UNIVERSITY OF ALBERTA

WAVES IN VISCOELASTIC AND THERMOELASTIC SOLIDS

by ©

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ABSTRACT

This thesis is concerned with the one-dimensional propagation of boundary-generated waves in two semi-infinite solids. In the first part of the thesis a viscoelastic solid obeying a single-integral constitutive functional is considered. Unlike in many other studies, the instantaneous elastic response function of the solid is not required to be strictly convex. An asymptotic solution is constructed and analyzed. In the second part of the thesis second sound propagation in elastic heat conductors is taken into consideration. First, a theory of thermoelasticity consistent with the second law of thermodynamics and the principle of material frame indifference is derived to describe second sound. Then, the one-dimensional propagation of second sound is examined in the linearized theory. Predictions of the linearized form of our theory, originally formulated by Lord and Shulman, are compared against those of the linearized theory of Green and Lindsay. It is found that predictions of the latter theory are unrealistic. Lastly, attention is turned to the nonlinear theory and implications of the material nonlinearity on the propagation of second sound is analyzed on the basis of the nonlinear geometric optics solution.

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CHAPTER I

INTRODUCTION

We are concerned with the propagation of waves in two materials. Both materials have memory but the processes they may undergo and the manners in which they remember the past are radically different. We are interested in constructing formal asymptotic solutions for our wave propagation problems to obtain not only qualitative but also approximate numerical information for comparison with experiments.

The first material we consider is a viscoelastic solid. In purely mechanical deformations of viscoelastic materials experience indicates that the stress at the present depends on the entire history of deformations but deformations which occurred in the recent past have more influence on the present stress than those which occurred in the distant past. In mathematical terms this fading memory is expressed by means of a constitutive functional that relates the present stress to the history of strain. A fading memory hypothesis is then invoked to impose certain smoothness requirements on the constitutive functional. We refer the reader to the recent article of Saut and Joseph [1] and references therein for detailed descriptions of fading memory hypotheses. Also of relevance to our work are the recent studies of single-integral constitutive relations by Gurtin and Hrusa [2,3] without recourse to any conventional fading memory hypothesis. In nonlinear deformations of viscoelastic materials fading memory manifests itself by introducing a dissipative mechanism competing against the destabilizing effect of nonlinearity of the material.

Available mathematical models that are capable of capturing the above mentioned competition between nonlinearity and dissipation can roughly be divided into the classes of rate type, single-integral type and multiple-integral type models. It is known that each class has advantages and disadvantages, and experiments indicate that none could be considered the best class of models since under certain loading regimes and for certain materials one may be more appropriate than the others whereas under different loading conditions or for some other materials another may become better suited. However, because of their simplicity for analytical and numerical studies, and flexibility for incorporating experimental data, single-integral models have received considerable attention in the past few decades. The reader can consult the recent monograph of Renardy, Hrusa and Nohel [4] and their more recent survey article [5] for detailed reviews of the related literature.

In the first part of this study we undertake an asymptotic analysis of one-dimensional propagation of small-amplitude, high-frequency disturbances by choosing a single-integral stress functional for the viscoelastic solid we consider. The physical problem we deal with describes the longitudinal propagation of small-amplitude, high-frequency disturbances entering into a homogeneous semi-infinite material that has been unstrained and at rest until a disturbance is applied at the boundary. We assume the constitutive functional obeys certain physically reasonable conditions guaranteeing that the material exhibits instantaneous elasticity and that the resulting system of integrodifferential equations *may be regarded* as *hyperbolic* with history dependent real characteristics; see [4].

Our main departure from most other studies is that we do not require the instantaneous elastic response function of the material to be strictly convex but rather allow an inflection point at the constant base state described above. We could have considered several inflection points but because of the local nature of our asymptotic analysis this is not necessary. Motivated by the concept of *local linear degeneration* related with a similar situation in the theory of hyperbolic partial differential equations, we call the instantaneous elastic response function locally linearly degenerate at the base state. In the context of hyperbolic systems of partial differential equations, problems suffering from local linear degeneracies have been considered by several authors; see, for example, Liu [6], Klainerman and Majda [7], and He and Moodie [8]. In the context of viscoelastic materials, the works of Dafermos [9], Nohel, Rogers and Tzavaras [10], and Warhola and Pipkin [11] are particularly related with the case when the instantaneous elastic response function is locally linearly degenerate at some constant base state. Some related experimental observations have been addressed in Nunziato et al. [12].

The second material of interest in this study is an elastic heat conductor that conducts heat by *second sound*. The concept of second sound originated in 1941 from Landau's attempt [13] to develop a theory of superfluidity for the superfluid liquid helium, He-2. Studying the propagation of sound according to his theory Landau showed that in He-2 there exist two sound velocities. The first of these is the velocity of the ordinary acoustic wave while the second the velocity of a thermal wave. This latter wave is called second sound.

In 1944, Peskhov [14] reported the experimental detection of second sound in He-2. In this report, on the basis of similarities between superfluidity and superconductivity, he argued that an analogous phenomenon might be observed in superconductors as well. Later, Peskhov [15] speculated further that second sound could be detected also in crystals. Following speculations of Peskhov, several attempts have been made to develop theories of second sound in solids. A detailed historical account on these attempts can be found in the survey article of Joseph and Preziosi [16]. In spite of many theoretical studies, no notable experimental effort was expended to detect second sound in solids until the mid-sixties.

Another concept related with thermal waves propagating at finite velocities is the concept of the "paradox of instantaneous propagation of thermal disturbances" predicted by the classical linear theory of heat conduction. In the classical theory Fourier's Law states that the heat flux is proportional to the temperature gradient. Therefore, according to Fourier's Law heat flow starts at the same instant a temperature gradient is generated. When Fourier's Law and the classical energy-temperature relation giving the specific internal energy as a linear function of the temperature are combined with the energy balance equation, the resulting equation is the classical parabolic heat equation suggesting that thermal disturbances diffuse with infinite velocity.

This concept was initiated in 1948 by Cattaneo [17] in his study concerning the kinetic theory of gases. Based on the observation we just explained briefly, Cattaneo argued that since thermal disturbances must propagate with finite velocities, this prediction of Fourier's theory is physically paradoxical. To eliminate the paradox, he then proposed an alternative law of heat conduction. Cattaneo's Law is a rate type constitutive equation relating the present heat flux to the past history of temperature gradient and implies simply that heat flow does not start instantaneously but establishes itself gradually with a relaxation time after a temperature gradient is created. When Fourier's Law is replaced with Cattaneo's Law in the classical theory of heat conduction, the resulting equation is the well known hyperbolic telegraphy equation whose solutions propagate at finite velocities.

In 1963, Chester [18] brought the two concepts together and suggested that Cattaneo's Law would provide a satisfactory mathematical model to describe second sound in crystals at low temperatures. In 1966, solving the linearized phonon Boltzman equation, Guyer and Krumhansl [19] derived a set of macroscopic equations as a model for second sound in crystals. The heat flux-temperature relation they determined deviates from Cattaneo's Law by terms involving spatial gradients of the heat flux and reduces to Cattaneo's Law when these terms are neglected. Shortly after, Guyer and his collaborators [20] announced the first successful demonstration of second sound in a solid, He-4. After this experiment, second sound has been observed in the dielectric crystals He-3 [21] and NaF [22-25], and also in Bi [26]. All these experiments have been performed at very low temperatures with "specially refined high-purity" crystals and "heat-pulse propagation" techniques have been employed. For other experiments reference is made to the survey article [16].

In the late sixties and early seventies, many attempts have been made to develop continuum theories capable of predicting thermal waves propagating at finite velocities for various types of materials; see Öncü and Moodie [27]. In the second part of the thesis we take two of these continuum theories into consideration as mathematical models for examination of second sound in crystals. These are the linear theory of Lord and Shulman [28] and the theory of Green and Lindsay [29] for thermoelastic materials. Our choice of thermoelasticity theories resides in the fact that experimentally observed thermal fluctuations in the above experiments, though at varying degrees, are coupled with deformations. In view of experiments we deal with the one-dimensional propagation of disturbances generated at the boundary of a homogeneous semi-infinite material body by disturbing the temperature.

The linear theory of Lord and Shulman [28] is a simple modification of the classical linear theory of thermoelasticity: Lord and Shulman simply replaced Fourier's Law with Cattaneo's Law to write down their constitutive equations.

Based on experimental data for NaF, Pao and Banerjee [30] showed that the predictions of the Lord and Shulman theory are in agreement with the experiment. The work done over the past two decades on the propagation of disturbances in the linear theory of Lord and Shulman now constitutes a considerably large literature. The number of studies on the propagation of disturbances in the Green and Lindsay theory is, on the other hand, comparatively small.

Green and Lindsay [29] developed their constitutive theory by exploiting a modified form of the Clausius-Duhem inequality as the mathematical statement of the second law of thermodynamics. By incorporating the rate of temperature into their list of independent constitutive variables, they obtained a set of constitutive equations which involve Fourier's Law as the constitutive equation for the heat flux for isotropic materials. We compare predictions of the linearized equations of the Green and Lindsay theory against those of the Lord and Shulman theory in the one-dimensional problem we set up and find that predictions of the former theory are physically unreasonable.

We then abandon the use of the theory of Green and Lindsay, and continue our analysis of the problem in a nonlinear theory we develop on the basis of thermodynamical arguments. We give our derivation prior to the formulation of the one-dimensional problem we examine. This is a three-dimensional development. Our nonlinear constitutive equations are extensions of the nonlinear constitutive equations recently developed by Coleman, Fabrizio and Owen [31] for rigid heat conductors to include elastic heat conductors. In our derivation we also provide a solution to the problem of lack of material frame invariance Cattaneo's Law suffers from. Our constitutive equations reduce to those of the theory of Lord and Shulman after linearization. In our analysis of the one-dimensional problem we concentrate our attention on the nonlinear influence of temperature on the propagation of second sound. In fact, the above noted experiments indicate a strong dependence of the material parameters on the temperature in the range of temperatures in which second sound occurs. The nonlinear influence of deformations, on the other hand, appears to be unimportant.

CHAPTER 2

ONE-DIMENSIONAL VISCOELASTIC SOLIDS

In this chapter we formulate the one-dimensional problem described in the introduction for the viscoelastic solid we wish to study. We start our formulation by delineating the appropriate governing equations. We then introduce the single-integral stress functional we choose for our viscoelastic solid and define the instantaneous elastic moduli. We conclude this chapter by discussing the properties of the resulting 2×2 system of Volterra integrodifferential equations for preparation to the asymptotic analysis of the next chapter.

2.1. Basic Equations

Consider the longitudinal motion of a one-dimensional viscoelastic solid that has been unstrained, at rest and occupied the interval $(0, \infty)$ for all time $t \leq 0$. Denote by $\chi(x,t)$ the motion of the solid, that gives the position at time t of the particle with the reference position $x \in (0, \infty)$, and suppose the solid is homogeneous with the constant mass density ρ_0 in its reference configuration. The velocity $v(x,t)$ and strain $u(x,t)$ are then defined by

$$v = \frac{\partial \chi}{\partial t}, \qquad u = \frac{\partial \chi}{\partial x} - 1. \tag{2.1}$$

It follows from (2.1) that

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{2.2}$$

and the law of conservation of linear momentum in the absence of body forces takes the form

$$\rho_0 \frac{\partial v}{\partial t} - \frac{\partial S}{\partial x} = 0, \tag{2.3}$$

where $S(x,t)$ is the Piola stress.

It will be necessary in what follows to deal with dimensionless variables. Consequently, we introduce the following nondimensionalization scheme:

$$\begin{aligned} &x^* = \tfrac{1}{L}\,x, \quad t^* = \tfrac{1}{T}\,t, \quad \chi^* = \tfrac{1}{L}\,\chi, \\ &v^* = \tfrac{T}{L}\,v, \quad S^* = \tfrac{T^2}{\rho_0 L^2}\,S, \end{aligned} \tag{2.4}$$

where L and T are appropriate length and time scales. We shall use these nondimensional variables exclusively but omit the asterisks for convenience. In terms of the dimensionless variables the equations (2.2) and (2.3) take the respective dimensionless forms

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{2.5}$$

$$\frac{\partial v}{\partial t} - \frac{\partial S}{\partial x} = 0. \tag{2.6}$$

We now suppose that the dimensionless Piola stress is determined by a single-integral functional through the relation

$$S = F(u^t), \tag{2.7}$$

where F is that single-integral functional which depends on the entire history of strain at x up to time t defined as

$$u^t(x,\tau) = u(x, t-\tau), \quad 0 \leq \tau < \infty. \tag{2.8}$$

The restriction of $u^t(x,\cdot)$ to $(0,\infty)$, that is,

$$u_r^t(x,\tau) = u(x, t-\tau), \quad 0 < \tau < \infty, \tag{2.9}$$

is called the past history of strain at x up to time t. We shall use $u(x,t)$ and $u^t(x,0)$ interchangeably to indicate the present value of the strain at x. The single-integral functional F will be specified in the next section.

Under the assumptions on the motion of the solid we have

$$u(x,t) = v(x,t) = 0, \quad x > 0, \quad t \leq 0. \tag{2.10}$$

Let us now suppose that a wave motion is set up at the boundary $x = 0$ with the boundary condition

$$u(0,t) = \varepsilon\sigma_0(t/\varepsilon^p) \quad \text{for all} \quad t \in (-\infty, \infty), \tag{2.11}$$

where $0 < \varepsilon \ll 1$ and p is an integer such that $1 \leq p \leq q$ for some integer $q \geq 1$. The function σ_0 vanishes for all $t \leq 0$ and is either smooth for all t or smooth for all t except for $t = 0$ where it is only Lipschitz continuous, that is, $\sigma_0'(0) \neq 0$ with the prime indicating differentiation with respect to the argument. Further, together with its first derivative, σ_0 is integrable on $(-\infty,\infty)$. Note that since prescribing the boundary velocity results in an equivalent problem, we deal only with the case of boundary prescribed strain. We avoid traction boundary conditions, which offer some technical difficulties, for expositional simplicity. Until further notice we assume $\sigma_0'(0)=0$.

2.2. The Stress Functional and the Instantaneous Elastic Moduli

Let us suppress the x dependence of the strain for convenience and limit our attention to the class of bounded strain histories which are functions from $[0,\infty)$ to D where D is an open, bounded and simply-connected neighbourhood of zero on the real line and $\overline{D}$ is the closure of D.

In this thesis we are concerned with the single-integral functionals given of the form

$$F(u^t)=G\big(u(t)\big)+\int_0^\infty a'(\tau)H\big(u(t),u(t-\tau)\big)d\tau, \tag{2.12}$$

or, equivalently,

$$F(u^t)=G\big(u(t)\big)+\int_{-\infty}^t a'(t-s)H\big(u(t),u(s)\big)ds, \tag{2.13}$$

where $s=t-\tau$, $0<\tau<\infty$ is the past time. In this functional a is a twice continuously differentiable, positive, decreasing and convex function on $[0,\infty)$ whereas with $n\geq 2$, G and H are n times continuously differentiable functions on $\overline{D}$ and $\overline{D}^2$, respectively. With no loss of generality, we set $a(\infty)=0$, normalize H according as $H(u,u)=0$ for all $u\in D$ and further suppose $G(0)=0$. Therefore, the solids we are dealing with are stress free in their strain free equilibrium states, though such a requirement is not necessary for our analysis. The function a is called the *memory function.*

Let us denote with $\boldsymbol{w}$ the vector defined as $\boldsymbol{w}=(w_1,w_2)^T=\big(u(t),u(s)\big)^T$ for fixed t and s, where the superscript T denotes the transpose. That G and H are n times continuously differentiable functions indicates they admit the following Taylor series expansions:

$$G(u)=\sum_{k=1}^{n}\widetilde{G}_k\,\frac{u^k}{k!}+o(|u|^n), \tag{2.14}$$

$$H(\boldsymbol{w})=\sum_{k=1}^{n}\sum_{\substack{i+j=k\\ i,j\geq 0}}\widetilde{H}_{ij}\,\frac{w_1^i}{i!}\,\frac{w_2^j}{j!}+o(|\boldsymbol{w}|^n), \tag{2.15}$$

where

$$G_k(u)=\frac{d^k}{du^k}\,G(u), \qquad k=1,2,\ldots,n, \tag{2.16}$$

$$H_{ij}(w_1,w_2)=\frac{\partial^{i+j}}{\partial w_1^i\partial w_2^j}\,H(w_1,w_2), \quad i+j=1,2,\ldots,n; \quad i,j\geq 0, \tag{2.17}$$

while $\widetilde{G}_k=G_k(0)$ and $\widetilde{H}_{ij}=H_{ij}(0,0)$.

The function $G(u)$ is called the *equilibrium elastic response function* of the material at the constant strain history $u^t(\tau)=u(t),\ 0\leq\tau<\infty$. The functions $G_k(u)$ are called the *k-th order equilibrium elastic moduli* at u. We assume that the first order equilibrium modulus obeys the condition

$$G_1(u)>0 \quad \text{for all} \quad u\in D, \tag{2.18}$$

which is compatible with the experiments at least near zero.

In view of the properties of the memory function a we require that the following *stability* condition is satisfied:

$$H_{01}(w_1, w_2) > 0 \quad \text{for all} \quad (w_1, w_2) \in D^2. \tag{2.19}$$

As will become apparent in the subsequent sections, this requirement ensures that the mechanism arising from the memory of the material is dissipative. We remark that since $H(u,u) = 0$ we have $H_{01}(u,u) = -H_{10}(u,u) > 0$.

The introduced properties of a, G and H are sufficient, but not necessary, for the material to exhibit instantaneous elasticity. In this matter we refer the reader to the articles of Coleman [32,33].

Let u^t be a given history and define a corresponding history u^t_c by

$$u^t_c(\tau) = \begin{cases} u^t(0) + \alpha, & \tau = 0, \\ u^t(\tau), & 0 < \tau < \infty. \end{cases} \tag{2.20}$$

Coleman [32] called u^t_c the *jump continuation* of u^t *with jump* α where α is a constant. It is evident that u^t_c describes a jump of size α imposed on u^t at time t.

The function E of α, defined as

$$E(\alpha; u^t) = F(u^t_c), \tag{2.21}$$

is called the *instantaneous elastic response function* of the material at the strain history u^t. Needless to say, this function depends not only on the jump α, but also on the entire history u^t. We remark that such a function is undefined unless a' is integrable; nonintegrability of a' indicates that the material does not exhibit an instantaneous elastic response but, as for the viscous fluids, a jump in the strain results in an infinite stress. Implications of this and weaker type of singularities in a' have been examined in the monograph [4] in detail.

The derivatives

$$E_k(u^t) = \frac{d^k}{d\alpha^k} E(\alpha; u^t)|_{\alpha=0}, \quad k = 1, 2, \ldots, n, \tag{2.22}$$

are called the *k-th order instantaneous elastic moduli* at u^t. By virtue of the fact that knowledge of the entire history u^t is equivalent to the knowledge of its past history u^t_r and its present u, it is possible to consider $F(u^t)$ as $F(u^t_r; u)$. With this observation, $E_k(u^t)$ can be defined alternatively as the usual partial derivatives of $F(u^t_r; u)$ with respect to the present value u by keeping the past history u^t_r constant:

$$E_k(u^t) = \frac{\partial^k}{\partial u^k} F(u^t_r; u). \tag{2.23}$$

When necessary, we shall refer to such derivatives of the functionals like F as the instantaneous derivatives. It follows from (2.13), (2.14), (2.15) and (2.23) that the $E_k(u^t)$ take the explicit forms

$$E_k(u^t) = G_k\big(u(t)\big) + \int_{-\infty}^{t} a'(t-s) H_{k0}\big(u(t), u(s)\big) ds. \tag{2.24}$$

The first order instantaneous elastic modulus is required to obey the condition

$$E_1(u^t) > 0 \quad \text{for all} \quad u^t \in D. \tag{2.25}$$

Like the condition (2.18) on $G_1(u)$, this condition on $E_1(u^t)$ is compatible with the experiments near zero. Further, with (2.25) it is guaranteed that the equations we consider may be regarded as hyperbolic and that they never change type (see [4]).

Let us Taylor expand E about $\alpha = 0$ and get

$$E(\alpha; u^t) = F(u^t) + \sum_{k=1}^{n} E_k(u^t)\,\frac{\alpha^t}{k!} + o(\alpha^n). \tag{2.26}$$

If $E_2(u^t) = 0$ for all $u^t \in D$, the sum in (2.26) terminates after the first term so that E is a linear function of α. Since this does not mean that the stress functional F is linear, we call the instantaneous elastic response function E *linearly degenerate* if this happens. If $E_2(u^t) \neq 0$ for all $u^t \in D$, then

$$E(\alpha; u^t) = F(u^t) + E_1(u^t)\alpha + \tfrac{1}{2}\,E_2(u^t)\alpha^2 + o(\alpha^2). \tag{2.27}$$

In this case we call the instantaneous elastic response function E *genuinely nonlinear*. On the other hand, if

$$\sum_{k=2}^{n} |E_k(u^t)| \neq 0 \quad \text{for all} \quad u^t \in D, \tag{2.28}$$

the instantaneous elastic response function E is not linearly degenerate but, though not necessarily genuinely, nonlinear, which we now assume (see [9]).

Suppose that there exists a strain history $\widehat{u}^t \in D$ where, requiring that $1 \leq q < n$, q is the smallest integer such that $E_{q+1}(\widehat{u}^t) \neq 0$ and that for all $u^t \in D$ other than $\widehat{u}^t$, $E_2(u^t) \neq 0$. Therefore, at $\widehat{u}^t$

$$E(\alpha; \widehat{u}^t) = F(\widehat{u}^t) + E_1(\widehat{u}^t)\alpha + \frac{1}{(q+1)!} E_{q+1}(\widehat{u}^t)\alpha^{q+1} + o(\alpha^{q+1}). \qquad (2.29)$$

It is clear that when $q = 1$, E is genuinely nonlinear. To accomodate such a situation, we then call the instantaneous elastic response function E ***locally linearly degenerate up to order*** $q - 1$ ***at the strain history*** $\widehat{u}^t$, reserving the term zeroth order local linear degeneracy to indicate genuine nonlinearity. It is easy to extend this definition to the case where there are several strain histories such as $\widehat{u}^t$. For simplicity in discussion we limit our attention to when $\widehat{u}^t$ is constant and further suppose $\widehat{u}^t(\tau) = 0$ for all $\tau \in [0, \infty)$. We remark that in shearing motions of viscoelastic materials it is necessary that $E_2(0) = 0$ (see [34]). At the constant strain history $\widehat{u}^t = 0$ we have $E_k(0) = \widetilde{E}_k$ where

$$\widetilde{E}_k = \widetilde{G}_k - a(0)\widetilde{H}_{k0}, \quad k = 1, 2, \ldots, n. \qquad (2.30)$$

Before closing this section we introduce the following functional for convenience

$$K(u^t | \partial_x u^t_r) = \int_{-\infty}^{t} a'(t-s) H_{01}(u(t), u(s)) \partial_x u(s) ds. \qquad (2.31)$$

We call this functional the *dissipation functional* for reasons which will become clear in what follows.

2.3. The System

It is the consequence of the ongoing discussion that the strain $u(x,t)$ and the velocity $v(x,t)$ satisfy the following system of Volterra integrodifferential equations:

$$\begin{aligned} u_{,t} - v_{,x} &= 0 \\ v_{,t} - E_1(u^t)u_{,x} &= K(u^t|u^t_{r,x}), \end{aligned} \qquad x > 0, \quad t \geq 0 \tag{2.32}$$

where subscripts following a comma indicate partial derivatives.

With a view to giving arguments to justify that (2.32) *may be regarded* as *hyperbolic* with history dependent characteristics, let us consider first the system of m quasilinear partial differential equations

$$\mathbf{u}_{,t} + A(\mathbf{u})\mathbf{u}_{,x} = 0, \tag{2.33}$$

where $\mathbf{u}$ is a vector function of x and t, and A a sufficiently smooth matrix function of $\mathbf{u} \in \mathcal{U}$ with $\mathcal{U}$ being a domain in $\mathbb{R}^m$. If the matrix A is the Jacobian matrix of a sufficiently smooth vector function f of $\mathbf{u} \in \mathcal{U}$, that is, $A(\mathbf{u}) = \mathrm{grad}_{\mathbf{u}} f(\mathbf{u})$, then (2.33) is conservative.

If for each $\mathbf{u} \in \mathcal{U}$, the matrix $A(\mathbf{u})$ has m real eigenvalues $\{\lambda_i(\mathbf{u})\}_{i=1}^m$ with the corresponding set of m linearly independent right eigenvectors $\{r_i(\mathbf{u})\}_{i=1}^m$, then the system (2.33) is *hyperbolic*. The system (2.33) is *strictly* *hyperbolic* if these eigenvalues are distinct for each $u \in \mathcal{U}$. The integral curves of the ordinary differential equations

$$\frac{dx}{dt} = \lambda_i(u), \quad i = 1, 2, \ldots, m, \tag{2.34}$$

are called the characteristics of (2.33). We assume from now on that (2.33) is strictly hyperbolic.

The i-th family of characteristics is called genuinely nonlinear in the sense of Lax [35] if

$$\text{grad}_u \lambda_i(u) \cdot r_i(u) \neq 0 \quad \text{for all} \quad u \in \mathcal{U}, \tag{2.35}$$

whereas it is called *linearly degenerate* in the sense of Lax if

$$\text{grad}_u \lambda_i(u) \cdot r_i(u) = 0 \quad \text{for all} \quad u \in \mathcal{U}. \tag{2.36}$$

If there is linear degeneration in the i-th characteristic family, then the first derivatives of solutions originating from smooth data do not blow up, to put it differently, waves do not break, in the i-th mode of propagation; see, for example, Lax [35]. If, on the other hand, the i-th characteristic family is genuinely nonlinear, these derivatives blow up in finite time in this mode of propagation.

A more complicated case is when one of the characteristic families, say the i-th, is neither linearly degenerate nor genuinely nonlinear. That is, there exists a nonempty proper subset $\mathcal{V}$ of $\mathcal{U}$ such that

$$\mathcal{V} = \{\boldsymbol{u} \in \mathcal{U} \mid \operatorname{grad}_{\boldsymbol{u}} \lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) = 0 \quad \text{at} \quad \boldsymbol{u}\}. \tag{2.37}$$

We call the i-th characteristic family *locally linearly degenerate* on $\mathcal{V}$. It has been shown by John [36] among others that although local linear degeneracy may improve the *lifespan* of smooth solutions in the i-th mode of propagation, it cannot prevent blow-up of the first derivatives.

Let us now consider the dissipative hyperbolic system

$$\boldsymbol{u}_{,t} + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_{,x} = \boldsymbol{b}(\boldsymbol{u}), \tag{2.38}$$

where $\boldsymbol{b}$ is a sufficiently smooth vector function of $\boldsymbol{u} \in \mathcal{U}$. It is well known that the implications of Lax's definitions concerning the characteristic families of the non-dissipative system (2.33) do not readily extend to the dissipative system (2.38). Setting $\boldsymbol{u} = (u, v)^T$ we see that the equations (2.32) mimic the dissipative system (2.38) with the history dependent coefficient matrix

$$\boldsymbol{A}(u^t) = \begin{pmatrix} 0 & -1 \\ -E_1(u^t) & 0 \end{pmatrix} \tag{2.39}$$

and right-hand side vector $\boldsymbol{b}(u^t|u^t_{r,x}) = \left(0, K(u^t|u^t_{r,x})\right)^T$. The matrix $\boldsymbol{A}(u^t)$ has the real and distinct eigenvalues

$$\lambda_2(u^t) = -\sqrt{E_1(u^t)} < 0 < \sqrt{E_1(u^t)} = \lambda_1(u^t), \tag{2.40}$$

with the corresponding right eigenvectors

$$r_1(u^t) = \big(1, -\sqrt{E_1(u^t)}\,\big), \quad r_2(u^t) = \big(1, \sqrt{E_1(u^t)}\,\big). \tag{2.41}$$

Therefore, the system (2.32) may be regarded as strictly hyperbolic with the real characteristics

$$\frac{dx}{dt} = \lambda_i(u^t), \quad i = 1, 2. \tag{2.42}$$

We remark that for these characteristics to be defined, $E_1(u^t)$ must exist which necessitates that a' must be an integrable function. The equations (2.24), (2.40) and (2.42) reveal formally that when a' is not integrable, the solutions of the system (2.32), as for the solutions of parabolic equations, propagate with infinite speeds (see [4]). Therefore, if a' is not integrable, the problem should not be viewed as hyperbolic.

It follows from (2.40) and (2.42) that if the instantaneous elastic response function E is genuinely nonlinear, then the characteristics of (2.32) depend on the entire history of the strain u^t. On the other hand, if E is linearly degenerate, the characteristics of (2.32) may depend only on the past history u^t_r and cannot depend on the present value u. It is then clear that the instantaneous directional derivatives

$$-\operatorname{grad}_u \lambda_2(u^t) \cdot r_2(u^t) = \frac{E_2(u^t)}{2\sqrt{E_1(u^t)}} = \operatorname{grad}_u \lambda_1(u^t) \cdot r_1(u^t), \tag{2.43}$$

vanish identically when E is linearly degenerate. Although recent studies (see, for example, Dafermos [9,37] and Hrusa [38]) suggest a strong connection between this and the concept of linear degeneracy in the sense of Lax, a detailed discussion of the nature of this connection falls beyond the scope of this thesis.

We conclude this section with a few remarks about the implications of the local linear degeneracy of E on the characteristics of (2.32). First, we define the following function of α at the strain history u^t:

$$\Lambda(\alpha; u^t) = \lambda_1(u^t_c) = -\lambda_2(u^t_c), \tag{2.44}$$

where u^t_c and α are given in (2.20). This function provides a measure for the size of the jumps in $\lambda_1(u^t)$ and $-\lambda_2(u^t)$ after a jump of size α imposed on the strain history u^t at the instant t. Because of the properties of F, Λ admits the expansion

$$\Lambda(\alpha; u^t) = \lambda_1(u^t) + \sum_{k=1}^{n-1} \Lambda_k(u^t)\,\frac{\alpha^k}{k!} + o(\alpha^{n-1}), \tag{2.45}$$

where

$$\Lambda_k(u^t) = \frac{d^k}{d\alpha^k}\,\Lambda(\alpha; u^t)|_{\alpha=0}, \quad k = 1, 2, \ldots, n-1. \tag{2.46}$$

Since E is locally linearly degenerate up to order $q-1$ at the strain history $\hat{u}^t(x,\tau) = 0$, $0 \leq \tau < \infty$, it follows from (2.24), (2.40), (2.45) and (2.46) that

$$\Lambda(\alpha; 0) = \lambda_1(0) + \frac{1}{q!}\,\widetilde{\Lambda}_q \alpha^q + o(\alpha^q), \tag{2.47}$$

where

$$\widetilde{\Lambda}_q = \Lambda_q(0) = \frac{\widetilde{E}_{q+1}}{2\sqrt{\widetilde{E}_1}} \,. \tag{2.48}$$

It can be seen from either of (2.45) or (2.47) that, unlike linear degeneracy, local linear degeneracy does not mean that the eigenvalues are independent of the present strain u. However, since $\Lambda(\alpha;0) - \lambda_1(0) = O(|\alpha|^q)$ it turns out that when the solutions perturbing a base state for which $u(x,t) = 0$ for all $t \leq 0$ are small, the larger the order of local linear degeneracy of E is, the smaller is the influence of the present values of these solutions on the eigenvalues of the system. As the asymptotic results of the next chapter will show, the waves originating from the boundary disturbance (2.11) do not break down unless the first derivative of the disturbance is sufficiently large. Specifically, if the disturbance is of order ε, then its first derivative must be of order ε^{-q+1} for a possible breakdown of the waves. Whether waves break or not then depends on the result of the competition between nonlinearity and dissipation.

CHAPTER 3

ASYMPTOTIC ANALYSIS OF THE VISCOELASTIC PROBLEM

In this chapter we present an asymptotic analysis of the problem formulated in the previous chapter by constructing single wave expansions through a systematic use of the method of multiple scales. These single wave expansions are uniformly valid over the time depending on the properties of the material and of the disturbances as long as the first derivatives remain bounded. We are interested in determining the conditions for the boundedness or the blow-up of these derivatives. The method we develop is based on the simple single wave expansions of nonlinear geometric optics for hyperbolic systems of partial differential equations. Methods based on these expansions have been rendered systematic procedures through the recent studies of Hunter and Keller [39] and Hunter, Majda and Rosales [40,41] among others and reviewed in the recent survey article of Majda [42].

The first section of this chapter is devoted to the formal construction of the single wave asymptotic expansions. In the second section the results are analyzed and comparisons are made with the findings of Coleman and Gurtin [34] about the growth and decay of one-dimensional acceleration waves in a general class of viscoelastic materials by specializing their results to the case we consider. Finally, in the last section, a simple but important numerical example is examined.

3.1. Geometric Optics Solution

In this section we are concerned with constructing a formal asymptotic solution $\widetilde{u}(x,t), \widetilde{v}(x,t)$ which uniformly approximates the solution $u(x,t), v(x,t)$ of the problem (2.32), (2.10) and (2.11) in its region of smoothness within terms of order ε^{p+1}. To be precise, we are concerned with constructing the formal single wave asymptotic solution of the form

$$\begin{aligned} \widetilde{u}(x,t) &= \varepsilon U\left(x, \tfrac{\phi}{\varepsilon^p}; \varepsilon\right), \\ \widetilde{v}(x,t) &= \varepsilon V\left(x, \tfrac{\phi}{\varepsilon^p}; \varepsilon\right), \end{aligned} \qquad 1 \le p \le q, \tag{3.1}$$

where, with

$$\theta(x,t) = \frac{\phi(x,t)}{\varepsilon^p}, \tag{3.2}$$

U and V are given by

$$\begin{aligned} U(x,\theta;\varepsilon) &= U_1(x,\theta) + \varepsilon^p U_{p+1}(x,\theta), \\ V(x,\theta;\varepsilon) &= V_1(x,\theta) + \varepsilon^p V_{p+1}(x,\theta). \end{aligned} \tag{3.3}$$

Here $\phi(x,t)$ is a phase function yet to be specified and it is required that the formal solution (3.1) satisfies (2.32) with errors of order ε^{p+1}.

It follows from (3.1) and the boundary condition (2.11) that

$$\phi(0,t) = t, \tag{3.4}$$

and recall that the solid we are dealing with is homogeneous. Consequently, we limit our attention to linear phase functions of the form

$$\phi(x,t) = t - \frac{x}{\lambda}, \tag{3.5}$$

where λ is a constant with the significance of the wave speed associated with the wave of phase ϕ. With the relation (3.5) and compatible with the history condition (2.10) we demand also that

$$U(x,\theta;\varepsilon) = V(x,\theta;\varepsilon) = 0 \quad \text{for all} \quad \phi \leq 0. \tag{3.6}$$

Regarding x and θ as independent variables by the method of multiple scales we now impose the following conditions on U and V:

i) U and V are smooth functions of x and θ,

ii) for each $x > 0$, U, V, $U_{,x}$, $V_{,x}$, $U_{,\theta}$ and $V_{,\theta}$ are bounded and integrable functions of θ.

The first step in the construction is the derivation of the equations which U and V satisfy within errors of order ε^{p+1}. It is clear from the transformation

$$(x,t) \mapsto (x,\theta), \tag{3.7}$$

that

$$\begin{aligned} \partial_x &\mapsto \partial_x - \frac{1}{\varepsilon^p \lambda}\,\partial_\theta, \\ \partial_t &\mapsto \frac{1}{\varepsilon^p}\,\partial_\theta. \end{aligned} \tag{3.8}$$

To determine how the integrals in (2.32) transform let us define

$$\begin{aligned} i(x,t) &= E_1\big(\widetilde{u}^t(x,\cdot)\big), \\ j(x,t) &= K\big(\widetilde{u}^t(x,\cdot)\big|\widetilde{u}^t_{r,x}(x,\cdot)\big). \end{aligned} \tag{3.9}$$

Suppressing the ε dependence of U for notational convenience, after appropriate changes of variables we find

$$\begin{aligned} i(x,t) &= I(x,\theta;\varepsilon), \\ j(x,t) &= J(x,\theta;\varepsilon), \end{aligned} \tag{3.10}$$

where

$$I(x,\theta;\varepsilon) = G_1\big(\varepsilon U(x,\theta)\big) + \varepsilon^p \int_{-\infty}^{\theta} a'(\varepsilon^p\theta - \varepsilon^p\varphi) H_{10}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big) d\varphi, \tag{3.11}$$

$$\begin{aligned} J(x,\theta;\varepsilon) = \varepsilon^{p+1} &\int_0^{\theta} a'(\varepsilon^p\theta - \varepsilon^p\varphi) H_{01}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big) U_{,x}(x,\varphi) d\varphi \\ &- \frac{\varepsilon}{\lambda} \int_0^{\theta} a'(\varepsilon^p\theta - \varepsilon^p\varphi) H_{01}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big) U_{,\varphi}(x,\varphi) d\varphi. \end{aligned} \tag{3.12}$$

In (3.12) the lower limits of the integrals are changed to zero because of (3.6).

We now claim that

$$I(x,\theta;\varepsilon) = \begin{cases} \widetilde{E}_1 + \frac{\varepsilon^q}{q!} \widetilde{E}_{q+1} U^q(x,\theta) + o(\varepsilon^q), & \text{if} \quad q \ge 1,\ p = q \\ \widetilde{E}_1 + o(\varepsilon^p), & \text{if} \quad q \ge 2,\ 1 \le p < q, \end{cases} \tag{3.13}$$

and

$$J(x,\theta;\varepsilon) = -\varepsilon \frac{a'(0)\widetilde{H}_{01}}{\lambda} U(x,\theta) + o(\varepsilon), \quad q \ge 1,\ 1 \le p \le q, \tag{3.14}$$

uniformly as $\varepsilon \to 0$.

To prove the claim let us deal with $I(x,\theta;\varepsilon)$ first. With this in mind, we now expand $G_1\big(\varepsilon U(x,\theta)\big)$ and $H_{10}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big)$ into the Taylor series

$$G_1\big(\varepsilon U(x,\theta)\big) = \widetilde{G}_1 + \sum_{i=1}^{p} \frac{\varepsilon^i}{i!}\, \widetilde{G}_{i+1} U^i(x,\theta) + o(\varepsilon^p), \tag{3.15}$$

$$H_{10}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big) = \widetilde{H}_{10} + \sum_{\substack{i+j=1\\ i,j\geq 0}}^{p} \frac{\varepsilon^{i+j}}{i!j!}\, \widetilde{H}_{i+1,j} U^i(x,\theta) U^j(x,\varphi) + o(\varepsilon^p). \tag{3.16}$$

Introducing these expansions into (3.11) and rearranging the terms we then get

$$I(x,\theta;\varepsilon) = I_1(x,\theta;\varepsilon) + I_2(x,\theta;\varepsilon), \tag{3.17}$$

where

$$\begin{aligned} I_1(x,\theta;\varepsilon) &= \sum_{i=0}^{p} \frac{\varepsilon^i}{i!} \{\widetilde{G}_{i+1} + \varepsilon^p \widetilde{H}_{i+1,0} \int_{-\infty}^{\theta} a'(\varepsilon^p\theta - \varepsilon^p\varphi)d\varphi\} U^i(x,\theta) + o(\varepsilon^p), \\ &= \sum_{i=0}^{p} \frac{\varepsilon^i}{i!} \{\widetilde{G}_{i+1} - a(0)\widetilde{H}_{i+1,0}\} U^i(x,\theta) + o(\varepsilon^p), \\ &= \sum_{i=0}^{p} \frac{\varepsilon^i}{i!}\, \widetilde{E}_{i+1} U^i(x,\theta) + o(\varepsilon^p), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} I_2(x,\theta;\varepsilon) = &\sum_{\substack{i+j=1\\ i\geq 0, j\geq 1}}^{p} \frac{\varepsilon^{i+j+p}}{i!j!} \\ &\times \widetilde{H}_{i+1,j} \{ \int_0^{\theta} a'(\varepsilon^p\theta - \varepsilon^p\varphi) U^j(x,\varphi) d\varphi \} U^i(x,\theta) + o(\varepsilon^{2p}), \end{aligned} \tag{3.19}$$

uniformly as $\varepsilon \to 0$. Although (3.18) is obvious, (3.19) needs justification. Indeed, it is implicit in (3.19) that

$$I_2(x,\theta;\varepsilon) = o(\varepsilon^p), \tag{3.20}$$

uniformly as $\varepsilon \to 0$. We now prove (3.20).

Let us define

$$W_j(x,\theta;\varepsilon) = \int_0^\theta a'(\varepsilon^p\theta - \varepsilon^p\varphi)U^j(x,\varphi)d\varphi, \quad j = 1,2,\ldots,p. \tag{3.21}$$

The order relation (3.20) follows if $W_j(x,\theta;\varepsilon) = O(1)$, $j = 1,2,\ldots,p$. But

$$\begin{aligned} |W_j(x,\theta;\varepsilon)| &\le \int_0^\theta |a'(\varepsilon^p\theta - \varepsilon^p\varphi)|\, |U(x,\varphi)|^j d\varphi \\ &\le |a'(\cdot)|_\infty |U(x,\cdot)|_\infty^{j-1} |U(x,\cdot)|_1, \end{aligned} \tag{3.22}$$

where $|\cdot|_\infty$ and $|\cdot|_1$ are the usual L_∞ and L_1 norms, respectively. This shows that $W_j(x,\theta;\varepsilon) = O(1)$, $j = 1,2,\ldots,p$. There then follows (3.20) and hence (3.13) so that it remains to verify (3.14).

To prove (3.14), let us rewrite (3.12) as

$$J(x,\theta;\varepsilon) = J_1(x,\theta;\varepsilon) - J_2(x,\theta;\varepsilon), \tag{3.23}$$

where

$$J_1(x,\theta;\varepsilon) = \varepsilon^{p+1} \int_0^\theta a'(\varepsilon^p\theta - \varepsilon^p\varphi) H_{01}(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)) U_{,x}(x,\varphi) d\varphi, \tag{3.24}$$

and

$$J_2(x,\theta;\varepsilon) = \frac{\varepsilon}{\lambda} \int_0^\theta a'(\varepsilon^p\theta - \varepsilon^p\varphi) H_{01}(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)) U_{,\varphi}(x,\varphi) d\varphi. \tag{3.25}$$

We show first that $J_1(x,\theta;\varepsilon) = o(\varepsilon^p)$ uniformly as $\varepsilon \to 0$. This requires a proof of

$$\varepsilon^{-(1+p)} J_1(x,\theta;\varepsilon) = O(1), \tag{3.26}$$

uniformly as $\varepsilon \to 0$. Let us define $H_{\max} = \max_{w \in \overline{D}^2} |H_{01}(w)|$ so that

$$\begin{aligned} |\varepsilon^{-(1+p)} J_1(x,\theta;\varepsilon)| &\le \int_0^\theta |a'(\varepsilon^p\theta - \varepsilon^p\varphi)| \, |H_{01}(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)| \, |U_{,x}(x,\varphi)| d\varphi \\ &\le H_{\max} \, |a'(\cdot)|_\infty \, |U_{,x}(x,\cdot)|_1. \end{aligned} \tag{3.27}$$

This proves (3.27) and therefore, (3.14) follows if

$$J_2(x,\theta;\varepsilon) = \varepsilon \, \frac{a'(0)\widetilde{H}_{01}}{\lambda} \, U(x,\theta) + o(\varepsilon), \quad q \ge 1, \quad 1 \le p \le q. \tag{3.28}$$

To prove (3.28), let us expand $H_{01}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big)$ into the series

$$H_{01}\big(\varepsilon U(x,\theta), \varepsilon U(x,\varphi)\big) = \sum_{i+j=0}^{p} \frac{\varepsilon^{i+j}}{i!j!} \, \widetilde{H}_{i,j+1} U^i(x,\theta) U^j(x,\varphi) + o(\varepsilon^p), \tag{3.29}$$

and insert this expansion into (3.25) to get

$$\begin{aligned} J_2(x,\theta;\varepsilon) = \frac{1}{\lambda} \sum_{i+j=0}^{p} & \frac{\varepsilon^{i+j+1}}{i!(j+1)!} \\ & \times \widetilde{H}_{i,j+1} \Big\{ \int_0^\theta a'(\varepsilon^p\theta - \varepsilon^p\varphi) [U^{j+1}(x,\varphi)]_{,\varphi} d\varphi \Big\} U^i(x,\theta) + o(\varepsilon^{p+1}). \end{aligned} \tag{3.30}$$

To make some progress we now define

$$Y_j(x,\theta;\varepsilon) = \int_0^\theta a'(\varepsilon^p\theta - \varepsilon^p\varphi) [U^{j+1}(x,\varphi)]_{,\varphi} d\varphi, \quad j = 0,1,2,\ldots,p. \tag{3.31}$$

Integrating (3.31) by parts we then find

$$Y_j(x,\theta;\varepsilon) = a'(0)U^{j+1}(x,\theta) + \varepsilon^p Z_j(x,\theta;\varepsilon), \tag{3.32}$$

where

$$Z_j(x,\theta;\varepsilon) = \int_0^\theta a''(\varepsilon^p\theta - \varepsilon^p\varphi)U^{j+1}(x,\varphi)d\varphi. \tag{3.33}$$

Reminding that a'' is bounded on $[0,\infty)$,

$$\begin{aligned} |Z_j(x,\theta;\varepsilon)| &\leq \int_0^\theta |a''(\varepsilon^p\theta - \varepsilon^p\varphi)|\ |U(x,\varphi)|^{j+1}d\varphi \\ &\leq |a''(\cdot)|_\infty |U(x,\cdot)|_\infty^j |U(x,\cdot)|_1, \end{aligned} \tag{3.34}$$

so that $Z_j(x,\theta;\varepsilon) = O(1)$ and therefore,

$$Y_j(x,\theta;\varepsilon) = a'(0)U^{j+1}(x,\theta) + O(\varepsilon^p), \tag{3.35}$$

uniformly as $\varepsilon \to 0$. Thus,

$$J_2(x,\theta;\varepsilon) = \sum_{i+j=0}^{p} \frac{\varepsilon^{i+j+1}}{i!(j+1)!} \frac{a'(0)\widetilde{H}_{i,j+1}}{\lambda} U^{i+j+1}(x,\theta) + o(\varepsilon^{p+1}), \tag{3.36}$$

uniformly as $\varepsilon \to 0$. From (3.36) and (3.23) follows (3.14) immediately.

Summarizing the results we distinguish two cases. These cases are

CASE 1. $q \geq 1,\ p = q$:

$$\begin{aligned} &U_{,\theta} + \tfrac{1}{\lambda} V_{,\theta} = \varepsilon^q V_{,x}, \\ &V_{,\theta} + \frac{\widetilde{E}_1}{\lambda} U_{,\theta} = \varepsilon^q \widetilde{E}_1 U_{,x} - \varepsilon^q \frac{\widetilde{E}_{q+1}}{\lambda q!} U^q U_{,\theta} - \varepsilon^q \frac{a'(0)\widetilde{H}_{01}}{\lambda} U + o(\varepsilon^q). \end{aligned} \tag{3.37}$$

CASE 2. $q \geq 2,\ 1 \leq p < q$:

$$U_{,\theta} + \frac{1}{\lambda} V_{,\theta} = \varepsilon^p V_{,x},$$
$$V_{,\theta} + \frac{\widetilde{E}_1}{\lambda} U_{,\theta} = \varepsilon^p \widetilde{E}_1 U_{,x} - \varepsilon^p \frac{a'(0)\widetilde{H}_{01}}{\lambda} U + o(\varepsilon^p). \tag{3.38}$$

It then follows from the assumption (3.3) and the above equations that for both cases the leading terms solve the

$O(1)$ PROBLEM:

$$U_{1,\theta} + \tfrac{1}{\lambda} V_{1,\theta} = 0,$$
$$V_{1,\theta} + \frac{\widetilde{E}_1}{\lambda} U_{1,\theta} = 0. \tag{3.39}$$

The equation (3.39) is satisfied provided we choose

$$U_1(x,\theta) = \sigma(x,\theta),$$
$$V_1(x,\theta) = -\lambda\sigma(x,\theta), \tag{3.40}$$

and determine the phase ϕ by solving the equation

$$1 - \frac{\widetilde{E}_1}{\lambda^2} = 0, \tag{3.41}$$

for the wave speed λ. Indeed, the equation (3.41) is the familiar eikonal equation of geometric optics. Since we are interested in the waves travelling into the region $x > 0$, we choose the positive root $\lambda = \sqrt{\widetilde{E}_1}$ so that the rays

$$x = \lambda(t - t_0), \tag{3.42}$$

associated with the phase ϕ move into the region $x > 0$ from the boundary $x = 0$ with increasing time. The amplitude function $\sigma(x, \theta)$ satisfying

$$\sigma(0, \theta) = \sigma_0(\theta), \tag{3.43}$$

from the boundary condition (2.11) is yet to be determined.

We now proceed to determine $\sigma(x, \theta)$ in each case separately.

CASE 1. $q \geq 1,\ p = q$:

It follows from (3.3), (3.37), (3.40) and (3.42) that in Case 1 the amplitude function $\sigma(x, \theta)$ and the functions $U_{q+1}(x, \theta)$ and $V_{q+1}(x, \theta)$ must be chosen so as to satisfy the

$O(\varepsilon^q)$ PROBLEM:

$$\begin{aligned} &U_{q+1,\theta} + \tfrac{1}{\lambda} V_{q+1,\theta} = -\lambda \sigma_{,x} \\ &V_{q+1,\theta} + \lambda U_{q+1,\theta} = \lambda^2 \sigma_{,x} - \frac{\widetilde{E}_{q+1}}{\lambda q!}\, \sigma^q \sigma_{,\theta} - \frac{a'(0)\widetilde{H}_{01}}{\lambda}\, \sigma. \end{aligned} \tag{3.44}$$

For the boundary condition (2.11) to be satisfied at the order ε^{q+1}, $U_{q+1}(x, \theta)$ must obey

$$U_{q+1}(0, \theta) = 0. \tag{3.45}$$

Suppose that $(0, x_s)$ is the region of space in which the solution of the problem remains smooth. Since we required that the leading order term of the formal solution provides a uniformly valid approximation to the solution of the problem in $(0, x_s)$ with errors of order ε^{q+1}, we will choose U_{q+1} and V_{q+1} in such a way that

$$|U_{q+1}(x,\theta)|,\ |V_{q+1}(x,\theta)| = O(1) \quad \text{for all} \quad x \in (0, x_s) \tag{3.46}$$

is also satisfied.

It follows from (3.44) that the amplitude function $\sigma(x,\theta)$ satisfies the transport equation

$$\sigma_{,x} - \nu_q \sigma^q \sigma_{,\theta} + \tfrac{\mu}{\lambda}\, \sigma = 0, \tag{3.47}$$

where

$$\nu_q = \frac{\tilde{E}_{q+1}}{2\lambda^3 q!}, \quad \mu = -\frac{a'(0)\widetilde{H}_{01}}{2\lambda^2} > 0. \tag{3.48}$$

Thus, the amplitude function $\sigma(x,\theta)$ is determined by the equation (3.47) and the initial condition (3.43). We defer the solution of (3.47) and (3.43) until the next section.

In view of the condition (3.45) we now choose

$$U_{q+1}(x,\theta) = 0. \tag{3.49}$$

Therefore, the equations (3.44) determine $V_{q+1}(x,\theta)$ as

$$V_{q+1,\theta} = -\lambda^2 \sigma_{,x} \tag{3.50}$$

or, upon integrating,

$$V_{q+1}(x,\theta) = -\lambda^2 \int_0^\theta \sigma_{,x}(x,\varphi)d\varphi. \tag{3.51}$$

This gives

$$|V_{q+1}(x,\theta)| \le \lambda^2 |\sigma_{,x}(x,\cdot)|_1 \quad \text{for all} \quad x \in (0, x_s), \tag{3.52}$$

so that the condition (3.46) is automatically satisfied.

To summarize, the formal solution in this case is given by

$$\begin{aligned} \widetilde{u}(x,t) &= \varepsilon\sigma(x,\theta), \\ \widetilde{v}(x,t) &= -\varepsilon\lambda\sigma(x,\theta) - \varepsilon^{q+1}\lambda^2 \textstyle\int_0^\theta \sigma_{,x}(x,\varphi)d\varphi, \end{aligned} \qquad q \ge 1, \tag{3.53}$$

uniformly for all $x \in (0, x_s)$ as $\varepsilon \to 0$, where $\sigma(x,\theta)$ is determined from (3.47) and (3.43) whereas

$$\theta = \frac{t - x/\lambda}{\varepsilon^q}, \quad \lambda = \sqrt{\widetilde{E}_1}. \tag{3.54}$$

We now turn our attention to Case 2.

CASE 2. $q \ge 2, \quad 1 \le p < q$:

In this case, the equations (3.38), (3.40) and (3.42), and the assumption (3.3) show that $\sigma(x,\theta)$, $U_{p+1}(x,\theta)$ and $V_{p+1}(x,\theta)$ must satisfy the

$O(\varepsilon^p)$ PROBLEM:

$$\begin{aligned} U_{p+1,\theta} + \tfrac{1}{\lambda} V_{p+1,\theta} &= -\lambda\sigma_{,x} \\ V_{p+1,\theta} + \lambda U_{p+1,\theta} &= \lambda^2\sigma_{,x} - \frac{a'(0)\widetilde{H}_{01}}{\lambda}\sigma, \end{aligned} \tag{3.55}$$

and

$$U_{p+1}(0,\theta) = 0, \tag{3.56}$$

which emerges from the boundary condition (2.11). Further, we require

$$|U_{p+1}(x,\theta)|,\ |V_{p+1}(x,\theta)| = O(1) \quad \text{for all} \quad x \in (0, x_s). \tag{3.57}$$

Since the selection procedure for this case is identical to that of Case 1, without duplicating the work we write down the following conclusion:

The formal asymptotic solution for Case 2 takes the form

$$\begin{aligned} &\tilde{u}(x,t) = \varepsilon\sigma(x,\theta), \\ &\tilde{v}(x,t) = -\varepsilon\lambda\sigma(x,\theta) - e^{p+1}\lambda^2 \int_0^\theta \sigma_{,x}(x,\varphi)d\varphi, \end{aligned} \qquad q \geq 2, \quad 1 \leq p < q, \tag{3.58}$$

uniformly for all $x \in (0, x_s)$ as $\varepsilon \to 0$ but this time $\sigma(x,\theta)$ solves the transport equation

$$\sigma_{,x} + \tfrac{\mu}{\lambda}\sigma = 0, \tag{3.59}$$

with μ as in (3.48), and satisfies the initial condition (3.43) where

$$\theta = \frac{t - x/\lambda}{\varepsilon^p}, \quad \lambda = \sqrt{\tilde{E}_1}\,. \tag{3.60}$$

In closing this section we note that although we have constructed the formal asymptotic solutions under the assumption of smoothness, this assumption may be relaxed. It is a consequence of the results of Coleman and Gurtin

[34] that when σ_0 is smooth everywhere except at $t = 0$ where σ_0' suffers a jump discontinuity, then jump discontinuities in the first derivatives of $u(x,t)$, $v(x,t)$ propagate along the surface $x = \lambda t$, which corresponds to the phase surface $\theta(x,t) = 0$. It is clear from the condition (2.10) that all of $u(x,t)$, $v(x,t)$, $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$, and their derivatives of all orders vanish ahead of this surface. In space-time regions of smoothness behind this surface, $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ are uniformly asymptotic respectively to $u(x,t)$ and $v(x,t)$. As will be seen in the next section, if jumps in the first derivatives of the asymptotic solution $\tilde{u}(x,t)$, $\tilde{v}(x,t)$ are allowed formally, then these jumps are identical to the corresponding jumps in the first derivatives of the solution $u(x,t)$, $v(x,t)$, that are obtained from the exact theory of acceleration waves.

3.2. Analysis of the Results

Our objective in this section is to determine the place and time of blow-up of the first derivatives of $\tilde{u}(x,t)$, $\tilde{v}(x,t)$, if they explode at all. Since $\tilde{u}(x,t)$, $\tilde{v}(x,t)$ are asymptotic to $u(x,t)$, $v(x,t)$ uniformly in regions of smoothness, blow-up of the first derivatives of $\tilde{u}(x,t)$, $\tilde{v}(x,t)$ indicates blow-up of the first derivatives of $u(x,t)$, $v(x,t)$. Therefore, the place and time of blow-up predicted by the formal asymptotic solution gives the approximate breaking place and time of the waves after which the formation of shock waves is expected. Also in this section, with $p = 1$ and $q \geq 1$, we compare our results with the results of Coleman and Gurtin [34] about the one-dimensional propagation of acceleration waves in a general class of viscoelastic materials by specializing their results to the case we consider. Based on this comparison we see that the formal asymptotic solution is uniformly valid even when the first derivatives suffer jump discontinuities across the wave front $\theta(x,t)=0$, which is *not* evident from the construction procedure of the previous section.

We start with examining the case $q \geq 1$ and $p = q$. Solving the problem (3.47) and (3.43) by the method of characteristics we find

$$\sigma(x,\theta) = \sigma_0(\xi)e^{-\mu x/\lambda}, \tag{3.61}$$

where

$$\theta = \frac{t - x/\lambda}{\varepsilon^q} = \xi + \frac{\lambda \nu_q}{\mu q}\,\sigma_0^q(\xi)(e^{-\mu q x/\lambda} - 1). \tag{3.62}$$

Since

$$\xi_{,\theta} = \frac{1}{1 + \frac{\lambda \nu_q}{\mu}\,\sigma_0^{q-1}(\xi)\sigma_0'(\xi)(e^{-\mu q x/\lambda} - 1)} = \frac{1}{\theta_{,\xi}}, \tag{3.63}$$

we have

$$\sigma_{,\theta}(x,\theta) = \frac{\sigma_0'(\xi)e^{-\mu x/\lambda}}{1 + \frac{\lambda \nu_q}{\mu}\,\sigma_0^{q-1}(\xi)\sigma_0'(\xi)(e^{-\mu q x/\lambda} - 1)}, \tag{3.64}$$

and, with (3.64), the equation (3.47) gives

$$\sigma_{,x}(x,\theta) = -\frac{\mu}{\lambda}\,\frac{1 - \frac{\lambda \nu_q}{\mu}\,\sigma_0^{q-1}(\xi)\sigma_0'(\xi)}{1 + \frac{\lambda \nu_q}{\mu}\,\sigma_0^{q-1}(\xi)\sigma_0'(\xi)(e^{-\mu q x/\lambda} - 1)}\,\sigma_0(\xi)e^{-\mu x/\lambda}. \tag{3.65}$$

It is now apparent from (3.53) and (3.61) to (3.65) that the condition for breaking of the waves is

$$\theta_{,\xi} = 1 + \frac{\lambda \nu_q}{\mu} \sigma_0^{q-1}(\xi)\sigma_0'(\xi)(e^{-\mu q x/\lambda} - 1) = 0. \tag{3.66}$$

Because $\theta = \xi$ and therefore, $\theta_{,\xi} = 1$ at $x = 0$, there exists some interval $(0, \tilde{x}_s)$ of x in which $\theta_{,\xi} > 0$. Presumably, $\tilde{x}_s$ is the smallest positive x at which $\theta_{,\xi} = 0$. If there does not exist such a positive x, then $\tilde{x}_s$ is infinity and thus, the formal asymptotic solution is uniformly valid for all $x > 0$ and $t > 0$, which implies that the waves do not break. A scrutiny of (3.66) shows that this is the case if

$$\nu_q \sigma_0^{q-1}(\xi)\sigma_0'(\xi) \le \frac{\mu}{\lambda} \quad \text{for all} \quad \xi > 0. \tag{3.67}$$

Otherwise, $\tilde{x}_s$ is given by

$$\tilde{x}_s = \min\{x > 0 | \, 1 + \frac{\lambda \nu_q}{\mu} \sigma_0^{q-1}(\xi)\sigma_0'(\xi)(e^{-\mu q x/\lambda} - 1) = 0\}, \tag{3.68}$$

from which also the corresponding $\tilde{\xi}_s$ is determined. Thus, the waves break at the approximate position $\tilde{x}_s$ and the approximate time $\tilde{t}_s$ which is computed from (3.62).

In the simpler case $q \ge 2$ and $1 \le p < q$, the solution of the problem (3.59) and (3.43) is

$$\sigma(x, \theta) = \sigma_0(\theta) e^{-\mu x/\lambda}, \tag{3.69}$$

where

$$\theta = \frac{t - x/\lambda}{\varepsilon^p}\,. \tag{3.70}$$

Thus, together with (3.69) and (3.70), (3.58) implies the boundedness of the first derivatives of $\widetilde{u}(x,t)$, $\widetilde{v}(x,t)$ for all $x > 0$ and $t > 0$.

The above results can be collected into the following proposition.

PROPOSITION. *Let $q \geq 1$ be the smallest integer such that $\widetilde{E}_{q+1} \neq 0$. Then*

i) if $q \geq 2$ and $1 \leq p < q$, then as $\varepsilon \to 0$ the single wave expansion given by (3.58), (3.69) and (3.70) is a uniformly valid asymptotic solution of the problem (2.32), (2.10) and (2.11) for all $x > 0$ and $t > 0$.

ii) if $q \geq 1$ and $p = q$, then as $\varepsilon \to 0$ the single wave expansion given by (3.54), (3.61) and (3.67) is a uniformly valid asymptotic solution of the problem (2.32), (2.10) and (2.11) for all $x > 0$ and $t > 0$ as long as the condition (3.67) is satisfied. If the condition (3.67) is violated, this expansion is a uniformly valid asymptotic solution of the problem in the region $(0, \widetilde{x}_s) \times (0, \widetilde{t}_s)$ where $\widetilde{x}_s$ and $\widetilde{t}_s$ are determined form (3.68) and (3.62), respectively.

The above proposition summarizes the relevant information on the breakdown of smooth solutions. We now turn attention to the comparison we promised in the opening of this section and fix $p = 1$.

A one-dimensional acceleration wave is a propagating singular surface $x = y(t)$ across which the first derivatives of $u(x,t)$ and $v(x,t)$ experience jump discontinuities but are continuous functions elsewhere while $u(x,t)$ and $v(x,t)$ are continuous everywhere. Suppose $\sigma_0'(0) \neq 0$ and so an acceleration wave is generated at $x = y(0) = 0$. Since this wave propagates into the region $(0, \infty)$ for the solid we consider, we have that $y'(t) > 0$ and that $u(x,t) = v(x,t) = 0$ for $x > y(t)$. For the viscoelastic solid obeying the constitutive relation (2.12), a result of Coleman and Gurtin [34] shows

$$\frac{d}{dt}\, y(t) = \lambda, \tag{3.71}$$

where λ is the previously obtained constant wave speed so that $y(t) = \lambda t$.

Let us denote with $[g]$ the jump discontinuity of the function $g(x,t)$ across the surface $x = \lambda t$. It follows from a theorem of Coleman and Gurtin [34] that the amplitude of the jump

$$f(t) = [v_{,t}] = -\lambda[u_{,t}], \tag{3.72}$$

satisfies the ordinary differential equation

$$\frac{df}{dt} = -\mu f - \nu_1 f^2, \tag{3.73}$$

where μ and ν_1 as in (3.48) with $q = 1$. Because of (2.11) and (3.72), f obeys the initial condition

$$f(0) = -\lambda \sigma_0'(0). \tag{3.74}$$

The solution of (3.73) and (3.74) is easily obtained to give

$$[v_{,t}] = -\lambda[u_{,t}] = \frac{-\lambda\sigma_0'(0)e^{-\mu t}}{1 + \frac{\lambda\nu_1}{\mu}\,\sigma_0'(0)(e^{-\mu t} - 1)}\,. \tag{3.75}$$

Whether this jump remains bounded or tends to infinity in finite time is not difficult to determine from (3.65). In particular, if $\widetilde{E}_2 = 0$ so that $\nu_1 = 0$, then it is bounded for all $t > 0$, since

$$[v_{,t}] = -\lambda[u_{,t}] = -\lambda\sigma_0'(0)e^{-\mu t}. \tag{3.76}$$

We observe from (3.61) to (3.65) for the case $q = 1$ and from (3.69) and (3.70) for the case $q > 1$ that $\sigma_{,x}(x,\theta)$ is continuous across $\theta(x,t) = 0$. Therefore, for both cases

$$[\widetilde{v}_{,t}] = -\lambda[\widetilde{u}_{,t}] = -\lambda\sigma_{,\theta}(x,0), \tag{3.77}$$

which is easily seen from (3.53) and (3.58), respectively. It then follows from (3.64) for $q = 1$ that

$$[\widetilde{v}_{,t}] = -\lambda[\widetilde{u}_{,t}] = \frac{-\lambda\sigma_0'(0)e^{-\mu x/\lambda}}{1 + \frac{\lambda\nu_1}{\mu}\,\sigma_0'(0)(e^{-\mu x/\lambda} - 1)}\,, \quad x = \lambda t. \tag{3.78}$$

That $\xi = 0$ if and only if $\theta = 0$ is correct as long as $\theta_{,\xi} > 0$ in some neighbourhood of $\theta = 0$. In fact, at the instant the condition $\theta_{,\xi} > 0$ is violated at $\theta = 0$, the above jumps explode. We also note

$$[\widetilde{v}_{,t}] = -\lambda[\widetilde{u}_{,t}] = -\lambda\sigma_0'(0)e^{-\mu x/\lambda}, \quad x = \lambda t, \tag{3.79}$$

obtained from (3.70) and (3.77) for $q > 1$.

Comparing (3.78) and (3.79) with (3.75) and (3.76) we see that the exact theory of acceleration waves and the asymptotic single wave expansion procedure agree in their predictions on the growth and decay of jumps in the first derivatives propagating along the wave front $\theta(x,t) = 0$. In particular, both approaches indicate that if the instantaneous elastic response function is linearly degenerate locally at the constant base state, that is, $\widetilde{E}_2 = 0$, then waves never break at the wave front. On the other hand, the asymptotic solution shows that under conditions stated in the proposition, waves break behind the wave front even when $\widetilde{E}_2 = 0$. Therefore, the asymptotic single wave expansion procedure gives the additional important information that even small amplitude solutions originating from smooth data, if they are sufficiently high frequency, may develop singularities in finite time as long as the instantaneous elastic response function is *not* linearly degenerate.

3.3. A Numerical Example

Let us assume that the stress functional F is given by

$$F\big(u^t(x,\cdot)\big) = f\big(u(x,t)\big) + \int_{-\infty}^{t} a'(t-s) f\big(u(x,s)\big) ds, \tag{3.80}$$

where $f(0) = 0$, $f'(u) > 0$ for all u and $0 < a(0) < 1$. This functional can be put in the form (2.13) by defining

$$G\big(u(x,t)\big) = \big(1 - a(0)\big)f\big(u(x,t)\big), \tag{3.81}$$

$$H\big(u(x,t), u(x,s)\big) = f\big(u(x,s)\big) - f\big(u(x,t)\big). \tag{3.82}$$

The significance of the requirement $0 < a(0) < 1$ can be seen from the identity (3.81) and the condition (2.18). Note that if $a(0) = 1$, then the functional (3.80) cannot describe the stress response of a viscoelastic solid but may describe that of a viscoelastic fluid; see Pipkin [43].

We present our example calculations by choosing the material functions f and a as

$$f(u) = u + ku^{q+1}; \quad k > 0, \qquad q \geq 1, \tag{3.83}$$

$$a(t) = me^{-rt}; \qquad 0 < m < 1, \quad r > 0, \tag{3.84}$$

and the boundary disturbance σ_0 as

$$\sigma_0(t/\varepsilon^q) = \begin{cases} \sin(t/\varepsilon^q), & 0 \leq t/\varepsilon^q \leq \pi, \\ 0, & \pi < t/\varepsilon^q. \end{cases} \tag{3.85}$$

Note that in view of the arguments given in the previous section, we have allowed jump discontinuities in σ_0'.

We examine the case $p = q$ only, since when $q \geq 2$ and $1 \leq p < q$ waves do not break. With the above choice of material functions and the boundary disturbance, we find from (3.58), (3.61) and (3.62) that the asymptotic solution $\tilde{u}(x,t)$ of the strain takes the form

$$\tilde{u}(x,t) = \begin{cases} \varepsilon \sin\xi e^{-mrx}, & 0 \le \xi \le \pi, \\ 0, & \xi < 0 \quad \text{and} \quad \xi > \pi \end{cases} \tag{3.86}$$

where

$$\frac{t-x}{\varepsilon^q} = \theta = \begin{cases} \xi + \frac{(q+1)k}{mrq} \sin^q \xi (e^{-mrqx} - 1), & 0 \le \xi \le \pi \\ \xi, & \xi < 0 \quad \text{and} \quad \xi > \pi. \end{cases} \tag{3.87}$$

Therefore, the inequality (3.67) indicates that waves never break if

$$(q+1)k \sin^{q-1}\xi \cos\xi \le mr; \quad 0 \le \xi \le \pi. \tag{3.88}$$

It is clear that this condition is always satisfied for $\frac{\pi}{2} \le \xi \le \pi$ regardless of the magnitudes of the material parameters. Let us assume that the condition (3.88) is violated in the range $0 \le \xi < \frac{\pi}{2}$.

For clarity in exposition we shall calculate the approximate breaking distance and time of the waves for $q = 1$ and $q = 2$. The implications of higher order nonlinearities, that is, $q > 2$, are similar to those for $q = 2$ qualitatively.

In the case $q = 1$ we have from (3.68)

$$\tilde{x}_s = \min\{x > 0 \,|\, 1 + \frac{2k}{mr} \cos\xi (e^{-mrx} - 1) = 0\}, \tag{3.89}$$

which occurs at $\xi = 0$. Therefore,

$$\widetilde{x}_s = -\frac{1}{mr}\,\ell n\,(1 - \frac{mr}{2k}), \tag{3.90}$$

and

$$\widetilde{t}_s = \widetilde{x}_s, \tag{3.91}$$

which is obtained from (3.87) with $x = \widetilde{x}_s$ and $\xi = 0$. This result clearly indicates that the waves generated by the boundary disturbance (3.85) break at the wave front $t = x$ located at $x = \widetilde{x}_s$.

For $q = 2$ the approximate breaking distance is given by

$$\widetilde{x}_s = \min\,\{x > 0 \big|\, 1 + \frac{3k}{2mr}\,\sin 2\xi (e^{-2mrx} - 1) = 0\}, \tag{3.92}$$

which occurs at $\xi = \frac{\pi}{4}$. This shows

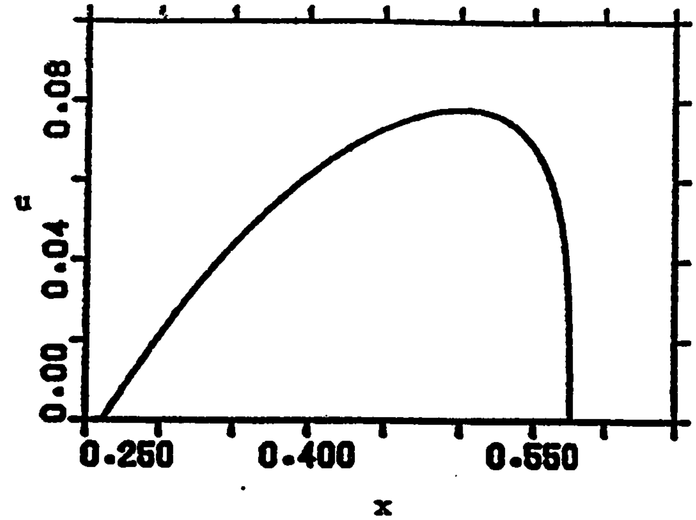
$$\widetilde{x}_s = -\frac{1}{2mr}\,\ell n\,(1 - \frac{mr}{3k}), \tag{3.93}$$

and

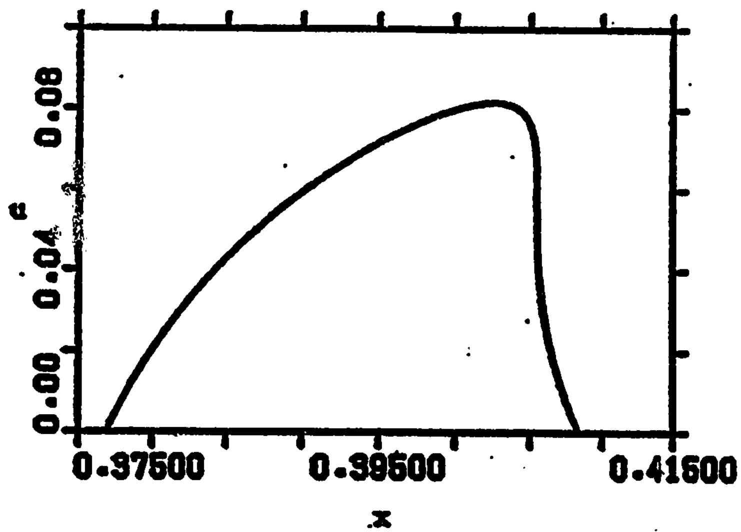
$$\widetilde{t}_s = \widetilde{x}_s + \varepsilon^2(\frac{\pi}{4} - \frac{1}{2}). \tag{3.94}$$

It is clear from (3.93) and (3.94) that this time the waves generated by the boundary disturbance (3.85) break behind the wave front $t = x$ located at $x = \widetilde{x}_s + \varepsilon^2(\frac{\pi}{4} - \frac{1}{2})$.

In Figs. 3.1-3.3, we display graphically the asymptotic solution $\widetilde{u}(x,t)$ of the strain given by (3.86) and (3.87) for the particular choice of $k = 1$, $m = 0.5$, $r = 1$ and $\varepsilon = 0.1$. In these figures the breaking place and time of the wave are $\widetilde{x}_s = t_s = 0.5754$ for $q = 1$ whereas for $q = 2$, these quantities are $\widetilde{x}_s = 0.4055$ and $\widetilde{t}_s = 0.4083$. In Fig. 3.1 we adjusted the time scaling to facilitate an easier comparison between the cases $q = 1$ and $q = 2$. In Fig. 3.3 we plotted also the corresponding linear asymptotic solution obtained by setting $k = 0$ in (3.87) for purposes of comparison. These figures depict how the nonlinearity of the material distorts the propagating disturbance. Fig. 3.2 displays a sequence of snapshots of the wave until it breaks. With these figures our analysis of the viscoelastic problem now concludes.

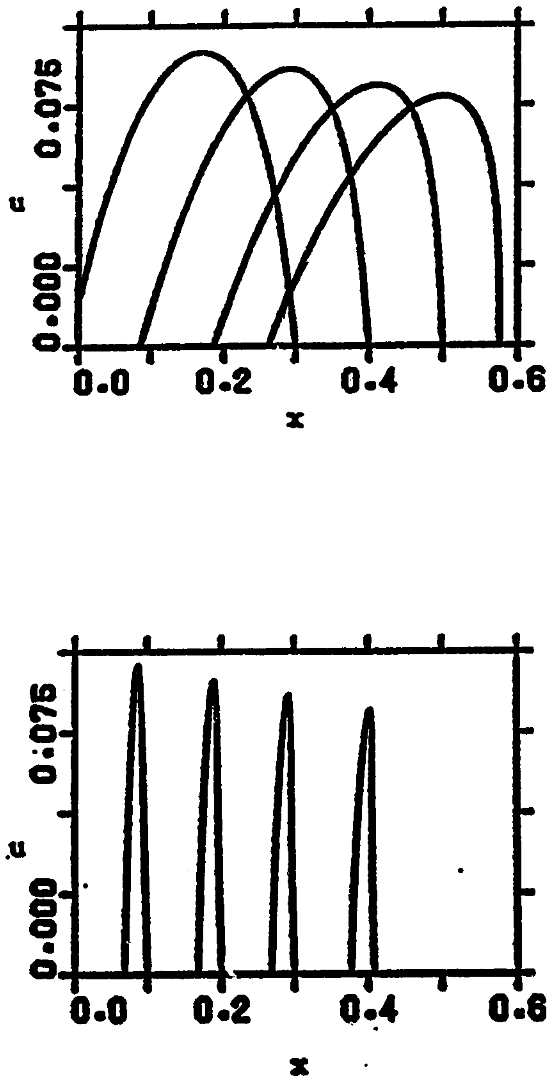


a)



b)

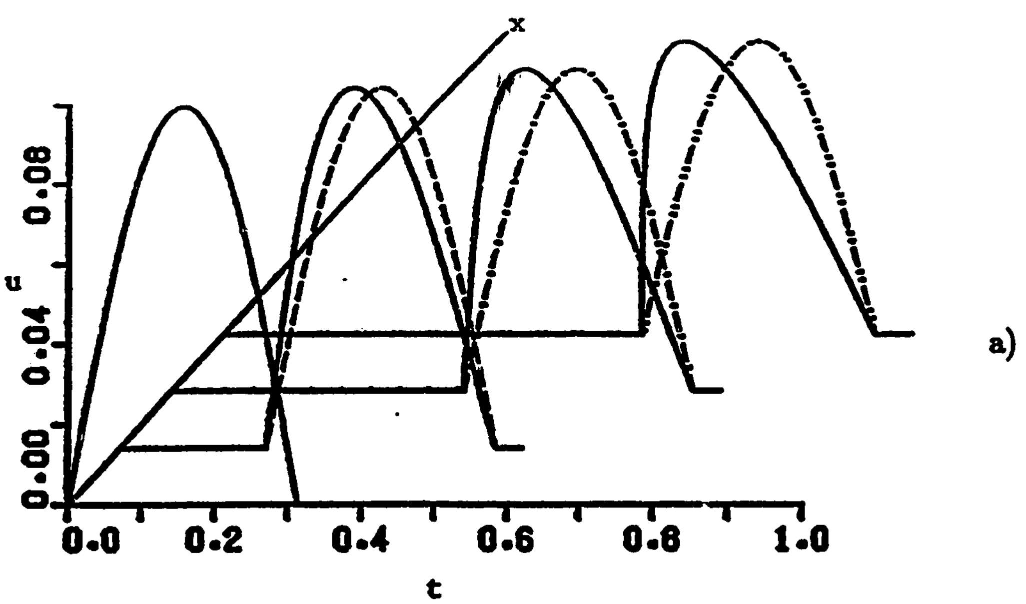
Fig. 3.1. Wave profile of strain at time of breaking:
a) for $q = 1$ where $\tilde{x}_s = 0.5754$, $\tilde{t}_s = 0.5754$,
b) for $q = 2$ where $\tilde{x}_s = 0.4055$, $\tilde{t}_s = 0.4083$.



a)

b)

Fig. 3.2. **Wave profiles of strain up to time of breaking:**
a) for $q = 1$ where $\tilde{x}_s = 0.5754$, $\tilde{t}_s = 0.5754$,
b) for $q = 2$ where $\tilde{x}_s = 0.4055$, $\tilde{t}_s = 0.4083$.



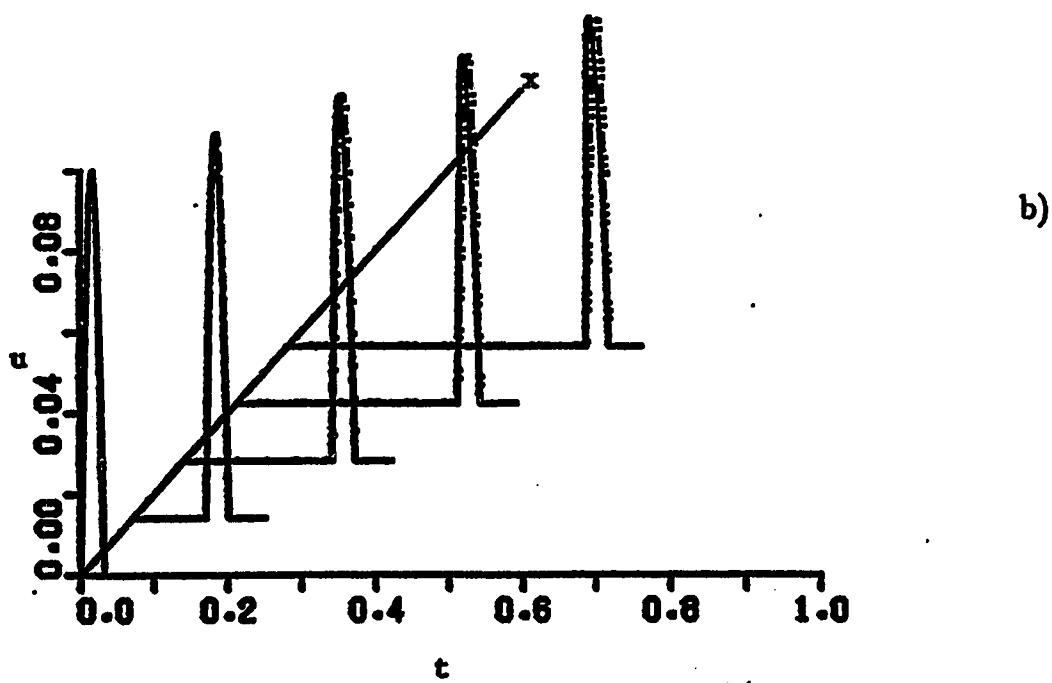


Fig. 3.3 Variation of strain with time:
a) for $q = 1$ at $x = 0, 0.2, 0.4, 0.5754$,
b) for $q = 2$ at $x = 0, 0.1, 0.2, 0.3, 0.4055$.

CHAPTER 4

CONSTITUTIVE RELATIONS FOR SECOND SOUND IN ELASTIC HEAT CONDUCTORS

As of this chapter we turn our attention to second sound thermoelasticity. Our objective in this chapter is to present a thermodynamical development of the nonlinear constitutive equations for an elastic heat conductor for which the heat flux obeys a nonlinear version of Cattaneo's Law. We give our development in three dimensions. The heat pulse propagation problem discussed in the introduction will be studied in later chapters after specializing the constitutive equations determined here for an isotropic material body whose departures from an equilibrium state are one-dimensional.

In our derivation of the thermodynamical restrictions on the constitutive equations we shall employ the methods of classical continuum thermodynamics in the manner of Coleman and Noll [44]. When deformations are ignored, the constitutive equations we derive yield those of Coleman, Fabrizio and Owen [31] that are valid for rigid heat conductors. The linearized forms of our constitutive equations are identical to the linear constitutive equations of the theory of Lord and Shulman [28]. In our formulation we also provide a simple solution to the problem of lack of material frame invariance from which the constitutive equations of Coleman, Fabrizio and Owen suffer.

We begin our derivation by recalling some preliminary notions of the classical continuum thermodynamics. Following this, we postulate a set of constitutive functions involving an evolution function for the heat flux. These functions are assumed to depend on the deformation gradient, absolute temperature, heat flux and temperature gradient. We investigate the restrictions imposed by the second law of thermodynamics on these constitutive functions in their generality. We then linearize the evolution function of the heat flux for the heat flux and temperature gradient obtaining a nonlinear version of Cattaneo's Law and determine further restrictions required by the second law for this special case. Lastly, we give a brief account on the requirements of the principle of material frame indifference for our constitutive functions.

4.1. Preliminary Notions

Let $\mathcal{E}$ denote a three-dimensional Euclidean point space. Also let the particles of a body B be identified with their positions $\boldsymbol{X} \in \mathcal{E}$ they occupy in a fixed reference configuration $\mathcal{B}$. Assume that a positive mass measure is assigned to B through the definition $m(\mathcal{P}) = \int_{\mathcal{P}} \rho_R dV$ where $m(\mathcal{P})$ is the mass of the subpart $\mathcal{P}$ of $\mathcal{B}$, and $\rho_R(\cdot) : \mathcal{B} \to (0, \infty)$ is the referential mass density.

We assume that the material comprising B is characterized by a given process class $\mathbb{P}(B)$ of B. It is convenient to consider the process class $\mathbb{P}(B)$ of B as a set of ordered 8-tuples of functions on $\mathcal{B} \times \mathbb{R}$

$$\pi = \{\mathcal{X}, \theta, e, \eta, \boldsymbol{S}, \boldsymbol{Q}, \boldsymbol{b}, r\} \in \mathbb{P}(B),$$

defined with respect to the reference configuration $\mathcal{B}$ and satisfying the laws of balance of linear momentum, balance of moment of momentum, balance of energy and imbalance of entropy, where $\boldsymbol{x} = \mathcal{X}(\boldsymbol{X}, t) \in \mathcal{E}$ is the *motion*, $\theta = \theta(\boldsymbol{X}, t) \in (0, \infty)$ the *absolute temperature*, $e = e(\boldsymbol{X}, t)$ the specific *internal energy* per unit mass, $\eta = \eta(\boldsymbol{X}, t)$ the specific *entropy* per unit mass, $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{X}, t)$ the *first Piola-Kirchhoff stress tensor*, $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{X}, t)$ the *referential heat flux vector*, $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{X}, t)$ the specific *body force* per unit mass and $r = r(\boldsymbol{X}, t)$ the *radiant heating* per unit mass. Any motion $\mathcal{X}(\cdot, t)$ of $\mathcal{B}$ is a continuous and almost everywhere invertible function from $\mathcal{B}$ into $\mathcal{E}$.

Let $\mathcal{X}(\cdot, \cdot) : \mathcal{B} \times \mathbb{R} \to \mathcal{E}$ be a given motion of $\mathcal{B}$. The *deformation gradient* $\boldsymbol{F}$ at $\boldsymbol{X}$ at time t is given by

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{X}, t) = \operatorname{Grad} \mathcal{X}(\boldsymbol{X}, t), \tag{4.1}$$

where Grad denotes the gradient with respect to $\boldsymbol{X}$. The *velocity* $\boldsymbol{v}$ of $\boldsymbol{X}$ at time t is determined by

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{X}, t) = \dot{\mathcal{X}}(\boldsymbol{X}, t), \tag{4.2}$$

where a superimposed dot denotes the material time derivative. We shall use Div to denote the divergence with respect to $\boldsymbol{X}$. To ensure the invertibility of $\mathcal{X}(\cdot, t)$ we assume that $J = \det \boldsymbol{F} > 0$ so that the motion is orientation preserving. Further, the law of conservation of mass requires that

$$\rho_R = J\rho, \tag{4.3}$$

where $\rho = \rho(\boldsymbol{X}, t)$ is the mass density of $\boldsymbol{X}$ at time t.

Under sufficient smoothness assumptions and in view of (4.3), the usual integral forms of the laws of balance of linear momentum, balance of moment of momentum, balance of energy and imbalance of entropy are equivalent to the local referential equations

$$\rho_R \dot{\boldsymbol{v}} = \operatorname{Div} \boldsymbol{S} + \rho_R \boldsymbol{b}, \tag{4.4}$$

$$\boldsymbol{F}\boldsymbol{S}^T = \boldsymbol{S}\boldsymbol{F}^T, \tag{4.5}$$

$$\rho_R \dot{e} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}} - \operatorname{Div} \boldsymbol{Q} + \rho_R r, \tag{4.6}$$

$$\rho_R \dot{\eta} \geq \rho_R r / \theta - \operatorname{Div}(\boldsymbol{Q}/\theta), \tag{4.7}$$

where $\boldsymbol{S} \cdot \dot{\boldsymbol{F}} = \operatorname{tr}(\boldsymbol{S}^T \dot{\boldsymbol{F}})$ and tr indicates the trace. The inequality (4.7) is also known as the entropy production inequality.

Let $\psi = \psi(\boldsymbol{X}, t)$ be the specific *free energy* per unit mass defined by

$$\psi = e - \eta\theta. \tag{4.8}$$

Then from (4.6) and (4.7) follows the dissipation inequality

$$\rho_R(\dot{\psi} + \eta\dot{\theta}) - \boldsymbol{S} \cdot \dot{\boldsymbol{F}} + \frac{1}{\theta} \boldsymbol{Q} \cdot \boldsymbol{G} \leq 0, \tag{4.9}$$

where $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{X}, t) = \operatorname{Grad} \theta(\boldsymbol{X}, t)$ is the temperature gradient with respect to the reference configuration $\mathcal{B}$.

4.2. Constitutive Assumptions and Thermodynamic Restrictions

Let $\mathcal{D}$ be an open, simply-connected domain consisting of quadruplets $(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G})$, and suppose that if $(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G})$ is in $\mathcal{D}$, then so is its corresponding "thermal equilibrium" state $(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0})$.

Assumption. *For every* $\pi \in \mathbb{P}(B)$ *the specific free energy* $\psi(\boldsymbol{X},t)$, *the specific entropy* $\eta(\boldsymbol{X},t)$, *the first Piola-Kirchhoff stress tensor* $\boldsymbol{S}(\boldsymbol{X},t)$, *and the time rate of the heat flux* $\dot{\boldsymbol{Q}}(\boldsymbol{X},t)$ *are given by continuously differentiable functions on* $\mathcal{D}$ *such that*

$$\psi = \hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}), \tag{4.10.a}$$

$$\eta = \hat{\eta}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}), \tag{4.10.b}$$

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}), \tag{4.10.c}$$

$$\dot{\boldsymbol{Q}} = \boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}). \tag{4.10.d}$$

Further the tensors $\partial_{\boldsymbol{Q}}\boldsymbol{H}(\cdot)$ *and* $\partial_{\boldsymbol{G}}\boldsymbol{H}(\cdot)$ *are non-singular.*

It is clear that once $\hat{\psi}(\cdot)$ and $\hat{\eta}(\cdot)$ are known, then the relation (4.8) gives the continuously differentiable function $\hat{e}(\cdot)$ determining $e(\boldsymbol{X},t)$ such that

$$e = \hat{e}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}). \tag{4.10.e}$$

The assumed properties of the ***heat flux evolution function*** $\boldsymbol{H}(\cdot)$ indicate that it is invertible for $\boldsymbol{Q}$ and also for $\boldsymbol{G}$. We denote the inverse of $\boldsymbol{H}(\cdot)$ with respect to $\boldsymbol{Q}$ with $\boldsymbol{Q}=\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\dot{\boldsymbol{Q}})$. Notice that

$$\partial_G \boldsymbol{H}^*(\cdot) = -[\partial_Q \boldsymbol{H}(\cdot)]^{-1}\partial_G \boldsymbol{H}(\cdot)$$

so that the tensor $\partial_G \boldsymbol{H}^*(\cdot)$ is also continuous and non-singular. Note also that in a materially inhomogeneous body the functions $\hat{\psi},\hat{\eta},\hat{\boldsymbol{S}}$ and $\boldsymbol{H}$ depend on $\boldsymbol{X}$ as well. For convenience this is not written but understood.

Given any motion $\mathcal{X}(\boldsymbol{X},t)$ and any temperature field $\theta(\boldsymbol{X},t)$ one uses the constitutive equations (4.10) to determine $e(\boldsymbol{X},t),\eta(\boldsymbol{X},t),\boldsymbol{S}(\boldsymbol{X},t)$ and $\boldsymbol{Q}(\boldsymbol{X},t)$, and the equations (4.4) and (4.6) determine $\boldsymbol{b}(\boldsymbol{X},t)$ and $r(\boldsymbol{X},t)$. Hence for any given motion and temperature field a corresponding process is constructed. The method of Coleman & Noll [44] is based on the postulate that every process π so constructed belongs to the process class $\mathbb{P}(B)$ of B. In view of the constitutive relations (4.10) this is equivalent to the

Dissipation Principle. *Given any motion and any temperature field, the process π constructed from the constitutive relations (4.10) belongs to the process class $\mathbb{P}(B)$ of B. Therefore the constitutive functions (4.10) are compatible with the second law of thermodynamics in the sense that they satisfy the dissipation inequality (4.9).*

Theorem 1. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) the free energy response function $\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G})$ *is independent of the temperature gradient* $\boldsymbol{G}$ *and determines the entropy and the first Piola-Kirchhoff stress through the relations*

$$\hat{\eta}(\boldsymbol{F},\theta,\boldsymbol{Q}) = -\partial_\theta\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q}), \quad \hat{\boldsymbol{S}}(\boldsymbol{F},\theta,\boldsymbol{Q}) = \rho_R\partial_{\boldsymbol{F}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q}); \tag{4.11}$$

(ii) the reduced dissipation inequality

$$\rho_R\theta\partial_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})\cdot\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G}) + \boldsymbol{Q}\cdot\boldsymbol{G} \leq 0, \tag{4.12}$$

is satisfied.

To prove the theorem, by the chain rule we obtain

$$\dot{\psi} = \partial_{\boldsymbol{F}}\hat{\psi}\cdot\dot{\boldsymbol{F}} + \partial_\theta\hat{\psi}\dot{\theta} + \partial_{\boldsymbol{Q}}\hat{\psi}\cdot\dot{\boldsymbol{Q}} + \partial_{\boldsymbol{G}}\hat{\psi}\cdot\dot{\boldsymbol{G}}. \tag{4.13}$$

Substituting this equation together with the constitutive relations (4.1) into the dissipation inequality (4.9) gives

$$(\rho_R\partial_{\boldsymbol{F}}\hat{\psi} - \hat{\boldsymbol{S}})\cdot\dot{\boldsymbol{F}} + \rho_R(\partial_\theta\hat{\psi} + \hat{\eta})\dot{\theta} + \rho_R\partial_{\boldsymbol{Q}}\hat{\psi}\cdot\boldsymbol{H} + \rho_R\partial_{\boldsymbol{G}}\hat{\psi}\cdot\dot{\boldsymbol{G}} + \frac{1}{\theta}\boldsymbol{Q}\cdot\boldsymbol{G} \leq 0. \tag{4.14}$$

In (4.13) and (4.14) we omitted the arguments for convenience. Theorem 1 now follows from demonstrating in the manner of Coleman & Noll [44] that $\dot{\boldsymbol{F}}, \dot{\theta}$ and $\dot{\boldsymbol{G}}$ may be assigned arbitrary values independently from the other variables.

Theorem 2. *The time derivative of the heat flux* $\dot{\boldsymbol{Q}}$ *vanishes for all thermal equilibrium states* $(\boldsymbol{F},\theta,\mathbf{0},\mathbf{0}) \in \mathcal{D}$ *and the tensor*

$$\boldsymbol{K}(\boldsymbol{F},\theta) = \partial_{\boldsymbol{Q}}\boldsymbol{H}(\boldsymbol{F},\theta,0,0)^{-1}\partial_{G}\boldsymbol{H}(\boldsymbol{F},\theta,\mathbf{0},\mathbf{0}), \tag{4.15}$$

is positive definite.

Let $\dot{\boldsymbol{Q}} = 0$ and recall that $\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\cdot)$ is the inverse of $\boldsymbol{H}(\boldsymbol{F},\theta,\cdot,\boldsymbol{G})$. It then follows from (4.12) that

$$\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\mathbf{0})\cdot\boldsymbol{G} \leq 0. \tag{4.16}$$

Fixing $\boldsymbol{F}$ and θ, define $f(\boldsymbol{G}) = \boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\mathbf{0})\cdot\boldsymbol{G}$. The inequality (4.16) shows that $f(\mathbf{0})$ is a maximum so that

$$\frac{d}{d\boldsymbol{G}}f(\boldsymbol{G}) = \boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\mathbf{0}) + \partial_{G}\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\mathbf{0})^T\boldsymbol{G}, \tag{4.17}$$

vanishes at $\boldsymbol{G} = \mathbf{0}$. Therefore $\boldsymbol{H}^*(\boldsymbol{F},\theta,\mathbf{0},\mathbf{0}) = \boldsymbol{H}(\boldsymbol{F},\theta,\mathbf{0},\mathbf{0}) = \mathbf{0}$. This proves the first claim of Theorem 2.

In order to show $\boldsymbol{K}(\boldsymbol{F},\theta)$ is positive definite, recall that $\partial_{G}\boldsymbol{H}^*(\cdot) = -\partial_{\boldsymbol{Q}}\boldsymbol{H}(\cdot)^{-1}\partial_{G}\boldsymbol{H}(\cdot)$, and $\partial_{G}\boldsymbol{H}^*(\cdot)$ is continuous and non-singular. Let $0 < G < C$ for some constant C where $G = |\boldsymbol{G}|$ is the magnitude of $\boldsymbol{G}$. The following statement is an immediate consequence of $\boldsymbol{H}^*(\boldsymbol{F},\theta,\mathbf{0},\mathbf{0}) = \mathbf{0}$ and the mean value theorem:

$$\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\mathbf{0}) = \partial_{G}H^*(\boldsymbol{F},\theta,\varepsilon\boldsymbol{G},\mathbf{0})\boldsymbol{G}, \quad 0 < \varepsilon < 1. \tag{4.18}$$

From (4.16), (4.18) and defining

$$\boldsymbol{K}^*(\boldsymbol{F},\theta,\boldsymbol{G}) = -\partial_{\boldsymbol{G}}\boldsymbol{H}^*(\boldsymbol{F},\theta,\boldsymbol{G},\boldsymbol{0})$$

we have $\boldsymbol{G}\cdot\boldsymbol{K}^*(\boldsymbol{F},\theta,\varepsilon\boldsymbol{G})\boldsymbol{G} \geq 0$. Let us rewrite this as $\boldsymbol{n}\cdot\boldsymbol{K}^*(\boldsymbol{F},\theta,\varepsilon G\boldsymbol{n})\boldsymbol{n} \geq 0$, where $\boldsymbol{n}$ is the unit vector $\boldsymbol{n} = \boldsymbol{G}/G$. As $\boldsymbol{G}$ is arbitrary, letting $G \to 0$ shows that $\boldsymbol{n}\cdot\boldsymbol{K}^*(\boldsymbol{F},\theta,\boldsymbol{0})\boldsymbol{n} \geq 0$ holds for all arbitrary unit vectors $\boldsymbol{n}$. Since $\boldsymbol{K}^*(\boldsymbol{F},\theta,\boldsymbol{G})$ is non-singular, it then follows that $\boldsymbol{K}(\boldsymbol{F},\theta) = \boldsymbol{K}^*(\boldsymbol{F},\theta,\boldsymbol{0})$ is positive definite.

4.3. Cattaneo's Law

In this section we investigate the case when $\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G})$ is linear in $\boldsymbol{Q}$ and $\boldsymbol{G}$. This is equivalent to expanding $\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{Q},\boldsymbol{G})$ about $(\boldsymbol{Q},\boldsymbol{G}) = (\boldsymbol{0},\boldsymbol{0})$, and keeping only the terms first order in $\boldsymbol{Q}$ and $\boldsymbol{G}$ in the expansion:

$$\dot{\boldsymbol{Q}} = \partial_{\boldsymbol{Q}}\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0})\boldsymbol{Q} + \partial_{\boldsymbol{G}}\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0})\boldsymbol{G}, \tag{4.19}$$

where we have used $\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}$.

For convenience in what follows we define

$$\boldsymbol{T}(\boldsymbol{F},\theta)^{-1} = -\partial_{\boldsymbol{Q}}\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0}), \quad \boldsymbol{Z}(\boldsymbol{F},\theta)^{-1} = -\partial_{\boldsymbol{G}}\boldsymbol{H}(\boldsymbol{F},\theta,\boldsymbol{0},\boldsymbol{0}). \tag{4.20}$$

It is clear from (4.15) and (4.20) that $\boldsymbol{K}(\boldsymbol{F},\theta) = \boldsymbol{T}(\boldsymbol{F},\theta)\boldsymbol{Z}(\boldsymbol{F},\theta)^{-1}$. Rewriting (4.1) as

$$\dot{\boldsymbol{Q}} = -\boldsymbol{T}(\boldsymbol{F},\theta)^{-1}\boldsymbol{Q} - \boldsymbol{Z}(\boldsymbol{F},\theta)^{-1}\boldsymbol{G}, \tag{4.21}$$

from the inequality (4.12) we get

$$-\rho_R\theta\partial_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})\cdot\boldsymbol{T}(\boldsymbol{F},\theta)^{-1}\boldsymbol{Q}-\rho_R\theta\partial_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})\cdot\boldsymbol{Z}(\boldsymbol{F},\theta)^{-1}\boldsymbol{G}+\boldsymbol{Q}\cdot\boldsymbol{G}\leq 0. \quad (4.22)$$

Since $\boldsymbol{G}$ and $\boldsymbol{Q}$ are arbitrary, this inequality is always satisfied if and only if

$$\partial_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})=\frac{1}{\rho_R\theta}\boldsymbol{Z}(\boldsymbol{F},\theta)^T\boldsymbol{Q}, \quad \rho_R\partial_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})\cdot\boldsymbol{T}(\boldsymbol{F},\theta)^{-1}\boldsymbol{Q}\geq 0. \qquad (4.23)$$

Noting $\partial^2_{\boldsymbol{Q}}\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q})$ is symmetric we then see that for the Dissipation Principle to hold $\boldsymbol{Z}(\boldsymbol{F},\theta)$ must be symmetric:

$$\boldsymbol{Z}(\boldsymbol{F},\theta)=\boldsymbol{Z}(\boldsymbol{F},\theta)^T. \qquad (4.24)$$

Notice that with (4.24), the second of (4.23) is equivalent to $\boldsymbol{Q}\cdot\boldsymbol{K}(\boldsymbol{F},\theta)^{-1}\boldsymbol{Q}\geq 0$ which obtains automatically by virtue of the fact that $\boldsymbol{K}(\boldsymbol{F},\theta)$ is positive definite.

From (4.8), (4.11), (4.23) and (4.24) we have the following theorem.

Theorem 3. *Let the evolution equation of the heat flux be given by the following form of Cattaneo's Law:*

$$\boldsymbol{T}(\boldsymbol{F},\theta)\dot{\boldsymbol{Q}}+\boldsymbol{Q}=-\boldsymbol{K}(\boldsymbol{F},\theta)\boldsymbol{G}. \qquad (4.25)$$

Then the Dissipation Principle is equivalent to the conditions:

(i) the tensor $\boldsymbol{K}(\boldsymbol{F},\theta)$ *is positive definite;*

(ii) the tensor $Z(F,\theta)$ *is symmetric;*

(iii) the response functions of the specific free energy, specific internal energy, specific entropy and first Piola-Kirchhoff stress are given by

$$\rho_R\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{Q}) = \rho_R\hat{\psi}_0(\boldsymbol{F},\theta) + \frac{1}{2\theta}\boldsymbol{Q}\cdot\boldsymbol{Z}(\boldsymbol{F},\theta)\boldsymbol{Q}, \tag{4.26}$$

$$\rho_R\hat{e}(\boldsymbol{F},\theta,\boldsymbol{Q}) = \rho_R\hat{e}_0(\boldsymbol{F},\theta) + \boldsymbol{Q}\cdot\boldsymbol{A}(\boldsymbol{F},\theta)\boldsymbol{Q}, \tag{4.27}$$

$$\rho_R\hat{\eta}(\boldsymbol{F},\theta,\boldsymbol{Q}) = \rho_R\hat{\eta}_0(\boldsymbol{F},\theta) + \boldsymbol{Q}\cdot\boldsymbol{B}(\boldsymbol{F},\theta)\boldsymbol{Q}, \tag{4.28}$$

$$\hat{\boldsymbol{S}}(\boldsymbol{F},\theta,\boldsymbol{Q}) = \hat{\boldsymbol{S}}_0(\boldsymbol{F},\theta) + \boldsymbol{Q}\cdot\boldsymbol{P}(\boldsymbol{F},\theta)\boldsymbol{Q}, \tag{4.29}$$

where

$$\begin{aligned} &\hat{\psi}_0(\boldsymbol{F},\theta)= \hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{0}), && \hat{e}_0(\boldsymbol{F},\theta)= \hat{\psi}_0(\boldsymbol{F},\theta) - \theta\partial_\theta\hat{\psi}_0(\boldsymbol{F},\theta),\\ &\hat{\eta}_0(\boldsymbol{F},\theta)= -\partial_\theta\hat{\psi}_0(\boldsymbol{F},\theta), && \hat{\boldsymbol{S}}_0(\boldsymbol{F},\theta)= \rho_R\partial_{\boldsymbol{F}}\hat{\psi}_0(\boldsymbol{F},\theta), \end{aligned} \tag{4.30}$$

while

$$\begin{aligned} &\boldsymbol{Z}(\boldsymbol{F},\theta)= \boldsymbol{K}(\boldsymbol{F},\theta)^{-1}\boldsymbol{T}(\boldsymbol{F},\theta), && \boldsymbol{A}(\boldsymbol{F},\theta)= -\frac{\theta^2}{2}\frac{\partial}{\partial\theta}\left[\frac{\boldsymbol{Z}(\boldsymbol{F},\theta)}{\theta^2}\right],\\ &\boldsymbol{B}(\boldsymbol{F},\theta)= -\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{\boldsymbol{Z}(\boldsymbol{F},\theta)}{\theta}\right], && \boldsymbol{P}(\boldsymbol{F},\theta)= \frac{1}{2\theta}\frac{\partial}{\partial\boldsymbol{F}}\boldsymbol{Z}(\boldsymbol{F},\theta). \end{aligned} \tag{4.31}$$

Note that in (4.29) we have used the notation $\boldsymbol{Q}\cdot\boldsymbol{P}\boldsymbol{Q}$ to indicate the second order tensor $\boldsymbol{E}$ with the Cartesian components $E_{ij} = P_{pqij}Q_pQ_q$ where summation over the repeated indices is implied.

4.4 Consequences of the Principle of Material Frame Indifference

The principle of material frame indifference states that the constitutive equations characterizing the response of a material must be invariant under a change of frame or observer. Accordingly, we demand that the constitutive relations (4.10) remain invariant under the change of frame

$$\boldsymbol{x} \rightarrow \boldsymbol{O}\boldsymbol{x} + \boldsymbol{c}, \tag{4.32}$$

where $\boldsymbol{O} = \boldsymbol{O}(t)$ is a proper orthogonal tensor and $\boldsymbol{c} = \boldsymbol{c}(t)$ is a vector. Under a change of frame, the scalars ψ, η and θ remain unchanged, $\boldsymbol{F}$ and $\boldsymbol{G}$ transform as

$$\begin{aligned} \boldsymbol{F} &\rightarrow \boldsymbol{O}\boldsymbol{F}, \\ \boldsymbol{G} &\rightarrow \boldsymbol{G}, \end{aligned} \tag{4.33}$$

and the first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ and the referential heat flux vector $\boldsymbol{Q}$ transform as

$$\boldsymbol{S} \rightarrow \boldsymbol{O}\boldsymbol{S}, \tag{4.34.a}$$

$$\boldsymbol{Q} \rightarrow \boldsymbol{Q}. \tag{4.34.b}$$

It is an immediate consequence of (4.34.b) that

$$\dot{\boldsymbol{Q}} \rightarrow \dot{\boldsymbol{Q}}. \tag{4.35}$$

It follows that the principle of material frame indifference is satisfied if and only if

$$\hat{\psi}(\boldsymbol{F}, \theta, \boldsymbol{Q}) = \hat{\psi}(\boldsymbol{O}\boldsymbol{F}, \theta, \boldsymbol{Q}), \tag{4.36.a}$$

$$\hat{\eta}(\boldsymbol{F}, \theta, \boldsymbol{Q}) = \hat{\eta}(\boldsymbol{O}\boldsymbol{F}, \theta, \boldsymbol{Q}), \tag{4.36.b}$$

$$\widehat{S}(F,\theta,Q) = O\widehat{S}(OF,\theta,Q), \tag{4.36.c}$$

$$H(F,\theta,Q,G) = H(OF,\theta,Q,G). \tag{4.36.d}$$

The reduced forms of the constitutive functions (4.10) can now be determined from (4.35) using methods described in Truesdell and Noll [45]. Limiting our attention to (4.25) and omitting details we find that with

$$T(F,\theta) = \overline{T}(C,\theta), \quad K(F,\theta) = \overline{K}(C,\theta), \tag{4.37}$$

the relation

$$\overline{T}(C,\theta)\dot{Q} + Q = -\overline{K}(C,\theta)G, \tag{4.38}$$

is a frame invariant form of (4.25), where $C = F^T F$ is the right Cauchy-Green deformation tensor.

With a view to comparing (4.38) with the form of Cattaneo's Law considered by Coleman, Fabrizio and Owen, let $q = q(x,t)$ be the current value of the heat flux in the present configuration $B_t = \mathcal{X}(B,t)$. They considered the following form of Cattaneo's Law:

$$\widetilde{T}(\theta)\dot{q} + q = -\widetilde{K}(\theta)g, \tag{4.39}$$

where, with grad denoting the gradient with respect to x, $g =$ grad $\theta(x,t)$ so that $g = F^{-T}G$. As pointed out by Coleman, Fabrizio and Owen, (4.39) is not frame invariant.

Recall that

$$\boldsymbol{Q} = J\boldsymbol{F}^{-1}\boldsymbol{q}, \tag{4.40}$$

so that differentiation gives

$$\dot{\boldsymbol{Q}} = J\boldsymbol{F}^{-1}\overset{\circ}{\boldsymbol{q}}, \tag{4.41}$$

where

$$\overset{\circ}{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{L}\boldsymbol{q} + \{\mathrm{tr}\,\boldsymbol{L}\}\boldsymbol{q}, \tag{4.42}$$

and $\boldsymbol{L} = \mathrm{grad}\,\boldsymbol{v}(\boldsymbol{x}, t)$ is the spatial gradient of the velocity. As it follows from (4.35) and (4.41) that $\overset{\circ}{\boldsymbol{q}}$ transforms as

$$\overset{\circ}{\boldsymbol{q}} \to \boldsymbol{O}\,\overset{\circ}{\boldsymbol{q}}, \tag{4.43}$$

it is frame invariant. Note that the above derivative $\overset{\circ}{\boldsymbol{q}}$ reduces to the upper convective derivative $\overset{\triangledown}{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \boldsymbol{L}\boldsymbol{q}$ for incompressible motions. It is straightforward to show that (4.38) is equivalent to

$$\widehat{T}(\boldsymbol{F}, \theta)\overset{\circ}{\boldsymbol{q}} + \boldsymbol{q} = -\widehat{K}(\boldsymbol{F}, \theta)\boldsymbol{g}, \tag{4.44}$$

where

$$\widehat{T}(\boldsymbol{F}, \theta) = \boldsymbol{F}\overline{T}(\boldsymbol{C}, \theta)\boldsymbol{F}^{-1}, \quad \widehat{K}(\boldsymbol{F}, \theta) = \frac{1}{J}\boldsymbol{F}\overline{K}(\boldsymbol{C}, \theta)\boldsymbol{F}^{T}. \tag{4.45}$$

Comparison shows that the essential deviation of our constitutive equation (4.44) from their constitutive equation (4.39) is the replacement of $\dot{\boldsymbol{q}}$ with $\mathring{\boldsymbol{q}}$.

CHAPTER 5

ONE-DIMENSIONAL THERMOELASTIC SOLIDS WITH SECOND SOUND

With this chapter we begin concerning ourselves with the one-dimensional departures of a semi-infinite, homogeneous, isotropic, thermoelastic material body from its reference configuration $(0,\infty)$. The points $x \in (0,\infty)$ of this reference configuration are used to identify the particles of the body and t to denote the time. The constant ρ_0 is the referential mass density of the material. The material body is assumed to have been unstressed, unstrained and at rest in this reference configuration with a uniform absolute temperature $\theta_0 > 0$ for all $t \leq 0$. For convenience we assume also that in the above described reference state the material has been free of internal energy.

Let $\chi(x,t)$ be the motion of the material body. Then the functions $w(x,t)$, $v(x,t)$ and $u(x,t)$ defined as

$$w = \chi - x, \quad v = \frac{\partial w}{\partial t}, \quad u = \frac{\partial w}{\partial x}, \tag{5.1}$$

are the displacement, velocity and strain, respectively. Consequently, the strain and velocity satisfy the compatibility equation

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0. \tag{5.2}$$

Let us now assume that there are no body forces nor external heat sources for the problems we are about to examine. The respective local forms of the laws of balance of linear momentum and balance of energy are then

$$\rho_0 \frac{\partial v}{\partial t} - \frac{\partial S}{\partial x} = 0, \tag{5.3}$$

$$\rho_0 \frac{\partial e}{\partial t} - S \frac{\partial v}{\partial x} + \frac{\partial q}{\partial x} = 0, \tag{5.4}$$

where $S(x,t)$ is the Piola stress, $e(x,t)$ the specific internal energy per unit mass and $q(x,t)$ the heat flux. To this list of functions we add the absolute temperature $\theta(x,t) > 0$ and the temperature difference $\Theta(x,t)$ given by

$$\Theta = \theta - \theta_0. \tag{5.5}$$

The above basic equations are to be completed by specifying the constitutive equations for the Piola stress, specific internal energy per unit mass and heat flux. In our analysis of the linear problem we take the theories of Lord and Shulman [28] and Green and Lindsay [29] into consideration. We introduce the linear constitutive equations of these theories in the next two sections, respectively. After this, we specialize the nonlinear constitutive equations we have determined in the previous chapter to the one-dimensional material body to be studied. We conclude this chapter by introducing the nondimensionalization scheme we employ in our asymptotic analysis.

5.1. Linear Constitutive Equations in the Theory of Lord and Shulman

In this theory the heat flux obeys Cattaneo's Law with the form

$$\tau_0 \frac{\partial q}{\partial t} + q = -\kappa_0 \frac{\partial \Theta}{\partial x}, \tag{5.6}$$

where

$$\tau_0 > 0, \quad \kappa_0 > 0, \tag{5.7}$$

are constants with the significance of thermal relaxation time and thermal conductivity, respectively.

The Piola stress and specific internal energy per unit mass are determined from the constitutive relations

$$S = (\lambda + 2\mu)u - (3\lambda + 2\mu)\alpha\Theta, \tag{5.8}$$

$$\rho_0 e = (3\lambda + 2\mu)\alpha\theta_0 u + \rho_0 c\Theta, \tag{5.9}$$

where λ and μ are the isothermal Lamé constants, α the coefficient of linear thermal expansion and c the specific heat per unit mass at constant strain. The constants λ, μ, α and c are required to obey the conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \alpha > 0, \quad c > 0. \tag{5.10}$$

5.2. Linear Constitutive Equations in the Theory of Green and Lindsay

In this theory the heat flux obeys Fourier's Law

$$q = -\kappa_0 \frac{\partial \Theta}{\partial x}, \tag{5.11}$$

while the Piola stress and specific internal energy per unit mass are given by the constitutive relations

$$S = (\lambda + 2\mu)u - (3\lambda + 2\mu)\alpha(\Theta + \tau_1 \frac{\partial \Theta}{\partial t}), \tag{5.12}$$

$$\rho_0 e = (3\lambda + 2\mu)\alpha\theta_0 u + \rho_0 c(\Theta + \tau_2 \frac{\partial \Theta}{\partial t}), \tag{5.13}$$

where $\kappa_0, \lambda, \mu, \alpha$ and c are as before. Green and Lindsay showed that if $\tau_2 \geq 0$, then thermodynamical requirements indicate $\tau_1 \geq \tau_2$. We suppose

$$\tau_2 > 0, \tag{5.14}$$

and define the nondimensional parameter β as

$$\tau_1 = \beta \tau_2, \tag{5.15}$$

so that $\beta \geq 1$. In our analysis we shall assume $\tau_2 = \tau_0$ which will ensure that both theories predict the same linear speed for the purely thermal second sound.

It may be mentioned that neither the positivity of τ_0 in the theory of Lord and Shulman, nor the positivity of τ_2 in the theory of Green and Lindsay is necessary for compatibility with thermodynamical requirements. On the other hand, if these conditions are not satisfied, then instabilities occur. This is easier to see in the theory of Lord and Shulman: if one sets the temperature gradient to a constant, then the relation (5.6) shows that when τ_0 is negative, the heat flux grows exponentially with time.

5.3. Nonlinear Constitutive Equations Based on Cattaneo's Law

As we have mentioned in the introduction, in view of the experimental studies cited there, we are interested mainly in the nonlinear influence of the temperature on the response of the material. Therefore, we assume that in the nonlinear case, the heat flux obeys Cattaneo's Law with the form

$$\tau(\theta)\,\frac{\partial q}{\partial t} + q = -\kappa(\theta)\,\frac{\partial \theta}{\partial x}\,, \tag{5.16}$$

and by doing so, neglect the influence of deformations on the thermal relaxation time $\tau(\theta) > 0$ and thermal conductivity $\kappa(\theta) > 0$. We suppose τ and κ are continuously differentiable functions of $\theta > 0$. The equation (5.16) can be written alternatively as

$$\frac{\partial q}{\partial t} + \frac{1}{\tau(\theta)}\,q = -\,\frac{1}{z(\theta)}\,\frac{\partial \theta}{\partial x}\,, \tag{5.17}$$

where

$$z(\theta) = \tau(\theta)/\kappa(\theta). \tag{5.18}$$

It follows from the assumption (5.16) and Theorem 3 of the previous chapter that the Piola stress and specific internal energy per unit mass are given by

$$S = S_0(u,\theta), \tag{5.19}$$

$$\rho_0 e = \rho_0 e_0(u,\theta) + a(\theta) q^2, \tag{5.20}$$

where

$$a(\theta) = \frac{1}{\theta}\, z(\theta) - \frac{1}{2}\, z'(\theta), \tag{5.21}$$

while the constitutive functions $S_0(u,\theta)$ and $e_0(u,\theta)$ obey the compatibility relation

$$\rho_0 \frac{\partial e_0}{\partial u}(u,\theta) = S_0(u,\theta) - \theta \frac{\partial S_0}{\partial \theta}(u,\theta). \tag{5.22}$$

We assume $S_0(u,\theta)$, $e_0(u,\theta)$ are twice continuously differentiable in u and θ, and note that by hypothesis $S_0(0,\theta_0) = 0$ and $e_0(0,\theta_0) = 0$.

Since $S_0(u,\theta)$ and $e_0(u,\theta)$ are twice continuously differentiable, they admit the following Taylor series expansions about $(u,\theta) = (0,\theta_0)$:

$$\begin{aligned} S_0(u,\theta) &= S_1 u + S_2\Theta + \frac{1}{2}\, S_{11}u^2 + S_{12}u\Theta + \frac{1}{2}\, S_{22}\Theta^2 + o(u^2,\Theta^2), \\ e_0(u,\theta) &= e_1 u + e_2\Theta + \frac{1}{2}\, e_{11}u^2 + e_{12}u\Theta + \frac{1}{2}\, e_{22}\Theta^2 + o(u^2,\Theta^2), \end{aligned} \tag{5.23}$$

where

$$S_1 = \frac{\partial S_0}{\partial u}(0,\theta_0), \quad S_2 = \frac{\partial S_0}{\partial \theta}(0,\theta_2),$$

$$S_{11} = \frac{\partial^2 S_0}{\partial u^2}(0,\theta_0), \quad S_{12} = \frac{\partial^2 S_0}{\partial u \partial \theta}(0,\theta_0), \quad S_{22} = \frac{\partial^2 S_0}{\partial \theta^2}(0,\theta_0),$$

$$e_1 = \frac{\partial e_0}{\partial u}(0,\theta_0), \quad e_2 = \frac{\partial e_0}{\partial \theta}(0,\theta_0),$$

$$e_{11} = \frac{\partial^2 e_0}{\partial u^2}(0,\theta_0), \quad e_{12} = \frac{\partial^2 e_0}{\partial u \partial \theta}(0,\theta_0), \quad e_{22} = \frac{\partial^2 e_0}{\partial \theta^2}(0,\theta_0), \qquad (5.24)$$

while Θ is the temperature difference given by (5.5). Neglecting the terms $o(u^2, \Theta^2)$ in the above gives the second order approximations to $S_0(u,\theta)$ and $e_0(u,\theta)$.

From the compatibility equation (5.22) we have the following relations

$$\rho_0 e_1 = -\theta_0 S_2, \quad \rho_0 e_{12} = -\theta_0 S_{12}, \quad \rho_0 e_{11} = S_1 - \theta_0 S_{12}. \qquad (5.25)$$

Guided by the linear constitutive relations (5.13) and (5.14) we set

$$S_1 = \lambda + 2\mu, \quad S_2 = -(3\lambda + 2\mu)\alpha, \quad e_2 = c, \qquad (5.26)$$

and define the dimensionless material parameters ν_1, ν_2, ν_3 and ν_4 as

$$\begin{aligned} \theta_0 S_2 S_{11} &= -\nu_1 S_1^2, \qquad \theta_0 S_{12} = \nu_2 S_1, \\ \theta_0 S_{22} &= \nu_3 S_2, \qquad \theta_0 e_{22} = \nu_4 e_2, \end{aligned} \qquad (5.27)$$

for convenience.

Let us now deal with Cattaneo's Law (5.16). Since $z(\theta)$ and $\tau(\theta)$ are continuously differentiable, the following Taylor series expansions about $\theta = \theta_0$ are valid:

$$
\begin{aligned}
z(\theta) &= z(\theta_0) + z'(\theta_0)\Theta + o(\Theta), \\
\tau(\theta) &= \tau(\theta_0) + \tau'(\theta_0)\Theta + o(\Theta).
\end{aligned}
\tag{5.28}
$$

In view of the linear relation (5.6) we set

$$
\tau(\theta_0) = \tau_0, \quad \kappa(\theta_0) = \kappa_0, \tag{5.29}
$$

and from (5.18) it follows that $z(\theta_0) = \tau_0/\kappa_0$. We shall use the nondimensional parameters ζ_1 and ζ_2 defined as

$$
\theta_0 z'(\theta_0) = \zeta_1 z(\theta_0), \quad \theta_0 \tau'(\theta_0) = \zeta_2 \tau(\theta_0), \tag{5.30}
$$

also for convenience in what follows.

It is now a straightforward matter to show that the second order approximations to the constitutive equations for the heat flux, Piola stress and specific internal energy per unit mass take the forms

$$
\frac{\partial q}{\partial t} + \frac{1}{\tau_0}\,(1 - \zeta_2\,\frac{\Theta}{\theta_0})q = -\,\frac{\kappa_0}{\tau_0}\,(1 - \zeta_1\,\frac{\Theta}{\theta_0})\,\frac{\partial \Theta}{\partial x}\,, \tag{5.31}
$$

$$
\begin{aligned}
S = (\lambda + 2\mu)u - (3\lambda + 2\mu)\alpha\Theta + \frac{1}{2}\,\nu_1\,\frac{(\lambda + 2\mu)^2}{(3\lambda + 2\mu)\alpha\theta_0}\,u^2 \\
+ \nu_2\,\frac{(\lambda + 2\mu)}{\theta_0}\,u\Theta - \frac{1}{2}\,\nu_3\,\frac{(3\lambda + 2\mu)\alpha}{\theta_0}\,\Theta^2,
\end{aligned}
\tag{5.32}
$$

$$\rho_0 e = (3\lambda + 2\mu)\alpha\theta_0 u + \rho_0 c\Theta + \frac{1}{2}\,(1-\nu_2)(\lambda+2\mu)u^2,$$

$$+\,\nu_3(3\lambda+2\mu)\alpha u\Theta + \frac{1}{2}\,\nu_4\,\frac{\rho_0 c}{\theta_0}\,\Theta^2 + (1-\frac{1}{2}\zeta_1)\,\frac{\tau_0}{\kappa_0\theta_0}\,q^2. \tag{5.33}$$

5.4. Nondimensionalization

In the rest of our analysis we employ the following nondimensionalization scheme:

$$x^* = \frac{\rho_0 c\bar{v}}{\kappa_0}\,x, \quad (t^*, \tau_0^*, \tau_2^*) = \frac{\rho_0 c\bar{v}^2}{\kappa_0}\,(t, \tau_0, \tau_2),$$

$$w^* = \frac{\rho_0 c\bar{v}(\lambda+2\mu)}{\kappa_0(3\lambda+2\mu)\alpha\theta_0}\,w, \quad v^* = \frac{\lambda+2\mu}{(3\lambda+2\mu)\alpha\theta_0\bar{v}}\,v, \quad u^* = \frac{\lambda+2\mu}{(3\lambda+2\mu)\alpha\theta_0}\,u,$$

$$S^* = \frac{1}{(3\lambda+2\mu)\alpha\theta_0}\,S, \quad e^* = \frac{1}{c\theta_0}\,e, \quad q^* = \frac{1}{\rho_0 c\bar{v}\theta_0}\,q, \quad \Theta^* = \frac{1}{\theta_0}\,\Theta \tag{5.34}$$

where $\bar{v}$ is an appropriate velocity scale. From now on we shall use these dimensionless variables but omit the asterisks for convenience.

The need for the following dimensionless parameters will arise in the sequel as well:

$$C_p = \frac{1}{\bar{v}}\,[\frac{\lambda+2\mu}{\rho_0}]^{1/2}, \tag{5.35}$$

$$C_t = [\frac{1}{\tau_0}]^{1/2} = [\frac{1}{\tau_2}]^{1/2}, \tag{5.36}$$

$$\delta = \frac{(3\lambda + 2\mu)^2 \alpha^2 \theta_0}{\rho_0 c(\lambda + 2\mu)} \, . \tag{5.37}$$

In the above the parameter δ is the well known thermoelastic coupling constant of the classical theory of thermoelasticity whereas C_p and C_t are the respective dimensionless velocities of the acoustic and second sound waves when the coupling is ignored; see, for example, Öncü and Moodie [46].

CHAPTER 6

ASYMPTOTIC ANALYSIS OF THE LINEAR THERMOELASTIC PROBLEMS

In this chapter we shall study the propagation of boundary generated disturbances according to the linear theories discussed in the previous chapter. With this in mind, we obtain from (5.1) to (5.4) and the nondimensionalization scheme (5.35) the following dimensionless forms of the linearized balance laws:

$$\frac{\partial^2 w}{\partial t^2} - C_v^2 \frac{\partial S}{\partial x} = 0, \tag{6.1}$$

$$\frac{\partial e}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{6.2}$$

Since the structures of the constitutive theories we consider are substantially different, we formulate the initial-boundary value problems for each case separately. We then construct the geometric optics expansions for these problems and compare the predictions of the theories both qualitatively and numerically. In our numerical computations we employ Padé approximants to extend the validity of these expansions beyond their radius of convergence.

6.1 Problem Formulation in the Theory of Lord and Shulman

It follows from the nondimensionalization scheme (5.34) and the equations (5.1), (5.5), (5.6), (5.8) and (5.9) that the dimensionless heat flux, Piola stress and specific internal energy obey the relations

$$\frac{\partial q}{\partial t} + C_t^2 q = -C_t^2 \frac{\partial \Theta}{\partial x}\,, \tag{6.3}$$

$$S = \frac{\partial w}{\partial x} - \Theta, \tag{6.4}$$

$$e = \delta \frac{\partial w}{\partial x} + \frac{\partial \Theta}{\partial x}\,, \tag{6.5}$$

respectively.

Let us assume for convenience that the dimensionless displacement $w(x,t)$ is generated by a thermoelastic potential $\Phi(x,t)$ according as

$$w = \frac{\partial \Phi}{\partial x}\,. \tag{6.6}$$

With this thermoelastic potential, the equations (6.1) to (6.6) give

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{C_p^2}\frac{\partial^2 \Phi}{\partial t^2} = \Theta, \tag{6.7}$$

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{1}{C_t^2}\frac{\partial^2 \Theta}{\partial t^2} - \frac{\partial \Theta}{\partial t} = \frac{\delta}{C_t^2}\frac{\partial^4 \Phi}{\partial t^2 \partial x^2} + \delta \frac{\partial^3 \Phi}{\partial t \partial x^2}\,, \tag{6.8}$$

and the constitutive relation (6.4) may be rewritten as

$$S = \frac{\partial^2 \Phi}{\partial x^2} - \Theta. \tag{6.9}$$

In view of the assumptions concerning the past of the thermoelastic body we consider, the equations (6.7) and (6.8) are to be solved subject to the initial conditions

$$\Theta(x,t) = \frac{\partial \Theta}{\partial t}(x,t) = \Phi(x,t) = \frac{\partial \Phi}{\partial t}(x,t) = 0, \quad x > 0,\ t \leq 0 \tag{6.10}$$

and an appropriate set of boundary conditions. Physically relevant boundary conditions consist of a thermal and a mechanical condition. In the theory of Lord and Shulman we deal with the boundary prescribed temperature and strain:

$$\Theta(0,t) = f(t)H(t), \quad \frac{\partial^2 \Phi}{\partial x^2}(0,t) = g(t)H(t), \tag{6.11}$$

where $H(t)$ is the Heaviside unit step function while the functions f and g are analytic on $[0,\infty)$. It is clear from (6.9) that the boundary conditions (6.11) are equivalent to

$$\Theta(0,t) = f(t)H(t), \quad S(0,t) = h(t)H(t), \tag{6.12}$$

if we choose $h(t) = g(t) - f(t)$. Since f and g are analytic on $[0,\infty)$ they admit the following Taylor series expansions about $t = 0$:

$$f(t) = \sum_{j=0}^{\infty} f_j \frac{t^j}{j!}, \quad g(t) = \sum_{j=0}^{\infty} g_j \frac{t^j}{j!}, \tag{6.13}$$

where

$$f_j = \frac{d^j}{dt^j} f(t)|_{t=0}, \quad g_j = \frac{d^j}{dt^j} g(t)|_{t=0}. \tag{6.14}$$

Prior to further study of the problem we eliminate either of Θ or Φ from the equations (6.7) and (6.8) to get

$$P_L(\partial_t, \partial_x)U = 0, \tag{6.15}$$

where $U = (\Theta, \Phi)^T$ and P_L is the partial differential operator

$$P_L(\partial_t, \partial_x) = \partial_x^4 - (\frac{1}{C_p^2} + \frac{1+\delta}{C_t^2})\partial_x^2\partial_t^2 + \frac{1}{C_p^2 C_t^2}\partial_t^4 - (1+\delta)\partial_t\partial_x^2 + \frac{1}{C_p^2}\partial_t^2. \quad (6.16)$$

It should be noted that one should not conclude from (6.15) that Θ and Φ are uncoupled, for they are coupled through the equations (6.7) and (6.8). We shall use one of them to determine the coupling between Θ and Φ.

Observe that the principal part of the operator P_L is

$$p_L(\partial_t, \partial_x) = \partial_x^4 - (\frac{1}{C_p^2} + \frac{1+\delta}{C_t^2})\, \partial_x^2\partial_t^2 + \frac{1}{C_p^2 C_t^2}\,\partial_t^4. \quad (6.17)$$

Therefore, the characteristic equation for P_L is

$$p_L(\omega, \xi) = 0, \quad (6.18)$$

where ω, ξ are scalars. The operator P_L is hyperbolic if for each real $\xi \neq 0$, all roots ω of the characteristic equation are real. It follows from (6.17) and (6.18) that

$$(\omega^2 - C_1^2\xi^2)(\omega^2 - C_2^2\xi^2) = 0, \quad (6.19)$$

where

$$C_\ell = \{\frac{C_t^2 + (1+\delta)C_p^2 + (-1)^{\ell+1}\,|\Delta|}{2}\}^{1/2}, \quad \ell = 1,2, \quad (6.20)$$

and

$$\Delta^2 = [C_t^2 - (1+\delta)C_p^2]^2 + 4\delta C_p^2 C_t^2. \quad (6.21)$$

It is trivial to see from (6.19) to (6.21) that the hyperbolicity condition is met. Indeed, P_L is strictly hyperbolic since for each $\xi \neq 0$, the roots of the characteristic equation are distinct. Consequently, the initial-boundary value problem we have formulated is hyperbolic so that its solution admits a geometric optics expansion.

6.2. Geometric Optics Solution in the Theory of Lord and Shulman

We proceed to solve the problem by representing $\boldsymbol{U}$ in terms of its geometric optics expansion

$$\boldsymbol{U}(x,t) = \sum_{j=0}^{\infty} \boldsymbol{U}_j(x) F_j(\phi), \quad \boldsymbol{U}_j \equiv 0,\ j < 0, \tag{6.22}$$

where

$$\boldsymbol{U}_j = (\Theta_j, \Phi_j)^T, \tag{6.23}$$

$$F_j' = F_{j-1}, \tag{6.24}$$

whereas

$$\phi(x,t) = t - x/\lambda. \tag{6.25}$$

Note that in view of the initial and boundary conditions, and the fact that the material is homogeneous, we have restricted our attention to linear phase functions of the form (6.25). The equation (6.24) enables us to relate all of the F_j to the wave form F_0 by successive integrations.

To determine λ and the amplitude functions U_j, we substitute U from (6.22) into (6.15), use (6.24) in the result, equate the coefficients of F_{j-4}, and get

$$
\begin{aligned}
&\Big\{\frac{1}{\lambda^4}-\Big(\frac{1}{C_p^2}+\frac{1+\delta}{C_t^2}\Big)\frac{1}{\lambda^2}+\frac{1}{C_p^2C_t^2}\Big\}U_j\\
&\qquad+\frac{2}{\lambda}\Big\{\frac{1}{C_p^2}+\frac{1+\delta}{C_t^2}-\frac{2}{\lambda^2}\Big\}U'_{j-1}+\Big\{\frac{1}{C_p^2}-\frac{1+\delta}{\lambda^2}\Big\}U_{j-1}\\
&\qquad+\Big\{\frac{6}{\lambda^2}-\Big(\frac{1}{C_p^2}+\frac{1+\delta}{C_t^2}\Big)\Big\}U''_{j-2}+\frac{2(1+\delta)}{\lambda}U'_{j-2}\\
&\qquad-\frac{4}{\lambda}U'''_{j-3}+U^{(iv)}_{j-4}=0,\quad j=0,1,2,\dots\,.
\end{aligned}
\tag{6.26}
$$

Setting $j=0$ in (6.26) we then obtain

$$
\Big\{\frac{1}{\lambda^4}-\Big(\frac{1}{C_p^2}+\frac{1+\delta}{C_t^2}\Big)\frac{1}{\lambda^2}+\frac{1}{C_p^2C_t^2}\Big\}U_0=0. \tag{6.27}
$$

Since, without loss of generality, we may require $U_0\not\equiv 0$, (6.27) reduces to the eikonal equation

$$
\frac{1}{\lambda^4}-\Big(\frac{1}{C_p^2}+\frac{1+\delta}{C_t^2}\Big)\frac{1}{\lambda^2}+\frac{1}{C_p^2C_t^2}=0. \tag{6.28}
$$

As the waves generated at the boundary must travel into the region $x>0$, we choose the positive roots $\lambda_\ell=C_\ell$, $\ell=1,2$ of this equation, where C_ℓ are given by (6.20) and (6.21). Thus, the phases for right-travelling waves are determined as

$$
\phi_\ell(x,t)=t-x/C_\ell,\quad \ell=1,2. \tag{6.29}
$$

This shows that in the theory of Lord and Shulman there are two wave components with their wave fronts located at

$$t = x/C_1, \qquad t = x/C_2, \tag{6.30}$$

and propagating through the medium at the finite speeds C_1 and C_2, respectively. From (6.20) and (6.21) we have $C_2 < (C_p, C_t) < C_1$. Therefore, if $C_p > C_t$, then as $\delta \to 0$, $C_1 \to C_p$ and $C_2 \to C_t$ so that the fast wave is quasi-elastic and the slow wave is quasi-thermal [46]. If $C_t > C_p$, then the roles of the fast and slow waves are reversed. However, in the materials in which second sound occurred, the former situation is always the case.

Having determined the existence of two wave components, we then consider the geometric optics expansion for U consisting of two expansions

$$U(x,t) = \sum_{j=0}^{\infty}\sum_{\ell=1}^{2} U_{\ell j}(x)F_j(t - x/C_\ell), \quad U_{\ell j} \equiv 0,\ j < 0,\ \ell = 1,2, \tag{6.31}$$

$$U_{\ell j} = (\Theta_{\ell j}, \Phi_{\ell j})^T, \tag{6.32}$$

where F_j satisfy (6.24) as before. Inserting this expansion into (6.15) and using (6.28) gives the so-called transport equations

$$\begin{aligned} U'_{\ell j} + W_\ell U_{\ell j} = Q_\ell \,\{ &(\frac{1}{C_1^2} + \frac{1}{C_2^2} - \frac{6}{C_\ell^2})U''_{\ell,j-1} - \frac{2(1+\delta)}{C_\ell}U'_{\ell,j-1} \\ &+ \frac{4}{C_\ell}U'''_{\ell,j-2} + (1+\delta)U''_{\ell,j-2} - U^{(iv)}_{\ell,j-3}\}, \\ &\ell = 1,2, \quad j \geq 0, \end{aligned} \tag{6.33}$$

where

$$Q_\ell = \frac{C_1^2 C_2^2}{C_\ell^2 - C_{3-\ell}^2}\, C_\ell, \tag{6.34}$$

and

$$W_\ell = \frac{C_\ell^2 - (1+\delta)C_p^2}{2C_p^2 C_\ell^2}\, Q_\ell. \tag{6.35}$$

With $j = 0$ the solution of (6.33) is

$$\boldsymbol{U}_{\ell 0}(x) = \overline{\boldsymbol{U}}_{\ell 0} e^{-W_\ell x}, \tag{6.36}$$

where $\overline{\boldsymbol{U}}_{\ell 0} = \boldsymbol{U}_{\ell 0}(0)$. Since

$$C_\ell^2 - (1+\delta)C_p^2 = \frac{\delta C_p^2 C_t^2}{C_\ell^2 - C_t^2}\,, \tag{6.37}$$

which follows from (6.20) and (6.21), (6.34) and (6.35) show that $W_\ell > 0$, $\ell = 1,2$. Therefore, in the theory of Lord and Shulman, any discontinuous change in $\boldsymbol{U}$ decays to zero as x tends to infinity.

It can be proved by induction on j that the amplitude coefficients $\boldsymbol{U}_{\ell j}$ have the form

$$\boldsymbol{U}_{\ell j}(x) = e^{-W_\ell x} \sum_{n=0}^{j} \boldsymbol{u}_{\ell n j} \frac{x^n}{n!}\,, \quad \ell = 1,2,\ j \ge 0, \tag{6.38}$$

where

$$\boldsymbol{u}_{\ell n j} = (t_{\ell n j}, \varphi_{\ell n j})^T. \tag{6.39}$$

The coefficients $u_{\ell nj}$ of (6.38) are given recursively by

$$u_{\ell nj} = \begin{cases} Q_\ell\{(\frac{1}{C_1^2}+\frac{1}{C_2^2}-\frac{6}{C_\ell^2})E_\ell^2 u_{\ell,n-1,j-1} - \frac{2(1+\delta)}{C_\ell}E_\ell u_{\ell,n-1,j-1} \\ \qquad + \frac{4}{C_\ell}E_\ell^3 u_{\ell,n-1,j-2} + (1+\delta)E_\ell^2 u_{\ell,n-1,j-2} - E_\ell^4 u_{\ell,n-1,j-3}\}, \\ \qquad 1 \le n \le j, \\ \overline{U}_{\ell j}, \qquad n=0,\ j\ge 0, \\ 0, \qquad n<0 \ \text{ or } \ n>j, \end{cases} \tag{6.40}$$

where E_ℓ is the weighted difference operator defined as

$$E_\ell u_{\ell nj} = u_{\ell,n+1,j} - W_\ell u_{\ell nj}. \tag{6.41}$$

To complete the solution it remains to determine $\overline{U}_{\ell j} = (\overline{\Theta}_{\ell j}, \overline{\Phi}_{\ell j})$ from the boundary conditions and one of the field equations (6.7) and (6.8). Inserting the expansion (6.31) into (6.7), equating the coefficients of $F_{j-2}(t-x/C_\ell)$ and using (6.36) to (6.40) in the result we find that (6.7) is satisfied if and only if

$$\begin{aligned}(\frac{1}{C_\ell^2}-\frac{1}{C_p^2})\overline{\Phi}_{\ell,j+2} + \frac{2W_\ell}{C_\ell}\overline{\Phi}_{\ell,j+1} + W_\ell^2\overline{\Phi}_{\ell j} - \overline{\Theta}_{\ell j} \\ = -\varphi_{\ell 2,j} + 2W_\ell\varphi_{\ell 1j} + \frac{2}{C_\ell}\varphi_{\ell,1,j+1},\end{aligned} \tag{6.42}$$

$$\ell = 1,2, \quad j = -2,-1,0,1,\ldots$$

It is the immediate consequence of (6.42) that $\overline{\Phi}_{\ell 0} = \overline{\Phi}_{\ell 1} = 0,\ \ell = 1,2$ so that the thermoelastic potential $\Phi(x,t)$ is a continuously differentiable function. This ensures the continuity of the displacement field $w(x,t)$.

Let us now insert the expansion (6.31) into the boundary conditions (6.11) to get

$$\sum_{\ell=1}^{2}\sum_{j=0}^{\infty}\overline{\Theta}_{\ell j}F_j(t)=\sum_{j=0}^{\infty}f_j\,\frac{t^j}{j!}\,H(t), \tag{6.43}$$

$$\sum_{\ell=1}^{2}\sum_{j=0}^{\infty}\{\frac{1}{C_\ell^2}\overline{\Phi}_{\ell,j+1}+\frac{2W_\ell}{C_\ell}\overline{\Phi}_{\ell,j+1}+W_\ell^2\overline{\Phi}_{\ell j}+\varphi_{\ell 2j}-2W_\ell\varphi_{\ell 1j}$$
$$-\frac{2}{C_\ell}\,\varphi_{\ell,1,j+1}\}F_j(t)=\sum_{j=0}^{\infty}g_j\,\frac{t^j}{j!}\,H(t), \tag{6.44}$$

where we have employed the Taylor series expansions (6.13) and (6.14). Upon choosing

$$F_j(t)=\frac{t^j}{j!}\,H(t),\quad j=0,1,2,\ldots, \tag{6.45}$$

we identify the coefficients of F_j as

$$\sum_{\ell=1}^{2}\overline{\Theta}_{\ell j}=f_j,\quad j=0,1,2,\ldots, \tag{6.46}$$

$$\sum_{\ell=1}^{2}\frac{1}{C_\ell^2}\,\overline{\Phi}_{\ell,j+2}=g_j-\sum_{\ell=1}^{2}B_{\ell j},\quad j=0,1,2,\ldots, \tag{6.47}$$

where

$$B_{\ell j}=\frac{2W_\ell}{C_\ell}\,\overline{\Phi}_{\ell j}+W_\ell^2\overline{\Phi}_{\ell,j-1}+\varphi_{\ell,2,j-1}-2W_\ell\varphi_{\ell,1,j-1}-\frac{2}{C_\ell}\,\varphi_{\ell 1j}. \tag{6.48}$$

Note that the above choice of F_j is consistent with the requirement (6.24).

The coefficients $\overline{\Theta}_{\ell j}$ and $\overline{\Phi}_{\ell j}$ can now be solved for from the equations (6.42), (6.46) and (6.47). The solution is

$$\overline{\Theta}_{\ell 0} = K_{3-\ell} A_{\ell 0}, \quad \ell = 1,2,$$
$$\overline{\Phi}_{\ell 2} = M_\ell A_{\ell 0}, \quad \ell = 1,2, \tag{6.49}$$

$$\overline{\Theta}_{\ell j} = K_{3-\ell} A_{\ell j} + K_\ell B_{\ell j} - K_{3-\ell} B_{3-\ell,j}, \quad \ell = 1,2, \quad j \geq 1,$$
$$\overline{\Phi}_{\ell,j+2} = M_\ell \Big(A_{\ell j} - \sum_{\ell=1}^{2} B_{\ell 2}\Big), \qquad \ell = 1,2, \quad j \geq 1, \tag{6.50}$$

where

$$K_\ell = \frac{C_\ell^2 (C_p^2 - C_{3-\ell}^2)}{C_p^2 (C_\ell^2 - C_{3-\ell}^2)}, \tag{6.51}$$

$$M_\ell = \frac{C_1^2 C_2^2}{C_{3-\ell}^2 - C_\ell^2}, \tag{6.52}$$

$$A_{\ell j} = \frac{(C_{3-\ell}^2 - C_p^2) g_j + C_p^2 f_j}{C_{3-\ell}^2}. \tag{6.53}$$

With the above results, construction of the geometric optics expansions of Θ and Φ are complete. These expansions are

$$\Theta(x,t) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{2} \Theta_{\ell j}(x) \, \frac{(t - x/C_\ell)^j}{j!} \, H(t - x/C_\ell), \tag{6.54}$$

$$\Phi(x,t) = \sum_{j=2}^{\infty} \sum_{\ell=1}^{2} \Phi_{\ell j}(x) \, \frac{(t - x/C_\ell)^j}{j!} \, H(t - x/C_\ell), \tag{6.55}$$

where $\Theta_{\ell j}$ and $\Phi_{\ell j}$ are defined by the expressions (6.38) to (6.41) and (6.49) to (6.53).

We conclude this section by presenting the discontinuities in u, S and Θ predicted by the theory of Lord and Shulman. These discontinuities, directly computed from (6.54) and (6.55), are

$$[u]_{t=x/C_\ell} = H_\ell e^{-W_\ell x}, \qquad \ell = 1,2, \tag{6.56}$$

$$[S]_{t=x/C_\ell} = \frac{C_\ell^2}{C_p^2} H_\ell e^{-W_\ell x}, \qquad \ell = 1,2, \tag{6.57}$$

$$[\theta]_{t=x/C_\ell} = \frac{C_p^2 - C_\ell^2}{C_p^2} H_\ell e^{-W_\ell x}, \quad \ell = 1,2, \tag{6.58}$$

where

$$H_\ell = \frac{(C_{3-\ell}^2 - C_p^2) g_0 + C_p^2 f_0}{C_{3-\ell}^2 - C_\ell^2}, \tag{6.59}$$

and $[\ ]_{t=x/C_\ell}$ indicates the discontinuity across $t = x/C_\ell$. We shall discuss the implications of the above jump conditions in later sections. At this point we mention only that if H_ℓ vanishes, then all of the field variables are continuous across $t = x/C_\ell$. It is interesting to note that this may happen even when the boundary disturbances to Θ and u are jointly discontinuous, that is, $f_0 \neq 0$ and $g_0 \neq 0$, but only at one of the wave fronts.

6.3. Problem Formulation in the Theory of Green and Lindsay

In the theory of Green and Lindsay, the dimensionless constitutive relations for the heat flux, Piola stress and specific internal energy obtained from

(5.1), (5.11), (5.12), (5.13) and (5.34) are

$$q = -\frac{\partial \Theta}{\partial x}, \tag{6.60}$$

$$S = \frac{\partial w}{\partial x} - (\Theta + \frac{\beta}{C_t^2}\frac{\partial \Theta}{\partial t}), \tag{6.61}$$

$$e = \delta\frac{\partial w}{\partial x} + (\theta + \frac{1}{C_t^2}\frac{\partial \Theta}{\partial t}), \tag{6.62}$$

respectively. Combining the above equations with (6.1) and (6.2) gives the equations to be satisfied by Θ and the thermoelastic potential Φ defined in (6.6). These equations are

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{C_p^2}\frac{\partial^2 \Phi}{\partial t^2} = \Theta + \frac{\beta}{C_t^2}\frac{\partial \Theta}{\partial t}, \tag{6.63}$$

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{1}{C_t^2}\frac{\partial^2 \Theta}{\partial t^2} - \frac{\partial \Theta}{\partial t} = \delta\frac{\partial^3 \Phi}{\partial t \partial x^2}. \tag{6.64}$$

Therefore, the problem to be solved in the theory of Green and Lindsay is described by (6.63), (6.64), (6.10) and (6.11). Since in this theory the boundary conditions (6.11) imply

$$S(0,t) = \{g(t) - f(t) - \frac{\beta}{C_t^2} f'(t)\}H(t) - \frac{\beta}{C_t^2} f(t)\delta(t)$$

where $\delta(t)$ is the Dirac delta function, (6.11) and (6.12) are not equivalent. In this theory we examine the case when the boundary conditions are (6.12) independently by assuming

$$h(t) = \sum_{j=0}^{\infty} h_j \frac{t^j}{j!}, \quad h_j = \frac{d^j}{dt^j} h(t)|_{t=0}. \tag{6.65}$$

As for the previous problem, we eliminate either of Θ or Φ from the governing equations obtaining

$$P_G(\partial_t, \partial_x)U = 0, \tag{6.66}$$

where

$$P_G(\partial_t, \partial_x) = \partial^4 x - \left(\frac{1}{C_p^2} + \frac{1+\delta\beta}{C_t^2}\right)\partial_x^2\partial_t^2 + \frac{1}{C_p^2 C_t^2}\,\partial_t^4 - (1+\delta)\partial_t\partial_x^2 + \frac{1}{C_p^2}\,\partial_t^3, \tag{6.67}$$

and $U = (\Theta, \Phi)^T,$. Comparison between (6.16) and (6.67) shows the operator P_G is also strictly hyperbolic. The characteristic equation associated with P_G is also of the form (6.19) but this time the characteristic speeds C_1 and C_2 are given by

$$C_\ell = \left\{\frac{C_t^2 + (1+\delta\beta)C_p^2 + (-1)^{\ell+1}|\Delta|}{2}\right\}^{1/2}, \quad \ell = 1, 2 \tag{6.68}$$

and

$$\Delta^2 = [C_t^2 - (1+\delta\beta)C_p^2]^2 + 4\delta\beta C_p^2 C_t^2. \tag{6.69}$$

From (6.20), (6.21), (6.68) and (6.69) we see that when $\beta = 1$, corresponding characteristic speeds of the two theories are identical. When $\beta > 1$, the faster (slower) speed of the theory of Green and Lindsay is larger (smaller) than that of the theory of Lord and Shulman.

6.4. Geometric Optics Solution in the Theory of Green and Lindsay

In the light of the geometric optics solution in the theory of Lord and Shulman, and because of the apparent similarities between P_L and P_G, we start from the outset with the expansion

$$U(x,t) = \sum_{j=0}^{\infty} \sum_{\ell=1}^{2} U_{\ell j}(x) F_j(t - x/\lambda_\ell), \quad U_{\ell_j} \equiv 0, \quad j < 0, \quad \ell = 1,2 \qquad (6.70)$$

where F_j satisfies (6.24) and $U_{\ell j} = (\Theta_{\ell j}, \Phi_{\ell j})^T$.

Proceeding as before we find λ_ℓ and $U_{\ell j}$ satisfy

$$\begin{aligned}
&\{\frac{1}{\lambda_\ell^4} - (\frac{1}{C_p^2} + \frac{1+\delta\beta}{C_t^2})\frac{1}{\lambda_\ell^2} + \frac{1}{C_p^2 C_t^2}\} U_{\ell j} \\
&\quad + \frac{2}{\lambda_\ell}\{\frac{1}{C_p^2} + \frac{1+\delta\beta}{C_t^2} - \frac{2}{\lambda_\ell^2}\} U'_{\ell,j-1} + \{\frac{1}{C_p^2} - \frac{1+\delta}{\lambda_\ell^2}\} U_{\ell,j-1} \\
&\quad + \{\frac{6}{\lambda_\ell^2}(\frac{1}{C_p^2} + \frac{1+\delta\beta}{C_t^2})\} U''_{\ell,j-2} + \frac{2(1+\delta)}{\lambda_\ell} U'_{\ell,j-2} \\
&\quad - \frac{4}{\lambda_\ell} U'''_{\ell,j-3} + U^{iv}_{\ell,j-4} = 0, \quad \ell = 1,2, \quad j = 0,1,2,\dots \qquad (6.71)
\end{aligned}$$

From (6.71) and without duplicating the work we reach the following conclusion:

For the right-travelling waves $\lambda_\ell = C_\ell$, $\ell = 1,2$ where C_ℓ are given by (6.68) and (6.69) whereas

$$U_{\ell j}(x) = e^{-W_\ell x} \sum_{n=0}^{j} u_{\ell n j} \frac{x^n}{n!}, \quad \ell = 1,2, \quad j \geq 0 \qquad (6.72)$$

where

$$
u_{\ell nj} = \begin{cases} Q_\ell\{(\frac{1}{C_1^2} + \frac{1}{C_2^2} - \frac{6}{C_\ell^2})E_\ell^2 u_{\ell,n-1,j-1} - \frac{2(1+\delta)}{C_\ell} E_\ell u_{\ell,n-1,j-1} \\ \qquad + \frac{4}{C_\ell} E_\ell^3 u_{\ell,n-1,j-2} + (1+\delta)E_\ell^2 u_{\ell,n-1,j-2} - E_\ell^4 u_{\ell,n-1,j-3}\}, \\ \qquad 1 \le n \le j, \\ \overline{U}_{\ell j}, \qquad n = 0,\ j \ge 0, \\ 0, \qquad n < 0 \quad \text{or} \quad n > j, \end{cases} \tag{6.73}
$$

$$
Q_\ell = \frac{C_1^2 C_2^2}{C_\ell^2 - C_{3-\ell}^2}\, C_\ell, \tag{6.74}
$$

$$
W_\ell = \frac{C_\ell^2 - (1+\delta)C_p^2}{2C_p^2 C_\ell^2}\, Q_\ell, \tag{6.75}
$$

$$
E_\ell\, u_{\ell nj} = u_{\ell,n+1,j} - W_\ell\, u_{\ell nj}, \tag{6.76}
$$

and $u_{\ell nj} = (t_{\ell nj}, \varphi_{\ell nj})^T$. From (6.68) we get

$$
C_\ell^2 - (1+\delta)C_p^2 = \frac{\delta C_p^2}{C_\ell^2 - C_t^2}\, \{(\beta - 1)C_\ell^2 + C_t^2\}, \tag{6.77}
$$

and we have $\beta \ge 1$. Thus, (6.74), (6.75) and (6.77) show $W_\ell > 0$, $\ell = 1, 2$. Hence, the theory of Green and Lindsay agrees with the theory of Lord and Shulman not only in predicting two wave components whose wave fronts are

$$
\begin{aligned} t &= x/C_1, \\ t &= x/C_2, \end{aligned} \tag{6.78}
$$

but also in predicting that jump discontinuities of any order across these wave fronts are attenuated with x. Indeed, when $\beta = 1$, like the corresponding wave speeds C_ℓ, the corresponding attenuation coefficients W_ℓ of the two theories are identical as well.

Let us now proceed to complete the solution by determining the initial values $\overline{U}_{\ell j} = (\overline{\Theta}_{\ell j}, \overline{\Phi}_{\ell j})^T$ of the amplitude coefficients. First, we insert (6.78) into one of the equations (6.63) or (6.64), say (6.63), and obtain

$$\left(\frac{1}{C_\ell^2} - \frac{1}{C_p^2}\right)\overline{\Phi}_{\ell,j+1} + \frac{2W_\ell}{C_\ell}\overline{\Phi}_{\ell j} + W_\ell^2 - \frac{\beta}{C_q^2}\overline{\Theta}_{\ell j} - \overline{\Theta}_{\ell,j-1}$$
$$= -\varphi_{\ell,2,j-1} + 2W_\ell\varphi_{\ell,1,j-1} + \frac{2}{C_\ell}\varphi_{\ell 1 j},$$
$$\ell = 1,2 \qquad j = -1,0,1,\dots . \tag{6.79}$$

Setting $j = -1$ in the above shows $\Phi_{\ell 0} = 0$, $\ell = 1,2$ so that for the boundary conditions under consideration, the thermoelastic potential $\Phi(x,t)$ is continuous. However (6.79) does not imply that $\overline{\Phi}_{\ell 1} = 0$, $\ell = 1,2$ and for this reason continuity of the displacement field $w(x,t)$ is not guaranteed in this theory. In fact, as it will be seen, the theory of Green and Lindsay allows for jump discontinuities in the displacement.

We now insert the expansion into the boundary conditions (6.11), employ the Taylor series expansions (6.13) and (6.14), choose

$$F_j(t) = \frac{t^j}{j!}H(t), \quad j = 0,1,2,\dots, \tag{6.80}$$

and find that

$$\sum_{\ell=1}^{2} \overline{\Theta}_{\ell j} = f_j, \quad j = 0, 1, 2, \ldots, \tag{6.81}$$

$$\sum_{\ell=1}^{2} \frac{1}{C_\ell^2} \overline{\Phi}_{\ell 1} = 0, \tag{6.82}$$

$$\sum_{\ell=1}^{2} \frac{1}{C_\ell^2} \overline{\Phi}_{\ell, j+1} = g_{j-1} - \sum_{\ell=1}^{2} B_{\ell j}, \quad j = 1, 2, 3, \ldots, \tag{6.83}$$

where

$$B_{\ell j} = \frac{2W_\ell}{C_\ell} \overline{\Phi}_{\ell j} + W_\ell^2 \overline{\Phi}_{\ell, j-1} + \varphi_{\ell,2,j-1} - 2W_\ell \varphi_{\ell,1,j-1} - \frac{2}{C_\ell} \varphi_{\ell,j}. \tag{6.84}$$

Note that we have $\dot{F}_{-1}(t) = \delta(t)$.

From (6.79), (6.81), (6.82) and (6.83) we get the following result:

$$\overline{\Theta}_{\ell 0} = \frac{C_p^2 - C_\ell^2}{C_{3-\ell}^2 - C_\ell^2} f_0, \qquad \ell = 1, 2, \tag{6.85}$$

$$\overline{\Phi}_{\ell 1} = \frac{\beta C_p^2 C_\ell^2}{C_t^2 (C_{3-\ell}^2 - C_\ell^2)} f_0, \qquad \ell = 1, 2, \tag{6.86}$$

$$\overline{\Theta}_{\ell j} = K_{3-\ell} A_{\ell j} - \frac{C_t^2}{\beta} \overline{\Theta}_{\ell, j-1} + K_\ell B_{\ell j} - K_{3-\ell} B_{3-\ell, j}, \quad \ell = 1, 2, \quad j \geq 1, \tag{6.87}$$

$$\overline{\Phi}_{\ell, j+1} = M_\ell (A_{\ell j} - \sum_{\ell=1}^{2} B_{\ell j}), \qquad \ell = 1, 2, \quad j \geq 1, \tag{6.88}$$

where

$$K_\ell = \frac{C_T^2}{\beta}\,\frac{C_\ell^2(C_p^2 - C_{3-\ell}^2)}{C_p^2(C_\ell^2 - C_{3-\ell}^2)}\,, \tag{6.89}$$

$$M_\ell = \frac{C_1^2 C_2^2}{C_{3-\ell}^2 - C_\ell^2}\,, \tag{6.90}$$

$$A_{\ell j} = \frac{(C_{3-\ell}^2 - C_p^2)g_{j-1} + C_p^2(f_{j-1} + \beta f_j/C_\ell^2)}{C_{3-\ell}^2}\,. \tag{6.91}$$

We have completed the construction of the geometric optics expansions for Θ and Φ in the theory of Green and Lindsay. These expansions are

$$\Theta(x,t) = \sum_{j=1}^{\infty}\sum_{\ell=1}^{2}\Theta_{\ell j}(x)\,\frac{(t - x/C_\ell)^j}{j!}\,H(t - x/C_\ell), \tag{6.92}$$

$$\Phi(x,t) = \sum_{j=1}^{\infty}\sum_{\ell=1}^{2}\Phi_{\ell j}(x)\,\frac{(t - x/C_\ell)^j}{j!}\,H(t - x/C_\ell), \tag{6.93}$$

where $\Theta_{\ell j}$ and $\Phi_{\ell j}$ are given by (6.73) and (6.74) together with (6.85) to (6.91).

The geometric optics expansions for w, u and S can be constructed from the expansions for Θ and Φ, and information on the discontinuities in all of the field variables can be extracted from these expansions. It is clear from (6.93) that the thermoelastic potential is continuous across each wave front. On the other hand, all other field variables suffer from jump discontinuities. Moreover, on arrival of the wave fronts u and S exhibit first a Dirac delta behaviour. This Dirac delta for S on the wave front $t = x/C_\ell$ is proportional to

$$\beta \frac{C_\ell^2}{C_t^2} D_\ell e^{-W_\ell x}, \tag{6.94}$$

where

$$D_\ell = \frac{f_0}{C_{3-\ell}^2 - C_\ell^2}. \tag{6.95}$$

Then S suffers from a jump discontinuity determined as

$$[S]_{t=x/C_\ell} = M_\ell(P_\ell + R_\ell x)e^{-W_\ell x}, \tag{6.96}$$

whereas the discontinuities for Θ and w are

$$[\Theta]_{t=x/C_\ell} = (C_\ell^2 - C_p^2)D_\ell e^{-W_\ell x}, \tag{6.97}$$

$$[w]_{t=x/C_\ell} = -\beta \frac{C_p^2}{C_t^2} C_\ell D_\ell e^{-W_\ell x}, \tag{6.98}$$

where

$$P_\ell = \frac{1}{C_p^2 C_{3-\ell}^2}\{\beta \frac{C_p^2}{C_t^2} f_1 + (C_{3-\ell}^2 + \beta \frac{C_1^2 + C_2^2 - 2(1+\delta)C_p^2}{C_t^2 C_\ell^2} M_\ell^2) f_0\}, \tag{6.99}$$

$$R_\ell = \beta \frac{Q_\ell}{C_T^2 C_{3-\ell}^2}\{(\frac{1}{C_{3-\ell}^2} - \frac{5}{C_\ell^2})W_\ell^2 + \frac{2(1+\delta)}{C_\ell} W_\ell\} f_0. \tag{6.100}$$

Now, we compare the above jump conditions with the jump conditions obtained for the theory of Lord and Shulman.

Firstly, it follows from (6.95) and (6.98) that according to the theory of Green and Lindsay, any discontinuous thermal disturbance produces discontinuities in the displacement field. As this implies that for such disturbances one portion of matter penetrates into another, the fundamental continuum hypothesis that matter is impenetrable is violated in the theory of Green and Lindsay. Furthermore, when the thermal disturbance has a jump discontinuity, it is seen from (6.94) and (6.95) that the stress takes infinite values on the wave fronts. That these are not the cases in the theory of Lord and Shulman are seen from (6.57) and (6.58). This is the first major disagreement between the two theories.

Secondly, (6.95) and (6.98) show that no discontinuous mechanical disturbance can generate discontinuities in the temperature: the temperature is always continuous unless the thermal disturbance is discontinuous. The theory of Lord and Shulman disagrees also with this prediction of the theory of Green and Lindsay by predicting discontinuities in the temperature generated by discontinuous mechanical disturbances; see (6.58) and (6.59).

Finally, according to the theory of Lord and Shulman, if both the mechanical and thermal disturbances are continuous, then all of the field variables are also continuous. On the other hand, it can be seen from (6.96), (6.99) and (6.100) that in the theory of Green and Lindsay such disturbances may generate discontinuities in the stress provided the first derivative of the thermal disturbance is discontinuous. Moreover, the signs of the jump discontinuities in stress depend on the relative sizes of the factors involved in (6.96) and may change with x.

We now turn our attention to the examination of the boundary conditions (6.12) we promised earlier. Since in this case the mechanical boundary condition is a traction boundary condition, the geometric optics expansion of the stress is needed. Using (6.61), (6.92) and (6.93) we get

$$S(x,t) = \sum_{j=0}^{\infty}\sum_{\ell=1}^{2} \frac{1}{C_p^2}\Phi_{\ell,j+1}(x)F_{j-1}(t - x/C_\ell). \tag{6.101}$$

Therefore, the boundary conditions (6.12) are satisfied if

$$\sum_{j=0}^{\infty}\sum_{\ell=1} \frac{1}{C_p^2}\overline{\Phi}_{\ell,j+1}F_{j-1}(t) = \sum_{j=0}^{\infty} h_j \frac{t^j}{j!} H(t), \tag{6.102}$$

and together with (6.45), the equation (6.43) holds. Matching the terms on either sides of (6.43) and (6.101) we then get

$$\sum_{\ell=1}^{2} \overline{\Theta}_{\ell j} = f_j, \quad j = 0,1,2,\dots, \tag{6.103}$$

$$\sum_{\ell=1}^{2} \frac{1}{C_p^2}\overline{\Phi}_{\ell 1} = 0, \tag{6.104}$$

$$\sum_{\ell=1}^{2} \frac{1}{C_p^2}\overline{\Phi}_{\ell,j+1} = h_{j-1}, \quad j = 1,2,3,\dots . \tag{6.105}$$

Repeating the previous solution procedure we obtain

$$\overline{\Theta}_{\ell 0} = \frac{C_{3-\ell}^2(C_p^2 - C_\ell^2)}{C_p^2(C_{3-\ell}^2 - C_\ell^2)} f_0, \quad \ell = 1,2, \tag{6.106}$$

$$\overline{\Phi}_{\ell 1} = \frac{\beta C_1^2 C_2^2}{C_\ell^2(C_{3-\ell}^2 - C_\ell^2)}\, f_0, \quad \ell = 1, 2, \tag{6.107}$$

$$\overline{\Theta}_{\ell j} = K_{3-\ell} A'_{\ell j} - \frac{C_\ell^2}{\beta}\overline{\Theta}_{\ell, j-1} + K_\ell B_{\ell j} - K_{3-\ell} B_{3-\ell, j}, \quad \ell = 1, 2, \quad j \geq 1, \tag{6.108}$$

$$\overline{\Phi}_{\ell, j+1} = M_j(A'_{\ell j} - \sum_{\ell=1}^{2} B_{\ell j}), \quad \ell = 1, 2, \quad j \geq 1, \tag{6.109}$$

where

$$A'_{\ell j} = \frac{(C_{3-\ell}^2 - C_p^2) h_{j-1} + C_{3-\ell}^2 (f_{j-1} + \beta f_j / C_\ell^2)}{C_{3-\ell}^2}, \tag{6.110}$$

whereas B_ℓ, K_ℓ and M_ℓ are defined by (6.84), (6.89) and (6.90), respectively. Therefore, the geometric optics solution for the boundary conditions (6.12) is also given by (6.92) and (6.93) but this time $\Theta_{\ell j}$ and $\Phi_{\ell j}$ are determined from (6.73), (6.74), (6.84), (6.89), (6.90) together with (6.106) to (6.110). Comparison shows that the previous discussion on the propagation of discontinuities for the boundary conditions (6.11) applies to the case in which the boundary conditions are (6.12) as well.

6.5. Numerical Results

In this section predictions of the theory of Green and Lindsay are compared numerically against the related predictions of the theory of Lord and Shulman. For purposes of comparison we consider a hypothetical material whose material parameters are $C_p = 1$, $C_t = 0.5$ and $\delta = 0.05$. For the theory of Green and Lindsay we take the cases $\beta = 1$ and $\beta = 3$. The boundary conditions are prescribed as

$$\Theta(0,t) = H(t), \quad S(0,t) = 0. \tag{6.111}$$

The numerical results presented in this section are obtained from the geometric optics solutions with the use of [10/10] Padé approximants. To explain how Padé approximants can be applied to the geometric optics expansions, we take the expansion for the temperature. At the wave front $t = x/C_\ell$ this expansion is

$$H(t - x/C_\ell) \sum_{j=0}^{\infty} \frac{1}{j!} \Theta_{\ell j}(x)(t - x/C_\ell)^j.$$

Defining the time elapsed after the passage of the wave front as $y = t - x/C_\ell$, we find that at a particular location x the above expansion has the formal power series representation

$$\sum_{j=0}^{\infty} \frac{1}{j!} \Theta_{\ell j} y^j, \quad y > 0. \tag{6.112}$$

That the linear geometric optics expansions are convergent has been proved by Ludwig [47] for analytic data. However, highly recursive nature of the amplitude coefficients defy determination of the radius of convergence by analytical means. We do this numerically.

For a given location x we sum the series (6.112) using 30 terms and then 40 terms. The numerical results obtained from summing 30 terms and

40 terms are in agreement for a certain range of t. This procedure provides us with a rough estimate of the length of time after which the series starts to diverge at a given location x.

To extend the validity of (6.112) beyond its radius of convergence, we employ Padé approximants. Let $P_L(y)$ and $Q_M(y)$ be polynomials of degree L and M, respectively. Then the L, M Padé approximant for (6.112) is defined as

$$[L/M] = \frac{P_L(y)}{Q_M(y)}\,, \tag{6.113}$$

where

$$\sum_{j=0}^{\infty} \frac{1}{j!}\,\Theta_{\ell j} y^j - \frac{P_L(y)}{Q_M(y)} = O(y^{L+M+1}), \tag{6.114}$$

with the normalization condition

$$Q_M(0) = 1. \tag{6.115}$$

We solve the coefficients of the numerator and denominator polynomials from the system of linear equations constructed from (6.114) and (6.115).

It is known from numerical experiments that near main diagonal Padé approximants exhibit relatively better convergence characteristics when compared to Padé approximants away from the main diagonal of the Padé table. For this reason, we restrict our choice to the $[L/L+J]$ Padé approximants where J may take the values $-1, 0$ or 1 depending on the behaviour of the boundary disturbances. If, for example, the boundary disturbances tend to zero as t

tends to infinity, then we choose the sequence of $[L/L+1]$ Padé approximants to approximate the wave front expansions. It should be mentioned, however, that choosing any one of the above indicated values for J would not affect the accuracy of the approximation drastically. We terminate the sequence at $L = 10$ to avoid excessive computation times and eliminate those terms whose singularities are either real or near real axis of the complex plane and lie in the range of approximation. We then choose the highest order Padé approximant from the remaining terms in the terminated sequence.

We display our numerical results for the temperature, displacement and stress in Figs. 6.1-6.3. In Fig. 6.2, the Dirac delta terms of the stress in the theory of Green and Lindsay have been neglected. We remark that at the first wave front the Dirac delta pulse is compressive whereas it is tensile at the other. This is compatible with the corresponding compressive and tensile jumps in the displacement.

It is seen from Fig. 6.1 that for the boundary disturbances (6.111), the two theories are in agreement on the behaviour of the temperature. This agreement is excellent shortly after the arrival of the second wave front. Indeed, when $\beta = 1$ they predict the same behaviour for the temperature. However, they are in complete disagreement on the behaviour of the displacement and stress for small times. Whereas the displacement is continuous for the theory of Lord and Shulman, it suffers from large jump discontinuities in the theory of Green and Lindsay. The size of these discontinuities increases with β. Moreover, a scrutiny of Figs. 6.2 and 6.3 shows that the predictions of the theory of Green and Lindsay on the displacement and stress are not always compatible: at station $x = 1$, immediately after the arrival of the second wave front, the stress for $\beta = 1$ is tensile whereas the displacement behaves in a compressive manner. It is also clear from these figures that both of the theories predict the same material response for larger times. This is because the time derivatives of the field variables diminish as t increases so that their influence becomes insignificant.

We now conclude this chapter with some remarks.

As was mentioned, Pao and Banerjee [30] showed the agreement between the experiment and the theory of Lord and Shulman. We showed several disagreements between the theory of Lord and Shulman, and the theory of Green and Lindsay. We showed further that the theory of Green and Lindsay violates its major premise by predicting discontinuities in the displacement. For this reason we now abandon the use of Green and Lindsay theory and adopt the nonlinear theory we proposed, that takes the theory of Lord and Shulman as a special case, to examine influences of the material nonlinearity.

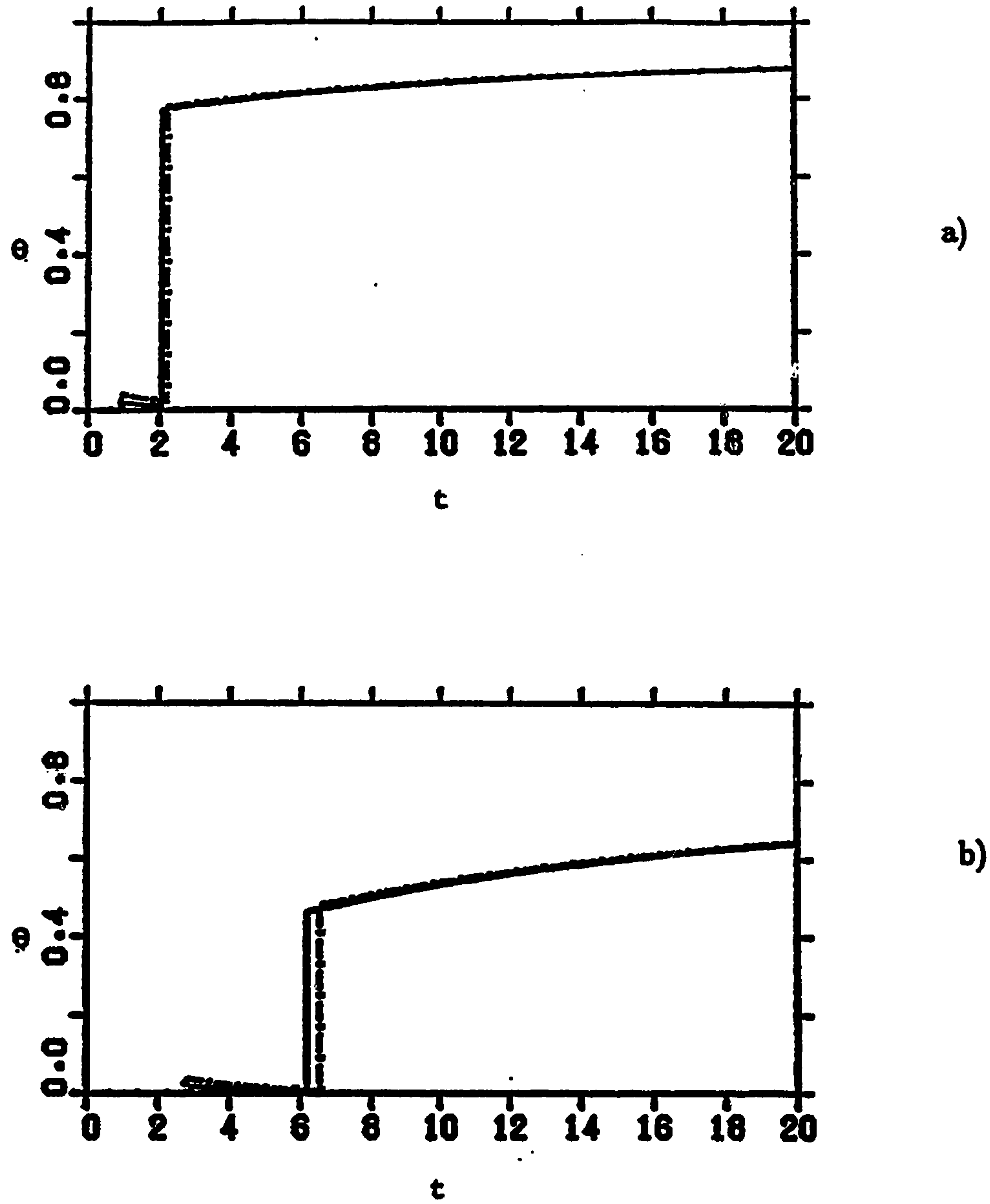


Fig. 6.1. Variation of temperature with time for the Lord and Shulman theory (—), the Green and Lindsay theory with $\beta = 1$ (---) and the Green and Lindsay theory with $\beta = 3$ (- · -): a) at $x = 1$, b) at $x = 3$.

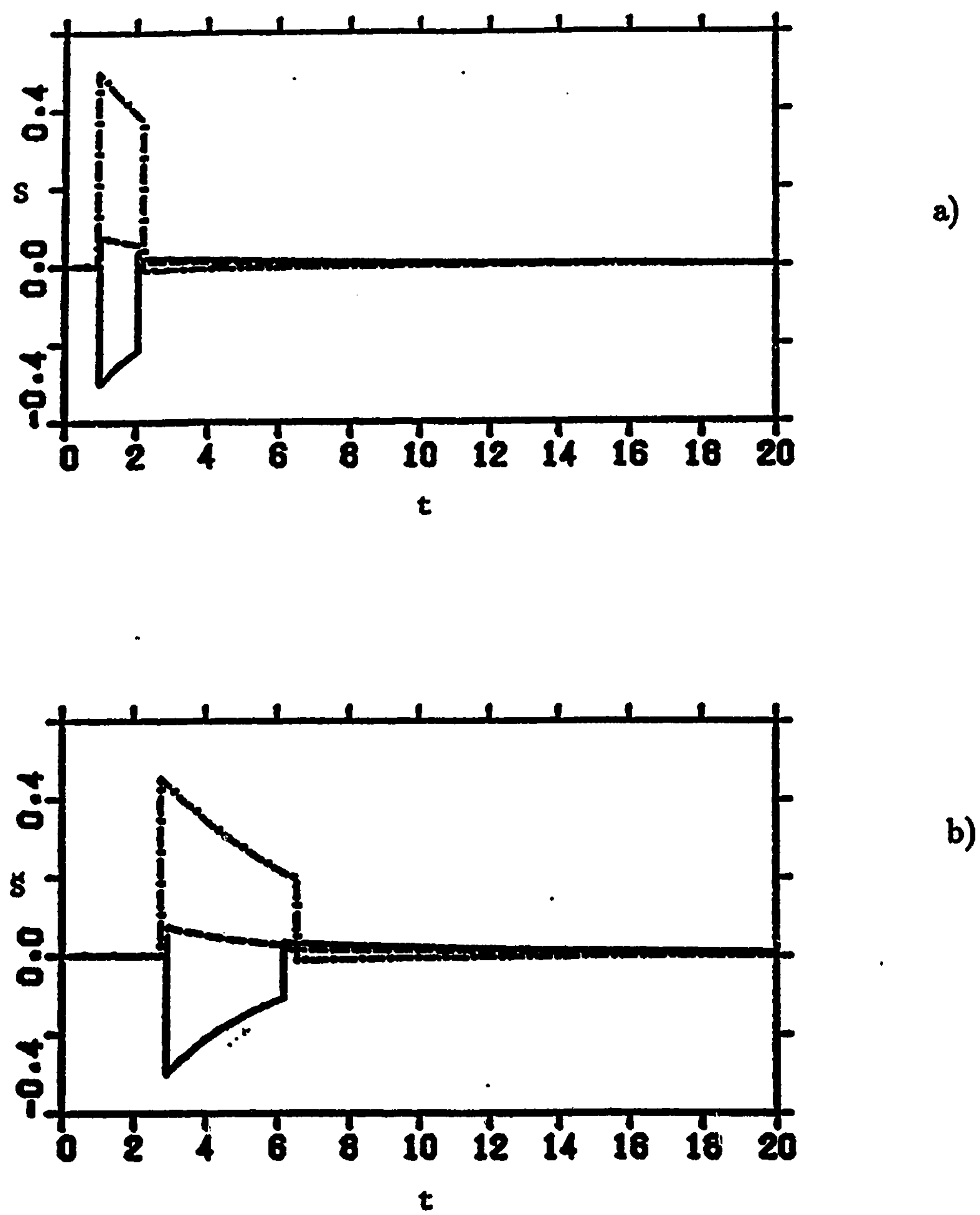


Fig. 6.2. Variation of stress with time
for the Lord and Shulman theory (—),
the Green and Lindsay theory with $\beta = 1$ (-–-)
and the Green and Lindsay theory with $\beta = 3$ (– · –):
a) at $x = 1$, b) at $x = 3$.

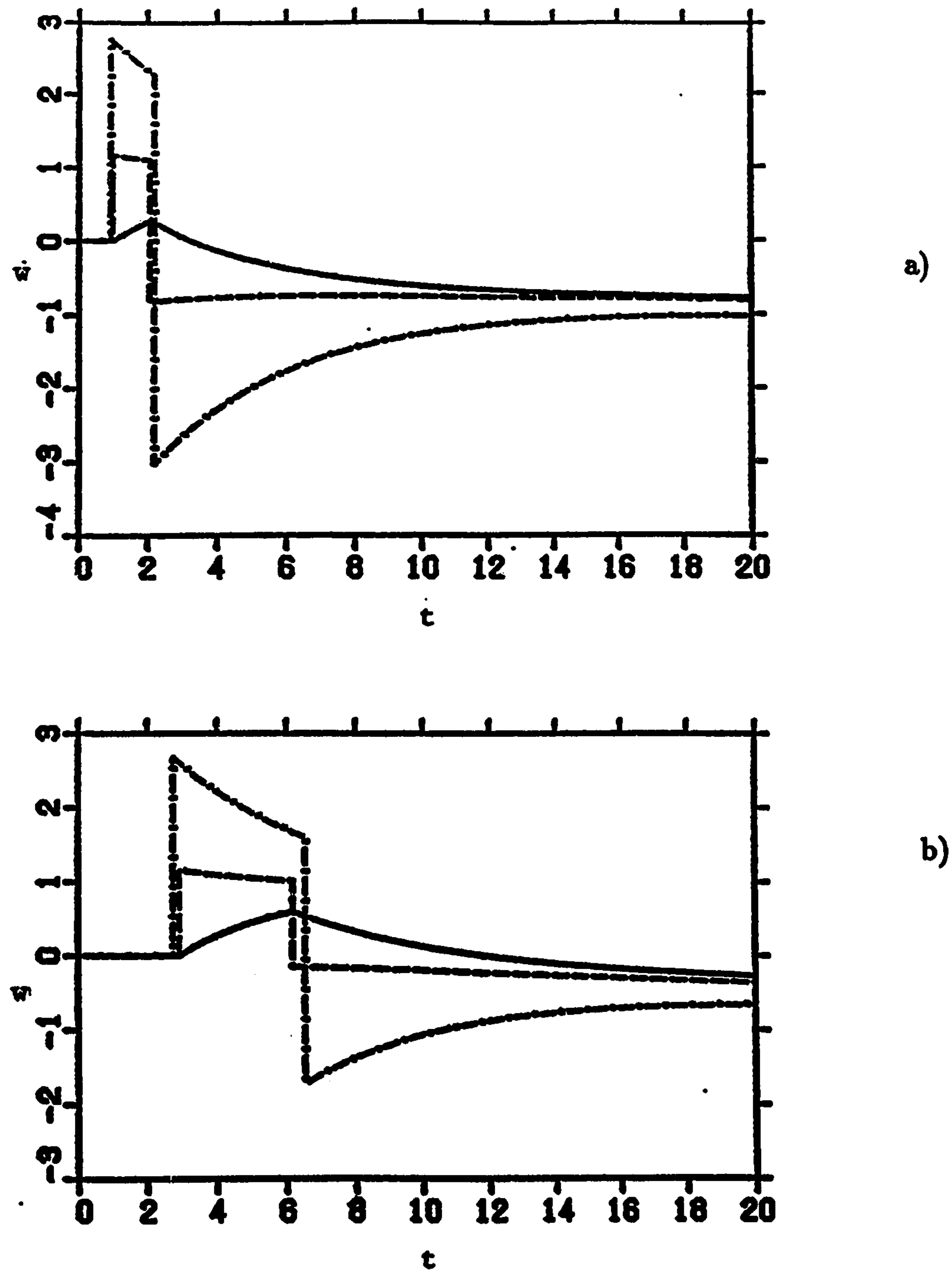


Fig. 6.3. Variation of displacement with time for the Lord and Shulman theory (—), the Green and Lindsay theory with $\beta = 1$ (-- --) and the Green and Lindsay theory with $\beta = 3$ (– · –): a) at $x = 1$, b) at $x = 3$.

CHAPTER 7

ASYMPTOTIC ANALYSIS

OF THE NONLINEAR THERMOELASTIC PROBLEM

We begin our analysis of the nonlinear problem by writing down the dimensionless forms of the balance laws obtained from (5.3), (5.4) and (5.34):

$$\frac{\partial v}{\partial t} - C_p^2 \frac{\partial S}{\partial x} = 0, \tag{7.1}$$

$$\frac{\partial e}{\partial t} - \delta S \frac{\partial v}{\partial x} + \frac{\partial q}{\partial x} = 0. \tag{7.2}$$

After nondimensionalization the compatibility equation (5.2) takes the form

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \tag{7.3}$$

whereas the second order constitutive equations are given by

$$\frac{\partial q}{\partial t} + C_t^2(1 - \zeta_1\Theta)\frac{\partial \Theta}{\partial x} + C_t^2(1 - \zeta_2\Theta)q = 0, \tag{7.4}$$

$$S = u - \Theta + \tfrac{1}{2}\,\nu_1 u^2 + \nu_2 u\Theta - \tfrac{1}{2}\,\nu_3\Theta^2, \tag{7.5}$$

$$e = \delta u + \Theta + \tfrac{1}{2}(1 - \nu_2)\delta u^2 + \nu_3\delta u\Theta + \tfrac{1}{2}\nu_4\Theta^2 + \frac{1}{C_t^2}(1 - \tfrac{1}{2}\zeta_1)q^2. \tag{7.6}$$

We will study the problem defined by the equations (7.1) to (7.6) subject to the initial conditions

$$u(x,t) = \Theta(x,t) = v(x,t) = q(x,t) = 0, \quad x > 0, \quad t \le 0, \tag{7.7}$$

and the boundary conditions

$$\Theta(0,t) = \varepsilon\Theta_0(t/\varepsilon), \quad u(0,t) = \varepsilon u_0(t/\varepsilon), \tag{7.8}$$

where $0 < \varepsilon << 1$ whereas Θ_0, u_0 vanish for all $t \leq 0$, and on $(-\infty,\infty)$ are bounded, integrable and smooth functions with integrable derivatives.

This chapter is organized as follows. In the first section some results of Majda and Rosales [40] and Majda and Artola [48] concerning the derivation of geometric optics solutions for a fairly general signalling problem in the quarter-space $x > 0$ and $t > 0$ are summarized. In the second section these results are specialized to the problem above, and in the next section, the influence of the material nonlinearity on the propagation of second sound is analyzed. Finally, in the last section, numerical results based on the nonlinear geometric optics solution are presented for a particular choice of the material parameters.

7.1. Nonlinear Geometric Optics for Hyperbolic Mixed Problems

Consider the strictly hyperbolic system

$$\boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_{,t} + \boldsymbol{B}(\boldsymbol{u})\boldsymbol{u}_{,x} = \boldsymbol{d}(\boldsymbol{u}), \quad x > 0, \quad t > 0, \tag{7.9}$$

where $\boldsymbol{u}$ is an n-dimensional vector function of x and t whereas the $n \times n$ matrices $\boldsymbol{A},\boldsymbol{B}$ and the n-dimensional vector $\boldsymbol{d}$ are sufficiently smooth functions of $\boldsymbol{u}$ in a domain $\mathcal{U} \subset \mathrm{I\!R}^n$. Further, $\det \boldsymbol{A} \neq 0$ for all $\boldsymbol{u} \in \mathcal{U}$.

Since (7.9) is strictly hyperbolic in $\mathcal{U}$, for each $\boldsymbol{u} \in \mathcal{U}$ the generalized eigenvalue problem

$$(\boldsymbol{B} - \hat{\lambda} \boldsymbol{A})\hat{\boldsymbol{r}} = \boldsymbol{0} \quad \text{and} \quad \hat{\boldsymbol{\ell}}(\boldsymbol{B} - \hat{\lambda} \boldsymbol{A}) = \boldsymbol{0}, \tag{7.10}$$

has n real and distinct eigenvalues $\{\hat{\lambda}_i(\boldsymbol{u})\}_{i=1}^n$ with the corresponding sets of linearly independent right eigenvectors $\{\hat{\boldsymbol{r}}_i(\boldsymbol{u})\}_{i=1}^n$ and left eigenvectors $\{\hat{\boldsymbol{\ell}}_i(\boldsymbol{u})\}_{i=1}^n$. The integral curves of the ordinary differential equations

$$\frac{dx}{dt} = \hat{\lambda}_i(\boldsymbol{u}), \quad i = 1, 2, \ldots, n, \tag{7.11}$$

are the characteristics of (7.9). We assume for all $\boldsymbol{u} \in \mathcal{U}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular. Consequently, each $\hat{\lambda}_i(\boldsymbol{u})$, $i = 1, 2, \ldots, n$, is nonzero so that $x = 0$ is not a characteristic. We order $\{\hat{\lambda}_i(\boldsymbol{u})\}_{i=1}^n$ as

$$\hat{\lambda}_n < \cdots < \hat{\lambda}_{m+1} < 0 < \hat{\lambda}_m < \cdots < \hat{\lambda}_2 < \hat{\lambda}_1, \tag{7.12}$$

and impose the normalizations

$$\hat{\boldsymbol{\ell}}_i \boldsymbol{A} \hat{\boldsymbol{r}}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{7.13}$$

where δ_{ij} is the Kronecker delta.

The signalling problem we are interested in consists of the system (7.9), the initial conditions

$$\boldsymbol{u}(x,t) = \boldsymbol{0}, \quad x > 0, \quad t \leq 0, \tag{7.14}$$

and the boundary conditions we shall introduce after we set the notation below. Since for $\boldsymbol{u} \in \mathcal{U}$ there are m characteristics entering into $x > 0,\ t > 0$ at the boundary $x = 0$, m boundary conditions must be specified; see, for example, Courant and Hilbert [49].

For any constant n-dimensional vector $\boldsymbol{v}$ we assume that

$$\boldsymbol{A}(\boldsymbol{u})\,\boldsymbol{v} = \boldsymbol{A}_0\boldsymbol{v} + \boldsymbol{A}_1(\boldsymbol{u},\boldsymbol{v}) + o(|\boldsymbol{u}|), \tag{7.15}$$

$$\boldsymbol{B}(\boldsymbol{u})\,\boldsymbol{v} = \boldsymbol{B}_0\boldsymbol{v} + \boldsymbol{B}_1(\boldsymbol{u},\boldsymbol{v}) + o(|\boldsymbol{u}|), \tag{7.16}$$

where $\boldsymbol{A}_0 = \boldsymbol{A}(0)$, $\boldsymbol{B}_0 = \boldsymbol{B}(0)$ whereas the bilinear forms $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are

$$\boldsymbol{A}_1 = \operatorname{grad}_{\boldsymbol{u}} \boldsymbol{A}(\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{0}}, \quad \boldsymbol{B}_1 = \operatorname{grad}_{\boldsymbol{u}} \boldsymbol{B}(\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{0}}. \tag{7.17}$$

To ensure $\boldsymbol{u} = \boldsymbol{0}$ is a solution of (7.9) we set $\boldsymbol{d}(\boldsymbol{0}) = \boldsymbol{0}$ and suppose

$$\boldsymbol{d}(\boldsymbol{u}) = \boldsymbol{D}\,\boldsymbol{u} + o(|\boldsymbol{u}|), \tag{7.18}$$

where $\boldsymbol{D} = \operatorname{grad}_{\boldsymbol{u}}\boldsymbol{d}(\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{0}}$. Finally, we define

$$\lambda_i = \widehat{\lambda}_i(\boldsymbol{0}), \quad \boldsymbol{r}_i = \widehat{\boldsymbol{r}}_i(\boldsymbol{0}), \quad \boldsymbol{\ell}_i = \widehat{\boldsymbol{\ell}}_i(\boldsymbol{0}), \quad i = 1, 2, \ldots, n \tag{7.19}$$

and note that $\boldsymbol{\ell}_i\,\boldsymbol{A}_0\,\boldsymbol{r}_j = \delta_{ij}$.

We are now ready to introduce the boundary conditions. Let $\boldsymbol{\mathcal{M}}$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. The boundary conditions considered by Majda and Artola [48] are

$$\boldsymbol{\mathcal{M}}(\boldsymbol{u}) = \varepsilon\,\boldsymbol{\mathcal{A}}(t, \frac{w_0 t}{\varepsilon}), \quad x = 0, \quad t > 0 \tag{7.20}$$

where w_0 is a constant and $\mathbf{A}\,(t,\varphi)$ is a prescribed bounded smooth m-dimensional vector function vanishing for $t \leq 0$ and having zero mean with respect to φ, meaning that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{A}\,(t,\varphi) d\varphi = 0. \tag{7.21}$$

In our summary we deal with the case

$$\mathcal{M}\,(\mathbf{u}) = \mathbf{N}\,\mathbf{u} \tag{7.22}$$

where $\mathbf{N}$ is a constant $m \times n$ matrix such that $\{\mathbf{N}\,\mathbf{r}_i\}_{i=1}^{m}$ is a set of m linearly independent vectors with the associated reciprocal vectors $\{\mathbf{R}_i\}_{i=1}^{m}$ defined as

$$\mathbf{R}_i\mathbf{N}\,\mathbf{r}_j = \delta_{ij}, \quad i,j = 1,2,\ldots,m. \tag{7.23}$$

We call any such $\mathbf{N}$ admissible for reasons which will become apparent shortly after; also see [48].

It follows from the results of [40] and [48] that

$$\tilde{\mathbf{u}}(x,t) = \varepsilon \sum_{j=1}^{n} \sigma_j(t,x,\varphi_j)\mathbf{r}_j$$

is the leading order geometric optics solution which approximates the solution of the problem (7.9), (7.14) and (7.20) within errors of order ε^2, where

$$\varphi_j = \frac{w_0 t + k_j x}{\varepsilon}, \quad w_0 = -\lambda_j k_j, \tag{7.24}$$

whereas $\sigma_j(t,x,\varphi_j)$, $j = 1,2,\ldots,n$ have zero mean with respect to φ_j and solve

$$\sigma_{j,t} + \lambda_j\sigma_{j,x} + \Gamma_j\sigma_j\sigma_{j,\varphi_j} - K_j\sigma_j + \sum_{\substack{p,q=1 \\ p\neq q \\ p\neq j\neq q}}^{n} \Gamma^j_{pq} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_p\left(\frac{1}{\lambda^p_{jq}}\varphi_j - \lambda^q_{jp}\xi\right)\sigma'_q(\lambda^p_{jq}\xi)d\xi = 0, \tag{7.25}$$

subject to the boundary conditions

$$\sigma_j\big|_{x=0} = R_j\big[A - \sum_{\ell=m+1}^{n} (N\,r_\ell)\sigma_\ell\big|_{x=0}\big], \quad j = 1,2,\ldots,m, \tag{7.26}$$

and the initial conditions

$$\sigma_j(t,x,\varphi_j) = 0, \quad x > 0, \quad t \leq 0, \quad j = 1,2,\ldots,n. \tag{7.27}$$

In the above integrals we suppressed the explicit x,t dependence of σ_p and σ_q for convenience, and, as usual, used the prime to denote the differentiation with respect to the entire argument. The constants appearing in (7.25) are the nonlinear self-interaction coefficient of the j-th wave

$$\Gamma_j = \ell_j[w_0\,A_1(r_j,r_j) + k_j\,B_1(r_j,r_j)], \tag{7.28}$$

the attenuation coefficient of the j-th wave

$$K_j = \ell_j D\,r_j, \tag{7.29}$$

the coefficient of the forcing on the j-th wave due to resonant interactions between the p-th and q-th waves

$$\Gamma^j_{pq} = \frac{1}{2}\,\boldsymbol{\ell}_j[w_0\,\boldsymbol{A}_1(\boldsymbol{r}_p,\boldsymbol{r}_q) + k_j\,\boldsymbol{B}_1(\boldsymbol{r}_p,\boldsymbol{r}_q)], \tag{7.30}$$

whereas the λ^p_{jq} are determined as

$$\lambda^p_{jq} = \frac{k_j - k_q}{k_p - k_q}\,. \tag{7.31}$$

Let us assume that each $\sigma_j(t,x,\varphi_j)$, $j = 1,2,\ldots,n$ is bounded with integrable derivatives. When this happens the integrals in (7.25) vanish, that is, there are no resonant interactions, so that only m waves are generated at $x = 0$. The leading order asymptotic solution becomes

$$\widetilde{\boldsymbol{u}}(x,t) = \varepsilon \sum_{j=1}^{m} \sigma_j(x,t,\varphi_j)\boldsymbol{r}_j, \tag{7.32}$$

where

$$\sigma_{j,t} + \lambda_i\sigma_{j,x} + \Gamma_j\sigma_j\sigma_{j,\varphi_j} - K_j\sigma_j = 0, \quad j = 1,2,\ldots,m, \tag{7.33}$$

subject to

$$\sigma_j\big|_{x=0} = \boldsymbol{R}_j\boldsymbol{A}, \tag{7.34}$$

and

$$\sigma_j(t,x,\varphi_j) = 0, \quad x > 0, \quad t \leq 0. \tag{7.35}$$

This is ensured if the boundary function $\mathcal{A}$ is further required to possess integrable derivatives.

7.2. Geometric Optics Solution

Here we intend to apply the method summarized in the previous section to our problem. To this end, we define $\boldsymbol{u} = (u, \Theta, v, q)^T$ and from (7.1) to (7.6) get

$$\boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_{,t} + \boldsymbol{B}(\boldsymbol{u})\boldsymbol{u}_{,x} = \boldsymbol{d}(\boldsymbol{u}), \quad x > 0, \quad t > 0, \tag{7.36}$$

where the only non-zero entries of the matrix $\boldsymbol{A} = (A_{ij})_{4\times4}$ are

$$\begin{aligned} A_{11} &= A_{33} = A_{44} = 1, \\ A_{22} &= 1 + \nu_3\delta u + \nu_4\Theta, \\ A_{24} &= \frac{1}{C_t^2}(2 - \zeta_1)q, \end{aligned} \tag{7.37}$$

and of the matrix $\boldsymbol{B} = (B_{ij})_{4\times4}$ are

$$\begin{aligned} B_{13} &= -B_{24} = -1, \\ B_{23} &= -\delta - (2 - \nu_2)\delta u + (1 - \nu_3)\delta\Theta, \\ B_{31} &= C_p^2 - C_p^2\nu_1 u - C_p^2\nu_2\Theta, \\ B_{32} &= C_p^2 - C_p^2\nu_2 u + C_p^2\nu_3\Theta, \\ B_{42} &= C_t^2 - C_t^2\zeta_1\Theta, \end{aligned} \tag{7.38}$$

whereas the only non-zero entry of the vector $\boldsymbol{d} = (d_i)_4$ is

$$d_4 = -C_t^2 q + C_t^2 \zeta_2 \Theta q. \tag{7.39}$$

We have

$$\boldsymbol{u}(x,t) = \boldsymbol{0}, \quad x > 0, \quad t \leq 0, \tag{7.40}$$

and

$$\boldsymbol{N}\,\boldsymbol{u}(0,t) = \varepsilon \mathcal{A}(t/\varepsilon), \tag{7.41}$$

where $\mathcal{A}(t/\varepsilon) = \left(u_0(t/\varepsilon), \theta_0(t/\varepsilon)\right)^T$ whereas

$$\boldsymbol{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{7.42}$$

From (7.37) and (7.38)

$$\boldsymbol{A}_0 = \operatorname{diag}(1,1,1,1), \tag{7.43}$$

and

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\delta & 1 \\ -C_p^2 & C_p^2 & 0 & 0 \\ 0 & C_t^2 & 0 & 0 \end{bmatrix}, \tag{7.44}$$

where $\boldsymbol{A}_0 = \boldsymbol{A}(\boldsymbol{0})$ and $\boldsymbol{B}_0 = \boldsymbol{B}(\boldsymbol{0})$. Thus at $\boldsymbol{u} = \boldsymbol{0}$ the generalized eigenvalue problem

$$(\boldsymbol{B} - \widehat{\lambda}\,\boldsymbol{A})\widehat{\boldsymbol{r}} = \boldsymbol{0} \quad \text{and} \quad \widehat{\boldsymbol{\ell}}\,(\boldsymbol{B} - \widehat{\lambda}\,\boldsymbol{A}) = 0$$

has the real and distinct eigenvalues $\lambda_i = \widehat{\lambda}_i(\mathbf{0}), \quad i = 1,2,3,4$ where

$$\lambda_1 = -\lambda_4 = C_1, \qquad \lambda_2 = -\lambda_3 = C_2, \tag{7.45}$$

and C_1, C_2 are given by (6.20) and (6.21). Since, as functions of $\boldsymbol{u}$, $\boldsymbol{A},\boldsymbol{B}$ and hence $\widehat{\lambda}_i$, $i = 1,2,3,4$ are smooth, there is a neighbourhood $\mathcal{U} \subset \mathbb{R}^4$ of $\boldsymbol{u} = \mathbf{0}$ in which (7.36) is strictly hyperbolic.

However, (7.36) may change type. For example for all $\boldsymbol{u} \in \mathcal{V}$, where

$$\mathcal{V} = \{\boldsymbol{u} \in \mathbb{R}^4 | 1 + \nu_3 \delta u + \nu_4 \Theta = 0\}, \tag{7.46}$$

det $\boldsymbol{A}(\boldsymbol{u}) = 0$ so that $\widehat{\lambda}(\boldsymbol{u}) = \infty$ is an eigenvalue. Then for any $\boldsymbol{u} \in \mathcal{V}$ the lines $t =$ const. are characteristics and consequently (7.36) is parabolic. On the other hand, when $\boldsymbol{u} \in \mathcal{V}$ at least one of u and Θ must be of $O(1)$, in which case the second order constitutive equations are not valid. As we are restricted to solutions remaining close to $\boldsymbol{u} = \mathbf{0}$, we limit our attention to $\boldsymbol{u} \in \mathcal{U}$ ensuring hyperbolicity of the problem.

After some algebra we find that the right and left eigenvectors corresponding to $\lambda_i = C_i$, $i = 1,2$, subject to $\boldsymbol{\ell}_i \boldsymbol{A}_0 \boldsymbol{r}_j = \delta_{ij}$, are

$$\boldsymbol{r}_i = \frac{1}{C_i}\left[C_i C_p^2, C_i(C_p^2 - C_i^2), -C_i^2 C_p^2, C_i^2(C_p^2 - C_i^2)\right]^T, \tag{7.47}$$

$$\boldsymbol{\ell}_i = \frac{1}{2C_i^2 C_p^2 (C_{3-i}^2 - C_i^2)}\left[C_p^2(C_i^2 - C_i^2), C_i C_p^2, C_i(C_i^2 - C_i^2), C_i C_p^2\right], \tag{7.48}$$

respectively. Further

$$\boldsymbol{N}\boldsymbol{r}_i = [C_p^2, C_p^2 - C_i^2]^T, \tag{7.49}$$

so that N is admissible since Nr_1 and Nr_2 are linearly independent. The reciprocal vectors determined from $R_i N r_j = \delta_{ij}$ are

$$R_i = \frac{1}{C_p^2(C_{3-i}^2 - C_i^2)}\,[C_{3-i}^2 - C_p^2, C_p^2], \quad i = 1, 2. \tag{7.50}$$

To specialize the results summarized in the previous section to our problem, let us observe from (7.8) and (7.41) that the boundary vector $A(t/\varepsilon)$ is smooth bounded integrable with integrable derivative on $(-\infty, \infty)$. Therefore, there are no resonant interactions and so only two waves are generated at the boundary $x = 0$. Since A does not depend on t independently it follows from the theory in Section 7.1 that the leading order geometric optics solution of the problem (7.36), (7.40) and (7.41) takes the form

$$\tilde{u}(x,t) = \varepsilon \sum_{i=1}^{2} \sigma_i(x, \varphi_i) r_i, \tag{7.51}$$

where

$$\varphi_i = \frac{t - x/C_i}{\varepsilon}, \tag{7.52}$$

and

$$\sigma_{i,x} + \frac{\Gamma_i}{C_i}\,\sigma_i \sigma_{i,\varphi_i} - \frac{K_i}{C_i}\,\sigma_i = 0, \quad i = 1, 2, \tag{7.53}$$

subject to

$$\sigma_i(0, \varphi) = f_i(\varphi), \quad \varphi > 0, \tag{7.54}$$

and

$$\sigma_i(x,\varphi) = 0, \quad x > 0, \quad \varphi \leq 0. \tag{7.55}$$

In the above

$$f_i(\varphi) = \frac{(C_{3-i}^2 - C_p^2)u_0(\varphi) + C_p^2\Theta_0(\varphi)}{C_p^2(C_{3-i}^2 - C_i^2)}, \quad i = 1,2, \tag{7.56}$$

$K_i = -C_iW_i$, where W_i, $i = 1,2$ are given by (6.34) and (6.35) while $\Gamma_i = C_iG_i$, $i = 1,2$ where

$$G_i = \frac{2(C_p^2 - C_i^2)^2 + \delta C_p^2(C_p^2 - 3C_i^2)}{2C_i(C_{3-i}^2 - C_i^2)} - \frac{C_p^4(C_i^2 - C_i^2)}{2C_i^3(C_{3-i}^2 - C_i^2)}\nu_1$$
$$- \frac{\delta C_p^4}{2C_i(C_{3-i}^2 - C_i^2)}\nu_2 + \frac{\delta C_p^2(C_p^2 - C_i^2)}{2C_i(C_{3-i}^2 - C_i^2)}\nu_3 + \frac{(C_p^2 - C_i^2)^2}{2C_i(C_{3-i}^2 - C_i^2)}\nu_4. \tag{7.57}$$

We have completed deriving the problems determining the leading order asymptotic approximation which approximates the solution of the original problem within errors of ε^2. In the next section we shall solve these problems to analyze the implications of what has been determined so far. In passing we remark that the nonlinear self-interaction coefficients G_i, $i = 1,2$ are independent from the material parameters ζ_1 and ζ_2 so that, in the second order theory we are working with, no knowledge of these parameters is necessary. However, this does not mean that the nonlinear dependence of the internal energy on the heat flux has no influence, for the first term in the right-hand side of (7.57) involves contributions from the heat flux in the constitutive relation (7.6).

7.3. Analysis of the Results

We concentrate our attention on the strain u and the temperature difference Θ only. The velocity v and the heat flux q behave in ways similar to those for the strain u and the temperature difference Θ, respectively.

It follows from the previous section that as $\varepsilon \to 0$, u and Θ are given by

$$u(x,t) = \varepsilon C_p^2 \sigma_1(x,\varphi_1) + \varepsilon C_p^2 \sigma_2(x,\varphi_2) + O(\varepsilon^2), \tag{7.58}$$

$$\Theta(x,t) = \varepsilon(C_p^2 - C_1^2)\sigma_1(x,\varphi_1) + \varepsilon(C_p^2 - C_2^2)\sigma_2(x,\varphi_2) + O(\varepsilon^2), \tag{7.59}$$

where

$$\varphi_i = \frac{t - x/C_i}{\varepsilon}, \quad i = 1,2, \tag{7.60}$$

whereas $\sigma_i(x,\varphi_i)$, $i = 1,2$ solve

$$\sigma_{i,x} + G_i \sigma_i \sigma_{i,\varphi_i} + W_i \sigma_i = 0, \tag{7.61}$$

subject to (7.54) and (7.55). Since $C_p > C_t$ and, therefore, as $\delta \to 0$, $C_1 \to C_p$ and $C_2 \to C_t$, the fast wave is quasi-elastic and the slow wave is quasi-thermal. This was discussed in our analysis of the linear problem in the previous chapter.

To determine whether these waves break or not, let us solve the problems given by (7.61), (7.54), (7.55) and (7.56). The solution is

$$\sigma_i(x,\varphi_i) = f_i(\xi_i)e^{-W_i x}, \quad i = 1,2, \tag{7.62}$$

where

$$\varphi_i = \frac{t - x/C_i}{\varepsilon} = \xi_i - \frac{G_i}{W_i} f_i(\xi_i)(e^{-W_i x} - 1). \tag{7.63}$$

In view of the discussion given in Section 3.2, we then conclude that the i-th wave breaks if the condition

$$\varphi_{i,\xi_i} = 1 - \frac{G_i}{W_i} f_i'(\xi_i)(e^{-W_i x} - 1) \neq 0, \tag{7.64}$$

is violated. As long as

$$-\frac{G_i}{W_i} f'(\xi_i) \leq 1, \quad i = 1,2 \tag{7.65}$$

neither of the waves break, since (7.64) holds. Otherwise, the i-th wave breaks at the approximate position $\tilde{x}_i$ given by

$$\tilde{x}_i = \min \left\{x > 0 \mid 1 - \frac{G_i}{W_i} f_i'(\xi_i)(e^{-W_i x} - 1) = 0\right\}, \quad i = 1,2, \tag{7.66}$$

from which the corresponding $\tilde{\xi}_i$ is determined as well. With $\tilde{x}_i$ and $\tilde{\xi}_i$ at hand, (7.63) then gives the approximate breaking time $\tilde{t}_i$. Consequently, the asymptotic solution ceases to be valid at the distance $\tilde{x}_s$ and the time $\tilde{t}_s$ where

$$\tilde{x}_s = \min\{\tilde{x}_1, \tilde{x}_2\}, \quad \tilde{t}_s = \min\{\tilde{t}_1, \tilde{t}_2\}. \tag{7.67}$$

Provided the material constants are determined and the disturbances are specified, valuable approximate quantitative information can be obtained from the above asymptotic solution with no essential difficulty. We shall demonstrate this for a physically reasonable set of material constants in the last section. In the rest of this section, however, we shall investigate what other information can be extracted from it without specifying the constants.

Let us look at the asymptotic solutions (7.58) and (7.59) more closely. Remember now that as $\delta \to 0$ we have $C_1 \to C_p$ and $C_2 \to C_t$. It then follows immediately that as $\delta \to 0$

$$u(x,t) \to \varepsilon C_p^2 \sigma_1\left(x, \frac{t - x/C_p}{\varepsilon}\right) + \varepsilon C_p^2 \sigma_2\left(x, \frac{t - x/C_t}{\varepsilon}\right) + O(\varepsilon^2), \tag{7.68}$$

$$\Theta(x,t) \to \varepsilon (C_p^2 - C_t^2)\sigma_2\left(x, \frac{t - x/C_t}{\varepsilon}\right) + O(\varepsilon^2). \tag{7.69}$$

The last of these tells us what we have displayed already in the first figure of the previous chapter: the main part of the temperature wave is carried by its quasi-thermal component, that is, by second sound. This information is not novel. To obtain novel information one should take the boundary disturbances also into consideration.

It will prove useful to rewrite the boundary disturbances given in (7.54) and (7.56) as

$$\sigma_1(0,\varphi) = \frac{(C_2^2 - C_p^2)u_0(\varphi) + C_p^2\Theta_0(\varphi)}{C_p^2(C_2^2 - C_1^2)}, \tag{7.70}$$

$$\sigma_2(0,\varphi) = \frac{(C_1^2 - C_p^2)u_0(\varphi) + C_p^2\Theta_0(\varphi)}{C_p^2(C_1^2 - C_2^2)}. \tag{7.71}$$

Suppose the boundary temperature is kept constant and the wave motion is set up by disturbing the strain. Since $\Theta_0 \equiv 0$, letting $\delta \to 0$ in (7.70) and (7.71) gives

$$\sigma_1(0,\varphi) \to \frac{u_0(\varphi)}{C_p^2}, \quad \sigma_2(0,\varphi) \to 0. \tag{7.72}$$

This shows that for such disturbances second sound is practically unobservable. Furthermore, (7.72), together with (7.68) and (7.69), shows also that

$$u(x,t) \to \varepsilon C_p^2 \sigma_1\big(x, \frac{t - x/C_p}{\varepsilon}\big) + O(\varepsilon^2), \quad \Theta(x,t) \to O(\varepsilon^2), \tag{7.73}$$

so that the entire wave motion is practically isothermal. In fact, in view of the apparent success of the theory of elasticity, this information is not so novel either, since the theory of elasticity relies on neglecting the thermal effects entirely when the disturbances are mechanical. Because of this we disregard purely mechanical disturbances until further notice and suppose the boundary temperature is perturbed.

Since it is supposed that $\Theta_0 \not\equiv 0$, letting $\delta \to 0$ in (7.70) and (7.71) gives

$$\sigma_1(0,\varphi) \to \frac{u_0(\varphi)}{C_p^2} + \frac{\Theta_0(\varphi)}{(C_t^2 - C_p^2)}, \quad \sigma_2(0,\varphi) \to \frac{\Theta_0(\varphi)}{(C_p^2 - C_t^2)}. \tag{7.74}$$

From the second of (7.74) we see once again that second sound is essentially unaffected by mechanical disturbances. On the other hand, the first of (7.74) indicates that a similar conclusion for first sound is not correct: the influence of thermal disturbances on first sound is unavoidable regardless of the strength of coupling. Prior to further discussion of this observation, we let $\delta \to 0$ also in (6.34) and (6.35) to get

$$W_1 \to 0, \quad W_2 \to \frac{C_t}{2}, \tag{7.75}$$

and recall that W_1 and W_2 are the attenuation coefficients of first sound and second sound, respectively. Therefore, although second sound is dissipated, first sound is practically not.

Let us put all of these observations together. In a purely mechanical experiment, in which thermal effects can be neglected entirely, strain measurements would give sufficient information concerning first sound. Consider now a heat-pulse experiment (see [20-26]) in which only the temperature is measured. In such a measurement the quasi-elastic component of the temperature wave could go unnoticed since the main part of this wave is carried by second sound.

Therefore, in such cases and as long as one is interested in temperature measurements only, it appears reasonable to neglect deformations entirely. In fact, this last claim does not contradict the previous conclusion that deformations are unavoidable, since neglecting them does not mean denying them. However, such a neglect leads to loss of valuable information that otherwise could have been obtained from strain measurements. A scrutiny of (7.68) and (7.74) shows that even the quasi-elastic component of the strain wave, that is, first sound, contains information regarding the speed of second sound. Indeed, it is the quasi-elastic component of the strain wave which is not dissipated and, therefore, which can be observed more conveniently. Based on these arguments we reach the following conclusion: taking deformations into consideration is worth the effort even when the coupling is small. Furthermore, there are many cases when the coupling is not so small so that both components of the temperature wave are observable (see [23-25]).

Lastly, we will deal with the nonlinear self-interaction coefficients G_i, $i = 1,2$ to investigate the influence of material nonlinearity. As before, we let $\delta \to 0$ and obtain from (7.57) the following result:

$$G_1 \to -\frac{C_p}{2}\,\nu_1, \quad G_2 \to \frac{C_p^2 - C_t^2}{C_t} + \frac{C_p^2 - C_t^2}{2C_t}\,\nu_4. \qquad (7.76)$$

This result hardly comes as a shock after the above observations. The nonlinearity of first sound comes essentially from the quadratic dependence of the stress on the strain. Likewise, the nonlinearity of second sound derives mainly from the quadratic dependence of the internal energy on the temperature and the heat flux. The cross terms in the temperature and the strain do not play significant roles in the nonlinear behavior of either of the waves.

7.4. A Numerical Example

We present our example calculations for the boundary disturbances

$$\Theta_0(\varphi) = \begin{cases} 2\varphi^2 \exp(1 - 4\varphi^2), & \varphi > 0, \\ 0, & \varphi \leq 0, \end{cases} \tag{7.77}$$

and

$$u_0(\varphi) = 0, \quad -\infty < \varphi < \infty. \tag{7.78}$$

This choice is motivated by the previously mentioned heat pulse experiments. The function in (7.77) describes a pulse of short duration that smoothly increases from zero to its peak value 0.5 at $\varphi = 0.5$ and thereof decreases asymptotically to zero as φ tends to infinity. It takes the value 0.1 at $\varphi = 1$ and thereafter rapidly becomes practically unobservable. Let us now fix $\varepsilon = 0.1$ and recall that at $x = 0$, $\varphi = t/\varepsilon$. Therefore, the duration of the above pulse is slightly longer than 0.1 nondimensional time units.

We now turn our attention to the material parameters and specify the first order constants as

$$C_p = 1, \quad C_t = 0.5, \quad \delta = 0.05. \tag{7.79}$$

The above choice is roughly in accord with the experimental data obtained for NaF at temperature of about $12°K$.

Next, we set

$$\nu_1 = 0, \quad \nu_2 = 1, \quad \nu_3 = 0. \tag{7.80}$$

This is solely a matter of convenience. The first two of the above offset the quadratic influence of the strain on the stress and internal energy, respectively, whereas the last offsets the quadratic dependence of the stress on the temperature. Since as $\delta \to 0$, $G_1 \to -C_p\nu_1/2$ and $\nu_1 = 0$, for us first sound is essentially linear so that we can concentrate on the nonlinear behaviour of second sound. No deep physical significance should be attached to the above values.

Lastly, we choose

$$\nu_4 = 3. \tag{7.81}$$

This choice is motivated by experimental observations. Hardy and Jaswal [22] determined that for NaF, the specific heat at constant strain at low temperatures is a cubic function of the absolute temperature. This indicates the above value for ν_4.

With the above disturbances and material parameters we find from (7.64), (7.65) and (7.66) that first sound remains smooth at least until second sound breaks at the approximate place $\widetilde{x}_s = 0.2213$ and the approximate time $\widetilde{t}_s = 0.5619$, which occurs at $\xi_2 = 0.7551$. Since the peak value of the pulse is attained at $\xi_2 = 0.5$, this indicates that second sound, generated by the disturbances (7.77) and (7.78), breaks backwards behind the peak value. Indeed, as long as $G_2 > 0$, (7.64), (7.65) and (7.66) show that second sound never breaks at any ξ_2 for which $f_2(\xi_2)$ is increasing. To put it differently, large values of second sound travel more slowly than smaller values of it. This result is in accord with the previously obtained result of Coleman, Fabrizio and Owen [31] concerning the purely thermal second sound: a thermal wave propagating in the direction of heat flux travels more slowly than one propagating in the opposite direction. That the above results are in agreement can be inferred easily from (7.64), (7.65) and (7.66) by observing that

$$q(x,t) = \varepsilon\,\frac{C_t^2}{C_1}(C_p^2 - C_1^2)\sigma_1(x,\varphi_1) + \varepsilon\frac{C_t^2}{C_2}(C_p^2 - C_2^2)\sigma_2(x,\varphi_2) + O(\varepsilon^2), \qquad (7.82)$$

where φ_i, $i = 1,2$ are given by (7.60), and comparing it with (7.59).

In Figs. 7.1 and 7.2 we depicted the geometric optics solutions for the temperature and strain. In these figures the linear geometric optics solution, obtained from the nonlinear solution by setting $G_1 = G_2 = 0$ and indicated by the broken curves, are also displayed for purposes of comparison. The already discussed qualitative behaviour of first and second sound waves are clearly seen from these figures. The qualitative resemblance between our Fig. 7.1.a. and Fig. 7.C. of Jackson and Walker [23] displaying the oscilloscope trace of the temperature detected at 12.7K in a high-purity NaF crystal of length 7.3 mm is interesting. However, it is unreasonable to reach a quick conclusion concerning the physical validity of our theory on the basis of this resemblance. To reach such a conclusion, or otherwise, more careful examinations of the material parameters and boundary disturbances should be performed and a larger number of comparisons with the experiments should be made.

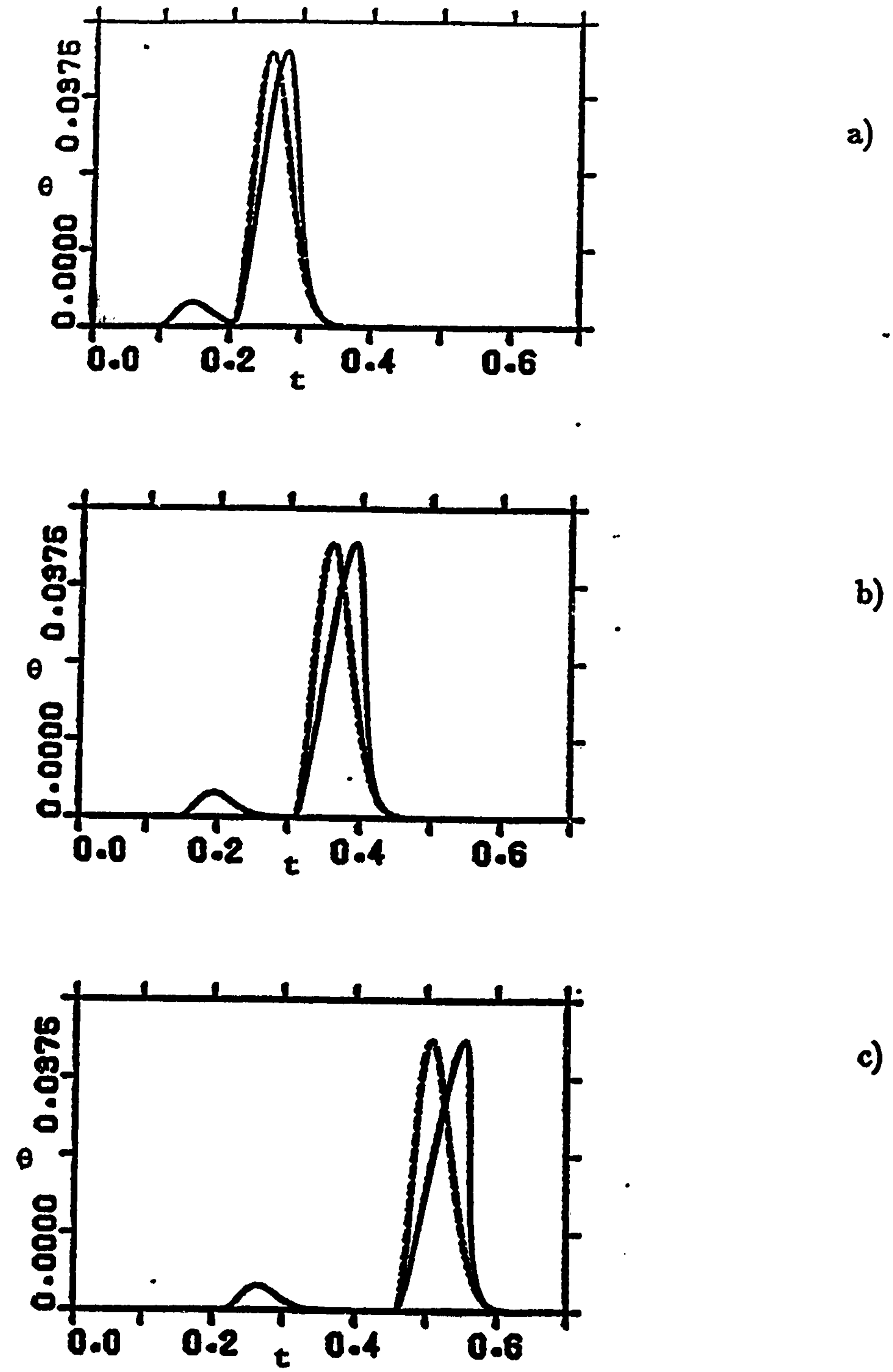
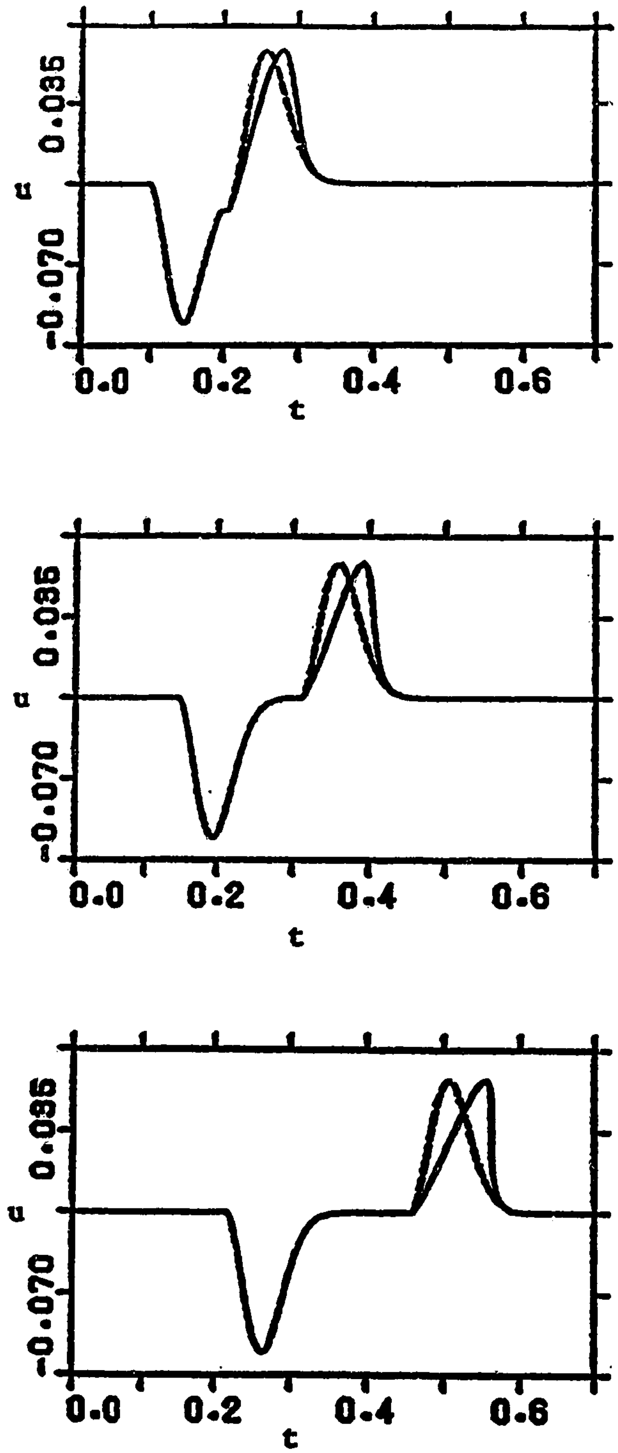


Fig. 7.1. Variation of temperature with time:
a) at $x = 0.1$, b) at $x = 0.15$, c) at $x = 0.2213$.



a)

b)

c)

Fig. 7.2. Variation of strain with time:
a) at $x = 0.1$, b) at $x = 0.15$, c) at $x = 0.2213$.

CHAPTER 8

SUMMARY AND CONCLUSIONS

We studied the one-dimensional propagation of boundary generated, small-amplitude, high-frequency waves in two semi-infinite solids. The leitmotif of our study has been geometric optics. We constructed asymptotic geometric optics solutions for our problems and demonstrated that valuable approximate quantitative information can be obtained from these solutions with no essential difficulty.

In the first part of our study we considered a viscoelastic solid. We limited our attention to that viscoelastic solid for which the present value of stress is given by a single-integral functional of the history of strain. We assumed that this functional obeys certain physically reasonable conditions ensuring that the solid exhibits instantaneous elasticity and that the resulting system of Volterra integrodifferential equations *may be regarded* as *hyperbolic*. We showed that although the strict convexity (or concavity) of the instantaneous elastic response function of the solid is sufficient for a possible break-down of the solutions originating from smooth data, it is not a necessary requirement. It should be mentioned that whether such a break-down occurs or not depends also on the strength of the dissipative mechanism introduced by the memory of the material. Our asymptotic results imply that even small amplitude solutions originating from smooth data, if they are sufficiently high frequency, may develop singularities in finite time as long as the instantaneous elastic response function is *not* linear. Such is the situation even when the instantaneous elastic response function possesses inflection points, in which case it is not strictly convex.

The second part of our study was devoted to examining the one-dimensional propagation of finite speed thermal waves in an elastic heat conductor. There are two notions related with such waves. The first is second sound, which was observed for the first time in superfluid liquid helium. The second is the paradox of instantaneous propagation of thermal disturbances predicted by Fourier's theory. There is no doubt that theories predicting finite speed for thermal disturbances provide satisfactory solutions to eliminate the paradox. From a practical point of view, however, this paradox does not pose a severe problem since for most materials "huge speeds and rapid relaxation restore diffusion even on the scale of the response time of modern oscilloscopes (Joseph and Preziosi [16])." On the other hand, second sound has been detected not only in superfluid liquid helium [14] but also in the dielectrics He-4 [20], He-3 [21], NaF [22-25] and in Bi [26] at low temperatures. Secondly, the recent discovery of high temperature superconductors, which can operate above the nitrogen boiling temperature, has revived some interest in second sound experiments in superconductors in the last three years or so. As pointed out by Peskhov [14], superconductors are potential candidates in which heat might be transported by second sound. The question whether heat is transported by second sound in high temperature superconductors or not, however, appears to be as yet unanswered, although an unsuccessful recent experiment has been addressed in the survey article by Flik [50]. These have been the motivations for our study.

The above mentioned experiments in crystals indicate that the speed of second sound depends on the temperature and that, though at varying degrees, detected thermal variations are coupled with deformations. Therefore, if one adopts the point of view of continuum mechanics, then a nonlinear theory of thermoelasticity capable of predicting finite speed for thermal disturbances provides a plausible mathematical model for studying the phenomenon. This led us to extend the thermodynamically consistent theory of Coleman, Fabrizio and Owen [31], valid for rigid materials, to include elastic materials. In our extension we also provided a solution to the lack of material frame invariance their constitutive equations suffer from.

We then considered the linear equations of our theory for a one-dimensional thermoelastic solid. The linearization of our equations gives the equations of the linear theory of Lord and Shulman [28]. We compared several predictions of these equations against the related predictions of the linearized equations of the theory of Green and Lindsay [29]. We found out that predictions of the latter theory are physically discomforting.

Lastly, we turned our attention to examining the nonlinear influence of the temperature on the propagation of second sound in the nonlinear theory we developed. The heat pulses generated in the above mentioned experiments are excellent examples of small-amplitude, high-frequency boundary disturbances.

This fact rendered the asymptotic geometric optics expansion procedure a good mathematical tool for studying the phenomenon. We analyzed the predictions of our theory on the basis of these expansions. One important implication of our theory is that larger values of second sound propagate more slowly than smaller values of it. Based on a temperature-rate front analysis Coleman, Fabrizio and Owen [31] determined that according to their theory a temperature-rate wave propagating in the direction of heat flux travels more slowly than one propagating in the opposite direction. This is in agreement with the above noted implication of our theory, which is an extension of theirs.

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