I do not approve of anything that tampers with natural ignorance. Ignorance is like a delicate exotic fruit; touch it and the bloom is gone.

The Importance of Being Earnest, Oscar Wilde

University of Alberta

Symmetries of Black Holes and D-Branes

by

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To

Erin

Abstract

Symmetry is one of the most important guiding principles in the formulation of modern physical theories and it also plays a major role in determining and constraining the dynamics of such theories. This is particularly true in the context of quantum field theory and string theory. In this thesis, symmetry aspects of two of the most important aspects of these theories are studied, namely black holes and D-branes.

Part I of the thesis focuses on several black hole solutions in four and higher dimensions. Specifically, the Kerr-(A)-de Sitter and the Myers-Perry metrics in all dimensions, some charged rotating supergravity black hole solutions in four and five dimensions, and a class of NUT charged black holes in several dimensions are studied. The separability of the Hamilton-Jacobi equation describing the propagation of classical particles and the Klein-Gordon equation describing the propagation of scalar fields in these spacetimes is analyzed. This analysis provides information regarding the spacetime symmetry group, and in many cases, non-trivial Killing tensors are found, whose existence is directly responsible for enhancement of symmetry that permits separation.

Part II of the thesis focuses on D-branes. In recent years, it has been realized that D-brane dynamics are heavily dominated by their charges. The macroscopic approach to D-brane charges involving K-theory and cohomology only calculates the charge groups, but not the explicit charges of the D-branes. Conformal field theory techniques can be used in a microscopic approach to determine D-brane charges. This calculation is explicitly carried out for a class of Wess-Zumino-Witten models describing string theory on Lie groups. Specifically, the D-brane charges of the group D_4 twisted by triality and the group E_6 twisted by charge conjugation are calculated explicitly. Along the way a number of non-trivial and surprising Lie theoretic identities are established and proved. The charges are also determined for the D-branes of the non-simply connected group $E_6/\mathbb{Z}_3$ twisted by charge conjugation.

Preface

The results presented in this Doctoral Thesis were obtained over the course of the author's Ph.D. program at The University of Alberta between 2002 and 2006. The presentation of this work is in accordance with the "Paper Format" regulations of the Faculty of Graduate Studies at the University of Alberta. The majority of results have been published in peer reviewed journals and appear in the following chapters:

Chapter 3 is based on

M. Vasudevan, K. A. Stevens, and D. N. Page, *Separability of the Hamilton-Jacobi and Klein-Gordon equations in Kerr-de Sitter metrics*, Class. Quant. Grav. 22 (2005) 339-352, gr-qc/0405125.

Chapter 4 is based on

M. Vasudevan, K. A. Stevens, and D.N. Page, *Particle Motion and scalar field propagation in Myers-Perry black hole spacetimes in all dimensions*, Class. Quant. Grav. 22 (2005) 1469-1482, gr-qc/0407030.

Chapter 5 is based on

M. Vasudevan, *Integrability of some charged rotating supergravity black hole solutions in four and five dimensions*, Phys. Lett. B 624 (2005) 287-296 , gr-qc/0507092.

Chapter 6 is based on

M. Vasudevan, *A note on particles and scalar fields in higher dimensional nutty spacetimes*, Phys. Lett. B632 (2006) 532-536, gr-qc/0511028.

Chapter 7 is based on

M. Vasudevan and K. A. Stevens, *Integrability of particle motion and scalar field propagation in Kerr-(anti) de Sitter black hole spacetimes in all dimensions*, Phys. Rev. D72 124008 (2005), gr-qc/0507096.

Chapter 11 is based on (following the usual convention in high energy physics of listing author names alphabetically)

T. Gannon and **M. Vasudevan**, *Charges of exceptionally twisted branes*, JHEP 07 (2005) 035, hep-th/0504006.

In addition, Chapter 12 contains completed sections of original research in progress which will appear at a future date.

Acknowledgements

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In addition, Professor Bruce Campbell has my gratitude for providing me an absolutely delightful education in string theory, quantum field theory, and particle physics. More importantly, the numerous discussions I have had with him between 2002 and 2004 have had great impact on my appreciation for physics and even more so for the power and beauty of differential and algebraic geometry. Professor Valeri Frolov taught me the physics on black holes in one of the best courses I have ever taken, and provided crucial advice when I needed it; for these he has my thanks. I would also like to thank Professors Richard Sydora and Andrej Czarnecki for advice and discussions at many points of my program.

Kory Stevens and I have had a remarkably productive collaboration based on our mutual research interests in black holes and higher dimensional physics. Discussions with him were a pleasure, especially over our very enjoyable games of chess. I also want to thank him for the many times he has helped me out with LaTeXand Maple difficulties.

Some of the other graduate students and postdoctoral fellows whose tenures at the University of Alberta overlapped with my program at various points deserve special recognition: Ian Blokland for his many visits to my office to discuss all aspects of physics and each other's research; David Shaw for useful discussions, and providing me with creative ways to procrastinate; Ernest Jankowski, for teaching me several aspects of supersymmetry and field theory; JP Archambault for providing moral support early in my program; Alexey Pak for useful discussions on field theory; Stephen Fairhurst, for patiently teaching me aspects of geometry and relativity, and helping me prepare for the candidacy exam; Ivan Booth for many stimulating arguments over topology; Tyler Luchko for helping with my computing handicaps; David Kubiznak for great talks on physics while hiking in Banff; and Robert Bryce for encouragement and advice in the concluding stage of my program

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The truly remarkable music of Gustav Mahler (particularly Symphony No. 2) and Peter Ilyich Tchaikovsky made several long stretches of great frustration over my research when I came upon stumbling blocks bearable; and I also do not know how I would have struggled through the arduous process of thesis typing without it.

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Most of all I want to thank Erin. It is hardly possible to describe the remarkable transition to joy that occurred in my life when she entered it. The importance of her optimism, belief in my abilities, encouragement, support, and ability to make me smile and laugh even during moments of extreme frustration cannot be emphasized enough. She is the love of my life and I owe her everything for making me the person that I am.

Table of Contents

List of Tables

List of Figures

Chapter 1

The Grand Scheme of Things

The concept of symmetry has become the fundamental guiding principle in the formulation and study of most aspects of theoretical physics, and particularly so in high energy physics and in theories of gravity. The importance of symmetry was essentially first appreciated in quantum field theory, where the so-called Landau-Ginzburg approach completely formulates the theory through symmetry demands on the fields of interest [1,2]. The lessons learned in quantum field theory have made the study of symmetries one of the most powerful tools in both the formulation and the dynamical structure of string theory.

Another lesson learned from quantum field theory is the extent to which the dynamics of the theory is dominated by solitonic structures when they are present. In quantum field theory, these are usually in the form of vortices, instantons, monopoles, domain walls, etc. [1,3,4]. In fact important phenomena like vacuum tunneling, and the Dirac quantization condition on charge are direct consequences of solitonic physics. In recent years, it has been learned that string theory formulations are incomplete without accommodating for solitons. In fact, it has been shown that string theories are completely inconsistent, inaccurate, and unpredictive if the solitonic sectors of the theory are ignored [5]. It has also been realized that solitonic structures are even more important than string dynamics itself in the large string tension limits of the theory [6, 7]. The two most important classes of solitons in string theory are black holes and D-branes. This thesis examines the symmetry structure and the resulting dynamical constraints for many important black holes and D-branes occurring in various limits of string theory.

The first part of this thesis deals with the study of several classical black holes in four and higher dimensions. The classical Hamilton-Jacobi equation describing the motion of massive and massless particles in these backgrounds is studied and is shown to be separable in several important situations. In addition, the Klein-Gordon equation describing the propagation of massive and massless scalar fields in these backgrounds is also studied, and separability is established for the same situations. In the process, expressions are obtained for Killing vectors that generate the various spacetime symmetries of these backgrounds, thereby obtaining information regarding the spacetime symmetry groups. In addition, non-trivial second-rank Killing tensors are found for many of these spacetimes, which provide the analogue of the Carter constant and permit separation of both equations. Killing tensors are "symmetries" on phase space which are conjectured to be very interesting quantities in string theory, particularly in the context of the so-called AdS/CFT correspondence.

The second part of the thesis deals with the study of D-brane charges of Wess-Zumino-Witten (WZW) models. WZW models describe string theory on a group manifold, and are of great current interest, as they are exactly solvable as Conformal Field Theories (CFT) [8]. In addition, some WZW models are exactly dual to string theory models of phenomenological interest. The dynamics of D-branes are constrained heavily by their conserved charges. The study of D-brane charges was initially investigated using the powerful geometric tools of K-theory, whereby it was realized that D-brane charges can be interpreted as instantons on the D-branes [9]. However, K-theoretic calculations turn out to provide information regarding the charge groups of the D-branes only, but not the actual charges of the D-branes specifically. A more "microscopic" and complete approach is to use a Boundary Conformal Field Theory (BCFT) calculation of the D-brane charges. This calculation is carried out in detail and the charges are determined for the charge conjugation twisted E_6 branes, and triality twisted D_4 branes, which completes an important previously missing section of the D-brane charge research literature. In addition, several non-trivial results regarding simple current symmetries of the affine Lie algebras in the context of WZW models are established. The charge calculations for D-branes on the non-simply connected group $E_6/\mathbb{Z}_3$ twisted by charge conjugation are also presented. This part also begins with a self-contained introduction to affine Lie algebras, WZW models and their fusion rules, and BCFT D-brane charge calculations. A short introduction to some methods of conformal field theory is also presented in an appendix.

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Part I

Symmetries of Higher Dimensional Black Hole Spacetimes

Chapter 2

Introduction

Solutions of the vacuum Einstein equations describing rotating black hole spacetimes in higher dimensions are of great current interest due to many recent developments in high energy physics and gravity. Models of spacetimes with large extra dimensions that have been proposed to deal with several questions arising in modern particle phenomenology (e.g. the hierarchy problem) naturally include such higher dimensional black hole solutions [1–3]. These models are also of interest in the context of mini-black hole production in high energy particle colliders, which would provide a window into non-perturbative gravitational physics [4, 5].

Higher dimensional black hole solutions also find a natural description in superstring and M-theory due to their 10 or 11 dimensional ambient spacetimes. Branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Solitonic objects in superstring theory frequently find a natural description in terms of higher dimensional black holes. In fact the black hole entropy calculation in string theory makes use of such a description where black holes are related to collections of D-branes. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6, 7].

With phenomenological interest now in a universe with nonzero cosmological constant, it is also important to consider spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an Anti-de Sitter background could give rise to interesting descriptions in terms of the CFT on the boundary leading to better understanding of the correspondence [8, 9]. There is also a very strong need to understand the structure of geodesics in the background of black holes in Anti-de Sitter backgrounds in the context of string theory and the AdS/CFT correspondence. This is due to the recent work in exploring black hole singularity structure using geodesics and correlators on the dual CFT on the boundary [10–15].

In this part of the thesis the separability of the Hamilton-Jacobi equation in many such spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons), is studied. Separation of the equation is explicitly demonstrated and carried out in these backgrounds for many cases. This explicit separation is used to obtain first-order equations of motion for both massive and massless particles in these backgrounds. The equations are obtained in a form that could be used for numerical study, and in the case of spacetimes with cosmological constant, also in the study of black hole singularity structure using geodesic probes and the AdS/CFT correspondence.

The Klein-Gordon equation describing the propagation of massive and massless scalar fields in these spacetimes is also studied. Separation is again explicitly shown for the same situations that the Hamilton-Jacobi equation is separable.

In many of these spacetimes, separation is possible for both equations due to the existence of a second-order non-trivial irreducible Killing tensor. These are generalizations of the Killing tensor in the Kerr black hole spacetime in four dimensions constructed in [16], which was subsequently described by Chandrasekhar as the "miraculous property of the Kerr metric". The Killing tensor provides an additional integral of motion necessary for complete integrability. The Killing vectors of the spacetimes, which are the generators of the spacetime symmetries, are explicitly constructed, and their role in the separability of both equations is demonstrated. By this procedure, information regarding the complete symmetry groups of these spacetimes is obtained.

The published text of the papers appears in the following six chapters, with very minor changes to correct errors and update the bibliographies.

Chapter 3 deals with the recently discovered Kerr-(Anti) de Sitter metrics in all dimensions [17,18]. Separation is carried out in the case where all the rotation parameters are equal. This also needs the restriction that the spacetime is odd-dimensional.

Chapter 4 deals with the Myers-Perry metrics describing rotating black holes in higher dimensions without a cosmological constant. Separation is carried out in the case where there are only two sets of possibly unequal rotation parameters. [19]

Chapter 5 deals with two rotating supergravity black hole solutions with charge in four and five dimensions. Separation is established in all cases.

Chapter 6 deals with a general class of rotating black hole spacetimes carrying NUT charge. Separation is established in all cases.

Chapter 7 deals again with the Kerr-(Anti) de Sitter metrics from Chapter 3. Separation is now established for the case where there are two sets of possibly unequal rotation parameters. This removes the restriction on dimensionality and is now applicable in all dimensions. This encompasses the results of Chapters 3 and 4. This also appears to be the most general case for this class of black holes where separability is possible.

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Chapter 3

Equal Parameter Kerr-de Sitter Metrics

3.1 Introduction

Solutions of the vacuum Einstein equations describing black hole solutions in higher dimensions are currently of great interest. This is mainly due to a number of recent developments in high energy physics. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (e.g. the hierarchy problem) [1] [2] [3]. These models allow for the existence of higher dimensional black holes which can be described classically. Also of interest in these models is the possibility of mini black hole production in high energy particle colliders which, if they occur, provide a window into non-perturbative gravitational physics [4] [5].

Superstring and M-Theory, which call for additional spacetime dimensions, naturally incorporate black hole solutions in higher dimensions (10 or 11). P-branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Black hole solutions in superstring theory are particularly relevant since they can be described as solitonic objects. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6] [7].

Astrophysically relevant black hole spacetimes are, to a very good approximation, described by the Kerr metric [8]. One generalization of the Kerr metric to higher dimensions is given by the Myers-Perry construction [9]. With interest now in a nonzero cosmological constant, it is worth studying spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an anti-de Sitter background could give rise to interesting descriptions in terms of the conformal field theory on the boundary leading to better understanding of the correspondence [10] [11]. The general Kerr-de Sitter metrics describing rotating black holes in the presence of a cosmological constant have been constructed explicitly in [12] [13].

In this paper we study the separability of the Hamilton-Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We also investigate the separability of the Klein-Gordon equation describing a spinless field propagating in this background. For both equations, separation is possible in some special cases due to the enlargement of the dynamical symmetry group underlying these metrics. We construct the separation of both equations explicitly in these cases. We also construct Killing vectors, which exist due to the additional symmetry, and which permit the separation of these equations. We also derive and study equations of motion for particles in these spacetimes.

3.2 Construction and Overview of the Kerr-de Sitter Metrics

A remarkable property of the Kerr metric is that it can be written in the so-called Kerr-Schild [14] form, where the metric $g_{\mu\nu}$ is given exactly by its linear approximation around the flat metric $\eta_{\mu\nu}$ as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{U}\left(k_\mu dx^\mu\right)^2 , \tag{3.1}$$

where k_μ is null and geodesic with respect to both the full metric $g_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$.

The Kerr-de Sitter metrics in all dimensions are obtained in [12] by using the de Sitter metric instead of the flat background $\eta_{\mu\nu}$, with coordinates chosen appropriately to allow for the incorporation of the Kerr metric via the null geodesic vectors k_μ. We quickly review the construction here.

We introduce $n = [D/2]$ coordinates μ_i subject to the constraint

$$\sum_{i=1}^{n} \mu_i^2 = 1 , \tag{3.2}$$

together with $N = [(D-1)/2]$ azimuthal angular coordinates ϕ_i, the radial coordinate r, and the time coordinate t. When the total spacetime dimension D is odd, $D =$

$2n+1 = 2N+1$, there are n azimuthal coordinates ϕ_i, each with period 2π. If D is even, $D = 2n = 2N+2$, there are only $N = n-1$ azimuthal coordinates ϕ_i. Define ϵ to be 1 for even D, and 0 for odd D.

The Kerr-de Sitter metrics ds^2 in D dimensions satisfy the Einstein equation

$$R_{\mu\nu} = (D-1)\,\lambda\, g_{\mu\nu}\,. \tag{3.3}$$

Define W and F as follows:

$$W \equiv \sum_{i=1}^{n} \frac{\mu_i^2}{1+\lambda\, a_i^2}\,, \qquad F \equiv \frac{r^2}{1-\lambda\, r^2}\,\sum_{i=1}^{n} \frac{\mu_i^2}{r^2+a_i^2}\,. \tag{3.4}$$

In D dimensions, the Kerr-de Sitter metrics are given by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}\,(k_\mu\, dx^\mu)^2\,, \tag{3.5}$$

where the de Sitter metric $d\bar{s}^2$, the null vector k_μ, and the function U are now given by

$$\begin{aligned} d\bar{s}^2 &= -W\,(1-\lambda\, r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{n} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, \mu_i^2\, d\phi_i^2 \\ &\quad + \frac{\lambda}{W\,(1-\lambda\, r^2)}\Big(\sum_{i=1}^{n} \frac{(r^2+a_i^2)\,\mu_i\, d\mu_i}{1+\lambda\, a_i^2}\Big)^2\,, \end{aligned} \tag{3.6}$$

$$k_\mu\, dx^\mu = W\, dt + F\, dr - \sum_{i=1}^{n-\epsilon} \frac{a_i\, \mu_i^2}{1+\lambda\, a_i^2}\, d\phi_i\,, \tag{3.7}$$

$$U = r^\epsilon \sum_{i=1}^{n} \frac{\mu_i^2}{r^2+a_i^2} \prod_{j=1}^{n-\epsilon} (r^2+a_j^2)\,. \tag{3.8}$$

In the even-dimensional case, where there is no azimuthal coordinate ϕ_n, there is also no associated rotation parameter; i.e., $a_n = 0$. Note that the null vector corresponding to the null one-form is

$$k^\mu\, \partial_\mu = -\frac{1}{1-\lambda\, r^2}\,\frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-\epsilon} \frac{a_i}{r^2+a_i^2}\,\frac{\partial}{\partial \phi_i}\,. \tag{3.9}$$

This is easily obtained by using the background metric to raise and lower indices rather than the full metric, since k is null with respect to both metrics.

For the purposes of analyzing the equations of motion and the Klein-Gordon equation, it is very convenient to work with the metric expressed in Boyer-Lindquist coordinates.

In these coordinates there are no cross terms involving the differential dr. In both even and odd dimensions, the Boyer-Lindquist form is obtained by means of the following coordinate transformation:

$$dt = d\tau + \frac{2M\,dr}{(1-\lambda\,r^2)(V-2M)}\,, \qquad d\phi_i = d\varphi_i - \lambda\,a_i\,d\tau + \frac{2M\,a_i\,dr}{(r^2+a_i^2)(V-2M)}\,. \quad (3.10)$$

In Boyer-Lindquist coordinates in D dimensions, the Kerr-de Sitter metrics are given by

$$\begin{aligned} ds^2 &= -W\,(1-\lambda r^2)\,d\tau^2 + \frac{U\,dr^2}{V-2M} + \frac{2M}{U}\Big(d\tau - \sum_{i=1}^{n-\epsilon}\frac{a_i\,\mu_i^2\,d\varphi_i}{1+\lambda\,a_i^2}\Big)^2 \\ &\quad + \sum_{i=1}^{n}\frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,d\mu_i^2 + \sum_{i=1}^{n-\epsilon}\frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,\mu_i^2\,(d\varphi_i - \lambda\,a_i\,d\tau)^2 \\ &\quad + \frac{\lambda}{W\,(1-\lambda r^2)}\Big(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1+\lambda\,a_i^2}\Big)^2\,, \end{aligned} \quad (3.11)$$

where V is defined here by

$$V \equiv r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{n-\epsilon}(r^2+a_i^2) = \frac{U}{F}\,, \quad (3.12)$$

Note that obviously $a_n = 0$ in the even dimensional case, as there is no rotation associated with the last direction.

3.3 Obtaining the Inverse Metric

Note that the metric is block diagonal in the (μ_i) and the (r, τ, φ_i) sectors and so can be inverted separately.

To deal with the (r, τ, φ_i) sector, the most efficient method is to use the Kerr-Schild construction of the metric. From (3.1) and using the fact that k is null, we can write

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{U}k^\mu k^\nu\,, \quad (3.13)$$

where η here is the de Sitter metric rather than the flat metric, and we raise and lower indices with η. Since the null vector k has no components in the μ_i sector, we can regard the above equation as holding true in the (r, τ, φ_i) sector with k null here as well. Then we can explicitly perform the coordinate transformation (3.10) (or rather its inverse) on the raised metric to obtain the components of $g^{\mu\nu}$ in Boyer-Lindquist coordinates in the (r, τ, φ_i) sector.

We get the following components for the (r, τ, φ_i) sector of $g^{\mu\nu}$:

$$
\begin{aligned}
g^{\tau r} &= g^{\varphi_i r} = 0\,, \\
g^{rr} &= \frac{V-2M}{U}\,, \\
g^{\tau\tau} &= Q - \frac{4M^2}{U(1-\lambda r^2)^2(V-2M)}\,, \\
g^{\tau\varphi_i} &= \lambda a_i Q - \frac{4M^2 a_i(1+\lambda a_i^2)}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)} - \frac{2M}{U}\frac{a_i}{(1-\lambda r^2)(r^2+a_i^2)}\,, \\
g^{\varphi_i\varphi_j} &= \frac{(1+\lambda a_i^2)}{(r^2+a_i^2)\mu_i^2}\delta^{ij} + \lambda^2 a_i a_j Q + Q^{ij} \\
&\quad + \frac{4M^2 a_i a_j(1+\lambda a_i^2)(1+\lambda a_j^2)}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)}\,,
\end{aligned}
\tag{3.14}
$$

where Q and Q^{ij} are defined to be

$$
Q = -\frac{1}{W(1-\lambda r^2)} - \frac{2M}{U}\frac{1}{(1-\lambda r^2)^2}\,, \tag{3.15}
$$

$$
\begin{aligned}
Q^{ij} &= \frac{-4M^2\lambda a_i a_j[(1+\lambda a_j^2)(r^2+a_i^2)+(1+\lambda a_i^2)(r^2+a_j^2)]}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)} \\
&\quad -\frac{2M}{U}\frac{a_i a_j}{(r^2+a_i^2)(r^2+a_j^2)} - \frac{2M\lambda a_i a_j}{U(1-\lambda r^2)}\left[\frac{1}{(r^2+a_i^2)}+\frac{1}{(r^2+a_j^2)}\right] \\
&\quad -\frac{4M^2 a_i a_j[(1+\lambda a_i^2)+(1+\lambda a_j^2)]}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)}\,.
\end{aligned}
\tag{3.16}
$$

These results were compared to previously known ones in the case of $\lambda = 0$ and showed agreement [15]. Also, we used the GRTensor package for Maple explicitly to check that this is the correct inverse metric [16].

Note that the functions W and U both depend explicitly on the μ_i's. Unless the (r, τ, φ_i) sector can be decoupled from the μ sector, complete separation is unlikely. If however, all the a_i's are equal, then the functions W and U are no longer μ dependent (taking the constraint into account). With unequal values of the rotation parameters a_i, separation does not seem to be possible in this coordinate system, and it is likely that a different coordinate system might be needed to analyze separability in those cases.

We will consider the case where all rotation parameters are equal: $a_i = a$. Then we explicitly show separability. Note that since $a_n = 0$ by definition for even dimensional cases, we will restrict our attention to odd dimensional spaces. In the discussions that follow, we explicitly set all rotation parameters equal, and assume that the spacetime dimensionality is odd.

Note that the μ sector metric is completely diagonal upon assuming that the rotation parameters are equal and upon imposing the constraint. Consider the last term in equation (3.11) in the case of odd dimensions with all $a_i = a$. In this case the term reads

$$\frac{\lambda}{W\left(1-\lambda r^2\right)} \frac{\left(r^2+a^2\right)}{\left(1+\lambda a^2\right)}\left(\sum_{i=1}^{n} \mu_i d\mu_i\right)^2 . \tag{3.17}$$

However, by differentiating the constraint (3.2) we get $\sum_i \mu_i d\mu_i = 0$. Hence upon imposing the constraint this term vanishes from the metric, and the corresponding term vanishes from the inverse metric (and thus in the Hamilton-Jacobi equation.)

Now that the μ_i's are constrained by (3.2), we can use independent coordinates. Since the constraint describes a unit $(n-1)$ sphere in μ space, the natural choice is to use spherical polar coordinates. We write

$$\mu_i = \left(\prod_{j=1}^{n-i} \sin\theta_j\right) \cos\theta_{n-i+1} , \tag{3.18}$$

with the understanding that the product is one when $i = n$ and that $\theta_n = 0$. The μ sector metric can then be written as

$$ds_\mu^2 = \frac{r^2+a^2}{1+\lambda a^2} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i-1} \sin^2\theta_j\right) d\theta_i^2 , \tag{3.19}$$

again with the understanding that the product is one when $i = 1$. This diagonal metric can be easily inverted to give

$$g^{\theta_i\theta_j} = \frac{\left(1+\lambda a^2\right)}{\left(r^2+a^2\right)} \frac{1}{\left(\prod_{k=1}^{i-1} \sin^2\theta_k\right)} \delta_{ij} . \tag{3.20}$$

3.4 The Hamilton-Jacobi Equation and Separation

The Hamilton-Jacobi equation in a curved background is given by

$$-\frac{\partial S}{\partial l} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{3.21}$$

where S is the action associated with the particle and l is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parametrized by proper time.

Using (3.14) and (3.20), we write the Hamilton-Jacobi equation in odd dimensions with all rotation parameters equal as

$$\begin{aligned}
-2\frac{\partial S}{\partial l} &= \frac{4M^2}{U(1-\lambda r^2)^2(V-2M)}\left[\frac{\partial S}{\partial \tau} - \frac{a(1+\lambda a^2)}{r^2+a^2}\sum_{i=1}^{n}\frac{\partial S}{\partial \varphi_i}\right]^2 \\
&\quad - \frac{4M}{U(1-\lambda r^2)}\frac{a}{(r^2+a^2)}\sum_{i=1}^{n}\frac{\partial S}{\partial \tau}\frac{\partial S}{\partial \varphi_i} - \frac{8M^2}{U(1-\lambda r^2)^2(V-2M)}\left(\frac{\partial S}{\partial \tau}\right)^2 \\
&\quad + \frac{V-2M}{U}\left(\frac{\partial S}{\partial r}\right)^2 + \sum_{ij=1}^{n} Q^{ij}\frac{\partial S}{\partial \varphi_i}\frac{\partial S}{\partial \varphi_j} + \frac{(1+\lambda a^2)}{(r^2+a^2)}\sum_{i=1}^{n}\frac{1}{\mu_i^2}\left(\frac{\partial S}{\partial \varphi_i}\right)^2 \\
&\quad + \frac{(1+\lambda a^2)}{(r^2+a^2)}\sum_{i=1}^{n-1}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\theta_k\right)}\left(\frac{\partial S}{\partial \theta_i}\right)^2 + Q\left[\frac{\partial S}{\partial \tau} + \lambda a\sum_{i=1}^{n}\frac{\partial S}{\partial \varphi_i}\right]^2
\end{aligned} \tag{3.22}$$

Note that here the μ_i are not coordinates, but simply notation defined by (3.18). The set of coordinates relevant to the problem is $(\tau, r, \varphi_i, \theta_j)$. Note also that the functions U, W, Q, and Q^{ij} are all now independent of the θ_i; i.e., in the Hamilton-Jacobi equation, the r sector has completely decoupled from the θ_i sector.

Now we can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{n} L_i\varphi_i + S_r(r) + \sum_{i=1}^{n-1} S_{\theta_i}(\theta_i). \tag{3.23}$$

τ and φ_i are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and those with rotation in the φ_i are the corresponding angular momenta L_i, all of which are conserved. Applying this ansatz to (3.22), we can separate out the overall θ dependence as

$$J_1^2 = \sum_{i=1}^{n} \left[\frac{L_i^2}{\left(\prod_{k=1}^{n-i} \sin^2\theta_k\right) \cos^2\theta_{n-i+1}} \right] + \sum_{i=1}^{n-1} \frac{1}{\left(\prod_{k=1}^{i-1} \sin^2\theta_k\right)} \left(\frac{dS_{\theta_i}}{d\theta_i}\right)^2 , \tag{3.24}$$

where J_1^2 is a constant. The separated r equation is

$$\begin{aligned} K &= m^2(r^2+a^2) + Q(r^2+a^2)\left[-E + \lambda a \sum_{i=1}^{n} L_i\right]^2 + \frac{(V-2M)(r^2+a^2)}{U}\left[\frac{dS_r}{dr}\right]^2 \\ &+ \frac{4M^2(r^2+a^2)}{U(1-\lambda r^2)^2(V-2M)}\left[E + \frac{a(1+\lambda a^2)}{r^2+a^2}\sum_{i=1}^{n} L_i\right]^2 \\ &- \frac{8M^2E^2(r^2+a^2)}{U(1-\lambda r^2)(V-2M)} + (r^2+a^2)\sum_{i,j=1}^{n} Q^{ij}L_iL_j + \frac{4MaE}{U(1-\lambda r^2)}\sum_{i=1}^{n} L_i , \end{aligned} \tag{3.25}$$

where this separation constant is $K = -(1+\lambda a^2)J_1^2$. At this point the (r, τ, φ_i) coordinates have been separated out. To show complete separation of the Hamilton-Jacobi equation we analyze the θ sector (3.24).

The pattern here is that of a Hamiltonian of a classical (non-relativistic) particle on the unit $(n-1)$ μ-sphere, with some potential dependent on the squares of the μ_i. This can easily be additively separated following the usual procedure, one angle at a time, and the pattern continues for all integers $n \geq 2$.

The separation has the following inductive form for $k = 1, \ldots, n-2$:

$$\begin{aligned} J_k^2 \sin^2\theta_k - \frac{L_{n-k+1}^2 \sin^2\theta_k}{\cos^2\theta_k} - \sin^2\theta_k \left(\frac{dS_{\theta_k}}{d\theta_k}\right)^2 &= J_{k+1}^2 , \\ J_{k+1}^2 = \sum_{i=k+1}^{n} \frac{L_{n-i+1}^2}{\left(\prod_{j=k+1}^{i-1} \sin^2\theta_j\right)\cos^2\theta_i} + \sum_{i=k+1}^{n-1} \frac{1}{\left(\prod_{j=k+1}^{i-1} \sin^2\theta_j\right)} \left(\frac{dS_{\theta_i}}{d\theta_i}\right)^2 & \end{aligned} \tag{3.26}$$

The final step of separation gives

$$J_{n-1}^2 = \frac{L_2^2}{\cos^2\theta_{n-1}} + \frac{L_1^2}{\sin^2\theta_{n-1}} + \left(\frac{dS_{\theta_{n-1}}}{d\theta_{n-1}}\right)^2 . \tag{3.27}$$

Thus, the Hamilton-Jacobi equation in odd dimensional Kerr-de Sitter space with all rotation parameters $a_i = a$ has the general separation

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{n} L_i \varphi_i + S_r(r) + \sum_{i=1}^{n-1} S_{\theta_i}(\theta_i)\,, \tag{3.28}$$

where the θ_i are the spherical polar coordinates on the unit $(n-1)$ sphere. $S_r(r)$ can be obtained by quadratures from (3.25), and the S_{θ_i} again by quadratures from (3.26) and (3.27).

3.5 The Equations of Motion

3.5.1 Derivation of the Equations of Motion

To derive the equations of motion, we will write the separated action S from the Hamilton-Jacobi equation in the following form:

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{n} L_i \varphi_i + \int^r \sqrt{R(r')}dr' + \sum_{i=1}^{n-1} \int^{\theta_i} \sqrt{\Theta_i(\theta_i')}d\theta_i' \tag{3.29}$$

where

$$\Theta_k = J_k^2 - \frac{J_{k+1}^2}{\sin^2\theta_k} - \frac{L_{n-k+1}^2}{\cos^2\theta_k}\,, \qquad k = 1, ..., n-1\,, \tag{3.30}$$

$$\begin{aligned} R \;=\; & -J_1^2 \frac{(1+\lambda a^2)U}{(V-2M)(r^2+a^2)} - \frac{QU}{(V-2M)}\left[-E + \lambda a \sum_{i=1}^{n} L_i\right]^2 \\ & -m^2 \frac{U}{(V-2M)} - \frac{4M^2}{(1-\lambda r^2)^2(V-2M)^2}\left[E + \frac{a(1+\lambda a^2)}{r^2+a^2}\sum_{i=1}^{n} L_i\right]^2 \\ & -\frac{4MaE}{(V-2M)(r^2+a^2)}\sum_{i=1}^{n} L_i - \frac{8M^2E^2}{(1-\lambda r^2)(V-2M)^2} - \frac{U}{(V-2M)}\sum_{i,j=1}^{n} Q^{ij}L_iL_j\,, \end{aligned} \tag{3.31}$$

where Q and Q^{ij} are functions of r given in (3.16) (with all $a_i = a$). For convenience, we define $J_n^2 = L_1^2$. (Note that J_n^2 is obviously not a new conserved quantity. It is simply written this way to facilitate the inductive definition given above for Θ_{n-1}).

To obtain the equations of motion, we differentiate S with respect to the parameters m^2, E, L_i, J_j^2 and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

$$\begin{aligned}
\frac{dr}{dl} &= \frac{(V-2M)\sqrt{R}}{U} \\
\frac{d\theta_i}{dl} &= \frac{(1+\lambda a^2)\sqrt{\Theta_i}}{(r^2+a^2)(\prod_{j=1}^{i-1}\sin^2\theta_j)} \qquad i=1,\ldots,n-1 \\
\frac{d\tau}{dl} &= 2Q(r^2+a^2)\left(E+\lambda a\sum_{i=1}^{n}L_i\right)-\frac{4Ma}{U(1-\lambda r^2)}\sum_{i=1}^{n}L_i \\
&\quad -\frac{8M^2(r^2+a^2)}{(1-\lambda r^2)^2(V-2M)}\left(E+\frac{a(1+\lambda a^2)}{(r^2+a^2)}\sum_{i=1}^{n}L_i\right)+\frac{16M^2E(r^2+a^2)}{U(1-\lambda r^2)(V-2M)}\,.
\end{aligned} \tag{3.32}$$

We can obtain n more equations of motion which give the $\frac{d\varphi_i}{dl}$ in terms of the r, θ_j coordinates by differentiating S with respect to the angular momenta L_i. However, these equations are not particularly illuminating, but can be written out explicitly if necessary by following this procedure.

3.5.2 Analysis of the Radial Equation

The worldline of particles in the Kerr-de Sitter backgrounds considered above are completely specified by the values of the conserved quantities E, L_i, J_j^2, and by the initial values of the coordinates. We will consider particle motion in the black hole exterior. Allowed regions of particle motion necessarily need to have positive value for the quantity R, owing to equation (3.32). We determine some of the possibilities of the allowed motion.

At large r, the dominant contribution to R, in the case of $\lambda = 0$, is $E^2 - m^2$. Thus we can say that for $E^2 < m^2$, we cannot have unbounded orbits, whereas for $E^2 > m^2$, such orbits are possible. For the case of nonzero λ, the dominant term at large r in R (or rather the slowest decaying term) is $\frac{m^2}{\lambda r^2}$. Thus in the case of the Kerr-anti-de Sitter background, only bound orbits are possible, whereas in the Kerr-de Sitter backgrounds, both unbounded and bound orbits may be possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose R as a quadratic in E as follows:

$$R = \alpha E^2 - 2\beta E + \gamma\,, \tag{3.33}$$

where

$$\begin{aligned}
\alpha &= -\frac{QU}{V-2M} - \frac{4M^2}{(1-\lambda r^2)^2(V-2M)^2} - \frac{8M^2}{(1-\lambda r^2)(V-2M)^2}, \\
\beta &= \left(\frac{QU\lambda a}{V-2M} + \frac{4M^2 a(1+\lambda a^2)}{(1-\lambda r^2)^2(V-2m)^2(r^2+a^2)} + \frac{2Ma}{(V-2M)(r^2+a^2)}\right)\sum_{i=1}^{n} L_i, \\
\gamma &= -\frac{J_1^2(1+\lambda a^2)U}{(V-2M)(r^2+a^2)} - \frac{QU\lambda^2 a^2}{V-2M}\left(\sum_{i=1}^{n} L_i\right)^2 - \frac{M^2 U}{V-2M} \\
&\quad -\frac{4M^2 a^2(1+\lambda a^2)^2}{(1-\lambda r^2)^2(V-2M)^2(r^2+a^2)^2}\left(\sum_{i=1}^{n} L_i\right)^2 - \frac{U}{V-2M}\sum_{ij=1}^{n} Q^{ij} L_i L_j . \qquad (3.34)
\end{aligned}$$

The turning points for trajectories in the radial motion (defined by the condition $R = 0$) are given by $E = V_{\pm}$ where

$$V_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha} . \qquad (3.35)$$

These functions, called the effective potentials [15], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.

3.5.3 Analysis of the Angular Equations

Another class of interesting motions possible describes motion at a constant value of θ_i. These motions are described by the simultaneous equations

$$\Theta_i(\theta_i = \vartheta_i) = \frac{d\Theta_i}{d\theta_i}(\theta_i = \vartheta_i) = 0 , \qquad (3.36)$$

where ϑ_i is the constant value of θ_i along this trajectory. These equations can be explicitly solved to give the relations

$$\begin{aligned}
\frac{J_{i+1}^2}{\sin^4\theta_i} &= \frac{L_{n-i-1}^2}{\cos^4\theta_i}, \\
J_i^2 &= \frac{J_{i+1}^2}{\sin^2\theta_i} + \frac{L_{n-i+1}^2}{\cos^2\theta_i}, \qquad i = 1, \dots, n-1 , \qquad (3.37)
\end{aligned}$$

where, as before, $J_n^2 = L_1^2$. Note that if $\vartheta_i = 0$, then $J_{i+1}^2 = 0$, and if $\vartheta_i = \pi/2$, then $L_{n-i+1}^2 = 0$.

Examining Θ_k in the general case, $\theta_k = 0$ can only be reached if $J_{k+1} = 0$, and

$\theta_k = \pi/2$ can be only be reached if $L_{n-k+1} = 0$. The orbit will completely be in the subspace $\theta_k = 0$ only if $J_k^2 = L_{n-i+1}^2$ and will completely be in the subspace $\theta_k = \pi/2$ only if $J_k^2 = J_{k+1}^2$.

Again these equations are in a form suitable for numerical analysis for specific values of the black hole and particle parameters.

3.6 Dynamical Symmetry

The general class of metrics discussed here are stationary and "axisymmetric"; i.e., $\partial/\partial\tau$ and $\partial/\partial\varphi_i$ are Killing vectors and have associated conserved quantities, $-E$ and L_i. In general if ξ is a Killing vector, then $\xi^\mu p_\mu$ is a conserved quantity, where p is the momentum. Note that this quantity is first order in the momenta.

With the assumption of odd dimensions and equality of all the a_i's, the spacetime acquires additional dynamical symmetry and more Killing vectors are generated. By setting the rotation parameters a_i's equal, we have complete symmetry between the various planes of rotation, and we can "rotate" one into another. The vectors that generate these transformations are the required Killing vectors. We will construct these explicitly. Parametrize the rotation planes as follows:

$$\begin{aligned} x_i &= r\mu_i \cos\varphi_i = r\left(\prod_{j=1}^{n-i}\sin\theta_j\right)\cos\theta_{n-i+1}\cos\varphi_i\,, \\ y_i &= r\mu_i \sin\varphi_i = r\left(\prod_{j=1}^{n-i}\sin\theta_j\right)\cos\theta_{n-i+1}\sin\varphi_i\,, \end{aligned} \tag{3.38}$$

again with the understanding that the product equals one when $i = n$ and that $\theta_n = 0$.

Define the rotation generators on the planes as

$$L_{ab} = a\partial_b - b\partial_a\,, \tag{3.39}$$

where a and b can be any x^i or y^j . The case of $a = x^i, b = y^i$ for same i is not interesting, as it simply represents rotation in φ_i, which is already known to generate a Killing vector. The L_{ab} themselves are obviously not Killing vectors (aside from the trivial cases just mentioned), but the combinations

$$\xi_{ij} = L_{x^i x^j} + L_{y^i y^j}\,, \qquad \rho_{ij} = L_{x^i y^j} + L_{x^j y^i}\,. \tag{3.40}$$

are Killing vectors. Explicit expressions for these in polar coordinates in the case of $n = 2$ can be found in [17] [18].

These additional Killing vectors exist, since the symmetry of the spacetime has been greatly enhanced by the equality of the rotation parameters. The $(U(1))^n$ spatial rotation symmetry, where each $U(1)$ is the rotational symmetry in one of the planes, has been increased to a $U(n)$ symmetry. This follows from the fact that we now have the additional symmetry of being able to rotate planes into one another.

The separation constants K in (3.25) and J_i^2 in (3.24) are conserved quantities that are quadratic in the associated momenta. So these quantities must be derived from a rank two Killing tensor $K^{\mu\nu}$ [19]. We will work with the J_i^2. (We can ignore K since it only differs from J_1^2 by a constant factor.) Any conserved quantity A that is second order in momenta is constructed from a Killing tensor as

$$A = K^{\mu\nu} p_\mu p_\nu = K^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \,. \tag{3.41}$$

Since the Hamilton-Jacobi equation can be fully separated, we should be able to construct Killing tensors explicitly. It turns out however that these Killing tensors are not irreducible; i.e., they can be constructed as linear combinations of tensor products of the Killing vectors present due to the increased symmetry.

Comparing (3.24), (3.26) and (3.27) with (3.41), where the conserved quantities are J_i^2, we can obtain the following Killing tensors:

$$\begin{aligned} K_{n-1}^{\mu\nu} &= \frac{1}{\sin^2\theta_{n-1}} \delta^\mu_{\varphi_1} \delta^\nu_{\varphi_1} + \frac{1}{\cos^2\theta_{n-1}} \delta^\mu_{\varphi_2} \delta^\nu_{\varphi_2} + \delta^\mu_{\theta_{n-1}} \delta^\nu_{\theta_{n-1}} \,, \\ K_k^{\mu\nu} &= \frac{1}{\sin^2\theta_k} K_{k+1}^{\mu\nu} + \frac{1}{\cos^2\theta_k} \delta^\mu_{\varphi_{n-k+1}} \delta^\nu_{\varphi_{n-k+1}} + \delta^\mu_{\theta_k} \delta^\nu_{\theta_k} \,, \quad k = 1, ..., n-2 \,, \end{aligned} \tag{3.42}$$

which can be written as

$$\begin{aligned} K_{n-k} &= \sum_{i=1}^{k+1} \partial_{\varphi_i} \otimes \partial_{\varphi_i} - \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} sym(\partial_{\varphi_i} \otimes \partial_{\varphi_j}) \\ &\quad + \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} \xi_{ij} \otimes \xi_{ij} + \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} \rho_{ij} \otimes \rho_{ij} \,, \quad k = 1, ..., n-1 \,, \end{aligned} \tag{3.43}$$

where $J_i^2 = K_i^{\mu\nu} p_\mu p_\nu$.

Therefore, as we can see from the form of the Killing tensors, they can explicitly be obtained from quadratic combinations of the Killing vectors ∂_{φ_i}, ξ_{ij}, and ρ_{ij}.

This is a demonstration of the fact that in this case separation of the Hamilton-

Jacobi equation is possible due to the enlargement of the symmetry group in the case of all $a_i = a$.

3.7 The Scalar Field Equation

Consider a scalar field Ψ with the action

$$S[\Psi] = -\frac{1}{2}\int d^D x\sqrt{-g}((\nabla\Psi)^2 + \alpha R\Psi^2 + m^2\Psi^2)\,, \tag{3.44}$$

where we have included a curvature dependent coupling. However, in the Kerr-(anti) de Sitter background, $R = \lambda$ is constant. As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein-Gordon equation in this background. We will simply set $\alpha = 0$ in the following. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = m^2\Psi\,. \tag{3.45}$$

As discussed by Carter [20], the assumption of separability of the Klein-Gordon equation usually implies separability of the Hamilton-Jacobi equation. Conversely, if the Hamilton-Jacobi equation does not separate, the Klein-Gordon equation seems unlikely to separate. We can also see this explicitly (as in the case of the Hamilton-Jacobi equation), since the (r, τ, φ_i) sector has coefficients in the equations that explicitly depend on the μ_i except in the case of all $a_i = a$. Thus, we will once again restrict our attention to the case of all $a_i = a$ in odd dimensional spacetimes.

Once again, we impose the constraint (3.2) and decompose the μ_i in terms of spherical coordinates as in (3.18). We calculate the determinant of the metric to be

$$g = -\frac{r^2(r^2+a^2)^{2n-2}}{(1+\lambda a^2)^{2n}}\prod_{j=1}^{n-1}\sin^{4n-4j-2}\theta_j\cos^2\theta_j\,. \tag{3.46}$$

For convenience we write $g = -PA$, where

$$P = \frac{r^2(r^2+a^2)^{2n-2}}{(1+\lambda a^2)^{2n}}\,, \qquad A = \prod_{j=1}^{n-1}\sin^{4n-4j-2}\theta_j\cos^2\theta_j\,. \tag{3.47}$$

Then the Klein-Gordon equation in this background (3.45) becomes

$$\begin{aligned} m^2\Psi &= Q\left[\frac{\partial}{\partial\tau}+\lambda a\sum_{i=1}^{n}\frac{\partial}{\partial\varphi_i}\right]^2\Psi+\frac{1}{\sqrt{P}}\partial_r\left(\sqrt{P}\frac{(V-2M)}{U}\frac{\partial\Psi}{\partial r}\right) \\ &+\frac{4M^2}{U(1-\lambda r^2)^2(V-2M)}\left[\frac{\partial}{\partial\tau}-\frac{a(1+\lambda a^2)}{r^2+a^2}\sum_{i=1}^{n}\frac{\partial}{\partial\varphi_i}\right]^2\Psi \\ &-\frac{4M}{U(1-\lambda r^2)}\frac{a}{(r^2+a^2)}\sum_{i=1}^{n}\frac{\partial^2\Psi}{\partial\tau\partial\varphi_i}-\frac{8M^2}{U(1-\lambda r^2)^2(V-2M)}\left(\frac{\partial^2\Psi}{\partial\tau^2}\right) \\ &+\sum_{ij=1}^{n}Q^{ij}\frac{\partial^2\Psi}{\partial\varphi_i\partial\varphi_j}+\frac{(1+\lambda a^2)}{(r^2+a^2)}\sum_{i=1}^{n}\frac{1}{\mu_i^2}\left(\frac{\partial^2\Psi}{\partial\varphi_i^2}\right) \\ &+\frac{1}{\sqrt{A}}\sum_{i,j=1}^{n-1}\partial_{\theta_i}\left(\sqrt{A}g^{\theta_i\theta_j}\frac{\partial\Psi}{\partial\theta_j}\right). \end{aligned} \tag{3.48}$$

We attempt the usual multiplicative separation for Ψ in the following form:

$$\Psi=e^{-iEt}e^{i\sum_i L_i\varphi_i}\Psi_\theta(\theta_1,...,\theta_{n-1})\Phi_r(r)\,. \tag{3.49}$$

Then the Klein-Gordon equation simplifies to give the following ordinary differential equation in r for $\Phi_r(r)$:

$$\begin{aligned} m^2\Phi_r &= -Q\left[E-\lambda a\sum_{i=1}^{n}L_i\right]^2\Phi_r+\frac{1}{\sqrt{P}}\frac{d}{dr}\left(\sqrt{P}\frac{(V-2M)}{U}\frac{d\Phi_r}{dr}\right)+\frac{(1+\lambda a^2)}{(r^2+a^2)}K_1\Phi_r \\ &-\frac{4MaE}{U(1-\lambda r^2)(r^2+a^2)}\sum_{i=1}^{n}L_i\Phi_r+\frac{8M^2E^2}{U(1-\lambda r^2)^2(V-2M)}\Phi_r \\ &-\sum_{ij=1}^{n}Q^{ij}L_iL_j\Phi_r-\frac{4M^2}{U(1-\lambda r^2)^2(V-2M)}\left[E+\frac{a(1+\lambda a^2)}{r^2+a^2}\sum_{i=1}^{n}L_i\right]^2\Phi_r\,. \end{aligned} \tag{3.50}$$

We have separated all the θ_i dependence into the separation constant K_1 given by

$$K_1=\frac{1}{\Psi_\theta}\sum_{i=1}^{n}\left[-\frac{L_i^2}{\mu_i^2}\right]+\sum_{i=1}^{n-1}\frac{1}{\Psi_\theta\sqrt{A}}\partial_{\theta_i}\left(\sqrt{A}g^{\theta_i\theta_i}\frac{\partial\Psi_\theta}{\partial\theta_i}\right)\,, \tag{3.51}$$

where we have used the fact that $g^{\theta_i\theta_j}$ is diagonal, and that the μ_i are functions of the θ_j given by (3.18).

Equation (3.50) separates out the r dependence of the Klein-Gordon equation, and gives the function $\Phi_r(r)$ when the differential equation is solved. We can also completely separate the θ_i sector. Again, assume a multiplicative separation of the form

$$\Psi_\theta = \Phi_{\theta_1}(\theta_1)...\Phi_{\theta_{n-1}}(\theta_{n-1}) \,. \tag{3.52}$$

The θ separation then reads as

$$K_1 = \sum_{i=1}^{k-1} A_i + \frac{K_k}{\prod_{j=1}^{k-1} \sin^2\theta_j} \,, \quad k = 1, ..., n-1 \,, \tag{3.53}$$

where

$$\begin{aligned} A_i = & \frac{1}{\Phi_{\theta_i} \cos\theta_i \sin^{2n-2i-1}\theta_i \prod_{k=1}^{i-1} \sin^2\theta_k} \frac{d}{d\theta_i}\left(\cos\theta_i \sin^{2n-2i-1}\theta_i \frac{d\Phi_{\theta_i}}{d\theta_i}\right) \\ & - \frac{L_{n-i+1}^2}{\cos^2\theta_i \prod_{j=1}^{i-1} \sin^2\theta_j} \,. \end{aligned} \tag{3.54}$$

Then we inductively have the complete separation of the θ_i dependence as

$$K_k = \frac{K_{k+1}}{\sin^2\theta_k} - \frac{L_{n-k+1}^2}{\cos^2\theta_k} + \frac{1}{\Phi_{\theta_k} \cos\theta_k \sin^{2n-2k-1}\theta_k} \frac{d}{d\theta_k}\left(\cos\theta_k \sin\theta_k \frac{d\Phi_{\theta_k}}{d\theta_k}\right) \,, \tag{3.55}$$

where $k = 1, ..., n-1$, and we use the convention $K_n = -L_1^2$.

As a result we can write the complete separation of the Klein-Gordon equation (3.48) in the Kerr-de Sitter background in odd dimensions with all rotation parameters equal as

$$\Psi = e^{-iEt} e^{i\sum_i L_i \varphi_i} \Phi_{\theta_1}(\theta_1)...\Phi_{\theta_{n-1}}(\theta_{n-1}) \Phi_r(r) \,, \tag{3.56}$$

where $\Phi(r)$ is obtained from (3.50), and the Φ_{θ_i}'s are the decomposition of the μ sector into eigenmodes in independent coordinates θ_i on the μ sphere.

Note that the separation of the Klein-Gordon equation in this geometry is again due to the fact that the symmetry of the space has been enlarged. (We can explicitly see the role of the Killing vectors again in the separation of the r equation from the θ sector in a very similar fashion to that in the Hamilton-Jacobi equation [20]).

Conclusions

We studied the separability properties of the Hamilton-Jacobi and the Klein-Gordon equations in the Kerr-de Sitter backgrounds. Separation in Boyer-Lindquist coordinates seems to be possible only for the case of an odd number of spacetime dimensions with all rotation parameters equal. This is possible due to the enlarged dynamical symmetry of the spacetime. We derive expressions for the Killing vectors that correspond to the additional symmetries. We also show that integrals of motion are obtained from reducible Killing tensors, which are themselves constructed from the angular Killing vectors. Thus we demonstrate the separability of the Hamilton-Jacobi and the Klein-Gordon equations as a direct consequence of the enhancement of symmetry. We also derive first-order equations of motion for classical particles in these backgrounds, and analyze the properties of some special trajectories.

Future work in this direction could include finding a suitable coordinate system to permit possible separation in an even number of spacetime dimensions. Different coordinates might also be required to study the cases of unequal rotation parameters, since separation does not seem likely in Boyer-Lindquist coordinates. The study of higher-spin field equations in these backgrounds could also prove to be of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results.

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Chapter 4

Two Parameter Myers-Perry Metrics

4.1 Introduction

Solutions of the vacuum Einstein equations describing black hole solutions in higher dimensions are currently of great interest. This is mainly due to a number of recent developments in high energy physics. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (e.g. the hierarchy problem) [1] [2] [3]. These models allow for the existence of higher dimensional black holes which can be described classically. Also of interest in these models is the possibility of mini black hole production in high energy particle colliders which, if they occur, provide a window into non-perturbative gravitational physics [4] [5].

Superstring and M-Theory, which call for additional spacetime dimensions, naturally incorporate black hole solutions in higher dimensions (10 or 11). P-branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Black hole solutions in superstring theory are particularly relevant since they can be described as solitonic objects. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6] [7].

Astrophysically relevant black hole spacetimes are, to a very good approximation, described by the Kerr metric [8]. The most natural generalization of the Kerr metric to higher dimensions, for zero cosmological constant, is given by the Myers-Perry construction [9]. (For a recent generalization with a cosmological constant, see [10], but a nonzero cosmological constant seems to thwart the type of separability demonstrated in the present paper, so here we shall take the cosmological constant to be zero.) The Myers-Perry metric also does not have charge, but since charged black holes are unlikely to occur in nature, we expect the Myers-Perry type black holes to be the most relevant type in spacetimes with extra dimensions.

In this paper, we analyze the separability of the Hamilton-Jacobi equation in Myers-Perry black hole backgrounds in all dimensions. We explicitly perform the separation in the case where there are only two sets of equal rotation parameters describing the black hole. We use this explicit separation to obtain first-order equations of motion for both massive and massless particles in these backgrounds. The equations are obtained in a form that could be used for numerical study.

We study the Klein-Gordon equation describing the propagation of a massive scalar field in this spacetime. Separation is again explicitly shown for the case of two sets of equal black hole rotation parameters. We construct the separation of both equations explicitly in these cases. We also construct Killing vectors, which exist due to the additional symmetry, and which permit the separation of these equations.

4.2 Overview of the Myers-Perry Metrics

The Myers-Perry metrics are vacuum solutions of Einstein's equations describing general rotating black hole spacetimes. The Kerr black hole in four dimensions needs an axis of rotation specified. In higher dimensions, this specification is no longer possible. Instead, we provide rotation parameters specifying rotations in various planes. As such, we use the construction described below.

We introduce $n = [D/2]$ coordinates μ_i subject to the constraint

$$\sum_{i=1}^{n} \mu_i^2 = 1 \,, \tag{4.1}$$

together with $N = [(D-1)/2]$ azimuthal angular coordinates ϕ_i, the radial coordinate r, and the time coordinate τ. When the total spacetime dimension D is odd, $D = 2n+1 = 2N+1$, there are $N = n$ azimuthal coordinates ϕ_i, each with period 2π. If D is even, $D = 2n = 2N+2$, there are only $N = n-1$ azimuthal coordinates ϕ_i. Define ϵ to be 1 for even D, and 0 for odd D, so $N = n - \epsilon$.

In Boyer-Lindquist coordinates in D dimensions, the Myers-Perry metrics are given by

$$\begin{aligned} ds^2 &= -d\tau^2 + \frac{U\,dr^2}{V-2M} + \frac{2M}{U}\left(d\tau - \sum_{i=1}^{n-\epsilon} a_i\,\mu_i^2\,d\phi_i\right)^2 \\ &\quad + \sum_{i=1}^{n}\left(r^2+a_i^2\right)\,d\mu_i^2 + \sum_{i=1}^{n-\epsilon}\left(r^2+a_i^2\right)\,\mu_i^2\,d\phi_i^2\,, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} U &= r^\epsilon \sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n-\epsilon}(r^2+a_j^2)\,, \\ F &= r^2\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\,, \\ V &= r^{\epsilon-2}\prod_{i=1}^{n-\epsilon}(r^2+a_i^2) = \frac{U}{F}\,. \end{aligned} \tag{4.3}$$

Note that obviously $a_n = 0$ in the even dimensional case, as there is no rotation associated with the last direction.

Since that the metric is block diagonal in the (μ_i) and the (r, τ, ϕ_i) sectors, these sectors can be inverted separately. To deal with the (r, τ, ϕ_i) sector, the most efficient method is to use the Kerr-Schild construction of the metric. For details on construction of the inverse metric using the Kerr-Schild form, see [11].

We get the following components for the (r, τ, ϕ_i) sector of $g^{\mu\nu}$:

$$\begin{aligned} g^{\tau r} &= g^{\phi_i r} = 0\,, \\ g^{rr} &= \frac{V-2M}{U}\,, \\ g^{\tau\tau} &= -1 - \frac{2MV}{U(V-2M)}\,, \\ g^{\tau\phi_i} &= -\frac{2MVa_i}{U(V-2M)(r^2+a_i^2)}\,, \\ g^{\phi_i\phi_j} &= \frac{1}{(r^2+a_i^2)\mu_i^2}\delta^{ij} - \frac{2MVa_ia_j}{U(V-2M)(r^2+a_i^2)(r^2+a_j^2)}\,. \end{aligned} \tag{4.4}$$

Note that the function U depends explicitly on the μ_i's. Unless the (r, τ, ϕ_i) sector can be decoupled from the μ sector, complete separation is unlikely. If however, all the $a_i = a$ for some non-zero value a, then the U are no longer μ dependent (taking the constraint into account) and separation seems likely. Note, however, that in this case we cannot deal with even dimensional spacetimes, since $a_n = 0$ is different from the other $a_i = a$.

We will actually work with a much more general case, in which separation works in both even and odd dimensional spacetimes. We consider the situation in which the set of rotation parameters a_i take on at most only two distinct values a and b ($a = b$ can be obtained as a special case). In even dimensions at least one of these values must be zero, since $a_n = 0$. As such in even dimensions we take $b = 0$ and a to be some (possibly different) value. In the odd dimensional case, there are no restrictions on the values of a and b. We adopt the convention

$$a_i = a \qquad \text{for} \quad i = 1, ..., m \quad , \qquad b_j = b \qquad \text{for} \quad j = 1, ..., p\,, \tag{4.5}$$

where $m + p = N + \epsilon = n$.

Since the μ_i's are constrained by (4.1), we need to use suitable independent coordinates instead. We use the following decomposition of the μ_i:

$$\mu_i = \lambda_i \sin\theta \quad \text{for} \quad i = 1, ..., m \quad , \qquad \mu_{j+m} = \nu_j \cos\theta \quad \text{for} \quad j = 1, ..., p\,, \tag{4.6}$$

where the λ_i and ν_j have to satisfy the constraints

$$\sum_{i=1}^{m} \lambda_i^2 = 1 \qquad , \qquad \sum_{j=1}^{p} \nu_j^2 = 1\,. \tag{4.7}$$

Since these constraints describe unit $(m-1)$ and $(p-1)$ dimensional spheres in the λ and ν spaces respectively, the natural choice is to use two sets of spherical polar coordinates. We write

$$\begin{aligned} \lambda_i &= \left(\prod_{k=1}^{m-i} \sin\alpha_k\right) \cos\alpha_{m-i+1}\,, \\ \nu_j &= \left(\prod_{k=1}^{p-j} \sin\beta_k\right) \cos\beta_{p-j+1}\,, \end{aligned} \tag{4.8}$$

with the understanding that the products are one when $i = m$ or $j = p$ respectively, and that $\alpha_m = 0$ and $\beta_p = 0$.

The μ sector metric can then be written as

$$ds_\mu^2 = \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta \sum_{i=1}^{m-1} \left(\prod_{k=1}^{i-1} \sin^2\alpha_k\right) d\alpha_i^2$$

$$+(r^2+b^2)\cos^2\theta\sum_{j=1}^{p-1}\left(\prod_{k=1}^{j-1}\sin^2\beta_k\right)d\beta_j^2\,, \tag{4.9}$$

again with the understanding that the products are one when $i = 1$ or $j = 1$. We use the definition

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,. \tag{4.10}$$

This diagonal metric can be easily inverted to give

$$\begin{aligned}
g^{\theta\theta} &= \frac{1}{\rho^2}\,, \\
g^{\alpha_i\alpha_j} &= \frac{1}{(r^2+a^2)\sin^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\alpha_k\right)}\delta_{ij}\,, \qquad i,j = 1,\ldots,m\,, \\
g^{\beta_i\beta_j} &= \frac{1}{(r^2+b^2)\cos^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\beta_k\right)}\delta_{ij}\,, \qquad i,j = 1,\ldots,p\,.
\end{aligned} \tag{4.11}$$

For the case of two sets of rotation parameters that we consider here, the following symbols will be extremely useful in addition to ρ^2:

$$\begin{aligned}
\Delta &= V - 2M\,, \\
\Pi &= \prod_{i=1}^{N}(r^2+a_i^2) = (r^2+a^2)^m(r^2+b^2)^{p-\epsilon}\,, \\
Z &= (r^2+a^2)(r^2+b^2)\,.
\end{aligned} \tag{4.12}$$

Note that these are functions of the variable r only. We note that $U = \frac{r^\epsilon\rho^2\Pi}{Z}$.

4.3 The Hamilton-Jacobi Equation and Separation

The Hamilton-Jacobi equation in a curved background is given by

$$-\frac{\partial S}{\partial l} = H = \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\,, \tag{4.13}$$

where S is the action associated with the particle and l is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parametrized by proper time.

We can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m} \Phi_i \phi_i + \sum_{i=1}^{p} \Psi_i \phi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1} S_{\beta_i}(\beta_i) . \quad (4.14)$$

τ and ϕ_i are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and the conserved quantity associated with rotation in each ϕ_i is the corresponding angular momentum Φ_i or Ψ_j. We also adopt the convention that $\Psi_p = 0$ in an even number of spacetime dimensions.

Using (4.4), (4.11), (4.12), and (4.14) we write the Hamilton-Jacobi equation (4.13) as

$$\begin{aligned}
-m^2 = & - \left(1 + \frac{2MZ}{r^2\rho^2\Delta}\right) E^2 + \frac{2Ma(r^2+b^2)}{r^2\rho^2\Delta} \sum_{i=1}^{m} E\Phi_i + \frac{2Ma(r^2+a^2)}{r^2\rho^2\Delta} \sum_{i=1}^{p} E\Psi_i \\
& + \frac{\Delta Z}{r^\epsilon \rho^2 \Pi} \left(\frac{dS_r}{dr}\right)^2 + \frac{1}{(r^2+a^2)} \sum_{i=1}^{m} \frac{\Phi_i^2}{\mu_i^2} + \frac{1}{(r^2+a^b)} \sum_{i=1}^{p} \frac{\Psi_i^2}{\mu_{i+m}^2} \\
& - \frac{2Ma^2(r^2+b^2)}{\Delta r^2 \rho^2 (r^2+a^2)} \sum_{i=1}^{m}\sum_{j=1}^{m} \Phi_i \Phi_j - \frac{2Mb^2(r^2+a^2)}{\Delta r^2\rho^2(r^2+b^2)} \sum_{i=1}^{p}\sum_{j=1}^{p} \Psi_i\Psi_j \\
& - \frac{4Mab}{\Delta r^2 \rho^2} \sum_{i=1}^{m}\sum_{j=1}^{p} \Phi_i \Psi_j + \sum_{i=1}^{m-1} \frac{1}{(r^2+a^2)\sin^2\theta \prod_{k=1}^{i-1}\sin^2\alpha_k} \left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2 \\
& + \sum_{i=1}^{p-1} \frac{1}{(r^2+b^2)\cos^2\theta \prod_{k=1}^{i-1}\sin^2\beta_k} \left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2 - \frac{1}{\rho^2}\left(\frac{dS_\theta}{d\theta}\right)^2 . \qquad (4.15)
\end{aligned}$$

Note that here the μ_i are not coordinates, but simply quantities defined by (4.6). We continue to use the convention defined for products of $\sin^2\alpha_i$ and $\sin^2\beta_j$ defined earlier. Separate the α_i and β_j coordinates from the Hamilton-Jacobi equation via

$$\begin{aligned}
J_1^2 &= \sum_{i=1}^{m} \left[\frac{\Phi_i^2}{\lambda_i^2} + \frac{1}{\prod_{k=1}^{i-1}\sin^2\alpha_k} \left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2 \right] , \\
L_1^2 &= \sum_{i=1}^{p} \left[\frac{\Psi_i^2}{\nu_i^2} + \frac{1}{\prod_{k=1}^{i-1}\sin^2\beta_k} \left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2 \right] , \qquad (4.16)
\end{aligned}$$

where J_1^2 and L_1^2 are separation constants. Then the remaining terms in the Hamilton-Jacobi equations can be explicitly separated to give ordinary differential equations for r

and θ:

$$\begin{aligned}
K = & - m^2r^2 + E^2\left(r^2 + \frac{2MZ}{r^2\Delta}\right) - \frac{\Delta Z}{r^\epsilon\Pi}\left(\frac{dS_r}{dr}\right)^2 - \frac{2Ma(r^2+b^2)}{r^2\Delta}\sum_{i=1}^{m} E\Phi_i \\
& - \frac{2Mb(r^2+a^2)}{r^2\Delta}\sum_{i=1}^{p} E\Psi_i + \frac{2Ma^2(r^2+b^2)}{\Delta r^2(r^2+a^2)}\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_i\Phi_j + \frac{4Mab}{\Delta r^2}\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_i\Psi_j \\
& + \frac{2Mb^2(r^2+a^2)}{\Delta r^2(r^2+b^2)}\sum_{i=1}^{p}\sum_{j=1}^{p}\Psi_i\Psi_j - \frac{r^2+b^2}{r^2+a^2}J_1^2 - \frac{r^2+a^2}{r^2+b^2}L_1^2, \qquad (4.17)\\
K = & \quad (m^2 - E^2)(a^2\cos^2\theta + b^2\sin^2\theta) + \left(\frac{dS_\theta}{d\theta}\right)^2 + \cot^2\theta J_1^2 + \tan^2\theta L_1^2, \qquad (4.18)
\end{aligned}$$

where K is a separation constant.

In order to show complete separation of the Hamilton-Jacobi equation, we analyze the α and β sectors in (4.16) and demonstrate separation of the individual α_i and β_j. The pattern here is that of a Hamiltonian of classical (non-relativistic) particles on the unit $(m-1)$-α and the unit $(p-1)$-β spheres, with some potential dependent on the squares of the μ_i. This can easily be additively separated following the usual procedure, one angle at a time, and the pattern continues for all integers $m, p \geq 2$.

The separation has the following inductive form for $k = 1, ..., m-2$, and $l = 1, ..., p-2$:

$$\begin{aligned}
\left(\frac{dS_{\alpha_k}}{d\alpha_k}\right)^2 &= J_k^2 - \frac{J_{k+1}^2}{\sin^2\alpha_k} - \frac{\Phi_{m-k+1}^2}{\cos^2\alpha_k}, \\
\left(\frac{dS_{\alpha_{m-1}}}{d\alpha_{m-1}}\right)^2 &= J_{m-1}^2 - \frac{\Phi_1^2}{\sin^2\alpha_{m-1}} - \frac{\Phi_2^2}{\cos^2\alpha_{m-1}}, \\
\left(\frac{dS_{\beta_l}}{d\beta_l}\right)^2 &= L_l^2 - \frac{L_{l+1}^2}{\sin^2\beta_l} - \frac{\Psi_{p-l+1}^2}{\cos^2\beta_l}, \\
\left(\frac{dS_{\beta_{p-1}}}{d\beta_{p-1}}\right)^2 &= L_{p-1}^2 - \frac{\Psi_1^2}{\sin^2\beta_{p-1}} - \frac{\Psi_2^2}{\cos^2\beta_{p-1}}. \qquad (4.19)
\end{aligned}$$

Thus, the Hamilton-Jacobi equation in the Myers-Perry rotating black hole background with two sets of possibly unequal rotation parameters has the general separation

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m}\Phi_i\phi_i + \sum_{i=1}^{p}\Psi_i\phi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1}S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1}S_{\beta_i}(\beta_i), \quad (4.20)$$

where the α_i and β_j are the spherical polar coordinates on the unit $(m-1)$ and unit

$(p-1)$ spheres respectively. $S_r(r)$ can be obtained by quadratures from (4.17). $S_\theta(\theta)$ by quadratures from (4.18), and the $S_{\alpha_i}(\alpha_i)$ and the $S_{\beta_j}(\beta_j)$ again by quadratures from (4.19).

4.4 The Equations of Motion

4.4.1 Derivation of the Equations of Motion

To derive the equations of motion, we will write the separated action S from the Hamilton-Jacobi equation in the following form:

$$\begin{aligned} S &= \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m}\Phi_i\phi_i + \sum_{i=1}^{p}\Psi_i\phi_i + \int^r \sqrt{R(r')}dr' + \int^\theta \sqrt{\Theta(\theta')}d\theta' \\ &+ \sum_{i=1}^{m-1}\int^{\alpha_i}\sqrt{A_i(\alpha_i')}d\alpha_i' + \sum_{i=1}^{p-1}\int^{\beta_i}\sqrt{B_i(\beta_i')}d\beta_i' , \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} A_k &= J_k^2 - \frac{J_{k+1}^2}{\sin^2\alpha_k} - \frac{\Phi_{m-k+1}^2}{\cos^2\alpha_k} , \qquad k = 1, ..., m-2 , \\ A_{m-1} &= J_{m-1}^2 - \frac{\Phi_1^2}{\sin^2\alpha_{m-1}} - \frac{\Phi_2^2}{\cos^2\alpha_{m-1}} , \\ B_k &= L_k^2 - \frac{L_{k+1}^2}{\sin^2\beta_k} - \frac{\Psi_{p-k+1}^2}{\cos^2\beta_k} , \qquad k = 1, ..., p-2 , \\ B_{p-1} &= L_{p-1}^2 - \frac{\Psi_1^2}{\sin^2\beta_{p-1}} - \frac{\Psi_2^2}{\cos^2\beta_{p-1}} , \end{aligned} \tag{4.22}$$

Θ is obtained from (4.18) as

$$\Theta = K + (E^2 - m^2)(a^2\cos^2\theta + b^2\sin^2\theta) - \cot^2\theta J_1^2 - \tan^2\theta L_1^2 , \tag{4.23}$$

and R is obtained from (4.17) as

$$\begin{aligned} \frac{\Delta Z}{\Pi r^\epsilon}R &= (E^2 - m^2)r^2 + \frac{2MZ}{r^2\Delta}E^2 - \frac{2Ma(r^2+b^2)}{r^2\Delta}\sum_{i=1}^{m}E\Phi_i \\ &- \frac{2Mb(r^2+a^2)}{r^2\Delta}\sum_{i=1}^{p}E\Psi_i + \frac{2Ma^2(r^2+b^2)}{\Delta r^2(r^2+a^2)}\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_i\Phi_j + \frac{4Mab}{\Delta r^2}\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_i\Psi_j \end{aligned}$$

$$+ \quad \frac{2Mb^2(r^2+a^2)}{\Delta r^2(r^2+b^2)} \sum_{i=1}^{p}\sum_{j=1}^{p} \Psi_i \Psi_j - \frac{r^2+b^2}{r^2+a^2} J_1^2 - \frac{r^2+a^2}{r^2+b^2} L_1^2 \,. \tag{4.24}$$

To obtain the equations of motion, we differentiate S with respect to the parameters $m^2, K, E, J_i^2, L_j^2, \Phi_i, \Psi_j$ and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

$$\begin{aligned}
\frac{\partial S}{\partial m^2} &= 0 \Rightarrow l = \int \frac{\Pi r^{\epsilon+2}}{\Delta Z} \frac{dr}{\sqrt{R}} + \int \frac{(a^2\cos^2\theta + b^2\sin^2\theta)d\theta}{\sqrt{\Theta}} \,, \\
\frac{\partial S}{\partial K} &= 0 \Rightarrow \int \frac{d\theta}{\sqrt{\Theta}} = \int \frac{\Pi r^{\epsilon}}{\Delta Z} \frac{dr}{\sqrt{R}} \,, \\
\frac{\partial S}{\partial J_1^2} &= 0 \Rightarrow \int \frac{d\alpha_1}{\sqrt{A_1}} = \int \frac{\Pi r^{\epsilon}}{\Delta Z} \frac{r^2+b^2}{r^2+a^2} \frac{dr}{\sqrt{R}} + \int \cot^2\theta \frac{d\theta}{\sqrt{\Theta}} \,, \\
\frac{\partial S}{\partial J_k^2} &= 0 \Rightarrow \int \frac{d\alpha_k}{\sqrt{A_k}} = \int \frac{1}{\sin^2\alpha_{k-1}} \frac{d\alpha_{k-1}}{\sqrt{A_{k-1}}} \,, \qquad k = 2, ..., m-2 \,, \\
\frac{\partial S}{\partial L_1^2} &= 0 \Rightarrow \int \frac{d\beta_1}{\sqrt{B_1}} = \int \frac{\Pi r^{\epsilon}}{\Delta Z} \frac{r^2+a^2}{r^2+b^2} \frac{dr}{\sqrt{R}} + \int \tan^2\theta \frac{d\theta}{\sqrt{\Theta}} \,, \\
\frac{\partial S}{\partial L_l^2} &= 0 \Rightarrow \int \frac{d\beta_l}{\sqrt{B_l}} = \int \frac{1}{\sin^2\alpha_{k-1}} \frac{d\beta_{l-1}}{\sqrt{B_{l-1}}} \,, \qquad l = 2, ..., p-2 \,.
\end{aligned} \tag{4.25}$$

We can obtain N more equations of motion for the variables ϕ by differentiating S with respect to the angular momenta Φ_i and Ψ_j. Another equation can also be obtained by differentiating S with respect to E involving the time coordinate τ. However, these equations are not particularly illuminating, but can be written out explicitly if necessary by following this procedure. It is often more convenient to rewrite these in the form of first-order differential equations obtained from (4.25) by direct differentiation with respect to the affine parameter. We only list the most relevant ones here:

$$\begin{aligned}
\rho^2 \frac{dr}{dl} &= \frac{\Delta Z}{\Pi r^{\epsilon}} \sqrt{R} \,, \\
\rho^2 \frac{d\theta}{dl} &= \sqrt{\Theta} \,, \\
(r^2+a^2)\frac{d\alpha_k}{dl} &= \frac{\sqrt{A_k}}{\sin^2\theta \prod_{i=1}^{k-1} \sin^2\alpha_i} \,, \qquad k = 1, ..., m-1 \,,
\end{aligned}$$

$$(r^2+b^2)\frac{d\beta_k}{dl} = \frac{\sqrt{B_l}}{\cos^2\theta\prod_{i=1}^{l-1}\sin^2\beta_i}\,, \qquad l=1,...,p-1\,, \tag{4.26}$$

4.4.2 Analysis of the Radial Equation

The worldline of particles in the Myers-Perry black hole backgrounds considered above are completely specified by the values of the conserved quantities $E, \Phi_i, \Psi_j, J_i^2, L_j^2$, and by the initial values of the coordinates. We will consider particle motion in the black hole exterior. Allowed regions of particle motion necessarily need to have positive value for the quantity R, owing to equation (4.26). At large r, the dominant contribution to R is $E^2 - m^2$. Thus we can say that for $E^2 < m^2$, we cannot have unbounded orbits, whereas for $E^2 > m^2$, such orbits are possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose R as a quadratic in E as follows:

$$R = \alpha E^2 - 2\beta E + \gamma\,, \tag{4.27}$$

where

$$\begin{aligned}
\alpha &= \frac{\Pi r^\epsilon}{\Delta Z}\left(r^2 + \frac{2MZ}{r^2\Delta}\right),\\
\beta &= -\frac{2M\Pi r^{\epsilon-2}}{\Delta^2 Z}\left(a(r^2+b^2)\sum_{i=1}^{m}E\Phi_i + b(r^2+a^2)\sum_{i=1}^{p}E\Psi_i\right), \\
\gamma &= -\frac{\Pi r^\epsilon}{\Delta R}\left[\frac{2M}{r^2\Delta}\left(a^2(r^2+b^2)\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_i\Phi_j + 2ab\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_i\Psi_j\right.\right.\\
&\quad\left.\left.+b^2(r^2+a^2)\sum_{i=1}^{p}\sum_{i=1}^{p}\Psi_i\Psi_j\right) - m^2r^2 - \frac{r^2+b^2}{r^2+a^2}J_1^2 - \frac{r^2+a^2}{r^2+b^2}L_1^2\right],
\end{aligned} \tag{4.28}$$

The turning points for trajectories in the radial motion (defined by the condition $R=0$) are given by $E = V_\pm$ where

$$V_\pm = \frac{\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha}\,. \tag{4.29}$$

These functions, called the effective potentials [12], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.

4.4.3 Analysis of the Angular Equations

Another class of interesting motions possible describes motion at a constant value of α_i or β_j. These motions are described by the simultaneous equations

$$A_i(\alpha_i = \alpha_{i0}) = \frac{dA_i}{d\alpha_i}(\alpha_i = \alpha_{i0}) = 0\,, \qquad i = 1, ..., m-1\,, \tag{4.30}$$

in the case of constant α_i motion, where α_{i0} is the constant value of α_i along this trajectory, or by the simultaneous equations

$$B_i(\beta_i = \beta_{i0}) = \frac{dB_i}{d\beta_i}(\beta_i = \beta_{i0}) = 0\,, \qquad i = 1, ..., p-1\,, \tag{4.31}$$

in the case of constant β_i motion, where β_{i0} is the constant value of β_i along this trajectory.

These equations can be explicitly solved. In the case of constant α_i motion, we get the relations

$$\begin{aligned} \frac{J_{i+1}^2}{\sin^4\alpha_i} &= \frac{\Phi_{m-i-1}^2}{\cos^4\alpha_i}\,, \\ J_i^2 &= \frac{J_{i+1}^2}{\sin^2\alpha_i} + \frac{\Phi_{m-i+1}^2}{\cos^2\alpha_i}\,, \qquad i = 1, ..., m-1\,. \end{aligned} \tag{4.32}$$

Note that if $\alpha_{i0} = 0$, then $J_{i+1}^2 = 0$, and if $\alpha_{i0} = \pi/2$, then $\Phi_{m-i+1}^2 = 0$. Similarly, in the case of constant β_i motion, we get the relations

$$\begin{aligned} \frac{L_{i+1}^2}{\sin^4\beta_i} &= \frac{\Psi_{p-i-1}^2}{\cos^4\beta_i}\,, \\ L_i^2 &= \frac{L_{i+1}^2}{\sin^2\beta_i} + \frac{\Psi_{p-i+1}^2}{\cos^2\beta_i}\,, \qquad i = 1, ..., p-1\,. \end{aligned} \tag{4.33}$$

Again if $\beta_{i0} = 0$, then $L_{i+1}^2 = 0$, and if $\beta_{i0} = \pi/2$, then $\Psi_{p-i+1}^2 = 0$.

Examining A_k in the general case, $\alpha_k = 0$ can only be reached if $J_{k+1} = 0$, and $\alpha_k = \pi/2$ can be only be reached if $\Phi_{m-k+1} = 0$. The orbit will completely be in the subspace $\alpha_k = 0$ only if $J_k^2 = \Phi_{m-i+1}^2$ and will completely be in the subspace $\alpha_k = \pi/2$ only if $J_k^2 = J_{k+1}^2$. Analogous results hold for constant β_i motion.

Again these equations are in a form suitable for numerical analysis for specific values of the black hole and particle parameters.

4.5 Dynamical Symmetry

The general class of metrics discussed here are stationary and "axisymmetric"; i.e., $\partial/\partial\tau$ and $\partial/\partial\varphi_i$ are Killing vectors and have associated conserved quantities, $-E$ and L_i. In general if ξ is a Killing vector, then $\xi^\mu p_\mu$ is a conserved quantity, where p is the momentum. Note that this quantity is first order in the momenta.

With the assumption of only two sets of possibly unequal rotation parameters, the spacetime acquires additional dynamical symmetry and more Killing vectors are generated. We have complete symmetry between the various planes of rotation characterized by the same value of rotation parameter $a_i = a$, and we can "rotate" one into another. Similarly, we have symmetry between the planes of rotation characterized by the same value of the rotation parameter $a_i = b$, and we can "rotate" these into one another as well. The vectors that generate these transformations are the required Killing vectors. The explicit construction of such Killing vectors is done in [11]. In this case, we get two independent sets of such Killing vectors, associated with the constant a and b value rotations.

These Killing vectors exist since the rotational symmetry of the spacetime has been greatly enhanced. In an odd number of spacetime dimensions, if $a \neq b$ and both are nonvanishing, then the rotational symmetry group is $U(m)\times U(p)$. If one of them is zero, but the other is nonzero (we take the nonzero one to be a), then the rotational symmetry group is $U(m) \times O(2p)$. In the case when $a = b \neq 0$, the rotational symmetry group is $U(m+p)$. In the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m+2p)$. In an even number of spacetime dimensions, $b = 0$ in the cases we have analyzed. If $a \neq 0$, then the rotational symmetry group is $U(m)\times O(2p-1)$, and in the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m+2p-1)$. Note that since these metrics are stationary, the full dynamical symmetry group is the direct product of $\mathbf{R}$ and the rotational symmetry group, where $\mathbf{R}$ is the additive group of real numbers parametrizing τ.

In addition to these reducible angular Killing tensors, we also obtain a non-trivial irreducible second-order Killing tensor, which permits the separation of the $r - \theta$ equations. This Killing tensor is a generalization of the result obtained in the five dimensional case in [12]. This is obtained from the separation constant K in (4.17) and (4.18). We choose to analyze the latter.

$$K = (m^2 - E^2)(a^2\cos^2\theta + b^2\sin^2\theta) + \cot^2\theta J_1^2 + \tan^2\theta L_1^2 + \left(\frac{\partial S}{\partial\theta}\right)^2. \tag{4.34}$$

The Killing tensor $K^{\mu\nu}$ is obtained from this separation constant (which is quadratic in the canonical momenta) using the relation $K = K^{\mu\nu}p_\mu p_\nu$. Its is then easy to see that

$$K^{\mu\nu} = (g^{\mu\nu} - \delta^\mu_\tau \delta^\nu_\tau)\left(a^2\cos^2\theta + b^2\sin^2\theta\right) + \cot^2\theta J_1^{\mu\nu} + \tan^2\theta L_1^{\mu\nu} + \delta^\mu_\theta \delta^\mu_\theta\,, \tag{4.35}$$

where $J_1^{\mu\nu}$ and $L_1^{\mu\nu}$ are the reducible Killing tensors associated with the α_1 and β_1 separation.

It is the existence of these additional Killing vectors and the nontrivial irreducible Killing tensor, due to the increased symmetry of the spacetime, which permits complete separation of the Hamilton-Jacobi equation.

4.6 The Scalar Field Equation

Consider a scalar field Ψ in a gravitational background with the action

$$S[\Psi] = -\frac{1}{2}\int d^D x\sqrt{-g}((\nabla\Psi)^2 + \alpha R\Psi^2 + m^2\Psi^2)\,, \tag{4.36}$$

where we have included a curvature-dependent coupling. However, the Myers-Perry background is Ricci flat since it is a solution to the vacuum Einstein equations, so $R = 0$. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = m^2\Psi\,. \tag{4.37}$$

As discussed by Carter [14], the assumption of separability of the Klein-Gordon equation usually implies separability of the Hamilton-Jacobi equation. Conversely, if the Hamilton-Jacobi equation does not separate, the Klein-Gordon equation seems unlikely to separate. We can also see this explicitly (as in the case of the Hamilton-Jacobi equation), since the (r, τ, ϕ_i) sector has coefficients in the equations that explicitly depend on the μ_i except when of all $a_i = a$, in which case separation seems likely. We will again consider the much more general case of two sets of possibly unequal sets of rotation parameters a and b. We continue using the same numbering conventions for the variables.

Once again, we impose the constraint (4.1) and decompose the μ_i in two sets of spherical polar coordinates as in (4.6) and (4.8). We calculate the determinant of the metric to be

$$g \;=\; -r^2\rho^4\Pi(r^2+a^2)^{m-2}(r^2+b^2)^{p-2}\sin^{4m-2}\theta\cos^{4p-2-2\epsilon}\theta$$

$$* \left[\prod_{j=1}^{m-1} \sin^{4m-4j-2}\alpha_j \cos^2\alpha_j\right] \left[\prod_{k=1}^{p-1} \sin^{4p-4k-2}\beta_k \cos^2\beta_k\right] \cos^{-2\epsilon}\beta_1 \,. \quad (4.38)$$

For convenience we write $g = -RTAB\rho^4$, where

$$\begin{aligned} R &= r^2\Pi(r^2+a^2)^{m-2}(r^2+b^2)^{p-2} \,, \\ T &= \sin^{4m-2}\theta \cos^{4p-2-2\epsilon}\theta \,, \\ A &= \prod_{j=1}^{m-1} \sin^{4m-4j-2}\alpha_j \cos^2\alpha_j \,, \\ B &= \prod_{k=1}^{p-1} \sin^{4p-4k-2}\beta_k \cos^2\beta_k \cos^{-2\epsilon}\beta_1 \,. \end{aligned} \quad (4.39)$$

Note that R and T are functions of r and θ only, and A and B only depend on the set of variables α_i and β_j respectively. Then the Klein-Gordon equation in this background (4.37) becomes

$$\begin{aligned} m^2\Psi &= \frac{1}{\rho^2\sqrt{R}}\partial_r\left(\sqrt{R}\frac{\Delta Z}{r^\epsilon\Pi}\partial_r\Psi\right) - \left[1+\frac{2MZ}{r^2\rho^2\Delta}\right]\partial_\tau^2\Psi + \frac{1}{r^2+a^2}\sum_{i=1}^{m}\frac{1}{\mu_i^2}\partial_{\phi_i}^2\Psi \\ &+ \frac{1}{r^2+b^2}\sum_{i=1}^{p-\epsilon}\frac{1}{\mu_{m+i}^2}\partial_{\phi_{i+m}}^2\Psi - \frac{2Ma^2(r^2+b^2)}{\Delta r^2\rho^2(r^2+a^2)}\sum_{i=1}^{m}\sum_{j=1}^{m}\partial_{\phi_i}\partial_{\phi_j}\Psi \\ &- \frac{2M}{r^2\rho^2\Delta}\left[a(r^2+b^2)\sum_{i=1}^{m}\partial_\tau\partial_{\phi_i}\Psi + b(r^2+a^2)\sum_{i=1}^{p-\epsilon}\partial_\tau\partial_{\phi_{m+i}}\Psi\right] \\ &- \frac{2Mb^2(r^2+a^2)}{\Delta r^2\rho^2(r^2+b^2)}\sum_{i=1}^{p-\epsilon}\sum_{j=1}^{p-\epsilon}\partial_{\phi_{i+m}}\partial_{\phi_{j+m}}\Psi - \frac{4Mab}{\Delta r^2\rho^2}\sum_{i=1}^{m}\sum_{j=1}^{p-\epsilon}\partial_{\phi_i}\partial_{\phi_{j+m}}\Psi \\ &+ \frac{1}{\rho^2\sqrt{T}}\partial_\theta\left(\sqrt{T}\partial_\theta\Psi\right) + \frac{1}{(r^2+a^2)\sin^2\theta}\left[\sum_{i=1}^{m-1}\frac{1}{\sqrt{A}}\partial_{\alpha_i}\left(\frac{\sqrt{A}}{\prod_{k=1}^{i-1}\sin^2\alpha_k}\partial_{\alpha_i}\Psi\right)\right] \\ &+ \frac{1}{(r^2+b^2)\cos^2\theta}\left[\sum_{i=1}^{p-1}\frac{1}{\sqrt{B}}\partial_{\beta_i}\left(\frac{\sqrt{B}}{\prod_{k=1}^{i-1}\sin^2\beta_k}\partial_{\beta_i}\Psi\right)\right] \,. \end{aligned} \quad (4.40)$$

We attempt the usual multiplicative separation for Ψ in the following form:

$$\Psi = \Phi_r(r)\Phi_\theta(\theta)e^{-iEt}e^{i\sum_i^m \Phi_i\phi_i}e^{i\sum_i^{p-\epsilon}\Psi_i\phi_{m+i}}\left(\prod_{i=1}^{m-1}\Phi_{\alpha_i}(\alpha_i)\right)\left(\prod_{i=1}^{p-1}\Phi_{\beta_i}(\beta_i)\right) \,. \quad (4.41)$$

The Klein-Gordon equation then completely separates. The r and θ equations are given as

$$
\begin{aligned}
K &= \frac{1}{\Phi_r\sqrt{R}}\frac{d}{dr}\left(\sqrt{R}\frac{\Delta Z}{r^\epsilon \Pi}\frac{d\Phi_r}{dr}\right) + r^2(E^2-m^2) + \frac{2MZE^2}{r^2\Delta} \\
&- \frac{2ME}{r^2\Delta}\left[a(r^2+b^2)\sum_{i=1}^{m}\Phi_i + b(r^2+a^2)\sum_{i=1}^{p-1\epsilon}\Psi_i\right] \\
&+ \frac{2M}{\Delta r^2}\left[\frac{a^2(r^2+b^2)}{(r^2+a^2)}\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_i\Phi_j + \frac{b^2(r^2+a^2)}{(r^2+b^2)}\sum_{i=1}^{p-\epsilon}\sum_{j=1}^{p-1\epsilon}\Psi_i\Psi_j \right. \\
&\left. +2ab\sum_{i=1}^{m}\sum_{j=1}^{p-\epsilon}\Phi_i\Psi_j\right], \\
-K &= \frac{1}{\Phi_\theta\sqrt{T}}\frac{d}{d\theta}\left(\sqrt{T}\frac{d\Phi_\theta}{d\theta}\right) + (E^2-m^2)(a^2\cos^2\theta + b^2\sin^2\theta) \\
&+ K_1\cot^2\theta + M_1\tan^2\theta\,,
\end{aligned} \tag{4.42}
$$

where K, K_1 and M_1 are separation constants. K_1 and M_1 encode all the α and β dependence respectively and are defined explicitly as follows:

$$
K_1 = \sum_{i=1}^{k-1} A_i + \frac{K_k}{\prod_{j=1}^{k-1}\sin^2\alpha_j}\,, \quad k = 1, ..., m-1\,, \tag{4.43}
$$

where

$$
\begin{aligned}
A_i &= \frac{1}{\Phi_{\alpha_i}\cos\alpha_i\sin^{2m-2i-1}\alpha_i\prod_{k=1}^{i-1}\sin^2\alpha_k}\frac{d}{d\alpha_i}\left(\cos\alpha_i\sin^{2m-2i-1}\alpha_i\frac{d\Phi_{\alpha_i}}{d\alpha_i}\right) \\
&-\frac{\Phi^2_{m-i+1}}{\cos^2\alpha_i\prod_{j=1}^{i-1}\sin^2\alpha_j}\,,
\end{aligned} \tag{4.44}
$$

and

$$
M_1 = \sum_{i=1}^{k-1} B_i + \frac{M_k}{\prod_{j=1}^{k-1}\sin^2\beta_j}\,, \quad k = 1, ..., p-1\,, \tag{4.45}
$$

and where

$$
\begin{aligned}
B_i &= \frac{1}{\Psi_{\beta_i}\cos\beta_i\sin^{2p-2i-1}\beta_i\prod_{k=1}^{i-1}\sin^2\beta_k}\frac{d}{d\beta_i}\left(\cos\beta_i\sin^{2p-2i-1}\beta_i\frac{d\Phi_{\beta_i}}{d\beta_i}\right) \\
&-\frac{\Psi^2_{p-i+1}}{\cos^2\beta_i\prod_{j=1}^{i-1}\sin^2\beta_j}\,,
\end{aligned} \tag{4.46}
$$

Then we inductively have the complete separation of the α_i dependence as

$$K_k = \frac{K_{k+1}}{\sin^2\alpha_k} - \frac{\Phi^2_{n-k+1}}{\cos^2\alpha_k} + \frac{1}{\Phi_{\alpha_k}\cos\alpha_k\sin^{2m-2k-1}\alpha_k}\frac{d}{d\alpha_k}\left(\cos\alpha_k\sin\alpha_k\frac{d\Phi_{\alpha_k}}{d\alpha_k}\right), \quad (4.47)$$

where $k = 1, ..., m-1$, and we use the convention $K_m = -\Phi_1^2$. Similarly, the complete separation of the β_i dependence is given inductively by

$$M_k = \frac{M_{k+1}}{\sin^2\beta_k} - \frac{\Psi^2_{p-k+1}}{\cos^2\beta_k} + \frac{1}{\Phi_{\beta_k}\cos\beta_k\sin^{2p-2k-1}\beta_k}\frac{d}{d\beta_k}\left(\cos\beta_k\sin\beta_k\frac{d\Phi_{\beta_k}}{d\beta_k}\right), \quad (4.48)$$

where $k = 1, ..., p-1$, and we use the convention $M_p = -\Psi_1^2$. These results agree with the previously known analysis in five dimensions [13].

At this point we have complete separation of the Klein-Gordon equation in the Myers-Perry black hole background in all dimensions with two sets of possibly unequal rotation parameters in the form given by (4.41) with the individual separation functions given by the ordinary differential equations above. Note that the separation of the Klein-Gordon equation in this geometry is again due to the fact that the symmetry of the space has been enlarged.

Conclusions

We studied the separability properties of the Hamilton-Jacobi and the Klein-Gordon equations in the Myers-Perry black hole backgrounds in all dimensions. Separation in Boyer-Lindquist coordinates is possible for the case of two possibly unequal sets of rotation parameters. This is due to the enlarged dynamical symmetry of the spacetime. We discuss the Killing vectors and reducible Killing tensors that exist in the spacetime. In addition we construct the nontrivial irreducible Killing tensor which explicitly permits complete separation. Thus we demonstrate the separability of the Hamilton-Jacobi and the Klein-Gordon equations as a direct consequence of the enhancement of symmetry. We also derive first-order equations of motion for classical particles in these backgrounds, and analyze the properties of some special trajectories.

Further work in this direction could include the study of higher-spin field equations in these backgrounds, which is of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results.

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Chapter 5

Some Charged Rotating Supergravity Black Holes

5.1 Introduction

Solutions of the vacuum Einstein equations describing black hole solutions in both four and higher dimensions are currently of great interest. This is mainly due to a number of recent developments in high energy physics. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (eat about.g. the hierarchy problem) [1–3]. These models allow for the existence of higher dimensional black holes which can be described classically. Also of interest in these models is the possibility of mini black hole production in high energy particle colliders which, if they occur, provide a window into non-perturbative gravitational physics [4, 5].

Superstring and M-Theory, which call for additional spacetime dimensions, naturally incorporate black hole solutions in higher dimensions (10 or 11). P-branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Black hole solutions in superstring theory are particularly relevant since they can be described as solitonic objects. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6, 7].

Astrophysically relevant black hole spacetimes are, to a very good approximation, described by the Kerr metric [8]. One generalization of the Kerr metric to higher dimensions is given by the Myers-Perry construction [9]. With interest now in a nonzero cosmological constant, it is worth studying spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an Anti-de Sitter background could give rise to interesting descriptions in terms of the conformal field theory on the boundary leading to better understanding of the correspondence [10, 11]. The general Kerr-de Sitter metrics describing rotating black holes in the presence of a cosmological constant have been constructed explicitly in [12, 13].

There is a strong need to understand explicitly the structure of geodesics in the background of black holes in Anti-de Sitter space in the context of string theory and the AdS/CFT correspondence. This is due to the recent work in exploring black hole singularity structure using geodesics and correlators in the dual CFT on the boundary [14–19]. The metrics mentioned above have so far proven to yield little or no information through an analysis of this sort. Black holes with charge are particularly interesting for this type of analysis, since the charges are reinterpreted as the R-charges of the dual theory. The spacetimes explored in this paper are exact solutions of supergravity in backgrounds with a cosmological constant and charges, and thus could be more suitable for this sort of geodesic analysis.

In this paper we work with the four-dimensional multicharge Kerr-Taub-NUT-(Anti) de Sitter solution of supergravity recently discovered by Chong, Cvetic, Lu, and Pope [20], as well as the $U(1)^3$ gauged Kerr-(Anti) de Sitter black hole solution of $\mathcal{N} = 2$ supergravity in five dimensions discovered by Cvetic, Lu, and Pope [21].

We study the separability of the Hamilton-Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We use this explicit separation to obtain first-order equations of motion for both massive and massless particles in these backgrounds. The equations are obtained in a form that could be used for numerical study, and also in the study of black hole singularity structure using geodesic probes and the AdS/CFT correspondence.

We also study the Klein-Gordon equation describing the propagation of a massive scalar field in these spacetimes. Separation again turns out to be possible with the usual multiplicative ansatz.

This paper greatly generalizes many of the results of [22, 23] for the Myers-Perry metric in five dimensions, [24] which separates the equations in the case of equal rotation parameters in the odd dimensional Kerr-(A) dS spacetimes, [25] which separates the equations for the general five dimensional Kerr-(A) dS spacetime with unequal rotation parameters, [26] which separates the equations in the case of two independent sets of rotation parameters in the Myers-Perry metrics in all dimensions, [27] which separates the equations in the case of two independent sets of rotation parameters in the Kerr-(A) dS metrics in all dimensions, and [28] which separates the equations in the case of a single non-zero rotation parameter for uncharged Kerr-Taub-NUT metrics in arbitrary dimensions. Some further work for other special cases were also done in [29] and [30].

Separation turns out to be possible for both equations in these metrics due to the existence of second-order Killing tensors, one of them non-trivial and irreducible. This is a generalization of the Killing tensor in the Kerr black hole spacetime in four dimensions constructed in [20] which was subsequently described by Chandrasekhar as the "miraculous property of the Kerr metric". A similar construction for the Myers-Perry metrics in higher dimensions has also been done [22,26], and for the Kerr-Taub-NUT metrics in arbitrary dimensions without charge and only one nonzero rotation parameter in [28]. The Killing tensors, in each case, provides an additional integral of motion necessary for complete integrability.

5.2 Overview of the Metrics

5.2.1 Four Dimensional Kerr-Taub-NUT Multicharge Gauged Solution of Supergravity

This metric was recently obtained by Chong, Cvetic, Lu, and Pope in [20]. The solution was obtained by starting out with the four dimensional Kerr-Taub-NUT metric, dimensionally reducing to three dimensions along the time direction, and then lifting back up after "dualizing". The metric is given by

$$ds^2 = -\frac{\Delta_r}{a^2 W}[adt + u_1 u_2 d\phi]^2 + \frac{\Delta_u}{a^2 W}[adt - r_1 r_2 d\phi]^2 + W\left[\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u}\right], \tag{5.1}$$

where

$$\begin{aligned} W &= r_1 r_2 + u_1 u_2 , \quad r_i = r + 2ms_i^2 , \quad u_i = u + 2ls_i^2 , \quad i = 1, 2 , \\ \Delta_r &= r^2 + a^2 - 2mr + g^2 r_1 r_2 (r_1 r_2 + a^2) , \\ \Delta_u &= -u^2 + a^2 + 2lu + g^2 u_1 u_2 (u_1 u_2 - a^2) , \end{aligned} \tag{5.2}$$

and we use the notation

$$s_i = \sinh \delta_i , \quad c_i = \cosh \delta_i , \quad i = 1, 2 . \tag{5.3}$$

Here δ_1 is the magnetic charge, δ_2 is the electric charge, l is the NUT parameter, a is the rotation parameter, and g is the gauge parameter. The cosmological constant Λ is given by $\Lambda = -g^2$. The ungauged solution is obtained by setting g to zero.

If the two charge parameters are set equal, $\delta_1 = \delta_2$, then the solution reduces to the charged AdS-Kerr-Taub-NUT solution of Einstein-Maxwell theory with a cosmological constant. To reduce to the usual coordinate system, we use the change of coordinate $u = a\cos\theta$. With l set to zero, we recover the metric found in [20] for a multicharge Kerr-(Anti) de Sitter black hole in gauged supergravity in four dimensions.

For future reference, we note the following expressions. The determinant of the metric is given by

$$g = -\frac{W^2}{a}\,. \tag{5.4}$$

The components of the inverse metric are

$$\begin{aligned} g^{tt} &= \frac{1}{\Delta_r \Delta_u W}[\Delta_r u_1^2 u_2^2 - \Delta_u r_1^2 r_2^2]\,, \quad g^{\phi\phi} = \frac{a^2}{\Delta_r \Delta_u W}[\Delta_r - \Delta_u]\,, \\ g^{t\phi} &= \frac{a}{\Delta_r \Delta_u W}[\Delta_r u_1 u_2 + \Delta_u r_1 r_2]\,, \quad g^{rr} = \frac{\Delta_r}{W}\,, \quad g^{uu} = \frac{\Delta_u}{W}\,. \end{aligned} \tag{5.5}$$

We also note that the functions Δ_r and Δ_u are functions of r and u only, respectively.

5.2.2 $U(1)^3$ Gauged Kerr-(Anti) de Sitter Black Hole Solution of $\mathcal{N} = 2$ Supergravity in Five Dimensions

This metric was recently obtained by Chong, Lu, and Pope in [21]. The metric is given by

$$\begin{aligned} ds^2 = &- \frac{Y - f_3}{R^2}dt^2 + \frac{r^2 R}{Y}dr^2 + R d\Omega_3^2 + \frac{f_1 - R^3}{R^2}(\sin^2\theta d\phi + \cos^2\theta d\psi)^2 \\ &- \frac{2f_2}{R^2}dt(\sin^2\theta d\phi + \cos^2\theta d\psi)\,, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} R &= r^2\left(\prod_{i=1}^{3} H_i\right)^{1/3}\,, \quad H_i = 1 + \frac{M s_i^2}{r^2}\,, \\ d\Omega_3^2 &= d\theta^2 + sin\theta^2 d\phi^2 + \cos^2\theta d\psi^2\,, \end{aligned} \tag{5.7}$$

and as before

$$s_i = \sinh\delta_i\,, \quad c_i = \cosh\delta_i\,, \quad i = 1, 2, 3\,. \tag{5.8}$$

The functions f_i, and Y are defined by

$$
\begin{aligned}
f_1 &= R^3 + Ma^2r^2 + M^2a^2\left[2\left(\prod_i c_i - \prod_i s_i\right)\prod_j s_j - \sum_{i<j} s_i^2 s_j^2\right] , \\
f_2 &= \gamma a \Lambda R^3 + Ma\left(\prod_i c_i - \prod_i s_i\right) r^2 + M^2 a \prod_i s_i , \\
f_3 &= \gamma^2 a^2 \Lambda^2 R^3 + Ma^2\Lambda\left[2\gamma\left(\prod_i c_i - \prod_i s_i\right) - \Sigma\right] r^2 \\
&\quad + Ma^2 - \Lambda\Sigma M^2 a^2\left[2(\left(\prod_i c_i - \prod_i s_i\right)\prod_j s_j - \sum_{i<j} s_i^2 s_j^2\right] + 2\lambda\gamma M^2 a^2 \prod_i s_i , \\
Y &= f_3 - \Lambda\Sigma R^3 + r^4 - Mr^2 ,
\end{aligned}
\tag{5.9}
$$

and

$$
\Sigma = 1 + \gamma^2 a^2 \Lambda. \tag{5.10}
$$

It is important to note that these are functions of the coordinate r only.

The parameter M is related to the mass of the black hole, the δ_i are the charges associated with each of the three $U(1)$ gauge groups, the gauge parameter g is related to the cosmological constant Λ via $\Lambda = -g^2$, a is the rotation parameter of the black hole (equal rotation parameters in the two independent planes was assumed in the derivation of the metric), and the constant γ is simply a redundant parameter which is useful to test several limits, but could be eliminated if necessary. This metric encompasses, as special limits, several previously known solutions such as the Klemm-Sabra BPS solution etc. More details about these limits can be found in [20].

In order to avoid long complicated expressions, we introduce the following functions to write the metric more compactly

$$
A(r) = \frac{Y - f_3}{R^2} , \quad W(r) = \frac{Y}{r^2 R} , \quad B(r) = \frac{f_1 - R^3}{R^2} , \quad C(r) = -\frac{f_2}{R^2} . \tag{5.11}
$$

Note that all of these are functions of the coordinate r only. The metric is then written compactly in the form

$$
\begin{aligned}
ds^2 = &- A(r)dt^2 + \frac{dr^2}{W(r)} + R d\Omega_3^2 + B(r)(\sin^2\theta d\phi + \cos^2\theta d\psi)^2 \\
&+ 2C(r)dt(\sin^2\theta d\phi + \cos^2\theta d\psi) .
\end{aligned}
\tag{5.12}
$$

The components of the inverse metric are

$$\begin{aligned}
g^{rr} &= W(r)\,, \\
g^{\theta\theta} &= \frac{1}{R}\,, \\
g^{tt} &= -\frac{B(r)+R}{r^2 W(r)}\,, \\
g^{t\phi} &= g^{t\psi} = \frac{C(r)}{r^2 W(r)}\,, \\
g^{\phi\phi} &= \frac{A(r)B(r)\cos^2\theta + A(r)R + C^2(r)\cos^2\theta}{Rr^2 W(r)\sin^2\theta}\,, \\
g^{\psi\psi} &= \frac{A(r)B(r)\sin^2\theta + A(r)R + C^2(r)\sin^2\theta}{Rr^2 W(r)\cos^2\theta}\,, \\
g^{\phi\psi} &= -\frac{A(r)B(r) + C^2(r)}{Rr^2 W(r)}\,.
\end{aligned} \tag{5.13}$$

We note for future reference the following identity which can easily be verified using Maple [32]

$$A(r)B(r) + A(r)R + C^2(r) = r^2 W(r)\,. \tag{5.14}$$

Finally, the determinant of the metric can be calculated to be

$$g = -r^2 R^2 \sin^2\theta\cos^2\theta\,, \tag{5.15}$$

where we need to make use of the identity given above repeatedly.

5.3 Integrals of Motion and the Hamilton-Jacobi Equation

The equations of motion of a test particle of mass m in a gravitational background described by a metric $g_{\mu\nu}$ are

$$\frac{D^2 x^\mu}{D\tau^2} = 0\,, \tag{5.16}$$

where $\frac{D}{D\tau}$ is the covariant derivative with respect to proper time τ. These equations can be derived from a Lagrangian

$$L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\,, \tag{5.17}$$

where a dot denotes a partial derivative with respect to an affine parameter λ. This can be chosen such that $\tau = m\lambda$.

The symmetries of the metric, if any, can provide us with some integrals of motion. For instance, if the metric is stationary, i.e. does not depend on the time t, then the energy is conserved. However, in most situations, sufficient number of integrals of motion do not exist. Also, using the Lagrangian formulation, sometimes certain integrals of motion are impossible to obtain even if they exist. Usually these are "second order" in the momenta such as the case of the Carter constant for the Kerr metric. Such additional integral of motion, which permit us in these cases to integrate the equations of motion completely, can be provided by the Hamilton-Jacobi equation (though a proper choice of coordinate system is necessary).

The Hamilton-Jacobi equation in a curved background is given by

$$-\frac{\partial S}{\partial \lambda} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}\,, \tag{5.18}$$

where S is the action associated with the particle and λ is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parametrized by proper time.

5.4 Particle Motion in the Four Dimensional Kerr-Taub-NUT Multicharge Gauged Solution of Supergravity

5.4.1 Separation of Variables

We can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2} m^2 \lambda - Et + L_\phi \phi + S_\theta(\theta) + S_r(r)\,. \tag{5.19}$$

t and ϕ are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and that with rotation in ϕ is the corresponding angular momentum L_ϕ. Then using the components of the inverse metric (5.5), the Hamilton-Jacobi equation (5.18) is written to be

$$\begin{aligned} -m^2 &= \frac{1}{\Delta_r \Delta_u W}[\Delta_r u_1^2 u_2^2 - \Delta_u r_1^2 r_2^2](-E)^2 + \frac{a^2}{\Delta_r \Delta_u W}[\Delta_r - \Delta_u] L_\phi^2 + \frac{\Delta_r}{W}\left[\frac{dS_r(r)}{dr}\right]^2 \\ &\quad + \frac{\Delta_u}{W}\left[\frac{dS_u(u)}{du}\right]^2 + \frac{2a}{\Delta_r \Delta_u W}[\Delta_r u_1 u_2 + \Delta_u r_1 r_2](-E) L_\phi\,. \end{aligned} \tag{5.20}$$

Now multiplying both sides by W, we can separate out the equation in the form

$$\begin{aligned} K &= \Delta_r \left[\frac{dS_r(r)}{dr}\right]^2 - \frac{1}{\Delta_r}[r_1 r_2 E + aL_\phi]^2 + m^2 r_1 r_2 \,, \\ K &= -\Delta_u \left[\frac{dS_u(u)}{du}\right]^2 - \frac{1}{\Delta_u}[u_1 u_2 E - aL_\phi]^2 - m^2 u_1 u_2 \,, \end{aligned} \tag{5.21}$$

where K is a constant of separation.

5.4.2 The Equations of Motion

To derive the equations of motion, we will write the separated action S from the Hamilton-Jacobi equation in the following form:

$$S = \frac{1}{2}m^2\lambda - Et + L_\phi \phi + \int^r \sqrt{\mathcal{R}(r')}dr' + \int^u \sqrt{U(u')}du' \,, \tag{5.22}$$

where

$$\begin{aligned} \Delta_r \mathcal{R}(r) &= K + \frac{1}{\Delta_r}[r_1 r_2 E + aL_\phi]^2 - m^2 r_1 r_2 \,, \\ \Delta_u U(u) &= -K - \frac{1}{\Delta_u}[u_1 u_2 E - aL_\phi]^2 - m^2 u_1 u_2 \,. \end{aligned} \tag{5.23}$$

To obtain the equations of motion, we differentiate S with respect to the parameters m^2, K, E, L_ϕ and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

$$\begin{aligned} \frac{\partial S}{\partial m^2} &= 0 \Rightarrow \lambda = \int r_1 r_2 \frac{dr}{\Delta_r \sqrt{\mathcal{R}}} + \int u_1 u_2 \frac{du}{\Delta_u \sqrt{U}} \,, \\ \frac{\partial S}{\partial K} &= 0 \Rightarrow \int \frac{du}{\Delta_u \sqrt{U}} = \int \frac{dr}{\Delta_r \sqrt{\mathcal{R}}} \,, \\ \frac{\partial S}{\partial L_\phi} &= 0 \Rightarrow \phi = \int (r_1 r_2 E + aL_\phi) \frac{dr}{\Delta_r^2 \sqrt{\mathcal{R}}} + \int (u_1 u_2 E - aL_\phi) \frac{du}{\Delta_u^2 \sqrt{U}} \,, \\ \frac{\partial S}{\partial E} &= 0 \Rightarrow t = \int r_1 r_2 (r_1 r_2 E + aL_\phi) \frac{dr}{\Delta_r^2 \sqrt{\mathcal{R}}} - \int u_1 u_2 (u_1 u_2 E - aL_\phi) \frac{du}{\Delta_u^2 \sqrt{U}} \end{aligned} \tag{5.24}$$

It is often more convenient to rewrite these in the form of first-order differential equations obtained from (5.24) by direct differentiation with respect to the affine parameter:

$$\begin{aligned}
W\frac{dr}{dl} &= \Delta_r\sqrt{\mathcal{R}}\,, \\
W\frac{du}{dl} &= \Delta_u\sqrt{U}\,, \\
W\frac{d\phi}{dl} &= \frac{r_1r_2E+aL_\phi}{\Delta_r}+\frac{u_1u_2E-aL_\phi}{\Delta_\theta}\,, \\
W\frac{dt}{dl} &= \frac{r_1r_2(r_1r_2E+aL_\phi)}{\Delta_r}-\frac{u_1u_2(u_1u_2E-aL_\phi)}{\Delta_\theta}\,.
\end{aligned} \tag{5.25}$$

5.4.3 Analysis of the Radial Equation

The worldline of particles in the background considered above are completely specified by the values of the conserved quantities E, L_ϕ, K, and by the initial values of the coordinates. We will consider particle motion in the black hole exterior. Allowed regions of particle motion necessarily need to have positive value for the quantity R, owing to equation (5.25). We determine some of the possibilities of the allowed motion.

We will consider the motion of a particle in the black hole exterior. Thus we can assume that $\Delta_r > 0$ for large r. At large r, the dominant contribution to $\mathcal{R}$, in the case of $\Lambda = 0$, is $E^2 - m^2$. Since $\Lambda = -g^2$, zero cosmological constant corresponds to the charged rotating black hole in four dimensions in ungauged supergravity. Here, we can thus say that for $E^2 < m^2$, we cannot have unbounded orbits, whereas for $E^2 > m^2$, such orbits are possible. For the case of nonzero Λ (i.e. also nonzero g which implies we are now considering gauged supergravity), the dominant term at large r in R (or rather the slowest decaying term) is $-m^2r^2$. Thus in the case of the Anti-de Sitter background (since $\Lambda = -g^2$ is negative), only bound orbits are possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose $\mathcal{R}$ as a quadratic in E as follows:

$$\mathcal{R} = \alpha E^2 - 2\beta E + \gamma\,, \tag{5.26}$$

where

$$\begin{aligned}
\alpha &= \frac{r_1^2r_2^2}{\Delta_r^2}\,, \\
\beta &= -\frac{r_1r_2aL_\phi}{\Delta_r^2}\,, \\
\gamma &= \frac{K-m^2r_1r_2}{\Delta_r}+\frac{a^2L_\phi^2}{\Delta_r^2}\,.
\end{aligned} \tag{5.27}$$

The turning points for trajectories in the radial motion (defined by the condition $\mathcal{R} = 0$) are given by $E = V_\pm$ where

$$V_\pm = \frac{\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha}. \tag{5.28}$$

These functions, called the effective potentials [22], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.

5.5 Particle Motion in the $U(1)^3$ Gauged Kerr-(Anti) de Sitter Black Hole Solution of $\mathcal{N} = 2$ Supergravity in Five Dimensions

We will only sketch the analysis of the separation of variables here, since the procedure for deriving equations of motion etc. is virtually identical to those of the metric above.

We can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2}m^2\lambda - Et + L_\phi\phi + L_\psi\psi + S_\theta(\theta) + S_r(r). \tag{5.29}$$

t, ϕ, and ψ are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and those with rotation in ϕ and ψ are the corresponding angular momenta L_ϕ and L_ψ. Then using the components of the inverse metric (5.13), the Hamilton-Jacobi equation (5.18) is written to be

$$\begin{aligned} -m^2 &= \frac{B(r)+R}{r^2W(r)}(-E)^2 + 2\frac{C(r)}{r^2W(r)}(-E)(L_\phi) + 2\frac{C(r)}{r^2W(r)}(-E)(L_\psi) \\ &+ \frac{A(r)B(r)\cos^2\theta + A(r)R + C^2(r)\cos^2\theta}{Rr^2W(r)\sin^2\theta}L_\phi^2 + \frac{1}{R}\left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 + \\ &+ \frac{A(r)B(r)\sin^2\theta A(r)R + C^2(r)\sin^2\theta}{Rr^2W(r)\cos^2\theta}L_\psi^2 + W(r)\left[\frac{dS_r(r)}{dr}\right]^2. \end{aligned} \tag{5.30}$$

After some algebraic manipulation and using some trigonometric identites we can write this as

$$-m^2 = W(r)\left[\frac{dS_r(r)}{dr}\right]^2 - 2\frac{C(r)E}{r^2W(r)}(L_\phi + L_\psi) - \frac{B(r)+R}{r^2W(r)}E^2$$

$$- \frac{A(r)B(r)+C^2(r)}{Rr^2W(r)}(L_\phi + L_\psi)^2 + \frac{1}{R}(\csc^2\theta L_\phi^2 + \sec^2\theta L_\psi^2) + \frac{1}{R}\left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 \quad (5.31)$$

In this form, the Hamilton-Jacobi equation can now be easily separated to give

$$\begin{aligned} -K &= \left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 + \csc^2\theta L_\phi^2 + \sec^2\theta L_\psi^2 , \\ K &= Rm^2 + W(r)R\left[\frac{dS_r(r)}{dr}\right]^2 - 2\frac{C(r)RE}{r^2W(r)}(L_\phi + L_\psi) - \frac{B(r)R+R^2}{r^2W(r)}E^2 \\ &- \frac{A(r)B(r)+C^2(r)}{r^2W(r)}(L_\phi + L_\psi)^2 , \end{aligned} \quad (5.32)$$

where K is a constant of separation.

To derive the equations of motion, the separated action S from the Hamilton-Jacobi equation is more conveniently written, as before, in the following form:

$$S = \frac{1}{2}m^2\lambda - Et + L_\phi\phi + L_\psi\psi + \int^r \sqrt{\mathcal{R}(r')}dr' + \int^\theta \sqrt{\Theta(\theta')}d\theta' , \quad (5.33)$$

where

$$\begin{aligned} RW(r)\mathcal{R}(r) &= K - Rm^2 + 2\frac{C(r)RE}{r^2W(r)}(L_\phi + L_\psi) + \frac{B(r)R+R^2}{r^2W(r)}E^2 \\ &+ \frac{A(r)B(r)+C^2(r)}{r^2W(r)}(L_\phi + L_\psi)^2 , \\ \Theta(\theta) &= -K - \csc^2\theta L_\phi^2 - \sec^2\theta L_\psi^2 . \end{aligned} \quad (5.34)$$

By following the same procedure as earlier, we can easily establish first-order equations of motion, a radial effective potential, etc. Since the derivation is remarkably similar, we will not reproduce the results here in the interests of being concise.

5.6 Dynamical Symmetry

The general class of metrics discussed here are stationary and "axisymmetric"; i.e., $\partial/\partial t$ and $\partial/\partial\phi$ (as well as $\partial/\partial\psi$ in the five dimensional $U(1)^3$ case) are Killing vectors and have associated conserved quantities, $-E$ and L_ϕ (and L_ψ). In general if ξ is a Killing vector, then $\xi^\mu p_\mu$ is a conserved quantity, where p is the momentum of the particle. Note that this quantity is first order in the momenta.

As mentioned earlier, the additional constant of motion K which allowed for complete integrability of the equations of motion is not related to a Killing vector from a cyclic coordinate. This constant is, rather, derived from a non-trivial irreducible second-order Killing tensor in both spacetimes, which permits the separation of the $r-\theta$ (or $r-u$) equations in both cases. These Killing tensors are generalizations of the Killing tensor obtained in four dimensions by Carter [31] and in five dimensions for the Myers-Perry metric in [22]. Killing tensors are not symmetries on configuration space, and cannot be derived from a Noether procedure, and are rather, symmetries on phase space. They obey a generalization of the Killing equation for Killing vectors (which do generate symmetries in configuration space by the Noether procedure) given by

$$K_{(\mu\nu;\rho)} = 0\,, \tag{5.35}$$

where K is any second order Killing tensor, and the parentheses indicate complete symmetrization of all indices.

The Killing tensors can be obtained from the expressions for the separation constant K in each case. If the particle has momentum p, then the Killing tensor $\mathcal{K}_{\mu\nu}$ is related to the constant K via

$$K = \mathcal{K}^{\mu\nu} p_\mu p_\nu = \mathcal{K}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}\,. \tag{5.36}$$

In both cases, we can use the expression in terms of the r equation or the u/θ equation. We will choose to work with the latter in both cases.

For the four dimensional Kerr-Taub-NUT metric analyzed above, the expression for K from (5.21) is

$$K \;=\; -\Delta_u \left[\frac{dS_u(u)}{du}\right]^2 - \frac{1}{\Delta_u}[u_1 u_2 E - aL_\phi]^2 - m^2 u_1 u_2\,. \tag{5.37}$$

Thus, from (5.36) we can easily read

$$\mathcal{K} = -\Delta_u \partial_u \otimes \partial_u - \frac{1}{\Delta_u}\left[u_1 u_2 \partial_t \otimes \partial_t + a^2 \partial_\phi \otimes \partial_\phi + sym(a u_1 u_2 \partial_t \otimes \partial_\phi)\right] \tag{5.38}$$

Since this Killing tensor is not a simple linear combination of Killing vectors, it is non-trivial and irreducible.

For the five dimensional $U(1)^3$ charged metric analyzed above, the expressions for K from (5.32) is

$$-K \;=\; \left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 + \csc^2\theta L_\phi^2 + \sec^2\theta L_\psi^2\,. \tag{5.39}$$

Thus, again from (5.36) we can read

$$\mathcal{K} = -\partial_\theta \otimes \partial_\theta - \frac{1}{\sin^2\theta}\partial_\phi \otimes \partial_\phi - \frac{1}{\cos^2\theta}\partial_\psi \otimes \partial_\psi \,. \tag{5.40}$$

This Killing tensor however turns out to be a reducible one. In this situation, since both rotation parameters, there is an additional Killing vector which represents the additional symmetry of being able to rotate each of the two rotation planes into each other. This Killing tensor can be obtained using linear combinations of outer products of this Killing vector. Further details and explicit constructions can be found in [26].

We can easily check using Maple [32], that the Killing tensors in both spacetimes do satisfy the Killing equation. It is the existence of these Killing tensors that allows for complete separation of the Hamilton-Jacobi equation.

5.7 The Scalar Field Equation

Consider a scalar field Ψ in a gravitational background with the action

$$S[\Psi] = -\frac{1}{2}\int d^D x\sqrt{-g}((\nabla\Psi)^2 + \alpha R\Psi^2 + m^2\Psi^2)\,, \tag{5.41}$$

where we have included a curvature dependent coupling. However, in these Kerr-(Anti) de Sitter backgrounds with charges, R is constant (proportional to the cosmological constant Λ). As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein-Gordon equation in this background. We will simply set $\alpha = 0$ in the following. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = m^2\Psi\,. \tag{5.42}$$

5.7.1 Massive Scalar Fields in the Four Dimensional Kerr-Taub-NUT Multicharge Gauged Solution of Supergravity

Using the explicit expressions for the components of the inverse metric (5.5) and the determinant (5.4), the Klein-Gordon equation for a massive scalar field in this spacetime can be written as

$$\begin{aligned} m^2\Psi &= \frac{1}{W}\left(\frac{1}{\Delta_u\Delta_r}[\Delta_r u_1^2 u_2^2 - \Delta_u r_1^2 r_2^2]\partial_t^2\Psi + \frac{2a}{\Delta_r\Delta_u}[\Delta_r u_1 u_2 + \Delta_u r_1 r_2]\partial_{t\phi}^2\Psi\right. \\ &+ \left.\frac{2a^2}{\Delta_r\Delta_u}[\Delta_r - \Delta_u]\partial_\phi^2\Psi + \partial_r(\Delta_r\partial_r\Psi) + \partial_u(\Delta_u\partial_u\Psi)\right)\,. \end{aligned} \tag{5.43}$$

(Note that this expression agrees with equation (16) in [28] with the notation $p = a\cos\theta$, $q = r$, $X = \Delta_u$, $Y = \Delta_r$, and $k = 0$ in four dimensions for the uncharged, i.e. $\delta_i = 0$, Kerr-Taub-NUT metrics. This is a good check for consistency.) We assume the usual multiplicative ansatz for the separation of the Klein-Gordon equation

$$\Psi = \Phi_r(r)\Phi_u(u)e^{-iEt}e^{iL_\phi \phi}\,. \tag{5.44}$$

Then we can easily separate out the r and u dependance as

$$\begin{aligned} K &= -\frac{1}{\Phi_u(u)}\frac{d}{du}\left(\Delta_u \frac{d\Phi_u(u)}{du}\right) + m^2 u_1 u_2 + \frac{u_1^2 u_2^2}{\Delta_u}E^2 - \frac{2au_1u_2}{\Delta_u}EL_\phi + \frac{2a^2L_\phi^2}{\Delta_u}\,, \\ K &= \frac{1}{\Phi_r(r)}\frac{d}{dr}\left(\Delta_r \frac{d\Phi_r(r)}{dr}\right) - m^2 r_1 r_2 - \frac{r_1^2 r_2^2}{\Delta_r}E^2 + \frac{2ar_1r_2}{\Delta_r}EL_\phi + \frac{2a^2L_\phi^2}{\Delta_r}\,, \end{aligned} \tag{5.45}$$

where K is again a separation constant. At this point we have completely separated out the Klein-Gordon equation for a massive scalar field in this spacetime.

5.7.2 Massive Scalar Fields in the $U(1)^3$ Gauged Kerr-(Anti) de Sitter Black Hole Solution of $\mathcal{N} = 2$ Supergravity in Five Dimensions

Using the explicit expressions for the components of the inverse metric (5.13) and the determinant (5.15), the Klein-Gordon equation for a massive scalar field in this spacetime can be written as

$$\begin{aligned} m^2\Psi &= -\frac{B(r)+R}{r^2W(r)}\partial_t^2\Psi + \frac{2C(r)}{r^2W(r)}(\partial_{t\phi}^2\Psi + \partial_{t\psi}\Psi) + \frac{1}{rR}\partial_r(rRW(r)\partial_r\Psi) \\ &+ \frac{A(r)B(r)\cos^2\theta + A(r)R + C^2(r)\cos^2\theta}{R\sin^2\theta r^2W(r)}\partial_\phi^2\Psi \\ &+ \frac{A(r)B(r)\sin^2\theta + A(r)R + C^2(r)\sin^2\theta}{R\cos^2\theta r^2W(r)}\partial_\psi^2\Psi - \frac{2[A(r)B(r)+C^2(r)]}{Rr^2W(r)}\partial_{\phi\psi}^2\Psi \\ &+ \frac{1}{R\sin\theta\cos\theta}\partial_\theta(\sin\theta\cos\theta\partial_\theta\Psi)\,. \end{aligned} \tag{5.46}$$

Again we assume the usual multiplicative ansatz for separation

$$\Psi = \Phi_r(r)\Phi_\theta(\theta)e^{-iEt}e^{iL_\phi\phi}e^{iL_\psi\psi}\,. \tag{5.47}$$

After extensive algebraic manipulation similar to that of the Hamilton-Jacobi equation, and the use of some trigonometric identities along the way, we find that the r and

θ equations decouple into the form

$$
\begin{aligned}
K &= -\frac{1}{\Phi_\theta(\theta)\sin\theta\cos\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\frac{d\Phi_\theta(\theta)}{d\theta}\right)+\frac{L_\phi^2}{\sin^2\theta}+\frac{L_\psi^2}{\cos^2\theta}\,, \\
K &= \frac{1}{\Phi_r(r)r}\frac{d}{dr}\left(rRW(r)\frac{d\Phi_r(r)}{dr}\right)+\frac{B(r)R+R^2}{r^2W(r)}E^2+\frac{2C(r)R}{r^2W(r)}(L_\phi+L_\psi) \\
&+ \frac{A(r)B(r)+C^2(r)}{r^2W(r)}(L_\phi+L_\psi)^2\,. \qquad (5.48)
\end{aligned}
$$

where K is again a separation constant. At this point we have completely separated out the Klein-Gordon equation for a massive scalar field in this spacetime.

We note the role of the Killing tensors in the separation terms of the Klein-Gordon equations in both spacetimes. In fact, the complete integrability of geodesic flow of both metrics via the Hamilton-Jacobi equation can be viewed as the classical limit of the statement that the Klein-Gordon equation in both metrics also completely separates.

Conclusions

We studied the complete integrability properties of the Hamilton-Jacobi and the Klein-Gordon equations in the background of two recently discovered rotating black hole solutions of supergravity with charge(s): the four dimensional Kerr-Taub-NUT Multicharge gauged supergravity solution, and the $U(1)^3$ gauged Kerr-(Anti) de Sitter black hole solution of $\mathcal{N}=2$ supergravity in five dimensions. Complete separation of both the Hamilton-Jacobi and Klein-Gordon equations in these backgrounds in Boyer-Lindquist-like coordinates is demonstrated. This is due to the enlarged dynamical symmetry of the spacetime. We construct the Killing tensors (one of them irreducible) in both spacetimes which explicitly permits complete separation. We also derive first-order equations of motion for classical particles in these backgrounds, and analyze the properties of some special trajectories. It should be emphasized that these complete integrability properties are a fairly non-trivial consequence of the specific form of the metrics, and generalize several such remarkable properties for other previously known metrics.

Further work in this direction could include the study of higher-spin field equations in these backgrounds, which is of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results. The geodesic equations presented can also readily be used in the study of black hole singularity structure in an AdS background using the AdS/CFT correspondence.

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Chapter 6

NUT-Charged Rotating Spacetimes

6.1 Introduction

Taub-NUT solutions arise in a very wide variety of situations in both string theory and general relativity. NUT-charged spacetimes, in general, are studied for their unusual properties which typically provide rather unique counterexamples to many notions in Einstein gravity. They are also widely studied in the context of issues of chronology protection in the AdS/CFT correspondence. Understanding the nature of geodesics in these backgrounds, as well as scalar field propagation, could prove to be very interesting in further exploration of these spacetimes.

There is a strong need to understand explicitly the structure of geodesics in the background of black holes in Anti-de Sitter space in the context of string theory and the AdS/CFT correspondence. This is due to the recent work in exploring black hole singularity structure using geodesics and correlators in the dual CFT on the boundary [1–6]. Black holes with charge are particularly interesting for this type of analysis since the charges are reinterpreted as the R-charges of the dual theory. The class of solutions dealt with in this paper also include black holes that carry both NUT and electric charges in various dimensions, and could prove very interesting in this sort of analysis.

In this paper we explore a very general metric describing a wide variety of spacetimes with NUT charge(s). In addition further metrics can also be obtained from these through various analytic continuations (which does not affect separability as demonstrated for these class of metrics). As such, the study of separability in this set of spacetimes encompasses the cases of both singly and multiply NUT-charged solutions, electrically and magnetically charged solutions with NUT parameter(s), solutions with a cosmological constant and NUT parameters(s), and time dependant bubble-like NUT-charged solutions. Many of these describe very interesting gravitational instantons. Some of these solutions include static backgrounds, while others are time-dependent and provide very interesting backgrounds for studying both string theory and general relativity. Some of these solutions, especially the bubble-like ones, are particularly interesting in the context of string theory as they arise in the context of topology changing processes. e.g. they show up as possible end states for Hawking evaporation., and they show up in transitions of black strings in closed string tachyon condensation.

We study the separability of the Hamilton-Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We use this explicit separation to obtain first-order equations of motion for both massive and massless particles in these backgrounds. The equations are obtained in a form that could be used for numerical study, and also in the study of black hole singularity structure using geodesic probes and the AdS/CFT correspondence. We also study the Klein-Gordon equation describing the propagation of a massive scalar field in these spacetimes. Separation again turns out to be possible with the usual multiplicative ansatz.

Separation is possible for both equations in these metrics due to the existence of non-trivial second-order Killing tensors. The Killing tensors, in each case, provides an additional integral of motion necessary for complete integrability.

There has been a lot of work recently dealing with geodesics and integrability in black hole backgrounds in higher dimensions both with and without the presence of a cosmological constant [7–16]. Of particular note in the context of this paper are [12, 14] which deal with black holes with NUT parameters in some special cases. This work extends, and generalizes, some of the results obtained in these papers.

6.2 Overview of the Metrics

The class of metrics dealt with in this paper, and their generalizations obtained via analytic continuations, have been constructed and analyzed in [17–22], as well as some references contained therein. We will very briefly describe the metrics, and some of the various types of spacetimes that can be obtained from them. As mentioned earlier, separability for all the metrics is addressed by dealing with the class we do here, since analytic continuations do not affect separability of either the Hamilton-Jacobi or Klein-Gordon equation (though they do affect the physical interpretations of the various variables and their associated conserved quantities).

The general spacetimes we study are described by the metrics

$$ds^2 = -F(r)\left[dt + \sum_{i=1}^{p} 2N_i f_i(\theta_i)d\phi_i\right]^2 + \frac{dr^2}{F(r)} + \sum_{i=1}^{p}(r^2 + N_i^2)(d\theta_i^2 + g_i^2(\theta_i)d\phi_i^2)\,. \quad (6.1)$$

A very general class of metrics in even dimensions where the (ϕ_i, θ_j) sector has the form $M_1 \times M_2 \times \ldots \times M_p$, with each M_i a two dimensional space of constant curvature δ_i. In this case the functions are given by

$$\begin{aligned}
\delta_i = 1: &\quad f_i(\theta_i) = -\cos\theta_i\,, \quad g_i^2(\theta_i) = \sin^2\theta_i\,,\\
\delta_i = 0: &\quad f_i(\theta_i) = -\theta_i\,, \quad g_i^2(\theta_i) = 1\,,\\
\delta_i = -1: &\quad f_i(\theta_i) = -\cosh\theta_i\,, \quad g_i^2(\theta_i) = \sinh^2\theta_i\,, \qquad (6.2)
\end{aligned}$$

and an expression for $F(r)$ can be found in [21] along with a detailed description. Generalizations to include electric charge are obtained by suitably modifying $F(r)$, and can be found in [20, 22]. Metrics describing "bubbles of nothing" also fall under this class and can be found in [19]. Examples of NUT-charged spacetimes in cosmological backgrounds also fall in this framework and can be found in [19].

For the purposes of analyzing separability, some odd dimensional NUT-charged spacetimes also fall under this category. For instance in five dimensions (i.e $p = 2$) a NUT charged spacetime is obtained by taking $g_2(\theta_2) = 0$ and $N_2 = 0$, i.e. a metric of the form

$$ds^2 = -F(r)(dt - 2N_1\cosh\theta_1 d\phi_1)^2 + \frac{dr^2}{F(r)} + (r^2 + N_1^2)(d\theta_1^2 + \sinh^2\theta_1 d\phi_1^2) + r^2 d\theta_2^2 \quad (6.3)$$

This describes a spacetime in an AdS background; similar dS and flat background spacetimes can be obtained by following the prescriptions in (6.2) while maintaining $g_2(\theta_2) = 0$ and $N_2 = 0$. Generalizations to higher odd dimensional spacetimes are obvious.

Various twists of these spacetimes can also be obtained through analytic continuations. For instance, using the prescriptions $t \to i\theta, \theta \to it$, we can obtain time-dependent bubbles. In five dimensions in an AdS background, some examples obtained via this prescription, and a few other suitable obvious variable redefinitions are

$$\begin{aligned}
ds^2 &= F(r)(d\theta_1 + 2N_1\cos t d\phi)^2 + \frac{dr^2}{F(r)} + (r^2 + N_1^2)(-dt^2 + \sin^2 t d\phi^2) + r^2 d\theta_2^2\,,\\
ds^2 &= F(r)(d\theta_1 + 2N_1\sinh\phi dt)^2 + \frac{dr^2}{F(r)} + (r^2 + N_1^2)(d\phi^2 - \cosh^2\phi dt^2) + r^2 d\theta_2^2\,.
\end{aligned}$$

$$
\begin{aligned}
ds^2 &= F(r)(d\theta_1 + 2N_1 \cosh\phi dt)^2 + \frac{dr^2}{F(r)} + (r^2 + N_1^2)(d\phi^2 - \sinh^2\phi dt^2) + r^2 d\theta_2^2 , \\
ds^2 &= F(r)(d\theta_1 + 2N_1 e^{\phi} dt)^2 + \frac{dr^2}{F(r)} + (r^2 + N_1^2)(d\phi^2 - e^{2\phi} dt^2) + r^2 d\theta_2^2 . \qquad (6.4)
\end{aligned}
$$

For future use, we give the determinant of the metric (6.1)

$$
g = -\prod_{i=1}^{p} (r^2 + N_i^2)^2 g_i^2(\theta_i) . \qquad (6.5)
$$

The components of the inverse metric are

$$
\begin{aligned}
g^{tt} &= \sum_{i=1}^{p} \frac{4N_i^2 f_i^2(\theta_i)}{(r^2 + N_i^2) g_i^2(\theta_i)} - \frac{1}{F(r)} , \\
g^{t\phi_i} &= -\frac{2N_i f_i(\theta_i)}{g_i^2(\theta_i)(r^2 + N_i^2)} , \\
g^{\phi_i\phi_j} &= \frac{\delta_{ij}}{(r^2 + N_i^2) g_i^2(\theta_i)} , \\
g^{rr} &= F(r) , \\
g^{\theta_i\theta_j} &= \frac{\delta_{ij}}{r^2 + N_i^2} . \qquad (6.6)
\end{aligned}
$$

These formulae are somewhat tedious to derive, but can be proved using a few Maple calculations, and then using mathematical induction [23].

6.3 The Hamilton-Jacobi Equation and Separability

The Hamilton-Jacobi equation in a curved background is given by

$$
-\frac{\partial S}{\partial \lambda} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} , \qquad (6.7)
$$

where S is the action associated with the particle and λ is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parametrized by proper time.

6.3.1 Separability

We can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2}m^2\lambda - Et + \sum_{i=1}^{p} L_{\phi_i}\phi_i + \sum_{i=1}^{p} S_{\theta_i}(\theta_i) + S_r(r)\,. \tag{6.8}$$

t and the ϕ_i are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and those with rotation in the ϕ_i are the corresponding angular momenta L_{ϕ_i}. Then, using the components of the inverse metric (6.6), the Hamilton-Jacobi equation (6.7) is written to be

$$\begin{aligned} -m^2 &= \sum_{i=1}^{p} \frac{4N_i^2 f_i^2(\theta_i)}{(r^2+N_i^2)g_i^2(\theta_i)}E^2 - \frac{E^2}{F(r)} - \sum_{i=1}^{p}\frac{4N_i f_i(\theta_i)}{(r^2+N_i^2)g_i^2(\theta_i)}(L_{\phi_i})(-E) \\ &+ \sum_{i=1}^{p}\frac{1}{(r^2+N_i^2)g_i^2(\theta_i)}L_{\phi_i}^2 + F(r)\left[\frac{dS_r(r)}{dr}\right]^2 + \sum_{i=1}^{p}\frac{1}{r^2+N_i^2}\left[\frac{dS_{\theta_i}(\theta_i)}{d\theta_i}\right]^2 \end{aligned} \tag{6.9}$$

After some manipulation, we can recursively separate out the equation into

$$\begin{aligned} -m^2 &= -\frac{E^2}{F(r)} + F(r)\left[\frac{dS_r(r)}{dr}\right]^2 + \sum_{i=1}^{p}\frac{K_i}{r^2+N_i^2}\,, \\ K_i &= \left[\frac{dS_{\theta_i}(\theta_i)}{d\theta_i}\right]^2 + \left[\frac{L_{\phi_i}+2N_i f_i(\theta_i)E}{g_i(\theta_i)}\right]^2\,. \end{aligned} \tag{6.10}$$

For future reference we will use the notation $K = \sum_{i=1}^{p} K_i$. Also note that for the metrics obtained through analytic continuations discussed earlier, the issue of separability is clearly not affected. However, for an analytic continuation of the form $t \to i\theta, \theta \to it$, we need to replace $E \to -iL_\theta$, and the energy is no longer conserved as we have a time-dependent background. However, now the angular momentum L_θ associated to θ is conserved. Similar substitutions need to be made for any other analytic continuations or variable redefinitions used to define the new metrics.

6.3.2 The Equations of Motion

To derive the equations of motion, we will write the separated action S from the Hamilton-Jacobi equation in the following form:

$$S = \frac{1}{2}m^2\lambda - Et + \sum_{i=1}^{p} L_{\phi_i}\phi_i + \int^r \sqrt{\mathcal{R}(r')}dr' + \sum_{i=1}^{p}\int^{\theta_i}\sqrt{\Theta_i(\theta_i')}d\theta_i'\,, \tag{6.11}$$

where

$$
\begin{aligned}
F(r)\mathcal{R}(r) &= -\sum_{i=1}^{p} \frac{K_i}{r^2+N_i^2} + \frac{E^2}{F(r)} - m^2 , \\
\Theta_i(\theta_i) &= K_i - \left[\frac{L_{\phi_i} + 2N_i f_i(\theta_i)E}{g_i(\theta_i)}\right]^2 .
\end{aligned} \tag{6.12}
$$

To obtain the equations of motion, we differentiate S with respect to the parameters m^2, K_i, E, L_{ϕ_i} and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

$$
\begin{aligned}
\frac{\partial S}{\partial m^2} &= 0 \Rightarrow \lambda = \int \frac{dr}{F(r)\sqrt{\mathcal{R}(r)}} , \\
\frac{\partial S}{\partial K_i} &= 0 \Rightarrow \int \frac{d\theta_i}{\sqrt{\Theta_i}} = \int \frac{1}{(r^2+N_i^2)} \frac{dr}{F(r)\sqrt{\mathcal{R}(r)}} , \\
\frac{\partial S}{\partial L_{\phi_i}} &= 0 \Rightarrow \phi_i = \int \frac{L_{\phi_i} + 2N_i f_i(\theta_i)E}{g_i^2(\theta_i)} \frac{d\theta_i}{\sqrt{\Theta_i(\theta_i)}} , \\
\frac{\partial S}{\partial E} &= 0 \Rightarrow t = \int \frac{E}{F^2(r)} \frac{dr}{\sqrt{\mathcal{R}(r)}} - \sum_{i=1}^{p} \int \frac{2N_i L_{\phi_i} f_i(\theta_i) + 4N_i^2 f_i^2(\theta_i)E}{g_i^2(\theta_i)} \frac{d\theta_i}{\sqrt{\Theta_i(\theta_i)}} .
\end{aligned} \tag{6.13}
$$

It is often more convenient to rewrite these in the form of first-order differential equations obtained from (6.13) by direct differentiation with respect to the affine parameter:

$$
\begin{aligned}
\frac{dr}{d\lambda} &= F(r)\sqrt{\mathcal{R}(r)} , \\
\frac{d\theta_i}{d\lambda} &= \frac{\sqrt{\Theta_i(\theta_i)}}{r^2+N_i^2} , \\
\frac{d\phi_i}{d\lambda} &= \frac{L_{\phi_i} + 2N_i f_i(\theta_i)E}{g_i^2(\theta_i)(r^2+N_i^2)} , \\
\frac{dt}{d\lambda} &= \frac{E}{F(r)} - \sum_{i=1}^{p} \frac{2N_i L_{\phi_i} f_i(\theta_i) + 4N_i^2 f_i^2(\theta_i)E}{g_i^2(\theta_i)(r^2+N_i^2)} .
\end{aligned} \tag{6.14}
$$

6.4 Dynamical Symmetry

The general class of metrics discussed here are stationary and "axisymmetric"; i.e., $\partial/\partial t$ and the $\partial/\partial\phi_i$ are Killing vectors and have associated conserved quantities, $-E$ and L_{ϕ_i}. In general, if ξ is a Killing vector, then $\xi^\mu p_\mu$ is a conserved quantity, where p is the momentum of the particle. Note that this quantity is first order in the momenta.

The additional constants of motion K_i which allowed for complete integrability of the equations of motion is not related to a Killing vector from a cyclic coordinate. These constants are, rather, derived from irreducible second-order Killing tensors in which permit the complete separation of equations. Killing tensors are not symmetries on configuration space and cannot be derived from a Noether procedure, but they are instead symmetries on phase space. They obey a generalization of the Killing equation for Killing vectors (which do generate symmetries in configuration space by the Noether procedure) given by

$$\mathcal{K}_{(\mu\nu;\rho)} = 0\,, \tag{6.15}$$

where $\mathcal{K}$ is any second order Killing tensor, and the parentheses indicate complete symmetrization of all indices.

The Killing tensors can be obtained from the expressions for the separation constants K_i in each case. If the particle has momentum p, then the Killing tensor $\mathcal{K}_{\mu\nu}$ is related to the constant K via

$$K = \mathcal{K}^{\mu\nu} p_\mu p_\nu = \mathcal{K}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}\,. \tag{6.16}$$

We can use the expression for the K_i in terms of the the θ_i equations.

For the Taub-NUT metrics analyzed above, the expression for K_i from (6.10) is

$$K_i = \left[\frac{dS_{\theta_i}(\theta_i)}{d\theta_i}\right]^2 + \left[\frac{L_{\phi_i} + 2N_i f_i(\theta_i) E}{g_i(\theta_i)}\right]^2\,. \tag{6.17}$$

Thus, from (6.16) we can readily read

$$\mathcal{K}_i = \partial_{\theta_i} \otimes \partial_{\theta_i} + \frac{1}{g_i(\theta_i)^2}\left[\partial_{\phi_i} \otimes \partial_{\phi_i} + 4N_i^2 f_i^2(\theta_i)\partial_t \otimes \partial_t - 2N_i f_i(\theta_i) sym(\partial_{\phi_i} \otimes \partial_t)\right] \tag{6.18}$$

We can easily check using Maple [23], that the Killings tensors do satisfy the Killing equation.

Note that if any of the NUT parameters N_k were zero, then the corresponding Killing tensor $\mathcal{K}_k$ would simply be the usual Killing tensor of the underlying two dimensional space M_k (which is a reducible one in the case of a homogenous constant curvature space M_k, as is the case for many situations here). In general, however, a non-zero NUT parameter N_k provides a nontrivial coupling between the (r, ϕ_i, θ_i) sectors, and the existence of the Killing vectors ∂_{ϕ_i} and ∂_t along is not enough to ensure complete separability. It is the existence of these nontrivial irreducible Killing tensors $\mathcal{K}_i$ that provides the addition separation constants K_i necessary for complete separation of each space M_i from another space M_j, as well as separation of the angular sectors completely from the radial sector. These tensors are irreducible since they are not simply linear combinations of tensor products of Killing vectors of the spacetime.

6.5 The Scalar Field Equation

Consider a scalar field Ψ in a gravitational background with the action

$$S[\Psi] = -\frac{1}{2}\int d^D x\sqrt{-g}((\nabla\Psi)^2 + \alpha R\Psi^2 + m^2\Psi^2)\,, \tag{6.19}$$

where we have included a curvature dependent coupling. However, in these (Anti)-de Sitter and flat backgrounds with charges, R is constant (proportional to the cosmological constant Λ). As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein-Gordon equation in this background. We will simply set $\alpha = 0$ in the following. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = m^2\Psi\,. \tag{6.20}$$

Using the explicit expressions for the components of the inverse metric (6.6) and the determinant (6.5), the Klein-Gordon equation for a massive scalar field in this spacetime can be written as

$$\begin{aligned} m^2\Psi \;=\; & \left[\sum_{i=1}^{p}\frac{4N_i^2 f_i^2(\theta_i)}{(r^2+N_i^2)g_i^2(\theta_i)} - \frac{1}{F(r)}\right]\partial_t^2\Psi - \sum_{i=1}^{p}\frac{4N_i f_i(\theta_i)}{g_i^2(\theta_i)(r^2+N_i^2)}\partial^2_{t\phi_i}\Psi \\ & + \sum_{i=1}^{p}\frac{1}{(r^2+N_i^2)g_i^2(\theta_i)}\partial^2_{\phi_i}\Psi + \frac{1}{\prod_{i=1}^{p}(r^2+N_i^2)}\frac{\partial}{\partial r}\left[\prod_{i=1}^{p}(r^2+N_i^2)F(r)\frac{\partial\Psi}{\partial r}\right] \\ & + \sum_{i=1}^{p}\frac{1}{(r^2+N_i^2)g_i(\theta_i)}\frac{\partial}{\partial\theta_i}\left[g_i(\theta_i)\frac{\partial\Psi}{\partial\theta_i}\right]\,. \end{aligned} \tag{6.21}$$

We assume the usual multiplicative ansatz for the separation of the Klein-Gordon equation

$$\Psi = \Phi_r(r)e^{-iEt}e^{i\sum_{i=1}^{p}L_{\phi_i}\phi_i}\prod_{i=1}^{p}\Phi_{\theta_i}(\theta_i)\,. \tag{6.22}$$

Then we can easily completely separate the Klein-Gordon equation as

$$\begin{aligned} K_i &= \frac{1}{g_i(\theta_i)\Phi_{\theta_i}(\theta_i)}\frac{d}{d\theta_i}\left[g_i(\theta_i)\frac{d\Phi_{\theta_i}(\theta_i)}{d\theta_i}\right] - \left[\frac{L_{\phi_i}+2N_i f_i(\theta_i)E}{g_i(\theta_i)}\right]^2 , \\ -m^2 &= \frac{1}{\prod_{i=1}^{p}(r^2+N_i^2)}\frac{d}{dr}\left[\prod_{i=1}^{p}(r^2+N_i^2)F(r)\frac{d\Phi_r(r)}{dr}\right] + \frac{E^2}{F(r)} + \sum_{i=1}^{p}\frac{K_i}{r^2+N_i^2} \end{aligned} \tag{6.23}$$

where the K_i are again separation constants. At this point we have completely separated out the Klein-Gordon equation for a massive scalar field in these spacetimes.

We note the role of the Killing tensors in the separation terms of the Klein-Gordon equations in these spacetimes. In fact, the complete integrability of geodesic flow of the metrics via the Hamilton-Jacobi equation can be viewed as the classical limit of the statement that the Klein-Gordon equation in these metrics also completely separates.

Conclusions

We studied the complete integrability properties of the Hamilton-Jacobi and the Klein-Gordon equations in the background of a very general class of Taub-NUT metrics in higher dimensions, which include the cases of both singly and multiply NUT-charged solutions, electrically and magnetically charged solutions with NUT parameter(s), solutions with a cosmological constant and NUT parameter(s), and time-dependent bubble-like NUT-charged solutions, and other very interesting gravitational instantons. Complete separation of both the Hamilton-Jacobi and Klein-Gordon equations in these backgrounds is demonstrated. This is due to the enlarged dynamical symmetry of the spacetime. We construct the Killing tensors in these spacetimes which explicitly permit complete separation. We also derive first-order equations of motion for classical particles in these backgrounds. It should be emphasized that these complete integrability properties are a fairly non-trivial consequence of the specific form of the metrics, and generalize several such remarkable properties for other previously known metrics.

Further work in this direction could include the study of higher-spin field equations in these backgrounds, which is of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results. The geodesic equations presented can also readily be used in the study of black hole singularity structure in an AdS background using the AdS/CFT correspondence.

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Chapter 7

Two Parameter Kerr-de Sitter Metrics

7.1 Introduction

A number of recent developments in high energy physics have generated great interest in vacuum solutions of Einstein equations describing higher dimensional black holes, and the properties of these spacetimes. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (e.g. the hierarchy problem) [1–3]. Higher dimensional black hole solutions arise naturally in such models. These models are also of interest in the context of mini-black hole production in high energy particle colliders, which would provide a window into non-perturbative gravitational physics [4, 5].

Superstring and M-theory also naturally give rise to higher dimensional black holes in their 10 or 11 dimensional ambient spacetimes. P-branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Solitonic objects in superstring theory frequently find a natural description in terms of higher dimensional black holes. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6, 7].

The Kerr metric describes astrophysically relevant black hole spacetimes, to a very good approximation [8]. One generalization of the Kerr metric to higher dimensions is given by the Myers-Perry construction [9]. With interest now in a nonzero cosmological constant, it is worth studying spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an Anti-de Sitter background could give rise to interesting descriptions in terms of the conformal field theory on the boundary leading to better understanding of the correspondence [10, 11]. The general Kerr-de Sitter metrics describing rotating black holes in the presence of a cosmological constant have been constructed explicitly in [12, 13].

There is also a very strong need to understand the structure of geodesics in the background of black holes in Anti-de Sitter backgrounds in the context of string theory and the AdS/CFT correspondence. This is due to the recent work in exploring black hole singularity structure using geodesics and correlators on the dual CFT on the boundary [14–19].

In this paper we study the separability of the Hamilton-Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We also investigate the separability of the Klein-Gordon equation describing a massive scalar field propagating in this background. We explicitly perform the separation in the case where there are only two sets of equal rotation parameters describing the black hole. We use this explicit separation to obtain first-order equations of motion for both massive and massless particles in these backgrounds. The equations are obtained in a form that could be used for numerical study, and also in the study of black hole singularity structure using geodesic probes and the AdS/CFT correspondence.

We also study the Klein-Gordon equation describing the propagation of a massive scalar field in this spacetime. Separation is again explicitly shown for the case of two sets of equal black hole rotation parameters. We construct the separation of both equations explicitly in these cases.

This paper greatly generalizes the results of [20,21] for the Myers-Perry metric in five dimensions, [22] which separates the equations in the case of equal rotation parameters in the odd dimensional Kerr-(A)dS spacetimes, and [23] which separates the equations in the case of two independent sets of rotation parameters in the Myers-Perry metrics in all dimensions, as well as some related results in five dimensional black hole spacetimes in [24, 25].

Separation is possible for both equations in this case due to the existence of a second-order non-trivial irreducible Killing tensor. This is a generalization of the Killing tensor in the Kerr black hole spacetime in four dimensions constructed in [26] which was subsequently described by Chandrasekhar as the "miraculous property of the Kerr metric". A similar construction for the Myers-Perry metrics in higher dimensions has also been done [20, 23]. The Killing tensor provides an additional integral of motion necessary for complete integrability. We also construct Killing vectors, which exist due to the additional symmetry, and which permit the separation of these equations.

7.2 Construction and Overview of the Kerr-de Sitter Metrics

One of the most useful properties of the Kerr metric is that it can be written in the Kerr-Schild [27] form, where the metric $g_{\mu\nu}$ is given exactly by its linear approximation around the flat metric $\eta_{\mu\nu}$ as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{U}\,(k_\mu dx^\mu)^2\,, \tag{7.1}$$

where k_μ is null and geodesic with respect to both the full metric $g_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$.

The Kerr-de Sitter metrics in all dimensions were obtained in [12] by using the de Sitter metric instead of the flat background $\eta_{\mu\nu}$, with coordinates chosen appropriately to allow for the incorporation of the Kerr metric via the null geodesic vectors k_μ. We quickly review the construction here.

In D-dimensional spacetime, we introduce $n = [D/2]$ coordinates μ_i, where $[i]$ denotes the integer part of i, subject to the constraint

$$\sum_{i=1}^{n} \mu_i^2 = 1\,, \tag{7.2}$$

together with $N = [(D-1)/2]$ azimuthal angular coordinates ϕ_i, the radial coordinate r, and the time coordinate t. When the total spacetime dimension D is odd, $D = 2n+1 = 2N+1$, there are n azimuthal coordinates ϕ_i, each with period 2π. If D is even, $D = 2n = 2N+2$, there are only $N = n-1$ azimuthal coordinates ϕ_i. Define ϵ to be 1 for even D, and 0 for odd D.

The Kerr-de Sitter metric ds^2 in D dimensional spacetime satisfies the Einstein equation with cosmological constant λ:

$$R_{\mu\nu} = (D-1)\,\lambda\, g_{\mu\nu}\,. \tag{7.3}$$

Define functions W and F as follows:

$$W \equiv \sum_{i=1}^{n} \frac{\mu_i^2}{1+\lambda\, a_i^2}\,, \qquad F \equiv \frac{r^2}{1-\lambda\, r^2} \sum_{i=1}^{n} \frac{\mu_i^2}{r^2+a_i^2}\,. \tag{7.4}$$

In D dimensions, the Kerr-de Sitter metrics are given by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}\,(k_\mu\, dx^\mu)^2\,, \tag{7.5}$$

where the de Sitter metric $d\bar{s}^2$, the null vector k_μ, and the function U are now given by

$$\begin{aligned} d\bar{s}^2 &= -W\,(1-\lambda\, r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{n} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2+a_i^2}{1+\lambda\, a_i^2}\, \mu_i^2\, d\phi_i^2 \\ &\quad + \frac{\lambda}{W\,(1-\lambda\, r^2)} \Big(\sum_{i=1}^{n} \frac{(r^2+a_i^2)\,\mu_i\, d\mu_i}{1+\lambda\, a_i^2} \Big)^2\,, \qquad (7.6) \\ k_\mu\, dx^\mu &= W\, dt + F\, dr - \sum_{i=1}^{n-\epsilon} \frac{a_i\, \mu_i^2}{1+\lambda\, a_i^2}\, d\phi_i\,, \qquad (7.7) \\ U &= r^\epsilon \sum_{i=1}^{n} \frac{\mu_i^2}{r^2+a_i^2} \prod_{j=1}^{n-\epsilon} (r^2+a_j^2)\,. \qquad (7.8) \end{aligned}$$

In the even-dimensional case, where there is no azimuthal coordinate ϕ_n, there is also no associated rotation parameter; i.e., $a_n = 0$. Note that the null vector corresponding to the null one-form is

$$k^\mu\, \partial_\mu = -\frac{1}{1-\lambda\, r^2}\, \frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-\epsilon} \frac{a_i}{r^2+a_i^2}\, \frac{\partial}{\partial \phi_i}\,. \tag{7.9}$$

This is easily obtained by using the background metric to raise and lower indices rather than the full metric, since k is null with respect to both metrics.

For the purposes of analyzing the equations of motion and the Klein-Gordon equation, it is very convenient to work with the metric expressed in Boyer-Lindquist coordinates. In these coordinates there are no cross terms involving the differential dr. In both even and odd dimensions, the Boyer-Lindquist form is obtained by means of the following coordinate transformation:

$$dt = d\tau + \frac{2M\, dr}{(1-\lambda\, r^2)(V-2M)}\,, \qquad d\phi_i = d\varphi_i - \lambda\, a_i\, d\tau + \frac{2M\, a_i\, dr}{(r^2+a_i^2)(V-2M)}\,. \tag{7.10}$$

In Boyer-Lindquist coordinates in D dimensions, the Kerr-de Sitter metrics are given by

$$\begin{aligned} ds^2 &= -W\,(1-\lambda r^2)\,d\tau^2 + \frac{U\,dr^2}{V-2M} + \frac{2M}{U}\Big(d\tau - \sum_{i=1}^{n-\epsilon}\frac{a_i\,\mu_i^2\,d\varphi_i}{1+\lambda\,a_i^2}\Big)^2 \\ &+ \sum_{i=1}^{n}\frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,d\mu_i^2 + \sum_{i=1}^{n-\epsilon}\frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,\mu_i^2\,(d\varphi_i - \lambda\,a_i\,d\tau)^2 \\ &+ \frac{\lambda}{W\,(1-\lambda r^2)}\Big(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1+\lambda\,a_i^2}\Big)^2\,, \end{aligned} \tag{7.11}$$

where V is defined here by

$$V \equiv r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{n-\epsilon}(r^2+a_i^2) = \frac{U}{F}\,, \tag{7.12}$$

Note that obviously $a_n = 0$ in the even dimensional case, as there is no rotation associated with the last direction.

7.3 Inverting the Kerr-(A)dS metric in all dimensions

We briefly review the process of inversion of the metric using the Kerr-Schild formalism. More extensive details of this type of procedure can be found in [22, 23]. This section will also help establish some useful notation and conventions for the rest of the paper. Note that the metric is block diagonal in the (μ_i) and the (r, τ, φ_i) sectors and so can be inverted separately.

To deal with the (r, τ, φ_i) sector, the most efficient method is to use the Kerr-Schild construction of the metric. From (7.1) and using the fact that k is null, we can write

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{U}k^\mu k^\nu\,, \tag{7.13}$$

where η here is the de Sitter metric rather than the flat metric, and we raise and lower indices with η. Since the null vector k has no components in the μ_i sector, we can regard the above equation as holding true in the (r, τ, φ_i) sector with k null here as well. Then we can explicitly perform the coordinate transformation (7.10) (or rather its inverse) on the raised metric to obtain the components of $g^{\mu\nu}$ in Boyer-Lindquist coordinates in the (r, τ, φ_i) sector.

We get the following components for the (r, τ, φ_i) sector of $g^{\mu\nu}$:

$$
\begin{aligned}
g^{\tau r} &= g^{\varphi_i r} = 0\,, \\
g^{rr} &= \frac{V-2M}{U}\,, \\
g^{\tau\tau} &= Q - \frac{4M^2}{U(1-\lambda r^2)^2(V-2M)}\,, \\
g^{\tau\varphi_i} &= \lambda a_i Q - \frac{4M^2 a_i(1+\lambda a_i^2)}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)} - \frac{2M}{U}\frac{a_i}{(1-\lambda r^2)(r^2+a_i^2)}\,, \\
g^{\varphi_i\varphi_j} &= \frac{(1+\lambda a_i^2)}{(r^2+a_i^2)\mu_i^2}\delta^{ij} + \lambda^2 a_i a_j Q + \frac{Q^{ij}}{U} \\
&\quad + \frac{4M^2 a_i a_j(1+\lambda a_i^2)(1+\lambda a_j^2)}{U(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)}\,,
\end{aligned} \tag{7.14}
$$

where Q and Q^{ij} are defined to be

$$
Q = -\frac{1}{W(1-\lambda r^2)} - \frac{2M}{U}\frac{1}{(1-\lambda r^2)^2}\,, \tag{7.15}
$$

$$
\begin{aligned}
Q^{ij} &= \frac{-4M^2\lambda a_i a_j[(1+\lambda a_j^2)(r^2+a_i^2)+(1+\lambda a_i^2)(r^2+a_j^2)]}{(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)} \\
&\quad - \frac{2M\lambda a_i a_j}{(1-\lambda r^2)}\left[\frac{1}{(r^2+a_i^2)} + \frac{1}{(r^2+a_j^2)}\right] - 2M\frac{a_i a_j}{(r^2+a_i^2)(r^2+a_j^2)} \\
&\quad - \frac{4M^2 a_i a_j[(1+\lambda a_i^2)+(1+\lambda a_j^2)]}{(1-\lambda r^2)^2(V-2M)(r^2+a_i^2)(r^2+a_j^2)}\,.
\end{aligned} \tag{7.16}
$$

These results were compared to previously known ones in the case of $\lambda = 0$ and showed agreement [20]. Also, we used the GRTensor package for Maple to explicitly check that this is the correct inverse metric [28].

Note that the functions W and U both depend explicitly on the μ_i's. Unless the (r, τ, φ_i) sector can be decoupled from the μ sector, complete separation is unlikely. If however, all the $a_i = a$ for some non-zero value a, then W and U are no longer μ dependent (taking the constraint into account) and separation seems likely. Note, however, that in this case we cannot deal with even dimensional spacetimes, since $a_n = 0$ is different from the other $a_i = a$. The analysis in this case has been done in detail in [22].

We will actually work with a much more general case, in which separation works in both even and odd dimensional spacetimes. We consider the situation in which the set of rotation parameters a_i take on at most only two distinct values a and b ($a = b$ can be obtained as a special case). In even dimensions at least one of these values must be zero, since $a_n = 0$. As such in even dimensions we take $b = 0$ and a to be some (possibly different) value. In the odd dimensional case, there are no restrictions on the values of a and b. We adopt the convention

$$a_i = a \quad \text{for} \quad i = 1, ..., m \quad , \quad a_{j+m} = b \quad \text{for} \quad j = 1, ..., p \, , \tag{7.17}$$

where $m + p = N + \epsilon = n$.

Since the μ_i's are constrained by (7.2), we need to use suitable independent coordinates instead. We use the following decomposition of the μ_i:

$$\mu_i = \lambda_i \sin\theta \quad \text{for} \quad i = 1, ..., m \quad , \quad \mu_{j+m} = \nu_j \cos\theta \quad \text{for} \quad j = 1, ..., p \, , \tag{7.18}$$

where the λ_i and ν_j have to satisfy the constraints

$$\sum_{i=1}^{m} \lambda_i^2 = 1 \quad , \quad \sum_{j=1}^{p} \nu_j^2 = 1 \, . \tag{7.19}$$

Since these constraints describe unit $(m-1)$ and $(p-1)$ dimensional spheres in the λ and ν spaces respectively, the natural choice is to use two sets of spherical polar coordinates. We write

$$\begin{aligned} \lambda_i &= \left(\prod_{k=1}^{m-i} \sin\alpha_k\right) \cos\alpha_{m-i+1} \, , \\ \nu_j &= \left(\prod_{k=1}^{p-j} \sin\beta_k\right) \cos\beta_{p-j+1} \, , \end{aligned} \tag{7.20}$$

with the understanding that the products are one when $i = m$ or $j = p$ respectively, and that $\alpha_m = 0$ and $\beta_p = 0$.

The μ sector metric can then be written as

$$\begin{aligned} ds_\mu^2 &= \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{r^2 + a^2}{\Sigma_a} \sin^2\theta \sum_{i=1}^{m-1} \left(\prod_{k=1}^{i-1} \sin^2\alpha_k\right) d\alpha_i^2 \\ &\quad + \frac{r^2 + b^2}{\Sigma_b} \cos^2\theta \sum_{j=1}^{p-1} \left(\prod_{k=1}^{j-1} \sin^2\beta_k\right) d\beta_j^2 \, , \end{aligned} \tag{7.21}$$

again with the understanding that the products are one when $i = 1$ or $j = 1$, and we use the definitions

$$\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \\
\Delta_\theta &= 1 + \lambda a^2\cos^2\theta + \lambda b^2\sin^2\theta\,, \\
\Sigma_a &= 1 + \lambda a^2\,, \\
\Sigma_b &= 1 + \lambda b^2\,, \\
Z &= r^\epsilon (r^2+a^2)^{m-1}(r^2+b^2)^{p-1-\epsilon}\,.
\end{aligned} \tag{7.22}$$

This diagonal metric can be easily inverted to give

$$\begin{aligned}
g^{\theta\theta} &= \frac{\Delta_\theta}{\rho^2}\,, \\
g^{\alpha_i\alpha_j} &= \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\alpha_k\right)}\delta_{ij}\,, \qquad i,j = 1,...,m\,, \\
g^{\beta_i\beta_j} &= \frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\beta_k\right)}\delta_{ij}\,, \qquad i,j = 1,...,p\,.
\end{aligned} \tag{7.23}$$

For the case of two sets of rotation parameters that we consider here, the following expressions will be extremely useful:

$$\begin{aligned}
U &= \rho^2 Z\,, \\
W &= \frac{\Delta_\theta}{\Sigma_a\Sigma_b}\,.
\end{aligned} \tag{7.24}$$

We note that both V and Z are functions of r only.

The following identity, which can be easily verified, will be crucial in the following:

$$Q = \frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)^2(V-2M)}\,. \tag{7.25}$$

7.4 The Hamilton-Jacobi Equation and Separation

The Hamilton-Jacobi equation in a curved background is given by

$$-\frac{\partial S}{\partial l} = H = \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}\,, \tag{7.26}$$

where S is the action associated with the particle and l is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parameterized by proper time.

We can attempt a separation of coordinates as follows. Let

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1} S_{\beta_i}(\beta_i)\,. \quad (7.27)$$

τ and φ_i are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy E, and the conserved quantity associated with rotation in each φ_i is the corresponding angular momentum Φ_i or Ψ_j. We also adopt the convention that $\Psi_p = 0$ in an even number of spacetime dimensions.

Using (7.14), (7.22), (7.23), and (7.27) we write the Hamilton-Jacobi equation (7.26) as

$$\begin{aligned}
-m^2 = \quad & \left[\frac{\Sigma_a\Sigma_b}{\lambda\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)} - \frac{4M^2}{\rho^2 Z(1-\lambda r^2)^2}\right] E^2 \\
+ \quad & 2\left[\frac{a\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{a\Sigma_a\Sigma_b}{\rho^2(1-\lambda r^2)} - \frac{4M^2 a\Sigma_a}{\rho^2 Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
- \quad & \left.\frac{2Ma}{\rho^2 Z(1-\lambda r^2)(r^2+a^2)} - \frac{2M\lambda a}{\rho^2 Z(1-\lambda r^2)^2}\right] \sum_{i=1}^{m}(-E)\Phi_i \\
+ \quad & 2\left[\frac{b\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{b\Sigma_a\Sigma_b}{\rho^2(1-\lambda r^2)} - \frac{4M^2 b\Sigma_b}{\rho^2 Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
- \quad & \left.\frac{2Mb}{\rho^2 Z(1-\lambda r^2)(r^2+b^2)} - \frac{2M\lambda b}{\rho^2 Z(1-\lambda r^2)^2}\right] \sum_{j=1}^{p}(-E)\Psi_j \\
+ \quad & \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\sum_{i=1}^{m}\frac{\Phi_i^2}{\lambda_i^2} + \frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\sum_{i=1}^{p}\frac{\Psi_i^2}{\nu_i^2} + \frac{\Delta_\theta}{\rho^2}\left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 \\
+ \quad & \frac{V-2M}{\rho^2 Z}\left[\frac{dS_r(r)}{dr}\right]^2 + \sum_{i=1}^{m-1}\frac{\Sigma_a}{(r^2+a^2)\sin^2\theta\prod_{k=1}^{i-1}\sin^2\alpha_k}\left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2 \\
+ \quad & \sum_{i=1}^{p-1}\frac{\Sigma_b}{(r^2+b^2)\cos^2\theta\prod_{k=1}^{i-1}\sin^2\beta_k}\left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2 \\
+ \quad & \sum_{i=1}^{m}\sum_{j=1}^{m}\left[\lambda^2 a^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right. \\
+ \quad & \left.\frac{4M^2a^2\Sigma_a^2}{\rho^2 Z(V-2M)(r^2+a^2)^2} + \frac{Q^{ij}}{\rho^2 Z}\right]\Phi_i\Phi_j \\
+ \quad & \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2 b^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{4M^2b^2\Sigma_a^2}{\rho^2 Z(V-2M)(r^2+b^2)^2} + \left.\frac{Q^{(i+m)(j+m)}}{\rho^2 Z}\right] \Psi_i\Psi_j \\
&+ \quad 2\sum_{i=1}^{m}\sum_{j=1}^{p}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right. \\
&+ \quad \frac{4M^2ab\Sigma_a\Sigma_b}{\rho^2 Z(V-2M)(r^2+a^2)(r^2+b^2)} + \left.\frac{Q^{i(j+m)}}{\rho^2 Z}\right] \Phi_i\Psi_j\,. \qquad (7.28)
\end{aligned}
$$

Note that here the λ_i and ν_j are not coordinates, but simply quantities defined by (7.20). We continue to use the convention defined for products of $\sin^2\alpha_i$ and $\sin^2\beta_j$ defined earlier. Separate the α_i and β_j coordinates from the Hamilton-Jacobi equation via

$$
\begin{aligned}
J_1^2 &= \sum_{i=1}^{m}\left[\frac{\Phi_i^2}{\lambda_i^2} + \frac{1}{\prod_{k=1}^{i-1}\sin^2\alpha_k}\left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2\right], \\
L_1^2 &= \sum_{i=1}^{p}\left[\frac{\Psi_i^2}{\nu_i^2} + \frac{1}{\prod_{k=1}^{i-1}\sin^2\beta_k}\left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2\right], \qquad (7.29)
\end{aligned}
$$

where J_1^2 and L_1^2 are separation constants. Then the remaining terms in the Hamilton-Jacobi equations can be explicitly separated to give ordinary differential equations for r and θ:

$$
\begin{aligned}
K = \quad & m^2r^2 - \left[\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)} + \frac{4M^2}{Z(1-\lambda r^2)^2}\right]E^2 + \frac{V-2M}{Z}\left[\frac{dS_r(r)}{dr}\right]^2 \\
&+ \quad 2\left[\frac{a\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda a}{Z(1-\lambda r^2)^2} + \frac{4M^2a\Sigma_a}{Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
&+ \quad \left.\frac{2Ma}{Z(1-\lambda r^2)(r^2+a^2)}\right]\sum_{i=1}^{m}(-E)\Phi_i \\
&+ \quad 2\left[\frac{b\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda b}{Z(1-\lambda r^2)^2} + \frac{4M^2b\Sigma_b}{Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
&+ \quad \left.\frac{2Mb}{Z(1-\lambda r^2)(r^2+b^2)}\right]\sum_{j=1}^{p}(-E)\Psi_j + \sum_{i=1}^{m}\sum_{j=1}^{m}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)}\right.\right. \\
&+ \quad \left.\left.\frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2a^2\Sigma_a^2}{Z(V-2M)(r^2+a^2)^2} - \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j \\
&+ \quad \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2b^2\Sigma_a^2}{Z(V-2M)(r^2+b^2)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \left.\frac{Q^{(i+m)(j+m)}}{Z}\right] \Psi_i\Psi_j + 2\sum_{i=1}^{m}\sum_{j=1}^{p}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
&- \left.\frac{4M^2 ab\Sigma_a\Sigma_b}{Z(V-2M)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+m)}}{Z}\right]\Phi_i\Psi_j \\
&+ \frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2 + \frac{\Sigma_b(r^2+a^2)}{r^2+b^2}L_1^2, \qquad (7.30)
\end{aligned}
$$

and

$$
\begin{aligned}
-K = \quad & m^2a^2\cos^2\theta + m^2b^2\sin^2\theta + \Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2 + \Sigma_a\cot^2\theta J_1^2 + \Sigma_b\tan^2\theta L_1^2 \\
&+ \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2 - 2\sum_{i=1}^{m}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i - 2\sum_{i=1}^{p}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j + 2\sum_{i=1}^{m}\sum_{j=1}^{p}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j\,, \qquad (7.31)
\end{aligned}
$$

where K is a separation constant.

In order to show complete separation of the Hamilton-Jacobi equation, we analyze the α and β sectors in (7.29) and demonstrate separation of the individual α_i and β_j. The pattern here is that of a Hamiltonian of non-relativistic classical particles on the unit $(m-1)$-α and the unit $(p-1)$-β spheres, with some potential dependent on the squares of the μ_i. This can easily be additively separated following the usual procedure, one angle at a time, and the pattern continues for all integers $m, p \geq 2$.

The separation has the following inductive form for $k = 1, ..., m-2$, and $l = 1, ..., p-2$:

$$
\begin{aligned}
\left(\frac{dS_{\alpha_k}}{d\alpha_k}\right)^2 &= J_k^2 - \frac{J_{k+1}^2}{\sin^2\alpha_k} - \frac{\Phi_{m-k+1}^2}{\cos^2\alpha_k}\,, \\
\left(\frac{dS_{\alpha_{m-1}}}{d\alpha_{m-1}}\right)^2 &= J_{m-1}^2 - \frac{\Phi_1^2}{\sin^2\alpha_{m-1}} - \frac{\Phi_2^2}{\cos^2\alpha_{m-1}}\,, \\
\left(\frac{dS_{\beta_l}}{d\beta_l}\right)^2 &= L_l^2 - \frac{L_{l+1}^2}{\sin^2\beta_l} - \frac{\Psi_{p-l+1}^2}{\cos^2\beta_l}\,, \\
\left(\frac{dS_{\beta_{p-1}}}{d\beta_{p-1}}\right)^2 &= L_{p-1}^2 - \frac{\Psi_1^2}{\sin^2\beta_{p-1}} - \frac{\Psi_2^2}{\cos^2\beta_{p-1}}\,. \qquad (7.32)
\end{aligned}
$$

Thus, the Hamilton-Jacobi equation in the Kerr-(Anti) de Sitter rotating black hole background in all dimensions with two sets of possibly unequal rotation parameters has the general separation

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1} S_{\beta_i}(\beta_i)\,, \quad (7.33)$$

where the α_i and β_j are the spherical polar coordinates on the unit $(m-1)$ and unit $(p-1)$ spheres respectively. $S_r(r)$ can be obtained by quadratures from (7.30), $S_\theta(\theta)$ by quadratures from (7.31), and the $S_{\alpha_i}(\alpha_i)$ and the $S_{\beta_j}(\beta_j)$ again by quadratures from (7.32).

7.5 The Equations of Motion

7.5.1 Derivation of the Equations of Motion

To derive the equations of motion, we will write the separated action S from the Hamilton-Jacobi equation in the following form:

$$\begin{aligned} S &= \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + \int^r \sqrt{R(r')}dr' + \int^\theta \sqrt{\Theta(\theta')}d\theta' \\ &\quad + \sum_{i=1}^{m-1} \int^{\alpha_i} \sqrt{A_i(\alpha_i')}d\alpha_i' + \sum_{i=1}^{p-1} \int^{\beta_i} \sqrt{B_i(\beta_i')}d\beta_i'\,, \end{aligned} \quad (7.34)$$

where

$$\begin{aligned} A_k &= J_k^2 - \frac{J_{k+1}^2}{\sin^2\alpha_k} - \frac{\Phi_{m-k+1}^2}{\cos^2\alpha_k}\,, \qquad k = 1, \dots, m-2\,, \\ A_{m-1} &= J_{m-1}^2 - \frac{\Phi_1^2}{\sin^2\alpha_{m-1}} - \frac{\Phi_2^2}{\cos^2\alpha_{m-1}}\,, \\ B_k &= L_k^2 - \frac{L_{k+1}^2}{\sin^2\beta_k} - \frac{\Psi_{p-k+1}^2}{\cos^2\beta_k}\,, \qquad k = 1, \dots, p-2\,, \\ B_{p-1} &= L_{p-1}^2 - \frac{\Psi_1^2}{\sin^2\beta_{p-1}} - \frac{\Psi_2^2}{\cos^2\beta_{p-1}}\,, \end{aligned} \quad (7.35)$$

Θ is obtained from (7.31) as

$$\begin{aligned} \Delta_\theta\Theta = &- m^2a^2\cos^2\theta - m^2b^2\sin^2\theta - \Sigma_a\cot^2\theta J_1^2 - \Sigma_b\tan^2\theta L_1^2 - \frac{\Sigma_a\Sigma_b}{\Delta_\theta}E^2 \\ &+ 2\sum_{i=1}^{m} \frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i + 2\sum_{i=1}^{m} \frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i - \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j \end{aligned}$$

$$- \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2 b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j - 2\sum_{i=1}^{m}\sum_{j=1}^{p}\frac{\lambda^2 ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j - K \tag{7.36}$$

and R is obtained from (7.30) as

$$\begin{aligned}
\frac{V-2M}{Z}R = & - m^2r^2 + \left[\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)} + \frac{4M^2}{Z(1-\lambda r^2)^2}\right]E^2 \\
& - 2\left[\frac{a\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda a}{Z(1-\lambda r^2)^2} + \frac{4M^2 a\Sigma_a}{Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
& + \left.\frac{2Ma}{Z(1-\lambda r^2)(r^2+a^2)}\right]\sum_{i=1}^{m}(-E)\Phi_i \\
& - 2\left[\frac{b\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda b}{Z(1-\lambda r^2)^2} + \frac{4M^2 b\Sigma_b}{Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
& + \left.\frac{2Mb}{Z(1-\lambda r^2)(r^2+b^2)}\right]\sum_{j=1}^{p}(-E)\Psi_j - \sum_{i=1}^{m}\sum_{j=1}^{m}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)}\right.\right. \\
& + \left.\left.\frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2a^2\Sigma_a^2}{Z(V-2M)(r^2+a^2)^2} - \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j \\
& - \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2b^2\Sigma_a^2}{Z(V-2M)(r^2+b^2)^2}\right. \\
& - \left.\frac{Q^{(i+m)(j+m)}}{Z}\right]\Psi_i\Psi_j - 2\sum_{i=1}^{m}\sum_{j=1}^{p}\left[\lambda^2ab\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
& - \left.\frac{4M^2ab\Sigma_a\Sigma_b}{Z(V-2M)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+m)}}{Z}\right]\Phi_i\Psi_j \\
& - \frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2 - \frac{\Sigma_b(r^2+a^2)}{r^2+b^2}L_1^2 + K\,.
\end{aligned} \tag{7.37}$$

To obtain the equations of motion, we differentiate S with respect to the parameters $m^2, K, E, J_i^2, L_j^2, \Phi_i, \Psi_j$ and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

$$\frac{\partial S}{\partial m^2} = 0 \Rightarrow l = \int \frac{Zr^2}{V-2M}\frac{dr}{\sqrt{R}} + \int\frac{(a^2\cos^2\theta + b^2\sin^2\theta)d\theta}{\Delta_\theta\sqrt{\Theta}}\,,$$

$$\begin{aligned}
\frac{\partial S}{\partial K} &= 0 \Rightarrow \int \frac{d\theta}{\Delta_\theta \sqrt{\Theta}} = \int \frac{Z}{V-2M}\frac{dr}{\sqrt{R}}\,, \\
\frac{\partial S}{\partial J_1^2} &= 0 \Rightarrow \int \frac{d\alpha_1}{\sqrt{A_1}} = \int \frac{Z}{V-2M}\frac{\Sigma_a(r^2+b^2)}{r^2+a^2}\frac{dr}{\sqrt{R}} + \int \frac{\Sigma_a \cot^2\theta d\theta}{\Delta_\theta\sqrt{\Theta}}\,, \\
\frac{\partial S}{\partial J_k^2} &= 0 \Rightarrow \int \frac{d\alpha_k}{\sqrt{A_k}} = \int \frac{1}{\sin^2\alpha_{k-1}}\frac{d\alpha_{k-1}}{\sqrt{A_{k-1}}}\,, \qquad k = 2, ..., m-2\,, \\
\frac{\partial S}{\partial L_1^2} &= 0 \Rightarrow \int \frac{d\beta_1}{\sqrt{B_1}} = \int \frac{Z}{V-2M}\frac{\Sigma_b(r^2+a^2)}{r^2+b^2}\frac{dr}{\sqrt{R}} + \int \frac{\Sigma_b \tan^2\theta d\theta}{\Delta_\theta\sqrt{\Theta}}\,, \\
\frac{\partial S}{\partial L_l^2} &= 0 \Rightarrow \int \frac{d\beta_l}{\sqrt{B_l}} = \int \frac{1}{\sin^2\beta_{k-1}}\frac{d\beta_{l-1}}{\sqrt{B_{l-1}}}\,, \qquad l = 2, ..., p-2\,.
\end{aligned} \tag{7.38}$$

We can obtain N more equations of motion for the variables φ_i by differentiating S with respect to the angular momenta Φ_i and Ψ_j. Another equation can also be obtained by differentiating S with respect to E involving the time coordinate τ. However, these equations are not particularly illuminating, but can be written out explicitly if necessary by following this procedure. It is often more convenient to rewrite these in the form of first-order differential equations obtained from (7.38) by direct differentiation with respect to the affine parameter. We only list the most relevant ones here:

$$\begin{aligned}
\rho^2\frac{dr}{dl} &= \frac{V-2M}{Z}\sqrt{R}\,, \\
\rho^2\frac{d\theta}{dl} &= \Delta_\theta\sqrt{\Theta}\,, \\
\frac{(r^2+a^2)}{\Sigma_a}\frac{d\alpha_k}{dl} &= \frac{\sqrt{A_k}}{\sin^2\theta\prod_{i=1}^{k-1}\sin^2\alpha_i}\,, \qquad k = 1, ..., m-1\,, \\
\frac{(r^2+b^2)}{\Sigma_b}\frac{d\beta_k}{dl} &= \frac{\sqrt{B_l}}{\cos^2\theta\prod_{i=1}^{l-1}\sin^2\beta_i}\,, \qquad l = 1, ..., p-1\,,
\end{aligned} \tag{7.39}$$

7.5.2 Analysis of the Radial Equation

Worldlines of particles in these backgrounds are completely specified by the values of the conserved quantities E, K, L_i^2, J_j^2, and by the initial values of the coordinates. We will consider particle motion in the black hole exterior. Allowed regions of particle motion necessarily need to have positive value for the quantity R, owing to equation (7.39). We determine some of the possibilities of the allowed motion.

At large radius r, the dominant contribution to R, in the case of $\lambda = 0$, is $E^2 - m^2$. Thus we can say that for $E^2 < m^2$, we cannot have unbounded orbits, whereas for $E^2 > m^2$, such orbits are possible. For the case of nonzero λ, the dominant term at large r in R (or rather the slowest decaying term) is $\frac{m^2}{\lambda r^2}$. Thus in the case of the Kerr-

Anti-de Sitter background, only bound orbits are possible, whereas in the Kerr-de Sitter backgrounds, both unbounded and bound orbits may be possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose R as a quadratic in E as follows:

$$R = \alpha E^2 - 2\beta E + \gamma \,, \tag{7.40}$$

where

$$\begin{aligned}
\alpha &= \frac{Z}{V-2M}\left[\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)} + \frac{4M^2}{Z(1-\lambda r^2)^2}\right], \\
\beta &= \frac{-Z}{V-2M}\left[\frac{a\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda a}{Z(1-\lambda r^2)^2} + \frac{4M^2 a\Sigma_a}{Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
&+ \left.\frac{2Ma}{Z(1-\lambda r^2)(r^2+a^2)}\right]\sum_{i=1}^{m}\Phi_i \\
&- \frac{Z}{V-2M}\left[\frac{b\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda b}{Z(1-\lambda r^2)^2} + \frac{4M^2 b\Sigma_b}{Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
&+ \left.\frac{2Mb}{Z(1-\lambda r^2)(r^2+b^2)}\right]\sum_{j=1}^{p}\Psi_j \,, \\
\gamma &= \left\{-\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\lambda^2 a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2a^2\Sigma_a^2}{Z(V-2M)(r^2+a^2)^2}\right.\right. \\
&- \left.\frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j - \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2 b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
&- \left.\frac{4M^2b^2\Sigma_a^2}{Z(V-2M)(r^2+b^2)^2} - \frac{Q^{(i+m)(j+m)}}{Z}\right]\Psi_i\Psi_j \\
&- \frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2 - \frac{\Sigma_b(r^2+a^2)}{r^2+a^2}L_1^2 + K - m^2r^2 \\
&- 2\sum_{i=1}^{m}\sum_{j=1}^{p}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
&- \left.\left.\frac{4M^2ab\Sigma_a\Sigma_b}{Z(V-2M)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+m)}}{Z}\right]\Phi_i\Psi_j\right\}\frac{Z}{V-2M}\,.
\end{aligned} \tag{7.41}$$

The turning points for trajectories in the radial motion (defined by the condition $R=0$) are given by $E = V_\pm$ where

$$V_\pm = \frac{\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha}\,. \tag{7.42}$$

These functions, called the effective potentials [20], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.

7.5.3 Analysis of the Angular Equations

Another class of interesting motions possible describes motion at a constant value of α_i or β_j. These are analogous to the same class of motions analyzed in [23]. We briefly summarize them here. These motions are described by the simultaneous equations

$$A_i(\alpha_i = \alpha_{i0}) = \frac{dA_i}{d\alpha_i}(\alpha_i = \alpha_{i0}) = 0\,, \qquad i = 1, ..., m-1\,, \tag{7.43}$$

in the case of constant α_i motion, where α_{i0} is the constant value of α_i along this trajectory, or by the simultaneous equations

$$B_i(\beta_i = \beta_{i0}) = \frac{dB_i}{d\beta_i}(\beta_i = \beta_{i0}) = 0\,, \qquad i = 1, ..., p-1\,, \tag{7.44}$$

in the case of constant β_i motion, where β_{i0} is the constant value of β_i along this trajectory.

These equations can be explicitly solved. In the case of constant α_i motion, we get the relations

$$\begin{aligned} \frac{J_{i+1}^2}{\sin^4\alpha_i} &= \frac{\Phi_{m-i-1}^2}{\cos^4\alpha_i}\,, \\ J_i^2 &= \frac{J_{i+1}^2}{\sin^2\alpha_i} + \frac{\Phi_{m-i+1}^2}{\cos^2\alpha_i}\,, \qquad i = 1, ..., m-1\,. \end{aligned} \tag{7.45}$$

Note that if $\alpha_{i0} = 0$, then $J_{i+1}^2 = 0$, and if $\alpha_{i0} = \pi/2$, then $\Phi_{m-i+1}^2 = 0$. Similarly, in the case of constant β_i motion, we get the relations

$$\begin{aligned} \frac{L_{i+1}^2}{\sin^4\beta_i} &= \frac{\Psi_{p-i-1}^2}{\cos^4\beta_i}\,, \\ L_i^2 &= \frac{L_{i+1}^2}{\sin^2\beta_i} + \frac{\Psi_{p-i+1}^2}{\cos^2\beta_i}\,, \qquad i = 1, ..., p-1\,. \end{aligned} \tag{7.46}$$

Again if $\beta_{i0} = 0$, then $L_{i+1}^2 = 0$, and if $\beta_{i0} = \pi/2$, then $\Psi_{p-i+1}^2 = 0$.

Examining A_k in the general case, $\alpha_k = 0$ can only be reached if $J_{k+1} = 0$, and $\alpha_k = \pi/2$ can be only be reached if $\Phi_{m-k+1} = 0$. The orbit will completely be in the subspace $\alpha_k = 0$ only if $J_k^2 = \Phi_{m-k+1}^2$ and will completely be in the subspace $\alpha_k = \pi/2$

only if $J_k^2 = J_{k+1}^2$. Analogous results hold for constant β_i motion.

Again these equations are in a form suitable for numerical analysis for specific values of the black hole and particle parameters.

7.6 Dynamical Symmetry

The spacetimes discussed here are stationary and "axisymmetric"; i.e., $\partial/\partial\tau$ and $\partial/\partial\varphi_i$ are Killing vectors and have associated conserved quantities, $-E$, Φ_i, and Ψ_i. In general, if η is a Killing vector, then $\eta^\mu p_\mu$ is a conserved quantity, where p is the momentum. Note that this quantity is first order in the momenta.

In the case of only two sets of possibly unequal rotation parameters, more Killing vectors exist since the spacetime acquires additional dynamical symmetry. We have complete symmetry between the various planes of rotation characterized by the same value of rotation parameter $a_i = a$, and we can "rotate" one into another. Similarly, we have symmetry between the planes of rotation characterized by the same value of the rotation parameter $a_i = b$, and we can "rotate" these into one another as well. The vectors that generate these transformations are the required Killing vectors. The explicit construction of such Killing vectors is done in [22]. In this case, we get two independent sets of such Killing vectors, associated with the constant a and b value rotations.

In an odd number of spacetime dimensions, if $a \neq b$ and both are nonvanishing, then the rotational symmetry group is $U(m) \times U(p)$. If one of them is zero, but the other is nonzero (we take the nonzero one to be a), then the rotational symmetry group is $U(m) \times O(2p)$. In the case when $a = b \neq 0$, the rotational symmetry group is $U(m+p)$. In the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m+2p)$. In an even number of spacetime dimensions, $b = 0$ in the cases we have analyzed. If $a \neq 0$, then the rotational symmetry group is $U(m) \times O(2p-1)$, and in the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m+2p-1)$. Note that since these metrics are stationary, the full dynamical symmetry group is the direct product of $\mathbf{R}$ and the rotational symmetry group, where $\mathbf{R}$ is the additive group of real numbers parameterizing τ.

We also obtain a non-trivial irreducible second-order Killing tensor, whose existence is the principal reason that permits the separation of the $r-\theta$ equations. This Killing tensor is a generalization of the result obtained in the five dimensional case in [20]. This is obtained from the separation constant K in (7.30) and (7.31). We choose to analyze the latter.

$$
\begin{aligned}
K = & - m^2a^2\cos^2\theta - m^2b^2\sin^2\theta - \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2 - \Sigma_a\cot^2\theta J_1^2 - \Sigma_b\tan^2\theta L_1^2 \\
& + 2\sum_{i=1}^{m}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i + 2\sum_{j=1}^{p}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i - \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j \\
& - \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j - 2\sum_{i=1}^{m}\sum_{j=1}^{p}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j - \Delta_\theta\left(\frac{\partial S}{\partial\theta}\right)^2 .
\end{aligned} \tag{7.47}
$$

The Killing tensor $K^{\mu\nu}$ is obtained from this separation constant (which is quadratic in the canonical momenta) using the relation $K = K^{\mu\nu}p_\mu p_\nu$. It is then easy to see that

$$
\begin{aligned}
K^{\mu\nu} = & - g^{\mu\nu}\left(a^2\cos^2\theta + b^2\sin^2\theta\right) - \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}\delta^\mu_\tau\delta^\nu_\tau - \Sigma_a\cot^2\theta J_1^{\mu\nu} - \Sigma_b\tan^2\theta L_1^{\mu\nu} \\
& - \Delta_\theta\delta^\mu_\theta\delta^\nu_\theta - \sum_{i=1}^{m}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_\tau\delta^\nu_{\varphi_i} + \delta^\mu_{\varphi_i}\delta^\nu_\tau) - \sum_{j=1}^{p}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_\tau\delta^\nu_{\varphi_{i+m}} + \delta^\mu_{\varphi_{i+m}}\delta^\nu_\tau) \\
& - \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\delta^\mu_{\varphi_i}\delta^\nu_{\varphi_j} - \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\delta^\mu_{\varphi_{i+m}}\delta^\nu_{\varphi_{j+m}} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{p}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_{\varphi_i}\delta^\nu_{\varphi_{j+m}} + \delta^\mu_{\varphi_{j+m}}\delta^\nu_{\varphi_i}) .
\end{aligned} \tag{7.48}
$$

where $J_1^{\mu\nu}$ and $L_1^{\mu\nu}$ are the reducible Killing tensors associated with the α and β separation.

The existence of these additional Killing vectors and of the nontrivial irreducible Killing tensor is the principal reason behind the complete separation of the Hamilton-Jacobi equation. The nontrivial Killing tensor, in particular, exists due to the detailed structure of the metrics under consideration and is a surprising result.

7.7 The Scalar Field Equation

Consider a scalar field Ψ in a gravitational background with the action

$$S[\Psi] = -\frac{1}{2}\int d^Dx\sqrt{-g}((\nabla\Psi)^2 + \alpha R\Psi^2 + m^2\Psi^2) , \tag{7.49}$$

where we have included a curvature dependent coupling. However, in the Kerr-(Anti) de Sitter background, $R = \lambda$ is constant. As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein-Gordon equation in this background. We will simply set $\alpha = 0$ in the following. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi) = m^2\Psi\,. \tag{7.50}$$

As discussed by Carter [26], the assumption of separability of the Klein-Gordon equation usually implies separability of the Hamilton-Jacobi equation. Conversely, if the Hamilton-Jacobi equation does not separate, the Klein-Gordon equation seems unlikely to separate. We can also see this explicitly (as in the case of the Hamilton-Jacobi equation), since the (r, τ, ϕ_i) sector has coefficients in the equations that explicitly depend on the μ_i except when of all $a_i = a$, in which case separation seems likely. We will again consider the much more general case of two sets of possibly unequal sets of rotation parameters a and b. We continue using the same numbering conventions for the variables.

Once again, we impose the constraint (7.2) and decompose the μ_i in two sets of spherical polar coordinates as in (7.18) and (7.20). We calculate the determinant of the metric to be

$$\begin{aligned} g &= \frac{-r^2\rho^4(r^2+a^2)^{2m-2}(r^2+b^2)^{2p-2-\epsilon}}{\Sigma_a^{2m}\Sigma_b^{2p-2\epsilon}}\sin^{4m-2}\theta\cos^{4p-2-2\epsilon}\theta \\ &\star\left[\prod_{j=1}^{m-1}\sin^{4m-4j-2}\alpha_j\cos^2\alpha_j\right]\left[\prod_{k=1}^{p-1}\sin^{4p-4k-2}\beta_k\cos^2\beta_k\right]\cos^{-2\epsilon}\beta_1\,. \end{aligned} \tag{7.51}$$

For convenience we write $g = -\frac{RTAB\rho^4}{\Sigma_a^{2m}\Sigma_b^{2p-2\epsilon}}$, where

$$\begin{aligned} R &= r^2(r^2+a^2)^{2m-2}(r^2+b^2)^{2p-2-\epsilon}\,, \\ T &= \sin^{4m-2}\theta\cos^{4p-2-2\epsilon}\theta\,, \\ A &= \prod_{j=1}^{m-1}\sin^{4m-4j-2}\alpha_j\cos^2\alpha_j\,, \\ B &= \prod_{k=1}^{p-1}\sin^{4p-4k-2}\beta_k\cos^2\beta_k\cos^{-2\epsilon}\beta_1\,. \end{aligned} \tag{7.52}$$

Note that R and T are functions of r and θ only, and A and B only depend on the set of variables α_i and β_j respectively. Then the Klein-Gordon equation in this background

(7.50) becomes

$$
\begin{aligned}
m^2\Psi &= \frac{1}{\rho^2\sqrt{R}}\partial_r\left(\sqrt{R}\frac{V-2M}{Z}\partial_r\Psi\right) + \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\sum_{i=1}^{m}\frac{1}{\lambda_i^2}\partial^2_{\varphi_i}\Psi \\
&+ \frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\sum_{i=1}^{p}\frac{1}{\nu_i^2}\partial^2_{\varphi_{i+m}}\Psi + \frac{1}{\rho^2\sqrt{T}}\partial_\theta\left(\sqrt{T}\Delta_\theta\partial_\theta\Psi\right) \\
&+ \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\left[\sum_{i=1}^{m-1}\frac{1}{\sqrt{A}}\partial_{\alpha_i}\left(\frac{\sqrt{A}}{\prod_{k=1}^{i-1}\sin^2\alpha_k}\partial_{\alpha_i}\Psi\right)\right] \\
&+ \frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\left[\sum_{i=1}^{p-1}\frac{1}{\sqrt{B}}\partial_{\beta_i}\left(\frac{\sqrt{B}}{\prod_{k=1}^{i-1}\sin^2\beta_k}\partial_{\beta_i}\Psi\right)\right] \\
&+ \left[\frac{\Sigma_a\Sigma_b}{\lambda\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)} - \frac{4M^2}{\rho^2 Z(1-\lambda r^2)^2}\right]\partial^2_\tau\Psi \\
&+ 2\left[\frac{a\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{a\Sigma_a\Sigma_b}{\rho^2(1-\lambda r^2)} - \frac{4M^2a\Sigma_a}{\rho^2 Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
&- \left.\frac{2M\lambda a}{\rho^2 Z(1-\lambda r^2)^2} - \frac{2Ma}{\rho^2 Z(1-\lambda r^2)(r^2+a^2)}\right]\sum_{i=1}^{m}\partial^2_{\tau\varphi_i}\Psi \\
&+ 2\left[\frac{b\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{b\Sigma_a\Sigma_b}{\rho^2(1-\lambda r^2)} - \frac{4M^2b\Sigma_b}{\rho^2 Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
&- \left.\frac{2M\lambda b}{\rho^2 Z(1-\lambda r^2)^2} - \frac{2Mb}{\rho^2 Z(1-\lambda r^2)(r^2+b^2)}\right]\sum_{j=1}^{p}\partial^2_{\tau\varphi_{m+j}}\Psi \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right. \\
&+ \left.\frac{4M^2a^2\Sigma_a^2}{\rho^2 Z(V-2M)(r^2+a^2)^2} + \frac{Q^{ij}}{\rho^2 Z}\right]\partial^2_{\varphi_i\varphi_j}\Psi \\
&+ \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right. \\
&+ \left.\frac{4M^2b^2\Sigma_a^2}{\rho^2 Z(V-2M)(r^2+b^2)^2} + \frac{Q^{(i+m)(j+m)}}{\rho^2 Z}\right]\partial^2_{\varphi_{i+m}\varphi_{j+m}}\Psi \\
&+ 2\sum_{i=1}^{m}\sum_{j=1}^{p}\left[\lambda^2ab\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda(1-\lambda r^2)} - \frac{2M}{\rho^2 Z(1-\lambda r^2)}\right)\right. \\
&+ \left.\frac{4M^2ab\Sigma_a\Sigma_b}{\rho^2 Z(V-2M)(r^2+a^2)(r^2+b^2)} + \frac{Q^{i(j+m)}}{\rho^2 Z}\right]\partial^2_{\varphi_i\varphi_{j+m}}\Psi\,. \qquad (7.53)
\end{aligned}
$$

We attempt the usual multiplicative separation for Ψ in the following form:

$$\Psi = \Phi_r(r)\Phi_\theta(\theta)e^{-iE\tau}e^{i\sum_i^m \Phi_i\varphi_i}e^{i\sum_i^p \Psi_i\varphi_{m+i}}\left(\prod_{i=1}^{m-1}\Phi_{\alpha_i}(\alpha_i)\right)\left(\prod_{i=1}^{p-1}\Phi_{\beta_i}(\beta_i)\right), \tag{7.54}$$

where we again adopt the convention that $\Psi_p = 0$ in the case of even dimensional spacetimes.

The Klein-Gordon equation then completely separates. The r and θ equations are given as

$$\begin{aligned}
K &= \frac{1}{\Phi_r\sqrt{R}}\frac{d}{dr}\left(\sqrt{R}\frac{V-2M}{Z}\frac{d\Phi_r}{dr}\right) + \left[\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)} + \frac{4M^2}{Z(1-\lambda r^2)^2}\right]E^2 \\
&+ \sum_{i=1}^m\sum_{j=1}^m\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right) - \frac{4M^2a^2\Sigma_a^2}{Z(V-2M)(r^2+a^2)^2}\right. \\
&- \left.\frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j + \sum_{i=1}^p\sum_{j=1}^p\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
&- \left.\frac{4M^2b^2\Sigma_a^2}{Z(V-2M)(r^2+b^2)^2} - \frac{Q^{(i+m)(j+m)}}{Z}\right]\Psi_i\Psi_j \\
&+ 2\sum_{i=1}^m\sum_{j=1}^p\left[\lambda^2ab\left(\frac{\Sigma_a\Sigma_b}{\lambda(1-\lambda r^2)} + \frac{2M}{Z(1-\lambda r^2)}\right)\right. \\
&- \left.\frac{4M^2ab\Sigma_a\Sigma_b}{Z(V-2M)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+m)}}{Z}\right]\Phi_i\Psi_j \\
&- 2\left[\frac{a\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda a}{Z(1-\lambda r^2)^2} + \frac{4M^2a\Sigma_a}{Z(1-\lambda r^2)^2(V-2M)(r^2+a^2)}\right. \\
&+ \left.\frac{2Ma}{Z(1-\lambda r^2)(r^2+a^2)}\right]\sum_{i=1}^m E\Phi_i \\
&- 2\left[\frac{b\Sigma_a\Sigma_b}{(1-\lambda r^2)} + \frac{2M\lambda b}{Z(1-\lambda r^2)^2} + \frac{4M^2b\Sigma_b}{Z(1-\lambda r^2)^2(V-2M)(r^2+b^2)}\right. \\
&+ \left.\frac{2Mb}{Z(1-\lambda r^2)(r^2+b^2)}\right]\sum_{j=1}^p E\Psi_j - \Sigma_a\frac{r^2+b^2}{r^2+a^2}\sum_{i=1}^m K_1\Phi_i^2 \\
&- \Sigma_b\frac{r^2+a^2}{r^2+b^2}\sum_{j=1}^p M_1\Psi_j^2 - m^2r^2 .
\end{aligned} \tag{7.55}$$

and

$$
\begin{aligned}
-K &= \frac{1}{\Phi_\theta\sqrt{T}}\frac{d}{d\theta}\left(\sqrt{T}\Delta_\theta\frac{d\Phi_\theta}{d\theta}\right) - \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2 - m^2(a^2\cos^2\theta + b^2\sin^2\theta) \\
&+ K_1\cot^2\theta + M_1\tan^2\theta - 2\lambda a^2\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{m}\sum_{j=1}^{m}\Phi_i\Phi_j - 2\lambda b^2\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{p}\sum_{j=1}^{p}\Psi_i\Psi_j \\
&- 4\lambda ab\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{m}\sum_{j=1}^{p}\Phi_i\Psi_j + 2\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{m}E\Phi_i + 2\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{j=1}^{p}E\Phi_j\,, \qquad (7.56)
\end{aligned}
$$

where K, K_1 and M_1 are separation constants. K_1 and M_1 encode all the α and β dependence respectively and are defined explicitly as follows:

$$
K_1 = \sum_{i=1}^{k-1}A_i + \frac{K_k}{\prod_{j=1}^{k-1}\sin^2\alpha_j}\,, \quad k = 1,...,m-1\,, \qquad (7.57)
$$

where

$$
\begin{aligned}
A_i &= \frac{1}{\Phi_{\alpha_i}\cos\alpha_i\sin^{2m-2i-1}\alpha_i\prod_{k=1}^{i-1}\sin^2\alpha_k}\frac{d}{d\alpha_i}\left(\cos\alpha_i\sin^{2m-2i-1}\alpha_i\frac{d\Phi_{\alpha_i}}{d\alpha_i}\right) \\
&- \frac{\Phi^2_{m-i+1}}{\cos^2\alpha_i\prod_{j=1}^{i-1}\sin^2\alpha_j}\,, \qquad (7.58)
\end{aligned}
$$

and

$$
M_1 = \sum_{i=1}^{k-1}B_i + \frac{M_k}{\prod_{j=1}^{k-1}\sin^2\beta_j}\,, \quad k = 1,...,p-1\,, \qquad (7.59)
$$

and where

$$
\begin{aligned}
B_i &= \frac{1}{\Psi_{\beta_i}\cos\beta_i\sin^{2p-2i-1}\beta_i\prod_{k=1}^{i-1}\sin^2\beta_k}\frac{d}{d\beta_i}\left(\cos\beta_i\sin^{2p-2i-1}\beta_i\frac{d\Phi_{\beta_i}}{d\beta_i}\right) \\
&- \frac{\Psi^2_{p-i+1}}{\cos^2\beta_i\prod_{j=1}^{i-1}\sin^2\beta_j}\,, \qquad (7.60)
\end{aligned}
$$

Then we inductively have the complete separation of the α_i dependence as

$$
K_k = \frac{K_{k+1}}{\sin^2\alpha_k} - \frac{\Phi^2_{n-k+1}}{\cos^2\alpha_k} + \frac{1}{\Phi_{\alpha_k}\cos\alpha_k\sin^{2m-2k-1}\alpha_k}\frac{d}{d\alpha_k}\left(\cos\alpha_k\sin\alpha_k\frac{d\Phi_{\alpha_k}}{d\alpha_k}\right)\,, \qquad (7.61)
$$

where $k = 1,...,m-1$, and we use the convention $K_m = -\Phi_1^2$. Similarly, the complete separation of the β_i dependence is given inductively by

$$M_k = \frac{M_{k+1}}{\sin^2 \beta_k} - \frac{\Psi^2_{p-k+1}}{\cos^2 \beta_k} + \frac{1}{\Phi_{\beta_k} \cos \beta_k \sin^{2p-2k-1} \beta_k} \frac{d}{d\beta_k} \left(\cos \beta_k \sin \beta_k \frac{d\Phi_{\beta_k}}{d\beta_k} \right) , \quad (7.62)$$

where $k = 1, ..., p-1$, and we use the convention $M_p = -\Psi_1^2$. These results agree with the previously known analysis in five dimensions [21].

At this point we have complete separation of the Klein-Gordon equation in the Kerr-(Anti) de Sitter black hole background in all dimensions with two sets of possibly unequal rotation parameters in the form given by (7.54) with the individual separation functions given by the ordinary differential equations above. Note that the separation of the Klein-Gordon equation in this geometry is again due to the existence of the non-trivial Killing tensor.

Conclusions

We studied the integrability properties of the Hamilton-Jacobi and the massive Klein-Gordon equations in the Kerr-(Anti) de Sitter black hole backgrounds in all dimensions. Complete separation of both equations in Boyer-Lindquist coordinates is possible for the case of two possibly unequal sets of rotation parameters. We discuss the Killing vectors and reducible Killing tensors that exist in the spacetime and also construct the nontrivial irreducible Killing tensor which explicitly permits complete separation. Thus we demonstrate the separability of the Hamilton-Jacobi and the Klein-Gordon equations as a direct consequence of the enhancement of symmetry. We also derive first-order equations of motion for classical particles in these backgrounds, and analyze the properties of some special trajectories.

Further work in this direction could include the study of higher-spin field equations in these backgrounds, which is of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results. The first order equations of motion presented here can also readily be used in the detailed study of black hole singularity structure in an AdS background geodesic probes and the AdS/CFT correspondence.

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Part II

D-Branes of Wess-Zumino-Witten Models

Chapter 8

The Virtues of "Lie"-ing

String theories that are of direct phenomenological relevance are notoriously difficult to handle for many reasons like non-perturbative limits, strong curvatures, strong couplings etc. As a consequence, it is desirable to study several "toy models" which exhibit similar features of interest as the real string theories, but are easier to understand and more tractable. Two of the most important classes of such models studied in recent years are the Wess-Zumino-Witten (WZW) models, and the matrix models. In this part of the thesis, we will work with symmetries and D-branes in the context of the former.

D-branes, and their charges, are extremely important aspects of string theory. D-branes cannot be ignored in any consistent string theory owing to their solitonic nature, and also since they are in some sense as natural as strings in certain sectors of the theory. Their charges heavily constrain their dynamics. For instance, a D-brane with a conserved charge may be stable against decay. Brane anti-brane annihilation is another situation where the charges are relevant, since the resulting products are constrained by charge conservation. In the target space approach, charges of D-branes in string theory are dealt with using the powerful geometric tools of K-theory and cohomology. This approach is essentially useful anytime the supergravity approximation can be trusted and is, thus, valid in many situations. However, one major drawback of this "macroscopic" approach is that it only provides information about the charge groups of the D-branes, and not the individual charges of the D-branes themselves. Also, there may be situations where the supergravity approximation is insufficient, and it may be useful to have another calculation method which could provide information about D-brane charges in these situations.

This other, "microscopic", approach has been developed in great detail for the situation of WZW models. WZW models describe string theory where the target space is some group manifold (i.e. a Lie group). WZW models exhibit many of the broad features we would like in a toy model of phenomenological string theory, and importantly for our purposes, they have very interesting D-branes and associated dynamics. Thus, understanding the microscopic D-brane charge calculation in the context of WZW models may perhaps shed light on similar calculations in other string theories. In addition, the study of WZW models has great intrinsic value in the context of mathematics, and particularly in knot theory and algebraic geometry.

WZW models are highly tractable since they are exactly solvable CFT's. In addition to the standard infinite dimensional conformal symmetry of any CFT, they also possess additional symmetries related to the affine Lie algebra of the underlying group manifold. Large amounts of symmetry translate to large numbers of constraints, which can be very effectively exploited to the properties of WZW models. The use of these symmetries provides a microscopic method of calculating D-brane charges. The remarkable thing about this method is that it is an exact CFT/string description, i.e. its answers can be trusted in all situations where the method is applicable. Thus, it provides information complementary to the K-theory approach. In addition, this method calculates the charges themselves and not just the charge groups.

The charge groups of D-branes on WZW models using K-theory have been calculated in [1] for full affine symmetry preserving D-branes and in [2] for D-branes that preserve the affine symmetry only up to some twist. Microscopic calculations have been done in many cases, and they agree with the K-theory calculations for the charge groups [3–7]. In addition, the microscopic calculation has been done for cases where the K-theory approach has not (yet) yielded information [8–10]. Several of these D-branes have been explicitly constructed, and their charges have been determined confirming the more abstract microscopic and K-theory calculations [11, 12]

Chapter 9 contains a very brief introduction to affine Lie algebras and WZW models. Chapter 10 presents the basic ideas behind the microscopic approach to D-branes and their charges. The main results of this part of the thesis appear in Chapter 11, where this approach is used to calculate the charges and charge groups for the triality-twisted D-branes of D_4 and the charge conjugation twisted D-branes of E_6. Chapter 12 contains results of a similar calculation for the case of charge conjugation twisted D-branes on the non-simply connected group $E_6/\mathbb{Z}_3$. The results of the calculations in this appendix will appear in a future paper that will also contain similar calculations for some other twisted non-simply connected groups. Appendix A contains a brief introduction to the very basic ideas of conformal field theory that are used throughout this part of the thesis.

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Chapter 9

Affine Lie Algebras & WZW Models for Dummies

9.1 Simple Lie Algebras

In many respects, the theory of affine Lie algebras is a very natural extension of the theory of simple Lie algebras, and as such, affine Lie algebras cannot be studied effectively on their own. In addition, the central interests in this part of the thesis are based on WZW models and their fusion rules. Fusion rules are naturally seen as truncations of tensor products of representations of the underlying Lie algebra. The very basics of the theory of simple Lie algebras are presented here. For further information, particularly in the context of CFT's and WZW models, an excellent reference is [1]. Other excellent references include [2, 3], which are highly recommended, as well as [4, 5].

A **Lie algebra** is a vector space $\mathfrak{g}$ that possesses an antisymmetric bilinear operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called a **commutator** or **Lie bracket**, satisfying the **Jacobi identity**:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \qquad \forall X, Y, Z \, . \tag{9.1}$$

A subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ which is itself a Lie algebra is called a Lie **subalgebra** of $\mathfrak{g}$ i.e. symbolically

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \, , \tag{9.2}$$

that is, $[x, y] \in \mathfrak{h} \, \forall x, y \in \mathfrak{h}$. If in addition to the above, $\mathfrak{h}$ satisfies the much stronger property

$$[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}\,, \tag{9.3}$$

then the subalgebra $\mathfrak{h}$ is said to an **ideal** of $\mathfrak{g}$ (or an invariant subalgebra). A Lie algebra is said to be **simple** if it has no proper ideals, and **semisimple** if it can be expressed as a direct sum of simple Lie algebras.

A **representation** (on V) of a Lie algebra $\mathfrak{g}$ is a linear mapping into $\mathfrak{gl}(V)$, the space of linear operators on a vector space V, which preserves the commutation relations of $\mathfrak{g}$. The dimension of V is known as the dimension of the representation. A representation is said to be **irreducible** if the matrices representing the elements of $\mathfrak{g}$ cannot all be brought into a block-diagonal form.

A Lie algebra is specified by giving a basis of generators [1] $\{J^a\}$ together with their commutation relations

$$[J^a, J^b] = \sum_c i f_c^{ab} J^c\,. \tag{9.4}$$

The constants f_c^{ab} are known as **structure constants**. The standard **Cartan-Weyl** basis is a preferred basis of generators constructed as follows. Find a maximal set of commuting Hermitean generators H^i, $1, ..., r$:

$$[H^i, H^j] = 0 \qquad i, j, = 1, ..., r\,. \tag{9.5}$$

These span a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ known as the **Cartan subalgebra**, and the dimension r of $\mathfrak{h}$ is known as the **rank** of the Lie algebra $\mathfrak{g}$. The remaining generators are chosen to be ones that diagonalize the Cartan subalgebra simultaneously with respect to the commutator i.e.

$$\left[H^i, E^\alpha\right] = \alpha^i E^\alpha\,, \tag{9.6}$$

where the vector $\alpha = (\alpha^1, ..., \alpha^r)$ is called a **root**, and E^α is the corresponding ladder operator. α is best thought of as an element of the dual space $\mathfrak{h}^*$ of the Cartan subalgebra via the mapping $\alpha(H^i) = \alpha^i$. By taking the adjoint of (9.6), we can see that $-\alpha$ is also a root with the corresponding ladder operator $E^{-\alpha} = (E^\alpha)^\dagger$. The set of all roots of $\mathfrak{g}$ is typically denoted by Δ.

[1] Note that the terminology is the standard one in high energy physics, but does not necessarily agree with the mathematical literature where the words basis and generator have slightly different meanings

A representation of special significance is the **adjoint representation** where the vector space V is chosen to be the Lie algebra $\mathfrak{g}$ itself regarded as a vector space. In the adjoint representation, the action of a generator X is represented by $ad(X)$ defined as

$$ad(X)Y = [X, Y]. \tag{9.7}$$

Using the Jacobi equation it is possible to show that if $\alpha + \beta \in \Delta$, then the commutator $[E^\alpha, E^\beta]$ is proportional to $E^{\alpha+\beta}$, and vanishes if $\alpha + \beta \notin \Delta$. Also when $\alpha = -\beta$, then $[E^\alpha, E^{-\alpha}]$ commutes with all the H^i, which is possible only if it is a linear combination of the generators of the Cartan subalgebra. Choosing the normalizations, and using the notation

$$\alpha \cdot H = \sum_{i=1}^{r} \alpha^i H^i, \qquad \alpha \cdot \beta = \sum_{i=1}^{r} \alpha^i \beta^j, \tag{9.8}$$

where α and β are both roots, the complete commutation relations of the Lie algebra can be cast into the form

$$\begin{aligned}
[H^i, H^j] &= 0, \\
[H^i, E^\alpha] &= \alpha^i E^\alpha, \\
[E^\alpha, E^\beta] &= N_{\alpha,\beta} E^{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \Delta. \\
&= \frac{2}{|\alpha|^2} \alpha \cdot H, \quad \text{if } \alpha = -\beta, \\
&= 0, \quad \text{otherwise},
\end{aligned} \tag{9.9}$$

where $N_{\alpha,\beta}$ is a number.

The **Killing Form** is a unique (up to normalization) inner product that can be defined for any Lie algebra $\mathfrak{g}$. This is the inner product with respect to which the adjoint map is skew symmetric. It can be shown that it is given by

$$K(X, Y) = \frac{1}{2h^\vee} Tr(ad(X)ad(Y)), \tag{9.10}$$

where the numerical factor is a convenient normalization, and $h^\vee$ will be defined below. By restriction, obviously this defines a norm on $\mathfrak{h}$ as well. Thus, a norm is also defined on the dual space $\mathfrak{h}^*$ consisting of roots (and weights, to be defined below). Further details on the exact construction can be found in [1]. Henceforth, it will be understood that (α, β) is the scalar product between the roots defined using the Killing form and $|\alpha|^2 = (\alpha, \alpha)$.

For an arbitrary representation, a basis $\{|\lambda>\}$ can always be found that simultaneously diagonalizes the Cartan subalgebra (as they are made up of commuting elements):

$$H^i|\lambda>=\lambda^i|\lambda> \ . \tag{9.11}$$

The eigenvalues λ^i build a vector $\lambda=(\lambda^1,...,\lambda^r)$, called a **weight**. Note that roots are simply weights in the adjoint representation. Weights also live in the dual space $\mathfrak{h}^*$. Using (9.6) it is possible to see that

$$H^iE^\alpha|\lambda>=\left[H^i,E^\alpha\right]|\lambda>+E^\alpha H^i|\lambda>=(\lambda^i+\alpha^i)E^\alpha|\lambda>, \tag{9.12}$$

so that $E^\alpha|\lambda>$, if non-zero, must be proportional to a state $|\lambda+\alpha>$. So essentially the operators E^α do indeed behave as ladder operators, similar to the operators $J^\pm$ in the case of angular momentum ($\mathfrak{su}(2)$) in quantum mechanics, or the ladder operators a and $a^\dagger$ for the simple harmonic oscillator. Using the various E^α the representation can be constructed using the basic commutation relations.

Pick a **base** of roots $\{\beta_1,...,\beta_r\}$ for $\mathfrak{h}^*$, so that any root can be expanded as $\alpha=\sum_{i=1}^r n_i\beta_i$ such that all the n_j's are integers and either all are greater than or equal to zero, or all are less than or equal to zero. Relative to this base an ordering can be defined: a root α is said to be positive if the first nonzero number in the sequence $(n_1,...,n_r)$ is positive. Denote the set of positive roots by Δ_+, and the set Δ_- of negative roots is defined similarly. A **simple root** α_i is defined to be a root that cannot be written as the sum of two positive roots. Clearly, there are necessarily r simple roots (where r is the rank of $\mathfrak{g}$).

A distinguished element of Δ is the **highest root** θ. It is the unique root for which in the expansion $\sum_i m_i\alpha_i$, the sum $\sum_i m_i$ is maximized. It is convenient to introduce the **coroots**:

$$\alpha_i^\vee=\frac{2\alpha_i}{(\alpha_i,\alpha_i)} \ . \tag{9.13}$$

The coefficients of the expansion of the highest root θ in the simple roots and coroots carry special names, and are respectively called the **marks** (a_i) or Coxeter labels and the **comarks** ($a_i^\vee$) or dual Coxeter labels:

$$\theta=\sum_{i=1}^r a_i\alpha_i=\sum_{i=1}^r a_i^\vee\alpha_i^\vee \ . \tag{9.14}$$

The weights are typically normalized by taking $|\theta|^2=2$. Clearly marks and co-marks are related by $a_i = 2a_i^\vee/|\alpha_i|^2$. Two very important quantities are the **Coxeter Number** h and the **Dual Coxeter Number** $h^\vee$ (which is the normalization factor that appears in the definition of the Killing form (9.10)), defined as

$$h = 1 + \sum_{i=1}^{r} a_i \,, \quad h^\vee = 1 + \sum_{i=1}^{r} a_i^\vee \,. \tag{9.15}$$

One of the most fundamental entities useful in the study of Lie algebras is the **Cartan Matrix** defined as

$$A_{ij} = (\alpha_i, \alpha_j^\vee) \,, \qquad 1 \le i, j \le r \,. \tag{9.16}$$

The entries of the Cartan matrix are necessarily integers. Its diagonal elements are all 2, and it is not symmetric in general. The off-diagonal terms are non-positive and can be only 0,-1,-2, or -3. The number A_{ij} characterizes how the $\mathfrak{su}(2)$ algebra generated by the operator E^{α_j} acts on the operator E^{α_i} through the commutation relations.

Weights and roots both live in the dual space of the Cartan subalgebra $\mathfrak{h}^*$. Weights can be expanded in terms of a basis of simple roots, but the coefficients are not integers, so a better basis to use is the one dual to the simple coroot basis. This is a set $\{\omega_i\}$ known as the **fundamental weights** of $\mathfrak{g}$ defined by

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \,. \tag{9.17}$$

The expansions coefficients λ_i of a weight λ in the fundamental weight basis are called **Dynkin labels**. Hence,

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \Leftrightarrow \lambda_i = (\lambda, \alpha_i^\vee) \,. \tag{9.18}$$

The Dynkin labels of weights in finite-dimensional irreducible representations are always integers, and such weights are said to be **integral** and a weight λ is specified in terms of its Dynkin labels as $\lambda = [\lambda_1, ..., \lambda_r]$. A useful fact to note is that the elements of the Cartan matrix of $\mathfrak{g}$ are the Dynkin labels of the simple roots i.e.

$$\alpha_i = \sum_{j=1}^{r} A_{ij} \omega_j \,, \qquad i = 1, ..., r \,. \tag{9.19}$$

While the Cartan-Weyl basis is the natural one to obtain information about the structure of the Lie algebra $\mathfrak{g}$ itself, there is a more natural basis to use in the study of the finite dimensional representations of $\mathfrak{g}$ known as the **Chevalley-Serre** basis. The generators are

$$e^i = E^{\alpha_i}\,, \quad f^i = E^{-\alpha_i}\,, \quad h^i = \alpha_i^\vee \cdot H\,. \tag{9.20}$$

The commutation relations between them (which can be easily obtained from the Cartan-Weyl commutation relations in combination with the definitions of the Chevalley-Serre generators) are:

$$\begin{aligned}
[h^i, h^j] &= 0\,, \\
[h^i, e^j] &= A_{ji} e^j\,, \\
[h^i, f^j] &= -A_{ji} f^j\,, \\
[e^i, f^j] &= \delta_{ij} h^j\,.
\end{aligned} \tag{9.21}$$

These do not reproduce all the commutation relations (since they only address the commutation of the ladder operators associated to the simple roots). The remaining commutation relations (which are present in the Cartan-Weyl commutation relations) are provided in terms of the **Serre relations**

$$\begin{aligned}
\left[ad(e^i)\right]^{1-A_{ji}} e^j &= 0\,, \\
\left[ad(f^i)\right]^{1-A_{ji}} f^j &= 0\,,
\end{aligned} \tag{9.22}$$

where $i \neq j$.

It is important to note that λ^i refers to an eigenvalue in the Cartan-Weyl basis, while the Dynkin labels λ_i are the eigenvalues in the Chevalley-Serre basis of the Cartan subalgebra, i.e.

$$h^i|\lambda> = \lambda_i|\lambda>\,, \tag{9.23}$$

and $\lambda_i \neq \lambda^i$ in most cases. A weight is said to be **dominant** if all of its Dynkin labels are non-negative integers. Finite dimensional representations of a Lie algebra are given by specifying its **highest weight**, which is dominant integral. The remaining weights are obtained by acting on it and its descendants using the Chevalley-Serre ladder operators e^i, f^j and taking the Serre relations (9.22) into account. The highest weight of the adjoint representation is θ. Dominant integral highest weight representations for simple Lie algebras give rise to irreducible representations.

All the information in the Cartan Matrix can be captured using **Dynkin diagrams**.

To every simple root α_i associate a node and join the nodes i and j by $A_{ij}A_{ji}$ lines. Hence orthogonal simple roots are not connected. In addition, an arrow is placed on the lines pointing from a longer root to a shorter root when they are of unequal size. The classification of finite dimensional simple Lie algebras boils down to a classification of Dynkin diagrams. It turns out that there are four infinite families:

-A_r, which are essentially the Lie algebras familiar as $\mathfrak{su}(r+1)$.

-B_r, which are essentially the Lie algebras familiar as $\mathfrak{so}(2r+1)$.

-C_r, which are essentially the Lie algebras familiar as $\mathfrak{sp}(2r)$.

-D_r, which are essentially the Lie algebras familiar as $\mathfrak{so}(2r)$.

In addition there are five exceptional Lie algebras knows as E_6, E_7, and E_8, as well as F_4 and G_2. The Dynkin diagrams for all of these is given in Fig. (9.1)[2]. Lie algebras all of whose roots are equal in size are said to be **simply laced**. There are no multiple lines present in any of their Dynkin diagrams. From Fig. (9.1), the simply laced Lie algebras are easily seen to the be families A_r, D_r, as well as E_6, E_7, and E_8.

The **quadratic form matrix** is defined by $F_{ij} = (\omega_i, \omega_j)$, using the inner product defined on $\mathfrak{h}^*$. In addition, we will also need the **Weyl vector** defined by

$$\rho = \sum_{i=1}^{r} \omega_i = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha\,. \tag{9.24}$$

Given an arbitrary root α , consider the operator s_α that acts on an arbitrary weight λ (remember that both roots and weights live in $\mathfrak{h}^*$) via the action

$$s_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha\,. \tag{9.25}$$

This corresponds to a reflection in root/weight space in the hyperplane orthogonal to α. It can be shown that if λ itself is taken to be a root, then $s_\alpha(\beta)$ is also a root. The set of all such reflections forms a group known as the **Weyl group** of $\mathfrak{g}$, usually denoted W. It is generated by the r **simple Weyl reflections** s_i defined by $s_i = s_{\alpha_i}$, in the sense that every element in $w \in W$ can be decomposed as $w = s_i s_j ... s_k$. The Weyl group is extremely important for many reasons, not least of which is in representation theory. Given a highest weight λ, the entire representation can be constructed by considering the action of the Weyl group on λ. Also of great use is the **shifted Weyl reflection** defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$. For an arbitrary $w \in W$ define $\epsilon(w) = (-1)^{l(w)}$, where $l(w)$ is the smallest number of simple reflections required to express w i.e. $w = s_{i_1} ... s_{i_l}$.

[2]Figure reproduced from [3] under the provisions of the Copyright Act. and with permission from Cambridge University Press.

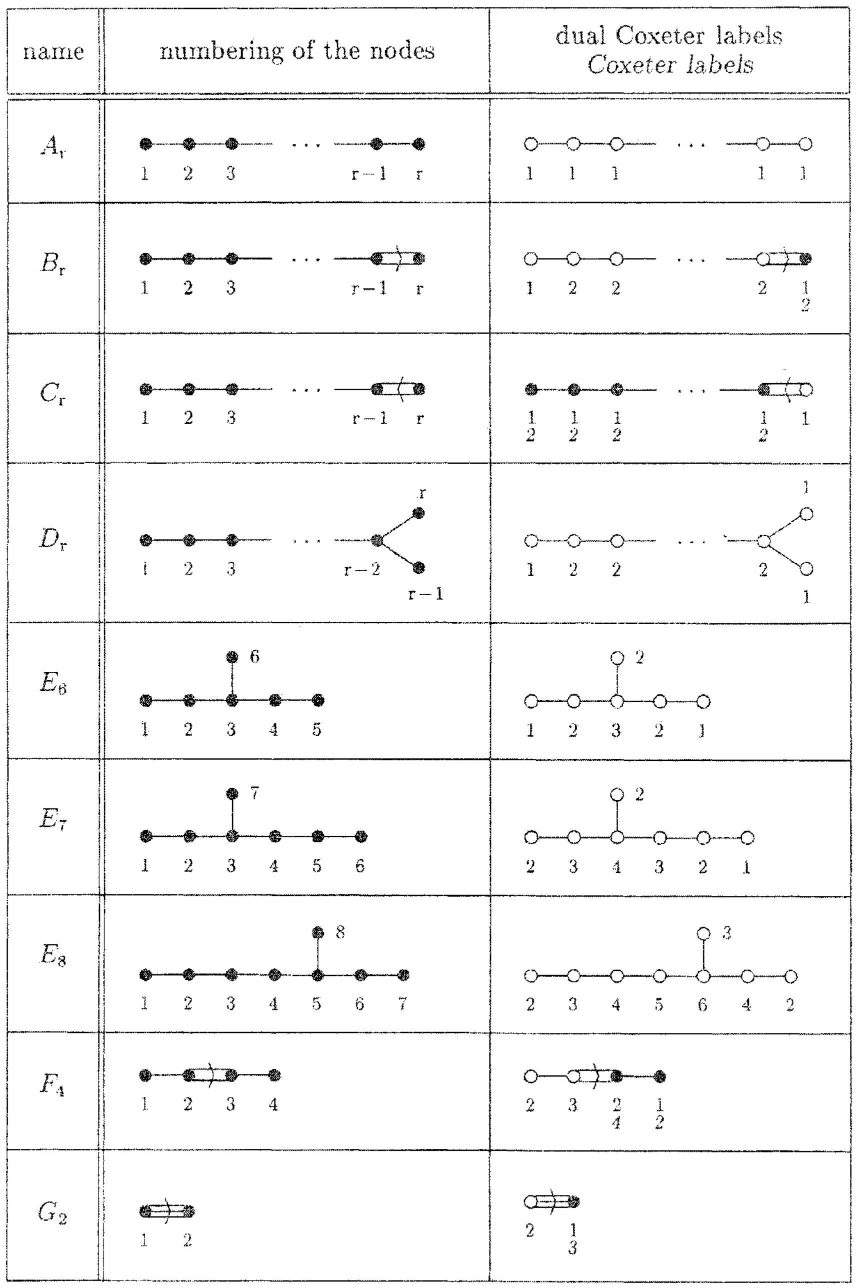

Figure 9.1: Dynkin diagrams for the finite dimensional simple Lie algebras

Given a representation of $\mathfrak{g}$ with highest weight λ, there is a lowest weight in the representation, which cannot further be acted on by the f^i's. There will be some element $w_0 \in W$ (not necessary a simple reflection) such that $w_0\lambda$ is the lowest weight. Turning the representation "upside down" produces the so-called **conjugate representation** (called charge conjugate in some contexts). Its highest weight is the negative of the lowest state of the original representation i.e. $\lambda^* = -(w_0\lambda)$. Conjugation corresponds to some symmetry of the Dynkin diagram of $\mathfrak{g}$. For instance for the A_n's there is a reflection symmetry (about the middle) of the Dynkin diagram which essentially amounts to reversing the order of the Dynkin labels, and this corresponds to the conjugate representation. Algebras whose Dynkin diagrams do not have any symmetries only have self-conjugate representations, i.e. $\lambda^* = -w_0\lambda = \lambda$.

The **character** of a highest weight representation λ is formally defined as

$$\chi_\lambda = \sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') e^{\lambda'}, \tag{9.26}$$

where Ω_λ is the set of all weights in the representation with highest weight λ, $\mathrm{mult}_\lambda(\lambda')$ is the multiplicity of the weight λ' in Ω_λ, and e^α denotes a formal exponential satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^\lambda(\xi) = e^{(\lambda,\xi)}$, where λ, μ, and ξ are arbitrary weights and on the right hand side of the second expression is a genuine exponential function of real numbers. It is quite a surprising fact that all the Lie theoretic information regarding a representation with highest weight λ is essentially encoded into χ_λ, while naively it may seem that the sum over weights in its definition wipes out the explicit structure of the representation. In fact, most Lie theory arguments tend to work with characters rather than the unwieldy representations themselves.

The famous **Weyl character formula** relates formal characters defined in terms of explicit sums over weights of a representation, to a sum over elements of the Weyl group and is given by

$$\chi_\lambda = \frac{\sum_{w\in W} \epsilon(w) e^{w\cdot(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w\cdot(\rho)}}. \tag{9.27}$$

Suppose a representation of $\mathfrak{g}$ has highest weight λ, then the **Weyl dimension formula** gives the dimension of the representation (i.e. the dimension of the vector space V on which the Lie algebra $\mathfrak{g}$ is represented as a subalgebra of $\mathfrak{gl}(V)$):

$$\dim[\lambda] = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}. \tag{9.28}$$

The Weyl dimension formula is derived by taking the limit $\lim_{t\to 0}\chi_\lambda(t\rho)$ and using L'Hôpital's rule on the expression in the Weyl character formula.

Tensor products of various representations of a Lie algebra $\mathfrak{g}$ will be very important subsequently, especially in the context of fusion rules of WZW models. Given two finite dimensional representations of g with highest weights λ and μ, the tensor product of these representations can be decomposed into irreducible representations as

$$\lambda\otimes\mu = \bigoplus_{\nu\in P_+} \mathcal{N}_{\lambda\mu}^{\nu}\nu\,, \tag{9.29}$$

where P_+ is the set of all dominant weights, and $\mathcal{N}_{\lambda\mu}^{\nu}$ called the **tensor product coefficient** gives the multiplicity of the representation ν in the decomposition of the tensor product $\lambda\otimes\mu$. There are very general methods such as the character method, Littlewood Richardson rules, Young tableaux etc. for calculating tensor product coefficients, many of which rely on the Weyl character formula. Further details can be found in [1–3].

9.2 Affine Lie Algebras

The basic philosophy behind affine Lie algebras is as follows. For every (finite) Lie algebra $\mathfrak{g}$ there is associated an affine extension $\hat{\mathfrak{g}}$ obtained by adding an extra node to the Dynkin diagram of $\mathfrak{g}$, which essentially corresponds to the highest root θ. The effect of adding this extra simple root is to make the root system of $\hat{\mathfrak{g}}$ infinite, and consequently, highest-weight representations are also infinite dimensional. However, the collection of these representations have an additional substructure to them in that they are organized by means of a new parameter called the level, usually denoted by k. For a fixed level, there is a finite number of highest weight representations, the so-called integrable representations. These representations have almost miraculous modular transformation properties. However, we adopt a slightly different approach in the initial construction of the affine Lie algebras, and display this aspect as well shortly thereafter.

Consider the situation where all the elements of $\mathfrak{g}$ are also Laurent polynomials in some variable z. The set of all such polynomials is denoted $\mathbb{C}[z, z^{-1}]$. This generalization is called the **loop algebra** $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ i.e. $\tilde{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[z, z^{-1}]$ with generators $J^a\otimes z^n$, where J^a are the generators of $\mathfrak{g}$. The name loop algebra is appropriate since $\tilde{\mathfrak{g}}$ could be regarded as the space of all polynomial mappings from the circle S^1 to $\mathfrak{g}$. The notation $J_n^a = J^a\otimes z^n$ will be used. Now $\tilde{\mathfrak{g}}$ is centrally extended by adding to it a **central element** $\hat{k}$ such that

$$\left[J^a_n, J^b_m\right] = \sum_c i f^{ab}_c J^c_{n+m} + \hat{k} n \delta_{ab} \delta_{n+m,0} \,,$$
$$\left[J^a_n, \hat{k}\right] = 0 \,, \tag{9.30}$$

where the commutation relations of the underlying simple Lie algebra $\mathfrak{g}$ (9.4) are inherited by the centrally extended loop algebra.

Finally a new operator, called a **derivation**, defined by

$$\ell_0 = -z\frac{d}{dz} \,, \tag{9.31}$$

which acts on the adjoint representation i.e. $[\ell_0, J^a_n] = -nJ^a_n$, is added to the centrally extended loop algebra. The resulting algebra,

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}\ell_0 \,, \tag{9.32}$$

is known as an **affine Lie algebra**. Clearly it is an infinite dimensional algebra since it has an infinite number of generators $\{J^a_n\}$, $n \in \mathbb{Z}$. $\mathfrak{g}$ is generally called the finite or zero-mode algebra, and it has generators J^a_0.

For the purposes of constructing the algebra and its representations, instead of using the set J^a_n, it is preferable to use H^i_n and E^α_n, which are the loop extended elements of the Cartan-Weyl basis, with the obvious commutation relations. **Affine weights** are characterized as $\hat{\lambda} = (\lambda; k_\lambda; n_\lambda)$, where λ is a finite weight of $\mathfrak{g}$, and k_λ and n_λ are the eigenvalues of $\hat{k}$ and ℓ_0. The algebra has sufficient degrees of freedom (e.g. linear deformations of ℓ_0) that the Killing form can be chosen to yield the inner product between two affine weights as

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda \,. \tag{9.33}$$

Affine roots are affine weights in the adjoint representation. In the adjoint representation the eigenvalue of $\hat{k}$ is 0 since $\hat{k}$ commutes with all the generators. So, roots of $\hat{\mathfrak{g}}$ are of the form $\hat{\beta} = (\beta; 0; n)$. Thus, the scalar product of two affine roots is simply the scalar product of the corresponding finite roots. The affine root associated to the generator E^α_n is clearly $(\alpha; 0; n)$. The generator $\delta = (0; 0; 1)$ is known as an **imaginary root**, since it has zero length i.e. $(\delta, \delta) = 0$. If we represent a finite root α in the space of affine weights as $\alpha = (\alpha; 0; 0)$, then every affine root is either of the form $\{\alpha + n\delta,\ n \in \mathbb{Z},\ \alpha \in \Delta\}$ or of the form $\{n\delta,\ n \in \mathbb{Z},\ n \neq 0\}$. The union of these two sets, i.e. the collection of all affine roots, is denoted $\hat{\Delta}$.

It can be shown that a basis of simple roots of $\hat{\mathfrak{g}}$ is given by α_i, $i = 1, ..., r$, the simple roots of $\mathfrak{g}$ together with $\alpha_0 = (-\theta; 0; 1) = -\theta + \delta$, where θ is the highest root of $\mathfrak{g}$. The set of positive roots is $\hat{\Delta}_+ = \{\alpha + n\delta | n > 0,\ \alpha \in \Delta\} \cup \{\alpha | \alpha \in \Delta_+\}$. This now provides a relation to the initial description of affine Lie algebras in terms of adding an extra node.

Now that a basis of simple roots and a scalar product have been defined, the **extended Cartan matrix** can be defined as

$$\hat{A}_{ij} = (\alpha_i, \alpha_j^\vee)\,, \quad 0 \le i, j \le r\,, \tag{9.34}$$

where the coroots are given by $\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2}\alpha = \frac{2}{|\alpha|^2}\alpha$. The addition of the extra simple root α_0 implies that the extended Cartan matrix has one extra row and column compared to the Cartan matrix of $\mathfrak{g}$. Similar to the construction of the Dynkin diagrams for simple Lie algebras, extended Dynkin diagrams can be constructed for these affine Lie algebras as well, which encodes all the information of the extended Cartan matrix. These are illustrated in the set (A) of Dynkin diagrams in Fig. (9.2)[3]. Clearly, these are obtained from the corresponding algebra by the addition of an extra node for α_0.

As mentioned earlier, associated to every affine Lie algebra, there is an extended Cartan matrix. However, in addition to the extended Cartan matrices obtained by the above procedure of going through centrally extending loop algebras, other extended Cartan matrices can be defined as well. These result in the so called **twisted affine Lie algebras** and are shown in set (B) in Fig. (9.2). The first label for each of these Dynkin diagrams indicates the finite algebra whose Cartan matrix has been extended to obtain the twisted algebra, and the superscript of 2 indicates that the affine algebra is twisted and is not readily constructed as an extended loop algebra as above, without modification of the procedure.

The second labeling for these Dynkin diagrams arises from the loop algebra technique used to construct the extended Lie algebra. As stated earlier, the loop algebra is the space of analytic mappings from S^1 to the Lie algebra $\mathfrak{g}$. If instead of using the boundary conditions $\mathcal{P}(e^{2\pi i}z) = \mathcal{P}(z)$, we impose the **twisted boundary conditions** $x \otimes \mathcal{P}(e^{2\pi i}z) = \omega(x) \otimes \mathcal{P}(z)$, for every x in $\mathfrak{g}$, where ω is an outer automorphism (a symmetry) of the (unextended) Dynkin diagram of $\mathfrak{g}$ on finite order N; i.e. N is the smallest integer such that $\omega^N = 1$. In this case the automorphism ω provides a natural $\mathbb{Z}_N$-grading of $\mathfrak{g}$, and consequently for the generators as well. The construction then proceeds similar to the affine Lie algebras constructed above, taking this grading into

[3]Figure reproduced from [3] under the provisions of the Copyright Act. and with permission from Cambridge University Press.

account as well. This procedure results in the twisted algebra. Thus, in this labeling the algebra symbol indicates the **horizontal subalgebra** whose N-folded centrally extended algebra results in the twisted affine Lie algebra and the superscript indicates the order N of the automorphism used.

The **affine marks** a_i and **affine comarks** $a_i^\vee$ are defined by

$$\sum_{i=0}^{r} a_i \hat{A}_{ij} = \sum_{i=0}^{r} \hat{A}_{ij} a_j^\vee = 0\,. \tag{9.35}$$

The **affine Coxeter** and **affine dual Coxeter** numbers are defined by $h = \sum_{i=0}^{r} a_i$ and $h^\vee = \sum_{i=0}^{r} a_i^\vee$ respectively. Through abuse of terminology, the label "affine" will typically be dropped when referring to all four of these quantities.

Completely analogous to the finite case, a Chevalley-Serre basis can be constructed, with commutation relations identical to (9.21) and (9.22), with indices now running from 0, instead of 1, to r, and the affine Cartan matrix is used instead.

The **affine fundamental weights** $\{\hat{\omega}_i\}$ are, as before, defined to be dual to the basis of coroots. The fundamental weights then turn out to be

$$\begin{aligned} \hat{\omega}_i &= (\omega_i; a_i^\vee; 0)\,, \quad 1 \leq i \leq r\,, \\ \hat{\omega}_0 &= (0; 1; 0)\,. \end{aligned} \tag{9.36}$$

The scalar product between the fundamental weights can be worked out to be $(\hat{\omega}_i, \hat{\omega}_j) = F_{ij}$ and $(\hat{\omega}_i, \hat{\omega}_0) = (\hat{\omega}_0, \hat{\omega}_0) = 0$ for $1 \leq i, j \leq r$. Again, weights are specified using Dynkin labels λ_i where i now runs from 0 to r via $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i$ and are usually written as $\hat{\lambda} = [\lambda_0, ..., \lambda_r]$. Also, the affine Weyl vector can be similarly defined as $\hat{\rho} = \sum_{i=0}^{r} \hat{\omega}_i$.

In an irreducible highest weight representation, $\hat{k}$ will be sent to a scalar multiple k of the identity. This number k plays a very important role, and is called the **level** of the representation. Any affine weight $\hat{\lambda}$ in a level k representation will satisfy

$$k = \sum_{i=0}^{r} a_i^\vee \lambda_i\,. \tag{9.37}$$

The representations of most interest in the case of affine Lie algebras are the so-called **integrable dominant highest weight representations**. As before, "dominant" implies that all Dynkin labels are non-negative. In addition, the integrability condition requires that

$$k \in \mathbb{Z}_+\,, \quad k \geq (\lambda, \theta)\,. \tag{9.38}$$

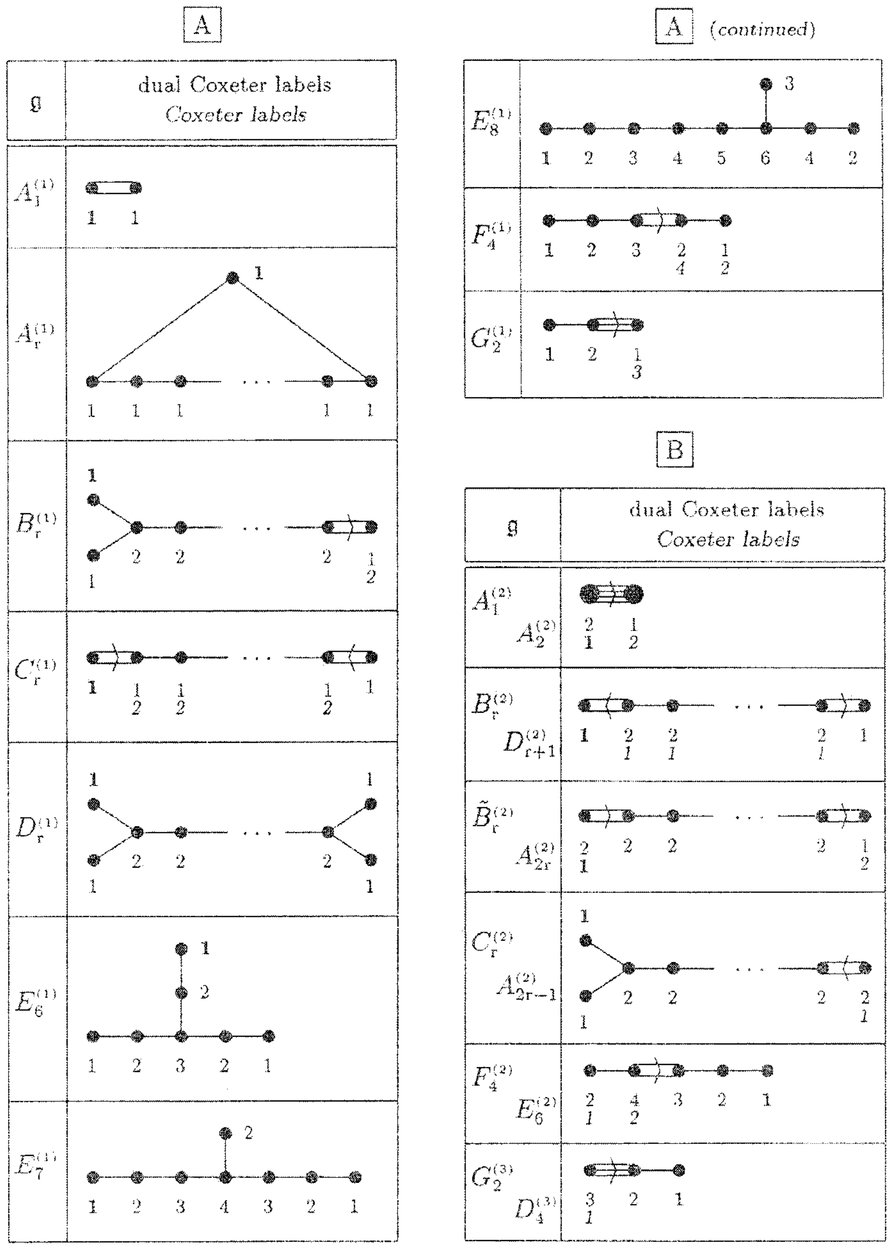

Figure 9.2: Dynkin diagrams for the affine Lie algebras

The latter condition ensures non-negativity of λ_0. The set of all dominant weights for a given level k will be denoted by P_+^k. The integrability condition also ensures that for any level k there are only a finite number of highest weights. Using the Chevalley-Serre generators and relations the representations can be explicitly constructed. The representations turn out necessarily to be infinite-dimensional. However, the integrability condition allows the representations to be effectively organized by the **grade**, the eigenvalue of L_0. At any grade, there are only a finite number of weights. This substructure of these representations allows greater control over the infinite dimensionality of the representation modules, and also provides a very effective way of dealing with their transformation properties. Explicit details of the representation theory will not concern us here, and can be found in [1,3,6].

The **affine Weyl group** $\hat{W}$ is generated in a similar fashion to the finite Weyl group, and is made of the "reflections" $s_{\hat{\alpha}} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha}$. The generator $s_{\hat{0}}$ corresponding to $\hat{\alpha}_0$ is in fact a reflection combined with a translation, and the affine Wely group has some additional structure organizing it in spite of being an infinite group, again dependent on the level k. Shifted affine Weyl reflections can be defined by $w \cdot \hat{\lambda} = w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$. $\epsilon(w)$ is also defined similar to the finite case as the parity of w in terms of simple Weyl elements.

Affine characters are defined by

$$\chi_{\hat{\lambda}} = \sum_{\hat{\lambda}' \in \Omega_{\hat{\lambda}}} \mathrm{mult}_{\hat{\lambda}}(\hat{\lambda}') e^{\hat{\lambda}'}, \tag{9.39}$$

which can be shown to be equivalent to

$$\chi_{\hat{\lambda}} = e^{m_{\hat{\lambda}}\delta} \frac{\sum_{w \in \hat{W}} \epsilon(w) e^{w \cdot (\hat{\lambda} + \hat{\rho})}}{\sum_{w \in \hat{W}} \epsilon(w) e^{w \cdot (\hat{\rho})}}, \tag{9.40}$$

where $m_{\hat{\lambda}}\delta$ is the so called **modular shift anomaly** given by

$$m_{\hat{\lambda}}\delta = \frac{|\hat{\lambda} + \hat{\rho}|^2}{2(k + h^\vee)} - \frac{|\hat{\rho}|^2}{2h^\vee} = \frac{|\lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee}. \tag{9.41}$$

Characters of affine Lie algebras encode all the information regarding their representation theory. In addition, they allow for easy analysis of several remarkable properties of affine Lie algebras, which would not be transparent if the representations are directly studied. The modular shift is essentially a normalization, but is going to be relevant for ensuring that these characters behave well under modular transformations to be discussed below.

9.3 Modular Transformations

Affine Lie algebras show up in the context of CFT's in many situations, not least of which are the WZW models which will be discussed in detail shortly. As is well known, CFT's have a a consistency condition that need to be satisfied at the one loop level, and this gives rise to the concept of modularity. At the one-loop level, we need to consider CFT's defined on a torus worldsheet. Tori are classified by their modular parameter τ, and the conformally equivalent classes are invariant under modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{9.42}$$

where a, b, c, and d are integers. That is, the **modular group** is $SL_2(\mathbb{Z})$. $SL_2(\mathbb{Z})$ has two generators $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. The the former simply interchanges the two non-trivial cycles of the torus, while latter corresponds to an operation on the torus known as a Dehn twist which cuts the torus, twists it and then reattaches it.

Any affine Lie algebras associated with the CFT's also need to have nice transformation properties under the modular group so that the one-loop consistency condition may be satisfied. As mentioned earlier, characters encode all the information about representations, and are particularly easy to work with in this context.

Given an affine weight (ξ, τ, t), under the action of the modular group through (9.42), the transformation of the weight is

$$(\xi, \tau, t) \to \left(\frac{\xi}{c\tau + d}; \frac{a\tau + b}{c\tau + d}; t + \frac{c|\xi|^2}{2(c\tau + d)}\right). \tag{9.43}$$

The matrices that specify the transformations between affine weights at level k generated by S and T of $SL_2(\mathbb{Z})$, are called the **modular S and T matrices** of $\hat{\mathfrak{g}}$ i.e.

$$\begin{aligned}
\chi_{\hat{\lambda}}(\xi; \tau + 1; t) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{T}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\xi; \tau; t), \\
\chi_{\hat{\lambda}}(\frac{\xi}{\tau}; \frac{-1}{\tau}; t + \frac{|\xi|^2}{2\tau}) &= \sum_{\hat{\mu} \in P_+^k} \mathcal{S}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\xi; \tau; t).
\end{aligned} \tag{9.44}$$

It is important to note that the transformation is only between dominant integrable weights at the same level k. As such, the matrices $\mathcal{S}$ and $\mathcal{T}$ are finite dimensional (since there are only a finite number of dominant weights at any given level), but their size increases with the level.

Explicit formulae for these matrices can be worked out. $\mathcal{T}$ is particularly simple:

$$\mathcal{T}_{\hat{\lambda}\hat{\mu}} = \delta_{\hat{\lambda}\hat{\mu}} e^{m_{\hat{\lambda}}\delta} , \tag{9.45}$$

i.e. $\tau \to \tau + 1$ simply introduces a phase change. $\mathcal{S}$ on the other hand, is non-diagonal and quite complicated

$$\mathcal{S}_{\hat{\lambda}\hat{\mu}} = \frac{i^{|\Delta_+|}\left[\det(\alpha_i^\vee, \alpha_j^\vee)\right]}{2(k+h^\vee)^{1/2}} \sum_{w\in W} \epsilon(w) e^{-2\pi i (w\cdot(\lambda+\rho),\mu+\rho)/(k+h^\vee)} , \tag{9.46}$$

where the sum is over the finite Weyl group, and the determinant is also only over the finite coroots. It is very important to note that both the $\mathcal{S}$ and $\mathcal{T}$ matrices are unitary.

It can be shown that $\mathcal{S}^2 = \mathcal{C}$ where $\mathcal{C}$ is the charge conjugation matrix with $\mathcal{C}\chi_{\hat{\lambda}} = \chi_{\hat{\lambda}^*}$. $\mathcal{C}$ itself acts on $\mathcal{S}$ very simply through complex conjugation of the matrix i.e. $\mathcal{CS} = \mathcal{SC} = \mathcal{S}^*$, or equivalently

$$\mathcal{S}^*_{\hat{\lambda}\hat{\mu}} = \mathcal{S}_{\hat{\lambda}^*\hat{\mu}} = \mathcal{S}_{\hat{\lambda}\hat{\mu}^*} , \tag{9.47}$$

where $\mathcal{S}^*$ is $\mathcal{S}$ with the matrix entries complex conjugated. It can also be shown that

$$\mathcal{S}_{\hat{\lambda}0} \geq \mathcal{S}_{00} > 0 , \tag{9.48}$$

where 0 is the state with Dynkin labels $[k, 0, .., 0]$, and is typically called the **vacuum**.

9.4 Wess-Zumino-Witten Models

CFT's are exactly solvable precisely because of the presence of a vast infinite dimensional symmetry algebra viz. the Virasoro algebra. The early study of CFT's, starting with the seminal work of Belavin, Polyakov, and Zamolodchikov [7], worked with the so-called minimal models, which basically implemented the Virasoro algebra in a "minimal fashion" with no other symmetries. Subsequently, attempts were made to define CFT's with sufficiently large symmetry algebras that the Virasoro algebra is included as a subalgebra, so that the tools of CFT would still be applicable, but richer structures would come into play. Some of the most important are the Wess-Zumino Witten models. Here the larger algebras are taken to be affine Lie algebras, and these naturally include the Virasoro algebra (as a Lie subalgebra of the universal enveloping algebra of the affine algebra), thereby preserving the infinite conformal symmetry. WZW models have also been the source of even richer CFT's such as coset CFT's obtained through taking cosets of the symmetry algebra of a WZW model, or CFT's with $\mathcal{W}$-algebra symmetry obtained through Hamiltonian reduction of WZW models.

Let G be a compact connected Lie group, and $\mathfrak{g}$ its simple Lie algebra. Suppose γ is a G-valued field on the complex plane i.e. we are considering the theory of a string with target space group manifold G. The Wess-Zumino-Witten (WZW) action is

$$S = \frac{k}{16\pi} \int d^2x \mathrm{Tr}(\partial^\mu \gamma^{-1} \partial_\mu \gamma) + k\Gamma \,, \tag{9.49}$$

where Γ will be discussed below [8]. The first term in the action is a so-called non-linear sigma model. It has an infrared stable fixed point, and when the dynamics of the theory lives at that point, the extra term Γ can be consistently added, which results in the infinite symmetry that gives rise to a CFT. Γ is known as the **Wess-Zumino** term:

$$\Gamma = \frac{-i}{24\pi} \int_B d^3y \epsilon_{\alpha\beta\delta} \mathrm{Tr}(\tilde{\gamma}^{-1}\partial^\alpha \tilde{\gamma}\tilde{\gamma}^{-1}\partial^\beta \tilde{\gamma}\tilde{\gamma}^{-1}\partial^\delta \tilde{\gamma}) \,. \tag{9.50}$$

This term is defined on a three-dimensional manifold B, such that its boundary is the compactification of our two-dimensional space (i.e. S^2) and $\tilde{\gamma}$ is the extension to B of the field γ. A natural question to ask is whether this extension is unique, or if it depends on the choice of manifold B thereby leading to an ambiguity in the definition of Γ. However, since the second homotopy group $\pi_2(G) = 0$, this implies that the extension is unique up to homeomorphism. In addition, to ensure that e^{iS} is single valued (for the purposes of the path integral), we need to have $k \in \mathbb{Z}_+$. This is essentially a Dirac quantization condition for this theory.

We will use complex coordinates on the plane henceforth. The WZW action has two conserved currents given by

$$J(z) = -k\partial\gamma\gamma^{-1} \,, \qquad \overline{J}(\overline{z}) = k\gamma^{-1}\overline{\partial}\gamma \,, \tag{9.51}$$

where $\partial = \partial_z$ and $\overline{\partial} = \partial_{\overline{z}}$. If we use the decomposition $J(z) = \sum_a J^a t^a$, where the t^a are the generators of the algebra $\mathfrak{g}$. Then the operator product expansion using the WZW action can be worked out as

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c i f_c^{ab} \frac{J^c(w)}{(z-w)} \,. \tag{9.52}$$

Further, doing a Laurent decomposition $J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J_n^a$ into modes and using standard CFT tools, we can obtain the commutation relations

$$\begin{aligned}
\left[J_n^a, J_m^b\right] &= \sum_c i f_c^{ab} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}\,, \\
\left[J_n^a, \overline{J}_m^b\right] &= 0\,, \\
\left[\overline{J}_n^a, \overline{J}_m^b\right] &= \sum_c i f_c^{ab} \overline{J}_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}\,.
\end{aligned} \tag{9.53}$$

which is essentially two copies of (9.30), the basic commutation relations for an affine Lie algebra i.e. we have one copy of $\hat{\mathfrak{g}}$ in the holomorphic sector, and another for the antiholomorphic sector. As is usual in CFT's we deal with the holomorphic sector. Also note that the constant k, which was essentially related to a winding number in the WZW action, is now interpreted in the full quantum theory as the level of the associated affine Lie algebra.

Using the so-called **Sugawara construction**, the stress tensor for the theory can be constructed:

$$T(z) = \frac{1}{2(k + h^\vee)} \sum_a \left(J^a J^a\right)(z)\,, \tag{9.54}$$

where $(\ldots)$ denotes the normal ordered product of fields. An operator product expansion of this T with itself gives the central charge as

$$c = \frac{k\,\mathrm{dim}\mathfrak{g}}{k + h^\vee}\,. \tag{9.55}$$

In addition we can form the operators

$$L_n = \frac{1}{2(k + h^\vee)} \sum_a \sum_m :J_m^a J_{n-m}^a:\,, \tag{9.56}$$

where $:\ldots:$ denotes operator normal ordering for operators.

These do indeed generate the Virasoro algebra and can be shown to satisfy the standard commutation relations

$$\begin{aligned}
\left[L_n, L_m\right] &= (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}\,, \\
\left[L_n, J_m^a\right] &= -mJ_{n+m}^a\,, \\
\left[J_n^a, J_m^b\right] &= \sum_c i f_c^{ab} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}\,,
\end{aligned} \tag{9.57}$$

where the last equation is from (9.53), and is included to show the complete commutation relations of the holomorphic sector. This explicitly illustrates the earlier claim that WZW models enlarge the Virasoro algebra symmetry of all CFT's.

Using these commutation relations, the representations of WZW models can be constructed. Further details can be found in [1]. Primary fields of WZW theories regarded as CFT's turn out to be the highest weight states of the affine algebra, i.e., we can label a specific representation of the WZW model states by giving a highest weight at level k. Typically the eigenvalue n of the operator L_0, known as the grade, is chosen to be 0 at the highest weight, and the commutation relations show that the action of the lowering operators likewise raise the grade in the descendents, as you would expect in the representation theory of affine Lie algebras. Note that this L_0 is simply a scalar multiple of the operator ℓ_0 defined as a derivation in the context of affine Lie algebras.

As mentioned earlier, it is typical to consider only one sector of the theory, usually the holomorphic sector. However, a very important question is regarding how to assemble the two sectors back together to make a complete theory. At the level of characters (which encode the complete information about the representations anyway, so there is no loss in information in just considering them), we can assemble a partition function for the theory:

$$\mathcal{Z}(\tau) = \sum_{\hat{\lambda},\, \hat{\mu} \in P_+^k} M_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\lambda}}(\tau) \overline{\chi}_{\hat{\mu}}(\tau)\,, \tag{9.58}$$

where the bar denotes a character in the antiholomorphic sector of the theory, and τ is the modular parameter on the torus. This implies that the Hilbert space of states of the full theory is assembled as

$$\mathcal{H} = \bigoplus_{\hat{\lambda},\, \hat{\mu} \in P_+^k} M_{\hat{\lambda}\hat{\mu}} \mathcal{H}_{\hat{\lambda}} \otimes \overline{\mathcal{H}}^*_{\hat{\mu}}\,. \tag{9.59}$$

The matrix $M_{\hat{\lambda}\hat{\mu}}$ that specifies how to combine the holomorphic and antiholomorphic sectors of the theory together to obtain the full theory is known as a **modular invariant**. Obviously, a good choice is to take M to the identity matrix, in which case it is referred to as the diagonal modular invariant. Non-diagonal modular invariants provide many interesting features not seen in the diagonal theories. Several of these non-diagonal modular invariants are known. Symmetries of the unextended Dynkin diagram always give rise to modular invariants. An example of this is to take $M = \mathcal{C}$ (where, as discussed earlier, charge conjugation is non-trivial if there is a symmetry of the unextended Dynkin diagram). Symmetries of the extended Dynkin diagram, may or may not give rise to non-trivial modular invariants. Usually there is some restriction on the level k, if it is possible at all. We will see some special cases later of such modular invariants in the context of twisted algebras. Symmetries of both the extended and unextended Dynkin diagrams give rise to an infinite family of modular invariants.

WZW models have extremely rich properties and hidden unexpected structure in the form of differential and algebraic constraints. Examples of these include the Knizhnik-Zamolodchikov equation, the Gepner-Witten equation, level-rank duality etc., all of which are still extremely interesting topics. In addition, they give rise to remarkable Lie and number theoretic phenomena, as well as extensive algebraic geometry in the form of concepts like the braid group, knot theory etc. Unfortunately, we will not have sufficient space to explore these here, but there are a number of excellent references available on the subject, including [1, 6, 9–11].

9.5 Fusion Rules and Simple Currents

In any Rational Conformal Field Theory (RCFT), the so-called **fusion coefficients** $\mathcal{N}_{\phi_i\phi_j\phi_k}$ count the number of independent couplings between the three primary fields ϕ_i, ϕ_j, and ϕ_k, i.e., $\mathcal{N}_{\phi_i\phi_j\phi_k}$ counts the multiplicity of the conjugate field ϕ_k^* in the Operator Product Expansion (OPE) of $\phi_i(z)$ with $\phi_j(w)$. Formally, we have

$$\phi_i \times \phi_j = \sum_{\phi_k^*} \mathcal{N}_{\phi_i\phi_j}^{\phi_k^*} \phi_k^* . \tag{9.60}$$

It is important to remember that in the above rule (and henceforth in the context of fusions), we are considering not just the field's ϕ_i, but rather all of its descendants as well, i.e., the OPE of any two descendants of ϕ_i and ϕ_j will also produce a field from the family of ϕ_k^* and its descendants with the same multiplicity $\mathcal{N}_{\phi_i\phi_j}^{\phi_k^*}$.

For a WZW model at level k with spectrum generating algebra $\hat{\mathfrak{g}}_k$, the primary fields are in one-to-one correspondence with the highest weight representations $\hat{\lambda} \in P_+^k$, and can, thus, simply be labeled by them. The fusion rules then take the form

$$\hat{\lambda} \times \hat{\mu} = \bigoplus_{\hat{\nu} \in P_+^k} \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} \hat{\nu} , \tag{9.61}$$

where it is understood that the fusion rules are specific to the level k.

There is a remarkable formula known as the **Verlinde formula** that allows for the calculation of the level k fusion coefficients $\mathcal{N}$ in terms of the level-k modular S matrix

(9.46):

$$\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} = \sum_{\hat{\alpha}\in P_+^k} \frac{\mathcal{S}_{\hat{\lambda}\hat{\alpha}}\mathcal{S}_{\hat{\mu}\hat{\alpha}}\mathcal{S}^*_{\hat{\nu}\hat{\alpha}}}{\mathcal{S}_{0\hat{\alpha}}}, \tag{9.62}$$

where 0 is the vacuum state $k\hat{\omega}_0$. This formula can be proven on some very general grounds and is also the source of the braiding relations of conformal blocks on a torus which gives rise to several very exciting knot theory results, including many of the results of Jones-Witten [9].

Unitarity of $\mathcal{S}$ immediately implies that $\mathcal{N}_{0\hat{\lambda}}^{\hat{\nu}} = \delta_{\hat{\lambda}}^{\hat{\nu}}$. In addition, (9.47) implies that

$$\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} = \mathcal{N}_{\hat{\lambda}\hat{\mu}\hat{\nu}^*} = \mathcal{N}_{\hat{\lambda}\hat{\nu}^*}^{\hat{\mu}^*}; \tag{9.63}$$

i.e., in WZW models (as in any RCFT), indices are raised and lowered using the charge conjugation matrix $\mathcal{C} = \mathcal{S}^2$.

In addition, an argument using the Weyl character formula results in the **Kac-Walton formula:**

$$\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} = \sum_{w\in\hat{W},\, w\cdot\nu\in P_+} \mathcal{N}_{\lambda\mu}^{w\cdot\nu}\epsilon(w), \tag{9.64}$$

where the coefficients $\mathcal{N}$ appearing on the right hand side are the tensor product coefficients (9.29) of the underlying finite algebra $\mathfrak{g}$ of $\hat{\mathfrak{g}}_k$, P_+ is the set of all dominant highest weights of $\mathfrak{g}$, and the sum is over the affine Weyl group elements whose action on ν gives a dominant finite weight of $\mathfrak{g}$ (i.e. in P_+). This formula relates fusion coefficients of WZW models with spectrum generating algebra $\hat{\mathfrak{g}}_k$ to tensor product coefficients of the finite algebra $\mathfrak{g}$. With some further work it can be shown that this formula roughly says that the fusion coefficients are the tensor product coefficients, but with an accommodation for level truncation. That is, a highest weight of $\mathfrak{g}$ is not taken into account, even if it shows up in the tensor product, if it cannot then be regarded as a level k dominant integral weight of $\mathfrak{g}$ (though in some cases it counts if there is a Weyl reflection of that weight that maps it into a dominant integral weight). Further details, including many calculational recipes, can be found in [1, 6]. In addition to some of the symmetries mentioned above, fusion rules can respect some other very important symmetries, which are related to symmetries of extended Dynkin diagrams. We now present a detailed discussion of these.

Many of the untwisted affine Lie algebras have symmetries associated with their (extended) Dynkin diagrams. These are shown in Fig. 9.3[4]. This in turn, in the obvious way, results in an action on the Dynkin labels of the weights of the affine Lie algebra. For instance, $\hat{A}_2^{(1)}$ has a cyclic symmetry A of its Dynkin diagram that acts on the Dynkin labels of weights via

$$A\,[\lambda_0, \lambda_1, \lambda_2] = [\lambda_2, \lambda_0, \lambda_1]\,. \tag{9.65}$$

The generalization to the higher $\hat{A}_n^{(1)}$'s is obvious, and likewise for the other algebras whose extended Dynkin diagrams have the symmetries shown in Fig. 9.3.

The set $\mathcal{O}(\hat{\mathfrak{g}})$ of symmetries of extended Dynkin diagrams is isomorphic to $B(G)$, the center of the group G of $\mathfrak{g}$ (obtained by exponentiating the elements of $\mathfrak{g}$). The center G is the normal subgroup of G consisting of all elements of G that commute with all the elements of the group.

The action of any outer automorphism A on the modular S-matrix can be shown to be

$$A\mathcal{S}_{\hat{\lambda}\hat{\mu}} = \mathcal{S}_{A(\hat{\lambda})\hat{\mu}} = S_{\hat{\lambda}\hat{\mu}}e^{-2\pi i(A\hat{\omega}_0, \hat{\mu})}\,. \tag{9.66}$$

The quantity appearing in the phase is known as the **charge** of the simple current is typically denoted $Q(\hat{\lambda}) = (A\hat{\omega}_0, \hat{\lambda})$.

Using this, and the Verlinde formula (9.62), the following properties of the fusion coefficients can be established:

$$\mathcal{N}_{A(\hat{\lambda})A'(\hat{\mu})}^{AA'(\hat{\nu})} = \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}}\,, \quad \mathcal{N}_{A(\hat{\lambda})\hat{\mu}}^{\hat{\nu}} = \mathcal{N}_{\hat{\lambda}A(\hat{\mu})}^{\hat{\nu}}\,. \tag{9.67}$$

A special case of the second equality is

$$\mathcal{N}_{A(0)\hat{\mu}}^{\hat{\nu}} = \mathcal{N}_{0A(\hat{\mu})}^{\hat{\nu}} = \delta_{A(\hat{\mu})}^{\hat{\nu}}\,; \tag{9.68}$$

that is, $A(0)$ acts simply as a permutation in the fusion rules; i.e., the OPE of $A(0)$ with any other primary field $\hat{\mu}$ only contains one primary field $\hat{\nu}$. Fields that act in this manner in fusion rules are known as **simple currents**. Usually, by abuse of terminology, the outer automorphism A itself is referred to as a simple current, and simple currents are said to act on weights through the action of this outer automorphism. In this context, the simple current is usually denoted by the letter J.

For WZW models based on simple Lie algebras, all simple currents arise from outer

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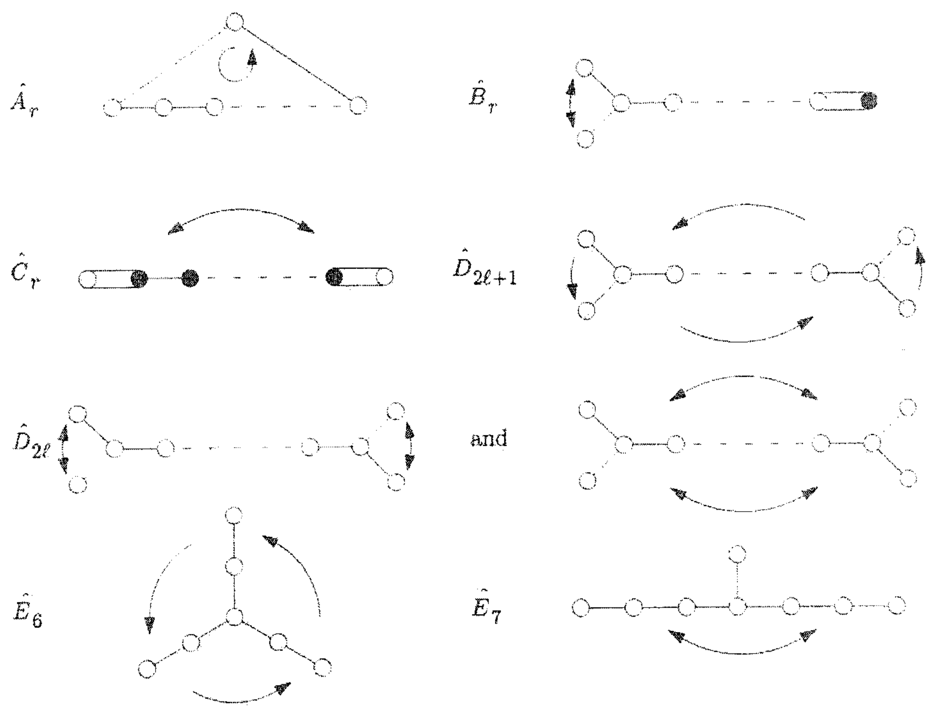

Figure 9.3: Outer automorphisms of affine Lie algebras

automorphisms, the only exception being a simple current that occurs only at level 2 for $\hat{E}_8^{(1)}$.

One further topic regarding fusion rules that merits discussion is the following. Since the fusion coefficients $\mathcal{N}$ are supposed to be multiplicities, they obviously need to be non-negative integers. However, this is not apparent from the Verlinde formula (9.62). Thus, so-called Non-Integer Matrix representations or NIM-reps of fusion algebras are of great interest since they can accurately represent fusions in WZW models.

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Chapter 10

BCFT Approach to D-branes

10.1 Boundaries in CFT

The standard picture of D-branes in string theory is of extended objects in space-time that can wrap around certain cycles in the target space geometry. From this point of view, the analysis of D-branes is tackled through powerful geometrical tools such as K-theory, cohomology etc. [1–3]. A second approach to D-branes is the so-called microscopic viewpoint. Here, D-branes are regarded as open string sectors that can be consistently added to closed string theory, and the analysis of D-branes is tackled through the equally powerful tools of boundary CFT. This is an exact string description and thus can be quite powerful. However, unfortunately, this description is only available at specific points in the moduli spaces of the target space geometries, such as orbifold points, etc. The macroscopic/geometrical viewpoint is more freely available, essentially whenever the supergravity approximation to string theory can be trusted. The two descriptions, in some sense, are "dual" to each other, and comparison of results from both approaches when possible provide insight into the structure of string theory, and in several situations one approach yields answers while the other fails. We will be concerned with the microscopic point of view here. We will start with a generic discussion of boundaries in CFT, then specialize to WZW models.

Closed string theory is defined on Riemann surfaces, which are closed compact world-sheets without boundary. In order to add D-branes to the closed string theory, we need to consistently add boundaries to closed Riemann surfaces. We can start with the sphere, since the behaviour of CFT's on the sphere uniquely determines its behaviour on all genus Riemann surfaces (whether it is a consistent CFT or not depends on whether it satisfies the one-loop constraint, i.e., whether it is well behaved under the modular group on the torus). The OPE defines an algebra of fields under fusion rules $\phi_a \times \phi_b = \sum_c \mathcal{N}_{ab}^c \phi_c$. Any potential boundary we add should respect this algebra of primary fields, and therefore, must define a homomorphism to $\mathbb{C}$ from the space of primary fields of the theory regarded as an algebra.

Every state in the spectrum $\mathcal{H}$ of the theory on the sphere defines such a homomorphism (by evaluation), and in fact, every such map arises from a suitable linear combination of such states. Thus, every boundary condition can be described by a coherent state in the full CFT. For a boundary condition labeled by λ, denote the corresponding boundary state by $||\lambda>$. The amplitude of fields in the presence of the boundary condition α is then given by expressions of the form $< \phi_1\phi_2 >_\lambda = < \phi_1\phi_2||\lambda>$, where the inner product is evaluated in $\mathcal{H}$. This is nothing more than the usual field/operator-state correspondence used in CFT.

However, not every linear homomorphism from the space of primary fields to $\mathbb{C}$ defines a boundary state. The coherent states that describe boundary conditions need to relate any symmetries in the closed theory properly at the boundary without breaking them in the bulk. If the boundary is taken to be along the real axis, then the relevant condition is that

$$S(z) = \rho(\overline{S}(\overline{z}))\,, \quad z \in \mathbb{R}\,, \tag{10.1}$$

where S and $\overline{S}$ are generators of the symmetry of the theory in the holomorphic and antiholomorphic sectors preserved by the boundary, and ρ is an automorphism of the algebra of primary fields that leaves the stress tensor invariant (i.e., does not change the CFT). If there are further symmetries of the CFT that need to be obeyed, similar conditions need to be respected by the generators of those symmetries at the boundary. S and $\overline{S}$ have Laurent decompositions

$$S(z) = \sum_{n\in\mathbb{Z}} \frac{S_n}{z^{n+h}}\,, \quad \overline{S}(\overline{z}) = \sum_{m\in\mathbb{Z}} \frac{\overline{S}_m}{\overline{z}^{m+\overline{h}}}\,, \tag{10.2}$$

where h and $\overline{h}$ are the conformal weights of S and $\overline{S}$.

The description above was on the description of the CFT in the plane/sphere with the boundary as the real axis. In order to get the description in terms of operators and boundary states, we transform to the picture on the cylinder via the conformal transformation $w = e^{-2\pi i z}$, and we use the fact that since this is a conformal transformation, a primary field $S(z)$ transforms as $S(z) \to w'(z)^h S(w(z))$. Now we are describing the CFT on the cylinder, and the boundary corresponds to a circle on the cylinder. Using the mode decompositions above, we can write the boundary condition (10.1) in terms of the boundary state in the operator picture as

$$\left(\sum_{n\in\mathbb{Z}} S_n w^n - (-1)^h \sum_{m\in\mathbb{Z}} \rho(\overline{S}_m)\overline{w}^{-m}\right) ||\lambda >= 0\,, \quad \text{for } |w| = 1\,. \tag{10.3}$$

Since this has to hold for all w such that $|w| = 1$, we obtain the **gluing condition**

$$\left(S_n - (-1)^h \rho(\overline{S}_{-n})\right) ||\lambda >= 0\,, \quad \forall n \in \mathbb{Z}\,. \tag{10.4}$$

The gluing condition needs to hold for any symmetries of the closed string sector. In particular, it must hold for conformal symmetries, in which case the modes S_n and $\overline{S}_m$ are, respectively, L_n and $\overline{L}_m$, the generators of the holomorphic and antiholomorphic sectors of the Virasoro algebras of the CFT. They both have conformal weights $h = \overline{h} = 2$. The gluing condition then reads

$$\left(L_n - \overline{L}_{-n}\right) ||\lambda >= 0\,, \quad \forall n \in \mathbb{Z}\,. \tag{10.5}$$

If the theory possesses sufficient symmetries, there may be enough constraints stemming from the various gluing conditions that the boundary state $||\alpha >$ may be determined uniquely (up to normalization and phase). Denote the symmetry algebra of the holomorphic sector of the CFT by $\mathcal{A}$ (and implicitly we assume that $\overline{\mathcal{A}} \cong \mathcal{A}$). Now, the full spectrum of the theory can be decomposed as $\mathcal{H} = \bigoplus_{i,j} M_{i,j}\mathcal{H}_i \otimes \overline{\mathcal{H}}_j$, where the decomposition is in terms of the individual spectra of the holomorphic and antiholomorphic sectors of the theory. Now, the modes that appear in (10.4) map each $\mathcal{H}_i \otimes \overline{\mathcal{H}}_j$ into itself. Thus, we can solve the gluing condition separately for each such summand. It turns out that a non-trivial solution can be found in the case where $\mathcal{H}_i$ is the conjugate representation of $\overline{\mathcal{H}}_j$. Thus, if a state exists solving the gluing condition (10.4) for every symmetry in $\mathcal{A}$, it is known as the **Ishibashi state** (unique up to normalization for each i in the sum):

$$||i >\in \mathcal{H}_i \otimes \overline{\mathcal{H}}_i\,, \quad \left(S_n - (-1)^{h_S} \rho(\overline{S}_{-n})\right) ||\lambda >= 0\,, \quad \forall n \in \mathbb{Z}\,, \forall S \in \mathcal{A}\,. \tag{10.6}$$

Since we only consider Rational CFT's, the spectrum $\mathcal{H}$ only contains a finite number of summands, and thus the number of Ishibashi states is also finite. This in turn means that there are only finitely many (if any at all) boundaries that can consistently be added to a given CFT. Every boundary state can be written in terms of the Ishibashi states as

$$||\lambda >= \sum_i B^i_\lambda ||i > , \tag{10.7}$$

for some constants B^i_λ. In addition, there are one-loop constraints to satisfy to ensure the theory is consistent. This is an important step, but we will not pursue the details of the most general construction, and further details can be found in [4, 5]. One of the most important of these is the Cardy condition, which needs to be satisfied. While it can be discussed in this general setting, we will restrict ourself to the case of WZW models discussed below.

10.2 WZW D-branes

The spectra of WZW models have the decomposition $\mathcal{H} = \bigoplus_{\hat\lambda,\,\hat\mu \in P^k_+} M_{\hat\lambda\hat\mu} \mathcal{H}_{\hat\lambda} \overline{\mathcal{H}}^*_{\hat\mu}$. Given a modular invariant M, a weight $\lambda \in P^k_+$ is said to be an **exponent** if $M_{\lambda\lambda} \neq 0$. From the arguments above, there is associated to every exponent λ, an Ishibashi state $||\lambda >$. Denote the set of all exponents λ of the modular invariant M (with multiplicity $M_{\lambda\lambda}$) $\mathcal{E}_M$.

WZW models have the full symmetry of the associated affine Lie algebra, which provide additional gluing conditions. In the simplest situation, the gluing conditions (10.4) for the affine symmetries read

$$\left(J^b_m + \overline{J}^b_{-m} \right) ||a >= 0 \,, \forall\, b \,\text{and}\, \forall\, m \in \mathbb{Z} \,. \tag{10.8}$$

Here, the J^a_m are the modes of the generators of the affine symmetry of the WZW models. In addition to these, the WZW model Ishibashi states also have to satisfy the gluing condition (10.5) arising from the conformal symmetry.

We shall partially fix the normalization of the Ishibashi states by requiring that

$$< \lambda || q^{\frac{1}{2}(L_0 + \overline{L}_0 - \frac{c}{12})} || \mu >= \delta_{\lambda\mu} \chi_\lambda(\tau) \,, \quad q = e^{2\pi i \tau} \,. \tag{10.9}$$

Every boundary state can then be written as a linear combination (10.7) of Ishibashi states

$$||a >= \sum_{\mu \in \mathcal{E}_m} \frac{\psi_{a\mu}}{\sqrt{S_{0\mu}}} ||\mu > \,. \tag{10.10}$$

We have adopted the standard convention of using lowercase Latin letters to denote boundary states. The factor $\sqrt{\mathcal{S}_{0\mu}}$ is just a convenient normalization of the coefficients. Given the above normalization of the Ishibashi states, finding the ψ matrix is equivalent to specifying a boundary state. In general, it turns out that ψ is square and unitary.

Not every linear combination (10.10) defines a boundary state. The allowed boundary states also need to satisfy a consistency condition known as the **Cardy condition**, which we now describe heuristically. The "overlap" between two boundary states is give by

$$< a||q^{\frac{1}{2}(L_0+L_0-\frac{c}{12})}||b>=\sum_{\mu\in\mathcal{E}_m}\frac{\psi^*_{a\mu}\psi_{b\mu}}{\mathcal{S}_{0\mu}}\chi_\mu(\tau)\,, \tag{10.11}$$

where we have used (10.9) and (10.10). Under the modular transformation $\tau\to -1/\tau$, this amplitude should be expressible in terms of a non-negative integer combination of characters i.e.

$$\sum_{\mu\in\mathcal{E}_m}\frac{\psi^*_{a\mu}\psi_{b\mu}}{\mathcal{S}_{0\mu}}\chi_\mu(-\frac{1}{\tau})=\sum_{\lambda\in P_+}\mathcal{N}^a_{\lambda b}\,\chi_\lambda(\tau)\,. \tag{10.12}$$

The $\mathcal{N}^a_{\lambda b}$ can easily shown to be

$$\mathcal{N}^a_{\lambda b}=\sum_{\mu\in\mathcal{E}_M}\frac{\psi^*_{a\mu}\mathcal{S}_{\lambda\mu}\psi_{b\mu}}{\mathcal{S}_{0\mu}}\,. \tag{10.13}$$

As this needs to be a non-negative set of integers, it forms a NIM-rep. If such a NIM-rep can be found, then the Cardy consistency condition is met.

As mentioned earlier, adding boundaries to a closed string theory corresponds to adding D-branes to the theory. Here we have associated a boundary state for every such boundary (provided consistency and gluing conditions can be met). Thus D-branes in the theory are labeled by the boundary states of the theory. Solutions to the gluing conditions (10.8) represent D-branes that preserve the full affine symmetry of the theory, and are known as **untwisted branes**.

In addition, there are other branes that can be constructed. One of these is the following set. Suppose ω is any outer automorphism of $\mathfrak{g}$ that is a symmetry of the unextended Dynkin diagram (i.e. of the finite algebra $\mathfrak{g}$, we are not discussing simple currents here). Associated to any such ω, a modular invariant can be constructed for the theory [6]. We want to find boundary states of a modular invariant that is built out of ω invariant states, i.e., we have an ω-twisted affine Lie algebra. The set of exponents $\mathcal{E}_\omega$ of the corresponding NIM-rep is the subset of P_+ consisting of representations invariant under ω. The gluing conditions satisfied here are

$$\left(J_m^b + \omega\left(\overline{J}_{-m}^b\right)\right) ||a>^{\omega} = 0\,, \forall\, b\,\text{and}\,\forall\, m \in \mathbb{Z}\,. \tag{10.14}$$

The construction above for the untwisted case can be repeated here in terms of twisted Ishibashi states $||\mu>^{\omega}$ which are defined for every $\mu \in \mathcal{E}_{\omega}$, i.e., for every $\mu \in P_+$ that satisfies $\omega(\mu) = \mu$. Similar normalization conditions, and the twisted version of the Cardy condition can be established leading to an identical expression for the NIM-reps $\mathcal{N}$. Such D-branes are known as **twisted D-branes**.

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Chapter 11

Charges of Exceptionally Twisted Branes

11.1 Introduction

Conserved charges of D-branes in string theory, to a very large part, determine their effective dynamics. As such, determining these charges and the associated charge groups provides significant information regarding the D-branes. For strings propagating on a group manifold, i.e. a g_k-WZW model, these charges can be determined using the underlying CFT [1]. WZW models possess an extremely rich variety of D-brane dynamics directly attributable to the additional affine Lie structure, which is preserved by the D-branes.

In addition to the standard untwisted branes, WZW models also possess D-branes which preserve the affine symmetry only up to a twist, the so-called "twisted" branes. For every automorphism ω of the finite dimensional Lie algebra $\overline{g}$ of the affine Lie algebra $\mathfrak{g}$, there exist ω-twisted D-branes. It is sufficient to consider outer automorphisms only, and as such, only automorphisms determined by symmetries of the Dynkin diagram of $\overline{\mathfrak{g}}$ [2,3]. Such twists exist for the A_n's, D_n's, and E_6, where ω in each case is an order two symmetry referred to as charge conjugation (or chirality flip in the case of D_n with n even), and for D_4, where ω is an order three symmetry referred to as triality. The microscopic analysis for twisted D-branes started with [4], and a study at large affine level was done in [5]. The charges and charge groups for the order-two twisted A_n and D_n D-branes have been calculated in [6] (up to some conjectures). This paper deals with the remaining cases of D_4 with triality and E_6 with charge conjugation.

The computations for D_4 and E_6 presented here are purely Lie theoretic, and are done from a "microscopic"/CFT point of view. These calculations provide confirmations for the results for the charge group obtained "macroscopically"/geometrically using K-theory [7]. However, the K-theoretic arguments only determine the charge group and not the charges themselves, so the calculations done here provide significantly more information about the D-branes.

We also prove some Lie theoretic identities which warrant further study. The most surprising, and likely important, of these are that G_2 and F_4 see the simple currents of A_2 and D_4, respectively. More precisely, for arbitrary choice of level k, the simple currents J^i of A_2 permute the integral weights a' of G_2 in such a way that

$$\dim_{G_2}(J^i a') = \dim_{G_2}(a') \mod M_{G_2}\,, \tag{11.1}$$

where M_{G_2} is an integer given next section. Similarly, the 4 simple currents J of D_4 permute the integral weights of F_4 in such a way that

$$\dim_{F_4}(Jb') = \dim_{F_4}(b') \mod M_{F_4}\,, \tag{11.2}$$

where likewise M_{F_4} is given next section.

We first provide a brief summary of the description of untwisted D-brane charges in CFT, as well as the order-two twisted D-branes of A_n and D_n. Subsequently, we deal with the exceptional cases of D_4 and E_6. The non-trivial Lie theoretic identities, which are needed along the way, are stated and proved in the appendices.

11.2 Overview of WZW D-Brane Charges in CFT

The WZW models of relevance here are the ones on simply connected compact group manifolds (partition function given by charge conjugation). D-branes that preserve the full affine symmetry are labelled by the level k integrable highest weight representations $P_+^k(\mathfrak{g})$ of the affine algebra $\mathfrak{g}$. They are solutions of the "gluing" condition

$$J(z) = \overline{J}(\overline{z})\,, z = \overline{z}\,, \tag{11.3}$$

where $J, \overline{J}$ are the chiral currents of the WZW model [8].

The charge q_μ of the D-brane labelled by μ satisfies

$$\dim(\lambda)\, \mathrm{q}_\mu = \sum_{\nu \in P_+^k(\mathfrak{g})} N_{\lambda\mu}^{\nu} \mathrm{q}_\nu \mod M\,, \tag{11.4}$$

where $\lambda \in P_+^k(\mathfrak{g})$, $N_{\lambda\mu}^{\nu}$ are the $\mathfrak{g}_k$-affine fusion rules, and $\dim(\lambda) = \dim(\overline{\lambda})$ denotes the dimension of the $\overline{\mathfrak{g}}$ representation whose highest weight is the finite part of the affine weight λ — in this paper we freely interchange the affine weight λ with its finite part $\overline{\lambda}$, which is unambiguous since the level will always be understood. For a finite level k, this relationship (11.4) is only true modulo some integer M, and the charge group of these D-branes is then $\mathbb{Z}/M\mathbb{Z}$, where M is the largest positive integer such that (11.4) holds. We are assuming here that the only common divisor of all the q_μ is 1 (if they do have a common divisor, then this factor can be divided out). Without loss of generality, we assume the normalization $\mathrm{q}_0 = 1$. If we take μ to be the trivial representation 0, then clearly

$$\mathrm{q}_\lambda = \dim(\lambda)\,. \tag{11.5}$$

The integer M is then the largest integer such that

$$\dim(\lambda)\dim(\mu) = \sum_{\nu \in P_+^k(g)} N_{\lambda\mu}^{\nu}\dim(\nu) \qquad \mathrm{mod}\, M \tag{11.6}$$

holds. It has been conjectured (and proved for the A_n and the C_n series) in [7,9,10] that the integer M is always of the form

$$M = \frac{k + h^\vee}{\gcd(k + h^\vee, L)}\,, \tag{11.7}$$

where $h^\vee$ is the dual Coxeter number of $\overline{\mathfrak{g}}$ and L is a k-independent integer given in Table 11.1[1].

WZW models also possess D-branes that only preserve the affine symmetry up to some twist. For every automorphism of the finite dimensional algebra $\overline{g}$, ω-twisted D-branes can be constructed. These are solutions of the "gluing" condition

$$J(z) = \omega \cdot \overline{J}(\overline{z})\,, z = \overline{z}\,, \tag{11.8}$$

where $J, \overline{J}$ are the chiral currents of the WZW model. These D-branes are labelled by the ω-twisted highest weight representations of $\mathfrak{g}_k$. The charge group is of the form $\mathbb{Z}_{M^\omega}$, where M^ω is the twisted analogue of the integer M from the untwisted case. The charge

[1] It was suggested initially in [10] that there were exceptional values for M at low levels k. However, this issue was subsequently resolved in [7], where it was proved that there are no exceptional cases using K-theory. We will show that there are no exceptional cases at low levels in both algebras, on the CFT and Lie theory side, when we prove uniqueness of the solutions.

Algebra	$h^\vee$	L
A_n	$n+1$	$\mathrm{lcm}(1,2,\dots,n)$
B_n	$2n-1$	$\mathrm{lcm}(1,2,\dots,2n-1)$
C_n	$n+1$	$2^{-1}\mathrm{lcm}(1,2,\dots,2n)$
D_n	$2n-2$	$\mathrm{lcm}(1,2,\dots,2n-3)$
E_6	12	$\mathrm{lcm}(1,2,\dots,11)$
E_7	18	$\mathrm{lcm}(1,2,\dots,17)$
E_8	30	$\mathrm{lcm}(1,2,\dots,29)$
F_4	9	$\mathrm{lcm}(1,2,\dots,11)$
G_2	4	$\mathrm{lcm}(1,2,\dots,5)$

Table 11.1: The dual Coxeter numbers and charge group integer L for the simple Lie algebras

carried by the D-brane labelled by the ω-twisted highest weight a has an integer charge q^ω_a, such that

$$\dim(\lambda)\,\mathrm{q}^\omega_a = \sum_b \mathcal{N}^b_{\lambda a}\,\mathrm{q}^\omega_b \mod M^\omega\,, \tag{11.9}$$

where $\mathcal{N}^b_{\lambda a}$ are the NIM-rep coefficients that appear in the Cardy analysis of these D-branes. M^ω is the largest integer such that (11.9) holds, again assuming that all the charges q^ω_a are relatively prime integers. However the difficulty in carrying over the analysis from the untwisted case is that there is no brane label a playing the role of the identity field, and thus we need to resort to a slightly different, and more complicated, analysis to determine the charges and M^ω.

It was suggested in [11,12] that the NIM-rep coefficients $\mathcal{N}^b_{\lambda a}$ are actually the twisted fusion rules that describe the WZW fusion of the twisted representation a with the untwisted representation λ to give the twisted representation b. Thus the conformal highest weight spaces of all three representations λ, a, and b form representations of the invariant horizontal subalgebra $\overline{\mathfrak{g}}^\omega$ that consists of the ω-invariant elements of $\overline{\mathfrak{g}}$ (For details on such matters, see [9]). The twisted fusion rules are a level k truncation of the tensor product coefficients of the horizontal subalgebra. This establishes a parallel with the untwisted case, where the untwisted fusion rules are the level k truncation of the tensor product coefficients of $\overline{\mathfrak{g}}$. Thus by analogy with (11.5), we can make the ansatz

$$\mathrm{q}^\omega_a = \dim_{\overline{\mathfrak{g}}^\omega}(a)\,, \tag{11.10}$$

i.e. the charge is simply the $\overline{\mathfrak{g}}^\omega$-Weyl dimension of the finite part of the twisted weight

a. Using this ansatz the integer M^{ω} was calculated in [6] for the chirality flip twisted A_n and D_n series, and it was also shown that, up to rescaling, (11.10) is the unique solution to (11.9).

The remaining non-trivial cases of triality twisted D_4 and charge conjugation twisted E_6 are dealt with in this paper, and require some nontrivial Lie theory, especially pertaining to twisted affine Lie algebras. The relevant background can be found in [9,11,13].

11.3 Triality Twisted D_4 Brane Charges

D_4 has five non-trivial conjugations, whose NIM-reps can all be determined from analyzing just the ones corresponding to chirality flip (which has already be done in [6,11]) and triality. The latter is an order three automorphism of the Dynkin diagram ω that sends the Dynkin labels $(\lambda_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to $(\lambda_0; \lambda_4, \lambda_2, \lambda_1, \lambda_3)$. Thus the relevant twisted algebra here is $D_4^{(3)}$ with a horizontal subalgebra G_2, labelling ω-invariant states. Thus boundary states are labelled by triples $(a_0; a_1, a_2)$ where the level $k = a_0 + 2a_1 + 3a_2$. In [11], it is shown how to express the twisted NIM-reps in terms of A_2 fusion rules at level $k+3$ via the branching $D_4 \supset G_2 \supset A_2$:

$$\mathcal{N}^b_{\lambda a} = \sum_{i=0}^{2} \sum_{\gamma''} b^{\lambda}_{\gamma''} \left(N^{b''}_{J^i\gamma'', a''} - N^{b''}_{J^i\gamma'', Ca''} \right) , \tag{11.11}$$

where C denotes charge conjugation in A_2, which takes a dominant A_2 weight to its dual by interchanging the finite Dynkin labels and J is the simple current of A_2 that acts by cyclic permutation of the Dynkin labels of the $A_2^{(1)}$ weights, and the $b^{\lambda}_{\gamma''}$ are the $D_4 \supset G_2 \supset A_2$ branching rules (see for example [14]). The relation between D_4 boundary states and the weights of $G_2^{(1)}$ and $A_2^{(1)}$ is given by the identifications of the appropriate Cartan subalgebras. Explicitly, we write [11]

$$a' = \iota(a_0; a_1, a_2) = (a_0 + a_1 + a_2 + 2; a_2, a_1) \in P_+^{k+2}(G_2) , \tag{11.12}$$

$$a'' = \iota' a' = \iota'\iota(a_0; a_1, a_2) = (a_0 + a_1 + a_2 + 2; a_2, a_1 + a_2 + 1) \in P_+^{k+3}(A_2) \tag{11.13}$$

In the following, level k-D_4 quantities (weights and boundary states) are unprimed, while the corresponding level $k+2$-G_2 weights and level $k+3$-A_2 weights are singly and doubly primed, respectively.

Following [6], we make the ansatz that the charge q^{ω}_a is the G_2 Weyl dimension of the horizontal projection (finite part) of the weight i.e.

$$\mathrm{q}_a^{\omega} = \dim_{G_2}(a')\,. \tag{11.14}$$

Then for an arbitrary dominant integral weight λ of D_4 the left hand side of (11.9) reads:

$$\begin{aligned}\dim_{D_4}(\lambda)\dim_{G_2}(a') &= \sum_{\gamma'} b^{\lambda}_{\gamma'}\dim_{G_2}(\gamma')\dim_{G_2}(a')\\ &= \sum_{\gamma'} b^{\lambda}_{\gamma'} \sum_{b'\in P_+^{k+2}(G_2)} N^{b'}_{\gamma' a'}\dim_{G_2}(b') \qquad \bmod M_{G_2}\,,\end{aligned} \tag{11.15}$$

where $b^{\lambda}_{\gamma'}$ are the $D_4 \supset G_2$ branching rules, and in the second line we have used (11.4) for the untwisted G_2 branes at level $k+2$. Now from Table 11.1 we know that at level $k+2$ M_{G_2} is the same as M_{D_4} at level k:

$$M_{G_2} = M_{D_4} = \frac{k+6}{\gcd(k+6, 2^2.3.5)}\,, \tag{11.16}$$

and so (11.15) holds $\bmod M_{D_4}$. Now G_2 fusion rules at level $k+2$ can be written in terms of the level $k+3$ fusion rules of A_2 following [15]

$$N^{b'}_{\gamma' a'} = \sum_{\gamma''} b^{\gamma'}_{\gamma''}\left[N^{b''}_{\gamma'' a''} - N^{b''}_{\gamma'' C a''}\right]\,, \tag{11.17}$$

where $b^{\gamma'}_{\gamma''}$ are the $G_2 \supset A_2$ branching rules. Using this and the fact that $\sum_{\gamma'} b^{\lambda}_{\gamma'} b^{\gamma'}_{\gamma''} = b^{\lambda}_{\gamma''}$, we rewrite the left hand side of (11.9) as

$$\text{L.H.S.} = \sum_{\gamma''} b^{\lambda}_{\gamma''} \sum_{b'\in P_+^{k+2}(G_2)} \left[N^{b''}_{\gamma'' a''} - N^{b''}_{\gamma'' C a''}\right]\dim_{G_2}(b') \qquad \bmod M_{D_4}\,. \tag{11.18}$$

In order to relate this to the right hand side of (11.9) where the summation is only over the boundary states of triality twisted D_4, we need to restrict the summation (11.18) somehow to the set $\mathcal{D} = Im(\iota'\iota)$ of images $b \mapsto b''$ under (11.12),(11.13). To do this, we first describe the relevant sets precisely.

An $A_2^{(1)}$ weight $(b_0''; b_1'', b_2'')$ belongs to $\mathcal{D}$, $J\mathcal{D}$, or $J^2\mathcal{D}$ respectively, if

$$\begin{aligned}\mathcal{D} &: \quad b_0'' > b_2'' > b_1'' \geq 0\,,\\ J\mathcal{D} &: \quad b_1'' > b_0'' > b_2'' \geq 0\,,\\ J^2\mathcal{D} &: \quad b_2'' > b_1'' > b_0'' \geq 0\,,\end{aligned} \tag{11.19}$$

where J is the A_2 simple current acting on $A_2^{(1)}$ weights by $J(a_0''; a_1'', a_2'') = (a_2''; a_0'', a_1'')$.

The set $\mathcal{G} = \iota'(P_+^{k+2}(G_2))$ of images of (11.13) (the set over which we are summing in (11.18)) only has the constraint $b_2'' > b_1'' \geq 0$. Thus a moment of thought will show that

$$\mathcal{G} = \mathcal{D} \cup J^2\mathcal{D} \cup CJ\mathcal{D} \cup \mathcal{B}\,, \tag{11.20}$$

where $\mathcal{B}$ consists of weights in $\mathcal{G}$ such that either $b_0'' = b_1''$ or $b_0'' = b_2''$. The following hidden symmetries are established in the appendices:

$$b' \in P^{k+2}(G_2) \Rightarrow \dim_{G_2}(Jb') = \dim_{G_2}(J^2b') = \dim_{G_2}(b') \mod M_{D_4}\,, \tag{11.21}$$

$$b'' \in \mathcal{B} \Rightarrow \dim_{G_2}(b') = 0 \mod M_{D_4}\,, \tag{11.22}$$

$$b' \in P^{k+2}(G_2) \Rightarrow \dim_{G_2}(Cb') = -\dim_{G_2}(b')\,, \tag{11.23}$$

where C and J act on $G_2^{(1)}$ weights through conjugation by ι':

$$C(b_0'; b_1', b_2') = (b_0'; b_1' + b_2' + 1, -b_2' - 2)\,, \tag{11.24}$$

$$J(b_0'; b_1', b_2') = (b_1' + b_2' + 1; b_0', b_1' - b_0' - 1)\,. \tag{11.25}$$

Here and elsewhere, we write '$\dim_{G_2}(a')$' even when a' is not dominant, by formally evaluating the Weyl dimension formula for G_2 at a'. The minus sign in (11.23) indicates that Cb' won't be a dominant G_2 weight when b' is — indeed, C belongs to the Weyl Group of G_2.

Using these, we can rewrite (11.18) as

$$\begin{aligned}\text{L.H.S.} &= \sum_{\gamma''} b_{\gamma''}^{\lambda} \Bigg[\sum_{b'' \in \mathcal{D}} [N_{\gamma'' a''}^{b''} - N_{\gamma'' C a''}^{b''}] \dim_{G_2}(b') \\ &+ \sum_{b'' \in J^2\mathcal{D}} [N_{\gamma'' a''}^{b''} - N_{\gamma'' C a''}^{b''}] \dim_{G_2}(b') \\ &+ \sum_{b'' \in J\mathcal{D}} [N_{\gamma'' a''}^{Cb''} - N_{\gamma'' C a''}^{Cb''}] \dim_{G_2}(Cb') \Bigg] \mod M_{D_4}\,,\end{aligned} \tag{11.26}$$

where we note that there is no contribution from $\mathcal{B}$ due to (11.22). Using (11.23), the symmetry $N_{\gamma'' a''}^{\nu''} = N_{C\gamma'' C a''}^{C\nu''}$, and the fact that the $D_4 \supset A_2$ branching has $b_{\gamma''}^{\lambda} = b_{C\gamma''}^{\lambda}$ for all γ'', to simplify the third sum in (11.26), we finally obtain

$$\text{L.H.S.} = \sum_{i=0}^{2} \sum_{\gamma''} b_{\gamma''}^{\lambda} \sum_{b'' \in J^i\mathcal{D}} [N_{\gamma'' a''}^{b''} - N_{\gamma'' C a''}^{b''}] \dim_{G_2}(b') \mod M_{D_4}\,. \tag{11.27}$$

But applying (11.11), (11.21), and the symmetries of fusion rules under the action of simple currents, we see that (11.27) also equals the right hand side of (11.9). Thus (11.9) is indeed satisfied by our ansatz

$$\mathrm{q}^{\omega}_{a} = \dim_{G_2}(a') \qquad \text{and} \qquad M^{\omega} = M_{D_4}\,, \tag{11.28}$$

that is, the charges are given by the Weyl dimension of the representation of the horizontal subalgebra, and the charge group is the same as in the untwisted case. As we show in Section 5, the charges are unique up to a rescaling by a constant factor.

11.4 Charge Conjugation Twisted E_6 Brane Charges

E_6 has a non-trivial order two symmetry of the Dynkin diagram that sends the Dynkin labels $(\lambda_0; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ to $(\lambda_0; \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_6)$. The relevant twisted algebra here is $E_6^{(2)}$, with a horizontal subalgebra F_4 labelling the ω invariant states. The boundary states are labelled by quintuples $(a_0; a_1, a_2, a_3, a_4)$ such that $k = a_0 + 2a_1 + 3a_2 + 4a_3 + 2a_4$. Again, in [11] it is shown how to express the twisted NIM-reps in terms of the untwisted fusion rules of D_4 at level $k+6$ via the branching $E_6 \supset F_4 \supset D_4$:

$$\mathcal{N}^{b}_{\lambda a} = \sum_{J}\sum_{\pi}\sum_{\gamma''} \epsilon(\pi)\, b^{\lambda}_{\gamma''}\, N^{b''}_{J\gamma'',\pi a''}\,, \tag{11.29}$$

where the sum over π is over all 6 conjugations of D_4 consisting of permutations of the 1st, 3rd, and 4th Dynkin labels, $b^{\lambda}_{\gamma''}$ are the $E_6 \supset F_4 \supset D_4$ branching rules and $\epsilon(\pi)$ is the parity of the permutation. The summation labelled by J is over the four simple currents of D_4: the identity, $J_v b'' = (b''_1; b''_0, b''_2, b''_4, b''_3)$, $J_s b'' = (b''_4; b''_3, b''_2, b''_1, b''_0)$, and $J_v J_s$. Note that each of these simple currents has order 2. The E_6 boundary states are related to $F_4^{(1)}$ and $D_4^{(1)}$ weights through the maps [11]

$$\begin{aligned} a' &= \iota(a_0; a_1, a_2, a_3, a_4) = (a_0 + a_1 + a_2 + a_3 + 3; a_4, a_3, a_2, a_1) \in P_+^{k+3}(F_4) && (11.30)\\ a'' &= \iota' a' = \iota'\iota(a_0; a_1, a_2, a_3, a_4) && (11.31)\\ &= (a_0 + a_1 + a_2 + a_3 + 3; a_1 + a_2 + a_3 + 2, a_4, a_3, a_2 + a_3 + 1) \in P_+^{k+6}(D_4)\,. \end{aligned}$$

These again correspond to the identification of the respective Cartan subalgebras. Unprimed quantities refer to level k-E_6 quantities, while their corresponding level $k+3$-F_4 and level $k+6$-D_4 weights are singly and doubly primed, respectively. Again, as explained in [11] and [15], these relations are established by examining the twisted version of the Verlinde formula.

Following [6] again, we take the ansatz that the charge q_a^ω is the F_4 Weyl dimension of the finite part of the weight, i.e.

$$\mathrm{q}_a^\omega = \dim_{F_4}(a')\,. \tag{11.32}$$

Then for an arbitrary dominant integral weight λ of E_6, the left hand side of (11.9) reads:

$$\begin{aligned}\dim_{E_6}(\lambda)\dim_{F_4}(a') &= \sum_{\gamma'} b^\lambda_{\gamma'}\dim_{F_4}(\gamma')\dim_{F_4}(a') \\ &= \sum_{\gamma'} b^\lambda_{\gamma'} \sum_{b'\in P_+^{k+3}(F_4)} N^{b'}_{\gamma' a'}\dim_{F_4}(b') \qquad \mathrm{mod}\, M_{F_4}\,,\end{aligned} \tag{11.33}$$

where $b^\lambda_{\gamma'}$ are the $E_6 \supset F_4$ branching rules, and in the second line we have used (11.4) for the untwisted F_4 branes at level $k+3$. From Table 11.1 we know that at level $k+3$ M_{F_4} is the same as M_{E_6} at level k:

$$M_{F_4} = M_{E_6} = \frac{k+12}{\gcd(k+12, 2^3.3^2.5.7.11)}\,, \tag{11.34}$$

and so (11.33) also holds $\mathrm{mod}\, M_{E_6}$.

Now F_4 fusion rules at level $k+3$ can be written in terms of the level $k+6$ fusion rules of D_4 following [15]

$$N^{b'}_{\gamma' a'} = \sum_{\gamma''}\sum_{\pi} \epsilon(\pi)\, b^{\gamma'}_{\gamma''}\, N^{b''}_{\gamma'' \pi a''}\,, \tag{11.35}$$

where $b^{\gamma'}_{\gamma''}$ are the $F_4 \supset D_4$ branching rules, and the $\pi \in S_3$ are as before. As explained in [11] and [15], this is obtained using the Verlinde formula by analyzing the subset of images of dominant integral weights under the branching. Now using this and the fact that $\sum_{\gamma'} b^\lambda_{\gamma'} b^{\gamma'}_{\gamma''} = b^\lambda_{\gamma''}$, we rewrite the left hand side of (11.9) as

$$\text{L.H.S.} = \sum_{\gamma''} \sum_{b'\in P_+^{k+3}(F_4)} \sum_{\pi} \epsilon(\pi)\, b^\lambda_{\gamma''}\, N^{b''}_{\gamma''\pi a''}\dim_{F_4}(b') \qquad \mathrm{mod}\, M_{E_6}\,. \tag{11.36}$$

In order to relate this to the right hand side of (11.9) where the summation is only over the boundary states of twisted E_6, we need to restrict the above summation somehow to the set $\mathcal{E} = Im(\iota'\iota)$ of images $b \mapsto b''$. To do this, we first describe the relevant sets precisely.

A $D_4^{(1)}$ weight $(b_0'';b_1'',b_2'',b_3'',b_4'')$ belongs to $\mathcal{E}, J_\nu\mathcal{E}, J_s\mathcal{E}$, or $J_\nu J_s\mathcal{E}$, where the J are the D_4 simple currents, if

$$\begin{aligned} \mathcal{E} \quad &: \quad b_0'' > b_1'' > b_4'' > b_3'' \geq 0\,, \\ J_\nu\mathcal{E} \quad &: \quad b_1'' > b_0'' > b_3'' > b_4'' \geq 0\,, \\ J_s\mathcal{E} \quad &: \quad b_4'' > b_3'' > b_0'' > b_1'' \geq 0\,, \\ J_\nu J_s\mathcal{E} \quad &: \quad b_3'' > b_4'' > b_1'' > b_0'' \geq 0\,. \end{aligned} \tag{11.37}$$

The set $\mathcal{F}$ of images of $P_+^{k+3}(F_4)$ under the $F_4 \supset D_4$ branching (the set over which we are summing in (11.36)) only has the constraints $b_1'' > b_4'' > b_3'' \geq 0$. Thus a moment of thought will show that

$$\mathcal{F} = \mathcal{E} \cup \pi_{143}J_\nu\mathcal{E} \cup \pi_{341}J_s\mathcal{E} \cup \pi_{413}J_\nu J_s\mathcal{E} \cup \mathcal{B}\,, \tag{11.38}$$

where π_{abc} is the D_4 conjugation taking the Dynkin labels 1,3 and 4 respectively to a, b and c, and $\mathcal{B}$ consists of weights in $\mathcal{F}$ such that either $b_0'' = b_1''$ or $b_0'' = b_2''$ or $b_0'' = b_4''$.

The following facts, where the π are the D_4 conjugations and the J are any of the D_4 simple currents, are proved in the appendices:

$$\begin{aligned} \dim_{F_4}(\pi b') &= \epsilon(\pi)\dim_{F_4}(b') \quad \forall b' \in P^{k+3}(F_4)\,, \qquad &(11.39) \\ \dim_{F_4}(Jb') &= \dim_{F_4}(b') \mod M_{F_4} \quad \forall b' \in P^{k+3}(F_4)\,, \qquad &(11.40) \\ \dim_{F_4}(b') &= 0 \mod M_{E_6} \quad \forall b'' \in \mathcal{B}\,. \qquad &(11.41) \end{aligned}$$

The action of the D_4 conjugations π and simple currents J on $F_4^{(1)}$ weights can be easily obtained by converting $F_4^{(1)}$ weights to $D_4^{(1)}$ weights using ι', applying π or J, and then converting back to $F_4^{(1)}$ using $\iota'^{-1}(d_0'';d_1'',d_2'',d_3'',d_4'') = (d_0'';d_2'',d_3'',d_4''-d_3''-1,d_1''-d_4''-1)$. As for G_2, we write '$\dim_{F_4}(a')$' even when a' is not dominant, by formally applying the Weyl dimension formula. The factor $\epsilon(\pi)$ in (11.39) is the parity ± 1, and as before, each π belongs to the Weyl Group of F_4.

Using these, we can rewrite (11.36) as

$$\begin{aligned} \text{L.H.S.} \;=\; & \sum_{\gamma''} b^\lambda_{\gamma''} \sum_\pi \epsilon(\pi) \Bigg[\sum_{b''\in\mathcal{E}} N_{\gamma''\pi a''}^{b''} \dim_{F_4}(b') \\ & + \sum_{b''\in J_\nu\mathcal{E}} N_{\gamma''\pi a''}^{\pi_{143}b''} \dim_{F_4}(\pi_{143}b') + \sum_{b''\in J_s\mathcal{E}} N_{\gamma''\pi a''}^{\pi_{341}b''} \dim_{F_4}(\pi_{341}b') \end{aligned}$$

$$+ \sum_{b'' \in J_\nu J_s \mathcal{E}} N_{\gamma''\pi a''}^{\pi_{413} b''} \dim_{F_4}(\pi_{413} b') \Bigg] \mod M_{E_6} , \tag{11.42}$$

where we note that there is no contribution from $\mathcal{B}$ due to (11.41). Exactly as for the argument of the previous section, the symmetries of the fusion rules under π and simple currents, the symmetry $b^\lambda_{\pi\gamma''} = b^\lambda_{\gamma''}$ of the branching rules, together with the hidden symmetries (11.39),(11.40) and the expression (11.29), show that

$$\text{L.H.S.} = \sum_{\gamma''} b^\lambda_{\gamma''} \sum_J \sum_\pi \epsilon(\pi) \sum_{b'' \in \mathcal{E}} N^{b''}_{J\gamma''\pi a''} \dim_{F_4}(b') = \text{R.H.S.} \mod M_{E_6} \tag{11.43}$$

Thus, again, (11.9) is indeed satisfied by our ansatz

$$\mathrm{q}^\omega_a = \dim_{F_4}(a') \qquad \text{and} \qquad M^\omega = M_{E_6} , \tag{11.44}$$

that is, the charges are once again given by the Weyl dimension of the representation of the horizontal subalgebra, and the charge group is the same as in the untwisted case. As we show next, the charges are unique up to a rescaling by a constant factor.

11.5 Uniqueness

We need to show that the solutions found to the charge equation (11.9) in both the D_4 and E_6 cases are unique up to rescaling. To this end it is sufficient to prove that if the charge equation is satisfied by a set of integers $\tilde{q}_a$ modulo some integer $\tilde{M}$, then

$$\tilde{q}_a = \dim(a')\, \tilde{q}_0 \mod \tilde{M} . \tag{11.45}$$

In this case, we can divide all charges by $\tilde{q}_0$, and the charge equation will still be satisfied if we also divide $\tilde{M}$ by $\gcd(\tilde{q}_0, \tilde{M})$. Finally, by an argument due to Fredenhagen [6] we get that $M' := \tilde{M}/\gcd(\tilde{q}_0, \tilde{M})$ must divide our M. Explicitly, by construction, M is the g.c.d. of the dimensions of the elements of the fusion ideal that quotients the representation ring in order to obtain the fusion ring. Since NIM-reps provide representations of the fusion ring, any element of the fusion ideal acts trivially i.e. $\dim(\alpha)\dim(a) = 0 \mod M'$ for any α in the fusion ideal. Thus, using the fact that the $\dim(a)$ are relatively prime integers, we see that M' must divide M. Thus any alternate solution $\tilde{q}_a, \tilde{M}$ to (11.9) which obeys (11.45), is just a rescaled version of our "standard" one q_a, M.

We will work with D_4, the proof for E_6 is similar and will be sketched at the end.

The $G_2 \subset D_4$ branching rules can be inverted: we can formally write

$$a' = \sum_{\lambda} \tilde{b}^a_{\lambda} \lambda \,, \tag{11.46}$$

where $\tilde{b}^a_\lambda$ are integers (possibly negative), $a' \in P_+(G_2)$, and the sum is over D_4 weights λ [15]. More precisely, (11.46) holds at the level of characters, where the domain of the D_4 ones is restricted to the ω-invariant vectors in the D_4 Cartan subalgebra, and the G_2 characters are evaluated at the image of those vectors by ι. To prove (11.46), it suffices to verify it for the G_2 fundamental weights, where we find $(1,0) = (0,1,0,0) - (1,0,0,0) + (0,0,0,0)$ and $(0,1) = (1,0,0,0) - (0,0,0,0)$. Since all other G_2 weights can be constructed from the fundamental ones by tensor products, every dominant G_2 weight can formally be inverted under the branching and written in terms of linear combinations of dominant integral D_4 weights. Then [2]

$$\begin{aligned} \dim_{G_2}(a')\, \tilde{q}_0 &= \sum_{\lambda} \tilde{b}^a_\lambda \dim_{D_4}(\lambda)\, \tilde{q}_0 \\ &= \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b'} \mathcal{N}^{b}_{\lambda,0} \tilde{q}_b \mod \tilde{M}\,, \end{aligned} \tag{11.47}$$

where we have used the charge equation, which the $\tilde{q}_a$ satisfy modulo $\tilde{M}$ by assumption. Now we use the expression (11.11) to write (11.47) in terms of A_2 fusions: we get

$$\text{R.H.S.} = \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b} \sum_{\gamma''} \sum_{j=0}^{2} b^{\lambda}_{\gamma''} \left[N^{b''}_{J^j\gamma'',0''} - N^{b''}_{J^j\gamma'',C0''} \right] \tilde{q}_b \,. \tag{11.48}$$

Now however, from (11.17) and [15], we can express this in terms of G_2 untwisted fusion rules, and using properties of the A_2 fusion rules under simple currents along the way, we obtain

$$\text{R.H.S.} = \sum_{\lambda} \tilde{b}^a_\lambda \sum_{b} \sum_{\gamma'} \sum_{j=0}^{2} b^{\lambda}_{\gamma'} N^{J^j b'}_{\gamma' 0'}\, \tilde{q}_b \mod \tilde{M} \,. \tag{11.49}$$

Note that $0'$ here is the G_2 vacuum (whereas $0''$, the image of $0'$ under ι', is not the A_2

[2] In the following, we use the fact that both (11.9) and (11.11) (and (11.29) in the case of E_6) remain true for all dominant weights λ and not just affine ones. These expressions were obtained using ratios of S-matrices (see [11]), which can be interpreted as Lie algebra characters in the case of finite weights, and thus the NIM-reps $\mathcal{N}$ can be continued to include all dominant weights. This continuation also removes any subtleties in the comparison to K-theory by evaluating the charge constraint equation for all dominant weights λ. We thus can also see there are no exceptional situations at low levels.

vacuum), and thus

$$\text{R.H.S.} = \sum_b \sum_{j=0}^{2} \delta_{a'}^{J^j b'} \tilde{q}_b = \tilde{q}_a \mod \tilde{M} , \tag{11.50}$$

where we have used the facts that b' is never fixed by J and that the sets $\mathcal{D}$ and $J\mathcal{D}$ are disjoint(see (11.25)). Thus (11.45), and with it uniqueness, is established.

The proof for E_6 is virtually identical, except now we use the expression for untwisted F_4 fusion rules in terms of D_4 fusion rules found in [15].

11.6 Conclusion

In this paper, we have shown that the charge groups of the triality twisted D_4 and the charge conjugation twisted E_6 branes are identical to those of the untwisted D-branes. This is in nice agreement with the K-theoretic calculation [7] and completes the exceptional cases not dealt with in [6]. Our calculations show that the charges of these twisted D-branes corresponding to the twisted representation a is the dimension of the highest weight space of the representation a. Thus from the string theoretic point of view, analogous to the situation with untwisted D-branes, the charge associated to the D-brane is the multiplicity of the ground state of the open string stretched between the fundamental $D0$-brane and the brane labelled by a in question. So, in the supersymmetric version of WZW models, the charge may be interpreted as an intersection index, motivating possible geometric interpretation of these results. The explicit computation of the charges is missing from the K-theoretic calculations, and has been supplied here.[3] There are no additional unproven conjectures made in this paper. All the arguments have been proved up to some conjectures needed from the untwisted cases (i.e. the content of Table 11.1 for D_4, G_2, E_6, and F_4).

A number of non-trivial, and somewhat surprising, Lie theoretic identities have been proved along the way. Some of the dimension formulae regarding the action of simple currents of a subalgebra on weights of the larger algebra indicate that there might exist interesting constraints on the larger algebra due to the underlying symmetry in the branchings. In some sense, the enlarged algebra "breaks" symmetries of the smaller algebra, but still "sees" the underlying symmetry (analogous to ideas of renormalization of quantum field theories with spontaneously broken symmetries.)

[3]During the preparation of this manuscript the work of [16] has come to our attention, in which similar results are derived using different methodology. In that work however, many of the numerical and Lie theoretic identities, which are explicitly proven here, are left as conjectures.

The Lie theoretic meaning of (11.23) and (11.39) is clear: the A_2 and D_4 conjugations C and $\pi \in S_3$ are elements of the Weyl groups of G_2 and F_4 respectively. The meaning of (11.21) and (11.40) is far less clear (though it has to do with the theory of equal rank subalgebras [17]), but it does suggest a far-reaching generalization whenever the Lie algebras share the same Cartan subalgebras — for example, $A_1 \oplus \cdots \oplus A_1$ (n copies) and C_n, or A_8 and E_8. Given any simple current J of any affine Lie algebra $\mathfrak{g}$ at level k, it is already surprising that Weyl dimensions for the horizontal subalgebra $\overline{\mathfrak{g}}$ see the action of J via $\dim_{\overline{\mathfrak{g}}}(J\lambda) = \pm\dim_{\overline{\mathfrak{g}}}(\lambda) \mod M_{g_k}$. Far more surprising is that, at least sometimes, if two Lie algebras $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}'$ share the same Cartan subalgebras, then the Weyl dimensions of the first sees the simple currents J' of the second: $\dim_{\overline{\mathfrak{g}}}(J'\lambda) = \pm\dim_{\overline{\mathfrak{g}}}(\lambda) \mod M_{\mathfrak{g}_k}$.

Acknowledgements

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11.7 Appendices

We will use the following fact for proofs in both the D_4 and E_6 cases.

Fact 1 *Suppose K, L, N are arbitrary integers. Write $M = \frac{K}{\gcd(K,L)}$, and let $L = \prod_p p^{\lambda_p}$ and $N = \prod_p p^{\nu_p}$ be prime decompositions. Suppose we have integers f_i, d_i such that both $\prod_i f_i$ and $\prod_i (f_i - d_i K)$ are divisible by N. Then*

$$\frac{\prod_i f_i}{N} = \frac{\prod_i (f_i - d_i K)}{N} \mod M \ , \tag{11.51}$$

provided that for each prime p dividing M such that $\lambda_p < \nu_p$, it is possible to find $0 \leq \alpha_{i,p} \leq \lambda_p$, such that $p^{\alpha_{i,p}}$ divides f_i for each i, and $\sum_i \alpha_{i,p} \geq \nu_p$.

The reason we can restrict to primes p dividing M is that p coprime to M are invertible modulo M, and so can be freely cancelled on both sides and ignored. For primes p dividing M, $a_i/p^{\alpha_{i,p}} = (a_i - \delta_i K)/p^{\alpha_{i,p}}$ holds $\mod M$, $\forall\, i$. If $\lambda_p \geq \nu_p$, choose $\alpha_{i,p}$ to be the exact power of p dividing a_i. The divisibility by N hypothesis will be automatically satisfied, because the products we will be interested in come from the Weyl dimension formula.

11.7.1 Appendix A: D_4 Dimension Formulae

For any integral weight $a'' = (a_0; a_1, a_2) \in P^{k+3}(A_2)$, we can substitute $\iota'^{-1}(a_0; a_1, a_2) = (a_0; a_1, a_2 - a_1 - 1) = a'$ into the Weyl dimension formula [18] of G_2, in order to express G_2-dimensions using A_2 Dynkin labels:

$$\begin{aligned} \dim_{G_2}(a') &= \frac{1}{120}(a_2 - a_1)(a_2+1)(2a_2+a_1+3)(a_1+1) \\ &\quad \times (a_2+a_1+2)(a_2+2a_1+3)\,. \end{aligned} \tag{11.52}$$

Theorem 1 $\forall\, a' \in P^{k+2}(G_2)$, $\dim_{G_2}(Ca') = -\dim_{G_2}(a')$.

This is an automatic consequence of the $a_1 \leftrightarrow a_2$ anti-symmetry of (7.2). In fact, C is in the Weyl group of G_2 and so more generally Theorem 1 follows from the anti-symmetry of the Weyl dimension formula under Weyl group elements.

Theorem 2 $\dim_{G_2}(a') = \dim_{G_2}(Ja') = \dim_{G_2}(J^2a') \mod M_{G_2} \quad \forall a' \in P^{k+2}(G_2)$.

Proof: Using (11.52) and ι', we get

$$\begin{aligned} \dim_{G_2}(J^2a') &= \frac{1}{120}(a_1+a_2+2)(a_1+2a_2+3-K)(a_2+1-K) \\ &\quad *(2a_1+3a_2+5-K)(a_1+3a_2+4-2K)(a_1+1+K)\,, \end{aligned}$$

where we put $K = k+6$ and used the fact that $k = a_0 + 2a_1 + 3a_2$. In the notation of Fact 1, here $N = 120 = 2^3.3.5$, $L = 60 = 2^2.3.5$. From Fact 1, it suffices to consider the primes p with $\nu_p > \lambda_p$, i.e. $p = 2$. To show that $p = 2$ always satisfies the condition of Fact 1, i.e. that the $\alpha_{i,2}$ can be found for any choice of a_i, it suffices to verify it separately for the 16 possible values of $a_1, a_2 \mod 2^2$. Though perhaps too tedious to check by hand, a computer does it in no time. The proof for $\dim_{G_2}(Ja')$ is now automatic from Theorem 1.

Theorem 3 *Given any $b' \in P^{k+2}(G_2)$, if $Cb' = J^ib'$ for some i, then* $\dim_{G_2}(b') = 0 \mod M_{G_2}$.

Proof: Write $b'' = (b_0; b_1, b_2) \in P^{k+3}(A_2)$. By Theorem 1, it suffices to consider the case where $b_0 = b_2$. In this case we can write $k+3 = b_1 + 2b_2$. Thus, again using (11.52) we have

$$\dim_{G_2}(b') = \frac{1}{120}(b_1+1)(b_2+1)(b_1+b_2+2)(3b_2+3-K)K(3b_2+3-2K)\,.$$

The proof now proceeds as in Theorem 2.

Of course given any weight $b'' \in \mathcal{B}$, $b' = \iota'^{-1}(b'')$ will obey the hypothesis of Theorem 3, and so (11.22) follows. Note that combining Theorems 1 and 2, we get that any weight b' as in Theorem 3 will obey $\dim_{G_2}(b') = -\dim_{G_2}(b') \mod M_{G_2}$. Thus Theorems 1 and 2 are almost enough to directly get Theorem 3 (and in fact imply it for all primes $p \neq 2$).

11.7.2 Appendix B: E_6 Dimension Formulae

As before, use ι' and the Weyl dimension formula for F_4 to write the (formal) Weyl dimension of an arbitrary F_4 integral weight $b' \in P(F_4)$ in terms of the Dynkin labels of the D_4 weight $b'' = \iota'(b') = (b''_0; b''_1, b''_2, b''_3, b''_4)$. For convenience write $a_i = b''_i + 1$. Then we obtain

$$\begin{aligned}
&\frac{1}{2^{15}3^7 5^4 7^2 11} a_1 a_2 a_3 a_4 (a_1+a_2)(a_1+a_3)(a_1+a_4)(a_2+a_3)(a_2+a_4)(a_3+a_4) \\
&\times(a_1-a_3)(a_1-a_4)(a_4-a_3)(a_1+a_2+a_3)(a_1+a_2+a_4)(a_2+a_3+a_4) \\
&\times(a_1+a_2+a_3+a_4)(a_1+2a_2+a_3)(a_1+2a_2+a_4)(2a_2+a_3+a_4) \\
&\times(a_1+2a_2+a_3+a_4)(2a_1+2a_2+a_3+a_4)(a_1+2a_2+2a_3+a_4) \\
&\times(a_1+2a_2+a_3+2a_4)\,.
\end{aligned} \tag{11.53}$$

Theorem 4 *For any F_4 weight $b' \in P(F_4)$ and any outer automorphism $\pi \in S_3$ of D_4,*

$$\dim_{F_4}(\pi b') = \epsilon(\pi) \dim_{F_4}(b') \quad . \tag{11.54}$$

This follows easily from the Weyl dimension formula (7.3) by explicitly using the action of the π on the weights. As with Theorem 1, it expresses the anti-symmetry of Weyl dimensions under the Weyl group.

Theorem 5 *For all $F_4^{(1)}$ weights $b' \in P^{k+3}(F_4)$,*

$$\dim_{F_4}(b') = \dim_{F_4}(J_v b') = \dim_{F_4}(J_s b') = \dim_{F_4}(J_v J_s b') \mod M_{F_4}\,. \tag{11.55}$$

From Theorem 4, it suffices to consider only J_v. The proof uses Fact 1 and an easy computer check for primes $p = 2, 3, 5, 7$, as in the proof of Theorem 2.

Theorem 6 *Let $b' \in P^{k+3}(F_4)$ be any $F_4^{(1)}$ weight satisfying $\pi b' = Jb'$, for any D_4 simple current J and any order-2 outer automorphism π of D_4. Then* $\dim_{F_4}(b') = 0 \mod M_{F_4}$.

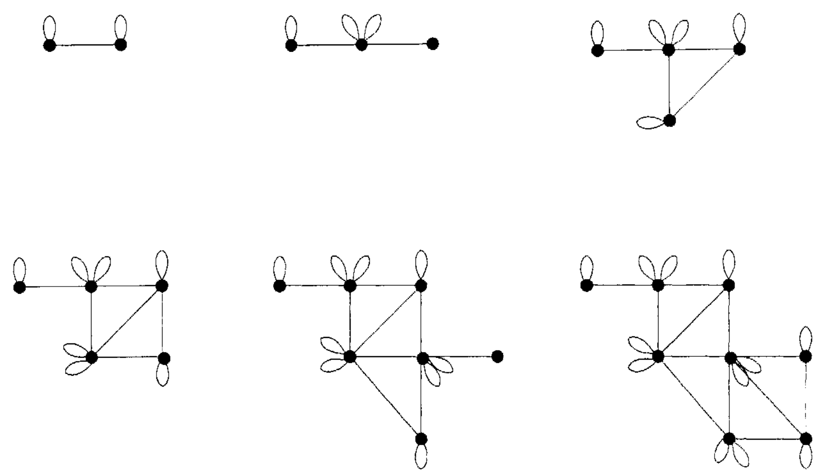

Figure 11.1: NIM-reps for D_4 with triality $k = 2, \ldots, 7$

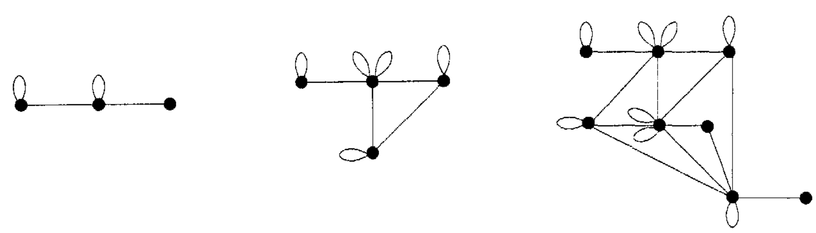

Figure 11.2: NIM-reps for E_6 with charge conjugation $k = 2, 3, 4$

Proof: The proof of this follows automatically from Theorem 4. Analogous to the situation for G_2, one of the factors in the dimension formula turns out to be K. So, accommodating the denominators as in Fact 1 will yield the term $\frac{K}{p^\alpha}$ for some $0 \leq \alpha \leq \lambda_p$. But this is 0 modulo M_{F_4}, for every prime p dividing M_{F_4}. Q.E.D.

11.7.3 Appendix C: NIM-Reps and Graphs

In this appendix we give explicit descriptions of some of the NIM-reps at low level for both D_4 and E_6 [11]. The NIM-rep graphs characterizing the matrix associated to the field $\lambda = \Lambda_1$ are given. The corresponding graph has vertices labeled by the rows (or columns) of $\mathcal{N}_\lambda$, and the vertex associated to i and j are linked by $(\mathcal{N}_\lambda)_{ij}$ lines.

11.7.4 Appendix D: A Sample Calculation

A suitable example to illustrate the situations considered is at level $k = 5$ for triality twisted D_4. In this case from Table 11.1, we get $M_{D_4} = 11$. The boundary states are labelled by triples $(a_0; a_1, a_2)$ such that $k = a_0 + 2a_1 + 3a_2$. The boundary weights then are

$$[5,0,0],[3,1,0],[2,0,1],[1,2,0],[0,1,1]\,, \tag{11.56}$$

whose G_2 Weyl dimensions are respectively, 1, 7, 14, 27, 64.

The relevant NIM-rep graph is illustrated in Figure 11.1. The charge equations (11.9) with $\lambda = \Lambda_1$ (fundamental representation of D_4 with dimension 8) thus are

$$\begin{aligned} 8\mathrm{q}_0 &= \mathrm{q}_0 + \mathrm{q}_1 \\ 8\mathrm{q}_1 &= \mathrm{q}_0 + 2\mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3 \\ 8\mathrm{q}_2 &= \mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3 + \mathrm{q}_4 \\ 8\mathrm{q}_3 &= \mathrm{q}_1 + \mathrm{q}_2 + 2\mathrm{q}_3 + \mathrm{q}_4 \\ 8\mathrm{q}_4 &= \mathrm{q}_2 + \mathrm{q}_3 + \mathrm{q}_4 \end{aligned} \tag{11.57}$$

The first three equations are identically true with $\mathrm{q}_0 = 1, \mathrm{q}_1 = 7, \mathrm{q}_2 = 14, \mathrm{q}_3 = 27$, and $\mathrm{q}_4 = 64$, and the last two equations are satisfied modulo $M_{D_4} = 11$.

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Chapter 12

A Twisted Non-Simply Connected Group

12.1 Introduction and Modular Invariants

The Hilbert space of CFT's is constructed from the holomorphic and antiholomorphic sectors by

$$\mathcal{H} = \bigoplus_{\lambda,\,\mu \in P_+^k} M_{\lambda\mu} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\mu^*, \tag{12.1}$$

and the resulting partition function for the full theory takes the form

$$\mathcal{Z}(\tau) = \sum_{\lambda,\,\mu \in P_+^k} M_{\lambda\mu} \chi_\lambda(\tau) \overline{\chi}_\mu(\tau)\,. \tag{12.2}$$

The matrix $M_{\lambda\mu}$ of entries is known as a modular invariant for the WZW theory [1, 2]. The theory described by taking $M = I$, the identity matrix, is known as the diagonal theory.

CFT's whose Hilbert space of states is described by the diagonal modular invariant define a consistent CFT that is defined on arbitrary Riemann surfaces. Other consistent modular invariants can be constructed from this theory through the use of global symmetries to either sector of the theory. For the WZW models based on the affine Lie algebra $\mathfrak{g}$, there are essentially two choices. Any symmetry of the unextended Dynkin diagram can be used to "twist" the modular invariant. These exist for A_n and E_6 corresponding to charge conjugation, for D_n corresponding to chirality flips, and for D_4 where there is the additional symmetry of triality [1,3]. For example, in the case of charge conjugation for the A_n's or E_6, the twisted modular invariant is simply $M = C$, the charge conjugation matrix. It can be shown that twisting by such discrete symmetries of the unextended Dynkin diagrams (twisting) always leads to a consistent modular invariant [4].

In addition, there are symmetries of the extended Dynkin diagrams which arise from outer automorphisms of the Lie algebra. Such symmetries of affine Lie algebras are known as simple currents. These exist for the A_n's, B_n's, C_n's, D_{2n+1}'s, E_6, E_7, and D_{2n}'s which possess two simple currents [1, 2]. In addition, there is a simple current at level 2 for E_8 which does not correspond to an outer automorphism and hence does not arise from a symmetry of the Dynkin diagram, but this will not concern us here. However, not every simple current defines a consistent modular invariant. There is an additional consistency condition imposed on the level. This can be described as follows

Suppose an affine Lie algebra $\mathfrak{g}$, has a simple current J arising from the outer automorphism of the extended Dynkin diagram. It can be shown that

$$J\mathcal{S}_{\lambda\nu} = \mathcal{S}_{J\lambda\mu} = \mathcal{S}_{\lambda\mu}e^{-2\pi i Q_J(\lambda)}\,, \tag{12.3}$$

where $Q(\lambda)$, known as the charge of the simple current J, has the explicit form

$$Q_J(\lambda) = (J\omega_0, \lambda)\,, \tag{12.4}$$

where ω_0 is the the zeroth fundamental weight of $\mathfrak{g}$. Suppose the order N of the simple current J is N i.e. N is the smallest positive integer such $J^N = A^N = 1$. Then the consistency condition requires

$$\frac{Nk}{2}|A\omega_0|^2 \in \mathbb{Z}\,. \tag{12.5}$$

Provided this condition is satisfied, the simple current modular invariant corresponding to J is given by

$$M_{\lambda\mu} = \sum_{p=1}^{N-1} \delta_{\lambda, J^p\mu}\delta_1\left(\left(J\omega_0, \mu + \frac{pk}{2}J\omega_0\right)\right)\,, \tag{12.6}$$

where $\delta_x(y)$ takes the value 1 if $\frac{y}{x}$ is an integer, and the value 0 otherwise [1, 2].

Such modular invariants arise in the study of non-simply connected groups. It is well known that the group of outer automorphisms $\mathcal{O}(\mathfrak{g})$ is isomorphic to the center $B(G)$ of the simply connected group G obtained by exponentiating the horizontal subalgebra $\overline{\mathfrak{g}}$.

A simple current of order N gives $\mathbb{Z}_N$ which is a subgroup of $\mathcal{O}(\mathfrak{g}) = B)G)$. Using the modular invariant arising from the simple current J, corresponds to defining the theory on the non-simply connected group manifold $G/\mathbb{Z}_N$. Strictly speaking, this is the case when N is prime. If the order N is not prime, then there is the additional option of using a simple current J^l where l is a divisor of N, in which case the theory is defined on the group manifold $G/\mathbb{Z}_{N/l}$. This occurs, for instance, for the A_n's [5].

If an affine Lie algebra $\mathfrak{g}$ possess both a twist and a simple current, clearly the two can be combined provided that the level consistency condition (12.5) can be met. The modular invariant then takes the form

$$M_{\lambda\mu} = \sum_{p=1}^{N-1} \delta_{\lambda, DJ^p\mu} \delta_1 \left(\left(J\omega_0, \mu + \frac{pk}{2} J\omega_0 \right) \right) , \tag{12.7}$$

where D denotes the twist (for example, C when we have charge conjugation)

In this chapter, we will be concerned with such modular invariants for E_6. E_6 has a charge conjugation symmetry C of the unextended Dynkin diagram which acts on the Dynkin labels of an affine weight μ via

$$C(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\mu_0, \mu_5, \mu_4, \mu_3, \mu_2, \mu_1, \mu_6) , \tag{12.8}$$

and an order 3 simple current J which acts on the Dynkin labels of the affine weights via

$$J(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\mu_1, \mu_5, \mu_4, \mu_3, \mu_6, \mu_0, \mu_2) , \tag{12.9}$$

with charge

$$Q_J(\mu) = (J0, \mu) = \frac{1}{3}[2\mu_1 + 4\mu_2 + 6\mu_3 + 5\mu_4 + 4\mu_5 + 3\mu_6] . \tag{12.10}$$

The vacuum $0 = k\omega_0$ clearly has $Q(0) = 0$. In addition, $J0 = k\omega_5$ has $Q(J0) = \frac{4k}{3}$ and $J^2 0 = k\omega_1$ has $Q(J^2 0) = \frac{2k}{3}$. These values will be important later.

For completeness, we note that the level k is given in terms of the Dynkin labels of an E_6 weight μ by

$$k = \mu_0 + \mu_1 + 2\mu_2 + 3\mu_3 + 2\mu_4 + \mu_5 + 2\mu_6 . \tag{12.11}$$

First of all, the consistency condition (12.5) can be checked with $N = 3$ and $J\omega_0 = \omega_5$

to give

$$\frac{3k}{2}|\omega_5|^2 = \frac{3k}{2}\frac{4}{3} \in \mathbb{Z}\,, \tag{12.12}$$

i.e. there is no restriction on the level k. Thus at any level k we can consistently have four possibilities: the untwisted simply-connected group E_6, which we will denote by $\mathcal{A}$; the twisted simply-connected group E_6^*, which we will denote by $\mathcal{A}^*$; the untwisted non-simply-connected group $E_6/\mathbb{Z}_3$, which we will denote $\mathcal{D}$; and finally, the twisted non-simply-connected group $(E_6/\mathbb{Z}_3)^*$, which we will denote $\mathcal{D}^*$. We will work with this last case. Remarkably it turns out that for any Lie algebra $\mathfrak{g}$, $\mathcal{D}^*$ is more tractable than $\mathcal{D}$ (when they can be defined), unlike the simply connected case where $\mathcal{A}$ is much simpler than $\mathcal{A}^*$ [6,7].

For $\mathcal{D}^*$, evaluating the inner product of the roots explicitly we find that

$$M_{\lambda\mu} = \delta_{\lambda,C\mu}\delta_1\left(Q(\mu)\right) + \delta_{\lambda,CJ\mu}\delta_1\left(Q(\mu)+\frac{2k}{3}\right) + \delta_{\lambda,CJ^2\mu}\delta_1\left(Q(\mu)+\frac{4k}{3}\right)\,. \tag{12.13}$$

12.2 States, Exponents, and NIM-reps

The first task is to find the corresponding NIM-rep, and the description of the exponents and the boundary states of the theory. We will first deal with the exponents. Recall that exponents are primaries λ of the CFT such that $M_{\lambda\lambda} \neq 0$, and they appear with a multiplicity $M_{\lambda\lambda}$ (since the Hilbert space will get that many copies of the corresponding tensor product between the holomorphic and antiholomorphic modules).

Consider any exponent of the $\mathcal{A}^*$ theory, i.e. C-invariant E_6 weights of the form

$$\mu = (\mu_0, \mu_1, \mu_2\mu_3, \mu_2, \mu_1, \mu_6)\,. \tag{12.14}$$

First consider the case $k =\neq 0$ mod 3. Using (12.9), (12.10), and (12.9) it is easy to see that

$$\begin{aligned} Q(J\mu) &= \frac{4k}{3} - 2\mu_1 - 3\mu_2 - 3\mu_3 - \mu_6\,, \\ Q(J^2\mu) &= \frac{2k}{3}\,. \end{aligned} \tag{12.15}$$

Then, by examining (12.13), with a bit of work (where we need to use the explicit form (12.14) of the weights), it is easy to see that μ, $J\mu$, and $J^2\mu$ are exponents, each with multiplicity 1. For μ, only the first δ term survives; for $J\mu$, only the second δ term survives, and for $J^2\mu$, only the third δ term survives, and in each case the argument of the other delta turns out to be an integer due to the explicit form of the charges calculated above, and thereby leading to a multiplicity of exactly one for each. In the case that k is a multiple of 3, in addition to the weights of this form, there are the additional weights that are both C and J invariant (known as J-fixed points) which have the Dynkin labels

$$\mu = (\mu_0, \mu_0, \mu_2, \mu_3, \mu_2, \mu_0, \mu_2) . \tag{12.16}$$

In this case, $Q(\mu) = Q(J\mu) = Q(J^2\mu) = 2\mu_0 + 4\mu_2 + 2\mu_3 \in \mathbb{Z}$, by the same procedure, it is easy to see that such weights are also exponents but with multiplicity 3, since all the terms will contribute. Thus, in all cases, the exponents can be neatly summarized to be of the form (μ, i) where μ is a C-invariant E_6 weight, and $0 \leq i \leq 2$ denotes the J orbit of such a weight.

There is a general construction by T. Gannon which suggests how to construct NIM-reps [8]. If $\mathcal{N}$ is a NIM-rep, and J a simple current of order N of some fusion ring, then the corresponding charge Q_J is a mapping $Q_J : P_+^k \to \frac{1}{N}\mathbb{Z}$. Then a new NIM-rep will be the N-fold covering with boundary states (x, j), where x is a boundary state of the old NIM-rep, and $j \in \mathbb{Z}_N$, and the NIM-rep itself is given by $\lambda \cdot (\hat{a}, i) = (\lambda \cdot \hat{a}, i + NQ_J(\lambda))$. Here, the notation on the right hand side is defined by $\lambda \cdot \hat{a} = \sum_{\hat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda,\hat{a}}^{\hat{b}} \hat{b}$, and the similar convention for the NIM-reps for the new theory on the left hand side is obvious. Essentially, the new NIM-rep will be of the form $\mathcal{N} \otimes \mathbb{Z}_N$. In this case, take the old theory to be $\mathcal{A}^*$, and the new theory to be $\mathcal{D}^*$. One additional requirement is that we would like to know whether our new NIM-rep is indecomposable/irreducible i.e. whether it can be expressed as a direct sum of smaller NIM-reps. The condition for an arbitrary NIM-rep to indecomposable is that the vacuum of the theory should have multiplicity 1. In this construction, this is equivalent to the requirement that the old NIM-rep is indecomposable and, in addition, no non-trivial power of the simple current J is itself an exponent of the old theory.

The NIM-rep constructed for the $\mathcal{D}^*$ theory is then

$$\lambda \cdot (\hat{a}, j) = (\lambda \cdot \hat{a}, j + 3Q(\lambda)) , \tag{12.17}$$

and the ψ matrix can easily be seen to be

$$\psi(\mathcal{D}^*)_{(\hat{a},p),(\mu,j)} = \frac{1}{\sqrt{3}} e^{2\pi i pj/3} \psi(\mathcal{A}^*)_{\hat{a},\mu} . \tag{12.18}$$

This can actually be guessed just from looking at the Verlinde formula, since the ψ

matrix essentially diagonalizes the NIM-reps.

By an argument similar to the one for the exponents, the boundary states are also seen to be of the form $(\hat{a}, i)$ with $0 \leq i \leq 2$, where $\hat{a}$ is a boundary state of the $\mathcal{A}^*$ theory i.e. a boundary state of the twisted simply-connected E_6 theory.

In addition, the vacuum, being a C invariant E_6 exponent, clearly has multiplicity 1 in the new NIM-rep (essentially $i = 0$ in the orbit $(0, i)$). This implies that the NIM-rep defined above is indecomposable. The other way to see this is that we know the NIM rep of the $\mathcal{A}^*$ theory is indecomposable. Its modular invariant is $M = C$ i.e. $M_{\lambda\mu} = \delta_{\lambda, C\mu}$. Clearly $J0 = k\omega_5$ and $J0 = k\omega_6$ are not exponents of this old NIM-rep.

12.3 D-brane Charges

Charges of D-branes in the $\mathcal{D}^*$ theory will be integers $q(\hat{a}, i)$ and M such that

$$\dim_{E_6}(\lambda) q(\hat{a}, j) = \sum_{(\hat{b}, i)} \mathcal{N}(\mathcal{D}^*)_{\lambda, (\hat{a}, j)}^{(\hat{b}, i)} q(\hat{b}, i) \mod M\,. \tag{12.19}$$

If all the $q(\hat{0}, i)$ are equal for $0 \leq i \leq 2$, then using the knowledge of the NIM-reps for $\mathcal{A}^*$, it is easy to see that any $q(\hat{a}, i)$ is independent of i, and must be a solution $q(\hat{a})$ of the $\mathcal{A}^*$ theory. However, the solutions for the D-brane charges of $\mathcal{A}^*$ are known to be $\dim_{F_4}(a)$ times a constant, which in this case is taken to be the common value $q(\hat{0}, i)$. In addition, the solutions for $\mathcal{A}^*$ are known to be unique [7]. Thus, two phenomena become apparent from this analysis. Firstly, the values $q(\hat{0}, i)$ uniquely determine any solution to (12.19). Secondly, the charge group for $\mathcal{D}^*$ must contain $\mathbb{Z}_{M_{E_6}}$, since it inherits a solution from the $\mathcal{A}$ theory, and we know that the $\mathcal{A}$ theory is solved modulo M_{E_6}. In what follows below, we will denote $M = M_{E_6}$.

However, the NIM-rep is given by $\mathcal{N}(\mathcal{D}^*) = \mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_3$. Thus, it is natural to expect that the additional $\mathbb{Z}_3$ will generate extra solutions to the charge equations. Since 3 is prime, $\mathbb{Z}_3$ does not have non-trivial subgroups. As a result the solution is constrained to be of the form

$$q(\hat{a}, i) = \dim_{F_4}(a) \left[Q + \frac{M}{D} Q_i \right]\,, \tag{12.20}$$

where $D = \gcd(3, M)$, $0 \leq Q < M$ and $0 \leq Q_i < D$. This solution reflects the tensored form of the NIM-reps.

If $D = 1$, clearly the second term is irrelevant in the charge equation, since everything is defined modulo M. Thus they vanish, and we simply recover the $\mathcal{A}^*$ case above (and

Q will be the common value $q(\hat{0}, i))$ giving the charge group $\mathbb{Z}_M$.

The other option is $D = 3$ (i.e. M is divisible by 3). In this case, the ansatz (12.20) on the left hand side of the charge equation (12.19) gives

$$\begin{aligned} \text{L.H.S} &= \dim_{E_6}(\lambda) q(\hat{a}, i) \\ &= \dim_{E_6}(\lambda) \dim_{F_4}(a) \left[Q + \frac{M}{D} Q_i \right] . \end{aligned} \tag{12.21}$$

Similarly, the ansatz (12.20) on the right hand side of the charge equation (12.19) gives

$$\begin{aligned} \text{R.H.S.} &= \sum_{(\hat{b},j)} \mathcal{N}(\mathcal{D}^*)_{\lambda(\hat{a},i)}^{(\hat{b},j)} q(\hat{b}, j) \\ &= \sum_{(\hat{b},j)} \mathcal{N}(\mathcal{D}^*)_{\lambda(\hat{a},i)}^{(\hat{b},j)} \dim_{F_4}(b) \left[Q + \frac{M}{D} Q_j \right] \\ &= \sum_{\hat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda\hat{a}}^{\hat{b}} \dim_{F_4}(b) \left[Q + \frac{M}{D} Q_{i+3Q(\lambda)} \right] , \\ &= \dim_{E_6}(\lambda) \dim_{F_4}(a) \left[Q + \frac{M}{D} Q_{i+3Q(\lambda)} \right] \mod M . \end{aligned} \tag{12.22}$$

where in the third line we have used the relation (12.17) between the NIM-reps of $\mathcal{A}^*$ and $\mathcal{D}^*$ and in the last line we have used the solution to $\mathcal{A}^*$.

Comparing the left and right hand sides, we need to satisfy

$$\dim_{E_6}(\lambda) \dim_{F_4}(a) \frac{M}{3} Q_i = \dim_{E_6}(\lambda) \dim_{F_4}(a) \frac{M}{3} Q_{i+3Q(\lambda)} \mod M . \tag{12.23}$$

Now, Q_0 can be taken to be 0 since only the relative values between the Q_i matter, and any unnecessary factor can be absorbed into the value of Q. Note that this is a matter of convenience, since Q_0 is no more physically privileged than either Q_1 or Q_2, and either of those could have been set to zero as the "ground state" value.

Exponents for the $\mathcal{A}^*$ theory are C invariant weights i.e. of the form $\mu = C\mu$. So the matrices $\mathcal{N}(\mathcal{A}^*)_\lambda$ and $\mathcal{N}(\mathcal{A}^*)_{C\lambda}$ must be equal. Also from the definition of conjugation as essentially inverting a representation we know that $\dim(\lambda) = \dim(C\lambda)$. Also, from explicitly computing the charges using (12.8) and (12.10), we can see that $3Q(\mu) = -3Q(C\mu)$ modulo 3. Thus using (12.19), we get

$$\dim_{E_6}(\lambda) q(\hat{0}, i) = \sum_{\hat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda,\hat{0}}^{\hat{b}} q(\hat{b}, i + 3Q(\lambda)) \mod M ,$$

$$\dim_{E_6}(C\lambda)q(\hat{0},i) = \sum_{\hat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda,\hat{0}}^{\hat{b}} q(\hat{b}, i - 3Q(\lambda)) \mod M\,, \tag{12.24}$$

which in turn gives us

$$\dim_{E_6}(\lambda)q(\hat{0},i) = \dim_{E_6}(\lambda)q(\hat{0}, i + 6Q(\lambda)) \mod M\,. \tag{12.25}$$

Now, for any $0 \le i < j < 3$, with $\gcd(3, j-i) = e$, the above gives us

$$\dim(\lambda)\left(q(\hat{0},i) - q(\hat{0},j)\right) = 0 \qquad \mod M\,, \tag{12.26}$$

for all λ with $\gcd(3Q(\lambda), 3) = e$. Define $D_e = \gcd\left(\frac{3}{e}, M\right)$ (i.e. 1 when $e = 3$ and 3 when $e = 1$ since we are considering the case $D = 3$ which implies that 3 is a factor of M.) $\dim(\omega_i)$ is divisible by 3 for any of the any of the (finite) fundamental weights ω_i of E_6 as can be seen from tables of dimensions for E_6 representations, and the definition (12.10) for the charge i.e. D_e also divides the dimensions of the fundamental representations as well. [9].

Any highest weight module L_λ can be written virtually (i.e. possibly with negative integer multiplicities) as a sum of tensor products of the representation modules of fundamental weights. Each term $L_{\omega_{i_1}} \otimes \cdots \otimes L_{\omega_{i_l}}$ will have the same "charge" $Q(\omega_{i_1} \otimes \cdots \otimes \omega_{i_l}) = \sum_j Q(\omega_{i_j})$ mod 1, and this will equal $Q(\lambda)$ mod 1. The dim of λ will be the sum, over all of these terms, of $\prod_j \dim(\omega_{i_j})$, and so will be a multiple of 3 unless it involves constant terms (arising from the scalar representation). Thus, it is easy to see that D_e will divide $\dim(\lambda)$. Thus, we get

$$q(\hat{0},i) = q(\hat{0},j) \mod \frac{M}{D_e}\,. \tag{12.27}$$

Finally, from the constraint (12.23), and using the above we get

$$Q_i = Q_j \qquad \mod \frac{M}{D_e}\,. \tag{12.28}$$

From our earlier claim we have $Q_0 = 0$ necessarily. As a result of the above, the charge group for the twisted non-simply connected case of C-twisted $E_6/\mathbb{Z}_3$ is

$$\mathbb{Z}_{M_{E_6}} \times \mathbb{Z}_3^2\,, \tag{12.29}$$

and the charges themselves are given by (12.20)

$$q(\hat{a}, i) = \dim_{F_4}(a) \left[Q + \frac{M}{D} Q_i \right] , \tag{12.30}$$

where $D = \gcd(3, M)$, $0 \leq Q < M$ and $0 \leq Q_i < D$.

12.4 Conclusion

The work in this chapter has been heavily influenced by the similar situation of the twisted non-simply connected versions of the A_n algebras worked out recently by M. Gaberdiel and T. Gannon. In fact, the development, for the most part, parallels the similar situation for $A_2/\mathbb{Z}_3$. This represents the first in a series of calculations involving twisted non-simply connected groups, which will be written up into a journal publication upon completion. In addition, M. Gaberdiel and T. Gannon have also discovered an intertwiner in the A_n case that relates the charges of the twisted non-simply connected groups to the untwisted non-simply connected groups. Since the calculations in the latter situation maybe be quite untractable, such an intertwiner could be used to obtain information regarding their charges using the more tractable calculations in the former case. It would be interesting to construct such intertwiners for charges of these remaining twisted non-simply connected groups as well.

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Chapter 14

At Last! Fortissimo!

The numbering of this concluding chapter was chosen specifically to avoid tempting fate when it comes to the defense. This chapter summarizes the basic results of both parts of the thesis.

Part I dealt with black holes in four and higher dimensions. Specifically, the separability of the Hamilton-Jacobi and Klein-Gordon equations in these backgrounds was analyzed and explicitly carried out in many cases. Along the way, information regarding the spacetime symmetry groups of these black holes was obtained, and in many cases, non-trivial second-rank Killing tensors that explicitly permit the separation were constructed. Chapter 3 worked with the recently discovered Kerr-(A) de Sitter metrics in the situation when all the rotation parameters of the black hole are taken to be equal (which is only possible in odd numbered dimensions). Chapter 4 worked with the Myers-Perry rotating black hole spacetimes in the situation when there are only two possibly unequal sets of rotation parameters. Chapter 7 addressed the situation of the Kerr-(A) de Sitter black holes in the situation where there are only two possibly unequal sets of rotation parameters. This result, which encompasses the situations of chapters 3 and 4, is thought to be the most generally separable situation for these spacetimes. More general separation is perhaps possible in coordinate systems besides Boyer-Lindquist, but seems unlikely based on the specific structure of the Killing vectors and tensors discovered here. Chapter 5 dealt with two multiply charged supergravity rotating black hole solutions in four and five dimensions, and chapter 6 worked with a very general class of NUT-charged rotating spacetimes. Separability was explicitly carried out for all these metrics, and in some cases non-trivial Killing tensors were found as well.

Part II of the thesis was based on symmetries of D-branes, and in particular the Conformal Field Theory (CFT) approach to D-brane charges of Wess-Zumino-Witten

(WZW) models which describe string theory on a Lie group. The CFT approach has a distinct advantage over the macroscopic description of D-brane charges using the geometrical tools available in the target space, since the CFT calculation, in addition to giving the charge groups of the D-branes, also gives the charges of the individual D-branes explicitly. Chapter 11 carried this calculation out for the D-branes of E_6 twisted by charge conjugation, and the D-branes of D_4 twisted by triality. In addition, this chapter also established some surprising phenomena relating to the behaviour of simple currents. Chapter 12 dealt with the D-brane charges of the non-simply connected group $E_6/\mathbb{Z}_3$ twisted by charge conjugation. Both chapters leave room for important continuations in the work. Chapter 11 can obviously be extended by studying simple currents in the context of equal rank subalgebras, and it is thought that this might also be able to explain the anomalous simple current of E_8 at level 2 in terms of simple currents of one of its equal rank subalgebras like A_8 or $A_1 \bigoplus E_7$. Chapter 12 can be extended by studying the D-brane charges on the other twisted non-simply connected groups. It would also be interesting to discover intertwiners similar to the A_n case that relates the charge groups between the twisted non-simply connected case and the untwisted non-simply connected case, thereby obtained information about the charges in the latter situation as well.

Both parts of the thesis addressed questions about the symmetries of several solitonic structures that occur in theories of high energy physics, particularly in string theory. The results established in this thesis provide important information regarding symmetries, charges, and conserved quantities that can be used to constrain their complicated dynamics. This work also raised several important questions which could lead to interesting research in the quest to understand black hole and D-brane dynamics.

Appendix A

Conformal Field Theory 101

CFT's have received great attention in the last two decades in theoretical physics owing to their extremely important uses in at least three different areas: as models for genuinely interacting quantum field theories; in describing the physics of critical phenomena; and in the fundamental formulation of string theory. Conformal field theories have also had great impact on various aspects of modern mathematics: Vertex Operator Algebras (OPEs), Borcherds algebras, knot theory, number theory, and low dimensional topology and geometry.

CFT's are essentially Euclidean quantum field theories with the additional property that their symmetry group contains, along with the Euclidean group of rotations and translations, local conformal transformations, i.e. transformations that preserve angles but not lengths. In higher than two dimensions, this additional constraint places severe constraints on the theory since only globally conformal transformations are available. However, in two dimensions the local conformal symmetry is of special importance in two dimensions since the corresponding symmetry algebra is infinite-dimensional in this case. As a consequence, two-dimensional conformal field theories have an infinite number of conserved quantities, and are completely solvable by symmetry considerations alone.

The brief presentation here will be from a very physical point of view, and does not do justice to the sophisticated mathematical methods associated to two-dimensional CFT's.

In D dimensions, the space of global conformal transformations is given by $SO(D+1,1)$. In two dimensions, we have the group $SO(3,1)$ as the global conformal group. We can give an explicit realization of this. Regard the two dimensional space as the complex plane $\mathbb{C}$ and consider a complex-valued function $f(z)$ that is supposed to be globally conformal. Clearly it should not have any branch points or essential singularities, since around a branch point the function is not uniquely defined, and around an essential singularity the function f sweeps the entire complex plane in an arbitrarily small neighborhood about the essential singularity (owing to Picard's theorem). Thus, in either case, the f would not be invertible. As a result, the only acceptable singularities are poles, and so f can be written as a ratio of polynomials $f(z) = P(z)/Q(z)$. If P has several distinct zeros, then the inverse image is not uniquely defined, and f would not be invertible. Moreover, if the one allowed zero z_0 of P has multiplicity n more than one, then the image of a small neighborhood of z_0 is wrapped n times around 0, and again f is not invertible. Thus P can only be a linear function of z. Similarly, the same argument shows Q must also only be a linear function of z, when we look at the behavior of f near ∞ as opposed to 0. Thus, the only global conformal transformations are of the form

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{A.1}$$

$ad - bc$ needs to be non-zero in order for f to be invertible. It is easy to see that the value can be chosen to be 1 without loss of generality. In addition, the reversal of all the signs on the numbers a, b, c, and d does not affect the transformation. Such functions f define the group $SL(2,\mathbb{C})/\mathbb{Z}_2$, which is well known to be isomorphic to the Lorentz group $SO(3,1)$ through the Weyl spinor representation.

However, in two dimensions, any holomorphic transformation provides a locally conformal transformation since it preserve angles. The space of infinitesimal generators of holomorphic functions is infinite dimensional, thereby making CFT in two-dimensions far richer. Any holomorphic infinitesimal transformation can be expressed as

$$z' = z + \epsilon(z), \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}. \tag{A.2}$$

We are considering the behavior of holomorphic functions near 0 without loss of generality, since any point of interest may be mapped to 0 by means of a global conformal transformation ,i.e., an element of $SL(2,\mathbb{C})$. Then, a field ϕ on $\mathbb{C}$ transforms under this infinitesimal mapping as $\phi'(z',\bar{z}') = \phi(z,\bar{z}) = \phi(z',\bar{z}') - \epsilon(z')\partial'\phi(z',\bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z',\bar{z}')$, or equivalently we can write

$$\delta\phi = \sum_n \left[c_n l_n \phi(z,\bar{z}) + \bar{c}_n \bar{l}_n \phi(z,\bar{z})\right], \tag{A.3}$$

where

$$l_n = -z^{n+1}\partial\,, \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}\,. \tag{A.4}$$

These are the generators of the infinitesimal conformal transformations on $\mathbb{C}$ and form the Witt algebra:

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m}\,, \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m}\,, \\ [l_n, \bar{l}_m] &= 0\,. \end{aligned} \tag{A.5}$$

The Witt algebra also contains the generators of global conformal transformations: l_{-1}, l_0, and l_1 and their complex conjugates. So far we have been considering spinless fields ϕ. Under a holomorphic transformation $w = w(z)$, a field with spin s and scaling dimension Δ transforms as

$$\phi'(w, \overline{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}} \phi(z, \overline{z})\,, \tag{A.6}$$

where $h = \frac{1}{2}(\Delta + s)$ and $\overline{h} = \frac{1}{2}(\Delta - s)$ are known as the holomorphic and antiholomorphic conformal dimensions. Fields that transform this way under global conformal transformations are known as quasi-primary fields. Fields that transform this way under all conformal transformations are known as primary fields. Technically, this definition should be rigorously defined using infinitesimal conformal transformations of the fields as done earlier. Primary fields (or primaries) are to CFT what the highest weight state is to a Lie algebra. Their transformation behavior is respected by correlations functions as well, i.e.,

$$< \phi_1(w_1, \overline{w}_1)...\phi_n(w_n, \overline{w}_n) >= \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{|w_i} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}_i}_{|\overline{w}_i} < \phi_1(z_1, \overline{z}_1)...\phi_n(z_n, \overline{z}_n) > \tag{A.7}$$

Clearly, the constraints placed by conformal symmetry should help us solve for such correlation functions, or at least that is the aim of CFT's: to be able to solve the theory from its abundant symmetries. For instance, conformal symmetry implies that two point functions are necessarily of the form

$$< \phi_1(z_1, \overline{z}_1)\phi_2(z_2, \overline{z}_2) >= \frac{C_{12}}{(z_{12})^{2h}(\overline{z}_{12})^{2\overline{h}}}, \text{ if } h_1 = h_2 = h, \text{ and } \overline{h}_1 = \overline{h}_2 = \overline{h}\,, \tag{A.8}$$

where $z_{12} = z_1 - z_2$. The two-point function vanishes if the conformal dimensions of the two fields do not match. Similar conditions can be used to constrain the three-point function uniquely as well. In addition, the holomorphic transformations constrain the dynamics of the theory through Ward identities.

Associated to any Lagrangian of the theory, a stress-energy tensor $T_{\mu\nu}$ can be constructed. Then, local translation, rotation and scaling invariance provide the Ward identities

$$\begin{aligned}
\frac{\partial}{\partial x^\mu} < T^\mu_\nu(x) X >&= \sum_{i=1}^n \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} < X > , \\
\epsilon_{\mu\nu} < T^{\mu\nu}(x) X >&= -i \sum_{i=1}^n s_i \delta(x - x_i) < X > , \\
< T^\mu_\mu(x) X >&= - \sum_{i=1}^n \delta(x - x_i) \Delta_i < X > \ .
\end{aligned} \tag{A.9}$$

Converting to complex coordinates and write $T(z) = -2\pi T_{zz}$, $\overline{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}}$. $T_{z\bar{z}} = T_{\bar{z}z} = 0$ owing to global scale invariance. The Ward identities can be rewritten

$$< T(z) X >= \sum_{i=1}^n \left[\frac{1}{z - w_i} \partial_{w_i} < X > + \frac{h_i}{(z - w_i)^2} < X > \right] + \text{reg} \, , \tag{A.10}$$

where regular terms are not relevant, since correlation functions will be put under contour integrals, and only terms with poles will contribute. The so-called conformal Ward identity provides information regarding the transformation of correlations functions under conformal mappings with infinitesimal parameter $\epsilon(z)$ (A.2):

$$\delta_{\epsilon,\bar{\epsilon}} < X >= -\frac{1}{2\pi i} \oint dz \epsilon(z) < T(z) X > + \text{c.c.} \, , \tag{A.11}$$

where the contour encloses the point of interest.

Referring to (A.10), for a primary ϕ with holomorphic dimensions h, we can write

$$T(z)\phi(w) \sim \frac{h}{(z - w)^2} \phi(w) + \frac{1}{z - w} \partial_w \phi(w) \, , \tag{A.12}$$

and its complex conjugate. It is understood that these expressions are valid only inside correlation functions. Such expressions which give information regarding the near distance behavior when two fields coincide are known as operator product expansions. The $\sim$ indicates modulo terms that are regular as $z \to w$, which are irrelevant for studying near distance behavior.

Typically the stress tensor is constructed from a Lagrangian, and its OPE's with the various fields of the theory are studied using contractions and Laurent expansions. This is the common way to find information about the behavior of a field, such as its conformal dimension. Note that the above OPE (A.12) is valid for primary fields that transform as (A.6) under all conformal mappings. Additional singular terms would appear for quasi-primary fields. The most important of these is the stress tensor itself which has the OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)\,, \tag{A.13}$$

which indicates that the field is quasi-primary with dimensions 2. This implies that the stress tensor actually transforms as

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w:z\}\right]\,. \tag{A.14}$$

where $\{w : z\}$ is the Schwartzian derivative of w with respect to z. The Schwartzian derivatives vanishes for linear fractional transformations (global conformal transformation), and thus T is a quasi-primary field, but not primary. The constant c, known as the central charge, is theory specific. Roughly speaking, it counts the number of degrees of freedom of the theory. For instance the theory of a free scalar field has $c = 1$, and that of a free Fermion is $c = 1/2$. c is also related to the vacuum energy for CFT on a curved manifold.

In addition to having the above description of CFT, it is desirable to have the so-called operator formalism. Initially, the theory is defined on a long cylinder, where time is goes from $-\infty$ to ∞ (i.e. goes along the flat direction of the cylinder), and space is compactified on the circle and runs from 0 to L. A Wick rotation is performed, and the cylinder is parameterized by a single complex coordinate $\xi = t + ix$. The cylinder is then mapped to the complex place $\mathbb{C}$ (or really the Riemann sphere S^2) via the mapping

$$z = e^{2\pi\xi/L}\,. \tag{A.15}$$

Past infinity (i.e. $t \to -\infty$) is mapped to the origin, future infinity (i.e. $t \to \infty$) is mapped to infinity on the Riemann sphere, and constant time sections of the cylinder are now circles on the complex plane (with the spatial coordinate x giving the angle around the circle). Also assume the existence of a vacuum state $|0>$. States are then defined from fields on $\mathbb{C}$ by $|\phi>_{in} = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0>$. Out states are similarly defined to the be Hermitian adjoint of these states. Any field ϕ on $\mathbb{C}$ can be decomposed into modes as

$$\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \frac{\phi_{m,n}}{z^{-m-h}\bar{z}^{-n-\bar{h}}}\,. \tag{A.16}$$

In order for the in and out states to be well defined, it is necessary to have $\phi_{m,n}|0>=0$ when $m>-h, n>-\bar{h}$. We adopt the usual practise of dealing with just the holomorphic sector, i.e., we simply write the holomorphic sectors and suppress the antiholomorphic fields and indices. Then, the above mode expansions could be written

$$\phi(z) = \sum_{m\in\mathbb{Z}} \frac{\phi_m}{z^{m+h}}\,. \tag{A.17}$$

Analogous to time ordering in quantum field theory, we order fields on $\mathbb{C}$ by radial ordering, since circles on $\mathbb{C}$ represent constant time sections, and larger radius corresponds to larger time coordinates. Radial ordering is defined by $\mathcal{R}\Phi_1(z)\Phi_2(w) = \Phi_1(z)\Phi_2(w)$ when $|z|>|w|$, and $\mathcal{R}\Phi_1(z)\Phi_2(w) = \Phi_2(w)\Phi_1(z)$ when $|z|<|w|$.

We can define the operators $A = \oint a(z)dz$ and $B = \oint b(z)dz$ associated to fields $a(z)$ and $b(z)$, where the integration is carried out over fixed time contours. The commutation relations of these operators can be related to the OPE's of the fields a and b via

$$[A,B] = \oint_0 dw \oint_w dz\, a(z)b(w)\,, \tag{A.18}$$

where the first integral is over a contour that includes the origin, and the second is over a contour that includes w.

For any infinitesimal conformal mapping $\epsilon(z)$ (A.2), we can define the charge that generates such a transformation,

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z)\,. \tag{A.19}$$

Using the conformal Ward identity (A.11), we can write

$$\delta_\epsilon\Phi(w) = [-Q_\epsilon, \Phi(w)]\,. \tag{A.20}$$

The stress tensor can be expanded into modes:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+1}}\,,\quad \bar{T}(\bar{z}) = \sum_{m\in\mathbb{Z}} \frac{\bar{L}_m}{\bar{z}^{m+1}}\,. \tag{A.21}$$

Using the OPE (A.13) and (A.18), we can write

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,, \\ [L_n \overline{L}_m] &= 0\,, \\ [\overline{L}_n, \overline{L}_m] &= (n-m)\overline{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,, \end{aligned} \tag{A.22}$$

known as the Virasoro algebra. This is the central extension of the Witt algebra, which generates infinitesimal conformal mappings on $\mathbb{C}$ and generates the corresponding infinitesimal conformal transformations on the space of fields and operators in the CFT.

Representations of the CFT are built out of the vacuum $|0>$ through the action of the Virasoro generators L_n and $\overline{L}_m$.

In addition to free fields, CFT's exist for interacting theories, where the OPE's produce highly singular terms. It is important to define a normal ordering correctly so that these singular terms are eliminated (since the VEV of a normal ordered term needs to be zero, singular expressions should not show up). This is correctly done by defining

$$(AB)(w) = \frac{1}{2\pi i}\oint \frac{dz}{z-w}A(z)B(w)\,, \tag{A.23}$$

which has the effect of removing all the singular terms in the OPE. A mode expansion of the terms (AB)(w) results in the corresponding operator normal ordering : ... : that is familiar from quantum field theory.

This concludes the very brief introduction to CFT which has touched all the topics necessary for the relevant CFT material in Part II of the thesis. Obviously, this barely touches the surface of CFT on both the physics and the mathematical sides. This subject is quite rich with ideas and phenomena in both fields, and is quite possibly one of the most fascinating areas representing the interplay between physics and modern mathematics that has come to be so crucial in recent years. Further details on the various aspects of CFT and complete descriptions of ideas only briefly touched upon in this appendix can be found in [1–10].

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