega_m = \rho_m/\rho_{crit}$ as cosmological parameters that describe the density of radiation and matter in the Universe.

- We measure the scalar field (and later, its fluctuations) in terms of the Planck mass $M_{pl} = G^{-1/2}$, introducing dimensionless $\varphi \to \varphi/M_{pl}$.
- We define the dimensionless potential of the scalar field $V \to V/(H_0^2 M_{pl}^2)$.

In this notation, the background system of equations to solve is

$$a' - \left[\Omega_m a + \Omega_r + \frac{8\pi}{3} a^4 \left(\frac{1}{2a^2}\varphi'^2 + V\right)\right]^{1/2} = 0, \quad (2.35)$$

$$\varphi'' + 2\frac{a'}{a}\varphi' + a^2 V_\varphi = 0. \quad (2.36)$$

The model properties of the scalar field are determined by the potential $V(\varphi)$. In this thesis we shall restrict our attention to inverse power law potentials where $V(\varphi) = \frac{\lambda}{n}\varphi^{-n}$, where λ is dimensionless, using units of $H_0^2 M_{pl}^{n+2}$. Here, we collect example calculations of the background properties in the model with $V = \frac{\lambda}{6}\varphi^{-6}$. These potentials have been shown to have the "tracking property" [30, 37].

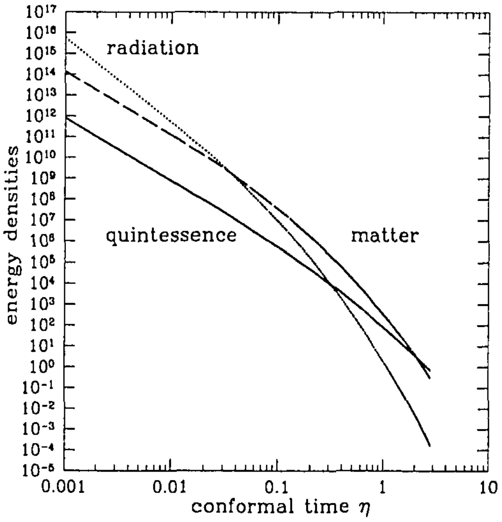
The concept of "tracking" behaviour of a quintessence field refers to two related properties:

- When energy contribution of the field φ is subdominant, there is a special solution of the equation of motion (2.36) following which the field exhibits an effective equation of state determined not only by its potential $V(\varphi)$ but also by the properties of the dominant energy component.
- This special solution is an attractor to a class of trajectories with arbitrary initial conditions $(\varphi(\eta_i), \varphi'(\eta_i)$. By itself, the tracking solution depends only on the parameters of the potential $V(\varphi)$. Once specified, they determine uniquely the value of the field and its derivative at any moment. This makes prediction of the model insensitive to a wide range of initial conditions for a scalar field.

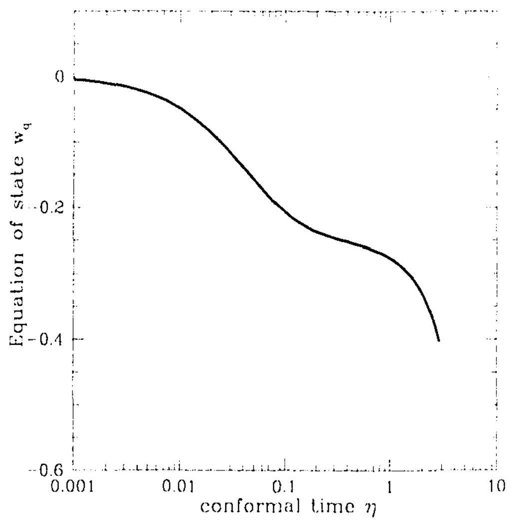
  The mathematical reason for the attractor properties lies in the large friction term in the equation of motion of the field, with the Hubble parameter, determined by the dominant component, as a coefficient.

The idea of using a scalar field with potential that exhibits the tracking behaviour is that it is possible to arrange conditions so that the energy of the field evolves similarily to the dominant component (under ideal tracking, it will scale exactly as a dominant component). This serves to alleviate the fine-tuning problem with a constant Λ-term, the scaling of which differs from matter and radiation energy density by a^3 or a^4, making them disparate by many orders of magnitudes in the early Universe.

In Figure 2.1(a) we plot the evolution of energy densities in three components, radiation, matter, and the scalar field with a potential $V(\varphi) = \frac{\lambda}{6}\varphi^{-6}$ potential for the cosmologically relevant case when the quintessence field is subdominant at early times. For this particular potential, the resulting law of change of quintessence en-



(a) Radiation, matter and quintessence are denoted by dotted, dashed and solid lines respectively. The time is in units of H_0^{-1}, the energy density is in units of $M_{pl}^2 H_0^2$.



(b) The equation of state parameter w_q of the quintessence field evolving through radiation, matter, and scalar field itself dominated stages

Figure 2.1: Time evolution of the energy density components in a quintessence model and equation of state of the scalar field.

ergy is $\rho_{sf} \propto a^{-3}$ during the radiation stage and $\rho_{sf} \propto a^{-2.25}$ during the matter stage, as we show analytically in Chapter 3. Figure 2.1(b) illustrates how the effective equation of state $w_q = p_{sf}/\rho_{sf}$ varies when the dominant component switches from radiation to matter. At the end, the quintessence field itself becomes dominant and w_q drops to more negative values.

Ideal tracking is not desirable, since then the quintessence field will always be subdominant, in conflict with modern observations. With power law potentials the field evolves slower than the dominant component, and at late times may dominate the energy density. Then, the Universe begins to accelerate, in accordance with observations. In the left panel of Figure 2.2 we plot the evolution of the Hubble parameter $\mathcal{H} = a'/a$ for several values of Ω_q. The accelerating expansion of the Universe corresponds to the growth of $\mathcal{H}$. Note that for $\Omega_q = 0.7$ the present day w_q

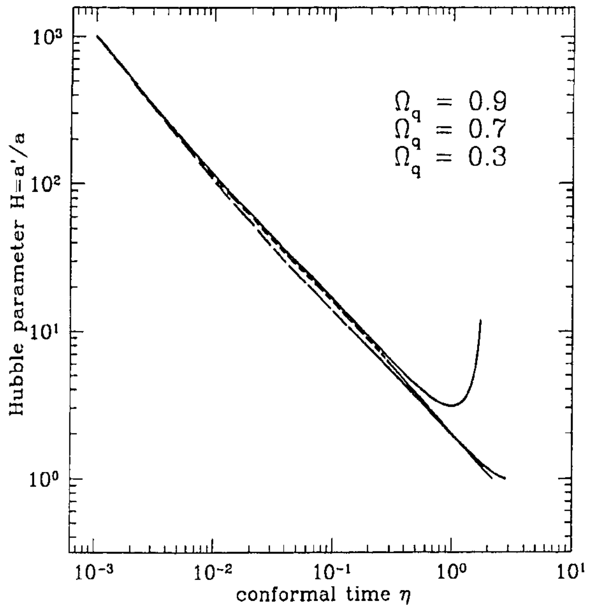


Figure 2.2: The evolution of Hubble parameter $\mathcal{H}$ for $\Omega_q = 0.9, 0.7, 0.3$ (from top to bottom). $\mathcal{H}$ is given in units of H_0 while conformal time is in units of H_0^{-1}. The curves stop at the present day value of the conformal time, which is smaller for larger values of Ω_q.

is still higher than -1. $w_q \approx -0.4$. This explains why in this model the Universe is just approaching the accelerating stage at the present moment, although the scalar field is already dominant. [1] This is in contrast to the constant Λ-term model which already accelerates now for $\Omega_\Lambda > 1/3$.

The behaviour of the scalar field φ for the discussed potential is given in Figure 2.3. The field starts with the initial value below Planck mass M_{pl} and slowly rolls down the potential increasing in value. The rate of the evolution is self-regulated to

[1] Einstein equations give $\ddot{a} = -1/2 \sum_i \Omega_i(1 + 3w_i)$ at the present time. For our two component case with just matter and quintessence contributing and $\Omega_m = 1 - \Omega_q$, we obtain $\Omega_q > -1/(3w_q)$ condition for acceleration $\ddot{a} > 0$.

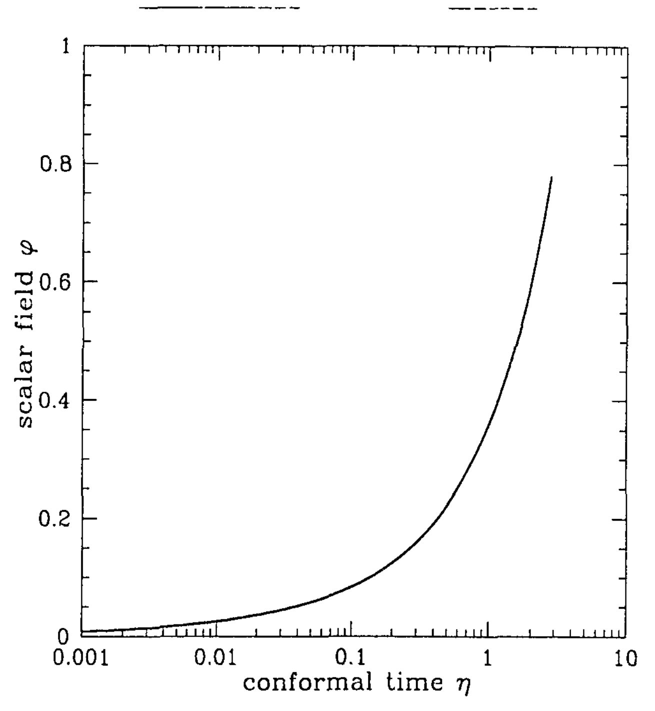


Figure 2.3: The unperturbed scalar field φ in M_{pl} units vs conformal time in H_0^{-1} units.

obey tracking behaviour until the scalar field starts to dominate the evolution.

At first glance, it may seem we have three parameters to describe the specific quintessence model- the energy scale λ and the initial values for $\varphi(\eta_i)$ and $\varphi'(\eta_i)$ to specify the field trajectory. However, tracking solutions are special solutions of the background equations and serve as an attractor to general solutions under widely varying initial conditions, [19, 30, 37]. This alleviates the need to fine-tune initial conditions in order to reproduce present day observables, a problem which persists in earlier models of quintessence [8, 29, 28]. [2]

In Figure 2.4 we present a "phase" diagram [3] which demonstrates the attractor properties of the tracking solution. For different initial values of φ and φ' we observe a convergence of the trajectories to the special tracker solution. When initial energy

[2]The study of stochastic behaviour of the quintessence field during inflation [22] showed that restriction on the inflationary models are required in order the quintessence initial values at the later FRW stage to fall into acceptable range

[3]The scalar field equation of motion depends explicitly on the scale factor and is not conservative. Therefore, the field trajectory in (φ, φ') plane is not determined completely by initial values, but also by the time moment at which these values are taken.

density of the field exceeds or is comparable to that of the dominant radiation component, the solution will diverge from the tracker trajectory. However, this case is of little direct interest, as it denotes a cosmology where quintessence dominates the Universe before matter, thereby preventing any astrophysical structures from forming.

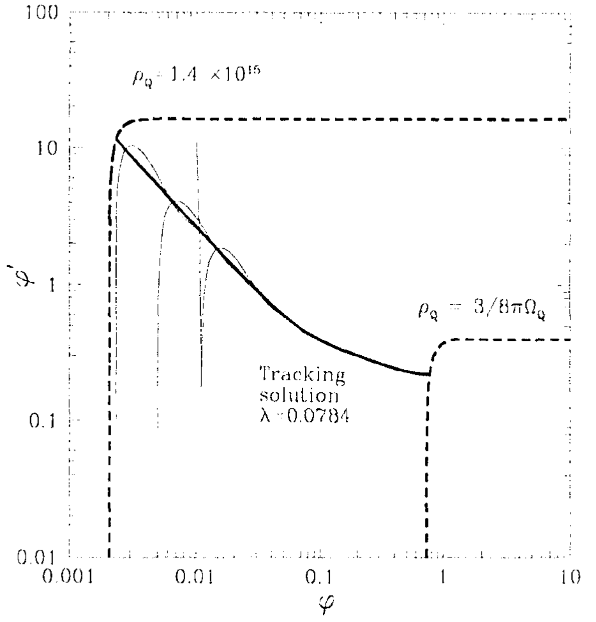


Figure 2.4: Tracking solution as an attractor in (φ, φ') "phase" space. Heavy solid line is the tracker trajectory. Dashed curves represent equal energy contours. The higher contour corresponds to the initial state of the quintessence taken at $\eta_i = 0.0001$, and the lower one marks the final energy reached at the present time. The parameter λ is chosen so that the tracking solution ends exactly at the required present energy density $\Omega_q = 0.7$. The parameter $\lambda = 0.0784$ is chosen so that the tracking solution ends exactly at the required present energy density $\Omega_q = 0.7$. Thin solid lines respresent trajectories of the field with different initial conditions at $\eta = 0.0001$.

Thus, as we shall see in the next chapter in detail, tracking behaviour corresponds to a unique relation at a given time between φ and φ' and φ and λ. So, whenever tracking behaviour is exhibited we are free to chose only one quintessence parameter, say the ratio of the energies in the scalar field and the dominant matter component.

However, if one sets Ω_m, the present day contribution of the quintessence field to its dimensionless energy must be equal to $1 - \Omega_m$ (with our choice of variables this is enforced implicitly by the requirement than that when $a = 1$ also we should have $a' = 1$). Thus, setting Ω_m, or, equivalently $\Omega_q = 1 - \Omega_m$, practically uniquely defines the background cosmology.

2.2 Cosmological perturbations

2.2.1 Metric perturbations

In order to study the growth of gravitational perturbations, we shall need the linearized Einstein equations in the background of a Friedman-Robertson-Walker (FRW) spacetime, which give us the evolution of any initial inhomogeneities present at the end of the inflationary epoch, and the onset of the radiation dominated era in the history of the universe.

In defining the perturbation to the metric, there are complicating issues related to the freedom of gauge or the choice of background coordinates. In this document, we use the "gauge invariant" formalism due to Bardeen [3], as formulated by Mukhanov et al [23], where metric variables are independent of the choice of coordinates. In this formalism, when the perturbed energy-momentum tensor is diagonal $\delta T^i_j \propto \delta^i_j$, scalar perturbations may be expressed in terms of a single gauge invariant variable Φ, and are given by

$$h_{\mu\nu} = \delta g_{\mu\nu} = a^2(\eta) \begin{bmatrix} 2\Phi & 0 \\ 0 & 2\Phi\delta_{ij} \end{bmatrix} . \tag{2.37}$$

From the following identity

$$g^{\mu\nu}\delta g_{\nu\rho} + \delta g^{\mu\nu} g_{\nu\rho} = 0 ,$$

we get

$$h^{\mu\nu} = \delta g^{\mu\nu} = -g^{\mu\sigma} g^{\nu\rho} \delta g_{\sigma\rho} . \tag{2.38}$$

This gives us

$$h^{\mu\nu} = -\frac{1}{a^2(\eta)} \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^{im} \end{bmatrix} \frac{1}{a^2(\eta)} \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^{jn} \end{bmatrix} a^2(\eta) \begin{bmatrix} 2\Phi & 0 \\ 0 & 2\Phi\gamma_{mn} \end{bmatrix} ,$$

$$= -\frac{1}{a^2(\eta)}\begin{bmatrix} 1 & 0 \\ 0 & -\gamma^{im} \end{bmatrix}\begin{bmatrix} 2\Phi & 0 \\ 0 & -2\Phi\gamma^j_m \end{bmatrix},$$
$$= -\frac{1}{a^2(\eta)}\begin{bmatrix} 2\Phi & 0 \\ 0 & 2\Phi\gamma^{ij} \end{bmatrix}.$$

Lowering one index, we get

$$h^\mu_\nu = \begin{bmatrix} 2\Phi & 0 \\ 0 & -2\Phi\delta^i_j \end{bmatrix}, \tag{2.39}$$

and on contracting these indices, we get

$$h^\mu_\mu = 2\Phi(1-\delta^i_i) = -4\Phi. \tag{2.40}$$

2.2.2 Perturbations in the affine connections

We are now faced with the task of computing the perturbations to the connection coefficients. The formula for a perturbation to the affine connection is given by

$$\begin{aligned} \delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\delta g^{\lambda\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\kappa\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right] \\ &+\frac{1}{2}g^{\lambda\kappa}\left[\frac{\partial h_{\kappa\nu}}{\partial x^\mu} + \frac{\partial h_{\kappa\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\kappa}\right]. \end{aligned} \tag{2.41}$$

With rearrangement, it assumes a friendlier form,

$$\delta\Gamma^\lambda_{\mu\nu} = -h^\lambda_\kappa\Gamma^\kappa_{\mu\nu} + \frac{1}{2}g^{\lambda\kappa}\left[\frac{\partial h_{\kappa\nu}}{\partial x^\mu} + \frac{\partial h_{\kappa\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\kappa}\right]. \tag{2.42}$$

The perturbations to the affine connections are then found to be (see details in Appendix A.3),

$$\begin{aligned} \delta\Gamma^0_{00} &= \Phi', \\ \delta\Gamma^0_{i0} &= \Phi_{,i}, \\ \delta\Gamma^0_{ij} &= -\left[4\mathcal{H}\Phi + \Phi'\right]\gamma_{ij}, \\ \delta\Gamma^i_{00} &= \gamma^{ik}\Phi_{,k}, \\ \delta\Gamma^i_{0j} &= -\Phi'\delta^i_j, \\ \delta\Gamma^i_{jl} &= -\Phi_{,j}\delta^i_l - \Phi_{,l}\delta^i_j + g^{ik}\Phi_{,k}g_{jl}. \end{aligned} \tag{2.43}$$

The components of $\delta\Gamma_\mu = \delta\Gamma^\nu_{\mu\nu}$ for the case of flat spatial section are

$$\begin{aligned}\delta\Gamma_0 &= -2\Phi' , \\ \delta\Gamma_i &= -2\Phi_{,j} .\end{aligned} \tag{2.44}$$

2.2.3 Perturbations in the Ricci tensor

From the connection coefficients $\Gamma^\mu_{\nu\kappa}$'s and its perturbations $\delta\Gamma^\mu_{\nu\kappa}$'s, we may now compose the perturbations to the Ricci tensor. The general formula for such a perturbation $\delta R_{\mu\nu}$ is given by

$$\delta R_{\mu\nu} = (\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - (\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} . \tag{2.45}$$

Substituting the requisite affine connections and their perturbations (given in Appendix A.4), we obtain the following,

$$\begin{aligned}\delta R_{00} &= -3\Phi'' - \nabla^2\Phi - 6\mathcal{H}\Phi' , \\ \delta R_{i0} &= -\left(2\Phi' + 2\mathcal{H}\Phi\right)_{,i} , \\ \delta R_{ij} &= \left[\Phi'' - \nabla^2\Phi + 6\mathcal{H}\Phi' + 4\left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi\right]\gamma_{ij} .\end{aligned} \tag{2.46}$$

2.2.4 The perturbed Einstein tensor

In order to calculate the perturbed Einstein equations we need the perturbation to the Einstein tensor $\delta G_{\mu\nu}$. Since the Einstein tensor $G_{\mu\nu}$ is composed of the Ricci tensor $R_{\mu\nu}$, and the Ricci scalar R, we proceed as follows. The Ricci scalar R is given by

$$R = R_{\mu\nu} g^{\mu\nu} . \tag{2.47}$$

Therefore, perturbing it, we get

$$\delta R = g^{\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu} R_{\mu\nu} . \tag{2.48}$$

The Einstein tensor is defined as

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2} R g^\mu_\nu . \tag{2.49}$$

Perturbing it, and combining with (2.48), we get

$$\delta G^{\mu}_{\nu} = g^{\mu\lambda}\delta R_{\lambda\nu} + \delta g^{\mu\lambda} R_{\lambda\nu} - \frac{1}{2}\left(g^{\rho\sigma}\delta R_{\rho\sigma} + \delta g^{\rho\sigma} R_{\rho\sigma}\right) g^{\mu}_{\nu}\,. \tag{2.50}$$

With rearrangement, this becomes

$$\delta G^{\mu}_{\nu} = g^{\mu\lambda}\delta R_{\lambda\nu} - \frac{1}{2}(g^{\rho\sigma}\delta R_{\rho\sigma})g^{\mu}_{\nu} + \delta g^{\mu\lambda} R_{\lambda\nu} - \frac{1}{2}(\delta g^{\rho\sigma} R_{\rho\sigma})g^{\mu}_{\nu}\,. \tag{2.51}$$

Evaluating the above expression for each component of the perturbed Einstein tensor, we get

$$\begin{aligned}
\delta G^{0}_{0} &= \frac{2}{a^2}\left[\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)\right], \\
\delta G^{0}_{i} &= \frac{2}{a^2}\left[\mathcal{H}\Phi + \Phi'\right]_{,i}, \\
\delta G^{i}_{j} &= -\frac{2}{a^2}\left[\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi\right].
\end{aligned} \tag{2.52}$$

The details of these calculations are given in Appendix A.5.

2.2.5 The perturbed Einstein equations

At last, we may now turn to the perturbed Einstein equations. The perturbed Einstein equations in gauge invariant form read

$$\delta G^{\mu}_{\nu} = 8\pi G\delta T^{\mu}_{\nu}\,. \tag{2.53}$$

Assuming a flat universe ($K = 0$), the time-time$(0, 0)$, time-space$(0, i)$, and space-space(i, i) parts respectively read

$$\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2\delta\rho\,, \tag{2.54}$$

$$\frac{1}{a}(a\Phi)'_{,i} = 4\pi G a^2(\rho + p)\delta U_i\,, \tag{2.55}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\delta p\,. \tag{2.56}$$

In the above equations, $\delta\rho$, δp and δU_i denote the perturbations to the energy density and pressure and fluid velocity in the conformal Newtonian gauge respectively.

2.3 Evolution of perturbations

In order to have an accurate understanding of the inhomogeneities in such a cosmology we need to calculate the evolution of the following variables;

1. Φ: the perturbation to the gravitational potential;

2. δ_m: the fractional perturbation to the energy density of matter, defined as

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m}. \tag{2.57}$$

3. δ_r: the fractional perturbation to the energy density of radiation, defined as

$$\delta_r \equiv \frac{\delta\rho_r}{\rho_r}. \tag{2.58}$$

4. v_m: the three-velocity potential of matter wrt the unperturbed background manifold, defined by

$$av_{m,i} = (\delta U_m)_i. \tag{2.59}$$

5. v_r: the three-velocity potential of radiation wrt the unperturbed background manifold, defined by

$$av_{r,i} = (\delta U_r)_i. \tag{2.60}$$

6. $\delta\varphi$: the perturbation to the scalar field value.

7. $\delta\varphi'$: the derivative of the perturbation to the scalar field wrt conformal time.

We shall decompose the perturbation equations using the set of eigenmodes of the Laplace operator. In a flat spacetime, these are the Fourier modes or plane waves $\{e(k)\}$, each of which corresponds to a spatial wavevector k_i. These modes form a complete orthonormal set of basis functions. The mode $\{e(k)\}$ corresponds to the eigenvalue $-k^2$ of the Laplacian i.e. $\nabla^2 e(k) = -k^2 e(k)$. The Fourier decomposed perturbation equations form a set of ordinary differential equations (ODEs), indexed or labelled by the wave number k, for the Fourier amplitudes, which themselves are functions of time only. This is a considerable simplification. Therefore, from now onwards, we shall implicitly assume we are working with individual spatial modes, of a certain wave number k, of the variables describing these inhomogeneities. For brevity, we shall not carry the index k through the equations.

For numerical stability we choose two different equations for the evolution of Φ. For $\eta \leq 0.05$, we use the time-time component of the perturbed Einstein equations

$$\begin{aligned}\Phi' &= \frac{a}{3a'}k^2\Phi - \frac{a'}{a}\Phi - \frac{1}{2}\Omega_r\delta_r/(aa') - \frac{1}{2}\Omega_m\delta_m/a', \\ &\quad -\frac{4\pi}{3}(\varphi'\delta\varphi' - \varphi'^2\Phi)\frac{a}{a'} - \frac{4\pi}{3}V_{,\varphi}\,\delta\varphi a^3/a'. \end{aligned} \tag{2.61}$$

For $\eta \geq 0.05$, we take the time-space component of the perturbed Einstein equations

$$\Phi' = -\frac{a'}{a}\Phi + \frac{3}{2}\frac{\Omega_m}{a}v_m + 2\frac{\Omega_r}{a^2}v_r + 4\pi\varphi'\delta\varphi. \tag{2.62}$$

The first equation behaves better numerically at small η, while with the second it is easier to compute the evolution of inhomogeneous Fourier modes $k \neq 0$ at later times since it does not contain any dependence on wave number.

With adiabatic initial conditions, the evolution of Φ with respect to conformal time can be characterized by the above set of equations, and for wave numbers $k = 0, 10, 50, 100$ and 200, is given in Figure 2.5(a).

As may be expected, higher k modes tend to vary more violently than lower k modes. The modes with higher wave numbers vary most rapidly before and during the time of recombination, and then settle down to a steady state value well into the matter dominated era. This trend is more apparent in a plot of the modes with respect to the log of conformal time in Figure 2.5(b).

So far we have merely restricted ourselves to the Einstein equations and its perturbations. They give us the evolution of the Hubble scale factor a and the gravitational potential Φ. The time evolution of the variables δ_m, δ_r,v_m,v_r follow from the vanishing covariant divergence of the energy-momentum tensor, which we now study in detail.

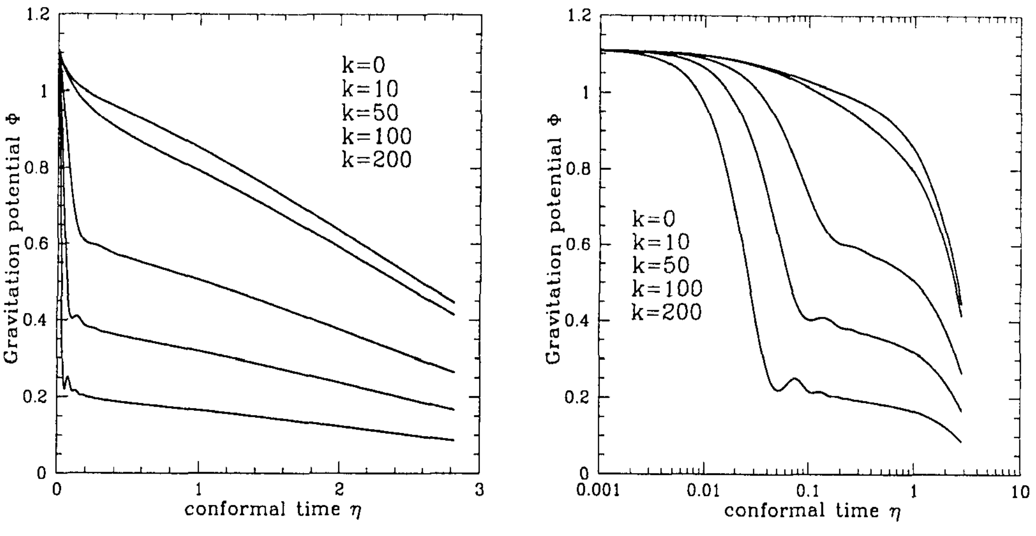
2.3.1 Conservation of energy in conformal time

In conformal time the velocity four-vector U^μ evaluated in the comoving reference frame is given by

$$U^\mu = [1/a, 0, 0, 0], \tag{2.63}$$

The unperturbed four-vector satisfies the relation

$$U^\mu U_\mu = U^\mu g_{\mu\nu} U^\nu = 1. \tag{2.64}$$



(a) Φ vs η for $k = 0, 10, 50, 100$ and 200 (from top to bottom).

(b) Φ vs $\log(\eta)$ for $k = 0, 10, 50, 100$ and 200 (from top to bottom).

Figure 2.5: Gravitational potential Φ for various wave numbers as a function of (a) conformal time and (b) log of conformal time.

With a perturbation vector δU^μ, in a perturbed metric, we have a similar relation

$$(U^\mu + \delta U^\mu)(g_{\mu\nu} + h_{\mu\nu})(U^\nu + \delta U^\nu) = 1\,,$$

thus

$$\delta U^\mu g_{\mu\nu} U^\nu + U^\mu g_{\mu\nu} \delta U^\nu + U^\mu h_{\mu\nu} U^\nu = 0\,.$$

Or, applying (2.63) we get

$$2a\delta U^0 + 2\Phi = 0\,,$$

which gives us

$$\delta U^0 = -\Phi/a\,. \tag{2.65}$$

Before we evaluate the covariant divergence of the energy-momentum tensor $T^{\mu\nu}$, we may, for convenience, compute the covariant divergence of $U^\mu U^\nu$, which is given by

$$(U^\mu U^\nu)_{:\nu} = (U^\mu U^\nu)_{,\nu} + \Gamma^\mu_{\lambda\nu} U^\lambda U^\nu + \Gamma_\lambda U^\lambda U^\mu\,.$$

For $\mu = 0$,

$$\begin{aligned}(U^0U^\nu)_{:\nu} &= \left(\frac{1}{a^2}\right)' + \Gamma^0_{00}\frac{1}{a^2} + \Gamma_0\frac{1}{a^2}, \\ &= 3\frac{a'}{a^3}. \end{aligned} \qquad (2.66)$$

The energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid is given by

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu - pg^{\mu\nu}. \qquad (2.67)$$

Vanishing covariant divergence of this tensor, $T^{\mu\nu}{}_{;\nu} = 0$, implies

$$(p+\rho)_{,\nu}U^\mu U^\nu + (p+\rho)(U^\mu U^\nu)_{:\nu} - p_{,\nu}g^{\mu\nu} = 0. \qquad (2.68)$$

The perturbation to the above, $\delta\left(T^{\mu\nu}{}_{;\nu}\right) = 0$, gives

$$\delta\rho' + \frac{3a'}{a}(\delta\rho + \delta p) = (\rho + p)\left[3\Phi' - a\nabla_i \delta U^i\right]. \qquad (2.69)$$

For details please refer Appendix A.6.

For radiation the equation of state is $p = \frac{1}{3}\rho$, and if we assume that there is no variation in the equation of state, the perturbations in pressure δp and density $\delta\rho$ are also related by $\delta p = \frac{1}{3}\delta\rho$. For scalar perturbations, we may substitute (2.60), to get

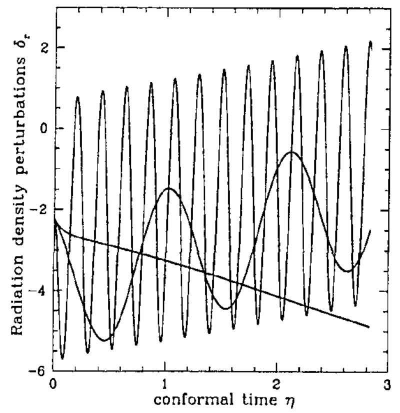
$$\delta\rho_r' + \frac{4a'}{a}\delta\rho_r = 4\Phi' + \frac{4}{3}\nabla^2 v_r. \qquad (2.70)$$

Since, for radiation $\rho_r' = -4\frac{a'}{a}\rho_r$, substituting (2.58), the equation for Fourier modes simplifies as follows

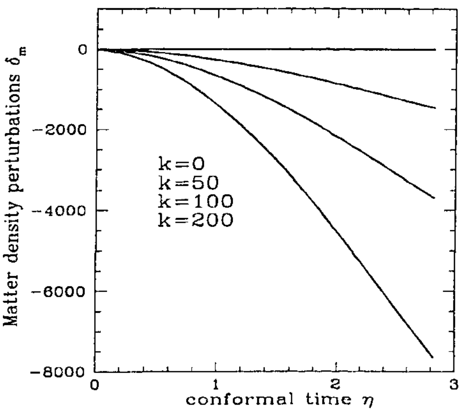
$$\delta_r' = 4\Phi' - \frac{4}{3}k^2 v_r. \qquad (2.71)$$

For various wave numbers the evolution of δ_r as governed by the above equation is given in Figure 2.6(a). It may be observed that the variations of the modes with lower wave numbers are steady and almost monotonic. The variations of the modes with higher wave numbers on the other hand, are more pronounced and sinusoidal in nature.

Similarly, for pressureless matter $\delta p, p = 0$. Since for matter $\rho_m' = -3\frac{a'}{a}\rho_m$,



(a) δ_r for wave numbers $k = 0, 10$ and 50 (in cH_0^{-1}), in order of increasing frequency of oscillation.



(b) δ_m for wave numbers $k = 0, 50, 100, 200$, from top to bottom.

Figure 2.6: The perturbation to energy densities of (a) radiation and (b) matter, δ_r and δ_m repsectively, vs conformal time for various wave numbers.

substituting $\delta_m = \delta\rho_m/\rho_m$, we get

$$\delta'_m = 3\Phi' - k^2 v_m \,. \tag{2.72}$$

For various wave numbers the evolution of δ_m, as governed by the above equation, gives us the Figure 2.6(b). For higher wave numbers the amplitudes of these perturbations is larger. The behaviour of the modes are qualitatively different from that in the case of radiation. This may be qualitatively explained as follows. Pressureless matter perturbations grow under the influence of self-gravity. This tendency is partially counteracted by the expansion of the universe. The growth of these perturbations is then governed by a power law [23] rather than an exponential, as would have been in the absence of an expanding universe.

2.3.2 Conservation of momentum in conformal time

We now turn to the spatial component of the vanishing covariant divergence of the energy-momentum tensor, $T^{i\nu}_{;\nu} = 0$. The perturbation to it, $\delta T^{i\nu}_{;\nu} = 0$, implies

$$\delta U_i{}' + \frac{p'}{p+\rho}\delta U_i = a\left[\Phi_{,j} + \frac{\delta p_{,j}}{p+\rho}\right]. \tag{2.73}$$

The intermediate steps are given in Appendix A.7. We may now replace the velocity by the velocity potential using (2.59). Since the partial derivative wrt x_i acts on all terms, we then get

$$v' + \left[\frac{p'}{p+\rho} + \frac{a'}{a}\right] v = \Phi + \frac{\delta p}{p+\rho}. \tag{2.74}$$

Radiation is characterized by the equation of state $p = \rho/3$ and the energy redshift relation $\rho' = -4\frac{a'}{a}\rho$. Applying these properties, the coefficient of v is seen to vanish. Furthermore, applying $\delta p = \delta\rho/3$, we obtain

$$v_r' = \frac{1}{4}\delta_r + \Phi. \tag{2.75}$$

The evolution of v_r as given by the above equation may be plotted with respect to conformal time as in Figure 2.7(a).

For pressureless matter all pressure terms $p, \delta p$ and their derivatives drop out to leave

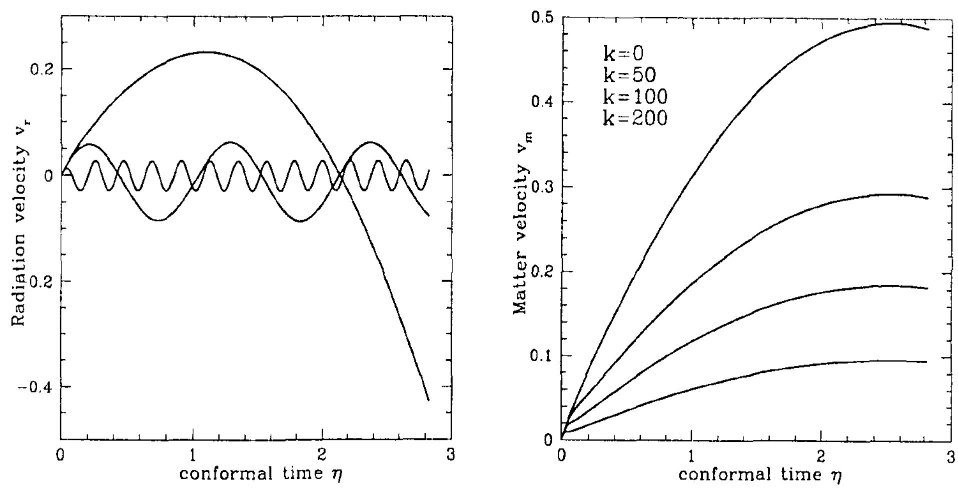
$$v_m' = -\frac{a'}{a}v_m + \Phi. \tag{2.76}$$

The evolution of v_m as given by the above equation may be plotted with respect to conformal time as in Figure 2.7(b).

2.3.3 Evolution of scalar field perturbations

If we take into consideration that the scalar field may not necessarily be homogeneous and may have evolving inhomogeneities, we must also derive the equation governing the perturbation to the scalar field. For this purpose we also need the perturbation to the Klein-Gordon equation. We know

$$\begin{aligned} \varphi^{;\mu} &= g^{\mu\nu}\varphi_{;\nu}, \\ &= g^{\mu\nu}\varphi_{,\nu}. \end{aligned} \tag{2.77}$$



(a) v_r for wave numbers $k = 0, 10$ and 50, in order of increasing frequency of oscillation.

(b) v_m for wave numbers $k = 0, 50, 100$ and 200 (from top to bottom).

Figure 2.7: The velocity potential perturbations of (a) radiation and (b) matter, v_r and v_m respectively, vs conformal time for various wave numbers.

Since for scalars, covariant derivatives may be replaced by partial derivatives, the perturbations to the derivative of the scalar field may be written as

$$\delta\varphi^{,\mu} = \delta g^{\mu\nu}\varphi_{,\nu} + g^{\mu\nu}\delta\varphi_{,\nu} \,. \tag{2.78}$$

For $\mu = 0$, this leads us to

$$\begin{aligned} \delta\varphi^{,0} &= -\frac{2\Phi}{a^2}\varphi' + \frac{1}{a^2}\delta\varphi' \,, \\ &= \frac{1}{a^2}[\delta\varphi' - 2\Phi\varphi'] \,. \end{aligned} \tag{2.79}$$

For $\mu = i$,

$$\delta\varphi^{,i} = -\frac{\gamma^{ij}}{a^2}\delta\varphi_{,j} \,. \tag{2.80}$$

Now, we are in a position to perturb the Klein-Gordon equation. We know

$$\varphi^{;\alpha}{}_{;\alpha} = \varphi^{,\alpha}{}_{;\alpha} \,,$$

$$= \frac{\partial \varphi^{,\alpha}}{\partial x^\alpha} + \Gamma_\alpha \varphi^{,\alpha} . \tag{2.81}$$

Therefore,

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) = \frac{\partial \delta\varphi^{,\alpha}}{\partial x^\alpha} + \delta\Gamma_\alpha \varphi^{,\alpha} + \Gamma_\alpha \delta\varphi^{,\alpha} . \tag{2.82}$$

This gives (see Appendix A.8)

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) = \frac{1}{a^2}\left[\delta\varphi'' + 2\frac{a'}{a}\delta\varphi' + 2\Phi a^2 V_{,\varphi} - 4\Phi'\varphi' + k^2\delta\varphi\right] . \tag{2.83}$$

So, the perturbed Klein-Gordon equation

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) + V_{,\varphi\varphi}\delta\varphi = 0 , \tag{2.84}$$

becomes

$$\delta\varphi'' + 2\frac{a'}{a}\delta\varphi' + a^2 V_{,\varphi\varphi}\delta\varphi + k^2\delta\varphi = -2a^2 V_{,\varphi}\Phi + 4\varphi'\Phi' . \tag{2.85}$$

Numerical solutions of quintessence inhomogeneities governed by the above equation shall be presented at the end of Chapter 3. There, it will be demonstrated that long-wave and short-wave perturbations exhibit markedly different behaviour.

This completes our discussion of all equations describing the evolution of inhomogeneities. The entire system of coupled differential equations outlined in this chapter when solved numerically, describes the growth of cosmological perturbations as functions of conformal time. All results presented in this thesis are computed in this manner.

Chapter 3

Analytical solutions

In this thesis, we shall approach the evolution of cosmological perturbations as a classical initial value problem. Therefore, together with the equations that govern the evolution of these perturbations, mentioned in the preceding chapter, we shall also need the initial values of these perturbations and their first derivatives. The derivation of these initial conditions, while important, are secondary, and have been inserted into Appendix A.9. In section 1 of this chapter we shall merely refer to these initial conditions to derive analytical solutions for background scalar field φ during the radiation and matter dominated eras. In particular, we derive simple power law solutions for the background scalar field and its energy density during these eras. In section 2 we shall turn to analytical solutions of the gravitational potential (Φ) in certain important limiting cases. They are, as follows, (i) the evolution of the zero mode (Φ when $k = 0$), in the absence of quintessence, valid for the entire history of the universe, (ii) the dominant modes of during the radiation dominated era and finally, (iii) the dominant modes during the matter dominated era. The focus of this thesis shall be to study the effect perturbations in quintessence have on the CMB anisotropies via the gravitational potential. To that end, in section 3, we shall derive a few analytical solutions for the evolution of these perturbations during the radiation dominated era for long and short wavelength modes separately. Later in section 3, the same is done for the matter dominated era. It is observed that long and short wavelength modes exhibit very different behaviour.

During the very early radiation dominated era, the energy content of the universe was dominated by radiation with matter being the subdominant component.

Therefore, during this era, we may approximate the first Friedman equation by ignoring the energy component of quintessence as follows

$$a'^2 = \Omega_m a + \Omega_r \,. \tag{3.1}$$

This has the solution

$$a = a_0 \eta(\eta + \eta_0) \,. \tag{3.2}$$

We get the following expressions

$$a_0 = \Omega_m / 4 \,, \tag{3.3}$$

$$\eta_0 = 4 \frac{\sqrt{\Omega_r}}{\Omega_m} \,. \tag{3.4}$$

The expression for the Hubble parameter, in conformal time, is then

$$\mathcal{H} = \frac{a'}{a} = \frac{2\eta + \eta_0}{\eta(\eta + \eta_0)} \,. \tag{3.5}$$

3.1 The background scalar field φ: Analytical solutions

In this section we shall derive a few analytical solutions for the evolution of the background (or unperturbed) scalar field in certain limiting cases. The scalar field is governed by the Klein-Gordon equation. This equation features a Hubble drag term which is dictated by the Friedman equation. Therefore, our task simplifies considerably if we solve the Klein-Gordon equation at epochs during which the Hubble term can be expressed simply. We shall show, in the succeeding sections, that in the radiation dominated era and the matter dominated era the Hubble factor is given by $\mathcal{H} = 2/\eta$ and $\mathcal{H} = 1/\eta$ respectively. This in turn leads to simple power law solutions of the scalar field φ in terms of the conformal time η during these two eras.

3.1.1 The scalar field φ during the radiation dominated era

During the radiation dominated era we may take $\eta \ll \eta_0$, and equations (3.2) and (3.3) simplify to

$$a = \sqrt{\Omega_r}\eta \,. \tag{3.6}$$

Substituting the above result into the Klein-Gordon equation, we get

$$\varphi'' + \frac{2}{\eta}\varphi' - \Omega_r \lambda \eta^2 \varphi^{-(n+1)} = 0\,, \tag{3.7}$$

where we assume the potential to have an inverse power law form as given by

$$V = \frac{\lambda \varphi^{-n}}{n}\,. \tag{3.8}$$

This equation has a special solution when the scalar field evolves as a power law given by

$$\varphi = A_r \eta^{\alpha_r}\,, \tag{3.9}$$

where

$$\begin{aligned} \alpha_r &= \frac{4}{n+2}\,, \\ A_r &= \left[\frac{\lambda \Omega_r (n+2)^2}{4(3n+10)}\right]^{\frac{1}{n+2}}\,. \end{aligned} \tag{3.10}$$

We may now use this solution for φ to calculate the energy of scalar field. The total energy of the scalar field may be written as

$$T_0^0 = \frac{1}{2a^2}\varphi'^2 + V(\varphi)\,. \tag{3.11}$$

Substituting the solution of the scalar field as given in equations (3.9) and (3.10), we get

$$T_0^0 = \frac{20 A_r^2}{n(n+2)\lambda}\eta^{-\frac{4n}{n+2}}\,. \tag{3.12}$$

The main result is then easy to read off. Apart from constants which characterize the potential of the scalar field and the energy densities of radiation, the total energy evolves as a simple power law of conformal time and is given by $T_0^0 \propto \eta^{-\frac{4n}{n+2}}$. For $n = 6$, this corresponds to $T_0^0 \propto \eta^{-3} \propto a^{-3}$.

This is the unique tracking solution of the system being considered. The condition for its existence is the dominance of the radiation energy over the energy in the quintessence field. Further, we shall consider perturbations of the scalar field φ, in particular, its homogeneous mode, where the value of the field is perturbed by the same amount everywhere in space. We will demonstrate that in the absence of metric perturbations, such homogeneous perturbations decay. Therefore, the tracking solution is an attractor for trajectories with perturbed initial conditions. In the presence of gravitational potential, the homogeneously perturbed system sets onto a neighbouring, coordinate-shifted tracking solution.

3.1.2 The scalar field φ during the matter dominated era

During the matter dominated era, we may assume $\eta \gg \eta_0$, but is not so large so that the scalar field starts to dominate the total energy content of the universe. Equations (3.2) and (3.3) then simplify to

$$a(\eta) = \frac{\Omega_m}{4}\eta^2 . \tag{3.13}$$

Substituting the above result into the Klein-Gordon equation, we get

$$\varphi'' + \frac{4}{\eta}\varphi' - \frac{\Omega_m^2\lambda}{16}\eta^4\varphi^{-(n+1)} = 0 . \tag{3.14}$$

Now, the special power-law tracking solutions is given by

$$\varphi = A_m\eta^{\alpha_m} , \tag{3.15}$$

where

$$\alpha_m = \frac{6}{n+2} , \tag{3.16}$$

$$A_m = \left[\frac{\lambda\Omega_m^2(n+2)^2}{96(5n+16)}\right]^{\frac{1}{n+2}} . \tag{3.17}$$

Proceeding as before, we see that the solution for the scalar field value leads to the following expression for the total energy of scalar field,

$$T_0^0 = \frac{768A_m^2}{n(n+2)\Omega_m^2}\eta^{-\frac{6n}{n+2}} . \tag{3.18}$$

Therefore, during the matter dominated era, the total energy of the scalar field $T_0^0 \propto \eta^{-\frac{6n}{n+2}}$. For $n = 6$, this dependence is $T_0^0 \propto \eta^{-9/2} \propto a^{-9/4}$.

3.2 Analytical solutions for the zero mode of Φ ($k = 0$)

We may begin by restating a few necessary equations in terms of a_0 and η_0 defined earlier in this chapter. The time-time linearized Einstein equation (2.54) becomes

$$-\frac{a'}{a}\Phi' - \left(\frac{a'}{a}\right)^2 \Phi - \frac{k^2}{3}\Phi = 2\frac{a_0}{a}\delta_m + \frac{a_0^2\eta_0^2}{2a^2}\delta_r\,. \tag{3.19}$$

The space-time part of the perturbed Einstein equations (2.55) becomes

$$\Phi' + \frac{a'}{a}\Phi \;=\; 6\frac{a_0}{a}v_m + \frac{2a_0^2\eta_0^2}{a^2}v_r\,. \tag{3.20}$$

The other equations for the variables $\delta_m, \delta_r, v_m, v_r$, given by (2.72), (2.71), (2.76) and (2.75), remain unmodified and are as follows

$$\delta_m' \;=\; 3\Phi' - k^2 v_m\,, \tag{3.21}$$

$$\delta_r' \;=\; 4\Phi' - \frac{4}{3}k^2 v_r\,, \tag{3.22}$$

$$v_m' \;=\; \Phi - \frac{a'}{a}v_m\,, \tag{3.23}$$

$$v_r' \;=\; \Phi + \frac{1}{4}\delta_r\,. \tag{3.24}$$

Now, we shall try to see how the gravitational potential Φ evolves with time. From the equations (3.19) and (3.20) we get

$$-\frac{k^2}{3}\Phi \;=\; 2\frac{a_0}{a}\left(\delta_m + 3\frac{a'}{a}v_m\right) + \frac{a_0^2\eta^2}{2a^2}\left(\delta_r + 4\frac{a'}{a}v_r\right)\,. \tag{3.25}$$

From the equations (3.22) and (3.24) we get the following 2^{nd} order ordinary differential equation (ODE) for v_r

$$v_r'' + \frac{k^2}{3}v_r \;=\; 2\Phi'\,. \tag{3.26}$$

For the special case when the wavenumber $k = 0$, equations (3.22) and (3.21) reduce to

$$\begin{aligned}\delta_r' &= 4\Phi'\,,\\ \delta_m' &= 3\Phi'\,.\end{aligned} \tag{3.27}$$

These equations have the following solutions

$$\begin{aligned}\delta_r &= 4\Phi - 6\Phi_0\,,\\ \delta_m &= 3\Phi - \frac{9}{2}\Phi_0\,,\end{aligned} \qquad (3.28)$$

where now $\Phi_0 = \Phi(k\eta \to 0)$. With a few intermediate steps hidden in Appendix A.10, we may judiciously combine the equations in this section to arrive at a first order ODE purely in terms of the gravitational potential Φ, given by

$$\Phi' + \Phi\left[\frac{3}{\eta} + \frac{3}{\eta+\eta_0} - \frac{1}{\eta+\eta_0/2}\right] = 3\left[\frac{3\eta^2 + 3\eta\eta_0 + \eta_0^2}{\eta(\eta+\eta_0)(2\eta+\eta_0)}\right]\Phi_0\,. \qquad (3.29)$$

The homogeneous solution to the above equation is given by

$$\Phi = C\left[\eta^{-3}(\eta+\eta_0)^{-3}(\eta+\eta_0/2)\right]\,. \qquad (3.30)$$

By the method of variation of constants we know

$$C' = \frac{3}{2}\frac{\eta^2(\eta+\eta_0)^2(3\eta^2+3\eta\eta_0+\eta_0^2)}{(\eta+\eta_0/2)^2}\Phi_0\,. \qquad (3.31)$$

Integrating the above expression for the value of C and substituting into (3.30) with the appropriate initial value, we get

$$\Phi(\eta) = \left[\frac{9}{10} + \frac{1}{10}\frac{\eta_0^2(\eta_0+\eta/2)}{(\eta+\eta_0)^3}\right]\Phi_0\,. \qquad (3.32)$$

The general solution drops out as it cannot satisfy the initial values. This solution may be substituted back into (3.28),(3.23) and (3.24) to get

$$\begin{aligned}\delta_r &= \left[-\frac{12}{5} + \frac{2}{5}\frac{\eta_0^2(\eta_0+\eta/2)}{(\eta+\eta_0)^3}\right]\Phi_0\,,\\ \delta_m &= \left[-\frac{9}{5} + \frac{3}{10}\frac{\eta_0^2(\eta_0+\eta/2)}{(\eta+\eta_0)^3}\right]\Phi_0\,,\\ v_r &= \left[\frac{3}{10} + \frac{1}{5}\frac{\eta_0(\eta_0+3\eta/4)}{(\eta+\eta_0)^2}\right]\Phi_0\,,\\ v_m &= \left[\frac{9}{20}\frac{\eta_0+2\eta/3}{\eta_0+\eta} + \frac{1}{20}\frac{\eta_0^2}{(\eta+\eta_0)^2}\right]\Phi_0\,.\end{aligned} \qquad (3.33)$$

Thus, for the homogeneous mode($k = 0$) in the absence of quintessence, we have exact solutions for the perturbation variables $\Phi, \delta_r, \delta_m, v_r$ and v_m. Combining the equations for δ_r and Φ above, we may also derive another expression which shall be useful in the formula for the temperature fluctuations of photons. This is given by

$$\Phi + \delta_r/4 = \left[\frac{3}{10} + \frac{1}{5}\frac{\eta_0^2(\eta_0 + \eta/2)}{(\eta + \eta_0)^3}\right]\Phi_0 \,. \tag{3.34}$$

3.2.1 The dominant modes during the radiation dominated era

In the radiation dominated era, we may use the approximations $a = a_0\eta\eta_0$ and $\eta \ll \eta_0$. With these approximations we can keep the most dominant terms on both sides of the Einstein equations. The time-time part (3.19) gives us

$$-\frac{k^2}{3}\Phi = \frac{1}{2\eta^2}\left(\delta_r + \frac{4}{\eta}v_r\right) \,. \tag{3.35}$$

The space-space part (3.20) gives us

$$\Phi' + \frac{1}{\eta}\Phi = \frac{2}{\eta^2}v_r \,. \tag{3.36}$$

The equation for vanishing covariant divergence remains

$$\frac{1}{4}\delta_r' + \frac{k^2}{3}v_r = \Phi' \,. \tag{3.37}$$

Differentiating (3.24) and combining with (3.37), we get

$$v_r'' + \frac{k^2}{3}v_r = 2\Phi' \,. \tag{3.38}$$

Combining equations (3.35) and (3.36)

$$-\frac{1}{\eta}\Phi' - \left(\frac{1}{\eta^2} + \frac{k^2}{3}\right)\Phi = \frac{1}{2\eta^2}\delta_r \,. \tag{3.39}$$

Eliminating δ_r from the above, using (3.24), we get

$$-\frac{1}{\eta}\Phi' + \left(\frac{1}{\eta^2} - \frac{k^2}{3}\right)\Phi = \frac{2}{\eta^2}v_r \,. \tag{3.40}$$

Eliminating v_r' from the above, using (3.36), we get

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{k^2}{3}\Phi = 0 . \tag{3.41}$$

This is a Bessel ODE of the first order and has the following general solution

$$\Phi = \frac{C_1}{(k\eta)^2}\left(\cos\frac{k\eta}{\sqrt{3}} - \frac{\sqrt{3}}{k\eta}\sin\frac{k\eta}{\sqrt{3}}\right) + \frac{C_2}{(k\eta)^2}\left(\sin\frac{k\eta}{\sqrt{3}} - \frac{\sqrt{3}}{k\eta}\cos\frac{k\eta}{\sqrt{3}}\right) \tag{3.42}$$

Applying the initial conditions as $k\eta \to 0$, we get $C_2 = 0$ and $\Phi(0) = -\frac{C_1}{9}$, which gives us $C_1 = -9\Phi_0$. The solution to Φ then becomes

$$\Phi(x) = \frac{3}{x^2}\left(\frac{\sin x}{x} - \cos x\right)\Phi_0 \tag{3.43}$$

where $x = \frac{k\eta}{\sqrt{3}}$.

3.2.2 The dominant modes during the matter dominated era

In the matter dominated era, we have $a = a_0\eta^2$. It follows $a' = 2a_0\eta$ and $a'/a = 2/\eta$. In this regime, the first Einstein equation becomes

$$-\frac{k^2}{3}\Phi = \frac{2}{\eta^2}\left(\delta_m + \frac{6}{\eta}v_m\right) + \frac{\eta_0^2}{2\eta^4}\left(\delta_r + \frac{8}{\eta}v_r\right) . \tag{3.44}$$

In the limit as $k \to 0$ we get $\delta_m = -\frac{6}{\eta}v_m$, and $\delta_r = -\frac{8}{\eta}v_r$. In the matter dominated era $\eta \gg \eta_0$ and the space-time part of the perturbed Einstein equations may be approximated as follows

$$\Phi' + \frac{a'}{a}\Phi = \frac{6a_0}{a}v_m . \tag{3.45}$$

The equation for v_m may be written out for convenience as

$$(av_m)' = a\Phi . \tag{3.46}$$

These two equations may be combined to eliminate v_m into the following single 2^{nd} order ODE given by

$$\Phi'' + 3\frac{a'}{a}\Phi' + \left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 - \frac{6a_0}{a}\right]\Phi = 0 . \tag{3.47}$$

The coefficient Φ conveniently drops off to zero to give

$$\Phi'' + 3\frac{a'}{a}\Phi' = 0\,. \tag{3.48}$$

Integrating this equation, we get

$$\begin{aligned}\Phi &= \Phi_m + \int \frac{C}{a^3}\,, \\ &= \Phi_m + \frac{C}{\eta^5}\,. \end{aligned} \tag{3.49}$$

Substituting this into the formula for v_m above, this gives us

$$v_m = \frac{1}{3}\Phi_m \eta\,. \tag{3.50}$$

For δ_m, we get

$$\delta_m = -\frac{k^2\eta^2}{6}\Phi_m\,. \tag{3.51}$$

3.3 Scalar field perturbations

We may now study the growth of scalar field perturbations. Though these perturbations were formerly denoted by $\delta\varphi$, for convenience we now denote them by φ_1. The perturbed Klein-Gordon equation (KGE) then reads

$$\varphi_1'' + k^2\varphi_1 + 2\mathcal{H}\varphi_1' + a^2 V_{,\varphi\varphi}\,\varphi_1 = -a^2 2V_{,\varphi}\,\Phi + 4\varphi'\Phi'\,. \tag{3.52}$$

For the inverse power law potential, the first and second order derivatives are given by $V_{,\varphi} = -\lambda\varphi^{-(n+1)}$ and $V_{,\varphi\varphi} = \lambda(n+1)\varphi^{-(n+2)}$.

3.3.1 Long wavelength scalar field perturbations (φ_1 as $k \to 0$) during the radiation dominated era

During the radiation dominated era the first constant mode for the gravitational potential Φ is dominant and the second forcing term involving Φ' may be ignored. So, $\Phi \approx$ *const.* In the long wavelength approximation $k^2\varphi_1 \to 0$ and from the perturbed KGE, we get

$$\varphi_1'' + \frac{2}{\eta}\varphi_1' + \frac{C_r}{\eta^2}\varphi_1 = D_r A_r \Phi \eta^{-2+\alpha_r}\,, \tag{3.53}$$

where $C_r = \frac{4(n+1)(3n+10)}{(n+2)^2}$ and $D_r = \frac{8(3n+10)}{(n+2)^2}$. This has the particular solution given by

$$\begin{aligned} \varphi_{1p} &= \frac{D_r}{\alpha_r(\alpha_r+3)+C_r} A_r \eta^{\alpha_r} \Phi\,, \\ &= \frac{2}{n+2}\left(A_r \eta^{\alpha_r}\right)\Phi = \frac{2}{n+2}\varphi\Phi\,, \end{aligned} \tag{3.54}$$

which grows with time proportionally to the background field. The homogeneous solution for the above equation denoted by φ_{1h} can be verified by the power series ansatz to be

$$\varphi_{1h} = \eta^{-1/2}\left[A\eta^{i\sqrt{4C_r-1}} + B\eta^{i\sqrt{4C_r-1}}\right]\,, \tag{3.55}$$

or, equivalently,

$$\varphi_{1h} = \eta^{-1/2}\left(A\cos\left[\log(\eta\sqrt{4C_r-1})\right] + B\sin\left[\eta\log(\sqrt{4C_r-1})\right]\right)\,. \tag{3.56}$$

The homogeneous modes decay with time and are subdominant wrt to the particular solution. This is typical of attractor solutions, where solutions with slightly varying initial conditions, converge to a certain attractor over time. In the absence of gravitational effects, any solution perturbed homogeneously from the tracking solution will approach it. In the presence of the gravitational potential Φ there is an offset to a neighbouring solution.

3.3.2 Short wavelength scalar field perturbations (φ_1 as $k \gg 1/\eta$) during the radiation dominated era

In the short wavelength approximation $k^2 \gg C/\eta^2$, the perturbed KGE reduces to

$$\varphi_1'' + \frac{2}{\eta}\varphi_1' + k^2\varphi_1 = D_r A_r \eta^{-2+\alpha}\Phi + 4\varphi'\Phi'\,. \tag{3.57}$$

The homogeneous solutions may be found by converting the homogeneous equation to Bessel's equation and are given by

$$\varphi_{1h}(\eta) = \frac{e^{ik\eta}}{\eta}\,, \varphi_{2h}(\eta) = \frac{e^{-ik\eta}}{\eta}\,. \tag{3.58}$$

For short wavelengths, the gravitational potential Φ is dominated by the second mode given in (3.43). Expanding in a series about $\eta = 0_+$ and retaining terms only till the order of $O(k\eta)$, we get the following for the complete forcing term,

$$\begin{aligned} F(\eta) &= -\frac{(12\alpha_r + 2D_r)}{k/\sqrt{3}} A_r \eta^{-5+\alpha_r} \cos\frac{k\eta}{\sqrt{3}}, \\ &+ (4\alpha_r + 2D_r) A_r \eta^{-4+\alpha_r} \sin\frac{k\eta}{\sqrt{3}} - (4\alpha_r k/\sqrt{3}) A_r \eta^{-3+\alpha_r} \cos\frac{k\eta}{\sqrt{3}} . \end{aligned} \quad (3.59)$$

For very short wavelengths we may approximate $k\eta \gg 1$. In such a case, the contribution from the 1st and 2nd terms become negligible in comparison to the third term and we may approximate the forcing term to be

$$F(\eta) = -L_3 k\eta^{-3+\alpha_r} \cos\frac{k\eta}{\sqrt{3}}, \quad (3.60)$$

where $L_3 = 4\alpha_r A_r/\sqrt{3}$. Substitute $\varphi_1 = u_1/\eta$ and the resulting equation may be approximated as follows

$$u_1'' + k^2 u_1 = -L_3 k\eta^{-2+\alpha_r} \cos\left(k\eta/\sqrt{3}\right). \quad (3.61)$$

This equation describes a simple harmonic oscillator (SHO) with a forcing term $F_s(\eta)$ which gradually decays with time and is given by

$$F_s(\eta) = -L_3 k\eta^{-2+\alpha_r} \cos(k\eta/\sqrt{3}). \quad (3.62)$$

The causal Green's function $G_s(\eta, \eta_1)$ for the SHO is given by

$$G_s(\eta, \eta_1) = \sin k(\eta - \eta_1). \quad (3.63)$$

Thus, the particular solution is

$$u_1(\eta) = \int_0^\eta G_s(\eta, \eta_1) F_s(\eta_1) d\eta_1, \quad (3.64)$$

which reduces to

$$\begin{aligned} u_1(\eta) &= -\sqrt{3} L_3 \int \eta_1^{-2+\alpha} \cos(k\eta_1/\sqrt{3}) \sin k(\eta - \eta_1) d\eta_1, \\ &= -\sqrt{3} L_3 \int \eta_1^{-2+\alpha} \left(\sin\left[k\eta - k(1 - \frac{1}{\sqrt{3}})\eta_1\right] + \sin\left[k\eta - k(1 + \frac{1}{\sqrt{3}})\eta_1\right]\right) d\eta_1 . \end{aligned}$$

Since the integrand is a highly oscillatory function for large k, the integral may be approximated by the method of stationary phase [5] to be

$$u_1(\eta) = -\sqrt{3}L_3\eta_0^{-2+\alpha}\left(\frac{\cos\left[k\eta - k(1-\frac{1}{\sqrt{3}})\eta_0\right]}{k(1-\frac{1}{\sqrt{3}})} + \frac{\cos\left[k\eta - k(1+\frac{1}{\sqrt{3}})\eta_0\right]}{k(1+\frac{1}{\sqrt{3}})}\right). \tag{3.65}$$

Ignoring the phase term corrections due to η_0, we get

$$u_1(\eta) \sim -\left[3\sqrt{3}L_3\eta_0^{-2+\alpha}\right]\frac{\cos(k\eta)}{k}. \tag{3.66}$$

Therefore, the dominant term in the scalar field inhomogeneity φ_1 for short wavelengths is given by

$$\varphi_1(\eta) \sim -K_1\frac{\cos(k\eta)}{k\eta}, \tag{3.67}$$

where K_1 is a constant whose value is set at the onset of the radiation stage and is given by $K_1 = [3\sqrt{3}L_3\eta_0^{-2+\alpha}]$. Though crude, and based on many approximations, it has a ready qualitative explanation: scalar field perturbations on short wavelengths are damped as $\sim \frac{1}{k\eta}$ and the forcing term is actually unimportant.

3.3.3 Long wavelength scalar field perturbations (φ_1 as $k \to 0$) during the matter dominated era

We may largely repeat the same analysis for the matter dominated era, where Φ, given by (3.49) is again nearly constant. In the long wave regime the perturbed KGE becomes

$$\varphi_1'' + \frac{4}{\eta}\varphi_1' + \frac{C_m}{\eta^2}\varphi_1 = D_m A_m \eta^{-2+\alpha_m}\Phi, \tag{3.68}$$

where $C_m = \frac{6(n+1)(5n+16)}{(n+2)^2}$ and $D_m = \frac{12(5n+16)}{(n+2)^2} = 2C_m/(n+1)$. The particular power-law solution of this equation is again

$$\begin{aligned}\varphi_{1p} &= \frac{D_m}{\alpha_m(\alpha_m+5)+C_m}A_m\eta^{\alpha_m}\Phi, \\ &= \frac{2}{n+2}\left(A_m\eta^{\alpha_m}\right)\Phi = \frac{2}{n+2}\varphi\Phi.\end{aligned} \tag{3.69}$$

We observe that the long-wave response to the gravitational potential given above is of the same form in the radiation and matter dominated eras and is independent of the type of dominant component. It may also be observed that the homogeneous solution for the scalar field perturbation is given by

$$\begin{aligned} \varphi_{1h} &= \frac{e^{i\sqrt{C_m-2}\eta}}{\eta^2}, \\ \varphi_{2h} &= \frac{e^{-i\sqrt{C_m-2}\eta}}{\eta^2}. \end{aligned} \tag{3.70}$$

Apart from an oscillatory factor, these general homogeneous solutions are damped out as $\sim \eta^{-2}$.

3.3.4 Short wavelength scalar field perturbations (φ_1 as $k \gg 1/\eta$) during the matter dominated era

In the short wavelength limit and in the matter dominated stage, the perturbed KGE may be approximated as

$$\varphi_1'' + \frac{4}{\eta}\varphi_1' k^2 \varphi_1 = F(\eta). \tag{3.71}$$

Substituting $\varphi_1 = u_1/\eta^2$ this again describes a SHO for u_1. We may yet again apply the Greens function in (3.63) and (3.64) to get

$$u_1 = \int \sin 2k(\eta - \eta_1) D_m \eta_1^{\alpha_m} d\eta_1 . \tag{3.72}$$

Since this is a highly oscillatory integral, we may justifiably approximate it using the method of stationary phase as in the radiation dominated era to get,

$$u_1 \sim \frac{\cos(k\eta)}{k}. \tag{3.73}$$

Therefore, the scalar field inhomogeneity is given by

$$\varphi_1 \sim \frac{\cos(k\eta)}{k\eta^2}. \tag{3.74}$$

This section may be now summarized as follows. Long wavelength perturbations exhibit attractor type solutions in both radiation and matter dominated eras. In the radiation dominated era the dominant solution grows as a power of the conformal time. Short wavelength perturbations to the scalar field on the other hand are severely damped down in both eras. During the radiation dominated era, they evolve as $\sim \frac{1}{k\eta}$, and during the matter dominated era they evolve as $\sim \frac{1}{k\eta^2}$. As an illustration of the foregoing analysis, numerical calculations of $\delta\varphi$ for the wave numbers $k = 0, 10, 50, 100$ and 200 are plotted in Figure 3.1. Here we see a pattern

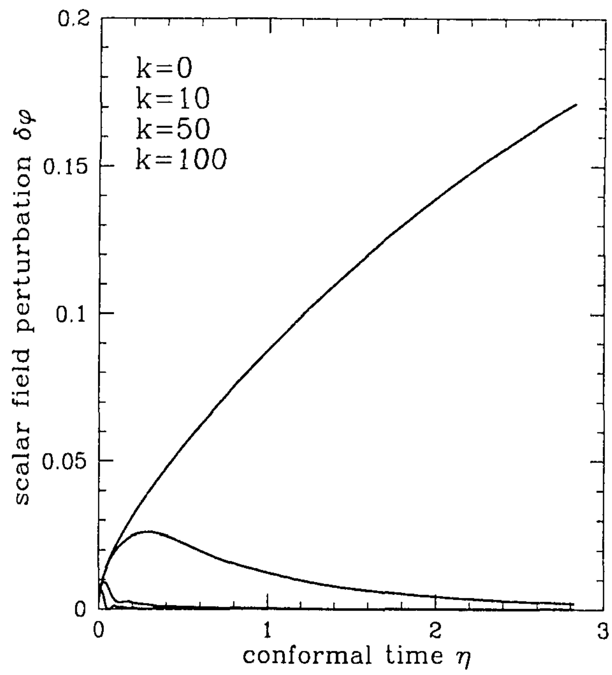


Figure 3.1: The evolution of the scalar field inhomogeneities for wave numbers $k = 0, 10, 50, 100$ and 200

similar to that for the gravitational potential Φ. The lowest wave number modes vary slowly. The higher wave number modes vary more violently, especially before the time of recombination. We also observe that long wavelength perturbations remain siginificant well into the matter dominated era, and till the present time. Short wavelength perturbations on the contrary are damped down and reach a steady state value close to zero during the matter dominated era. These modes therefore have negligible effect on the evolution of Φ. More relevantly for this thesis, their effect on photon temperature fluctuations, and therefore CMB anisotropies to be introduced in the next chapter, shall also be negligible.

Chapter 4

Quintessence and large scale CMB anisotropy

4.1 The Cosmic Microwave Background

The discovery of the Cosmic Microwave Background (CMB) by Penzias and Wilson marks the birth of modern, precise cosmology. The CMB has a perfect blackbody spectrum at a temperature of $T_0 = 2.726 \pm 0.010K$ (95 % Cl) and temperature anisotropies at the level of one part in 10^5 [16]. The big bang cosmology, almost uniquely amongst cosmological models, predicts such a radiation background. In the big bang model the Universe undergoes an expansion from some initial singularity and light from distant sources is redshifted in proportion to distance. Correspondingly, more distant sources emitted the light a longer time ago, when the universe was smaller. Due to the expansion of the universe the wavelengths of photons were stretched and particle number densities dropped leading to the low temperatures and photon densities observed today. This explains why the spectrum is thermal at $2.7K$, a temperature much lower than other matter in the universe. Conversely, extrapolating backwards in time, we may infer that the Universe began in a hot dense state. At sufficiently high temperatures, interactions between particles were sufficiently rapid for the universe to be in thermal equilibrium. Adiabatic cooling from the expansion preserves such a spectrum and this explains the blackbody nature of the thermal spectrum.

The high degree of isotropy observed in the CMB is more puzzling. In the early universe radiation interacted with matter through Compton scattering. After the time of recombination, at redshift $z \sim 10^3$, photons no longer had the energy to keep hydrogen photo-ionized. Therefore the time of recombination also marks the time the CMB last interacted with matter. At such early times, the patches of sky off which the CMB last scattered should not have been in causal contact. This apparent violation of causality has been dubbed *the horizon problem.* The theory of inflation proposed by Guth [15] alleviates it by postulating a very early phase of rapid expansion that separates originally causally connected regions by vast distances necessary to account for the large scale isotropy of the CMB.

Measurements of the thermal nature and isotropy of the CMB support the overall hot big bang model. Anisotropies of the CMB on the other hand carry information about the fluctuations which led to structure formation in the universe. A few facts and figures may be illuminating. Hubble's law states that the observed redshift scales with distance as $z = H_0 d$ due to the uniform expansion of the Universe. The Hubble constant $H_0 = 100 h kms^{-1} Mpc^{-1}$ where observations require $h \sim 0.7$. H_0 also sets the expansion time scale $H_0^{-1} \sim 10h^{-1} Gyr$ and thereby determines the age of the Universe.

In general relativity, mass tends to decelerate the expansion and a higher energy density implies a younger universe. The mass is usually parameterized by Ω_m which is the energy density in units of the critical density $\rho_{crit} = 3H_0^2/8\pi G = 1.879 * 10^{-29} h^2 g/cm^3$. In this thesis, we must also consider the possibility that vacuum energy and pressure in the form of a cosmological constant, or a quintessence scalar field, can provide an acceleration to the expansion of the Universe. The ratio of the energy density of this component to ρ_{crit} may similarly be expressed as $\Omega_{q,\Lambda}$, where the subscripts q and Λ denote quintessence and the cosmological constant respectively. For a spatially flat universe, $\Omega_m + \Omega_{q,\Lambda} = 1$. Dynamical measurements of the mass in the halo of galaxies implies $\Omega_m \geq 0.1 - 0.3$. Luminous matter in the form of stars in the central part of galaxies account for $\Omega_* \sim 0.04$ of the critical density. This implies the existence of a significant amount of nonbaryonic dark matter. Collisionless dark matter, unlike baryonic matter, does not undergo dissipative processes. The CMB energy density $\Omega_r h^2 = 2.38 * 10^{-5} \theta_{2.7}^4$ where $\theta_{2.7} = T_0/2.7K$. Although negligible today, in the early universe it increases in importance relative to the energy density ρ_m, since $\rho_r/\rho_m \propto 1 + z$, due to the redshift. The photon density is thus fixed through the CMB temperature.

4.2 The Sachs-Wolfe effect

Large-scale anisotropies are not affected by any local microphysics at last scattering. At the time of recombination, the perturbations responsible for these anisotropies were on scales far larger than could be connected by causal processes. Since in this thesis, we are only concerned with large-scale anisotropies, we restrict our attention to the primary mechanism which generates them, namely, the Sachs-Wolfe effect. In order to derive it we shall work in the conformal Newtonian Gauge with coordinates (η, x^i).

On large scales a fractional change in photon temperature in the CMB may be caused by two distinct mechanisms;

1. An intrinsic temperature fluctuation on the surface of last scattering during recombination denoted here by $\frac{\Delta T}{T}_{rec}$, T being the temperature, due to a local over(under) density of photons.

2. A temperature fluctuation incurred during the photon's journey from the surface of last scattering to the observer, denoted by $\frac{\Delta T}{T}_{jour}$.

Thus, the total fractional change in temperature is given by

$$\frac{\Delta T}{T} = \frac{\Delta T}{T}_{rec} + \frac{\Delta T}{T}_{jour} . \tag{4.1}$$

The formula for $\frac{\Delta T}{T}_{rec}$ is easy to derive. The energy density of photons ρ_r is proportional to T^4. Then, for a fractional change in energy density $\delta_r = \frac{\delta\rho_r}{\rho_r}$ at the surface of last scattering, we have

$$\frac{\Delta T}{T}_{rec} = \frac{1}{4}\delta_r . \tag{4.2}$$

The derivation of the formula for $\frac{\Delta T}{T}_{jour}$ is a bit more involved. We begin by noting that the CMB has an almost perfect blackbody spectrum with a distribution function given by

$$f(q) = \frac{1}{e^{q/kT} - 1} , \tag{4.3}$$

where q and T are the total momentum magnitude and temperature respectively. Since the CMB cools adiabatically, we may infer that the distribution function remains unchanged. Therefore, at any instant of time during the journey

$$\frac{\Delta T}{T}(\eta) = \frac{\Delta q}{q}(\eta) . \tag{4.4}$$

The total change in momentum Δq may be expressed as $\Delta q = \frac{dq}{d\eta}d\eta$. In order to get the cumulative temperature fluctuation due to the entire journey, we may integrate the above expression wrt η, to get

$$\frac{\Delta T}{T}_{jour} = \int_{\eta_{rec}}^{\eta_0} \frac{1}{q}\frac{dq}{d\eta}d\eta\,. \tag{4.5}$$

We may now proceed to convert the above expression into a more convenient one. In the absence of perturbations, the particle momentum "redshifts" as $1/a$. We assume collisions do not alter the energy of a species, and do not create or destroy particles, as from Thomson scattering.

It is convenient to make the following definitions:

$$\begin{aligned} \mathbf{q} &= a\mathbf{p}\,, \\ \mathbf{n} &= \frac{\mathbf{q}}{q}\,(unit\,vector)\,, \\ \epsilon &= aE = (q^2 + m^2a^2)^{1/2}\,. \end{aligned}$$

To get the deviation from the unperturbed "redshift" $p \propto 1/a$, we must evaluate $\frac{dq}{d\eta}$. To do so, we need to relate (E, p^i), the four-momentum components in the locally orthonormal frame to those in the conformal Newtonian coordinate system denoted by P^μ. The necessary relations are

$$\begin{aligned} P^0 &= g_{00}^{1/2}E = a(1+\Phi)E\,, \\ P^i &= g_{ii}^{1/2}p^i = a(1-\Phi)p^i\,, \end{aligned}$$

The geodesic equation for a photon is given by

$$\frac{dP^\mu}{d\eta} = g^{\mu\nu}\left(\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)\frac{P^\alpha P^\beta}{P^0}\,, \tag{4.6}$$

Evaluating the time component along the particle trajectory $\frac{dx^i}{d\eta} = n^i$, we find to 1^{st} order,

$$\frac{1}{q}\frac{dq}{d\eta} = \frac{\partial\Phi}{\partial\eta} - \frac{\epsilon}{q}n^i\frac{\partial\Phi}{\partial x^i}\,. \tag{4.7}$$

For a photon $\epsilon = q$, since

$$\frac{d\Phi}{d\eta} = \frac{\partial\Phi}{\partial\eta} + \frac{dx^i}{d\eta}\frac{\partial\Phi}{\partial x^i}\,,$$

$$= \frac{\partial \Phi}{\partial \eta} + n^i \frac{\partial \Phi}{\partial x^i} . \tag{4.8}$$

We may use the above to eliminate the troublesome second term in (4.7) to get

$$\frac{1}{q}\frac{dq}{d\eta} = 2\frac{\partial \Phi}{\partial \eta} - \frac{d\Phi}{d\eta} . \tag{4.9}$$

Combining the above with (4.5), we get

$$\begin{aligned} \frac{\Delta T}{T}_{jour} &= \int_{\eta_{LS}}^{\eta_0} \frac{1}{q}\frac{dq}{d\eta} d\eta , \\ &= \Phi(x_{rec}, \eta_{rec}) - \Phi(x_0, \eta_0) + 2\int_{\eta_{rec}}^{\eta_0} \frac{\partial \Phi(ex, \eta)}{\partial \eta} d\eta . \end{aligned} \tag{4.10}$$

The second term of this expression does not contribute to the anisotropy and may be dropped. Summing up the terms given in (4.2) and (4.10) we get the following for the total temperature fluctuation

$$\frac{\Delta T}{T} = \left(\frac{1}{4}\delta_r + \Phi\right)(x_{rec}, \eta_{rec}) + 2\int_{\eta_{rec}}^{\eta_0} \frac{\partial \Phi(ex, \eta)}{\partial \eta} d\eta . \tag{4.11}$$

It is conventional to denote the first term, which is due to values on the surface of last scattering (SLS) the Naive Sachs-Wolfe (NSW) effect. Correspondingly, the integral term, due to the properties of the intervening medium between the SLS and the observer, is called the Integrated Sachs-Wolfe (ISW) effect. We use this expression to evaluate the temperature fluctuations and thereafter, the CMB anisotropies in this document. The standard expression most often used in literature [18],[20], which differs from that above is now stated.

In the low wavelength limit, well after the onset of the matter dominated era, we shall demonstrate in the next chapter that

$$\delta_r \approx -\frac{8}{3}\Phi , \tag{4.12}$$

As a further approximation, the lower limit integral can be taken to be 0 instead of η_{rec}. Therefore, the total temperature fluctuation now goes as

$$\frac{\Delta T}{T} = \frac{1}{3}\Phi(x_{rec}, \eta_{rec}) + 2\int_0^{\eta_0} \frac{\partial \Phi(ex, \eta)}{\partial \eta} d\eta . \tag{4.13}$$

This convention was first used by Kofman and Starobinskii [18] and later popularized by Liddle and Lyth [20]. We stress the fact that this often quoted result is an approximation to the more rigorous result earlier. Wherever possible, we shall try to show the divergence between the exact result and the approximate one. A pedagogical derivation for the Naive Sachs-Wolfe effect is given below. Potential perturbations at last scattering have two effects:

1. They redshift the photons we see so that an overdensity cools the background as the photons climb out of the potential wells $\frac{\Delta T}{T} = \Phi$.

2. They cause time dilation at the last scattering surface, so that we are looking at a younger and hence hotter universe where there is an overdensity of photons. The time dilation is $\frac{dt}{t} = \Phi$. Assuming recombination occurs at the time of matter domination, the scale factor $a \propto t^{2/3}$ and $T \propto 1/a$, which produces a couterterm $\frac{\Delta T}{T} = -\frac{2}{3}\Phi$. The net effect is thus: $\frac{\Delta T}{T} = \frac{1}{3}\Phi$.

4.3 CMB angular power spectra

The Sachs-Wolfe effect we have just derived is applicable to a single photon. We shall now try to quantify the observed temperature fluctuations field $\frac{\Delta T}{T}$ with a statistical measure: the angular autocorrelation function of the field of temperature fluctuations. This field inhabits a universe that is isotropic and homogeneous in its large scale properties. This suggests that the autocorrelation function should also be homogeneous even though it is a field that describes inhomogeneities.

As a preparatory step, we must decompose these fluctuations into a set of modes. Since in a flat comoving geometry the Fourier basis form a complete set of basis functions, we shall take the Fourier transform of (4.11) over space, to get

$$\begin{aligned}\frac{\Delta T}{T}(\hat{q},\hat{k}) &= a_k\Big[(\delta_r/4+\Phi)(\tau_{rec},k)e^{ik.\hat{q}(\tau_0-\tau_{rec})} \\ &\quad +2\int_{\tau_{rec}}^{\tau_0} d\tau\Phi'(\tau,k)e^{ik.\hat{q}(\tau_0-\tau)}\Big]. \qquad (4.14)\end{aligned}$$

Here, a_k denotes the randomization operator. It is used to denote that the coefficient of each mode is a random variable. In this context, it denotes that denotes that each mode has a random phase. The mean of each coefficient is zero. We may now use the expansion for a plane wave in terms of spherical harmonics

$$e^{ik.\hat{q}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^m(\hat{k}) Y_l^{m*}(\hat{q}), \tag{4.15}$$

to get,

$$\begin{aligned} \frac{\Delta T}{T}(\hat{q},\hat{k}) &= a_k \Bigg[4\pi \sum_{lm} i^l \{(\delta_r/4 + \Phi)(\tau_{rec},k) j_l(k\tau_0 - \tau_{rec}) \;, \\ &\quad +2\int_{\tau_{rec}}^{\tau_0} d\tau \Phi'(\tau,k) j_l(k\tau_0 - \tau)\} Y_l^m(\hat{k}) Y_l^{m*}(\hat{q}) \Bigg] \,. \end{aligned} \tag{4.16}$$

For economy, we may rewrite the above as

$$\frac{\Delta T}{T}(\hat{q},\hat{k}) = a_k \left[4\pi \sum_{ml} i^l \{NSW(k) + ISW(k)\} Y_l^m(\hat{k}) Y_l^{m*}(\hat{q}) \right], \tag{4.17}$$

where the $NSW(k)$ and $ISW(k)$ are the Naive and Integrated Sachs-Wolfe terms respectively, defined as,

$$\begin{aligned} NSW(k) &= (\delta_r/4 + \Phi)(\tau_{rec},k) j_l(k\tau_0 - \tau_{rec}), \\ ISW(k) &= 2\int_{\tau_{rec}}^{\tau_0} d\tau \Phi'(\tau,k) j_l(k\tau_0 - \tau) \,. \end{aligned} \tag{4.18}$$

The autocorrelation of the fluctuation modes $\frac{\Delta T}{T}(k)$ in terms of the coefficients of spherical harmonics is then given by

$$\begin{aligned} \left\langle \frac{\Delta T}{T}_{lm}(\hat{k}), \frac{\Delta T^*}{T}_{l'm'}(\hat{k'}) \right\rangle &= \int \langle a_k, a^*_{k'} \rangle \Big[16\pi^2 i^{l'-l} \{NSW(k') + ISW(k')\} \\ &\quad \{NSW(k) + ISW(k)\} Y_{l'm'}(\hat{k}') Y_{lm}(\hat{k}) d^3k d^3k' \Big] \,. \end{aligned}$$

The angle brackets denote an averaging over the normalization volume V. Apart from the statistical isotropy of the temperature fluctuation field, we might also make the reasonable assumption that the phases of the different Fourier modes $\frac{\Delta T}{T}(k)$ are uncorrelated and random. This corresponds to treating the initial disturbance as a form of random noise, analogous to Johnson noise in electrical circuits. Each mode is uncorrelated to the other. A given mode has nonzero variance, so

$$\langle a_k, a^*_{k'} \rangle = P_a(k) \delta^3(k - k') \,. \tag{4.19}$$

where δ^3 is the three dimensional Dirac delta-function, and $P_a(k)$ is the spectrum of the modes, and is known as the Harrison-Peebles-Zeldovich spectrum or more briefly the Zeldovich spectrum. This spectrum has the property that $k^3 P_a(k)$ remains constant and is hence scale-invariant. This property is explained by Guth's theory of inflation [15]. However, we resist the temptation to digress and proceed with the implications of such a spectrum. We get, apart from a few constants,

$$\left\langle \frac{\Delta T}{T}_{lm}(\hat{k}), \frac{\Delta T^*}{T}_{l'm'}(\hat{k}) \right\rangle = \int \frac{1}{k^3} \left[i^{l'-l} \{NSW(k) + ISW(k)\}^2 Y_{l'm'}(\hat{k}) Y_{lm}(\hat{k}) d^3k \right] .$$

For convenience, we may perform the above integral over k in spherical coordinates. The measure may be written $d^3k = k^2 dk d\Omega_k$. Since the spherical harmonics satisfy the orthonormality criterion,

$$\int Y_{l'm'}(\hat{k}) Y_{lm}(\hat{k}) d\Omega_k = \delta_{l'l}\delta_{mm'} \, , \tag{4.20}$$

the relation (4.3) simplifies to

$$\left\langle \frac{\Delta T}{T}_{lm}(\hat{k}), \frac{\Delta T^*}{T}_{l'm'}(\hat{k}) \right\rangle = i^{l'-l} \delta_{l'l}\delta_{mm'} \int \{NSW(k) + ISW(k)\}^2 \frac{dk}{k} . \tag{4.21}$$

We observe that only the diagonal components are nonzero. It is customary to denote these components by C_l. The definition for the coefficients C_l then follows,

$$C_l = \int \{NSW(k) + ISW(k)\}^2 \frac{dk}{k} . \tag{4.22}$$

It is noteworthy that any azimuthal dependence represented by the index m has disappeared from the expression. These components are a function only of the angular separation represented by the index l. The coefficients C_l completely define the angular autocorrelation function. The C_l's thus defined are variously called "Multipoles", "Angular Spectra", "CMB spectra" and even very misleadingly "Power spectra". In the remainder of the thesis we shall adhere to the most conventional term: "Multipoles". In the convention of Liddle and Lyth, which we mentioned earlier in this chapter, (refer (4.13)), the Naive and Integrated Sachs-Wolfe terms have the following approximations

$$NSW_{ll}(k) = \frac{1}{3}\Phi(\tau_{rec}, k) j_l(k\tau_0 - \tau_{rec}) \, ,$$

$$ISW_{ll}(k) \;=\; 2\int_0^{\tau_0} d\tau \Phi'(\tau,k) j_l(k\tau_0 - \tau)\,. \tag{4.23}$$

The corresponding multipoles are then defined by

$$C_{l,ll} = \int \{NSW_{ll}(k) + ISW_{ll}(k)\}^2 \frac{dk}{k}\,. \tag{4.24}$$

These different expressions for the Sachs-Wolfe effect can result in markedly different

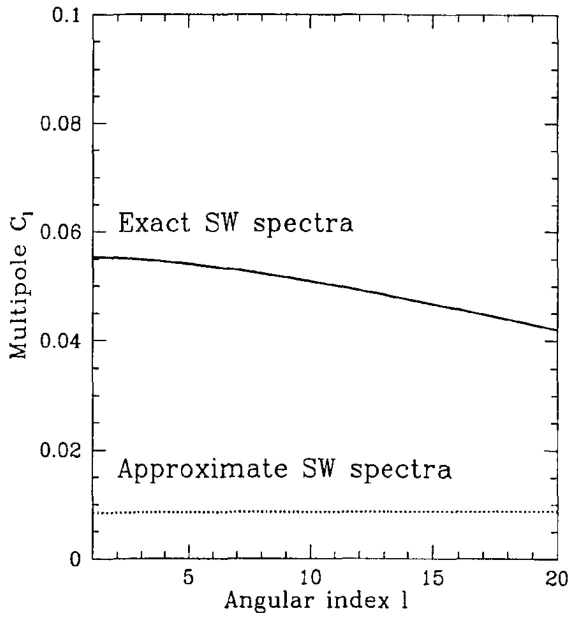


Figure 4.1: Comparison of CMB spectra for a purely matter dominated universe, with Sachs-Wolfe terms and their standard approximations given in Liddle and Lyth

angular power spectra, as evident from the Figure 4.1. This corresponds to the hypothetical case of a purely matter dominated universe ($\Omega_m = 1.0$). This case is in conflict with experimental data but has the advantage of having closed form solutions and is therefore widely regarded as a benchmark in literature. In [18] it was proven that under approximations which are strictly valid only for long wavelengths well into the matter dominated stage, this case corresponds to flat angular power spectra.

4.4 Approximations to C_l for the dipole, $l = 1$

In this section we shall derive a few useful approximations for the dipole component of the correlation function of large scale CMB anisotropies. Their purpose is twofold, to serve as a check for numerical results and to aid comparison amongst various expressions that exist for angular power spectra. We highlight two separate cases (i) the approximate expressions for the generation of anisotropies given in equations (4.13),(4.23) and the corresponding multipoles (4.24), (ii) the exact expressions thereof, given in equations (4.11),(4.18),(4.22).

The power spectra corresponding to the two cases are shown in the above figure. The observed angular power spectra for case (i) widely quoted from [18] and [20] remain flat. In contrast, the power spectra for case (ii) fall steeply, We now proceed to explain this difference. As a first step we shall try to explain the difference in the values of C_l for $l = 1$ using suitable asymptotic analysis, Since for $l = 1$ the peak of the spherical Bessel function is at a very low value, only the lowest wavelengths contribute to the value of C_l. For low values of the argument, the spherical Bessel function may be replaced by its asymptotic expression

$$j_l(x) = \frac{x^l}{(2l+1)!!}\,. \tag{4.25}$$

For $l = 1$, this gives

$$j_1(x) = \frac{x}{3}\,. \tag{4.26}$$

4.4.1 Case (i) Liddle and Lyth's approximation for the Sachs-Wolfe effect

For this case the expression for Sachs-Wolfe effect, with I_{ll} denoting the temperature fluctuation, as given in (4.13), is

$$I_{ll} = \frac{1}{3}\Phi(\eta_{rec})j_l(k(\eta_0 - \eta_{rec})) + 2\int_0^{\eta_0} \Phi'(\eta)j_l(k(\eta_0 - \eta))d\eta\,. \tag{4.27}$$

For low k this can be approximated as

$$I_{ll} = \frac{k}{3}\left[\frac{1}{3}\Phi(\eta_{rec})(\eta_0 - \eta_{rec}) + 2\int_0^{\eta_0} \Phi'(\eta)(\eta_0 - \eta)d\eta\right]\,. \tag{4.28}$$

The integrated Sachs-Wolfe term may be decomposed as follows

$$\int_0^{\eta_0} \Phi'(\eta)(\eta_0 - \eta)d\eta = \int_0^{\eta_0} \Phi'(\eta)(\eta_0 - \eta_{rec})d\eta - \int_0^{\eta_0} \Phi'(\eta)(\eta - \eta_{rec})d\eta \,. \qquad (4.29)$$

The second integral can be ignored on the heuristic consideration that the potential Φ varies at the time of recombination, (whereabout $\eta - \eta_{rec} \sim 0$) and is otherwise constant. Thus, either factor in the integrand is always negligible. Therefore,

$$\begin{aligned} \int_0^{\eta_0} \Phi'(\eta)(\eta_0 - \eta)d\eta &\approx \int_0^{\eta_0} \Phi'(\eta)(\eta_0 - \eta_{rec})d\eta \,, \\ &\approx [\Phi(\eta_0) - \Phi(0)](\eta_0 - \eta_{rec}) \,. \end{aligned} \qquad (4.30)$$

Substituting back into the integral and dropping the pre-factor of $k/3$, we get,

$$I_{ll} = \left[\frac{1}{3}\Phi(\eta_{rec}) + 2(\Phi(\eta_0) - \Phi(0))\right](\eta_0 - \eta_{rec}) \,. \qquad (4.31)$$

We now have an approximate expression for the Sachs-Wolfe effect purely in terms of the values of the potential Φ at times $\eta = \eta_{rec}$ (time of recombination) and η_0 (present time). For $k = 0$, the numerical computation of Φ gives us the values $\Phi(\eta_{rec}) = 1.02679$, $\Phi(\eta_0) = 0.99975$, and $\Phi(0) = 1.1058$. These values are in excellent agreement with those obtained from the analytical solution for the homogeneous mode given in Chapter 3, which therefore serves as a further check. The resulting value for I_{ll} is

$$I_{ll} = 0.13015(\eta_0 - \eta_{rec}) \,. \qquad (4.32)$$

We may repeat the above analysis for the second case.

4.4.2 Case (ii) The exact expression for the Sachs-Wolfe effect

For this case, the expression for Sachs-Wolfe effect, with I denoting the temperature fluctuation, is as given in (4.11)

$$I = (\frac{\delta_r}{4} + \Phi)(\eta_{rec})j_l(k(\eta_0 - \eta_{rec})) + 2\int_{\eta_{rec}}^{\eta_0} \Phi'(\eta)j_l(k(\eta_0 - \eta))d\eta \,. \qquad (4.33)$$

Decomposing the integral term we proceed similarly as before. The expression for I can be approximated as, after dropping the pre-factor of $k/3$,

$$I = [(\delta_r/4 - \Phi)(\eta_{rec}) + 2\Phi(\eta_0)]\,(\eta_0 - \eta_{rec}) \,. \qquad (4.34)$$

For $k = 0$ we have observed $\delta_r/4 - \Phi$ is a constant wrt time, and for $k = 0$ equals $-5/3$. Hence, we get

$$I = [-5/3 + 2\Phi(\eta_0)]\,(\eta_0 - \eta_{rec})\,. \tag{4.35}$$

For $k = 0$, the numerical value of $\Phi(\eta = 0) = 1.1058$ gives us the following value

$$I = 0.28818(\eta_0 - \eta_{rec})\,. \tag{4.36}$$

The terms I and I_{ll} squared are the integrands for the angular moments C_l. The ratios of the squares should give us the relative ratios of the coefficient C_l in the two cases. Thus $(I/I_{ll})^2 \approx 4.9$, which is very close to the observed value in the figure. This explains the differences in the values of C_l for $l = 1$.

4.5 Approximations to C_l for multipoles $l \geq 1$

The arguments employed in the former section are valid only in the low frequency regime ($k \to 0$). Consequently they cannot be applied to approximate the coefficients C_l for $l > 1$. This is unfortunate, as it denies us concrete numerical estimates for the ratios of C_l in the two cases for higher values of l. We are therefore forced to be more vague. The CMB spectra for the exact case (ii) falls off with increasing l, while that for the more popular approximate case (i) remains flat. This difference will only be explained qualitatively.

We observe that the coefficients C_l have a convenient decomposition

$$C_l = C_{l,NSW} + C_{l,inter} + C_{l,ISW}\,, \tag{4.37}$$

where the component terms are defined as

$$\begin{aligned} C_{l,NSW} &= \int NSW(k)^2 \frac{dk}{k}\,, \\ C_{l,inter} &= \int 2NSW(k)ISW(k)\frac{dk}{k}\,, \\ C_{l,ISW} &= \int ISW(k)^2 \frac{dk}{k}\,. \end{aligned} \tag{4.38}$$

Thus the first and third terms are the contributions to the angular moments purely from the Naive and Integrated Sachs-Wolfe effects, in that order, and are positive definite. The second term is the contribution to the angular moment due to the interference of the Naive and Integrated Sachs-Wolfe effects, and hence the subscript.

In the Liddle and Lyth approximation, the coefficients $C_{l,ll}$ have a similar decomposition

$$C_{l,ll} = C_{l,llNSW} + C_{l,llinter} + C_{l,llISW} , \tag{4.39}$$

with the component terms defined in exact analogy to equations (4.38).

The essence of our argument consists of comparing each individual component in the exact case (4.37) to its counterpart in the approximation (4.39). If we observe that the ratios of the individual terms fall off, we may safely infer, that the ratio of the totals also fall off.

But before we do so, we need a few useful results concerning spherical Bessel functions. The spherical Bessel function satisfies the following equation:

$$x^2 j_l'' + 2x j_l' + \left(x^2 - (l^2 + l)\right) j_l = 0 . \tag{4.40}$$

At an extremal point x, the derivative vanishes, $j_l' = 0$. Therefore

$$\frac{j_l''}{j_l} = \left[\frac{l^2 + l}{x^2} - 1\right] . \tag{4.41}$$

The Bessel function may be written in the form of an exponential

$$j_l(x) = e^{-S(x)} . \tag{4.42}$$

It then follows

$$\begin{aligned} j_l'(x) &= -S'(x) e^{-S(x)} , \\ j_l''(x) &= \left[-S''(x) + S'^2(x)\right] e^{-S(x)} . \end{aligned} \tag{4.43}$$

Therefore, at an extremal point x, where $S'(x) = 0$, we get

$$j_l''(x) = -S''(x) e^{-S(x)} . \tag{4.44}$$

Combining the above equations for $j_l(x)$ and $j_l''(x)$ at an extremal point, we get

$$S''(x) = -\frac{j_l''(x)}{j_l(x)} , \tag{4.45}$$

or,

$$S''(x) = 1 - \frac{l^2 + l}{x^2}\,. \tag{4.46}$$

We are now in a position to derive an approximate expression for the separate contributions to the coefficients of the angular power spectra C_l, namely, the pure Naive Sachs-Wolfe, interference and pure Integrated Sachs-Wolfe terms separately.

4.5.1 Pure Naive Sachs-Wolfe term

We begin with the pure Naive Sachs-Wolfe term which is the most straightforward. In general this term may be written as

$$C_{l,NSW} = \int_0^{\infty} [\Phi + \delta_r/4]^2(k) j_l^2(k(\eta_0 - \eta_r)) \frac{dk}{k}\,. \tag{4.47}$$

In the above expressions Φ and δ_r refer to the gravitational potential and perturbation in the radiation energy density at the time of recombination, henceforth denoted by η_r, respectively. If we substitute the exponential form of $j_l(x)$ given in (4.42), we get

$$C_{l,NSW} = \int_0^{\infty} [\Phi + \delta_r/4]^2(k) e^{-2S(k(\eta_0 - \eta_r))} \frac{dk}{k}\,. \tag{4.48}$$

We may change the dummy variable to $x = k(\eta_0 - \eta_r)$. This gives us

$$C_{l,NSW} = \int_0^{\infty} [\Phi + \delta_r/4]^2 \left(\frac{x}{\eta_0 - \eta_r}\right) e^{-2S(x)} \frac{dx}{x}\,. \tag{4.49}$$

The function $j_l(x)$ has a maximum at a position say x_m which is dependent on the index l. These maxima vary approximately as $x_m \sim (l + 1/2)$. The exact values of these maxima have been found numerically. We begin by using the Taylor expansion of $S(x)$ till the second order in the neighbourhood of x_m in the argument of the exponent. Then the integral becomes

$$C_{l,NSW} = \int_0^{\infty} [\Phi + \delta_r/4]^2 \left(\frac{x}{\eta_0 - \eta_r}\right) e^{-2[S(x_m) + \frac{S''(x_m)}{2}(x - x_m)^2]} \frac{dx}{x}\,. \tag{4.50}$$

Since the function $j_l(x)$ is rapidly vanishing outside the neighbourhood of x_m, the above integral may be approximated therefore, as given in [5], as follows

$$C_{l,NSW} = [\Phi + \delta_r/4]^2 \left(\frac{x_m}{\eta_0 - \eta_r}\right) \frac{e^{-2S(x_m)}}{x_m} \int_0^{\infty} e^{-S''(x_m)(x - x_m)^2} dx\,. \tag{4.51}$$

The exponential outside the integral may be substituted using (4.42) and the integral which is now a Gaussian may be evaluated to give,

$$C_{l,NSW} = [\Phi + \delta_r/4]^2 \left(\frac{x_m}{\eta_0 - \eta_r}\right) \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{S''(x_m)}} \,. \tag{4.52}$$

Substituting (4.46) for $S''(x)$ and regrouping terms, we get

$$C_{l,NSW} = [\Phi + \delta_r/4]^2 \left(\frac{x_m}{\eta_0 - \eta_r}\right) \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}} \,. \tag{4.53}$$

If we repeat the above procedure with the Liddle and Lyth approximants to the Naive Sachs-Wolfe terms, we get

$$C_{l,llNSW} = \frac{\Phi^2(0, \frac{x_m}{\eta_0})}{9} \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}} \,. \tag{4.54}$$

The ratio of these expressions may then be found to be

$$\frac{C_{l,NSW}}{C_{l,llNSW}} = \frac{[\Phi + \delta_r/4]^2 (\eta_r, \frac{x_m}{\eta_0 - \eta_r})}{\Phi^2(0, \frac{x_m}{\eta_0})/9} \,. \tag{4.55}$$

4.5.2 Pure Integrated Sachs-Wolfe term

The pure integrated Sachs-Wolfe term $ISW(k)$ for an arbitrary wave number k is given by;

$$ISW(k) = \int_{\eta_r}^{\eta_0} \Phi'(\eta, k) j_l(k(\eta_0 - \eta)) d\eta \,. \tag{4.56}$$

By the mean value theorem

$$ISW(k) = [\Phi(\eta_r, k) - \Phi(\eta_0, k)] j_l(k(\eta_0 - \eta_m)) \,, \tag{4.57}$$

where η_m is the mean value, as yet undetermined, in the range $[\eta_r, \eta_0]$. But since the gravitational potential varies most markedly till well into the matter dominated era, in the earliest part of this interval we may assume $\eta_m \approx \eta_r$. This is very true for lower wave numbers. Now, the contribution to the l^{th} multipole of the angular power spectra becomes

$$C_{l,ISW} = \int_0^\infty [\Phi(\eta_i, k) - \Phi(\eta_0, k)]^2 j_l^2(k(\eta_0 - \eta_m)) \frac{dk}{k} \,. \tag{4.58}$$

If we proceed to approximate the above integral as before, with the Pure Naive Sachs-Wolfe term, we get

$$C_{l,ISW} = \left[\Phi(\eta_r, \frac{x_m}{\eta_0 - \eta_m}) - \Phi(\eta_0, \frac{x_m}{\eta_0 - \eta_m})\right]^2 \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}} . \tag{4.59}$$

This expression contains η_m which is an unknown value, and cannot be evaluated directly. We may now approximate $\eta_m \approx \eta_r$ to get

$$C_{l,ISW} = \left[\Phi(\eta_r, \frac{x_m}{\eta_0 - \eta_r}) - \Phi(\eta_0, \frac{x_m}{\eta_0 - \eta_r})\right]^2 \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}} . \tag{4.60}$$

For the Liddle and Lyth approximation, we apply the mean value theorem to the term $ISW_{ll}(k)$, to get

$$ISW_{ll}(k) = [\Phi(0,k) - \Phi(\eta_0, k)] j_l \left(k(\eta_0 - \eta_m)\right) . \tag{4.61}$$

The contribution to the angular multipole, when approximated similarly, becomes

$$C_{l,llISW} = \left[\Phi(0, \frac{x_m}{\eta_0}) - \Phi(\eta_0, \frac{x_m}{\eta_0})\right]^2 \frac{j_l^2(x_m)}{x_m} \sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}} . \tag{4.62}$$

The ratio of these contributions is found to be

$$\frac{C_{l,ISW}}{C_{l,llISW}} = \frac{\left[\Phi(\eta_r, \frac{x_m}{\eta_0 - \eta_r}) - \Phi(\eta_0, \frac{x_m}{\eta_0 - \eta_r})\right]^2}{\left[\Phi(0, \frac{x_m}{\eta_0}) - \Phi(\eta_0, \frac{x_m}{\eta_0})\right]^2} . \tag{4.63}$$

4.5.3 Interference term

Now, we turn to the contribution to the angular correlation function from the interference of the NSW and ISW terms, which is given by

$$C_{l,inter} = 2 \int_0^\infty ISW(k) NSW(k) \frac{dk}{k} . \tag{4.64}$$

From the expression we have derived for $ISW(k)$ with η_m approximated as η_r, we get

$$ISW(k) = [\Phi(\eta_r, k) - \Phi(\eta_0, k)] \, j_l(k(\eta_0 - \eta_r)) . \tag{4.65}$$

Substituting this value into the expression $C_{l,inter}$, we get

$$C_{l,inter} = 2\int_0^\infty [\Phi + \delta_r/4](k)[\Phi(\eta_r, k) - \Phi(\eta_0, k)] j_l^2(k(\eta_0 - \eta_r))\frac{dk}{k}. \qquad (4.66)$$

As before with the purely NSW and ISW terms we proceed as is given in [5] and we arrive at the following approximate expression

$$\begin{aligned} C_{l,inter} &= 2[\Phi + \delta_r/4](\frac{x_m}{\eta_0 - \eta_r})\left[\Phi(\eta_r, \frac{x_m}{\eta_0 - \eta_r}) - \Phi(\eta_0, \frac{x_m}{\eta_0 - \eta_r})\right] \\ &\quad \frac{j_l^2(x_m)}{x_m}\sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}}. \end{aligned} \qquad (4.67)$$

Proceeding analogously as above for the Liddle and Lyth approximations with the Naive and Integrated Sachs-Wolfe terms $NSW_{ll}(k)$ and $ISW_{ll}(k)$ defined as in equations (4.23), we get

$$\begin{aligned} C_{l,llinter} &= \frac{2}{3}\Phi(0, \frac{x_m}{\eta_0})[\Phi(0, \frac{x_m}{\eta_0}) - \Phi(\eta_0, \frac{x_m}{\eta_0})] \\ &\quad \frac{j_l^2(x_m)}{x_m}\sqrt{\frac{\pi}{1 - \frac{l^2+l}{x_m^2}}}. \end{aligned} \qquad (4.68)$$

Thus the ratio of the interference terms comes out to be

$$\frac{C_{l,inter}}{Cl_{l,llinter}} = \frac{[\Phi + \delta_r/4](\eta_r, \frac{x_m}{\eta_0 - \eta_r})\left[\Phi(\eta_r, \frac{x_m}{\eta_0 - \eta_r}) - \Phi(\eta_0, \frac{x_m}{\eta_0 - \eta_r})\right]}{\frac{1}{3}\Phi(0, \frac{x_m}{\eta_0})[\Phi(0, \frac{x_m}{\eta_0}) - \Phi(\eta_0, \frac{x_m}{\eta_0})]}. \qquad (4.69)$$

To summarize, in this section we have derived approximate ratios for the pure NSW, interference and pure ISW contributions to the multipoles C_l of the angular autocorrelation function for temperature fluctuations, for the exact, rigorous expression for the Sachs-Wolfe effect to the Liddle and Lyth approximation. Separately, these ratios show a falloff for higher l. Therefore, we may infer that the multipole C_l for the exact expression shall also fall with higher l compared to the multipole $C_{l,ll}$ in the Liddle and Lyth approximation. Since this is what we observe in Figure 4.1, our numerical calculations of these CMB spectra are therefore validated.

4.6 Effect of perturbations in quintessence on large scale CMB anisotropies

In this section we discuss the difference between the effects various forms of quintessence have on the large-scale CMB anisotropies. Large-scale CMB anisotropies are governed by the long-wave behaviour of cosmological perturbations, the most important of them being the gravitational potential. The main effect of the Λ-term like component is the deviation of Φ from constant behaviour, observed in a purely matter dominated universe, at late epochs, which leads to non-zero ISW effect [18].

Thus, we begin this discussion comparing the evolution of the gravitational potential Φ as a function of conformal time in different quintessence models. The first case we study as the benchmark is the standard Λ term. In Figure 4.2 we plot the evolution of Φ for $k = 0$ for several values of $\Omega_q = \Omega_\Lambda$. Since Λ quintessence has a

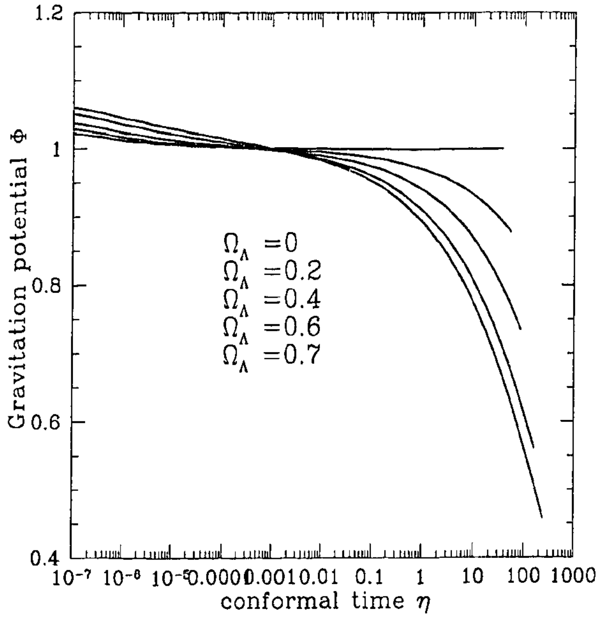


Figure 4.2: Evolution of the gravitational potential Φ for $\Omega_\Lambda = 0, 0.2, 0.4, 0.6, 0.7$ Lambda quintessence vs log of conformal time

strictly negative equation of state with $w_p = -1$, we may infer this "counteracts" gravitational infall. The gravitational potential Φ, is in naive terms an indicator of the growth of gravitational instabilities under their own self gravity. We may therefore infer, that for increasing Ω_Λ, which represents an increasing influence of Λ quintessence, Φ shall then decrease. Since Λ quintessence begins to dominate the energy ratio of the universe at relatively late times, we may also expect the expected decrement to be more pronounced at later times. This is indeed the case as evident from Figure 4.2.

The second case, that of a dynamical homogeneous scalar field, which acts like quintessence is very much similar, as one can see from Figure 4.3. Though its

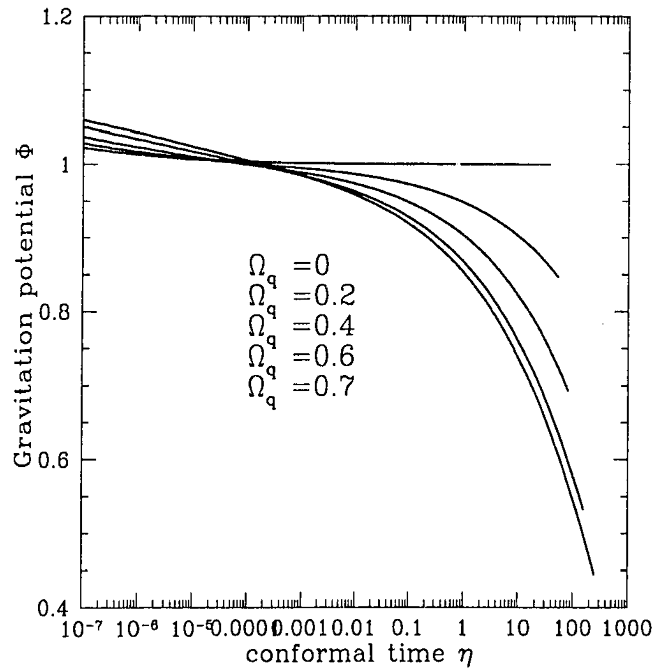


Figure 4.3: Evolution of the gravitational potential Φ for $\Omega_q = 0, 0.2, 0.4, 0.6, 0.7$ scalar field quintessence vs log of conformal time

equation of state is variable and not identically -1, it is negative and much the same reasoning applies as for Λ quintessence. The evolution of φ for a certain value of Ω_q in both Λ quintessence, and dynamical homogeneous quintessence, is qualitatively very similar. For a purely matter dominated era, when $\Omega_q = 0$, the gravitational potential Φ drops sharply as before the time of recombination, then remains constant at the value unity as predicted by the analytical solution, for the zero mode. As Ω_q increases, quintessence is more influential at late times. The growth of gravitational potential Φ is more strongly counteracted. Therefore for larger values of Ω_q we observe larger fall offs from unity. In addition, the falloff, while not exactly equal to that observed for Λ quintessence with the same value of Ω_q, is certainly comparable.

Finally we consider the case of scalar field quintessence with perturbations. Here we observe that the effect of quintessence on Φ is qualitatively the same, but that it is much less pronounced. The results for long-wave evolution of Φ in this case are given in Figure 4.4. When $\Omega_q = 0$ the gravitational potential falls off and remains

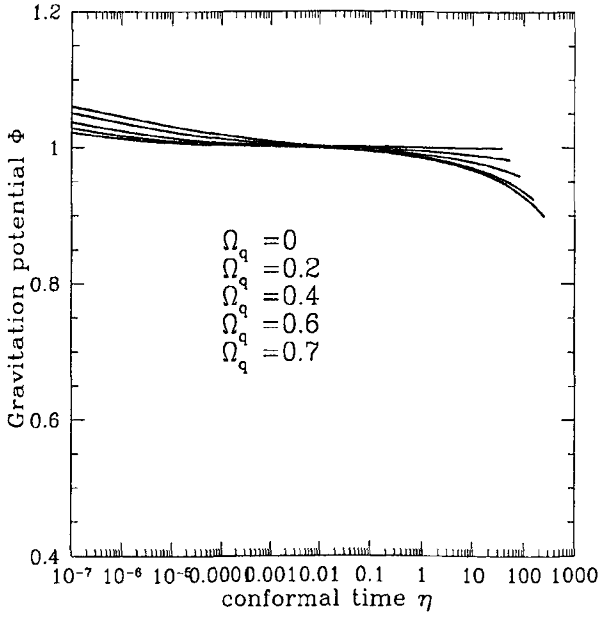


Figure 4.4: Evolution of the gravitational potential Φ for $\Omega_q = 0, 0.2, 0.4, 0.6, 0.7$ scalar field quintessence with perturbations vs log of conformal time

constant at unity as expected from the analytical solution. As Ω_q increases, the expected fall off from unity is observed, but is much less pronounced, as may be observed from Figure 4.4. Thus, inspite of the presence of quintessence the evolution of Φ is much more similar to that in a purely matter dominated universe. Therefore, we reach an important conclusion. The new degree of freedom which can develop imhomogeneities in the quintessence model allows the amplitude of the perturbed gravitational potential to be maintained.

In Figure 4.5 we compare these three cases for $\Omega_q = 0.7$ corresponding to the value preffered by modern observations (e.g., from the results of WMAP consortium and SNIa analysis). For Λ quintessence and homogeneous quintessence the

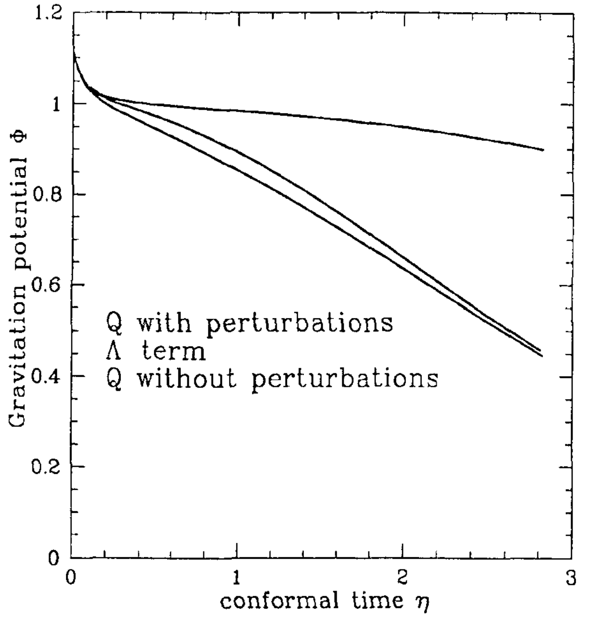


Figure 4.5: Evolution of the zero mode gravitational potential Φ for $\Omega_q = 0.7$ for dynamical quintessence with and without perturbations and Λ quintessence

curves for Φ are very close. For quintessence with perturbations, the curve for Φ is markedly higher, showing much less late time decrease than for the two preceding cases. We shall invoke this observation later to explain the difference between their corresponding CMB spectra.

It is worthwhile asking, since the zero mode of the gravitational potential for quintessence with perturbations is so markedly different from those of Lambda quintessence and homogeneous quintessence (i.e. without perturbations), if we shall also see a similar difference for higher wave numbers (i.e. shorter wavelengths). In our analysis of perturbations of the scalar field in Chapter 3, we saw short wavelength perturbations evolve as $\frac{1}{k\eta}$ in the radiation dominated era and as $\frac{1}{k\eta^2}$ in the matter dominated era. We may therefore argue, since short wavelength scalar pertubations are damped out, at such wavelengths the gravitational potential Φ shall evolve unaffected by perturbations in the quintessence field. Consequently, Φ for quintessence with perturbations, homogeneous quintessence (without perturbations), and Λ quintessence should be very similar. This is indeed borne out by exact numerical calculations shown in Figure 4.6.

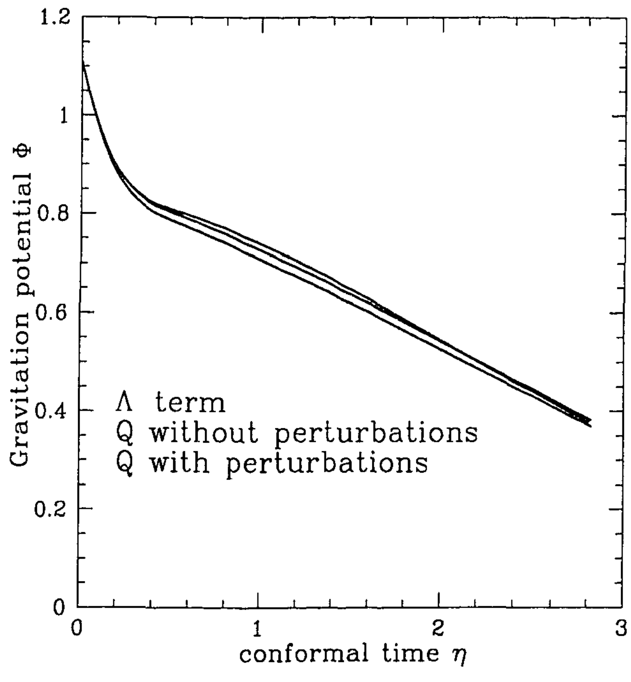


Figure 4.6: Evolution of short wavelength ($k = 20$) the gravitational potential Φ for $\Omega_q = 0.7$ for dynamical quintessence with and without perturbations and Λ quintessence

We have so far established that differences between quintessence with perturbations on the one hand, and homogeneous quintessence and Λ quintessence on the other, manifest themselves only for gravitational perturbations at long wavelengths. We may now turn to central result of this thesis, the comparison of the CMB anisotropy spectra for quintessence like scalar fields with evolving inhomogeneities, quintessence like scalar fields without inhomogeneities and the benchmark Λ quintessence. Our numerical calculations of the large-angle (low l) multipoles, using the exact expressions for the Sachs-Wolfe effect (4.11) and for the value $\Omega_q = 0.7$ are shown in Figure 4.7.

We observe that the CMB spectra for quintessence with inhomogeneities varies markedly from that of homogeneous quintessence at low l. At high values of l, the CMB spectra for both cases merge and become indistinguishable. This may be explained as follows. As we had established earlier in this chapter, the l^{th} multipole C_l is most sensitive to the gravitational potential with wavenumbers $k \sim l/2$. Consequently, the lowest multipoles reflect the low k or long wavelength modes in Φ.

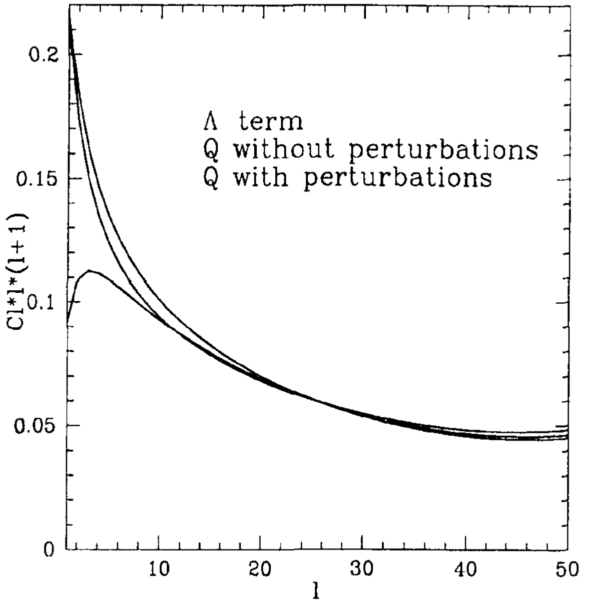


Figure 4.7: CMB spectra of Lambda term, scalar field quintessence with and without perturbations

Correspondingly, the higher multipoles reflect the high k or short wavelength modes in Φ. Earlier in this section, using both numerical illustrations and analytical results, we had observed that only the gravitational perturbations at long wavelengths (low k) are affected by evolving perturbations in the scalar field. It follows then, that only large angle multipoles C_l, for low values of l, should differ when evolving perturbations are considered in quintessence, as observed in the comparative plot of the angular spectra in Figure 4.7.

The existence of a measurable effect on large-angle CMB anisotropy is very important for the prospect of restricting the choice of quintessence models observationally. In future work we plan to compare the predictions of the models with different potentials and/ or kinetic terms (the last class may lead to qualitatively different results) and compare these predictions with state of the art CMB observations using full-scale statistical analysis.

Chapter 5

Conclusions

5.1 Conclusions

This thesis considers the effect of quintessence scalar fields in large-scale (low l) CMB anisotropies. Particular emphasis is paid to the fact that a realistic quintessence scalar field should, to some extent, have evolving inhomogeneities. The development of the thesis has been as follows. Chapter 1 contains a brief introduction to the subject of quintessence. Therein, the idea of quintessence and its relevance to modern cosmology are introduced without any formalism. The formalism necessary, that of linearized Einstein equations and the other equations which govern the perturbation variables are given in Chapter 2, together with exact numerical solutions to these equations. Example numerical simulations are used to illustrate the acceleration in the expansion in the universe caused by quintessence. A few important properties of the inverse power law potential governing the scalar field are also outlined in this chapter. The 'tracking' property, whereby the energy density of the quintessence largely tracks the dominant constituent at various cosmological eras, and the existence of 'attractor' like solutions for the background scalar field are outlined with numerical simulations and analytical arguments. These properties make the inverse power law model for quintessence robust to a wide range of initial conditions, thereby alleviating the need to fine-tune conditions to get the desired observables. Analytical solutions to these equations in a few limiting cases are given in Chapter 3. These include the zero mode of the gravitational potential Φ, its dominant modes during the radiation and matter dominated stages and the inhomogeneities of the scalar field during these stages.

The theory of large scale anisotropies and the statistical technique used to analyse these anisotropies of the CMB are developed in Chapter 4. The differences between the exact expression for the generation of temperature fluctuations in the CMB, and a popular approximation to it widely used in literature are highlighted. Semi-analytical expressions for the resulting differences between the angular multipoles in these two cases are also derived. The benchmark case of a purely matter dominated universe is then used to validate numerical results obtained for CMB power spectra.

Later in Chapter 4, we turn to the main topic of the thesis, namely, the effect of the Lambda term or Λ quintessence and dynamical quintessence with and without inhomogeneities on CMB anisotropies. We establish that the existence of inhomogeneities in the scalar field affects the gravitational potential only at long wavelengths. Therefore, the existence of inhomogeneities in the scalar field affects the large-angle multipoles in the CMB spectra, as verifed by numerical calculations. This opens possibilities to place constraints on possible quintessence models from modern CMB observations.

Our research can, and should be extended in the following directions:

Currently, only adiabatic perturbations have been considered. Since we have three primary components for the energy content of the universe, radiation, pressureless matter and quintessence, we must consider the evolution of the two other, isocurvature modes of perturbations, which keep the total energy unperturbed. For, a general perturbation shall be the sum of these three types of perturbation. The well-known isocurvature mode which corresponds to relative perturbations in energy densities of radiation and matter in the early Universe has been studied in the literature (e.g., see [23]). The 2^{nd} isocurvature component, which appears due to introduction of new dynamical degrees of freedom in the form of a scalar field, its evolution during various cosmological stages, and its effect on the CMB spectra require new investigation.

Results for the background scalar field, its evolving perturbations and the gravitational potential have been derived assuming an inverse power law potential for the background scalar field. Such a potential has been considered because it exhibits the 'tracking' property. We believe our results will generally hold for a wide class of quintessence potentials with 'tracking' properties. However, this deserves closer scrutiny. Also, we plan to analyse the theories with modified kinetic terms.

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Appendix A

A.1 Background affine connections in conformal time

As a prelude to computing the perturbation to the Einstein or Ricci tensors, we must first compute the affine connections for our background metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - \gamma_{ik}dx^i dx^k\right) \tag{A.1}$$

For $\lambda = 0$ (corresponding to conformal time),

$$\begin{aligned} \Gamma^0_{\mu\nu} &= \frac{1}{2}g^{0\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\kappa\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right], \\ &= \frac{1}{2}g^{00}\left[\frac{\partial g_{0\nu}}{\partial x^\mu} + \frac{\partial g_{0\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^0}\right]. \end{aligned} \tag{A.2}$$

For $\mu = 0$,

$$\Gamma^0_{0\nu} = \frac{1}{2}g^{00}\left[\frac{\partial g_{00}}{\partial x^\nu}\right]. \tag{A.3}$$

For $(\mu, \nu) = (0, 0)$,

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^0}, \\ &= \frac{a'}{a}. \end{aligned} \tag{A.4}$$

For $(\mu, \nu) = (0, i)$,

$$\Gamma^0_{0i} = 0. \tag{A.5}$$

For $(\mu,\nu)=(i,j)$,

$$\begin{aligned}\Gamma^0_{ij} &= -\frac{1}{2}g^{0\kappa}\frac{\partial g_{ij}}{\partial x^\kappa}\,,\\ &= -\frac{1}{2}g^{00}\frac{\partial g_{ij}}{\partial x^0}\,.\end{aligned} \tag{A.6}$$

Substituting $g_{ij}=-a^2(\eta)\gamma_{ij}$, we get

$$\Gamma^0_{ij}=\frac{a'}{a}\gamma_{ij}\,. \tag{A.7}$$

For $\lambda=i$ (a spatial index),

$$\begin{aligned}\Gamma^i_{\mu\nu} &= \frac{1}{2}g^{i\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^\mu}+\frac{\partial g_{\kappa\mu}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right],\\ &= \frac{1}{2}g^{il}\left[\frac{\partial g_{l\nu}}{\partial x^\mu}+\frac{\partial g_{l\mu}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^l}\right].\end{aligned} \tag{A.8}$$

For $\mu=0$,

$$\Gamma^i_{0\nu} = \frac{1}{2}g^{il}\left[\frac{\partial g_{l\nu}}{\partial x^0}-\frac{\partial g_{0\nu}}{\partial x^l}\right]. \tag{A.9}$$

For $\nu=0$,

$$\Gamma^i_{00}=0\,. \tag{A.10}$$

For $\nu=j$,

$$\begin{aligned}\Gamma^i_{0j} &= \frac{1}{2}g^{il}\frac{\partial g_{lj}}{\partial x^0}\,,\\ &= \frac{a'}{a}\delta^i_j\,.\end{aligned} \tag{A.11}$$

For $\mu=j$,

$$\Gamma^i_{j\nu} = \frac{1}{2}g^{i\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^j}+\frac{\partial g_{\kappa j}}{\partial x^\nu}-\frac{\partial g_{j\nu}}{\partial x^\kappa}\right]. \tag{A.12}$$

For $\nu=0$,

$$\Gamma^i_{j0} = \frac{1}{2}g^{i\kappa}\frac{\partial g_{\kappa j}}{\partial x^0}$$

$$= \frac{a'}{a}\delta^i_j\,. \tag{A.13}$$

For $\nu = k$,

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}g^{i\kappa}\left[\frac{\partial g_{\kappa k}}{\partial x^j} + \frac{\partial g_{\kappa j}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\kappa}\right], \\
&= \frac{1}{2}g^{il}[\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}], \\
&= \frac{1}{2}\gamma^{il}\left[\frac{\partial \gamma_{lk}}{\partial x^j} + \frac{\partial \gamma_{lj}}{\partial x^k} - \frac{\partial \gamma_{jk}}{\partial x^l}\right], \\
&= \widetilde{\Gamma}^i_{jk}\,.
\end{aligned} \tag{A.14}$$

We also define

$$\begin{aligned}
\Gamma_\mu &\equiv \Gamma^\nu_{\mu\nu}\,, \\
\Gamma_0 &= 4\frac{a'}{a}\,, \\
\Gamma_i &= \widetilde{\Gamma}^k_{ik}\,.
\end{aligned} \tag{A.15}$$

A.2 Elements of the Ricci tensor in FRW spacetimes

The Ricci tensor is defined as follows

$$R_{\mu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\sigma\kappa} - \Gamma^\sigma_{\mu\kappa}\Gamma^\lambda_{\sigma\lambda}\,. \tag{A.16}$$

For $(\mu, \kappa) = (0, 0)$,

$$\begin{aligned}
R_{00} &= \frac{\partial \Gamma_0}{\partial x^0} - \frac{\partial \Gamma^\lambda_{00}}{\partial x^\lambda} + \Gamma^\sigma_{0\lambda}\Gamma^\lambda_{\sigma 0} - \Gamma^\sigma_{00}\Gamma_\sigma\,, \\
&= \frac{\partial \Gamma_0}{\partial x^0} - \frac{\partial \Gamma^0_{00}}{\partial x^0} + \Gamma^0_{0\lambda}\Gamma^\lambda_{00} + \Gamma^i_{0\lambda}\Gamma^\lambda_{i0} - \Gamma^0_{00}\Gamma_0\,, \\
&= \frac{\partial \Gamma_0}{\partial x^0} - \frac{\partial \Gamma^0_{00}}{\partial x^0} + \Gamma^0_{00}\Gamma^0_{00} + \Gamma^i_{0j}\Gamma^j_{i0} - \Gamma^0_{00}\Gamma_0\,, \\
&= \frac{\partial}{\partial \eta}(4\mathcal{H}) - \frac{\partial}{\partial \eta}\mathcal{H} + \mathcal{H}^2 + \mathcal{H}\delta^i_j\mathcal{H}\delta^j_i - 4\mathcal{H}^2\,, \\
&= 3\mathcal{H}'\,.
\end{aligned} \tag{A.17}$$

For $(\mu,\kappa)=(i,j)$,

$$\begin{aligned}
R_{ij} &= \frac{\partial\Gamma_i}{\partial x^j} - \frac{\partial\Gamma^\lambda_{ij}}{\partial x^\lambda} + \Gamma^\sigma_{i\lambda}\Gamma^\lambda_{\sigma j} - \Gamma^\sigma_{ij}\Gamma^\lambda_{\sigma\lambda}\,,\\
&= \frac{\partial\widetilde{\Gamma}_i}{\partial x^j} - \frac{\partial\Gamma^0_{ij}}{\partial x^0} - \frac{\partial\widetilde{\Gamma}^k_{ij}}{\partial x^k} + \Gamma^0_{i\lambda}\Gamma^\lambda_{0j} + \Gamma^m_{i\lambda}\Gamma^\lambda_{mj} - \Gamma^0_{ij}\Gamma_0 - \widetilde{\Gamma}^m_{ij}\widetilde{\Gamma}_m\,,\\
&= \frac{\partial\widetilde{\Gamma}_i}{\partial x^j} - \frac{\partial\widetilde{\Gamma}^k_{ij}}{\partial x^k} + \widetilde{\Gamma}^m_{ik}\widetilde{\Gamma}^k_{mj} - \widetilde{\Gamma}^m_{ij}\widetilde{\Gamma}_m + \left[\Gamma^0_{i0}\Gamma^0_{0j} + \Gamma^0_{ik}\Gamma^k_{0j} + \Gamma^m_{i0}\Gamma^0_{mj} - \Gamma^0_{ij}\Gamma_0 - \frac{\partial\Gamma^0_{ij}}{\partial x^0}\right],\\
&= \widetilde{R}_{ij} + \left[\mathcal{H}\gamma_{ik}\mathcal{H}\delta^k_j + \mathcal{H}\delta^m_i\mathcal{H}\gamma_{jm} - \mathcal{H}\gamma_{ij}(4\mathcal{H}) - \mathcal{H}'\gamma_{ij}\right],\\
&= \widetilde{R}_{ij} - \left[\mathcal{H}' + 2\mathcal{H}^2\right]\gamma_{ij}\,. \qquad \text{(A.18)}
\end{aligned}$$

For maximally symmetric spaces, the spatial curvature tensor is given by

$$\widetilde{R}_{ij} = -2K\gamma_{ij}\,. \qquad \text{(A.19)}$$

We then get

$$R_{ij} = -\left[\mathcal{H}' + 2\mathcal{H}^2 + 2K\right]\gamma_{ij}\,. \qquad \text{(A.20)}$$

A.3 Perturbations in the affine connections

For $\lambda = 0$,

$$\delta\Gamma^0_{\mu\nu} = -2\Phi\Gamma^0_{\mu\nu} + \frac{1}{2a^2}\left[\frac{\partial h_{0\nu}}{\partial x^\mu} + \frac{\partial h_{0\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^0}\right]. \qquad \text{(A.21)}$$

For $(\mu,\nu)=(0,0)$,

$$\begin{aligned}
\delta\Gamma^0_{00} &= -2\Phi\Gamma^0_{00} + \frac{1}{2a^2}\frac{\partial h_{00}}{\partial x^0}\,,\\
&= -2\Phi\frac{a'}{a} + \frac{1}{2a^2}\left(2\Phi a^2\right)'\,,\\
&= \Phi'\,. \qquad \text{(A.22)}
\end{aligned}$$

For $(\mu,\nu)=(i,0)$,

$$\begin{aligned}
\delta\Gamma^0_{i0} &= \frac{1}{2}g^{0\kappa}\frac{\partial h_{\kappa 0}}{\partial x^i}\,,\\
&= \frac{1}{2a^2}\frac{\partial h_{00}}{\partial x^i}\,,
\end{aligned}$$

$$= \Phi_{,i} \,. \tag{A.23}$$

For $(\mu,\nu)=(i,j)$,

$$\begin{aligned}
\delta\Gamma^0_{ij} &= -2\Phi\Gamma^0_{ij} - \frac{1}{2a^2}\left(2a^2\Phi\gamma_{ij}\right)', \\
&= -\left[4\Phi\frac{a'}{a} + \Phi'\right]\gamma_{ij} \,.
\end{aligned} \tag{A.24}$$

For $\lambda = i$,

$$\delta\Gamma^i_{\mu\nu} = -h^i_k\Gamma^k_{\mu\nu} + \frac{1}{2}g^{ik}\left[\frac{\partial h_{k\nu}}{\partial x^\mu} + \frac{\partial h_{k\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^k}\right]. \tag{A.25}$$

For $(\mu,\nu)=(0,0)$,

$$\begin{aligned}
\delta\Gamma^i_{00} &= -h^i_k\Gamma^k_{00} + \frac{1}{2}g^{ik}\left[-\frac{\partial h_{00}}{\partial x^k}\right], \\
&= -\frac{1}{2}g^{ik}\frac{\partial}{\partial x^k}(2a^2\Phi), \\
&= \frac{1}{2a^2}\gamma^{ik}\frac{\partial}{\partial x_k}(a^2 2\Phi), \\
&= \gamma^{ik}\Phi_{,k} \,.
\end{aligned} \tag{A.26}$$

For $(\mu,\nu)=(0,j)$,

$$\begin{aligned}
\delta\Gamma^i_{0j} &= -h^i_k\Gamma^k_{0j} + \frac{1}{2}g^{ik}\frac{\partial h_{kj}}{\partial x^0}, \\
&= -h^i_k\frac{a'}{a}\delta^k_j + \frac{1}{2}g^{ik}\left(2a^2\Phi\right)'\gamma_{kj}, \\
&= 2\Phi\frac{a'}{a}\delta^i_j - \frac{1}{a^2}\gamma^{ik}\left(2aa'\Phi + a^2\Phi'\right)\gamma_{kj}, \\
&= -\Phi'\delta^i_j \,.
\end{aligned} \tag{A.27}$$

For $(\mu,\nu)=(j,l)$,

$$\delta\Gamma^i_{jl} = -h^i_k\Gamma^k_{jl} + \frac{1}{2}g^{ik}\left[\frac{\partial h_{kl}}{\partial x^j} + \frac{\partial h_{kj}}{\partial x^l} - \frac{\partial h_{jl}}{\partial x^k}\right]. \tag{A.28}$$

Since $h_{kl} = -2\Phi g_{kl}$, we get

$$\delta\Gamma^i_{jl} = 2\Phi\Gamma^i_{jl} + \frac{1}{2}g^{ik}\left[(-2\Phi)\left(\frac{\partial\gamma_{kl}}{\partial x^j} + \frac{\partial h_{kj}}{\partial x^l} - \frac{\partial h_{jl}}{\partial x^k}\right)\right.$$

$$
\begin{aligned}
& -2\ \left(\Phi_{,j}g_{kl} + \Phi_{,l}g_{kj} + \Phi_{,k}\, g_{jl}\right)\Big]\ , \\
= \ & 2\Phi\Gamma^i_{jl} - 2\Phi\Gamma^i_{jl} \\
& -\left(\Phi_{,j}\delta^i_l + \Phi_{,l}\delta^i_j - \Phi_{,k}g^{ik}g_{jl}\right)\ , \\
= \ & -\left(\Phi_{,j}\delta^i_l + \Phi_{,l}\delta^i_j - g^{ik}\Phi_{,k}g_{jl}\right)\ .
\end{aligned} \tag{A.29}
$$

A.4 Perturbations in the Ricci tensor

For $(\mu, \nu) = (0, 0)$ we get

$$
\delta R_{00} = (\delta\Gamma^\lambda_{0\lambda})_{;0} - (\delta\Gamma^\lambda_{00})_{;\lambda}\,. \tag{A.30}
$$

We evaluate the first and second terms, which we call T_1 and T_2, separately.

$$
\begin{aligned}
T_1 &= \frac{\partial(\delta\Gamma^\lambda_{0\lambda})}{\partial x^0} - \Gamma^\beta_{00}\delta\Gamma^\lambda_{\beta\lambda}\,, \\
&= \frac{\partial(\delta\Gamma_0)}{\partial x^0} - \Gamma^0_{00}\delta\Gamma_0\,, \\
&= -2\Phi'' + \frac{a'}{a}(2\Phi')\,. \\
T_2 &= \frac{\partial(\delta\Gamma^\lambda_{00})}{\partial x^\lambda} - \Gamma^\beta_{0\lambda}\delta\Gamma^\lambda_{\beta 0} - \Gamma^\beta_{\lambda 0}\delta\Gamma^\lambda_{0\beta} + \Gamma^\lambda_{\beta\lambda}\delta\Gamma^\beta_{00}\,, \\
&= \frac{\partial(\delta\Gamma^\lambda_{00})}{\partial x^\lambda} - 2\Gamma^\beta_{0\lambda}\delta\Gamma^\lambda_{0\beta} + \Gamma^\lambda_{\beta\lambda}\delta\Gamma^\beta_{00}\,.
\end{aligned}
$$

We call the successive terms in the above expression $T_{21,22,23}$ and evaluate them separately.

$$
\begin{aligned}
T_{21} &= \frac{\partial(\delta\Gamma^\lambda_{00})}{\partial x^\lambda}\,, \\
&= \frac{\partial(\delta\Gamma^0_{00})}{\partial x^0} + \frac{\partial(\delta\Gamma^i_{00})}{\partial x^i}\,, \\
&= \Phi'' + \gamma^{ij}\Phi_{,i,j}\,. \\
T_{22} &= 2\Gamma^\beta_{0\lambda}\delta\Gamma^\lambda_{0\beta}\,, \\
&= 2\left[\Gamma^0_{0\lambda}\delta\Gamma^\lambda_{00} + \Gamma^i_{0\lambda}\delta\Gamma^\lambda_{0i}\right]\,, \\
&= 2\left[\frac{a'}{a}\Phi' + \frac{a'}{a}\delta^i_j\Gamma^j_{0i}\right]\,, \\
&= 2\left[\frac{a'}{a}\Phi' - 3\frac{a'}{a}\Phi'\right]\,,
\end{aligned}
$$

$$
\begin{aligned}
&= -4\frac{a'}{a}\Phi' . \\
T_{23} &= 2\Gamma^{\lambda}_{0\lambda}\delta\Gamma^{0}_{00} , \\
&= 4\frac{a'}{a}\Phi' .
\end{aligned}
$$

Now summing them up, we have,

$$
\begin{aligned}
T_2 &= T_{21} - T_{22} + T_{23} \\
&= \Phi'' + \gamma^{ij}\Phi_{,i,j} + 8\frac{a'}{a}\Phi' .
\end{aligned}
$$

This then gives us

$$
\delta R_{00} = -3\Phi'' - \nabla^2\Phi - 6\frac{a'}{a}\Phi' .
$$

Similarly, for $(\mu, \nu) = (i, 0)$,

$$
\delta R_{i0} = (\delta\Gamma^{\lambda}_{i\lambda})_{:0} - (\delta\Gamma^{\lambda}_{i0})_{:\lambda} .
$$

Proceeding similarly as before, we call the first and second terms $T_{1,2}$ respectively. They are evaluated separately.

$$
\begin{aligned}
T_1 &= \frac{\partial(\delta\Gamma^{\lambda}_{i\lambda})}{\partial x^0} - \Gamma^{\beta}_{i0}\delta\Gamma^{\lambda}_{\beta\lambda} , \\
&= \frac{\partial}{\partial x^0}(-2\Phi_{,i}) - \Gamma^{j}_{i0}\partial\Gamma^{\lambda}_{j\lambda} , \\
&= \frac{\partial}{\partial \eta}(-2\Phi_{,i}) - \frac{a'}{a}\delta\Gamma^{\lambda}_{i\lambda} , \\
&= -2\Phi'_{,i} + 2\frac{a'}{a}\Phi_{,i} . \\
T_2 &= \frac{\partial(\delta\Gamma^{\lambda}_{i0})}{\partial x^{\lambda}} - \Gamma^{\beta}_{i\lambda}\delta\Gamma^{\lambda}_{\beta 0} - \Gamma^{\beta}_{\lambda 0}\delta\Gamma^{\lambda}_{i\beta} + \Gamma^{\lambda}_{\beta\lambda}\delta\Gamma^{\beta}_{i0} .
\end{aligned}
$$

The successive terms are called $T_{21,22,23,24}$.

$$
\begin{aligned}
T_{21} &= \frac{\partial(\delta\Gamma^{\lambda}_{i0})}{\partial x^{\lambda}} , \\
&= \frac{\partial(\delta\Gamma^{0}_{i0})}{\partial x^0} + \frac{\partial(\delta\Gamma^{k}_{i0})}{\partial x^k} , \\
&= \Phi'_{,i} - \frac{\partial}{\partial x^k}(\Phi'\delta^k_i) ,
\end{aligned}
$$

$$
\begin{aligned}
&= 0\,. \\
T_{22} &= \Gamma^{\beta}_{i\lambda}\delta\Gamma^{\lambda}_{\beta 0}\,, \\
&= \left[\Gamma^{0}_{i\lambda}\delta\Gamma^{\lambda}_{00} + \Gamma^{k}_{i\lambda}\delta\Gamma^{\lambda}_{k0}\right], \\
&= \left[\Gamma^{0}_{il}\delta\Gamma^{l}_{00} + \Gamma^{k}_{i0}\delta\Gamma^{0}_{k0} + \Gamma^{k}_{il}\delta\Gamma^{l}_{k0}\right], \\
&= \left[\frac{a'}{a}\gamma_{il}(\gamma^{lk}\Phi_{,k}) + \frac{a'}{a}\delta^{k}_{i}\Phi_{,k} + \Gamma^{k}_{il}(-\Phi'\delta^{l}_{k})\right], \\
&= \left[2\frac{a'}{a}\Phi_{,i} + \Gamma^{k}_{il}(-\Phi'\delta^{l}_{k})\right], \\
&= \left[2\frac{a'}{a}\Phi_{,i} - \Phi'\Gamma^{k}_{ik}\right].
\end{aligned}
$$

Since for a flat spacetime we have $\Gamma^{k}_{ik} = \tilde{\Gamma}^{k}_{ik} = 0$, the above equation simplifies to

$$T_{22} = 2\frac{a'}{a}\Phi_{,i}\,.$$

The remaining terms are given by

$$
\begin{aligned}
T_{23} &= \Gamma^{\beta}_{0\lambda}\delta\Gamma^{\lambda}_{i\beta}\,, \\
&= \Gamma^{\beta}_{00}\delta\Gamma^{0}_{i\beta} + \Gamma^{\beta}_{0l}\delta\Gamma^{l}_{i\beta}\,, \\
&= \frac{a'}{a}\Phi_{,i} + \frac{a'}{a}\delta^{j}_{l}\delta\Gamma^{l}_{ij}\,, \\
&= \frac{a'}{a}\Phi_{,i} + \frac{a'}{a}(-3\Phi_{,i})\,, \\
&= -2\frac{a'}{a}\Phi_{,i}\,. \\
T_{24} &= \Gamma^{\lambda}_{\beta\lambda}\delta\Gamma^{\beta}_{i0}\,, \\
&= \Gamma^{\lambda}_{0\lambda}\delta\Gamma^{0}_{i0} + \Gamma^{\lambda}_{l\lambda}\delta\Gamma^{l}_{i0}\,.
\end{aligned}
$$

Since $\Gamma_0 = 4\frac{a'}{a}$ and for spatially flat universes $\Gamma_l = 0$ we have,

$$T_{24} = 4\frac{a'}{a}\Phi_{,i}\,.$$

Summing these pieces, we get

$$T_2 = 4\frac{a'}{a}\Phi_{,i}\,.$$

This gives us

$$\delta R_{i0} = -\left(2\Phi' + 2\frac{a'}{a}\Phi\right)_{,i}.$$

For $(\mu,\nu) = (i,j)$,

$$\delta R_{ij} \;=\; (\delta\Gamma^{\lambda}_{i\lambda})_{:j} - (\delta\Gamma^{\lambda}_{ij})_{:\lambda}.$$

Proceeding in exactly the same manner, with all terms being defined analogously,

$$\begin{aligned}
T_1 &= \frac{\partial(\delta\Gamma^{\lambda}_{i\lambda})}{\partial x^j} - \Gamma^{\beta}_{ij}\delta\Gamma^{\lambda}_{\beta\lambda},\\
&= -2\Phi_{,i,j} - \Gamma^{0}_{ij}\delta\Gamma^{\lambda}_{0\lambda}\,(\text{inflatspace}),\\
&= 2\frac{a'}{a}\gamma_{ij}\Phi' - 2\Phi_{,i,j},\\
T_2 &= \frac{\partial(\delta\Gamma^{\lambda}_{ij})}{\partial x^\lambda} - \Gamma^{\beta}_{i\lambda}\delta\Gamma^{\lambda}_{\beta j} - \Gamma^{\beta}_{\lambda j}\delta\Gamma^{\lambda}_{i\beta} + \Gamma^{\lambda}_{\beta\lambda}\delta\Gamma^{\beta}_{ij},\\
T_{21} &= \frac{\partial(\delta\Gamma^{\lambda}_{ij})}{\partial x^\lambda},\\
&= \frac{\partial(\delta\Gamma^{0}_{ij})}{\partial x^0} + \frac{\partial(\delta\Gamma^{k}_{ij})}{\partial x^k},\\
&= -\left[4\left(\mathcal{H}\Phi' + \mathcal{H}'\Phi\right) + \Phi''\right]\gamma_{ij} - \left[2\Phi_{,i,j} - g^{lk}\Phi_{,l,k}g_{ij}\right],\\
&= -\left[4\left(\mathcal{H}\Phi' + \mathcal{H}'\Phi\right) + \Phi'' - \nabla^2\Phi\right]\gamma_{ij} - 2\Phi_{,i,j},\\
T_{22} &= \Gamma^{\beta}_{i\lambda}\delta\Gamma^{\lambda}_{\beta j},\\
&= \left[\Gamma^{0}_{i\lambda}\delta\Gamma^{\lambda}_{0j} + \Gamma^{l}_{i\lambda}\delta\Gamma^{\lambda}_{lj}\right],\\
&= \left[\Gamma^{0}_{il}\partial\Gamma^{l}_{0j} + \Gamma^{l}_{i0}\partial\Gamma^{0}_{lj}\right],\\
&= -\left(2\mathcal{H}\Phi' + 4\mathcal{H}^2\Phi\right)\gamma_{ij},\\
T_{23} &= \Gamma^{\beta}_{\lambda j}\delta\Gamma^{\lambda}_{i\beta},\\
&= \Gamma^{0}_{\lambda j}\delta\Gamma^{\lambda}_{i0} + \Gamma^{l}_{\lambda j}\delta\Gamma^{\lambda}_{il},\\
&= \Gamma^{0}_{kj}\delta\Gamma^{k}_{i0} + \Gamma^{l}_{0j}\delta\Gamma^{0}_{il},\\
&= -\left(2\mathcal{H}\Phi' + 4\mathcal{H}^2\Phi\right)\gamma_{ij},\\
T_{24} &= \Gamma^{\lambda}_{\beta\lambda}\delta\Gamma^{\beta}_{ij},\\
&= \Gamma^{\lambda}_{0\lambda}\delta\Gamma^{0}_{ij} + \Gamma^{\lambda}_{k\lambda}\delta\Gamma^{k}_{ij},\\
&= -\left(16\mathcal{H}^2\Phi + 4\mathcal{H}\Phi'\right)\gamma_{ij}.
\end{aligned}$$

Summing them up for T_2, we get

$$\begin{aligned} T_2 &= T_{21} - T_{22} - T_{23} + T_{24}\,, \\ &= -\left[4\left(\mathcal{H}'\Phi + \mathcal{H}\Phi' + 2\mathcal{H}^2\Phi\right) + \Phi'' - \nabla^2\Phi\right]\gamma_{ij} - 2\Phi_{,i,j}\,. \end{aligned}$$

Then, we get

$$\delta R_{ij} = \left[\Phi'' - \nabla^2\Phi + 6\mathcal{H}\Phi' + 4\left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi\right]\gamma_{ij}\,. \tag{A.31}$$

A.5 The perturbed Einstein tensor

The perturbed Einstein tensor is given by

$$\delta G^\mu_\nu = g^{\mu\lambda}\delta R_{\lambda\nu} - \frac{1}{2}\left(g^{\rho\sigma}\delta R_{\rho\sigma}\right)g^\mu_\nu + \delta g^{\mu\lambda}R_{\lambda\nu} - \frac{1}{2}\left(\delta g^{\rho\sigma}R_{\rho\sigma}\right)g^\mu_\nu\,. \tag{A.32}$$

We may call the sum of the first and second terms T_1, and the sum of the last two terms into T_2. For convenience, we evaluate them separately and then sum them up. T_2 may be simplified using the identity

$$\begin{aligned} \delta g^{\rho\sigma} &= -\frac{1}{a^2(\eta)}\begin{bmatrix} 4\Phi & 0 \\ 0 & 0 \end{bmatrix} + 2\Phi g^{\rho\sigma}\,, \\ &= -4\Phi g^{00} g^\rho_0 g^\sigma_0 + 2\Phi g^{\rho\sigma}\,. \end{aligned} \tag{A.33}$$

For $(\mu,\nu) = (0,0)$, this gives us

$$\begin{aligned} T_2 &= \delta g^{0\lambda}R_{\lambda 0} - \frac{1}{2}\left(-4\Phi g^{00}g^\rho_0 g^\sigma_0 + 2\Phi g^{\rho\sigma}\right)R_{\rho\sigma}\,, \\ &= -2\Phi g^{00}R_{00} + 2\Phi g^{00}R_{00} - \Phi g^{\rho\sigma}R_{\rho\sigma}\,, \\ &= -\Phi R\,, \\ &= \Phi\frac{6\left(\mathcal{H}' + \mathcal{H}^2\right)}{a^2}\,. \end{aligned} \tag{A.34}$$

Now,

$$\begin{aligned} \delta G^0_0 &= g^{0\lambda}\delta R_{\lambda 0} - \frac{1}{2}(g^{00}\delta R_{00} + g^{ij}\delta R_{ij}) + T_2\,, \\ &= \frac{1}{2}(g^{00}\delta R_{00} - g^{ij}\delta R_{ij}) + T_2\,, \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2a^2}\left[4\nabla^2\Phi - 12\mathcal{H}\Phi' - 12(\mathcal{H}' + 2\mathcal{H}^2)\Phi\right] + \frac{6\Phi\left(\mathcal{H}' + \mathcal{H}^2\right)}{a^2}\,, \\
&= \frac{2}{a^2}\left[\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right)\right]\,. \qquad \text{(A.35)}
\end{aligned}
$$

For $(\mu,\nu) = (0,i)$, since the elements g^0_i are zero, the expression for δG^0_i simplifies to

$$
\begin{aligned}
\delta G^0_i &= g^{0\lambda}\delta R_{\lambda i} + \delta g^{0\lambda} R_{\lambda i}\,, \\
&= g^{00}\delta R_{0i} + \delta g^{00} R_{0i}\,, \\
&= \frac{2}{a^2}\left[\mathcal{H}\Phi + \Phi'\right]_{,i}\,. \qquad \text{(A.36)}
\end{aligned}
$$

For $(\mu,\nu) = (i,j)$,

$$
\delta G^i_j = g^{il}\delta R_{lj} - \frac{1}{2}\left(g^{00}\delta R_{00} + g^{lk}\delta R_{lk}\right) g^i_j + T_2\,. \qquad \text{(A.37)}
$$

T_2 can be simplified as follows

$$
\begin{aligned}
T_2 &= \delta g^{i\lambda} R_{\lambda j} - \frac{1}{2}\left(\delta g^{\rho\sigma} R_{\rho\sigma}\right) g^i_j\,, \\
&= 2\Phi g^{il} R_{lj} - \frac{1}{2}\left(-2\Phi g^{00} R_{00} + 2\Phi g^{lk} R_{lk}\right) g^i_j\,. \qquad \text{(A.38)}
\end{aligned}
$$

When $i \neq j$ all elements in te above expression are evidently zero. For $i = j$, we can use the identity

$$
g^{il} R_{lj} = \frac{1}{3} g^{jl} R_{lj}\,, \qquad \text{(A.39)}
$$

to get

$$
\begin{aligned}
T_2 &= \Phi\left(g^{00} R_{00} - \frac{1}{3} g^{lk} R_{lk}\right)\,, \\
&= \Phi\left(\frac{4}{3} G^0_0 + \frac{1}{3} R\right)\,, \\
&= \frac{2\Phi}{a^2}\left(\mathcal{H}^2 - \mathcal{H}'\right)\,. \qquad \text{(A.40)}
\end{aligned}
$$

Now, we proceed to evaluate T_1 which is given by

$$
T_1 = g^{il}\delta R_{lj} - \frac{1}{2}\left(g^{00}\delta R_{00} + g^{lk}\delta R_{lk}\right) g^i_j\,. \qquad \text{(A.41)}
$$

For $i \neq j$, all terms are obviously zero. For $i = j$, the first term can be simplified using (A.39) to give

$$\begin{aligned} T_1 &= \frac{1}{3}g^{lk}\delta R_{lk} - \frac{1}{2}\left(g^{00}\delta R_{00} + g^{lk}\delta R_{lk}\right)g^i_j\,, \\ &= -\frac{1}{2}g^{00}\delta R_{00} - \frac{1}{6}g^{lk}\delta R_{lk}\,, \\ &= -\frac{1}{2a^2}\delta R_{00} - \frac{1}{6a^2}\gamma^{lk}\delta R_{lk}\,, \\ &= -\frac{2}{a^2}\left[\Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi\right]. \end{aligned} \tag{A.42}$$

Combining the expressions for T_1 and T_2 we get

$$\delta G^i_j = -\frac{2}{a^2}\left[\Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi\right]. \tag{A.43}$$

A.6 Conservation of energy in conformal time

The energy-momentum tensor $T^{\mu\nu}$ for a perfect fluid is given by

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu - pg^{\mu\nu}\,. \tag{A.44}$$

Therefore, vanishing covariant divergence $T^{\mu\nu}{}_{;\nu} = 0$ implies ($g^{\mu\nu}{}_{;\nu} = 0$)

$$(p+\rho)_{,\nu}U^\mu U^\nu + (p+\rho)(U^\mu U^\nu)_{;\nu} - p_{,\nu}g^{\mu\nu} = 0\,. \tag{A.45}$$

We may call the terms in the above expression $T_{1,2,3}$ respectively. We calculate the perturbations in these terms separately and then sum them up. The unperturbed four-velocity components are given by $U_0 = a$, $U^0 = 1/a$, $U_i = U^i = 0$. Therefore,

$$\begin{aligned} \delta T_1 &= (\delta p + \delta\rho)_{,\nu}U^\mu U^\nu + (p+\rho)_{,\nu}(U^\mu \delta U^\nu + \delta U^\mu U^\nu)\,, \\ \delta T_2 &= (\delta p + \delta\rho)\left(U^\mu U^\nu\right)_{;\nu} + (p+\rho)\delta\left[(U^\mu U^\nu)_{;\nu}\right]\,, \\ \delta T_3 &= \delta p_{,\nu}g^{\mu\nu} + p_{,\nu}\delta g^{\mu\nu}\,. \end{aligned} \tag{A.46}$$

For $\mu = 0$,

$$\begin{aligned} \delta T_1 &= (\delta p + \delta\rho)'\frac{1}{a^2} - (p+\rho)'\frac{2\Phi}{a^2}\,, \qquad (A.47) \\ \delta T_2 &= (\delta p + \delta\rho)(U^0 U^\nu)_{;\nu} + (p+\rho)_{,\nu}\delta(U^0 U^\nu)_{;\nu}\,. \end{aligned}$$

For convenience, we call the two terms in the above expression δT_{21} and δT_{22} respectively. Applying (2.66), the first term becomes

$$\delta T_{21} = 3\frac{a'}{a}(\delta p + \delta\rho)\,.$$

Evaluating the second term is a bit more tedious.

$$\begin{aligned}
\delta T_{22} &= \delta(U^\mu U^\nu)_{;\nu}\,, \\
&= \delta U^\mu_{,\nu} U^\nu + U^\mu_{,\nu}\delta U^\nu + \delta U^\mu U^\nu_{,\nu} + U^\mu \delta U^\nu_{,\nu} + \delta\Gamma_\lambda U^\mu U^\lambda \\
&\quad + \Gamma_\lambda \delta U^\mu U^\lambda + \Gamma_\lambda U^\mu \delta U^\lambda + \delta\Gamma^\mu_{\lambda\nu} U^\lambda U^\nu \\
&\quad + \Gamma^\mu_{\lambda\nu}\delta U^\lambda U^\nu + \Gamma^\mu_{\lambda\nu} U^\lambda \delta U^\nu\,.
\end{aligned}$$

For $\mu = 0$,

$$\begin{aligned}
\delta T_{22} &= \delta U^{0\prime} U^0 + U^{0\prime}\delta U^0 + \delta U^0 U^{0\prime} + U^0\left(\delta U^{0\prime} + \delta U^i_{,i}\right) + \delta\Gamma_0 U^0 U^0 \\
&\quad + \Gamma_0 \delta U^0 U^0 + \left(\Gamma_0 U^0 \delta U^0 + \Gamma_i U^0 \delta U^i\right) + \delta\Gamma^0_{00} U^0 U^0\,, \\
&\quad + 2\left(\Gamma^0_{00}\delta U^0 U^0 + \Gamma^0_{i0} U^0 \delta U^i\right)\,.
\end{aligned}$$

Substituting $\Gamma^0_{i0} = 0$, $\Gamma_i = 0$ and regrouping all terms, we get

$$\begin{aligned}
\delta T_{22} &= 2U^0\delta U^{0\prime} + 2U^{0\prime}\delta U^0 + U^0\nabla_i\delta U^i + \left(\delta\Gamma_0 + \delta\Gamma^0_{00}\right)U^0U^0 + 2\left(\Gamma_0 + \Gamma^0_{00}\right)\delta U^0 U^0\,, \\
&= \frac{2}{a}(-\Phi/a)' - 2\frac{a'}{a^2}(-\Phi/a) + U^0\nabla_i\delta U^i + (-\Phi')\frac{1}{a^2} + 10\frac{a'}{a}(-\Phi/a^2)\,, \\
&= -3\frac{\Phi'}{a^2} - 6\frac{\Phi a'}{a^3} + \frac{1}{a}\nabla_i\delta U^i\,.
\end{aligned}$$

Summing δT_{21} and δT_{22}, we then have

$$\delta T_2 = (\delta p + \delta\rho)\,\frac{3a'}{a^3} + (p+\rho)\left(-3\frac{\Phi'}{a^2} - 6\frac{\Phi a'}{a^3} + \frac{1}{a}\nabla_i\delta U^i\right)\,. \tag{A.48}$$

The third term δT_3, for $\mu = 0$, simplifies to

$$\delta T_3 = \delta p'\frac{1}{a^2} - p'\frac{2\Phi}{a^2}\,. \tag{A.49}$$

Combining three terms in

$$\delta T_1 + \delta T_2 - \delta T_3 = 0\,,$$

and rearranging terms, gives us

$$\delta\rho' + \frac{3a'}{a}(\delta\rho + \delta p) = (\rho + p)\left(3\Phi' - a\nabla_i \delta U^i\right) + 2\Phi\left[\rho' + \frac{3a'}{a}(\rho + p)\right].$$

The coefficient of Φ (i.e. the term in brackets on the right hand side) vanishes due to conservation of energy. This leaves us

$$\delta\rho' + \frac{3a'}{a}(\delta\rho + \delta p) = (\rho + p)\left(3\Phi' - a\nabla_i \delta U^i\right). \tag{A.50}$$

A.7 Conservation of momentum in conformal time

For $\mu = i$, vanishing covariant divergence $T^{i\nu}{}_{;\nu} = 0$ of the energy-momentum tensor implies

$$(p + \rho)_{,\nu} U^i U^\nu + (p + \rho)(U^i U^\nu)_{;\nu} - p_{,\nu} g^{i\nu} = 0. \tag{A.51}$$

We again call the successive terms $T_{1,2,3}$. As before, we compute the perturbations in these terms separately and then sum them up. The perturbation to the first term is given by

$$\delta T_1 = (\delta p + \delta\rho)_{,\nu} U^i U^\nu + (p + \rho)_{,\nu}(U^i \delta U^\nu + \delta U^i U^\nu). \tag{A.52}$$

Since the spatial components U^i are zero in any comoving reference frame, the above expression simplifies to

$$\delta T_1 = (p + \rho)' \frac{\delta U^i}{a}. \tag{A.53}$$

The perturbation to T_2 can be written as

$$\delta T_2 = (\delta p + \delta\rho)(U^i U^\nu)_{;\nu} + (p + \rho)\delta(U_i U^\nu)_{;\nu}. \tag{A.54}$$

Since $U^i = 0$, the first term drops out. For the second term, we shall first evaluate

$$\begin{aligned} \delta(U^i U^\nu)_{;\nu} &= \delta U^i_{,\nu} U^\nu + U^i_{,\nu} \delta U^\nu + \delta U^i U^\nu_{,\nu} + U^i \delta U^\nu_{,\nu} + \delta\Gamma_\lambda U^i U^\lambda \\ &\quad + \Gamma_\lambda \delta U^i U^\lambda + \Gamma_\lambda U^i \delta U^\lambda + \delta\Gamma^i_{\lambda\nu} U^\lambda U^\nu \\ &\quad + \Gamma^i_{\lambda\nu} \delta U^\lambda U^\nu + \Gamma^i_{\lambda\nu} U^\lambda \delta U^\nu. \end{aligned} \tag{A.55}$$

All terms containing U_i and its derivatives drop off and since $\Gamma^i_{00} = 0$, we get the following, much simplified expression

$$\begin{aligned}\delta(U^i U^\nu)_{;\nu} &= \delta U^{i\prime} U^0 + \delta U^i U^{0\prime} + \Gamma_0 \delta U^i U^0 + \delta\Gamma^i_{00} U^0 U^0 + 2\Gamma^i_{0j}\delta U^j U^0 \,, \\ &= \left(\frac{1}{a}\delta U^{i\prime} + 5\frac{a'}{a^2}\delta U^i + \frac{1}{a^2}\gamma^{ij}\Phi_{,j}\right) .\end{aligned}$$

Substituting in (A.54), we get

$$\delta T_2 = (p+\rho)\left(\frac{1}{a}\delta U^{i\prime} + 5\frac{a'}{a^2}\delta U^i + \frac{1}{a^2}\gamma^{ij}\Phi_{,j}\right) . \tag{A.56}$$

The third term is given by

$$\delta T_3 = \delta p_{,\nu} g^{i\nu} + p_{,\nu}\delta g^{i\nu} .$$

Since the pressure defined on the unperturbed manifold has only zero spatial derivatives, $p_{,i} = 0$, the above expression simplifies to

$$\delta T_3 = -\frac{\delta p_{,j}}{a^2}\gamma^{ij} .$$

Combining these three terms as in

$$\delta T_1 + \delta T_2 - \delta T_3 = 0 ,$$

we get, after using the background equation of motion

$$(p+\rho)\left[\frac{\delta U^{i\prime}}{a} + 2\frac{a'}{a}\frac{\delta U^i}{a}\right] + p'\frac{\delta U^i}{a} = -\left[(p+\rho)\Phi_{,j} + \delta p_{,j}\right]\frac{\gamma^{ij}}{a^2} . \tag{A.57}$$

A shorter form is obtained if use covariant spatial components of four-velocity, $U^i = -1/a^2\gamma^{ij}U_j$, $\delta U^i = -1/a^2\gamma^{ij}\delta U_j$,

$$-\left[(p+\rho)\frac{\delta U_j'}{a} + p'\frac{\delta U_j}{a}\right]\frac{\gamma^{ij}}{a^2} = -\left[(p+\rho)\Phi_{,j} + \delta p_{,j}\right]\frac{\gamma^{ij}}{a^2} , \tag{A.58}$$

which gives

$$\delta U_j' + \frac{p'}{p+\rho}\delta U_j = a\left[\Phi_{,j} + \frac{\delta p_{,j}}{p+\rho}\right] . \tag{A.59}$$

A.8 Perturbing the Klein-Gordon equation

A perturbation of the Klein-Gordon equation has the form

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) + \delta\left(V_{,\varphi}\right) = 0 \tag{A.60}$$

We start with

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) = \frac{\partial\delta\varphi^{,\alpha}}{\partial x^{\alpha}} + \delta\Gamma_{\alpha}\varphi^{,\alpha} + \Gamma_{\alpha}\delta\varphi^{,\alpha}. \tag{A.61}$$

We again call these terms $T_{1,2,3}$ respectively. The first term is given by

$$\begin{aligned} T_1 &= \frac{\partial\delta\varphi^{,0}}{\partial x^0} + \frac{\partial\delta\varphi^{,i}}{\partial x^i}, \\ &= \left[\frac{1}{a^2}(\delta\varphi' - 2\Phi\varphi')\right]' - \frac{\gamma^{ij}\delta\varphi^{,i,j}}{a^2}, \\ &= \frac{1}{a^2}\left[\delta\varphi'' - 2\frac{a'}{a}\delta\varphi' - 2\Phi(\varphi'' - 2\frac{a'}{a}\varphi') - \gamma^{ij}\delta\varphi_{,i,j}\right]. \end{aligned} \tag{A.62}$$

The second term is evaluated as follows

$$\begin{aligned} T_2 &= \Gamma_{\alpha}\delta\varphi^{,\alpha}, \\ &= \Gamma_0\delta\varphi^{,0}, \\ &= \frac{4a'}{a^3}\left[\delta\varphi' - 2\Phi\varphi'\right]. \end{aligned} \tag{A.63}$$

And the third term

$$\begin{aligned} T_3 &= \delta\Gamma_{\alpha}\varphi^{,\alpha}, \\ &= \delta\Gamma_0\varphi^{,0}, \\ &= -\frac{2\Phi'}{a^2}\varphi'. \end{aligned} \tag{A.64}$$

When we sum them up, we obtain

$$\delta\left(\varphi^{;\alpha}{}_{;\alpha}\right) = \frac{1}{a^2}\left[\delta\varphi'' + 2\frac{a'}{a}\delta\varphi' - 2\Phi(\varphi'' + 2\frac{a'}{a}\varphi') - 4\Phi'\varphi' - \nabla^2\delta\varphi\right]. \tag{A.65}$$

Simplifying the coefficient of Φ using the unperturbed equation we finally obtain Klein-Gordon equation in the first order

$$\frac{1}{a^2}\left[\delta\varphi'' + 2\frac{a'}{a}\delta\varphi' + 2\Phi a^2 V_{,\varphi} - 4\Phi'\varphi' - \nabla^2\delta\varphi\right] + V_{,\varphi\varphi}\delta\varphi = 0\,. \qquad \text{(A.66)}$$

A.9 Initial conditions

We may begin by restating a few necessary equations in terms of a_0 and η_0 defined in Chapter 2. The time-time linearized Einstein equation is given by

$$-\frac{a'}{a}\Phi' - \left(\frac{a'}{a}\right)^2\Phi - \frac{k^2}{3}\Phi = 2\frac{a_0}{a}\delta_m + \frac{a_0^2\eta_0^2}{2a^2}\delta_r\,. \qquad \text{(A.67)}$$

The other equations for the variables $\delta_m, \delta_r, v_m, v_r$ remain unmodified and are as follows

$$\delta_m' = 3\Phi' - k^2 v_m\,, \qquad \text{(A.68)}$$

$$\delta_r' = 4\Phi' - \frac{4}{3}k^2 v_r\,, \qquad \text{(A.69)}$$

$$v_m' = \Phi - \frac{a'}{a}v_m\,, \qquad \text{(A.70)}$$

$$v_r' = \Phi + \frac{1}{4}\delta_r\,. \qquad \text{(A.71)}$$

The space-time part of the perturbed Einstein equations states

$$\Phi' + \frac{a'}{a}\Phi = 6\frac{a_0}{a}v_m + \frac{2a_0^2\eta_0^2}{a^2}v_r\,. \qquad \text{(A.72)}$$

In order to determine the initial values of the functions $\Phi, \delta_m, \delta_r, v_m, v_r$ as $\eta \to 0$, we must Taylor expand a few functions in series of η/η_0. Since we have $a = a_0\eta(\eta + \eta_0)$ and $a' = a_0(2\eta + \eta_0)$, we get

$$\frac{a'}{a} = \frac{1}{\eta}\left[1 + \frac{\eta}{\eta_0} - \left(\frac{\eta}{\eta_0}\right)^2 \ldots\,,\right.$$

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{\eta^2}\left[1 + 2\frac{\eta}{\eta_0} - \left(\frac{\eta}{\eta_0}\right)^2 \ldots\,,\right.$$

$$\frac{a_0}{a} = \frac{1}{\eta\eta_0}\left[1 - \frac{\eta}{\eta_0} + \left(\frac{\eta}{\eta_0}\right)^2 \ldots\,,\right.$$

$$\left(\frac{a_0}{a}\right)^2 = \frac{1}{\eta\eta_0}\left[1 - 2\frac{\eta}{\eta_0} + 3\left(\frac{\eta}{\eta_0}\right)^2 \ldots \right. \tag{A.73}$$

Now, if we substitute the following formulae

$$\begin{aligned} \Phi &= \Phi^n \eta^n , \\ \delta_r &= \delta_r^n \eta^n , \\ \delta_m &= \delta_m^n \eta^n , \\ v_r &= v_r^n \eta^n , \\ v_m &= v_m^n \eta^n , \end{aligned} \tag{A.74}$$

where repeated indices denote summation, into the above equations, we get the following from (A.67),

$$\begin{aligned} -\Phi^0 &= \frac{1}{2}\delta_r^0 , \\ -2\Phi^1 - \frac{2}{\eta_0}\Phi^0 &= \frac{2}{\eta_0}\delta_m^0 + \frac{1}{2}\delta_r^1 - \frac{1}{\eta_0}\delta_r^0 , \\ -\frac{k^2}{3}\Phi^0 - 3\Phi^2 - \frac{3}{\eta_0}\Phi^1 + \frac{1}{\eta_0^2}\Phi^0 &= \frac{2}{\eta_0}\delta_m^1 - \frac{2}{\eta_0^2}\delta_m^0 + \frac{1}{2}\delta_r^2 - \frac{1}{\eta_0}\delta_r^1 + \frac{3}{2\eta_0^2}\delta_r^0 \end{aligned} \tag{A.75}$$

Similarly, from (A.68), we get

$$\begin{aligned} \delta_m^1 + k^2 v_m^0 &= 3\Phi^1 , \\ 2\delta_m^2 + k^2 v_m^1 &= 6\Phi^2 . \end{aligned} \tag{A.76}$$

From (A.69), we get

$$\begin{aligned} \delta_r^1 + \frac{4}{3}k^2 v_r^0 &= 4\Phi^1 , \\ 2\delta_r^2 + \frac{4}{3}k^2 v_r^1 &= 8\Phi^2 . \end{aligned} \tag{A.77}$$

From (A.70), we get

$$\begin{aligned} v_m^0 &= 0 , \\ 2v_m^1 &= \Phi^0 , \\ 3v_m^2 + \frac{1}{\eta_0}v_m^1 &= \Phi^1 . \end{aligned} \tag{A.78}$$

From (A.71), we get

$$\begin{aligned} v_r^1 &= \frac{1}{4}\delta_r^0 + \Phi^0 \,, \\ 2v_r^2 &= \frac{1}{4}\delta_r^1 + \Phi^1 \,. \end{aligned} \tag{A.79}$$

From (A.72), we get

$$\begin{aligned} v_r^0 &= 0 \,, \\ \Phi_0 &= \frac{6}{\eta_0}v_m^0 + 2v_r^1 - \frac{4}{\eta_0}v_r^0 \,, \\ 2\Phi^1 + \frac{1}{\eta_0}\Phi^0 &= \frac{6}{\eta_0}v_m^1 - \frac{6}{\eta_0^2}v_m^0 + 2v_r^2 - \frac{4}{\eta_0}v_r^1 + \frac{6}{\eta_0^2}v_r^0 \,, \\ 3\Phi^2 + \frac{1}{\eta_0}\Phi^1 - \frac{1}{\eta_0^2}\Phi^0 &= \frac{6}{\eta_0}v_m^2 - \frac{6}{\eta_0^2}v_m^1 + 2v_r^3 - \frac{4}{\eta_0}v_r^2 + \frac{6}{\eta_0^2}v_r^1 \,. \end{aligned} \tag{A.80}$$

We may define two convenient initial values as follows

$$\begin{aligned} C_1 &= \Phi^0 \,, \\ C_2 &= \delta_m^0 - \Phi^0 \,. \end{aligned} \tag{A.81}$$

In terms of these constants, we get for the first set of coefficients,

$$\begin{aligned} \delta_r^0 &= -2C_1 \,, \\ v_m^1 &= \frac{1}{2}C_1 \,, \\ v_r^1 &= \frac{1}{2}C_1 \,, \\ \Phi^0 &= C_1 \,, \\ \delta_m^0 &= C_2 - \frac{3}{2}C_1 \,. \end{aligned} \tag{A.82}$$

In terms of these constants, we get for the second set of coefficients,

$$\begin{aligned} \delta_r^1 &= -\frac{1}{\eta_0}(C_1 + 2C_2) \,, \\ v_m^2 &= -\frac{1}{4\eta_0}(C_1 + \frac{2}{3}C_2) \,, \\ v_r^2 &= -\frac{1}{4\eta_0}(C_1 + 2C_2) \,, \end{aligned}$$

$$\Phi^1 = -\frac{1}{4\eta_0}(C_1 + 2C_2)\,,$$
$$\delta_m^1 = -\frac{3}{4\eta_0}(C_1 + 2C_2)\,. \tag{A.83}$$

In terms of these constants, we get for the third set of coefficients,

$$\delta_r^2 = \frac{9}{5\eta_0^2}(C_1 + 2C_2) - \frac{7}{15}k^2 C_1\,,$$
$$v_m^3 = \frac{3}{10\eta_0^2}(C_1 + \frac{8}{9}C_2) - \frac{1}{120}k^2 C_1\,,$$
$$v_r^3 = \frac{3}{10\eta_0^2}(C_1 + 2C_2) - \frac{1}{20}k^2 C_1\,,$$
$$\Phi^2 = \frac{9}{20\eta_0^2}(C_1 + 2C_2) - \frac{1}{30}k^2 C_1\,,$$
$$\delta_m^2 = \frac{27}{20\eta_0^2}(C_1 + 2C_2) - \frac{7}{20}k^2 C_1\,. \tag{A.84}$$

The above ordinary differential equations (ODEs) have two separate solutions corresponding to two different initial conditions, namely, the so-called adiabatic and isocurvature modes. The generic solution for any wave-number k can, therefore, be written as the sum of these distinct modes:

$$f(C_1, C_2) = A_k f(C_1', C_2') + B_k f(C_1'', C_2'') \tag{A.85}$$

For the adiabatic mode we may take $C_2' = 0$ and $C_1' = 10/9$. This satisfies the condition for adiabaticity which states $\delta_m = \frac{3}{4}\delta_r$. For the isocurvature mode we may take $C_2'' = -5$ and $C_1'' = 0$ which satisfies the condition $\delta\rho_m = -\delta\rho_r$.

A.10 Zero mode of Φ (k=0)

The solutions (3.28) may be used to eliminate terms with k^2 and Φ' from (3.25) to get

$$\frac{a'}{a}\left[6\frac{a_0}{a}v_m + 2\frac{a_0^2\eta_0^2}{a^2}v_r\right] = -\left[6\frac{a_0}{a} + \frac{a_0^2\eta_0^2}{a^2}\right]\Phi + \left[9\frac{a_0}{a} + 3\frac{a_0^2\eta^2}{a^2}\right]\Phi_0\,. \tag{A.86}$$

Using (A.70) and (A.71) to eliminate the terms with v_r and v_m we finally arrive at an ODE purely in terms of the gravitational potential, which is given by

$$\Phi' + \Phi\left[\frac{a'}{a} + 6\frac{a_0}{a'} + 2\frac{a_0^2\eta_0^2}{aa'}\right] = \left[9\frac{a_0}{a'} + 3\frac{a_0^2\eta^2}{aa'}\right]\Phi_0\,. \tag{A.87}$$

This equation may be further simplified, using (3.3) and (3.4), to get

$$\Phi' + \Phi\left[\frac{3}{\eta} + \frac{3}{\eta+\eta_0} - \frac{1}{\eta+\eta_0/2}\right] = 3\left[\frac{3\eta^2 + 3\eta\eta_0 + \eta_0^2}{\eta(\eta+\eta_0)(2\eta+\eta_0)}\right]\Phi_0\,. \tag{A.88}$$

This equation can be easily solved. The solution is given in the main text by the formula (3.32).