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ON THE ASYMPTOTIC NORMALITY OF A LINEAR COMBINATION OF INDUCED ORDER STATISTICS WITH APPLICATION TO A CONDITIONAL QUANTILE ESTIMATOR

by

SURYA PRAKASH UPADRASTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled ON THE ASYMPTOTIC NORMALITY OF LINEAR COMBINATION OF INDUCED ORDER STATISTICS WITH APPLICATION TO A CONDITIONAL QUANTILE ESTIMATOR Submitted by SURYA PRAKASH UPADRASTA in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematical Statistics.

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Dedicated to

my parents Mr. and Mrs. R.S. UPADRASTA

and

my uncle Mr. S.N. GODAVARTI

ABSTRACT

Let (X_i, Y_i), $i=1,\ldots,n$ be independent and identically distributed random variables with a bivariate continuous distribution function $H(x,y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ be the order statistics of X - observations and let $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ denote, respectively, the Y-observations associated with the above ordered $X_{(i:n)}$'s. The $Y_{[i:n]}$'s have been referred to in the literature as "induced" or "concomitants of" order statistics. In this thesis we are concerned with the asymptotic normality of linear combinations of induced order statistics with double weights, namely, weights which depend both on the ranks of X's as well as the ranks of Y's. This work extends the results of Stigler (1974) and Yang (1981). The method employed is the "Projection method" introduced by Hajek (1968).

We consider statistics of the form

$$S_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$$

where $J(u,v)$, $0 \leq u, v \leq 1$, is a suitable bivariate score function satisfying certain regularity conditions (see Theorem 3.1.1). Our results specialize to those of Stigler (1974) if we set $J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) = J\left(\frac{R_{in}}{n}\right)$ and to those of Yang (1981) if $J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) = J\left(\frac{i}{n}\right)$. In Part I of this thesis, it is proved that, under the regularity conditions referred to above,

$$P\left(\frac{S_n - ES_n}{\sigma(S_n)} \leq t\right) \to \Phi(t), \quad -\infty < t < \infty \text{ as } n \to \infty,$$

where $\Phi(t)$ is the standard normal distribution function. The approach developed in Part I is then used in Part II to establish the asymptotic normality of a conditional quantile (Kernel) estimator proposed by Mehra (198[illegible]).

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CHAPTER I

INTRODUCTION AND SUMMARY

1.1. THE STATISTICS S_n AND $Q_{n.x_o}(\lambda)$. Let (X_i, Y_i), $i=1,\ldots,n$, be independent bivariate random variables (rv's) with common continuous bivariate distribution function $H(x,y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(n:n)}$ denote the X-variate order statistics, and $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ denote the corresponding Y-variates associated, respectively, with the above ordered $X_{(i:n)}$'s. The $Y_{[i:n]}$'s have been termed in the literature as induced order statistics or concomitants of order statistics. Asymptotic distribution theory of functions of these induced order statistics, including their appropriate linear combination have been studied by several authors (e.g., Bhattacharya (1974), David, O'Connell et al. (1977), and Yang (1977) and (1981)). Their results are useful in application to non-parametric testing and estimation problems concerning the regression function $m(x) = E(Y|X=x)$ as well as to certain Y-variate prediction and selection problems based on information contained in X-variate order statistics. In Part I of this thesis, containing Chapters II and III, we study the asymptotic normality of a linear combination of induced order statistics $Y_{[i:n]}$'s, with double weights, namely, S_n defined by (1.1.1) below, where the weights depend on the ranks of $X_{(i:n)}$'s (among the X-observations) and ranks of $Y_{[i:n]}$'s (among the Y-observations):

$$(1.1.1) \qquad S_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}$$

where R_{in} is the rank of $Y_{[i:n]}$ among the Y-observations and $J(.,.)$ is a two dimensional score function satisfying certain regularity conditions (see Theorem 3.1.1 below and also [11]). These results are useful in applications to the problems of testing and estimation of parameters of conditional distribution of Y given $X = x$. Our results extend those of Yang (1981) and Stigler (1974).

We also study in Part II of this thesis, containing chapters IV and V, the asymptotic normality of a closely related statistic $Q_{n,x_o}(\lambda)$, defined by (1.1.2) below, which is a nonparametric estimator of the λth quantile, $0 < \lambda < 1$, of the conditional distribution of Y given $X = x_o$, proposed by Mehra (1986):

$$(1.1.2) \qquad Q_{n,x_o}(\lambda) = \frac{1}{na_n^2} \sum_{i=1}^{n} K\left(\frac{G_n(X_{(i:n)})-G_n(x_o)}{a_n}, \frac{F_{n,x_o}(Y_{[i:n]})-\lambda}{a_n}\right)Y_{[i:n]}$$

$$\equiv \frac{1}{na_n^2} \sum_{i=1}^{n} K\left(\frac{G_n(X_i)-G_n(x_o)}{a_n}, \frac{F_{n,x_o}(Y_i)-\lambda}{a_n}\right)Y_i,$$

$$-\infty < x_o < \infty, \ 0 < \lambda < 1$$

where $\{a_n\}$ is a sequence of positive numbers which goes to zero as $n \to \infty$,

$$(1.1.3) \qquad G_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \le x)$$

is the marginal empirical distribution function of the X-observations,

$$(1.1.4) \qquad F_{n,x_o}(y) = \frac{1}{na_n} \sum_{j=1}^{n} K^*\left(\frac{G_n(X_j)-G_n(x_o)}{a_n}\right)I(Y_j \le y)$$

is the "conditional empirical distribution function" of Y-observations given $X = x_o$, and $K^*(.)$ and $K(.,.)$ are, respectively, a univariate and a bivariate kernel functions satisfying certain regularity conditions (see (4.1.3) to (4.1.6)). $Q_{n,x_o}(\lambda)$, defined by (1.1.2), is related to the class of statistics S_n defined by (1.1.1) in the sense that it is also a linear combination of induced order statistics with double weights, but is not covered by that class since the weights in $Q_{n,x_o}(\lambda)$ depend on the ranks of $X_{(i:n)}$ (among X's) and on the "generalized ranks" of $Y_{[i:n]}$ (among Y's) unlike in (1.1.1) where the weights depend on the ranks of $X_{(i:n)}$'s and the ranks of $Y_{[i:n]}$'s only.

The methods of proof employed in proving our main Theorems in the two cases are similar in that both are based on the "Projection Technique" introduced by Hajek (1968).

1.2 A BRIEF SUMMARY OF THE RESULTS. In Chapter II, we approximate S_n by a sum of independent and identically distributed (iid) rv's using the "Projection Technique" and prove some preliminary results which assist us in Chapter III.

Chapter III is concerned with the asymptotic normality of S_n for which we make use of the results obtained in Chapter II, and with obtaining the results in Theorem 1 of Yang (1981) and those in Theorem 2 of Stigler (1974) as special cases.

In Chapter IV, we prove the weak consistency of $Q_{n,x_o}(\lambda)$.

Chapter V deals with the asymptotic normality of $Q_{n,x_o}(\lambda)$ utilising the results obtained in Chapter IV. The results of Chapter IV and Chapter V are derived under certain regularity conditions on the sequence $\{a_n\}$, the kernel functions K^* and K, and the joint distribution $H(x,y)$, $-\infty < x, y < \infty$ of (X,Y).

1.3. NOTATION AND TERMINOLOGY. Most of the notation is introduced in the text as and when necessary. However, the following terms should be noted, as they appear more frequently in the text.

(i) C_1, C_2 etc., denote absolute positive constants, not necessarily representing the same value in each appearance.

(ii) The inverse of a function is defined to be left continuous:

$$G^{-1}(t) = \inf \{x : G(x) \geq t\}.$$

(iii) $K_1'(\theta_1,\theta_2)$, $K_1''(\theta_1,\theta_2)$ etc., (or $K_1^{(m)}(\theta_1,\theta_2)$, $m = 1,2,\ldots$) denote respectively, the first order partial derivative, second order partial derivative etc., of the function K with respect to the first argument, evaluated at (θ_1,θ_2). However the notation $K_1^{(m)}(\theta_1,\theta_2)$ is used in case of higher order derivatives only, for convenience. $K_2'(\theta_1,\theta_2)$, $K_2''(\theta_1,\theta_2)$ etc., and $K_2^{(m)}(\theta_1,\theta_2)$, are similarly defined. $K_{12}^{(m_1,m_2)}(\theta_1,\theta_2)$ denote the (m_1+m_2)-th order mixed derivative of K obtained by differentiating m_1 times with respect to the first argument and m_2 times with respect to the second argument, and evaluated at (θ_1,θ_2).

(iv) $g'(\theta)$, $g''(\theta)$ etc., (or $g^{(m)}(\theta)$, $m=1,2,\ldots$) denote the first derivative, second derivative etc., of g evaluated at θ.

(v) For any real numbers x and y, $x \wedge y$ indicates $\min(x,y)$.

(vi) The order symbols O, o, O_p and o_p are used in their standard notation.

(vii) $I(A)$ denotes the indicator function of the set A, and $\square$ signals the end of a proof. The following abbreviations are also used:

$$\sum_i \equiv \sum_{i=1}^{n} \quad \text{unless, otherwise specified,}$$

$$\int = \int_{-\infty}^{\infty} \quad \text{unless, otherwise specified.}$$

CHAPTER II

PROJECTION $\hat{S}_n'$ OF S_n AND VARIANCE OF $\hat{S}_n$

The object of Chapters II and III is to establish the asymptotic normality of statistics of the form S_n (see 2.1.1. below) which is a linear combination of induced order statistics with double weights, weights depending on the ranks of the observations. In the present Chapter namely, Chapter II, we shall prove some preliminary results concerning S_n which assist us in proving the asymptotic normality of S_n (in a properly normalized form) in Chapter III. A brief outline of the method of proof employed in proving the asymptotic normality of S_n, namely, Hajek's projection approach is given in Section 2.1. In Section 2.2 we find the projection $\hat{S}_n$ of S_n and the asymptotic variance of $\hat{S}_n$ is obtained in Section 2.3.

2.1 PRELIMINARIES. Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be iid rv's, with common continuous bivariate distribution function $H(x,y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ denote the X-variate order-statistics and $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ denote the corresponding Y-variate induced order statistics. We consider statistics of the form

$$(2.1.1) \qquad S_n = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}$$

where R_{in} is the rank of $Y_{[i:n]}$ when $Y_{[i:n]}$'s, $1 \leq i \leq n$, are arranged in ascending order of the values and $J(u,v)$, $0 \leq u, v \leq 1$ is a suitable bounded score function satisfying certain regularity conditions.

Our method of proof in proving the asymptotic normality of S_n is the same as that employed by Stigler [17] and Yang [22], namely, the use of the Projection technique introduced originally by Hajek. This method enables us to demonstrate that S_n can be suitably approximated in mean square by its projection $\hat{S}_n$ (defined below), as $n \to \infty$. $\hat{S}_n$ being a sum of independent r.v.'s with finite second moments, the above approximation and the standard central limit theorem gives the asymptotic normality of S_n. For convenience, we state Hajek's [7] projection lemma here as lemma 2.1.1.

Lemma 2.1.1. (Hajek's (1968) Projection Lemma). Let $Z_1, Z_2, \ldots, Z_n$ be independent r.v.'s and F be the Hilbert space of almost sure equivalence classes of square integrable statistics depending on $Z_1, \ldots, Z_n$. Let $\mathcal{S}$ be the closed linear subspace of F consisting of statistics of the form $L = \sum_{i=1}^{n} \ell_i(Z_i)$, where ℓ_i are functions such that $E[\ell_i^2(Z_i)] < \infty$. Then, if $S \in F$, the projection of S on $\mathcal{S}$ is given by

$$\hat{S} = \sum_{i=1}^{n} E(S|Z_i) - (n-1)E(S). \tag{2.1.2}$$

Thus $E(\hat{S}) = E(S)$ and

$$E(S-\hat{S})^2 = \sigma^2(S) - \sigma^2(\hat{S}). \tag{2.1.3}$$

2.2 PROJECTION $\hat{S}_n$ OF S_n. Following Lemma 2.1.1 we define the projection $\hat{S}_n$ of S_n as

$$\hat{S}_n = \sum_{k=1}^{n} E(S_n|X_k, Y_k) - (n-1)E(S_n). \tag{2.2.1}$$

<u>Lemma 2.2.1</u> <u>The projection</u> $\hat{S}_n$ <u>of</u> S_n <u>(as defined by</u> (2.2.1) <u>is given by</u>

$$(2.2.2)\quad \hat{S}_n = n^{-1}\sum_{k=1}^{n} Y_k\Big[\sum_{i=1}^{n}\sum_{s=1}^{n} J\Big(\frac{i}{n},\frac{s}{n}\Big)p_{i-1,s-1}^{(n-1)}(X_k,Y_k)\Big]$$

$$+\, n^{-1}\sum_{k=1}^{n}\Bigg[(n-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\Big(\frac{i+1}{n},\frac{s}{n}\Big)p_{i-1,s-1}^{(n-2)}(u,v)\Big] \times I(X_k\le u, Y_k>v)\,v\,dH(u,v)$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\Big(\frac{i+1}{n},\frac{s+1}{n}\Big)p_{i-1,s-1}^{(n-2)}(u,v)\Big]I(X_k\le u,Y_k\le v)\,v\,dH(u,v)$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\Big(\frac{i}{n},\frac{s}{n}\Big)p_{i-1,s-1}^{(n-2)}(u,v)\Big]I(X_k>u,Y_k>v)\,v\,dH(u,v)$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\Big(\frac{i}{n},\frac{s+1}{n}\Big)p_{i-1,s-1}^{(n-2)}(u,v)\Big]I(X_k>u,Y_k\le v)\,v\,dH(u,v)\Bigg]$$

$$-\,\eta_n,$$

<u>where</u> $\eta_n = (n-1)E(S_n)$, <u>is a constant depending</u> on n, J <u>and</u> H, <u>and</u>

$$(2.2.3)\quad p_{i,s}^{(n)}(u,v) = P[\text{exactly } i\ X\text{'s are} \le u \text{ and exactly } s\ Y\text{'s are} < v$$

out of n independently and identically distributed r.v.'s (X_k,Y_k), $k = 1,2,\dots,n]$

$$= \sum_{k=0}^{\min(i,s)} \frac{n!}{k!(i-k)!(s-k)!(n-i-s+k)!}\,\theta_1^{k}\,\theta_2^{i-k}\,\theta_3^{s-k}\,\theta_4^{n-i-s+k}$$

<u>with</u> $\theta_1 = P(X_i\le u, Y_i\le v)$, $\theta_2 = P(X_i\le u, Y_i>v)$,

$\theta_3 = P(X_i>u, Y_i\le v)$ <u>and</u> $\theta_4 = P(X_i>u, Y_i>v)$, for $i, s = 0, 1, 2, \ldots, n$.

PROOF. From (2.1.2) we have

$$S_n = n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{n} E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} | X_k, Y_k] - (n-1)E(S_n) \tag{2.2.4}$$

$$= n^{-1} \sum_{k=1}^{n} \sum_{i=1}^{n} E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} | X_k, Y_k] - (n-1)E(S_n).$$

We shall first evaluate the conditional expectations involved in (2.2.4). For $1 < i < n$, we may write

$$E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} | X_n = x_n, Y_n = y_n] \tag{2.2.5}$$

$$= E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} I(X_{(i-1:n-1)} \le X_n \le X_{(i:n-1)}) | X_n = x_n, Y_n = y_n]$$

$$+ E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} I(X_n < X_{(i-1:n-1)}) | X_n = x_n, Y_n = y_n]$$

$$+ E[J(\tfrac{i}{n}, \tfrac{R_{in}}{n}) Y_{[i:n]} I(X_n > X_{(i:n-1)}) | X_n = x_n, Y_n = y_n]$$

$$= (A) + (B) + (C)$$

Since $R_{in} = \sum_{j=1}^{n} I(Y_j \le Y_{[i:n]})$, using the obvious implications

$$X_{(i-1:n-1)} \le X_n \le X_{(i:n-1)} \Rightarrow X_{(i:n)} = X_n \Rightarrow Y_{[i:n]} = Y_n,$$

$$(A) = y_n E[J(\tfrac{i}{n}, \tfrac{\sum_{1}^{n} I(Y_j \le y_n)}{n}) I(X_{(i-1:n-1)} \le x_n \le X_{(i:n-1)}) | X_n = x_n, Y_n = y_n]$$

continued

$$= y_n E\left[J\left(\frac{i}{n}, \frac{1+\sum_{1}^{n-1} I(Y_j \le y_n)}{n}\right) I(X_{(i-1:n-1)} \le x_n \le X_{(i:n-1)})\right]$$

$$= y_n \sum_{s=0}^{n-1} J\left(\frac{i}{n}, \frac{s+1}{n}\right) p_{i-1,s}^{(n-1)}(x_n, y_n)$$

$$= y_n \sum_{s=1}^{n} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(x_n, y_n);$$

$$\textcircled{B} = E\left[J\left(\frac{i}{n}, \frac{\sum_{j=1}^{n} I(Y_j \le Y_{[i:n]})}{n}\right) Y_{[i:n]} I(X_n < X_{(i-1:n-1)}) \,\Big|\, X_n = x_n, Y_n = y_n\right]$$

$$= E\left[J\left(\frac{i}{n}, \frac{I(y_n \le Y_{[i:n]}) + \sum_{j=1}^{n-1} I(Y_j \le Y_{[i:n]})}{n}\right) Y_{[i:n]} I(x_n < X_{(i-1:n-1)}) \,\Big|\, X_n = x_n, Y_n = y_n\right]$$

from which, using the implication

$$X_n < X_{(i-1:n-1)} \Rightarrow X_{(i:n)} = X_{(i-1:n-1)} \Rightarrow Y_{[i:n]} = Y_{[i-1:n-1]},$$

we obtain

$$\textcircled{B} = E\left[J\left(\frac{i}{n}, \frac{I(y_n \le Y_{[i-1:n-1]}) + \sum_{j=1}^{n-1} I(Y_j \le Y_{[i-1:n-1]})}{n}\right) Y_{[i-1:n-1]} I(x_n < X_{(i-1:n-1)})\right]$$

$$= (n-1) \iint v E\left[J\left(\frac{i}{n}, \frac{I(y_n \le v) + \sum_{j=1}^{n-1} I(Y_j \le v)}{n}\right) I\left(\sum_{j=1}^{n-1} I(X_j < u) = i-2\right)\right] I(x_n < u)\, dH(u,v)$$

$$= (n-1) \iint v\left[\sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v)\right] I(y_n > v) I(x_n < u)\, dH(u,v)$$

$$+ (n-1) \iint v\left[\sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s+1}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v)\right] I(y_n \le v) I(x_n < u)\, dH(u,v)$$

$$= (n-1) \iint \left[\sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-2,s-1}^{(n-2)}(u,v)\right] I(x_n \le u, y_n > v)\, v\, dH(u,v)$$

continued

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s+1}{n}\big)p^{(n-2)}_{i-2,s-1}(u,v)\Big]I(x_n \le u, y_n \le v)\,v\,dH(u,v),$$ and

proceeding on the same lines as for B, we find that

$$\text{(C)} = (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s}{n}\big)p^{(n-2)}_{i-1,s-1}(u,v)\Big]I(x_n > u,\ y_n > v)\,v\,dH(u,v)$$

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s+1}{n}\big)p^{(n-2)}_{i-1,s-1}(u,v)\Big]I(x_n > u,\ y_n \le v)\,v\,dH(u,v).$$

Hence for $1 < i < n$, we have from (2.2.5) and the expressions for (A), (B) and (C),

$$\text{(2.2.6)}\qquad E\Big[J\big(\tfrac{i}{n},\tfrac{R_{in}}{n}\big)Y_{[i:n]}\Big|X_n,Y_n\Big]$$

$$= Y_n\sum_{s=1}^{n} J\big(\tfrac{i}{n},\tfrac{s}{n}\big)p^{(n-1)}_{i-1,s-1}(X_n,Y_n)$$

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s}{n}\big)p^{(n-2)}_{i-2,s-1}(u,v)\Big]I(X_n \le u, Y_n > v)\,v\,dH(u,v)$$

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s+1}{n}\big)p^{(n-2)}_{i-2,s-1}(u,v)\Big]I(X_n \le u, Y_n \le v)\,v\,dH(u,v)$$

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s}{n}\big)p^{(n-2)}_{i-1,s-1}(u,v)\Big]I(X_n > u, Y_n > v)\,v\,dH(u,v)$$

$$+\ (n-1)\iint\Big[\sum_{s=1}^{n-1} J\big(\tfrac{i}{n},\tfrac{s+1}{n}\big)p^{(n-2)}_{i-1,s-1}(u,v)\Big]I(X_n > u, Y_n \le v)\,v\,dH(u,v).$$

Now, for $i = 1$, we may write

$$E\Big[J\big(\tfrac{1}{n},\tfrac{R_{1n}}{n}\big)Y_{[1:n]}\Big|X_n = x_n, Y_n = y_n\Big]$$

continued

$$= E[J(\frac{1}{n}, \frac{R_{1n}}{n})Y_{[1:n]}I(X_n \leq X_{(1:n-1)}) | X_n = x_n, Y_n = y_n]$$

$$+ E[J(\frac{1}{n}, \frac{R_{1n}}{n})Y_{[1:n]}I(X_n > X_{(1:n-1)}) | X_n = x_n, Y_n = y_n]$$

$$= \textcircled{D} + \textcircled{E}.$$

It is easy to see that

$$\textcircled{D} = y_n \sum_{s=1}^{n} J(\frac{1}{n}, \frac{s}{n}) p_{0,s-1}^{(n-1)}(x_n, y_n) \quad \text{and}$$

$$\textcircled{E} = (n-1) \iint [\sum_{s=1}^{n-1} J(\frac{1}{n}, \frac{s}{n}) p_{0,s-1}^{(n-2)}(u,v)] I(x_n > u, y_n > v) v dH(u,v)$$

$$+ (n-1) \iint [\sum_{s=1}^{n-1} J(\frac{1}{n}, \frac{s+1}{n}) p_{0,s-1}^{(n-2)}(u,v)] I(x_n > u, y_n \leq v) v dH(u,v),$$

so that,

(2.2.7) $$E[J(\frac{1}{n}, \frac{R_{1n}}{n})Y_{[1:n]} | X_n, Y_n]$$

$$= Y_n \sum_{s=1}^{n} J(\frac{1}{n}, \frac{s}{n}) p_{0,s-1}^{(n-1)}(X_n, Y_n)$$

$$+ (n-1) \iint [\sum_{s=1}^{n-1} J(\frac{1}{n}, \frac{s}{n}) p_{0,s-1}^{(n-2)}(u,v)] I(X_n > u, Y_n > v) v dH(u,v)$$

$$+ (n-1) \iint [\sum_{s=1}^{n-1} J(\frac{1}{n}, \frac{s+1}{n}) p_{0,s-1}^{(n-2)}(u,v)] I(X_n > u, Y_n \leq v) v dH(u,v).$$

For $i = n$, we may write

$$E[J(1, \frac{R_{nn}}{n})Y_{[n:n]} | X_n, Y_n]$$

$$= E[J(1, \frac{R_{nn}}{n})Y_{[n:n]}I(X_n > X_{(n-1:n-1)}) | X_n, Y_n]$$

$$+ E[J(1, \frac{R_{nn}}{n})Y_{[n:n]}I(X_n \leq X_{(n-1:n-1)}) | X_n, Y_n].$$

Again, it is easy to see that

(2.2.8) $E[J(1, \frac{R_{nn}}{n})Y_{[n:n]}|X_n, Y_n]$

$$= Y_n \sum_{s=1}^{n} J(1,\frac{s}{n})p_{n-1,s-1}^{(n-1)}(X_n,Y_n)$$

$$+ (n-1) \iint [\sum_{s=1}^{n-1} J(1,\frac{s}{n})p_{n-2,s-1}^{(n-2)}(u,v)] I(X_n \le u, Y_n > v) dH(u,v)$$

$$+ (n-1) \iint [\sum_{s=1}^{n-1} J(1, \frac{s+1}{n})p_{n-2,s-1}^{(n-2)}(u,v)] I(X_n \le u, Y_n \le v) dH(u,v).$$

A symmetric argument shows that when conditioned on any (X_k, Y_k), the expressions for the conditional expectations in (2.2.6), (2.2.7) and (2.2.8) remain valid with (X_n, Y_n) replaced by (X_k, Y_k) for $k = 1, 2, \ldots, n-1$.

Now (2.2.2) follows by using the above expressions (2.2.6) to (2.2.8) in (2.2.4) and this proves the lemma. □

The following lemma is useful in section 2.3.

Lemma 2.2.2. Let $K(.,.)$ be any bounded continuous function. Let (X_i, Y_i), $i = 1, \ldots, n$ be independently identically distributed with a continuous d.f. $H(x,y)$, $-\infty < x, y < \infty$ with marginal d.f.'s of X and Y denoted by $G(x)$ and $F(y)$, respectively. Also for any fixed real numbers u and v let $p_{i,s}^{(n)}(u,v)$ be as defined by (2.2.3) above.

Then, for fixed $-\infty < u, v < \infty$,

(i) $\sum_{i=0}^{n} \sum_{s=0}^{n} K(\frac{i}{n}, \frac{s}{n}) p_{is}^{(n)}(u,v) \to K(G(u), F(v))$ as $n \to \infty$.

(ii) $\sum_{i=0}^{n} \sum_{s=0}^{n} K(a_n(\frac{i}{n}) + b_n, c_n(\frac{s}{n}) + d_n) p_{i,s}^{(n)}(u,v) \to K(G(u), F(v))$, where sequences of constants $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ satisfy

$a_n \to 1$, $c_n \to 1$, $b_n \to 0$ and $d_n \to 0$ as $n \to \infty$.

PROOF: First note that the r.v.'s $T_1 = \{\# \text{ X's} < u\}$ and $T_2 = \{\# \text{ Y's} < v\}$ are Binomially distributed with parameters $(n, G(u))$ and $(n, F(v))$ respectively, and that

$$p_{i,s}^{(n)}(u,v) = P(T_1 = i, T_2 = s).$$

Let $B_n = \{(i,s): |\frac{i}{n} - G(u)| \le n^{-1/3}, |\frac{s}{n} - F(v)| \le n^{-1/3}\}$.

Write

$$(2.2.9) \qquad \left|\sum_{i=0}^{n}\sum_{s=0}^{n} K(\tfrac{i}{n}, \tfrac{s}{n}) p_{i,s}^{(n)}(u,v) - K(G(u),F(v))\right|$$

$$\le \sum_{i=0}^{n}\sum_{s=0}^{n} \left| K(\tfrac{i}{n}, \tfrac{s}{n}) - K(G(u),F(v))\right| p_{i,s}^{(n)}(u,v)$$

$$= \sum\sum_{(i,s)\in B_n} \left| K(\tfrac{i}{n}, \tfrac{s}{n}) - K(G(u), F(v))\right| p_{i,s}^{(n)}(u,v)$$

$$+ \sum\sum_{(i,s)\in B_n^c} \left| K(\tfrac{i}{n}, \tfrac{s}{n}) - K(G(u),F(v))\right| p_{i,s}^{(n)}(u,v)$$

Now

$$\sum\sum_{(i,s)\in B_n} \left|K(\tfrac{i}{n}, \tfrac{s}{n}) - K(G(u),F(v))\right| p_{i,s}^{(n)}(u,v) \le w(K; n^{-1/3}, n^{-1/3}) \sum\sum_{(i,s)\in B_n} p_{i,s}^{(n)}(u,v)$$

$$\le w(K; n^{-1/3}, n^{-1/3})$$

where

$$(2.2.10) \qquad w(K;u,v) = \text{modulus of continuity of } K \text{ at } (u,v)$$

$$= \sup_{\substack{|x_1-x_2| \le u \\ |y_1-y_2| \le v}} |K(x_1,y_1) - K(x_2,y_2)|$$

But $w(K;n^{-\alpha},n^{-\alpha}) \to 0$ as $n \to \infty$ for $\alpha > 0$, since K is a continuous function. Hence the first term on the R.H.S of (2.2.9) $\to 0$ as $n \to \infty$.

$$\sum\sum_{(i,s)\in B_n^c} \left|K\left(\frac{i}{n}, \frac{s}{n}\right) - K(G(u),F(v))\right| p_{i,s}^{(n)}(u,v) \le 2 \sup_{x,y} |K(x,y)| \sum\sum_{(i,s)\in B_n^c} p_{i,s}^{(n)}(u,v)$$

$$= 2 \sup_{x,y} |K(x,y)| p\left(\left|\frac{T_1}{n} - G(u)\right| \le n^{-1/3}, \left|\frac{T_2}{n} - F(v)\right| \le n^{-1/3}\right)^c$$

$$\le 2 \sup_{x,y} |K(x,y)| \left[P\left(\left|\frac{T_1}{n} - G(u)\right| > n^{-1/3}\right) + P\left(\left|\frac{T_2}{n} - F(v)\right| > n^{-1/3}\right)\right]$$

Applying Tchebychev's inequality to the binomial probabilities on the R.H.S. and since K is assumed bounded we find that the R.H.S. $\to 0$ as $n \to \infty$. Thus both terms on the R.H.S. of (2.2.9) go to 0 as $n \to \infty$ and hence (i) of the lemma follows.

(ii) We write

$$(2.2.11) \quad \left|\sum_{i=0}^{n} \sum_{s=0}^{n} K\left(a_n \cdot \frac{i}{n} + b_n, c_n \cdot \frac{s}{n} + d_n\right) p_{i,s}^{(n)}(u,v) - K(G(u),F(v))\right|$$

$$\le \sum_{i=0}^{n} \sum_{s=0}^{n} \left|K\left(a_n \cdot \frac{i}{n} + b_n, c_n \cdot \frac{s}{n} + d_n\right) - K\left(\frac{i}{n},\frac{s}{n}\right)\right| p_{i,s}^{(n)}(u,v)$$

$$+ \left|\sum_{i=0}^{n} \sum_{s=0}^{n} K\left(\frac{i}{n},\frac{s}{n}\right) p_{i,s}^{(n)}(u,v) - K(G(u),F(v))\right|$$

$$= (1) + (2), \text{ say.}$$

(2) goes to zero as $n \to \infty$ by part (i) of the lemma.

Now

$$(1) \le \sum_{i=0}^{n} \sum_{s=0}^{n} W\left(K; \left|(a_n - 1)\frac{i}{n} + b_n\right|, \left|(c_n - 1)\frac{s}{n} + d_n\right|\right) p_{i,s}^{(n)}(u,v)$$

where w is the modulus of continuity as defined by (2.2.10). Since $W(K;u,v)$ is nondecreasing in both u and v for any continuous function K, we can write

$$(1) \leq \sum_{i=0}^{n} \sum_{s=0}^{n} W(K;|a_n-1|+|b_n|, |c_n-1|+|d_n|)p_{i,s}^{(n)}(u,v)$$

$$= W(K;|a_n-1|+|b_n|, |c_n-1|+|d_n|)$$

which goes to zero as $n \to \infty$ since $a_n \to 1$, $c_n \to 1$, $b_n \to 0$, $d_n \to 0$ as $n \to \infty$ and K is a continuous function. Hence (ii) of the lemma follows from (2.2.11). □

2.3 ASYMPTOTIC VARIANCE OF $\hat{S}_n$. We shall demonstrate in Chapter III that the statistics $T_n = n^{1/2}(S_n - ES_n)$ and $\hat{T}_n = n^{1/2}(\hat{S}_n - E\hat{S}_n)$ are asymptotically equivalent in mean square by showing that $E(T_n - \hat{T}_n)^2 = Var(T_n) - Var(\hat{T}_n) \to 0$ as $n \to \infty$. As a first step towards that, in this section we find the asymptotic variance of $\hat{T}_n$.

Let Q_n, $\bar{Q}_n$, Q_n^*, $\bar{Q}_n^*$ and $\tilde{Q}_n$ denote the following expressions:

$$Q_n(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J(\frac{i+1}{n}, \frac{s}{n})p_{i-1,s-1}^{(n-2)}(u,v)$$

(2.3.1)
$$Q_n^*(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} J(\frac{i}{n}, \frac{s}{n})p_{i-1,s-1}^{(n-2)}(u,v)$$

$$\tilde{Q}_n(u,v) = \sum_{i=1}^{n} \sum_{s=1}^{n} J(\frac{i}{n}, \frac{s}{n})p_{i-1,s-1}^{(n-1)}(u,v)$$

$$(2.3.1)\qquad \bar{Q}_n(u,v) = \sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\left(\frac{i+1}{n}, \frac{s+1}{n}\right)p_{i-1,s-1}^{(n-2)}(u,v)$$

$$\bar{Q}_n^*(u,v) = \sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J\left(\frac{i}{n}, \frac{s+1}{n}\right)p_{i-1,s-1}^{(n-2)}(u,v).$$

Using the above notation and the fact that

$$I(X_k \leq u, Y_k > v) = I(X_k \leq u) - I(X_k \leq u, Y_k \leq v)$$

$$I(X_k > u, Y_k > v) = 1 - I(X_k \leq u) - I(X_k > u, Y_k \leq v)$$

$$I(X_k > u, Y_k \leq v) = I(Y_k \leq v) - I(X_k \leq u, Y_k \leq v),$$

(2.2.2) may be rewritten as

$$(2.3.2)\qquad \hat{S}_n = n^{-1}\sum_{k=1}^{n} Y_k\bar{Q}_n(X_k,Y_k)$$

$$+ n^{-1}\sum_{k=1}^{n}(n-1)\{\iint[Q_n(u,v) - Q_n^*(u,v)]I(X_k \leq u)v\,dH(u,v)$$

$$+ \iint [\bar{Q}_n(u,v) - Q_n(u,v) + Q_n^*(u,v) - \bar{Q}_n^*(u,v)]I(X_k \leq u, Y_k \leq v)v\,dH(u,v)$$

$$+ \iint[\bar{Q}_n^*(u,v) - Q_n^*(u,v)]I(Y_k \leq v)v\,dH(u,v)$$

$$+ \iint Q_n^*(u,v)v\,dH(u,v)\} - \eta_n.$$

We make the following assumptions on J:

(2.3.3) $J(u,v)$, $0 \leq u, v \leq 1$ is a continuous function.

(2.3.4) $J(\cdot,\cdot)$ possesses bounded continuous partial derivatives up to the third order.

LEMMA 2.3.1. Suppose that J satisfies assumptions (2.3.3) and (2.3.4). Then for any real numbers u,v such that $0 < G(u), F(v) < 1$ the following assertions hold:

(2.3.5)

(a) $\tilde{Q}_n(u,v) \to J(G(u), F(v))$ as $n \to \infty$.

(b) $(n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] \to J_2'(G(u),F(v))$ as $n \to \infty$.

(c) $(n-1)[\bar{Q}_n^*(u,v) - Q_n^*(u,v)] \to J_2'(G(u),F(v))$ as $n \to \infty$.

(d) $(n-1)[Q_n(u,v) - Q_n^*(u,v)] \to J_1'(G(u),F(v))$ as $n \to \infty$.

(In (2.3.5) above $J_1'(.,.)$ and $J_2'(.,.)$ denote, respectively, the first order partial derivatives w.r. to the first and second arguments. The notation $J_1''(.,.)$, $J_2''(.,.)$ and $J_{12}''(.,.)$ used below have a similar obvious connotation.)

PROOF:

(a) Write $\tilde{Q}_n(u,v) = \sum_{i=1}^{n} \sum_{s=1}^{n} J(\frac{i}{n}, \frac{s}{n}) p_{i-1,s-1}^{(n-1)}(u,v)$

$$= \sum_{i=0}^{n-1} \sum_{s=0}^{n-1} J(\frac{i+1}{n}, \frac{s+1}{n}) p_{i,s}^{(n-1)}(u,v).$$

The result follows by using (ii) of lemma 2.2.2.

(b) $\bar{Q}_n(u,v) - Q_n(u,v) = \sum_{i=1}^{n-1} \sum_{s=1}^{n-1} [J(\frac{i+1}{n}, \frac{s+1}{n}) - J(\frac{i+1}{n}, \frac{s}{n})] \, p_{i-1,s-1}^{(n-2)}(u,v).$

By Taylor's expansion of $J(\frac{i+1}{n}, \frac{s+1}{n})$ around the point $(\frac{i+1}{n}, \frac{s}{n})$ and since the second derivatives are assumed bounded, we have

$$\bar{Q}_n(u,v) - Q_n(u,v) = \frac{1}{n}\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J_2'\left(\frac{i+1}{n}, \frac{s}{n}\right)p_{i-1,s-1}^{(n-2)}(u,v) + O\left(\frac{1}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=0}^{n-2}\sum_{s=0}^{n-2} J_2'\left(\frac{i+2}{n}, \frac{s+1}{n}\right)p_{i,s}^{(n-2)}(u,v) + O\left(\frac{1}{n^2}\right),$$

so that

(2.3.6) $$(n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] = \frac{n-1}{n}\sum_{i=0}^{n-1}\sum_{s=0}^{n-1} J_2'\left(\frac{i+2}{n}, \frac{s+1}{n}\right)p_{i,s}^{(n-2)}(u,v) + O\left(\frac{1}{n}\right).$$

Since J_2' is assumed bounded and continuous, lemma (2.2.2) applies to the first term on the R.H.S. of (2.3.6) and we have

$$(n-1)[\bar{Q}_n(u,v) - Q_n(u,v)] \to J_2'(G(u),F(v)), \quad \text{as} \quad n \to \infty.$$

(c) follows exactly on the same lines as (b).

(d) From (2.3.1) we have

$$Q_n(u,v) - Q_n^*(u,v) = \sum_{i=1}^{n-1}\sum_{s=1}^{n-1}\left[J\left(\frac{i+1}{n}, \frac{s}{n}\right) - J\left(\frac{i}{n}, \frac{s}{n}\right)\right]p_{i-1,s-1}^{(n-2)}(u,v)$$

Since the second derivatives of J are assumed bounded, the Taylor expansion of $J\left(\frac{i+1}{n}, \frac{s}{n}\right)$ around $\left(\frac{i}{n}, \frac{s}{n}\right)$ yields

$$Q_n(u,v) - Q_n^*(u,v) = \frac{1}{n}\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J_1'\left(\frac{i}{n}, \frac{s}{n}\right)p_{i-1,s-1}^{(n-2)}(u,v) + O\left(\frac{1}{n^2}\right),$$

so that

(2.3.7) $$(n-1)[Q_n(u,v) - Q_n^*(u,v)] = \frac{n-1}{n}\sum_{i=1}^{n-1}\sum_{s=1}^{n-1} J_1'\left(\frac{i}{n}, \frac{s}{n}\right)p_{i-1,s-1}^{(n-2)}(u,v) + O\left(\frac{1}{n}\right).$$

Since J_1' is bounded and continuous, lemma 2.2.2 applies to the first term on the R.H.S. of (2.3.7) again as in the case of (2.3.6) and we have

$$(n-1)[Q_n(u,v) - Q^*(u,v)] \to J_1'(G(u),F(v)), \quad \text{as} \quad n \to \infty. \qquad \square$$

LEMMA 2.3.2. Under the assumptions of lemma 2.3.1 we have

(2.3.8) $$\lim_{n\to\infty} \mathrm{Var}(\hat{T}_n) = \sigma^2(J,H)$$

where

(2.3.9) $$\sigma^2(J,H) = \iint J^2(G(u),F(v))v^2dH(u,v)$$
$$+ \iiiint [J_1'(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))[G(u_1\wedge u_2) - G(u_1)G(u_2)]$$
$$+ J_2'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))[F(v_1\wedge v_2)-F(v_1)F(v_2)]$$
$$+ 2J(G(u_1),F(v_1))J_1'(G(u_2), F(v_2))[I(u_1\le u_2)-G(u_2)]$$
$$+ 2J(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))[I(v_1\le v_2)-F(v_2)]$$
$$+ 2J_1'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))[H(u_1,v_2)-G(u_1)F(v_2)]$$
$$- J(G(u_1),F(v_1))J(G(u_2),F(v_2))]v_1v_2dH(u_1,v_1)dH(u_2,v_2).$$

PROOF: Since (X_k,Y_k), $k = 1,2,\ldots,n$ are all independent, from equation (2.3.2), we have

(2.3.10) $$\mathrm{Var}(\hat{T}_n) = n\,\mathrm{Var}(\hat{S}_n)$$
$$= \mathrm{Var}(A_k+B_k+C_k+D_k)$$

where

(2.3.11) $$A_k = Y_k\tilde{Q}_n(X_k,Y_k)$$
$$B_k = \iint (n-1)[Q_n(u,v) - Q_n^*(u,v)]I(X_k\le u)v\,dH(u,v)$$

$$C_k = \iint (n-1)[\bar{Q}_n(u,v) - Q_n(u,v) + Q_n^*(u,v) - \bar{Q}_n^*(u,v)]I(X_k \le u, Y_k \le u)vdH(u,v)$$

$$D_K = \iint (n-1)[\bar{Q}_n^*(u,v) - Q_n^*(u,v)]I(Y_k \le v)vdH(u,v).$$

We shall first obtain the expressions in the limit, for each of the variance and covariance terms on the R.H.S. and then put them all together to obtain the final limiting expression for $\mathrm{Var}(T_n)$.

$$\begin{aligned}\mathrm{Var}(A_k) &= \iint v^2[\tilde{Q}_n(u,v)]^2 dH(u,v) - \left(\iint v\tilde{Q}_n(u,v)dH(u,v)\right)^2 \\ &= \iint v^2[\tilde{Q}_n(u,v)]^2 dH(u,v) \\ &\quad - \iiiint \tilde{Q}_n(u_1,v_1)\tilde{Q}_n(u_2,v_2)v_1v_2 dH(u_1,v_1)dH(u_2,v_2).\end{aligned}$$

Since $|\tilde{Q}_n(u,v)| \le \sup_{x,y} |J(x,y)| < \infty$, and $EY^2 < \infty$, an application of Lebesgue Dominated Convergence Theorem (LDCT) along with Lemma 2.3.1 gives

$$(2.3.12) \quad \lim_{n\to\infty} \mathrm{Var}(A_k) = \iint J^2(G(u),F(v))v^2 dH(u,v) - \iiiint J(G(u_1),F(v_1))J(G(u_2),F(v_2)) \times v_1v_2 dH(u_1,v_1)dH(u_2,v_2).$$

Now on using Fubini's theorem to take the expectation inside the integrals we see that

$$(2.3.13) \quad \mathrm{Var}(B_k) = \iiiint (n-1)[Q_n(u_1,v_1) - Q_n^*(u_1,v_1)](n-1)[Q_n(u_2,v_2) - Q_n^*(u_2,v_2)] [G(u_1 \wedge u_2) - G(u_1)G(u_2)]v_1v_2\, dH(u_1,v_1)dH(u_2,v_2).$$

By assumption (2.3.4) on J and (2.3.7), we see that $(n-1)[Q_n(u,v) - Q_n^*(u,v)]$ is $O(1)$ and since $E(Y^2) < \infty$ LDCT applies in (2.3.13). Hence an application of LDCT along with lemma 2.3.1 yields

(2.3.14) $\lim_{n\to\infty} \mathrm{Var}(B_k)$

$$= \iiiint J_1'(G(u_1),F(v))J_1'(G(u_2),F(v_2))[G(u_1 \wedge u_2)-G(u_1)G(u_2)]$$

$$v_1 v_2 dH(u_1,v_1)dH(u_2,v_2).$$

Similarly, an application of LDCT along with lemma 2.3.1, also gives

(2.3.15) $\lim_{n\to\infty} \mathrm{Var}(C_k) = \lim_{n\to\infty} \iiiint (n-1)[\bar{Q}_n(u_1,v_1) - Q_n(u_1,v_1)+Q_n^*(u_1,v_1)-\bar{Q}_n^*(u_1,v_1)]$

$$\times (n-1)[\bar{Q}_n(u_2,v_2) - Q_n(u_2,v_2) + Q_n^*(u_2,v_2) - \bar{Q}_n^*(u_2,v_2)]$$

$$\times [H(u_1 \wedge u_2; v_1 \wedge v_2) - H(u_1,v_1)H(u_2,v_2)]$$

$$\times v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$= 0,$$

(2.3.16) $\lim_{n\to\infty} \mathrm{Var}(D_k) = \lim_{n\to\infty} \iiiint (n-1)[\bar{Q}_n^*(u_1,v_1) - Q_n^*(u_1 v_1)]$

$$\times (n-1)[\bar{Q}_n^*(u_2,v_2) - Q_n^*(u_2,v_2)]$$

$$\times [F(v_1 \wedge v_2) - F(v_1)F(v_2)]v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$= \iiiint J_2'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$

$$\times [F(v_1 \wedge v_2) - F(v_1)F(v_2)]v_1 v_2\, dH(u_1,v_1)dH(u_2,v_2).$$

Now from (2.3.11), we have

$$E(A_k \cdot B_k) = \iint [v_1\tilde{Q}_n(u_1,v_1)][(n-1)\iint \{Q_n(u_2,v_2) - Q_n^*(u_2,v_2)\}I(u_1 \le u_2)]$$
$$\times\ dH(u_1,v_1)v_2dH(u_2,v_2)$$
$$= \iiiint \tilde{Q}_n(u_1,v_1)(n-1)[Q_n(u_2,v_2) - Q_n^*(u_2,v_2)]I(u_1 \le u_2)v_1v_2$$
$$dH(u_1,v_1)dH(u_2,v_2),$$

and

$$E(A_k).E(B_k) = (\iint v_1\tilde{Q}_n(u_1,v_1)dH(u_1,v_1))$$
$$\times\ (\iint (n-1)[Q_n(u_2,v_2) - Q_n^*(u_2,v_2)]G(u_2)v_2dH(u_2,v_2))$$
$$= \iiiint \tilde{Q}_n(u_1,v_1)(n-1)[Q_n(u_2,v_2) - Q_n^*(u_2,v_2)]G(u_2)v_1v_2$$
$$\times\ dH(u_1,v_1)dH(u_2,v_2).$$

Again applying LDCT and Lemma 2.3.1 yields

(2.3.17) $$\lim_{n\to\infty} \text{Cov}(A_k,B_k) = \lim_{n\to\infty} \iiiint \tilde{Q}_n(u_1,v_1)(n-1)[Q_n(u_2,v_2)-Q_n^*(u_2,v_2)]$$
$$\times\ [I(u_1 \le u_2) - G(u_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$
$$= \iiiint J(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))$$
$$\times\ [I(u_1 \le u_2)-G(u_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2).$$

On the same lines, we can see that

(2.3.18) $$\lim_{n\to\infty} \text{Cov}(A_k,D_k) = \iiint J(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$
$$\times\ [I(v_1 \le v_2) - F(v_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2),$$

$$\lim_{n\to\infty} \mathrm{Cov}(A_k, C_k) = 0$$

(2.3.19)
$$\lim_{n\to\infty} \mathrm{Cov}(B_k, C_k) = 0$$

$$\lim_{n\to\infty} \mathrm{Cov}(C_k, D_k) = 0$$

(2.3.20)
$$\lim_{n\to\infty} \mathrm{Cov}(B_k, D_k) = \lim_{n\to\infty} \iiiint (n-1)[Q_n(u_1,v_1) - Q_n^*(u_1,v_1)]$$
$$\times (n-1)[\bar{Q}_n^*(u_2,v_2) - Q_n^*(u_2,v_2)][H(u_1,v_2) - G(u_1)F(v_2)]$$
$$\times v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$
$$= \iiiint J_1'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$
$$\times [H(u_1,v_2) - G(u_1)F(v_2)]v_1 v_2 dH(u_1,v_1)dH(u_2,v_2).$$

The lemma now follows by using (2.3.12) and (2.3.14) to (2.3.20) with (2.3.10). □

CHAPTER III

ASYMPTOTIC NORMALITY OF S_n

In this chapter we prove the asymptotic normality of S_n as defined by (2.1.1), in a properly normalized form. This we shall do by first establishing the asymptotic equivalence in mean square of T_n and $\hat{T}_n$ and then proving the asymptotic normality of $\hat{T}_n$ where T_n and $\hat{T}_n$ are as defined in section 2.2 (also given by (3.1.1) and (3.1.2) below). A sequence of rv's $\{X_n\}$ is asymptotically normal with mean μ_n and variance σ_n^2 if $\sigma_n > 0$ for all n sufficiently large and

$$\frac{X_n-\mu_n}{\sigma_n} \to N(0,1) \text{ in distribution as } n \to \infty$$

i.e. $\left|P\left(\frac{X_n-\mu_n}{\sigma_n} \leq t\right) - \Phi(t)\right| \to 0$, as $n \to \infty$ for $-\infty < t < \infty$,

where $\Phi(t)$ is the standard normal distribution function.

In section 3.1. we state our main results and establish the asymptotic equivalence of T_n and $\hat{T}_n$ while using the results obtained in Chapter II. The proofs of the main results are given in sections 3.2 and 3.3.

3.1 ASYMPTOTIC EQUIVALENCE OF S_n AND $\hat{S}_n$.

Let S_n and $\hat{S}_n$ be as given by (2.1.1) and (2.3.2) respectively and let

(3.1.1) $$T_n = n^{1/2}(S_n - E(S_n))$$

(3.1.2) $$\hat{T}_n = n^{1/2}(\hat{S}_n - E(\hat{S}_n))$$

as defined in section 2.2. In this section we establish the asymptotic equivalence of T_n and $\hat{T}_n$, in mean square. But we first state the main result of this chapter, namely, the asymptotic normality of T_n as Theorem 3.1.1 and the two important special cases considered by Yang [22] and Stigler [17] as Theorem 3.1.2 below.

Theorem 3.1.1. Assume that $E(X^2) < \infty$ and $E(Y^2) < \infty$. Let $J(u,v)$, $0 \leq u, v \leq 1$, be a bivariate score function satisfying assumptions (2.3.3) and (2.3.4). Then, if H is continuous and $\sigma^2(J,H) > 0$

$$\left(\frac{S_n - E(S_n)}{\sigma(S_n)}\right) \rightarrow N(0,1),$$

as $n \rightarrow \infty$, where $\sigma^2(J,H) = \lim_{n\rightarrow\infty} n \operatorname{Var}(S_n)$ is as given by (2.3.9).

Theorem 3.1.2. If the function J in Theorem 3.1.1 above satisfies

(a) $J(u,v) = J(v)$, $0 \leq v \leq 1$, then the assumptions and conclusions of Theorem 3.1.1 reduce to those of Theorem 2 of Stigler [17], with $\sigma^2(J,F)$ given by

$$(3.1.3) \qquad \sigma^2(J,F) = \iint J(F(v_1))J(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]dv_1 dv_2 \; ;$$

(b) $J(u,v) = J(u)$, $0 \leq u \leq 1$, then the assumptions and conclusions of Theorem 3.1.1 reduce to those of Theorem 1 of Yang [22], under the additional assumption that $m(x) = E[Y|X = x]$, $-\infty < x < \infty$ is a function of bounded variation, with $\sigma^2(J,H) \equiv \sigma^2(J,G)$ given by

$$\text{(3.1.4)} \qquad \sigma^2(J,H) = \int J^2(G(u))\sigma^2(u)dG(u) + \iint[G(u_1 \wedge u_2) - G(u_1)G(u_2)] \times J(G(u_1))J(G(u_2))dm(u_1)dm(u_2)$$

where $\sigma^2(u) = \mathrm{Var}(Y|X = u)$.

COROLLARY 3.1.1. Under the conditions of Theorem 3.1.1

$$\frac{S_n - \mu(J,H)}{\sigma(S_n)} \to N(0,1) \text{ in distribution, as } n \to \infty$$

where

$$\mu(J,H) = \iint J(G(u),F(v))\, v\,dH(u,v).$$

The proofs of Theorem 3.1.1 and Corollary 3.1.1 are given in section 3.2 and the proof of Theorem 3.1.2 is given in section 3.3.

LEMMA 3.1.1. Suppose assumptions (2.2.3) and (2.2.4) on J are satisfied. We then have

$$\lim_{n\to\infty} \mathrm{Var}(T_n) = \lim_{n\to\infty} n\,\mathrm{Var}(S_n) = \sigma^2(J,H)$$

where $\sigma^2(J,H)$ is as given by (2.3.9).

PROOF. First note that —

$$\text{(3.1.[illegible])} \qquad n.\mathrm{Var}(S_n) = n^{-1}\Big[\sum_{i=1}^{n} \mathrm{Var}\Big(J\big(\tfrac{i}{n}, \tfrac{R_{in}}{n}\big)Y[i:n]\Big) + \sum_{i<i'} \mathrm{Cov}\Big(J\big(\tfrac{i}{n}, \tfrac{R_{in}}{n}\big)Y_{[i:n]}, J\big(\tfrac{i'}{n}, \tfrac{R_{i'n}}{n}\big)Y_{[i':n]}\Big) + \sum_{i>i'} \mathrm{Cov}\Big(J\big(\tfrac{i}{n}, \tfrac{R_{in}}{n}\big)Y_{[i:n]}, J\big(\tfrac{i'}{n}, \tfrac{R_{i'n}}{n}\big)Y_{[i':n]}\Big)\Big].$$

Now setting $r(Y_j)$ = rank of Y_j among the n Y's, $r(X_j)$ = rank of X_j among the n X's, we can see that

$$(3.1.6)\qquad E[J(\tfrac{i}{n},\tfrac{R_{in}}{n})Y_{[i:n]}] = nE(J(\tfrac{i}{n},\tfrac{r(Y_n)}{n})Y_n I(r(X_n)=i))$$

$$= n\iint [\sum_{s=1}^{n} J(\tfrac{i}{n},\tfrac{s}{n})p^{(n-1)}_{i-1,s-1}(u,v)]v\,dH(u,v),$$

where $p^{(n)}_{i,j}(u,v)$ is defined by (2.2.3), so that

$$(3.1.7)\qquad Var(J(\tfrac{i}{n},\tfrac{R_{in}}{n})Y_{[i:n]}) = n\iint [\sum_{s=1}^{n} J^2(\tfrac{i}{n},\tfrac{s}{n})p^{(n-1)}_{i-1,s-1}(u,v)]v^2 dH(u,v)$$

$$- n^2\{\iint[\sum_{s=1}^{n} J(\tfrac{i}{n},\tfrac{s}{n})p^{(n-1)}_{i-1,s-1}(u,v)]v\,dH(u,v)\}^2$$

$$= n\iint \sum_{s=1}^{n} J^2(\tfrac{i}{n},\tfrac{s}{n})p^{(n-1)}_{i-1,s-1}(u,v)]v^2 dH(u,v)$$

$$- n^2 \iiiint[\sum_{s=1}^{n} J(\tfrac{i}{n},\tfrac{s}{n})p^{(n-1)}_{i-1,s-1}(u_1,v_1)]$$

$$\times [\sum_{s'=1}^{n} J(\tfrac{i}{n},\tfrac{s'}{n})p^{(n-1)}_{i-1,s'-1}(u_2,v_2)]v_1 v_2 dH(u_1,v_1)$$

$$dH(u_2,v_2).$$

Further for $i < i'$

$$(3.1.8)\quad E\Big(J\big(\tfrac{i}{n},\tfrac{R_{in}}{n}\big)Y_{[i:n]}\,J\big(\tfrac{i'}{n},\tfrac{R_{i'n}}{n}\big)Y_{[i':n]}\Big)$$

$$= n(n-1)E\Big[J\big(\tfrac{i}{n},\tfrac{r(Y_n)}{n}\big)Y_n\,J\big(\tfrac{i'}{n},\tfrac{r(Y_{n-1})}{n}\big)Y_{n-1}$$

$$I\big(r(X_n)=i,\ r(X_{n-1})=i'\big)\Big]$$

$$= n(n-1)\ E\Big[J\big(\tfrac{i}{n},\tfrac{r(Y_n)}{n}\big)Y_n\,J\big(\tfrac{i'}{n},\tfrac{r(Y_{n-1})}{n}\big)Y_{n-1}I(Y_n<Y_{n-1})$$

$$I\big(r(X_n)=i,\ r(X_{n-1})=i'\big)\Big]$$

$$+\ n(n-1)E\Big[J\big(\tfrac{i}{n},\tfrac{r(Y_n)}{n}\big)Y_n\,J\big(\tfrac{i'}{n},\tfrac{r(Y_{n-1})}{n}\big)Y_{n-1}I(Y_n>Y_{n-1})$$

$$I\big(r(X_n)=i,\ r(X_{n-1})=i'\big)\Big]$$

$$= n(n-1)\iiiint\limits_{\substack{u_1\le u_2\\ v_1\le v_2}}\Big[\sum_{s<s'}\sum J\big(\tfrac{i}{n},\tfrac{s}{n}\big)\,J\big(\tfrac{i'}{n},\tfrac{s'}{n}\big)p^{11}_{isi's'}(u_1,v_1,u_2,v_2)\Big]v_1v_2\ dH(u_1,v_1)dH(u_2,v_2)$$

$$+\ n(n-1)\iiiint\limits_{\substack{u_1<u_2\\ v_1>v_2}}\Big[\sum_{s>s'}\sum J\big(\tfrac{i}{n},\tfrac{s}{n}\big)\,J\big(\tfrac{i'}{n},\tfrac{s'}{n}\big)p^{12}_{isi's'}(u_1,v_1,u_2,v_2)\Big]v_1v_2\ dH(u_1,v_1)dH(u_2,v_2)$$

where

$$(3.1.9)\quad p^{11}_{isi's'}(u_1,v_1,u_2,v_2) = P\big[\text{exactly } (i-1)\ X\text{'s} < u_1,\ (i'-2)\ X\text{'s} < u_2,$$

$$(s-1)Y\text{'s} < v_1,\ (s'-2)\ Y\text{'s} < v_2,$$

$$\text{out of } (n-2) \text{ iid } (X,Y) \text{ pairs}\big],$$

$$\text{for } u_1 < u_2,\ v_1 < v_2;\ \ i < i',\ s < s'$$

and

$$(3.1.10)\qquad p^{12}_{isi's'}(u_1,v_1,u_2,v_2) = P[\text{exactly } (i-1)\ X\text{'s} < u_1,\ (i'-2)X\text{'s} < u_2,\ (s'-1)Y\text{'s} < v_2,\ (s-2)Y\text{'s} < v_1 \text{ out of } (n-2) \text{ iid } (X,Y) \text{ pairs}]$$

$$\text{for } u_1 < u_2,\ v_1 > v_2;\ i < i',\ s > s'.$$

Observe that (3.1.10) is easily obtained from (3.1.9) by interchanging v_1 and v_2 and s and s'. (The explicit expressions of these probabilities can be obtained using the multinomial distribution in the same way as the probabilities i (2.2.3) are obtained using the Binomial distribution. However, since we do not require the explicit expressions in our arguments they are omitted). Now it follows from (3.1.6) and (3.1.8) that for $i < i'$

$$(3.1.11)\qquad \mathrm{Cov}\Big(J\big(\tfrac{i}{n}, \tfrac{R_{in}}{n}\big)Y_{[i:n]},\ J\big(\tfrac{i'}{n}, \tfrac{R_{i'n}}{n}\big)Y_{[i':n]}\Big)$$

$$= n(n-1)\Bigg[\underset{\substack{u_1<u_2\\ v_1<v_2}}{\iiiint} \Big[\sum_{s<s'}\sum J\big(\tfrac{i}{n}, \tfrac{s}{n}\big) J\big(\tfrac{i'}{n}, \tfrac{s'}{n}\big) p^{11}_{isi's'}(u_1,v_1,u_2,v_2)\Big] \times v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$+ \underset{\substack{u_1<u_2\\ v_1>v_2}}{\iiiint} \Big[\sum_{s>s'}\sum J\big(\tfrac{i}{n}, \tfrac{s}{n}\big) J\big(\tfrac{i'}{n}, \tfrac{s'}{n}\big) p^{12}_{isi's'}(u_1,v_1,u_2,v_2)\Big] \times v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)\Bigg]$$

$$- n^2 \iiiint \Big[\sum_{s=1}^{n} J\big(\tfrac{i}{n}, \tfrac{s}{n}\big) p^{(n-1)}_{i-1,s-1}(u_1,v_1)\Big]$$

$$\times \Big[\sum_{s'=1}^{n} J\big(\tfrac{i'}{n}, \tfrac{s'}{n}\big) p^{(n-1)}_{i'-1,s'-1}(u_2,v_2)\Big] v_1 v_2 dH(u_1,v_1)dH(u_2,v_2),$$

and similarly for $i > i'$ we obtain

(3.1.12) $$\mathrm{Cov}\left(J\left(\frac{i}{n}, \frac{R_{in}}{n}\right)Y_{[i:n]}, J\left(\frac{i'}{n}, \frac{R_{i'n}}{n}\right)Y_{[i':n]}\right)$$

$$= n(n-1) \iiiint\limits_{\substack{u_1>u_2\\ v_1<v_2}} \left[\sum_{s<s'}\sum J\left(\frac{i}{n},\frac{s}{n}\right) J\left(\frac{i'}{n},\frac{s'}{n}\right)p^{21}_{isi's'}(u_1,v_1,u_2,v_2)\right] \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ n(n-1) \iiiint\limits_{\substack{u_1>u_2\\ v_1>v_2}} \left[\sum_{s>s'}\sum J\left(\frac{i}{n},\frac{s}{n}\right)J\left(\frac{i'}{n},\frac{s'}{n}\right)p^{22}_{isi's'}(u_1,v_1,u_2,v_2)\right] \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$- n^2 \iiiint \left[\sum_{s=1}^{n} J\left(\frac{i}{n},\frac{s}{n}\right)p^{(n-1)}_{i-1,s-1}(u_1,v_1)\right]$$

$$\times \left[\sum_{s'=1}^{n} J\left(\frac{i'}{n},\frac{s'}{n}\right)p^{(n-1)}_{i'-1,s'-1}(u_2,v_2)\right]v_1v_2dH(u_1,v_1)dH(u_2,v_2),$$

where the probabilities $p^{21}_{isi's'}(u_1,v_1,u_2,v_2)$ are defined as those obtained by interchanging u_1 and u_2 and i and i' in the definitions of $p^{12}_{isi's'}(u_1,v_1,u_2,v_2)$ and $p^{22}_{isi's'}(u_1,v_1,u_2,v_2)$ (given by (3.1.9) and (3.1.10)) respectively. Now using (3.1.7), (3.1.11) and (3.1.12) in (3.1.5) we obtain

(3.1.13) $$n\,\mathrm{Var}(S_n) = \iint \left[\sum_{i=1}^{n}\sum_{s=1}^{n} J^2\left(\frac{i}{n}, \frac{s}{n}\right)p^{(n-1)}_{i-1,s-1}(u,v)\right]v^2dH(u,v)$$

$$+ (n-1) \iiiint\limits_{\substack{u_1<u_2\\ v_1<v_2}} \left[\sum_{i<i'}\sum \sum_{s<s'}\sum J\left(\frac{i}{n}, \frac{s}{n}\right)J\left(\frac{i'}{n},\frac{s'}{n}\right)p^{11}_{isi's'}(u_1,v_1,u_2,v_2)\right] \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

continued

$$+ (n-1) \iiiint_{\substack{u_1<u_2 \\ v_1>v_2}} \Big[\sum_{i<i'}\sum\ \sum_{s>s'}\sum J(\tfrac{i}{n},\tfrac{s}{n})\ J(\tfrac{i'}{n},\tfrac{s'}{n}) p^{12}_{isi's'}(u_1,v_1,u_2,v_2) \Big] \times v_1 v_2 dH(u_1,v_1) dH(u_2,v_2)$$

$$+ (n-1) \iiiint_{\substack{u_1>u_2 \\ v_1<v_2}} \Big[\sum_{i>i'}\sum\ \sum_{s<s'}\sum J(\tfrac{i}{n},\tfrac{s}{n})\ J(\tfrac{i'}{n},\tfrac{s'}{n}) p^{21}_{isi's'}(u_1,v_1,u_2,v_2) \Big] \times v_1 v_2 dH(u_1,v_1) dH(u_2,v_2)$$

$$+ (n-1) \iiiint_{\substack{u_1>u_2 \\ v_1>v_2}} \Big[\sum_{i>i'}\sum\ \sum_{s>s'}\sum J(\tfrac{i}{n},\tfrac{s}{n})\ J(\tfrac{i'}{n},\tfrac{s'}{n}) p^{22}_{isi's'}(u_1,v_1,u_2,v_2) \Big] \times v_1 v_2 dH(u_1,v_1) dH(u_2,v_2)$$

$$- n \iiiint \Big[\sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{s=1}^{n} \sum_{s'=1}^{n} J(\tfrac{i}{n},\tfrac{s}{n})\ J(\tfrac{i'}{n},\tfrac{s'}{n}) p^{(n-1)}_{i-1,s-1}(u_1,v_1) p^{(n-1)}_{i'-1,s'-1}(u_2,v_2) \Big] \times v_1 v_2 dH(u_1,v_1) dH(u_2,v_2).$$

By Taylor's expansion of $J(\frac{i}{n},\frac{s}{n})$ around the point $(G(u_1), F(v_1))$ and that of $J(\frac{i'}{n},\frac{s'}{n})$ around $(G(u_2), F(v_2))$, we may write

$$J(\tfrac{i}{n},\tfrac{s}{n}) = J(G(u_1), F(v_1)) + (\tfrac{i}{n} - G(u_1)) J_1'(G(u_1), F(v_1))$$
$$+ (\tfrac{s}{n} - F(v_1)) J_2'(G(u_1), F(v_1))$$
$$+ \frac{(\frac{i}{n} - G(u_2))^2}{2} J_1''(G(u_1), F(v_1))$$
$$+ \frac{(\frac{s}{n} - F(v_1))^2}{2} J_2''(G(u_1), F(v_1))$$
$$+ (\tfrac{i}{n} - G(u_1))(\tfrac{s}{n} - F(v_1)) J_{12}''(G(u_1), F(v_1)) + R_n \tag{3.1.14}$$

and

$$J\left(\frac{i'}{n},\frac{s'}{n}\right) = J\left[G(u_2),\ F(v_2)\right] + \left(\frac{i'}{n} - G(u_2)\right)J_1'\left(G(u_2),F(v_2)\right) + \left(\frac{s'}{n} - F(v_2)\right)J_2'\left(G(u_2),F(v_2)\right) + \frac{\left(\frac{i'}{n} - G(u_2)\right)^2}{2}J_1''\left(G(u_2),F(v_2)\right) + \frac{\left(\frac{s'}{n} - F(v_2)\right)^2}{2}J_2''\left(G(u_2),F(v_2)\right) + \left(\frac{i'}{n} - G(u_2)\right)\left(\frac{s'}{n} - F(v_2)\right)J_{12}''\left(G(u_2),F(v_2)\right) + R_n', \tag{3.1.15}$$

where R_n and R_n' include all higher order terms.

Also observe that

$$\sum_{i=1}^{n}\sum_{s=1}^{n} p_{i-1,s-1}^{(n-1)}(u,v) = 1 \tag{3.1.16}$$

$$\sum_{i<i'}\sum_{s<s'} p_{isi's'}^{11}(u_1,v_1,u_2,v_2) = \sum_{i<i'}\sum_{s>s'} p_{isi's'}^{12}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i>i'}\sum_{s<s'} p_{isi's'}^{21}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i>i'}\sum_{s>s'} p_{isi's'}^{22}(u_1,v_1,u_2,v_2)$$

$$= 1,$$

and for fixed $i = 1,2,\ldots,n-1$,

$$\sum_{i'=i+1}^{n} \sum_{s<s'}\sum p^{11}_{isi's'}(u_1,v_1,u_2,v_2) = P[\text{exactly } (i-1)\ X\text{'s} < u_1 \text{ out of of } (n-2) \text{ i.i.d. pairs}]$$

$$= p^{(n-2)}_{i-1}(u_1),$$

for fixed i, i' with $i < i'$, $i = 1,2,\ldots,n-1$, $i' = 2,3,\ldots,n$,

$$\sum_{s<s'}\sum p^{11}_{isi's'}(u_1,v_1,u_2,v_2) = P[\text{exactly } (i-1)\ X\text{'s} < u_1, (i'-2)\ X\text{'s} < u_2 \text{ out of } (n-2) \text{ i.i.d. pairs}]$$

$$= p^{11}_{ii'}(u_1,u_2),$$

for fixed i', s', $i' = 2,3,\ldots,n$, $s' = 2,3,\ldots,n$

$$\sum_{i<i'}\sum_{s<s'} p^{11}_{isi's'}(u_1,v_1u_2,v_2) = P[\text{exactly } (i'-2)\ X\text{'s} < u_2, (s'-2)\ Y\text{'s} < v_2 \text{ out of } (n-2) \text{ i.i.d. pairs}]$$

$$= p^{11}_{i's'}(u_2,v_2),$$

and for fixed $i = 1,2,\ldots,n$,

$$\sum_{s=1}^{n} p^{(n-1)}_{i-1,s-1}(u,v) = P[\text{exactly } (i-1)\ X\text{'s} < u \text{ out of } (n-1) \text{ i.i.d pairs}],$$

$$= p^{(n-1)}_{i-1}(u)$$

and similar relations hold for the probabilities $p^{12}_{isi's'}(u_1,v_1,u_2,v_2)$, $p^{21}_{isi's'}(u_1,v_1,u_2,v_2)$ and $p^{22}_{isi's'}(u_1,v_1,u_2,v_2)$. Now using the expansions given by (3.1.14) and (3.1.15) in place of $J(\frac{i}{n},\frac{s}{n})$ and $J(\frac{i'}{n},\frac{s'}{n})$, respectively, in (3.1.13) and also the relations of the type given by (3.1.16) therein, we obtain the resulting expression (on collecting similar terms) as the sum of the expressions (F), (G), (H), (I), (J), (K), (L), (M) given below:

The sum of terms involving $J^2(G(u), F(v))$, viz.,

$$(3.1.17) \qquad \text{Ⓕ} = \iint J^2(G(u), F(v))v^2 dH(u,v).$$

The sum of terms involving $J(G(u_1), F(v_1))J(G(u_2), F(v_2))$, viz.,

$$(3.1.18) \qquad \text{Ⓖ} = (n-1) \iiiint J(G(u_1),F(v_1))J(G(u_2),F(v_2))v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$- n \iiiint J(G(u_1),F(v_1))J(G(u_2),F(v_2))v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$= - \iiiint J(G(u_1),F(v_1))J(G(u_2),F(v_2))v_1 v_2 dH(u_1,v_1)dH(u_2,v_2).$$

Let $T_1 = \sum_{i=1}^{n-2} I(X_i \le u_1) = \#\{$ of X's $\le u_1$ out of $(n-2)$ iid observations$\}$.

Hence T_1 is Binomial $(n-2, G(u_1))$ and

$$\sum_{i=0}^{n-2} [i-(n-2)G(u_1)]p_i^{(n-2)}(u_1) = E(T_1 - ET_1) = 0.$$

This together with (3.1.16) yields the relations

$$(3.1.19) \qquad \sum_{i<i'}\sum\sum_{s<s'}\sum \left(\frac{i}{n} - G(u_1)\right)p^{11}_{isi's'}(u_1,v_1,u_2,v_2) = \sum_{i=1}^{n-1}\left(\frac{i}{n} - G(u_1)\right)p^{(n-2)}_{i-1}(u_1)$$

$$= \frac{1}{n}\left(1-2G(u_1)\right),$$

$$\sum_{i<i'}\sum\sum_{s>s'}\sum \left(\frac{i}{n} - G(u_1)\right)p^{12}_{isi's'}(u_1,v_1,u_2,v_2) = \sum_{i=1}^{n-1}\left(\frac{i}{n} - G(u_1)\right)p^{(n-2)}_{i-1}(u_1)$$

$$= \frac{1}{n}\left(1-2G(u_1)\right),$$

$$\sum_{i>i'}\sum_{s<s'}\left(\frac{i}{n} - G(u_1)\right)p^{12}_{isi's'}(u_1,v_1,u_2,v_2) = \sum_{i=2}^{n}\left(\frac{i}{n} - G(u_1)\right)p^{(n-2)}_{i-2}(u_1)$$

$$= \frac{1}{n}\left(2-2G(u_1)\right),$$

$$\sum_{i>i'}\sum_{s>s'}\left(\frac{i}{n} - G(u_1)\right)p^{22}_{isi's'}(u_1,v_1,u_2,v_2) = \sum_{i=2}^{n}\left(\frac{i}{n} - G(u_1)\right)p^{(n-2)}_{i-2}(u_1)$$

$$= \frac{1}{n}\left(2-2G(u_1)\right),$$

and

$$\sum_{i}\sum_{i'}\sum_{s}\sum_{s'}\left(\frac{i}{n} - G(u_1)\right)p^{(n-1)}_{i-1,s-1}(u_1,v_1)p^{(n-1)}_{i'-1,s'-1}(u_2,v_2)$$

$$= \sum_{i=1}\left(\frac{i}{n} - G(u_1)\right)p^{(n-1)}_{i-1}(u_1) = \frac{1}{n}\left(1-G(u_1)\right).$$

Using (3.1.19) we obtain the sum of terms involving $J_1'\left(G(u_1),F(v_1)\right)J\left(G(u_2),F(v_2)\right)$ as

(3.1.20) (H) $= (n-1)\iiiint\limits_{\substack{u_1<u_2\\ v_1<v_2}} J_1'\left(G(u_1),F(v_1)\right)J\left(G(u_2),F(v_2)\right) \times \frac{1}{n}\left(1-2G(u_1)\right)v_1v_2\,dH(u_1,v_1)\,dH(u_2,v_2)$

$+ (n-1)\iiiint\limits_{\substack{u_1<u_2\\ v_1>v_2}} J_1'\left(G(u_1),F(v_1)\right)J\left(G(u_2),F(v_2)\right) \times \frac{1}{n}\left(1-2G(u_1)\right)v_1v_2\,dH(u_1,v_1)\,dH(u_2,v_2)$

$+ (n-1)\iiiint\limits_{\substack{u_1>u_2\\ v_1<v_2}} J_1'\left(G(u_1),F(v_1)\right)J\left(G(u_2),F(v_2)\right) \times \frac{1}{n}\left(2-2G(u_1)\right)v_1v_2\,dH(u_1,v_1)\,dH(u_2,v_2)$

continued

$$+ (n-1) \iiint_{\substack{u_1>u_2\\ v_1>v_2}} J_1'(G(u_1),F(v_1))J(G(u_2),F(v_2)) \times \frac{1}{n}(2-2G(u_1))v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$- n \iiiint J_1'(G(u_1),F(v_1))J(G(u_2),F(v_2)) \times \frac{1}{n}(1-G(u_1))v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$= \frac{n-1}{n} \iiiint_{u_1<u_2} J_1'(G(u_1),F(v_1))J(G(u_2),F(v_2)) \times(-G(u_1))v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ \frac{n-1}{n} \iiiint_{u_1>u_2} J_1'(G(u_1),F(v_1))J(G(u_2),F(v_2)) \times(1-G(u_1))\, v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$- \frac{1}{n} \iiiint J_1'(G(u_1),F(v_1))(G(u_2),F(v_2))(1-G(u_1)) \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$= \frac{n-1}{n} \iiiint J_1'(G(u_1),F(v_1))J(G(u_2),F(v_2))[I(u_1>u_2)-G(u_1)] \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ O(1/n).$$

A similar argument reduces the sum of the terms involving $J_1'(G(u_2,F(v_2))J(G(u_1),F(v_1))$ to

$$(3.1.21) \quad \textcircled{I} = \frac{n-1}{n} \iiiint J_1'(G(u_2),F(v_2))J(G(u_1),F(v_1))[I(u_1<u_2)-G(u_2)] \times v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ O(1/n).$$

(Note that the expression in (3.1.21) is exactly the same as that of (3.1.20) with (u_1,v_1) and (u_2,v_2) interchanged) and the sum of the terms involving $J_2'(G(u_1),F(v_1))J(G(u_2),F(v_2))$ and $J_2'(G(u_2),F(v_2))J(G(u_1),F(v_1))$ to

$$(3.1.22)\quad \textcircled{J} = \frac{2(n-1)}{n} \iiiint J_2'(G(u_2),F(v_2))J(G(u_1),F(v_1)) \times [I(v_1<v_2) - F(v_2)]v_1 v_2 dH(u_1,v_1)dH(u_2,v_2) + O(1/n).$$

Since

$$(3.1.23)\quad \sum_{i<i'}\sum_{s<s'}\sum\sum (\tfrac{i}{n} - G(u_1))(\tfrac{i'}{n} - G(u_2))p^{11}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i<i'}\sum_{s>s'}\sum\sum (\tfrac{i}{n} - G(u_1))(\tfrac{i'}{n} - G(u_2))p^{12}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i<i'}\sum (\tfrac{i}{n} - G(u_1))(\tfrac{i'}{n} - G(u_2))p^{11}_{ii'}(u_1,u_2)$$

$$= \sum_{i=1}^{n-1}\sum_{i'=i+1}^{n} (\tfrac{i}{n} - G(u_1))(\tfrac{i'}{n} - G(u_2))p^{11}_{ii'}(u_1,u_2)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{i'=i+1}^{n-1} [(i-1) - (n-2)G(u_1)][(i'-2)-(n-2)G(u_2)]p^{11}_{ii'}(u_1,u_2) + \frac{(1-2G(u_1))(2-2G(u_2))}{n^2}$$

$$= \frac{1}{n^2}\,\mathrm{Cov}\Big(\sum_{i=1}^{n-2} I(X_i\le u_1),\ \sum_{j=1}^{n-2} I(X_j\le u_2)\Big) + O(\tfrac{1}{n^2})$$

continued

$$= \frac{n-2}{n^2}[G(u_1 \wedge u_2) - G(u_1)G(u_2)] + O(\frac{1}{n^2}),$$

and similarly

(3.1.24)
$$\sum_{i>i'}\sum_{s<s'}\sum\sum (\frac{i}{n} - G(u_1))(\frac{i'}{n} - G(u_2))p^{21}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i>i'}\sum_{s>s'}\sum\sum (\frac{i}{n} - G(u_1))(\frac{i'}{n} - G(u_2))p^{22}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \frac{n-2}{n^2}[G(u_1 \wedge u_2)-G(u_1)G(u_2)] + O(\frac{1}{n^2})$$

and

(3.1.25)
$$\sum_{i}\sum_{i'}\sum_{s}\sum_{s'} (\frac{i}{n} - G(u_1))(\frac{i'}{n} - G(u_2))p^{(n-1)}_{i-1,s-1}(u_1,v_1)p^{(n-1)}_{i'-1,s'-1}(u_2,v_2)$$

$$= \sum_{i=1}^{n} (\frac{i}{n} - G(u_1))p^{(n-1)}_{i-1}(u_1) \cdot \sum_{i'=1}^{n} (\frac{i'}{n} - G(u_2))p^{(n-1)}_{i'-1}(u_2)$$

$$= \frac{(1-G(u_1))(1-G(u_2))}{n^2}$$

$$= O(\frac{1}{n^2}).$$

We see that the sum of the terms involving $J_1'(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))$ reduces to

(3.1.26)
$$\textcircled{K} = \frac{(n-1)(n-2)}{n^2} \iiiint J_1'(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))$$

$$\times [G(u_1 \wedge u_2)-G(u_1)G(u_2)]v_1 v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$+ O(\frac{1}{n}).$$

Similarly the sum of the terms involving $J_2'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$ by use of relations of the type (3.1.23) to (3.1.25) with $(\frac{i}{n}-G(u_1))(\frac{i'}{n}-G(u_2))$ replaced by $(\frac{s}{n}-F(v_1))(\frac{s'}{n}-F(v_2))$, reduces to

$$(3.1.27)\qquad \textcircled{L} = \frac{(n-1)(n-2)}{n^2} \iiiint J_2'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2)) \times [F(v_1 \wedge v_2)-F(v_1)F(v_2)]v_1 v_2 dH(u_1,v_1)dH(u_2,v_2) + O(1/n).$$

Further, we have the following relations (3.1.28) to (3.1.30) which are similar to those of (3.1.23) to (3.1.25):

$$(3.1.28)\qquad \sum_{i<i'}\sum \sum_{s<s'}\sum (\frac{i}{n}-G(u_1))(\frac{s'}{n}-F(v_2))p^{11}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i=1}^{n-1}\sum_{s'=2}^{n}(\frac{i}{n}-G(u_1))(\frac{s'}{n}-F(v_2))p^{11}_{is'}(u_1,v_2)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{s'=2}^{n}[(i-1)-(n-2)G(u_1)][(s'-2)-(n-2)F(v_2)]p^{11}_{is'}(u_1,v_2)$$

$$+ \frac{(1-2G(u_1))(2-2F(v_2))}{n^2}$$

$$= \frac{1}{n^2}\operatorname{Cov}(\sum_{i=1}^{n-2} I(X_i \le u_1), \sum_{j=1}^{n-2} I(Y_j \le v_2)) + O(\frac{1}{n^2})$$

$$= \frac{n-2}{n^2}[H(u_1,v_2)-G(u_1)F(v_2)] + O(\frac{1}{n^2}),$$

and similarly

(3.1.29) $$\sum_{i<i'}\sum\ \sum_{s>s'}\sum \left(\frac{i}{n} - G(u_1)\right)\left(\frac{s'}{n} - F(v_2)\right)p^{12}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \frac{n-2}{n^2}[H(u_1,v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

$$\sum_{i>i'}\sum\ \sum_{s<s'}\sum \left(\frac{i}{n} - G(u_1)\right)\left(\frac{s'}{n} - F(v_2)\right)p^{21}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \frac{n-2}{n^2}[H(u_1,v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

$$\sum_{i>i'}\sum\ \sum_{s>s'}\sum \left(\frac{i}{n} - G(u_1)\right)\left(\frac{s'}{n} - F(v_2)\right)p^{22}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \frac{n-2}{n^2}[H(u_1,v_2) - G(u_1)F(v_2)] + O\left(\frac{1}{n^2}\right),$$

and

(3.1.30) $$\sum_{i}\sum_{i's}\sum_{s'}\sum \left(\frac{i}{n} - G(u_1)\right)\left(\frac{s'}{n} - F(v_2)\right)p^{(n-1)}_{i-1,s-1}(u_1,v_1)p^{(n-1)}_{i'-1,s'-1}(u_2,v_2)$$

$$= \sum_{i=1}^{n}\left(\frac{i}{n} - G(u_1)\right)p^{(n-1)}_{i-1}(u_1)\ \sum_{s'=1}^{n}\left(\frac{s'}{n} - F(v_2)\right)p^{(n-1)}_{s'-1}(v_2)$$

$$= \frac{(1-G(u_1))(1-F(v_2))}{n^2}$$

$$= O\left(\frac{1}{n^2}\right).$$

Now using the relations (3.1.28) to (3.1.30) and the similar kind which we get when $\left(\frac{i}{n} - G(u_1)\right)\left(\frac{s'}{n} - F(v_2)\right)$ there is replaced by $\left(\frac{i'}{n} - G(u_2)\right)\left(\frac{s}{n} - F(v_1)\right)$, the sum of terms involving $J_1'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$ and $J'_1(G(u_2), F(v_2))J_2'(G(u_1),F(v_2))$ reduces to

$$(3.1.31) \quad \textcircled{M} = \frac{2(n-1)(n-2)}{n^2} \iiiint J_1'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$

$$\times \left[H(u_1,v_2)-G(u_1)F(v_2)\right]dH(u_1,v_1)dH(u_2,v_2)$$

$$+ O(1/n).$$

By similar arguments as above we can see that all other terms will be of order $O(n^{-1/2})$ or higher. Since the arguments involved are essentially the same we shall only deal with two of those terms here, namely, $J_1''(G(u_1), F(v_1))J(G(u_2),F(v_2))$ term and $J_1''(G(u_1)F(v_1))J_2'(G(u_2),F(v_2))$ term:

Since

$$\sum_{i<i'}\sum_{s<s'}\sum\sum \left(\frac{i}{n} - G(u_1)\right)^2 p^{11}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i<i'}\sum_{s>s'}\sum\sum \left(\frac{i}{n} - G(u_1)\right)^2 p^{12}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i=1}^{n-1} \left(\frac{i}{n} - G(u_1)\right)^2 p^{(n-2)}_{i-1}(u_1)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n-1}\left[(i-1)-(n-2)G(u_1)\right]^2 p^{(n-2)}_{i-1}(u_1) + \frac{(1-2G(u_1))^2}{n^2}$$

$$= \frac{(n-2)G(u_1)(1-G(u_1))}{n^2} + O\left(\frac{1}{n^2}\right),$$

and similarly

$$\sum_{i>i'}\sum_{s<s'}\sum\sum \left(\frac{i}{n} - G(u_1)\right)^2 p^{21}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i>i'}\sum_{s>s'}\sum\sum \left(\frac{i}{n} - G(u_1)\right)^2 p^{22}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$= \sum_{i=2}^{n} \left(\frac{i}{n} - G(u_1)\right)^2 p^{(n-2)}_{i-2}(u_1)$$

$$= \frac{(n-2)G(u_1)(1-G(u_1))}{n^2} + O\left(\frac{1}{n^2}\right),$$

and

$$\sum_{i}\sum_{i's}\sum_{s'}\sum\left(\frac{i}{n}-G(u_1)\right)^2 p^{(n-2)}_{i-1,s-1}(u_1,v_1)p^{(n-2)}_{i'-1,s'-1}(u_2,v_2)$$

$$=\sum_{i=1}^{n}\left(\frac{i}{n}-G(u_1)\right)^2 p^{(n-1)}_{i-1}(u_1)$$

$$\doteq \frac{(n-1)G(u_1)(1-G_1(u_1))}{n^2}+O\left(\frac{1}{n^2}\right),$$

the sum of the terms involving $J_1''(G(u_1),F(v_1))J(G(u_2),F(v_2))$ reduces to

$$\frac{(n-1)(n-2)}{n^2}\iiiint J_1''(G(u_1),F(v_1))J(G(u_2),F(v_2))G(u_1)(1-G(u_1))v_1v_2 \, dH(u_1,v_1)dH(u_2,v_2)$$

$$-\frac{n-1}{n}\iiiint J_1''(G(u_1),F(v_1))J(G(u_2),F(v_2))G(u_1)(1-G(u_1))v_1v_2 \, dH(u_1,v_1)dH(u_2,v_2)$$

$$+O\left(\frac{1}{n}\right)$$

$$=O\left(\frac{1}{n}\right).$$

The term involving $J_1''(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$ from the second integral on the R.H.S. of (3.1.13) is

$$I=(n-1)\iiiint\left[\sum_{i<i'}\sum_{s<s'}\sum\sum\left(\frac{i}{n}-G(u_1)\right)^2\left(\frac{s'}{n}-F(v_2)\right)p^{11}_{isi's'}(u_1,v_1,u_2,v_2)\right]$$

$$\cdot J_1''(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))v_1v_2dH(u_1,v_1)dH(u_2,v_2).$$

Write

$$(3.1.32)\quad P=\sum_{i<i'}\sum_{s<s'}\sum\sum\left(\frac{i}{n}-G(u_1)\right)^2\left(\frac{s'}{n}-F(v_2)\right)p^{11}_{isi's'}(u_1,v_1,u_2,v_2)$$

$$=\sum_{i=1}^{n-1}\sum_{s'=2}^{n}\left(\frac{i}{n}-G(u_1)\right)^2\left(\frac{s'}{n}-F(v_2)\right)p^{11}_{is'}(u_1,v_2)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n-1} \sum_{s'=2}^{n} [(i-1)-(n-2)G(u_1) + (1-2G(u_1))]^2$$
$$[(s'-2)-(n-2)F(v_2)+(2-2F(v_2))] p_{is'}^{11}(u_1,v_2).$$

Now if $T_1 = \sum_{i=1}^{n-2} I(X_i \leq u_1)$ and $T_2 = \sum_{j=1}^{n-2} I(Y_j \leq v_2)$, then T_1 is Binomial $(n-2, G(u_1))$, T_2 is Binomial $(n-2, F(v_2))$ and we have the relations

(3.1.33) $$\sum_{i=1}^{n-1} [(i-1)-(n-2)G(u_1)] p_{i-1}^{(n-2)}(u_1) = E(T_1-ET_1) = 0,$$

$$\sum_{s'=2}^{n} [(s'-2)-(n-2)F(v_2)] p_{s'-2}^{(n-2)}(v_2) = E(T_2-ET_2) = 0,$$

$$\sum_{i=1}^{n-1} [(i-1)-(n-2)G(u_1)]^2 p_{i-1}^{(n-2)}(u_1) = Var(T_1) = (n-2)G(u_1)(1-G(u_1)),$$

$$\sum_{s'=2}^{n} [(s'-2)-(n-2)F(v_2)]^2 p_{s'-2}^{(n-2)}(v_2) = Var(T_2) = (n-2)F(v_2)(1-F(v_2)),$$

$$\sum_{i=1}^{n-1} \sum_{s'=2}^{n} [(i-1)-(n-2)G(u_1)][(s'-2)-(n-2)F(v_2)] = Cov(T_1,T_2)$$
$$= (n-2)[H(u_1,v_2)-G(u_1)F(v_2)],$$

and $$\left| \sum_{i=1}^{n-1} \sum_{s'=2}^{n} [(i-1)-(n-2)G(u_1)]^2 [(s'-2)-(n-2)F(v_2)] p_{is'}^{11}(u_1,v_2) \right|$$

$$= | E[(T_1-ET_1)^2(T_2-ET_2)] |$$

$$\leq [E(T_1-ET_1)^4 \ E(T_2-ET_2)^2]^{1/2}$$

$$= [O(n^2)O(n)]^{1/2} = O(n^{3/2}).$$

Using the relations (3.1.16) and (3.1.33) in (3.1.32) we see that

$$P = \frac{1}{n^3}\sum_{i=1}^{n-1}\sum_{s'=2}^{n}[(i-1)-(n-2)G(u_1)]^2[(s'-2)-(n-2)F(v_2)]p_{1s'}^{11}(u_1,v_2)$$

$$+\frac{(2-2F(v_2))}{n^3}\sum_{i=1}^{n}[(i-1)-(n-2)G(u_1)]^2 p_{i-1}^{(n-2)}(u_1)$$

$$+\frac{2(1-2G(u_1))}{n^3}\sum_{i=1}^{n-1}\sum_{s'=2}^{n}[(i-1)-(n-2)G(u_1)][(s'-2)-(n-2)F(v_2)]p_{1s'}^{11}(u_1,v_2)$$

$$+\frac{2(1-2G(u_1))^2(2-2F(v_2))}{n^3}$$

$$= O(n^{-3/2}) + O(n^{-2}) + O(n^{-2}) + O(n^{-3})$$

and consequently $I = O(n^{-1/2})$ since J_1'', J_2' are bounded and $E\,Y^2 < \infty$. On the same lines it is easy to see that the term involving $J_1''(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$ from every integral on the R.H.S. of (3.1.13) is $O(n^{-1/2})$.

All the other terms involving the higher order derivatives of J can be handled exactly as the $J_1''(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$ term and can be shown to be asymptotically negligible . Hence (3.1.17), (3.1.18), (3.1.20) to (3.1.22), (3.1.26) to (3.1.27) and (3.1.31) together yields

(3.1.34) $$n\,\mathrm{Var}(S_n) = \iint J^2(G(u),F(v))v^2 dH(u,v)$$

$$- \iiiint J(G(u_1),F(v_1))J(G(u_2),F(v_2))v_1v_2 dH(u_1,v_1)dH(u_2,v_2)$$

$$+\frac{2(n-1)}{n}\iiiint J(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))$$

$$\times [I(u_1<u_2)-G(u_2)]v_1v_2 dH(u_1,v_1)dH(u_2,v_2)$$

continued

$$+ \frac{2(n-1)}{n} \iiiint J(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$

$$\times [I(v_1<v_2)-F(v_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ \frac{(n-1)(n-2)}{n^2} \iiiint J_1'(G(u_1),F(v_1))J_1'(G(u_2),F(v_2))$$

$$\times [G(u_1 \wedge u_2)-G(u_1)G(u_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ \frac{(n-1)(n-2)}{n^2} \iiiint J_2'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$

$$\times [F(v_1 \wedge v_2)-F(v_1)F(v_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ \frac{2(n-1)(n-2)}{n^2} \iiiint J_1'(G(u_1),F(v_1))J_2'(G(u_2),F(v_2))$$

$$\times [H(u_1,v_2)-G(u_1)F(v_2)]v_1v_2dH(u_1,v_1)dH(u_2,v_2)$$

$$+ O(n^{-1/2}).$$

The Lemma now follows by taking limit, as $n \to \infty$, in (3.1.34). □

From lemma 2.3.2, lemma 3.1.1 and equation (2.1.3) it now follows that

$$\lim_{n\to\infty} E(T_n-\hat{T}_n)^2 = \lim_{n\to\infty}[Var(T_n) - Var(\hat{T}_n)] = 0$$

and hence T_n and $\hat{T}_n$ are asymptotically equivalent in mean square.

3.2 ASYMPTOTIC NORMALITY OF S_n. In this section we shall prove Theorem 3.1.1 which establishes the asymptotic normality of S_n.

PROOF OF THEOREM 3.1.1. Since $T_n = n^{1/2}(S_n-ES_n)$ and $\hat{T}_n = n^{1/2}(\hat{S}_n-E\hat{S}_n)$ are asymptotically equivalent in mean square, it suffices to prove that $\hat{T}_n$ is asymptotically normally distributed (e.g. see Corollary 2, page 349 of [10]). Using (2.2.2) and the notation set out in (2.3.1), we can write $\hat{S}_n$ (see (2.3.2)) and, therefore $\hat{T}_n$ as

$$(3.2.1) \qquad \hat{T}_n = n^{1/2}(\hat{S}_n - E\hat{S}_n) = n^{-1/2}\sum_{k=1}^{n} Z_{kn}$$

where for $k = 1,2,\ldots,n,$

$$\begin{aligned} Z_{kn} &= Y_k\tilde{Q}_n(X_k,Y_k) - \iint\tilde{Q}_n(u,v)v\,dH(u,v) \\ &\quad + \iint (n-1)[Q_n(u,v)-Q_n^*(u,v)][I(X_k\le u)-G(u)]v\,dH(u,v) \\ &\quad + \iint (n-1)[\bar{Q}_n(u,v)-Q_n(u,v)+Q_n^*(u,v)-\bar{Q}_n^*(u,v)] \\ &\qquad\qquad \times [I(X_k\le u,Y_k\le v)-H(u,v)]v\,dH(u,v) \\ &\quad + \iint (n-1)[\bar{Q}_n^*(u,v)-Q_n^*(u,v)][I(Y_k\le v)-F(v)]v\,dH(u,v). \end{aligned}$$

Now define for $k = 1,2,\ldots,n,$

$$\begin{aligned} Z_k &= Y_kJ(G(X_k),F(Y_k)) - \iint J(G(u),F(v))v\,dH(u,v) \\ &\quad + \iint J_1'(G(u),F(v))[I(X_k\le u)-G(u)]v\,dH(u,v) \\ &\quad + \iint J_2'(G(u),F(v))[I(Y_k\le v) - F(v)]v\,dH(u,v). \end{aligned}$$

Then

$$\begin{aligned} (3.2.2) \qquad \mathrm{Var}\Big(n^{-1/2}\sum_{k=1}^{n} Z_{kn} - n^{-1/2}\sum_{k=1}^{n} Z_k\Big) &= \mathrm{Var}(Z_{1n} - Z_1) \\ &= \mathrm{Var}\big[Y_1[\tilde{Q}_n(X_1,Y_1) - J(G(X_1),F(Y_1))] \\ &\quad - \iint[\tilde{Q}_n(u,v) - J(G(u), F(v))]v\,dH(u,v) \end{aligned}$$

continued

$$+ \iint [(n-1)(Q_n(u,v) - Q_n^*(u,v)) - J_1'(G(u),F(v))]$$

$$\times [I(X_1 \le u) - G(u)] v dH(u,v)$$

$$+ \iint (n-1)[\bar{Q}_n(u,v) - Q_n(u,v) + Q_n^*(u,v) - \bar{Q}_n^*(u,v)]$$

$$\times [I(X_1 \le u, Y_1 \le v) - H(u,v)] v dH(u,v)$$

$$+ \iint [(n-1)(\bar{Q}_n^*(u,v) - Q_n^*(u,v)) - J_2'(G(u),F(v))]$$

$$\times [I(Y_1 \le v) - F(v)] v dH(u,v)].$$

It follows from arguments similar to those used in the proof of Lemma 2.3.2 (see (2.3.5)) that the right hand side of (3.2.2) goes to 0 as $n \to \infty$, which implies that $n^{-1/2} \sum_{k=1}^{n} Z_{kn}$ and $n^{-1/2} \sum_{k=1}^{n} Z_k$ are asymptotically equivalent in mean square. But the Z_k's are i.i.d. with mean 0 and finite variance $\sigma^2(J,H)$. So by the standard central limit theorem $n^{-1/2} \sum_{k=1}^{n} Z_k$ is asymptotically $N(0,\sigma^2(J,H))$ and the proof of Theorem 3.1.1 is complete. □

<u>PROOF OF COROLLARY</u> 3.1.1. From (2.1.1) and (3.1.6) we have

(3.2.3) $$E(S_n) = n^{-1} \sum_{i=1}^{n} E\left[J\left(\frac{i}{n}, \frac{R_{in}}{n}\right) Y_{[i:n]}\right]$$

$$= \iint v \left[\sum_{i=1}^{n} \sum_{s=1}^{n} J\left(\frac{i}{n}, \frac{s}{n}\right) p_{i-1,s-1}^{(n-1)}(u,v)\right] dH(u,v)$$

where $p_{i,s}^{(n)}(u,v)$ is as defined by (2.2.3). Taylor's expansion of $J\left(\frac{i}{n},\frac{s}{n}\right)$ about $(G(u),F(v))$ yields

$$E(S_n) = \iint \sum_{i=1}^{n} \sum_{s=1}^{n} \{J(G(u),F(v)) + (\tfrac{i}{n} - G(u))J_1'(G(u),F(v)) + (\tfrac{s}{n} - F(v))J_2'(G(u),F(v)) + (\tfrac{i}{n} - G(u))^2 J_1''(\Delta_{in},\Delta_{sn}) + (\tfrac{s}{n} - F(v))^2 J_2''(\Delta_{in},\Delta_{sn}) + (\tfrac{i}{n} - G(u))(\tfrac{s}{n} - F(v)) J_{12}''(\Delta_{in},\Delta_{sn})\} p_{i-1,s-1}^{(n-1)}(u,v)\, v\, dH(u,v)$$

where Δ_{in} lies between $\frac{i}{n}$ and $G(u)$ and Δ_{sn} lies between $\frac{s}{n}$ and $F(v)$. On using the appropriate relations from (3.1.33), and since J_1'', J_2'', J_{12}'' are bounded, $E|Y| < \infty$, we see that

(3.2.4) $$E(S_n) = \iint J(G(u),F(v))\, v\, dH(u,v) + O(\tfrac{1}{n}) = \mu(J,H) + O(\tfrac{1}{n})$$

and it follows that

(3.2.5) $$n^{1/2}\left(E(S_n) - \mu(J,H)\right) \to 0 \text{ as } n \to \infty. \quad \square$$

Now in view of (3.2.5), $E(S_n)$ can be replaced by $\mu(J,H)$ in Theorem 3.1.1.

3.3. TWO IMPORTANT SPECIAL CASES. In this section we shall prove Theorem 3.1.2 which covers the results of Stigler [17] and Yang [22] as special cases of Theorem 3.1.1.

<u>PROOF OF THEOREM</u> 3.1.2.

First observe that our statistic S_n itself reduces to that of Stigler (1974) and Yang (1981) under the assumptions $J(x,y) = J(y)$, $0 \leq x, y \leq 1$, and $J(x,y) = J(x)$, $0 \leq x, y \leq 1$, respectively.

For convenience we will write $V(S_n)$, $V_S(S_n)$ and $V_Y(S_n)$ to denote the expressions for the asymptotic variance of the statistic $n^{1/2}S_n$ as considered in this thesis, in Stigler [17] and in Yang [22] respectively. Also let $\sigma^2(x) = \mathrm{Var}(Y|X=x)$, $m(x) = E(Y|X = x)$.

We will first prove part (b): Under the assumption $J(x,y) = J(x)$, $0 \le x, y \le 1$, $V(S_n)$ given by (2.3.9) reduces to

(3.3.1)
$$\begin{aligned} V(S_n) = & \iint J^2(G(u))v^2 dH(u,v) + \iiiint J(G(u_1))J(G(u_2))v_1v_2 dH(u_1,v_1) dH(u_2,v_2) \\ & + 2\iiiint J(G(u_1))J'(G(u_2))[I(u_1 \le u_2)-G(u_2)]v_1v_2 dH(u_1,v_1) dH(u_2,v_2) \\ & + \iiiint J'(G(u_1))J'(G(u_2))[G(u_1 \wedge u_2)-G(u_1)G(u_2)]v_1v_2 dH(u_1,v_1)dH(u_2,v_2) \\ = & \int J^2(G(u))(\sigma^2(u) + m^2(u))dG(u) \\ & - \iint J(G(u_1))J(G(u_2))m(u_1)m(u_2)dG(u_1)dG(u_2) \\ & + 2\iint J(G(u_1))J'(G(u_2))[I(u_1 \le u_2) - G(u_2)]m(u_1)m(u_2) dG(u_1)dG(u_2) \\ & + \iint J'(G(u_1))J'(G(u_2))[G(u_1 \wedge u_2)-G(u_1)G(u_2)]m(u_1)m(u_2) dG(u_1)dG(u_2). \end{aligned}$$

Now from Yang [22], we have

(3.3.2)
$$\begin{aligned} V_Y(S_n) = & \int J^2(G(u))\sigma^2(u)dG(u) \\ & + \iint [G(u_1 \wedge u_2) - G(u_1)G(u_2)] \times J(G(u_1))J(G(u_2))dm(u_1)dm(u_2). \end{aligned}$$

Consider first

$$(3.3.3) \qquad \iint[G(u_1 \wedge u_2) - G(u_1)G(u_2)]J(G(u_1))J(G(u_2))dm(u_1)dm(u_2)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{u_2} G(u_1)(1-G(u_2))J(G(u_1))J(G(u_2))dm(u_1)dm(u_2)$$

$$+ \int_{-\infty}^{\infty}\int_{u_2}^{\infty} G(u_2)(1-G(u_1))J(G(u_1))J(G(u_2))dm(u_1)dm(u_2).$$

$$= I_1 + I_2.$$

Now

$I_1 = \int_{-\infty}^{\infty}(\int_{-\infty}^{u_2} G(u_1)J(G(u_1))dm(u_1))J(G(u_2))(1-G(u_2))dm(u_2)$, which expression upon integration by parts for the inside integral above reduces to

$$= \int m(u_2)J^2(G(u_2))G(u_2)(1-G(u_2))dm(u_2)$$

$$- \iint I(u_1 \le u_2)[J(G(u_1)) + G(u_1)J'(G(u_1))]m(u_1)J(G(u_2))(1-G(u_2)) \, dm(u_2)dG(u_1)$$

$$= \int m(u_2)J^2(G(u_2))G(u_2)(1-G(u_2))dm(u_2)$$

$$- \int_{-\infty}^{\infty}[\int_{u_1}^{\infty} J(G(u_2))(1-G(u_2))dm(u_2)][J(G(u_1))+G(u_1)J'(G(u_1))]m(u_1) \, dG(u_1).$$

Using integration by parts again to the inside integral of the second term in the above expression we see that

$$(3.3.4) \qquad I_1 = \int m(u_2)J^2(G(u_2))\, G(u_2)(1-G(u_2))dm(u_2)$$

$$- \int_{-\infty}^{\infty}\{-m(u_1)J(G(u_1))(1-G(u_1)) - \int_{u_1}^{\infty} m(u_2)[-J(G(u_2))+(1-G(u_2))J'(G(u_2))] \, dG(u_2)\}$$

$$[J(G(u_1)) + G(u_1)J'(G(u_1))]m(u_1)dG(u_1)$$

continued

$$= \int m(u_2)J^2(G(u_2))G(u_2)(1-G(u_2))dm(u_2)$$

$$+ \int J^2(G(u_1))m^2(u_1)(1-G(u_1))dG(u_1)$$

$$+ \int J(G(u_1))J'(G(u_1))G(u_1)(1-G(u_1))m^2(u_1)dG(u_1)$$

$$- \iint_{u_1 \le u_2} J(G(u_1))J(G(u_2))m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$- \iint_{u_1 \le u_2} J(G(u_2))J'(G(u_1))G(u_1)m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$+ \iint_{u_1 \le u_2} J(G(u_1))J'(G(u_2))(1-G(u_2))m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$+ \iint_{u_1 \le u_2} J'(G(u_1))J'(G(u_2))G(u_1)(1-G(u_2))m(u_1)m(u_2)dG(u_1)dG(u_2).$$

Proceeding exactly on the same lines we can see that

(3.3.5)
$$I_2 = -\int m(u_2)J^2(G(u_2))G(u_2)(1-G(u_2))dm(u_2)$$

$$+ \int J^2(G(u_1))m^2(u_1)G(u_1)dG(u_1)$$

$$- \int J(G(u_1))J'(G(u_1))m^2(u_1)G(u_1)(1-G(u_1))dG(u_1)$$

$$- \iint_{u_1 > u_2} J(G(u_1))J(G(u_2))m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$- \iint_{u_1 > u_2} J(G(u_1))J'(G(u_2))G(u_2)m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$+ \iint_{u_1 > u_2} J(G(u_2))J'(G(u_1))(1-G(u_1))m(u_1)m(u_2)dG(u_1)dG(u_2)$$

$$+ \iint_{u_1 > u_2} J'(G(u_1))J'(G(u_2))G(u_2)(1-G(u_1))m(u_1)m(u_2)dG(u_1)dG(u_2).$$

Now using (3.3.3) to (3.3.5) in (3.3.2) we see that

$$V(S_n) = V_Y(S_n)$$

where $V(S_n)$ is as given by (3.3.1).

(a) First observe that under the assumption $J(x,y) = J(y)$, $0 \le x \le y \le 1$ $V(S_n)$ given by (2.3.9) reduces to

(3.3.6) $$V(S_n) = \int J^2(F(v))v^2 dF(v) - \iint J(F(v_1))J(F(v_2))v_1 v_2 dF(v_1)dF(v_2)$$
$$+ 2\iint J(F(v_1))J'(F(v_2))[I(v_1 \le v_2) - F(v_2)]v_1 v_2 dF(v_1)dF(v_2)$$
$$+ \iint J'(F(v_1))J'(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]v_1 v_2 dF(v_1)dF(v_2),$$

and from Stigler [17]

(3.3.7) $$V_S(S_n) = \iint J(F(v_1))J(F(v_2))[F(v_1 \wedge v_2) - F(v_1)F(v_2)]du_1 dv_1.$$

Now by writing $m(u) = u$ and replacing G by F in (3.3.3) the expression on the L.H.S. of (3.3.3) reduces to that of (3.3.7) and consequently it is easy to see that equations (3.3.3) to (3.3.5) with the corresponding changes yields

$$V(S_n) = V_S(S_n). \quad \square$$

CHAPTER IV

A CONSISTENT CONDITIONAL QUANTILE ESTIMATOR

This chapter essentially deals with the consistency of a nonparametric estimator $Q_{n,x_o}(\lambda)$, of the λ^{th} quantile $0 < \lambda < 1$, proposed by Mehra [12] (See 4.1.1 below) of the conditional distribution of Y given $X = x_o$. Certain preliminary results obtained in this regard are also referred to in Chapter V. In section 4.1 the main result of this chapter, namely, the consistency of the estimator $Q_{n,x_o}(\lambda)$ is stated along with a brief outline of the method of proof and a useful decomposition of $Q_{n,x_o}(\lambda)$ is obtained which is basic for all the subsequent results. In section 4.2 we define the conditional empirical process and show that its supremum is stochastically bounded under certain conditions. Though this result is required here only in proving our main theorem, it may also be of some independent interest. Sections 4.3 and 4.4 are devoted to prove some further preliminary results and the proof of the main result is given in Section 4.5.

4.1 THE ESTIMATOR $Q_{n,x_o}(\lambda)$. Let (X_1,Y_1), $(X_2,Y_2),\ldots,(X_n,Y_n)$ be independent and identically distributed rv's with absolutely continuous cumulative distribution function (cdf) $H(x,y)$, and continuous joint probability density function $h(x,y)$, $-\infty < x, y < \infty$. Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ denote the X-variate order statistics and $Y_{[1:n]}, Y_{[2:n]},\ldots,Y_{[n:n]}$ denote the corresponding Y-variate induced order statistics. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \to 0$ as $n \to \infty$. The following estimator $Q_{n,x_o}(\lambda)$ is proposed by Mehra [12] as a nonparametric kernel estimator of the λ^{th} quantile of the conditional distribution of Y given $X = x_o$ (for $-\infty < x_o < \infty$, $0 < \lambda < 1$):

$$(4.1.1.)\qquad Q_{n,x_o}(\lambda) = \frac{1}{na_n^2}\sum_{i=1}^{n} K\left(\frac{G_n(X_{(i:n)}) - G_n(x_o)}{a_n}, \frac{F_{n,x_o}(Y_{[i:n]})-\lambda}{a_n}\right)Y_{[i:n]}$$

$$= \frac{1}{na_n^2}\sum_{i=1}^{n} K\left(\frac{G_n(X_i)-G_n(x_o)}{a_n}, \frac{F_{n,x_o}(Y_i)-\lambda}{a_n}\right) Y_i,$$

where $G_n(.)$ is the marginal empirical distribution function of X-observations,

$F_{n,x_o}(.)$ is the "conditional empirical distribution function" of Y-observations given $x = x_o$ defined by

$$(4.1.2)\qquad F_{n,x_o}(t) = \frac{1}{na_n}\sum_{j=1}^{n} K^*\left(\frac{G_n(X_j)-G_n(x_o)}{a_n}\right) I(Y_j \le t),$$

and $K^*(.)$, $K(.,.)$ are, respectively, a univariate and bivariate kernel density functions satisfying the following conditions:

(4.1.3) $K(.,.)$ is a bivariate continuous probability density function (pdf) with $\nu(4 \le \nu \le 6)$ bounded partial derivatives and has compact support,

(4.1.4) $K^*(.)$ is a univariate continuous probability density function with $\nu^*(2 \le \nu^* \le 4)$ bounded derivatives and has compact support,

(4.1.5) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\, k(u,v)\,du\,dv = 0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v\, k(u,v)\,du\,dv$,

(4.1.6.) $\int_{-\infty}^{\infty} t\, K^*(t)\,dt = 0.$

Let $G, g(F,f)$ denote, respectively, the marginal cdf and pdf of X (of Y). Also, let $F_x(f_x)$ denote the conditional cdf(pdf) of Y given $X = x$. The λ^{th} quantile of the conditional distribution of Y given $X = x_o$ is defined by

(4.1.7) $$Q_{x_o}(\lambda) = F_{x_o}^{-1}(\lambda) = \inf\{t: F_{x_o}(t) \geq \lambda\},\ 0 < \lambda < 1.$$

The main object of this chapter is to prove the consistency of the estimator $Q_{n,x_o}(\lambda)$ viz. to prove that $Q_{n,x_o}(\lambda) \xrightarrow{pr} Q_{x_o}(\lambda)$ as $n \to \infty$. Our method of proof is as follows: First, we shall decompose $Q_{n,x_o}(\lambda)$, by the Taylor expansion using the differentiability properties of K, as

(4.1.8) $$Q_{n,x_o}(\lambda) = I_1 + I_2 + I_3 + R_n$$

where I_1 is the "constant" term, I_2 and I_3 are the first order derivative trerms and R_n includes all the terms with second and higer order derivatives of K. Using the differentiability properties of K^*, the I_3 term is further decomposed by the Taylor expansion to $I_{31} + I_{32} - I_{30} + R'_n$, where I_{31} and I_{32} are, respectively, the "constant" and first order derivative terms with respect to K^*, I_{30} is that part of I_3 term which does not include K^* and R'_n includes all the terms with second and higher order derivatives of K^*, so that we have the final decomposition of $Q_{n,x_o}(\lambda)$ given by

(4.1.9) $$Q_{n,x_o}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30} + R_n + R'_n$$

$$= S_{n,x_o}(\lambda) + R_n + R'_n.$$

(For the exact expressions of I_1, I_2, I_{31}, I_{32}, I_{30}, R_n and R_n' (see (4.1.20) to (4.1.24)) . From (4.1.9) we write

(4.1.10) $\quad Q_{n,x_o}(\lambda) - Q_{x_o}(\lambda) = [S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)] + R_n + R_n'$ and then show

that each of the terms R_n, R_n' and $[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)]$ converge to zero in probability as $n \to \infty$, which proves the consistency of $Q_{n,x_o}(\lambda)$. Convergence of $[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)]$ to zero in probability is proved by showing that $na_n^2 \cdot \mathrm{Var}(S_{n,x_o}(\lambda) - Q_{x_o}(\lambda))$ goes to a finite value as $n \to \infty$ and that $E(S_{n,x_o}(\lambda) - Q_{x_o}(\lambda))$ goes to zero as $n \to \infty$. The result that $na_n^2 \, \mathrm{Var}(S_{n,x_o}(\lambda))$ goes to a finite value as $n \to \infty$ is required in Chapter V to prove the asymptotic normality of $Q_{n,x_o}(\lambda)$ in a properly normalized form.

We shall assume that $E(X^2) < \infty$ and $E(Y^2) < \infty$ throughout Chapters IV and V. We also make the following assumptions on the joint distribution function $H(x,y)$:

(4.1.11) $H(.,.)$ has bounded partial derivatives up to the fourth order in some neighborhood of $(x_o, F_{x_o}^{-1}(\lambda))$.

(4.1.12) $f_{x_o}(.)$ has two bounded derivaties in some neighborhood of $F_{x_o}^{-1}(\lambda)$.

(4.1.13) $h(.,.)$ is positive in some neighborhood of $(x_o, F_{x_o}^{-1}(\lambda))$.

(4.1.14) $f_x(y)$ is bounded uniformly in x and y, $-\infty < x < \infty$, $-\infty < y < \infty$.

(4.1.15) $H_1'(x,y)$, $H_1''(x,y)$, $H_1'''(x,y)$ and $H_1''''(x,y)$ exist for all $y \in \mathbb{R}$ and x in some neighborhood of x_o and are uniformly bounded in y.

We now state the main result of this chapter.

THEOREM 4.1.1. *Let* $Q_{n,x_o}(\lambda)$, $0 < \lambda < 1$ *be as defined by* (4.1.1). *Suppose assumptions* (4.1.3) *to* (4.1.6) *on* K *and* K^* *are satisfied with* $\nu = 4$ *and* $\nu^* = 3$ *and assumptions* (4.1.11) *to* (4.1.15) *on the joint distribution* H *are also satisfied. Then for each sequence* $\{a_n\}$ *of positive numbers with* $a_n \to 0$, $na_n^4 \to \infty$, *but* $na_n^5 \to 0$ *we have*

$$Q_{n,x_o}(\lambda) \to Q_{x_o}(\lambda) \text{ in probability, as } n \to \infty .$$

For the proof of the theorem, given in Section 4.5, we first obtain the representation (4.1.9) of $Q_{n,x_o}(\lambda)$, and prove some preliminary results given as lemmas 4.1.1, 4.2.1, 4.3.1 and 4.4.1 to 4.4.4.

Now observe that $Q_{n,x_o}(\lambda)$ defined by (4.1.1) may be written as

$$Q_{n,x_o}(\lambda) = a_n^{-2} \iint K\left(\frac{G_n(x_1) - G_n(x_o)}{a_n}, \frac{F_{n,x_o}(y_1)-\lambda}{a_n}\right) y_1 \, dH_n(x_1,y_1) \tag{4.1.16}$$

where H_n is the bivariate empirical distribution function of (X,Y). Since K is four times differentiable, the Taylor expansion of K about

$\left(\frac{G(x_1)- G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$ yields

(4.1.17) $$Q_{n,x_o}(\lambda) = a_n^{-2}\iint K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1)$$

$$+ a_n^{-3}\iint[G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)]$$

$$K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1)$$

$$+ a_n^{-3}\iint[F_{n,x_o}(y_1)- F_{x_o}(y_1)]K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1)$$

$$+ \sum_{m=2}^{3}\frac{a_n^{-2-m}}{m!}\iint[(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o))\frac{\partial}{\partial t_1}$$

$$+ (F_{n,x_o}(y_1) - F_{x_o}(y_1))\frac{\partial}{\partial t_2}]^m$$

$$K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1))$$

$$+ \frac{a_n^{-6}}{4!}\iint[G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)]\frac{\partial}{\partial t_1}$$

$$+ (F_{n,x_o}(y_1)-F_{x_o}(y_1))\frac{\partial}{\partial t_2}]^4 K(\Delta_{x_1,n}, \Delta_{y_1,n})y_1 dH_n(x_1,y_1)$$

$$= I_1 + I_2 + I_3 + \sum_{m=2}^{4}\eta_{nm}$$

where $K_i'(\theta_1,\theta_2)$ is the first order partial derivative of K with respect to the i^{th} argument evaluated at (θ_1,θ_2), $i=1, 2$,

$\Delta_{x_1,n}$ lies between $a_n^{-1}[G_n(x_1)-G_n(x_o)]$ and $a_n^{-1}[G(x_1)-G(x_o)]$,

$\Delta_{y_1,n}$ lies between $a_n^{-1}[F_{n,x_o}(y_1)-\lambda]$ and $a_n^{-1}[F_{x_o}(y_1)-\lambda]$.

Similarly, since K^* is three times differentiable, by the Taylor expansion of K^*, $F_{n,x_o}(t)$ defined by (4.1.2) may be expressed as

$$(4.1.18)\quad F_{n,x_o}(t) = a_n^{-1} \iint K^*\left(\frac{G(x_2) - G(x_o)}{a_n}\right) I(y_2 \le t)\, dH_n(x_2,y_2)$$

$$+ a_n^{-2} \iint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]$$

$$K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) I(y_2 \le t)\, dH_n(x_2,y_2)$$

$$+ \frac{a_n^{-3}}{2!} \iint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^2$$

$$K^{*\prime\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) I(y_2 \le t)\, dH_n(x_2,y_2)$$

$$+ \frac{a_n^{-4}}{3!} \iint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^3 K^{*\prime\prime\prime}(\Delta_{x_2,n})$$

$$I(y_2 \le t)\, dH_n(x_2,y_2)$$

where $K^{*\prime}$, $K^{*\prime\prime}$, $K^{*\prime\prime\prime}$ are, respectively, the first, second and third derivatives of K^*, and $\Delta_{x_2,n}$ lies between $a_n^{-1}[G_n(x_2) - G(x_o)]$ and $a_n^{-1}[G(x_2) - G(x_o)]$. In view of (4.1.18), I_3 of (4.1.17) can be written as

$$(4.1.19)\quad I_3 = a_n^{-4} \iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1) dH_n(x_1,y_1) dH_n^*(x_2,y_2)$$

$$+ a_n^{-5} \iiiint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]$$

$$K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1) dH_n(x_1,y_1) dH_n(x_2,y_2)$$

$$+ \frac{a_n^{-6}}{2} \iiiint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^2$$

$$K^{*''}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1) dH_n(x_1,y_1) dH_n(x_2,y_2)$$

$$+ \frac{a_n^{-7}}{3!} \iiiint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^3$$

$$K^{*'''}(\Delta_{x_2,n})\, K_2'\left(\frac{G(x_1-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1) dH_n(x_1,y_1) dH_n(x_2,y_2)$$

$$- a_n^{-3} \iint F_{x_o}(y_1) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 dH_n(x_1,y_1)$$

$$= I_{31} + I_{32} + \sum_{m=2}^{3} \zeta_{nm} - I_{30}.$$

From (4.1.17) and (4.1.19) we now have the representation of $Q_{n,x_o}(\lambda)$ as

(4.1.20) $$Q_{n,x_o}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30} + \sum_{m=2}^{3} \zeta_{nm} + \sum_{m=2}^{4} \eta_{nm}$$

$$= S_{n,x_o}(\lambda) + \sum_{m=2}^{3} \zeta_{nm} + \sum_{m=2}^{4} \eta_{nm}$$

where $S_{n,x_o}(\lambda)$ is given by (4.1.21), ζ_{nm} are given by (4.1.22) to (4.1.23) and η_{nm} are given by (4.1.24) to (4.1.25) below:

(4.1.21) $$S_{n,x_o}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30}$$

$$= a_n^{-2}\iint K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1)$$

$$+ a_n^{-3}\iint [G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)]$$

$$K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1)$$

$$+ a_n^{-4}\iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1) dH_n(x_1,y_1) dH_n(x_2,y_2)$$

continued

$$+a_n^{-5}\iiiint[G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_2 \le y_1)dH_n(x_1,y_1)$$

$$dH_n(x_2,y_2)$$

$$-a_n^{-3}\iint F_{x_o}(y_1)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1).$$

(4.1.22) $$\zeta_{n2} = \frac{a_n^{-6}}{2!}\iiiint K^{*\prime\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1)[G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^2$$

$$dH_n(x_1,y_1)dH_n(x_2,y_2).$$

(4.1.23) $$\zeta_{n3} = \frac{a_n^{-7}}{3!}\iiiint K^{*\prime\prime\prime}(\Delta_{x_2,n})\,K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1)[G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^3$$

$$dH_n(x_1,y_1)dH_n(x_2,y_2).$$

(4.1.24) $$\eta_{nm} = \frac{a_n^{-2-m}}{m!}\iint\Big[\big(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)\big)\frac{\partial}{\partial t_1} +$$

$$\big(F_{n,x_o}(y_1) - F_{x_o}(y_1)\big)\frac{\partial}{\partial t_2}\Big]^m$$

$$K\left(\frac{G(x_1)-G(x_o)}{a_n},\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 dH_n(x_1,y_1),$$

for $m = 2,3$.

$$(4.1.25) \qquad \eta_{n4} = \frac{a_n^{-6}}{4!} \iint [(G_n(x_1) - G_n(x_1) - G(x_1) + G(x_0)) \frac{\partial}{\partial t_1}$$

$$+ (F_{n,x_0}(y_1) - F_{x_0}(y_1)) \frac{\partial}{\partial t_2}]^4 K(\Delta_{x_1,n}, \Delta_{y_1,n}) y_1 dH_n(x_1, y_1).$$

LEMMA 4.1.1. Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \to 0$ as $n \to \infty$ and u and v be real numbers such that $|u|$, $|v| \leq 1$. Then the following Taylor's expansions are valid for sufficiently large n, say, $n > N$:

(i) $F_{x_0}^{-1}(\lambda + a_n v) = F_{x_0}^{-1}(\lambda) + a_n v D_1 + a_n^2 v^2 D_2 + O(a_n^3)$,

(ii) $F_{G^{-1}(G(x_0) + a_n u)}(F_{x_0}^{-1}(\lambda + a_n v)) = \lambda + a_n u D_3 + a_n v + a_n^2 u^2 D_4$

$$+ a_n^2 uv D_5 + O(a_n^3),$$

(iii)
$$\frac{h[G^{-1}(G(x_0) + a_n u), F_{x_0}^{-1}(\lambda + a_n v)]}{g[G^{-1}(G(x_0) + a_n u)]\, f_{x_0}[F_{x_0}^{-1}(\lambda + a_n v)]}$$

$$= 1 + a_n u D_6 + a_n^2 u^2 D_7 + a_n^2 uv D_8 + O(a_n^3),$$

where $\quad D_1 = D_1(x_0, \lambda) = \dfrac{1}{f_{x_0}(F_{x_0}^{-1}(\lambda))}$,

$$(4.1.26) \qquad D_2 = D_2(x_0, \lambda) = -f'_{x_0}(F_{x_0}^{-1}(\lambda))/2f_{x_0}^3(F_{x_0}^{-1}(\lambda))$$

$$D_3 = D_3(x_0, \lambda) = \frac{1}{g^2(x_0)}[H_1''(x_0, F_{x_0}^{-1}(\lambda)) - \lambda g'(x_0)],$$

continued

$$D_4 = D_4(x_o,\lambda) = \frac{1}{2g^4(x_o)}\left[g(x_o)H_1'''((x_o),F_{x_o}^{-1}(\lambda)) - 3g'(x_o)\right.$$

$$\left.\times H_1''(x_o,F_{x_o}^{-1}(\lambda)) - \lambda(g''(x_o)g(x_o) - 3(g'(x_o))^2)\right],$$

$$D_5 = D_5(x_o,\lambda) = \frac{H_{21}''(x_o,F_{x_o}^{-1}(\lambda))}{h(x_o,F_{x_o}^{-1}(\lambda))g(x_o)} - \frac{g'(x_o)}{g^2(x_o)},$$

(4.1.26)

$$D_6 = D_6(x_o,\lambda) = \frac{h_1'(x_o,F_{x_o}^{-1}(\lambda))}{h(x_o,F_{x_o}^{-1}(\lambda))g(x_o)} - \frac{g'(x_o)}{g^2(x_o)},$$

$$D_7 = D_7(x_o,\lambda) = \frac{1}{2g^3(x_o)} \quad \frac{g(x_o)h_1''(x_o,F_{x_o}^{-1}(\lambda)) - 3g'(x_o)h_1'(x_o,F_{x_o}^{-1}(\lambda))}{h(x_o,F_{x_o}^{-1}(\lambda))}$$

$$- \frac{g''(x_o)g(x_o) - 3(g'(x_o))^2}{g(x_o)},$$

$$D_8 = D_8(x_o,\lambda) = \frac{h_{12}''(x_o,F_{x_o}^{-1}(\lambda)) + g'(x_o)f_{x_o}'(F_{x_o}^{-1}(\lambda))}{h^2(x_o,F_{x_o}^{-1}(\lambda))}.$$

PROOF: (i) Taylor's expansion of $F_{x_o}^{-1}$ about λ yields

$$F_{x_o}^{-1}(\lambda+a_n v) = F_{x_o}^{-1}(\lambda) + \frac{a_n v}{f_{x_o}(F_{x_o}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_o}^3(F_{x_o}^{-1}(\lambda))} f'_{x_o}(F_{x_o}^{-1}(\lambda)) + O(a_n^3).$$

(ii) First, note that we can write

$$F_x(y) = \frac{H_1'(x,y)}{g(x)} \quad \text{and hence}$$

$$F_{G^{-1}(G(x_o)+a_n u)}(F_{x_o}^{-1}(\lambda+a_n v)) = \frac{H_1'[G^{-1}(G(x_o)+a_n u),\ F_{x_o}^{-1}(\lambda+a_n v)]}{g[G^{-1}(G(x_o)+a_n u)]}.$$

By the Taylor expansion of the inside functions first,

$$= \frac{H_1'\left[x_o + \frac{a_n u}{g(x_o)} - \frac{a_n^2 u^2}{2g^3(x_o)} g'(x_o) + O(a_n^3),\ F_{x_o}^{-1}(\lambda) + \frac{a_n v}{f_{x_o}(F_{x_o}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_o}^3(F_{x_o}^{-1}(\lambda))} f'_{x_o}(F_{x_o}^{-1}(\lambda)) + O(a_n^3)\right]}{g\left[x_o + \frac{a_n u}{g(x_o)} - \frac{a_n^2 u^2}{2g^3(x_o)} g'(x_o) + O(a_n^3)\right]}.$$

Now expanding H_1' about $(x_o, F_{x_o}^{-1}(\lambda))$ and g about x_o,

$$= \left[H_1'(x_o, F_{x_o}^{-1}(\lambda)) + \left(\frac{a_n u}{g(x_o)} - \frac{a_n^2 u^2\, g'(x_o)}{2g^3(x_o)}\right) H_1''(x_o, F_{x_o}^{-1}(\lambda))\right.$$

continued

$$+ \left[\frac{a_n v}{f_{x_o}(F_{x_o}^{-1}(\lambda))} - \frac{a_n^2 v^2}{2f_{x_o}^3(F_{x_o}^{-1}(\lambda))} f'_{x_o}(F_{x_o}^{-1}(\lambda)) \right] H''_{12}(x_o, F_{x_o}^{-1}(\lambda)) + \frac{a_n^2 u^2}{2g^2(x_o)} H'''_1(x_o, F_{x_o}^{-1}(\lambda))$$

$$+ \frac{a_n^2 v^2}{2f_{x_o}^2(F_{x_o}^{-1}(\lambda))} H'''_{122}(x_o, F_{x_o}^{-1}(\lambda)) + \frac{a_n^2 uv}{h(x_o, F_{x_o}^{-1}(\lambda))} H'''_{112}(x_o, F_{x_o}^{-1}(\lambda)) + O(a_n^3) \Bigg]$$

$$\left[g(x_o) + \frac{a_n u}{g(x_o)} g'(x_o) - \frac{a_n^2 u^2}{2g^3(x_o)} (g'(x_o))^2 + \frac{a_n^2 u^2}{2g^2(x_o)} g''(x_o) + O(a_n^3) \right]^{-1}$$

$$= \left[\frac{H'_1(x_o, F_{x_o}^{-1}(\lambda))}{g(x_o)} + \frac{a_n u}{g^2(x_o)} H''_1(x_o, F_{x_o}^{-1}(\lambda)) + \frac{a_n v}{h(x_o, F_{x_o}^{-1}(\lambda))} H''_{12}(x_o, F_{x_o}^{-1}(\lambda)) \right.$$

$$+ \frac{a_n^2 u^2}{2g^4(x_o)} [g(x_o) H'''_1(x_o, F_{x_o}^{-1}(\lambda)) - g'(x_o) H''_1(x_o, F_{x_o}^{-1}(\lambda))] + \frac{a_n^2 v^2}{2h(x_o, F_{x_o}^{-1}(\lambda)) f_{x_o}^2(F_{x_o}^{-1}(\lambda))}$$

$$\left[f_{x_o}(F_{x_o}^{-1}(\lambda)) H'''_{122}(x_o, F_{x_o}^{-1}(\lambda)) - f'_{x_o}(F_{x_o}^{-1}(\lambda)) H''_{12}(x_o, F_{x_o}^{-1}(\lambda)) \right]$$

$$\left. + \frac{a_n^2 uv H'''_{112}(x_o, F_{x_o}^{-1}(\lambda))}{h(x_o, F_{x_o}^{-1}(\lambda)) g(x_o)} + O(a_n^3) \right] \left[1 - \frac{a_n u}{g^2(x_o)} g'(x_o) - \frac{a_n^2 u^2}{2g^4(x_o)} [g''(x_o) g(x_o) - 3(g'(x_o))^2] + O(a_n^3) \right]$$

$$= \lambda + \frac{a_n u}{g^2(x_o)} (H''_1(x_o, F_{x_o}^{-1}(\lambda)) - \lambda g'(x_o)) + \frac{a_n v}{h(x_o, F_{x_o}^{-1}(\lambda))} H''_{12}(x_o, F_{x_o}^{-1}(\lambda)) + \frac{a_n^2 u^2}{2g^4(x_o)}$$

$$\left[g(x_o) H'''_1(x_o, F_{x_o}^{-1}(\lambda)) - 3H''_1(x_o, F_{x_o}^{-1}(\lambda)) g'(x_o) - \lambda (g''(x_o) g(x_o) - 3(g'(x_o))^2) \right]$$

continued

$$+ \frac{a_n^2 v^2}{2h(x_o,F^{-1}(\lambda)) f^2_{x_o}(F^{-1}_{x_o}(\lambda))} \Big[f_{x_o}(F^{-1}_{x_o}(\lambda)) H'''_{122}(x_o, F^{-1}_{x_o}(\lambda))$$

$$- f'_{x_o}(F^{-1}_{x_o}(\lambda)) H''_{12}(x_o, F^{-1}_{x_o}(\lambda)) \Big] \pm a_n^2 uv \left[\frac{H'''_{112}(x_o, F^{-1}_{x_o}(\lambda))}{h(x_o, F^{-1}_{x_o}(\lambda)) g(x_o)} - \frac{g'(x_o)}{g^2(x_o)} \right] + O(a_n^3).$$

Since

$$H_{12}(x,y) = h(x,y)$$

$$H'''_{122}(x,y) = \frac{\partial}{\partial y} h(x,y) = \frac{\partial}{\partial y}[g(x) f_x(y)] = g(x) f'_x(y),$$

the coefficient of $a_n v$ is equal to one and that of $a_n^2 v^2$ is equal to zero in the above expression and hence is (ii) of the lemma.

(iii) Proceeding on the same lines as in (ii), we have

$$\frac{h[G^{-1}(G(x_o)+a_n u), F^{-1}_{x_o}(\lambda+a_n v)]}{g[G^{-1}(G(x_o)+a_n u)] f_{x_o}[F^{-1}_{x_o}(\lambda+a_n v)]}$$

$$= \left[h(x_o, F^{-1}_{x_o}(\lambda)) + \left(\frac{a_n u}{g(x_o)} - \frac{a_n^2 u^2 g'(x_o)}{2g^3(x_o)}\right) h'_1(x_o, F^{-1}_{x_o}(\lambda)) + \left[\frac{a_n v}{f_{x_o}(F^{-1}_{x_o}(\lambda))} \right.\right.$$

$$\left. - \frac{a_n^2 v^2}{2 f^3_{x_o}(F^{-1}_{x_o}(\lambda))} f'_{x_o}(F^{-1}_{x_o}(\lambda)) \right] h'_2(x_o, F^{-1}_{x_o}(\lambda)) + \frac{a_n^2 u^2}{2g^2(x_o)} h''_1(x_o, F^{-1}_{x_o}(\lambda))$$

continued

$$+ \frac{a_n^2 v^2}{2 f^2_{x_o}(F^{-1}_{x_o}(\lambda))} h_2''(x_o, F^{-1}_{x_o}(\lambda)) + \frac{a_n^2 uv}{h(x_o, F^{-1}_{x_o}(\lambda))} h_{12}''(x_o, F^{-1}_{x_o}(\lambda)) + O(a_n^3) \Bigg]$$

$$\Bigg[g(x_o) + \frac{a_n u}{g(x_o)} g'(x_o) - \frac{a_n^2 u^2}{2g^3(x_o)} (g'(x_o))^2 + \frac{a_n^2 u^2}{2g^2(x_o)} g''(x_o) + O(a_n^3) \Bigg]^{-1} \Bigg[f_{x_o}(F^{-1}_{x_o}(\lambda))$$

$$+ \frac{a_n v}{f_{x_o}(F^{-1}_{x_o}(\lambda))} f'_{x_o}(F^{-1}_{x_o}(\lambda)) - \frac{a_n^2 v^2}{2 f^3_{x_o}(F^{-1}_{x_o}(\lambda))} (f'_{x_o}(F^{-1}_{x_o}(\lambda)))^2 + \frac{a_n^2 v^2}{2 f^2_{x_o}(F^{-1}_{x_o}(\lambda))} f''_{x_o}(F^{-1}_{x_o}(\lambda))$$

$$+ O(a_n^3) \Bigg]^{-1}$$

$$= \Bigg[\frac{h(x_o, F^{-1}_{x_o}(\lambda))}{g(x_o) f_{x_o}(F^{-1}_{x_o}(\lambda))} + \frac{a_n u}{g(x_o)} \frac{h_1'(x_o, F^{-1}_{x_o}(\lambda))}{h(x_o, F^{-1}_{x_o}(\lambda))} + \frac{a_n v}{f_{x_o}(F^{-1}_{x_o}(\lambda))} \frac{h_2'(x_o, F^{-1}_{x_o}(\lambda))}{h(x_o, F^{-1}_{x_o}(\lambda))}$$

$$+ \frac{a_n^2 u^2}{2g^3(x_o)} \left[\frac{g(x_o) h_1''(x_o, F^{-1}_{x_o}(\lambda)) - g'(x_o) h_1'(x_o, F^{-1}_{x_o}(\lambda))}{h(x_o, F^{-1}_{x_o}(\lambda))} \right] + \frac{a_n^2 v^2}{2 f^3_{x_o}(F^{-1}_{x_o}(\lambda))}$$

$$\left[\frac{f_{x_o}(F^{-1}_{x_o}(\lambda)) h_2''(x_o, F^{-1}_{x_o}(\lambda)) - f'_{x_o}(F^{-1}_{x_o}(\lambda)) h_2'(x_o, F^{-1}_{x_o}(\lambda))}{h(x_o, F^{-1}_{x_o}(\lambda))} \right] + \frac{a_n^2 uv}{h^2(x_o, F^{-1}_{x_o}(\lambda))} h_{12}''(x_o, F^{-1}_{x_o}(\lambda))$$

$$+ O(a_n^3) \Bigg]$$

continued

$$\Big[1-\frac{a_n u}{g^2(x_o)}g'(x_o)-\frac{a_n v}{f^2_{x_o}(F^{-1}_{x_o}(\lambda))}f'_{x_o}(F^{-1}_{x_o}(\lambda))-\frac{a_n^2u^2}{2g^4(x_o)}[g''(x_o)g(x_o)-3(g'(x_o))^2]$$

$$-\frac{a_n^2v^2}{2f^4_{x_o}(F^{-1}_{x_o}(\lambda))}[f''_{x_o}(F^{-1}_{x_o}(\lambda))f_{x_o}(F^{-1}_{x_o}(\lambda))-3(f'_{x_o}(F^{-1}_{x_o}(\lambda)))^2]$$

$$+\frac{a_n^2uv}{h^2(x_o,F^{-1}_{x_o}(\lambda))}g'(x_o)f'_{x_o}(F^{-1}_{x_o}(\lambda))+O(a_n^3)\Big]$$

$$=1+\frac{a_n u}{g(x_o)}\left[\frac{h'_1(x_o,F^{-1}_{x_o}(\lambda))}{h(x_o,F^{-1}_{x_o}(\lambda))}-\frac{g'(x_o)}{g(x_o)}\right]+\frac{a_n v}{f_{x_o}(F^{-1}_{x_o}(\lambda))}\left[\frac{h'_2(x_o,F^{-1}_{x_o}(\lambda))}{h(x_o,F^{-1}_{x_o}(\lambda))}-\frac{f'_{x_o}(F^{-1}_{x_o}(\lambda))}{f_{x_o}(F^{-1}_{x_o}(\lambda))}\right]$$

$$+\frac{a_n^2u^2}{2g^3(x_o)}\left[\frac{g(x_o)h''_1(x_o,F^{-1}_{x_o}(\lambda))-3g'(x_o)h'_1(x_o,F^{-1}_{x_o}(\lambda))}{h(x_o,F^{-1}_{x_o}(\lambda))}-\frac{g''(x_o)g(x_o)-3(g'(x_o))^2}{g(x_o)}\right.$$

$$+\frac{a_n^2v^2}{2f^3_{x_o}(F^{-1}_{x_o}(\lambda))}\left[\frac{f_{x_o}(F^{-1}_{x_o}(\lambda))h''_2(x_o,F^{-1}_{x_o}(\lambda))-3f'_{x_o}(F^{-1}_{x_o}(\lambda))h'_2(x_o,F^{-1}_{x_o}(\lambda))}{h(x_o,F^{-1}_{x_o}(\lambda))}\right.$$

$$\left.-\frac{f''_{x_o}(F^{-1}_{x_o}(\lambda))f_{x_o}(F^{-1}_{x_o}(\lambda))-3(f'_{x_o}(F^{-1}_{x_o}(\lambda))^2)}{f_{x_o}(F^{-1}_{x_o}(\lambda))}\right]+\frac{a_n^2uv}{h^2(x_o,F^{-1}_{x_o}(\lambda))}$$

$$[h''_{12}(x_o,F^{-1}_{x_o}(\lambda))+g'(x_o)f'_{x_o}(F^{-1}_{x_o}(\lambda))]+O(a_n^3).$$

Since $H(x,y) = g(x).f_x(y)$,

$$h_2'(x,y) = g(x)f_x'(y) \quad \text{and} \quad h_2''(x,y) = g(x)f_x''(y),$$

we see that the coefficients of $a_n v$ and $a_n^2 v^2$ in the above expression are both equal to zero and consequently (iii) of the lemma follows. □

4.2. THE CONDITIONAL EMPIRICAL PROCESS $\{\Pi_{n,x_o}(y), -\infty<y<\infty\}$. In this section we shall apply some results of Stute [18], [19] and the criterion of tightness of a sequence of random functions to show that the supremum of the conditional empirical process defined below ((4.2.1)) is stochastically bounded.

Let (X_i, Y_i), $i = 1,2,\ldots,n$, be iid with absolutely continuous cdf $H(x,y)$ and joint density $h(x,y)$. Let $G(x)$, $F(y)$, $F_x(y)$ denote the marginal cdf of X, marginal cdf of Y and the conditional cdf of Y given $X = x$ respectively. We define the conditional empirical process $\Pi_{n,x_o}(y)$ by

$$(4.2.1) \qquad \Pi_{n,x_o}(y) = (na_n)^{1/2}[F_{n,x_o}(y) - F_{x_o}(y)], \ -\infty < y < \infty,$$

where $F_{n,x_o}(y) = \frac{1}{na_n} \sum_{j=1}^{n} K^*\left(\frac{G_n(X_j)-G_n(x_o)}{a_n}\right) I(Y_j \le y)$ as defined by (4.1.2).

LEMMA 4.2.1. *Suppose assumptions* (4.1.11) *to* (4.1.15) *on the joint distribution* H *are satisfied and* K^* *satisfies assumption* (4.1.4) *with* $\nu^* = 2$. *Let* $\{a_n\}$ *be any sequence of positive numbers such that* $a_n \to 0$, $na_n^4 \to \infty$ *but* $na_n^5 \to 0$. *We then have*

$$(4.2.2) \qquad \sup_{-\infty<y<\infty} |\Pi_{n,x_o}(y)| = O_p(1)$$

where $\Pi_{n,x_o}(y)$ is defined by (4.2.1).

PROOF: Define $Z_i = F_{x_o}(Y_i)$, $i = 1,\ldots,n$ and let $t = F_{x_o}(y), -\infty<y<\infty$. We observe that

$$0 \le t \le 1,$$

and $F^*_{x_o}(t) = P(Z \le t \mid X = x_o) = P(Y \le F^{-1}_{x_o}(t) \mid X = x_o) = F_{x_o}(F^{-1}_{x_o}(t)) = t.$

Hence to study the conditional empirical process $\{\Pi_{n,x_o}(y): -\infty < y < \infty\}$ obtained from (X_i,Y_i), $i = 1,2,\ldots,n$ we may as well study the transformed conditional empirical process $\{\Pi^*_{n,x_o}(t): 0 \le t \le 1\}$ obtained from (X_i,Z_i), $i = 1,2,\ldots,n$, namely,

$$(4.2.3) \qquad \Pi^*_{n,x_o}(t) = (na_n)^{1/2}\,[F^*_{n,x_o}(t) - F^*_{x_o}(t)] = (na_n)^{1/2}[F^*_{n,x_o}(t)-t],$$

where

$$(4.2.4) \qquad F^*_{n,x_o}(t) = \frac{1}{na_n}\sum_{j=1}^{n} K^*\left(\frac{G_n(X_j)-G_n(x_o)}{a_n}\right)I(Z_j \le t), \qquad 0 \le t \le 1.$$

So it suffices to show that $\sup_{0\le t\le 1} |\Pi^*_{n,x_o}(t)| = O_p(1)$. Let H^* and H^*_n denote, respectively, the joint cdf and the bivariate empirical distribution function of (X,Z). Observe that (4.2.3) can be rewritten as

$$(4.2.5)\qquad \Pi^*_{n,x_o}(t) = (na_n)^{1/2}\Big[a_n^{-1}\iint K^*\Big(\frac{G_n(x)-G_n(x_o)}{a_n}\Big) I(z\le t)\,dH^*_n(x,z) - t\Big].$$

Since K^* is twice differentiable, the Taylor expansion of K^* in (4.2.5) yields

$$(4.2.6)\qquad \Pi^*_{n,x_o}(t) = (na_n)^{1/2}\Big[a_n^{-1}\iint K^*\Big(\frac{G(x)-G(x_o)}{a_n}\Big) I(z\le t)\,dH^*_n(x,z) - t\Big]$$

$$+ (na_n)^{1/2}a_n^{-2}\iint[G_n(x)-G_n(x_o)-G(x)+G(x_o)]K^{*\prime}\Big(\frac{G(x)-G(x_o)}{a_n}\Big) I(z\le t)\,dH^*_n(x,z)$$

$$+ (na_n)^{1/2}a_n^{-3}\iint[G_n(x)-G_n(x_o)-G(x)+G(x_o)]^2\,\frac{K^{*\prime\prime}(\Delta_{x,n})}{2}\, I(z\le t)\,dH^*_n(x,z)$$

$$= (na_n)^{1/2}a_n^{-1}\iint K^*\Big(\frac{G(x)-G(x_o)}{a_n}\Big) I(z\le t)\,[dH^*_n(x,z)-dH^*(x,z)]$$

$$+ (na_n)^{1/2}\Big[a_n^{-1}\iint K^*\Big(\frac{G(x)-G(x_o)}{a_n}\Big) I(z\le t)\,dH^*(x,z) - t\Big]$$

$$+ (na_n)^{1/2}a_n^{-2}\iint[G_n(x)-G(x_o)-G(x)+G(x_o)]K^{*\prime}\Big(\frac{G(x)-G(x_o)}{a_n}\Big) I(z\le t)\,dH^*_n(x,z)$$

$$+ (na_n)^{1/2} a_n^{-3} \iint [G_n(x)-G_n(x_o)-G(x)+G(x_o)]^2 \frac{K^{*''}(\Delta_{x,n})}{2} I(z\le t)dH_n^*(x,z)$$

$$= U_{n,x_o}(t) + V_{n,x_o}(t) + W_{n,x_o}(t) + P_{n,x_o}(t), \quad \text{say,}$$

where $\Delta_{x,n}$ lies between $a_n^{-1}(G_n(x)-G_n(x_o))$ and $a_n^{-1}(G(x)-G(x_o))$. We shall prove the lemma by showing that each one of $\sup_{0\le t\le 1} | U_{n,x_o}(t) |$, $\sup_{0\le t\le 1} |V_{n,x_o}(t)|$, $\sup_{0\le t\le 1} |W_{n,x_o}(t)|$ and $\sup_{0\le t\le 1} | P_{n,x_o}(t) |$ is $O_p(1)$. We consider $P_{n,x_o}(t)$ first.

$$(4.2.7) \quad \sup_{0\le t\le 1} | P_{n,x_o}(t) | \le (na_n)^{1/2} a_n^{-3} \iint [G_n(x)-G_n(x_o)-G(x)+G(x_o)]^2 \left| \frac{K^{*''}(\Delta_{x,n})}{2} \right| dH_n^*(x,z)$$

By the same argument as used in lemma 1 of Stute [18], it follows that the R.H.S. of (4.2.7) converges to 0 in probability as $n \to \infty$. Because of the frequent reliance on this argument with respect to several other terms also in the latter part of this chapter, we reproduce it here for convenience, once for all. Since K^* has compact support, say, $[-1,1]$ without loss of generality, the above expansion of $F_{n,x_o}^*(t)$ holds true with integration restricted to those x for which $| G_n(x) - G_n(x_o) | < a_n$. The Devoretzky-Keifer-Wolfowitz (1956) bound for the tails of $\sup_x | G_n(x)-G(x)|$

yields that for given $\varepsilon > 0$ there exists some finite $C > 0$ such that $\sup_x |G_n(x)-G(x)| \leq C.n^{-1/2}$ for all n, up to an event of probability less than or equal to ε. Let this set be denoted by A_ε. On A_ε, $|G_n(x)-G_n(x_o)|<a_n$ implies

$$|G(x)-G(x_o)| = |G(x)-G_n(x)+G_n(x)-G_n(x_o)+G_n(x_o)-G(x_o)|$$

$$\leq a_n + 2C.n^{-1/2} \leq C_1 a_n.$$

for some $C_1 < \infty$ (since $(n^{1/2}a_n)^{-1}$ goes to zero, as $n \to \infty$).

Lemma 2.3 in Stute [19] asserts that

(4.2.8) $$B_{n,x_o} = \sup_{\{x: |G(x)-G(x_o)| \leq C_1 a_n\}} (na_n)^{1/2} |G_n(x)-G_n(x_o)-G(x)+G(x_o)|$$

is $O_p(1)$ as $n \to \infty$. Hence, from (4.2.6) we have

(4.2.9) $$\sup_{0\leq t\leq 1} | P_{n,x_o}(t) | \leq (na_n^3)^{-1/2} B^2_{n,x_o} \iint \left|\frac{K^{*''}(\Delta_{x,n})}{2}\right| dH_n^*(x,Z),$$ on the set A_ε. Since $K^{*''}$ is bounded, B_{n,x_o} is $O_p(1)$, $na_n^3 \to \infty$ and ε is arbitrary, the R.H.S. term in (4.2.9) converges to zero in probability as $n \to \infty$ and hence so does $\sup_{0\leq t\leq 1} | P_{n,x_o}(t) |$. Thus we have

(4.2.10) $$\sup_{0\leq t\leq 1} | P_{n,x_o}(t) | = o_p(1).$$

We have $\sup_{0\le t\le 1} | W_{n,x_o}(t) | \le a_n^{-1} \iint (\frac{n}{a_n})^{1/2} |G_n(x)-G_n(x_o)-G(x)+G(x_o)|$

$$|K^{*'}(\frac{G(x)-G(x_o)}{a_n})| dH_n^*(x,z).$$

By the same argument as used at (4.2.7) above, we can write

$$\text{(4.2.11)} \qquad \sup_{0\le t\le 1} |W_{n,x_o}(t)| \le B_{n,x_o} \cdot T_{n,x_o}$$

where B_{n,x_o} is as given by (4.2.8) and

$$T_{n,x_o} = a_n^{-1} \iint |K^{*'}(\frac{G(x)-G(x_o)}{a_n})| dH_n^*(x,z) = \frac{1}{na_n} \sum_{i=1}^{n} |K^{*'}(\frac{G(X_i)-G(x_o)}{a_n})|.$$

Now, by the Markov inequality, for any $M > 0$

$$P(|T_{n,x_o}| > M) \le \frac{E|T_{n,x_o}|}{M} = \frac{\frac{1}{a_n}\int |K^{*'}(\frac{G(x)-G(x_o)}{a_n})| dG(x)}{M}$$

$$\le \frac{\int |K^{*'}(u)| du}{M} \le \frac{C}{M},$$

since $K^{*'}$ is bounded and vanishes outside a bounded interval because of assumption (4.1.4). Hence for any $\varepsilon > 0$, the R.H.S. quantity can be made $< \varepsilon$ by choosing M sufficiently large, which proves that T_{n,x_o} is $O_p(1)$. Since B_{n,x_o} is also $O_p(1)$, it now follows from (4.2.11) that

$$\text{(4.2.13)} \qquad \sup_{0\le t\le 1} |W_{n,x_o}(t)| = O_p(1).$$

Observe that $a_n^{-1}\int K^*(\frac{G(x)-G(x_o)}{a_n})\, dG(x) = \int_{-G(x_o)/a_n}^{(1-G(x_o))/a_n} K^*(u)du.$

Since K^* has bounded support ([-1,1]) $0 < G(x_o) < 1$ and $a_n \to 0$ as $n \to \infty$, the last integral is equal to one for sufficiently large n, say, for $n > n^*$. Hence we may write for $n > n^*$

$$(4.2.14) \quad V_{n,x_o}(t) = (na_n)^{1/2} a_n^{-1} \int K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) [F_x^*(t)-t]dG(x)$$

$$= (na_n)^{1/2} \int_{-1}^{1} K^*(u) [F^*_{G^{-1}(G(x_o)+a_n u)}(t) - t]du$$

where the last equality follows upon transformation of variables. Now, since H^* is four times differentiable, which follows from assumption (4.1.15), Taylor's expansion of $F^*_{G^{-1}(G(x_o)+a_n u)}(t)$ as in (iii) of lemma 4.1.1 yields

$$(4.2.15) \quad V_{n,x_o}(t) = (na_n)^{1/2} \int_{-1}^{1} K^*(u)[F^*_{x_o}(t)+a_n uD_3(x_o,t)+a_n^2u^2D_4(x_o,t)+O(a_n^3)-t]du$$

where (see (4.1.26))

$$D_3(x_o,t) = \frac{1}{g^2(x_o)}[H_1^{*''}(x_o,t)-tg'(x_o)]$$

and

$$D_4(x_o,t) = \frac{1}{2g^4(x_o)}[g(x_o)H_1^{*'''}(x_o,t)-g'(x_o)H_1^{*''}(x_o,t)-tg''(x_o)g(x_o)$$

$$-3t(g'(x_o))^2-2g'(x_o)H_1^{*''}(x_o,t)].$$

Since $\sup_{0<t<1} |D_4(x_o,t)| < \infty$ (by assumptions (4.1.11) to (4.1.15)), $F^*_{x_o}(t) = t$, $\int uK^*(u)du = 0$ and $na_n^5 \to 0$ as $n \to \infty$, it follows from (4.2.15) that

$$\sup_{0\le t\le 1} | V_{n,x_o}(t) | = O(1).$$

Hence it remains to show that $\sup_{0\le t\le 1} | U_{n,x_o}(t) | = O_p(1)$. This we shall do by proving the tightness of the sequence $\{U_{n,x_o}\}$ on the D[0,1]. The required result will then follow from Theorem 15.2 of [2].

The tightness will be proved by verifying the conditions of Theorem 15.5 of [2]. Since $U_{n,x_o}(o) = 0$ for all n, by the continuity assumption on F, condition (i) of Theorem 15.5 of [2] is trivially satisfied. Hence, it suffices to show that condition (ii) of Theorem 15.5 of [2] is also satisfied, namely, for each positive ε and η there exists a δ, $0 < \delta < 1$, such that

(4.2.16) $$P\{W(U_{n,x_o},\delta) > \varepsilon\} \leq \eta$$

for all sufficiently large n, where $W_U(\delta) = W(U,\delta)$ is the modulus of continuity of the function U as defined by (8.1) of [2].

Now fix ε and η. Suppose $0 \leq s \leq t \leq 1$. From (4.2.6) we have

(4.2.17) $$U_{n,x_o}(t) - U_{n,x_o}(s) = (na_n)^{-1/2} \sum_{i=1}^{n} [K^*(\frac{G(X_i)-G(x_o)}{a_n})I(s \leq Z_i \leq t) - \int\int K^*(\frac{G(x)-G(x_o)}{a_n})I(s \leq z \leq t) dH^*(x,z)]$$

$$= (na_n)^{-1/2} \sum_{i=1}^{n} (T_i - ET_i)$$

where $$T_i = K^*(\frac{G(X_i)-G(x_o)}{a_n}) I(s \leq Z_i \leq t)$$

Since T_i's are all iid rv's

(4.2.18) $$E(U_{n,x_o}(t) - U_{n,x_o}(s))^4 = \frac{1}{n^2a_n^2}[n E(T_1 - ET_1)^4 + 3n(n-1)\{E(T_1-ET_1)^2\}^2].$$

Since T_i's are all nonnegative rv's, we see that

$$(4.2.19)\quad E(T_1 - ET_1)^4 \le E(T_1^4) + 6E(T_1^2)\, E^2(T_1)$$

$$= \int K^{*4}\left(\frac{G(x_1)-G(x_o)}{a_n}\right)\left(F_x^*(t)-F_x^*(s)\right)dG(x)$$

$$+ 6\left(\int K^{*2}\left(\frac{G(x)-G(x_o)}{a_n}\right)\left(F_x^*(t)-F_x^*(s)\right) dG(x)\right.$$

$$\left.\left(\int K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)\left(F_x^*(t)-F_x^*(s)\right)dG(x)\right)^2\right..$$

Since the conditional density $f_x(y)$ is assumed to be bounded uniformly in x and y, we may write

$$(4.2.20)\quad \left| F_x^*(t) - F_x^*(s) \right| \le C_1(t-s) \quad \text{where} \quad C_1 = \sup_{x,z} \left|f_x^*(z)\right|.$$

By (4.2.20) and the fact that K^* is bounded it follows from (4.2.19) that

$$E(T_1 - ET_1)^4 \le C_1(t-s) \int\left\{K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)\right\}^4 dG(x) + C_2(t-s)\int\left\{K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)\right\}^2 dG(x)$$

$$\le C_3 a_n (t-s)$$

where the last inequality follows upon transformation of variables. Note that the constant C_3 depends on the bounds of K^* and $f_x^*(z)$ only. We also have,

$$(4.2.22)\quad E(T_1-ET_1)^2 \le E(T_1^2) = \int\left\{K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)\right\}^2\left(F_x^*(t)-F_x^*(s)\right)dG(x)$$

$$\le C_4 a_n (t-s),$$

where the constant $C_4 > 0$ again depends on the bounds of K^* and $f_x^*(z)$ only.

Now, from (4.2.19), (4.2.21) and (4.2.22), we have

$$E(U_{n,x_o}(t)-U_{n,x_o}(s))^4 \leq \frac{C_3}{na_n}(t-s) + 3C_4^2(t-s)^2$$

$$\leq \frac{C_5}{na_n}(t-s) + C_5(t-s)^2.$$

Hence for any $0 \leq s, t \leq 1$,

(4.2.23) $$E|U_{n,x_o}(t)-U_{n,x_o}(s)|^4 \leq \frac{C_5}{na_n}|t-s|+C_5|t-s|^2,$$

and, if $\frac{\varepsilon}{na_n} < |t-s|$ (with $\varepsilon<1$), we have

(4.2.24) $$E|U_{n,x_o}(t) - U_{n,x_o}(s)|^4 \leq \frac{2C_5}{\varepsilon}(t-s)^2,$$

where the constant C_5 depends on the bounds of K^* and $f_x^*(z)$ only. Now suppose that p is a number satisfying $\frac{\varepsilon}{na_n} \leq p$, and consider the rv's $U_{n,x_o}(s+ip)-U_{n,x_o}(s+(i-1)p)$, $i = 1,2,\ldots,m$, where m is a positive integer. By theorem 12.2 of [2]

(4.2.25) $$P\{\max_{i\leq m}|U_{n,x_o}(s+ip)-U'_{n,x_o}(s)|\geq\lambda\} \leq \frac{C^*}{\varepsilon\lambda^4} m^2p^2.$$

For $s \le t \le s+p$,

(4.2.26)
$$U_{n,x_o}(t)-U_{n,x_o}(s) = (na_n)^{-1/2}\sum_{i=1}^{n}\left[K^*\left(\frac{G(x_i)-G(x_o)}{a_n}\right)I(Z_i\le t) - \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Z\le t)\,dH^*(x,z)\right] - U_{n,x_o}(s)$$

$$\le (na_n)^{-1/2}\sum_{i=1}^{n}\left[K^*\left(\frac{G(x_i)-G(x_o)}{a_n}\right)I(Z_i\le s+p) - \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Z\le s+p)\,dH^*(x,z) + \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Z\le s+p)\,dH^*(x,z) - \iint K\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Z\le t)\,dH^*(x,z)\right] - U_{n,x_o}(s)$$

$$= \left[U_{n,x_o}(s+p)-U_{n,x_o}(s)\right] + \left(\frac{n}{a_n}\right)^{1/2}\iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(t\le Z\le s+p)\,dH^*(x,z)$$

$$\le \left|U_{n,x_o}(s+p)-U_{n,x_o}(s)\right| + \left(\frac{n}{a_n}\right)^{1/2}\iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(s\le Z\le s+p)\,dH^*(x,z)$$

Also,

$$(4.2.27)\quad -[U_{n,x_o}(t)-U_{n,x_o}(s)] \le \left(\frac{n}{a_n}\right)^{1/2} \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(s\le z\le t)\,dH^*(x,z)$$

$$- (na_n)^{-1/2}\left[\sum_{i=1}^{n} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) I(Z_i\le t) - \sum_{i=1}^{n} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) I(Z_i\le s)\right]$$

$$= \left(\frac{n}{a_n}\right)^{1/2} \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(s\le z\le t)\,dH^*(x,z)$$

$$- (na_n)^{-1/2} \sum_{i=1}^{n} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) I(s\le Z_i\le t)$$

$$\le \left(\frac{n}{a_n}\right)^{1/2} \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(s\le z\le s+p)\,dH^*(x,z)$$

$$\le \left(\frac{n}{a_n}\right)^{1/2} \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(s< z< s+p)\,dH^*(x,z)$$

$$+ |U_{n,x_o}(s+p)-U_{n,x_o}(s)|.$$

Hence from (4.2.26) and (4.2.27) we have

$$(4.2.28)\quad |U_{n,x_o}(t)-U_{n,x_o}(s)| \le |U_{n,x_o}(s+p)-U_{n,x_o}(s)| + \left(\frac{n}{a_n}\right)^{1/2} \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(s\le z\le s+p)\,dH^*(x,z)$$

for $s \le t \le s_k + p$.

But the second term on the R.H.S. of (4.2.28) is

$$= \left(\frac{n}{a_n}\right)^{1/2} \int K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)\left(F_x^*(s+p)-F_x^*(s)\right)dG(x)$$

$$\leq (na_n)^{1/2} p\, C_1,$$

by (4.2.20) and the fact that K^* is a density function. Hence we have

$$(4.2.29) \qquad |U_{n,x_o}(t)-U_{n,x_o}(s)| \leq |U_{n,x_o}(s+p)-U_{n,x_o}(s)| + (na_n)^{1/2}\, pC_1$$

for $s \leq t \leq s+p$. But (4.2.29) implies

$$(4.2.30) \qquad \sup_{s\leq t\leq s+mp} |U_{n,x_o}(t)-U_{n,x_o}(s)| \leq 3 \max_{i\leq m} |U_{n,x_o}(s+ip)-U_{n,x_o}(s)| (na_n)^{1/2} pC_1.$$

Now suppose that $\frac{\varepsilon}{na_n} < p < \frac{\varepsilon}{(na_n)^{1/2}.C_1}$.

Then (4.2.25) applies and it follows by (4.2.30) that

$$(4.2.31) \qquad P\{\sup_{s\leq t\leq s+mp} |U_{n,x_o}(t)-U_{n,x_o}(s)| \geq 4\varepsilon\} \leq \frac{C^*}{\varepsilon^5} m^2 p^2.$$

Choose δ so that $\frac{C^*\delta}{\varepsilon^5} < \eta$. Then from (4.2.31) we have

$$(4.2.32) \qquad P\{\sup_{s\leq t\leq s+\delta} |U_{n,x_o}(t)-U_{n,x_o}(s)| \geq 4\varepsilon\} \leq \eta\, \delta,$$

provided there exists a 'p' and an integer 'm' such that $mp = \delta$, and

$\frac{\varepsilon}{na_n} < p < \frac{\varepsilon}{(na_n)^{1/2}C_1}$ is satisfied or equivalently, there exists an integer 'm' with $(\delta/\varepsilon)(na_n)^{1/2}C_1 < m < (\delta/\varepsilon)na_n$, which is true for all sufficiently large n, since $na_n \to \infty$. For given ε and η, therefore, there exists a δ such that (4.2.32) holds for sufficiently large n. But this implies (4.2.16) by the Corollary to Theorem 8.3 of [2]. This completes the proof of tightness of $\{U_{n,x_o}\}$ and hence the lemma. □

4.3 ASYMPTOTIC EQUIVALENCE OF $Q_{n,x_o}(\lambda)$ AND $S_{n,x_o}(\lambda)$ IN DISTRIBUTION. In this section we shall prove that $Q_{n,x_o}(\lambda)$ and $S_{n,x_o}(\lambda)$, as given by (4.1.20) and (4.1.21) respectively, are asymptotically equivalent in distribution by showingh that $Q_{n,x_o}(\lambda) - S_{n,x_o}(\lambda)$ converges to zero in probability as $n \to \infty$. From (4.1.20) we have

$$Q_{n,x_o}(\lambda) - S_{n,x_o}(\lambda) = \sum_{m=2}^{3} \zeta_{nm} + \sum_{m=2}^{4} \eta_{nm}$$

where ζ_{nm}'s are given by (4.1.22) to (4.1.23) and η_{nm} are given by (4.1.24) to (4.1.25). Hence it suffices to show that each of the terms ζ_{n2}, ζ_{n3}, η_{n2}, η_{n3} and η_{n4} converge to zero in probability as $n \to \infty$. Lemma 4.2.1 is particularly useful in this regard.

LEMMA 4.3.1. (a) <u>Suppose assumptions</u> (4.1.11) <u>to</u> (4.1.13) <u>are satisfied and let</u> $\{a_n\}$ <u>be a sequence of positive numbers with</u> $a_n \to 0$ <u>and</u> $na_n^4 \to \infty$. <u>We then have</u>

$$\zeta_{nm} \xrightarrow{pr} 0 \text{ as } n \to \infty \text{ for } m = 2,3$$

<u>where</u> ζ_{n2} <u>and</u> ζ_{n3} <u>are given by</u> (4.1.22) <u>and</u> (4.1.23) <u>respectively.</u>

(b) Suppose assumptions (4.1.11) to (4.1.15) are satisfied and let $\{a_n\}$ be a sequence of positive numbers with $a_n \to 0$, $na_n^4 \to \infty$ but $na_n^5 \to 0$. We then have

$$\eta_{nm} \xrightarrow{pr} 0 \text{ as } n \to \infty \text{ for } m = 2,3 \text{ and } 4,$$

where η_{nm}, $m = 2,3$ are given by (4.1.24) and η_{n4} is given by (4.1.25).

PROOF: (a) Since $K^{*''}$ is bounded, from (4.1.22) we have

$$|\zeta_{n2}| \leq Ca_n^{-6} \iiiint [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^2$$

$$|K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})| |y_1| dH_n(x_1,y_1) dH_n(x_2,y_2).$$

By the same argument as used at (4.2.7), we further have

$$(4.3.1) \qquad |\zeta_{n2}| \leq \frac{C}{na_n^5} B^2_{n,x_o} \iint |K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}) y_1| dH_n(x_1,y_1)$$

where B_{n,x_o} is given by (4.2.8).

Now, we can write

$$(4.3.2) \qquad \frac{1}{na_n^5} \iint |K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}) y_1| dH_n(x_1,y_1)$$

$$= \frac{1}{n} \sum_{i=1}^{n} |\frac{1}{na_n^5} K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}) Y_i|$$

$$= \frac{1}{n} \sum T_i = \bar{T}_n$$

and by the Markov inequality, we have

(4.3.3) $$p(\bar{T}_n > \varepsilon) \le \frac{E(T_1)}{\varepsilon}.$$

But $$E(T_1) = \frac{1}{na_n^5} \iint \left|K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)\right| |y_1| h(x_1,y_1)dx_1dy_1$$

$$\le \frac{1}{na_n^3} \iint |K_2'(u_1,v_1)| |F_{x_o}^{-1}(\lambda+a_nv_1)| \frac{h[G^{-1}(G(x_o)+a_nu_1), F_{x_o}^{-1}(\lambda+a_nv_1)]}{g[G^{-1}(G(x_o)+a_nu_1)] f_{x_o}[F_{x_o}^{-1}(\lambda+a_nv_1)]} du_1dv_1$$

where the last inequality follows by transformation of variables $\frac{G(x_1)-G(x_o)}{a_n} = u_1$ and $\frac{F_{x_o}(y_1)-\lambda}{a_n} = v_1$, and using the fact that $P\{G^{-1}(G(X)) \neq X, F_{x_o}^{-1}(F_{x_o}(Y)) \neq Y\} = 0$. Hence using the Taylor expansions from lemma 4.1.1, we see that

(4.3.4) $$E(T_1) \le \frac{1}{na_n^3} \iint |K_2'(u_1,v_1)| [F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$
$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)] du_1dv_1$$
$$= O\left(\frac{1}{na_n^3}\right).$$

Since $na_n^3 \to \infty$, from (4.3.3) and (4.3.4) we have

(4.3.5) $$\bar{T}_n \xrightarrow{pr} 0, \text{ as } n \to \infty.$$

Since B_{n,x_o} is $O_p(1)$ up to an event of probability less than ε and ε is arbitrary, from (4.3.1) and (4.3.5) it now follows that

$$\zeta_{n2} \xrightarrow{pr} 0 \quad \text{as } n \to \infty.$$

The argument used at (4.2.7), when applied to (4.1.23) yields

$$|\zeta_{n3}| \le \frac{1}{n^{3/2}a_n^{11/2}} B^3_{n,x_o} \iiiint |K^{*'''}(\Delta_{x_2,n})K_2'(\frac{F(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})|$$

$$|y_1| I(y_2 \le y_1) dH_n(x_1,y_1) dH_n(x_2,y_2).$$

Since $K^{*'''}$ and K_2' are bounded we can further write

$$(4.3.5) \qquad |\zeta_{n3}| \le \frac{C}{n^{3/2}a_n^{11/2}} B^3_{n,x_o} \iint |y_1| dH_n(x_1,y_1).$$

Since $na_n^4 \to \infty$, $\lim_{n\to\infty} \text{Sup} \iint |y_1| dH_n(x_1,y_1) < \infty$ with probability one, and B_{n,x_o} is $O_p(1)$ up to an event of probability less than ε, but ε is arbitary, it follows that

$$\zeta_{n3} \xrightarrow{pr} 0 \quad \text{as } n \to \infty.$$

(b) From (4.1.24), observe that the terms involved in η_{nm}, for a fixed $m(m=2,3)$, are of the following three different types:

$$\eta_{nm}(A) = \frac{a_n^{-2-m}}{m!} \iint [G_n(x)-G_n(x_o)-G(x)+G(x_o)]^m K_1^{(m)}(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n})$$

$$y dH_n(x,y),$$

$$\eta_{nm}(B) = \frac{a_n^{-2-m}}{m!} \iint [F_{n,x_o}(y)-F_{x_o}(y)]^m K_2^{(m)}(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}) y dH_n(x,y),$$

and
$$\eta_{nm}(C) = \frac{a_n^{-2-m}}{m!} \iint [G_n(x)-G_n(x_o)-G(x)+G(x_o)]^{m_1} [F_{n,x_o}(y)-F_{x_o}(y)]^{m_2}$$

$$K_{12}^{(m_1,m_2)}(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}) y dH_n(x,y)$$

where $m_1+m_2 = m$, $1 < m_1, m_2 < m$,

$K_i^{(m)}(\theta_1,\theta_2)$ is the m^{th} order partial derivative of $K(.,.)$ with respect to the i^{th} argument evaluated at (θ_1,θ_2), $i=1,2$,

$K_{12}^{(m)}(\theta_1,\theta_2)$ is the m^{th} mixed derivative of $K(.,.)$, obtained by differentiating m_1 times with respect to the first argument and m_2 times with respect to the second argument, and evaluated at (θ_1,θ_2).

Using the argument applied at (4.2.7), we have

$$|\eta_{nm}(A)| \le \left(\frac{1}{na_n^{1+\frac{4}{m}}}\right)^{\frac{m}{2}} B^m_{n,x_0} \iint \left|K_1^{(m)}\left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}\right)y\right| dH_n(x,y).$$

Since $na_n^{1+\frac{4}{m}} \to \infty$ for $m = 2,3$, by the same argument as used in part (a) for the term ζ_{n3}, we see that

$$\eta_{nm}(A) \to 0 \text{ is probability, for } m = 2,3.$$

Now, $$|\eta_{nm}(B)| \le \frac{1}{n^{m/2}a_n^{\frac{3m+4}{2}}} L^m_{n,x_0} \iint \left|K_2^{(m)}\left(\frac{G(x)-G(x_0)}{a_n}, \frac{F_{x_0}(y)-\lambda}{a_n}\right)y\right| dH_n(x,y),$$

where

(4.3.6) $$L_{n,x_0} = \sup_{-\infty<y<\infty} (na_n)^{1/2}|F_{n,x_0}(y)-F_{x_0}(y)|$$

From lemma 4.2.1, we have that $L_{n,x_o} = O_p(1)$. Applying the same argument as used in part (a) for the term ζ_{n2} and observing that $n^{\frac{m}{2}} a_n^{\frac{3m}{2}} \to \infty$ for $m = 2,3$, it follows that

$$\eta_{nm}(B) \to 0 \text{ in probability, for } m = 2,3.$$

Using the argument applied at (4.2.7) and lemma 4.2.1, we have

$$(4.3.7) \quad |\eta_{nm}(C)| \le \frac{a_n^{-2-m}}{m!}\left(\frac{a_n}{n}\right)^{\frac{m_1}{2}} \left(\frac{1}{na_n}\right)^{\frac{m_2}{2}} B_{n,x_o}^{m_1} L_{n,x_o}^{m_2} \iint \left|K_2^{(m_1,m_2)}\left(\frac{G(x)-G(x_o)}{a_n}\right)y\right| dH_n(x,y),$$

where both B_{n,x_o} and L_{n,x_o} are $O_p(1)$. Consider the case $m_1 = 1$, $m_2 = m-1$ in (4.3.7) first. Then (4.3.7) becomes

$$(4.3.8) \quad |\eta_{nm}(C)| \le \frac{1}{n^{\frac{m}{2}} a_n^{\frac{3m+2}{2}}} B_{n,x_o} L_{n,x_o}^{m-1} \iint \left| K_2^{(1,m-1)}\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)y\right| dH_n(x,y).$$

Again, by the same argument as used in part (a) for the term ζ_{n3} and observing that $n^{\frac{m}{2}} a_n^{\frac{3m+2}{2}} \to \infty$ for $m = 2,3$, it follows that

$$\eta_{nm}(c) \to 0 \text{ in probability, for } m = 2,3 \text{ with } m_1 = 1,\ m_2 = m-1.$$

For all the other values of m_1 and m_2 such that $m_1 + m_2 = m$, $m_1 > 1$, (for each fixed m), the constant term on the R.H.S. of (4.3.8) will be of the form $\frac{1}{n^{(m-1)/2} a_n^{\alpha}}$ with $\alpha < \frac{3m}{2}$. Hence, convergence of $n^{1/2} a_n \cdot \eta_{nm}(C)$ to zero is probability, with those values of m_1, m_2 automatically follows, once it is established with $m_1 = 1$, $m_2 = m-1$. Therefore, we have

$$\eta_{nm}(C) \to 0 \text{ in probability, for } m = 2,3.$$

It remains to show that η_{n4} goes to zero in probability. First, from (4.1.25) observe that η_{n4} again has the same types of terms as η_{n2} or η_{n3}, with $K_1^{(m)}\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$, $K_2^{(m)}\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$ and $K_{12}^{(m_1,m_2)}\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$ replaced by $K_1^{(m)}(\Delta_{x,n}, \Delta_{y,n})$, $K_2^{(m)}(\Delta_{x,n}, \Delta_{y,n})$ and $K_{12}^{(m_1,m_2)}(\Delta_{x,n}, \Delta_{y,n})$ respectively, namely ,

$$\eta_{n4}(A) = \frac{a_n^{-6}}{4!}\iint[G_n(x)-G_n(x_o)-G(x)+G(x_o)]^4 K_1^{(4)}(\Delta_{x,n}, \Delta_{y,n})\,y\,dH_n(x,y),$$

$$\eta_{n4}(B) = \frac{a_n^{-6}}{4!}\iint[F_{n,x_o}(y)-F_{x_o}(y)]^4 K_2^{(4)}(\Delta_{x,n}, \Delta_{y,n})\,y\,dH_n(x,y),$$

and $\eta_{n4}(\overline{C}) = \frac{a_n^{-6}}{4!}\iint[G_n(x)-G_n(x_o)-G(x)+G(x_o)]^{m_1}[F_{n,x_o}(y)-F_{x_o}(y)]^{m_2}$

$$K_{12}^{(m_1,m_2)}(\Delta_{x,n},\Delta_{y,n})y\,dH_n(x,y)$$

where $m_1 + m_2 = 4$, $m_1 \geq 1$ and $m_2 \geq 1$.

The convergence in probability to zero of $\eta_{n4}(A)$ follows on the same lines as that of ξ_{n3} in part (a) above. Now

(4.3.9) $$|\eta_{n4}(B)| \leq \frac{1}{n^2 a_n^8} L_{n,x_o}^4 \iint |K_2^{(4)}(\Delta_{x,n},\Delta_{y,n})y|\,dH_n(x,y),$$

where L_{n,x_o} is as defined by (4.2.29), and is $0_p(1)$ by lemma 4.2.1. Since $K_2^{(4)}$ is bounded, $\lim_{n\to\infty} \sup\iint|y|\,dH_n(x,y) < \infty$ and $na_n^4 \to \infty$, from (4.3.9) it follows that $\eta_{n4}(B) \to 0$ in probability.

As before (in the case of η_{nm} for $m < 4$), for $\eta_{n4}(C)$ we need only consider the case $m_1 = 1$, and $m_2 = m-1 = 3$. The same argument leading to (4.3.8) yields

(4.3.10) $$|\eta_{n_4}(C)| \leq \frac{1}{n^2 a_n^7} B_{n,x_o} L_{n,x_o}^3 \iint |K_2^{(4)}(\Delta_{x,n},\Delta_{y,n})y|\,dH_n(x,y).$$

It now follows from (4.3.10) that $\eta_{n4}(C)$ converges to zero in probability just as that of $\eta_{n4}(B)$ followed from (4.3.8). This completes the proof of part (b) of the lemma. Hence the lemma. □

4.4 ASYMPTOTIC VARIANCE OF $n^{1/2}a_n . S_{n,x_o}(\lambda)$. — In this section we shall show that $\mathrm{var}(n^{1/2}a_n . S_{n,x_o}(\lambda))$ converges to a finite value as $n \to \infty$, where $S_{n,x_o}(\lambda)$ is as given by (4.1.21). First observe that $S_{n,x_o}(\lambda)$ may be rewritten as

$$(4.4.1) \quad S_{n,x_o}(\lambda) = \frac{1}{na_n^2}\sum_{i=1}^{n} K\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$+ \frac{1}{na_n^3}\sum_{i=1}^{n}\left[G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)\right]$$

$$K_1'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$+ \frac{1}{n^2a_n^4}\sum_{i=1}^{n}\sum_{j=1}^{n} K^*\left(\frac{G(X_j)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)$$

$$Y_i I(Y_j \le Y_i)$$

$$+ \frac{1}{n^2a_n^5}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[G_n(X_j)-G_n(x_o)-G(X_i)+G(x_o)\right]$$

$$K^{*\prime}\left(\frac{G(X_j)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)$$

$$Y_i I(Y_j \le Y_i)$$

$$- \frac{1}{na_n^3}\sum_{i=1}^{n} F_{x_o}(Y_i)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$= I_1 + I_2 + I_{31} + I_{32} - I_{30}.$$

We shall further rewrite $I_{31} = I_{311} + I_{312}$ and $I_{32} = I_{321} + I_{322}$ so that

(4.4.2) $$S_{n,x_o}(\lambda) = I_1 + I_2 + I_{311} + I_{312} + I_{321} + I_{322} - I_{30}$$

where I_{311}, I_{312}, I_{321} and I_{322} are given below by (4.4.3), (4.4.4), (4.4.5) and (4.4.6) respectively:

(4.4.3) $$I_{311} = \frac{1}{n^2 a_n^4} \sum_{i=1}^{n} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i .$$

(4.4.4) $$I_{312} = \frac{1}{n^2 a_n^4} \sum\sum_{i \neq j} K^*\left(\frac{G(X_j)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i I(Y_j \leq Y_i) .$$

(4.4.5) $$I_{321} = \frac{1}{n^2 a_n^5} \sum_{i=1}^{n} \left[G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)\right]$$

$$K^{*\prime}\left(\frac{G(X_i)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i .$$

(4.4.6) $$I_{322} = \frac{1}{n^2 a_n^5} \sum\sum_{i \neq j} \left[G_n(X_j)-G_n(x_o)-G(X_j)+G(x_o)\right] K^{*\prime}\left(\frac{G(X_j)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i I(Y_j \leq Y_i) .$$

LEMMA 4.4.1 Let I_2, I_{311}, I_{321} and I_{322} be as defined by (4.4.1), (4.4.3), (4.4.5) and (4.4.6) respectively. Suppose K and K^* satisfy assumptions (4.1.3) to (4.1.6) with $\nu = 1$, $\nu^* = 1$ and the joint distribution H satisfies assumptions (4.1.11) to (4.1.13). Then for any sequence $\{a_n\}$ of positive numbers with $a_n \to 0$ and $na_n^4 \to \infty$, we have

(i) $\mathrm{var}(n^{1/2} a_n . I_2) \to 0$ as $n \to \infty$.

(ii) $\mathrm{var}(n^{1/2} a_n . I_{311}) \to 0$ as $n \to \infty$.

(iii) $\mathrm{var}(n^{1/2} a_n . I_{321}) \to 0$ as $n \to \infty$.

(iv) $\mathrm{var}(n^{1/2} a_n . I_{322}) \to 0$ as $n \to \infty$.

PROOF: (i) We first obtain the following conditional expectations by easy computations:

(4.4.7) $$E(G_n(X_1)-G_n(x_o)-G(X_1)-G(x_o)\,|\,X_1=x_1)$$

$$= \frac{1}{n} E\Big[1+\sum_{i=2}^{n} I(X_i \le x_1) - I(x_1 \le x_o) - \sum_{i=2}^{n} I(X_i \le x_o) - n(G(x_1)-G(x_o))\Big]$$

$$= \frac{1}{n}\big[I(x_1 > x_o) - (G(x_1)-G(x_o))\big]$$

(4.4.8) $$E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))^2\,|\,X_1 = x_1]$$

$$= \frac{1}{n^2} E\Big\{\sum_{i=2}^{n}\big[I(X_i \le x_1)-I(X_i \le x_o)-(G(x_1)-G(x_o))\big] + \big[I(x_1>x_o)-(G(x_1)-G(x_o))\big]\Big\}^2$$

$$= \frac{n-1}{n^2}|G(x_1)-G(x_o)|\,(1-|G(x_1)-G(x_o)|) + \frac{1}{n^2}\big[I(x_1>x_o)-(G(x_1)-G(x_o))\big]^2.$$

(4.4.9) $$E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))(G_n(X_2)-G_n(x_o)-G(X_2)+G(x_o)) \mid X_1=x_1, X_2=x_2]$$

$$= \frac{1}{n^2}E\Big[\Big[1+I(x_2\le x_1)+\sum_{i=3}^{n} I(X_i\le x_1)-I(x_1\le x_o)-I(x_2\le x_o)-\sum_{i=3}^{n} I(X_i\le x_o) - n(G(x_1)-G(x_o))\Big]$$

$$\Big[1+I(x_1\le x_2)+\sum_{j=3}^{n} I(X_j\le x_2)-I(x_1\le x_o)-I(x_2\le x_o)-\sum_{j=3}^{n} I(X_j\le x_o) - n(G(x_2)-G(x_o))\Big]\Big| X_1=x_1, X_2=x_2\Big]$$

$$= \frac{n-2}{n^2}\ [G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(x_2\wedge x_o)+G(x_o)-(G(x_1)-G(x_o))(G(x_2)-G(x_o))]$$

$$+\frac{1}{n^2}\sum_{i=1}^{2}(I(x_i\le x_1)-I(x_i\le x_o)-G(x_1)+G(x_o))\sum_{j=1}^{2}(I(x_j\le x_2)-I(x_j\le x_o)-G(x_2)+G(x_o)).$$

From (4.4.1), we have,

(4.4.10) $$\mathrm{Var}(n^{1/2}a_n I_2) = \frac{1}{na_n^4}\Big[\sum_i \mathrm{Var}(t_i)+\sum_{i\ne j}\sum \mathrm{Cov}(t_i,t_j)\Big]$$

$$= \frac{1}{na_n^4}[n\mathrm{Var}(t_1)+n(n-1)\ \mathrm{Cov}(t_1,t_2)]$$

where

(4.4.11) $$t_i = [G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)]K_1'\Big(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\Big)Y_i.$$

Now

(4.4.12) $$E(t_1) = \iint E[(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o))\mid X_1]$$

$$K_1'\Big(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\Big)y_1\,dH(x_1,y_1)$$

$$= \frac{1}{n}\iint [I(x_1 > x_o) - (G(x_1) - G(x_o))] K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 \, dH(x_1, y_1),$$

by (4.4.7)

$$= \frac{1}{n}\iint [I(x_1 > x_o) - (G(x_1) - G(x_o))] K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 h(x_1, y_1) \, dx_1 \, dy_1 .$$

Now, using the transformation of variables,

$$(4.4.13) \qquad \frac{G(x_1)-G(x_o)}{a_n} = u_1, \quad \frac{F_{x_o}(y_1)-\lambda}{a_n} = v_1$$

and the fact that $P\{G^{-1}(G(X)) \neq X,\ F_{x_o}^{-1}(F_{x_o}(Y)) \neq Y\} = 0$, we can write

$$E(t_1) = \int_{-\lambda/a_n}^{1-\lambda/a_n} \int_{\frac{-G(x_o)}{a_n}}^{\frac{1-G(x_o)}{a_n}} \frac{a_n^2}{n} [I(u_1 \geq 0) - a_n u_1] K_1'(u_1, v_1) F_{x_o}^{-1}(\lambda + a_n v_1)$$

$$\frac{h[G^{-1}(G(x_o)+a_n u_1), F_{x_o}^{-1}(\lambda + a_n v_1)]}{g[G^{-1}(G(x_o)+a_n u_1)]\ f_{x_o}[F_{x_o}^{-1}(\lambda + a_n v_1)]} \, du_1 \, dv_1 .$$

Since $K(.,.)$ has compact support, say, $[-1,1] \times [-1,1]$, without loss of generality, $0 < G(x_o) < 1$, $0 < \lambda < 1$ and $a_n \to 0$, there exists N' depending on x_o, λ and G only, such that for $n > N'$, the region $\left(-\frac{G(x_o)}{a_n}, \frac{1-G(x_o)}{a_n}\right) \times \left(\frac{-\lambda}{a_n}, \frac{1-\lambda}{a_n}\right)$ includes the whole support of K in it. Hence, with the transformation of variables (4.4.13), $E(t_1)$ may be written as

(4.4.14) $$E(t_1) = \frac{a_n^2}{n}\int_{-1}^{1}\int_{-1}^{1}[I(u_1\geq 0)-a_n u_1]K_1'(u_1,v_1)F_{x_o}^{-1}(\lambda+a_n v_1)\,\frac{h[G^{-1}(G(x_o)+a_n u_1),F_{x_o}^{-1}(\lambda+a_n v_1)]}{g[G^{-1}(G(x_o)+a_n u_1)]f_{x_o}[F_{x_o}^{-1}(\lambda+a_n v_1)]}\,du_1dv_1$$

for $n > N'$. Using Taylor expansions as given by lemma (4.1.1) in (4.4.14), which are valid for all $n > N$, we can write for $n > n_0 = \max(N,N')$

(4.4.15) $$E(t_1) = \frac{a_n^2}{n}\int_{-1}^{1}\int_{-1}^{1}[I(u_1\geq 0)-a_n u_1]K_1'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]$$
$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]du_1dv_1$$
$$= O\left(\frac{a_n^2}{n}\right) \text{ as } n\to\infty.$$

Similarly we have

$$E(t_1^2) = \iint E[(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o))^2|X_1]K_1'^2\left(\frac{G(x_1)-G(x_o)}{a_n},\frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1^2dH(x_1,y_1)$$
$$= \frac{1}{n^2}\iint[(n-1)|G(x_1)-G(x_o)|((1-|G(x_1)-G(x_o)|))+(I(x_1>x_o)-G(x_1)+G(x_o))^2]$$
$$K_1'^2\left(\frac{G(x_1)-G(x_o)}{a_n},\frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1^2dH(x_1,y_1),\ \text{by (4.4.8)}.$$

Upon transformation of variab[illegible]4.13) and using lemma 4.1.1, we get for n [illegible] n_o

$$(4.4.16)\quad E(t_1^2) = \frac{a_n^2}{n^2}\int_{-1}^{1}\int_{-1}^{1}\left[(n-1)|a_n u_1|(1-|a_n u_1|) + (I(u_1>0)-a_n u_1)^2\right]K_1'^2(u_1,v_1)$$
$$[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]$$
$$[1+a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)]dudv$$
$$= O(\frac{a_n^3}{n}).$$

Now

$$(4.4.17)\quad E(t_1 t_2) = \iiiint E[(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o))$$
$$(G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o))|X_1,X_2]$$
$$K_1'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})K_1'(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n})$$
$$y_1 y_2 h(x_1,y_1)h(x_2,y_2)dx_1dy_1dx_2dy_2$$
$$= \frac{n-2}{n^2}\iiiint[G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(x_2\wedge x_o)+G(x_o)$$
$$-(G(x_1)-G(x_o))(G(x_2)-G(x_o))]K_1'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})$$
$$K_1'(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n})y_1 y_2 h(x_1,y_1)h(x_2,y_2)$$
$$dx_1dy_1dx_2dy_2$$
$$+ O(\frac{1}{n^2}),$$

on using (4.4.9) and since K_1' is bounded, $E(Y^2) < \infty$. Now, we make the transformation of variables similar to that of (4.4.13), namely, (4.4.18) $$\frac{G(x_1)-G(x_o)}{a_n} = u_1 , \quad \frac{F_{x_o}(y_1)-\lambda}{a_n} = v_1 , \quad \frac{G(x_2)-G(x_o)}{a_n} = u_2 \quad \text{and}$$

$$\frac{F_{x_o}(y_2)-\lambda}{a_n} = v_2 .$$

Observe that

$$G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(x_2\wedge x_o)+G(x_o) = \begin{cases} -\left(G(x_2)-G(x_o)\right) & \text{if } x_1<x_2<x_o \\ -\left(G(x_1)-G(x_o)\right) & \text{if } x_2<x_1<x_o \\ 0 & \text{if } x_1<x_o<x_2 \\ 0 & \text{if } x_2<x_o<x_1 \\ \left(G(x_1)-G(x_o)\right) & \text{if } x_o<x_1<x_2 \\ \left(G(x_2)-G(x_o)\right) & \text{if } x_o<x_2<x_1 , \end{cases}$$

so that after the transformation of variables

$$G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(x_2\wedge x_o)+G(x_o) = \begin{cases} -a_n u_2 & \text{if } u_1<u_2<0 \\ -a_n u_1 & \text{if } u_2<u_1<0 \\ 0 & \text{if } u_1<0<u_2 \\ 0 & \text{if } u_2<0<u_1 \\ a_n u_1 & \text{if } 0<u_1<u_2 \\ a_n u_2 & \text{if } 0<u_2<u_1 , \end{cases}$$

which may be written as

$$G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(xc_2\wedge x_o)+G(x_o) = a_n(|u_1|)\wedge|u_2|)I(u_1u_2>0).$$

Also, as explained at (4.4.12), after the transformation of variables (4.4.18), the multiple integral in (4.4.17) may be considered as over the set $[-1,1]^4$, for $n > N'$. Hence we have for $n > N'$

$$E(t_1t_2) = \frac{n-2}{n^2} a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [(|u_1| \wedge |u_2|) I(u_1u_2>0) - a_n^2 u_1 u_2]$$

$$K_1'(u_1 v_1) K_1'(u_2, v_2)\, F_{x_o}^{-1}(\lambda + a_n v_1) F_{x_o}^{-1}(\lambda + a_n v_2)$$

$$\frac{h[G^{-1}(G(x_o)+a_n u_1), F_{x_o}^{-1}(\lambda+a_n v_1)]}{g[G^{-1}(G(x_o)+a_n u_1)] f_{x_o}[F_{x_o}^{-1}(\lambda+a_n v_1)]}$$

$$\frac{h[G^{-1}(G(x_o)+a_n u_2), F_{x_o}^{-1}(\lambda+a_n v_2)]}{g[G^{-1}(G(x_o)+a_n u_2)] f_{x_o}[F_{x_o}^{-1}(\lambda+a_n v_2)]}$$

$$du_1 dv_1 du_2 dv_2 .$$

Using the Taylor expansions from lemma (4.1.1), we can write, for $n > n_o$

(4.4.19)
$$E(t_1t_2) = \frac{n-2}{n^2} a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [(|u_1| \wedge |u_2|) I(u_1u_2>0) - a_n^2 u_1 u_2] K_1'(u_1, v_1)$$

$$K_1'(u_2, v_2)$$

$$[F_{x_o}^{-1}(\lambda) + a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]$$

$$[F_{x_o}^{-1}(\lambda) + a_n v_2 D_1 + a_n^2 v_2^2 D_2 + O(a_n^3)]$$

$$[1 + a_n u_2 D_6 + a_n^2 u_2^2 D_7 + a_n^2 u_2 v_2 D_8 + O(a_n^3)]$$

$$du_1 dv_1 du_2 dv_2$$

$$+ O(\frac{1}{n^2})$$

$$= O(\frac{a_n^5}{n}) + O(\frac{a_n^6}{n^2}) + O(\frac{1}{n^2}).$$

Also, from (4.4.15)

(4.4.20) $$E(t_1)E(t_2) = o(\frac{a_n^4}{n^2}).$$

Now from (4.4.15), (4.4.16), (4.4.19), (4.4.20), and (4.4.10), assertion (i) of the lemma follows upon observing that $a_n \to 0$ and $na_n^4 \to \infty$.

(ii) From (4.4.3), we have

(4.4.21) $$\mathrm{Var}(n^{1/2}a_n I_{311}) = \frac{1}{n^3 a_n^6}\left[\sum \mathrm{Var}(\ell_i)\right] = \frac{1}{n^2 a_n^6}\mathrm{Var}(\ell_1),$$

where

(4.4.22) $$\ell_i = K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(y_i)-\lambda}{a_n}\right)Y_i.$$

Therefore,

$$E(\ell_1^2) = \iint\left\{K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right)\right\}^2 K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1^2 dH(x_1,y_1),$$

and since K^* and K_2' are bounded and $E(Y_1^2) < \infty$ we have

$$\mathrm{Var}(\ell_1) \le E(\ell_1^2) < \infty,$$

and assertion (ii) of the lemma now follows from (4.4.21) since $na_n^3 \to \infty$.

(iii) From (4.4.5) we have

(4.4.23) $$\mathrm{Var}(n^{1/2}a_n I_{321}) = \frac{1}{n^3 a_n^8}\left[\sum_i \mathrm{Var}(t_{ii}) + \sum_{i\neq j}\sum \mathrm{Cov}(t_{ii},t_{jj})\right]$$

$$= \frac{1}{n^3 a_n^8}\left[n\,\mathrm{Var}(t_{11}) + n(n-1)\,\mathrm{Cov}(t_{11},t_{22})\right],$$

where $t_{ii} = [G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)]\, K^{*'}(\frac{G(X_i)-G(x_o)}{a_n})$

$$K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i$$

As in (4.4.16) we find that

$$(4.4.24)\quad E(t_{11}^2) = \iint E[(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o))^2|X_1]$$

$$\left\{K_2'^{\,2}(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})\right.$$

$$\left.\{K^{*'}(\frac{G(x_1)-G(x_o)}{a_n})\}^2 y_1^2\, dH(x_1,y_1)\right.$$

$$= O(\frac{a_n^3}{n}).$$

Since $Var(t_{11}) \le E(t_{11}^2)$

$$|Cov(t_{11},t_{22})| \le [Var(t_{11}).\ Var(t_{22})]^{1/2} = Var(t_{11}),$$

and $na_n^4 \to \infty$, assertion (iii) of the lemma now follows from (4.4.23) and (4.4.24).

(iv) We first obtain the following conditional expectations which are very similar to (4.4.7) to (4.4.9) and whose computations are straightforward:

$$(4.4.25)\quad E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))|X_1 = x_1,\ X_2 = x_2]$$

$$= \frac{1}{n}\sum_{i=1}^{2}[I(x_i \le x_1) - I(x_i \le x_o)-G(x_1)+G(x_o)]$$

$$(4.4.26)\quad E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))\,|\,X_1=x_1,X_2=x_2,X_3=x_3]$$

$$=\frac{1}{n}\sum_{i=1}^{3}[I(x_i\le x_1)-I(x_i\le x_o)-G(x_1)+G(x_o)]$$

$$(4.4.27)\quad E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))\,|\,X_1=x_1,X_2=x_2,X_3=x_3,X_4=x_4]$$

$$=\frac{1}{n}\sum_{i=1}^{4}[I(x_i\le x_1)-I(x_i\le x_o)-G(x_i)+G(x_o)]$$

$$(4.4.28)\quad E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))^2\,|\,X_1=x_1,X_2=x_2]$$

$$=\frac{n-2}{n^2}\,|G(x_1)-G(x_o)|\,(1-|G(x_1)-G(x_o)|)$$

$$+\frac{1}{n^2}\Big\{\sum_{i=1}^{2}[I(x_i\le x_1)-I(x_i\le x_o)-G(x_1)+G(x_o)]\Big\}^2$$

$$(4.4.29)\quad E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o))^2\,|\,X_1=x_1,X_2=x_2,X_3=x_3]$$

$$=\frac{n-3}{n^2}\,|G(x_1)-G(x_o)|\,(1-|G(x_1)-G(x_o)|)$$

$$+\frac{1}{n^2}\Big\{\sum_{i=1}^{3}[I(x_i\le x_1)-I(x_i\le x_o)-G(x_1)+G(x_o)]\Big\}^2$$

(4.4.30) $$E[(G_n(X_1)-G_n(x_o) - G(X_1)+G(x_o))(G_n(X_2)-G_n(x_o)-G(X_2)+G(x_o)) \mid X_1=x_1, X_2=x_2, X_3=x_3]$$

$$= \frac{n-3}{n^2}[G(x_1 \wedge x_2)-G(x_1 \wedge x_o)-G(x_2 \wedge x_o)+G(x_o) - (G(x_1)-G(x_o))(G(x_2)-G(x_o))]$$

$$+ \frac{1}{n^2}\sum_{i=1}^{3}[I(x_i \le x_1)-I(x_i \le x_o)-G(x_1)+G(x_o)] \sum_{j=1}^{3}[I(x_j \le x_2) - I(x_j \le x_o)-G(x_2)+G(x_o)]$$

(4.4.31) $$E[(G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o)) (G_n(X_2)-G_n(x_o)-G(X_2) + G(x_o)) \mid X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4]$$

$$= \frac{n-4}{n^2}[G(x_1 \wedge x_2)-G(x_1 \wedge x_o)-G(x_2 \wedge x_o)+G(x_o) -(G(x_1)-G(x_o))(G(x_2)-G(x_o))]$$

$$+ \frac{1}{n^2}\sum_{i=1}^{4}[I(x_i \le x_1)-I(x_i \le x_o)-G(x_1)+G(x_o)] \sum_{j=1}^{4}[I(x_j \le x_2) -I(x_j \le x_o)-G(x_2)+G(x_o)].$$

Now, from (4.4.6), we have

$$\operatorname{var}(n^{1/2}a_n I_{322}) = \frac{1}{n^3 a_n^8}\Big[\sum_{i \ne j}\sum \operatorname{var}(t_{ij}) + \sum_{\substack{(i,j)\ne(k,\ell) \\ i\ne j,\ k\ne \ell}}\sum\sum \operatorname{cov}(t_{ij}, t_{k\ell})\Big]$$

where t_{ij}s are given by (4.4.33). We can write

(4.4.32) $\mathrm{Var}(n^{1/2} a_n I_{322})$

$$= \frac{1}{n^3 a_n^8}\Big[\sum_{i \neq j}\sum \mathrm{Var}(t_{ij}) + \sum_{i \neq j \neq k}\sum\sum \mathrm{Cov}(t_{ij}, t_{kj}) + \sum_{i \neq j \neq \ell}\sum\sum \mathrm{Cov}(t_{ij}, t_{i\ell})$$

$$+ \sum_{i \neq j}\sum \mathrm{Cov}(t_{ij}, t_{ji}) + \sum_{i \neq j \neq \ell}\sum\sum \mathrm{Cov}(t_{ij}, t_{j\ell}) + \sum_{i \neq j \neq k}\sum\sum \mathrm{Cov}(t_{ij}, t_{k}$$

$$+ \sum_{i \neq j \neq k \neq \ell}\sum\sum\sum \mathrm{Cov}(t_{ij}, t_{k\ell})\Big]$$

$$= \frac{1}{n^3 a_n^8}\big[n(n-1)\mathrm{Var}(t_{12}) + n(n-1)(n-2)\mathrm{Cov}(t_{12}, t_{13})$$

$$+ n(n-1)(n-2)\mathrm{Cov}(t_{13}, t_{23}) + n(n-1)\mathrm{Cov}(t_{12}, t_{21})$$

$$+ n(n-1)(n-2)\mathrm{Cov}(t_{12}, t_{31}) + n(n-1)(n-2)\mathrm{Cov}(t_{12}, t_{23})$$

$$+ n(n-1)(n-2)(n-3)\mathrm{Cov}(t_{12}, t_{34})\big]$$

$$= \frac{(n-1)}{n^2 a_n^8}\mathrm{Var}(t_{12}) + \frac{(n-1)(n-2)}{n^2 a_n^8}\mathrm{Cov}(t_{12}, t_{13}) + \frac{(n-1)(n-2)}{n^2 a_n^8}\mathrm{Cov}(t_{13}, t_{23}$$

$$+ \frac{(n-1)}{n^2 a_n^8}\mathrm{Cov}(t_{12}, t_{21}) + \frac{(n-1)(n-2)}{n^2 a_n^8}\mathrm{Cov}(t_{12}, t_{31})$$

$$+ \frac{(n-1)(n-2)}{n^2 a_n^8}\mathrm{Cov}(t_{12}, t_{23}) + \frac{(n-1)(n-2)(n-3)}{n^2 a_n^8}\mathrm{Cov}(t_{12}, t_{34})$$

$$= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 ,$$

with

$$(4.4.33)\quad t_{ij} = [G_n(X_j)-G_n(x_o)-G(X_j)+G(x_o)]K^{*'}\left(\frac{G(X_j)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i I(Y_j \le Y_i).$$

Therefore

$$E(t_{12}) = \iiiint E[(G_n(x_2)-G(x_o)-G(x_2)+G(x_o))|X_1,X_2]$$

$$K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)y_1$$

$$I(y_2 \le y_1)dH(x_1,y_1)dH(x_2,y_2)..$$

$$= \frac{1}{n}\iiint\left(\sum_{i=1}^{2}[I(x_i \le x_1)-I(x_i \le x_o)-G(x_1)+G(x_o)]\right)$$

$$K_2'\left(\frac{G(x_1-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$

$$y_1 F_{x_2}(y_1)dH(x_1,y_1)dG(x_2), \text{ by } (4.4.25).$$

Now, transformation of variables (4.4.13) and an application of lemma (4.1.1) yields, for $n > n_o$

$$(4.4.34)\quad E(t_{12}) = \frac{a_n^3}{n}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\sum_{i=1}^{2}[I(u_i \le u_1)-I(u_i \le 0)-a_n u_1]\right)K_2'(u_1 v_1)K^{*'}(u_2)$$

$$[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]$$

$$[\lambda+a_n u_2 D_3+a_n v_1+a_n^2 u_2^2 D_4+a_n^2 u_2 v_1 D_5+O(a_n^3)]$$

$$[1+a_n u_1 D_6+a_n u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]\, du_1 dv_1 du_2$$

$$= O\left(\frac{a_n^3}{n}\right).$$

On the same lines, using (4.4.27) we find that

$$(4.4.35) \qquad E(t_{12}^2) = O\left(\frac{a_n^3}{n}\right) + O\left(\frac{a_n^3}{n^2}\right).$$

Hence, we have

$$(4.4.36) \qquad \operatorname{var}(t_{12}) = O\left(\frac{a_n^3}{n}\right).$$

Using (4.4.33) we obtain

$$(4.4.37) \quad E(t_{12}t_{13}) = \iiiint\!\!\iint E\big[\big(G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)\big)\big(G_n(x_3)-G_n(x_o)-G(x_3)+G(x_o)\big)\,|X_1,X_2,X_3\big]$$
$$K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$
$$K^{*\prime}\left(\frac{G(x_3)-G(x_o)}{a_n}\right)y_1^2 I(y_2<y_1)I(y_3<y_1)\,dH(x_1,y_1)\,dH(x_2,y_2)\,dH(x_3,y_3)$$
$$= \iiiint \Big\{\frac{n-3}{n^2}\big[G(x_2\wedge x_3)-G(x_2\wedge x_o)-G(x_3\wedge x_o)+G(x_o) - \big(G(x_2)-G(x_o)\big)\big(G(x_3)-G(x_o)\big)\big]\Big\}$$
$$K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^{*\prime}\left(\frac{G(x_3)-G(x_o)}{a_n}\right)$$
$$K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1^2 F_{x_2}(y_1)F_{x_3}(y_1)$$
$$dH(x_1,y_1)\,dG(x_2)\,dG(x_3)$$
$$+ O\left(\frac{1}{n^2}\right),$$

where the last equality follows on using (4.4.30).

Now, by transformation of variables (4.4.18) and an application of lemma (4.1.1) in (4.4.37) yields for $n > n_o$,

$$(4.4.38)\quad E(t_{12}t_{13}) = \frac{n-3}{n^2}a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [(|u_2| \wedge |u_3|) I(u_2u_3>0) - a_n [illegible] u_3]$$

$$K^{*'}(u_2)K^{*'}(u_3)K_2^{'2}(u_1,v_1)[\bar{F}_{x_o}^{-1}(\lambda)+a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]^2$$

$$[\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)]$$

$$[\lambda + a_n u_3 D_3 + a_n v_1 + a_n^2 u_3^2 D_4 + a_n^2 u_3 v_1 D_5 + O(a_n^3)]$$

$$[1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)] du_1 dv_1 du_3 du_2$$

$$+ O(\frac{a_n^4}{n^2})$$

$$= O(\frac{a_n^5}{n}) .$$

Hence, from (4.4.34) and (4.4.38) it follows that

$$(4.4.39)\quad \mathrm{Cov}(t_{12},t_{13}) = O(\frac{a_n^5}{n}) + O(\frac{a_n^6}{n^2}) = O(\frac{a_n^5}{n}).$$

Proceeding on the same lines, we find that

$$\mathrm{Cov}(t_{13},t_{23}) = O(\frac{a_n^6}{n}),$$

$$(4.4.40)\quad \mathrm{Cov}(t_{12},t_{21}) = O(\frac{a_n^5}{n}),$$

$$\mathrm{Cov}(t_{12},t_{31}) = O(\frac{a_n^6}{n}), \text{ and}$$

$$\mathrm{Cov}(t_{12},t_{23}) = O(\frac{a_n^6}{n}) .$$

Since $a_n \to 0$ and $na_n^4 \to \infty$, it follows from (4.4.36), (4.4.39), (4.4.40) and (4.4.32) that each of the terms A_1 to A_6 goes to zero as $n \to \infty$. So, it suffices to show that $A_7 \to 0$ as $n \to \infty$, where we have

(4.4.41) $$A_7 = \frac{(n-1)(n-2)(n-3)}{n^2 a_n^8}[E(t_{12}t_{34})-E(t_{12})E(t_{34})], \text{ from (4.4.32).}$$

Using (4.4.33) we obtain

(4.4.42) $$E(t_{12}t_{34}) = \int\int\int\int\int\int\int\int E[(G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o))$$

$$(G_n(x_4)-G_n(x_o)-G(x_4)+G(x_o))|X_1,X_2,X_3,X_4]$$

$$K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^{*\prime}\left(\frac{G(x_4)-G(xo)}{a_n}\right)$$

$$K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)$$

$$y_1y_3I(y_2\le y_1)I(y_4\le y_3)dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)dH(x_4,y_4)$$

$$= \int\int\int\int\int\int\{\frac{n-4}{n^2}[G(x_2\wedge x_4)-G(x_2\wedge x_o)-G(x_4\wedge x_o)+G(x_o)$$

$$-(G(x_2)-G(x_o))(G(x_4)-G(x_o))] + o(\frac{1}{n^2})\}$$

$$K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^{*\prime}\left(\frac{G(x_4)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3F_{x_2}(y_1)F_{x_4}(y_3)dH(x_1,y_1)$$

$$dH(x_3,y_3)dG(x_2)dG(x_4)$$

where the last equality follows upon using (4.4.31).

Upon transformation of variables (4.4.18) and applying lemma 4.1.1, we have, for $n > n_o$,

(4.4.43) $$E(t_{12}t_{34}) = \frac{n-4}{n^2} a_n^6 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [a_n(|u_2|\wedge|u_4|)I(u_2u_4>0)-a_n^2u_2u_4]$$

$$K^{*'}(u_2)K^{*'}(u_4)K_2'(u_1,v_1)K_2'(u_3,v_3)$$

$$[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$

$$[F_{x_o}^{-1}(\lambda)+a_nv_3D_1+a_n^2v_1^2D_2+O(a_n^3)]$$

$$[\lambda+a_nu_2D_3+a_nv_1+a_n^2u_2^2D_4+a_n^2u_2v_1D_5+O(a_n^3)]$$

$$[\lambda+a_nu_4D_3+a_nv_3+a_n^2u_4^2D_4+a_n^2u_4v_3D_5+O(a_n^3)]$$

$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]$$

$$[1+a_nu_3D_6+a_nu_3^2D_7+a_nu_3v_3D_8+O(a_n^3)]du_1dv_1du_3dv_3du_2du_4$$

$$+ O(\frac{a_n^6}{n^2}).$$

On collecting terms in powers of a_n from (4.4.43) we have

(4.4.44) $$E(t_{12}t_{34}) = \frac{n-4}{n^2} a_n^7 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (|u_2|\wedge u_4|)I(u_2u_4>0)K^{*'}(u_2)$$

$$K^{*'}(u_4)K_2'(u_1,v_1)K_2'(u_3,v_3)$$

$$\lambda^2(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_3dv_3du_2du_4$$

$$+ \frac{n-4}{n^2} a_n^8 \int\int\int\int\int\int(|u_2|\wedge|u_4|)I(u_2u_4>0)K^{*'}(u_2)K^{*'}(u_4)$$

$$K_2'(u_1v_1)K_2'(u_3,v_3)[\lambda^2F_{x_o}^{-1}(\lambda)(v_1D_1+v_3D_1)$$

$$+ \lambda(F_{x_o}^{-1}(\lambda))^2(u_2D_3+v_1+u_4D_3+v_3)$$

$$+\lambda^2(F_{x_o}^{-1}(\lambda))^2(u_1D_6+u_3D_6)]$$

$$du_1dv_1du_3dv_3du_2du_4$$

continued

(4.4.44) $$-\frac{n-4}{n^2}a_n^8 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} u_2u_4K^{*'}(u_2)K^{*'}(u_2)K^{*'}(u_4)K_2'(u_1,v_1)K_2'(u_3,v_3)$$

$$\lambda^2(F_{x_0}^{-1}(\lambda))^2du_1dv_1du_3dv_3du_2du_4$$

$$+ O(\frac{a_n^9}{n}) + O(\frac{a_n^6}{n}) \quad \text{for } n > n_0.$$

Since $\int_{-1}^{1}\int_{-1}^{1} K_2'(u,v)dudv$ is equal to zero, the integrals on the R.H.S. of (4.4.44) are all equal to zero and it now follows from (4.4.34), (4.4.41) and (4.4.44) th

$$A_7 = O(a_n) + O(\frac{1}{na_n^2})$$

which goes to zero as $n \to \infty$, upon observing that $a_n \to 0$ and $na_n^2 \to \infty$. This proves assertion (iv) of the lemma and hence the proof of the lemma 4.4.1 is complete. □

Next, we shall show that variance of $n^{1/2}a_n(I_{312}-I_{30})$ converges to a finite value.

<u>LEMMA</u> 4.4.2 <u>Under the hypothesis of Lemma</u> 4.4.1, $Var[n^{1/2}a_n(I_{312}-I_{30})]$ <u>converges to</u> $\sigma_1^2(x_0,\lambda)$, <u>where</u>

(4.4.45) $$\sigma_1^2(x_0,\lambda) = (F_{x_0}^{-1}(\lambda))^2\int\int K^{*2}(u)K_2^2(v)dudv$$

$$K_2(v) = \int K(u,v)du$$

<u>and</u> I_{312}, I_{30} <u>are given by</u> (4.4.44) <u>and</u> (4.4.1) <u>respectively.</u>

<u>PROOF:</u> We shall write

(4.4.46) $$Var[n^{1/2}a_n(I_{312}-I_{30})] = Var(n^{1/2}a_nI_{312}) + Var(n^{1/2}a_nI_{30})$$

$$-2\,Cov(n^{1/2}a_nI_{312}, n^{1/2}a_nI_{30}).$$

From (4.4.4) we have

$$\mathrm{Var}(n^{1/2}a_n I_{312}) = \frac{1}{n^3 a_n^6}\Big[\sum_{i\neq j}\sum \mathrm{Var}(\ell_{ij}) + \sum_{\substack{(i,j)\neq(k,m)\\ i\neq j \quad k\neq m}}\sum\sum\sum \mathrm{Cov}(\ell_{ij},\ell_{km})\Big]$$

where

$$(4.4.47)\qquad \ell_{ij} = K^*\Big(\frac{G(X_j)-G(x_o)}{a_n}\Big)K_2^*\Big(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\Big)Y_i I(Y_j \le Y_i).$$

As in (4.4.32), we can write

$$(4.4.48)\qquad \mathrm{Var}(n^{1/2}a_n I_{312})$$

$$= \frac{1}{n^3 a_n^6}\big[n(n-1)\mathrm{Var}(\ell_{12}) + n(n-1)(n-2)\,\mathrm{Cov}(\ell_{12},\ell_{13})$$

$$+ n(n-1)(n-2)\mathrm{Cov}(\ell_{13},\ell_{23}) + n(n-1)\mathrm{Cov}(\ell_{12},\ell_{21})$$

$$+ n(n-1)(n-2)\mathrm{Cov}(\ell_{12},\ell_{31}) + n(n-1)(n-2)\mathrm{Cov}(\ell_{12},\ell_{23})$$

$$+ n(n-1)(n-2)(n-3)\mathrm{Cov}(\ell_{12},\ell_{34})\big]$$

$$= \frac{n-1}{n^2 a_n^6}\mathrm{Var}(\ell_{12}) + \frac{(n-1)(n-2)}{n^2 a_n^6}\mathrm{Cov}(\ell_{12},\ell_{13}) + \frac{(n-1)(n-2)}{n^2 a_n^6}\mathrm{Cov}(\ell_{13},\ell_{23})$$

$$+ \frac{(n-1)}{n^2 a_n^6}\mathrm{Cov}(\ell_{12},\ell_{21}) + \frac{(n-1)(n-2)}{n^2 a_n^6}\mathrm{Cov}(\ell_{12},\ell_{31})$$

$$+ \frac{(n-1)(n-2)}{n^2 a_n^6}\mathrm{Cov}(\ell_{12},\ell_{23}) + \frac{(n-1)(n-2)(n-3)}{n^2 a_n^6}\mathrm{Cov}(\ell_{12},\ell_{34})$$

$$= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7.$$

First, observe that $\mathrm{Cov}(\ell_{12},\ell_{34})$ is zero, since ℓ_{12} and ℓ_{34} are functions of two independent sets of rv's. Hence,

$$(4.4.49)\qquad B_7 = 0.$$

Now, using (4.4.47)

$$E(\ell_{12}^2) = \iiiint K^{*2}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1^2 I(y_2 \le y_1)$$

$$dH(x_1,y_1)dH(x_2,y_2)$$

$$= \iiint K^{*2}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1^2 F_{x_2}(y_1)$$

$$dH(x_1,y_1)dG(x_2).$$

Upon transformation of variables (4.4.18) and using lemma 4.1.1, we easily see that

(4.4.50) $E(\ell_{12}^2) = O(a_n^3)$ and hence $Var(\ell_{12}) = O(a_n^3)$.

Consequently it follows from (4.4.48) that

(4.4.51) $B_1 = O(\frac{1}{na_n^3})$, which goes to zero as $n \to \infty$.

Since $Var(\ell_{12}) = Var(\ell_{21})$, by Schwartz's inequality, we have $|Cov(\ell_{12},\ell_{21})| \le Var(\ell_{12})$ and it follows from (4.4.48) and (4.4.50) that

(4.4.52) $B_4 = O(\frac{1}{na_n^3})$, which again goes to zero as $n \to \infty$.

Hence, we now need to compute the terms B_2, B_3, B_5 and B_6 only. We have from (4.4.48)

(4.4.53) $B_2 = \frac{(n-1)(n-2)}{n^2 a_n^6}[E(\ell_{12}\ell_{13}) - E(\ell_{12})E(\ell_{13})]$.

We obtain

(4.4.54) $E(\ell_{12})E(\ell_{13})$

$$= \int\int\int\int\int\int\int\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^*\left(\frac{G(x_4)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)$$

$$y_1y_3I(y_2\le y_1)I(y_4\le y_3)dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)dH(x_4,y_4)$$

$$= \int\int\int\int\int\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^*\left(\frac{G(x_4)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3F_{x_2}(y_1)F_{x_4}(y_3)dH(x_1,y_1)$$

$$dH(x_3,y_3)dG(x_2)dG(x_4).$$

Upon usual transformation of variables as in (4.4.18) and applying lemma 4.1.1, and collecting terms up to order a_n^6, we get, for $n > n_o$,

$$E(\ell_{12})E(\ell_{13}) = a_n^6 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K^*(u_4)K_2'(u_1,v_1)K_2'(u_3,v_3)\lambda^2(F_{x_o}^{-1}(\lambda))^2$$

$$du_1dv_1du_3dv_3du_2du_4$$

$$+ O(a_n^7)$$

$$= a_n^6\lambda^2(F_{x_o}^{-1}(\lambda))^2\left(\int_{-1}^{1}\int_{-1}^{1}K^*(u_2)K^*(u_4)du_2du_4\right)\left(\int_{-1}^{1}\int_{-1}^{1}K_2'(u_1,v_1)du_1dv_1\right)^2$$

$$+ O(a_n^7)$$

$$= O(a_n^7),$$

since $\int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)du_1dv_1 = 0.$

From (4.4.47), we write

$$(4.4.56)\quad E(\ell_{12}\ell_{13}) = \iiiint\!\!\iint K_2^{\prime 2}\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$

$$K^*\left(\frac{G(x_3)-G(x_o)}{a_n}\right)y_1^2 I(y_2 \le y_1)I(y_3 \le y_1)$$

$$dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)$$

$$= \iiiint K_2^{\prime 2}\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$

$$K^*\left(\frac{G(x_3)-G(x_o)}{a_n}\right)y_1^2 F_{x_2}(y_1)F_{x_3}(y_1)dH(x_1,y_1)$$

$$dG(x_2)dG(x_3).$$

By transformation of variables (4.4.18) and using lemma 4.1.1 we get, for $n > n_o$,

$$E(\ell_{12}\ell_{13}) = a_n^4 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K^*(u_3)K_2^{\prime 2}(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+0(a_n^3)]^2$$

$$[\lambda+a_n u_2 D_3+a_n v_1+a_n^2 u_2^2 D_4+a_n^2 u_2 v_1 D_5+0(a_n^3)]$$

$$[\lambda+a_n u_3 D_3+a_n v_1+a_n^2 u_3^2 D_4+a_n^2 u_3 v_1 D_5+0(a_n^3)]$$

$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+0(a_n^3)]du_1 dv_1 du_2 du_3.$$

On collecting terms in powers of a_n from the R.H.S. integral while using the hypothesis $\int K^*(u)du = 1$, $\int uK^*(u)du = 0$ and using the resulting expression along with (4.4.54) in (4.4.53) we obtain

$$(4.4.58)\quad B_2 = \frac{(n-1)(n-2)}{n^2a_n^2}\iint K_2'^2(u_1,v_1)\lambda^2(F_{x_o}^{-1}(\lambda))^2\,du_1dv_1$$

$$+\frac{(n-1)(n-2)}{n^2a_n}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1 + \lambda^2(F_{x_o}^{-1}(\lambda))^2u_1D_6]du_1dv_1$$

$$+\frac{(n-1)(n-2)}{n^2}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1^2D_2+\lambda^2v_1^2D_1^2 + \lambda^2(F_{x_o}^{-1}(\lambda))^2(u_1^2D_7+u_1v_1D_8)+4\lambda F_{x_o}^{-1}(\lambda)v_1^2D_1+2\lambda^2F_{x_o}^{-1}(\lambda)v_1u_1D_1D + (F_{x_o}^{-1}(\lambda))^2v_1^2+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1u_1D_6]du_1dv_1$$

$$+\frac{(n-1)(n-2)}{n^2}\iint K_2'^2(u_1,v_1)2(F_{x_o}^{-1}(\lambda))^2D_4du_1dv_1(\int u_2^2K^*(u_2)du_2)$$

$$+O(a_n).$$

Now, $Cov(\ell_{13},\ell_{23}) = E(\ell_{13}\ell_{23}) - E(\ell_{13})E(\ell_{23})$ and

$$(4.4.59)\quad E(\ell_{13}\ell_{23}) = \iiiint\!\!\iint K^{*2}\left(\frac{G(x_3)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\frac{F_{x_o}(y_1)-\lambda}{a_n}\right) K_2'\left(\frac{G(x_2)-G(x_o)}{a_n},\frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2I(y_3\le y_1\wedge y_2) dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)$$

$$= 2\iiiint\!\!\iint K^{*2}\left(\frac{G(x_3)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n},\frac{F_{x_o}(y_1)-\lambda}{a_n}\right) K_2'\left(\frac{G(x_2)-G(x_o)}{a_n},\frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2I(y_3\le y_1)I(y_1\le y_2) dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3).$$

After transformation of variables as in (4.4.18) and using Taylor expansions from lemma 4.1.1, we get, for $n > n_o$

(4.4.60) $$E(\ell_{13}\ell_{23}) = 2a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \{K^*(u_3)\}^2 K_2'(u_1,v_1)K_2'(u_2,v_2)I(v_1 \le v_2)$$

$$[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]$$

$$[F_{x_o}^{-1}(\lambda)+a_n v_2 D_1+a_n^2 v_2^2 D_2+O(a_n^3)]$$

$$[\lambda+a_n u_3 D_3+a_n \bar{v}_1+a_n^2 u_3^2 D_4+a_n^2 u_3 v_1 D_5+O(a_n^3)]$$

$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]$$

$$[1+a_n u_2 D_6+a_n^2 u_2^2 D_7+a_n^2 u_2 v_2 D_8+O(a_n^3)]$$

$$du_1 dv_1 du_2 dv_2 du_3$$

$$= 2a_n^5 \lambda(F_{x_o}^{-1}(\lambda))^2 \left(\iiiint_{\{v_1 \le v_2\}} K_2'(u_1,v_1)K_2'(u_2,v_2) du_1 dv_1 du_2 dv_2 \right)$$

$$\int \{K^*(u_3)\}^2 du_3$$

$$+ 2a_n^6 \iiiiint\!\!\int_{\{v_1 \le v_2\}} \{K^*(u_3)\}^2 K_2'(u_1,v_1)K_2'(u_2,v_2)$$

$$[\lambda(v_1+v_2)D_1+(F_{x_o}^{-1}(\lambda))^2(u_3 D_3+v_1)$$

$$+\lambda(F_{x_o}^{-1}(\lambda))^2(u_1+u_2)D_6]$$

$$+ O(a_n^7) .$$

Since $\iint K_2'(u,v)dudv = 0$, we observe that

(4.4.61) $$2\iiiint_{\{v_1\le v_2\}} K_2'(u_1,v_1)K_2'(u_2,v_2)du_1dv_1du_2dv_2 = \iiiint K_2'(u_1,v_1)K_2'(u_2,v_2)du_1dv_1du_2dv_2$$

$$= \left(\iint K_2(u_1,v_1)du_1dv_1\right)^2 = 0,$$

(4.4.62) $$2\iiiint_{\{v_1\le v_2\}}(v_1+v_2)K_2'(u_1,v_2)\,K_2'(u_2,v_2)du_1dv_1du_2dv_2 = \iiiint(v_1+v_2)K_2'(u_1,v_1)K_2'(u_2,v_2)du_1dv_1du_2dv_2$$

$$= 0,$$

(4.4.63) $$2\iiiint_{\{v_1\le v_2\}}(u_1+u_2)K_2'(u_1,v_1)K_2'(u_2,v_2)dv_1du_1du_2dv_2 = \iiiint(u_1+u_2)K_2'(u_1,v_1)K_2'(u_2,v_2)du_1dv_1du_2dv_2$$

$$= 0.$$

Hence, from (4.4.54) and (4.4.60) to (4.4.63), we have

(4.4.64) $$\mathrm{Cov}(\ell_{13},\ell_{23}) = 2(F_{X_0}^{-1}(\lambda))^2a_n^6\iiiint_{\{v_1\le v_2\}} v_1K_2'(u_1,v_1)K_2'(u_2,v_2)(\int[K^*(u_3)]^2du_3)du_1dv_1du_2dv_2$$

$$+ O(a_n^7).$$

Therefore, from (4.4.48), it follows that

(4.4.65) $$B_3 = \frac{(n-1)(n-2)}{n^2}2(F_{X_0}^{-1}(\lambda))^2\iiiint_{\{v_1\le v_2\}} v_1K_2'(u_1,v_1)K_2'(u_2,v_2)(\int[K^*(u_3)]^2du_3)du_1dv_1du_2dv_2,$$

$$+ O(a_n).$$

From (4.4.47) we have

$$(4.4.66)\quad E(\ell_{12}\ell_{31}) = \int\int\int\int\int\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)I(y_1\le y_3)$$

$$dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)$$

$$= \int\int\int\int\int\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_1\le y_3)F_{x_2}(y_1)dH(x_1,y_1)$$

$$dH(x_3,y_3)dG(x_2),$$

and note that

$$(4.4.67)\quad E(\ell_{12}\ell_{31}) = E(\ell_{12}\ell_{23}).$$

Again, transformation of variables as in (4.4.18) and an application of lemma 4.1, in (4.4.66) yields, for $n > n_o$

$$E(\ell_{12}\ell_{31}) = a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1\le v_3)$$

$$[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$

$$[F_{x_o}^{-1}(\lambda)+a_nv_3D_1+a_n^2v_3^2D_2+O(a_n^3)]$$

$$[\lambda+a_nu_2D_3+a_nv_1+a_n^2u_2^2D_4+a_n^2u_2v_1D_5+O(a_n^3)]$$

continued

$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]$$

$$[1+a_n u_3 D_6+a_n^2 u_3^2 D_7+a_n^2 u_3 v_3 D_8+O(a_n^3)]$$

$$du_1 dv_1 du_3 dv_3 du_2,$$

which we may write as

$$E(\ell_{12}\ell_{31}) = a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \lambda(F_{x_o}^{-1}(\lambda))^2 K^*(u_2)K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1\leq v_3)\, du_1 dv_1 du_3 dv_3 du_2$$

$$+ a_n^6 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1\leq v_3)$$

$$[\lambda F_{x_o}^{-1}(\lambda)(v_1+v_3)D_1+(F_{x_o}^{-1}(\lambda))^2(u_2 D_3+v_1)$$

$$+\lambda(F_{x_o}^{-1}(\lambda))^2(u_1+u_3)D_6]du_1 dv_1 du_3 dv_3 du_2$$

$$+ O(a_n^7) .$$

Since $\int_{-1}^{1} K^*(u)du = 1$, $\int_{-1}^{1} uK^*(u)du = 0$, by hypothesis, the above expression reduces to

(4.4.68) $$E(\ell_{12}\ell_{31}) = a_n^5 \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1\leq v_3)\lambda(F_{x_o}^{-1}(\lambda))^2\, du_1 dv_1 du_3 dv_3$$

$$+ a_n^6 \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1\leq v_3)$$

$$[\lambda F_{x_o}^{-1}(\lambda)(v_1+v_3)D_1+(F_{x_o}^{-1}(\lambda))^2 v_1+\lambda(F_{x_o}^{-1}(\lambda))^2$$

$$(u_1+u_3)D_6]du_1 dv_1 du_3 dv_3$$

$$+ O(a_n^7), \qquad \text{for } n > n_o.$$

Hence, it follows from (4.4.48), (4.4.54) and (4.4.67) that

(4.4.69) $$B_5 + B_6 = \frac{(n-1)(n-2)}{n^2 a_n} \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1 \le v_3)\, 2\lambda(F_{X_0}^{-1}(\lambda))^2 du_1 dv_1 du_3 dv_3$$

$$+ \frac{(n-1(n-2)}{n^2} \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1 \le v_3) [2\lambda F_{X_0}^{-1}(\lambda)(v_1+v_3)D_1 + 2(F_{X_0}^{-1}(\lambda))^2 v_1 + 2\lambda(F_{X_0}^{-1}(\lambda))^2(u_1+u_3)D_6]du_1 dv_1 du_3 dv_3$$

$$+ O(a_n).$$

Therefore

(4.4.70) $$\mathrm{Var}(n^{1/2} a_n I_{312})$$

$$= \frac{(n-1)(n-2)}{n^2 a_n^2} \iint K_2'^2(u_1,v_1)\lambda^2(F_{X_0}^{-1}(\lambda))^2 du_1 dv_1$$

$$+ \frac{(n-1)(n-2)}{n^2 a_n} \iint K_2'^2(u_1,v_1)[2\lambda^2 F_{X_0}^{-1}(\lambda)v_1 D_1 + 2\lambda(F_{X_0}^{-1}(\lambda))^2 v_1 + \lambda^2(F_{X_0}^{-1}(\lambda))u_1 D_6]du_1 dv_1$$

$$+ \frac{(n-1)(n-2)}{n^2} \iint K_2'^2(u_1,v_1)[2\lambda^2 F_{X_0}^{-1}(\lambda)v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2(F_{X_0}^{-1}(\lambda))^2(u_1^2 D_7 + u_1 v_1 D_8)$$

$$+ 4\lambda F_{X_0}^{-1}(\lambda)v_1^2 D_1 + 2\lambda^2 F_{X_0}^{-1}(\lambda)v_1 u_1 D_1 D_6 + (F_{X_0}^{-1}(\lambda))^2 v_1^2 + 2\lambda(F_{X_0}^{-1}(\lambda))^2 v_1 u_1 D_6]$$

continued

$$+ \frac{(n-1)(n-2)}{n^2} \iint K_2'^2(u_1,v_1)2\lambda(F_{x_o}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2)du_2)$$

$$+ \frac{(n-1)(n-2)}{n^2} 2(F_{x_o}^{-1}(\lambda))^2 \iiiint v_1 I(v_1 \le v_2) K_2'(u_1,v_1)K_2'(u_2,v_2)du_1 dv_1 du_2 dv_2$$

$$(\int \{K^*(u_3)\}^2 du_3)$$

$$+ \frac{(n-1)(n-2)}{n^2 a_n} 2\lambda(F_{x_o}^{-1}(\lambda))^2 \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1 \le v_3)du_1 dv_1 du_3 dv_3$$

$$+ \frac{(n-1)(n-2)}{n^2} \iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_3,v_3)I(v_1 \le v_3)$$

$$[2\lambda F_{x_o}^{-1}(\lambda)(v_1+v_3)D_1 + 2(F_{x_o}^{-1}(\lambda))^2 v_1 + 2\lambda(F_{x_o}^{-1}(\lambda))^2(u_1+u_3)D_6]$$

$$du_1 dv_1 du_3 dv_3$$

$$+ O(\frac{1}{na_n^3}) + O(a_n).$$

Now, from (4.4.1) we have

$$I_{30} = \frac{1}{na_n^3} \sum_{i=1}^n F_{x_o}(Y_i)\, K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i, \quad \text{and hence}$$

$$(4.4.71) \quad Var(n^{1/2} a_n I_{30}) = \frac{na_n^2}{n^2 a_n^6} [\sum_i Var(p_i)]$$

$$= \frac{1}{a_n^4} Var(p_1),$$

where

$$(4.4.72) \quad p_i = K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i F_{x_o}(Y_i).$$

Therefore

$$E(p_1^2) = \iint K_2'^2(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})y_1^2 F_{x_o}^2(y_1)dH(x_1,y_1).$$

By transformation of variables (4.4.13) and application of lemma 4.1.1, for $n > n_o$, we obtain

(4.4.73) $$E(p_1^2) = a_n^2 \int_{-1}^{1}\int_{-1}^{1} K_2'^2(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]^2 [\lambda+a_n v_1]^2 [1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]du_1 dv_1$$

$$= a_n^2 \iint K_2'^2(u_1,v_1)(F_{x_o}^{-1}(\lambda))^2\lambda^2 du_1 dv_1$$

$$+ a_n^3 \iint K_2'^2(u_1,v_1)[2\lambda^2 F_{x_o}^{-1}(\lambda)v_1 D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2 v_1+\lambda^2(F_{x_o}^{-1}(\lambda))^2 u_1 D_6]du_1 dv_1$$

$$+ a_n^4 \iint K_2'^2(u_1,v_1)[2\lambda^2 F_{x_o}^{-1}(\lambda)v_1^2 D_2+\lambda^2 v_1^2 D_1^2+(F_{x_o}^{-1}(\lambda))^2 v_1^2$$
$$+\lambda^2(F_{x_o}^{-1}(\lambda))^2(u_1^2 D_7+u_1 v_1 D_8)+4\lambda F_{x_o}^{-1}(\lambda)v_1^2 D_1$$
$$+2\lambda^2 F_{x_o}^{-1}(\lambda)v_1 u_1 D_1 D_6+2\lambda(F_{x_o}^{-1}(\lambda))^2 v_1 u_1 D_6]$$

$$+ O(a_n^5).$$

Also

$$[E(p_1)]^2 = \iiiint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_2'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right) y_1 y_2 F_{x_o}(y_1)F_{x_o}(y_2)dH(x_1,y_1)dH(x_2,y_2).$$

Again by transformation of variables (4.4.18) and using lemma 4.1.1, we get, for $n > n_o$

(4.4.74) $$[E(p_1)]^2 = a_n^4 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)K_2'(u_2,v_2)[F_{X_0}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]$$

$$[F_{X_0}^{-1}(\lambda)+a_n v_2 D_1+a_n^2 v_2^2 D_2+O(a_n^3)][\lambda+a_n v_1]$$

$$[\lambda+a_n v_2][1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]$$

$$[1+a_n u_2 D_6+a_n^2 u_2^2 D_7+a_n^2 u_2 v_2 D_8+O(a_n^3)]du_1 dv_1 du_2 dv_2$$

$$= a_n^4 \lambda^2 (F_{X_0}^{-1}(\lambda))^2 \iiiint K_2'(u_1,v_1)K_2'(u_2,v_2)du_1 dv_1 du_2 dv_2$$

$$+ O(a_n^5)$$

$$= O(a_n^5), \text{ since } \iint K_2'(u,v)dudv = 0.$$

Hence, from (4.4.71), (4.4.73) and (4.4.74) we have, for $n > n_0$,

(4.4.75) $$\mathrm{Var}(n^{1/2}a_n I_{30}) = \frac{1}{a_n^2}\iint K_2'^{\,2}(u_1,v_1)\lambda^2(F_{X_0}^{-1}(\lambda))^2 du_1 dv_1$$

$$+ \frac{1}{a_n}\iint K_2'^{\,2}(u_1,v_1)[2\lambda^2 F_{X_0}^{-1}(\lambda)v_1 D_1+2\lambda(F_{X_0}^{-1}(\lambda))^2 v_1$$

$$+\lambda^2(F_{X_0}^{-1}(\lambda))^2 u_1 D_6]du_1 dv_1$$

$$+ \iint K_2'^{\,2}(u_1,v_1)[2\lambda^2 F_{X_0}^{-1}(\lambda)v_1^2 D_2+\lambda^2 v_1^2 D_1^2+(F_{X_0}^{-1}(\lambda))^2 v_1^2$$

$$+ \lambda^2(F_{X_0}^{-1}(\lambda))^2(u_1^2 D_7+u_1 v_1 D_8)+4\lambda F_{X_0}^{-1}(\lambda)v_1^2 D_1$$

$$+ 2\lambda^2 F_{X_0}^{-1}(\lambda)v_1 u_1 D_1 D_6+2\lambda(F_{X_0}^{-1}(\lambda))^2 v_1 u_1 D_6]$$

$$+ O(a_n).$$

Now, observe that

$$(4.4.76)\quad \mathrm{Cov}(n^{1/2}a_n I_{312}, n^{1/2}a_n I_{30}) = na_n^2 \mathrm{Cov}\Big(\frac{1}{n^2a_n^4}\sum_{i\neq j}\sum \ell_{ij}, \frac{1}{na_n^3}\sum_m p_m\Big)$$

$$= \frac{1}{n^2a_n^5}\sum_{i\neq j}\sum\sum_m \mathrm{Cov}(\ell_{ij}, p_m)$$

$$= \frac{1}{n^2a_n^5}\Big[\sum_{i\neq j}\sum \mathrm{Cov}(\ell_{ij},p_i) + \sum_{i\neq j}\sum \mathrm{Cov}(\ell_{ij},p_j)\Big]$$

$$= \frac{1}{n^2a_n^5}[n(n-1)\mathrm{Cov}(\ell_{12},p_1) + n(n-1)\mathrm{Cov}(\ell_{21},p_1)]$$

$$= \frac{n-1}{na_n^5}\mathrm{Cov}(\ell_{12},p_1) + \frac{n-1}{na_n^5}\mathrm{Cov}(\ell_{21},p_1)$$

where ℓ_{ij}'s are defined by (4.4.47) and p_m's by (4.4.72). We shall now compute the covariances that appear in the last expression:

$$(4.4.77)\quad E(\ell_{12}p_1) = E\Big[K^*\Big(\frac{G(X_2)-G(x_o)}{a_n}\Big)K_2'^{\,2}\Big(\frac{G(X_1)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_1)-\lambda}{a_n}\Big)Y_1^2F_{x_o}(Y_1)I(Y_2\leq Y_1)\Big]$$

$$= \iiiint K^*\Big(\frac{G(x_2)-G(x_o)}{a_n}\Big)K_2'^{\,2}\Big(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\Big)y_1^2F_{x_o}(y_1)I(y_2\leq y_1)$$

$$dH(x_1,y_1)dH(x_2,y_2)$$

$$= \iiint K^*\Big(\frac{G(x_2)-G(x_o)}{a_n}\Big)K_2'^{\,2}\Big(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\Big)$$

$$y_1^2F_{x_o}(y_1)F_{x_2}(y_1)dH(x_1,y_1)dG(x_2).$$

By the transformation of variables (4.4.18) together with lemma 4.1.1 yields for $n > n_o$

$$E(\ell_{12}p_1) = a_n^3 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2^{'2}(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1 a_n^2 v_1^2 D_2+O(a_n^3)]^2$$

$$[\lambda+a_n v_1]\{\lambda+a_n u_2 D_3+a_n v_1+a_n^2 u_2^2 D_4+a_n^2 u_2 v_1 D_5+O(a_n^3)]$$

$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]du_1 dv_1 du_2.$$

Considering the terms up to order a_n^5 and recalling that $\int K^*(u)du = 1$, $\int uK^*(u)du = 0$, we have

$$(4.4.78) \quad E(\ell_{12}p_1) = a_n^3 \int\int K_2^{'2}(u_1,v_1)\lambda^2(F_{x_o}^{-1}(\lambda))^2 du_1 dv_1$$

$$+ a_n^4 \int\int K_2^{'2}(u_1,v_1)[2\lambda^2 F_{x_o}^{-1}(\lambda)v_1 D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2 v_1+\lambda^2(F_{x_o}^{-1}(\lambda))^2$$

$$u_1 D_6]du_1 dv_1$$

$$+ a_n^5 \int\int K_2^{'2}(u_1,v_1)[2\lambda^2 F_{x_o}^{-1}(\lambda)v_1^2 D_2+\lambda^2 v_1^2 D_1^2+\lambda^2(F_{x_o}^{-1}(\lambda))^2(u_1^2 D_7$$

$$+u_1 v_1 D_8)+4\lambda F_{x_o}^{-1}(\lambda)v_1^2 D_1+2\lambda^2 F_{x_o}^{-1}(\lambda)v_1 u_1 D_1 D_6$$

$$+(F_{x_o}^{-1}(\lambda))^2 v_1^2+2\lambda(F_{x_o}^{-1}(\lambda))^2 v_1 u_1 D_6]du_1 dv_1$$

$$+ a_n^5 \int\int K_2^{'2}(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2)du_2)$$

$$+ O(a_n^6).$$

Similarly,

$$E(\ell_{21}p_1) = E[K^*(\frac{G(X_1)-G(x_o)}{a_n})K_2'(\frac{G(X_1)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_1)-\lambda}{a_n})$$

$$K_2'(\frac{G(X_2)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_2)-\lambda}{a_n})Y_1Y_2I(Y_1\le Y_2)F_{x_o}(Y_1)]$$

$$= \iiiint K^*(\frac{G(x_1)-G(x_o)}{a_n})K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})$$

$$K_2'(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n})y_1y_2I(y_1\le y_2)F_{x_o}(y_1)dH(x_1,y_1)dH(x_2,y_2)$$

$$= a_n^4\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K_2'(u_1,v_1)K_2'(u_2,v_2)[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$

$$[F_{x_o}^{-1}(\lambda)+a_nv_2D_1+a_n^2v_2^2D_2+O(a_n^3)][\lambda+a_nv_1]$$

$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]$$

$$[1+a_nu_2D_6+a_n^2u_2^2D_7+a_n^2u_2v_2D_8+O(a_n^3)]du_1dv_1du_2dv_2,$$

for $n > n_o$.

So, we have

$$(4.4.79)\quad E(\ell_{21}p_1) = a_n^4\iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_2,v_2)I(v_1\le v_2)\lambda(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_2dv_2$$

$$+ a_n^5\iiiint K^*(u_1)K_2'(u_1,v_1)K_2'(u_2,v_2)I(v_1\le v_2)[\lambda F_{x_o}^{-1}(\lambda)(v_1+v_2)D_1$$

$$+(F_{x_o}^{-1}(\lambda))^2v_1+\lambda(F_{x_o}^{-1}(\lambda))^2(u_1+u_2)D_6]du_1dv_1du_2dv_2$$

$$+ O(a_n^6)\ .$$

Also,

$$(4.4.80)\quad E(\ell_{12})E(p_1) = \iiiint\!\!\iint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)$$

$$F_{x_o}(y_3)dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3).$$

By transformation of variables as in (4.4.18) and applying lemma 4.1.1 we easily see that, for $n > n_o$

$$E(\ell_{12})E(p_3) = a_n^5 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)K_2'(u_3,v_3)\lambda^2(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_3dv_3du_2$$

$$+ O(a_n^6).$$

But the first term on the R.H.S. expression is zero since $\iint K_2'(u,v)dudv = 0.$ Hence, we have

$$(4.4.81)\quad E(\ell_{12})E(p_1) = O(a_n^6).$$

Now, (4.4.78), (4.4.79) and (4.4.81) in (4.3.76) give

$$(4.4.82)\quad Cov(n^{1/2}a_nI_{312}, n^{1/2}a_nI_{30})$$

$$= \frac{n-1}{na_n^2}\iint K_2'^2(u_1,v_1)\lambda^2(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ \frac{n-1}{na_n}\iint K_2'^2(u_1,v_1)\left[2\lambda^2F_{x_o}^{-1}(\lambda)v_1D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1+\lambda^2(F_{x_o}^{-1}(\lambda))^2u_1D_6\right]$$

$$du_1dv_1$$

continued

$$+ \frac{n-1}{n} \iint K_2'^2(u_1,v_1)[2\lambda^2 F_{X_0}^{-1}(\lambda) v_1^2 D_2 + \lambda^2 v_1^2 D_1^2 + \lambda^2 (F_{X_0}^{-1}(\lambda))^2 (u_1^2 D_7 + u_1 v_1 D_8)$$

$$+ 4\lambda F_{X_0}^{-1}(\lambda) v_1^2 D_1 + 2\lambda^2 F_{X_0}^{-1}(\lambda) v_1 u_1 D_1 D_6 + (F_{X_0}^{-1}(\lambda))^2 v_1^2$$

$$+ 2\lambda (F_{X_0}^{-1}(\lambda))^2 v_1 u_1 D_6] du_1 dv_1$$

$$+ \frac{n-1}{n} \iint K_2'(u_1,v_1) \lambda (F_{X_0}^{-1}(\lambda))^2 D_4 du_1 dv_1 (\int u_2^2 K^*(u_2) du_2)$$

$$+ \frac{n-1}{na_n} \iiiint K^*(u_1) K_2'(u_1,v_1) K_2'(u_2,v_2) I(v_1 \le v_2) \lambda (F_{X_0}^{-1}(\lambda))^2 du_1 dv_1 du_2 dv_2$$

$$+ \frac{n-1}{n} \iiiint K^*(u_1) K_2'(u_1,v_1) K_2'(u_2,v_2) I(v_1 \le v_2)$$

$$[\lambda F_{X_0}^{-1}(\lambda)(v_1+v_2) D_1 + (F_{X_0}^{-1}(\lambda))^2 v_1 + \lambda (F_{X_0}^{-1}(\lambda))^2 (u_1+u_2) D_6]$$

$$du_1 dv_1 du_2 dv_2$$

$$+ O(a_n).$$

From (4.4.70), (4.4.75), (4.4.82) and (4.4.46), it follows that

$$\mathrm{Var}[n^{1/2} a_n (I_{312} - I_{30})]$$

$$= \frac{(n-1)(n-2)}{n^2} 2(F_{X_0}^{-1}(\lambda))^2 \iiiint v_1 I(v_1 \le v_2) K_2'(u_1,v_1) K_2'(u_2,v_2) du_1 dv_1 du_2 dv_2 (\int K^{*2}(u_3) du_3)$$

$$+ O(a_n) + O(\frac{1}{n}) + (\frac{1}{na_n}) + O(\frac{1}{na_n^2}) + O(\frac{1}{na_n^3}),$$

and by letting $n \to \infty$, we have

$$(4.4.83) \quad \lim_{n\to\infty} \mathrm{Var}[n^{1/2} a_n (I_{312} - I_{30})]$$

$$= 2(F_{X_0}^{-1}(\lambda))^2 (\int K^{*2}(u_3) du_3) \iiiint v_1 I(v_1 \le v_2) K_2'(u_1,v_1) K_2'(u_2,v_2) du_1 dv_1 du_2 dv_2.$$

But, $I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} v_1 I(v_1 \le v_2) K_2'(u_1,v_1) K_2'(u_2,v_2)\,du_1\,dv_1\,du_2\,dv_2$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} v_1 K_2'(u_1,v_1)\Big(\int_{v_1}^{1} K_2'(u_2,v_2)\,dv_2\Big)du_1\,dv_1\,du_2$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} v_1 K_2'(u_1,v_1)\big(-K(u_2,v_1)\big)du_1\,dv_1\,du_2,$$

since $K(u,1) = 0$ for all u, by hypothesis (see (4.1.3)).

Therefore, $I = -\int_{-1}^{1}\int_{-1}^{1} v_1 K_2(v_1) K_2'(u_1,v_1)\,du_1\,dv_1$

$$= -\int_{-1}^{1}\Big[\int_{-1}^{1}(v_1 K_2(v_1)) K_2'(u_1,v_1)\,dv_1\Big]du_1$$

$$= \int_{-1}^{1}\int_{-1}^{1} K(u_1,v_1)\big[v_1 K_2'(v_1) + K_2(v_1)\big]du_1\,dv_1, \quad \text{where } K_2(v) = \int_{-1}^{1} K(u,v)\,du,$$

on integration by parts and using (4.1.3). Integrating out u_1, we get

$$I = \int_{-1}^{1} K_2(v_1)\big[v_1 K_2'(v_1) + K_2(v_1)\big]dv_1$$

$$= \int_{-1}^{1} v_1 K_2(v_1) K_2'(v_1)\,dv_1 + \int_{-1}^{1} K_2^2(v_1)\,dv_1.$$

Using integration by parts to the first integral on the R.H.S. and using (4.1.3) we obtain

(4.4.84) $\quad \iiiint v_1 I(v_1 \le v_2) K_2'(u_1,v_1) K_2'(u_2,v_2)\,du_1\,dv_1\,du_2\,dv_2 = \frac{1}{2}\int_{-1}^{1} K_2^2(v_1)\,dv_1$

and hence from (4.4.83)

$$\lim_{n\to\infty} \mathrm{Var}\big[n^{1/2} a_n (\hat{I}_{312} - I_{30})\big] = F_{x_0}^{-1}(\lambda))^2 \iint K^{*2}(u) K_2^2(v)\,du\,dv.$$

This completes the proof of lemma 4.4.2. □

LEMMA 4.4.3. Under the hypothesis of lemma 4.4.1 $Var(n^{1/2}a_n I_1)$ converges to $(F_{x_o}^{-1}(\lambda))^2 \iint K^2(u,v)du dv$ as $n \to \infty$ where I_1 is as given in (4.4.1).

PROOF: From (4.4.1)

(4.4.85) $$Var(n^{1/2}a_n I_1) = \frac{1}{na_n^2} Var(\sum q_i)$$

$$= \frac{1}{a_n^2} Var(q_1),$$

where

(4.4.86) $$q_i = K\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i .$$

Therefore

$$E(q_1^2) = \iint K^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1^2 \; dH(x_1,y_1)$$

and by making the transformation of variables (4.4.13) and applying lemma 4.1.1, we can write, for $n > n_o$

(4.4.87) $$E(q_1^2) = a_n^2 \int_{-1}^{1}\int_{-1}^{1} K^2(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1+a_n^2 v_1^2 D_2+O(a_n^3)]^2$$

$$[1+a_n u_1 D_6+a_n^2 u_1^2 D_7+a_n^2 u_1 v_1 D_8+O(a_n^3)]du_1 dv_1$$

$$= a_n^2(F_{x_o}^{-1}(\lambda))^2 \iint K^2(u_1,v_1)du_1 dv_1 + O(a_n^3).$$

By similar computation, we easily see that

(4.4.88) $E^2(q_1) = O(a_n^4)$.

Hence, from (4.4.85), (4.4.86) and (4.4.88), we have

$$\mathrm{Var}(n^{1/2}a_n I_1) = (F_{x_o}^{-1}(\lambda))^2 \iint K^2(u,v)\,du\,dv + O(a_n),$$

from which the lemma follows immediately by letting $n \to \infty$. □

In order to find the limiting value of $\mathrm{Var}(n^{1/2}a_n S_{n,x_o}(\lambda))$ from (4.4.2), all the required variance terms have been computed in lemmas 4.4.1 to 4.4.3. The only covariance term that we need to compute is $\mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n(I_{312}-I_{30}))$, since it follows from lemma 4.4.1 that all the remaining covariances go to zero as n goes to ∞.

<u>LEMMA</u> 4.4.4. <u>Under the hypothesis of lemma 4.1.1,</u> $\mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n(I_{312}-I_{30}))$ <u>converges to</u> $\sigma_{12}(x_o,\lambda)$ <u>as</u> $n \to \infty$ <u>where</u>

(4.4.89) $\sigma_{12}(x_o,\lambda) = -(F_{x_o}^{-1}(\lambda))^2 \iint K^*(u)K_2(v)K(u,v)\,du\,dv$

<u>and</u> I_1, I_{312}, I_{30} <u>are as given by</u> (4.4.1), (4.4.4) <u>and</u> (4.4.1) <u>respectively.</u>

<u>PROOF</u>: We shall write

(4.4.90) $\mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n(I_{312}-I_{30}))$

$$= \mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n I_{312}) - \mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n I_{30}).$$

As in (4.4.76), we have

(4.4.91) $\text{Cov}(n^{1/2}a_nI_1, n^{1/2}a_nI_{312}) = na_n^2.\text{Cov}(\frac{1}{na_n^2}\sum_m q_m, \frac{1}{n^2a_n^4}\sum_{i\neq j}\sum \ell_{ij})$

$$= \frac{1}{n^2a^4}[n(n-1)\text{Cov}(q_1,\ell_{12})+n(n-1)\text{Cov}(q_1,\ell_{21})]$$

$$= \frac{n-1}{na_n^4}\text{Cov}(q_1,\ell_{12}) + \frac{n-1}{na_n^4}\text{Cov}(q_1,\ell_{21})$$

where q_m's are defined by (4.4.86) and ℓ_{ij}'s are defined by (4.4.47). Proceeding exactly as in (4.4.76), we have

$$E(\ell_{12}q_1) = \iiint K^*(\frac{G(x_2-G(x_o)}{a_n})K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})$$

$$K(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})y_1^2F_{x_2}(y_1)dH(x_1,y_1)dG(x_2).$$

Transformation of variables (4.4.18) together with lemma 4.1.1 yields, for $n > n_o$

(4.4.92) $E(\ell_{12}q_1) = a_n^3\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)K(u_1,v_1)$

$$[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]^2$$

$$[\lambda+a_nu_2D_3+a_nv_1+a_n^2u_2^2D_4+a_n^2u_2v_1D_5+O(a_n^3)]$$

$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]du_1dv_1du_2$$

$$= a_n^3\iint K_2'(u_1,v_1)K(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ a_n^4\iint K_2'(u_1,v_1)K(u_1,v_1)[2\lambda F_{x_o}^{-1}(\lambda)v_1D_1+(F_{x_o}^{-1}(\lambda))^2v_1+\lambda(F_{x_o}^{-1}(\lambda))^2u_1D_6]du_1dv_1$$

$$+ O(a_n^5),$$

where the last equality follows by expansion and using the facts $\int_{-1}^{1} K^*(u)du = 1$, $\int_{-1}^{1} uK^*(u)du = 0$.

$$(4.4.93)\quad E(\ell_{21}q_1) = \int\int\int\int K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)$$
$$K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1y_2I(y_1\le y_2)dH(x_1,y_1)dH(x_2,y_2).$$

Again, the usual transformation of variables (4.4.48) together with lemma 4.1.1 yields, for $n > n_o$,

$$(4.4.94)\quad E(\ell_{21}q_1) = a_n^4\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$
$$[F_{x_o}^{-1}(\lambda)+a_nv_2D_1+a_n^2v_2^2D_2+O(a_n^3)]$$
$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]$$
$$[1+a_nu_2D_6+a_n^2u_2^2D_7+a_n^2u_2v_2D_8+O(a_n^3)]du_1dv_1du_2dv_2$$
$$= a_n^4\int\int\int\int K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_2dv_2 + O(a_n^5).$$

Also,

$$(4.4.95)\quad E(\ell_{12})E(q_3) = \int\int\int\int\int\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)dH(x_1,y_1)$$
$$dH(x_2,y_2)dH(x_3,y_3).$$

After the usual transformation of variables, we can easily see that

$$(4.4.96)\quad E(\ell_{12})E(q_3) = O(a_n^5).$$

Now, using (4.4.92) to (4.4.96) in (4.4.91), we get

$$(4.4.97)\quad \mathrm{Cov}(n^{1/2}a_nI_1, n^{1/2}a_nI_{312})$$
$$= \frac{n-1}{n}\iint K_2'(u_1,v_1)K(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2 du_1dv_1$$
$$+ \frac{n-1}{n}\iint K_2'(u_1,v_1)K(u_1,v_1)[2\lambda F_{x_o}^{-1}(\lambda)v_1D_1+(F_{x_o}^{-1}(\lambda))^2v_1$$
$$+\lambda(F_{x_o}^{-1}(\lambda))^2u_1D_6]du_1dv_1$$
$$+ \frac{n-1}{n}\iiiint K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_2dv_2$$
$$+ O(a_n).$$

Observe that

$$(4.4.98)\quad \mathrm{Cov}(n^{1/2}a_nI_1, n^{1/2}a_nI_{30}) = na_n^2\,\mathrm{Cov}\left(\frac{1}{na_n^2}\sum q_i, \frac{1}{na_n^3}\sum p_j\right)$$
$$= \frac{1}{na_n^3}\sum_i\sum_j \mathrm{Cov}(q_i,p_j) = \frac{1}{na_n^3}\sum_i \mathrm{Cov}(q_i,p_i)$$
$$= \frac{1}{a_n^3}\,\mathrm{Cov}(q_1,p_1),$$

where q_i's are defined by (4.4.86) and p_j's are defined by (4.4.72).

We have
$$E(q_1p_1) = \iint K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$y_1^2F_{x_o}(y_1)dH(x_1,y_1).$$

Transformation of variables (4.4.13) together with lemma 4.1.1 gives, for $n > n_o$,

(4.4.99) $$E(q_1p_1) = a_n^2\int_{-1}^{1}\int_{-1}^{1} K(u_1,v_1)K_2'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]^2$$

$$[\lambda+a_nv_1][1+a_nu_1D_6+a_n^2u_1^2D_7+a_nu_1v_1D_8+O(a_n^3)]du_1dv_1$$

$$= a_n^2\iint K(u_1,v_1)K_2'(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ a_n^3\iint K(u_1,v_1)K_2'(u_1,v_1)[2\lambda F_{x_o}^{-1}(\lambda)v_1D_1+(F_{x_o}^{-1}(\lambda))^2v_1+\lambda(F_{x_o}^{-1}(\lambda))^2 u_1D_6]du_1dv_1$$

$$+ O(a_n^4).$$

Also,

$$E(q_1)E(p_1) = \iiiint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)$$

$$y_1y_2F_{x_o}(y_1)dH(x_1,y_1)dH(x_2,y_2),$$

and by the usual transformation of variables, we immediately see that

(4.4.100) $$E(q_1)E(p_1) = O(a_n^4).$$

It now follows from (4.4.98), (4.4.99) and (4.4.100) that

(4.4.101) $$\mathrm{Cov}(n^{1/2}a_nI_1,n^{1/2}a_nI_{30})$$

$$= \frac{1}{a_n}\iint K(u_1,v_1)K_2'(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ \iint K(u_1,v_1)K_2'(u_1,v_1)[2\lambda F_{x_o}^{-1}(\lambda)v_1D_1+(F_{x_o}^{-1}(\lambda))^2v_1+\lambda(F_{x_o}^{-1}(\lambda))^2 u_1D_6]du_1dv_1$$

$$+ O(a_n).$$

(4.4.97) and (4.4.101) in (4.4.90) yield

$$\mathrm{Cov}(n^{1/2}a_nI_1, n^{1/2}a_n(I_{312}-I_{30}))$$

$$= \frac{n-1}{n}\iiiint K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)(F_{X_0}^{-1}(\lambda))^2du_1dv_1du_2dv_2$$

$$+ O(a_n) + O(\tfrac{1}{n}) + O(\tfrac{1}{na_n}),$$

and by letting $n \to \infty$, we have

(4.4.102)
$$\lim_{n\to\infty} \mathrm{Cov}(n^{1/2}a_nI_1, n^{1/2}a_n(I_{312}-I_{30}))$$

$$= (F_{X_0}^{-1}(\lambda))^2 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K_2(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)\,du_1dv_1du_2dv_2.$$

But as in (4.4.83) we can write

(4.4.103)
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)du_1dv_1du_2dv_2$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K(u_1,v_1)\Big(\int_{v_1}^{1} K_2'(u_2,v_2)dv_2\Big)du_1dv_1du_2$$

$$= -\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K(u_1,v_1)K(u_2,v_1)du_1dv_1dv_2,$$

since $K(u,1) = 0$ for all u,

$$= -\int_{-1}^{1}\int_{-1}^{1} K^*(u_1)K(u_1,v_1)K_2(v_1)du_1dv_1,$$

by integrating out u_2.

Hence we have (4.4.89) and the proof of the lemma 4.4.4 is complete. □

Now, from equation (4.4.2) and lemmas 4.4.1 to 4.4.4, it follows that

(4.4.104) $$\lim_{n\to\infty} \mathrm{Var}(n^{1/2}a_n S_{n,x_o}(\lambda)) = \lim_{n\to\infty}[\mathrm{Var}(n^{1/2}a_n I_1) + \mathrm{Var}(n^{1/2}a_n(I_{312}-I_{30}))$$

$$+ 2\mathrm{Cov}(n^{1/2}a_n I_1, n^{1/2}a_n(I_{312}-I_{30}))]$$

$$= (F_{x_o}^{-1}(\lambda))^2 \iint K^2(u,v)dudv + (F_{x_o}^{-1}(\lambda))^2 \iint K^{*2}(u)K_2^2(v)dudv$$

$$- 2(F_{x_o}^{-1}(\lambda))^2 \iint K^*(u)K_2(v)K(u,v)dudv$$

$$= (\ {}_{x_o}^{-1}(\lambda))^2 \iint [K(u,v)-K^*(u)K_2(v)]^2 dudv.$$

4.5. CONSISTENCY OF $Q_{n,x_o}(\lambda)$. We now give the proof of Theorem 4.1.1 in this section which proves the consistency of $Q_{n,x_o}(\lambda)$.

PROOF OF THEOREM 4.1.1. From (4.1.19) we can write

(4.5.1) $$Q_{n,x_o}(\lambda) - Q_{x_o}(\lambda) = [S_{n,x_o}(\lambda)-Q_{x_o}(\lambda)] + \sum_{m=2}^{3} \zeta_{nm} + \sum_{m=2}^{4} \eta_{nm}.$$

Because of lemma 4.3.1 it remains to show that $[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)]$ converges to zero in probability as $n \to \infty$. Relation (4.4.104) implies that $\mathrm{Var}[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)]$ goes to zero as $n \to \infty$. Hence it suffices to show that $E[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)]$ goes to zero as $n \to \infty$. From (4.4.2) we have

$$S_{n,x_o}(\lambda) = I_1 + I_2 + I_{311} + I_{312} + I_{321} + I_{322} - I_{30}$$

so that we can write

(4.5.2) $$E[S_{n,x_o}(\lambda) - Q_{x_o}(\lambda)] = E(I_1-Q_{x_o}(\lambda)) + E(I_2) + E(I_{311})$$

$$+ E(I_{312}-I_{30}) + E(I_{321}) + E(I_{322}).$$

We shall now show that each of the expectations on the R.H.S. of (4.5.2) goes to zero as $n \to \infty$.

We have

(4.5.3) $$E(I_2) = \frac{1}{na_n^3} \sum E(t_i)$$

where t_i's are as given by (4.4.10). We also have, from (4.4.11),

$$E(t_i) = \frac{1}{n} \iint [I(x_1 > x_0) - (G(x_1) - G(x_0))] K_1'\left(\frac{G(x_1)-G(x_0)}{a_n}, \frac{F_{x_0}(y_1)-\lambda}{a_n}\right) y_1 \, dH(x_1, y_1)$$

$$= O\left(\frac{1}{n}\right)$$

since K_1' is bounded and $E|Y| < \infty$. So it follows from (4.5.3) that

(4.5.4) $$E(I_2) = O\left(\frac{1}{na_n^3}\right)$$

which goes to zero as $n \to \infty$ since $na_n^3 \to \infty$. From (4.4.3) we have

(4.5.5) $$E(I_{311}) = \frac{1}{n^2 a_n^4} \sum E(\ell_i)$$

where $\ell_i = K^*\left(\frac{G(X_1)-G(x_0)}{a_n}\right) K_2'\left(\frac{G(X_i)-G(x_0)}{a_n}, \frac{F_{x_0}(Y_i)-\lambda}{a_n}\right) Y_i$.

Since K^*, K_2' are bounded and $E|Y| < \infty$ it follows from (4.5.5) that

(4.5.6) $$E(I_{311}) = O\left(\frac{1}{na_n^4}\right)$$

which goes to zero as $n \to \infty$ because $na_n^4 \to \infty$.

From (4.4.4) we have

$$(4.5.7) \qquad E(I_{312}) = \frac{1}{n^2 a_n^4} \sum_{i \neq j}\sum E(\ell_{ij}) = \frac{n-1}{n a_n^4} E(\ell_{12})$$

where $\ell_{ij} = K^*\left(\frac{G(X_j)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i I(Y_j \leq Y_i)$.

Hence

$$E(\ell_{12}) = \iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_2 \leq y_1)$$

$$dH(x_1,y_1)dH(x_2,y_2)$$

$$= \iiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 F_{x_2}^{*}(y_1)$$

$$dH(x_1,y_1)dG(x_2).$$

Now, by transformation of variables (4.4.18), in the last integral and using Taylor expansions from lemma 4.1.1, we have for $n > n_o$

$$(4.5.8) \qquad E(\ell_{12}) = a_n^3 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)]$$

$$[\lambda + a_n u_2 D_3 + a_n v_1 + a_n^2 u_2^2 D_4 + a_n^2 u_2 v_1 D_5 + O(a_n^3)]$$

$$[1 + a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)]du_1 dv_1 du_2$$

$$= a^3 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)\lambda F_{x_o}^{-1}(\lambda)du_1 dv_1 du_2$$

$$+ a_n^4 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)[\lambda v_1 D_1 + F_{x_o}^{-1}(\lambda)v_1 + \lambda F_{x_o}^{-1}(\lambda)u_1 D_6]du_1 dv_1 du_2$$

$$+ O(a_n^5).$$

Since $\int_{-1}^{1} K_2'(u,v)dudv = 0$, the first integral in the last expression vanishes and from (4.5.7) and (4.5.8) we have

(4.5.9) $$E(I_{312}) = \frac{n-1}{n} \int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)[\lambda v_1 D_1 + F_{x_o}^{-1}(\lambda)v_1 + \lambda F_{x_o}^{-1}(\lambda)u_1 D_6]du_1 dv_1$$

$$+ O(a_n).$$

Now from (4.1.1) we write

(4.5.10) $$E(I_{30}) = \frac{1}{na_n^3} \sum E(p_i) = \frac{1}{a_n^3} E(p_1)$$

where $$p_i = K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i F_{x_o}(Y_i).$$

Therefore

$$E(p_1) = \iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 F_{x_o}(y_1)dH(x_1,y_1).$$

By making transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we see that, for $n > n_o$

(4.5.11) $$E(p_1) = a_n^2 \int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_n v_1 D_1 + a_n^2 v_1^2 D_2 + O(a_n^3)][\lambda + a_n v_1]$$

$$[1+a_n u_1 D_6 + a_n^2 u_1^2 D_7 + a_n^2 u_1 v_1 D_8 + O(a_n^3)]du_1 dv_1$$

$$= a_n^2 \int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)\lambda F_{x_o}^{-1}(\lambda)du_1 dv_1$$

$$+ a_n^3 \int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)[\lambda v_1 D_1 + F_{x_o}^{-1}(\lambda)v_1 + \lambda F_{x_o}^{-1}(\lambda)u_1 D_6]du_1 dv_1$$

$$+ O(a_n^4).$$

As in (4.5.8), the first integral in the last expression vanishes and we have from (4.5.10) and (4.5.11)

$$(4.5.12)\quad E(I_{30}) = \int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)[\lambda v_1 D_1 + F_{x_o}^{-1}(\lambda)v_1 + \lambda F_{x_o}^{-1}(\lambda)u_1 D_6]du_1 dv_1$$

$$+ O(a_n).$$

From (4.5.9) and (4.5.12) it now follows that

$$(4.5.13)\quad E(I_{312}'-I_{30}) = O(\tfrac{1}{n}) + O(a_n)$$

which goes to zero as $n \to \infty$.

From (4.4.5) we have

$$(4.5.14)\quad E(I_{321}) = \frac{1}{n^2 a_n^5} \sum E(t_{ii})$$

where $t_{ii} = [G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)]K^{*'}(\frac{G(X_i)-G(x_o)}{a_n})$

$$K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i.$$

Hence for $i=1,2,\ldots,n$

$$(4.5.15)\quad E(t_{ii}) = \iint \frac{1}{n}[I(x_1>x_o)-G(x_1)-G(x_o)]K^{*'}(\frac{G(x_1)-G(x_o)}{a_n})$$

$$K_2'(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n})y_1 dH(x_1,y_1)$$

$$= O(\tfrac{1}{n}),$$

since $K^{*'}$, K_2' are bounded and $E|Y| < \infty$. From (4.5.14) and (4.5.15) we now have

(4.5.16) $$E(I_{321}) = O\left(\frac{1}{n^2 a_n^5}\right)$$

which goes to zero as $n \to \infty$. From (4.4.6) we write

(4.5.17) $$E(I_{322}) = \frac{1}{n^2 a_n^5} \sum_{i \neq j}\sum E(t_{ij}) ,$$

where t_{ij}'s are given by (4.4.32) and from (4.4.33) we have $E(t_{ij}) = O\left(\frac{a_n^3}{n}\right)$ for $i,j = 1,\ldots,n$, $i \neq j$. Hence it follows from (4.5.17) that

(4.5.18) $$E(I_{322}) = O\left(\frac{1}{na_n^2}\right)$$

which goes to zero as $n \to \infty$.

Now it remains to show that $E(I_1 - Q_{x_o}(\lambda)) \to 0$ as $n \to \infty$. From (4.4.1) we have

$$I_1 = \frac{1}{na_n^2} \sum K\left(\frac{G(x_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

and hence

$$E(I_1) - Q_{x_o}(\lambda) = \frac{1}{a_n^2} \iint K\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)y\,dH(x,y) - Q_{x_o}(\lambda).$$

By transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we get, for $n > n_o$

$$(4.5.19) \qquad E(I_1) - Q_{x_0}(\lambda) = \int_{-1}^{1}\int_{-1}^{1} K(u,v)\left[F_{x_0}^{-1}(\lambda)+a_n vD_1+a_n^2v^2D_2+O(a_n^3)\right]$$

$$\left[1+a_n uD_6+a_n^2u^2D_7+a_n^2uvD_8+O(a_n^3)\right]du_1dv_1$$

$$- Q_{x_0}(\lambda).$$

Since $\int\int K(u,v)dudv = 1$, $\int\int uK(u,v)dudv = 0 = \int\int vK(u,v)dudv$ and (by assumptions (4.1.3) to (4.1.5)) and $F_{x_0}^{-1}(\lambda) \equiv Q_{x_0}(\lambda)$ it follows that

$$(4.5.20) \qquad E(I_1) - Q_{x_0}(\lambda) = O(a_n^2)$$

which goes to zero as $n \to \infty$.

Now (4.5.4), (4.5.6), (4.5.13), (4.5.16), (4.5.18) and (4.5.20) together implies that $E[S_{n,x_0}(\lambda) - Q_{x_0}(\lambda)] \to 0$ as $n \to \infty$. This completes the proof of Theorem 4.1.1. □

CHAPTER V

ASYMPTOTIC NORMALITY OF $Q_{n,x_o}(\lambda)$

In this chapter we prove the asymptotic normality of the conditional quantile estimator $Q_{n,x_o}(\lambda)$, in a suitability normalized form where $Q_{n,x_o}(\lambda)$ is defined by (4.1.1). We continue to use the representation (4.1.9) of $Q_{n,x_o}(\lambda)$, namely,

$$Q_{n,x_o}(\lambda) = S_{n,x_o}(\lambda) + R_n + R_n'$$

where $S_{n,x_o}(\lambda)$ is as given by (4.1.21) but the higher order terms R_n and R_n' are comprised of more terms now, than before. First we show that the higher order terms (after proper normalization of $Q_{n,x_o}(\lambda)$) converge to zero in probability, as n goes to ∞, so that it then suffices to prove the asymptotic normality of $S_{n,x_o}(\lambda)$ (in the normalized form). Our method of proof of this is essentially the same as that used in the case of the statistic $\frac{1}{n}\sum_{i=1}^{n} J((\frac{i}{n},\frac{R_{in}}{n})Y_{[i:n]}$ in Chapters 2 and 3. We shall use Hajek's projection lemma (lemma 2.1.1) to approximate $S_{n,x_o}(\lambda)$ in mean square by a sum of iid rv's and the asymptotic normality of the latter is then proved along classical lines. In section 5.1, the main result of this chapter, namely, the asymptotic normality of $Q_{n,x_o}(\lambda)$ is stated as Theorem 5.1.1. and Corollory 5.1.1. In section 5.2. we establish the asymptotic equivalence in mean square of $S_{n,x_o}(\lambda)$ to that of a sum of iid rv's, while using some of the results obtained in chapter 4. The proofs of the Theorem and the Corollary are given in section 5.3.

5.1. PROJECTION $\hat{S}_{n,x_o}(\lambda)$ OF $S_{n,x_o}(\lambda)$. In this section we state our main resuls of this chapter as Theorem 5.1.1 and Corollary 5.1.1 and obtain the projection of $S_{n,x_o}(\lambda)$.

THEOREM 5.1.1. Let $Q_{n,x_o}(\lambda)$, $0 < \lambda < 1$ be as defined by (4.1.1) with K and K^* satisfying assumptions (4.1.3) to (4.1.6) with $\nu = 6$ and $\nu^* = 4$. Suppose the joint distribution function H satisfies assumptions (4.1.11) to (4.1.15). Then for each sequence $\{a_n\}$ of positive numbers with $a_n \to 0$, $na_n^4 \to \infty$ but $na_n^5 \to 0$ we have

(5.1.1) $$n^{1/2}a_n[Q_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}] \to N(0, \sigma_o^2(x_o,\lambda))$$ in distribution as $n \to \infty$, provided $\sigma_o^2(x_o,\lambda) > 0$, where

(5.1.2) $$\bar{Q}_{n,x_o}(\lambda) = a_n^{-2}\iint K\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right) y\,dH(x,y)$$

(5.1.3) $$\sigma_o^2(x_o,\lambda) = (F_{x_o}^{-1}(\lambda))^2 \iint [K(u,v) - K^*(v)K_2(v)]^2 du\,dv$$ and

(5.1.3 a) $$K_2(v) = \int K(u,v)du.$$

COROLLARY 5.1.1 Under the hypothesis of Theorem 5.1.1 we have

(5.1.4) $$n^{1/2}a_n[Q_{n,x_o}(\lambda) - Q_{x_o}(\lambda)] \to N(0, \sigma_o^2(x_o,\lambda))$$ in distribution as $n \to \infty$.

The proofs of Theorem 5.1.1 and Corollary 5.1.1 are given in section 5.3.

First observe that since K' is six times differentiable and K^* is four times differentiable (under the hypothesis of Theorem 5.1.1) the following representation of $Q_{n,x_o}(\lambda)$ can be obtained exactly on the same lines as (4.1.20) was obtained:

$$(5.1.5)\qquad Q_{n,x_o}(\lambda) = S_{n,x_o}(\lambda) + \sum_{m=2}^{4} \zeta_{nm} + \sum_{m=2}^{6} \eta_{nm}$$

where $S_{n,x_o}(\lambda)$ is as given by (4.1.21), η_{nm} (m=2,3,4 and 5) is given by (4.1.24) and

$$(5.1.6)\qquad \eta_{n6} = \frac{a_n^{-8}}{6!} \iint \Big[\big(G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)\big)\frac{\partial}{\partial t_1} + \big(F_{n,x_o}(y_1) - F_{x_o}(y_1)\big)\frac{\partial}{\partial t_2}\Big]^6 K(\Delta_{x_1,n}, \Delta_{y_1,n})\, y_1\, dH_n(x_1,y_1)\ ,$$

$$(5.1.7)\qquad \zeta_{nm} = \frac{a_n^{-4-m}}{m!} \iiiint K^{*(m)}\Big(\frac{G(x_2)-G(x_o)}{a_n}\Big) K_2'\Big(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\Big) y_1 I(y_2 \le y_1)\, [G_n(x_2)-G(x_o)-G(x_2)+G(x_o)]^m\, dH_n(x_1,y_1)\, dH_n(x_2,y_2),$$

for $m = 2$, and 3,

$$(5.1.8)\qquad \zeta_{n4} = \frac{a_n^{-8}}{4!} \iiiint K^{*(4)}(\Delta_{x_2,n}) K_2'\Big(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\Big) y_1 I(y_2 \le y_1) \cdot [G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)]^4\, dH_n(x_1,y_1)\, dH_n(x_2,y_2).$$

The following lemma shows that the higher order terms ζ_{nm}'s and η_{nm}'s in the normalized form of $Q_{n,x_o}(\lambda)$ converge to zero in probability as $n \to \infty$.

LEMMA 5.1.1. (a) Suppose assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $\nu = 6$, $\nu^* = 4$ and assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers with $a_n \to 0$ and $na_n^4 \to \infty$. We then have

$$n^{1/2} a_n . \zeta_{nm} \to 0 \text{ in probability, as } n \to \infty \text{ for } m = 2, 3 \text{ and } 4$$

where ζ_{nm} are given by (5.1.7) to (5.1.8).

(b) Suppose assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $\nu = 6$, $\nu^* = 4$ and assumptions (4.1.11) to (4.1.15) on the joint distribution H are satisfied. Let $\{a_n\}$ be a sequence of positive numbers such that $a_n \to 0$, $na_n^4 \to \infty$ but $na_n^5 \to 0$. We then have

$$n^{1/2} a_n \eta_{nm} \to 0 \text{ in probability, as } n \to \infty \text{ for } m = 2, \ldots, 6$$

where η_{nm} are given by (4.1.24) for $m = 2, 3, 4$ and 5 and η_{n6} is given by (5.1.6).

PROOF: The proof of this lemma runs parallel to that of lemma 4.3.1 and hence is omitted. □

LEMMA 5.1.2. Let $S_{n,x_o}(\lambda)$ be as defined by (4.4.1). Suppose assumptions (4.1.3) to (4.1.4) on K and K^* are satisfied with $\nu = 1$ and $\nu^* = 1$.

Then the projection $\hat{S}_{n,x_o}(\lambda)$ of $S_{n,x_o}(\lambda)$, as defined by (2.1.1) is given by

(5.1.9) $\hat{S}_{n,x_o}(\lambda)$

$$= \sum \frac{1}{na^2} K\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$+ \sum \frac{1}{n^2a_n^3} K_1'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)[I(X_i>x_o)-G(X_i)+G(x_o)]Y_i$$

$$+ \sum \frac{n-1}{n^2a_n^3} \iint K_1'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)[I(X_i\le x)-I(X_i\le x_o)-(G(x)-G(x_o))]\, ydH(x,y)$$

$$+ \sum \frac{1}{n^2a_n^4} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$+ \sum \frac{n-1}{n^2a_n^4} K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)yI(Y_i\le y)dH(x,y)$$

$$+ \sum \frac{n-1}{n^2a_n^4} K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Y_i\ge y)dH(x,y)$$

$$+ \sum \frac{1}{n^3a_n^5} K^{*\prime}\left(\frac{G(X_i)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) [I(X_i>x_o)-(G(X_i)-G(x_o))]Y_i$$

$$+ \sum \frac{n-1}{n^3a_n^5} K^{*\prime}\left(\frac{G(X_i)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)yI(Y_i\le y) [I(X_i>x)-I(X_i\le x_o)+I(x>x_o)-2(G(X_i)-G(x_o))]dH(x,y)$$

$$+ \sum \frac{n-1}{n^3a_n^5} K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i \iint K^{*\prime}\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Y_i>y) [I(X_i\le x)-I(X_i\le x_o)+I(x>x_o)-2(G(x)-G(x_o))]dH(x,y)$$

continued

$$+ \sum \frac{n-1}{n^3 a_n^5} \iint K^{*'}\left(\frac{G(x)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$$

$$[I(X_i \le x) - I(X_i \le x_o) - (G(x)-G(x_o))]\, y\, dH(x,y)$$

$$+ \sum \frac{(n-1)(n-2)}{n^3 a_n^5} \iiiint K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$[I(X_i \le x_2) - I(X_i \le x_o) - (G(x_2)-G(x_o))]\, y_1 I(y_2 \le y_1)$$

$$dH(x_1,y_1)\, dH(x_2,y_2)$$

$$- \sum \frac{1}{n a_n^3} K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i F_{x_o}(Y_i) + C$$

$$= S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 + S_9 + S_{10} + S_{11} - S_{12} + C,$$

where C is a constant depending on K^*, K, x_o, λ and n.

PROOF: Since $S_{n,x_o}(\lambda) = I_1 + I_2 + I_{31} + I_{32} - I_{30}$. (see 4.4.1), by the linearity property of the projection operator, we can write

(5.1.10) $$\hat{S}_{n,x_o}(\lambda) = \hat{I}_1 + \hat{I}_2 + \hat{I}_{31} + \hat{I}_{32} - \hat{I}_{30}$$

where $\hat{I}_j$ denotes the projection (as defined by 2.1.1.) of I_j.

First, observe that both I_1 and I_{30} being sums of iid rv's, we have

(5.1.11) $$\hat{I}_1 = I, \text{ and } \hat{I}_{30} = I_{30}.$$

From (4.4.1), we have

$$I_2 = \frac{1}{na_n^3}\sum[G_n(X_i)-G_n(x_o)-G(X_i)+G(x_o)]K_1'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$= \frac{1}{na_n^3}\sum t_i.$$

By (2.1.1),

(5.1.12) $$\hat{I}_2 = \sum_j E(I_2|X_j,Y_j) - (n-1)E(I_2).$$

We now write

(5.1.13) $$E(I_2|X_n,Y_n) = (na_n^3)^{-1}E(t_n+\sum_{i\neq n}t_i|X_n,Y_n)$$

$$= (na_n^3)^{-1}E(t_n|X_n,Y_n)+(n-1)(na_n^3)^{-1}E(t_1|X_n,Y_n),$$

where

(5.1.14) $$E(t_n|X_n=x_n,Y_n=y_n) = K_1'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)y_n$$

$$E[G_n(x_n)-G_n(x_o)-G(x_n)+G(x_o)|X_n=x_n]$$

$$= K_1'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)y_n$$

$$\frac{1}{n}[I(x_n>x_o)-(G(x_n)-G(x_o))]$$

by (4.4.7), and

$$(5.1.15)\quad E(t_1|X_n=x_n,Y_n=y_n) = E\{[G_n(X_1)-G_n(x_o)-G(X_1)+G(x_o)] K_1'\left(\frac{G(X_1)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_1)-\lambda}{a_n}\right)Y_1|X_n=x_n,Y_n=y_n\}$$

$$= \iint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 E[(G_n(x_1)-G_n(x_o) -G(x_1)+G(x_o))|X_n=x_n,X_1=x_1]dH(x_1,y_1)$$

$$= \iint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)\frac{1}{n}[I(x_1>x_o)+I(x_n\le x_1) -I(x_n\le x_o)-2(G(x_1)-G(x_o))]y_1 dH(x_1,y_1)$$

by (4.4.25).

Hence, from (5.1.13), (5.1.14) and (5.1.15), we have

$$(5.1.16)\quad E(I_2|X_n,Y_n)$$

$$= (n^2a_n^3)^{-1}K_1'\left(\frac{G(X_n)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_n)-\lambda}{a_n}\right)Y_n[I(X_n>x_o)-(G(X_n)-G(x_o))]$$

$$+ (n-1)(n^2a_n^3)^{-1}\iint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)[I(x_1>x_o)+I(X_n\le x_1) -I(X_n\le x_o)-2(G(x_1)-G(x_o))]y_1 dH(x_1,y_1).$$

A symmetric argument gives the same expression for $E(I_2|X_j,Y_j)$, $j = 1,2,\ldots,n-1$, as on the R.H.S. of (5.1.16) with (X_n,Y_n) replaced by (X_j,Y_j). Hence it follows (5.1.16) that

(5.1.17) $$\hat{I}_2 = (n^2a_n^3)^{-1} \sum K_1'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i\left[I(X_i>x_o)-(G(X_i)-G(x_o))\right]$$

$$+ (n-1)(n^2a_n^3)^{-1}\sum \iint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)\left[I(X_i\le x_1)-I(X_i\le x_o)\right.$$

$$\left.-(G(x_1)-G(x_o))\right]y_1 dH(x_1,y_1)$$

$$+ C_2$$

where C_2 is a constant (i.e. non-random) into which we have included a part of the constant term from the R.H.S. of (5.1.16) for convenience, viz.

$$C_2 = \frac{n(n-1)}{n^2a_n^3}\iint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)\left[I(x_1>x_o)-(G(x_1)-G(x_o))\right]$$

$$y_1 dH(x_1,y_1)$$

$$- (n-1)E(I_2).$$

We now proceed on the same lines as above to find $\hat{I}_{31}$ and $\hat{I}_{32}$.

By (2.1.1) we have

(5.1.18) $$\hat{I}_{31} = \sum_m E(I_{31}|X_m,Y_m) - (n-1)E(I_{31})$$

where

$$I_{31} = (n^2a_n^4)^{-1} \sum_{ij}\sum K^*\left(\frac{G(X_j)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i I(Y_j\le Y_i)$$

$$= (n^2a_n^4)^{-1} \sum_{ij}\sum \ell_{ij}.$$

We shall write

(5.1.19) $E(I_{31}|X_n,Y_n)$

$$= (n^a a_n^4)^{-1} E[\ell_{nn} + \sum_{i \neq n} \ell_{in} + \sum_{j \neq n} \ell_{nj} + \sum_{i \neq n} \ell_{ii} + \sum_{\substack{i \neq j \\ i,j \neq n}} \ell_{ij} | X_n, Y_n]$$

$$= (n^2 a_n^4)^{-1} E(\ell_{nn}|X_n,Y_n) + (n-1)(n^2 a_n^4)^{-1} (\ell_{1n}|X_n,Y_n)$$

$$+ (n-1)(n^2 a_n^4)^{-1} E(\ell_{n1}|X_n,Y_n) + (n-1)(n^2 a_n^4)^{-1} E(\ell_{11}|X_n,Y_n)$$

$$+ (n-1)(n-2)(n^2 a_n^4)^{-1} E(\ell_{12}|X_n,Y_n).$$

It is easy to see that

$$E(\ell_{nn}|X_n,Y_n) = K^*\left(\frac{G(X_n)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(X_n)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_n)-\lambda}{a_n}\right) Y_n,$$

$$E(\ell_{1n}|X_n,Y_n) = K^*\left(\frac{G(X_n)-G(x_o)}{a_n}\right) \iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 I(Y_n \leq y_1) dH(x_1,y_1),$$

(5.1.20) $$E(\ell_{n1}|X_n,Y_n) = K_2'\left(\frac{G(X_n)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_n)-\lambda}{a_n}\right) Y_n \iint K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right) I(y_1 \leq Y_n) dH(x_1,y_1)$$

$$E(\ell_{11}|X_n,Y_n) = \iint K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 dH(x_1,y_1)$$

and

$$E(\ell_{12}|X_n,Y_n) = \iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(y_2 \le y_1)dH(x_1,y_1)dH(x_2,y_2).$$

Using (5.1.20) in (5.1.19) we obtain $E(I_{31}|X_n,Y_n)$. We also observe that $E(I_{31}|X_n,Y_n)$, $m = 1,2,\ldots,n-1$, can be obtained by replacing (X_n,Y_n) with (X_m,Y_m) in the expression for $E(I_{31}|X_n,Y_n)$. It then follows from (5.1.18) that

$$(5.1.21) \quad \hat{I}_{31} = (n^2a_n^4)^{-1} \sum K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$+ (n-1)(n^2a_n^4)^{-1} \sum K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$y_1 I(Y_i \le y_1)dH(x_1,y_1)$$

$$+ (n-1)(n^2a_n^4)^{-1} \sum K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i \iint K^*\left(\frac{G(x_1)-G(x_o)}{a_n}\right)$$

$$I(Y_i \ge y_1)dH(x_1,y_1)$$

$$+ C_3,$$

where C_3 is the constant term (non-random).

Again by (2.1.1) we have

$$(5.1.22) \qquad \hat{I}_{32} = \sum_m E(I_{32}|X_m,Y_m) - (n-1)E(\text{[illegible]}$$

where
$$I_{32} = (n^2a_n^5)^{-1} \sum\sum_{ij} [G_n(X_j)-G_n(x_o)-G(X_j)+G(x_o)]K^{*\prime}\left(\frac{G(X_j)-G(x_o)}{a_n}\right)$$
$$K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) \text{[illegible]} Y_i)$$
$$= (n^2a_n^5)^{-1} \sum\sum_{ij} t_{ij}.$$

As in (5.1.19) we can write

$$(5.1.23) \quad E(I_{32}|X_n,Y_n) = (n^2a_n^5)^{-1}E(t_{nn}|X_n,Y_n)+(n-1)(n^2a_n^5)^{-1}E(t_{1n}|X_n,Y_n)$$
$$+ (n-1)(n^2a_n^5)^{-1}E(t_{n1}|X_n,Y_n) + (n-1)(n^2a_n^5)^{-1}E(t_{11}|X_n,Y_n)$$
$$+ (n-1)(n-2)(n^2a_n^5)^{-1}E(t_{12}|X_n,Y_n).$$

We see that

$$E(t_{nn}|X_n=x_n,Y_n=y_n) = K^{*\prime}\left(\frac{G(x_n)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)y_n$$
$$E[G_n(x_n)-G_n(x_o)-G(x_n)+G(x_o)|X_n=x_n]$$
$$= K^{*\prime}\left(\frac{G(x_n)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)$$
$$y_n \frac{1}{n}[I(x_n>x_o)-G(x_n)+G(x_o)], \quad \text{by (4.4.7)},$$

$E(t_{1n}|X_n = x_n, Y_n = y_n)$

$$= K^{*'}\left(\frac{G(x_n)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_n \le y_1)$$

$$E[G_n(x_n)-G_n(x_o)-G(x_n)+G(x_o)|X_n=x_n, X_1=x_1]dH(x_1,y_1)$$

$$= K^{*'}\left(\frac{G(x_n)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_n \le y_1)$$

$$\frac{1}{n}[I(x_n>x_o)+I(x_1\le x_n)-I(x_1\le x_o)-2\{G(x_n)-G(x_o)\}]dH(x_1,y_1)$$

by (4.4.25),

$E(t'_{n1}|X_n=x_n, Y_n=y_n)$

$$= K_2'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)y_n \iint K^{*'}\left(\frac{G(x_1)-G(x_o)}{a_n}\right)I(y_1 \le y_n)$$

$$E[G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)|X_n=x_n, X_1=x_1]$$

$$dH(x_1,y_1)$$

$$= K_2'\left(\frac{G(x_n)-G(x_o)}{a_n}, \frac{F_{x_o}(y_n)-\lambda}{a_n}\right)y_n \iint K^{*'}\left(\frac{G(x_1)-G(x_o)}{a_n}\right)I(y_1 \le y_n)$$

$$\frac{1}{n}[I(x_1>x_o)+I(x_n\le x_1)-I(x_n\le x_o)-2\{G(x_1)-G(X_o)\}]dH(x_1,y_1)$$

by (4.4.25) ,

$E(t_{11}|X_n=x_n, Y_n=y_n)$

$$= \iint K^{*'}\left(\frac{G(x_1)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1$$

$$E[G_n(x_1)-G_n(x_o)-G(x_1)+G(x_o)|X_n=x_n, X_1=x_1]dH(x_1,y_1)$$

$$= \iint K^{*'}\left(\frac{G(x_1)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1$$

$$\frac{1}{n}[I(x_1>x_o)+I(x_n\le x_1)-I(x_n\le x_o)-2(G(x_1)-G(x_o))]$$

$$dH(x_1,y_1)$$

and

$$E(t_{12}|X_n = x_n, Y_n=y_n)$$

$$= \iiiint K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_2\le y_1)$$

$$E[G_n(x_2)-G_n(x_o)-G(x_2)+G(x_o)|X_n=x_n, X_1=x_1, X_2=x_2]$$

$$dH(x_1,y_1)dH(x_2,y_2)$$

$$= \iiiint K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1 I(y_2\le y_1)$$

$$\frac{1}{n}[I(x_2>x_o)+I(x_n\le x_2)+I(x_1\le x_2)-I(x_n\le x_o)-I(x_1\le x_o)-3(G(x_2)-G(x_o))]$$

$$dH(x_1,y_1)dH(x_2,y_2), \text{ by } (4.4.26).$$

Now, as before using the above expressions in (5.1.23), to obtain the expression for $E(I_{32}|X_n,Y_n)$ and observing that the same expression with (X_n,Y_n) replaced by (X_j,Y_j) gives $E(I_{32}|X_j,Y_j)$ for $j = 1,\ldots,n-1$, it follows from (5.1.22) that

$$(5.1.24)\quad \hat{I}_{32} = (n^3a_n^5)^{-1} \sum K^{*'}\left(\frac{G(X_i)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$[I(X_i>x_o)-G(X_i)+G(x_o)]$$

continued

$$
\begin{aligned}
&+ (n-1)(n^3a^5)^{-1} \sum K^{*\prime}\left(\frac{G(X_i)-G(x_o)}{a_n}\right) \iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) \\
&\qquad y_1 I(Y_i \le y_1)\left[I(X_i > x_o) + I(X_i \ge x_1) - I(x_1 \le x_o) \right. \\
&\qquad\qquad \left. - 2\left(G(X_i)-G(x_o)\right)\right] dH(x_1,y_1) \\
&+ (n-1)(n^3a_n^5)^{-1} \sum K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right) Y_i \iint K^{*}\left(\frac{G(x_1)-G(x_o)}{a_n}\right) I(Y_i \ge y_1) \\
&\qquad \left[I(X_i \le x_1) - I(X_i \le x_o) - \left(G(x_1)-G(x_o)\right)\right] dH(x_1,y_1) \\
&+ (n-1)(n^3a_n^5)^{-1} \sum \iint K^{*\prime}\left(\frac{G(x_1)-G(x_o)}{a_n}\right) K_2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 \\
&\qquad \left[I(X_i \le x_1) - I(X_i \le x_o) - \left(G(x_1)-G(x_o)\right)\right] dH(x_1,y_1) \\
&+ (n-1)(n-2)(n^3a_n^5)^{-1} \sum \iiiint K^{*\prime}\left(\frac{G(x_2)-G(x_o)}{a_n}\right) k_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) \\
&\qquad y_1 I(y_2 \le y_1)\left[I(X_i \le x_2) - I(X_i \le x_o) - \left(G(x_2)-G(x_o)\right)\right] \cdot \\
&\qquad\qquad dH(x_1,y_1)\, dH(x_2,y_2) \\
&+ C_4
\end{aligned}
$$

where C_4 is the constant term.

Now (5.1.11), (5.1.17), (5.1.21) and (5.1.24) in (5.1.10) gives (5.1.9). This completes the proof of the lemma. □

5.2 ASYMPTOTIC EQUIVALENCE OF $S_{n,x_o}(\lambda)$ and $\hat{S}_{n,x_o}(\lambda)$. In this section we prove that $n^{1/2}a_n\, S_{n,x_o}(\lambda)$ is asymptotically equivalent in mean square to $n^{1/2}a_n\, \hat{S}_{n,x_o}(\lambda)$.

By lemma 2.1.1 we have

(5.2.1) $E[n^{1/2}a_n(S_{n,x_o}(\lambda)-\hat{S}_{n,x_o}(\lambda))]^2 = Var[n^{1/2}a_n S_{n,x_o}(\lambda)] - Var[n^{1/2}a_n\hat{S}_{n,x_o}(\lambda)]$

and by (4.4.104) we have

$$\lim_{n\to\infty} Var[n^{1/2}a_n . S_{n,x_o}(\lambda)] = \sigma_o^2(x_o,\lambda)$$

where $\sigma_o^2(x_o,\lambda)$ is as given by (5.1.3).

Hence to prove the asymptotic equivalence of $n^{1/2}a_n S_{n,x_o}(\lambda)$ and $n^{1/2}a_n\hat{S}_{n,x_o}(\lambda)$ all we need to show is that $\lim_{n\to\infty} Var[n^{1/2}a_n\hat{S}_{n,x_o}(\lambda)] = \sigma_o^2(x_o,\lambda)$.

First observe that since K^*, $K^{*'}$, $K(.,.)$, $K_1'(.,.)$, $K_2'(.,.)$, are all bounded, $E(Y)^2 < \infty$ and $na_n^4 \to \infty$, from (5.1.15) we see that

(5.2.2) $$Var(n^{1/2}a_n S_2) \le \frac{n^2a_n^2}{n^4a_n^6} E\{K_1'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}) [I(X_i>x_o) - G(X_i)+G(x_o)]Y_i\}^2$$

$$= O(\frac{1}{n^2a_n^4}), \text{ which goes to zero as } n\to\infty,$$

(5.2.3) $$Var(n^{1/2}a_n S_4) \le \frac{n^2a_n^2}{n^4a_n^8} E\{K^*(\frac{G(X_i)-G(x_o)}{a_n})K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i\}^2$$

$$= O(\frac{1}{n^2a_n^6}), \text{ which goes to zero as } n\to\infty,$$

(5.2.4) $$Var(n^{1/2}a_n S_7) \le \frac{n^2a_n^2}{n^6a_n^{10}} E\{K^{*'}(\frac{G(X_i)-G(x_o)}{a_n})K_2'(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n})Y_i [I(X_i>x_o)-G(X_i)+G(x_o)]\}^2$$

$$= O(\frac{1}{n^4a_n^8}), \text{ which goes to zero as } n\to\infty,$$

$$(5.2.5)\quad \operatorname{Var}(n^{1/2}a_nS_8) \le \frac{n^2a_n^2(n-1)^2}{n^6a_n^{10}} E\{K^{*'}\left(\frac{G(X_i)-G(x_o)}{a_n}\right)\iint K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$$

$$y[I(X_i>x)-I(X_i\le x_o)+I(x>x_o)-2(G(X_i)-G(x_o))]$$

$$I(Y_i\le y)dH(x,y)\}^2$$

$$= O\left(\frac{1}{n^2a_n^8}\right), \text{ which goes to zero as } n\to\infty,$$

$$(5.2.6)\quad \operatorname{Var}(n^{1/2}a_nS_9) \le \frac{n^2a_n^2(n-1)^2}{n^6a_n^{10}} E\{K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i$$

$$\iint K^{*'}\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Y_i>y)[I(X_i\le x)-I(X_i\le x_o)$$

$$+I(X>x_o)-2(G(x)-G(x_o))]dH(x,y)\}^2$$

$$= O\left(\frac{1}{n^2a_n^8}\right), \text{ which goes to zero as } n\to\infty,$$

and

$$(5.2.7)\quad \operatorname{Var}(n^{1/2}a_nS_{10}) = \frac{n^2a_n^2(n-1)^2}{n^6a_n^{10}} E\{\iint K^{*'}\left(\frac{G(x)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)$$

$$[I(X_i\le x)-I(X_i\le x_o)-(G(x)-G(x_o))]dH(x,y)\}$$

$$= \left(\frac{1}{n^2a_n^8}\right), \text{ which goes to zero as } n\to\infty.$$

LEMMA 5.2.1. *Let S_3 and S_{11} be as given in (5.1.9). Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied and assumptions (4.1.3) to (4.1.6) on K and K^* are satisfied with $\nu = 1$, $\nu^* = 1$. Then for any sequence $\{a_n\}$ of positive numbers with $a_n \to 0$ and $na_n^4 \to \infty$, we have*

(5.2.8).

$$\text{(a)}\quad \lim_{n\to\infty} \mathrm{Var}(n^{1/2}a_nS_3) = 0.$$

$$\text{(b)}\quad \lim_{n\to\infty} \mathrm{Var}(n^{1/2}a_nS_{11}) = 0.$$

PROOF.

(a) Since $S_3 = \frac{n-1}{n^2a_n^3}\sum\iint K_1'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$

$$[I(X_i\le x)-I(X_i<x_o)-(G(x)-G(x_o))]y\,dH(x,y),$$

we have $E(S_3) = 0$

and

$$\mathrm{Var}(n^{1/2}a_nS_3) = \frac{n^2a_n^2(n-1)^2}{n^4a_n^6}E\Big\{\iint K_1'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$[I(X_i\le x)-I(X_i\le x_o)-(G(x)-G(x_o))]y\,dH(x,y)\Big\}^2$$

$$= \frac{(n-1)^2}{n^2a_n^4}\iiiint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_1'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2$$

$$E\{[I(X_i\le x_1)-I(X_i\le x_o)-(G(x_1)-G(x_o))]$$

$$[I(X_i\le x_2)-I(X_i\le x_o)-(G(x_2)-G(x_o))]\}dH(x_1,y_1)dH(x_2,y_2)$$

$$= \frac{(n-1)^2}{n^2a_n^4}\iiiint K_1'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K_1'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2$$

$$[G(x_1\wedge x_2)-G(x_1\wedge x_o)-G(x_2\wedge x_o)+G(x_o)-(G(x_1)-G(x_o))(G(x_2)-G(x_o))]$$

$$dH(x_1,y_1)dH(x_2,y_2).$$

Observe that the integral in the last expression is the same as that in (4.4.17) from which it follows that it is of order $O(a_n^5)$. Hence

$$\operatorname{Var}(n^{1/2}a_n S_3) = \frac{(n-1)^2}{n^2 a_n^4} O(a_n^5) = O(a_n),$$

and since $a_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \operatorname{Var}(n^{1/2}a_n S_3) = 0 .$$

(b) From (5.1.9) we have

$$S_{11} = \frac{(n-1)(n-2)}{n^3 a_n^5} \sum_i Y_i , \quad \text{where}$$

$$Y_i = \iiiint K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$[I(X_i \le x_2) - I(X_i \le x_o) - (G(x_2)-G(x_o))]$$

$$y_1 I(y_2 \le y_1)\, dH(x_1,y_1)\, dH(x_1,y_1)\, dH(x_2,y_2).$$

Therefore

$$\operatorname{Var}(n^{1/2}a_n S_{11}) = \frac{(n-1)^2(n-2)^2}{n^4 a_n^8} \operatorname{Var}(Y_1)$$

and

$$\operatorname{Var}(Y_1) = \idotsint K^{*'}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^{*'}\left(\frac{G(x_4)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right) y_1 y_3 I(y_2 \le y_1) I(y_4 \le y_3)$$

$$[G(x_2 \wedge x_4) - G(x_2 \wedge x_o) - G(x_4 \wedge x_o) + G(x_o) - (G(x_2)-G(x_o))(G(x_4)-G(x_o))]$$

$$dH(x_1,y_1)\, dH(x_2,y_2)\, dH(x_3,y_3)\, dH(x_4,y_4).$$

But the last integral is similar to that in (4.4.42) and it follows from the computations therein that

$$Var(Y_1) = O(a_n^9)$$

and consequently we have

$$Var(n^{1/2}a_n S_{11}) = O(a_n)$$

which goes to zero as $n \to \infty$. □

LEMMA 5.2.2. *Suppose assumptions (4.1.11) to (4.1.13) on the joint distribution H are satisfied and S_5 is as in (5.1.5). Then*

$$Var(n^{1/2}a_n . S_5) \to \sigma_1^2(x_o,\lambda) \text{ , as } n \to \infty$$

where $\sigma_1^2(x_o,\lambda) = (F_{x_o}^{-1}(\lambda))^2 \iint K^*(u)K_2^2(v)dudv$, *as given by* (4.3.45).

PROOF. From (5.1.9) we have

$$S_5 = \frac{n-1}{n^2 a_n^4} \sum \alpha_i \text{, where}$$

(5.2.9) $$\alpha_i = K^*\left(\frac{G(X_i)-G(x_o)}{a_n}\right) \iint K_2\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right) yI(Y_i \le y)dH(x,y).$$

Therefore,

(5.2.10) $$Var(n^{1/2}a_n S_5) = \frac{(n-1)^2}{n^2 a_n^6} Var(\alpha_1)$$

and

$$(5.2.11)\quad E(\alpha_1^2) = \iiiiiint K^{*2}\left(\frac{G(x_3)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K_2'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2I(y_3\le y_1\wedge y_2)dH(x_1,y_1)$$
$$dH(x_2,y_2)dH(x_3,y_3),$$

$$(5.2.12)\quad E^2(\alpha_1) = \iiiiiiiint K^{*}\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^{*}\left(\frac{G(x_4)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)I(y_4\le y_3)$$
$$dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3)dH(x_4,y_4).$$

Observe that the integrals in (5.2.11) and (5.2.12) are exactly those in (4.4.59) corresponding to $E(\ell_{13}\ell_{23})$ and (4.4.54) corresponding to $E(\ell_{13})E(\ell_{23})$ respectively. Hence, it follows from (4.4.64) that

$$\begin{aligned}\mathrm{Var}(\alpha_1) &\doteq \mathrm{Cov}(\ell_{13},\ell_{23})\\ &= 2(F_{x_o}^{-1}(\lambda))^2a_n^6 \iiiint v_1I(v_1\le v_2)K_2'(u_1,v_1)K_2'(u_2,v_2)(K^{*2}(u_3)du_3)\,du_1dv_1du_2dv_2\\ &\quad + o(a_n^7).\\ &= (F_{x_o}^{-1}(\lambda))^2a_n^6\iint K^{*2}(u)K_2^2(v)dudv + o(a_n^7),\end{aligned}$$

by (4.4.84), and consequently we have from (5.2.10)

$$\mathrm{Var}(n^{1/2}a_nS_5) = \frac{(n-1)^2}{n^2}(F_{x_o}^{-1}(\lambda))^2\iint K^{*2}(u)K_2^2(v)dudv$$
$$+ O(a_n).$$

Therefore

$$(5.2.13)\qquad \lim_{n\to\infty} \mathrm{Var}\left(n^{1/2}a_nS_5\right) = \left(F_{x_o}^{-1}(\lambda)\right)^2 \iint K^{*2}(u)K_2^2(v)\,du\,dv. \qquad \square$$

<u>LEMMAS</u> 5.2.3. <u>Under the hypothesis of lemma</u> 5.2.1, <u>we have</u>

$$(5.2.14)\qquad \lim_{n\to\infty} \mathrm{Var}\left[n^{1/2}a_n\left(S_6-S_{12}\right)\right] = 0.$$

<u>PROOF</u>. From (5.1.9) we have $S_6 = \frac{n-1}{n^2a_n^4}\sum \beta_i$, where

$$(5.2.15)\qquad \beta_i = K_2'\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)I(Y_i \geq y)\,dH(x,y),$$

so that

$$(5.2.16)\qquad \mathrm{Var}\left(n^{1/2}a_nS_6\right) = \frac{(n-1)^2}{n^2a_n^6}\mathrm{Var}(\beta_1)$$

and

$$(5.2.17)\qquad E(\beta_1^2) = \iiiint\!\!\iint K_2'^2\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)$$
$$K^*\left(\frac{G(x_3)-G(x_o)}{a_n}\right)y_1^2 I(y_2\leq y_1)I(y_3\leq y_1)\,dH(x_1,y_1)\,dH(x_2,y_2)$$
$$dH(x_3,y_3),$$

$$(5.2.18)\quad E^2(\beta_1) = \iiiint\!\!\iiiint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K^*\left(\frac{G(x_3)-G(x_o)}{a_n}\right)$$

$$K_2'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)K^*\left(\frac{G(x_4)-G(x_o)}{a_n}\right)$$

$$y_1y_2I(y_3 \le y_1)I(y_4 \le y_2)dH(x_1,y_1)dH(x_2,y_2)$$

$$\cdot dH(x_3,y_3)dH(x_4,y_4).$$

Observe that the integral in (5.2.18) is same as that in (4.4.54) from where it follows that

$$(5.2.19)\qquad E^2(\beta_1) = O(a_n^7).$$

We also observe that the integral in (5.2.17) is the same as that in (4.4.56) and hence it follows from (4.4.56), (5.2.19) and (5.2.16) that

$$(5.2.20)\quad \mathrm{Var}(n^{1/2}a_nS_6)$$

$$= \frac{(n-1)^2}{n^2a_n^2}\iint K_2'^2(u_1,v_1)\lambda^2(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ \frac{(n-1)^2}{n^2a_n}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1+\lambda^2(F_{x_o}^{-1}(\lambda))^2 u_1D_6]du_1dv_1$$

$$+ \frac{(n-1)^2}{n^2}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1^2D_2+\lambda^2v_1^2D_1^2+\lambda^2(F_{x_o}^{-1}(\lambda))^2$$

$$(u_1^2D_7+u_1v_1D_8)+4\lambda F_{x_o}^{-1}(\lambda)v_1^2D_1+2\lambda^2F_{x_o}^{-1}(\lambda)v_1u_1D_1D_6$$

$$+(F_{x_o}^{-1}(\lambda))^2v_1{}^2+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1u_1D_6]$$

$$+\frac{(n-1)^2}{n^2}\iint K_2'^2(u_1,v_1)2\lambda(F_{x_o}^{-1}(\lambda))^2D_4du_1dv_1(\int u_2^2K^*(u_2)du_2)$$

$$+ O(a_n).$$

Since $S_{12} \equiv I_{30}$, we have $\mathrm{Var}(n^{1/2}a_n S_{12})$ as given by (4.4.75).

We observe that

(5.2.21) $$\mathrm{Cov}(n^{1/2}a_n S_6,\ n^{1/2}a_n S_{12})$$

$$= na_n^2 \ \mathrm{Cov}\left(\frac{n-1}{n^2 a_n^4} \sum \beta_i \ ,\ \frac{1}{na_n^3} \sum p_j\right)$$

$$= \frac{(n-1)}{na_n^5} \ \mathrm{Cov}(\beta_1, p_1),$$

where β_i's are defined by (5.2.15) and p_j's by (4.4.72).

Now,

(5.2.22) $$E(\beta_1 p_1) = E\left[K_2'\left(\frac{G(X_1)-G(x_o)}{a_n}\ ,\ \frac{F_{x_o}(Y_1)-\lambda}{a_n}\right)Y_1 \iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right) I(Y_1 \geq y)\, dH(x,y)\right.$$

$$\left. \times\ K_2'\left(\frac{G(X_1)-G(x_o)}{a_n}\ ,\ \frac{F_{x_o}(Y_1)-\lambda}{a_n}\right) Y_1 F_{x_o}(Y_1)\right]$$

$$= \iiiint K_2'^{\,2}\left(\frac{G(x_1)-G(x_o)}{a_n}\ ,\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right) y_1^2 I(y_2 \leq y_1)$$

$$F_{x_o}(y_1)\, dH(x_1,y_1)\, dH(x_2,y_2)$$

and

(5.2.23) $$E(\beta_1)E(p_1) = \iiiiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}\ ,\ \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}\ ,\ \frac{F_{x_o}(y_3)-\lambda}{a_n}\right) y_1 y_3 I(y_2 \leq y_1) F_{x_o}(y_3)$$

$$dH(x_1,y_1)\, dH(x_2,y_2)\, dH(x_3,y_3).$$

Since the integrals in (5.2.22) and (5.2.23) are the same as those in (4.4.77) and (4.4.80) respectively, it follows from (4.4.78), (4.4.81) and (5.2.21) that

(5.2.24) $\quad \mathrm{Cov}(n^{1/2}a_nS_6, n^{1/2}a_nS_{12})$

$$= \frac{n-1}{na_n^2}\iint K_2'^2(u_1,v_1)\lambda^2(F_{x_o}^{-1}(\lambda))^2du_1dv_1$$

$$+ \frac{n-1}{na_n}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1D_1+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1+\lambda^2(F_{x_o}^{-1}(\lambda))^2u_1D_6]\,du_1dv_1$$

$$+ \frac{n-1}{n}\iint K_2'^2(u_1,v_1)[2\lambda^2F_{x_o}^{-1}(\lambda)v_1^2D_2+\lambda^2v_1^2D_1^2+\lambda^2(F_{x_o}^{-1}(\lambda))^2(u_1^2D_7+u_1v_1D_8)+4\lambda F_{x_o}^{-1}(\lambda)v_1^2D_1+2\lambda^2F_{x_o}^{-1}(\lambda)v_1u_1D_1D_6$$
$$+(F_{x_o}^{-1}(\lambda))^2v_1^2+2\lambda(F_{x_o}^{-1}(\lambda))^2v_1u_1D_6]du_1dv_1$$

$$+ \frac{n-1}{n}\iint K_2'^2(u_1,v_1)\lambda(F_{x_o}^{-1}(\lambda))^2D_4du_1dv_1(\int u_2^2K^*(u_2)du_2)$$

$$+ O(a_n).$$

Since $\mathrm{Var}[n^{1/2}a_n(S_6-S_{12})] = \mathrm{Var}(n^{1/2}a_nS_6) + \mathrm{Var}(n^{1/2}a_nS_{12}) - 2\mathrm{Cov}(n^{1/2}a_nS_6, n^{1/2}a_nS_{12})$,

it follows from from (5.2.20), (4.4.75) and (5.2.24) that

$$\mathrm{Var}[n^{1/2}a_n(S_6-S_{12})] = O(\frac{1}{n^2a_n^2}) + O(\frac{1}{n^2a_n}) + O(\frac{1}{n^2}) + O(a_n).$$

The lemma now follows immediately on letting $n \to \infty$ in the above expression, since $na_n^4 \to \infty$ and $a_n \to 0$ as $n \to \infty$. □

LEMMA 5.2.4 Under the hypothesis of lemma 5.2.1, we have

(5.2.25) $$\lim_{n\to\infty} \mathrm{Var}(n^{1/2} a_n \hat{S}_{n,x_o}(\lambda)) = \sigma_o^2(x_o, \lambda)$$

where $\sigma_o^2(x_o, \lambda) = (F_{x_o}^{-1}(\lambda))^2 \iint [K(u,v) - K^*(u) K_2(v)]^2 du dv$, as given by (5.1.3).

PROOF. From (5.2.1) to (5.2.7), lemmas 5.2.1 to 5.2.3, clearly it suffices to confine attention to the terms $n^{1/2} a_n S_1$ and $n^{1/2} a_n S_5$ in order to find the limiting expression for $\mathrm{Var}(n^{1/2} a_n \hat{S}_{n,x_o}(\lambda))$.

Since $S_1 \equiv I_1$, from lemma 4.4.3, we have

(5.2.26) $$\lim_{n\to\infty} \mathrm{Var}(n^{1/2} a_n S_1) = (F_{x_o}^{-1}(\lambda))^2 \iint K^2(u,v) du dv.$$

We also have from lemma 5.2.2

(5.2.27) $$\lim_{n\to\infty} \mathrm{Var}(n^{1/2} a_n S_5) = (F_{x_o}^{-1}(\lambda))^2 \iint K^{*^2}(u) K_2^2(v) du dv.$$

Now, observe that

(5.2.28) $$\mathrm{Cov}(n^{1/2} a_n S_1, n^{1/2} a_n S_5)$$

$$= n a_n^2 \mathrm{Cov}\left(\frac{1}{n a_n^2} \sum q_i, \frac{n-1}{n^2 a_n^4} \sum \alpha_j\right)$$

$$= \frac{n-1}{n a_n^4} \mathrm{Cov}(q_1, \alpha_1)$$

where $\alpha_i = K^*\left(\frac{G(X_i) - G(x_o)}{a_n}\right) \iint K_2\left(\frac{G(x) - G(x_o)}{a_n}, \frac{F_{x_o}(y) - \lambda}{a_n}\right) y I(Y_i \le y) dH(x,y)$, as defined by (5.2.9) and

$$q_i = K\left(\frac{G(X_i)-G(x_o)}{a_n}, \frac{F_{x_o}(Y_i)-\lambda}{a_n}\right)Y_i, \text{ as defined by (4.4.86).}$$

Therefore

$$(5.2.29)\quad E(q_1\alpha_1) = \iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right)y_1y_2I(y_2\le y_1)\,dH(x_1,y_1)$$
$$dH(x_2,y_2)$$

and

$$(5.2.30)\quad E(q_1)E(\alpha_1) = \iiiiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)$$
$$dH(x_1,y_1)dH(x_2,y_2)dH(x_3,y_3).$$

As before, once again observe that the integrals in (5.2.29) and (5.2.30) are exactly the same as those in (4.4.93) and (4.4.95) respectively. Consequently, it follows from (4.4.94) and (4.4.96) that

$$(5.2.31)\quad Cov(q_1,\alpha_1) = a_n^4\iiiint K^*(u_1)K_2'(u_2,v_2)K(u_1,v_1)I(v_1\le v_2)(F_{x_o}^{-1}(\lambda))^2du_1dv_1du_2dv_2 + O(a_n^5)$$
$$= -a_n^4(F_{x_o}^{-1}(\lambda))^2\iint K^*(u)K_2(v)K(u,v)\,du\,dv + O(a_n^5), \text{ by (4.4.103)}$$

Hence from (5.2.28) and (5.2.31) we have

$$\mathrm{Cov}(n^{1/2}a_n S_1, n^{1/2}a_n S_5) = -\frac{(n-1)}{n}\,(F_{x_o}^{-1}(\lambda))^2 \iint K^*(u)K_2(v)K(u,v)\,du\,dv + O(a_n)$$

and therefore

(5.2.32) $$\lim_{n\to\infty} \mathrm{Cov}(n^{1/2}a_n S_1, n^{1/2}a_n S_5) = -(F_{x_o}^{-1}(\lambda))^2 \iint K^*(u)K_2(v)K(u,v)\,du\,dv.$$

Now from equations (5.2.1) to (5.2.6) and lemmas 5.2.1 to 5.2.3 we have

(5.2.33) $$\lim_{n\to\infty} \mathrm{Var}(n^{1/2}a_n \hat{S}_{n,x_o}(\lambda)) = \lim_{n\to\infty} \mathrm{Var}(n^{1/2}a_n S_1 + n^{1/2}a_n S_5),$$

and the lemma now follows from (5.2.26), (5.2.27) and (5.2.32). □

Because of (4.4.104) and (5.2.25), by taking the limit as $n \to \infty$ in (5.2.1) we have

$$\lim_{n\to\infty} E[n^{1/2}a_n(S_{n,x_o}(\lambda) - \hat{S}_{n,x_o}(\lambda))]^2 = 0.$$

i.e. $n^{1/2}a_n S_{n,x_o}(\lambda)$ and $n^{1/2}a_n \hat{S}_{n,x_o}(\lambda)$ are asymptotically equivalent in mean square.

5.3. ASYMPTOTIC NORMALITY OF $Q_{n,x_o}(\lambda)$. In this section we give proofs of Theorem 5.1.1 and Corollary 5.1.1.

PROOF OF THEOREM 5.1.1. First observe that lemma 5.1.1 implies that $n^{1/2}a_n(Q_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ has the same asymptotic distribution as $n^{1/2}a_n(S_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$. But $n^{1/2}a_n(\hat{S}_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ and $n^{1/2}a_n(S_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ are asymptotically equivalent in mean square, which implies that $n^{1/2}a_n(Q_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ has the same asymptotic distribution as $n^{1/2}a_n(\hat{S}_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$.

Now, we further show that $n^{1/2}a_n(\hat{S}_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ has indeed the same asymptotic distribution as $n^{1/2}a_n(S_1+S_5-\bar{Q}_{n,x_o}(\lambda))$ where S_1 and S_5 are as defined in (5.1.9), by showing that $[n^{1/2}a_n\hat{S}_{n,x_o}(\lambda) - n^{1/2}a_n(S_1+S_5)]$ converges to zero in probability as $n \to \infty$. From (5.1.9) we can write

$$n^{1/2}a_n\hat{S}_{n,x_o}(\lambda) - n^{1/2}a_n(S_1+S_5)$$

$$= n^{1/2}a_n[S_2+S_3+S_4+(S_6-S_{12})+S_7+S_8+S_9+S_{10}+S_{11}].$$

From equations (5.2.1) to (5.2.7), lemma 5.2.1 and lemma 5.2.3, we already have $\mathrm{Var}(n^{1/2}a_nS_2)$, $\mathrm{Var}(n^{1/2}a_nS_3)$, $\mathrm{Var}(n^{1/2}a_nS_4)$, $\mathrm{Var}(n^{1/2}a_n(S_6-S_{12}))$, $\mathrm{Var}(n^{1/2}a_nS_7),\ldots,\mathrm{Var}(n^{1/2}a_nS_{11})$ all going to zero as $n \to \infty$. Hence it suffices to show that the corresponding expectations also go to zero as $n \to \infty$. Using the expressions for S_i's from (5.1.9), first note that

$$E(S_3) = 0 = E(S_{10}) = E(S_{11}).$$

Now

$$E(n^{1/2}a_nS_2) = (n^{1/2}a_n^2)^{-1}\iint K_1'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)[I(x>x_o)-(G(x)-G(x_o))]\, y\,dH(x,y)$$

which goes to zero as $n \to \infty$, since K_1' is bounded $E|Y| < \infty$ and $na_n^4 \to \infty$, and similarly

$$E(n^{1/2}a_nS_7) = O\left(\frac{1}{n^{3/2}a_n^4}\right), \text{ goes to zero as } n \to \infty.$$

$$E(n^{1/2}a_nS_4) = (n^{1/2}a_n^3)^{-1}\iint K^*\left(\frac{G(x)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right)y\,dH(x,y).$$

By transformation of variables (4.4.13) and application of lemma 4.1.1, for $n > n_o$ the above integral may be written as

$$= a_n^2 \int_{-1}^{1}\int_{-1}^{1} K^*(u)K_2'(u,v)\left[F_{x_o}^{-1}(\lambda)+a_nvD_1+a_n^2v^2D_2+O(a_n^3)\right]$$

$$\left[1+a_nuD_6+a_n^2u^2D_7+a_n^2uvD_8+O(a_n^3)\right]du\,dv$$

$$= O(a_n^2).$$

Hence we have $E(n^{1/2}a_nS_4) = O(n^{-1/2}a_n^{-1})$ which goes to zero as $n \to \infty$.

Proceeding exactly on the same lines as above, we easily see that

$$E(n^{1/2}a_nS_8) = \frac{n-1}{n^{3/2}a_n^4}\,O(a_n^3) = O\left(\frac{1}{n^{1/2}a_n}\right)$$

and
$$E(n^{1/2}a_nS_9) = \frac{n-1}{n^{3/2}a_n^4}\,O(a_n^3) = O\left(\frac{1}{n^{1/2}a_n}\right)$$

which go to zero as $n \to \infty$.

We now consider $E(n^{1/2}a_n(S_6-S_{12}))$. First we obtain

$$E(S_6) = \frac{n-1}{na_n^4}\iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1I(y_2\le y_1)$$
$$dH(x_1,y_1)dH(x_2,y_2)$$

$$= \frac{n-1}{na_n^4}\iiiint K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1F_{x_2}(y_1)$$
$$dH(x_1,y_1)dG(x_2).$$

After the transformation of variables (4.4.18) followed by the use of Taylor expansions from lemma 4.1.1, we see that for $n > n_o$

(5.3.1) $$E(S_6) = \frac{n-1}{na_n}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K^*(u_2)K_2'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)]$$

$$[\lambda+a_nu_2D_3+a_nv_1+a_n^2u_2^2D_4+a_nu_2v_1D_5+O(a_n^3)]$$

$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]du_1dv_1du_2dv_2.$$

$$= \frac{n-1}{n}\iint K_2'(u_1,v_1)[\lambda v_1D_1+F_{x_o}^{-1}(\lambda)v_1+\lambda F_{x_o}^{-1}(\lambda)u_1D_6]du_1dv_1$$

$$+\frac{n-1}{n}a_n\iint K_2'(u_1,v_1)[\lambda v_1^2D_2+\lambda F_{x_o}^{-1}(\lambda)(u_1^2D_7+u_1v_1D_8)+v_1^2D_1+\lambda v_1u_1D_6D_1$$

$$+F_{x_o}^{-1}(\lambda)v_1u_1D_6]du_1dv_1$$

$$+O(a_n^2),$$

where the last equality in (5.3.1) follows by expanding in powers of a_n and utilising the hypothesis $\int_{-1}^{1}K^*(u)du = 1$, $\int_{-1}^{1}uK^*(u)du = 0$, $\int_{-1}^{1}K_2'(u,v)dv = 0$.

Similarly, we find

$$E(S_{12}) = \frac{1}{a_n^3}\iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)y_1F_{x_o}(y_1)dH(x_1,y_1).$$

$$= \frac{1}{a_n}\int_{-1}^{1}\int_{-1}^{1} K_2'(u_1,v_1)[F_{x_o}^{-1}(\lambda)+a_nv_1D_1+a_n^2v_1^2D_2+O(a_n^3)][\lambda+a_nv_1]$$

$$[1+a_nu_1D_6+a_n^2u_1^2D_7+a_n^2u_1v_1D_8+O(a_n^3)]du_1dv_1$$

for $n > n_o$, by using transformation of variables (4.4.13) and applying lemma 4.1.1.

Therefore, we have

$$(5.3.2) \qquad E(S_{12}) = \iint K_2'(u_1,v_1)\left[\lambda v_1 D_1 + F_{x_o}^{-1}(\lambda)v_1 + \lambda F_{x_o}^{-1}(\lambda)u_1 D_6\right]$$

$$+ a_n \iint K_2'(u_1,v_1)\left[\lambda v_1^2 D_2 + \lambda F_{x_o}^{-1}(\lambda)(u_1^2 D_7 + u_1 v_1 D_8) + v_1^2 D_1 + \lambda v_1 u_1 D_1 D_6\right.$$

$$\left. + F_{x_o}^{-1}(\lambda) v_1 u_1 D_6\right] du_1 dv_1$$

$$+ O(a_n^2).$$

From (5.3.1) and (5.3.2), we have

$$E(S_6 - S_{12}) = O(\tfrac{1}{n}) + O(\tfrac{a_n}{n}) + O(a_n^2)$$

and hence

$$E(n^{1/2} a_n(S_6 - S_{12})) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we have established that $n^{1/2}a_n(\hat{S}_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ has the same asymptotic distribution as that of $n^{1/2}a_n(S_1 + S_5 - \bar{Q}_{n,x_o}(\lambda))$. Now by the series of implications mentioned in the beginning of the proof it is clear that $n^{1/2}a_n(Q_{n,x_o}(\lambda) - \bar{Q}_{n,x_o}(\lambda))$ has the same asymptotic distribution as that of $n^{1/2}a_n(S_1 + S_5 - \bar{Q}_{n,x_o}(\lambda))$ and hence it suffices to prove the asymptotic normality of the latter. We have

$$n^{1/2}a_n(S_1+S_5-\bar{Q}_{n,x_o}(\lambda))$$

$$= n^{1/2}a_n\ \Big[\ \frac{1}{na_n^2}\sum K\Big(\frac{G(X_i)-G(x_o)}{a_n},\ \frac{F_{x_o}(Y_i)-\lambda}{a_n}\Big)Y_i$$

$$+\ \frac{n-1}{n^2a_n^4}\sum K^*\Big(\frac{G(X_i)-G(x_o)}{a_n}\Big)\iint K_2'\Big(\frac{G(x)-G(x_o)}{a_n},\ \frac{F_{x_o}(y)-\lambda}{a_n}\Big)yI(Y_i\le y)dH(x,y)$$

$$-\ \frac{1}{a_n^2}\iint K\Big(\frac{G(x)-G(x_o)}{a_n},\ \frac{F_{x_o}(y)-\lambda}{a_n}\Big)ydH(x,y)\ \Big]$$

which may be rewritten as

$$(5.3.3)\qquad T_n = n^{1/2}a_n(S_1+S_5-\bar{Q}_{n,x_o}(\lambda))$$

$$= \sum\ \Big[\frac{1}{n^{1/2}a_n}K\Big(\frac{G(X_i)-G(x_o)}{a_n},\ \frac{F_{x_o}(Y_i)-\lambda}{a_n}\Big)Y_i$$

$$+\ \frac{n-1}{n^{3/2}a_n^3}\ K^*\Big(\frac{G(X_i)-G(x_o)}{a_n}\Big)\iint K_2'\Big(\frac{G(x)-G(x_o)}{a_n},\frac{F_{x_o}(y)-\lambda}{a_n}\Big)yI(Y_i\le y)dH(x,y)$$

$$-\ \frac{1}{n^{1/2}a_n}\iint K\Big(\frac{G(x)-G(x_o)}{a_n},\ \frac{F_{x_o}(y)-\lambda}{a_n}\Big)ydH(x,y)\ \Big]$$

$$= \sum\ [p_n(X_i,Y_i) + q_n(X_i,Y_i)-C_n],\quad \text{say,}$$

where C_n is the non-random term.

Observe that for each n, T_n' in a sum of iid rv's with

$\sigma_n^2 = \operatorname{Var}[n^{1/2}a_n(S_1+S_5-\bar{Q}_{n,x_o}(\lambda))] = \operatorname{Var}[n^{1/2}a_n(S_1+S_5)] + \sigma_o^2(x_o,\lambda)$ as shown in lemma 5.2.4. So assuming $\sigma_o^2(x_o,\lambda)>0$, it suffices to show that the array defining the T_n's satisfies the Lindeberg condition, i.e. it suffices to show that for every $\varepsilon > 0$

(5.3.4) $$\frac{n}{\sigma_n^2} \iint_{A_n} (p_n(x,y)+q_n(x,y) - \mu(p_n) - \mu(q_n))^2 dH(x,y) \to 0, \text{ as } n \to \infty$$

where

(5.3.5) $$\mu(p_n) = E(p_n(X,Y)), \quad \mu(q_n) = E(q_n(X,Y))$$

and

(5.3.6) $$A_n = \{|p_n(x,y)+q_n(x,y)-\mu(p_n)-\mu(q_n)|>\varepsilon\sigma_n\}$$

Since σ_n^2 converges to a positive constant it suffices to prove that

(5.3.7) $$g_n(\varepsilon) = n \iint_{A_n} (p_n(x,y)+q_n(x,y) - \mu(p_n)-\mu(q_n))^2 dH(x,y)$$

goes to 0 as $n \to \infty$.

Now observe that

(5.3.8) $$g_n(\varepsilon) \le 4n \iint_{A_n} p_n^2(x,y)dH(x,y) + 4n \iint_{A_n} q_n^2(x,y)dH(x,y) + 4n\mu^2(p_n) + 4n\mu^2(q_n).$$

From (5.3.3) and (5.3.5) we have

$$\mu^2(p_n) = \frac{1}{na_n^2}\iiiint K\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)K\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right) y_1y_2dH(x_1,y_1)dH(x_2,y_2).$$

But the integral above is of order $O(a_n^4)$, which may easily be seen by the usual transformation of variables (4.4.18) and consequently we have

(5.3.9) $n.\mu^2(p_n) = O(a_n^2)$, which goes to 0 as $n \to \infty$.

Similarly

$$\mu^2(q_n) = \frac{(n-1)^2}{n^3a_n^6}\cdot\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int K^*\left(\frac{G(x_2)-G(x_o)}{a_n}\right)K^*\left(\frac{G(x_4)-G(x_o)}{a_n}\right)K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$
$$K_2'\left(\frac{G(x_3)-G(x_o)}{a_n}, \frac{F_{x_o}(y_3)-\lambda}{a_n}\right)y_1y_3I(y_2\le y_1)I(y_4\le y_3)dH(x_1,y_1)$$
$$dH(x_2,y_2)dH(x_3,y_3)dH(x_4,y_4),$$

and the last integral is exactly the same as that in (4.4.54) which was shown to be of order $O(a_n^7)$. Hence we have

(5.3.10) $n\mu^2(q_n) = O(a_n)$, which goes to 0 as $n \to \infty$.

Since

(5.3.11) $$A_n \subset A_{1n} \cup A_{2n} \cup A_{3n}$$

where

$$A_{1n} = \{x,y: |p_n(x,y)| \geq \frac{\varepsilon\sigma_n}{3}\}$$

$$A_{2n} = \{x,y: |q_n(x,y)| \geq \frac{\varepsilon\sigma_n}{3}\}$$

$$A_{3n} = \{x,y: |\mu(p_n)+\mu(q_n)| \geq \frac{\varepsilon\sigma_n}{3}\},$$

we have

(5.3.12) $$n \iint_{A_n} p_n^2(x,y)\,dH(x,y) \leq \iint_{A_{1n}} np_n^2(x,y)\,dH(x,y) + \iint_{A_{2n}} np_n^2(x,y)\,dH(x,y) + \iint_{A_{3n}} np_n^2(x,y)\,dH(x,y).$$

Further

(5.3.13) $$A_{1n} = \{x,y: |\frac{1}{n^{1/2}a_n} K(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n})y| \geq \frac{\varepsilon\sigma_n}{3}\}$$

$$\subset \{x,y: |y| \geq \delta\sigma_n(n^{1/2}a_n)\} = B_{1n}, \text{ say,}$$

where $\delta = \frac{\varepsilon}{3M}$ and $M = \sup_{x,y}|K(x,y)|$.

Since $\sigma_n^2 \to$ positive constant, $n^{1/2}a_n \to \infty$, the sequence of sets

(5.3.14) $\{B_{1n}\} \to \phi$, the empty set, as $n \to \infty$.

Next, we have

$$A_{2n} = \{x,y: |\frac{n-1}{n^{3/2}a_n^3} K^*(\frac{G(x)-G(x_o)}{a_n}) \iint K_2(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}) y_1 I(y \leq y_1)\,dH(x_1,y_1)| \geq \frac{\varepsilon\sigma_n}{3}\}.$$

Since

$$\left| \iint K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 I(y \le y_1)\, dH(x_1,y_1) \right|$$

$$\le \iint \left| K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right) y_1 \right| dH(x_1,y_1)$$

$$= O(a_n^2),$$

(which maybe easily seen by the routine transformation of variables explained by (4.4.13) and application of lemma 4.1.1) and K^* is bounded it follows that the sequence of sets

(5.3.15) $\{A_{2n}\} \to \phi$ as $n \to \infty$.

Since both $\mu(p_n)$ and $\mu(q_n)$ go to 0 as $n \to \infty$, which may be seen from (5.3.9) and (5.3.10), we also have that the sequence of sets

(5.3.16) $\{A_{3n}\} \to \phi$ as $n \to \infty$.

Thus from (5.3.12) and (5.3.13) to (5.3.16) we obtain

(5.3.17) $$n\iint_{A_n} p_n^2(x,y)\,dH(x,y) \le \iint_{B_{1n}} np_n^2(x,y) + \iint_{A_{2n}} p_n^2(x,y)\,dH(x,y) + \iint_{A_{3n}} np_n^2(x,y)\,dH(x,y)$$

and

(5.3.18) $$n\iint_{A_n} q_n^2(x,y)\,dH(x,y) \le \iint_{B_{1n}} nq_n^2(x,y)\,dH(x,y) + \iint_{A_{2n}} nq_n^2(x,y)\,dH(x,y)$$

$$+ \iint_{A_{3n}} nq_n^2(x,y)\,dH(x,y).$$

where $\{B_{1n}\}$, $\{A_{2n}\}$ and $\{A_{3n}\}$ all $\to \phi$ as $n \to \infty$.

From (5.3.17), consider

$$\iint_{B_{1n}} np_n^2(x,y)\,dH(x,y) = \frac{1}{a_n^2} \iint_{B_{1n}} K^2\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right) y^2\,dH(x,y).$$

By transformation of variables (4.4.13) and using Taylor expansions from lemma 4.1.1, we get

(5.3.19) $$\iint_{B_{1n}} np_n^2(x,y)\,dH(x,y) = \iint_{E_{1n}} K^2(u,v)\left(F_{x_o}^{-1}(\lambda)\right)^2 du\,dv + O(a_n)$$

where $E_{1n} = \{u,v: |F_{x_o}^{-1}(\lambda + a_n v)| \ge \delta\sigma_n(n^{1/2}a_n)\}$.

Since K^2 is integrable and the set E_{1n} converges to ϕ, the integral on the R.H.S. of (5.3.19) goes to 0. The second term goes to zero since $a_n \to 0$ and consequently

$$\iint_{B_{1n}} p_n(x,y)\,dH(x,y) \to 0 \text{ as } n \to \infty.$$

The other two integrals on the R.H.S. of (5.3.17) can be handled exactly in the same manner and thus we have

(5.3.20) $$n\iint_{A_n} p_n^2(x,y)\,dH(x,y) \to 0 \text{ as } n \to \infty.$$

To show that $n\iint_{A_n} q_n^2(x,y)\,dH(x,y)$ goes to zero, we shall again deal with the first integral on the R.H.S. of (5.3.18) only since the others can be handled on the very same lines.

Now

(5.3.21) $$n\iint_{B_{1n}} q_n^2(x,y)\,dH(x,y)$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{1}{a_n^6} \iiiint\!\!\iint_{B_{1n}} K^{*2}\left(\frac{G(x)-G(x_o)}{a_n}\right) K_2'\left(\frac{G(x_1)-G(x_o)}{a_n}, \frac{F_{x_o}(y_1)-\lambda}{a_n}\right)$$

$$K_2'\left(\frac{G(x_2)-G(x_o)}{a_n}, \frac{F_{x_o}(y_2)-\lambda}{a_n}\right) y_1 y_2 I(y\le y_1) I(y\le y_2)$$

$$dH(x_1,y_1)\,dH(x_2,y_2)\,dH(x,y)$$

$$= \left(\frac{n-1}{n}\right)^2 \iiiint\!\!\iint_{E_{1n}} K^{*2}(u) K_2'(u_1,v_1) K_2'(u_2,v_2) I(v\le v_1 \wedge v_2) \left(F_{x_o}^{-1}(\lambda)\right)^2$$

$$du_1\,dv_1\,du_2\,dv_2\,du\,dv$$

$$+ O(a_n),$$

where the last equality in (5.3.21) follows after the transformation of variables x, y, x_1, y_1, x_2, y_2 to u, v, u_1, v_1, u_2, v_2 as in (4.4.18), followed by the use of Taylor expansions from lemma 4.4.1. Since the set E_{1n} converges to ϕ as $n \to \infty$ and $a_n \to 0$ as $n \to \infty$, we have

$$n \iint_{B_{1n}} q_n^2(x,y)\,dH(x,y) \to 0 \quad \text{, as } n \to \infty.$$

Hence we have

(5.3.22) $$n\iint_{A_n} q_n^2(x,y)\,dH(x,y) \to 0 \quad \text{, as } n \to \infty$$

and it now follows from (5.3.9), (5.3.10), (5.3.20), (5.3.22) and (5.3.8) that $g_n(\varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. The proof of the Theorem is complete. □

<u>PROOF OF COROLLARY</u> 5.1.1:

Observe that

$$E(I_1) = a_n^{-2} \iint K\left(\frac{G(x)-G(x_o)}{a_n}, \frac{F_{x_o}(y)-\lambda}{a_n}\right) y\,dH(x,y) = \bar{Q}_{n,x_o}.$$

Hence from (4.5.20) we have

$$\bar{Q}_{n,x_o}(\lambda) - Q_{x_o}(\lambda) = O(a_n^2)$$

and consequently

$$(5.3.23) \qquad n^{1/2} a_n [\bar{Q}_{n,x_o}(\lambda) - Q_{x_o}(\lambda)] = O(n^{1/2} a_n^3)$$

which goes to zero as $n \to \infty$ since $na_n^5 \to 0$. The Corollary is now a consequence of Theorem 5.1.1 and (5.3.23). □

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