Weak Laplace principles on topological spaces

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Abstract

We address the missing analog of vague convergence in the weak-converge-large-deviations analogy. Specifically, we introduce the weak Laplace principle and show it implies both the well-known weak LDP and the Laplace principle lower bound. Both the weak LDP and weak Laplace principle hold in settings where there is no exponential tightness and imply the LDP when exponential tightness holds as well as the Laplace principle when, in addition, the space is Polish. As a side effect, we also generalize Bryc's lemma. Whereas vague convergence is only defined on locally compact spaces, our definition of the weak Laplace principle holds on general topological spaces.

Key words: Large Deviation Principle, Laplace Principle, Weak LDP, Probability Measure, Topological Space, Homeomorphism, Cadlag Process, Projective Limit.

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1 Introduction

Large deviation principles (LDPs) imply the convergence of scaled log-Laplace functionals, the latter often being referred to as the Laplace principle and the implication as Varadhan's lemma (see Varadhan [17]). Bryc's lemma (see Bryc [2]), a converse of Varadhan's lemma, establishes that the convergence of these scaled log-Laplace functionals implies the LDP *assuming* exponential tightness. A second converse of Varadhan's lemma establishes that the LDP holds when the Laplace principle holds *with* limits being characterized in terms of a good rate function (see Dupuis and Ellis [8]). These three results are among the most important and useful basic results in large deviation theory. Herein, we define a weak Laplace principle, whose relationship to the full Laplace principle is analogous to the relationship of vague convergence to weak convergence of probability measures on locally compact spaces. We explore the connections between weak forms of the Laplace and large deviation principles and give simple examples. The application to larger problems will be handled in future work.

Suppose E is a topological space, $\{X_n\}$ is a sequence of E-valued random variables and $I : E \to [0,\infty]$ is a lower semicontinuous function, implying that $\{x : I(x) \le c\}$ is *closed* for each $c \in [0,\infty)$. Then, $\{X_n\}$ satisfies the (full) LDP with *rate function* I if:

$$\limsup_{n\to\infty} \frac{1}{n} \log P(X_n \in F) \le -\inf_{x\in F} I(x) \quad \text{for each closed set } F; \text{ and} \tag{1.1}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n \in O) \ge -\inf_{x\in O} I(x) \quad \text{for each open set } O. \tag{1.2}$$

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I is called *good* if $\{x : I(x) \leq c\}$ is in fact *compact* for each $c \in [0, \infty)$. An LDP is equivalent to the assertion that for each set $A \in \mathcal{B}(E)$ (the Borel σ-algebra)

$$-\inf_{x \in A^o} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P(X_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log P(X_n \in A) \leq -\inf_{x \in \overline{A}} I(x),$$

where A^o ($\overline{A}$) is the interior (closure) of A. In the case that E is a Polish space and I is good, the LDP upper bound (1.1) and lower bound (1.2) are also respectively equivalent to the Laplace principle upper bound and lower bound:

$$\limsup_{n \to \infty} \frac{1}{n} \log E[e^{-n\phi(X_n)}] \leq -\inf_{x \in E} \{\phi(x) + I(x)\} \quad \forall \phi \in \overline{C}(E);$$
$$\liminf_{n \to \infty} \frac{1}{n} \log E[e^{-n\phi(X_n)}] \geq -\inf_{x \in E} \{\phi(x) + I(x)\} \quad \forall \phi \in \overline{C}(E),$$

where $\overline{C}(E)$ denotes the space of continuous bounded $\mathbb{R}$-valued functions on E. Indeed, Varadhan showed the LDP implies (both bounds) of the Laplace principle under more general conditions and this implication is referred to as Varadhan's lemma. The reverse, Laplace-principle-to-LDP implication under the above conditions is a converse to Varadhan's lemma, but there is second converse to Varadhan's lemma:

Lemma 1 *(Bryc) Suppose E is a completely regular topological space, i) $\Lambda(\phi) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{n\phi(X_n)}]$ exists for each $\phi \in \overline{C}(E)$ and ii)* exponential tightness *holds, i.e., for any $a > 0$, there is a compact $K_a \subset E$ such that $\limsup_{n\to\infty} \frac{1}{n} \log P(X_n \in K_a^c) \leq -a$. Then, the LDP holds with good rate function*

$$I(x) = \sup_{\phi \in \overline{C}(E)} \{\phi(x) - \Lambda(\phi)\} = -\inf_{\phi \in \overline{C}(E), \phi(x) = 0} \Lambda(\phi). \tag{1.3}$$

The two converses of Varadhan's lemma differ in what is known a priori: the limits in terms of a good rate function for the Laplace-principle-to-LDP implication or exponential tightness of $\{X_n\}$ in the case of Bryc's lemma. Bryc [2] gave results where the limits $\Lambda(\phi)$ need only exist for ϕ in a subset of $\overline{C}(E)$ and [7, Theorem

4.4.10], [10, Proposition 3.20] generalize Bryc's lemma. However, these results require exponential tightness, which can be difficult to establish directly.

Often when there is an escape of mass neither a full LDP nor a Laplace principle hold (cf. [7, page 7] and Section 6 herein) and one must settle for a *weak* LDP where (1.1) is only proved for all compact sets K instead of closed sets F. Indeed, to show a weak LDP with a rate function I, one need only show (1.2) and for any $\alpha \in [0, \infty)$ that

$$\limsup_{n \to \infty} \frac{1}{n} \log P(X_n \in K) \leq -\alpha$$

for each compact $K \subset \{x \in E : I(x) > \alpha\}$. The weak LDP can be used to attain the *full* LDP when exponential tightness holds (see [7, Lemma 1.2.18]). Our first main result, Theorem 19, generalizes Bryc's lemma by establishing the *weak* LDP without requiring exponential tightness while only requiring the limits $\Lambda(\phi)$ to exist for a small class of functions ϕ. This result is then re-interpretted in Theorem 22 to give a new, small rate-function-determining class of functions that can be used to establish the LDP. For our second main result, Theorem 26, we define the weak Laplace principle (see (5.1) below) and use this to obtain weak LDPs and LDPs. Our motivation for both main results stems from Blount and Kouritzin [1], who used homemorphic methods to transfer weak convergence on test functions to completely regular spaces, and Dawson and Gärtner [6], whose projective limit large deviation result lifts our LDPs from finite dimensional spaces to infinite dimensional spaces. A key idea is to split the test functions ϕ into inner and outer functions, which is common practice for random measures and distributions. The inner functions need only strongly separate points while the outer functions need only be rate function determining on compact subsets of $\mathbb{R}^k$. These methods are used to establish a full LDP on a Hausdorff compactification $\overline{E}$ of E from which we obtain a weak LDP on E by the following simple result (proved within by the proof of Theorem 19 and

Remark 21).

Lemma 2 *Suppose that E is imbedded into a compact space $\overline{E}$, $\{X_n\}$ lives on E and satisfies the LDP on $\overline{E}$ with rate function I^*. Then, $\{X_n\}$ satisfies the weak LDP on E with rate function $I^0 \doteq I^*|_E$. Moreover, if in addition $I^*(x) = \infty$ for all $x \notin E$, then $\{X_n\}$ satisfies the LDP on E with rate function I^0 and I^0 is good.*

We discuss the relationship of our work with the variational approach in the next section, which can be skipped by the uninterested reader. Our notation is provided in Section 3. Section 4 contains series/entropy conditions for weak LDP and our weak version of Bryc's lemma. These are results on inferring weak LDPs for random variables $\{X_n\}$ on general topological spaces from weak Laplace convergence or from weak LDPs of $\{g(X_n)\}$ for a small collection of test functions g. In Section 5, we define the weak Laplace principle and prove that it implies the weak LDP. In other words, we show that convergence of a small class of Laplace functionals to a limit characterized by a rate function implies a weak LDP with this rate function. Section 6 houses simple motivating examples.

2 Relationship with the Variational Approach

As mentioned in the introduction, the LDP and the Laplace principle are equivalent in the Polish, good-rate-function case. Representation theorems can used to prove Laplace principles for certain stochastic partial differential equations driven by (possibly-infinitely-dimensional) Brownian motion or Poisson random measures (see e.g. Budhiraja and Dupuis [3] as well as Budhiraja, Dupuis and Maroulas [4]). This beautiful approach also uses weak convergence and stochastic control type arguments. However, it is only known to work on Polish spaces, it requires strong solutions to stochastic equations driven by processes with known Laplace principles and it does not allow escape of mass (even in the diminishing manner demonstrated herein). One interesting aspect of this approach that relates to our work is that the Laplace principle proof is often broken into the upper and lower bounds (see e.g. the proofs of [3, Theorem 4.4] and [4, Theorem 5]). Our results can reduce the burden of showing these bounds, especially the lower one, when E is Polish and I is good.

While the LDP-Laplace-principle equivalence is only known for Polish spaces and good rate functions, our results hold more generally. This generality is important since the LDP for random variables $\{X_n\}$ living on non-Polish Lusin spaces or nuclear Fréchet space duals is of interest (see e.g. Xiong [19]). In the related problems of tightness and weak convergence for random variables on completely regular spaces it proved profitable to first establish these properties for the real random variables $\{g(X_n)\}$ for a suitable class of test functions g (see Jakubowski [13], Blount and Kouritzin [1]) and then *transfer* these properties to $\{X_n\}$.

3 Notation and Background

$(E, \mathcal{T})$ or just E will denote a topological space, $\mathcal{B}(E)$ or $\mathcal{B}(\mathcal{T})$ will be the Borel sets, $\mathcal{P}(E)$ will be the space of Borel probability measures equipped with the weak topology, and $M(E)$, $B(E)$, $C(E)$, $\overline{C}(E)$ will denote the Borel measurable, bounded measurable, continuous, and continuous bounded $\mathbb{R}$-valued functions on E, respectively. Our product spaces will be given the product topology.

Definition 3 *Let $(E, \mathcal{T})$ be a topological space and $\mathcal{M} \subset M(E)$. Then: (i) $\mathcal{M}$ separates points (s.p.) if for $x \neq y \in E$ there is a $g \in \mathcal{M}$ with $g(x) \neq g(y)$. (ii) $\mathcal{M}$ strongly separates points (s.s.p.) if, for every $x \in E$ and neighborhood O_x of x, there* *is a finite collection* $\{g_1, \ldots, g_k\} \subset \mathcal{M}$ *such that* $\inf_{y \notin O_x} \max_{1 \le l \le k} |g_l(y) - g_l(x)| > 0$.

We also use s.s.p. below for *strongly separating points* and *strongly separate points*, depending upon which is the correct English usage.

If $\mathcal{M}$ s.s.p., then for any x and neighborhood O_x there are $\varepsilon > 0$ and $\{g_1, \ldots, g_k\} \subset \mathcal{M}$ such that $\{y \in E : \max_{1 \le l \le k} |g_l(y) - g_l(x)| < \varepsilon\} \subset O_x$. Thus, $\mathcal{M}$ s.s.p. implies $\mathcal{M}$ s.p. (in a Hausdorff space) and the basis

$$\mathbb{B}^{\mathcal{M}} \doteq \{\{y \in E : \max_{1 \le l \le k} |g_l(y) - g_l(x)| < \varepsilon\},\ g_1, \ldots, g_k \in \mathcal{M},\ \varepsilon > 0,\ x \in E, k \in \mathbb{N}\}$$

defines a topology $\mathcal{T}^{\mathcal{M}}$ on E that is finer than the original topology. This yields the following simple lemma ([1, Lemma 1]):

Lemma 4 *Let E be Hausdorff, $\mathcal{M} \subset M(E)$ and $G(x) \doteq (g(x))_{g \in \mathcal{M}}$. Then, G has a continuous inverse $G^{-1} : G(E) \subset \mathbb{R}^{\mathcal{M}} \to E$ if and only if $\mathcal{M}$ s.s.p. Hence, G is an imbedding of E in $\mathbb{R}^{\mathcal{M}}$ if and only if $\mathcal{M} \subset C(E)$ and $\mathcal{M}$ s.s.p.*

Proof. If $\mathcal{M}$ s.s.p., then $\mathcal{M}$ s.p., so G^{-1} exists. Moreover, $\mathcal{T} \subset \mathcal{T}^{\mathcal{M}}$, so G^{-1} is continuous. ■

An immediate consequence of Lemma 4 is that $\mathcal{T} = \mathcal{T}^{\mathcal{M}}$ when $\mathcal{M} \subset C(E)$ s.s.p. Given a collection $\mathcal{M} \subset M(E)$ that does not necessarily s.s.p., one can still define $\mathcal{T}^{\mathcal{M}}$ through the basis $\mathbb{B}^{\mathcal{M}}$ and find that $\mathcal{T}^{\mathcal{M}}$ may differ from $\mathcal{T}$. For our LDP results, it will be helpful to look at the s.s.p. property from another angle:

Lemma 5 *(cf. [1, Lemma 4]). Suppose $(E, \mathcal{T})$ is a Hausdorff space and $\mathcal{M} \subset M(E)$. Then, $\mathcal{M}$ s.s.p. if and only if for any net $\{x_i\}_{i \in I} \subset E$ and point $x \in E$, one has that $g(x_i) \to g(x)$ for all $g \in \mathcal{M}$ implies that $x_i \to x$ in E.*

There is a class of functions that is particularly important for large deviations.

Definition 6 *(cf. [10, Definition 3.15]) Let E be a topological space. Define*

$$\Xi_{\{X_n\}} = \left\{ f \in \overline{C}(E) : \Lambda(f) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}] \text{ exists} \right\}.$$

$D \subset \overline{C}(E)$ is rate function determining *(r.f.d.) if whenever E-valued random variables sequence $\{X_n\}$ is exponentially tight and $D \subset \Xi_{\{X_n\}}$, $\{X_n\}$ satisfies the LDP with good rate function*

$$I(x) = \sup_{f\in D}\{f(x) - \Lambda(f)\} \quad \forall\, x \in E.$$

Assume that E is a completely regular space, i.e., Hausdorff and $C(E)$ separates points from closed sets, meaning for any closed $F \subset E$ and any point $x \notin F$, there is a $f \in C(E)$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in F$. [1] We will use the following simple fact:

Lemma 7 *Suppose E is Hausdorff. Then, $\overline{C}(E)$ s.s.p. if and only if it separates points from closed sets. Hence, E is completely regular if and only if $\overline{C}(E)$ s.s.p.*

Proof. Suppose $\overline{C}(E)$ separates points from closed sets. Then, by the imbedding theorem (see Munkres [15, Theorem 4-4.2]), $G(x) \doteq (g(x))_{g\in\overline{C}(E)}$ is an imbedding into $\mathbb{R}^{\overline{C}(E)}$. Hence, $\overline{C}(E)$ s.s.p. by Lemma 4. Now, let $\overline{C}(E)$ s.s.p. so $G(x) \doteq (g(x))_{g\in\overline{C}(E)}$ is a homeomorphism into a precompact subset $G(E)$ of $\mathbb{R}^{\overline{C}(E)}$. Let F_k be the class of Friedrichs' mollifiers on $\mathbb{R}^k$ for $k \in \mathbb{N}$ and π_α be the projection function from $\mathbb{R}^{\overline{C}(E)}$ to $\mathbb{R}^\alpha$ for $\alpha \subset \overline{C}(E)$. Define

$$\mathcal{U} \doteq \{f \circ \pi_\alpha : f \in F_{|\alpha|}, \alpha \subset \overline{C}(E) \text{ with finite cardinality } |\alpha|\}.$$

Then, $\mathcal{U}$ separates points from closed sets on $\mathbb{R}^{\overline{C}(E)}$. Hence, $\{u \circ G : u \in \mathcal{U}\}$ separates points from closed sets on E. □

[1] Some authors exclude the Hausdorff property from the completely regular definition.

$\overline{C}(E)$ is r.f.d. by Bryc's lemma (cf. [7, Theorem 4.4.2]). We will generalize Bryc's lemma by coming up with a much smaller class of functions that is r.f.d. Our functions will be composite functions, where the inner functions map into some compact subset of $\mathbb{R}^k$ and the class of outer functions is r.f.d. on each such compact subset of $\mathbb{R}^k$.

Definition 8 *A family $D = \{D_k\}_{k=1}^{\infty}$ is* well-determining *(w.d.) if for each $k \in \mathbb{N}$: (a) $D_k \subset C(\mathbb{R}^k)$ and (b) $D_k|_{G_k} \doteq \{f|_{G_k} : f \in D_k\}$ is r.f.d. for any compact subset G_k of $\mathbb{R}^k$. (For clarity, Definition 6 holds for each compact $E = G_k$ with $D = D_k|_{G_k}$.)*

We can replace the abstract r.f.d. notion with a verifiable condition in our (outer function) w.d. class:

Definition 9 *Let $x \in \mathbb{R}^k$ and $\varepsilon \in (0,1)$. A function $h \in C(\mathbb{R}^k)$ is an x,ε-highpoint function (h.f.) if $h(x) \geq \left(\sup_{0<|y-x|<\varepsilon} h(y) - \varepsilon\right) \vee \left(\sup_{\varepsilon \leq |y-x| < \frac{1}{\varepsilon}} h(y) + \frac{1}{\varepsilon}\right)$.*

Such functions have a maximum near x and then quickly drop off.

Lemma 10 *$\{D_k\}_{k=1}^{\infty}$ is w.d. if each $D_k \subset C(\mathbb{R}^k)$ contains an x,ε-h.f. for any $x \in \mathbb{R}^k, \varepsilon > 0$.*

Proof. Let G_k be a compact subset of $\mathbb{R}^k$ and $\{X_n\}$ be a sequence of G_k-valued random variables such that $D_k|_{G_k} \subset \Xi_{\{X_n\}}$. Then

$$\Lambda(f) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}]$$

exists for all $f \in D_k$. It suffices to show that (cf. [10, Theorem 3.7 and Proposition 3.8]) for any subsequence along which the LDP holds, the corresponding rate function is given by

$$I(x) = \sup_{f \in D_k} \{f(x) - \Lambda(f)\} \ \ \forall\, x \in G_k.$$

Suppose $\{X_{n_j}\}$ is a subsequence satisfying the LDP with good rate function J. Denote

$$\Gamma(g) \doteq \lim_{j\to\infty} \frac{1}{n_j} \log E[e^{n_j g(X_{n_j})}] \quad \forall\, g \in C(G_k).$$

Then, $J(x) = -\inf_{g\in C(G_k),g(x)=0} \Gamma(g)$ (cf. (1.3)).

Let $x \in G_k$, $\delta > 0$ and $g \in C(G_k)$ satisfy $g(x) = 0$. Take $\varepsilon \in (0, \delta)$ small enough that: i) $g(y) \geq -\delta$ for $|y - x| < \varepsilon$, ii) $g(y) \geq -\frac{1}{\varepsilon}$ for all $y \in G_k$, and iii) $G_k \subset B(x, \frac{1}{\varepsilon})$. Now, let $h \in D_k$ be an x, ε-h.f. and $f(y) \doteq h(y) - h(x)$ for $y \in G_k$. Then, it is easy to see that $f(y) \leq g(y) + 2\delta$ so

$$\Gamma(g) \geq \Gamma(f) - 2\delta = \Lambda(h) - h(x) - 2\delta.$$

By the arbitrariness of δ and g, we get $J(x) = \sup_{f\in D_k}\{f(x) - \Lambda(f)\}$. This completes the proof. □

Definition 11 *A family $D = \{D_k\}_{k=1}^{\infty}$ is* well-isolating *(w.i.) if for each $k \in \mathbb{N}$: (a) $D_k \subset C(\mathbb{R}^k)$ and (b) D_k contains an x, ε-h.f. for any $x \in \mathbb{R}^k, \varepsilon > 0$.*

By Lemma 10, we know that if $D = \{D_k\}_{k=1}^{\infty}$ is w.i. then it is w.d.

Example 12 *For $k \in \mathbb{N}$ and $x \in \mathbb{R}^k$, we define $c_k^i(x) \doteq x_i$ for $1 \leq i \leq k$. Define*

$$\begin{aligned} D_k^1 &\doteq \{\text{the smallest algebra over } \mathbb{Q} \text{ generated by } \{1, c_k^i, 1 \leq i \leq k\}\} \\ &= \{f : f \text{ is a polynomial on } \mathbb{R}^k \text{ with rational coefficients}\}, \text{ and} \\ D_k^2 &\doteq \{\text{the smallest linear lattice over } \mathbb{Q} \text{ generated by } \{c_k^i, 1 \leq i \leq k\}\}. \end{aligned}$$

(A linear lattice over $\mathbb{Q}$ is defined by the properties: (i) $g_1, g_2 \in \mathcal{L} \Rightarrow g_1 \wedge g_2 \in \mathcal{L}$ and (ii) $\alpha_1, \alpha_2 \in \mathbb{Q}$, $g \in \mathcal{L} \Rightarrow \alpha_1 + \alpha_2 g \in \mathcal{L}$.)

Then, $D^1 \doteq \{D_k^1\}_{k=1}^{\infty}$ and $D^2 \doteq \{D_k^2\}_{k=1}^{\infty}$ are w.i. by the Stone-Weierstrass theorem (cf. [12, Theorem 4.45 and Lemma 4.49]) and hence w.d.

4 The Weak Bryc's Lemma

In this paper, the term 'LDP' means 'full LDP'; whereas 'weak LDP' is defined in [7, page 7] as well as the introduction. Bryc showed that full Laplace convergence implies the LDP. In this section, we show that weak Laplace convergence implies the weak LDP.

Definition 13 *Weak Laplace convergence means the following (A-D) hold.*

(A) $(E, \mathcal{T})$ is a topological space.

(B) $\{X_n\}$ is a sequence of E-valued random variables on probability space $(\Omega, \mathcal{F}, P)$.

(C) $\mathcal{M} \subset B(E)$ s.p. and s.s.p.

(D) There exists a w.d. class $D = \{D_k\}_{k=1}^{\infty}$ such that either

$$\text{i) } \Lambda(g) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{ng(X_n)}] \text{ exists for all } g \in \mathcal{G}, \text{ or} \tag{4.1}$$

$$\text{ii) } \{g(X_n)\} \text{ satisfies the weak LDP in } \mathbb{R} \text{ for all } g \in \mathcal{G}, \tag{4.2}$$

where $\mathcal{G} \doteq \{f(g_1, \ldots, g_k) : f \in D_k, g_1, \ldots, g_k \in \mathcal{M}, k \in \mathbb{N}\}$.

Remark 14 *1) If E is Hausdorff, then we need not assume $\mathcal{M}$ s.p. in (C) as s.s.p. implies s.p. on Hausdorff spaces. 2) If E is completely regular, then $\mathcal{M} = \overline{C}(E)$ fulfils (C) and $D_k = C(\mathbb{R}^k)$ can be used in (D). This completely-regular, $\mathcal{M} = \overline{C}(E)$ case is the setting for Bryc's lemma, which can be considered a weak LDP result (see [7, Lemmas 4.4.5 and 4.4.6]). 3) If $\mathcal{M} \subset B(E)$ is an algebra or a linear lattice over $\mathbb{Q}$ such that (4.1) exists for all $g \in \mathcal{M}$, then letting D be respectively the D^1 or D^2 defined in Example 12, we find that (4.1) exists for all $g \in \mathcal{G}$. Therefore (D) holds. 4) Similar to domains of martingale problems, $\mathcal{G}$ must be large enough to capture complex spatial interactions in a collection of scalar LDPs or the scalar limits assumed in (D).*

Below we introduce two conditions that imply (4.1) and will be useful in the applications to follow in Section 6. Denote $\theta_n \doteq PX_n^{-1}$ for $n \in \mathbb{N}$.

(D_1) For any $\varepsilon > 0$ and $g \in \mathcal{G}$, there exists N such that for every $M \geq N$, $k > 0$

$$\sum_{n=M}^{M+k} \frac{1}{n+1} \left(\log \frac{E[e^{(n+1)g(X_n)}]}{E[e^{(n+1)g(X_{n+1})}]} \right) < \varepsilon.$$

(D_2) For any $\varepsilon > 0$, there exists N such that for every $M \geq N$, $k > 0$

$$\sum_{n=M}^{M+k} \frac{1}{n+1} \sup_{x \in E} \left(\log \frac{d\theta_n}{d\theta_{n+1}}(x) \right) < \varepsilon.$$

Lemma 15 *(D_2) implies (D_1) and (D_1) implies (4.1).*

Remark 16 *Since Condition (D_2) does not depend upon g, it implies the weak LDP by the weak LDP version of Bryc's lemma (see [7, Lemmas 4.4.5 and 4.4.6]) and our argument below. We will show that (D_1) is also sufficient for a weak LDP. Any term in the sums in (D_1) and (D_2) could be negative, which will only help satisfy the condition.*

Proof. $(D_2) \Rightarrow (D_1)$: We need only show that

$$\log \left(\frac{E[e^{(n+1)g(X_n)}]}{E[e^{(n+1)g(X_{n+1})}]} \right) \leq \int_E \left(\log \frac{d\theta_n}{d\theta_{n+1}} \right) \frac{e^{(n+1)g}}{\int_E e^{(n+1)g} d\theta_n} d\theta_n.$$

Letting $d\gamma_n = \frac{e^{(n+1)g} d\theta_n}{\int_E e^{(n+1)g} d\theta_n}$, we obtain by [8, Proposition 1.4.2] that

$$\begin{aligned} \log E[e^{(n+1)g(X_n)}] &= \sup_{\gamma \in \mathcal{P}(E)} \left\{ -R(\gamma \| \theta_n) + \int_E (n+1) g d\gamma \right\} \\ &= -R(\gamma_n \| \theta_n) + \int_E (n+1) g d\gamma_n \end{aligned}$$

and

$$\log E[e^{(n+1)g(X_{n+1})}] \geq -R(\gamma_n \| \theta_{n+1}) + \int_E (n+1) g d\gamma_n,$$

where $R(\gamma\|\theta) = \int_E \left(\log \frac{d\gamma}{d\theta}\right) d\gamma$ denotes the relative entropy. Therefore,

$$\begin{aligned}
\log\left(\frac{E[e^{(n+1)g(X_n)}]}{E[e^{(n+1)g(X_{n+1})}]}\right) &\leq -R(\gamma_n\|\theta_n) + R(\gamma_n\|\theta_{n+1}) \\
&= -\int_E \left(\log \frac{d\gamma_n}{d\theta_n}\right) d\gamma_n + \int_E \left(\log \frac{d\gamma_n}{d\theta_{n+1}}\right) d\gamma_n \\
&= \int_E \left(\log \frac{d\theta_n}{d\theta_{n+1}}\right) d\gamma_n.
\end{aligned}$$

$(D_1) \Rightarrow$ (4.1): Suppose (4.1) does not hold. Denote $a_n = \frac{1}{n}\log E[e^{ng(X_n)}]$. Then, there exist two subsequences $\{n_p\}$ and $\{m_q\}$, $\varepsilon > 0$ and $R > 0$ such that

$$a_{n_p} - a_{m_q} > \varepsilon \tag{4.3}$$

whenever $p \geq R$ and $q \geq R$.

If (D_1) holds, then there exists an N such that for every $M \geq N$, $k > 0$

$$\sum_{n=M}^{M+k} \frac{1}{n+1}\left(\log \frac{E[e^{(n+1)g(X_n)}]}{E[e^{(n+1)g(X_{n+1})}]}\right) < \varepsilon. \tag{4.4}$$

We fix a $p \geq R$ satisfying $n_p \geq N$ and a $q \geq R$ satisfying $m_q > N \vee n_p$. Note that by Jensen's inequality and (4.4) we get

$$\begin{aligned}
a_{n_p} - a_{m_q} &= \sum_{t=n_p}^{m_q-1} \{a_t - a_{t+1}\} \\
&= \sum_{t=n_p}^{m_q-1} \left\{\frac{1}{t}\log E[e^{tg(X_t)}] - \frac{1}{t+1}\log E[e^{(t+1)g(X_{t+1})}]\right\} \\
&\leq \sum_{t=n_p}^{m_q-1} \left\{\frac{1}{t+1}\log E[e^{(t+1)g(X_t)}] - \frac{1}{t+1}\log E[e^{(t+1)g(X_{t+1})}]\right\} \\
&= \sum_{t=n_p}^{m_q-1} \frac{1}{t+1}\left(\log \frac{E[e^{(t+1)g(X_t)}]}{E[e^{(t+1)g(X_{t+1})}]}\right) \\
&< \varepsilon,
\end{aligned}$$

which contradicts (4.3). ■

There is no need to explicitly establish limits if (D_1) or (D_2) is used. Now, we show that (4.1) and (4.2) are equivalent and (A-D) give us a *full* LDP for vectors.

Lemma 17 *Suppose (A-C) hold. Then, the following are equivalent: a) (4.1), b) (4.2) and c)* $\{(g_1(X_n), \ldots, g_k(X_n))\}$ *satisfies the LDP in* $\mathbb{R}^k$ *with the good rate function*

$$J_k(z) = \begin{cases} \sup_{f \in D_k} \{f(z) - \Lambda(f(g_1, \ldots, g_k))\}, & \text{if } z \in F_k, \\ \infty, & \text{otherwise}, \end{cases}$$

for all $g_1, \ldots, g_k \in \mathcal{M}$*, where* $F_k \doteq \overline{(g_1, \ldots, g_k)(E)}$ *is the* $\mathbb{R}^k$*-closure of* $(g_1, \ldots, g_k)(E)$*.*

Proof. Suppose (4.2) holds. Then, g is bounded so $\{g(X_n)\}$ satisfies the LDP in a compact $K_g \subset \mathbb{R}$ with some good rate function I_g. Hence, by the Laplace principle

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{ng(X_n)}] = -\inf_{x \in K_g} \{I_g(x) - x\}$$

and (4.1) holds. Now, suppose (4.1) holds. Let $g_1, \ldots, g_k \in \mathcal{M}$. Since each g_i is bounded, $\{(g_1(X_n), \ldots, g_k(X_n))\}$ is exponentially tight in F_k. For each $f \in D_k$,

$$\Lambda(f(g_1, \ldots, g_k)) = \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(g_1(X_n), \ldots, g_k(X_n))}]$$

exists by (4.1). Hence, the sequence $\{(g_1(X_n), \ldots, g_k(X_n))\}$ of F_k-valued random variables satisfies the LDP with the good rate function

$$I_k(z) = \sup_{f \in D_k} \{f(z) - \Lambda(f(g_1, \ldots, g_k))\}, \quad z \in F_k.$$

Then, $J_k(z) \doteq \begin{cases} I_k(z), & \text{if } z \in F_k, \\ \infty, & \text{otherwise} \end{cases}$ is a good rate function on $\mathbb{R}^k$ and $\{(g_1(X_n), \ldots, g_k(X_n))\}$ satisfies the LDP in $\mathbb{R}^k$ with the J_k by [7, Lemma 4.1.5(a)]. Finally, (4.2) holds by the contraction principle. ■

We now use mapping methods, the Dawson-Gärtner theorem and Lemma 17 to extend our LDP for vectors to an LDP for random variables on (potentially) infinite dimensional spaces. We define a measurable bijection $G : E \to G(E) \subset \mathbb{R}^{\mathcal{M}}$ by $G(x) \doteq (g(x))_{g\in\mathcal{M}}$. By Lemmas 4 and 7, $(E, \mathcal{T}^{\mathcal{M}})$ is completely regular and G is a homeomorphism when E is equipped with $\mathcal{T}^{\mathcal{M}}$. Then, (by [15, page 239]) $(E, \mathcal{T}^{\mathcal{M}})$ can be densely imbedded into a compact Hausdorff space $\overline{E}$, referred to as the Stone-Čech compactification, and G can be extended to a homeomorphism $\Gamma = (\overline{g})_{g\in\mathcal{M}} : \overline{E} \to \overline{G(E)}$, the closure of $G(E)$ in $\mathbb{R}^{\mathcal{M}}$. Next, we let $\overline{g}$ denote the extension of g from this homeomorphism and infer an LDP for $\{X_n\}$ on $\overline{E}$.

Proposition 18 *$\{X_n\}$ satisfies the LDP in $\overline{E}$ with good rate function I^*, where*

$$I^*(x) = \sup_{g\in\mathcal{G}}\{\overline{g}(x) - \Lambda(g)\} \quad \forall x \in \overline{E}, \tag{4.5}$$

when (A)-(D) hold.

Proof. Idea: 1) Transfer the LDP for each $\{\phi(X_n) = (g_1(X_n), \ldots, g_{k_\phi}(X_n))\}$ on $\mathbb{R}^k$ to an LDP on a subset Z_ϕ of $\mathbb{R}^{\mathcal{M}}$ via homeomorphism. 2) Define distributions on the projective limit $\mathcal{X}$, a subset of product space of the Z_ϕ over all possible ϕ that preserves the notion that some ϕ contain others. 3) Transfer these LDPs on Z_ϕ to $\mathcal{X}$ via the Dawson-Gärtner theorem. 4) Transfer the LDP to product space $\mathbb{R}^{\mathcal{M}}$ via homeomorphism W from the projective limit $\mathcal{X}$ to this product space. 5) Finally, use our homeomorphism Γ to transfer the LDP from $\mathbb{R}^{\mathcal{M}}$ to $\overline{E}$.

Notation: Throughout $\phi(x) = (g_1(x), \ldots, g_{k_\phi}(x))$ for some $k_\phi \in \mathbb{N}$ and $g_1, \ldots, g_{k_\phi} \in \mathcal{M}$.

Step 1: Define Z_ϕ, create homeomorphism and transfer LDPs to Z_ϕ.

Let Φ be the family of all *finite* subsets of $\mathcal{M}$ and

$$Z_\phi \doteq \{z = (z_g)_{g\in\mathcal{M}} \in \mathbb{R}^{\mathcal{M}} : z_g = 0 \text{ if } g \notin \phi\},$$

equipped with the subspace topology of $\mathbb{R}^{\mathcal{M}}$, for all $\phi \in \Phi$. Define the homeomorphism $\tau_\phi : Z_\phi \to \mathbb{R}^{k_\phi}$, $\tau_\phi((x_g)_{g\in\mathcal{M}}) = (x_g)_{g\in\phi}$. One finds $\{\tau^{-1}\phi(X_n)\}$ satisfies the LDP on Z_ϕ with the good rate function $J_{k_\phi} \circ \tau_\phi$ by Lemma 17 and the contraction principle.

Step 2: Define projective system, its limit $\mathcal{X}$ and the distributions on $\mathcal{X}$.

For $\phi, \psi \in \Phi$ with $\phi \subseteq \psi$ and $z = (z_g)_{g\in\mathcal{M}} \in Z_\psi$; we define $p_{\phi,\psi} : Z_\psi \to Z_\phi$ by $(p_{\phi,\psi}(z))_g = z_g$ if $g \in \phi$ and $(p_{\phi,\psi}(z))_g = 0$ otherwise. We also define $\mathcal{Z} \doteq \prod_{\phi\in\Phi} Z_\phi$ with the product topology and $\mathcal{X} = \{\zeta \in \mathcal{Z} : \zeta_\phi = p_{\phi,\psi}(\zeta_\psi) \text{ if } \phi \subseteq \psi\}$ with the subspace topology of $\mathcal{Z}$. Then, $(\Phi, \subseteq)$ is a partially ordered, right-filtering set, $(Z_\psi, p_{\phi,\psi})_{\phi\subseteq\psi\in\Phi}$ is a projective system and $\mathcal{X}$ is its projective limit. For $\phi \in \Phi$, we let q_ϕ be the coordinate map from $\mathcal{Z}$ to Z_ϕ and define $p_\phi \doteq q_\phi|_{\mathcal{X}}$. Finally, we define the homeomorphism $W : \mathcal{X} \to \mathbb{R}^{\mathcal{M}}$, $x \to W(x) = (W(x)_g)_{g\in\mathcal{M}}$ by

$$W(x)_g \doteq (p_{\{g\}}(x))_g.$$

For $n \in \mathbb{N}$, we define $Q_n^* \doteq P(H(G(X_n)))^{-1}$ on $\mathcal{X}$, where $H \doteq W^{-1}$.

Step 3: LDP on projective limit $\mathcal{X}$.

$\{Q_n^* p_\phi^{-1}\}$ satisfies the LDP on Z_ϕ with the good rate function $J_{k_\phi} \circ \tau_\phi$ by Step 1. By the Dawson-Gärtner theorem (cf. [7, Theorem 4.6.1]), $\{Q_n^*\}$ satisfies the LDP with the good rate function

$$J^*(x) = \sup_{\phi\in\Phi}\{J_{k_\phi} \circ \tau_\phi \circ p_\phi(x)\}, \quad x \in \mathcal{X}.$$

Step 4: LDP on product space $\mathbb{R}^{\mathcal{M}}$.

By the contraction principle $\{Q_n\}$, defined by $Q_n = Q_n^* H$, satisfies the LDP in $\mathbb{R}^{\mathcal{M}}$ with the good rate function $J \doteq J^* \circ H$, which satisfies

$$J(z) = \sup_{(z_{g_1},\ldots,z_{g_k})} \sup_{f\in D_k} \{f(z_{g_1},\ldots,z_{g_k}) - \Lambda(f(g_1,\ldots,g_k))\} \;\; \forall z = \{z_g\}_{g\in\mathcal{M}} \in \overline{G(E)}. \quad (4.6)$$

Note that if $z = (z_g)_{g\in\mathcal{M}} \in \overline{G(E)}$, then $(z_{g_1},\ldots,z_{g_k}) \in \overline{(g_1,\ldots,g_k)(E)}$ for any $g_1,\ldots,g_k \in \mathcal{M}$ so the above definition makes sense.

Step 5: LDP on compact Hausdorff space $\overline{E}$ by restriction and homeomorphism. $J(z) = \infty$ if $z \notin \overline{G(E)}$ by (1.2) and the fact $Q_n(\overline{G(E)}) = 1$ for $n \in \mathbb{N}$. Thus, $\{Q_n\}$ satisfies the LDP in $\overline{G(E)}$ with the good rate function $J|_{\overline{G(E)}}$ by [7, Lemma 4.1.5(b)]. To simplify notation, we still use J to denote $J|_{\overline{G(E)}}$. Define $I^* \doteq J\circ\Gamma$, where Γ is defined above the proposition. By the contraction principle, $\{X_n\}$ satisfies the LDP in $\overline{E}$ with the good rate function I^*. Finally, I^* has the form (4.5) by (4.6). ■

When we strengthen (C) so $\mathcal{M} \subset \overline{C}(E)$, we can infer a weak LDP on the original space $(E,\mathcal{T})$ from our LDP on $\overline{E}$. The following is our first main result.

Theorem 19 $\{X_n\}$ *satisfies the weak LDP in E with rate function*

$$I^0(x) = \sup_{g\in\mathcal{G}}\{g(x) - \Lambda(g)\} \;\; \forall x \in E,$$

when (A)-(D) hold and $\mathcal{M} \subset \overline{C}(E)$.

Proof. We define $\overline{P}_n \doteq PX_n^{-1}$ and $P_n \doteq PX_n^{-1}$ when X_n is considered $\overline{E}$ and E-valued respectively. By Proposition 18, $\{\overline{P}_n\}$ satisfies the LDP in the compact Hausdorff space $\overline{E}$ with rate function I^*. We show that $\{P_n\}$ satisfies the weak LDP with rate function $I^0 \doteq I^*|_E$.

Let $G \subset E$ and $G' \subset \overline{E}$ be open and satisfy $G = G' \cap E$. (1.2) holds as follows:

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n}\log P_n(G) &= \liminf_{n\to\infty} \frac{1}{n}\log \overline{P}_n(G') \\ &\geq -\inf_{x\in G'} I^*(x) \\ &\geq -\inf_{x\in G} I^*(x)\end{aligned}$$

$$= -\inf_{x \in G} I^0(x).$$

Let $\alpha \in [0, \infty)$ and K be a compact subset of E such that $K \subset \{x \in E : I^0(x) > \alpha\}$. Note that K is also a compact subset of $\overline{E}$. Then, by the LDP of $\{\overline{P}_n\}$ on $\overline{E}$, we get

$$\limsup_{n \to \infty} \frac{1}{n} \log P_n(K) \leq -\alpha.$$

Therefore $\{X_n\}$ satisfies the weak LDP in E with the rate function I^0. ■

Remark 20 *Theorem 19 provides a non-trivial generalization of Bryc's lemma that does not require exponential tightness. Our method might be distinguished from those of [7, Theorem 4.4.10] or [10, Proposition 3.20)] by the fact that we imbed our random variables into a compact space to avoid requiring exponential tightness.*

Remark 21 *With respect to the previous two results, we can think of I^* as the extended rate function for the weak LDP of $\{X_n\}$. If it can be determined that $I^*(x) = \infty$ for all $x \notin E$, then Theorem 19 actually gives a full LDP on E and the rate function I^0 is good: Let $F \subset E$ and $F' \subset \overline{E}$ be closed and satisfy $F = F' \cap E$. Then,*

$$\begin{aligned}
\limsup_{n \to \infty} \frac{1}{n} \log P_n(F) &= \liminf_{n \to \infty} \frac{1}{n} \log \overline{P}_n(F') \\
&\leq -\inf_{x \in F'} I^*(x) \\
&\leq -\inf_{x \in F} I^*(x) \ \text{since } I^*(E^c) = \infty \\
&= -\inf_{x \in F} I^0(x).
\end{aligned}$$

Goodness of I^0 follows from the goodness of I^*.

This leads to an important result, which really is just another way of looking at our first main result.

Theorem 22 *Let E be completely regular, $\mathcal{M} \subset \overline{C}(E)$ s.s.p. and $D = \{D_k\}_{k=1}^{\infty}$ be w.i. Then, $\mathcal{G} \doteq \{f(g_1, \ldots, g_k) : f \in D_k, g_1, \ldots, g_k \in \mathcal{M}, k \in \mathbb{N}\}$ is r.f.d. on E.*

Proof. Let $\{X_n\}$ be a sequence of exponentially tight E-valued random variables such that $D \subset \Xi_{\{X_n\}}$. By Proposition 18, $\{X_n\}$ satisfies the LDP in $\overline{E}$ with good rate function $I^*(x) = \sup_{g\in\mathcal{G}}\{\overline{g}(x) - \Lambda(g)\}$. To show that $\mathcal{G}$ is r.f.d. on E, by Theorem 19 and Remark 21, we need only show that exponential tightness implies $I^*(x) = \infty$ for all $x \notin E$. Let $x \notin E$. Given $0 < \delta < 1$, let compact $K_\delta \subset E$ satisfy $\limsup_{n\to\infty} \frac{1}{n} \log P(X_n \in K_\delta^c) \leq -\frac{1}{\delta}$. Now, there exist $0 < \varepsilon < \delta$ and $\overline{\phi} = (\overline{g}_1, ..., \overline{g}_{k_\phi})$ for some $k_\phi \in \mathbb{N}$ and $g_1, \dots, g_{k_\phi} \in \mathcal{M}$ such that

$$\varepsilon < \sup_{y\in K_\delta} |\overline{\phi}(y) - \overline{\phi}(x)| < \frac{1}{\varepsilon}.$$

Let $h \in D_{k_\phi}$ be an $\overline{\phi}(x), \varepsilon$-h.f. and $f(y) \doteq h(y) - h(\overline{\phi}(x))$ for $y \in \mathbb{R}^{k_\phi}$. Then,

$$\limsup_{n\to\infty} \frac{1}{n} \log E\left[e^{nf\circ\phi(X_n)}\right] \leq \lim_{n\to\infty} \frac{1}{n} \log\{e^{n\varepsilon}e^{-\frac{n}{\delta}} + e^{-\frac{n}{\varepsilon}}\} \leq \delta - \frac{1}{\delta}.$$

Hence,

$$I^*(x) \geq f(\overline{\phi})(x) - \Lambda(f \circ \phi) \geq \frac{1}{\delta} - \delta \tag{4.7}$$

and the result follows by the arbitrariness of δ. □

Remark 23 *Actually, by Theorem 19 and [7, Lemma 1.2.18], we can show that Theorem 22 holds if we use w.d. instead of w.i. However, we chose to use the highpoint functions within the proof as they are an easily verifiable class that is still very general.*

In Theorem 19, we imposed the extra condition that $\mathcal{M} \subset \overline{C}(E)$ to ensure that the topologies $\mathcal{T} = \mathcal{T}^{\mathcal{M}}$. If $\mathcal{M}$ is not a subset of $\overline{C}(E)$ we still have the following:

Corollary 24 *Suppose (A)-(D) hold and $\mathcal{M}$ is countable. Then, $\{X_n\}$ satisfies the weak LDP in $(E, \mathcal{T}^{\mathcal{M}})$ with rate function I^0 given in Theorem 19. Moreover, we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n \in U) \geq -\inf_{x\in U} I^0(x) \;\; \forall \text{ open subset } U \text{ of } (E, \mathcal{T}).$$

Proof. Each $P_n \doteq PX_n^{-1}$ is a probability measure on both $(E, \mathcal{T}^{\mathcal{M}})$ and $(E, \mathcal{T})$ (cf. [1, Lemma 3]). Then, the assertions follow from Theorem 19 applied to $(E, \mathcal{T}^{\mathcal{M}})$ and the fact that $(E, \mathcal{T})$ has a coarser topology than $(E, \mathcal{T}^{\mathcal{M}})$. ■

We have an immediate corollary in terms of the Laplace principle:

Corollary 25 $\{X_n\}$ *satisfies the Laplace principle lower bound with rate function*

$$I^0(x) = \sup_{g\in\mathcal{G}} \{g(x) - \Lambda(g)\} \;\; \forall x \in E,$$

when (A)-(D) hold, E is Polish and either $\mathcal{M} \subset \overline{C}(E)$ or $\mathcal{M}$ is countable.

5 The Weak Laplace Principle and its Ramifications

Now, we consider the situation where the rate function on E is known. Besides (A)-(C) of Section 4, we assume throughout this section our weak Laplace principle:

(D') I is a rate function on E and there is a w.i. class $\{D_k\}_{k=1}^{\infty}$ such that either

$$\text{i) } \lim_{n\to\infty} \frac{1}{n} \log E[e^{-ng(X_n)}] = -\inf_{y\in\mathbb{R}} \left\{ y + \inf_{x\in E:\, g(x)=y} I(x) \right\} \;\; \forall g \in \mathcal{G}, \tag{5.1}$$

or ii)

$$\limsup_{n\to\infty} \frac{1}{n} \log P(g(X_n) \in K) \leq - \inf_{x\in E:\, g(x)\in K} I(x) \;\; \forall \text{ compact } K \subset \mathbb{R}, g \in \mathcal{G}, \tag{5.2}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log P(g(X_n) \in O) \geq - \inf_{x\in E:\, g(x)\in O} I(x) \;\; \forall \text{ open } O \subset \mathbb{R}, g \in \mathcal{G}, \tag{5.3}$$

where $\mathcal{G} \doteq \{f(g_1, \ldots, g_k) : f \in D_k, g_1, \ldots, g_k \in \mathcal{M}, k \in \mathbb{N}\}$.

Theorem 26 *Suppose (A)-(C) and (D') hold. Then,* *(a)* $\{X_n\}$ *satisfies the weak LDP in* $(E, \mathcal{T})$ *with rate function* I *if* $\mathcal{M} \subset \overline{C}(E)$.

(b) $\{X_n\}$ *satisfies the weak LDP in* $(E, \mathcal{T}^{\mathcal{M}})$ *with rate function* I *and*

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n \in U) \geq -\inf_{x\in U} I(x) \quad \forall \text{ open } U \subset (E, \mathcal{T}) \text{ if } \mathcal{M} \text{ is countable.}$$

In either case, we have the Laplace lower bound

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{-n\phi(X_n)}] \geq -\inf_{x\in E}\{\phi(x) + I(x)\} \quad \forall \phi \in \overline{C}(E).$$

Remark 27 *Conditions (5.2) and (5.3) constitute a weak LDP for each* $\{g(X_n)\}$*. Hence, we have established conditions on* $\mathcal{G}$ *that allow transfer of the weak LDP from test functions to the underlying random object.*

Proof. (a) For $g \in \mathcal{G}$ and $y \in \mathbb{R}$, we define

$$I_g(y) \doteq \inf_{x\in E:\, g(x)=y} I(x).$$

Following the argument of the proof of [8, Theorem 1.2.1], we can show (5.2, 5.3) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{-ng(X_n)}] = -\inf_{y\in\mathbb{R}}\{y + I_g(y)\},$$

i.e., (5.1) holds. By (A-C) and either (5.1) or (5.2, 5.3), $\{X_n\}$ satisfies the weak LDP by Theorem 19. To complete the proof, we show the rate function is I.

By (4.6) and the fact $\Lambda(g) = \sup_{y\in\mathbb{R}}\{y - I_g(y)\}$ for $g \in \mathcal{G}$, $\{Q_n \doteq P(G(X_n))^{-1}\}$ satisfies the LDP in $\mathbb{R}^{\mathcal{M}}$ with good rate function:

$$J(z) = \sup_{(z_{g_1},\dots,z_{g_k})} \sup_{f\in D_k} \{f(z_{g_1}, \dots, z_{g_k}) - \sup_{x\in E} N_k(f, x)\} \tag{5.4}$$

if $z = (z_g)_{g\in\mathcal{M}} \in \overline{G(E)}$, where

$$\begin{aligned} N_k(f, x) \doteq\, & f(g_1(x), \dots, g_k(x)) \\ & - \inf\{I(\tilde{x}) : \tilde{x} \in E \text{ and } f(g_1(\tilde{x}), \dots, g_k(\tilde{x})) = f(g_1(x), \dots, g_k(x))\} \end{aligned}$$

for $g_1, \ldots, g_k \in \mathcal{M}$, $f \in D_k$ and $x \in E$.

We now show $J(z) = I_G(z) \doteq I(G^{-1}(z))$ for $z \in G(E)$: Suppose $z = (z_g)_{g \in \mathcal{M}} \in G(E)$ and $x^* = G^{-1}(z)$ so $I_G(z) = I(x^*)$. We first suppose that $I(x^*) < \infty$. Let $\varepsilon > 0$. Then by the lower semicontinuity of I on E and (C), there exist $g_1, \ldots, g_k \in \mathcal{M}$ such that for any $\tilde{x} \in E$ satisfying $|g_j(\tilde{x}) - z_{g_j}| = |g_j(\tilde{x}) - g_j(x^*)| < \varepsilon$ for all $1 \le j \le k$, one has that $I(\tilde{x}) \ge I(x^*) - \varepsilon$. Let $0 < \varepsilon < 1$, $h \in D_k$ be a $(z_{g_1}, ..., z_{g_k}), \varepsilon$-h.f. and $f(y) \doteq h(y) - h(z_{g_1}, ..., z_{g_k}) - \varepsilon$ for $y \in \mathbb{R}^k$. Then, $f(z_{g_1}, \ldots, z_{g_k}) = -\varepsilon$, $\sup_{w \in \overline{(g_1,\ldots,g_k)(E)}} f(w) \le 0$ and

$$\sup_{w \in \overline{(g_1,\ldots,g_k)(E)} \cap B^c((z_{g_1},\ldots,z_{g_k}),\varepsilon)} f(w) < -\frac{1}{\varepsilon}. \tag{5.5}$$

(5.5) implies that

$$\text{if } w \in \overline{(g_1, \ldots, g_k)(E)} \text{ satisfies } f(w) \ge -\frac{1}{\varepsilon}, \text{ then } w \in B((z_{g_1}, \ldots, z_{g_k}), \varepsilon). \tag{5.6}$$

Then, by (5.4), we get

$$\begin{aligned}
J(z) &\ge f(z_{g_1}, \ldots, z_{g_k}) - \sup_{x \in E} N_k(f, x) \\
&\ge -\varepsilon - \sup_{x: f(g_1(x),\ldots,g_k(x)) < -\frac{1}{\varepsilon}} N_k(f, x) \vee \sup_{x: f(g_1(x),\ldots,g_k(x)) \ge -\frac{1}{\varepsilon}} N_k(f, x) \\
&\ge -\varepsilon - \sup_{x: f(g_1(x),\ldots,g_k(x)) < -\frac{1}{\varepsilon}} N_k(f, x) \vee \left(- \inf_{\tilde{x}: (g_1(\tilde{x}),\ldots,g_k(\tilde{x})) \in B((z_{g_1},\ldots,z_{g_k}),\varepsilon)} I(\tilde{x}) \right) \\
&\ge -\varepsilon - \left\{ \left(-\frac{1}{\varepsilon} \right) \vee \left(-(I(x^*) - \varepsilon) \right) \right\} \\
&\to I(x^*) = I_G(z) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where the fact $\sup_{w \in \overline{(g_1,\ldots,g_k)(E)}} f(w) \le 0$ and (5.6) were used in the third inequality.

If $I(x^*) = \infty$, we follow the argument in the previous paragraph, starting with $I(\tilde{x}) > \frac{1}{\varepsilon}$ for all $\tilde{x} \in E$ satisfying $\max_{j \le k} |g_j(\tilde{x}) - g_j(x^*)| < \varepsilon$ for some $g_1, \ldots, g_k \in \mathcal{M}$ to show that $J(z) = \infty$. Hence, $J(z) \ge I_G(z)$ for $z \in G(E)$.

By choosing $x = x^* = G^{-1}(z)$ in (5.4), we see that

$$\begin{aligned} J(z) &\leq \sup_{(z_{g_1},\ldots,z_{g_k})} \sup_{f\in D_k} \inf\{I(\tilde{x}) : \tilde{x} \in E,\ f(g_1(\tilde{x}),\ldots,g_k(\tilde{x})) = f(g_1(x^*),\ldots,g_k(x^*))\} \\ &\leq I(x^*) = I_G(z). \end{aligned}$$

Thus, $J(z) = I_G(z)$ for $z \in G(E)$ and the rate function I^0 associated with the weak LDP of $\{X_n\}$, given in the proof of Theorem 19, is equal to I.

(b) The assertions follow from Corollary 24 and the above characterization of the rate function associated with the weak LDP of $\{X_n\}$. ■

The next result gives a common, convenient way to choose $\{D_k\}$ and $\mathcal{M}$.

Corollary 28 *Suppose (i) D_k is the set of polynomials on $\mathbb{R}^k$; (ii) E is a metric space and $\mathcal{M}$ is the space of uniformly continuous functions with bounded support, or E is a locally compact metric space and $\mathcal{M}$ is the space of continuous functions with compact support; and (iii) $\{X_n\}$ is a sequence of E-valued random variables on probability space $(\Omega, \mathcal{F}, P)$. Then,*
(a) $\{X_n\}$ satisfies the weak LDP in E if (D) holds.
(b) $\{X_n\}$ satisfies the weak LDP in E with rate function I if (D') holds.

Proof. This follows from Theorem 19, Theorem 26 and [1, Equation (4)]. ■

6 Motivation and Examples

In this section, we provide simple motivation for our main results, first in the finite dimensional setting, then in the random measure setting and finally in the stochastic process setting.

Example 29 *(i) We consider the geometric Brownian motion*

$$dX_t = \alpha X_t dt + \beta X_t dW_t, \quad X_0 = x > 0.$$

It can be shown that (cf. [5, (8.11.49)]) X_t has the transition density

$$p(t,x,y) = \frac{1}{y\sqrt{2\pi\beta^2 t}} e^{-\frac{\left[\log y - \log x - \left(\alpha - \frac{\beta^2}{2}\right)\right]^2}{2\beta^2 t}}, \quad t > 0. \tag{6.1}$$

Define $X_n(\omega) = X(n,\omega)$, $n \in \mathbb{N}$. Then

$$\log \frac{d\theta_n}{d\theta_{n+1}} \le \frac{1}{2} \log\left(1 + \frac{1}{n}\right) \le \frac{1}{2n}.$$

Therefore (D_2) holds and $\{X_n\}$ satisfies the weak LDP.

(ii) Let $X(t,\omega)$ be the standard d-dimensional Cauchy process starting at $x \in \mathbb{R}^d$. Define $X_n(\omega) = X(n,\omega)$, $n \in \mathbb{N}$. Then

$$d\theta_n = \frac{c_d n}{(n^2 + |x-y|^2)^{(d+1)/2}} dy$$

for some positive constant c_d. Hence

$$\log \frac{d\theta_n}{d\theta_{n+1}} \le \frac{d+1}{2} \log\left(1 + \frac{3}{n}\right) \le \frac{3(d+1)}{2n}.$$

Therefore (D_2) holds and $\{X_n\}$ satisfies the weak LDP.

Example 30 *Let Z be a standard normal random variable on $\mathbb{R}$ and $\{Y_n\}$ be a sequence of random variables on $\mathbb{R}^2$ satisfying $\lim_{n\to\infty}\{\inf_{\omega\in\Omega} |Y_n(\omega)|\} = \infty$. Define*

$$X_n \doteq \begin{cases} (\sqrt{n}Z, 0), & \text{with probability } 1/2, \\ Y_n, & \text{with probability } 1/2. \end{cases}$$

Let $C_c(\mathbb{R}^2)$ be the space of continuous functions on $\mathbb{R}^2$ with compact support and D_k be the polynomials on $\mathbb{R}^k$. Then, for $f \in D_k$, $g_1, \ldots, g_k \in C_c(\mathbb{R}^2)$ and sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned}
\log \frac{E[e^{(n+1)f\circ(g_1,\ldots,g_k)(X_n)}]}{E[e^{(n+1)f\circ(g_1,\ldots,g_k)(X_{n+1})}]} &= \log \frac{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n}Z,0))}] + f(0,\ldots,0)}{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n+1}Z,0))}] + f(0,\ldots,0)} \\
&\leq \log \left(\frac{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n}Z,0))}]}{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n+1}Z,0))}]} \vee 1 \right) \\
&= \left(\log \frac{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n}Z,0))}]}{E[e^{(n+1)f\circ(g_1,\ldots,g_k)((\sqrt{n+1}Z,0))}]} \right) \vee 0 \\
&= \left(\log \frac{n^{-1/2} \int_{-\infty}^{\infty} e^{(n+1)f\circ(g_1,\ldots,g_k)((y,0))} e^{-y^2/2n} dy}{(n+1)^{-1/2} \int_{-\infty}^{\infty} e^{(n+1)f\circ(g_1,\ldots,g_k)((y,0))} e^{-y^2/2(n+1)} dy} \right) \vee 0 \\
&\leq \frac{1}{2} \log \left(1 + \frac{1}{n} \right) \\
&\leq \frac{1}{2n}.
\end{aligned}$$

Therefore, (D_1) holds and $\{X_n\}$ satisfies the weak LDP by Lemma 15 and Corollary 28.

The limits $\Lambda(g) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{ng(X_n)}]$ may not exist for some $g \in \overline{C}(\mathbb{R}^2)$ and thus the weak LDP version of Bryc's lemma (see [7, Lemmas 4.4.5 and 4.4.6]) does not apply. For instance, let $Y_n \equiv (0, n)$ and $g \in \overline{C}(\mathbb{R}^2)$ satisfying

$$g((x_1, n)) = \begin{cases} 1, & \text{if } x_1 \in \mathbb{R} \text{ and } n \text{ is an even number,} \\ 2, & \text{if } x_1 \in \mathbb{R} \text{ and } n \text{ is an odd number.} \end{cases}$$

Then

$$\limsup_{n\to\infty} \frac{1}{n} \log E[e^{ng(X_n)}] \geq 2, \quad \liminf_{n\to\infty} \frac{1}{n} \log E[e^{ng(X_n)}] \leq 1.$$

Therefore the limit $\Lambda(g)$ does not exist.

We consider application of Theorem 19 to random measures. Let E be a topological space. For $\nu \in \mathcal{P}(E)$ and $\gamma \in \overline{C}(E)$, define $\hat{\gamma}(\nu) = \int_E \gamma d\nu$.

Corollary 31 *Suppose E is a topological space; $\{X_n\}$ is a sequence of $\mathcal{P}(E)$-valued random variables; $D = \{D_k\}_{k=1}^{\infty}$ is w.d., $\mathcal{H} \subset \overline{C}(E)$ s.p., s.s.p. and is closed under multiplication; either $\mathcal{H}$ is countable or E has a countable base; and*

$$\Lambda\left(f \circ (\hat{g}_1, \ldots, \hat{g}_k)\right) \doteq \lim_{n\to\infty} \frac{1}{n} \log E[e^{n(f\circ(\hat{g}_1,\ldots,\hat{g}_k)(X_n))}] \text{ exists } \forall f \in D_k,\ g_1, \ldots, g_k \in \mathcal{H}.$$

Then, $\{X_n\}$ satisfies the weak LDP in $\mathcal{P}(E)$ with rate function

$$I(\mu) = \sup_{\phi = f\circ(\hat{g}_1,\ldots,\hat{g}_k), f\in D_k, g_i\in\mathcal{H}} \{\phi(\mu) - \Lambda\left(f \circ (\hat{g}_1, \ldots, \hat{g}_k)\right)\}.$$

Proof. This follows from Theorem 19 and [1, Theorems 6 and 11 (a)]. ■

We also give a known-rate-function version of Corollary 31 for convenience.

Corollary 32 *Suppose E is a topological space; $\{X_n\}$ is a sequence of $\mathcal{P}(E)$-valued random variables; $D = \{D_k\}_{k=1}^{\infty}$ is w.i., $\mathcal{H} \subset \overline{C}(E)$ s.p., s.s.p. and is closed under multiplication; either $\mathcal{H}$ is countable or E has a countable base; I is a rate function on $\mathcal{P}(E)$; and*

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{-n(f\circ(\hat{g}_1,\ldots,\hat{g}_k)(X_n))}] = -\inf_{y\in\mathbb{R}} \{y + \inf_{\mu: f\circ(\hat{g}_1,\ldots,\hat{g}_k)(\mu)=y} I(\mu)\}$$

$\forall f \in D_k,\ g_1, \ldots, g_k \in \mathcal{H}$. Then, $\{X_n\}$ satisfies the weak LDP with rate function I.

Proof. This follows from Theorem 26 and [1, Theorems 6 and 11 (a)]. ■

We now illustrate Corollary 31. Let $E = [0,\infty)$ and $\mathcal{H} = C_c(E)$, the continuous functions on E with compact support, so $\mathcal{H}$ s.p. and s.s.p. Let D_k be the polynomials on $\mathbb{R}^k$. Furthermore, for $n \in \mathbb{N}$, we let $\{\xi_i\}_{i=1}^n$ be n particles on $[0,\infty)$ such that with probability $\frac{n-1}{n}$ all particles are at site $\{0\}$ and with probability $\frac{1}{n}$ all particles are independently and uniformly distributed over $[n, n+1]$. Let X_n be the empirical measure of $\{\xi_i\}_{i=1}^n$ on $[0,\infty)$ so $\hat{g}(X_n) = \frac{1}{n}\sum_{i=1}^{n} g(\xi_i)$ for $g \in \mathcal{H}$.

Then, for all $f \in D_k$ and $g_1, \ldots, g_k \in \mathcal{H}$, we obtain by the fact $\lim_{n\to\infty} \frac{1}{n}\log\left(\frac{n-1}{n}\right) = 0$ and [7, Lemma 1.2.15] that

$$\begin{aligned}
\Lambda(f \circ (\hat{g}_1, \ldots, \hat{g}_k)) &= \lim_{n\to\infty} \frac{1}{n} \log E[e^{n(f\circ(\hat{g}_1,\ldots,\hat{g}_k)(X_n))}] \\
&= \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{n-1}{n} e^{nf(g_1(0),\ldots,g_k(0))} + \frac{1}{n} e^{nf(0,\ldots,0)}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log\left(e^{nf(g_1(0),\ldots,g_k(0))} + \frac{1}{n-1} e^{nf(0,\ldots,0)}\right) \\
&= \max\left\{\lim_{n\to\infty} \frac{1}{n} \log e^{nf(g_1(0),\ldots,g_k(0))}, \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{n-1} e^{nf(0,\ldots,0)}\right\} \\
&= f(g_1(0), \ldots, g_k(0)) \vee f(0, \ldots, 0),
\end{aligned}$$

so $\{X_n\}$ satisfies the weak LDP in $\mathcal{P}([0,\infty))$ by Corollary 31 with rate function

$$\begin{aligned}
I(\mu) &= \sup_{\phi = f\circ(\hat{g}_1,\ldots,\hat{g}_k), f\in D_k, g_i\in\mathcal{H}} \{\phi(\mu) - f(g_1(0), \ldots, g_k(0)) \vee f(0, \ldots, 0)\} \\
&= \begin{cases} 0, & \text{if } \mu = \delta_0, \\ \infty, & \text{if } \mu \neq \delta_0. \end{cases}
\end{aligned}$$

Here, we used the fact that if $\mu \neq \delta_0$, then $\mu((\varepsilon, \frac{1}{\varepsilon})) > 0$ for some $\varepsilon > 0$. Notice $\Lambda(-f \circ (\hat{g}_1, \ldots, \hat{g}_k)) \neq -f(g_1(0), \ldots, g_k(0)) = -\inf_{\mu\in\mathcal{P}(E)}\{f \circ (\hat{g}_1, \ldots, \hat{g}_k)(\mu) + I(\mu)\}$ as would be predicted by the Laplace principle. However, it does satisfy the Laplace lower bound and our weak Laplace principle (see (5.1)) since:

$$\Lambda(-f \circ (\hat{g}_1, \ldots, \hat{g}_k)) = -\inf_{y\in\mathbb{R}}\{y + \inf_{\mu:\, f\circ(\hat{g}_1,\ldots,\hat{g}_k)(\mu)=y} I(\mu)\}$$

$$= - \inf_{z \in (\hat{g}_1, \ldots, \hat{g}_k)(\mathcal{P}(E))} \{ f(z) + \inf_{\mu : (\mu(g_1), \ldots, \mu(g_k)) = z} I(\mu) \}.$$

Next, we show that $\{X_n\}$ cannot satisfy the full LDP and hence neither satisfies the Laplace principle nor is exponentially tight. Let $\gamma \in \overline{C}([0,\infty))$ be supported on $[1,\infty)$ and satisfy $\gamma(x) = 1$ for $x \geq 2$ and set $F = \{\mu \in \mathcal{P}(E) : \int_0^\infty \gamma(x)\, \mu(\mathrm{d}x) \geq 1\}$. Then, F is closed in the weak topology and $P(X_n \in F) = P(X_n([2,\infty)) \geq 1) = \frac{1}{n}$ for $n \geq 2$, so $\lim_{n\to\infty} \frac{1}{n} \log (P(X^n \in F)) = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{n} = 0$. But $\inf_{\mu \in F} I(\mu) = \infty$ so the upper bound and hence the full LDP cannot hold. Therefore, this weak LDP could not be obtained directly from the Laplace principle, Bryc's Lemma, Dembo and Zeitouni [7, Theorem 4.4.10], nor Feng and Kurtz [10, Proposition 3.17]. Indeed, the limits $\Lambda(\hat{g})$ do not even exist for some $g \in \overline{C}(E)$: Let $g = \sin(\pi x)$ so $P(\hat{g}(X_n) > 0) = 0$ for $n = 1, 3, \ldots$ and $P(\hat{g}(X_n) \geq \frac{1}{\sqrt{2}}) = \frac{1}{2n}$ for $n = 2, 4, \ldots$, which implies $E[e^{n\hat{g}(X_n)}] < 1$ for $n = 1, 3, \ldots$ and $E[e^{n\hat{g}(X_n)}] \geq \frac{n-1}{n} + \frac{1}{2n} e^{n/\sqrt{2}}$ for $n = 2, 4, \ldots$ Thus, $\frac{1}{n} \log E[e^{n\hat{g}(X_n)}]$ oscillates, $\Lambda(\hat{g})$ does not exist and even the weak LDP version of Bryc's lemma does not apply.

While this simple example is contrived to demonstrate the need for new results, our real motivation shares some of the same features. In 1967, Watanabe [18] showed a strong law of large numbers (as $t \to \infty$) for a class of supercritical branching Markov processes in the non-ergodic case. The limit, in the sense of vague convergence, was randomly-scaled Lebesgue measure. Hence, his result says that the mass blows up and escapes to infinity but does so in such a regular manner that the (non-exponentially) time-scaled version of the empirical measure converges to Lebesgue measure. Recently, we proved a superprocess analog for Watanabe's result (see Kouritzin and Ren [14]). Our motivation is to develop weak LDPs for superprocesses where there is escape of mass and the state space might not be metrizable (see Fitzsimmons [11]).

We turn our attention to cadlag processes under the measurability constraint that they can be treated as $D_E[0,\infty)$-valued random variables with the Skorohod topology. This holds, for example, for any cadlag process if E is a separable metric space (see Ethier and Kurtz [9, Proposition 3.7.1]). In this $D_E[0,\infty)$ setting, there are two homeomorphisms of interest: The first homeomorphism is defined by:

$$\tilde{G} : D_E[0,\infty) \to D_{\mathbb{R}^{\mathcal{M}}}[0,\infty),\ \tilde{G}(x)(t) = G(x(t))\ \ \forall x \in D_E[0,\infty), t \geq 0,$$

where $G = (g)_{g\in\mathcal{M}}$ is our familiar homeomorphism. We just fixed time and applied our familiar homeomorphism pointwise. For the second homeomorphism $\widehat{G} : D_E[0,\infty) \to (D_{\mathbb{R}}[0,\infty))^{\mathcal{M}'}$ we fix a test function instead of time. It is defined by $\widehat{G} = (\tilde{g})_{g\in\mathcal{M}'}$, where $\tilde{g}(X)(t) = g(X(t))$. It is a quirk of the Skorohod topology that the class of functions $\mathcal{M}'$ used in the definition of this second homeomorphism may have to be larger than the class of functions $\mathcal{M}$ for the first homeomorphism. As is shown in Jakubowski [13, Theorems 1.7 and 4.3(ii)], it is sufficient for $\mathcal{M}'$ to s.s.p. *and* be closed under addition. Next, suppose that $\{h_k\}_{k=1}^{\infty} \subset \overline{C}(D_{\mathbb{R}}[0,\infty))$ s.s.p. [2] so $\{h_k \circ \pi_\alpha : k \in \mathbb{N}, \alpha \in \mathcal{M}'\}$ s.s.p. on $(D_{\mathbb{R}}[0,\infty))^{\mathcal{M}'}$ by Lemma 5. Then, the combined $H \circ \widehat{G} : D_E[0,\infty) \to \mathbb{R}^{\infty\times\mathcal{M}'}$ homeomorphism, defined by $(h_k \circ \pi_\alpha \circ \hat{G})_{k\in\mathbb{N},\alpha\in\mathcal{M}'}$, is of our usual type and uses the composite functions $\{h_k \circ \tilde{g}\}_{k\in\mathbb{N},g\in\mathcal{M}'}$, which must then s.s.p. on $D_E[0,\infty)$ by Lemma 4. We will use these facts below.

For processes, Theorem 19 can be used to obtain a weak LDP analog of [1, Theorem 10], giving an alternative to showing exponential compact containment or exponential tightness, which might not be true when one is only interested in a *weak* LDP.

[2] We only need a countable number of functions since $D_{\mathbb{R}}[0,\infty))$ is separable.

Theorem 33 *Let E be a Hausdorff space and $\{X_n\}$ be a sequence of $D_E[0,\infty)$-valued random variables. Suppose that $\mathcal{M} \subset C(E)$ s.s.p. and $\{(g_1,\ldots,g_k)\circ X_n\}$ satisfies the LDP in $D_{\mathbb{R}^k}[0,\infty)$ with a good rate function for each $g_1,\ldots,g_k \in \mathcal{M}$. Then, $\{X_n\}$ satisfies the weak LDP in $D_E[0,\infty)$.*

Proof. Note that $D_E[0,\infty)$ is Hausdorff. Let $\mathcal{M}'$ be the collection of finite sums of functions in $\mathcal{M}$ and $\{h_k\}_{k=1}^{\infty} \subset \overline{C}(D_{\mathbb{R}}[0,\infty))$ s.s.p. Then, it follows from the above discussion that the collection $\{h_k \circ \tilde{g}\}_{k\in\mathbb{N},g\in\mathcal{M}'}$ s.s.p. on $D_E[0,\infty)$, where $\tilde{g} : D_E[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is defined by $\tilde{g}(x)(t) = g(x(t))$ for $t \geq 0$. Define $\mathcal{H} \doteq \{\prod_{i=1}^{m}(h^i \circ \tilde{g^i}) : g^i \in \mathcal{M}', h^i \in \{h_k\}_{k=1}^{\infty}, m \in \mathbb{N}\}$. Then, by assumption and the contraction principle, we know that $\{\sum_{i=1}^{l} \alpha_i f_i(X_n)\}$ satisfies the LDP with a good rate function for any $\alpha_i \in \mathbb{R}$ and any $f_i \in \mathcal{H}$. Note that $\mathcal{G} \doteq \{\sum_{i=1}^{l} \alpha_i f_i : \alpha_i \in \mathbb{R}, f_i \in \mathcal{H}, 1 \leq i \leq l, l \in \mathbb{N}\}$ is an algebra. Hence, $\{X_n\}$ satisfies the weak LDP in $D_E[0,\infty)$ by Theorem 19. ■

The following theorem and its corollary provide an extension of Feng and Kurtz [10, Theorem 4.28] from Polish spaces to completely regular spaces.

Theorem 34 *Let E be a Hausdorff space and $\{X_n\}$ be a sequence of exponentially tight $D_E[0,\infty)$-valued random variables. Suppose that $\mathcal{M} \subset C(E)$ s.s.p. and $\{g_1(X_n(t_1)),\ldots,g_k(X_n(t_1));\ldots;g_1(X_n(t_m)),\ldots,g_k(X_n(t_m))\}$ satisfies the LDP in $\mathbb{R}^{k\times m}$ for each $0 \leq t_1 < t_2 < \cdots < t_m$ and $g_1, g_2, \ldots, g_k \in \mathcal{M}$. Then $\{X_n\}$ satisfies the LDP in $D_E[0,\infty)$.*

Proof. Let $g_1,\ldots,g_m \in \mathcal{M}$. Then, $\{(g_1,\ldots,g_k)\circ X_n\}$ is exponentially tight in $D_{\mathbb{R}^k}[0,\infty)$ by the continuity of $x \to (g_1,\ldots,g_k)\circ x$. By assumption and [10, Theorem 4.28], we know that $\{(g_1,\ldots,g_k)\circ X_n\}$ satisfies the LDP in $D_{\mathbb{R}^k}[0,\infty)$ with a good rate function. Therefore, the proof is completed by Theorem 33. ■

Corollary 35 *Let E be a completely regular space and $\{X_n\}$ be a sequence of expo-*

nentially tight $D_E[0,\infty)$-valued random variables. Suppose that $\{(X_n(t_1), X_n(t_2), \ldots, X_n(t_m))\}$ satisfies the LDP in E^m with a good rate function for each $0 \le t_1 < t_2 < \cdots < t_m$. Then $\{X_n\}$ satisfies the LDP in $D_E[0,\infty)$.

Proof. By Lemma 7, a Hausdorff space is completely regular if and only if some $\mathcal{M} \subset \overline{C}(E)$ s.s.p. Hence, the corollary follows from Theorem 34 and the contraction principle. ■

By combining Theorem 33 and Schied [16, Theorem 1] (cf. also [10, Theorem 4.4]), we obtain the following path space LDP criterion. Our result improves Schied's result, which only gives exponential tightness.

Theorem 36 *Let E be a completely regular space with metrizable compacts and $\{X_n\}$ be a sequence of $D_E[0,\infty)$-valued random variables. Then, $\{X_n\}$ is exponentially tight and satisfies the LDP in $D_E[0,\infty)$ if and only if the following two conditions are fulfilled:*

(i) (Exponential compact containment) For every $a, T > 0$ there is a compact $K_{a,T} \subset E$ such that $\limsup_{n\to\infty} \frac{1}{n} \log P(\exists t \in [0,T] : X_n(t) \notin K_{a,T}) \le -a$.
(ii) There exists a family $\mathcal{M} \subset C(E)$, which s.s.p. and is closed under addition, such that $\{g(X_n)\}$ is exponentially tight in $D_{\mathbb{R}}[0,\infty)$ for each $g \in \mathcal{M}$ and $\{(g_1,\ldots,g_k)\circ X_n\}$ satisfies the LDP in $D_{\mathbb{R}^k}[0,\infty)$ for each $g_1,\ldots,g_k \in \mathcal{M}$.

Proof. This is a direct consequence of [16, Theorem 1] and Theorem 33. ■

Corollary 37 *Let E be a completely regular space with metrizable compacts and $\{X_n\}$ be a sequence of $D_E[0,\infty)$-valued random variables. Suppose that the following two conditions are fulfilled:*

(i) (Exponential compact containment) For every $a, T > 0$ there is a compact $K_{a,T} \subset$

E *such that* $\limsup_{n\to\infty} \frac{1}{n} \log P(\exists t \in [0,T] : X_n(t) \notin K_{a,T}) \leq -a.$

(ii) There exists a family $\mathcal{M} \subset C(E)$*, which s.s.p. and is closed under addition, such that* $\{g(X_n)\}$ *is exponentially tight in* $D_{\mathbb{R}}[0,\infty)$ *for each* $g \in \mathcal{M}$ *and* $\{g_1(X_n(t_1)),\ldots,g_k(X_n(t_1));\ldots;g_1(X_n(t_m)),\ldots,g_k(X_n(t_m))\}$ *satisfies the LDP in* $\mathbb{R}^{k\times m}$ *for each* $0 \leq t_1 < t_2 < \cdots < t_m$ *and* $g_1, g_2, \ldots, g_k \in \mathcal{M}$*.*

Then, $\{X_n\}$ *is exponentially tight and satisfies the LDP in* $D_E[0,\infty)$*.*

Proof. This is a direct consequence of [16, Theorem 1] and Theorem 34. ■

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