University of Alberta

Volume distribution and the geometry of high-dimensional random polytopes

by

Peter Pivovarov

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Department of Mathematical and Statistical Sciences

Fall 2010
Edmonton, Alberta

Examining Committee

Nicole Tomczak-Jaegermann, Mathematical and Statistical Sciences

Alexander Litvak, Mathematical and Statistical Sciences

Vladimir Troitsky, Mathematical and Statistical Sciences

Anthony To-Ming Lau, Mathematical and Statistical Sciences

Vlad Yaskin, Mathematical and Statistical Sciences

Lorna Stewart, Computing Sciences

Elisabeth Werner, Mathematics, Case Western Reserve University

Abstract

This thesis is based on three papers on selected topics in Asymptotic Geometric Analysis.

The first paper is about the volume of high-dimensional random polytopes; in particular, on polytopes generated by Gaussian random vectors. We consider the question of how many random vertices (or facets) should be sampled in order for such a polytope to capture significant volume. Various criteria for what exactly it means to capture significant volume are discussed. We also study similar problems for random polytopes generated by points on the Euclidean sphere.

The second paper is about volume distribution in convex bodies. The first main result is about convex bodies that are (i) symmetric with respect to each of the coordinate hyperplanes and (ii) in isotropic position. We prove that most linear functionals acting on such bodies exhibit super-Gaussian tail-decay. Using known facts about the mean-width of such bodies, we then deduce strong lower bounds for the volume of certain caps. We also prove a converse statement. Namely, if an *arbitrary* isotropic convex body (not necessarily satisfying the symmetry assumption (i)) exhibits similar cap-behavior, then one can bound its mean-width.

The third paper is about random polytopes generated by sampling points according to multiple log-concave probability measures. We prove related estimates for random determinants and give applications to several geometric inequalities; these include estimates on the volume-radius of random zonotopes and Hadamard's inequality for random matrices.

Acknowledgements

A number of years ago, I made the three-hour trek from Calgary and started studying mathematics with Professor Nicole Tomczak-Jaegermann. That decision has enriched my life immeasurably. I have benefited on many levels from countless hours of discussions, her creative ideas and perspective. She has provided invaluable guidance on my research as it has taken shape. I also appreciate her careful reading and thoughtful input on various manuscripts and research proposals (not to mention crafting n reference letters, with $n \to \infty$). She has always actively encouraged me to broaden my mathematical interests and seek out new connections; especially affording me the opportunity for a number of visits abroad. I am deeply appreciative for all that she has done.

I thank my supervisory committee, Professors Sasha Litvak and Vladimir Troitsky for helpful advice; I also learned a great deal in their graduate courses. Additionally, I am grateful to Professor Litvak for many helpful discussions and all the work he has done on my behalf.

I am indebted to the organizers of the *Phenomena in High-Dimensions* (PHD) European network (MRTN- CT-2004-511953) - Professor Alain Pajor, in particular - for my participation in various network activities; specifically, my stays as an Early-Stage Researcher in Athens, Greece, from February to May 2008; in Warsaw, Poland, from April to June 2007; in Paris, France, from May to July 2006 for the PHD thematic program; as well as various workshops and conferences of the network. I express gratitude to my host in Athens, Professor Apostolos Giannopoulos for his warm hospitality and help with all things mathematical and otherwise. I also thank Professors Piotr Mankiewicz and Rafal Latala in Warsaw for making my stay such a positive experience.

Over the last few years, I have learned a great deal from discussions with friends, student colleagues, and professors. I especially thank Radek Adamczak for many fruitful exchanges of ideas, as well as Marton Naszodi, Omar Rivasplata, Steven Taschuk and Ben Willson. Many thanks also to Professors Herman König and Grigoris Paouris.

I thank my parents, Ivan and Inge Pivovarov, and also my brothers Juraj and Viktor.

Lastly, I express my deepest gratitude to Carly for her continual support and everlasting patience.

My Ph.D studies were supported by the Natural Sciences and Engineering Research Council of Canada (Post-Graduate Scholarship); the Alberta Heritage Scholarship Fund (Ralph Steinhauer Award of Distinction); the Killam Trusts (Izaak W. Killam Graduate Scholarship); and the Department of Mathematical and Statistical Sciences at the University of Alberta (Josephine M. Mitchell Scholarship).

Contents

List of Figures

CHAPTER 1

Introduction

1.1 Asymptotic Geometric Analysis

One does not have to look far to find examples of the peculiar behavior of volume in high dimensions. For instance, consider the n-dimensional cube $[-1,1]^n$ and Euclidean ball B_2^n. The ball has volume $\mathrm{vol}(B_2^n) = \pi^{n/2}/\Gamma(n/2+1)$, which for large n is of the order $(2\pi e/n)^{n/2}$. Though the ball touches the cube in each of its $2n$-faces, the proportion of volumes is a minuscule

$$\frac{\mathrm{vol}(B_2^n)}{\mathrm{vol}([-1,1]^n)} \simeq \left(\frac{\pi e}{2n}\right)^{n/2}, \quad \text{as } n \to \infty.$$

Put another way, if a point is sampled randomly in the cube $[-1,1]^n$, it will miss the ball with probability about $1-(\pi e/(2n))^{n/2}$. This high-dimensional property is usually incorporated in a two-dimensional picture by drawing the ball as in Figure 1.1.

Where exactly does the volume in the cube concentrate? What about other convex bodies?

The distribution of volume in convex bodies is a well-studied topic in *Asymptotic Geometric Analysis*. The latter field is concerned with various aspects of convex bodies and especially the characteristic behavior that emerges when the dimension tends to infinity.

Probabilistic methods play a key role in the theory. Since V.D. Milman's seminal use of the *concentration of measure phenomenon* in his approach to Dvoretzky's Theorem [32], sophisticated methods have been developed, spawning numerous directions of research (see, e.g., [34], [19], [28]).

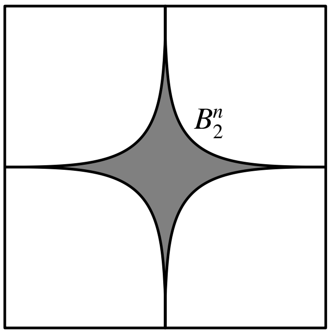


Figure 1.1: The proportion of volume the ball occupies in the cube

Random convex bodies - more precisely, random polytopes - have also played an important role. E.D. Gluskin was the first to use such bodies in this context, showing that they exhibit essential differences in shape [20]. Similar probabilistic methods have been used to great effect (see, e.g., [29]) and random polytopes appear in a variety of contexts.

This thesis is a selection of topics in Asymptotic Geometric Analysis. Each of Chapters 2 to 4 are self-contained papers based on the articles [39], [38], and [37], respectively. A common thread is the geometry of high-dimensional random polytopes. These can be generated by sampling N points $X_1,\ldots,X_N$ independently according to a probability measure μ on $\mathbb{R}^n$, and forming their convex hull

$$K_N := \operatorname{conv}\{X_1,\ldots,X_N\}. \tag{1.1}$$

Chapter 2, loosely speaking, addresses the following problem: How many points $N = N(n)$ are needed for K_N to capture significant volume as $n \to \infty$? Of course, the meaning of "significant volume" depends on the model of randomness and one wants the smallest such N. This work was motivated by results of Dyer, Füredi and McDiarmid [13] who answered the question when the X_i are drawn independently from the cube $[-1,1]^n$.

Chapter 3 is about the distribution of volume in high-dimensional convex bodies. This work is connected to recent research emanating from J. Bourgain's approach to a famous problem about isotropic constants of convex bodies.

In Chapter 4, we further explore the relation between the volume of random polytopes in convex bodies and isotropic constants.

The remainder of this chapter is to serve as an introduction to all three papers. Along the way, we indicate how some of our results fit within the theory. In Chapter 5, we state some further developments (current joint work between this author and G. Paouris) and open problems. Chapter 6 summarizes our main results.

1.1.1 Convex bodies as probability spaces

If A and B are subsets of $\mathbb{R}^n$, their *Minkowski sum* is the set $A+B := \{a+b : a \in A, b \in B\}$ and, if $\lambda \in \mathbb{R}$, we set $\lambda A = \{\lambda a : a \in A\}$.

The Brunn-Minkowski inequality governs how volume behaves with respect to Minkowski addition.

Theorem 1.1.1. *Let A and B be compact sets in $\mathbb{R}^n$ and let $0 \leq \lambda \leq 1$. Then*

$$\mathrm{vol}\,(\lambda A + (1-\lambda)B) \geq \mathrm{vol}\,(A)^{\lambda}\,\mathrm{vol}\,(B)^{1-\lambda}. \tag{1.2}$$

In other words, the logarithm of $\mathrm{vol}\,(\cdot)$ is a concave function. See [14] for an extensive survey on the latter inequality and its many uses.

If $K \subset \mathbb{R}^n$ is a convex body with $\mathrm{vol}\,(K) = 1$, a probability measure can be associated to K by defining

$$\mathrm{vol}\,|_K(A) := \mathrm{vol}\,(A \cap K)$$

for Borel measurable sets $A \subset \mathbb{R}^n$. The Brunn-Minkowksi inequality implies that

$$\mathrm{vol}\,|_K(\lambda A + (1-\lambda)B) \geq \mathrm{vol}\,|_K(A)^{\lambda}\,\mathrm{vol}\,|_K(B)^{1-\lambda} \tag{1.3}$$

for any compact $A, B \subset \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. In fact, (1.3) is a defining property for an important class of measures.

Definition 1.1.2. A Borel measure μ on $\mathbb{R}^n$ is said to be *log-concave* if for any $\lambda \in [0,1]$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda} \tag{1.4}$$

for all compact $A, B \subset \mathbb{R}^n$.

Many properties of convex bodies also hold for log-concave measures and some authors choose to work in this more general setting. For simplicity, we will focus most of this chapter on convex bodies. We discuss properties of log-concave measures in subsequent chapters, as needed. (See [23] and the references therein for further information on log-concave measures.)

Tail-decay of linear functionals

Suppose K is a convex body with $\mathrm{vol}(K) = 1$. From the probabilistic viewpoint, functions $f : K \to \mathbb{R}$ are viewed as random variables; in particular, the Euclidean norm $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ and, for each θ on the Euclidean sphere S^{n-1}, the linear functional $\langle \cdot, \theta \rangle$, defined by

$$\langle x, \theta \rangle = x_1\theta_1 + \ldots + x_n\theta_n \quad (x \in K).$$

For bounded measurable functions $f : K \to \mathbb{R}$, define

$$\|f\|_p := \|f\|_{L_p(K)} = \left(\int_K |f(x)|^p dx \right)^{1/p} \tag{1.5}$$

One consequence of the Brunn-Minkowksi inequality is the following proposition (which follows from Borell's Lemma, e.g., [34, Appendix III]).

Proposition 1.1.3. *Let $K \subset \mathbb{R}^n$ be a convex body with* $\mathrm{vol}(K) = 1$ *and let $\theta \in S^{n-1}$. Then*

$$\mathrm{vol}\left(\{x \in K : |\langle x, \theta \rangle| \geq t \|\langle \cdot, \theta \rangle\|_2\}\right) \leq 2e^{-t/C}, \tag{1.6}$$

for any $t \geq 1$, where $C > 0$ is an absolute constant.

Thus in each direction θ, the volume of K outside the slab $\{x \in \mathbb{R}^n : |\langle x, \theta \rangle| < t \|\langle \cdot, \theta \rangle_2\|\}$ decays at an exponential rate. This is one reason we often draw convex bodies in a hyperbolic form, as in Figure 1.2 (see also [31] for a discussion of such pictures of convex bodies).

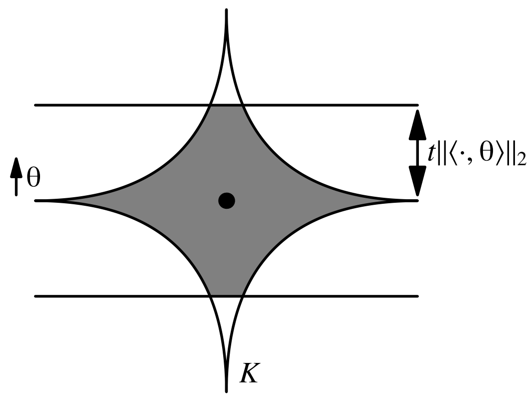


Figure 1.2: Tail-decay of linear functionals

The estimate from Proposition 1.1.3 is of the right order for some convex bodies. For instance, consider the cross-polytope $B_1^n = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$, where $e_1, \ldots, e_n$ is the standard unit vector basis for $\mathbb{R}^n$. Let c_n be such that $\mathrm{vol}(c_nB_1^n) = 1$ (then $c_n/n \to 1/2e$ as $n \to \infty$). If $\theta = e_n$, a routine calculation shows that for each $t \geq 1$,

$$\mathrm{vol}(\{x \in c_nB_1^n : \langle x, e_n\rangle \geq t\|\langle\cdot,\theta\rangle\|_2\}) \simeq e^{-ct},$$

where $c > 0$ is an absolute constant. (Here the notation $A \simeq B$ means $cA \leq B \leq CA$ for absolute constants $c, C > 0$.)

For some convex bodies, the functionals $\langle\cdot,\theta\rangle$ exhibit a Gaussian-type tail-decay. For instance, consider the Euclidean ball B_2^n and let d_n be such that $\mathrm{vol}(d_nB_2^n) = 1$ (then $d_n/\sqrt{n} \to (2\pi e)^{-1/2}$ as $n \to \infty$). Then for each $\theta \in S^{n-1}$ and $t \geq 1$,

$$\mathrm{vol}(\{x \in d_nB_2^n : \langle x, \theta\rangle \geq t\|\langle\cdot,\theta\rangle\|_2\}) \simeq e^{-ct^2}, \tag{1.7}$$

where $c > 0$ is an absolute constant.

The rate of tail-decay of linear functionals has important consequences for the geometry of the body, which will be explored in subsequent sections. When studying tail-decay, we typically assume that the body is in a suitable position.

1.1.2 Isotropic convex bodies and isotropic constants

Definition 1.1.4. A convex body K is *isotropic* if $\mathrm{vol}(K) = 1$, its center of mass is the origin, i.e.,

$$\int_K \langle x, \theta\rangle\, dx = 0 \text{ for each } \theta \in S^{n-1} \tag{1.8}$$

and there is a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2\, dx = L_K^2 \text{ for each } \theta \in S^{n-1}. \tag{1.9}$$

In the probabilistic interpretation, all functionals $\langle\cdot,\theta\rangle$ are centered and have the same variance.

The constant L_K is called the *isotropic constant*. If $K \subset \mathbb{R}^n$ is a convex body with center of mass at the origin, then there is a linear image TK of K such that TK is isotropic. Moreover, T is unique up to orthogonal transformations. Thus L_K can be defined for any convex body and it is an affine-invariant.

Note that (1.9) implies

$$\int_K |x|^2 dx = nL_K^2. \tag{1.10}$$

Put another way, if a vector x is sampled randomly in an isotropic convex body K (according to the measure $\mathrm{vol}|_K$) then its expected length is about $\sqrt{n}L_K$. Another important fact is the following proposition.

Proposition 1.1.5. *Let $K \subset \mathbb{R}^n$ be a convex body with* $\mathrm{vol}(K) = 1$ *and center of mass at the origin. Then K is isotropic if and only if*

$$\int_K |x|^2 dx \leq \int_{SK} |x|^2 dx \textit{ for each } S \in SL(n). \tag{1.11}$$

As a sample of an isotropic convex body, see Figure 1.3.

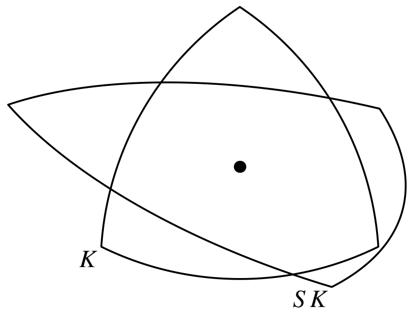


Figure 1.3: A convex body K in isotropic position

It is known that the isotropic constants of convex bodies admit a uniform lower bound. More precisely, for any convex body $K \subset \mathbb{R}^n$, one has

$$L_K \geq L_{B_2^n} \geq \frac{1}{\sqrt{2\pi e}}. \tag{1.12}$$

A uniform upper bound is a long-standing open problem.

Conjecture 1.1.6 (Uniform bound for isotropic constants)**.** *There exists an absolute constant C such that for any integer $n \geq 1$ and for any convex body $K \subset \mathbb{R}^n$,*

$$L_K \leq C. \tag{1.13}$$

Upper bounds for the isotropic constant L_K were first studied by J. Bourgain and work on the conjecture has spawned many directions of research. The latter conjecture is also known as the Hyperplane Conjecture or Slicing Problem (due to an equivalent formulation involving the volume of central sections of K, first considered by J. Vaaler and studied by K. Ball and D. Hensley). As all of our relevant results are stated in terms of isotropic constants, we will not discuss other equivalent formulations beyond this point; we refer the reader to [3], [33] or [18] and the references therein for further information.

Known results

In [8], J. Bourgain proved that if K is an isotropic convex body in $\mathbb{R}^n$, then $L_K \leq Cn^{1/4}\log n$. Presently, the best known result is due to B. Klartag, who has shown that $L_K \leq Cn^{1/4}$.

There are many classes of convex bodies for which the conjecture has a positive solution, for instance, unconditional convex bodies [7], [33], zonoids and duals of zonoids [4], unit balls of Schatten norms [27], ψ_2-bodies [9], and others [22], [30]; more recently, for various random polytopes [26], [11], [1] and polytopes with few vertices [2]. We define and discuss some of these classes in subsequent sections.

1.1.3 Distribution of volume in isotropic convex bodies

Despite the lack of understanding of the isotropic constant L_K, recent years have seen quite striking results about the distribution of the Euclidean norm on an isotropic convex body K; in particular, on how $|\cdot| : K \to R$ deviates from $\sqrt{n}L_K$, due to G. Paouris [36].

Theorem 1.1.7. *There exists an absolute constant $C > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then for all $t \geq 1$,*

$$\operatorname{vol}\left(\{x \in K : |x| \geq C\sqrt{n}L_K t\}\right) \leq e^{-\sqrt{n}t}. \tag{1.14}$$

Thus the volume of K lying outside the Euclidean ball $C\sqrt{n}L_K B_2^n$ decays exponentially fast.

Another breakthrough concerns how the Euclidean norm concentrates around $\sqrt{n}L_K$, due to B. Klartag, from [24] (improving upon [23]).

Theorem 1.1.8. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $0 \leq \varepsilon \leq 1$. Then*

$$\operatorname{vol}\left(\{x \in K : ||x| - \sqrt{n}L_K| \geq \varepsilon\sqrt{n}L_K\}\right) \leq C\exp\left(-c\varepsilon^{\tau}n^{\kappa}\right), \tag{1.15}$$

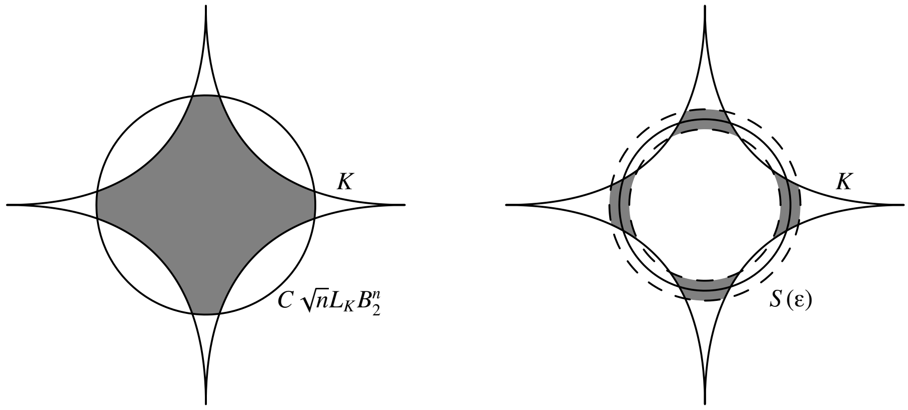
where τ, κ, c and C are positive absolute constants.

In other words, if $S(\varepsilon)$ is the shell

$$S(\varepsilon) := \{x \in \mathbb{R}^n : (1-\varepsilon)\sqrt{n}L_K \leq |x| \leq (1+\varepsilon)\sqrt{n}L_K\}, \tag{1.16}$$

then Klartag's theorem implies that, for large n, most of the volume of K lies inside $S(\varepsilon)$.

We have included a diagram to compare the two results, see See Figure 1.4.



(a) Large deviations of the Euclidean norm (b) ε-concentration of the Euclidean norm

Figure 1.4: Comparison of the theorems of Paouris and Klartag (right)

1.1.4 Sub-Gaussian tail-decay and its implications

Definition 1.1.9. Let K be a convex body with $\mathrm{vol}(K) = 1$ and center of mass at the origin. We say that $\theta \in S^{n-1}$ is a *sub-Gaussian* direction for K, if there is a constant b such that

$$\mathrm{vol}\left(\{x \in K : |\langle \cdot, \theta\rangle| \geq t\|\langle \cdot, \theta\rangle\|_2\}\right) \leq 2\exp\left(-t^2/2b^2\right) \quad \forall t \geq 1. \tag{1.17}$$

When the body K is clear from the context, we simply say that θ is sub-Gaussian. If the value of the constant b is important, we say that θ is sub-Gaussian with constant b. The terminology comes from the comparison with a standard Gaussian (normal) random variable γ with mean-zero and variance one. The density corresponding to γ is $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. One can check that $\mathbb{P}(|\gamma| \geq t) \leq 2e^{-t^2/2}$.

A major reason for interest in the sub-Gaussian tail-decay of linear functionals on isotropic convex bodies is the following theorem due to J. Bourgain, from [9] (based on ideas from [8]).

Theorem 1.1.10. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body and let $b > 0$. Suppose that each $\theta \in S^{n-1}$ is sub-Gaussian with constant b as in (1.17). Then*

$$L_K \leq Cb \log b, \tag{1.18}$$

where C is an absolute constant.

Recently, N. Dafnis and G. Paouris [12] have shown that (1.18) can be improved to $L_K \leq Cb\sqrt{\log b}$.

For an arbitrary convex body, it turns out to be quite difficult to show that there is even *one* direction θ that exhibits sub-Gaussian tail-decay (1.17). More precisely, consider the following the question, first posed by V.D. Milman.

Question 1.1.11. *Is there a constant $c > 0$ such that for any integer $n \geq 1$ and any convex body $K \subset \mathbb{R}^n$ with $\operatorname{vol}(K) = 1$, there is a direction $\theta \in S^{n-1}$ for which*

$$\operatorname{vol}\left(\{x \in K : |\langle x, \theta\rangle| \geq t\|\langle \cdot, \theta\rangle\|_2\}\right) \leq 2\exp\left(-ct^2\right), \quad \forall t \geq 1? \tag{1.19}$$

Up to a logarithmic factor, it was answered in the affirmative by B. Klartag [25].

Theorem 1.1.12. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then there exists $\theta \in S^{n-1}$ such that*

$$\operatorname{vol}\left(\{x \in K : |\langle x, \theta\rangle| \geq t\|\langle \cdot, \theta\rangle\|_2\}\right) \leq 2\exp\left(-ct^2/\log^{\tau}(t+1)\right), \tag{1.20}$$

for all $t \geq 1$, where c and τ are positive absolute constants.

Klartag's proof gave $\tau = 5$. A. Giannopoulos, A. Pajor and G. Paouris [16] gave an alternate proof with $\tau = 2$.

Mean-width of isotropic convex bodies

The search for sub-Gaussian tail-decay of linear functionals is connected to a well-known problem about the mean-width of isotropic convex bodies. For a convex body $K \subset \mathbb{R}^n$, denote its support function by

$$h_K(\theta) := \sup_{x \in K} \langle x, \theta\rangle, \quad (\theta \in S^{n-1}).$$

The width of K in the direction of θ is the quantity $w(K,\theta) = h_K(\theta) + h_K(-\theta)$ and the mean-width of K is

$$w(K) = \int_{S^{n-1}} w(K,\theta) d\sigma(\theta) = 2 \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Conjecture 1.1.13. *There is an absolute constant C such that for any integer $n \geq 1$ and for any isotropic convex body $K \subset \mathbb{R}^n$,*

$$w(K) \leq C\sqrt{n \log n} L_K. \tag{1.21}$$

Presently, the best known upper bound is the following:

Theorem 1.1.14. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then*

$$w(K) \leq Cn^{3/4} L_K, \tag{1.22}$$

where C is an absolute constant.

We include a sketch of the proof of the latter theorem and further remarks on the problem of bounding the mean-width in Chapter 3. For now, we mention only the relation with sub-Gaussian tail-decay. In [35, Lemma 4.2], it is proved that if K is an isotropic convex body and $\theta \in S^{n-1}$ is sub-Gaussian with constant b then

$$\max\{h_K(\theta), h_K(-\theta)\} \leq Cb\sqrt{n} L_K, \tag{1.23}$$

where C is an absolute constant. Thus if one could show that "most" directions are sub-Gaussian (or nearly so), then one would obtain a better bound on the mean-width. As discussed above, it is non-trivial to establish even the existence of a single such θ.

1.1.5 Unconditional convex bodies

For convex bodies exhibiting certain symmetries many of the themes discussed above are well-understood.

Definition 1.1.15. A convex body $K \subset \mathbb{R}^n$ is 1-*unconditional* if for each $x = (x_1, \ldots, x_n) \in K$ and each choice of signs $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, n$, the vector $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$ belongs to K.

Thus if K is isotropic and 1-unconditional then K satisfies the following conditions:

U1) $\mathrm{vol}(K) = 1$.

U2) If $x = (x_i) \in K$ then $[-|x_1|, |x_1|] \times \ldots \times [-|x_n|, |x_n|] \subset K$.

U3) $\int_K x_j^2 dx = L_K^2$.

It is well-known that isotropic constants of 1-unconditional convex bodies are uniformly bounded [7], [33]. The following formulation is from [6].

Proposition 1.1.16. *If $K \subset \mathbb{R}^n$ is a 1-unconditional isotropic convex body then $L_K \leq 1/\sqrt{2}$.*

Bobkov and Nazarov [6] initiated the study of sub-Gaussian behavior of functionals on 1-unconditional bodies and proved the following proposition.

Proposition 1.1.17. *Let $K \subset \mathbb{R}^n$ be a 1-unconditional isotropic convex body and let θ_d be the diagonal direction $\theta_d = \frac{1}{\sqrt{n}}(1, \ldots, 1)$. Then for each $t \geq 1$,*

$$\mathrm{vol}(\{x \in K : |\langle x, \theta_d \rangle| \geq t\}) \leq 2e^{-ct^2}, \tag{1.24}$$

where $c > 0$ is an absolute constant.

The isotropic constant $L_K = \|\langle \cdot, \theta \rangle\|_2$ has been omitted from the statement of the latter proposition since 1-unconditional bodies have uniformly bounded isotropic constants. In a subsequent paper, Bobkov and Nazarov studied sub-Gaussian behavior for directions other than the main diagonal and proved the following theorem (from [5]).

Theorem 1.1.18. *There exist positive numerical constants c_1, c_2 and t_0 with the following property. For any integer $n \geq 1$ and for any 1-unconditional isotropic convex body $K \subset \mathbb{R}^n$, the σ-measure of the set of $\theta \in S^{n-1}$ such that*

$$\mathrm{vol}(\{x \in K : |\langle x, \theta \rangle| \geq t\}) \leq \exp(-c_2 t^2 / \log t), \quad \forall t \geq t_0, \tag{1.25}$$

is at least $1 - n^{-c_1}$. Moreover, c_1 can be chosen arbitrarily large at the expense of c_2 and t_0.

Thus the tail-decay of most $\langle \cdot, \theta \rangle$ are nearly sub-Gaussian.

In Chapter 3, we prove a complement to Theorem 1.1.18 and show that 1-unconditional isotropic convex bodies have many *super-Gaussian* directions (analogous to (1.17) but with the reverse inequality). We also prove related estimates that have implications for the mean-width of an arbitrary isotropic convex body.

1.1.6 Random polytopes and isotropic constants

Isotropic constants of random polytopes

As mentioned in the introduction, a major success in Asymptotic Geometric Analysis has been the use of random polytopes in solutions to various long-standing open problems. Their potential as counter-examples to Conjecture 1.1.6 was recently studied by Klartag and Kozma [26]. As a sample result, we mention the following theorem for polytopes generated by Gaussian random vectors of the form $X = (\gamma_1, \dots, \gamma_n)$, where γ_i are independent $N(0,1)$ random variables.

Theorem 1.1.19. *Let $N > n$ and let $X_1, \dots X_N$ be independent Gaussian random vectors in $\mathbb{R}^n$. Set*

$$G_N := \operatorname{conv}\{X_1, \dots, X_N\}.$$

Then, with probability at least $1 - C_1 e^{-c_1 n}$,

$$L_{G_N} \leq C, \tag{1.26}$$

where C, C_1 and c_1 are positive absolute constants.

For polytopes generated by points sampled independently and uniformly on the sphere S^{n-1}, a similar result was proved by D. Alonso-Gutierrez [1].

Subsequent research has examined random polytopes generated by points in an isotropic convex body $K \subset \mathbb{R}^n$. Let $X_1, \dots, X_N$ be independent random vectors distributed uniformly in K and K_N their convex hull

$$K_N := \operatorname{conv}\{X_1, \dots, X_N\}. \tag{1.27}$$

In [11], N. Dafnis, A. Giannopoulos and O. Guedon asked the following question about the relation between the isotropic constants of K_N and K.

Question 1.1.20. *Is it true that, with probability tending to 1 as $n \to \infty$, one has $L_{K_N} \leq CL_K$, where $C > 0$ is a constant independent of K, n and N?*

They gave an affirmative answer for the class of 1-unconditional isotropic convex bodies.

Theorem 1.1.21. *Let K be a 1-unconditional isotropic convex body in $\mathbb{R}^n$. For every $N > n$ consider K_N as defined by (1.27) If $cn < N \leq e^{cn}$, then*

$$\mathbb{P}\left(L_{K_N} \leq C\right) \geq 1 - e^{c_1 n} \tag{1.28}$$

If $n < N \leq cn$, then

$$\mathbb{P}\left(L_{K_N} \leq C\right) \geq 1 - e^{-c_2 n/\log n}, \tag{1.29}$$

where c, c_1, c_2, C, are absolute constants.

The reason for the two different probabilities is due to the lack of suitable volume estimates in the case $n < N \leq cn$. One of the results in Chapter 4 improves on volume estimates used in the proof of the latter theorem. Volume estimates for K_N involve various subtleties and have direct implications for the boundedness of L_K, which we discuss further in the next section. Before doing so, we mention also that isotropic constants of arbitrary (non-random) polytopes were studied in [2]; in particular, the following theorem is proved.

Theorem 1.1.22. *Let $K \subset \mathbb{R}^n$ be an n-dimensional polytope with N vertices. Then*

$$L_K \leq C\sqrt{\frac{N}{n}}. \tag{1.30}$$

The volume of random polytopes in isotropic convex bodies

If $K \subset \mathbb{R}^n$ is an isotropic convex body and K_N is the random polytope defined in (1.27), then the precise dependence of $\mathbb{E}\operatorname{vol}(K_N)$ on the dimension n, the number of points N and the isotropic constant L_K is closely related to the boundedness of L_K.

In a series of papers ([17], [15], [11], and [10]), A. Giannopoulos and various coauthors studied how the volume of K_N depends on n, N and L_K. We summarize their results in the following theorem.

Theorem 1.1.23. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body and let $n < N \leq e^n$. Let K_N be the random polytope defined by (1.27). Then*

$$\frac{c_1\sqrt{\log(2N/n)}}{\sqrt{n}} \leq \mathbb{E}\operatorname{vol}\left(K_N\right)^{1/n} \leq \frac{C_1\sqrt{\log(2N/n)}}{\sqrt{n}} L_K. \tag{1.31}$$

The $\sqrt{\log(2N/n)}\sqrt{n}$ factor appears in both the upper and lower bound. The crucial difference is that the isotropic constant L_K does not appear in the lower bound. Since $K_N \subset K$ and $\mathrm{vol}\,(K) = 1$, lower estimates for $\mathbb{E}\,\mathrm{vol}\,(K_N)^{1/n}$ in terms of L_K immediately lead to upper bounds for L_K. (In fact, for $N = n+1$, it is also well-known that the lower bound includes L_K, which we discuss further in Chapter 4).

The conjectured bound (equivalent to Conjecture 1.1.6) is the following.

Conjecture 1.1.24. *There is an absolute constant* $c_1 > 0$ *such that*

$$\mathbb{E}\,\mathrm{vol}\,(K_N)^{1/n} \geq \min\left\{c_1 \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} L_K, 1\right\}. \tag{1.32}$$

Why does L_K disappear in the lower bound in Theorem 1.1.23? We will discuss one of the key ingredients. If K is a convex body of volume one and $p > 0$, set

$$\mathbb{E}_p(K,N) = \int_K \cdots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{x_1,\ldots,x_N\})^p \, dx_N \ldots dx_1. \tag{1.33}$$

Giannopoulos and Tsolomitis [17] (extending a result of Groemer [21]) proved that for each $p > 0$,

$$\mathbb{E}_p(K,N) \geq \mathbb{E}_p(\overline{B_2^n}, N), \tag{1.34}$$

where $\overline{B_2^n}$ is the Euclidean ball of volume one. The key element in such arguments is *Steiner symmetrization*, which is discussed in the Appendix. Thus by reducing this to the case of the Euclidean ball, one loses the dependence on the isotropic constant.

Chapter 4 is devoted to various related volume estimates.

REFERENCES

[1] D. Alonso-Gutiérrez. On the isotropy constant of random convex sets. *Proc. Amer. Math. Soc.*, 136(9):3293–3300, 2008.

[2] D. Alonso-Gutierrez, J. Bastero, J. Bernues, and P. Wolff. On the isotropy constant of projections of polytopes. *J. Funct. Anal.*, 258:1452–1465, 2010.

[3] K. Ball. Logarithmically concave functions and sections of convex sets in $\mathbf{R}^n$. *Studia Math.*, 88(1):69–84, 1988.

[4] K. Ball. Normed spaces with a weak-Gordon-Lewis property. In *Functional analysis (Austin, TX, 1987/1989)*, volume 1470 of *Lecture Notes in Math.*, pages 36–47. Springer, Berlin, 1991.

[5] S. G. Bobkov and F. L. Nazarov. Large deviations of typical linear functionals on a convex body with unconditional basis. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 3–13. Birkhäuser, Basel, 2003.

[6] S. G. Bobkov and F. L. Nazarov. On convex bodies and log-concave probability measures with unconditional basis. In *Geometric Aspects of Functional Analysis*, volume 1807 of *Lecture Notes in Math.*, pages 53–69. Springer, Berlin, 2003.

[7] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. *Amer. J. Math.*, 108(6):1467–1476, 1986.

[8] J. Bourgain. On the distribution of polynomials on high-dimensional convex sets. In *Geometric Aspects of Functional Analysis (1989–90)*, volume 1469 of *Lecture Notes in Math.*, pages 127–137. Springer, Berlin, 1991.

[9] J. Bourgain. On the isotropy-constant problem for ψ_2-bodies. In *Geometric Aspects of Functional Analysis*, volume 1807 of *Lecture Notes in Math.*, pages 114–121. Springer, Berlin, 2003.

[10] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. *J. Funct. Anal.*, 257(9):2820–2839, 2009.

[11] N. Dafnis, O. Guédon, and A. Giannopoulos. On the isotropic constant of random polytopes. *Adv. Geom.* to appear.

[12] N. Dafnis and G. Paouris. Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture. preprint, available at `http://www.math.tamu.edu/∼grigoris`.

[13] M. E. Dyer, Z. Füredi, and C. McDiarmid. Volumes spanned by random points in the hypercube. *Random Structures Algorithms*, 3(1):91–106, 1992.

[14] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405 (electronic), 2002.

[15] A. Giannopoulos, M. Hartzoulaki, and A. Tsolomitis. Random points in isotropic unconditional convex bodies. *J. London Math. Soc. (2)*, 72(3):779–798, 2005.

[16] A. Giannopoulos, A. Pajor, and G. Paouris. A note on subgaussian estimates for linear functionals on convex bodies. *Proc. Amer. Math. Soc.*, 135(8):2599–2606 (electronic), 2007.

[17] A. Giannopoulos and A. Tsolomitis. Volume radius of a random polytope in a convex body. *Math. Proc. Cambridge Philos. Soc.*, 134(1):13–21, 2003.

[18] A. A. Giannopoulos. Notes on isotropic convex bodies, 2003. Available at `http://users.uoa.gr/~apgiannop/`.

[19] A. A. Giannopoulos and V. D. Milman. Euclidean structure in finite dimensional normed spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 707–779. North-Holland, Amsterdam, 2001.

[20] E. D. Gluskin. The diameter of the Minkowski compactum is roughly equal to n. *Funktsional. Anal. i Prilozhen.*, 15(1):72–73, 1981.

[21] H. Groemer. On the mean value of the volume of a random polytope in a convex set. *Arch. Math. (Basel)*, 25:86–90, 1974.

[22] M. Junge. Hyperplane conjecture for quotient spaces of L_p. *Forum Math.*, 6(5):617–635, 1994.

[23] B. Klartag. A central limit theorem for convex sets. *Invent. Math.*, 168(1):91–131, 2007.

[24] B. Klartag. Power-law estimates for the central limit theorem for convex sets. *J. Funct. Anal.*, 245(1):284–310, 2007.

[25] B. Klartag. Uniform almost sub-Gaussian estimates for linear functionals on convex sets. *Algebra i Analiz*, 19(1):109–148, 2007.

[26] B. Klartag and G. Kozma. On the hyperplane conjecture for random convex sets. *Israel J. Math.*, 170:253–268, 2009.

[27] H. König, M. Meyer, and A. Pajor. The isotropy constants of the Schatten classes are bounded. *Math. Ann.*, 312(4):773–783, 1998.

[28] M. Ledoux. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[29] P. Mankiewicz and N. Tomczak-Jaegermann. Quotients of finite-dimensional Banach spaces; random phenomena. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1201–1246. North-Holland, Amsterdam, 2003.

[30] E. Milman. Dual mixed volumes and the slicing problem. *Adv. Math.*, 207(2):566–598, 2006.

[31] V. Milman. Surprising geometric phenomena in high-dimensional convexity theory. In *European Congress of Mathematics, Vol. II (Budapest, 1996)*, volume 169 of *Progr. Math.*, pages 73–91.

[32] V. D. Milman. A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.*, 5(4):28–37, 1971.

[33] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric Aspects of Functional Analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.

[34] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[35] G. Paouris. Ψ_2-estimates for linear functionals on zonoids. In *Geometric aspects of functional analysis*, volume 1807 of *Lecture Notes in Math.*, pages 211–222. Springer, Berlin, 2003.

[36] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.

[37] P. Pivovarov. On determinants and the volume of random polytopes in isotropic convex bodies. *Geom. Dedicata*. 14 pages, to appear.

[38] P. Pivovarov. On the volume of caps and bounding the mean-width of an isotropic convex body. *Math. Proc. Cambridge Philos. Soc.* 15 pages, to appear.

[39] P. Pivovarov. Volume thresholds for Gaussian and spherical random polytopes and their duals. *Studia Math.*, 183(1):15–34, 2007.

CHAPTER 2

Volume threshold problems for random polytopes[1]

2.1 Introduction

A remarkable result due to M.E. Dyer, Z. Füredi, and C. McDiarmid gives a threshold for the expected volume of random polytopes generated by vertices of the cube $[-1,1]^n$. Specifically, let μ be the uniform probability measure on $\{-1,1\}$ and let $Z=(z_1,\ldots,z_n)$ be a random vector whose coordinates are independent and identically distributed according to μ. Consider $N=N(n)$ independent random vectors $Z_1,\ldots,Z_N$, each with the same distribution as Z, and form their convex hull $C_N = \mathrm{conv}\{Z_1,\ldots,Z_N\}$. In [7], a threshold value for N is established at which C_N captures significant volume in the following sense: for each $\varepsilon>0$, we have

$$\frac{\mathbb{E}[\mathrm{vol}\,(C_N)]}{\mathrm{vol}\,([-1,1]^n)} \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } N \leq (2/\sqrt{e}-\varepsilon)^n, \\ 1 & \text{if } N \geq (2/\sqrt{e}+\varepsilon)^n. \end{cases} \tag{2.1}$$

The corresponding result for the case when μ is uniform on $[-1,1]$ is also proved. Their method has since been significantly generalized; namely, D. Gatzouras and A. Giannopoulos, in [8], obtain analogous results for a large class of compactly supported probability measures μ on $\mathbb{R}$.

[1] A version of this chapter has been published. P. Pivovarov. Volume thresholds for Gaussian and spherical random polytopes and their duals. Studia Math., 183(1):1534, 2007.

We consider similar problems for Gaussian random polytopes and polytopes generated by random points on the Euclidean sphere.

In the Gaussian case, let $\gamma_1, \dots, \gamma_n$ be independent $\mathrm{N}(0,1)$ random variables and let $g = (\gamma_1, \dots, \gamma_n)$. Consider $N = N(n)$ independent copies of g, say, $g_1, \dots, g_N$ and form their convex hull

$$K_N := \operatorname{conv}\{g_1, \dots, g_N\}. \tag{2.2}$$

For instance, in $\mathbb{R}^2$, some samples of K_N are shown in Figure 2.1.

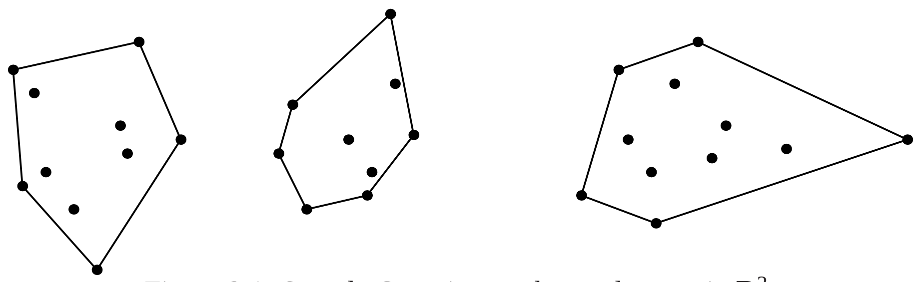


Figure 2.1: Sample Gaussian random polytopes in $\mathbb{R}^2$.

The Gaussian measure is not compactly supported and thus if one is to consider the analogous threshold problem, the following question arises.

Question 2.1.1. *What does it mean for K_N to capture significant volume?*

There are are a number of ways of answering the latter question. For instance, the Gaussian measure is rotationally-invariant and one suitable answer may be to intersect K_N with a Euclidean ball and study the proportion of volume lying inside the ball, as suggested in the figure.

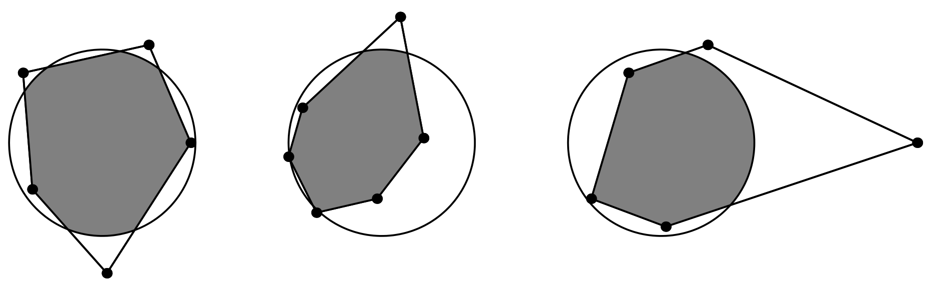


Figure 2.2: Intersection with a Euclidean ball.

On the other hand, the Gaussian measure is a product and thus another fitting answer is to intersect K_N with a cube. Of course, there are many other answers - intersecting with other convex bodies; the *Gaussian measure* of K_N is equally natural. In fact, using a recent concentration result of B. Klartag about log-concave measures, we give a rather satisfactory answer to this question. Our considerations include intersecting K_N with any convex body in a suitable position. The statement of the main result is §2.2.

Gaussian polytopes can also be generated by random facets, i.e., if the g_i's are Gaussian random vectors as above, then we can consider

$$K_N' := \{x \in \mathbb{R}^n : \langle g_i, x\rangle \leq 1 \text{ for each } i = 1, \ldots, N\}. \tag{2.3}$$

Such polytopes exhibit similar threshold phenomena, corresponding in a natural way to those for K_N. In this case, our arguments do not invoke duality and use only elementary properties of the random vectors involved.

We also study the analogous problem for random polytopes generated by points on the Euclidean sphere. The threshold for N in the spherical case is super-exponential in the dimension n, which corresponds to known results about approximation of the ball by polytopes [3]. The results in [7] are exponential in n and the authors of [8] considered only measures for which there is an exponential threshold in n.

We follow the same approach as that of Dyer, Füredi and McDiarmid. The tools developed in [7] have a simple realization in our setting; this simplicity nicely illustrates the geometry behind the method. The lack of independence of coordinates in the spherical case presents no difficulty as the argument depends more on geometric considerations than on probabilistic techniques such as the theory of large deviations, as in [7] and [8].

Lastly, a few words on notation. We shall denote the canonical Euclidean norm on $\mathbb{R}^n$ by $|\cdot|$; B_2^n the Euclidean ball; Lebesgue measure on $\mathbb{R}^n$ by $\mathrm{vol}(\cdot)$; the unit sphere S^{n-1}.

Isotropic log-concave measures

We start with a few basic facts about log-concave measures.

A Borel probability measure ν on $\mathbb{R}^n$ is said to be *log-concave* if for any $\lambda \in [0,1]$,

$$\nu(\lambda A + (1-\lambda)B) \geq \nu(A)^{\lambda}\nu(B)^{1-\lambda} \tag{2.4}$$

for all compact $A, B \subset \mathbb{R}^n$. Here $\lambda A + (1-\lambda)B := \{\lambda a + (1-\lambda)b : a \in A, b \in B\}$. Log-concave measures have a surprisingly simple characterization.

Definition 2.1.2. A function $f : \mathbb{R}^n \to [0,\infty)$ is *log-concave* if for any $\lambda \in [0,1]$ and and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}.$$

The next proposition is due to Borell ([5]).

Proposition 2.1.3. *Let ν be a Borel probability measure on $\mathbb{R}^n$ and suppose that ν is not supported on any proper affine subspace. Then ν is log-concave if and only if there is a log-concave function $g : \mathbb{R}^n \to [0,\infty)$ such that*

$$\nu(B) = \int_B g(x)dx.$$

Basic examples of log-concave measures include the uniform distribution on a convex body K of $\text{vol}(K) = 1$ and also standard n-dimensional Gaussian measure $N(0, I_n)$, i.e., the density of which is

$$f(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}.$$

A probability measure ν on $\mathbb{R}^n$ is *isotropic* if its center of mass is the origin, i.e.,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle d\nu(x) = 0, \qquad \text{for each } \theta \in S^{n-1}, \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\nu(x) = 1 \qquad \text{for each } \theta \in S^{n-1}. \tag{2.6}$$

Note the difference in normalization in (2.6) and that for isotropic convex bodies (cf. 1.9). We are assuming the variance of each functional is 1. This is the standard normalization for log-concave measures.

We will make essential use of the following theorem due to B. Klartag (the analogue of Theorem 1.1.8 for log-concave measures).

Theorem 2.1.4. *Let ν be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then for all $0 \leq \varepsilon \leq 1$,*

$$\nu\{x \in \mathbb{R}^n : ||x| - \sqrt{n}| \geq \varepsilon\sqrt{n}\} \leq C \exp\left(-c\varepsilon^{\tau} n^{\kappa}\right), \tag{2.7}$$

where τ, κ, c and C are positive absolute constants.

2.2 Threshold phenomena for Gaussian random polytopes

As in the introduction, let $\gamma_1, \ldots, \gamma_n$ be independent Gaussian $\mathrm{N}(0,1)$ random variables. Denote the standard unit vector basis in $\mathbb{R}^n$ by $e_1, \ldots, e_n$. Consider the random vector $g = \sum_{i=1}^n \gamma_i e_i$; then g satisfies $\mathbb{E}|g| \approx \sqrt{n}$. Let $N = N(n) > n$ be an integer and consider N independent random vectors $g_1, \ldots, g_N$, each with the same distribution as g.

In this section, we state and prove the main theorems for Gaussian polytopes K_N (generated by random vertices as in (2.2)) and K'_N (generated by random facets as in (2.3)).

To state the main theorem, we need some relevant notation.

Denote by Φ the cumulative distribution function of a standard $\mathrm{N}(0,1)$ random variable, i.e.,

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx \quad (a \in \mathbb{R}), \tag{2.8}$$

as in the figure.

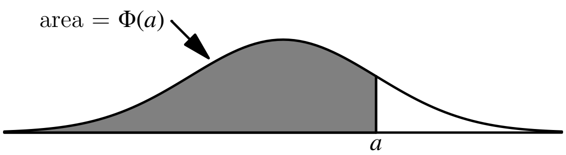


Figure 2.3: Cumulative distribution function of a standard Gaussian.

For simplicity of notation, we denote the reciprocal of $1 - \Phi$ by Ψ, i.e.,

$$\Psi(a) := \frac{1}{1 - \Phi(a)}. \tag{2.9}$$

Theorem 2.2.1. *Let K_N be the random polytope defined in (2.2). Let ν be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $0 < \varepsilon < 1$. Then as $n \to \infty$,*

$$\mathbb{E}\nu(K_N) \longrightarrow \begin{cases} 0 & \text{if } N \le \Psi((1-\varepsilon)\sqrt{n}), \\ 1 & \text{if } N \ge \Psi((1+\varepsilon)\sqrt{n}). \end{cases} \tag{2.10}$$

Theorem 2.2.2. *Let K'_N be the random polytope defined in (2.3). Let ν be an isotropic* *log-concave probability measure on* $\mathbb{R}^n$. *Let* $0 < \varepsilon < 1$. *Then as* $n \to \infty$,

$$\mathbb{E}\nu(nK_N') \longrightarrow \begin{cases} 1 & \text{if } n < N \leq \Psi((1-\varepsilon)\sqrt{n}) \\ 0 & \text{if } N \geq \Psi((1+\varepsilon)\sqrt{n}). \end{cases} \tag{2.11}$$

The proof of Theorem 2.2.1 is in §2.2.1; Theorem 2.2.2 in §2.2.2.

2.2.1 Random vertices

Preparatory results

In this section we define some of the tools that are used in [7]; see also [2] for an overview of related concepts and their use in the study of random polytopes.

Let μ be a probability measure on $\mathbb{R}^n$ (or S^{n-1}) and suppose that X is a random vector distributed according to μ, (i.e., $\mathbb{P}(X \in A) = \mu(A)$ for measurable sets A). For $x \in \mathbb{R}^n$, set

$$q_\mu(x) := \inf\{\mathbb{P}(X \in H) : H \text{ is a closed halfspace containing } x\}.$$

Let $X_1, \ldots, X_N$ be independent random vectors distributed according to μ and set

$$\mathcal{K}_N := \operatorname{conv}\{X_1, \ldots, X_N\}.$$

Lemma 2.2.3. *Let* $x \in \mathbb{R}^n$. *Then* $\mathbb{P}(x \in \mathcal{K}_N) \leq N q_\mu(x)$

Proof. Let H be a halfspace containing x. If none of $X_1, \ldots, X_N$ belong to H then $\mathcal{K}_N$ lies in $\mathbb{R}^n \backslash H$ and hence $x \notin \mathcal{K}_N$. Consequently,

$$\{x \in \mathcal{K}_N\} \subset \bigcup_{i=1}^{N} \{X_i \in H\}.$$

See Figure 2.4. Since H was an arbitrary halfspace containing x, the result follows. □

Before continuing with tools for the Gaussian case, we mention an important connection when μ is the uniform measure on a convex body.

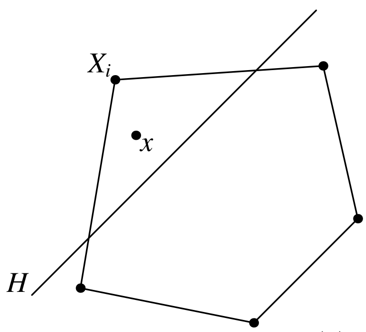


Figure 2.4: Estimating $q_\mu(x)$

Remark 2.2.4. Suppose $\mu = \mu_K$ is the uniform measure on a convex body $K \subset \mathbb{R}^n$ with $\mathrm{vol}(K) = 1$ and write $q = q_\mu$. Then the set

$$K_\delta := \{x \in K : q(x) \geq \delta\}. \tag{2.12}$$

is called the *floating body* of K. The latter set plays an important role in asymptotic results concerning $\mathbb{E}\,\mathrm{vol}(\mathcal{K}_N)$ when $N \to \infty$ and n is fixed; in particular, a well-known result from [4] is

$$c_1 \,\mathrm{vol}\left(K_{1/N}\right) \leq \mathbb{E}\,\mathrm{vol}(\mathcal{K}_N) \leq C_1(n)\,\mathrm{vol}\left(K_{1/N}\right) \tag{2.13}$$

for all $N \geq C_2(n)$ (here c_1 is an absolute constant and $C_1(n), C_2(n)$ depend on n); see also the survey [2]. For recent observations concerning (2.13), in particular, the dependence of $C_2(n)$ on n, see [6, Remark 2.4], which makes use of the results in [10].

Gaussian setting

In this section, we assume $g_1, \ldots, g_N$ are Gaussian random vectors and $K_N = \mathrm{conv}\{g_1, \ldots, g_N\}$.

For the Gaussian measure we can actually calculate $q(x)$. For a closed set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, let $d(x,A) := \inf\{|x-a| : a \in A\}$.

Lemma 2.2.5. *(a) If H is a halfspace with $d := d(0,H) > 0$, then $\mathbb{P}(g \in H) = 1 - \Phi(d)$.*

(b) For each $x \in \mathbb{R}^n$, we have $q(x) = 1 - \Phi(|x|)$.

Proof. (a) The density of g with respect to Lebesgue measure is $f(x) := (2\pi)^{-n/2} e^{-|x|^2/2}$.

By rotational invariance we may assume that $H := \{x \in \mathbb{R}^n : x_1 \geq d\}$. Consequently,

$$\mathbb{P}(g \in H) = \int_H f(x)dx = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x_n^2/2} dx_n\right)^{n-1} \left(\frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-x_1^2/2} dx_1\right) = 1 - \Phi(d).$$

(b) If $x = 0$ then $q(0) = 1/2 = 1 - \Phi(0)$. Suppose that $x \neq 0$. Let $H(x)$ be the halfspace bounded by the tangent hyperplane to $|x|B_2^n$ at x and which does not contain 0. Then by part (a) we have

$$1 - \Phi(|x|) = \mathbb{P}(g \in H(x)) \geq q(x).$$

Conversely, let H be any halfspace containing x. Set $d = d(0, H)$. If $d = 0$ then $\mathbb{P}(g \in H) \geq 1/2 \geq 1 - \Phi(|x|)$. If $d > 0$ then

$$\mathbb{P}(g \in H) = 1 - \Phi(d) \geq 1 - \Phi(|x|)$$

since $d \leq |x|$. It follows that $q(x) \geq 1 - \Phi(|x|)$. □

A consequence of Lemma 2.2.5 is the following simple observation:

$$RB_2^n = \{x \in \mathbb{R}^n : q(x) \geq 1 - \Phi(R)\} \tag{2.14}$$

for any $R > 0$. Equality (2.14) allows us to use an argument from [7] to establish the next lemma; we include the proof for completeness.

Lemma 2.2.6. *Let $R > 0$. Then*

$$\mathbb{P}(RB_2^n \subset K_N) \geq 1 - 2\binom{N}{n}(\Phi(R))^{N-n}. \tag{2.15}$$

Proof. For any $J \subset \{1, \ldots, N\}$ with $|J| = n$, the set $\{g_j\}_{j \in J}$ is linearly (hence affinely) independent almost surely. In particular, the affine hull of $\{g_j\}_{j \in J}$ is a hyperplane almost surely. Let us now define the event E_J: one of the two halfspaces H determined by $\{g_j\}_{j \in J}$ contains K_N and $\mathbb{P}(g \notin H) \geq 1 - \Phi(R)$.

Suppose $x \in RB_2^n \backslash K_N$. Then there exists $J \subset \{1, \ldots, N\}$ with $|J| = n$ such that one of the two half spaces H determined by $\{g_j\}_{j \in J}$ contains K_N but excludes x. But then x belongs to the complimentary halfspace $\widetilde{H}$ and so

$$\mathbb{P}(g \notin H) \geq q(x) \geq 1 - \Phi(R)$$

since $|x| \leq R$. It follows that

$$\{RB_2^n \not\subset K_N\} \subset \bigcup_{\substack{J\subset\{1,\ldots,n\} \\ |J|=n}} E_J.$$

Thus if we set $D = \{1,\ldots,n\}$ we have

$$\mathbb{P}(RB_2^n \not\subset K_N) \leq \binom{N}{n} \mathbb{P}(E_D).$$

Now let us estimate $\mathbb{P}(E_D)$ by conditioning on $g_1,\ldots,g_n$. Let H and $\widetilde{H}$ denote the two halfspaces generated by $g_1,\ldots,g_n$. If $\mathbb{P}(g \notin H) \geq 1-\Phi(R)$ then

$$\mathbb{P}(g_j \in H : j = n+1,\ldots,N) \leq (\Phi(R))^{N-n}$$

and similarly for $\widetilde{H}$. It follows that

$$\mathbb{P}(E_D | g_1,\ldots,g_n) \leq 2(\Phi(R))^{N-n}.$$

Now since

$$\mathbb{P}(E_D) = \mathbb{E}(1_{E_D}) = \mathbb{E}(\mathbb{E}(1_{E_D}|g_1,\ldots,g_n)) = \mathbb{E}(\mathbb{P}(E_D|g_1,\ldots,g_n))$$

we obtain $\mathbb{P}(E_D) \leq 2(\Phi(R))^{N-n}$ and hence

$$\mathbb{P}(RB_2^n \not\subset K_N) \leq 2\binom{N}{n}(\Phi(R))^{N-n}.$$

□

Lemma 2.2.7. *Let ν be a Borel probability measure on $\mathbb{R}^n$ and let B be a Borel subset of $\mathbb{R}^n$. Then*

$$\nu(B)\mathbb{P}(B \subset K_N) \leq \mathbb{E}\nu(K_N \cap B) \leq N\nu(B)\sup_{x\in B}(1-\Phi(|x|)).$$

Proof. Note that

$$\mathbb{E}\,\mathrm{vol}(K_N \cap B) = \mathbb{E}\int_B 1_{\{x\in K_N\}}dx = \int_B \mathbb{P}(x \in K_N)\,dx.$$

The upper bound follows from Lemma 2.2.3 and Lemma 2.2.5 and the lower bound is obvious. □

Estimates for Gaussian tail-decay

As in §2.2, we shall use the following standard notation:

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx \quad (a \in \mathbb{R}). \tag{2.16}$$

Note that for $a > -1$ we have the approximation (see [13])

$$\frac{2}{a + (a^2+4)^{1/2}} \leq \sqrt{2\pi} \exp(a^2/2)(1 - \Phi(a)) \leq \frac{4}{3a + (a^2+8)^{1/2}}. \tag{2.17}$$

Recall that $\Psi(a) = \frac{1}{1-\Phi(a)}$. The following lemma will be useful in subsequent calculations.

Lemma 2.2.8. *Let $a \leq b$. Then*

$$\Psi(a)(1 - \Phi(b)) \leq 2 \exp(a^2/2 - b^2/2) \tag{2.18}$$

and

$$\Psi(b)(1 - \Phi(a)) \geq (1/2) \exp(b^2/2 - a^2/2). \tag{2.19}$$

Proof. Applying (2.17), we have

$$\Psi(a)(1 - \Phi(b)) \leq \frac{2(a + (a^2+4)^{1/2})}{3b + (b^2+8)^{1/2}} \exp(a^2/2 - b^2/2) \leq 2 \exp(a^2/2 - b^2/2)$$

and

$$\Psi(b)(1 - \Phi(a)) \geq \frac{2(3b + (b^2+8)^{1/2})}{4(a + (a^2+4)^{1/2})} \exp(b^2/2 - a^2/2) \geq (1/2) \exp(b^2/2 - a^2/2).$$

□

Proof of Theorem 2.2.1. Let $t_n = (1 - \varepsilon/2)\sqrt{n}$ and let $R_n = (1 + \varepsilon/2)\sqrt{n}$. Set

$$B = R_n B_2^n \backslash t_n B_2^n.$$

Assume first that $N \le \Psi((1-\varepsilon)\sqrt{n}))$. Observe that

$$\mathbb{E}\nu(K_N) \le \nu(t_n B_2^n) + \mathbb{E}\nu(K_N \cap B) + \nu(\mathbb{R}^n \backslash R_n B_2^n).$$

By Theorem 2.1.4, it suffices to show that

$$\mathbb{E}\nu(K_N \cap B) \to 0$$

as $n \to \infty$. To this end, apply Lemma 2.2.7 and Lemma 2.2.3 to obtain

$$\begin{aligned} \mathbb{E}\nu(K_N \cap B) &\le N\nu(B) \sup_{x\in B}(1-\Phi(x)) \\ &\le \Psi((1-\varepsilon)\sqrt{n})(1-\Phi((1-(\varepsilon/2))\sqrt{n}) \\ &\le 2\exp\left((1-\varepsilon)^2 n/2 - (1-(\varepsilon/2))^2 n/2)\right), \end{aligned}$$

where we used Lemma 2.2.8 to obtain the last inequality. The latter term tends to 0 as $n \to \infty$.

Assume now that $N \ge \Psi((1+\varepsilon)\sqrt{n})$. Lemma 2.2.6 implies that

$$\begin{aligned} \mathbb{P}(R_n B_2^n \not\subset K_N) &\le 2\binom{N}{n}(\Phi(R_n))^{N-n} \qquad (2.20) \\ &\le 2(eN/n)^n \exp((N-n)\ln\Phi(R_n)) \\ &= 2\exp\Big(n\ln(eN/n) + (N-n)\ln\Phi(R_n)\Big). \end{aligned}$$

Note that $1/2 < \Phi(R) < 1$ and hence for $n > 2e$ the latter expression is less than

$$2\exp\Big(n\ln N + N\ln\Phi(R_n)\Big) \le 2\exp\Big(n\ln N - N(1-\Phi(R_n))\Big), \qquad (2.21)$$

where we have used the estimate $\ln x \le x - 1$.

For convenience of notation, set $r_n := (1+\varepsilon)\sqrt{n}$. Without loss of generality we may assume that $N = \lceil\Psi(r_n)\rceil$, where $\lceil x\rceil$ denotes the smallest integer larger than x. Appealing to (2.17) yields

$$\begin{aligned} n\ln N &\le n\ln\Big((\sqrt{2\pi}/2)(r_n + (r_n^2+4)^{1/2})\exp(r_n^2/2)\Big) \\ &\le n\ln(\sqrt{2\pi}(r_n+1)) + nr_n^2/2 \\ &\le nr_n^2 \end{aligned}$$

provided that

$$\ln(\sqrt{2\pi}(r_n+1)) \leq r_n^2/2. \tag{2.22}$$

Applying Lemma 2.2.8 gives

$$N(1-\Phi(R_n)) \quad \geq \quad (1/2)\exp((r_n^2-R_n^2)/2).$$

Thus if n satisfies (2.22) and $n > 2e$, we can apply Lemma 2.2.7 to obtain

$$\begin{aligned}
\mathbb{E}\nu(K_N) &\geq \mathbb{P}(R_nB_2^n \subset K_N)\,\nu(R_nB_2^n) \qquad (2.23)\\
&\geq \left(1-2\exp\left(nr_n^2-(1/2)\exp\left(r_n^2/2-R_n^2/2\right)\right)\right)\nu(R_nB_2^n)\\
&= \left(1-2\exp\left((1+\varepsilon)^2n^2-(1/2)\exp\left((1+\varepsilon)^2n/2-n/2\right)\right)\right)\nu(R_nB_2^n) \qquad (2.24)
\end{aligned}$$

The latter term tends to 1 as $n \to \infty$, which completes the proof. □

2.2.2 Random facets

In this section we prove Theorem 2.2.2, i.e., the threshold result for $\nu(nK_N')$, where

$$K_N' = \{x \in \mathbb{R}^n : \langle g_i, x\rangle \leq 1 \text{ for each } i = 1,\dots,N\}.$$

Lemma 2.2.9. *For each* $x \in \mathbb{R}^n\backslash\{0\}$ *we have*

$$\mathbb{P}\left(x \in K_N'\right) = \left(\Phi(1/|x|)\right)^N.$$

Proof. By independence and rotational invariance of the g_i's, we have

$$\begin{aligned}
\mathbb{P}\left(x \in K_N'\right) &= \mathbb{P}(\langle x, g_i\rangle \leq 1 \text{ for each } i = 1,\dots,N)\\
&= (\mathbb{P}(\langle x, g_1\rangle \leq 1))^N\\
&= (\mathbb{P}(\gamma_1 \leq 1/|x|))^N \qquad (\gamma_1 \sim \mathrm{N}(0,1))\\
&= (\Phi(1/|x|))^N.
\end{aligned}$$

□

Lemma 2.2.10. *Suppose that* ν *is a Borel probability measure on* $\mathbb{R}^n$. *Let* $0 < t < R$ *and set*

$B = RB_2^n \backslash tB_2^n$. *Then for each n we have*

$$\nu(B)\Phi(n/R)^N \leq \mathbb{E}\nu((nK_N') \cap B) \leq \nu(B)\Phi(n/t)^N. \tag{2.25}$$

Proof. Argue as in the proof of Lemma 2.2.7 and apply Lemma 2.2.9. □

Remark 2.2.11. Let $a > 0$. The identity $\Phi(a)^N = \exp(N \ln \Phi(a))$ and the estimate

$$x - 1 - (x-1)^2 \leq \ln x \leq x - 1 \qquad x \in [1/2, 1] \tag{2.26}$$

imply that

$$\Phi(a)^N \geq \exp\left(-N(1-\Phi(a)) - N(1-\Phi(a))^2\right) \tag{2.27}$$

and

$$\Phi(a)^N \leq \exp\left(-N(1-\Phi(a))\right). \tag{2.28}$$

These estimates will be used in conjunction with Lemma 2.2.10.

Proof of Theorem 2.2.2. Let $s_n = (1-\varepsilon/2)^{-1}\sqrt{n}$ and let $r_n = (1+\varepsilon/2)^{-1}\sqrt{n}$. Set

$$B = s_n B_2^n \backslash r_n B_2^n$$

Assume first that $N \geq \Psi((1+\varepsilon)\sqrt{n})$. Observe that

$$\mathbb{E}\nu(nK_N') \leq \nu(r_n B_2^n) + \mathbb{E}\nu((nK_N') \cap B) + \nu(\mathbb{R}^n \backslash s_n B_2^n).$$

By Theorem 2.1.4, it suffices to show that

$$\mathbb{E}\nu((nK_N') \cap B) \to 0$$

as $n \to \infty$.

By Lemma 2.2.10,

$$\begin{aligned} \mathbb{E}\nu((nK_N') \cap B) &\leq \nu(B)\Phi(n/r_n)^N & (2.29)\\ &\leq \nu(B)\exp\left(-N(1-\Phi(n/r_n))\right). & (2.30) \end{aligned}$$

Without loss of generality, assume that $N = \lceil \Psi((1+\varepsilon)\sqrt{n}) \rceil$. By Lemma 2.2.8,

$$N(1-\Phi(n/r_n)) \quad \geq \quad (1/2)\exp\left((1+\varepsilon)^2 n/2 - (1+\varepsilon/2)^2 n/2\right) \tag{2.31}$$

The latter term tends to ∞ as $n \to \infty$ and hence $\mathbb{E}\nu((nK'_N) \cap B) \to 0$ as $n \to \infty$.

Assume now that $N \leq \Psi((1-\varepsilon)\sqrt{n})$. Lemma 2.2.8 implies that

$$N(1-\Phi(n/s_n)) \leq 2\exp\Big((1-\varepsilon)^2 n/2 - (1-\varepsilon/2)^2 n/2\Big) \to 0, \tag{2.32}$$

as $n \to \infty$ and hence

$$N(1-\Phi(n/s_n))^2 \to 0, \tag{2.33}$$

as $n \to \infty$.

Therefore, invoking Lemma 2.2.10 and (2.27), we get

$$\begin{aligned} \mathbb{E}\nu((nK'_N) \cap B) \quad &\geq \quad \nu(B)(\Phi(n/s_n))^N \\ &\geq \quad \nu(B)\exp\left(-N(1-\Phi(n/s_n)) - N(1-\Phi(n/s_n))^2\right). \end{aligned}$$

The latter term tends to 1 as $n \to \infty$ (2.32) and (2.33). This completes the proof. $\square$

2.2.3 Intersecting with Euclidean balls

As mentioned in the introduction, the results of the previous section generalize those of [11]. In this section, we state two results from the latter article. The proofs are omitted because they are similar to those of the last section (and the complete proofs appear in [11]).

Theorem 2.2.12. *Let* $\kappa > 0$, $c > 0$ *and let* $0 < \varepsilon < c$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\,(K_N \cap cn^{\kappa} B_2^n)]}{\mathrm{vol}\,(cn^{\kappa} B_2^n)} \longrightarrow \begin{cases} 0 & \text{if } N \leq \Psi((c-\varepsilon)n^{\kappa}), \\ 1 & \text{if } N \geq \Psi((c+\varepsilon)n^{\kappa}). \end{cases} \tag{2.34}$$

Corollary 2.2.13. *Let* $\kappa > 0$, $c > 0$ *and let* $0 < \varepsilon < c$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\left(K'_N \cap (cn^{\kappa})^{-1} B_2^n\right)]}{\mathrm{vol}\,((cn^{\kappa})^{-1} B_2^n)} \longrightarrow \begin{cases} 1 & \text{if } n < N \leq \Psi((c-\varepsilon)n^{\kappa}), \\ 0 & \text{if } N \geq \Psi((c+\varepsilon)n^{\kappa}). \end{cases} \tag{2.35}$$

2.3 Threshold phenomena for random polytopes on the sphere

Let σ denote Haar measure on S^{n-1} and let $u_1, \ldots, u_N$ be independent random vectors distributed according to σ, where $N = N(n) > n$ and set

$$L_N := \operatorname{conv}\{u_1, \ldots, u_N\}.$$

Theorem 2.3.1. *Let* $0 < R < 1$ *and let* $0 < \varepsilon < 1$*. Then, as* $n \to \infty$*,*

$$\frac{\mathbb{E}[\operatorname{vol}(L_N \cap RB_2^n)]}{\operatorname{vol}(RB_2^n)} \longrightarrow \begin{cases} 0 & \text{if } N \leq \exp\Big((1-\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\Big), \\ 1 & \text{if } N \geq \exp\Big((1+\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\Big). \end{cases} \tag{2.36}$$

One can see that as $R \to 1$, more than exponentially many points are needed to capture the volume of the entire ball.

Theorem 2.3.2. *Let* $0 < \varepsilon < 1$*. Then, as* $n \to \infty$*,*

$$\frac{\mathbb{E}[\operatorname{vol}(L_N)]}{\operatorname{vol}(B_2^n)} \longrightarrow \begin{cases} 0 & \text{if } N \leq \exp\Big((1-\varepsilon)(n-1)\ln\sqrt{n}\Big), \\ 1 & \text{if } N \geq \exp\Big((1+\varepsilon)(n-1)\ln\sqrt{n}\Big). \end{cases} \tag{2.37}$$

The proofs of the above theorems are in §2.3.1. The complementary results for polytopes generated by random facets are Theorems 2.3.12 and 2.3.13 in Section 2.3.2. See also the comments preceding Theorem 2.3.13.

Related results

Theorems 2.3.1 and 2.3.2 complement existing research on polytopes generated by points in the ball. In particular, in [3] the authors consider the following quantity

$$V(n,N) := \frac{\max\{\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_N\}) : x_1, \ldots, x_N \in B_2^n\}}{\operatorname{vol}(B_2^n)}$$

and derive upper and lower bounds for $V(n,N)$ when $N = N(n)$ is a function of n, specifically when N is linear, polynomial and exponential in n. They also show that for any $\alpha > 0$,

$$e^{-n^{1-2\alpha}} < V(n, n^{\alpha(n-1)}) < e^{-n^{1-2\alpha}/2}. \tag{2.38}$$

Thus half of Theorem 2.3.2 (i.e., when $N \leq n^{(1-\varepsilon)(n-1)/2}$ follows from the upper estimate in (2.38). However, we also give a short direct proof. We have chosen to use the exp notation because the main calculations in our proof take place "in the exponent." The configuration of points leading to the lower estimate in (2.38) is non-random. Theorem 2.3.2 shows that $N \geq n^{(1+\varepsilon)(n-1)/2}$ many random points are sufficient to have $V(n,N) \to 1$ as $n \to \infty$.

In [9], Müller derives an asymptotic formula for the difference $\mathrm{vol}(B_2^n) - \mathbb{E}\,\mathrm{vol}(L_N)$; the asymptotic treatment is for n fixed and $N \to \infty$. A major extension of Müller's result is in [12], to which we refer the reader for further results in this direction.

2.3.1 Random vertices

We will use the notation defined in Section 2.3. For $x \in B_2^n$, set

$$q(x) := \inf\{\mathbb{P}(u \in H) : H \text{ is a halfspace containing } x\}.$$

For $v \in S^{n-1}$ and $0 \leq R \leq 1$ set

$$C(R,v) := \{x \in S^{n-1} : \langle x, v\rangle \geq R\}. \tag{2.39}$$

Since we are interested in surface area, we will omit the reference to v and write $C(R) := C(R,v)$. Upper and lower estimates for $\sigma(C(R))$ are standard calculations. Such estimates, however, are not commonly stated in the form that best serves our purpose and thus we have included the proofs.

Area of spherical caps

Let α be the angle of the cap (2.39), by which we mean $\cos\alpha = R$. Fix $0 < t < \alpha$. Let H be a hyperplane at distance $\cos t$ from the origin. Then $B_2^n \cap H$ is an $(n-1)$-dimensional Euclidean ball of radius $\sin t$. Thus if we let σ denote Haar measure on S^{n-1} then

$$\sigma(C(R)) = \frac{\int_0^\alpha \mathrm{vol}_{n-2}(\partial(\sin t B_2^{n-1}))dt}{\int_0^\pi \mathrm{vol}_{n-2}(\partial\left(\sin t B_2^{n-1}\right))dt} = \frac{\int_0^\alpha \sin^{n-2} t\,dt}{\int_0^\pi \sin^{n-2} t\,dt}.$$

Let $I_n := \int_0^{\pi/2} \sin^n t dt$. Integrating by parts gives $I_n = \frac{n-1}{n} I_{n-2}$. The latter recurrence and Stirling's formula may be used to verify that $\sqrt{n} I_n \to \sqrt{\pi/2}$; in fact, for $n \geq 3$ we have

$$\frac{1}{2}\sqrt{\frac{2\pi}{n}} \leq \int_0^{\pi} \sin^{n-2} t dt \leq 2\sqrt{\frac{2\pi}{n}}. \tag{2.40}$$

Lemma 2.3.3. *Let $R \in (0,1)$. Then for each $n \geq 3$, we have*

$$\sigma(C(R)) \geq \frac{(1-R^2)^{(n-1)/2}}{6\sqrt{n}}. \tag{2.41}$$

Proof. Observe that

$$\int_0^{\alpha} \sin^{n-2} t dt \geq \int_0^{\alpha} \sin^{n-2} t \cos t dt = \frac{\sin^{n-1} \alpha}{n-1}.$$

Applying (2.40) and noting that $\sin\alpha = \sqrt{1-R^2}$ yields the result. □

Lemma 2.3.4. *Let $R \in \big(0,1\big)$. Then for each $n \geq 3$, we have*

$$\sigma(C(R)) \leq 3(1-R^2)^{(n-1)/2}. \tag{2.42}$$

Proof. Assume first that $1/\sqrt{2} < R < 1$. Using the inequality

$$1 - \cos t \leq 2\sin^2 t \cos t \qquad (t \in [0, \pi/4]), \tag{2.43}$$

and recalling that $R = \cos\alpha$, we have

$$\begin{aligned} \int_0^{\alpha} \sin^{n-2} t dt &= \int_0^{\alpha} \sin^{n-2} t \cos t dt + \int_0^{\alpha} \sin^{n-2} t (1-\cos t) dt \\ &\leq \int_0^{\alpha} \sin^{n-2} t \cos t dt + 2\int_0^{\alpha} \sin^n t \cos t dt \\ &= \frac{\sin^{n-1}\alpha}{n-1} + \frac{2\sin^{n+1}\alpha}{n+1} \\ &\leq \frac{3\sin^{n-1}\alpha}{n-1}. \end{aligned}$$

Applying again (2.40) and noting that $\sin\alpha = \sqrt{1-R^2}$ gives the result. Finally, for $0 < R \leq 1/\sqrt{2}$, one may argue, for example, as in the proof of [1, Lemma 2.2]. □

Analogues of tools used in the Gaussian case

Lemma 2.3.5. *a. Let H be a halfspace with $d := d(0,H)$. If $d \in (0,1]$ then $\mathbb{P}(u \in H) = \sigma(C(d))$.*

b. For $x \in B_2^n$, we have $q(x) = \sigma(C(|x|))$.

Lemma 2.3.6. *Let $R \in (0,1)$. For each $n > 2e$, the inclusion $RB_2^n \subset L_N$ holds with probability greater than $1 - 2\exp\big(n \ln N - N\sigma(C(R))\big)$.*

Proof. The proof is analogous to that of Lemma 2.2.6 and the estimates starting with (2.20) and ending with (2.21). □

Lemma 2.3.7. *Let B be a measurable subset of B_2^n. Then*

$$\mathrm{vol}\,(B)\,\mathbb{P}\,(B \subset L_N) \le \mathbb{E}\,\mathrm{vol}\,(L_N \cap B) \le N\,\mathrm{vol}\,(B) \sup_{x \in B} \sigma(C(|x|)).$$

Proof. Argue as in the proof of Lemma 2.2.7 and apply Lemma 2.2.3 and Lemma 2.3.5. □

We now have all the tools for proving Theorem 2.3.1.

Proof of Theorem 2.3.1. Assume first that

$$N \le \exp\Big((1-\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\Big) \tag{2.44}$$

Let $t := \sqrt{1-(1-R^2)^{(1-\varepsilon/2)}}$ so that $0 < t < R$. Set $B := RB_2^n \backslash tB_2^n$ and write

$$L_N \cap RB_2^n = \Big(L_N \cap tB_2^n\Big) \cup \Big(L_N \cap B\Big).$$

Since

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(tB_2^n)}{\mathrm{vol}\,(RB_2^n)} = \lim_{n\to\infty} (t/R)^n = 0, \tag{2.45}$$

we need only show that $\lim_{n\to\infty} \mathbb{E}\,\mathrm{vol}\,(L_N \cap B)\,/\,\mathrm{vol}\,(RB_2^n) = 0$.

By Lemma 2.3.4 and the fact that $(1-\varepsilon/2)\ln\big(\sqrt{1-R^2}\big) = \ln\big(\sqrt{1-t^2}\big)$, we have

$$\begin{aligned} \sigma(C(t)) &\le 3\exp\Big((n-1)\ln\big(\sqrt{1-t^2}\big)\Big) \\ &= 3\exp\Big(-(1-\varepsilon/2)(n-1)\ln\big(1/\sqrt{1-R^2}\big)\Big), \end{aligned} \tag{2.46}$$

for all $n \geq 3$.

Thus by Lemma 2.3.7 and (2.46), we have

$$\begin{aligned} \frac{\mathbb{E}\,\mathrm{vol}\,(L_N \cap B)}{\mathrm{vol}\,(RB_2^n)} &\leq N\sigma(C(t)) \\ &\leq 3\exp\left(-(\varepsilon/2)(n-1)\ln(1/\sqrt{1-R^2})\right) \longrightarrow 0, \end{aligned} \tag{2.47}$$

as $n \to \infty$.

Assume now that

$$N \geq \exp\left((1+\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\right). \tag{2.48}$$

Then by Lemma 2.3.7 we have

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L_N \cap RB_2^n)}{\mathrm{vol}\,(RB_2^n)} \geq \mathbb{P}\,(RB_2^n \subset L_N).$$

For convenience of notation, let us set $r = \sqrt{1-R^2}$. Without loss of generality, we may assume that $N = \lceil \exp((1+\varepsilon)(n-1)\ln(1/r)) \rceil$, where $\lceil x \rceil$ denotes the smallest integer larger than x. Lemma 2.3.3 implies that

$$\begin{aligned} N\sigma(C(R)) &\geq \exp\left((1+\varepsilon)(n-1)\ln(1/r) - (n-1)\ln(1/r) - \ln(6\sqrt{n})\right) \\ &\geq \exp\left((\varepsilon/2)(n-1)\ln\big(1/r\big)\right), \end{aligned}$$

for all n satisfying

$$\ln(6\sqrt{n}) \leq (\varepsilon/2)(n-1)\ln(1/r). \tag{2.49}$$

Thus if n satisfies (2.49) and $n > 2e$, Lemma 2.3.6 gives us

$$\mathbb{P}\,(RB_2^n \not\subset L_N) \leq 2\exp\Big(2n^2\ln\big(1/r\big) - \exp\big((\varepsilon/2)(n-1)\ln(1/r)\big)\Big) \longrightarrow 0, \tag{2.50}$$

as $n \to \infty$, which completes the proof of Theorem 2.3.1.

□

Remark 2.3.8. The rate of convergence in Theorem 2.3.1 can be obtained from lines (2.45), (2.47), and (2.50).

Remark 2.3.9. In Theorem 2.3.1, we may replace ε by ε_n where $(\varepsilon_n)_{n\geq 1} \subset (0,1)$ with $\varepsilon_n \to 0$ provided that (ε_n) satisfies (2.45), (2.47), (2.49) and (2.50). One may verify that $\varepsilon_n = n^{-\gamma}$, for any fixed $\gamma \in (0,1)$, serves this purpose.

Proof of Theorem 2.3.2. We shall use the following elementary fact:

$$\lim_{n\to\infty}(1-n^{-\beta})^{n/2} = \begin{cases} 0 & \text{if } 0<\beta<1, \\ 1 & \text{if } \beta>1. \end{cases} \tag{2.51}$$

Assume first that

$$N \le \exp\Big((1-\varepsilon)(n-1)\ln\sqrt{n}\Big). \tag{2.52}$$

Let $\beta := 1-\varepsilon/2$ and set $R_n := \sqrt{1-n^{-\beta}}$. Let $B := B_2^n \backslash R_n B_2^n$ and write

$$L_N = \Big(L_N \cap R_n B_2^n\Big) \cup \Big(L_N \cap B\Big).$$

By (2.51), we have

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(R_n B_2^n)}{\mathrm{vol}\,(B_2^n)} = \lim_{n\to\infty} R_n^n = 0, \tag{2.53}$$

and thus we need only show that $\lim_{n\to\infty} \mathbb{E}\,\mathrm{vol}\,(L_N\cap B)\,/\,\mathrm{vol}\,(B_2^n) = 0$.

By Lemma 2.3.4 and the fact that $\sqrt{1-R_n^2} = n^{-\beta/2}$, we obtain

$$\begin{aligned} \sigma(C(R_n)) &\le 3\exp\left((n-1)\ln\big(\sqrt{1-R_n^2}\big)\right) \\ &= 3\exp\big(-(1-\varepsilon/2)(n-1)\ln\sqrt{n}\big) \end{aligned} \tag{2.54}$$

for all $n \ge 3$.

Thus by Lemma 2.3.7 and (2.54), we have

$$\begin{aligned} \frac{\mathbb{E}\,\mathrm{vol}\,(L_N\cap B)}{\mathrm{vol}\,(B_2^n)} &\le N\sigma(C(R_n)) \\ &\le 3\exp\Big(-(\varepsilon/2)(n-1)\ln\sqrt{n}\Big) \longrightarrow 0, \end{aligned} \tag{2.55}$$

as $n\to\infty$.

Let us now assume that

$$N \ge \exp\Big((1+\varepsilon)(n-1)\ln\sqrt{n}\Big). \tag{2.56}$$

Let $\gamma = 1+\varepsilon/2$ and set $r_n := \sqrt{1-n^{-\gamma}}$. Applying Lemma 2.3.7, we get

$$\begin{aligned}\frac{\mathbb{E}\,\mathrm{vol}\,(L_N)}{\mathrm{vol}\,(B_2^n)} &\geq \frac{\mathbb{E}\,\mathrm{vol}\,(L_N \cap r_n B_2^n)}{\mathrm{vol}\,(B_2^n)} \\ &\geq r_n^n \cdot \mathbb{P}\,(r_n B_2^n \subset L_N)\,.\end{aligned}$$

Using (2.51), we have

$$\lim_{n\to\infty} r_n^n = 1, \tag{2.57}$$

and thus we need only prove that $\mathbb{P}\,(r_n B_2^n \subset L_N) \to 1$ as $n \to \infty$. Without loss of generality, we may assume that $N = \lceil \exp\left((1+\varepsilon)(n-1)\ln\sqrt{n}\right)\rceil$, where $\lceil x \rceil$ denotes the smallest integer larger than x. Using Lemma 2.3.3 and the fact that $\sqrt{1-r_n^2} = n^{-\gamma/2}$, we get

$$\begin{aligned} N\sigma(C(r_n)) &\geq \exp\left((1+\varepsilon)(n-1)\ln\sqrt{n} + (n-1)\ln\sqrt{1-r_n^2} - \ln(6\sqrt{n})\right) \\ &= \exp\left((1+\varepsilon)(n-1)\ln\sqrt{n} - (1+\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n})\right) \\ &\geq \exp((\varepsilon/4)(n-1)\ln\sqrt{n}), \end{aligned}$$

for all n satisfying

$$\ln(6\sqrt{n}) \leq (\varepsilon/4)(n-1)\ln\sqrt{n}. \tag{2.58}$$

Thus if n satisfies (2.58) and $n > 2e$, Lemma 2.3.6 yields

$$\mathbb{P}\,(r_n B_2^n \not\subset L_N) \leq 2\exp\left(2n^2\ln\sqrt{n} - \exp\left((\varepsilon/4)(n-1)\ln\sqrt{n}\right)\right) \longrightarrow 0, \tag{2.59}$$

as $n \to \infty$, which completes the proof of (2.37). □

Remark 2.3.10. The rate of convergence in Theorem 2.3.2 can be obtained from lines (2.53), (2.55), and (2.59).

Remark 2.3.11. In Theorem 2.3.2, we may replace ε by ε_n where $(\varepsilon_n)_{n\geq 1} \subset (0,1)$ with $\varepsilon_n \to 0$ provided that (ε_n) satisfies (2.53), (2.55), (2.57) - (2.59). One can check that $\varepsilon_n = 1/\ln(\ln n)$ works.

2.3.2 Random facets

In this section we discuss counterparts to Theorems 2.3.1 and 2.3.2 for polytopes generated by random facets. We shall use the notation defined in Section 2.3. Set

$$L_N' := \{x \in \mathbb{R}^n : \langle u_i, x\rangle \le 1 \text{ for each } i = 1, \ldots, N\}.$$

Theorem 2.3.12. *Let* $0 < R < 1$ *and let* $0 < \varepsilon < 1$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}[\mathrm{vol}\left(L_N' \cap R^{-1}B_2^n\right)]}{\mathrm{vol}\,(R^{-1}B_2^n)} \longrightarrow \begin{cases} 1 & \text{if } n < N \le \exp\Big((1-\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\Big), \\ 0 & \text{if } N \ge \exp\Big((1+\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})\Big). \end{cases} \tag{2.60}$$

Next, we turn our attention to threshold results for the entire body L_N'. Since $L_N' \supset B_2^n$, it is natural to consider the quantity

$$\frac{\mathrm{vol}\,(B_2^n)}{\mathbb{E}\,\mathrm{vol}\,(L_N')}.$$

In fact, $\mathbb{E}\,\mathrm{vol}\,(L_N') = \infty$. To see this, let $1 = t < s$, set $B = sB_2^n \backslash B_2^n$ and apply Lemma 2.3.15:

$$\mathbb{E}\,\mathrm{vol}\left(L_N' \cap B\right) \quad \ge \quad (1/2)^N \,\mathrm{vol}\,(sB_2^n \backslash B_2^n).$$

Thus if n is fixed, $\mathbb{E}\,\mathrm{vol}\,(L_N' \cap B) \to \infty$ as $s \to \infty$. Nevertheless, we can still prove the following threshold result.

Theorem 2.3.13. *Let* $0 < \varepsilon < 1$.

a. There exists a sequence $(t_n)_{n=1}^\infty = (t_n(\varepsilon))_{n=1}^\infty$ *with* $t_n > 1$ *and* $\lim_{n\to\infty} t_n = 1$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(B_2^n)}{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap t_n B_2^n)} = 0 \quad \text{if } n < N \le \exp\Big((1-\varepsilon)n\ln\sqrt{n}\Big). \tag{2.61}$$

b. There exists a sequence $(R_n)_{n=1}^\infty = (R_n(\varepsilon))_{n=1}^\infty$ *with* $R_n > 1$ *and* $\lim_{n\to\infty} R_n = \infty$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}\,(B_2^n)}{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap R_n B_2^n)} = 1 \quad \text{if } N \ge \exp\Big((1+\varepsilon)n\ln\sqrt{n}\Big). \tag{2.62}$$

We prove only Theorem 2.3.13 (the proof of Theorem 2.3.12 is similar and appears in [11]).

Recall that the notation for a spherical cap $C(R)$ was introduced in (2.39).

Lemma 2.3.14. *For each $x \in \mathbb{R}^n \backslash B_2^n$ we have*

$$\mathbb{P}\left(x \in L'_N\right) = (1 - \sigma(C(1/|x|)))^N. \tag{2.63}$$

Proof. Let $x \in \mathbb{R}^n \backslash B_2^n$. Observe first that

$$\begin{aligned} \{\theta \in S^{n-1} : \langle \theta, x \rangle \leq 1\} &= \{\theta \in S^{n-1} : \langle \theta, x/|x| \rangle \leq 1/|x|\} \\ &= S^{n-1} \backslash \operatorname{int} C(1/|x|, x/|x|), \end{aligned} \tag{2.64}$$

where $\operatorname{int} A$ denotes the interior of A.

By independence of the u_i's, we have

$$\begin{aligned} \mathbb{P}\left(x \in L'_N\right) &= \mathbb{P}\left(\langle u_i, x \rangle \leq 1 \text{ for each } i = 1, \ldots, N\right) \\ &= \left(\mathbb{P}\left(\langle u_1, x \rangle \leq 1\right)\right)^N \\ &= \left(1 - \sigma(C(1/|x|))\right)^N. \end{aligned}$$

□

Lemma 2.3.15. *Let $1 \leq t < s$ and set $B := sB_2^n \backslash tB_2^n$. Then for each n we have*

$$\operatorname{vol}(B)\,(1 - \sigma(C(1/s)))^N \leq \mathbb{E}\operatorname{vol}\left(L'_N \cap B\right) \leq \operatorname{vol}(B)\,(1 - \sigma(C(1/t)))^N. \tag{2.65}$$

Proof. Argue as in the proof of Lemma 2.2.7 and apply Lemma 2.3.14. □

Remark 2.3.16. Let $0 \leq a \leq 1$. The identity $(1 - \sigma(C(a))^N = \exp(N \ln(1 - \sigma(C(a))))$ and (2.26) imply that

$$(1 - \sigma(C(a)))^N \geq \exp\left(-N\sigma(C(a)) - N\sigma(C(a))^2\right) \tag{2.66}$$

and

$$(1 - \sigma(C(a)))^N \leq \exp\left(-N\sigma(C(a))\right). \tag{2.67}$$

These estimates will be used in conjunction with Lemma 2.3.15.

Proof of Theorem 2.3.13. Assume first that (2.52) holds. Without loss of generality we shall assume that $N = \lfloor \exp((1 - \varepsilon)(n - 1) \ln \sqrt{n}) \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

Let $(t_n)_{n=2}^{\infty} \subset (1,\infty)$ be any sequence satisfying the following conditions:

(i) $\lim_{n\to\infty} t_n = 1$.

(ii) $\lim_{n\to\infty} t_n^n = \infty$.

(iii) $\lim_{n\to\infty} N\sigma(C(1/t_n)) = 0$.

For instance, let $t_n := 1/\sqrt{1-n^{-(1-\varepsilon/2)}}$. Then $t_n > 1$, $t_n \to 1$ as $n \to \infty$ and t_n satisfies condition (ii) by (2.51) (in the proof of Theorem 2.3.2).

To see that (iii) is satisfied, apply Lemma 2.3.4 and $\ln\sqrt{1-(1/t_n)^2} = -(1-\varepsilon/2)\ln\sqrt{n}$ to get

$$\begin{aligned}\sigma(C(1/t_n)) &\leq 3\exp\left((n-1)\ln\sqrt{1-(1/t_n)^2}\right)\\ &= 3\exp\left(-(1-\varepsilon/2)(n-1)\ln\sqrt{n}\right)\end{aligned}$$

and thus

$$N\sigma(C(1/t_n)) \leq 3\exp\left(-(\varepsilon/2)(n-1)\ln\sqrt{n}\right) \longrightarrow 0 \tag{2.68}$$

as $n \to \infty$ and hence also

$$N\sigma(C(1/t_n))^2 \longrightarrow 0 \tag{2.69}$$

as $n \to \infty$.

Set $B = t_n B_2^n \setminus B_2^n$. Since

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap t_n B_2^n)}{\mathrm{vol}\,(B_2^n)} = \frac{\mathrm{vol}\,(B_2^n) + \mathbb{E}\,\mathrm{vol}\,(L_N' \cap B)}{\mathrm{vol}\,(B_2^n)} = 1 + \frac{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap B)}{\mathrm{vol}\,(B_2^n)}, \tag{2.70}$$

it suffices to prove that

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap B)}{\mathrm{vol}\,(B_2^n)} \longrightarrow \infty \tag{2.71}$$

as $n \to \infty$.

By Lemma 2.3.15, we have

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L_N' \cap B)}{\mathrm{vol}\,(B_2^n)} \geq (t_n^n - 1)(1-\sigma(C(1/t_n)))^N.$$

The latter term tends to ∞ as $n \to \infty$ by our choice of (t_n), (2.66), (2.68), and (2.69).

Let us assume now that (2.56) holds. Without loss of generality, we shall assume that $N = \lceil \exp((1+\varepsilon)(n-1)\ln\sqrt{n})\rceil$, where $\lceil x\rceil$ denotes the smallest integer larger than x.

Before defining conditions for choosing the sequence R_n, we introduce an auxiliary sequence. Let $(r_n)_{n=2}^{\infty} \subset (1,\infty)$ be any sequence such that

(a) $\lim_{n\to\infty} r_n = 1$.

(b) $\lim_{n\to\infty} r_n^n = 1$.

(c) $\lim_{n\to\infty} N\sigma(C(1/r_n)) = \infty$.

For instance, let $r_n := 1/\sqrt{1-n^{-(1+\varepsilon/2)}}$. Then $r_n > 1$, $r_n \to 1$ as $n\to\infty$ and, by (2.51), condition (b) also holds. By Lemma 2.3.3 and the fact that $\ln\sqrt{1-(1/r_n)^2} = -(1+\varepsilon/2)\ln\sqrt{n}$, condition (c) is satisfied since

$$\begin{aligned}\sigma(C(1/r_n)) &\geq \exp\left((n-1)\ln\sqrt{1-(1/r_n)^2} - \ln(6\sqrt{n})\right)\\ &= \exp\left(-(1+\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n})\right)\end{aligned}$$

and hence

$$\begin{aligned}N\sigma(C(1/r_n)) &\geq \exp\left((\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n})\right)\\ &\geq \exp\left((\varepsilon/4)(n-1)\ln\sqrt{n}\right) \qquad (2.72)\end{aligned}$$

provided that

$$\ln(6\sqrt{n}) \leq (\varepsilon/4)(n-1)\ln\sqrt{n}. \qquad (2.73)$$

Now let $(R_n)_{n=2}^{\infty} \subset (1,\infty)$ be any sequence such that

(A) $R_n > r_n$ for each n.

(B) $\lim_{n\to\infty} R_n = \infty$.

(C) $\lim_{n\to\infty} R_n^n(1-\sigma(C(1/r_n)))^N = 0$.

For instance, choose (R_n) such that $n\ln R_n \leq (1/2)\exp\left((\varepsilon/4)(n-1)\ln\sqrt{n}\right)$. In this case, if n satisfies (2.73), then (2.67) and (2.72) imply that

$$R_n^n(1-\sigma(C(1/r_n)))^N \leq \exp\left(n\ln R_n - \exp((\varepsilon/4)(n-1)\ln\sqrt{n})\right) \to 0 \qquad (2.74)$$

as $n \to \infty$.

Set $B = R_n B_2^n \backslash B_2^n$. Since

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L'_N \cap R_n B_2^n)}{\mathrm{vol}\,(B_2^n)} = \frac{\mathrm{vol}\,(B_2^n) + \mathbb{E}\,\mathrm{vol}\,(L'_N \cap B)}{\mathrm{vol}\,(B_2^n)} = 1 + \frac{\mathbb{E}\,\mathrm{vol}\,(L'_N \cap B)}{\mathrm{vol}\,(B_2^n)}, \tag{2.75}$$

it suffices to prove that

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L'_N \cap B)}{\mathrm{vol}\,(B_2^n)} \longrightarrow 0 \text{as } n \to \infty. \tag{2.76}$$

Writing $B = R_n B_2^n \backslash r_n B_2^n \cup r_n B_2^n \backslash B_2^n$ and applying Lemma 2.3.15 twice gives

$$\frac{\mathbb{E}\,\mathrm{vol}\,(L'_N \cap B)}{\mathrm{vol}\,(B_2^n)} \leq (r_n^n - 1) + R_n^n (1 - \sigma(C(1/r_n)))^N. \tag{2.77}$$

The right-hand side of the latter inequality tends to 0 by our choice of (r_n) and (R_n). □

REFERENCES

[1] K. Ball. An elementary introduction to modern convex geometry. Flavors of Geometry, pages 1–58. MSRI Publications, Cambridge University Press, Cambridge, 1997.

[2] I. Bárány. Random polytopes, convex bodies, and approximation. In *Stochastic geometry*, volume 1892 of *Lecture Notes in Math.*, pages 77–118. Springer, Berlin, 2007.

[3] I. Bárány and Z. Füredi. Approximation of the sphere by polytopes having few vertices. *Proc. Amer. Math. Soc.*, 102(3):651–659, 1988.

[4] I. Bárány and D. G. Larman. Convex bodies, economic cap coverings, random polytopes. *Mathematika*, 35(2):274–291, 1988.

[5] C. Borell. Convex set functions in d-space. *Period. Math. Hungar.*, 6(2):111–136, 1975.

[6] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. *J. Funct. Anal.*, 257(9):2820–2839, 2009.

[7] M.E. Dyer, Z. Füredi, and C. McDiarmid. Volumes spanned by random points in the hypercube. *Random Structures and Algorithms*, 3(1):91–106, 1992.

[8] D. Gatzouras and A. Giannopoulos. Threshold for the volume spanned by random points with independent coordinates. *Israel J. Math.*, 169:125–153, 2009.

[9] J.S. Müller. Approximation of the ball by random polytopes. *Journal of Approximation Theory*, 63:198–209, 1990.

[10] P. Paouris and E. Werner. Relative entropy of cone measures and L_p centroid bodies. Preprint, available at `http://www.math.tamu.edu/~grigoris`.

[11] P. Pivovarov. Volume thresholds for Gaussian and spherical random polytopes and their duals. *Studia Math.*, 183(1):15–34, 2007.

[12] C. Schütt and E. Werner. Polytopes with vertices chosen randomly from the boundary of a convex body. volume 1807 of *Lecture Notes in Mathematics*, pages 241–422. Springer-Verlag, 2001-2002.

[13] S.J. Szarek and E. Werner. A nonsymmetric correlation inequality for Gaussian measure. *Journal of Multivariate Analysis*, 68:193–211, 1999.

CHAPTER 3

On the volume of caps and bounding the mean-width of an isotropic convex body[1]

3.1 Introduction

Let K be a convex body in $\mathbb{R}^n$ with volume $\mathrm{vol}\,(K) = 1$ and suppose its center of mass is the origin. As is commonly done in Asymptotic Geometric Analysis, we treat K as a probability space. In particular, for each unit vector θ, we view the linear functional $\langle \cdot, \theta \rangle : K \to \mathbb{R}$ given by

$$\langle x, \theta \rangle = x_1\theta_1 + \ldots + x_n\theta_n, \quad (x \in K),$$

as a random variable on K. Motivated by Bourgain's approach to the Hyperplane Conjecture [4], (cf. §1.1.4 of the introductory chapter), recent research has focused on the distribution of the functionals $\langle \cdot, \theta \rangle$. In particular, efforts have been made to show that for any such K, there exists a direction θ which exhibits *sub-Gaussian* tail-decay, meaning that,

$$\mathrm{vol}\,(\{x \in K : |\langle x, \theta \rangle| > t\|\langle \cdot, \theta \rangle\|_2\}) \leq e^{-ct^2} \tag{3.1}$$

[1]A version of this chapter has been accepted for publication. P. Pivovarov. On the volume of caps and bounding the mean-width of an isotropic convex body. Math. Proc. Cambridge Philos. Soc.

for all $t \geq 1$, where $c > 0$ is an absolute constant and $\|\langle\cdot,\theta\rangle\|_2^2 = \int_K |\langle x,\theta\rangle|^2 dx$. The papers [7] and [15] contain the latest developments, as well as further motivation and history (cf. also §1.1.4).

In this paper, we consider bounds involving the reverse inequality, namely *super-Gaussian* estimates of the form

$$\operatorname{vol}(\{x \in K : |\langle x,\theta\rangle| > t\|\langle\cdot,\theta\rangle\|_2\}) \geq e^{-ct^2}, \tag{3.2}$$

for $t > 0$ in some suitable range. Such estimates are garnering increased attention, as in [14] and [20], and are the starting point for our discussion.

Our first main result concerns super-Gaussian directions for convex bodies that are isotropic and 1-unconditional (cf. §1.1.5). Thus we assume that $K \subset \mathbb{R}^n$ satisfies the following conditions:

1) $\operatorname{vol}(K) = 1$.
2) If $x = (x_i) \in K$ then $[-|x_1|,|x_1|] \times \ldots \times [-|x_n|,|x_n|] \subset K$.
3) $\int_K x_j^2 dx = L_K^2$.

For such bodies, there are many super-Gaussian directions, which we gauge in terms of the Haar measure σ on the sphere S^{n-1}.

Proposition 3.1.1. *There exists an absolute constant $C \geq 1$ such that for any integer $n \geq 1$, and any 1-unconditional isotropic convex body $K \subset \mathbb{R}^n$, the σ-measure of the set of $\theta \in S^{n-1}$ such that*

$$\operatorname{vol}(\{x \in K : |\langle x,\theta\rangle| \geq t\}) \geq \exp(-Ct^2) \tag{3.3}$$

whenever

$$C \leq t \leq \frac{\sqrt{n}}{C\log n}, \tag{3.4}$$

is at least $1 - 2^{-n}$.

The isotropic constant $L_K = \|\langle\cdot,\theta\rangle\|_2$ has been omitted from the statement since such bodies satisfy $1/\sqrt{2\pi e} \leq L_K \leq 1/\sqrt{2}$ (see, e.g., [3]).

Proposition 3.1.1 complements a theorem of Bobkov and Nazarov [2] who treated the case of sub-Gaussian directions, (cf. Theorem 1.1.18, §1.1.5).

As a corollary, we get lower bounds for the volume of caps defined in terms of the width of the body, measured in terms of the support function $h_K(\theta) := \sup_{x\in K}\langle x,\theta\rangle$.

Corollary 3.1.2. *Let $\beta > 0$. Then there is a constant $\widetilde{C} = \widetilde{C}(\beta)$ such that for any integer $n \geq 1$, and any 1-unconditional isotropic convex body $K \subset \mathbb{R}^n$, the σ-measure of the set of $\theta \in S^{n-1}$ satisfying*

$$\mathrm{vol}\left(\{x \in K : |\langle x,\theta\rangle| \geq \varepsilon h_K(\theta)\}\right) \geq \exp(-C\widetilde{C}\varepsilon^2 n\log n) \tag{3.5}$$

whenever

$$\frac{C}{\sqrt{\widetilde{C}n\log n}} \leq \varepsilon \leq \frac{1}{C\sqrt{\widetilde{C}}\log^{3/2} n} \tag{3.6}$$

is at least $1-2^{-n}-2n^{-\beta}$, where $\widetilde{C} = 3(\beta+1)$ and C is the constant from Proposition 3.1.1.

For us, the motivation for bounding the volume of the caps in (3.5) comes from a paper by Giannopoulos and Milman [10] involving approximation of a convex body by a random polytope. Corollary 3.1.2 shows that for 1-unconditional bodies, one has better estimates in most directions. Such estimates, in turn, are intimately related to mean-width, which leads us to the second purpose of this paper.

Finding the correct upper bound for the mean-width of an *arbitrary* isotropic convex body (not necessarily 1-unconditional) is a problem well-known to specialists. It has numerous connections and implications (some of which we review below). We connect the latter problem with volume estimates for caps, similar to (3.5), and give a sufficient condition under which one can bound the mean-width. Our approach may be of independent interest since it involves approximating a convex body by a random polytope with relatively few vertices.

The paper is organized as follows. The proofs of Proposition 3.1.1 and Corollary 3.1.2 are in §3.2, the first three subsections of which point out the key ingredients. The observations about mean-width are contained in §3.3, the main result being Proposition 3.3.9.

Lastly, a few words on notation and viewpoint. Our results are most meaningful when the dimension n is large. Throughout, $c, c_1, C, C', \ldots$, etc. denote absolute constants (in particular, independent of n and K). The symbol $|\cdot|$ will serve the dual role of the standard Euclidean norm on $\mathbb{R}^n$ and also the absolute value of a scalar, the use of which will be clear from the context; for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_{i\leq n}|x_i|$.

3.2 Super-Gaussian estimates in 1-unconditional isotropic convex bodies

We begin by isolating the key ingredients in the proof of Proposition 3.1.1. Our proof generalizes an argument due to Schmuckenschläger [25, Proposition 3.4], who showed that the diagonal direction $\theta_d = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$ is super-Gaussian for the unit ball of ℓ_p^n for $1 \leq p \leq \infty$. Our first step is to pass from θ_d to a large subset of directions with "well-spread" coordinates.

3.2.1 Well-spread vectors on the sphere

The next lemma identifies the set of directions for which we will establish estimate (3.3) in Proposition 3.1.1. Similar facts have been used in various problems (e.g., the use of "incompressible" vectors as in [24]). We include a proof for completeness.

Throughout, we use the following notation

$$[n] := \{1,\ldots,n\}. \tag{3.7}$$

Lemma 3.2.1. *There exist absolute constants $C_1 > 0$ and $\kappa > 0$ such that for any integer $n \geq 1$, the set*

$$\Theta := \left\{\theta \in S^{n-1} | \exists I = I(\theta) \subset [n] \text{ with } \#I \geq \kappa n : \frac{1}{C_1\sqrt{n}} \leq |\theta_i| \leq \frac{C_1}{\sqrt{n}} \ \forall\, i \in I\right\} \tag{3.8}$$

has σ-measure at least $1 - 2^{-n}$.

For the proof, we will use the following standard facts.

Lemma 3.2.2. *There exists an absolute constant $c' > 0$ such that for any integer $n \geq 1$, the set*

$$\Theta' := \{\theta \in S^{n-1} : c'\sqrt{n} \leq \|\theta\|_1 \leq \sqrt{n}\} \tag{3.9}$$

has σ-measure at least $1 - 2^{-n}$.

Lemma 3.2.2 follows from, e.g., [19, §2.3 & §5.3]; alternatively, one can use [23, Theorem 6.1].

The second fact we need is the Paley-Zygmund inequality.

Lemma 3.2.3. *If Z is a random variable with finite variance, then*

$$\mathbb{P}(Z \geq t\mathbb{E}Z) \geq (1-t)^2 \frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2}. \tag{3.10}$$

For a proof, see, e.g., [18, Lemma 3.5].

Proof of Lemma 3.2.1. Fix $\theta \in \Theta'$ (from Lemma 3.2.2) and write $\alpha_i := |\theta_i|\sqrt{n}$. Without loss of generality, we may assume that the α_i are distinct. Let Z be a random variable such that $\mathbb{P}(Z = \alpha_i) = 1/n$. Then

$$\mathbb{E}Z = \frac{1}{n}\sum_{i=1}^{n} \alpha_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} |\theta_i|$$

and hence $c' \leq \mathbb{E}Z \leq 1$. By Markov's inequality, for any $\lambda > 0$, we have

$$\mathbb{P}(Z > \lambda) \leq \mathbb{P}(Z > \lambda\mathbb{E}Z) \leq \frac{1}{\lambda}$$

and hence we obtain

$$\#\{i \in [n] : \alpha_i \leq \lambda\} \geq (1-\lambda^{-1})n. \tag{3.11}$$

Next, observe that

$$\mathbb{E}Z^2 = \frac{1}{n}\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \theta_i^2 = 1.$$

By the Paley-Zygmund inequality (Lemma 3.2.3), we have

$$\mathbb{P}\left(Z \geq c'/2\right) \geq \mathbb{P}(Z \geq (1/2)\mathbb{E}Z) \geq (c')^2/4.$$

and therefore

$$\#\{i \in [n] : \alpha_i \geq c'/2\} \geq (c')^2 n/4. \tag{3.12}$$

By (3.11) and (3.12), we conclude the result. □

3.2.2 Main probabilistic ingredients

The symmetries exhibited by 1-unconditional convex bodies have a very useful probabilistic interpretation. Namely, let $X = (x_1, \ldots, x_n)$ be a random vector distributed uniformly in a 1-unconditional convex body. Let $\varepsilon_1, \ldots, \varepsilon_n$, be independent Rademacher random variables, i.e.,

$$\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2, \quad i = 1, \dots, n. \tag{3.13}$$

Then X and $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$ have the the same distribution. The latter fact allows us to use several properties of Rademacher random variables. The first is the Contraction Principle; see, e.g., [16, Theorem 4.4].

Theorem 3.2.4. *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher random variables. Let $x_1, \dots, x_n$ be elements of a Banach space B and let $\alpha_1, \dots, \alpha_n$ be real numbers such that $|\alpha_i| \le 1$ for all $i = 1, \dots, n$. Then for any $t > 0$,*

$$\mathbb{P}\left(\|\sum_{i=1}^{n} \alpha_i \varepsilon_i x_i\| > t \right) \le 2\mathbb{P}\left(\|\sum_{i=1}^{n} \varepsilon_i x_i\| > t \right). \tag{3.14}$$

The second ingredient is the following theorem about super-Gaussian estimates for Rademacher sums, which can be found in [16, §4.1].

Theorem 3.2.5. *There is an absolute constant $C_2 \ge 1$ such that if $\varepsilon_1, \dots, \varepsilon_n$ are independent Rademacher random variables (as in (3.13)) and if $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ satisfy*

$$C_2 |\xi| \le s \le \frac{|\xi|^2}{C_2 \|\xi\|_\infty}, \tag{3.15}$$

then

$$\mathbb{P}\left(\sum_{i=1}^{n} \varepsilon_i \xi_i \ge s \right) \ge \exp(-C_2 s^2 / |\xi|^2). \tag{3.16}$$

To show that each $\theta \in \Theta$ (Lemma 3.2.1) satisfies the super-Gaussian estimate (3.3), Theorem 3.2.4 will be used to pass to subspaces $E_I := \mathrm{span}\{e_i : i \in I\}$ on which we have control of the coordinates of θ. To use Theorem 3.2.5, we will need volume estimates for certain sets involving the $|\cdot|$ and $\|\cdot\|_\infty$ norms on the orthogonal projection of K onto E_I. This is done in the next section.

3.2.3 Projections and retention of volume

Here we prove a lemma which gives a uniform lower bound for the volume of certain sets that will be used in conjunction with Theorems 3.2.4 and 3.2.5. We emphasize that it is a general fact, true for *arbitrary* isotropic convex bodies and *arbitrary* subspaces (not just unconditional bodies and coordinate subspaces as we need here).

For $1 \le \ell \le n$, let $\mathcal{G}_{n,\ell}$ denote the set of all ℓ-dimensional subspaces of $\mathbb{R}^n$; for $E \in \mathcal{G}_{n,\ell}$, let P_E be the orthogonal projection onto E.

Lemma 3.2.6. *There exist positive absolute constants C', C'' and c such that for each integer $n \ge 1$, for any isotropic convex body $K \subset \mathbb{R}^n$, any $\ell \in [n]$ and any $E \in \mathcal{G}_{n,\ell}$, the intersection of the sets*

$$K'_E := \{x \in K : (1/C')\sqrt{\ell}L_K \le |P_E x| \le C'\sqrt{\ell}L_K\} \tag{3.17}$$

and

$$K''_E := \{x \in K : \|P_E x\|_\infty \le C'' L_K \log n\}, \tag{3.18}$$

say $K_E := K'_E \cap K''_E$, has volume greater than c.

The proof relies on two basic facts. See, for instance, [8, Proposition 2.5.1] and [8, Proposition 2.1.1].

Fact 3.2.7. There exists an absolute constant C_3 such that if $n \ge 1$, $K \subset \mathbb{R}^n$ is an isotropic convex body and $\mathcal{N}$ is a finite subset of B_2^n, then

$$\int_K \max_{\theta \in \mathcal{N}} |\langle x, \theta \rangle| dx \le C_3 L_K \log(\#\mathcal{N}).$$

Fact 3.2.8. There exists an absolute constant C_4 such that if $n \ge 1$, $K \subset \mathbb{R}^n$ is a convex body of volume one, and if f is a semi-norm, then

$$\left(\int_K f^p(x) dx \right)^{1/p} \le C_4 p \int_K f(x) dx \quad \text{for all } p \ge 1. \tag{3.19}$$

Proof of Lemma 3.2.6. Let $K \subset \mathbb{R}^n$ be an isotropic convex body, $\ell \in [n]$ and $E \in \mathcal{G}_{n,\ell}$. Then

$$\int_K |P_E x| dx \le \left(\int_K |P_E x|^2 dx \right)^{1/2} = \sqrt{\ell} L_K.$$

By Markov's inequality, for any $\tau > 0$, we have

$$\mathrm{vol}\left(\{x \in K : |P_E x| \le \tau\sqrt{\ell}L_K\} \right) \ge 1 - \frac{1}{\tau^2}. \tag{3.20}$$

Setting $c_1 := (2C_4)^{-1}$, where C_4 is the constant from Fact 3.2.8, we have

$$\int_K |P_E x| dx \geq c_1 \left(\int_K |P_E x|^2 dx \right)^{1/2} = c_1 \sqrt{\ell} L_K.$$

Applying the Paley-Zygmund inequality (Lemma 3.2.3), we get

$$\mathrm{vol}\left(\{ x \in K : |P_E x| \geq (c_1/2) \sqrt{\ell} L_K \} \right) \geq c_1^2/4. \tag{3.21}$$

Taking into account (3.20) and (3.21), we determine that there are positive absolute constants C' and $c > 0$ for which

$$\mathrm{vol}\left(K_E'\right) = \{ x \in K : (1/C') \sqrt{\ell} L_K \leq |P_E x| \leq C' \sqrt{\ell} L_K \} \geq 2c. \tag{3.22}$$

To conclude, set $C'' := C_3/c$, where C_3 is the constant from Fact 3.2.7. Since

$$\|P_E x\|_\infty = \max_{i \leq n} |\langle P_E x, e_i \rangle| = \max_{i \leq n} |\langle x, P_E e_i \rangle|,$$

we can apply Markov's inequality and Fact 3.2.7 to obtain

$$\mathrm{vol}\left(K_E''\right) = \mathrm{vol}\left(\{ x \in K : \|P_E x\|_\infty \leq C'' L_K \log n \} \right) \geq 1 - c.$$

Thus

$$\mathrm{vol}\left(K_E\right) = \mathrm{vol}\left(K_E' \cap K_E''\right) \geq c,$$

which concludes the proof. □

3.2.4 Proofs of the cap estimates

Here we combine the results of the previous sections to complete the proofs of Proposition 3.1.1 and Corollary 3.1.2.

Proof of Proposition 3.1.1. Assume K is a 1-unconditional isotropic convex body in $\mathbb{R}^n$. Consider C_1, κ and Θ from Lemma 3.2.1. Set $\ell := \lfloor \kappa n \rfloor$, the largest integer less than κn. Fix $\theta \in \Theta$ so that

$$\frac{1}{C_1 \sqrt{n}} \leq |\theta_i| \leq \frac{C_1}{\sqrt{n}}$$

for all $i \in I$, where $I = I(\theta) \subset [n]$ and $|I| = \ell$. Set

$$E(I) := \mathrm{span}\{e_i : i \in I\},$$

where the e_i's are the standard unit vector basis for $\mathbb{R}^n$. By Lemma 3.2.6, the intersection $K'_{E(I)} \cap K''_{E(I)}$ has volume $\mathrm{vol}\left(K_{E(I)}\right) \geq c$.

Let $X = (x_1, \dots, x_n)$ be a random vector distributed uniformly in K. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher random variables (cf. (3.13)). Then X and $(\varepsilon_1 X_1, \dots, \varepsilon_n X_n)$ have the same distribution. Denote the probability measure corresponding to X, namely $\mathrm{vol}(\cdot|_K)$, by $\mathbb{P}_K$; by $\mathbb{P}_\varepsilon$ the product-measure corresponding to $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. Then

$$\begin{aligned}
\mathrm{vol}\left(\{x \in K : |\langle x, \theta\rangle| > t\}\right) &= \mathbb{P}_K\left(|\sum_{i=1}^n \theta_i x_i| > t\right) \\
&= \mathbb{P}_K \otimes \mathbb{P}_\varepsilon\left(|\sum_{i=1}^n \varepsilon_i \theta_i x_i| > t\right) \\
&= \int_K \mathbb{P}_\varepsilon\left(|\sum_{i=1}^n \varepsilon_i \theta_i x_i| > t\right) dx \\
&\geq (1/2)\int_K \mathbb{P}_\varepsilon\left(|\sum_{i\in I} \varepsilon_i \theta_i x_i| > t\right) dx \quad \text{(by Thm. 3.2.4)} \\
&\geq (1/2)\int_{K_{E(I)}} \mathbb{P}_\varepsilon\left(|\sum_{i\in I} \varepsilon_i \theta_i x_i| > t\right) dx. \qquad (3.23)
\end{aligned}$$

Fix $x \in K_{E(I)}$, and set $y = (\theta_i x_i)_{i\in I}$. Then, by definition of $K_{E(I)}$ and Θ,

$$\|y\|_\infty \leq \frac{C_1}{\sqrt{n}}\|P_{E(I)}x\|_\infty \leq \frac{C_1 C'' L_K \log n}{\sqrt{n}}$$

and

$$\frac{\sqrt{\ell}L_K}{C_1 C'\sqrt{n}} \leq \frac{1}{C_1\sqrt{n}}|P_{E(I)}x| \leq |y| \leq \frac{C_1}{\sqrt{n}}|P_{E(I)}x| \leq \frac{C_1 C'\sqrt{\ell}L_K}{\sqrt{n}}.$$

Since K is 1-unconditional, $L_K \leq 1/\sqrt{2}$, (e.g., [3]). Moreover, for any convex body K, $L_K \geq L_{B_2^n} \geq 1/\sqrt{2\pi e}$. Recalling that $\ell = \lfloor \kappa n \rfloor$, we conclude that there exist absolute constants $A_1 > 1$ and $A_2 > 1$ such that

$$\|y\|_\infty \leq \frac{A_1 \log n}{\sqrt{n}}$$

and

$$\frac{1}{A_2} \leq |y| \leq A_2.$$

Let C_2 be the constant from Theorem 3.2.5. At this point we can determine the constant C asserted in Proposition 3.1.1: take $C := A_1 A_2^2 C_2$. Then our assumption (3.4) implies

$$C_2|y| \leq t \leq \frac{|y|^2}{C_2\|y\|_\infty},$$

making (3.23) ripe for an application of Theorem 3.2.5:

$$\begin{aligned}
(1/2)\int_{K_{E(I)}} \mathbb{P}_\varepsilon\left(|\sum_{i\in I}\varepsilon_i\theta_i x_i| > t\right)dx &\geq (1/2)\int_{K_{E(I)}} \exp(-C_2t^2/|y|^2)dx \\
&\geq (c/2)\exp(-C_2A_2^2t^2) \\
&\geq (c/2)\exp(-Ct^2),
\end{aligned}$$

where we have used the notation $y = (\theta_i x_i)_{i\in I}$ as above. Since $c/2 \geq e^{-t^2}$ for t large enough, we can recover the proposition as stated simply by adjusting the constants. □

To prove Corollary 3.1.2, we will need two additional results.

Lemma 3.2.9. *For any $M \in (0,1)$, the set*

$$\Theta_1 := \{\theta \in S^{n-1} : \|\theta\|_\infty \leq M\}$$

has σ-measure at least $1 - 2ne^{-nM^2/2}$.

Proof. Using the well-known estimate

$$\sigma(\theta \in S^{n-1} : |\langle e_1, \theta\rangle| > M) \leq 2e^{-nM^2/2}, \tag{3.24}$$

(see, e.g., [1, Lemma 2.2]), we have

$$\begin{aligned}
\sigma\left(\theta \in S^{n-1} : \exists i \leq n : |\langle e_i, \theta\rangle| > M\right) &\leq n\sigma\left(\theta \in S^{n-1} : |\langle e_1, \theta\rangle| > M\right) \\
&\leq 2ne^{-nM^2/2}.
\end{aligned}$$

□

Another result, due to Bobkov and Nazarov [3, Propositions 2.4], will also be of use.

Proposition 3.2.10. *Let K be a 1-unconditional isotropic convex body in $\mathbb{R}^n$. Then*

$$K \subset \sqrt{3/2}nB_1^n.$$

Proof of Corollary 3.1.2. Let $\beta > 0$. Apply Lemma 3.2.9 with $M_n := \sqrt{\frac{2(\beta+1)\log n}{n}}$ so that $\sigma(\Theta_1) \geq 1 - 2n^{-\beta}$. By Proposition 3.2.10,

$$h_K(\theta) \leq \sqrt{3/2}n\|\theta\|_\infty$$

for each $\theta \in S^{n-1}$. Thus

$$\begin{aligned} \sigma\left(\theta \in S^{n-1} : h_K(\theta) \leq \sqrt{3(\beta+1)n\log n}\right) &\geq \sigma(\Theta_1) \\ &\geq 1 - 2n^{-\beta}. \end{aligned}$$

Let Θ be the set from Lemma 3.2.1. As the proof of Proposition 3.1.1 shows, any element of Θ satisfies the super-Gaussian estimate (3.3). Thus if $\theta \in \Theta \cap \Theta_1$, we have

$$\begin{aligned} \mathrm{vol}\left(\{x \in K : |\langle x, \theta\rangle| \geq \varepsilon h_K(\theta)\}\right) &\geq \mathrm{vol}\left(\{x \in K : |\langle x, \theta\rangle| \geq \varepsilon\sqrt{3(\beta+1)n\log n}\}\right) \\ &\geq \exp(-C\widetilde{C}\varepsilon^2 n\log n), \end{aligned}$$

(where $\widetilde{C} = 3(\beta+1)$) provided that

$$C \leq \varepsilon\sqrt{3(\beta+1)n\log n} \leq \frac{\sqrt{n}}{C\log n},$$

where C is the constant from Proposition 3.1.1. □

3.2.5 Comparison with recent results

As we mentioned in the introduction, super-Gaussian estimates have recently been studied by B. Klartag [14] and G. Paouris [20]. We state only the results from Klartag's paper as the latter is still in preparation.

In [14], a Borel probability measure μ on $\mathbb{R}^n$ is said to be *decent* if $\mu(E) \leq (1/n)\dim E$ for any subspace $E \subset \mathbb{R}^n$. In particular, any absolutely continuous probability measure on $\mathbb{R}^n$ is decent. The following is from [14, Cor. 1.4].

Proposition 3.2.11. *There exists a sequence $R_n \to \infty$ with the following property: Let μ be a decent probability measure on $\mathbb{R}^n$. Then, there exists a non-zero linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\mu(\{x \in \mathbb{R}^n : \phi(x) \geq tM\}) \geq c\exp(-Ct^2) \quad \textit{for all } 0 \leq t \leq R_n$$

and

$$\mu(\{x \in \mathbb{R}^n : \phi(x) \leq -tM\}) \geq c\exp(-Ct^2) \quad \textit{for all } 0 \leq t \leq R_n,$$

where $M > 0$ is a median, that is,

$$\mu(\{x \in \mathbb{R}^n |\phi(x)| \leq M\}) \geq 1/2 \quad \textit{and } \mu(\{x \in \mathbb{R}^n : |\phi(x)| \geq M\}) \geq 1/2$$

and $c, C > 0$ are universal constants. Moreover, one may take $R_n = c(\log n)^{1/4}$.

If one makes additional assumptions on the “position” of the measure, a similar statement holds for “most” functionals ϕ; we refer the reader to the article for the precise result.

3.3 On the mean-width of an isotropic convex body

For a convex body $K \subset \mathbb{R}^n$, denote its support function by

$$h_K(\theta) := \sup_{x \in K} \langle x, \theta \rangle, \quad (\theta \in S^{n-1}).$$

The width of K in the direction of θ is the quantity $w(K,\theta) = h_K(\theta) + h_K(-\theta)$ and the mean-width of K is

$$w(K) = \int_{S^{n-1}} w(K,\theta) d\sigma(\theta) = 2 \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

Recall Urysohn’s inequality (see, e.g., [23, Corollary 1.4]).

Proposition 3.3.1. *Let $K \subset \mathbb{R}^n$ be a convex body. Then*

$$w(K) \geq 2 \left(\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(B_2^n)} \right)^{1/n}.$$

In particular, if $K \subset \mathbb{R}^n$ is a convex body with $\mathrm{vol}(K) = 1$, then $w(K) \geq c\sqrt{n}$, where $c > 0$ is an absolute constant. On the other hand, the following theorem is often referred to as a "reverse-Urysohn inequality".

Theorem 3.3.2. *Let K be a convex body in $\mathbb{R}^n$. Then there exists an affine image TK of K of volume one such that*

$$w(TK) \leq C\sqrt{n}\log n, \tag{3.25}$$

where C is an absolute constant.

The proof is a combination of results due to Figiel and Tomczak-Jaegermann [6] and Pisier [22]. The position associated to the latter fact is referred to as ℓ-position and plays an important role in Asymptotic Geometric Analysis; see, e.g., the survey [11, §2.3].

In this section, we discuss upper bounds for the mean-width of a convex body in isotropic position; recall the latter assumption entails $\mathrm{vol}(K) = 1$, the center of mass of K is the origin and

$$\int_K \langle x,\theta\rangle^2\,dx = L_K^2 \text{ for each } \theta \in S^{n-1}. \tag{3.26}$$

Currently, the best-known upper bound is the following.

Theorem 3.3.3. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then*

$$w(K) \leq Cn^{3/4}L_K, \tag{3.27}$$

where C is an absolute constant.

The latter estimate follows from Dudley's entropy estimate as in [9, Theorem 5.6] and the covering number bound from [17, Lemma 4]; a proof is in [12]. Included below is a sketch of the proof for the sake of completeness. Before doing so, we will require the following fact [13, Theorem 4.1].

Lemma 3.3.4. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. Then*

$$L_K B_2^n \subset K \subset (n+1)L_K B_2^n. \tag{3.28}$$

Proof of Theorem 3.3.3. For each $t > 0$, let $N(K,tB_2^n)$ be the smallest number of translates of tB_2^n whose union covers K, i.e.,

$$N(K,tB_2^n) := \min\left\{N \,|\, \exists x_1,x_2,\ldots,x_N \in K : K \subset \bigcup_{i=1}^{N}(x_i + tB_2^n)\right\}. \tag{3.29}$$

By Dudley's entropy bound,

$$w(K) \leq \frac{C}{\sqrt{n}} \int_0^\infty \sqrt{\log N(K, tB_2^n)} dt, \tag{3.30}$$

where C is an absolute constant. Note that $N(K, tB_2^n) = 1$ if $t \geq (n+1)L_K$ (by Lemma 3.3.4). Thus, since K is assumed to be isotropic, we in fact have

$$w(K) \leq \frac{C}{\sqrt{n}} \int_0^{(n+1)L_K} \sqrt{\log N(K, tB_2^n)} dt. \tag{3.31}$$

One can bound the covering number $N(K, tB_2^n)$ using a special case of [17, Lemma 4], which states that for any convex body $K \subset \mathbb{R}^n$, one has

$$N(K, tB_2^n) \leq \exp\left(C_1 n M_2(K)/t\right), \tag{3.32}$$

where

$$M_2(K) := \frac{1}{\mathrm{vol}\,(K)} \int_K |x| dx \tag{3.33}$$

In our case, K being isotropic implies that

$$M_2(K) \leq \left(\int_K |x|^2 dx\right)^{1/2} = \sqrt{n} L_K. \tag{3.34}$$

Thus

$$\begin{aligned} w(K) &\leq \frac{C}{\sqrt{n}} \int_0^{(n+1)L_K} \left(C_1 n M_2(K)/t\right)^{1/2} dt \\ &\leq C_2 n^{1/4} L_K^{1/2} \int_0^{(n+1)L_K} t^{-1/2} dt \\ &\leq C_3 n^{3/4} L_K. \end{aligned}$$

□

Remark 3.3.5. The bound (3.27) can also be derived easily using more recent tools, namely results of Paouris on L_q-centroid bodies in [21] (see also [7, §2 (in particular, (2.2) and Lemma 2.5)].

It is known that sub-Gaussian estimates such as (3.1) also have implications for the width of K (cf. §1.1.4, in particular (1.23)).

In the next section, we offer another condition, related to lower bounds for caps similar to (3.5), under which one can bound the mean-width.

3.3.1 Bounding the mean-width via random polytopes

Throughout this section, we assume that K is an isotropic convex body in $\mathbb{R}^n$ (as in (3.26)), $X_1,\ldots,X_N$ are independent random vectors distributed uniformly in K; K_N their convex hull:

$$K_N := \operatorname{conv}\{X_1,\ldots,X_N\}; \tag{3.35}$$

$\mathbb{P}$ the associated product measure on $\otimes_{i=1}^N K$.

Lemma 3.3.6. *Let $t \geq 1$ and suppose that $n < N \leq e^{\sqrt{n}t/2}$. Then*

$$\mathbb{P}\left(w(K_N) \leq \overline{C}\sqrt{\log N} L_K t\right) \geq 1 - e^{-\sqrt{n}t/2}, \tag{3.36}$$

where $\overline{C} > 0$ is an absolute constant.

Proof. Let $u_1,\ldots,u_N$ be points on the sphere S^{n-1}. Then, using (3.24) in a standard way, we have

$$\int_{S^{n-1}} \max_{i\leq N} |\langle u_i,\theta\rangle| d\sigma(\theta) \leq \frac{C_1'\sqrt{\log N}}{\sqrt{n}}, \tag{3.37}$$

where C_1' is an absolute constant.

By [21, Theorem 1.1], we have

$$\mathbb{P}\left(|X_i| \leq C_2'\sqrt{n}L_K t \text{ for each } i = 1,\ldots,N\right) \geq 1 - e^{-\sqrt{n}t/2},$$

where C_2' is an absolute constant. Assume now that $0 < |X_i| \leq C_2'\sqrt{n}L_K t$ and write $X_i' = X_i/|X_i|$. Then

$$\begin{aligned} w(K_N) &\leq 2\int_{S^{n-1}} \max_{i\leq N} |\langle X_i,\theta\rangle| d\sigma(\theta) \\ &\leq 2C_2'\sqrt{n}L_K t \int_{S^{n-1}} \max_{i\leq N} |\langle X_i',\theta\rangle| d\sigma(\theta) \\ &\leq \overline{C}\sqrt{\log N} L_K t, \end{aligned}$$

where we used (3.37) for the last inequality and $\overline{C} = 2C_1'C_2'$. □

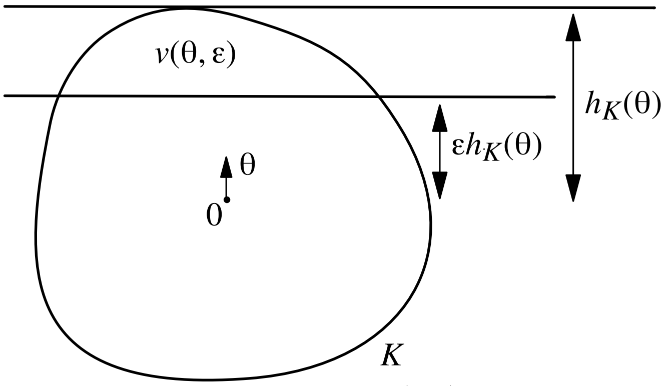


Figure 3.1: The volume $v(\theta,\varepsilon)$ of a cap.

Remark 3.3.7. See [5, Proposition 3.3] for further observations about the mean-width of the random polytope K_N; in particular, the relation to the width of L_q-centroid bodies.

Next, we use an idea of Giannopoulos and Milman from [10, Lemma 5.1]. For each $\varepsilon \in (0,1)$ and $\theta \in S^{n-1}$, let

$$v(\theta,\varepsilon) := \mathrm{vol}\left(\{x \in K : \langle x,\theta\rangle \geq \varepsilon h_K(\theta)\}\right), \tag{3.38}$$

as in Figure 3.1.

Lemma 3.3.8. *Let* $\varepsilon > 0$. *Then*

$$\mathbb{P}\left(h_{K_N}(\theta) < \varepsilon h_K(\theta)\right) \leq \exp\left(-Nv(\theta,\varepsilon)\right).$$

Proof. By definition,

$$\mathrm{vol}\left(\{x \in K : \langle x,\theta\rangle < \varepsilon h_K(\theta)\}\right) = 1 - v(\theta,\varepsilon),$$

hence

$$\mathbb{P}\left(\max_{j\leq N}\langle X_j,\theta\rangle < \varepsilon h_K(\theta)\right) = (1-v(\theta,\varepsilon))^N \leq \exp(-Nv(\theta,\varepsilon)).$$

□

3.3.2 Sufficient conditions for bounding the mean-width

In this section, we prove that one can bound the mean-width of an isotropic convex body under a certain hypothesis; namely, that in "most" directions θ, the volume of the caps

$v(\theta,\varepsilon)$ (cf. (3.38)) is suitably large. "Most" in this case is meant with respect to the Haar measure σ on the sphere S^{n-1}, and is quantified by a certain constant; expressly, let C_0 be the smallest constant such that for any positive integer n and any isotropic convex body $K \subset \mathbb{R}^n$,

$$\left(\int_{S^{n-1}}\left(\max_{x\in K}|\langle x,\theta\rangle|\right)^2 d\sigma(\theta)\right)^{1/2} \leq C_0 \int_{S^{n-1}} \max_{x\in K}|\langle x,\theta\rangle| d\sigma(\theta), \tag{3.39}$$

By Fact 3.2.8, C_0 is an absolute constant. It will play a role in the formulation of the proposition.

Proposition 3.3.9. *Let n be a positive integer and K an isotropic convex body in $\mathbb{R}^n$. Let $\alpha \geq 1$, $\varepsilon \in (0,1)$ and $p \in [1,2]$. Let $v(\theta,\varepsilon)$ be the volume of the cap defined in (3.38) and C_0 as in (3.39). If $4\alpha\varepsilon^p\sqrt{n} \geq 1$ and*

$$\sigma\left(\{\theta \in S^{n-1} : v(\theta,\varepsilon) \geq e^{-\alpha\varepsilon^p n}\}\right) \geq 1 - \frac{1}{16C_0^2}, \tag{3.40}$$

then

$$w(K) \leq \widehat{C}\alpha^{3/2}\varepsilon^{3p/2-1} n L_K, \tag{3.41}$$

where $\widehat{C}$ is an absolute constant.

Before proving the proposition, we give several remarks to illustrate its potential utility and emphasize the important ranges for α, ε and p.

Remark 3.3.10. The argument from [10, Lemma 5.1] shows that for *every* $\theta \in S^{n-1}$ and *every* $\varepsilon \in (0,1)$, one has

$$v(\theta,\varepsilon) \geq \frac{c}{n^2}(1-\varepsilon)^n,$$

where $c > 0$ is an absolute constant. But $(c/n^2)(1-\varepsilon)^n \geq e^{-3\varepsilon n}$ provided that $\log(n^2/c)/n \leq \varepsilon \leq 1/2$. Hence (3.40) holds with $\alpha = 3$, $\varepsilon = n^{-1/2}$, and $p = 1$, in which case the proposition recovers the known estimate:

$$w(K) \leq C n^{3/4} L_K,$$

with C an absolute constant.

Remark 3.3.11. If (3.40) holds with

$$\alpha = C_4' \log n, \qquad \varepsilon = \frac{1}{n^{1/4}\log^{1/2} n}, \qquad p = 2, \tag{3.42}$$

one would obtain the optimal bound

$$w(K) \leq C\sqrt{n \log n} L_K,$$

where C is an absolute constant.

Remark 3.3.12. Corollary 3.1.2 shows that (3.40) holds with α, ε and p as in the previous remark (3.42) for all 1-unconditional isotropic convex bodies (with a stronger measure estimate). Note, however, that we have used Proposition 3.2.10 (the upper-bound on the width) to prove Corollary 3.1.2. Nevertheless, this shows that (3.40) holds with the values in (3.42) for a large class of convex bodies.

Proof of Proposition 3.3.9. Let $t = 4\alpha\varepsilon^p\sqrt{n}$ so that (by assumption) $t \geq 1$. Set $N = e^{\sqrt{n}t/2}$ and suppose that $X_1, \dots, X_N$ are independent random vectors distributed uniformly in K and, as in (3.35), K_N is their convex hull. By Lemma 3.3.6, we have

$$w(K_N) \leq \overline{C}\sqrt{\log N} L_K t \tag{3.43}$$

with probability at least $1 - e^{-\sqrt{n}t/2}$.

On the other hand, we can use Lemma 3.3.8 and an approximation argument, as in [10, Theorem 5.2], to bound the width of K by that of K_N. For convenience, denote the set appearing in (3.40) by $A(\alpha, \varepsilon, p)$. A standard volume argument shows that for any $\eta \in (0,1)$, there exists an η-net $\mathcal{N} \subset A(\alpha, \varepsilon, p)$, i.e., a finite set satisfying the condition

$$\forall \theta \in A(\alpha, \varepsilon, p), \exists \theta_0 \in \mathcal{N} \text{ such that } |\theta - \theta_0| < \eta,$$

with cardinality $\#\mathcal{N} \leq (3/\eta)^n$. In particular, for $\eta = \varepsilon/4(n+1)$, let us fix one such η-net $\mathcal{N} \subset A(\alpha, \varepsilon, p)$ with cardinality

$$\#\mathcal{N} \leq (12(n+1)/\varepsilon)^n. \tag{3.44}$$

Claim 3.3.13.

$$\mathbb{P}\left(\exists \theta \in A(\alpha, \varepsilon, p) : h_{K_N}(\theta) < \frac{\varepsilon}{2} h_K(\theta)\right) \leq \mathbb{P}\left(\exists \theta_0 \in \mathcal{N} : h_{K_N}(\theta_0) \leq \varepsilon h_K(\theta_0)\right). \tag{3.45}$$

Proof of Claim 3.3.13. Suppose that there exists $\theta \in A(\alpha, \varepsilon, p)$ such that

$$h_{K_N}(\theta) < (\varepsilon/2) h_K(\theta).$$

Choose $\theta_0 \in \mathcal{N}$ such that $|\theta - \theta_0| < \eta$. The claim then follows from

$$\begin{aligned} h_{K_N}(\theta_0) &\le h_{K_N}(\theta) + h_{K_N}(\theta_0 - \theta) \\ &\le (\varepsilon/2) h_K(\theta) + h_K(\theta_0 - \theta) \\ &\le (\varepsilon/2) h_K(\theta_0) + (\varepsilon/2) h_K(\theta - \theta_0) + h_K(\theta_0 - \theta) \\ &\le (\varepsilon/2) h_K(\theta_0) + 2(n+1) L_K \eta && \text{(by (3.28))} \\ &\le (\varepsilon/2) h_K(\theta_0) + 2(n+1)\eta h_K(\theta_0) && \text{(by (3.28))} \\ &\le \varepsilon h_K(\theta_0) && (\eta = \varepsilon/(4(n+1))). \end{aligned}$$

□

Claim 3.3.13 and Lemma 3.3.8 yield

$$\begin{aligned} \mathbb{P}\left(\exists\, \theta \in A(\alpha,\varepsilon,p) : h_{K_N}(\theta) < \frac{\varepsilon}{2} h_K(\theta)\right) &\le \#\mathcal{N} \max_{\theta_0 \in \mathcal{N}} \exp(-N\nu(\theta_0,\varepsilon)) \\ &\le \left(\frac{12(n+1)}{\varepsilon}\right)^n \exp\left(-e^{\alpha\varepsilon^p n/2}\right). \end{aligned}$$

At this point a remark on the possible range of ε is in order. Our desired conclusion (3.41) is a triviality if $\alpha^{3/2}\varepsilon^{3p/2-1} > 1$ (by the diameter bound (3.28)); hence we may assume $\alpha^{3/2}\varepsilon^{3p/2-1} \le 1$, in which case our assumption $4\alpha\varepsilon^p\sqrt{n} \ge 1$ yields the restriction $\varepsilon \ge 1/(8n^{3/4})$. Thus the latter probability is at most

$$\begin{aligned} (96(n+1))^{2n} \exp\left(-e^{\alpha\varepsilon^p n/2}\right) &\le \exp\left(2n\log(96(n+1)) - e^{\sqrt{n}/8}\right) \\ &\le \exp\left(-(1/2)e^{\sqrt{n}/8}\right), \end{aligned}$$

provided that n satisfies $2n\log(96(n+1)) \le (1/2)e^{\sqrt{n}/8}$. Therefore

$$h_K(\theta) \le 2\varepsilon^{-1} h_{K_N}(\theta) \text{ for each } \theta \in A(\alpha,\varepsilon,p) \tag{3.46}$$

with probability at least $1 - \exp\left(-e^{\sqrt{n}/8}/2\right)$.

Thus if K_N satisfies both (3.43) and (3.46), we have

$$\begin{aligned}\int_{A(\alpha,\varepsilon,p)} h_K(\theta)d\sigma(\theta) &\leq 2\varepsilon^{-1}\int_{A(\alpha,\varepsilon,p)} h_{K_N}(\theta)d\sigma(\theta)\\ &\leq 2\varepsilon^{-1}w(K_N)\\ &\leq 2\varepsilon^{-1}\overline{C}\sqrt{\log N}L_K t.\end{aligned}$$

While on the compliment $A(\alpha,\varepsilon,p)^{\mathrm{c}} = S^{n-1}\backslash A(\alpha,\varepsilon,p)$,

$$\begin{aligned}\int\limits_{A(\alpha,\varepsilon,p)^{\mathrm{c}}} h_K(\theta)d\sigma(\theta) &\leq \left(\int\limits_{S^{n-1}}\left(\max_{x\in K}|\langle x,\theta\rangle|\right)^2 d\sigma(\theta)\right)^{1/2}\sqrt{\sigma(A(\alpha,\varepsilon,p)^{\mathrm{c}})}\\ &\leq C_0 w(K)\sqrt{\sigma(A(\alpha,\varepsilon,p)^{\mathrm{c}})}\\ &\leq w(K)/4.\end{aligned}$$

Combining the latter estimates,

$$\begin{aligned}w(K) &= 2\int_{A(\alpha,\varepsilon,p)} h_K(\theta)d\sigma(\theta) + 2\int_{A(\alpha,\varepsilon,p)^{\mathrm{c}}} h_K(\theta)d\sigma(\theta)\\ &\leq 4\varepsilon^{-1}\overline{C}\sqrt{\log N}L_K t + w(K)/2,\end{aligned}$$

hence

$$w(K) \leq \widehat{C}\alpha^{3/2}\varepsilon^{3p/2-1}nL_K,$$

with $\widehat{C}$ an absolute constant. □

REFERENCES

[1] K. Ball. An elementary introduction to modern convex geometry. In *Flavors of geometry*, volume 31 of *Math. Sci. Res. Inst. Publ.*, pages 1–58. Cambridge Univ. Press, Cambridge, 1997.

[2] S. G. Bobkov and F. L. Nazarov. Large deviations of typical linear functionals on a convex body with unconditional basis. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 3–13. Birkhäuser, Basel, 2003.

[3] S. G. Bobkov and F. L. Nazarov. On convex bodies and log-concave probability measures with unconditional basis. In *Geometric Aspects of Functional Analysis*, volume 1807 of *Lecture Notes in Math.*, pages 53–69. Springer, Berlin, 2003.

[4] J. Bourgain. On the distribution of polynomials on high-dimensional convex sets. In *Geometric aspects of functional analysis (1989–90)*, volume 1469 of *Lecture Notes in Math.*, pages 127–137. Springer, Berlin, 1991.

[5] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. to appear in Journal of Functional Analysis.

[6] T. Figiel and N. Tomczak-Jaegermann. Projections onto Hilbertian subspaces of Banach spaces. *Israel J. Math.*, 33(2):155–171, 1979.

[7] A. Giannopoulos, A. Pajor, and G. Paouris. A note on subgaussian estimates for linear functionals on convex bodies. *Proc. Amer. Math. Soc.*, 135(8):2599–2606 (electronic), 2007.

[8] A. A. Giannopoulos. Notes on isotropic convex bodies, 2003. available at http://users.uoa.gr/ apgiannop/.

[9] A. A. Giannopoulos and V. D. Milman. Mean width and diameter of proportional sections of a symmetric convex body. *J. Reine Angew. Math.*, 497:113–139, 1998.

[10] A. A. Giannopoulos and V. D. Milman. Concentration property on probability spaces. *Adv. Math.*, 156(1):77–106, 2000.

[11] A. A. Giannopoulos and V. D. Milman. Asymptotic convex geometry: short overview. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 87–162. Kluwer/Plenum, New York, 2004.

[12] M. Hartzoulaki. *Probabilistic methods in the theory of convex bodies*. PhD thesis, University of Crete, 2003.

[13] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995.

[14] B. Klartag. On nearly radial marginals of high-dimensional probability measures. *J. Eur. Math. Soc.* (to appear).

[15] B. Klartag. Uniform almost sub-Gaussian estimates for linear functionals on convex sets. *Algebra i Analiz*, 19(1):109–148, 2007.

[16] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.

[17] A. E. Litvak, V. D. Milman, and A. Pajor. The covering numbers and "low M^*-estimate" for quasi-convex bodies. *Proc. Amer. Math. Soc.*, 127(5):1499–1507, 1999.

[18] A. E. Litvak, A. Pajor, M. Rudelson, and N. Tomczak-Jaegermann. Smallest singular value of random matrices and geometry of random polytopes. *Adv. Math.*, 195(2):491–523, 2005.

[19] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[20] G. Paouris. Super-Gaussian estimates for log-concave measures. in preparation.

[21] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.

[22] G. Pisier. Holomorphic semigroups and the geometry of Banach spaces. *Ann. of Math. (2)*, 115(2):375–392, 1982.

[23] G. Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[24] M. Rudelson and R. Vershynin. The Littlewood-Offord problem and invertibility of random matrices. *Adv. Math.*, 218(2):600–633, 2008.

[25] M. Schmuckenschläger. A concentration of measure phenomenon on uniformly convex bodies. In *Geometric Aspects of Functional Analysis (Israel, 1992–1994)*, volume 77 of *Oper. Theory Adv. Appl.*, pages 275–287. Birkhäuser, Basel, 1995.

CHAPTER 4

On determinants and the volume of random polytopes[1]

4.1 Introduction

Recent research in the theory of high-dimensional convex bodies has focused on random polytopes generated by points in an isotropic convex body. Recall that a convex body $K \subset \mathbb{R}^n$ is isotropic if it has volume one, center of mass at the origin and there is a constant L_K such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \text{for each } \theta \in S^{n-1}. \tag{4.1}$$

Any convex body $K \subset \mathbb{R}^n$ has an affine image which is isotropic and the isotropic constant L_K is an affine-invariant (see, e.g., [20]; cf. also §1.1.2). We generate a random polytope in K by sampling independent random vectors uniformly in K, say $X_1, \ldots, X_N$, and forming their (absolute) convex hull:

$$K_N := \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}. \tag{4.2}$$

The volume of K_N has been studied in several articles; for the most recent developments, see [5] and the references cited therein. There is also recent interest in the isotropic constants L_{K_N} of such polytopes [4] in which estimates for $\operatorname{vol}(K_N)$ play an important role (cf. §1.1.6).

[1] A version of this chapter has been accepted for publication. P. Pivovarov. On determinants and the volume of random polytopes in isotropic convex bodies. Geom. Dedicata.

Our first result involves lower bounds for the volume $\mathrm{vol}(K_N)$ when $N = n$. This case is treated in [4]. The approach taken in the latter article is to reduce the problem to the case when K is the Euclidean ball (via Steiner symmetrization). Our first proposition goes in a different direction and involves more general random polytopes.

Proposition 4.1.1. *Let $K^{(1)}, \ldots, K^{(n)}$ be isotropic convex bodies in $\mathbb{R}^n$. Let $X_1, \ldots, X_n$ be independent random vectors such that X_i is uniformly distributed in $K^{(i)}$ for $i = 1, \ldots, n$. Then with probability at least $1 - e^{-n}$,*

$$\mathrm{vol}\left(\mathrm{conv}\{\pm X_1, \ldots, \pm X_n\}\right) \geq \left(\frac{c_1}{n}\right)^{n/2} \prod_{i=1}^{n} L_{K^{(i)}}, \tag{4.3}$$

where c_1 is a positive absolute constant.

There are three improvements on [4, Proposition 2.2(ii)]: the vectors need not be sampled in the same body, the isotropic constants appear in the lower bound and the estimate on the probability is stronger. Lower bounds involving L_K are of interest because of the potential implications for the boundedness of L_K; at present it is unknown whether or not L_K is bounded above by an absolute constant (independent of K and the dimension n); for the most recent developments on this problem, see [12] and the references therein (cf. §1.1.2).

The proof of Proposition 4.1.1 involves a novel way of bounding the determinant $\det[X_1 \ldots X_n]$ and we give several applications.

Firstly, in the special case when each X_i is sampled in the Euclidean ball, we get an immediate proof of a known formula for $\mathbb{E}\,\mathrm{vol}\left(\mathrm{conv}\{\pm X_1, \ldots, \pm X_n\}\right)^q$ for $-1 < q < \infty$.

A second application involves zonotopes, i.e., Minkowski sums of line segments, of the form

$$Z_N := \sum_{i=1}^{N} [-X_i, X_i],$$

where the X_i are independent random vectors such that X_i is distributed uniformly in an isotropic convex body $K^{(i)}$ and $[-X_i, X_i] := \{\lambda X_i : -1 \leq \lambda \leq 1\}$. The volume of Z_N is considered in [3] and the argument also involves a reduction to the Euclidean ball via Steiner symmetrization; another example in which information about the isotropic constants is lost. We give an elementary direct estimate for the expected volume $\mathbb{E}\,\mathrm{vol}(Z_N)^{1/n}$ which retains information about the isotropic constants $L_{K^{(i)}}$ and improves a result from [3].

Our last application concerns the sharpness of Hadamard's determinant inequality for random matrices.

As with many results concerning convex bodies, our results actually hold in the more general setting of log-concave measures. We briefly recall the relevant definitions in §4.2. The analogue of Proposition 4.1.1 in the log-concave case is proved in §4.3.1. A similar result holds for random simplices, i.e., for $\operatorname{conv}\{X_1,\ldots,X_{n+1}\}$, see Proposition 4.3.6 below. §4.3.2 contains the special case of the Euclidean ball. §4.4 & 4.5 treat the volume of zonotopes Z_N and Hadamard's inequality, respectively.

Lastly, we mention some notation and conventions. Our results are most meaningful when the dimension n is large. Throughout, $c, c_1, C, C', \ldots$, etc. denote absolute constants (in particular, independent of n and the given measures). The symbol $|\cdot|$ will serve multiple roles, including the standard Euclidean norm on $\mathbb{R}^n$ and the absolute value of a scalar, the use of which will be clear from the context.

4.2 Isotropicity and marginals of log-concave measures

Here we list some definitions and basic facts concerning log-concave measures. We refer the reader to the introductory pages of [13] and the references listed there for a more complete treatment.

Recall that a measure μ on $\mathbb{R}^n$ is said to be log-concave if for any $\lambda \in [0,1]$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda} \tag{4.4}$$

for all compact $A, B \subset \mathbb{R}^n$. Here $\lambda A + (1-\lambda)B := \{\lambda a + (1-\lambda)b : a \in A, b \in B\}$.

Similarly, a function $f : \mathbb{R}^n \to [0,\infty)$ is log-concave if for any $\lambda \in [0,1]$ and and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}.$$

Basic examples of log-concave measures include Lebesgue measure, standard Gaussian measure and, most importantly for us, uniform distribution on a convex body $K \subset \mathbb{R}^n$:

$$\mu(A) = \frac{\operatorname{vol}(A \cap K)}{\operatorname{vol}(K)}$$

for any measurable $A \subset \mathbb{R}^n$. A theorem of Borell [2] characterizes log-concave measures that are not supported on any proper affine subspace as those that are absolutely continuous with respect to Lebesgue measure and have log-concave densities (cf. Proposition 2.1.3 in Chapter 2).

A probability measure μ on $\mathbb{R}^n$ is isotropic if its center of mass is the origin, i.e.,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0, \quad \text{for each } \theta \in S^{n-1}, \tag{4.5}$$

and

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x) = 1 \quad \text{for each } \theta \in S^{n-1}. \tag{4.6}$$

In particular, if μ is isotropic then for any subspace $E \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |P_E x|^2 d\mu(x) = \dim(E), \tag{4.7}$$

where P_E denotes the orthogonal projection onto E.

It is known that for any probability measure μ on $\mathbb{R}^n$ that is not supported on a proper affine subspace there exists an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu \circ T^{-1}$ is isotropic.

Suppose now that μ is a log-concave probability measure on $\mathbb{R}^n$. If E is a subspace of $\mathbb{R}^n$, then the marginal of μ with respect to E is the measure $\mu \circ P_E^{-1}$ on E defined by

$$\mu \circ P_E^{-1}(A) = \mu(\{x \in \mathbb{R}^n : P_E x \in A\})$$

for measurable $A \subset E$. One can check that $\mu \circ P_E^{-1}$ is itself log-concave and, if μ is isotropic then so too is $\mu \circ P_E^{-1}$.

We will also make use of the following lemma.

Lemma 4.2.1. *Let μ be an isotropic log-concave probability on $\mathbb{R}^n$. Let $0 < s < t$ and let $\theta \in S^{n-1}$. Then*

$$\mu(\{x \in \mathbb{R}^n : s < \langle x, \theta \rangle < t\}) \le t - s$$

The proof can be found in [13, §2].

Remark 4.2.2. Note the difference between the definitions of isotropicity for convex bodies (cf. (4.1)) and for probability measures (cf. (4.6)). In particular, if X is a random vector distributed uniformly in an isotropic convex body $K \subset \mathbb{R}^n$ (as defined by (4.1)), then X/L_K is distributed according to an isotropic log-concave probability measure.

Moment comparisons

We will use some well-known facts about comparison of moments.

Lemma 4.2.3. *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Suppose that $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $-1 < q < 0 < p < \infty$. Then*

$$\frac{1}{C'p}\left(\int_{\mathbb{R}^n}\|x\|^p d\mu(x)\right)^{1/p} \leq \int_{\mathbb{R}^n}\|x\|d\mu(x) \leq \frac{4e}{1+q}\left(\int_{\mathbb{R}^n}\|x\|^q d\mu(x)\right)^{1/q}, \tag{4.8}$$

where $C' \geq 1$ is an absolute constant.

The left-most inequality is standard (it can be proved by applying Borell's lemma, e.g., [21, Appendix III]; in fact, it holds for semi-norms). The right-most inequality is due to Guedon [10]. For related developments on negative moments, see [15], [16] and [23].

For the reader's convenience, we isolate a particular case of Lemma 4.2.3 used below.

Lemma 4.2.4. *Let X be a random vector distributed according to an isotropic log-concave probability measure μ on $\mathbb{R}^n$. Let $1 \leq \ell \leq n$ and let $E \subset \mathbb{R}^n$ be a subspace with $\dim E = \ell$. Then the random variable*

$$Y := \frac{|P_E X|}{\sqrt{\ell}}$$

satisfies

$$\mathbb{E}|Y|^{-1/2} \leq C, \tag{4.9}$$

where $C > 0$ is an absolute constant.

Proof. Observe that

$$\begin{aligned}
\int_{\mathbb{R}^n}|P_E x|^{-1/2}d\mu(x) &= \int_E |x|^{-1/2}d\mu\circ P_E^{-1}(x) \\
&\leq (8e)^{1/2}\left(\int_E |x|d\mu\circ P_E^{-1}(x)\right)^{-1/2} && \text{(by Lemma 4.2.3)} \\
&\leq C\left(\int_E |x|^2 d\mu\circ P_E^{-1}(x)\right)^{-1/4} && \text{(by Lemma 4.2.3)} \\
&= C(\sqrt{\ell})^{-1/2}. && \text{(by the isotropicity of } \mu\circ P_E^{-1})
\end{aligned}$$

Here C is an absolute constant that depends only C' from Lemma 4.2.3. $\square$

4.3 Random determinants

We begin with an elementary lemma about determinants of random matrices with independent columns distributed according to isotropic probability measures (no assumption of log-concavity).

Lemma 4.3.1. *Let $\mu_1, \ldots, \mu_n$ be isotropic probability measures on $\mathbb{R}^n$ (as in (4.5) and (4.6)). Let $X_1, \ldots, X_n$ be independent random vectors such that X_i is distributed according to μ_i for each $i = 1, \ldots, n$. Suppose also that $X_1, \ldots, X_n$ are linearly independent with probability one. Then*

$$\mathbb{E}|\det[X_1, \ldots, X_n]|^2 = n!. \tag{4.10}$$

Proof. Set

$$V_0 := \{0\}, \text{ and } V_k := \operatorname{span}\{X_1, \ldots, X_k\} \text{ for } k = 1, \ldots, n-1. \tag{4.11}$$

Note that

$$|\det[X_1, \ldots, X_n]| = |X_1||P_{V_1^\perp} X_2| \cdots |P_{V_{n-2}^\perp} X_{n-1}||P_{V_{n-1}^\perp} X_n|. \tag{4.12}$$

Apply Fubini's Theorem iteratively, integrating first with respect to X_n, then X_{n-1} and so on. At each stage, use the isotropicity condition (4.7). □

Remark 4.3.2. In the case when all μ_i are equal to the uniform measure on a convex body $K \subset \mathbb{R}^n$, the latter lemma is a well-known fact attributed to Blaschke (see, e.g., [8]). Our argument is somewhat shorter as we avoid brute-force expansion of the determinant.

4.3.1 Random cross-polytopes and simplices

We now formulate and prove the analogue of Proposition 4.1.1 for log-concave measures.

Proposition 4.3.3. *Let $\mu_1, \ldots, \mu_n$ be isotropic log-concave probability measures on $\mathbb{R}^n$. Let $X_1, \ldots, X_n$ be independent random vectors such that X_i is distributed according to μ_i for $i = 1, \ldots, n$. Then with probability at least $1 - e^{-n}$,*

$$\operatorname{vol}(\operatorname{conv}\{\pm X_1, \ldots, \pm X_n\}) \geq \left(\frac{c_1}{n}\right)^{n/2}, \tag{4.13}$$

where c_1 is a positive absolute constant.

Proposition 4.1.1 in the introduction follows immediately by Remark 4.2.2.

Proof. For each $k = 0, 1, \ldots, n-1$, let V_k be as in (4.11). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator mapping the standard unit vector basis (e_i) to (X_i), i.e., $Te_i = X_i$ for $i = 1, \ldots, n$. Then $\mathrm{conv}\{\pm X_1, \ldots, \pm X_n\} = T[\mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}]$ and

$$\mathrm{vol}(\mathrm{conv}\{\pm X_1, \ldots, \pm X_n\}) = \frac{2^n}{n!} |\det T|. \tag{4.14}$$

For each $k = 1, \ldots, n$, set

$$Y_k := \frac{|P_{V_{k-1}^\perp} X_k|}{\sqrt{n-k+1}}.$$

Note that $Y_k > 0$ with probability 1. Suppose now that X_1, X_2,..., X_{k-1} are fixed. Let $\mathbb{E}_k$ denote expectation in X_k (with $X_1, \ldots, X_{k-1}$ fixed). Letting C denote the constant from Lemma 4.2.4, we have

$$\mathbb{E}_k Y_k^{-1/2} \leq C.$$

Applying Fubini's Theorem iteratively (integrating first with respect to X_n, then X_{n-1} and so on, as in the proof of Lemma 4.3.1), we obtain

$$\mathbb{E} \prod_{k=1}^n Y_k^{-1/2} \leq C^n. \tag{4.15}$$

Setting $c_1 := (eC)^{-2}$ and using (4.12), we have

$$\begin{aligned}
\mathbb{P}\left(|\det T| < c_1^n \sqrt{n!} \right) &= \mathbb{P}\left(\prod_{k=1}^n |P_{V_{k-1}^\perp} X_k| < c_1^n \sqrt{n!} \right) \\
&= \mathbb{P}\left(\prod_{k=1}^n Y_k < c_1^n \right) \\
&= \mathbb{P}\left(\prod_{k=1}^n Y_k^{-1/2} > c_1^{-n/2} \right) \\
&\leq e^{-n},
\end{aligned}$$

where the inequality follows from Markov, (4.15) and our choice of c_1. The proof now follows from (4.14). □

Remark 4.3.4. By Lemma 4.3.1 and (4.14), the lower bound (4.13) captures the correct dependence on n.

For subsequent use, we isolate one consequence of the bound for $\det[X_1, \ldots, X_n]$ given in latter proof.

Corollary 4.3.5. *Let $\mu_1, \dots, \mu_n$ be isotropic log-concave probability measures on $\mathbb{R}^n$. Let $X_1, \dots, X_n$ be independent random vectors such that X_i is distributed according to μ_i for $i = 1, \dots, n$. Then*

$$\mathbb{E}|\det[X_1, \dots, X_n]|^{1/n} \geq c'' \sqrt{n},$$

where c'' is a positive absolute constant.

In the case when all μ_i are equal to the uniform distribution on a convex body $K \subset \mathbb{R}^n$, the latter corollary appears in [20, §3.7]. The benefit of our argument is that we avoid direct expansion of the determinant.

There is an analogue of Proposition 4.3.3 for random simplices.

Proposition 4.3.6. *Let $\mu_1, \dots, \mu_{n+1}$ be isotropic log-concave probability measures on $\mathbb{R}^n$. Suppose that $X_1, \dots, X_{n+1}$ are independent random vectors such that X_i is distributed according to μ_i for each $i = 1, \dots, n+1$. Then with probability at least $1 - \overline{C} e^{-c'n}$,*

$$\mathrm{vol}\left(\mathrm{conv}\left\{X_1, \dots, X_{n+1}\right\}\right) \geq \left(\frac{\widetilde{c_1}}{n}\right)^{n/2},$$

where $\widetilde{c_1}$, $\overline{C}$ and c' are absolute constants.

For the proof, we follow the argument given in [4, Proposition 2.2(ii)], which is based on [14, Lemma 3.3].

For clarity of exposition, we will prove two lemmas about the volume of arbitrary (non-random) simplices involving a reduction to the symmetric case. The first is a consequence of the Rogers-Shepard difference body inequality [24]: for any convex body $K \subset \mathbb{R}^n$,

$$\mathrm{vol}\left(K - K\right) \leq \binom{2n}{n} \mathrm{vol}\left(K\right). \tag{4.16}$$

Lemma 4.3.7. *Let $x_1, \dots, x_{n+1} \in \mathbb{R}^n$ be affinely independent points. Then*

$$\mathrm{vol}\left(\mathrm{conv}\left\{x_1, \dots, x_{n+1}\right\}\right) \geq 4^{-n} \mathrm{vol}\left(\mathrm{conv}\left\{\pm(x_i - x_{n+1})\right\}_{i=1}^n\right). \tag{4.17}$$

Proof. Set $W = \{0, x_1 - x_{n+1}, \ldots, x_n - x_{n+1}\}$. Then by (4.16), we have

$$\begin{aligned}
\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_{n+1}\}) &= \operatorname{vol}(\operatorname{conv} W) \\
&\geq 4^{-n} \operatorname{vol}(\operatorname{conv} W - \operatorname{conv} W) \\
&= 4^{-n} \operatorname{vol}(\operatorname{conv}(W - W)) \\
&\geq 4^{-n} \operatorname{vol}(\operatorname{conv}\{\pm(x_i - x_{n+1})\}_{i=1}^{n}).
\end{aligned}$$

□

Lemma 4.3.8. *Suppose that $x_1, \ldots, x_{n+1}$ are affinely independent. Suppose also that $v \in \mathbb{R}^n$ satisfies $\langle v, x_i \rangle = 1$ for each $i = 1, \ldots, n$. Then*

$$\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_{n+1}\}) \geq 4^{-n} |1 - \langle v, x_{n+1} \rangle| \operatorname{vol}(\operatorname{conv}\{\pm x_1, \ldots, \pm x_n\}). \tag{4.18}$$

Proof. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by $F(x) = x - \langle v, x \rangle x_{n+1}$. Then

$$F[\operatorname{conv}\{\pm x_1, \ldots, \pm x_n\}] = \operatorname{conv}\{\pm(x_i - x_{n+1})\}_{i=1}^{n}$$

and hence

$$|\det(F)| \operatorname{vol}(\operatorname{conv}\{\pm x_1, \ldots, \pm x_n\}) = \operatorname{vol}(\operatorname{conv}\{\pm(x_i - x_{n+1})\}_{i=1}^{n}). \tag{4.19}$$

The proof now follows from Lemma (4.3.7) the fact that $|\det(F)| = |1 - \langle v, x_{n+1} \rangle|$. □

Proof of Proposition 4.3.6. With probability one, $X_1, \ldots, X_n$ are linearly independent and $X_1, \ldots, X_{n+1}$ are affinely independent. Thus we can define $V = V(X_1, \ldots, X_n)$ by $\langle V, X_i \rangle = 1$ for each $i = 1, \ldots, n$.

By Paouris' theorem [22, Theorem 1.1 & §8] (cf. Theorem 1.1.7) formulated for log-concave measures, we have

$$\mathbb{P}(|X_1| \leq C_0 n) \geq 1 - e^{-c_0 n}.$$

where C_0 and c_0 are positive absolute constants. Next, note that $1 = \langle V, X_1 \rangle \leq |V||X_1|$ and hence

$$\mathbb{P}(|V| \geq 1/(C_0 n)) \geq 1 - e^{-c_0 n}. \tag{4.20}$$

Observe that

$$\begin{aligned}\mathbb{P}\left(|1-\langle V,X_{n+1}\rangle|<e^{-n}\right) &= \mathbb{E}_{X_1,\ldots,X_n}\mathbb{P}_{X_{n+1}}(1-e^{-n}<\langle V,X_{n+1}\rangle<1+e^{-n})\\ &= \mathbb{E}_{X_1,\ldots,X_n}\mathbb{P}_{X_{n+1}}\left(\frac{1-e^{n}}{|V|}<\langle V/|V|,X_{n+1}\rangle<\frac{1+e^{-n}}{|V|}\right)\\ &\leq 2C_0ne^{-n}+e^{-c_0n},\end{aligned}$$

where the inequality follows from Lemma 4.2.1 and (4.20).

Thus by Lemma 4.3.7, we have

$$\mathrm{vol}\left(\mathrm{conv}\{X_1,\ldots,X_{n+1}\}\right)\geq 4^{-n}|1-\langle V,X_{n+1}\rangle|\,\mathrm{vol}\left(\mathrm{conv}\{\pm X_1,\ldots,\pm X_n\}\right). \tag{4.21}$$

By Proposition 4.3.3, we have

$$\mathbb{P}\left(\mathrm{vol}\left(\mathrm{conv}\{\pm X_1,\ldots,\pm X_n\}\right)\geq (c_1/n)^{n/2}\right)\geq 1-e^{-n}.$$

Thus with probability at least $1-e^{-n}-2C_o ne^{-n}-e^{-c_0n}$, we have

$$\mathrm{vol}\left(\mathrm{conv}\{X_1,\ldots,X_{n+1}\}\right)\geq\left(\frac{c_1}{(4e)^2n}\right)^{n/2}. \tag{4.22}$$

Finally, choose absolute constants $\overline{C}$ and c' such that $e^{-n}+2C_o ne^{-n}+e^{-c_0n}<\overline{C}e^{-c'n}$. □

4.3.2 Moment formulas for random cross-polytopes in the ball

Since the Euclidean ball plays a unique role in volume estimates for random polytopes (as in, e.g., [4]), we give an elementary direct proof of the special case of Proposition 4.1.1 when each $K^{(i)}$ is the Euclidean ball of volume one $\overline{B_2^n}$. Our argument improves the estimate on the probability given in [4, Proposition 2.2(ii)] and unifies the approach for such volume problems. In the process, we also get a short proof of a known formula for $\mathbb{E}\,\mathrm{vol}\left(\mathrm{conv}\{\pm X_i\}_{i=1}^n\right)^q$, where the X_i are independent random vectors in $\overline{B_2^n}$ and $-1<q<\infty$.

As usual, we will denote the volume of the Euclidean ball of radius one in $\mathbb{R}^n$ by ω_n; the Haar measure on S^{n-1} by σ.

Lemma 4.3.9. *Let $1\leq k\leq n$. Suppose that $E\subset\mathbb{R}^n$ is a subspace of dimension k and let P_E*

denote the orthogonal projection onto E. Then for any $q \in (-k, \infty)$,

$$\int_{S^{n-1}} |P_E\theta|^q d\sigma(\theta) = \frac{k\Gamma(\frac{k+q}{2})\Gamma(1+\frac{n}{2})}{n\Gamma(\frac{n+q}{2})\Gamma(1+\frac{k}{2})}. \tag{4.23}$$

The proof involves a standard rotational-invariance argument.

Proof. By rotational invariance of σ, we may assume that $E = \text{span}\{e_1, \dots, e_k\}$. Using polar coordinates, we have

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |P_E x|^q e^{-|x|^2/2} dx &= \frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} |x|^q e^{-|x|^2/2} dx_k \dots dx_1 \\ &= \frac{k\omega_k}{(2\pi)^{k/2}} \int_0^\infty r^{k+q-1} e^{-r^2/2} dr \\ &= \frac{2^{q/2} k\Gamma(\frac{k+q}{2})}{2\Gamma(1+\frac{k}{2})} \end{aligned}$$

but also

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |P_E x|^q e^{-|x|^2/2} dx &= \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^\infty r^{n+q-1} e^{-r^2/2} dr \int\limits_{S^{n-1}} |P_E\theta|^q d\sigma(\theta) \\ &= \frac{2^{q/2} n\Gamma(\frac{n+q}{2})}{2\Gamma(1+\frac{n}{2})} \int\limits_{S^{n-1}} |P_E\theta|^q d\sigma(\theta). \end{aligned}$$

□

Proposition 4.3.10. *Let $X_1, \dots, X_n$ be independent random vectors uniformly distributed in the Euclidean ball of volume one $\overline{B_2^n}$. Then for any $q \in (-1, \infty)$,*

$$\mathbb{E}\,\text{vol}\,(\text{conv}\,\{\pm X_i\}_{i=1}^n)^q = \left(\frac{2^n\Gamma(1+\frac{n}{2})}{\pi^{n/2} n!}\right)^q \left(\frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+\frac{n+q}{2})}\right)^n \prod_{k=1}^n \frac{\Gamma(\frac{k+q}{2})}{\Gamma(\frac{k}{2})}. \tag{4.24}$$

Similar facts have appeared in the literature in several places (via various methods); see, e.g., [19], [25], [17], [18]. The important range for us is $q \in (-1, 0)$.

Proof. Let $E \subset \mathbb{R}^n$ be a subspace of dimension k. Integrating in polar coordinates and applying Lemma 4.3.9, we have

$$\begin{aligned}\int_{\overline{B_2^n}} |P_E x|^q dx &= n\omega_n \int_0^{\omega_n^{-1/n}} r^{n+q-1} dr \int_{S^{n-1}} |P_E \theta|^q d\sigma(\theta) \\ &= \omega_n^{-q/n} \frac{k}{n+q} \frac{\Gamma(\frac{k+q}{2})\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n+q}{2})\Gamma(1+\frac{k}{2})}.\end{aligned}$$

The proposition follows from the determinant formulas (4.12), (4.14) and Fubini's theorem (integrating first with respect to X_n, then X_{n-1} and so on). □

Proposition 4.3.11. *Let $X_1, \ldots, X_n$ be independent random vectors uniformly distributed in $\overline{B_2^n}$. Then with probability at least $1-e^{-n}$,*

$$|\mathrm{conv}\{\pm X_1, \ldots, \pm X_n\}| \geq \left(\frac{c_2}{n}\right)^{n/2}, \tag{4.25}$$

where c_2 is a positive absolute constant.

Proof. By Proposition 4.3.10, Stirling's formula and the fact that

$$\lim_{x\to\infty} \frac{\Gamma(x+\alpha)}{x^\alpha \Gamma(x)} = 1 \quad (\alpha \in \mathbb{R}),$$

there is an absolute constant $\widetilde{C}$ such that

$$\mathbb{E}\,\mathrm{vol}\,(\mathrm{conv}\{\pm X_i\}_{i=1}^n)^{-1/2} \leq (\widetilde{C}n)^{n/4}. \tag{4.26}$$

Set $c_2 := (e^4\widetilde{C})^{-1}$. Then

$$\begin{aligned}\mathbb{P}\left(\mathrm{vol}\,(\mathrm{conv}\{\pm X_i\}_{i=1}^n) < (c_2/n)^{n/2}\right) &= \mathbb{P}\left(\mathrm{vol}\,(\mathrm{conv}\{\pm X_i\}_{i=1}^n)^{-1/2} > (c_2/n)^{-n/4}\right) \\ &\leq (c_2/n)^{n/4} \mathbb{E}\,\mathrm{vol}\,(\mathrm{conv}\{\pm X_i\}_{i=1}^n)^{-1/2} \\ &\leq e^{-n}.\end{aligned}$$

□

4.4 Zonotopes and a geometric inequality

Definition 4.4.1. A Minkowski sum of line segments $S_1,\dots,S_N$ in $\mathbb{R}^n$, $Z := \sum_{i=1}^N S_i$ is called a *zonotope*.

We will consider zonotopes generated by line segments of the form $[-x,x] = \{\alpha x : -1 \le \alpha \le 1\}$ or $[0,x] = \{\alpha x : 0 \le \alpha \le 1\}$, where $x \in \mathbb{R}^n$.

If $x_1,\dots,x_N \in \mathbb{R}^n$, and $M : \mathbb{R}^N \to \mathbb{R}^n$ is the linear operator defined by $Me_i = x_i$ for $i = 1,\dots,N$, then the zonotope $Z = \sum_{i=1}^N [-x_i,x_i]$ is the image of the cube B_∞^n under M since

$$MB_\infty^N = \left\{\sum_{i=1}^N \lambda_i x_i : \lambda = (\lambda_i) \in B_\infty^N\right\} = \left\{\sum_{i=1}^N \lambda_i x_i : |\lambda_i| \le 1, i = 1,\dots,N\right\} = \sum_{i=1}^N [-x_i,x_i].$$

In this section we discuss the volume of random zonotopes and their application to a multi-integral norm first considered by Bourgain, Meyer, Milman, and Pajor [3]. Since the latter article deals with convex bodies, we will work exclusively with convex bodies (for ease of comparison with the results from [3] and to make clear the exact dependence on the isotropic constants of the associated convex bodies, which are “hidden” in the log-concave setting).

Let $V_1,\dots,V_N$ be convex bodies of volume one in $\mathbb{R}^n$ where $N \ge n$. For $p \ge 0$, let $\mathcal{I}_p(V_1,\dots,V_N)$ be the expected p-th power of the volume of the zonotope $\sum_{i=1}^N [0,X_i]$, where the X_i's are independent random vectors with X_i distributed uniformly in V_i, i.e.,

$$\mathcal{I}_p(V_1,\dots,V_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}\left(\sum_{i=1}^N [0,x_i]\right)^p dx_1 \dots dx_N.$$

If $V_i = V$ for each $i = 1,\dots,N$, we will use the notation $\mathcal{I}_p(V,N)$ instead of $\mathcal{I}_p(V_1,\dots,V_N)$.

In [3, Theorem 1.3], it is proved, via Steiner symmetrization (see Appendix) that if $V_1,\dots,V_N$ are convex bodies of volume one, then for each $p \ge 0$,

$$\mathcal{I}_p(V_1,\dots,V_N) \ge \mathcal{I}_p(\overline{B_2^n},N); \tag{4.27}$$

also, ([3, Lemma 2.6]) for $p = 1/n$,

$$\mathcal{I}_{1/n}(\overline{B_2^n},N) \ge \frac{\widetilde{c_3}N}{\sqrt{n}},$$

where $\widetilde{c_3} > 0$ is an absolute constant (here, as above, $\overline{B_2^n}$ is the Euclidean ball of volume one).

If we assume that each V_i is isotropic, we can use Corollary 4.3.5 to estimate $I_{1/n}(V_1,\dots,V_N)$ directly; we thus retain information about the isotropic constants.

Proposition 4.4.2. *Let $V_1,\dots,V_N$ be isotropic convex bodies in $\mathbb{R}^n$. Then*

$$I_{1/n}(V_1,\dots,V_N) \geq \frac{\widetilde{c_4}N}{\sqrt{n}} \left(\frac{1}{\binom{N}{n}} \sum_{\substack{I\subset\{1,\dots,N\}\\|I|=n}} \left(\prod_{i\in I} L_{V_i}\right)^{1/n} \right),$$

where $\widetilde{c_4} > 0$ is an absolute constant.

Proof. Let $X_1,\dots,X_N$ be independent random vectors such that X_i is distributed uniformly in V_i for each $i = 1,\dots,N$. For each $I \subset \{1,\dots N\}$ with $|I| = n$, set $d_I := |\det[X_i]_{i\in I}|$. By Corollary 4.3.5 (and Remark 4.2.2), we have

$$\mathbb{E}d_I^{1/n} \geq \widetilde{c_4}\sqrt{n} \left(\prod_{i\in I} L_{V_i}\right)^{1/n}, \tag{4.28}$$

where $\widetilde{c_4}$ is an absolute constant. Using the zonotope volume formula

$$\mathrm{vol}\left(\sum_{i=1}^{N}[0,X_i]\right) = \sum_{\substack{I\subset\{1,\dots,N\}\\|I|=n}} |\det[X_i]_{i\in I}| \tag{4.29}$$

(see, e.g., [20, pg 73]), together with concavity of $x \mapsto x^{1/n}$, we have

$$\begin{aligned} I_{1/n}(V_1,\dots,V_N) &= \mathbb{E}\left(\sum_{|I|=n} d_I\right)^{1/n} \\ &\geq \binom{N}{n}^{1/n-1} \sum_{|I|=n} \mathbb{E}d_I^{1/n} \\ &\geq \frac{\widetilde{c_4}N}{\sqrt{n}} \left(\binom{N}{n}^{-1} \sum_{\substack{I\subset\{1,\dots,N\}\\|I|=n}} \left(\prod_{i\in I} L_{V_i}\right)^{1/n} \right). \end{aligned}$$

□

Remark 4.4.3. In the latter proposition, if $V_i = V$ for each $i = 1,\dots,N$, then $I_{1/n}(V,N) \geq \widetilde{c_4} N L_V/\sqrt{n}$. On the other hand, setting $I_0 = \{1,\dots,n\}$, and applying Jensen's inequality, we have

$$I_{1/n}(V,N) = \mathbb{E}\Big(\sum_{|I|=n} d_I\Big)^{1/n} \leq \binom{N}{n}^{1/n} (\mathbb{E}d_{I_0})^{1/n} \leq \widetilde{C_4} N L_V/\sqrt{n},$$

where $\widetilde{C_4}$ is an absolute constant (for the second inequality, we have used Lemma 4.3.1, Remark 4.2.2 and Stirling's formula).

A multi-integral norm

Suppose now that $V_1,\dots,V_N$ and K are centrally-symmetric convex bodies in $\mathbb{R}^n$, i.e., $V_i = -V_i$ and $K = -K$. For $\lambda = (\lambda_i) \in \mathbb{R}^N$, let

$$\|\lambda\| := \int_{V_1} \cdots \int_{V_N} \Big\| \sum_{i=1}^N \lambda_i x_i \Big\|_K dx_N \dots dx_1. \tag{4.30}$$

In [3, Theorem 1.4], it is proved that, in the case $N = n$, if $V_i = V$ for each $i = 1,\dots,n$, $\mathrm{vol}(V) = \mathrm{vol}(K) = 1$ and if $X = (\mathbb{R}^n, \|\cdot\|_K)$ has cotype q, then

$$\|\lambda\| \geq c_q \sqrt{n} \left(\prod_{i=1}^n |\lambda_i|\right)^{1/n} L_V, \tag{4.31}$$

where c_q is a constant that depends on the cotype-q constant of X.

For more recent developments, see [9], where a lower ℓ_2 bound for $\|\cdot\|$ is established (not involving the isotropic constant L_V); see also [7].

Using our Proposition 4.4.2, we can prove (4.31) without the cotype assumption on $X = (\mathbb{R}^n, \|\cdot\|_K)$. For the proof, we use the following proposition ([3, Proposition 2.1]).

Proposition 4.4.4. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body with* $\mathrm{vol}(K) = 1$ *and let $x_1,\dots,x_N \in \mathbb{R}^n$. Let $\varepsilon_1,\dots,\varepsilon_N$ be independent random variables with* $\mathbb{P}(\varepsilon_i = 1/2) = \mathbb{P}(\varepsilon_i = -1/2)$. *Then*

$$\mathbb{E}_\varepsilon \Big\| \sum_{i=1}^N \varepsilon_i x_i \Big\|_K \geq \frac{\widetilde{c_5}\sqrt{n}}{\sqrt{N}} \mathrm{vol}\left(\sum_{i=1}^N [-x_i, x_i]\right)^{1/n},$$

where $\widetilde{c_5}$ is an absolute constant.

Proposition 4.4.5. *Let $V_1, \dots, V_N$ and K be centrally-symmetric convex bodies of volume one in $\mathbb{R}^n$. Suppose also that V_i is isotropic for each $i = 1, \dots, N$. Then for each $\lambda \in \mathbb{R}^N$,*

$$\|\lambda\| \geq c\sqrt{N}\left(\binom{N}{n}^{-1} \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=n}} \left(\prod_{i\in I}|\lambda_i|\right)^{1/n} \left(\prod_{i \in I} L_{V_i}\right)^{1/n}\right),$$

where c is a positive absolute constant.

Proof. Fix $\lambda \in \mathbb{R}^N$. Let $X_1 \dots, X_N$ be independent random vectors such that X_i is distributed uniformly in V_i for each $i = 1, \dots, N$. Let $\mathbb{E}$ denote expectation in $X_1, \dots, X_N$. Let $\varepsilon_1, \dots, \varepsilon_N$ be independent random variables (also independent of $X_1, \dots, X_N$) such that $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$ and let $\mathbb{E}_\varepsilon$ denote expectation in $\varepsilon_1, \dots, \varepsilon_N$. For each $I \subset \{1, \dots N\}$ with $|I| = n$, set $d_{I,\lambda} := |\det[\lambda_i X_i]_{i\in I}|$. Then

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=1}^N \lambda_i X_i\right\|_K &= \mathbb{E}\left(\mathbb{E}_\varepsilon \left\|\sum_{i=1}^N \varepsilon_i \lambda_i X_i\right\|_K\right) \\
&\geq \frac{\widetilde{c_5}\sqrt{n}}{\sqrt{N}}\mathbb{E}\left(\mathrm{vol}\left(\sum_{i=1}^N [-\lambda_i X_i, \lambda_i X_i]\right)\right)^{1/n} && \text{(by Prop. 4.4.4)} \\
&= \frac{2\widetilde{c_5}\sqrt{n}}{\sqrt{N}}\mathbb{E}\left(\sum_{|I|=n} d_{I,\lambda}\right)^{1/n} && \text{(cf. (4.29))} \\
&\geq \frac{2\widetilde{c_5}\sqrt{n}}{\sqrt{N}}\binom{N}{n}^{1/n-1} \sum_{|I|=n} \mathbb{E} d_{I,\lambda}^{1/n} && \text{(concavity of } x \mapsto x^{1/n}) \\
&\geq c\sqrt{N}\left(\binom{N}{n}^{-1} \sum_{|I|=n} \left(\prod_{i\in I}|\lambda_i|\right)^{1/n} \left(\prod_{i\in I} L_{V_i}\right)^{1/n}\right), && \text{(cf. (4.28))}
\end{aligned}$$

where $\widetilde{c_5}$ and c are positive absolute constants.

□

Corollary 4.4.6. *Let $V_1, \dots, V_n$ and K be centrally-symmetric convex bodies of volume one in $\mathbb{R}^n$. Suppose also that V_i is isotropic for each $i = 1, \dots, n$. Then for any $\lambda \in \mathbb{R}^n$,*

$$\|\lambda\| \geq c\sqrt{n}\left(\prod_{i=1}^n |\lambda_i|\right)^{1/n} \left(\prod_{i=1}^n L_{V_i}\right)^{1/n},$$

where $c > 0$ is an absolute constant.

Remark 4.4.7. In the latter corollary, if $V_i = V$ for each $i = 1,\ldots,n$, then the isotropicity assumption on V may be dropped (since, for $X_1,\ldots,X_n \in V$, $|\det[TX_1,\ldots,TX_n]| = |\det[X_1\ldots,X_n]|$ for any $T \in SL(n)$).

4.5 Hadamard's inequality for matrices with independent log-concave columns

Let A be a matrix with columns $A_1,\ldots,A_n$. Hadamard's inequality states that

$$|\det A| \leq \prod_{i=1}^{n} |A_i|; \tag{4.32}$$

when each A_i is non-zero, equality holds if and only if the A_i are orthogonal. The ratio

$$h(A) := \frac{|\det A|}{\prod_{i=1}^{n} |A_i|} \tag{4.33}$$

has been studied for various random matrices A (e.g., [11], [6], [1]). For instance, the case when the A_i are uniformly distributed on S^{n-1} is examined by Dixon in [6], where he computes the mean and variance of $\log h(A)$ and proves that for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1/4-\varepsilon} e^{-n/2} \leq h(A) \leq n^{-1/4+\varepsilon} e^{-n/2}\right) = 1.$$

In this case, of course Lemma 4.3.9 above and the determinant formula (4.12) give

$$\mathbb{E} h(A)^q = \frac{n!}{n^n} \left(\frac{\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n+q}{2})}\right)^n \prod_{k=1}^{n} \frac{\Gamma(\frac{k+q}{2})}{\Gamma(1+\frac{k}{2})} \tag{4.34}$$

for $-1 < q < \infty$. One can also calculate $\mathbb{E}\log h(A)$ by using (4.34) and the fact that

$$\exp(\mathbb{E}\log h(A)) = \lim_{q\to 0} (\mathbb{E} h(A)^q)^{1/q}.$$

More generally, suppose that μ is a log-concave probability measure on $\mathbb{R}^n$ with center of mass at the origin. Let $T \in GL(n)$ be such that $\mu \circ T^{-1}$ is isotropic and set $S = T/|\det T|^{1/n}$. Then, as in [20, pg 70] (cf. §1.1.2, Proposition 1.1.5), we have

$$\int_{\mathbb{R}^n} |x|^2 d\mu \circ S^{-1}(x) \leq \int_{\mathbb{R}^n} |x|^2 d\mu(x).$$

Thus if $A_1, \dots, A_n$ are independent random vectors distributed according to μ, then

$$\mathbb{E}|\det A|^2 = \mathbb{E}|\det(SA)|^2 \le \mathbb{E}\prod_{i=1}^{n}|SA_i|^2 \le \mathbb{E}\prod_{i=1}^{n}|A_i|^2.$$

In other words, Hadamard's inequality will give the best bound if μ is isotropic. As in Proposition 4.3.3, we can also consider the case when the A_i are not necessarily identically distributed.

Proposition 4.5.1. *Let $\mu_1, \dots, \mu_n$ be isotropic log-concave measures. Suppose that $A_1, \dots, A_n$ are independent random vectors such that A_i is distributed according μ_i for each $i = 1, \dots, n$. Let A be the matrix $A = [A_1 \cdots A_n]$ and let $h(A)$ be as defined in (4.33). Then*

$$\mathbb{P}\left(h(A)^{1/n} \in [c_1', c_2']\right) \ge 1 - 2e^{-c_3' n} \tag{4.35}$$

where $0 < c_1' < c_2' < 1$ and $c_3' > 0$ are absolute constants.

Proof. Let C' be the constant from Lemma 4.2.3 and set

$$B := 2(4C')^2\sqrt{n}B_2^n.$$

By Markov's inequality, we have

$$\mu(\mathbb{R}^n \backslash B) \le \frac{1}{4(4C')^4}. \tag{4.36}$$

Let $E \subset \mathbb{R}^n$ be a subspace of dimension k for some $k \in \{0, \dots, n-1\}$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n \backslash B} |P_E x|^2 d\mu(x) &\le \mu(\mathbb{R}^n \backslash B)^{1/2} \left(\int_{\mathbb{R}^n} |P_E x|^4 d\mu(x)\right)^{1/2} \\
&\le \mu(\mathbb{R}^n \backslash B)^{1/2}(4C')^2 k && \text{(by Lemma 4.2.3)} \\
&\le k/2, && \text{(by (4.36))}
\end{aligned}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|P_E x|^2}{|x|^2} d\mu(x) &\geq \frac{1}{4(4C')^4 n} \int_B |P_E x|^2 d\mu(x) \\ &= \frac{1}{4(4C')^4 n} \left(\int_{\mathbb{R}^n} |P_E x|^2 d\mu(x) - \int_{\mathbb{R}^n \setminus B} |P_E x|^2 d\mu(x) \right) \\ &\geq \frac{k}{8(4C')^4 n}.\end{aligned}$$

Thus setting $C_3 = 8(4C')^4$, we have

$$\int_{\mathbb{R}^n} \frac{|P_{E^\perp} x|^2}{|x|^2} d\mu(x) = 1 - \int_{\mathbb{R}^n} \frac{|P_E x|^2}{|x|^2} d\mu(x) \leq 1 - \frac{k}{C_3 n}.$$

As in the proof of Lemma 4.3.1 (cf. (4.12)), we apply Fubini's theorem (integrating first with respect to X_n then X_{n-1} and so on) to obtain

$$\mathbb{E} h(A)^2 \leq \prod_{k=0}^{n-1} (1 - k/(C_3 n)) \leq \left(\frac{1}{n} \sum_{k=0}^{n-1} (1 - k/(C_3 n)) \right)^n \leq \left(1 - \frac{1}{4C_3} \right)^n.$$

Set $c_4 := 1 - 1/(4C_3)$ and observe that for any $1 < \alpha < c_4^{-1}$, we have

$$\mathbb{P}\left(h(A) > (\alpha c_4)^{n/2} \right) \leq \alpha^{-n}. \tag{4.37}$$

We know turn our attention to the reverse bound. Letting C denote the constant from Lemma 4.2.4 and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|x|^{1/4}}{|P_{E^\perp} x|^{1/4}} d\mu(x) &\leq \left(\int_{\mathbb{R}^n} |x|^{1/2} d\mu(x) \right)^{1/2} \left(\int_{\mathbb{R}^n} |P_{E^\perp} x|^{-1/2} d\mu(x) \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^n} |x|^2 d\mu(x) \right)^{1/8} C^{1/2} (n-k)^{-1/8} \text{ (cf. proof of Lemma 4.2.4)} \\ &\leq C^{1/2} \left(\frac{n}{n-k} \right)^{1/8}.\end{aligned}$$

Integrating as above, we conclude that

$$\mathbb{E} h(A)^{-1/4} \leq C^{n/2} \left(n^n / n! \right)^{1/8} \leq (e^{1/8} C^{1/2})^n.$$

Finally, for any $\beta > 1$, we have

$$\mathbb{P}\left(h(A) < (\beta e^{1/2}C^2)^{-n}\right) = \mathbb{P}\left(h(A)^{-1/4} > (\beta^{1/4}e^{1/8}C^{1/2})^n)\right) \le \beta^{-n/4}.$$

This concludes the proof of the proposition. □

REFERENCES

[1] J. Abbott and T. Mulders. How tight is Hadamard's bound? *Experiment. Math.*, 10(3):331–336, 2001.

[2] C. Borell. Convex set functions in *d*-space. *Period. Math. Hungar.*, 6(2):111–136, 1975.

[3] J. Bourgain, M. Meyer, V. Milman, and A. Pajor. On a geometric inequality. In *Geometric Aspects of Functional Analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 271–282. Springer, Berlin, 1988.

[4] N. Dafnis, A. Giannopoulos, and O. Guedon. On the isotropic constant of random polytopes. *Advances in Geometry*, 2009. to appear.

[5] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. *Journal of Functional Analysis*, 257:2820–2839, 2009.

[6] J. D. Dixon. How good is Hadamard's inequality for determinants? *Canad. Math. Bull.*, 27(3):260–264, 1984.

[7] A. Giannopoulos, M. Hartzoulaki, and A. Tsolomitis. Random points in isotropic unconditional convex bodies. *J. London Math. Soc. (2)*, 72(3):779–798, 2005.

[8] A. Giannopoulos and A. Tsolomitis. Volume radius of a random polytope in a convex body. *Math. Proc. Cambridge Philos. Soc.*, 134(1):13–21, 2003.

[9] E. Gluskin and V. Milman. Geometric probability and random cotype 2. In *Geometric Aspects of Functional Analysis*, volume 1850 of *Lecture Notes in Math.*, pages 123–138. Springer, Berlin, 2004.

[10] O. Guédon. Kahane-Khinchine type inequalities for negative exponent. *Mathematika*, 46(1):165–173, 1999.

[11] C. R. Johnson and M. Newman. How bad is the Hadamard determinantal bound? *J. Res. Nat. Bur. Standards Sect. B*, 78B:167–169, 1974.

[12] B. Klartag. On convex perturbations with a bounded isotropic constant. *Geom. Funct. Anal.*, 16(6):1274–1290, 2006.

[13] B. Klartag. A central limit theorem for convex sets. *Invent. Math.*, 168(1):91–131, 2007.

[14] B. Klartag and G. Kozma. On the hyperplane conjecture for random convex sets. *Israel J. Math.*, 170:253–268, 2009.

[15] B. Klartag and R. Vershynin. Small ball probability and Dvoretzky's theorem. *Israel J. Math.*, 157:193–207, 2007.

[16] R. Latała and K. Oleszkiewicz. Small ball probability estimates in terms of widths. *Studia Math.*, 169(3):305–314, 2005.

[17] A. M. Mathai. Random p-content of a p-parallelotope in Euclidean n-space. *Adv. in Appl. Probab.*, 31(2):343–354, 1999.

[18] M. W. Meckes. Volumes of symmetric random polytopes. *Arch. Math. (Basel)*, 82(1):85–96, 2004.

[19] R. E. Miles. Isotropic random simplices. *Advances in Appl. Probability*, 3:353–382, 1971.

[20] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.

[21] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[22] G. Paouris. Concentration of mass on convex bodies. *Geom. Funct. Anal.*, 16(5):1021–1049, 2006.

[23] G. Paouris. Small ball probability estimates for log-concave measures, 2009. Preprint.

[24] C. A. Rogers and G. C. Shephard. The difference body of a convex body. *Arch. Math. (Basel)*, 8:220–233, 1957.

[25] H. Ruben. The volume of an isotropic random parallelotope. *J. Appl. Probab.*, 16(1):84–94, 1979.

CHAPTER 5

Concluding remarks and open problems

5.1 Concluding remarks

A common thread in Asymptotic Geometric Analysis is the use of probabilistic methods, especially in proving existence theorems. The key to using such methods often lies in deducing useful information from "average behavior." Milman's proof of Dvoretzky's Theorem, and the use of concentration of measure, is the archetypal example.

We conclude this thesis with several general comments on how we have effectively used averaging techniques, especially in the context of isotropic convex bodies. While these may be transparent to experts, we point out several examples for the benefit of the non-specialist reader.

Deducing useful information from average behavior is ubiquitous in Chapter 3, especially in the proof of Proposition 3.1.1. For instance, in the notation of said proposition, one transfers a question about a 1-unconditional isotropic convex body K to Rademacher random variables ε_i by averaging over K (see steps in §3.2.4)

$$\mathrm{vol}\left(\{x \in K : |\langle x, \theta\rangle| > t\}\right) = \int_K \mathbb{P}_\varepsilon \left(|\sum_{i=1}^{n} \varepsilon_i \theta_i x_i| > t \right) dx. \tag{5.1}$$

To get the strongest result, one wants (a) the largest range of t and (b) the largest set of $\theta \in S^{n-1}$ for which a super-Gaussian estimate holds.

In order to use Theorem 3.2.5, one needs to compare the ℓ_2^n and ℓ_∞^n norms for a "typical" vector of the form $y = (\theta_i x_i)$. To satisfy the assumption (3.15) from Theorem 3.2.5 one essentially wants

$$|y| \geq f(n)\|y\|_\infty,$$

for "most" y with $f(n)$ as large as possible. How do each of $x \in K$ and $\theta \in S^{n-1}$ behave on average? Well,

$$\int_{S^{n-1}} \|\theta\|_\infty d\sigma(\theta) \approx \frac{\sqrt{\log n}}{\sqrt{n}}.$$

In this case, the average of $\|\theta\|_\infty$ is a bit too large. By passing to coordinate subspaces of the form $E_I = \text{span}\{e_i\}_{i\in I}$, we can in fact guarantee $\|P_{E_I}\theta\|_\infty \approx 1/\sqrt{n}$, while still ensuring that such θ occupy a significant portion of the sphere. This is the content of Lemma 3.2.1 (the proof of which also involves deducing information from the average of the ℓ_1^n norm on the sphere).

On the other hand, introducing the projection P_{E_I}, we must determine the effect on $x \in K$. Of course, we have plenty of information about the behavior on average; namely, isotropic position entails

$$\int_K |P_E x|^2 dx = \dim(E) L_K^2 \tag{5.2}$$

for *any* subspace $E \subset \mathbb{R}^n$. Thus by Fact 3.2.7 and (5.2), one should have

$$\int_K |P_E x| dx \geq f(n) \int_K \|P_E x\|_\infty dx$$

for any proportional dimensional subspace E with $f(n)$ about $\sqrt{n}/\log n$. Passing from the average is then done precisely in Lemma 3.2.6. Thus most steps in the proof are done by first asking how the quantity under question behaves on average and transferring properties accordingly.

Of course, Chapter 4 contains many more examples of deducing information from average behavior.

5.2 Recent developments

In this thesis we mentioned two results in which Steiner symmetrization (see Appendix) is used. In particular, in the notation of Chapter 4, let $V_1, \ldots, V_N$ be convex bodies of volume one. Then for each $p \geq 0$, the expected p-th power of the volume of the random zonotope

$$I_p(V_1,\dots,V_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}\left(\sum_{i=1}^{N}[0,x_i]\right)^p dx_1 \dots dx_N$$

is minimized when each $V_i = \overline{B_2^n}$, the Euclidean ball of volume one; i.e.,

$$I_p(V_1,\dots,V_N) \geq I_p(\overline{B_2^n},\dots,\overline{B_2^n}). \tag{5.3}$$

Similarly, for the expected volume of a random polytope in a convex body $K \subset \mathbb{R}^n$,

$$\mathbb{E}_p(K,N) = \int_K \cdots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{x_1,\dots,x_N\})^p\, dx_N \dots dx_1,$$

one has

$$\mathbb{E}_p(K,N) \geq \mathbb{E}_p(\overline{B_2^n},N). \tag{5.4}$$

These two formulas have since been unified in current joint work by G. Paouris and this author. For the reader's interest, we will state the generalization.

For $x_1,\dots,x_N \in \mathbb{R}^n$, let $M = M(x_1,\dots,x_N) : \mathbb{R}^N \to \mathbb{R}^n$ be the operator defined by

$$Me_i = x_i \quad \text{for each } i = 1,\dots,N,$$

where the e_i's are the standard unit vector basis for $\mathbb{R}^N$. Thus the matrix of M with respect to $\{e_i\}$ has x_i as its i-th column.

Let $B \subset \mathbb{R}^N$ be an arbitrary compact, convex set. For $p > 0$, let

$$\mathbb{E}_p(K,N,B) := \left(\int_K \dots \int_K \mathrm{vol}\,(M(x_1,\dots,x_N)B)^p\, dx_N \dots dx_1\right)^{\frac{1}{pn}}. \tag{5.5}$$

Theorem 5.2.1. *With the preceding notation, we have*

$$\mathbb{E}_p(K,N,B) \geq \mathbb{E}_p(\overline{B_2^n},N,B). \tag{5.6}$$

Thus if $N > n$ and and $B = \mathrm{conv}\,\{e_1,\dots,e_N\}$, then

$$M(x_1,\dots,x_N)B = \mathrm{conv}\,\{x_1,\dots,x_N\}. \tag{5.7}$$

In this case, inequality (5.6) is just (5.4).

If $N \geq n$ and $B = B_\infty^N$, then

$$M(x_1, \ldots, x_N)B = \sum_{i=1}^{N} [-x_i, x_i],$$

and hence inequality (5.6) recovers (5.3).

5.3 Further research

Let K_N be the random polytope

$$K_N = \operatorname{conv}\{X_1, \ldots, X_N\}, \tag{5.8}$$

where $X_1, \ldots, X_N$ are independent random vectors distributed uniformly in an isotropic convex body K. We close with a few remarks on Conjecture 1.1.24 mentioned in the introductory chapter, namely, estimating $\mathbb{E}\operatorname{vol}(K_N)^{1/n}$ from below.

A problem at the heart of this research involves the volume of the convex hull of N arbitrary (non-random) points in $\mathbb{R}^n$. Sharp *upper* bounds have been known for almost twenty years now. (e.g., [3], [4], [1]; see also [2]).

Theorem 5.3.1. *If $x_1, \ldots, x_N \in \mathbb{R}^n$ and $|x_i| \leq \alpha\sqrt{n}$, then*

$$\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_N\})^{1/n} \leq \frac{C\alpha\sqrt{\log(2N/n)}}{\sqrt{n}}, \tag{5.9}$$

where C is an absolute constant.

The only known examples that illustrate the sharpness for the full range of $n < N \leq e^n$ are random (as far as I know); for instance, if $x_1, \ldots, x_N$ are sampled uniformly in the Euclidean ball $\overline{B_2^n}$ or if they are Gaussian vectors, then the lower bound for $\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_N\})^{1/n}$ is of the same order as the upper bound (5.9). Finding a suitable characterization for the convex hull of arbitrary points to have the maximum possible volume would complement the current literature and may lead to insights for the random polytope $K_N \subset K$ as defined above; in particular, in understanding the proper dependence on the isotropic constant L_K.

Perhaps placing the volume problem for the random polytope K_N in a more general framework will yield new insights. For convenience, let K_N now denote the symmetric polytope

$$K_N = \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}. \tag{5.10}$$

Let B_p^N is the unit ball in ℓ_p^N. Let $M : \mathbb{R}^N \to \mathbb{R}^n$ be the random matrix defined by $Me_i = X_i$, $i = 1, \ldots, N$ (where the e_i's are the standard unit vector basis in $\mathbb{R}^N$). Thus $K_N = MB_1^N$. How does $\operatorname{vol}\left(MB_p^N\right)$ behave as $p \to 1$? Since $MB_1^N = K_N \subset K$, we have $\operatorname{vol}(MB_1^n) \leq 1$ and any lower bound for $\operatorname{vol}\left(MB_1^N\right)$ in terms of L_K leads to an immediate upper bound for L_K. On the other hand, the upper bound $\operatorname{vol}\left(MB_p^N\right) \leq 1$ is not necessarily true for $p > 1$. In particular, the implications for the boundedness of L_K are not as strong; hence this may be a more tractable problem. This is related to estimates for the volume of (non-random) p-zonotopes in [5].

REFERENCES

[1] K. Ball and A. Pajor. Convex bodies with few faces. *Proc. Amer. Math. Soc.*, 110(1):225–231, 1990.

[2] I. Bárány and Z. Füredi. Computing the volume is difficult. *Discrete Comput. Geom.*, 2(4):319–326, 1987.

[3] B. Carl and A. Pajor. Gelfand numbers of operators with values in a Hilbert space. *Invent. Math.*, 94(3):479–504, 1988.

[4] E. D. Gluskin. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces. *Mat. Sb. (N.S.)*, 136(178)(1):85–96, 1988.

[5] Y. Gordon and M. Junge. Volume formulas in L_p-spaces. *Positivity*, 1(1):7–43, 1997.

CHAPTER 6

Thesis summary

In this thesis we considered various problems in *Asymptotic Geometric Analysis*. The main focus of the field is the geometry of convex sets in n-dimensional space. In contrast to Classical Geometry, which usually involves the familiar two and three dimensional spaces, the main interest here is geometry in higher dimensions. The key focus is on *high-dimensional phenomena* - where the characteristic behavior reveals itself only when the dimension is suitably large (hence the use of the term *asymptotic*). High-dimensional systems are ubiquitous in mathematics and applied fields and precise descriptions of high-dimensional phenomena are of broad interest. Understanding and, more importantly, *quantifying* such phenomena can be a challenge as our low-dimensional intuition is of little use. One such topic, with highly counter-intuitive results, is the *distribution of volume in convex sets*. In the introductory chapter, we surveyed recent developments and outlined how the results of this thesis fit within the theory. Chapters 2 to 4 are self-contained papers based on the articles [6], [5] and [4], respectively. Each paper addresses a certain aspect of the behavior of volume in high-dimensional convex sets. In this chapter, we summarize our main results.

A common theme underlying all papers in this thesis is the geometry of high-dimensional *random polytopes*. One way to generate such objects is to sample N points independently according to a probability measure on n-dimensional Euclidean space and form their convex hull. Sample polytopes, generated by randomly selecting vertices of the three-dimensional cube, are shown in Figure 6.1. Chapters 2 and 4 are largely devoted to understanding properties of polytopes. In Chapter 3, random polytopes are used more as a tool to analyze the geometry of general convex bodies.

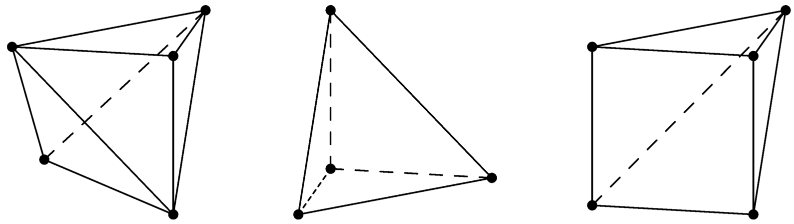


Figure 6.1: Sample random polytopes generated by vertices of the cube.

Chapter 2 addresses the following (somewhat loosely stated) question.

Question 6.0.2. *How many points N should be sampled for a random polytope to capture significant volume?*

Of course, the meaning of "significant volume" depends on the model of randomness and one wants the smallest such N, typically a critical threshold value. Here the interest is when the dimension n is large and N depends on n. This work was motivated by results of Dyer, Furedi and McDiarmid [2] (who answered the question when the points are drawn from the n-dimensional cube) and a subsequent generalization due to Gatzouras and Giannopoulos [3]. Both of the latter papers consider random models with particular characteristics (the random vectors involved have compact support and independent coordinates). In Chapter 2, we treat random models that lack these features. For instance, if one samples according to n-dimensional Gaussian measure, i.e., multi-variate normal distribution (which is not compactly supported), the corresponding polytopes are not uniformly bounded. In this case, there are a number of possible criteria for capturing volume. For example, "How many points should be sampled to capture the volume of a Euclidean ball?" (a natural choice as Gaussian measure is invariant under rotations). For clarity, we have included sample two-dimensional polytopes and the portion of volume of a ball that they capture in Figure 6.2. On the other hand, one can ask, "How many points are needed to capture the volume of a cube?" (Gaussian measure is, after all, a product-measure.) Of course, there are many possible answers, e.g., capturing volume in other convex bodies; taking the Gaussian measure of the polytope is equally natural. Theorem 2.2.1 provides a rather satisfactory answer, including all of the aforementioned criteria; in fact, we determine the number of Gaussian vectors needed to capture not only different volumes but more general notions of size (according to log-concave measures). In §2.3, we also consider polytopes generated by sampling vertices from the n-dimensional sphere. In this case, the coordinates of the random vectors lack independence. We also provide a natural complement to the

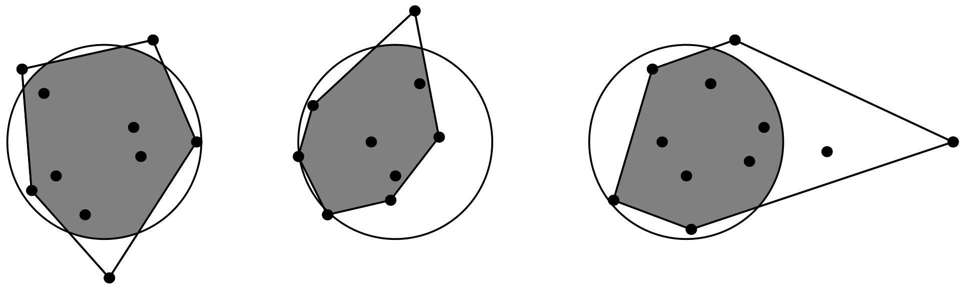


Figure 6.2: Sample Gaussian polytopes and the portion of volume inside Euclidean balls.

latter results by proving corresponding theorems for polytopes generated by random faces (as opposed to just random vertices).

Chapter 3 is devoted to the distribution of volume in convex bodies. Fundamental questions in the theory can be loosely stated as (i) What parts of a high-dimensional convex body account for its size? (ii) Why does volume seem to concentrate in places that contradict our low-dimensional intuition? If K is a convex body in n-dimensional space, we can measure the distribution of volume by considering caps as shown in Figure 6.3. The cap

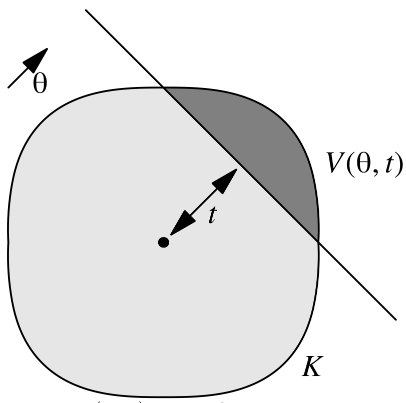


Figure 6.3: A cap $V(\theta,t)$ in K for gauging decay of volume.

$V(\theta,t)$ of height t in the direction of θ gauges how the volume of K decays as t increases. Precise *upper bounds* for the volume of $V(\theta,t)$ are closely connected to several difficult open problems. Perhaps the most famous is Conjecture 1.1.6 (on the uniform boundedness of isotropic constants, discussed in §1.1.2). Motivated by Fields Medalist J. Bourgain's approach to the latter conjecture, there has been much research on upper bounds for the volume of the caps $V(\theta,t)$. Chapter 3 is somewhat of a departure from previous research in that we show that the *reverse* estimates, i.e., *lower bounds* for the volume of such caps are also important. The first proposition in Chapter 3 involves sharp lower bounds for the volume of $V(\theta,t)$ when K is a convex body exhibiting certain symmetries (symmetric under coordinate reflections). In §3.3, we discuss the problem of bounding the mean-width of an isotropic convex body, a natural question about a classical parameter. Proposition 3.3.9, the main result of §3.3, reveals that cap-estimates similar to those for $V(\theta,t)$ are at the heart of the problem. The proof is of independent interest as it involves approximation of a convex body by a random polytope with relatively few vertices.

Chapter 4 further explores the volume of random polytopes. In this case, the polytopes are generated by sampling points $X_1,\ldots,X_N$ (independently) in an arbitrary isotropic convex body K and forming their convex hull, say, $K_N = \text{conv}\{X_1,\ldots,X_N\}$ as shown in Figure 6.4. Rather than calculating threshold values as in Chapter 2, the focus is on precise estimates

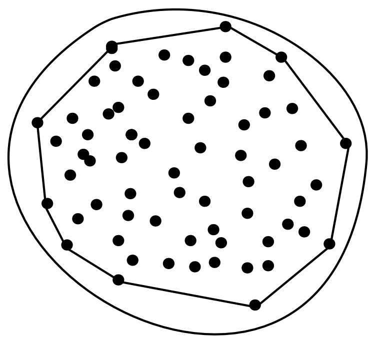


Figure 6.4: A random polytope K_N in K.

for the expected volume of K_N in terms of three parameters: the dimension n, the number of points N, and the isotropic constant L_K of K. Determining the correct dependence is genuinely difficult; in fact, equivalent to resolving Conjecture 1.1.6 (mentioned in the previous paragraph). The difficulty lies in capturing the correct dependence on all three parameters, n, N, and L_K, simultaneously. Giannopoulos and his coauthors [1] have sharp results in n and N but at the expense of L_K. In Chapter 4 we obtain the correct dependence on all three parameters but only when N is small relative to n. Our results improve on previous estimates in several ways. Firstly, the model considered in §4.3 allows one to sample the points from multiple convex bodies (or log-concave measures). Surprisingly, the more general model yields cleaner proofs and more accurate estimates. The results can also be phrased in terms of determinants of random matrices and thus have applications to several geometric inequalities, e.g., the volume of zonotopes and Hadamard's inequality for random matrices.

In Chapter 5, we conclude with recent developments and open problems. In particular, we discuss placing the volume problem for the random polytope K_N (mentioned in the previous paragraph) in a more general framework.

REFERENCES

[1] N. Dafnis, A. Giannopoulos, and A. Tsolomitis. Asymptotic shape of a random polytope in a convex body. *J. Funct. Anal.*, 257(9):2820–2839, 2009.

[2] M. E. Dyer, Z. Füredi, and C. McDiarmid. Volumes spanned by random points in the hypercube. *Random Structures Algorithms*, 3(1):91–106, 1992.

[3] D. Gatzouras and A. Giannopoulos. Threshold for the volume spanned by random points with independent coordinates. *Israel J. Math.*, 169:125–153, 2009.

[4] P. Pivovarov. On determinants and the volume of random polytopes in isotropic convex bodies. *Geom. Dedicata*. 14 pages, to appear.

[5] P. Pivovarov. On the volume of caps and bounding the mean-width of an isotropic convex body. *Math. Proc. Cambridge Philos. Soc.* 15 pages, to appear.

[6] P. Pivovarov. Volume thresholds for Gaussian and spherical random polytopes and their duals. *Studia Math.*, 183(1):15–34, 2007.

CHAPTER 7

Appendix: Steiner symmetrization

In this appendix we discuss Steiner symmetrization, which is used in the proofs of several theorems mentioned in previous chapters.

Definition 7.0.3. Let K be a convex body and H a hyperplane. The *Steiner symmetral* $S_H(K)$ of K with respect to H is defined by the following procedure: For each straight line L orthogonal to H such that $K \cap L \neq \emptyset$, shift the line segment $K \cap L$ along the line L until its midpoint is in H. The union of all such line segments is $S_H(K)$; see Figure 7.1.

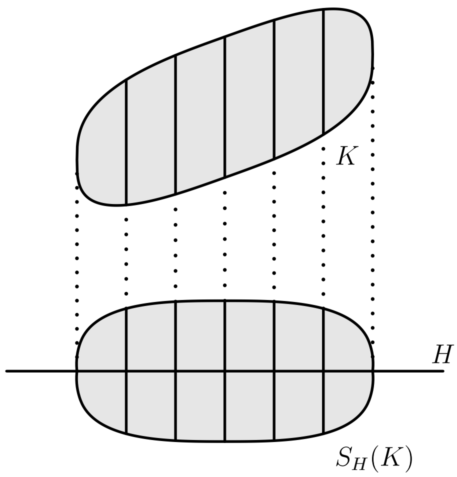


Figure 7.1: Steiner symmetrization

The key fact in proving (1.34) and (4.27) is that both quantities decrease under Steiner symmetrization of the associated convex bodies.

One can then invoke a classical fact, due to Gross [1].

Theorem 7.0.4. *Let $K \subset \mathbb{R}^n$ be a convex body of volume one. Then there is a sequence of successive Steiner symmetrizations of K which converges to the Euclidean ball of volume one $\overline{B_2^n}$ in the Hausdorff metric.*

We refer the reader to [2] for further information on Steiner symmetrization.

REFERENCES

[1] W. Groß. Die Minimaleigenschaft der Kugel. *Monatsh. Math. Phys.*, 28(1):77–97, 1917.

[2] P. M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.