Modelling and Estimation of Lévy driven Ornstein Uhlenbeck processes

by

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Abstract

This dissertation is concerned with the parameter estimation problem for Ornstein-Uhlenbeck processes and Vasicek models and the product formula for multiple Itô integrals of Lévy processes.

In the first part of the thesis, we study the parameter estimation for Ornstein-Uhlenbeck processes driven by the double exponential compound Poisson process. In chapter 23 a method of moments using ergodic theory is proposed to construct ergodic estimators for the double exponential Ornstein-Uhlenbeck process, where the process is observed at discrete time instants with time step size h. We further also show the existence and uniqueness of the function equations to determine the estimators for fixed time step size h. Also, we show the strong consistency and the asymptotic normality of the estimators. Furthermore, we propose a simulation method of the double exponential Ornstein-Uhlenbeck process and perform some numerical simulations to demonstrate the effectiveness of the proposed estimators.

In the next chapter, we consider the parameter estimation problem for Vasicek model driven by the compound Poisson process with double exponential jumps as discussed in Chapter 4. Here we discuss the construction of least square estimators for drift parameters based on continuous time observations.

In the last chapter of the dissertation, we show the derivation of the product formula for finitely many multiple stochastic integrals of Lévy process, expressed in terms of the associated Poisson random measure. A short proof is found that uses properties of exponential vectors and polarization techniques.

Preface

This dissertation is based on two published papers listed below

- Chapter 3 of the dissertation is joint work with Prof. Yaozhong Hu and has been published as "Ergodic Estimators of double exponential Ornstein-Uhlenbeck processes" in the Journal of Computational and Applied Mathematics.
- Chapter 4 of the dissertation is joint work with Prof. Yaozhong Hu and is a work in progress. This work in chapter 4 is in the editing stage and will soon be submitted for publication.
- Chapter 5 of the dissertation is a joint work with Prof. Yaozhong Hu and Dr. Nishant Agrawal and has been published as "General Product Formula of Multiple Integrals of Lévy Process" in the journal of Stochastic Analysis.

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Chapter 1

Summary

This dissertation consists of the work done during my Ph.D. under the supervision of Prof. Yaozhong Hu. This dissertation discusses topics of stochastic calculus, Lévy process, Poisson random measures, parameter estimation, and applications.

Chapter 2 introduces briefly the background and preliminaries of Ornstein Uhlenbeck type processes and stochastic calculus for jump-diffusion processes which are used in Chapters 3, 4, and 5 respectively.

Chapter 3 initiates the study of the parameter estimation for Ornstein-Uhlenbeck processes driven by the double exponential compound Poisson process. In this chapter, a method of moments based on ergodic theory is discussed and proposed to construct ergodic estimators for the double exponential Ornstein-Uhlenbeck process, where the process is observed at discrete time instants with time step size h. Further in this chapter the existence and uniqueness of the function equations to determine the estimators for fixed time step size h is also discussed. The main results are in Section 3.4 concerning the strong consistency and the asymptotic normality of the estimators. Furthermore, in Section 3.5 a simulation method of the double exponential Ornstein-Uhlenbeck process is proposed, and numerical simulations to demonstrate the effectiveness of the proposed estimators are carried out there.

The results of Chapter 3 are further extended in Chapter 4 in which the parameter estimation problem for the Vasicek model driven by the compound Poisson process with double exponential jumps are discussed. In this chapter, estimators are constructed by using the least square techniques for drift parameters based on continuous time observations.

TO deepen my understanding of stochastic analysis, I have also made a contribution to the theory of stochastic analysis of Poisson random measures which has been discussed in Chapter 5. Chapter 5 is based on the derivation of a product formula for finitely many multiple stochastic integrals of Lévy process, expressed in terms of the associated Poisson random measure. In the chapter, it has been shown that the formula is compact. The proof is short and uses exponential vectors and polarization techniques

Chapters 3 to 5 are based on the following works which are listed below.

- Ergodic estimators of double exponential Ornstein–Uhlenbeck processes. with Yaozhong Hu. Journal of Computational and Applied Mathematics, Volume 434, Issue C, Dec 2023
- Parameter Estimation for Vasicek Model with double exponential jump. (In editing for submission)
- General product formula of multiple Integrals of L ́evy process, with Yaozhong Hu and Nishant Agrawal, Journal of Stochastic Analysis: Vol. 1, No. 3 , Article 3.

1.1 Summary of Works

1.1.1 Summary on Ergodic estimators of double exponential Ornstein–Uhlenbeck processes

Consider the following Ornstein-Ulenbeck process described by the following Langevin equation

$$dX_t = -\theta X_t dt + \sigma dZ_t\,, \quad t \in [0, \infty), \quad X_0 = x_0\,. \tag{1.1}$$

the process $Z_t = \sum_{i=1}^{N_t} Y_i$ is the compound Poisson process with double exponential jumps. This process X_t depends on the parameters θ, σ, p (or q), η, λ, and φ. In this work, it is assumed that the process $\{X_t; t \geq 0\}$ can be observed at discrete time instants $t_j = jh$, where $h > 0$ is some observation time interval. We use the discrete observation data $\{X_{t_j}; j = 1, 2, \dots n\}$ to estimate the parameters θ, σ, p, η, λ, and

φ. Given Z_t, a compound Poisson process with double exponential jumps, a unique solution to the equation (1.1) is given by

$$X_t = e^{-\theta t} x_0 + \sigma \int_0^t e^{-\theta(t-s)} dZ_s \,. \tag{1.2}$$

If $\theta > 0$, then the double exponential Ornstein-Uhlenbeck process X_t converges in law to the random variable $\mathbb{X}_o = \sigma \int_0^\infty e^{-\theta s} dZ_s$ which exists a square-integrable random variable.

The explicit form of the distribution of $\mathbb{X}_o$ is hard to obtain. So, it is hard to compute $\mathbb{E}(f(\mathbb{X}_o))$ for general f. But when f has some particular form, namely, when $f(x) = e^{\iota \xi x}$, then the computation of $\mathbb{E}(f(\mathbb{X}_o))$ is much simplified.

Theorem 1.1.1. *Let X_t be the double exponential Ornstein-Uhlenbeck process with initial condition $x_0 \in \mathbb{R}$. Then for any $h \in \mathbb{R}_+, u, v \in \mathbb{R}$, we have almost surely (denoting $t_j = jh$)*

$$\begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n e^{iuX_{t_j}} = \Big(\frac{\eta}{\eta - iu\sigma}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\eta}{\eta + iu\varphi}\Big)^{\frac{q\lambda}{\theta}} \\ \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \exp\left[iuX_{t_j} + ivX_{t_j+h}\right] \\ \qquad = \Big(\frac{\eta}{\eta - i\sigma(u+ve^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi}{\varphi + i\sigma(u+ve^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}} \\ \qquad\quad \cdot \Big(\frac{\eta - i\sigma e^{-\theta h} v}{\eta - i\sigma v}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi + ie^{-\theta h}\sigma v}{\varphi + i\sigma v}\Big)^{\frac{q\lambda}{\theta}} . \end{cases} \tag{1.3}$$

We shall use the above theorem to estimate all the parameters η, θ, φ, λ, σ, and p by replacing the limits in (1.3) by their the empirical characteristic functions $\hat{\Psi}_{1,n}(u)$ and $\hat{\Psi}_{2,n}(u)$ defined as follows

$$\begin{cases} \hat{\Psi}_{1,n}(u,v) := \frac{1}{n} \sum_{j=1}^n \exp iuX_{t_j} \,; \\ \hat{\Psi}_{2,n}(u,v) = \frac{1}{n} \sum_{j=1}^n \exp(iuX_{t_j} + ivX_{t_j+h}) \,. \end{cases} \tag{1.4}$$

Since $\mathbb{E}|\mathbb{X}_0|^m < \infty$ and $\mathbb{E}|\mathbb{X}_0\mathbb{X}_h|^m < \infty$ for all m we know (e.g. [15, Theorem 1.1]) that (1.3) hold true for moment functions, in particular, we shall choose $f = x, x^2, x^3, g(x,y) = xy$. Evaluating the moments from the characteristic functions allows to obtain the estimators of parameters by solving the system of equations. We summarize the above discussions as the following theorem about the existence and uniqueness of the parameter estimators and their strong consistency results.

Theorem 1.1.2. *From the observation data, we denote $\mu_{k,n}$, $k = 1, 2, 3, 4$ by (refer Section 4.2,(3.3)). Then $\hat{\theta}_n$ is given by*

$$\hat{\theta}_n = \frac{1}{h} \ln \Big(\frac{\mu_{2,n} - \mu_{1,n}^2}{\mu_{4,n} - \mu_{1,n}^2} \Big) \tag{1.5}$$

and $f_k, k = 1, 2, 3$ by (3.10). If (3.15) has a unique solution $\hat{p}_n$ on $(0, 1)$, namely,

$$\begin{aligned} &(1-\hat{p}_n)^2 \left(f_1 \hat{p}_n + \sqrt{\hat{p}_n (1-\hat{p}_n)(f_2 - f_1^2)} \right)^3 \\ &\quad + \hat{p}_n^2 \left(f_1 - f_1 \hat{p}_n - \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)} \right)^3 - f_3 \hat{p}_n^2 (1-\hat{p}_n)^2 = 0 \end{aligned} \tag{1.6}$$

and if $\hat{p}_n$ is a continuous function of f_1, f_2, f_3, then (3.5)-(3.8) has a unique solution $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)$ given by (3.16), (3.17) and

$$\begin{cases} \hat{\rho}_n = \dfrac{f_1 \hat{p}_n + \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)}}{\hat{p}_n}, \\ \hat{\xi}_n = \dfrac{\hat{p}_n \hat{\rho}_n - f_1}{1 - \hat{p}_n}. \end{cases} \tag{1.7}$$

Define

$$\hat{\eta}_n := \frac{1}{\hat{\rho}_n}, \quad \hat{\varphi}_n := \frac{1}{\hat{\xi}_n}. \tag{1.8}$$

If $(\theta, \eta, \varphi, p)$ are the true parameters, namely, if the double exponential process X_t satisfies (1.2) with the above parameters and with $\lambda = \sigma = 1$, and if (3.15) has a unique solution when f_1, f_2, f_3 are replaced by their limits as $n \to \infty$, then when $n \to \infty$, $(\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n) \to (\theta, \eta, \varphi, p)$ almost surely.

Further the central limit theorem for the ergodic estimators $\hat{\Theta}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n)$ is also proved. The goal is to prove that $\sqrt{n}(\hat{\Theta}_n - \Theta)$, where $\Theta = (\theta, \eta, \varphi, p)$ converges in law to a mean zero normal vector and to find the asymptotic covariance matrix. Let

$$\begin{cases} g(x,y) = (g_1(x,y), g_2(x,y), g_3(x,y), g_4(x,y))^T \,, \\ g_1(x,y) = x, \quad g_2(x,y) = x^2, \quad g_3(x,y) = x^3, \quad g_4(x,y) = xy \end{cases}$$

and

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4)\,, \quad \text{where} \quad \mu_k = \mathbb{E}[g_k(\mathbb{X}_o, \mathbb{X}_h)], \quad k = 1,2,3,4\,.$$

Denote

$$\mu_n = (\mu_{1,n}\,, \mu_{2,n}\,, \mu_{3,n}\,, \mu_{4,n})\,,$$

where $\mu_{k,n}$, $k = 1,2,3,4$ are defined by (3.3).

Theorem 1.1.3. *Denote* $\Theta = (\theta, \eta, \varphi, p)$ *and* $\hat{\Theta}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n)$. *If* $\hat{p}_n$ *is a continuous function of* f_1, f_2, f_3 *and if* (3.15) *has a unique solution when* f_1, f_2, f_3 *are replaced by their limits as* $n \to \infty$, *then as* $n \to \infty$ *we have*

$$\sqrt{n}(\hat{\Theta}_n - \Theta) \xrightarrow{d} \mathcal{N}(0, \Sigma) \tag{1.9}$$

where

$$\Sigma = \left((\nabla h)^{-1} \nabla \tilde{h} \right)^T A\, (\nabla h)^{-1} \nabla \tilde{h}\,. \tag{1.10}$$

This chapter is concluded by numerical simulations which are shown to validate the ergodic estimators. To do so a distributional decomposition to exactly simulate the double exponential Ornstein-Uhlenbeck process is proposed following the idea of [29], where the exact simulation of Gamma Ornstein-Uhlenbeck process is studied. First, we have the following result. Without loss of generality, we can assume $\sigma = 1$.

Theorem 1.1.4. *Let* X_t *be the double exponential Ornstein-Uhlenbeck process given by* (1.2). *For any* $t, t_1 > 0$, *the Laplace transform of* X_{t+t_1} *conditioning on* X_t *is given by*

$$\begin{aligned} \mathbb{E}[e^{iuX_{t+t_1}} | X_t] = e^{-iuwX_t} \exp \Big[\frac{-\lambda p}{\theta} \int_0^\infty (1 - e^{-ius}) \int_1^{1/w} \eta v e^{-s\eta v} \frac{1}{v} dv ds \\ - \frac{\lambda q}{\theta} \int_{-\infty}^0 (1 - e^{-ius}) \int_1^{1/w} \phi v e^{s\phi v} \frac{1}{v} dv ds \Big]\,, \end{aligned} \tag{1.11}$$

where $w = e^{-\theta t_1}$.

Corollary 1.1.5 (Exact Simulation via Decomposition Approach)**.** *Let N be a Poisson random variable of rate λh and let $\{S_k\}_{k=1,2,\dots}$ be i.i.d random variables following a mixture of double exponential distribution*

$$\begin{aligned} f_{S_k}(y) =& p\eta e^{\theta h U} e^{-\eta e^{\theta h U} y} I_{y\geq 0} + q\phi e^{\theta h U} e^{\phi e^{\theta h U} y} I_{y<0} \, , \\ & \forall \; k = 1\, , 2\, , \dots \, , \end{aligned} \tag{1.12}$$

where $U \stackrel{d}{=} \mathcal{U}[0,1]$ is the uniform distribution on $[0,1]$. Then

$$X_{t+h} \stackrel{d}{=} X_t e^{-\theta h} + \sum_{k=1}^{N} S_k \, . \tag{1.13}$$

The above formula (1.13) enables us to simulate the process X_t by the exact decomposition approach.

1.1.2 Summary on Parameter Estimation for Vasicek Model with double exponential jump

The model is expressed in the form of the following stochastic differential equation (SDE),

$$dX_t = (\mu - \theta X_t)dt + d\tilde{L}_t \tag{1.14}$$

$$X_0 = 0 \tag{1.15}$$

The first term $(\mu - \theta X_t)dt$ represents the drift term. The parameter θ gives the reversion speed of the stochastic component. The long-term mean is given $\frac{\mu}{\theta}$. Here the Vasicek model is driven by compensated Lêvy process $(\tilde{L}_t, t \geq 0)$ refer (2.2), where

$$L_t = \sum_{i=1}^{N_t} Y_i$$

is the double exponential compound Poisson process and the compensated double exponential compound Poisson Process $\hat{L}_t$ is given by

$$\hat{L}_t = L_t - \lambda t \mathbb{E}[Y_1] \tag{1.16}$$

The goal is to construct least square estimators under continuous observations. We find the estimators by minimizing the following contrast function

$$\Phi(\theta, \mu) = \min_{\theta,\mu} \int_0^T |\dot{X}_t - (\mu - \theta X_t)|^2 dt \tag{1.17}$$

$$\tag{1.18}$$

Upon minimizing the contrast function we obtain expressing with integrals of the form $\int_0^T X_t dX_t$. Such integrals can be interpreted as Young integrals. Using the ergodicity of X_t and using BDG inequality (Lemma 2.1[35]) we get the following result

Theorem 1. The estimators $\hat{\theta}_T$ and $\hat{\mu}_T$ given by

$$\hat{\theta}_T = \theta + \frac{\tilde{L}_T \int_0^T X_t dt - T \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}$$

$$\hat{\mu}_T = \mu + \frac{\tilde{L}_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}$$

converge $a.s.$ to θ and μ respectively as $T \to \infty$.

1.1.3 Summary on General Product formula of multiple integrals of Lévy process

The product formula for two multiple integrals of Brownian motion is known since the work of [10, Section 4] and the general product formula can be found for instance in [18, chapter 5]. In Chapter 5, a general formula for the product of m multiple integrals of the Poisson random measure associated with (purely jump) Lévy process is obtained. The formula is in a compact form and it is reduced to the Shigekawa's formula when $m = 2$ and when the Lévy process is reduced to Brownian motion.

When $m = 2$, we have the following example

Example 1.1.6. If $m = 2$, then $\kappa_m = 2^2 - 1 - 2 = 1$. To shorten the notations we can write $q_1 = n$, $q_2 = m$, $f_1 = f_n$, $f_2 = g_m$, $l_{\alpha_1} = l$, $n_{\beta_1} = k$. Thus, $\chi(1, \vec{l}, \vec{n}) = \chi(2, \vec{l}, \vec{n}) = l + k$ and $|q| + |\vec{n}| - |\chi(\vec{l}, \vec{n})| = n + m + k - 2(l + k) = n + m - 2l - k$. Hence the formula (2.12) becomes the following. If

$$f_n \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} n}$$

and

$$g_m \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} m},$$

then

$$I_n(f_n)I_m(g_m) = \sum_{\substack{k,l \in \mathbb{Z}_+ \\ k+l \le m \wedge n}} \frac{n!m!}{l!k!(n-k-l)!(m-k-l)!} I_{n+m-2l-k}\left(f_n \hat{\otimes}_{k,l} g_m\right),$$

where $\mathbb{Z}_+$ denotes the set of non negative integers and

$$\begin{aligned} & f_n \hat{\otimes}_{k,l} g_m(s_1, z_1, \cdots, s_{n+m-k-2l}, z_{n+m-k-2l}) \\ & = \text{symmetrization of} \int_{\mathbb{T}^l} f_n(s_1, z_1, \cdots, s_{n-l}, z_{n-l}, t_1, y_1, \cdots, t_l, y_l) \\ & \qquad g_m(s_1, z_1, \cdots, s_k, z_k, s_{n-l+1}, \cdots, z_{n-l+1}, \cdots, \\ & \qquad s_{n+m-k-2l}, z_{n+m-k-2l}, t_1, z_1, \cdots, t_l, z_l) dt_1 \nu(dz_1) \cdots dt_l \nu(dz_l). \end{aligned} \tag{1.19}$$

In the chapter, a product formula for finitely many multiple stochastic integrals of Lévy process, expressed in terms of the associated Poisson random measure is derived. The chapter gives proof of the following result.

Theorem 1.1.7. *Let $q_1, \cdots, q_m$ be positive integers greater than or equal to 1. Let*

$$f_k \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} q_k}, \quad k = 1, \cdots, m.$$

Then

$$\prod_{k=1}^{m} I_{q_k}(f_k) = \sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\le q_1 \\ \cdots \\ \chi(m,\vec{l},\vec{n})\le q_m}} \frac{\prod_{k=1}^{m} q_k!}{\prod_{\alpha=1}^{\kappa_m} l_{\mathbf{i}_\alpha}! \prod_{\beta=1}^{\kappa_m} \mu_{\mathbf{j}_\beta}! \prod_{k=1}^{m}(q_k - \chi(k,\vec{l},\vec{n}))!} I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{j_{\kappa_m}}}(f_1,\cdots,f_m)) \tag{1.20}$$

where we recall

$$|q| = q_1 + \cdots + q_m \quad \text{and} \quad |\chi(\vec{l},\vec{n})| = \chi(1,\vec{l},\vec{n}) + \cdots + \chi(m,\vec{l},\vec{n})\,.$$

Please refer to (5.2) for the details in the notions.

Chapter 2

Preliminaries

In this chapter, we will briefly discuss some background on the Ornstien-Uhlenbeck processes and Lévy processes.

2.1 Lévy Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a proability spae.

Definition 2.1. A filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ is a family of σ-algebras $\mathcal{F}_t$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s < t$. Note that $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in\mathbb{R}} \mathcal{F}_t\right)$.

Definition 2.2. Let us have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$. Then a one-dimensional process $Z = Z(t), t \geq 0$ is called a Lévy process if the following holds true :

1. $Z(0) = 0 \;\; \mathbb{P}\; a.s.$,

2. $Z_{t+s} - Z_t$ is independent of $\mathcal{F}_t$ for every $s, t \geq 0$.

3. Z has stationary increments, $i.e$, $Z_{t+s} - Z_t$ and Z_s have the same law for every $s, t \geq 0$,

4. Z is continuous in probability , i.e for every $t \geq 0$ and $\epsilon > 0$ we have

$$\lim_{s \to t} \mathbb{P}\{|Z(t) - Z(s)| > \epsilon\} = 0,$$

5. Z is càdlàg upto a modification.

Let $T > 0$ be a positive number and let $\{Z(t) = Z(t, \omega), 0 \leq t \leq T\}$ be a Lévy process on some probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$. The jump of the process Z at time t is defined by

$$\Delta Z(t) := Z(t) - Z(t-) \quad \text{if } \Delta Z(t) \neq 0\,.$$

Given $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, then $\mathcal{B}(\mathbb{R}_0)$ is the the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we then define the Poisson random measure $N : [0, T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with the Lévy process Z by

$$N(t, U) := \sum_{0 \leq s \leq t} \chi_U(\Delta Z(s))\,, \tag{1.1}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of Z is defined by

$$\nu(U) := \mathbb{E}[N(1, U)] \tag{1.2}$$

and the compensated jump measure $\tilde{N}$ is defined by

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt\,, \tag{1.3}$$

where ν satisfies

$$\int \min\{1, x^2\}\nu(dx) < \infty.$$

There are real-life cases where the trajectories of Z have infinitely many jumps of small size, and its occurrence can be seen often in financial modeling. However, the Levý measure always follows the above equality. For any t, let $\mathcal{F}_t$ be the σ-algebra generated $\mathbb{N}(ds, dz)$, $z \in \mathbb{R}_0$, $s \leq t$.

A stochastic process $f = f(t, z)$, $t \geq 0$, $z \in \mathbb{R}_0$, is $\mathcal{F}$-adapted if for all $t \geq 0$ and for all $z \in \mathbb{R}_0$, the random variable $f(t, z) = f(t, z, \omega)$, $\omega \in \Omega$, is $\mathcal{F}_t$-measurable. Also, if f satisfies

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} f^2(t, z)\nu(dz)dt\right] < \infty \text{ for some } T > 0, \tag{1.4}$$

we can see that the process $Z_n(t)$ is a martingale in $L^2(P)$, where $Z_n(t)$ is defined as

$$Z_n(t) := \int_0^t \int_{|z| \geq \frac{1}{n}} f(s, z)\widetilde{N}(ds, dz), \quad 0 \leq t \leq T,$$

and it's limit

$$Z(t) := \lim_{n \to \infty} Z_n(t) := \int_0^t \int_{\mathbb{R}_0} f(s, z)\widetilde{N}(ds, dz), \quad 0 \leq t \leq T, \tag{1.5}$$

in $L^2(P)$ is also a martingale. This also results in the the Itô isometry,

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} f(t, z)\widetilde{N}(dt, dz)^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} f^2(t, z)\nu(dz)dt\right]. \tag{1.6}$$

In the next chapters, the parameter estimation problem for the Ornstein-Uhlenbeck (OU) process and the Vasicek model driven by the double exponential jump diffusion process are discussed. In this chapter, some preliminaries of the OU process and the Vasicek model are presented below.

Definition 2.3. An Ornstein-Uhlenbeck (OU) process driven by a Lévy process $(Z_t)_{t\geq 0}$ is defined to be the stochastic process satisfying the SDE

$$dX_t = -\theta X_t dt + \sigma dZ_t\,, \quad t \in [0,\infty), \quad X_0 = x_0\,. \tag{1.7}$$

Here $\theta > 0$ and $\sigma > 0$ are parameters of the OU process. Such stochastic processes are also referred to as Ornstein-Uhlenbeck-type processes.

The solution of the SDE can be given by

$$\begin{aligned} e^{\theta t} X_t &= e^{\theta s} X_s + \int_s^t e^{\theta u} \theta X_u du + \int_s^t e^{\theta u} dX_u \\ &= e^{\theta s} X_s + \int_s^t e^{\theta u} \theta X_u du + \int_s^t e^{\theta u} (-\theta X_u du + \sigma dZ_u) \\ &= e^{\theta s} X_s + \int_s^t e^{\theta u} \theta X_u du - \int_s^t e^{\theta u} \theta X_u du + \sigma \int_s^t e^{\theta u} dZ_u \\ &= e^{\theta s} X_s + \sigma \int_s^t e^{\theta u} dZ_u. \end{aligned}$$

This gives the solution,

$$X_t(\omega) = e^{-\theta t} x_0 + \sigma \int_0^t e^{-\theta(t-s)} dZ_s(\omega). \tag{1.8}$$

The Vasicek model, originally introduced by Oldrich A. Vasicek in 1977, is a mathematical model used to describe the evolution of interest rates over time. It is a stochastic model that assumes that interest rates follow a mean-reverting process. In its basic form, the Vasicek model is driven by a Brownian motion or a Wiener process. However, it can be extended to incorporate other stochastic processes, including Levy processes.

Definition 2.4. The Vasicek model driven by Lévy Process is a stochastic model that follows the following stochastic differential equation (SDE),

$$dX_t = (\mu - \theta X_t)dt + dZ_t \tag{1.9}$$

$$X_0 = 0 \tag{1.10}$$

The first term $(\mu - \theta X_t)dt$ represents the drift term. Here $\theta > 0$ and $\mu > 0$ are parameters of the above process.

The solution to the above SDE can be written as

$$X_t = \frac{\mu}{\theta}(1 - e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} dZ_s$$

Definition 2.5. The characteristic function $\hat{P}_X(z)$ of a random variable X whose distribution function is P_X on $\mathbb{R}$ is defined by

$$\hat{P}_X(z) = \mathbb{E}(e^{izX}) = \int_{\mathbb{R}} e^{izx} P_X dx.$$

Definition 2.6. Let $(Y_n, n \geq 1)$ be a sequence of independent real-valued random variables with distribution f. Let (N_t) be the Poisson process with rate $\lambda > 0$, independent of $\{Y_i, i = 1, 2, \ldots\}$. Then the process

$$Z_t = \sum_{i=1}^{N_t} Y_i$$

is called compound Poisson process.

Remark 2.1.1. When $(Y_n, n \geq 1)$ follows the distribution

$$f_Y(x) = p\eta e^{-\eta x} I_{[x \geq 0]} + q\varphi e^{\varphi x} I_{[x<0]}$$

where the parameters p, q, η, φ are positive and $p + q = 1$. Then the process $Z_t = \sum_{i=1}^{N_t} Y_i$ is called the double exponential compound Poisson process.

Remark 2.1.2. For a step function $g(u) = \sum_{j=1}^{n} a_j 1_{(u_{j-1},u_j]}(u)$ with $s = u_0 < u_1, \cdots < u_n = t$, and Lévy process (Z_t) the following holds true,

$$\mathbb{E}\Big[\exp\Big(iz\int_s^t g(u)dZ_u(\omega)\Big)\Big] = \exp\Big[\int_s^t \Psi(g(u)z)du\Big]$$

where $z \in \mathbb{R}$ and $\Psi(z) = \ln \hat{P}_{Z_1}(z)$, and $\hat{P}$ is the characteristic function of Z_1. By approximation, the above is also true for any real continuous function $g(u)$ on $[s,t]$. Therefore we have

$$\mathbb{E}[e^{iuX_t}] = \exp\Big[ie^{-\theta t}x_0 u + \int_0^t \Psi(\sigma e^{-\theta s}u)ds\Big].$$

Theorem 2.1.3 ([12]Lévy -Khintchine Formula in one-dimension)**.** *Let Z be a Lévy process on $\mathbb{R}$. Then the process $(Z_t)_{t\geq 0}$ for each t is infinitely divisible distribution and its characteristic function $\hat{P}_Z(z)$ is given by the Lévy -Khintchine Formula,*

$$\hat{P}_{Z_t}(z) = \exp\Big(t\Big(-\frac{1}{2}Az^2 + iyz + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_D(x)\rho(dx)\Big)\Big) \tag{1.11}$$

$z \in \mathbb{R}$, where $A \geq 0$ and $D = \{x : |x| \leq 1\}$, ρ is a measure on $\mathbb{R}$ satisfying

$$\rho(\{0\}) = 0$$

$$\int_{\mathbb{R}}(|x|^2 \wedge 1)\rho(dx) < \infty\,.$$

Here $(tA, t\rho, ty)$ is called the characteristic triplet of Z_t. We can also refer to as (A, ρ, y), the characteristic triplet of the Lévy process Z.

Definition 2.7. A map $P_{s,t}(x, B)$ of $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ for $0 \leq s \leq t < \infty$ is called a transition function on $\mathbb{R}$ if

1. for any fixed x, it is a probability measure as a mapping of B.

2. for any fixed B, $x \mapsto P_{s,t}(x, B)$ is measurable.

3. $P_{s,s}(x, B) = \delta_x(B)$ for $s \geq 0$

4. also

$$\int_{\mathbb{R}} P_{s,t}(x, dy) P_{t,y}(y, B) = P_{s,u}(x, B), \quad 0 \leq s \leq t \leq u,$$

5. if, in addition, we have that $P_{s+h,t+h}(x, B)$ does not depend on h, then it is called a temporally homogeneous transition function and it is given by

$$P_t(x, B) = P_{s,s+t}(x, B), \quad s \geq 0.$$

Lemma 2.1.4 (Sato[30], Lemma 17.1)**.** *Let $\{Z_t\}$ be a Lévy process on $\mathbb{R}$ generated by (G, ρ, β). Let $\theta \in R$, then there exists a temporally homogeneous transition function $P_t(x, B)$ on R such that*

$$\int_{R^d} e^{izy} P_t(x, dy) = \exp[ie^{-\theta t} xz + \int_0^t \psi(\sigma e^{-\theta s} z) ds] \tag{1.12}$$

where $\psi(z) = ln\hat{P}_Z(z) = lnE[e^{izZ}]$. For each t, $P_t(x, \cdot)$ is infinitely divisible with the generating characteristic triplet $(A_t, v_t, \gamma_{t,x})$

$$A_t = \int_0^t e^{-2\theta s} ds G \tag{1.13}$$

$$v_t(B) = \int_{R^d} \rho(dy) \int_0^t I_B(e^{-\theta s} y) ds \tag{1.14}$$

$$\gamma_{t,x} = e^{-\theta t} x + \int_0^t \sigma e^{-\theta s} x ds + \int_R \rho(dy) \int_0^t e^{-\theta s} y (I_D(e^{-\theta s} y) - I_D(y)) ds \tag{1.15}$$

$D = \{x : |x| \leq 1\}$

Definition 2.8. Let $\theta > 0$ $\{Z_t\}$ is a Lévy process on $\mathbb{R}$ generated by (G, ρ, β), the temporally homogeneous Markov process with transition function $\{P_t(x, B)\}$ is called process of Ornstein Unhlenbeck type generated by (G, ρ, β, θ)

Theorem 2 (Sato[30], Theorem 17.9). Fix $\theta > 0$, If ρ satisfies,

$$\int_{x|>2} log|x|\rho dx < \infty$$

the process of Ornstein-Unhlenbeck type on $\mathbb{R}$ generated by (G, ρ, β, θ) has the limit distribution μ with

$$\hat{\mu}(z) = \exp[\int_0^\infty \psi(e^{-\theta s} z) dz]$$

The distribution μ is self decomposable and the generating triplet (A, v, γ) of μ is given by ,

$$A = \frac{1}{2\theta} G$$

$$v(B) = \frac{1}{\theta} \int_R \rho(dy) \int_0^\infty I_B(e^{-\theta s} y) ds$$

$$\gamma = \frac{1}{\theta}\beta + \frac{1}{\theta} \int_{|y|>1} \frac{y}{|y|} \rho(dy)$$

2.2 Stochastic Calculus for Jump Diffusion Process

Theorem 2.2.1 ([2]Lévy -Itô decomposition theorem). *Let Z be a Lèvy process. Then $Z = Z(t), t \geq 0$ has the following integral representation*

$$Z(t) = a_1 t + \sigma W(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz),$$

here $a_1, \sigma \in \mathbb{R}$ are constants, $\tilde{N}(ds, dz)$ as defined in 1.3 *and $W = W(t), t \geq 0$ is a standard Wiener process.*

If $\int_{|z|\geq 1} |z|^2 \nu(dz) < \infty$, then the above expression can be written in the form

$$Z(t) = at + \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz).$$

Definition 2.9. Consider the process $X(t)$ of the form

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz),$$

where $\alpha(t), \beta(t)$, and $\gamma(t, z)$ are predictable processes such that, for all $t > 0$, $z \in \mathbb{R}_0$. If,

$$\int_0^t [|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s, z)\nu(dz)]ds < \infty,$$

then the stochastic process $X(t)$ is well-defined and is a local martingale. Such processes are called Itô's - Lèvy process.

Theorem 2.2.2 (Itô formula[12]). *Let $X = X(t), t \geq 0$, be the Itô–Lèvy process and let $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0, \infty) \times \mathbb{R})$ and define*

$$Y(t) := f(t, X(t)), t \geq 0.$$

Then the process $Y = Y(t), t \geq 0$, is also an Lèvy-Itô process and its differential form is given by

$$\begin{aligned} dY(t) = & \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))\alpha(t)dt \\ & + \frac{\partial f}{\partial x}(t, X(t))\beta(t)dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\beta^2(t)dt \\ & + \int_{\mathbb{R}_0} [f(t, X(t) + \gamma(t, z)) - f(t, X(t)) - \frac{\partial f}{\partial x}(t, X(t))\gamma(t, z)]\nu(dz)dt+ \\ & + \int_{\mathbb{R}_0} [f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)]\tilde{N}(dt, dz). \end{aligned} \tag{2.1}$$

In case of multidimensional Itô–Lévy process where there is an L-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_L(t))$, $t \geq 0$, and M independent compensated Poisson random measure $\tilde{N}(dt, dz) = (\tilde{N}_1(dt, dz_1), \ldots, \tilde{N}_M(dt, dz_M))^T$, $t \geq 0$, the Itô–Lévy process is of the form

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{(\mathbb{R}_0)^M} \gamma(t, z)\widetilde{N}(dt, dz), \quad t \geq 0. \tag{2.2}$$

where $\alpha(t), \beta(t)$, and $\gamma(t, z)$ are predictable processes .

Theorem 2.2.3 (The multidimensional Itô formula[12]). *Let $X = X(t), t \geq 0$, be an n-dimensional Itô–Lévy process of the form* (2.2)*. Let $f : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and define*

$$Y(t) := f(t, X(t)), \quad t \geq 0.$$

Note that the function f is in $C^{1,2}((0, \infty) \times \mathbb{R}^n)$ Then the process $Y = Y(t), t \geq 0$, is a one-dimensional Itô–Lévy process and its differential form is given by

$$\begin{aligned}
dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X(t))\alpha_i(t)dt \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{L} \frac{\partial f}{\partial x_i}(t, X(t))\beta_{ij}(t)dW_j(t) + \frac{1}{2}\sum_{i=1}^{n} \sum_{j=1}^{L} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X(t))(\beta\beta^T)_{ij}(t)dt \\
&+ \sum_{k=1}^{M} \int_{\mathbb{R}_0} \Big[f(t, X(t) + \gamma^k(t, z)) - f(t, X(t)) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X(t))\gamma_{ik}(t, z)\Big]\nu_k(dz_k)dt \\
&+ \sum_{k=1}^{M} \int_{\mathbb{R}_0} \Big[f(t, X(t^-) + \gamma_k(t, z)) - f(t, X(t^-))\Big]\tilde{N}_k(dt, dz_k),
\end{aligned} \tag{2.3}$$

where γ_k is the column number k of the $n \times k$ matrix $\gamma = \begin{bmatrix} \gamma_{1k} \\ \vdots \\ \gamma_{nk} \end{bmatrix}$.

Example 2.2.1. *Let $h \in \mathcal{L}^2([0, T])$ be a cadlág real function. Let us consider a one-dimensional stochastic differential equation for the cadlág process $Z = Z(t), t \in [0, T]$*

of the form,

$$dZ(t) = Z(t^-)\int_{\mathbb{R}_0}(e^{h(t)z}-1)\tilde{N}(dt,dz).$$

We argue that the solution to this equation is

$$Z(t) = \exp\{X(t)\}, \quad t\in[0,T].$$

where

$$X(t) = \Big\{\int_0^T\int_{\mathbb{R}_0} h(s)z\tilde{N}(ds,dz) - \int_0^T\int_{\mathbb{R}_0}[e^{h(s)z}-1-h(s)z]\nu(dz)ds\Big\}.$$

In particular $Y(t), t\in[0,T]$ *is a local martingale. To show this we apply the one-dimensional Itô formula to* $Z(t) = f(t,X(t)), t\in[0,T]$ *with* $f(t,x)=e^x$ *and* X_t *as given above. Since we have* $\frac{\partial f}{\partial t}(t,X(t)) = 0$, $\alpha(t) = -\int_{\mathbb{R}_0}[e^{h(s)z}-1-h(s)z]\nu(dz)$, $\beta(t)=0$ *from* (2.2.2) *defined in Then we get,*

$$\begin{aligned} dZ(t) = &-\exp\{X(t)\}\int_{\mathbb{R}_0}[e^{h(s)z}-1-h(t)z]\nu(dz)dt \\ &+\int_{\mathbb{R}_0}[\exp\{X(t)+h(t)\}-\exp\{X(t)\}-\exp\{X(t)\}h(t)z]\nu(dz)dt \\ &+\int_{\mathbb{R}_0}[\exp\{X(t^-)+h(t)\}-\exp\{X(t^-)\}]\tilde{N}(dt,dz), \end{aligned}$$

which gives us

$$dZ(t) = Z(t^-)\int_{\mathbb{R}_0}(e^{h(t)z}-1)\tilde{N}(dt,dz)$$

as required.

Let us consider a pure jump Lêvy process given by

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz), \ \ t \geq 0,$$

We consider that $\eta(t)$ is adapted to the σ-algebras $\mathcal{F}_t$.

Lemma 2.2.4. *[2] The set of all random variables of the form*

$$\{f(\eta(t_1)\ldots\eta(t_n)) : t_i \in [0,t], 1 \leq i \leq n; f \in \mathbf{C}_0^\infty(\mathbb{R}^n)\, n = 1, 2\ldots\}$$

is dense in the subspace $\mathcal{L}^2(\mathcal{F}_T, \mathbf{P}) \subset \mathbf{L}^2(\mathbf{P})$ of $\mathcal{F}_T$-measurable square-integrable random variables.

Lemma 2.2.5. *[2] Given the Doleans-Dade exponentials of the form*

$$\exp\Big\{\int_0^T \int_{\mathbb{R}_0} h(t)z_{\chi[0,R]}(z)\tilde{N}(ds, dz) - \int_0^T \int_{\mathbb{R}_0} [e^{h(t)z_{\chi[0,R]}(z)} - 1 - h(t)z_{\chi[0,R]}(z)]\nu(dz)dt\Big\},$$

where $h \in \mathbf{C}(0,T)$, $R > 0$ are dense in the space of $\mathcal{F}_T$-measurable square-integrable random variables $\mathcal{L}^2(\mathcal{F}_T, \mathbf{P})$.

Theorem 2.2.6. *(Itô representation theorem)[2] Let $U \in L^2(\mathbf{P})$ be $\mathcal{F}_T$ measurable random variable. Then U can be written as*

$$U = \mathbb{E}[U] + \int_0^T \int_{\mathbb{R}_0} f(t,z)\tilde{N}(ds, dz), \tag{2.4}$$

where $f = f(t,z), t \geq 0, z \in \mathbb{R}_0$ is a unique predictable process such that

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} f(t,z)^2 \nu(dz)dt\Big] < \infty.$$

Proof Let U be represented as process $U = V(T)$, where

$$V(t) = \exp\Big(\int_0^t \int_{\mathbb{R}_0} h(s)z\chi[0,R](z)\tilde{N}(ds, dz) - \int_0^t \int_{\mathbb{R}_0} [e^{h(s)z}\chi[0,R](z) - 1 - h(s)z\chi[0,R](z)]\nu(dz)ds\Big), \quad t \in [0,T],$$

for some $h \in C(0,\infty)$. Clearly, U is a Wick/Doléans–Dade exponential. With Itô formula and Example 1.2.1, the differential form $V(t)$ can be written as

$$dV(t) = V(t-)\left(\int_{R_0} e^{h(t)z}\chi[0,R](z) - 1\right)\tilde{N}(dt,dz).$$

Therefore, U can be written as

$$U = V(T) = V(0) + \int_0^T dV(t) = 1 + \int_0^T \int_{\mathbb{R}_0} V(t-)\left(e^{h(t)z}\chi[0,R](z) - 1\right)\tilde{N}(dt,dz).$$

Therefore

$$f(t,z) = V(t-)\left(e^{h(t)z}\chi[0,R](z) - 1\right).$$

also,

$$E\left[V^2(T)\right] = 1 + E\left[\int_0^T \int_{\mathbb{R}_0} V^2(t-)\left(e^{h(t)z}\chi[0,R](z) - 1\right)^2 \nu(dz)dt\right].$$

Let U be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Using 2.2.4 we can choose a sequence U_n of linear combinations of Doléan–Dade exponentials such that $U_n \to U$ in $L^2(P)$. Then we have

$$U_n = E[U_n] + \int_0^T \int_{\mathbb{R}_0} f_n(t,z)\tilde{N}(dt,dz),$$

for all $n = 1, 2, \ldots$, such that

$$E[U_n^2] = (E[U_n])^2 + \int_0^T \int_{\mathbb{R}_0} f_n^2(t,z)\nu(dz)dt < \infty.$$

Using Itô isometry we can see that f_n, $n = 1, 2, \ldots$ is a Cauchy sequence in $L^2(P \times \lambda \times \nu)$ and it converges to a limit f in $L^2(P \times \lambda \times \nu)$ and

$$
\begin{aligned}
U = \lim_{n\to\infty} U_n &= \lim_{n\to\infty}\left(E[U_n] + \int_0^T \int_{\mathbb{R}_0} f_n(t,z)\tilde{N}(dt,dz)\right)\\
&= E[U] + \int_0^T \int_{\mathbb{R}_0} f(t,z)\tilde{N}(dt,dz).
\end{aligned}
$$

■

2.3 Wiener -Itô Chaos Expansion

Let us use $\mathbb{T}$ to represent $[0,T]\times\mathbb{R}_0$ to simplify notation.

Let

$$
\left(L^2(\mathbb{T},\lambda\times\nu)\right)^{\otimes n} \subseteq L^2\left(\mathbb{T}^n,(\lambda\times\nu)^n\right)
$$

be the space of deterministic real functions f such that

$$
\|f\|^2_{L^{2,n}} = \int_{\mathbb{T}^n} f^2(t_1,z_1,\cdots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n) < \infty\,,
$$

where $\lambda(dt)=dt$ is the Lebesgue measure.

Definition 2.10. The symmetrization $\tilde{f}$ of f is defined by

$$
\tilde{f}(t_1,z_1,\cdots,t_n,z_n) = \frac{1}{n!}\sum_{\sigma} f(t_{\sigma_1},z_{\sigma_1},\cdots,t_{\sigma_n},z_{\sigma_n}),
$$

where $\sigma=(\sigma_1,\ldots,\sigma_n)$ and the sum is taken over all the permutation of σ.

In the above, a function $f\in L^{2,n}$ is called symmetric if $f=\tilde{f}$. Also space of all symmetric functions in $L^{2,n}$ can be denoted by $\hat{L}^{2,n}$.

Define

$$
G_n := \left\{\left(t_1,z_1,\cdots,t_n,z_n\right) : 0\le t_1\le\cdots\le t_n\le T, z_i\in\mathbb{R}_0, i=1,2,\cdots,n\right\}
$$

and let $L^2(G_n)$ be the set of all real functions g on G_n such that

$$\|g\|_{\tilde{L}^2(G_n)} := \Big(\int\limits_{G_n} g^2(t_1, z_1, \cdots, t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n) \Big)^{1/2} < \infty.$$

Also, for any $f \in \hat{L}^{2,n}$, we have $f_{|G_n} \in \hat{L}^2(G_n)$ and

$$\|f\|^2_{L^2} = n! \|f\|^2_{L^2(G_n)}.$$

Definition 2.11. For any $g \in L^2(G_n)$, the n-fold iterated integral $J_n(g)$ is the random variable in $L^2(P)$ defined as

$$J_n(g) := \int\limits_0^T \int\limits_{\mathbb{R}_0} \cdots . \int\limits_0^{t_2^-} \int\limits_{\mathbb{R}_0} g(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n).$$

We set $J_0(g) = g$ for any $g \in \mathbb{R}$.

For any $f \in \hat{L}^{2,n}$, we can write the multiple Wiener-Itô integral as

$$I_n(f) := \int_{\mathbb{T}^n} f(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) = n! J_n(f). \tag{3.1}$$

Also for any $g \in \hat{L}^{2,n}$ and $f \in \hat{L}^{2,n}$, the following relation holds true

$$\mathbb{E}[I_m(g) I_n(f)] = \begin{cases} 0, & \text{if } n \neq m \\ (g, f)_{L^{2,n}}, & \text{if } n = m \end{cases}$$

where

$$(g, f)_{L^{2,n}} = \int_{\mathbb{T}^n} g(t_1, z_1, \cdots, t_n, z_n) f(t_1, z_1, \cdots, t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n).$$

Theorem 2.3.1. *[2](Wiener-Itô chaos expansion for Lévy process) Let $\mathcal{F}_T = \sigma(\eta(t), 0 \leq t \leq T)$ be σ - algebra generated by the **Lévy** process η.*

Let $F \in L^2(\Omega, \mathcal{F}_T, P)$ be an $\mathcal{F}_T$ measurable square integrable random variable. Then F admits the following chaos expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n)\,, \tag{3.2}$$

where $f_n \in \hat{L}^{2,n}, n = 1, 2, \cdots$ and where we denote $I_0(f_0) := f_0 = \mathbb{E}(F)$. Moreover, we have

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n\,!\|f_n\|^2_{L^{2,n}}\,. \tag{3.3}$$

Proof By theorem 1.2.6, a predictable process $\theta(t_1, z_1) \in \mathbb{T}$ exists such that for $F \in L^2(\Omega, \mathcal{F}_T, P)$, F can be written as

$$F = E[F] + \int_0^T \int_{\mathbb{R}^0} \theta_1(t_1, z_1)\tilde{N}(dt_1, dz_1),$$

also

$$\|F\|^2_{L^2(P)} = (E[F])^2 + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} \theta_1^2(t_1, z_1)\, dt_1\, \nu(dz_1)\Big] < \infty.$$

Applying theorem 1.2.6 on $\theta_1(t_1, z_1)$, for almost all $(t_1, z_1) \in \mathbb{T}$, there exists a predictable process $\theta_2(t_1, z_1, t_2, z_2)$, where $(t_2, z_2) \in [0, t_1] \times \mathbb{R}_0$, such that

$$\theta_1(t_1, z_1) = E[\theta_1(t_1, z_1)] + \int_0^T \int_{\mathbb{R}_0} \theta_2(t_1, z_1, t_2, z_2)\tilde{N}(dt_2, dz_2).$$

This allows us to write F as

$$\begin{aligned} F = E[F] &+ \int_0^T \int_{\mathbb{R}_0} E\left[\theta_1(t_1, z_1)\tilde{N}(dt_1, dz_1)\right] \\ &+ \int_0^T \int_{\mathbb{R}_0} \int_0^{t_1^-} \int_{\mathbb{R}_0} \theta_2(t_1, z_1, t_2, z_2)\tilde{N}(dt_2, dz_2)\tilde{N}(dt_1, dz_1). \end{aligned}$$

Let's define

$$g_0 := E[F]$$
$$g_1(t_1, z_1) := E[\theta_1(t_1, z_1)], \quad (t_1, z_1) \in \mathbb{T}.$$

Similarly we can repeat the above same argument for (t_2, z_2) and again for new integrands generated through the process $\theta_2(t_1, z_1, t_2, z_2)$. This will give

$$F = \sum_{n=0}^{k-1} J(g_n) + \int_{G_k} \theta_k(t_1, z_1, \ldots, t_k, z_k) \tilde{N}^{\otimes k}(dt, dz),$$

where

$$G_k := \{(t_1, z_1, \cdots, t_k, z_k) : 0 \le t_1 \le \cdots \le t_k \le T, z_i \in \mathbb{R}_0, i = 1, 2, \cdots, k\}.$$

refer to Theorem 1.10 [2] to see that the residual term $\int_{G_k} \theta_k(t_1, z_1, \ldots, t_k, z_k) \tilde{N}^{\otimes k}(dt, dz)$ converges to 0 in $L^2(\Omega, \mathcal{F}_T, P)$.

This gives the following chaos expansion

$$F = \sum_{n=0}^{\infty} J(g_n).$$

Define $f_n := \tilde{g}_n$ by extending the function g_n on the whole $\mathbb{T}^n$ such that $g_n := 0$ on $\mathbb{T}^n \backslash G^n$. Then

$$I_n(f_n) = n! J_n(f_n) = n! J_n(\tilde{g}_n) = J_n(g_n).$$

Thus,we have

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

■

Example 2.3.2. Let $F = V(T)$, where

$$V(t) = \exp\left(\int_0^t \int_{\mathbb{R}_0} h(s)z\,\widetilde{N}(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \left(e^{h(s)z} - 1 - h(s)z\right)\nu(dz)\,ds\right), \quad t \in [0,T]$$

here $h \in L^2([0,T])$ is a càdlàg real function. Clearly, V is a Wick/Doléans–Dade exponential. Then, by the Itô formula, we have

$$dV(t) = V(t-)\int_{\mathbb{R}_0} \left(e^{h(t)z} - 1\right)\widetilde{N}(dt,dz).$$

Therefore

$$V(T) = 1 + \int_0^T \int_{\mathbb{R}_0} V(t-)\left(e^{h(t)z} - 1\right)\widetilde{N}(dt,dz).$$

Repeating the above iteration again for $V(t-)$ and so on again, the following expansion is obtained

$$\begin{aligned} V(T) &= \sum_{k=0}^{n-1} I_n(f_n) \\ &+ \int_0^T \int_0 \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} V(t_1^-) \prod_{i=1}^{k} \left(e^{h(t_i)z_i} - 1\right)\widetilde{N}(dt_1,dz_1)\cdots\widetilde{N}(dt_k,dz_k), \end{aligned}$$

where

$$\begin{aligned} f_n(t_1,z_1,\ldots,t_n,z_n) &:= \frac{1}{n!}\prod_{i=1}^{n}\left(e^{h(t_i)z_i - 1}\right) \\ &= \frac{1}{n!}\left(\prod_{i=1}^{n}\left(e^{h(t)z-1}\right)\right)^{\otimes n}(t_1,z_1,\ldots,t_n,z_n), \end{aligned}$$

which leads to the chaos expansion

$$V(T) = \sum_{n=0}^{\infty} I_n(f_n),$$

with convergence in $L^2(P)$. To prove this, we need to verify that

$$\mathbb{E}\left[\int_0^T \int_0 \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} V(t_1^-)\left(e^{h(t)z}-1\right)^{\otimes k} \widetilde{N}^{\otimes k}(dt,dz)\right] \to 0, \quad k \to \infty.$$

This follows from the estimate

$$\int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2} \int_{\mathbb{R}_0} \mathbb{E}\left[V^2(t_1^-)\left(e^{h(t)z}-1\right)^{\otimes k}\right] \nu(dz_1)dt_1 \cdots \nu(dz_k)dt_k$$
$$\leq \frac{\mathbb{E}\left[V^2(T)\right]}{k!}\left(\int_0^T \int_{\mathbb{R}_0} \left(e^{h(t_1)z_1-1}\right)^2 \nu(dz_1)dt_1\right)^k \quad \to 0 \quad \text{as } k \to \infty.$$

Chapter 3

Ergodic Estimators of double exponential Ornstein-Uhlenbeck processes

In this chapter, we study the parameter estimation for Ornstein-Uhlenbeck processes driven by the double exponential compound Poisson process. A method of moments using ergodic theory is proposed to construct ergodic estimators for the double exponential Ornstein-Uhlenbeck process, where the process is observed at discrete time instants with time step size h.

The existence and uniqueness of the function equations to determine the estimators for fixed time step size h is also shown. Also, we show the strong consistency and the asymptotic normality of the estimators. Furthermore, we propose a simulation method of the double exponential Ornstein-Uhlenbeck process and perform some numerical simulations to demonstrate the effectiveness of the proposed estimators.

3.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a right continuous family of increasing σ-algebras $(\mathcal{F}_t, t \geq 0)$ satisfying the usual condition ([19]). We denote the expectation on this probability space by $\mathbb{E}$. Recently, there have been a very successful applications of double exponential jump processes to finance and insurance, we refer to [13, 16, 17, 21] and relevant references for further details. To apply these models to more specific situations, we need to estimate the parameters appeared in the model from the historical data. It seems there has been no work on this direction yet. This motivates us to study the parametric estimation problem for the double exponential Ornstein-Uhlenbeck process. To introduce this process let us recall the following concept. Let $(Y_n, n \geq 1)$ be a sequence of independent real-valued random variables with the following probability density function

$$f_Y(x) = p\eta e^{-\eta x} I_{[x \geq 0]} + q\varphi e^{\varphi x} I_{[x<0]}\,, \tag{1.1}$$

where the parameters p, q, η, φ are positive and $p+q=1$. Let N_t be the Poisson process with rate $\lambda > 0$, independent of $\{Y_i, i = 1, 2, \ldots\}$. Then the process $Z_t = \sum_{i=1}^{N_t} Y_i$ is called the double exponential compound Poisson process. The double exponential compound Poisson process is a particular Lévy process. The stochastic calculus with respect to this process falls in the framework of the stochastic calculus for general Lévy processes. For more details, we refer to [30] whose results will be used freely.

Let us consider the following double-exponential Ornstein-Ulenbeck process given by the following Langevin equation driven by the double exponential compound Poisson process Z_t:

$$dX_t = -\theta X_t dt + \sigma dZ_t\,, \quad t \in [0, \infty), \quad X_0 = x_0\,. \tag{1.2}$$

Of course, the integral form of this equation can be written as

$$X_t = x_0 - \theta \int_0^t X_s ds + \sigma Z_t\,. \tag{1.3}$$

As we observe, this process X_t depends on the parameters θ, σ, p (or q), η, λ, and φ. In this chapter we assume that the process $\{X_t; t \geq 0\}$ can be observed at discrete time instants $t_j = jh$, where $h > 0$ is some observation time interval. We want to use the discrete observation data $\{X_{t_j}; j = 1, 2, \ldots n\}$ to estimate the parameters θ, σ, p, η, λ, and φ. To construct such estimators, we shall use the ergodic theorem $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(X_{t_j}) = \int_{\mathbb{R}} f(x)\mu(dx)$, where μ is the limiting distribution of X_t. It appears that with appropriate choices of different f we shall have sufficient number of equations so that we may be able to find all the parameters. However, the limiting distribution depends on the parameters in such a way (e.g. (2.13)) that one cannot decouple them. For this reason and motivated by [18], we also get involved the ergodic theorem of the form $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(X_{t_j}, X_{t_{j+1}}) = \int_{\mathbb{R}} g(x,y)\nu(dx,dy)$, where $\nu(dx,dy)$ is the limiting distribution of (X_t, X_{t+h}). After finding the distribution of μ and ν we shall use the moment functions (e.g. $f(x) = x^n$ etc) to obtain appropriate equations for the ergodic estimators to satisfy.

The existence, local uniqueness and global uniqueness of the system is the immediate problem after the obtention of the equations. We shall address this elementary and challenging problem and prove that when the sample size is sufficiently large we shall have the existence and uniqueness of a local solution. For the global uniqueness we reduce the problem to another one of finding zero for a real valued function of one variable, where the mean value theorem can be used. The strong consistency and asymptotic normality of our ergodic estimators are also given.

We propose an exact decomposition simulation algorithm for our double exponential Ornstein-Ulenbeck process to validate our approach. As a consequence of the exact decomposition, we can write the distribution of X_{t+h} given X_t as a sum of non stochastic function and a mixed compound Poisson process. After discussing the algorithm we simulate the data from (1.3) assuming some given values of θ, p, η, and φ. Then we apply the estimators to estimate these parameters. The numerical results show that our estimators converge fast to the true parameters.

The chapter is organized as follows.

In Section 2, we give some preliminaries and some basic results for our double exponential Ornstein-Uhlenbeck process. We also obtain the explicit form of the characteristic functions of limiting distributions μ and ν mentioned earlier.

In Section 3, the ergodic estimators for all the parameters in the double exponential Ornstein-Uhlenbeck process are constructed. The local existence, uniqueness and the global uniqueness of the system of equations that determine these ergodic estimators are discussed.

In Section 4, we obtain the joint asymptotic normality of the the estimators.

In Section 5, we discuss the exact decomposition algorithm for simulating the double exponential OU process.

In Section 6 we perform some numerical simulations to validate our results which demonstrate the effectiveness of our estimators.

Section 7 contains the computation of a covariance matrix that appeared in our theorems.

3.2 Preliminaries

Given Z_t a compound Poisson process with double exponential jumps, a unique solution to the equation (1.2) is given by

$$X_t = e^{-\theta t} x_0 + \sigma \int_0^t e^{-\theta(t-s)} dZ_s \,. \tag{2.1}$$

If $\theta > 0$, then the double exponential Ornstein-Uhlenbeck process X_t converges in law to the random variable $\mathbb{X}_o = \sigma \int_0^\infty e^{-\theta s} dZ_s$. If the process starts at the stationary distribution i.e., the initial condition X_0 has the same law of $\mathbb{X}_o$ and if X_0 is independent of the process Z_t, then X_t is a stationary process.

It is well-known from [30, Theorem 17.5]) that the double exponential process $\{X_t, t \geq 0\}$ is ergodic. Namely, we have the following result from [28, Theorem 8.1].

Proposition 3.2.1. *[20, Theorem 3.8] Let $f : \mathbb{R} \to \mathbb{R}$ be measurable such that $\mathbb{E}|f(\mathbb{X}_o)| < \infty$. Then for any initial condition $x_0 \in \mathbb{R}$ and for any $h \in \mathbb{R}_+$, we have (denoting $t_j = jh$)*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n f(X_{t_j}) = \mathbb{E}(f(\mathbb{X}_o)) \qquad a.s. \tag{2.2}$$

The explicit form of the distribution of $\mathbb{X}_o$ is hard to obtain. So, it is hard to compute $\mathbb{E}(f(\mathbb{X}_o))$ for general f. But when f has some particular form, namely, when $f(x) = e^{\iota \xi x}$, then the computation of $\mathbb{E}(f(\mathbb{X}_o))$ is much simplified.

Evaluation of the limiting characteristic functions

Proposition 3.2.2. *[30] Let Z_t be the double exponential compound Poisson process and let $0 < s < t < \infty$. Then for any real valued continuous function $g(u)$ on $[s,t]$ we have*

$$\mathbb{E}\Big[\exp\Big(iz\int_s^t g(u)dZ_u(\omega)\Big)\Big] = \exp\Big[\int_s^t \Psi(g(u)z)du\Big], \quad \forall\, z \in \mathbb{R}\,, \tag{2.3}$$

where

$$\Psi(u) = \log \hat{P}_{Z_1}(u) = \log \mathbb{E}\Big[e^{iuZ_1}\Big] = \lambda \int_{\mathbb{R}} e^{iuy} f_Y(y)dy - \lambda \tag{2.4}$$

with f_Y being given by (1.3).

Proof We follow the idea of [30, Section 17]. Let us first compute the characteristic function of Z_t.

$$\begin{aligned}
\hat{P}_{Z_t}(u) :=& \mathbb{E}\Big[e^{iuZ_t}\Big] = \mathbb{E}\Big[e^{iu\sum_{j=1}^{N_t} Y_j}\Big] \\
=& \sum_{n=0}^{\infty} E\Big[e^{iu\sum_{j=1}^{n} Y_j} | N_t = n\Big] P(N_t = n) \\
=& \sum_{n=0}^{\infty} \frac{(\lambda \cdot t)^n}{n!} e^{-\lambda}\Big(\mathbb{E}(e^{iuY_1})\Big)^n \\
=& e^{[\lambda t \mathbb{E}(e^{iuY_1}) - \lambda]} \\
=& \exp\left[\lambda t \int_{\mathbb{R}} e^{iuy} f_Y(y)dy - \lambda\right] ,
\end{aligned}$$

where $f_Y(y)$ is the double exponential density defined by (1.3). When $t = 1$ we have (2.4).

■

Now we are going to compute the characteristic function of the limiting distribution of $\mathbb{X}_o$. From Equation (2.1) and Proposition 3.2.2 it follows

$$
\begin{aligned}
\mathbb{E}[e^{iuX_t}] &= \exp\Big[ie^{-\theta t}x_0u + \int_0^t \Psi(\sigma e^{-\theta s}u)ds\Big] \\
&= \exp\left[ie^{-\theta t}x_0u + \lambda\int_0^t \left[\int_{\mathbb{R}} e^{i\sigma e^{-\theta s}uy} f_Y(y)dy - 1\right] ds\right] \\
&= \exp\left[ie^{-\theta t}x_0u + \lambda I_{1,t}\right],
\end{aligned}
\tag{2.5}
$$

where $I_{1,t} = \int_0^t [I_{2,s} - 1]ds$ and $I_{2,s}$ is defined and computed as follows.

$$
\begin{aligned}
I_{2,s} &= \int_{\mathbb{R}} e^{i\sigma e^{-\theta s}uy} f_Y(y)dy \\
&= \int_{\mathbb{R}} e^{i\sigma e^{-\theta s}uy}\left[p\eta e^{-\eta y} I_{[y\geq 0]} + q\varphi e^{\varphi y} I_{[y<0]}\right] dy \\
&= p\eta\int_0^\infty e^{i\sigma e^{-\theta s}uy} e^{-\eta y}dy + q\varphi\int_{-\infty}^0 e^{i\sigma e^{-\theta s}uy}e^{\varphi y}dy \\
&= \frac{p\eta}{\eta - i\sigma u e^{-\theta s}} + \frac{q\varphi}{\varphi + i\sigma u e^{-\theta s}},
\end{aligned}
$$

where in the above second identity we used the explicit form of f_Y given by (1.3).

Thus

$$
\begin{aligned}
I_{1,t} &= \int_0^t [I_{2,s} - 1]ds \\
&= \int_0^t \Big(\frac{p\eta}{\eta - i\sigma e^{-\theta s}u} + \frac{q\varphi}{\varphi + i\sigma e^{-\theta s}u} - 1\Big)ds \\
&= \frac{p}{\theta}\ln\Big(\frac{\eta - i\sigma e^{-\theta t}u}{\eta - i\sigma u}\frac{1}{e^{-\theta t}}\Big) + \frac{q}{\theta}\ln\Big(\frac{\varphi + ie^{-\theta t}\sigma u}{\varphi + i\sigma u}\frac{1}{e^{-\theta t}}\Big) - t \\
&= \ln\left[\Big(\frac{\eta - i\sigma e^{-\theta t}u}{\eta - i\sigma u}\frac{1}{e^{-\theta t}}\Big)^{\frac{p}{\theta}} \cdot \Big(\frac{\varphi + ie^{-\theta t}\sigma u}{\varphi + i\sigma u}\frac{1}{e^{-\theta t}}\Big)^{\frac{q}{\theta}} \cdot e^{-t}\right] \\
&= \ln\left[\Big(\frac{\eta - i\sigma e^{-\theta t}u}{\eta - i\sigma u}\Big)^{\frac{p}{\theta}} \cdot \Big(\frac{\varphi + ie^{-\theta t}\sigma u}{\varphi + i\sigma u}\Big)^{\frac{q}{\theta}}\right],
\end{aligned}
\tag{2.6}
$$

where in the above last identity, we used $p + q = 1$. Consequently, we have as $t \to \infty$

$$
\lim_{t\to\infty} I_{1,t} = \ln\left[\Big(\frac{\eta}{\eta - i\sigma u}\Big)^{\frac{p}{\theta}} \cdot \Big(\frac{\varphi}{\varphi + i\sigma u}\Big)^{\frac{q}{\theta}}\right].
$$

This combined with (2.5) yields

$$\lim_{t\to\infty} \mathbb{E}[e^{iuX_t}] = \Big(\frac{\eta}{\eta - i\sigma u}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi}{\varphi + i\sigma u}\Big)^{\frac{q\lambda}{\theta}} . \tag{2.7}$$

In other words, we have

$$\mathbb{E}\left[e^{iu\mathbb{X}_o}\right] = \lim_{t\to\infty} \mathbb{E}[e^{iuX_t}] = \Big(\frac{1}{1 - iu\frac{\sigma}{\eta}}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{1}{1 + iu\frac{\sigma}{\varphi}}\Big)^{\frac{q\lambda}{\theta}} . \tag{2.8}$$

As we notice the above characteristic function (2.8) uniquely determines the probability distribution function of $\mathbb{X}_o$. This formula also means that the invariant random variable $\mathbb{X}_o$ depends on $\frac{\sigma}{\eta}, \frac{\varphi}{\eta}, \frac{\lambda}{\theta}$ and then we cannot separate the parameters θ, σ, η, φ, λ and p.

Motivated by the works of [18, 22] we use the multi-time ergodic theorem to find more parameters. Our theoretical basis is the following general ergodic result, which is a consequence of [15, Theorem 1.1].

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(X_{t_j}, X_{t_j+h}) = \mathbb{E}\left[g\Big(\mathbb{X}_0, \mathbb{X}_h\Big)\right] \tag{2.9}$$

where $\mathbb{X}_t$ satisfies the Langevin equation (1.2) with the initial condition $\mathbb{X}_0 = \mathbb{X}_o$, namely, $d\mathbb{X}_t = -\theta \mathbb{X}_t dt + \sigma dZ_t$ and $\mathbb{X}_0$ has the invariant measure given by (2.13). The right hand side of (2.17) is hard to compute for general g. So we shall compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \exp\left[iuX_{t_j} + ivX_{t_j+h}\right] = \mathbb{E}\left[\exp\Big(iu\mathbb{X}_0 + iv\mathbb{X}_h\Big)\right] \tag{2.10}$$

for arbitrary $u, v \in \mathbb{R}$. In fact, we shall evaluate the above quantity by evaluating $\lim_{t\to\infty} \mathbb{E}[e^{i(uX_t+vX_{t+h})}]$. We shall still use the formula (2.3) to do our computations. As we see we can assume $X_0 = 0$. Thus,

$$X_t(\omega) = \sigma \int_0^t e^{-\theta(t-s)} dZ_s(\omega)\,; \quad X_{t+h}(\omega) = \sigma \int_0^{t+h} e^{-\theta(t+h-s)} dZ_s(\omega)\,.$$

Therefore,

$$
\begin{aligned}
uX_t(\omega) + vX_{t+h}(\omega) =& \sigma \int_0^t (ue^{-\theta(t-s)} + ve^{-\theta(t+h-s)}) dZ_s \\
& + \sigma \int_t^{t+h} ve^{-\theta(t+h-s)} dZ_s \,.
\end{aligned} \tag{2.11}
$$

Because of the independent increment property of the double exponential compound Poisson process Z_t, we have

$$
\begin{aligned}
\mathbb{E}\Big[\exp\Big(iuX_t + ivX_{t+h}\Big)\Big] &= \mathbb{E}\Big[\exp\Big(i\int_0^t \sigma e^{-\theta(t-s)}(u + ve^{-\theta h}) dZ_s\Big)\Big] \cdot \\
& \quad \mathbb{E}\Big[i\int_t^{t+h} \sigma ve^{-\theta(t+h-s)} dZ_s\Big] \\
&=: I_{3,t} \cdot I_{4,t} \,,
\end{aligned} \tag{2.12}
$$

where $I_{3,t}$ and $I_{4,t}$ denote the above first and second expectations. Similar to (2.6), we have

$$
\begin{aligned}
I_{3,t} =& \exp\Big[\frac{p\lambda}{\theta}\ln\Big(\frac{\eta - i\sigma(e^{-\theta t}u + ve^{-\theta(t+h)})}{\eta - i\sigma(u + ve^{-\theta h})}\frac{u + ve^{-\theta h}}{ue^{-\theta t} + ve^{-\theta(t+h)}}\Big) \\
& + \frac{q\lambda}{\theta}\ln\Big(\frac{\varphi + i\sigma(e^{-\theta t}u + ve^{-\theta(t+h)})}{\varphi + i\sigma(u + ve^{-\theta h})} \\
& \cdot \frac{u + ve^{-\theta h}}{e^{-\theta t}u + ve^{-\theta(t+h)})}\Big) - \ln(e^{\lambda t})\Big]
\end{aligned} \tag{2.13}
$$

and

$$
I_{4,t} = \Big(\frac{\eta - i\sigma v}{\eta - i\sigma e^{-\theta h} v}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi + ie^{-\theta h}\sigma v}{\varphi + i\sigma v}\Big)^{\frac{q\lambda}{\theta}} \,. \tag{2.14}
$$

It may be a bit strange to see that $I_{4,t}$ is independent of t. But this is because of the independent increment property of the process Z_t. In fact, we see easily that $\int_t^{t+h} \sigma ve^{-\theta(t+h-s)} dZ_s$ has the same law as that of $\int_0^h \sigma ve^{-\theta(h-s)} dZ_s$. It is easy to verify

$$
\lim_{t\to\infty} I_{3,t} = \Big(\frac{\eta}{\eta - i\sigma(u + ve^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi}{\varphi + i\sigma(u + ve^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}} \,. \tag{2.15}
$$

Hence, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(iu\mathbb{X}_0+iv\mathbb{X}_h\right)\right]=&\lim_{t\to\infty}\mathbb{E}\Big[\exp\left(iuX_t+ivX_{t+h}\right)\Big]\\ =&\Big(\frac{\eta}{\eta-i\sigma(u+ve^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{\varphi}{\varphi+i\sigma(u+ve^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}}\\ &\cdot\Big(\frac{\eta-i\sigma e^{-\theta h}v}{\eta-i\sigma v}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{\varphi+ie^{-\theta h}\sigma v}{\varphi+i\sigma v}\Big)^{\frac{q\lambda}{\theta}}.\end{aligned}\tag{2.16}$$

We summarize (2.2), (2.13), (2.10), (2.10) as the following theorem.

Theorem 3.2.3. *Let X_t be the double exponential Ornstein-Uhlenbeck process with initial condition $x_0\in\mathbb{R}$. Then for any $h\in\mathbb{R}_+,u,v\in\mathbb{R}$, we have almost surely (denoting $t_j=jh$)*

$$\begin{cases}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}e^{iuX_{t_j}}=\Big(\frac{\eta}{\eta-iu\sigma}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{\eta}{\eta+iu\varphi}\Big)^{\frac{q\lambda}{\theta}}\\ \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\exp\left[iuX_{t_j}+ivX_{t_j+h}\right]\\ \qquad=\Big(\frac{\eta}{\eta-i\sigma(u+ve^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{\varphi}{\varphi+i\sigma(u+ve^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}}\\ \qquad\quad\cdot\Big(\frac{\eta-i\sigma e^{-\theta h}v}{\eta-i\sigma v}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{\varphi+ie^{-\theta h}\sigma v}{\varphi+i\sigma v}\Big)^{\frac{q\lambda}{\theta}}.\end{cases}\tag{2.17}$$

3.3 Estimation of the parameters η, θ, φ and p

We have assumed now that the double exponential Ornstein-Uhlenbeck process can be observed at discrete time. Hence we have the availability of the observation data $\{X_{t_j},j=1,\cdots,n\}$, where $t_j=jh$ for some given observation time interval length h. Presumably Theorem 3.2.3 can be used to estimate all the parameters η, θ, φ, λ, σ, and p by replacing the limits in (2.17) by their the empirical characteristic functions

$\hat{\Psi}_{1,n}(u)$ and $\hat{\Psi}_{2,n}(u)$ defined as follows

$$\begin{cases} \hat{\Psi}_{1,n}(u,v) := \frac{1}{n}\sum_{j=1}^{n} \exp iuX_{t_j}\,; \\ \hat{\Psi}_{2,n}(u,v) = \frac{1}{n}\sum_{j=1}^{n} \exp(iuX_{t_j} + ivX_{t_j+h})\,. \end{cases} \tag{3.1}$$

For any given pair (u, v) although $\hat{\Psi}_{1,n}(u, v)$ depends only on u we write it as a function of u, v for convenience. Since we have 6 parameters, it may be possible for us to choose appropriately 6 pairs of (u_k, v_k) such that the 6 parameters can be determined by

$$\begin{cases} \Big(\frac{\eta}{\eta - iu_k\sigma}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\eta}{\eta + iu_k\varphi}\Big)^{\frac{q\lambda}{\theta}} = \hat{\Psi}_{1,n}(u_k, v_k)\,, \quad k = 1, \cdots, m, \\ \Big(\frac{\eta}{\eta - i\sigma(u_k + v_k e^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi}{\varphi + i\sigma(u_k + v_k e^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}} \\ \qquad \cdot \Big(\frac{\eta - i\sigma e^{-\theta h} v_k}{\eta - i\sigma v_k}\Big)^{\frac{p\lambda}{\theta}} \Big(\frac{\varphi + ie^{-\theta h}\sigma v_k}{\varphi + i\sigma v_k}\Big)^{\frac{q\lambda}{\theta}} = \hat{\Psi}_{2,n}(u_k, v_k)\,, \\ \qquad\qquad k = m+1, \cdots, 6\,, \end{cases} \tag{3.2}$$

where m is some integer between 1 and 6. For any given pair (u, v), the empirical characteristic functions $\hat{\Psi}_{1,n}(u, v)$ and $\hat{\Psi}_{1,n}(u, v)$ are known since we have the available observation data. Thus (3.2) is a system of function equations on the parameters η, θ, φ, λ, σ, and p. With appropriate choice of (u_k, v_k) we believe we should be able to use (3.2) to estimate all the above six parameters. However, it is still difficult for us to argue if this system of equations have a global unique solution or not although this system of nonlinear function equations (3.2) is explicit and appears to be quite simple as well. We shall assume $\lambda = \sigma = 1$ since we want to deal with the global uniqueness of the system (3.2). This allows us to have only four parameters: η, θ, φ, and p. If we choose four different values of (u_k, v_k), we should be able to obtain a system of four equations for the four unknowns. However, it is still difficult to argue the global uniqueness for the obtained system. So we are proposing an alternative method. Since (2.17) holds true for all $(u, v) \in \mathbb{R}$ we can obtain explicit formulas for the moments and then we use the moments to identity the parameters. Since $\mathbb{E}|\mathbb{X}_0|^m < \infty$ and $\mathbb{E}|\mathbb{X}_0\mathbb{X}_h|^m < \infty$ for all m we know (e.g. [15, Theorem 1.1]) that (2.2) and (2.17) hold true for moment functions, in particular, we shall choose $f = x, x^2, x^3, g(x, y) = xy$. Thus the system of four equations we choose to obtain the estimators for η, θ, φ, and

p are

$$\begin{cases} \mathbb{E}[\mathbb{X}_o] \approx \mu_{1,n}, \quad \text{where } \mu_{1,n} := \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} X_{t_j}\,, \\ \mathbb{E}[\mathbb{X}_o^2] \approx \mu_{2,n}, \quad \text{where } \mu_{2,n} := \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} X_{t_j}^2\,, \\ \mathbb{E}[\mathbb{X}_o^3] \approx \mu_{3,n}, \quad \text{where } \mu_{3,n} := \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} X_{t_j}^3\,, \\ \mathbb{E}[\mathbb{X}_o\mathbb{X}_h] \approx \mu_{4,n}, \quad \text{where } \mu_{4,n} := \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} X_{t_j}X_{t_j+h}\,. \end{cases} \tag{3.3}$$

With discrete time observations of the double exponential Ornstein-Uhlenbeck process X_t the right hand sides of (3.3) (namely, $\mu_{i,n}, i = 1,2,3,4$) are known. The left hand sides of (3.3) are functions of the parameters η, θ, φ, and p. We need first to find out how they depend on the four parameters explicitly and then solve this system to construct the ergodic estimators $\hat{\eta}_n$, $\hat{\theta}_n$, $\hat{\varphi}_n$, and $\hat{p}_n$ for the parameters. Let us also emphasize that (3.3) are not equations for the true parameters but they are equations for the ergodic estimators.

Now let us find the explicit forms for the left hand sides of (3.2). Let $\rho = \frac{\sigma}{\eta}$ and $\xi = \frac{\sigma}{\varphi}$. From the identities (2.13) and (2.10), we see by the expression of moments through characteristic function (e.g. Corollary 1 to Theorem 2.3.1 in

[25])

$$
\begin{cases}
\mathbb{E}[\mathbb{X}_o] = \frac{1}{i}\frac{\partial}{\partial u}\mathbb{E}[e^{iu\mathbb{X}_o}]\Big|_{u=0} \\
\quad = \frac{1}{i}\frac{\partial}{\partial u}\Big(\frac{1}{1-iu\rho}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{1}{1+iu\xi}\Big)^{\frac{q\lambda}{\theta}}\Big|_{u=0} \\
\quad = \frac{\lambda}{\theta}\Big[p\rho - q\xi\Big]\,; \\
\mathbb{E}[\mathbb{X}_o^2] = \frac{1}{i^2}\frac{\partial^2}{\partial u^2}\mathbb{E}[e^{iu\mathbb{X}_o}]\Big|_{u=0} \\
\quad = \frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] + \frac{\lambda}{\theta}\Big[p\rho - q\xi\Big]^2 \\
\quad = \frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] + \mathbb{E}[\mathbb{X}_o]^2\,; \\
\mathbb{E}[\mathbb{X}_o^3] = \frac{1}{i^3}\frac{\partial^3}{\partial u^3}\mathbb{E}[e^{iu\mathbb{X}_o}]\Big|_{u=0} \\
\quad = \frac{2\lambda}{\theta}\Big[p\rho^3 - q\xi^3\Big] + \Big(\frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] + \frac{\lambda}{\theta}\Big[p\rho - q\xi\Big]^2\Big)\Big(\frac{\lambda}{\theta}\Big[p\rho - q\xi\Big]\Big) \\
\quad + 2\frac{\lambda}{\theta}\Big[p\rho - q\xi\Big]\Big(\frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big]\Big) \\
\quad = \frac{2\lambda}{\theta}\Big[p\rho^3 - q\xi^3\Big] + \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] + 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)\,; \\
\mathbb{E}[\mathbb{X}_o\mathbb{X}_h] = \frac{1}{i^2}\frac{\partial}{\partial v}\frac{\partial}{\partial u}\mathbb{E}\Big[\exp\Big(iu\mathbb{X}_0 + iv\mathbb{X}_h\Big)\Big]\Big|_{u=0,v=0} \\
\quad = \frac{1}{i^2}\frac{\partial}{\partial v}\frac{\partial}{\partial u}\Big(\frac{1}{1-i\rho(u+ve^{-\theta h})}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{1}{1+i\xi(u+ve^{-\theta h})}\Big)^{\frac{q\lambda}{\theta}} \\
\qquad \cdot\Big(\frac{1-i\rho e^{-\theta h}v}{1-i\rho v}\Big)^{\frac{p\lambda}{\theta}}\Big(\frac{1+ie^{-\theta h}\xi v}{1+i\xi v}\Big)^{\frac{q\lambda}{\theta}}\Big|_{u=0,v=0} \\
\quad = e^{-\theta h}\frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] + \frac{\lambda^2}{\theta^2}\Big[p\rho - q\xi\Big]^2 \\
\quad = e^{-\theta h}\frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] + \mathbb{E}[\mathbb{X}_o]^2
\end{cases}
\quad . \quad (3.4)
$$

An elementary simplification yields (noticing $\lambda = 1$)

$$\frac{1}{\theta}\Big[p\rho - q\xi\Big] = \mu_{1,n}\,, \tag{3.5}$$
$$\frac{1}{\theta}\Big[p\rho^2 + q\xi^2\Big] = \mu_{2,n} - \mu_{1,n}^2\,, \tag{3.6}$$
$$\frac{2}{\theta}\Big[p\rho^3 - q\xi^3\Big] = \mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\,, \tag{3.7}$$
$$\frac{1}{\theta}e^{-\theta h}\Big[p\rho^2 + q\xi^2\Big] = \mu_{4,n} - \mu_{1,n}^2\,. \tag{3.8}$$

Thus we have the explicit form (3.5)-(3.8) for (3.3). Now we want to solve this system of function equations (e.g. (3.5)-(3.8)). Dividing (3.6) by (3.8) gives

$$\hat{\theta}_n = \frac{1}{h}\ln\Big(\frac{\mu_{2,n} - \mu_{1,n}^2}{\mu_{4,n} - \mu_{1,n}^2}\Big)\,. \tag{3.9}$$

Now we use the three equations (3.5)-(3.7) to solve for the remaining three unknowns p, ρ, ξ (noticing $q = 1 - p$). Denote

$$\begin{cases} f_1 = \hat{\theta}_n\mu_{1,n}\,, \\ f_2 = \hat{\theta}_n\left(\mu_{2n} - \mu_{1,n}^2\right)\,, \\ f_3 = \dfrac{\hat{\theta}_n}{2}\left(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\right)\,. \end{cases} \tag{3.10}$$

Thus we have

$$\begin{cases} p\rho - (1-p)\xi = f_1 \\ p\rho^2 + (1-p)\xi^2 = f_2 \\ p\rho^3 - (1-p)\xi^3 = f_3 \end{cases} \tag{3.11}$$

The first equation in (3.11) yields

$$\xi = \frac{p\rho - f_1}{1-p}\,. \tag{3.12}$$

Substituting to the second equation in (3.11) we have

$$p\rho^2 - 2f_1 p\rho + f_1^2 - f_2(1-p) = 0\,.$$

Solving for ρ, we have

$$\rho = \frac{f_1 p \pm \sqrt{p(1-p)(f_2 - f_1^2)}}{p}\,. \tag{3.13}$$

Recalling $\sigma = 1$, $\rho = \frac{1}{\eta}$ and $\xi = \frac{1}{\varphi}$ we have

$$f_2 - f_1^2 = p\rho^2 + q\xi^2 - (p\rho - q\xi)^2 = p(1-p)\rho^2 + q(1-q)\xi^2 + 2pq\rho\xi > 0$$

so the discriminant defining ρ (ie (3.13)) is nonnegative. Moreover, since $\xi = \frac{1}{\varphi}$, we see from (3.12) that $\frac{p\rho - f_1}{1-p} = \frac{1}{\varphi}$ which means

$$\rho = \frac{f_1}{p} + \frac{1-p}{p\varphi} > f_1\,.$$

Thus in (3.13), we should take the positive sign to obtain

$$\rho = \frac{f_1 p + \sqrt{p(1-p)(f_2 - f_1^2)}}{p}\,. \tag{3.14}$$

Now we substitute ξ given by (3.12) into the third equation in (3.11) to obtain

$$p\rho^3 - (1-p)\left(\frac{p\rho - f_1}{1-p}\right)^3 = f_3\,.$$

This means

$$f_3(1-p)^2 = p(1-p)^2\rho^3 + (f_1 - p\rho)^3\,.$$

Finally we substitute ρ in the above equation by (3.14) to obtain one function equation for only one unknown p:

$$\begin{aligned}(1-p)^2\left(f_1 p + \sqrt{p(1-p)(f_2 - f_1^2)}\right)^3 \\ +p^2\left(f_1 - f_1 p - \sqrt{p(1-p)(f_2 - f_1^2)}\right)^3 - f_3 p^2 (1-p)^2 = 0\,.\end{aligned} \tag{3.15}$$

This equation depends on f_1, f_2, f_3 is computed from the observation data of the double exponential Ornstein-Ulenbeck process. It is still hard to know if this function equation has a unique global solution or not. However, since it contains only one equation for one unknown we can plot the graph of the function (we denote the left-hand side of (3.15) by $h(p), 0 < p < 1$) to see if $h(p)$ has a unique solution on the interval $0 < p < 1$ or not. Below is the graph of $h(p)$ for $0 < p < 1$ with values of $f_1 = 0.0895, f_2 = 0.1025, f_3 = 0.0693$ computed from the simulated path of double exponential Ornstein Uhlenbeck process with $h = 0.02$, $\eta = 1.2$, $\varphi = 1.6$ and $\theta = 2.0$, $\sigma = \lambda = 1$.

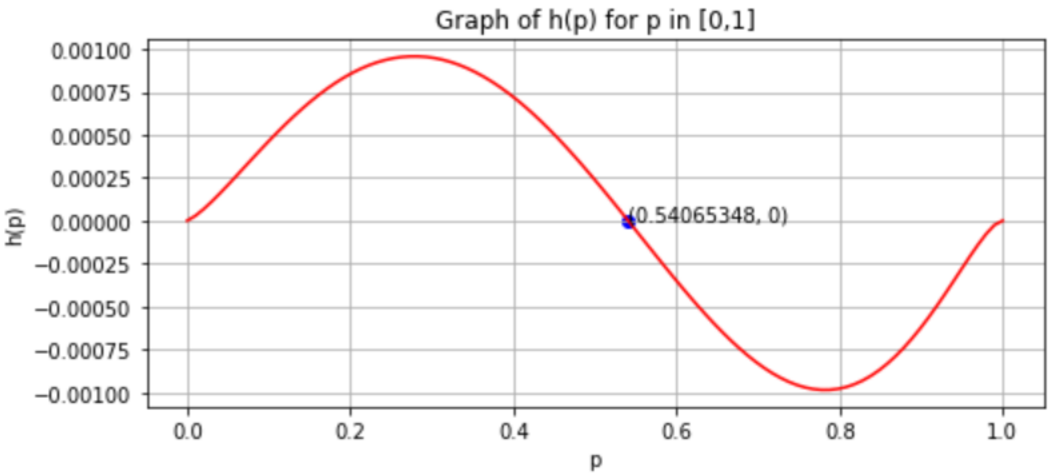


Figure 3.1: Function h(p) for p in [0,1]

We summarize the above discussions as the following theorem about the existence and uniqueness of the parameter estimators and their strong consistency results.

Theorem 3.3.1. *From the observation data, we denote* $\mu_{k,n}$, $k = 1, 2, 3, 4$ *by* (3.3). *Then* $\hat{\theta}_n$ *is given by* (3.9), *namely*

$$\hat{\theta}_n = \frac{1}{h} \ln \left(\frac{\mu_{2,n} - \mu_{1,n}^2}{\mu_{4,n} - \mu_{1,n}^2} \right) \tag{3.16}$$

and $f_k, k = 1, 2, 3$ *by* (3.10). *If* (3.15) *has a unique solution* $\hat{p}_n$ *on* $(0, 1)$, *namely,*

$$(1 - \hat{p}_n)^2 \left(f_1 \hat{p}_n + \sqrt{\hat{p}_n (1 - \hat{p}_n)(f_2 - f_1^2)} \right)^3 \tag{3.17}$$
$$+ \hat{p}_n^2 \left(f_1 - f_1 \hat{p}_n - \sqrt{\hat{p}_n (1 - \hat{p}_n)(f_2 - f_1^2)} \right)^3 - f_3 \hat{p}_n^2 (1 - \hat{p}_n)^2 = 0$$

and if $\hat{p}_n$ *is a continuous function of* f_1, f_2, f_3*, then* (3.5)-(3.8) *has a unique solution* $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)$ *given by* (3.16)*,* (3.17) *and*

$$\begin{cases} \hat{\rho}_n = \dfrac{f_1\hat{p}_n + \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)}}{\hat{p}_n} \,, \\ \hat{\xi}_n = \dfrac{\hat{p}_n\hat{\rho}_n - f_1}{1-\hat{p}_n} \,. \end{cases} \tag{3.18}$$

Define

$$\hat{\eta}_n := \frac{1}{\hat{\rho}_n}\,, \quad \hat{\varphi}_n := \frac{1}{\hat{\xi}_n}\,. \tag{3.19}$$

If $(\theta, \eta, \varphi, p)$ *are the true parameters, namely, if the double exponential process* X_t *satisfies* (1.2) *with the above parameters and with* $\lambda = \sigma = 1$*, and if* (3.15) *has a unique solution when* f_1, f_2, f_3 *are replaced by their limits as* $n \to \infty$*, then when* $n \to \infty$*,* $(\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n) \to (\theta, \eta, \varphi, p)$ *almost surely.*

Proof For any fixed n, it is clear that f_1, f_2, f_3 are continuous function of $\mu_{k,n}$, $k = 1,2,3,4$. So, $\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n$ are continuous functions of $\mu_{k,n}$, $k = 1,2,3,4$. Since $\mu_{k,n}$, $k = 1,2,3,4$ have limits as $n \to \infty$, we then see $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)$ have limits $(\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p})$. However, by the above argument, for each n, $\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n$ satisfy (3.5)-(3.8). Taking the limits of this system of equations we see $(\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p})$ satisfies

$$\begin{cases} \dfrac{1}{\hat{\theta}}\Big[\hat{p}\hat{\rho} - (1-\hat{p})\xi\Big] = \lim\limits_{n\to\infty} \mu_{1,n}\,, \\ \dfrac{1}{\hat{\theta}}\Big[\hat{p}\hat{\rho}^2 + (1-\hat{p})\hat{\xi}^2\Big] = \lim\limits_{n\to\infty} \big[\mu_{2,n} - \mu_{1,n}^2\big]\,, \\ \dfrac{2}{\hat{\theta}}\Big[\hat{p}\hat{\rho}^3 - (1-\hat{p})\hat{\xi}^3\Big] = \lim\limits_{n\to\infty} \big[\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\big]\,, \\ \dfrac{1}{\hat{\theta}}e^{-\hat{\theta}h}\Big[\hat{p}\hat{\rho}^2 + (1-\hat{p})\hat{\xi}^2\Big] = \lim\limits_{n\to\infty} \big[\mu_{4,n} - \mu_{1,n}^2\big]\,. \end{cases} \tag{3.20}$$

Since (3.15) has a unique solution when f_1, f_2, f_n are replaced by their limits as $n \to \infty$, by the same argument as above we can show (3.20) has a unique solution. Obviously, (θ, ξ, ρ, p) satisfy (3.20). Thus $(\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p}) = (\theta, \xi, \rho, p)$. This means that when $n \to \infty$, $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n) \to (\theta, \xi, \rho, p)$ almost surely and hence we obtain that when $n \to \infty$, $(\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n) \to (\theta, \eta, \varphi, p)$ almost surely. ■

Remark 3.3.2. The estimators $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)$ defined in the above theorem are called the ergodic estimators of the parameters (θ, ξ, ρ, p). The above theorem states that these ergodic estimators are uniquely determined and are strongly consistent.

Remark 3.3.3. The existence and uniqueness of the equation (3.15) depends on the values of f_1, f_2, f_3 which are from the real data. The function

$$\begin{aligned} h(p) = (1-p)^2 \left(f_1 p + \sqrt{p(1-p)(f_2 - f_1^2)} \right)^3 \\ +p^2 \left(f_1 - f_1 p - \sqrt{p(1-p)(f_2 - f_1^2)} \right)^3 - f_3 p^2 (1-p)^2 \end{aligned} \tag{3.21}$$

on the left hand side of equation may have no zero on $(0,1)$ for some values of f_1, f_2, f_3. For example, when f_1, f_2 are fixed, then when $f_3 \to \infty$, then $h(p) \to -\infty$, which suggests that (3.15) may have no zero for $p \in (0,1)$. However, once the data are given the problem of existence and uniqueness of $p \in (0,1)$ can be known by graph the function $h(p)$. For the data obtained by simulation in Section 6, we graph $h(p)$ in figure 1 which clearly demonstrates that in this case the equation (3.15) has a unique solution in $(0,1)$.

3.4 Joint asymptotic behavior of all the obtained estimators

In this section, we shall prove the central limit theorem for our ergodic estimators $\hat{\Theta}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n)$. Our goal is to prove that $\sqrt{n}(\hat{\Theta}_n - \Theta)$, where $\Theta = (\theta, \eta, \varphi, p)$ converges in law to a mean zero normal vector and to find the asymptotic covariance matrix. Let

$$\begin{cases} g(x,y) = (g_1(x,y), g_2(x,y), g_3(x,y), g_4(x,y))^T, \\ g_1(x,y) = x, \quad g_2(x,y) = x^2, \quad g_3(x,y) = x^3, \quad g_4(x,y) = xy \end{cases}$$

and

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4)\,, \quad \text{where} \quad \mu_k = \mathbb{E}[g_k(\mathbb{X}_o, \mathbb{X}_h)], \quad k = 1,2,3,4\,.$$

Denote

$$\mu_n = (\mu_{1,n}\,, \mu_{2,n}\,, \mu_{3,n}\,, \mu_{4,n})\,,$$

where $\mu_{k,n}$, $k = 1,2,3,4$ are defined by (3.3).

First, we have the following central limiting result.

Lemma 3.4.1. *Let μ_n, μ and g be defined as above. Then as $n \to \infty$, we have*

$$\sqrt{n}(\mu_n - \mu) \xrightarrow{d} N(0, A)\,, \tag{4.1}$$

with the 4×4 covariance matrix A being given by

$$A = \left(\sigma_{g_i g_j}\right)_{1 \le i,j \le 4}\,, \tag{4.2}$$

where $\sigma_{g_i g_j}$, $1 \le i, j \le 4$ will be given in the appendix.

Proof We shall use the Cramer-Wold device (e.g. [14, Theorem 29.4]). For any $a = (a_1, a_2, a_3, a_4)^T \in \mathbb{R}^4$, consider $a^T \mu_n = \sum_{k=1}^4 a_k \mu_{k,n}$. By [27, Theorem 2.6] and [26], the double exponential Ornstein-Uhlenbeck process $\{X_t\}$ is exponentially β-mixing. Since the exponential β-mixing implies the exponential α-mixing, by the central limit theorem (e.g. [23, Theorem 18.6.2]) for stationary process with exponential α-mixing, we have

$$\sqrt{n} a^T (\mu_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma_a^2)\,, \tag{4.3}$$

Since $a \in \mathbb{R}^4$ is arbitrary, we prove the lemma through the Cramer-Wold device.

■

Denote

$$\begin{cases} h_1(\theta,\xi,\rho,p) = \dfrac{1}{\theta}\Big[p\rho - (1-p)\xi\Big]\,, \\ h_2(\theta,\xi,\rho,p) = \dfrac{1}{\theta}\Big[p\rho^2 + (1-p)\xi^2\Big]\,, \\ h_3(\theta,\xi,\rho,p) = \dfrac{1}{\theta}\Big[p\rho^3 - (1-p)\xi^3\Big]\,, \\ h_4(\theta,\xi,\rho,p) = \dfrac{1}{\theta}e^{-\theta h}\Big[p\rho^2 + (1-p)\xi^2\Big]\,. \end{cases}$$

and

$$\begin{cases} \tilde{h}_1(\mu_1, \mu_2, \mu_3, , \mu_4) = \mu_1 \,, \\ \tilde{h}_2(\mu_1, \mu_2, \mu_3, \mu_4) = \mu_2 - \mu_1^2 \,, \\ \tilde{h}_3(\mu_1, \mu_2, \mu_3, \mu_4) = \mu_3 - \mu_2\mu_1 - 2\mu_1(\mu_2 - \mu_1^2) \,, \\ \tilde{h}_4(\mu_1, \mu_2, \mu_3, \mu_4) = \mu_4 - \mu_1^2 \,. \end{cases}$$

Set

$$h = (h_1, h_2, h_3, h_4)^T \quad \text{and} \quad \tilde{h} = (\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4)^T \,.$$

We compute the partial derivative of $\tilde{h}$ with respect to μ to obtain

$$\begin{aligned} &\frac{\partial \tilde{h}_1}{\partial \mu_1} = 1 \,, \quad \frac{\partial \tilde{h}_1}{\partial \mu_2} = \frac{\partial \tilde{h}_1}{\partial \mu_3} = \frac{\partial \tilde{h}_1}{\partial \mu_4} = 0 \\ &\frac{\partial \tilde{h}_2}{\partial \mu_1} = -2\mu_1 \,, \quad \frac{\partial \tilde{h}_2}{\partial \mu_2} = 1 \,, \quad \frac{\partial \tilde{h}_1}{\partial \mu_3} = \frac{\partial \tilde{h}_1}{\partial \mu_4} = 0 \\ &\frac{\partial \tilde{h}_3}{\partial \mu_1} = -3\mu_2 - 6\mu_1^2 \,, \quad \frac{\partial \tilde{h}_3}{\partial \mu_2} = -3\mu_1 \,, \quad \frac{\partial \tilde{h}_3}{\partial \mu_3} = 1 \,, \quad \frac{\partial \tilde{h}_3}{\partial \mu_4} = 0 \\ &\frac{\partial \tilde{h}_4}{\partial \mu_1} = -2\mu_1 \,, \quad \frac{\partial \tilde{h}_4}{\partial \mu_4} = 1 \,, \quad \frac{\partial \tilde{h}_4}{\partial \mu_2} = \frac{\partial \tilde{h}_4}{\partial \mu_3} = 0 \end{aligned}$$

We compute the partial derivatives of h with respect to the parameters to obtain

$$\begin{aligned} &\frac{\partial h_1}{\partial p} = \frac{1}{\theta}(\rho + \xi) \,, \quad \frac{\partial h_2}{\partial p} = \frac{1}{\theta}(\rho^2 - \xi^2) \,, \quad \frac{\partial h_3}{\partial p} = \frac{1}{\theta}(\rho^3 + \xi^3) \\ &\frac{\partial h_4}{\partial p} = \frac{1}{\theta}e^{-\theta h}(\rho^2 - \xi^2) \,, \quad \frac{\partial h_1}{\partial \rho} = \frac{1}{\theta}(p - q\xi) \,, \quad \frac{\partial h_2}{\partial \rho} = \frac{1}{\theta}(2p\rho + q\xi^2) \\ &\frac{\partial h_3}{\partial \rho} = \frac{1}{\theta}(3p\rho^2 - q\xi^3) \,, \quad \frac{\partial h_4}{\partial \rho} = \frac{1}{\theta}e^{-\theta h}(2p\rho + q\xi^2) \,, \quad \frac{\partial h_1}{\partial \xi} = \frac{1}{\theta}(p\rho - q) \\ &\frac{\partial h_2}{\partial \xi} = \frac{1}{\theta}(p\rho^2 + 2q\xi) \,, \quad \frac{\partial h_3}{\partial \xi} = \frac{1}{\theta}(p\rho^3 - 3q\xi^2) \,, \quad \frac{\partial h_4}{\partial \xi} = \frac{1}{\theta}e^{-\theta h}(p\rho^2 + 2q\xi) \\ &\frac{\partial h_1}{\partial \theta} = \frac{-1}{\theta}(p\rho - q\xi) \,, \quad \frac{\partial h_2}{\partial \theta} = \frac{-1}{\theta}(p\rho^2 + q\xi^2) \,, \quad \frac{\partial h_3}{\partial \theta} = \frac{-1}{\theta}(p\rho^3 - q\xi^3) \\ &\frac{\partial h_4}{\partial \theta} = \frac{-1}{\theta}e^{-\theta h}(p\rho^2 + q\xi^2)\left[\frac{1}{\theta} + 1\right] \end{aligned}$$

Let us denote the matrix

$$\nabla_\Theta h(\Theta) = \begin{pmatrix} \frac{\partial h_1}{\partial p} & \frac{\partial h_1}{\partial \rho} & \frac{\partial h_1}{\partial \xi} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial p} & \frac{\partial h_2}{\partial \rho} & \frac{\partial h_2}{\partial \xi} & \frac{\partial h_2}{\partial \theta} \\ \frac{\partial h_3}{\partial p} & \frac{\partial h_3}{\partial \rho} & \frac{\partial h_3}{\partial \xi} & \frac{\partial h_3}{\partial \theta} \\ \frac{\partial h_4}{\partial p} & \frac{\partial h_4}{\partial \rho} & \frac{\partial h_4}{\partial \xi} & \frac{\partial h_4}{\partial \theta} \end{pmatrix}$$

Then we have the following result.

Theorem 3.4.2. *Denote* $\Theta = (\theta, \eta, \varphi, p)$ *and* $\hat{\Theta}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{p}_n)$. *If* $\hat{p}_n$ *is a continuous function of* f_1, f_2, f_3 *and if* (3.15) *has a unique solution when* f_1, f_2, f_3 *are replaced by their limits as* $n \to \infty$, *then as* $n \to \infty$ *we have*

$$\sqrt{n}(\hat{\Theta}_n - \Theta) \xrightarrow{d} \mathcal{N}(0, \Sigma) \tag{4.4}$$

where

$$\Sigma = \left((\nabla h)^{-1} \nabla \tilde{h}\right)^T A \, (\nabla h)^{-1} \nabla \tilde{h} \,. \tag{4.5}$$

Proof It is easy to see that $h, \tilde{h} : \mathbb{R}^4 \to \mathbb{R}^4$ defined as above are smooth mappings. Using these two mappings, we can write the system (3.5)-(3.8) to determine the ergodic estimators Θ_n

$$h(\Theta_n) = \tilde{h}(\mu_n) \,. \tag{4.6}$$

From Theorem 3.3.1, it follows that h has inverse h^{-1} so that

$$\Theta_n = (h^{-1} \circ \tilde{h})(\mu_n) \,.$$

By Lemma 3.4.1 and the Delta method, we see that

$$\sqrt{n}(\hat{\Theta}_n - \Theta) \xrightarrow{d} \mathcal{N}(0, \Sigma) \tag{4.7}$$

where

$$\begin{aligned} \Sigma =& (\nabla_\mu (h^{-1} \circ \tilde{h}))^T A \nabla_\mu (h^{-1} \circ \tilde{h}) \\ =& \left((\nabla h)^{-1} \nabla \tilde{h}\right)^T A \, (\nabla h)^{-1} \nabla \tilde{h} \,. \end{aligned}$$

This proves the theorem. ■

3.5 Exact Simulation for the double exponential Ornstein-Uhlenbeck process

Before we give some numerical simulations to validate our ergodic estimators, in this section we propose a distributional decomposition to exactly simulate the double exponential Ornstein-Uhlenbeck process. We follow the idea of [29], where the exact simulation of Gamma Ornstein-Uhlenbeck process is studied. First, we have the following result. Without loss of generality we can assume $\sigma = 1$.

Theorem 3.5.1. *Let X_t be the double exponential Ornstein-Uhlenbeck process given by* (1.2)*. For any $t, t_1 > 0$, the Laplace transform of X_{t+t_1} conditioning on X_t is given by*

$$\begin{aligned}\mathbb{E}[e^{iuX_{t+t_1}}|X_t] = e^{-iuwX_t} \exp\Big[\frac{-\lambda p}{\theta}\int_0^\infty (1-e^{-ius})\int_1^{1/w} \eta v e^{-s\eta v}\frac{1}{v}dvds \\ -\frac{\lambda q}{\theta}\int_{-\infty}^0 (1-e^{-ius})\int_1^{1/w} \phi v e^{s\phi v}\frac{1}{v}dvds\Big],\end{aligned} \tag{5.1}$$

where $w = e^{-\theta t_1}$.

Proof Recall the Ψ defined by (2.4) and the formula (2.5). We can write the characteristic function of $X_{t_1} = \sigma\int_t^{t+t_1} e^{-(t+t_1-s)}dZ_s$ as

$$\begin{aligned}\mathbb{E}[e^{iuX_{t_1}}] &= \exp\Big[\int_t^{t+t_1}\Psi(\sigma e^{-\theta(t+t_1-s)}u)ds\Big] \\ &= \exp\Big[\int_0^{t_1} -\lambda\Big(1-\mathbb{E}\Big(e^{(i\sigma e^{-\theta(t_1-s)}uY_1)}\Big)\Big)ds\Big].\end{aligned} \tag{5.2}$$

Denote $\hat{h}(z) = \mathbb{E}\Big(e^{zY_1}\Big)$. The Laplace transform of X_{t+t_1} conditioning on X_t is

$$\begin{aligned}\mathbb{E}[e^{iuX_{t+t_1}}|X_t] =& e^{-iuwX_t}\exp\Big[\int_t^{t+t_1} -\lambda\Big(1-\hat{h}(i\sigma e^{-\theta(t+t_1-s)}u)\Big)ds\Big] \\ =& e^{-iuwX_t}\exp\Big[\int_0^{t_1} -\lambda\Big(1-\hat{h}(i\sigma e^{-\theta s}u)\Big)ds\Big].\end{aligned} \tag{5.3}$$

Let $ue^{-\theta s} = x$, then for $\sigma = 1$, we have

$$\int_0^{t_1} \left(1 - \hat{h}(iue^{-\theta s})\right) ds = \frac{1}{\theta} \int_u^{ue^{-\theta t_1}} \frac{-(1 - \hat{h}(ix))}{x} dx$$
$$= -\left(I_1 + I_2\right),$$

where

$$I_1 = \frac{1}{\theta} \int_{uw}^{u} \frac{1}{x} \int_0^{\infty} (1 - e^{-ixy}) \eta p e^{-\eta y} dy dx\,,$$
$$I_2 = \frac{1}{\theta} \int_{uw}^{u} \frac{1}{x} \int_{-\infty}^{0} (1 - e^{-ixy}) \phi q e^{\phi y} dy dx\,.$$

The first term I_1 can be written

$$I_1 = \frac{1}{\theta} \int_0^{\infty} \frac{(1 - e^{-ius})}{s} \int_s^{s/w} \eta p e^{-\eta y} dy ds$$
$$= \frac{p}{\theta} \int_0^{\infty} \frac{(1 - e^{-ius})}{1} \frac{e^{-\eta s} - e^{-\eta s/w}}{s} dy$$
$$= \frac{p}{\theta} \int_0^{\infty} \frac{(1 - e^{-ius})}{1} \int_{\eta}^{\eta/w} e^{-sv} dv ds$$
$$= \frac{p}{\theta} \int_0^{\infty} \frac{(1 - e^{-ius})}{1} \int_1^{1/w} \eta v e^{-s\eta v} \frac{1}{v} dv ds\,.$$

The second term I_2 can be written as

$$I_2 = \frac{1}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-ius})}{s} \int_s^{s/w} \phi q e^{-\eta y} dy ds$$
$$= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-ius})}{1} \frac{e^{\phi s} - e^{\phi s/w}}{s} dy$$
$$= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-ius})}{1} \int_{\phi}^{\phi/w} e^{sv} dv ds$$
$$= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-ius})}{1} \int_1^{1/w} \phi v e^{s\phi v} \frac{1}{v} dv ds\,.$$

This gives us (5.1), proving the theorem. ∎

Since the second exponential factor on the right hand side of (5.1) is the characteristic function of the compound Poisson process we have

Corollary 3.5.2 (Exact Simulation via Decomposition Approach). *Let N be a Poisson random variable of rate λh and let $\{S_k\}_{k=1,2,\ldots}$ be i.i.d random variables following a mixture of double exponential distribution*

$$\begin{aligned} f_{S_k}(y) =& p\eta e^{\theta h U} e^{-\eta e^{\theta h U} y} I_{y \geq 0} + q\phi e^{\theta h U} e^{\phi e^{\theta h U} y} I_{y<0} \,, \\ & \forall \; k = 1\,, 2\,, \ldots \,, \end{aligned} \tag{5.4}$$

where $U \overset{d}{=} \mathcal{U}[0,1]$ is the uniform distribution on $[0,1]$. Then

$$X_{t+h} \overset{d}{=} X_t e^{-\theta h} + \sum_{k=1}^{N} S_k \,. \tag{5.5}$$

The above formula (5.5) enables us to simulate the process X_t by the exact decomposition approach.

3.6 Numerical results

To validate our estimators discussed in Section 4, we perform some numerical simulations. We choose the values of $p = 0.6$, $\eta = 1.2$, $\varphi = 1.6$ and $\theta = 2.0$ (and $\lambda = \sigma = 1$). With these parameters, we simulate the double exponential Ornstein-Uhlenbeck process using the exact decomposition algorithm given by (5.5). A simulated sample is displayed in Figure 3.2. Figures 3.3 and 3.4 plot the assumed values versus the values by the ergodic estimators. Table 1 lists the approximation of ergodic estimators to the true parameters as the time becomes larger. It demonstrates that the rate of convergence is quite faster.

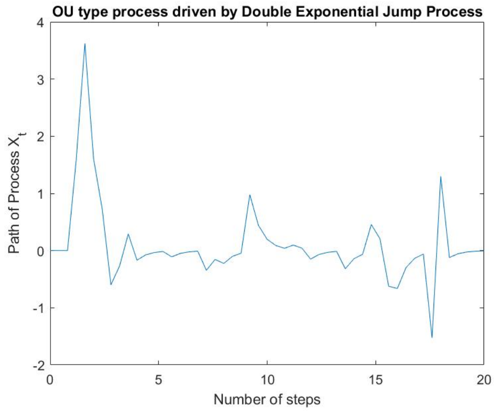
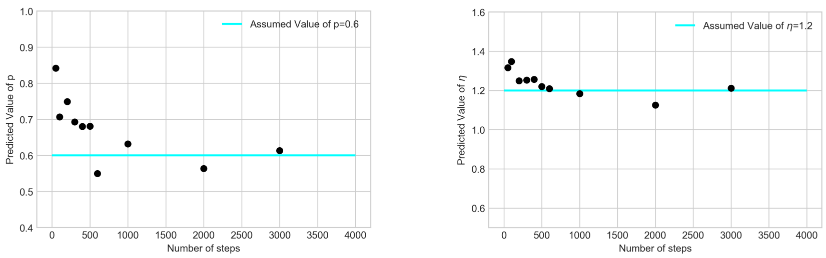


Figure 3.2: Simulated sample path for a double exponential Orntein-Uhlenbeck process with T=20, Nsteps=50, h=0.4 $\eta = 1.2$, $\varphi = 1.6$ and $\theta = 2.0$, $\sigma = \lambda = 1$



(a) Estimated Values of p

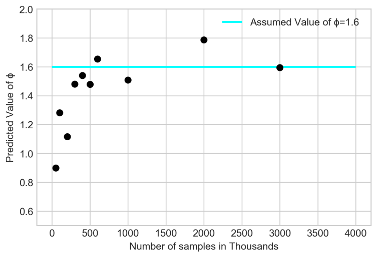
(b) Estimated Values of η

Figure 3.3: Assumed versus estimated values

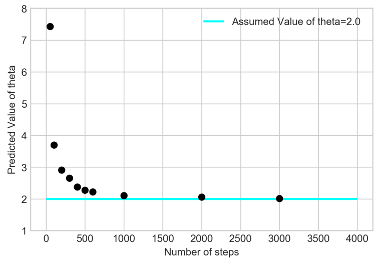
The table 3.1 shows the estimated values of the parameters p, η, ϕ and θ with different number of steps N and fixed $h = 0.02$ and $T = Nh$

Time	Number of steps	$p = 0.6$	$\eta = 1.2$	$\phi = 1.6$	$\theta = 2.0$
1	50	0.8421	1.31677	0.8995	7.4297
2	100	0.7070	1.3477	1.2816	3.6995
4	200	0.74925	1.2498	1.1164	2.9127
6	300	0.6928	1.2532	1.4803	2.6587
8	400	0.6804	1.2571	1.5397	2.3808
10	500	0.6812	1.2204	1.4793	2.2743
12	600	0.5500	1.2089	1.6546	2.2217
20	1000	0.6320	1.1836	1.5078	2.1066
40	2000	0.5635	1.1255	1.7866	2.0631
60	3000	0.6135	1.2112	1.5940	2.0128

Table 3.1: Assumed Values and Estimated values of the parameters with different number of steps N and fixed $h = 0.02$ and $T = Nh$



(a) Estimated Values of φ



(b) Estimated Values of θ

Figure 3.4: Assumed versus estimated values

In our theoretical analysis, we assume that $\lambda = \sigma = 1$. However, in applications, λ, σ are usually unknown. To estimate λ, σ, we can introduce two more moment equations and solve them by numerically to find all parameters $p, \theta, \xi, \rho, \lambda, \sigma$. Here to show the same, we have also computed the fourth and fifth moments along with (3.4). This gives us

$$
\begin{aligned}
\mathbb{E}[\mathbb{X}_o^4] &= \frac{1}{i^4}\frac{\partial^4}{\partial u^4}\mathbb{E}[e^{iu\mathbb{X}_o}]\Big|_{u=0} \\
&= \frac{6\lambda}{\theta}\Big[p\rho^4 + q\xi^4\Big] + \mathbb{E}[\mathbb{X}_o]^4 + 3(\mathbb{E}[\mathbb{X}_o])^2(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) \\
&+ \mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^3] - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)) + 3(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)\mathbb{E}[\mathbb{X}_o^2] \\
&+ \mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) \\
\mathbb{E}[\mathbb{X}_o^5] &= \frac{1}{i^5}\frac{\partial^5}{\partial u^5}\mathbb{E}[e^{iu\mathbb{X}_o}]\Big|_{u=0} \\
&= \frac{24\lambda}{\theta}\Big[p\rho^5 - q\xi^5\Big] + \mathbb{E}[\mathbb{X}_o]^5 + 4\mathbb{E}[\mathbb{X}_o]^3(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) \\
&+ 3\mathbb{E}[\mathbb{X}_o]^2(\mathbb{E}[\mathbb{X}_o^3] - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)) \\
&+ (\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)(3\mathbb{E}[\mathbb{X}_o]^3 + 6(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)) \\
&+ 5\mathbb{E}[\mathbb{X}_o]\Big[\mathbb{E}[\mathbb{X}_o^4] - \mathbb{E}[\mathbb{X}_o]^4 - 3(\mathbb{E}[\mathbb{X}_o])^2(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) - 3(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)\mathbb{E}[\mathbb{X}_o^2] \\
&- \mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) - \mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^3] - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2))\Big] \\
&+ 2(\mathbb{E}[\mathbb{X}_o^3] - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2))(3\mathbb{E}[\mathbb{X}_o^2] + 3(\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)) \\
&+ (\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2)\Big[3\mathbb{E}[\mathbb{X}_o]^3 + 6\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) \\
&+ 9\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2) + 3(\mathbb{E}[\mathbb{X}_o]^3 - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]^2))\Big] \\
&+ (\mathbb{E}[\mathbb{X}_o^3] - \mathbb{E}[\mathbb{X}_o^2]\mathbb{E}[\mathbb{X}_o] - 2\mathbb{E}[\mathbb{X}_o](\mathbb{E}[\mathbb{X}_o^2] - \mathbb{E}[\mathbb{X}_o]))(\mathbb{E}[\mathbb{X}_o^2] + \mathbb{E}[\mathbb{X}_o]^2)
\end{aligned}
$$

We can solve the following system of equations numerically to obtain the parameters $p, \theta, \xi, \rho, \lambda$,

$$\begin{aligned}
\frac{\lambda}{\theta}\Big[p\rho - q\xi\Big] &= \mu_{1,n}\,, \\
\frac{\lambda}{\theta}\Big[p\rho^2 + q\xi^2\Big] &= \mu_{2,n} - \mu_{1,n}^2\,, \\
\frac{2\lambda}{\theta}\Big[p\rho^3 - q\xi^3\Big] &= \mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\,, \\
\frac{\lambda}{\theta}e^{-\theta h}\Big[p\rho^2 + q\xi^2\Big] &= \mu_{4,n} - \mu_{1,n}^2\,, \\
\frac{6\lambda}{\theta}\Big[p\rho^4 + q\xi^4\Big] &= \mu_{5,n} - \mu_{1,n}^4 - 3\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2) \\
&\quad - 3\mu_{1,n}(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)) \\
&\quad - 3(\mu_{2,n} - \mu_{1,n}^2)\mu_{2,n} - \mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\,,
\end{aligned}$$

$$\begin{aligned}
\frac{24\lambda}{\theta}\Big[p\rho^5 - q\xi^5\Big] &= \mu_{6,n} - \mu_{1,n}^5 - 4\mu_{1,n}^3(\mu_{2,n} - \mu_{1,n}^2) \\
&\quad - 3\mu_{1,n}^2(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)) \\
&\quad - (\mu_{2,n} - \mu_{1,n}^2)(3\mu_{1,n}^3 + 6(\mu_{2,n} - \mu_{1,n}^2)) \\
&\quad - 5\mu_{1,n}\Big[\mu_{5,n} - \mu_{1,n}^4 - 3\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2) \\
&\quad - 3\mu_{1,n}(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)) \\
&\quad - 3(\mu_{2,n} - \mu_{1,n}^2)\mu_{2,n} - \mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\Big] \\
&\quad - 2(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2))(3\mu_{1,n}^2 + 3(\mu_{2,n} - \mu_{1,n}^2)) \\
&\quad - (\mu_{2,n} - \mu_{1,n}^2)\Big[3\mu_{1,n}^3 + 15\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2) \\
&\quad + 3(\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)\Big] \\
&\quad - (\mu_{3,n} - \mu_{2,n}\mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)(\mu_{1,n}^2 + \mu_{2,n})
\end{aligned}$$

where $\mu_{5,n} := \frac{1}{n}\sum_{j=1}^n X_{t_j}^4$ and $\mu_{6,n} := \frac{1}{n}\sum_{j=1}^n X_{t_j}^5$ We have numerically estimated the parameters $p, \theta, \xi, \rho, \lambda$ by solving first five nonlinear equations from simulated data. Similarly, we can solve for all the parameters including σ numerically using Newton's Raphson method [31] and obtain the parameters.

Time	Number of steps	$p = 0.6$	$\eta = 1.2$	$\phi = 1.6$	$\theta = 2.0$	λ=1.0
1	50	0.8218	1.5087	1.0445	6.061	1.3198
2	100	0.6203	1.5123	1.7117	3.6881	1.6826
4	200	0.5865	1.4139	2.1101	2.9641	1.5514
6	300	0.5619	1.2951	2.2689	2.5361	1.6127
8	400	0.5820	1.3251	1.8637	2.5552	1.3424
10	500	0.5761	1.4336	1.9873	2.3187	1.5333
12	600	0.5591	1.4874	2.1702	2.2577	1.6498
20	1000	0.5580	1.2395	1.9293	2.1609	1.1983
40	2000	0.6115	1.1146	1.3757	2.0413	0.8927
60	3000	0.5880	1.1839	1.7118	2.0039	1.0807

Table 3.2: Assumed Values and Estimated values of the parameters with different number of steps N and fixed $h = 0.02$ and $T = Nh$

Chapter 4

Parameter Estimation for Vasicek Model with double exponential jump

In this chapter, we consider the parameter estimation problem for Vasicek model driven by the compound Poisson process with double exponential jumps. Here we will construct least square estimators for parameters based on continuous time observations.

4.1 Introduction

The Vasicek model is a stochastic model used in finance to describe the evolution of interest rates over time. It is a single-factor, continuous-time model that assumes interest rates follow a mean-reverting process. The model has found wide application in fixed-income valuation, risk management, and interest rate derivatives pricing. For more details refer to [34]

The model is expressed in the form of the following stochastic differential equation (SDE),

$$dX_t = (\mu - \theta X_t)dt + d\tilde{L}_t \tag{1.1}$$

$$X_0 = 0 \tag{1.2}$$

The first term $(\mu - \theta X_t)dt$ represents the drift term. The parameter θ gives the reversion speed of the stochastic component. The mean-reversion in finance can be interpreted as the fluctuation in the stochastic price is around the mean and the prices only peak temporarily. These peaks can be explained as a result of unforeseen circumstances such as outages or shortages due to supply and demand.

This leads to the exploration of parameter estimation problems for the Vasicek model as it is of great significance in econometrics. when $\mu = 0$, the process (1.1) becomes the well known Ornstein-Uhlenbeck process. The parameter estimation problem of the Ornstein-Uhlenbeck driven by compound Poisson process with double exponential jumps has been discussed in Chapter 3. Parameter estimation for fractional Vasicek models and Ornstein Uhlenbeck process using least square estimators has been extensively studied in [32] and [33].
In this chapter, we want to discuss the parameter estimation for θ and μ when the Vasicek model is driven by compensated Lêvy process $(\hat{L}_t, t \geq 0)$, where

$$L_t = \sum_{i=1}^{N_t} Y_i$$

and $(Y_n, n \geq 1)$ is a sequence of independent real-valued random variables with the following probability density function

$$f_Y(x) = p\eta e^{-\eta x} I_{[x\geq 0]} + q\varphi e^{\varphi x} I_{[x<0]} \,, \tag{1.3}$$

where the parameters p, q, η, φ are positive and $p + q = 1$.
Also N_t is the Poisson process with rate $\lambda > 0$, independent of $\{Y_i, i = 1, 2, \ldots\}$.

Here the compensated Lévy process is a compensated double exponential compound Poisson Process $\hat{L}_t$ is given by

$$\tilde{L}_t = L_t - \lambda t \mathbb{E}[Y_1] \tag{1.4}$$

Note that $\tilde{L}_t$ is a martingale $w.r.t$ filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

4.2 Preliminaries

Let us have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a right continuous and increasing family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ and $\hat{L}_t$ be the compensated double exponential compound Poisson Process. We aim to study the parameter estimation for the Vasicek model driven by the compensated Levy process, which is given by the following stochastic equation

$$dX_t = (\mu - \theta X_t)dt + d\tilde{L}_t \tag{2.1}$$

$$X_0 = 0. \tag{2.2}$$

Here we assume $\mu > 0$ and $\theta > 0$ and $X_0 = 0$ a.s. Our goal is to construct least square estimators under continuous observations.Let $\dot{X}_t$ denote the differentiation of X_t with respect to t. We will find the estimators by minimizing the following contrast function

$$\Phi(\theta, \mu) = \min_{\theta,\mu} \int_0^T |\dot{X}_t - (\mu - \theta X_t)|^2 dt \tag{2.3}$$

Upon minimizing the contrast function we obtain expressing with integrals of the form $\int_0^T X_t dX_t$. Such integrals can be interpreted as Young integrals.

Young [36] introduced the Riemann–Stieltjes integral as follows. Suppose that $f, g : [0, T] \to \mathbb{R}$ are Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$ for fixed $T > 0$. Then the Young integral $\int_0^T f_s dg_s$ exists. If $\alpha = \beta$, $\alpha, \beta \in (0, 1)$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1, the integrals $\int_0^t \frac{\partial\phi}{\partial f}(f_u, g_u)\, df_u$ and $\int_0^t \frac{\partial\phi}{\partial g}(f_u, g_u)\, dg_u$ exist in the Young sense, and we have the following change of variables formula

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial\phi}{\partial f}(f_u, g_u)\, df_u + \int_0^t \frac{\partial\phi}{\partial g}(f_u, g_u)\, dg_u, \quad 0 \leq t \leq T. \tag{2.4}$$

refer [36] for more details.

4.3 Least Square Estimators μ and θ

Let $\dot{X}_t$ denote the differentiation of X_t with respect to t. Then, upon minimizing the contrast funtion $\Phi(\theta, \mu)$ we get

$$\Phi(\theta,\mu) = \min_{\theta,\mu} \int_0^T |\dot{X}_t - (\mu - \theta X_t)|^2 dt \tag{3.1}$$

$$= \min_{\theta,\mu} \Big(-2\mu \int_0^T \dot{X}_t dt + 2\theta \int_0^T X_t \dot{X}_t dt + \int_0^T \dot{X}_t^2 dt \tag{3.2}$$

$$+ \int_0^T \mu^2 dt + \int_0^T \theta^2 X_t^2 dt - 2\int_0^T \mu\theta X_t dt \Big). \tag{3.3}$$

Minimizing w.r.t to μ will give us

$$\frac{\partial\Phi}{\partial\mu} = -2\int_0^T \dot{X}_t dt + 2\mu \int_0^T dt - 2\int_0^T \theta X_t dt = 0$$

and minimizing $\Phi(\theta, \mu)$ w.r.t θ gives us

$$\frac{\partial\Phi}{\partial\theta} = 2\int_0^T X_t \dot{X}_t dt + 2\theta \int_0^T X_t^2 dt - 2\mu \int_0^T X_t dt = 0$$

.

Therefore, the minimum is attained when

$$\hat{\mu_T} = \frac{\hat{\theta}_T \int_0^T X_t dt + X_T}{T} \tag{3.4}$$

and

$$\hat{\theta}_T = \frac{X_T \int_0^T X_t dt - T\int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}. \tag{3.5}$$

Upon substituting $\hat{\theta}_T$ we get

$$\hat{\mu}_T = \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}. \tag{3.6}$$

The solution to the SDE (2.1) can be written as

$$X_t = \frac{\mu}{\theta}(1 - e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} d\tilde{L}_s =: X_1(t) + X_2(t)\,.$$

Here $X_2(t)$ satisfies

$$dX_2(t) = -\theta X_2(t)dt + d\tilde{L}_t\,.$$

It is easy to see from the expression of $X(t)$ that from $\theta > 0$, X_t is asymptotically stationary and ergodic and as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T X_1(t)dt \xrightarrow{a.s.} \mathbb{E}[X_1(\infty)] = \frac{\mu}{\theta}\,.$$

Also we have,

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T X_1^2(t)dt \xrightarrow{a.s.} \mathbb{E}[X_1^2(\infty)].$$

Since,

$$\begin{aligned} \mathbb{E}(X_t^2) &= \mathbb{E}(X_1^2(t)) + 2\mathbb{E}(X_1(t)X_2(t)) + \mathbb{E}(X_2^2(t)) \\ &= \frac{\mu^2}{\theta^2} + 0 + \int_0^t e^{-2\theta(t-s)}\nu ds, \end{aligned}$$

we get

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{T}\int_0^T X_1^2(t)dt &= \lim_{t\to\infty} \Big(\frac{\mu^2}{\theta^2} + 0 + \int_0^t e^{-2\theta(t-s)}\nu ds\Big) \\ &\xrightarrow{a.s.} \frac{\mu^2}{\theta^2} + \frac{\nu}{2\theta}, \end{aligned}$$

where $\nu = \lambda\mathbb{E}[Y_1]$ is the Lévy measure of L. Further, We can write

$$\begin{aligned}
X_t &= \frac{\mu}{\theta}(1-e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} d\tilde{L}_s \\
\int_0^T X_t dt &= \int_0^T \Big[\frac{\mu}{\theta}(1-e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} d\tilde{L}_s\Big] dt \\
&= \frac{\mu}{\theta}[T + \frac{e^{-\theta T}-1}{\theta}] + \int_0^T \int_0^t e^{-\theta t} e^{\theta u} d\tilde{L}_u dt \\
&= \frac{\mu}{\theta}[T + \frac{e^{-\theta T}-1}{\theta}] + \int_0^T \int_u^T e^{-\theta t} dt e^{\theta u} d\tilde{L}_u \\
&= \frac{\mu}{\theta}[T + \frac{e^{-\theta T}-1}{\theta}] + \int_0^T \frac{1}{\theta}(e^{-\theta u} - e^{-\theta T}) e^{\theta u} d\tilde{L}_u \\
&= \frac{\mu}{\theta}[T + \frac{e^{-\theta T}-1}{\theta}] + \frac{L_T}{\theta} - \int_0^T \frac{e^{-\theta T} e^{\theta u}}{\theta} d\tilde{L}_u \\
&= \frac{1}{\theta}\Big(\mu T - \frac{\mu}{\theta} + \tilde{L}_T\Big) + e^{-\theta T}\Big[\frac{\mu}{\theta^2} - \frac{\int_0^T e^{\theta u} d\tilde{L}_u}{\theta}\Big].
\end{aligned}$$

Therefore we can write the integral as

$$\int_0^T X_t dt = \frac{1}{\theta}\Big(\mu T - \frac{\mu}{\theta} + \tilde{L}_T\Big) + e^{-\theta T}\Big[\frac{\mu}{\theta^2} - \frac{\int_0^T e^{\theta u} d\tilde{L}_u}{\theta}\Big]. \tag{3.7}$$

We have

$$
\begin{aligned}
\hat{\theta}_T &= \frac{X_T \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{X_T \int_0^T X_t dt - T \int_0^T X_t [(\mu - \theta X_t) dt + d\tilde{L}_t]}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{X_T \int_0^T X_t dt - T\mu \int_0^T X_t dt + T\theta \int_0^T X_t^2 dt - T \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{T\theta \int_0^T X_t^2 dt - T \int_0^T X_t d\tilde{L}_t + X_T \int_0^T X_t dt - T\mu \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{T\theta \int_0^T X_t^2 dt - T \int_0^T X_t d\tilde{L}_t + (\frac{\mu}{\theta}(1 - e^{-\theta T}) + \int_0^t e^{-\theta(T-u)} d\tilde{L}_u) \int_0^T X_t dt - T\mu \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{T\theta \int_0^T X_t^2 dt - T \int_0^T X_t d\tilde{L}_t + (\frac{\mu}{\theta} - \mu T - \frac{\mu}{\theta} e^{-\theta T} + e^{-\theta T} \int_0^t e^{-\theta u} d\tilde{L}_u) \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}.
\end{aligned}
$$

From (3.7) we can write,

$$
\begin{aligned}
\hat{\theta}_T &= \frac{T\theta \int_0^T X_t^2 dt - T \int_0^T X_t d\tilde{L}_t + (\tilde{L}_T - \theta \int_0^T X_t dt) \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \theta + \frac{L_T \int_0^T X_t dt - T \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}.
\end{aligned}
$$

Similarly, we can evaluate $\hat{\mu}$ as,

$$\begin{aligned}
\hat{\mu}_T &= \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t[(\mu - \theta X_t)dt + d\tilde{L}_t]}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \frac{-\mu(\int_0^T X_t dt)^2 + \int_0^T X_t^2 dt[X_T + \theta \int_0^T X_t dt] - \int_0^T X_t d\tilde{L}_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \mu + \frac{\tilde{L}_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}.
\end{aligned}$$

This allows us to write $\hat{\theta}_T$ and $\hat{\mu}_T$ as the following

$$\begin{aligned}
\hat{\theta}_T &= \theta + \frac{\tilde{L}_T \int_0^T X_t dt - T \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \theta + \frac{\frac{\tilde{L}_T}{T} \frac{\int_0^T X_t dt}{T} - \frac{\int_0^T X_t d\tilde{L}_t}{T}}{\frac{\int_0^T X_t^2 dt}{T} - (\frac{\int_0^T X_t dt}{T})^2}. \\
\hat{\mu}_T &= \mu + \frac{\tilde{L}_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\
&= \mu + \frac{\frac{\tilde{L}_T}{T} \frac{\int_0^T X_t^2 dt}{T} - \frac{\int_0^T X_t dt}{T} \frac{\int_0^T X_t d\tilde{L}_t}{T}}{\frac{\int_0^T X_t^2 dt}{T} - (\frac{\int_0^T X_t dt}{T})^2}.
\end{aligned}$$

Clearly we know $\lim_{T\to\infty} \frac{\int_0^T X_t^2 dt}{T} < \infty$ and $\lim_{T\to\infty} \frac{\int_0^T X_t dt}{T} < \infty$

We want to find the limit of $\frac{\tilde{L}_T}{T}$ and $\frac{\int_0^T X_t d\tilde{L}_t}{T}$ as $t \to \infty$. To find the limit of $\frac{\tilde{L}_T}{T}$ and $\frac{\int_0^T X_t d\tilde{L}_t}{T}$ as $t \to \infty$, the following result is needed. Let K_t be the compensated Poisson stochastic integral process given by:

$$K_t = \int_0^t \int_{\mathbb{R}_0} g(s, y)(N(ds, dy) - \nu_s(dy)ds), \quad t \in \mathbb{R}^+$$

of the predictable integrand $g(s, y)$. Then the following extension of BDG inequality holds,

Lemma 4.3.1. *[35] Consider the compensated Poisson stochastic integral process $(K_t)_{t\in\mathbb{R}^+}$ of a predictable integrand $g(s,y)$. Then for all $p \geq 2$, we have*

$$E\left((K_t^*)^p\right) = \frac{2}{p}(40p)^{\frac{p}{2}}\Big(\frac{p^2 e}{2}\Big)^{p(\log_2 p)/2}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|g(s,y)|^p\nu_s(dy)ds\Big] + 2^p\sum_{k=1}^{[\log_2 p]-1}\frac{p^{pk}}{2^k}\Big(\frac{e}{2}\Big)^{\frac{kp}{2}}\mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}_0}(g(s,y))^{2^k}\nu_s(dy)ds\Big)^{\frac{p}{2^k}}\Big], \tag{3.8}$$

where

$$K_t^* = \sup_{s\in[0,t]}|K_s|.$$

The proof is given in Lemma 2.1[35]. The above results allow us to write

$$\mathbb{E}\sup_{0\leq s\leq t}|K_s|^p \leq C_p\mathbb{E}\int_0^t|g(s,y)|^p\nu(ds,dy) + \sum_{k=1}^{[\log_2 p]-1}C_{k,p}\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}(g(s,y)^{2^k}\nu(ds,dy)\right)^{p/2^k}\right].$$

Applying this inequality to $K_t = \tilde{L}_t = \int_0^t\int_{\mathbb{R}_0} y\tilde{N}(ds,dy)$ yields

$$\mathbb{E}\sup_{0\leq s\leq T}|K_s|^p \leq C_p\mathbb{E}\int_0^T\int_{\mathbb{R}_0}|y|^p\nu(ds,dy) + \sum_{k=1}^{[\log_2 p]-1}C_{k,p}\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0}(y^{2^k}\nu(ds,dy)\right)^{p/2^k}\right]$$

for any $p \geq 2$.

In case of the compensated double exponential compound Poisson Process $\hat{L}_t$ where $\hat{L}_t = L_t - \lambda t\mathbb{E}[Y]$, we have the Lévy measure $\nu(ds,dy) = ds\nu(dy) = ds\lambda\mathbb{E}[Y_1]$, and

$$\sup_{s\geq 0}\int_{\mathbb{R}_0}|y|^p\nu(dy) < \infty.$$

Then

$$\mathbb{E}\sup_{0\leq s\leq T}|K_s|^p \leq C_pT + \sum_{k=1}^{[\log_2 p]-1}C_{k,p}\left[T^{p/2^k}\right] \leq C_pT^{p/2},$$

when T is large. Now consider

$$\begin{aligned} P(K_n^*/n \geq n^{-\lambda}) &= P((K_n^*/n)^p \geq n^{-p\lambda}) \\ &\leq n^{p\lambda}\mathbb{E}\left[(K_n^*/n)^p\right] \\ &\leq n^{p\lambda} n^{-\frac{1}{2}p} = n^{-(\frac{1}{2}-\lambda)p} . \end{aligned}$$

So when $\lambda < 1/2$, we can choose p such that

$$\sum_{n=1}^{\infty} P(K_n^*/n \geq n^{-\lambda}) \leq \sum_{n=1}^{\infty} n^{-(\frac{1}{2}-\lambda)p} < \infty . \tag{3.9}$$

Then by Borel-Cantelli lemma we have

$$\lim_{n\to\infty} K_n^*/n = 0 .$$

Since

$$K_T \leq K_{[T]+1}^* ,$$

this implies

$$\lim_{T\to\infty} \frac{|K_T|}{T} \leq \lim_{T\to\infty} \frac{K_{[T]+1}^*}{[T]+1} \frac{[T]+1}{T} = 0 .$$

This gives us

$$\lim_{T\to\infty} \frac{\tilde{L}_T}{T} = 0.$$

Using Lemma 3.3.1 [35], we also have

$$\lim_{T\to\infty} \frac{\int_0^T X_t d\tilde{L}_t}{T} = 0.$$

Hence, we have,

$$\lim_{T\to\infty} \frac{\tilde{L}_T \int_0^T X_t dt - T \int_0^T X_t d\tilde{L}_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} = 0. \tag{3.10}$$

This gives us

$$\lim_{T\to\infty} \hat{\theta_T} \xrightarrow{a.s.} \theta \tag{3.11}$$

Similarly, since $\lim_{T\to\infty} \frac{\tilde{L}_T}{T} = 0$ and $\lim_{T\to\infty} \frac{\int_0^T X_t d\tilde{L}_t}{T} = 0$, we gave

$$\lim_{T\to\infty} \frac{\tilde{L}_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t d\tilde{L}_t}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} = 0 \tag{3.12}$$

which gives us

$$\lim_{T\to\infty} \hat{\mu_T} \xrightarrow{a.s.} \mu \tag{3.13}$$

Thus we have the following result,

Theorem 3. The estimators $\hat{\theta}_T$ and $\hat{\mu}_T$ given by

$$\hat{\theta}_T = \theta + \frac{\tilde{L}_T \int_0^T X_t dt - T\int_0^T X_t d\tilde{L}_t}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}$$

$$\hat{\mu}_T = \mu + \frac{\tilde{L}_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t d\tilde{L}_t}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}$$

converge $a.s.$ to θ and μ respectively as $T \to \infty$.

Chapter 5

General Product formula of multiple integrals of Lévy process

In this chapter, we derive a product formula for finitely many multiple stochastic integrals of Lévy process, expressed in terms of the associated Poisson random measure. The formula is compact. The proof is short and uses the exponential vectors and polarization techniques.

5.1 Introduction

Stochastic analysis of nonlinear functionals of Lévy processes (including Brownian motion and Poisson process) have been studied extensively and found many applications. There have been already many standard books on this topic [1, 8, 9]. In the analysis of nonlinear Wiener (Brownian) functional the Wiener-Itô chaos expansion to expand a nonlinear functional of Brownian motion into the sum of multiple Wiener-Itô integrals is a fundamental contribution to the field. The product formula to express the product of two (or more) multiple integrals as linear combinations of some other multiple integrals is one of the important tools ([3]). It plays an important role in stochastic analysis, e.g. Malliavin calculus ([3, 7]).

The product formula for two multiple integrals of Brownian motion is known since the work of [10, Section 4] and the general product formula can be found for instance in [3, chapter 5]. In this chapter we give a general formula for the product of m multiple integrals of the Poisson random measure associated with (purely jump) Lévy process. The formula is in a compact form and it reduced to the Shigekawa's formula when $m = 2$ and when the Lévy process is reduced to Brownian motion.

When $m = 2$ a similar formula was obtained in [4], where the multiple integrals is with respect to the Lévy process itself (ours is with respect to the associated Poisson random measure which has better properties). To obtain their formula in [4] Lee and Shih use white noise analysis framework. In this work, we only use the classical framework in hope that this work is accessible to a different spectrum of readers.

The product formula for multiple Wiener-Itô integrals of the Brownian motion plays an important role in many applications such as in U-statistics [5]. We hope similar things may happen. But we are not pursuing this goal in the current chapter. Our formula is for purely jump Lévy process. It can be combined with the classical formulas [3, 5, 7, 10] so that it holds for general Lévy process (including the continuous component).

This chapter is organized as follows. In Section 2, we recall some preliminaries on Lévy process, the associated Poisson random measure, multiple integrals. We also state our main result in this section. In Section 3, we give the proof of the formula.

5.2 Preliminary and main results

Let $T > 0$ be a positive number and let $\{\eta(t) = \eta(t, \omega), 0 \le t \le T\}$ be a Lévy process on some probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ satisfying the usual condition. This means that $\{\eta(t)\}$ has independent and stationary increment and the sample path is right continuous with left limit. Without loss of generality, we assume $\eta(0) = 0$. If the process $\eta(t)$ has all moments for any time index t, then presumably, one can use the polynomials of the process to approximate any nonlinear functional of the process $\{\eta(t), 0 \le t \le T\}$. However, it is more convenient to use the associated Poisson random measure to carry out the stochastic analysis of these nonlinear functionals.

The jump of the process η at time t is defined by

$$\Delta\eta(t) := \eta(t) - \eta(t-) \quad \text{if } \Delta\eta(t) \neq 0 \,.$$

Denote $\mathbb{R}_0 := \mathbb{R} \backslash \{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we then define the Poisson random measure $N : [0, T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with the Lévy process η by

$$N(t,U) := \sum_{0\le s\le t} \chi_U(\Delta\eta(s))\,, \tag{2.1}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of η is defined by

$$\nu(U) := \mathbb{E}[N(1,U)] \tag{2.2}$$

and the compensated jump measure $\tilde{N}$ is defined by

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt\,. \tag{2.3}$$

The stochastic integral $\int_{\mathbb{T}} f(s,z)\tilde{N}(ds,dz)$ is well-defined for a predictable process $f(s,z)$ such that $\int_{\mathbb{T}} \mathbb{E}|f(s,z)|^2\nu(dz)ds < \infty$, where and throughout this chapter we use $\mathbb{T}$ to represent the domain $[0,T]\times\mathbb{R}_0$ to simplify notation.

Let

$$\hat{L}^{2,n} := \left(L^2(\mathbb{T},\lambda\times\nu)\right)^{\otimes n} \subseteq L^2\left(\mathbb{T}^n,(\lambda\times\nu)^n\right)$$

be the space of symmetric, deterministic real functions f such that

$$\|f\|^2_{\hat{L}^{2,n}} = \int_{\mathbb{T}^n} f^2(t_1,z_1,\cdots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n) < \infty\,,$$

where $\lambda(dt) = dt$ is the Lebesgue measure. In the above the symmetry means that

$$\begin{aligned} &f(t_1,z_1,\cdots,t_i,z_i,\cdots,t_j,z_j,\cdots,t_n,z_n)\\ &\qquad = f(t_1,z_1,\cdots,t_j,z_j,\cdots,t_i,z_i,\cdots,t_n,z_n)\end{aligned}$$

for all $1\le i<j\le n$. For any $f\in\hat{L}^{2,n}$ the multiple Wiener-Itô integral

$$I_n(f) := \int_{\mathbb{T}^n} f(t_1,z_1,\cdots,t_n,z_n)\tilde{N}(dt_1,dz_1)\cdots\tilde{N}(dt_n,dz_n) \tag{2.4}$$

is well-defined. The importance of the introduction of the associated Poisson measure and the multiple Wiener-Itô integrals are in the following theorem which means that any square integrable nonlinear functional F of the Lévy process η can be expanded as sum of multiple Wiener-Itô integrals.

Theorem 5.2.1 (`Wiener-Itô chaos expansion for Lévy process`). *Let* $\mathcal{F}_T = \sigma(\eta(t), 0 \le t \le T)$ *be the* σ*-algebra generated by the* ***Lévy*** *process* η.

Let $F \in L^2(\Omega, \mathcal{F}_T, P)$ *be an* $\mathcal{F}_T$ *measurable square integrable random variable. Then* F *admits the following chaos expansion:*

$$F = \sum_{n=0}^{\infty} I_n(f_n)\,, \tag{2.5}$$

where $f_n \in \hat{L}^{2,n}, n = 1, 2, \cdots$ *and where we denote* $I_0(f_0) := f_0 = \mathbb{E}(F)$. *Moreover, we have*

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n\,!\|f_n\|^2_{\hat{L}^{2,n}}\,. \tag{2.6}$$

This chaos expansion theorem is one of the fundamental results in stochastic analysis of Lévy processes. It has been widely studied in particular when η is the Brownian motion (Wiener process). We refer to [18], [7], [8] and references therein for further reading.

To state our main result of this chapter, we need some notation. Fix a positive integer $m \ge 2$. Denote

$$\Upsilon \;=\; \Upsilon_m = \{\mathbf{i} = (i_1, \cdots, i_\alpha),\ \alpha = 2, \cdots, m\,,\ 1 \le i_1 < \cdots < i_\alpha \le m\} \tag{2.7}$$

where $\alpha = |\mathbf{i}|$ is the length of the multi-index $\mathbf{i}$ (we shall use α, β to denote a natural number). It is easy to see that the cardinality of Υ is $\kappa_m := 2^m - 1 - m$. Denote $\vec{\mathbf{i}} = (\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m})$, which is an unordered list of the elements of Υ, where $\mathbf{i}_\beta \in \Upsilon$. We use $\vec{l} = (l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}})$ to denote a multi-index of length κ_m associated with Υ, where $l_{\mathbf{i}_\alpha}$, $1 \le \alpha \le \kappa_m$ are nonnegative integers. $\vec{l}$ can be regarded as a function from Υ to $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. Denote

$$\begin{cases} \Omega = \left\{\vec{l}, \vec{n} : \Upsilon \to \mathbb{Z}_+\right\} \qquad \text{and for any } \vec{l},\ \vec{n} \in \Omega\,, \\ \chi(k, \vec{l}, \vec{n}) = \sum_{1 \le \alpha \le \kappa_m} \left[l_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}} + n_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}}\right]. \end{cases} \tag{2.8}$$

The above χ on the right hand side refers to the indicate function. Denote $\chi(\vec{l}, \vec{n}) = (\chi(1, \vec{l}, \vec{n}), \cdots, \chi(m, \vec{l}, \vec{n}))$. The conventional notations such as $|\vec{l}| = l_{\mathbf{i}_1} + \cdots + l_{\mathbf{i}_{\kappa_m}}$;

$\vec{l}! = l_{\mathbf{i}_1}! \cdots l_{\mathbf{i}_{\kappa_m}}!$ and so on are in use. Notice that we use $l_{\mathbf{i}_1}$ instead of l_1 to emphasize that the $l_{\mathbf{i}_1}$ corresponds to $\mathbf{i}_1$. For $\mathbf{i} = (i_1, \cdots, i_\alpha), \mathbf{j} = (j_1, \cdots, j_\beta) \in \Upsilon$, and non negative integers μ and ν denote

$$\begin{aligned} \hat{\otimes}_{\mathbf{i}}^{\mu}(f_1, \cdots, f_m) &= \int_{([0,T]\times\mathbb{R}_0)^\mu} f_{i_1}((s_1, z_1), \cdots, (s_\mu, z_\mu), \cdots) \hat{\otimes} \cdots \\ &\qquad \hat{\otimes} f_{i_\alpha}((s_1, z_1), \cdots, (s_\mu, z_\mu), \cdots) ds_1 \nu(dz_1) \cdots \\ &\qquad ds_\mu \nu(dz_\mu)\ f_1 \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{i_\alpha} \cdots \hat{\otimes} f_m , \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} V_{\mathbf{j}}^{\nu}(f_1, \cdots, f_m) &= f_{j_1}((s_1, z_1), \cdots, (s_\nu, z_\nu), \cdots) \hat{\otimes} \cdots \\ &\quad \hat{\otimes} f_{j_\beta}((s_1, z_1), \cdots, (s_\nu, z_\nu), \cdots) f_1 \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{j_1} \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{j_\beta} \cdots \hat{\otimes} f_m , \end{aligned} \tag{2.10}$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\hat{f}_{j_1}$ means that the function f_{j_1} is removed from the list. Let us emphasize that both $\hat{\otimes}_{\mathbf{i}}^{\mu}$ and $V_{\mathbf{j}}^{\nu}$ are well-defined when the lengths of $\mathbf{i}$ and $\mathbf{j}$ are one. However, we shall not use $\hat{\otimes}_{\mathbf{i}}^{\mu}$ when $|\mathbf{i}| = 1$ and when $|\mathbf{j}| = 1$, $V_{\mathbf{j}}^{\nu}(f_1, \cdots, f_m) = f_1 \hat{\otimes} \cdots \hat{\otimes} f_m$ (namely, the identity operator). For any two elements $\vec{l} = (l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}})$ and $\vec{n} = (\mu_{\mathbf{j}_1}, \cdots, \mu_{\mathbf{j}_{\kappa_m}})$ in Ω, denote

$$\begin{aligned} \hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}} &= \hat{\otimes}_{\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}}} = \hat{\otimes}_{\mathbf{i}_1}^{l_{\mathbf{i}_1}} \cdots \hat{\otimes}_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_{\kappa_m}}} , \\ V_{\vec{\mathbf{j}}}^{\vec{n}} &= V_{\mathbf{j}_1, \cdots, \mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1}, \cdots, \mu_{j_{\kappa_m}}} = V_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} V_{\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_{\kappa_m}}} . \end{aligned} \tag{2.11}$$

Now we can state the main result of the chapter.

Theorem 5.2.2. *Let $q_1, \cdots, q_m$ be positive integers greater than or equal to 1. Let*

$$f_k \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} q_k} , \quad k = 1, \cdots, m .$$

Then

$$\prod_{k=1}^{m} I_{q_k}(f_k) = \sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\le q_1 \\ \cdots \\ \chi(m,\vec{l},\vec{n})\le q_m}} \frac{\prod_{k=1}^{m} q_k!}{\prod_{\alpha=1}^{\kappa_m} l_{\mathbf{i}_\alpha}! \prod_{\beta=1}^{\kappa_m} \mu_{\mathbf{j}_\beta}! \prod_{k=1}^{m}(q_k - \chi(k,\vec{l},\vec{n}))!}$$

$$I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1,\cdots\,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{j_{\kappa_m}}}(f_1,\cdots,f_m))\,, \tag{2.12}$$

where we recall

$$|q| = q_1 + \cdots + q_m \quad \text{and} \quad |\chi(\vec{l},\vec{n})| = \chi(1,\vec{l},\vec{n}) + \cdots + \chi(m,\vec{l},\vec{n})\,.$$

Remark 5.2.3. The above formula looks sophisticated and it may be understood in the following manner. There are two types of contraction operation involved in the above formula. The first one is the *integration contraction*: we choose certain subset of functions $f_{i_1},\cdots,f_{i_\alpha}$ and we choose μ variables (throughout the chapter for simplicity we call a pair (s,z) as one variable) in each of these chosen functions and set them to be the same: $(s_1,z_1),\cdots,(s_\mu,z_\mu)$ and we integrate with respect to these variables (with respect to the product measure of $ds\nu(dz)$) as in (2.9). The second one is the *simple contraction* without integration: we also choose certain subset of functions $f_{j_1},\cdots,f_{j_\beta}$ and let ν variables in all of these functions be the same: $(s_1,z_1),\cdots,(s_\nu,z_\nu)$, as in (2.10). We just concatenate the remaining variables: The concatenation of function $g_1(x_1,\cdots,x_{n_1})$, $\cdots$, $g_m(x_1,\cdots,x_{n_m})$ means

$$g_1(x_{1,1},\cdots,x_{1,n_1})\cdots g_m(x_{m,1},\cdots,x_{m,n_m})\,.$$

All the variables not integrated out with respect to $ds\nu(dz)$ will be integrated with respect to the Poisson random measure. The summation in the formula (2.12) is over all the possible two contraction operations. See the following examples 5.2.5-5.2.6 for more explanation.

Remark 5.2.4. If the index $\vec{n}$ does not appear, then there will be no operator V. In this case the formula (2.12) becomes [18, formula 5.3.5], which is the product formula for finitely many multiple integrals of Brownian motion.

Example 5.2.5. If $m = 2$, then $\kappa_m = 2^2 - 1 - 2 = 1$. To shorten the notations we can write $q_1 = n$, $q_2 = m$, $f_1 = f_n$, $f_2 = g_m$, $l_{\alpha_1} = l$, $n_{\beta_1} = k$. Thus, $\chi(1, \vec{l}, \vec{n}) = \chi(2, \vec{l}, \vec{n}) = l + k$ and $|q| + |\vec{n}| - |\chi(\vec{l}, \vec{n})| = n + m + k - 2(l + k) = n + m - 2l - k$. Hence the formula (2.12) becomes the following. If

$$f_n \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} n}$$

and

$$g_m \in \left(L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz))\right)^{\hat{\otimes} m} ,$$

then

$$I_n(f_n) I_m(g_m) = \sum_{\substack{k,l \in \mathbb{Z}_+ \\ k+l \le m \wedge n}} \frac{n!m!}{l!k!(n-k-l)!(m-k-l)!} I_{n+m-2l-k}\left(f_n \hat{\otimes}_{k,l} g_m\right) ,$$

where $\mathbb{Z}_+$ denotes the set of non negative integers and

$$\begin{aligned} & f_n \hat{\otimes}_{k,l} g_m(s_1, z_1, \cdots, s_{n+m-k-2l}, z_{n+m-k-2l}) \\ & = \text{symmetrization of } \int_{\mathbb{T}^l} f_n(s_1, z_1, \cdots, s_{n-l}, z_{n-l}, t_1, y_1, \cdots, t_l, y_l) \\ & \qquad g_m(s_1, z_1, \cdots, s_k, z_k, s_{n-l+1}, \cdots, z_{n-l+1}, \cdots, \\ & \qquad s_{n+m-k-2l}, z_{n+m-k-2l}, t_1, z_1, \cdots, t_l, z_l) dt_1 \nu(dz_1) \cdots dt_l \nu(dz_l). \end{aligned} \tag{2.13}$$

Example 5.2.6. If $m = 3$, then $\kappa_m = 2^3 - 1 - 3 = 4$. The set

$$\Upsilon_3 = \{\mathbf{i}_1 = (1,2), \mathbf{i}_2 = (2,3), \mathbf{i}_3 = (1,3), \mathbf{i}_4 = (1,2,3)\}$$

We also write

$$l_{\mathbf{i}_1} = l_{12}\,,\ l_{\mathbf{i}_2} = l_{23}\,,\ l_{\mathbf{i}_3} = l_{13}\,,\ l_{\mathbf{i}_4} = l_{123}\,,$$
$$\mu_{\mathbf{j}_1} = k_{12}\,, \mu_{\mathbf{j}_2} = k_{23}\,, \mu_{\mathbf{j}_3} = k_{13}\,, \mu_{\mathbf{j}_4} = k_{123}\,.$$

Thus,

$$\begin{aligned}
\chi(1,\vec{l},\vec{n}) &= l_{12} + l_{13} + l_{123} + k_{12} + k_{13} + k_{123}\,,\\
\chi(2,\vec{l},\vec{n}) &= l_{12} + l_{23} + l_{123} + k_{12} + k_{23} + k_{123}\,,\\
\chi(3,\vec{l},\vec{n}) &= l_{13} + l_{23} + l_{123} + k_{13} + k_{23} + k_{123}\,,
\end{aligned}$$

and

$$\begin{aligned}
|q| + |\vec{n}| - |\chi(\vec{l},\vec{n})| &= q_1 + q_2 + q_3 - 2l_{12} - 2l_{23} - 2l_{13} - 3l_{123}\\
&\qquad -k_{12} - k_{23} - k_{13} - 2k_{123}\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&I_{q_1}(f_1)I_{q_2}(f_2)I_{q_3}(f_3) \qquad (2.14)\\
&= \sum_{\substack{l_{ij},k_{ij}\ge 0\\ \chi(i,\vec{l},\vec{n})\le q_i\,,i=1,2,3}} \frac{q_1!q_2!q_3!}{l_{12}!l_{13}!l_{23}!l_{123}!k_{12}!k_{13}!k_{23}!k_{123}!\prod_{r=1}^{3}(q_i-\chi(i,\vec{l},\vec{n}))!}\\
&\qquad \cdot I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}\Big(\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\ \hat{\otimes}\, V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3)\Big)\,. \qquad (2.15)
\end{aligned}$$

The above contraction operator $\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\ \hat{\otimes}\, V_{\vec{\mathbf{j}}}^{\vec{n}}$ is given as follows:

$$\begin{aligned}
&\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\ \hat{\otimes}\, V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3)(\vec{s_1},\vec{s_2},\vec{s_3},\vec{s}_{12},\vec{s}_{13},\vec{s}_{23},\vec{s}_{123})\\
&\quad = \text{symmetrization of } \int_{\mathbb{T}^{|l|}} f_1(\vec{r}_{12},\vec{r}_{13},\vec{r}_{123},\vec{s}_{12},\vec{s}_{13},\vec{s}_{123},\vec{s_1})\\
&\qquad f_2(\vec{r}_{12},\vec{r}_{23},\vec{r}_{123},\vec{s}_{12},\vec{s}_{23},\vec{s}_{123},\vec{s_2})f_3(\vec{r}_{13},\vec{r}_{23},\vec{r}_{123},\vec{s}_{13},\vec{s}_{23},\vec{s}_{123},\vec{s_3})\\
&\qquad \nu(d\vec{r}_{12})\nu(d\vec{r}_{13})\nu(d\vec{r}_{23})\nu(d\vec{r}_{123})\,, \qquad (2.16)
\end{aligned}$$

where (denoting $|l| = l_{12} + l_{13} + l_{23} + l_{123}$ and $|k| = k_{12} + k_{13} + k_{23} + k_{123}$)

$$\begin{aligned}
\vec{r}_{12} &= ((s_1, z_1), \cdots, (s_{l_{12}}, z_{l_{12}})) , \\
\vec{r}_{13} &= \left((s_{l_{12}+1}, z_{l_{12}+1}), \cdots, (s_{l_{12}+l_{13}}, z_{l_{12}+l_{13}})\right) , \\
\vec{r}_{23} &= \left((s_{l_{12}+l_{13}+1}, z_{l_{12}+l_{13}+1}), \cdots, (s_{l_{12}+l_{13}+l_{23}}, z_{l_{12}+l_{13}+l_{23}})\right) , \\
\vec{r}_{123} &= \left((s_{l_{12}+l_{13}+l_{23}+1}, z_{l_{12}+l_{13}+l_{23}+1}), \cdots, (s_{|l|}, z_{|l|})\right) , \\
\vec{s}_{12} &= \left((s_{|l|+1}, s_{|l|+1}), \cdots, (s_{|l|+k_{12}}, z_{|l|+k_{12}})\right) , \\
\vec{s}_{13} &= \left((s_{|l|+k_{12}+1}, z_{|l|+k_{12}+1}), \cdots, (s_{|l|+k_{12}+k_{13}}, z_{|l|+k_{12}+k_{13}})\right) , \\
\vec{s}_{23} &= \left((s_{|l|+k_{12}+k_{13}+1}, z_{|l|+k_{12}+k_{13}+1}), \cdots, \right. \\
&\qquad \left. (s_{|l|+k_{12}+k_{13}+k_{23}}, z_{|l|+k_{12}+k_{13}+k_{23}})\right) , \\
\vec{s}_{123} &= \left((s_{|l|+k_{12}+k_{13}+k_{23}+1}, z_{|l|+k_{12}+k_{13}+k_{23}+1}), \cdots, (s_{|l|+|k|}, z_{|l|+|k|})\right) ;
\end{aligned}$$

for $i = 1, 2, 3$, $\vec{s_i}$ represents the remaining variables in f_i and there are $q_i - \chi(i, \vec{l}, \vec{n})$ variables (we count every pair (s, z) as one variable) in $\vec{s}_i$. In (2.16), the variables marked as $\vec{r}$ are integrated out. The total number of variables appeared in all $\vec{s}$ is

$$\begin{aligned}
|q| + |\vec{n}| - |\chi(\vec{l}, \vec{n})| &= q_1 + q_2 + q_3 - 2l_{12} - 2l_{23} - 2l_{13} - 3l_{123} \\
&\quad -k_{12} - k_{23} - k_{13} - 2k_{123}
\end{aligned}$$

and they will be integrated with respect to the Poisson random measure as a multiple integral.

Remark 5.2.7. When η is the Brownian motion, the product formula (2.13) is known since [10] (see e.g. [18, Theorem 5.6] for a formula of the general form (2.13)) and is given by

$$I_n(f_n) I_m(g_m) = \sum_{l=0}^{n \wedge m} \frac{n! m!}{l!(n-l)!(m-l)!} I_{n+m-2l}\left(f_n \hat{\otimes}_l g_m\right) . \tag{2.17}$$

It is a "special case" of (2.12) when $k = 0$.

5.3 Proof of Theorem 5.2.2

We shall prove the main result (Theorem 5.2.2) of this chapter. We shall prove this by using the polarization technique (see [18, Section 5.2]). First, let us find the Wiener-Itô chaos expansion for the *exponential functional* (random variable) of the form

$$\begin{aligned} Y(T) &= \mathcal{E}(\rho(s,z)) \\ &:= \exp\left\{\int_{\mathbb{T}} \rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}} \left(e^{\rho(s,z)} - 1 - \rho(s,z)\right)\nu(dz)ds\right\} \end{aligned} \tag{3.1}$$

where $\rho(s,z) \in \hat{L}^2 := \hat{L}^{2,1} = L^2(\mathbb{T}, \nu(dz) \otimes \lambda(dt))$. An application of Itô formula (see e.g. [8]) yields

$$Y(T) = 1 + \int_0^T \int_{\mathbb{R}_0} Y(s-)\Big[\exp\left(\rho(s,z)\right) - 1\Big]\tilde{N}(ds,dz)\,.$$

Repeatedly using this formula, we obtain the chaos expansion of $Y(T)$ as follows.

$$\begin{aligned} \mathcal{E}(\rho(s,z)) &= \exp\left\{\int_{\mathbb{T}} \rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}} \left(e^{\rho(s,z)} - 1 - \rho(s,z)\right)\nu(dz)ds\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n)\,, \end{aligned} \tag{3.2}$$

where the convergence is in $L^2(\Omega, \mathcal{F}_T, P)$ and

$$f_n = f_n(s_1, z_1, \cdots, s_n, z_n) = (e^{\rho} - 1)^{\hat{\otimes} n} = \prod_{i=1}^{n} \left(e^{\rho(s_i,z_i)} - 1\right). \tag{3.3}$$

We shall first make critical application of the above expansion formula (3.2)-(3.3). For any functions $p_k(s,z) \in \hat{L}^2$ (in what follows when we write k we always mean $k = 1, 2, \cdots, m$ and we shall omit $k = 1, 2, \cdots, m$), we denote

$$\rho_k(u_k, s, z) = \log(1 + u_k p_k(s,z))\,, \tag{3.4}$$

From (3.2)-(3.3), we have (consider u_k as fixed real numbers)

$$\mathcal{E}(\rho_k(u_k, s, z)) = \sum_{n=0}^{\infty} \frac{1}{n!} u_k^n I_n(f_{k,n}), \tag{3.5}$$

where

$$f_{k,n} = \frac{1}{u_k^n} \prod_{i=0}^{n} (e^{\rho_k(u_k, s_i, z_i)} - 1) = p_k^{\otimes n} = \prod_{i=1}^{n} p_k(s_i, z_i) \tag{3.6}$$

It is clear that

$$\prod_{k=1}^{m} \mathcal{E}(\rho_k(u_k, s, z)) = \sum_{q_1, \cdots, q_m = 0}^{\infty} \frac{1}{q_1! \cdots q_m!} u_1^{q_1} \cdots u_m^{q_m} I_{q_1}(f_{1,q_1}) \cdots I_{q_m}(f_{m,q_m}) \tag{3.7}$$

where f_{k,q_k}, $k = 1, \cdots, m$ are defined by (3.6). On the other hand, from the definition of the exponential functional (3.1), we have

$$\begin{aligned}
&\prod_{k=1}^{m} \mathcal{E}(\rho_k(u_k, s, z)) \\
&= \prod_{k=1}^{m} \exp\Big\{ \int_{\mathbb{T}} \rho_k(u_k, s, z) \tilde{N}(dz, ds) \Big\} \qquad (3.8) \\
&\exp\Big\{ - \int_{\mathbb{T}} \Big(e^{\rho_k(u_k, s, z)} - 1 - \rho_k(u_k, s, z) \Big) \nu(dz) ds \Big\} \\
&= \exp\Big\{ \int_{\mathbb{T}} \sum_{k=1}^{m} \rho_k(u_k, s, z) \tilde{N}(dz, ds) \\
&- \int_{\mathbb{T}} \Big(e^{\sum_{k=1}^{m} \rho_k(u_k, s, z)} - 1 - \sum_{k=1}^{m} \rho_k(u_k, s, z) \Big) \nu(dz) ds \Big\} \\
&\cdot \exp\Big\{ \int_{\mathbb{T}} e^{\sum_{k=1}^{m} \rho_k(u_k, s, z)} - \sum_{k=1}^{m} e^{\rho_k(u_k, s, z)} + m - 1 \Big) \nu(dz) ds \Big\} \\
&=: A \cdot B \qquad (3.9)
\end{aligned}$$

where A and B denote the above first and second exponential factors.

The first exponential factor A is an exponential functional of the form (3.1). Thus, again by the chaos expansion formula (3.2)-(3.3), we have

$$A = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(u_1, \cdots, u_m)), \tag{3.10}$$

where

$$h_n(u_1, \cdots, u_m) = \prod_{i=0}^{n} (e^{\sum_{k=1}^{m} \rho_k(u_k, s_i, z_i)} - 1). \tag{3.11}$$

By the definition of q_k, we have

$$\sum_{k=1}^{m} \rho_k(u_k, s_i, z_i) \quad = \quad \log \prod_{k=1}^{m} (1 + u_k p_k(s_i, z_i)).$$

Or

$$\begin{aligned} h_n(u_1, \cdots, u_m) \quad &= \quad \left(\left[\prod_{k=1}^{m} (1 + u_k p_k) - 1 \right] \right)^{\hat{\otimes} n} \\ &= \quad \mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)} \prod_{i=1}^{n} \left[\prod_{k=1}^{m} (1 + u_k p_k(s_i, z_i)) - 1 \right], \end{aligned}$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)}$ denotes the symmetriization with respect to $(s_1, z_1), \cdots, (s_n, z_n)$. Define

$$S = \{\mathbf{j} = (j_1, \cdots, j_\beta),\ \beta = 1, \cdots, m,\ 1 \le j_1 < \cdots < j_\beta \le m\}.$$

The cardinality of S is $|S| = \tilde{\kappa}_m := 2^m - 1$. We shall freely use the notations introduced in Section 2. Denote also

$$u_{\mathbf{j}} = u_{j_1} \cdots u_{j_\beta}, \quad p_{\mathbf{j}}(s, z) = p_{j_1}(s, z) \cdots p_{j_\beta}(s, z) \quad (\text{for } \mathbf{j} = (j_1, \cdots, j_\beta) \in S).$$

We have

$$h_n(u_1,\cdots,u_m)=\left(\sum_{\mathbf{j}\in S}u_{\mathbf{j}}p_{\mathbf{j}}\right)^{\hat{\otimes}n}=\sum_{|\vec{\mu}|=n}\frac{|\vec{\mu}|!}{\vec{\mu}!}u_{\vec{\mathbf{j}}}^{\vec{\mu}}p_{\vec{\mathbf{j}}}^{\hat{\otimes}\vec{\mu}}$$

$$=\sum_{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n}\frac{n!}{\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde{\kappa}_m}}!}u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes}p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}},$$

where $\vec{\mu}:S\to\mathbb{Z}_+$ is a multi-index and we used the notation

$$u_{\vec{\mathbf{j}}}^{\vec{\mu}}=u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}};\qquad p_{\vec{\mathbf{j}}}^{\hat{\otimes}\vec{\mu}}=p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes}p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}.$$

Inserting the above expression into (3.10) we have

$$A=\sum_{n=0}^{\infty}\sum_{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n}\frac{1}{\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde{\kappa}_m}}!}u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}I_n(p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes}p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}})\tag{3.12}$$

Now we consider the second exponential factor in (3.9):

$$B=\exp\left\{\int_{\mathbb{T}}\left(e^{\sum_{k=1}^m\rho_k(u_k,s,z)}-\sum_{k=1}^m e^{\rho_k(u_k,s,z)}+m-1\right)\nu(dz)ds\right\}$$

$$=\exp\left\{\sum_{\mathbf{i}\in\Upsilon}u_{\mathbf{i}}\int_{\mathbb{T}}p_{\mathbf{i}}(s,z)\nu(dz)ds\right\},$$

where Υ is defined by (2.7) (which is a subset of S such that $|\mathbf{j}|\geq 2$). Thus,

$$B=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{\mathbf{i}\in\Upsilon}u_{\mathbf{i}}\int_{\mathbb{T}}p_{\mathbf{i}}(s,z)\nu(dz)ds\right)^n$$

$$=\sum_{n=0}^{\infty}\sum_{l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa_m}}=n}\frac{1}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!}u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}}\cdots u_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_{\kappa_m}}}\left(\int_{\mathbb{T}}p_{\mathbf{i}_1}(s,z)\nu(dz)ds\right)^{l_{\mathbf{i}_1}}$$

$$\cdots\left(\int_{\mathbb{T}}p_{\mathbf{i}_{\kappa_m}}(s,z)\nu(dz)ds\right)^{l_{\mathbf{i}_{\kappa_m}}},\tag{3.13}$$

where $\vec{l} \in \Omega$ is a multi-index. Combining (3.12)-(3.13), we have

$$AB = \sum_{n,\tilde{n}=0}^{\infty} \sum_{\substack{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n \\ l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa_m}}\tilde{n}}} \frac{1}{\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde{\kappa}_m}}!l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}$$
$$u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}}\cdots u_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}\kappa_m}} B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}}\,, \tag{3.14}$$

where

$$B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}} := \left(\int_{\mathbb{T}} p_{\mathbf{i}_1}(s,z)\nu(dz)ds\right)^{l_{\mathbf{i}_1}}\cdots$$
$$\left(\int_{\mathbb{T}} p_{\mathbf{i}_{\kappa_m}}(s,z)\nu(dz)ds\right)^{l_{\mathbf{i}_{\kappa_m}}} I_n(p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes}p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa}_m}}})\,. \tag{3.15}$$

To get an expression for $B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}}$ we use the notations (2.9)-(2.10) and (2.11). Then

$$B_{\mathbf{j},\tilde{\mathbf{j}},n_{\mathbf{j}},\tilde{n}_{\mathbf{j}}} = I_n(\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\hat{\otimes} V_{\vec{\mathbf{j}}}^{\vec{\mu}}(p_1^{\otimes n_{\mathbf{i}_1}},\cdots,p_m^{\otimes n_m}))\,. \tag{3.16}$$

To compare the coefficients of $u_1^{n_{\mathbf{i}_1}}\cdots u_m^{n_m}$, we need to express the right hand side of (3.14) as a power series of $u_1,\cdots,u_m$. For $k=1,\cdots,m$ denote

$$\tilde{\chi}(k,\vec{l},\vec{\mu}) = \sum_{1\le\alpha\le\kappa_m} l_{\mathbf{i}_\alpha} I_{\{\mathbf{i}_\alpha \text{ contains } k\}} + \sum_{1\le\beta\le\tilde{\kappa}_m} \mu_{\mathbf{j}_\beta} I_{\{\mathbf{j}_\beta \text{ contains } k\}}\,. \tag{3.17}$$

Combining (3.9), (3.14) and (3.16), we have

$$\sum_{q_1,\cdots,q_m=0}^{\infty} \frac{u_1^{q_1}\cdots u_m^{q_m}}{q_1!\cdots q_m!} I_{q_1}(p_1^{\otimes q_1})\cdots I_{q_m}(p_m^{\otimes q_m})$$
$$= \sum_{n,\tilde{n}=0}^{\infty} \sum_{\substack{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa}_m}}=n \\ l_{\mathbf{i}_1}+\cdots+l_{\kappa_m}=\tilde{n} \\ \tilde{\chi}(k,\vec{l},\vec{\mu})=q_k,k=1,\ldots,m}} \frac{u_1^{q_1}\cdots u_m^{q_m}}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde{\kappa}_m}}!}$$
$$I_n(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}}\hat{\otimes} V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}(p_1^{\otimes q_1},\cdots,p_m^{\otimes q_m}))\,. \tag{3.18}$$

Comparing the coefficient of $u_1^{q_1}\cdots u_m^{q_m}$, we have

$$\prod_{k=1}^m I_{q_k}(p_k^{\otimes q_k}) = \sum_{\substack{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde\kappa_m}\in S\\ \mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}\in\Upsilon}} \sum_{\tilde\chi(k,\vec l,\vec\mu)=q_k,k=1,\dots,m} \frac{q_1!\cdots q_m!}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa_m}}!\mu_{\mathbf{j}_1}!\cdots\mu_{\mathbf{j}_{\tilde\kappa_m}}!}$$
$$I_n(\hat\otimes_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}}\hat\otimes V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\tilde\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\tilde\kappa_m}}}(p_1^{\otimes q_1},\cdots,p_m^{\otimes q_m}))\,. \tag{3.19}$$

Notice that when $|\mathbf{j}|=1$, namely, $\mathbf{j}=(k), k=1,\cdots,m$, then $V_{\mathbf{j}}^{\mu}(f_1,\cdots,f_m)=f_1\hat\otimes\cdots\hat\otimes f_m$. We can separate these terms from the remaining ones, which will satisfy $|\mathbf{j}|\ge 2$. Thus, the remaining multi-indices $\mathbf{j}$'s consists of the set Υ. We can write a multi-index $\vec\mu : S\to\mathbb{Z}_+$ as $\vec\mu=(n_{(1)},\cdots,n_{(m)},\vec n)$, where $\vec n\in\Upsilon$. We also observe $q_k=\tilde\chi(k,\vec l,\vec\mu)=n_{(k)}+\chi(k,\vec l,\vec n)$. After replacing $\vec\mu$ by $\vec n$, (3.19) gives (2.12). This proves Theorem 5.2.2 for $f_k=p_k^{\otimes q_k}$, $k=1,\cdots,m$. By polarization technique (see e.g. [18, Section 5.2]), we also know the identity (2.12) holds true for $f_k=p_{k,1}\otimes\cdots\otimes p_{k,q_k}$, $p_{k,q_k}\in L^2([0,T]\times\mathbb{R}_0,ds\times\nu(dz))$, $k=1,\cdots,m$. Because both sides of (2.12) are multi-linear with respect to f_k, we know (2.12) holds true for

$$f_k=\sum_{\ell=1}^{\nu_k} c_{k,\ell}p_{k,1,\ell}\otimes\cdots\otimes p_{k,q_k,\ell}\,,\qquad k=1,\cdots,m\,,$$

where $c_{k,\ell}$ are constants, $p_{k,k',\ell}\in L^2([0,T]\times\mathbb{R}_0,ds\times\nu(dz))$, $k=1,\cdots,m,k'=1,\cdots,q_k$ and $\ell=1,\cdots,\nu_k$. Finally, the identity (2.12) is proved by a routine limiting argument.

Chapter 6

Application of double exponential Ornstein Uhlenbeck Process in Finance

The Orntein Uhlenbeck process has been introduced as a more sophisticated model for Brownian Motion that captures the effect of friction in the motion (Uhlenbeck and Orntein 1930)[46]. This process has since been widely used in evolutionary biology, physics, and finance.
In this chapter, we briefly mention some other applications of the Ornstein Uhlenbeck process driven by the double exponential compound Poisson process[refer to Chapters 3 and 4] in the area of Finance.

6.1 Pair Trading strategy with OU process

A pair trade refers to a trade that consists of matching a long position with a short position for two stocks that have a high correlation value. Pair trading was introduced by research scientists in Morgan Stanley [37]. The idea of using this strategy is to reduce the overall exposure to the market risk. For more applications and studies of pair trading strategy, refer to [38]. In this section, we will briefly discuss the application of the double exponential Ornstein Uhlenbeck process in the modeling of pair trading strategy.

Given two stocks U and V with prices $S_U(t)$ and $S_V(t)$ respectively, the spread is given by

$$X_t = \ln\left(\frac{S_U(t)}{S_U(0)}\right) - \ln\left(\frac{S_V(t)}{S_V(0)}\right), \quad t \geq 0\,. \tag{1.1}$$

This spread dynamics can be modelled using double-exponential Ornstein Uhlenbeck process where the process X_t is given by the following Langevin equation :

$$dX_t = (\mu - \theta X_t)dt + dZ_t\,, \quad t \in [0,\infty), \quad X_0 = x\,. \tag{1.2}$$

with mean reversion speed $\theta \in \mathbb{R}$ and mean-reversion level $\mu \in \mathbb{R}$. The solution to the SDE (1.2) can be written as

$$X_t = xe^{-\theta t} + \frac{\mu}{\theta}(1 - e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} dZ_s.$$

Details on double exponential and hyper-exponential jump-diffusion processes have been discussed in [40], [17]. Here the double-exponential Ornstein Uhlenbeck process is considered due to its effectiveness in approximating the stock data better since it also captures the sudden jumps in stock prices.

The stochastic process which is a double-exponential compound Poisson process is given by

$$Z_t = \sum_{i=1}^{N_t} Y_i\,,$$

where $(Y_n, n \geq 1)$ is a sequence of independent real-valued random variables with distribution f given by

$$f_Y(x) = p\eta e^{-\eta x} I_{[x\geq 0]} + q\varphi e^{\varphi x} I_{[x<0]}\,, \tag{1.3}$$

where the parameters p, q, η, φ are positive and $p + q = 1$. Here N_t is the Poisson process with rate $\lambda > 0$, independent of $\{Y_i, i = 1, 2, \ldots\}$. The constants p and q are the probabilities associated with an upward and downward jump respectively.
The moment-generating function of double exponential jump Y is given by

$$\mathbb{E}[e^{uY}] = p\frac{\eta}{\eta - s} + q\frac{\varphi}{\varphi + s}.$$

The moment-generating function of the compound Poisson process is given by

$$\mathbb{E}[e^{uZ_t}] = e^{\lambda t(p\frac{\eta}{\eta - s} + q\frac{\varphi}{\varphi + s} - 1)}\,. \tag{1.4}$$

The goal is to maximize the expected return in a unit time. To do so, consider the entry and exit points a and $b(a, b \in \mathbb{R})$ and it is safe to assume that $a < b$ W.L.O.G. The first passage time $\tau_{b,a}$ is defined as the time over which the return takes place, i.e,

$$\tau_{b,a} = \inf\{t \geq 0 | X_t \geq b\}$$

for $X_0 = a$. The entry and exit of the position is made when the spread crosses the threshold. This implies that the trading party enters at $t_0 = \inf\{t \geq 0|X_t \leq a\}$ and exits for the first time when $t_1 = \inf\{t \geq 0|X_t \geq b\}$. Let R be the function of the return that depends on the entry and exit signal point a and b and also the cost of transaction $c \geq 0$. This gives an optimization problem

$$\max_{a,b} \frac{R(b, a, c)}{\mathbb{E}[\tau_{b,a}]}\,. \tag{1.5}$$

This leads to the problem of approximation the distribution of the first passage time $\tau_{b,a}$. The analytical solutions of the first passage time has been discussed in [41].
To calculate the first passage time on real stock data, we can fit the Ornstein-Uhlenbeck process driven by a double-exponential jump by looking at the time series historical data of the pairs of stocks. Applying parameter estimation techniques from Chapter 3 and Chapter 4 can help to estimate the parameters μ and θ by observing the real data at discrete time events and can be used further for benchmarking the model and backtesting.

6.2 OU short rate process

Modeling stochastic interest rates is very important for the banks. There have been extensive studies and literature available that discuss and bifurcate between the forward rate models and short rate models [refer to [42],[43]].
For modeling short rates, the more extensively used models are the Vasicek Model, the Hull-White model, and Cox-Ingersoll Ros(CIR)[44]. In this section, the short rate process modeled by the double exponential Ornstein Uhlenbeck process is discussed.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a right continuous family of increasing σ-algebras $(\mathcal{F}_t, t \geq 0)$ satisfying the usual condition ([19]). Let $\mathbb{Q}$ be a risk-neutral probability measure and $T > 0$ be fixed time. Then the short rate process $\{R_t\}_{t\in[0,T]}$ is given by

$$R_t = a(t) + \sum_{k=1}^{n} X_t^k \tag{2.1}$$

where $a(t)$ if a differentiable real-valued function and X_t^k is modelled using double-exponential Ornstein Uhlenbeck process given by :

$$dX_t^k = -\theta_k X_t^k dt + \sigma_k dZ_t^k\,, \quad t \in [0, \infty), \quad X_0^k = x_k \geq 0\,. \tag{2.2}$$

Proposition 6.2.1. *([45]) For $0 \leq s \leq t \leq T$,*

$$\mathbb{E}_{\mathbb{Q}}(R_t|\mathcal{F}_s) = a(t) + \sum_{k=1}^{n} \left(X_k e^{-\lambda_k(t-s)} + \sigma_k \frac{1 - e^{-\lambda_k(t-s)}}{\lambda_k} \int_{D_k} z\, d\nu_k(z) \right),$$
$$Var_{\mathbb{Q}}(R_t|\mathcal{F}_s) = \sum_{k=1}^{n} \frac{\sigma_k^2(1 - e^{-2\lambda_k(t-s)})}{2\lambda_k^2} \int_{D_k} z^2\, d\nu_k(z),$$

where the short rate process R_t satisfies (2.1)*, ν_k is the Lévy measure associated with the compound Poisson process and $D_k = [\epsilon_k^1, \epsilon_k^1]$ with $0 < \epsilon_k^1 < \epsilon_k^1$.*

For more details on the Proposition we refer to [45]. Parameter estimation techniques from Chapter 3 for X_t^k can be used to estimate the expected value for the short rate process and also to estimate the variance for the short rate process R_t.

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Appendix A

Matlab codes to the simulation of Ornstein Uhlenbeck process in Chapter 3

The following code was used to simulate the paths of double exponential Ornstein Uhlenbeck process using Exact simulation via the decomposition approach.

```
function S_avg= OU_jump_Mr23(p,q,eta,phi,Nsteps,lambda,
   Npaths,T,sigma,theta)
%S=zeros(Nsteps+1,Npaths)  ;
Sum1=zeros(1,Npaths);
Sum2=zeros(1,Npaths);
Sum3=zeros(1,Npaths);
Sum4=zeros(1,Npaths);
Sum5=zeros(1,Npaths);
Sum6=zeros(1,Npaths);

s1=zeros(1,Npaths);
s2=zeros(1,Npaths);
```

```
s3=zeros(1,Npaths);
s4=zeros(1,Npaths);
s5=zeros(1,Npaths);
s6=zeros(1,Npaths);
n=zeros(Nsteps+1,Npaths);
for k=1:Npaths
    for i=1:Nsteps+1
        n(i,k)=poissrnd(lambda*(T/Nsteps));
    end
end

path=zeros(1,Npaths);
path(1,:)=0;

for k = 1:Npaths
    for j=1:Nsteps
        %path(j+1,k)=-theta* path(j,k)*T/Nsteps+sigma*(
           doubleexpo1(p,q,eta,phi,n(j+1,k))-doubleexpo1
           (p,q,eta,phi,n(j,k)));
        path(j+1,k)=exp(-theta*(T/Nsteps))* path(j,k)+
           doublexp(p,q,eta,phi,n(j+1,k),theta,T,Nsteps)
           ;
        %path(j+1,k)=exp(-theta*(T/Nsteps))* path(j,k)+
           sigma*((exp(-theta*(T/Nsteps))-1)*(
           doubleexpo1(p,q,eta,phi,n(j+1,k))))/theta;
        s1(1,k)=s1(1,k)+path(j+1,k);
        s2(1,k)=s2(1,k)+path(j+1,k)^2;
        s3(1,k)=s3(1,k)+path(j+1,k)^3;
        s4(1,k)=s4(1,k)+path(j,k)*path(j+1,k);
```

```
            s5(1,k)=s5(1,k)+path(j+1,k)^4;
            s6(1,k)=s6(1,k)+path(j+1,k)^5;
        Sum1(1,k)=s1(1,k)/Nsteps;
        Sum2(1,k)=s2(1,k)/Nsteps-(Sum1(1,k)^2);
        Sum3(1,k)=s3(1,k)/Nsteps-2*Sum1(1,k)*Sum2(1,k)-Sum1
           (1,k)*(s2(1,k)/Nsteps);
        Sum4(1,k)=s4(1,k)/Nsteps-(Sum1(1,k)^2);
        Sum5(1,k)=s5(1,k)/Nsteps-Sum1(1,k)^4-3*(Sum1(1,k)^2)
           *Sum2(1,k)-3*Sum1(1,k)*Sum3(1,k)-Sum2(1,k)*3*Sum2
           (1,k)-Sum1(1,k)*Sum2(1,k);
        %Sum5(1,k)=s5(1,k)/Nsteps-Sum1(1,k)^4-3*(Sum1(1,k)
           ^2)*(Sum2(1,k)-Sum1(1,k)^2)-3*Sum1(1,k)*(Sum3(1,k
           )-Sum2(1,k)*Sum1(1,k)-2*Sum1(1,k)*(Sum2(1,k)-Sum1
           (1,k)^2))-(Sum2(1,k)-Sum1(1,k)^2)*3*Sum2(1,k)-
           Sum1(1,k)*(Sum2(1,k)-Sum1(1,k)^2);
        Sum6(1,k)=s6(1,k)/Nsteps-Sum1(1,k)^5-4*Sum1(1,k)^3*(
           Sum2(1,k))-3*Sum1(1,k)^2*Sum3(1,k)-2*Sum2(1,k)
           *(3*Sum1(1,k)^2+6*Sum2(1,k))-3*Sum1(1,k)*Sum5(1,k
           )-Sum3(1,k)*(3*Sum1(1,k)^2+3*Sum2(1,k))-Sum2(1,k)
           *(3*Sum1(1,k)^3+6*Sum1(1,k)*Sum2(1,k)+9*Sum1(1,k)
           *Sum2(1,k)+3*Sum3(1,k))-2*Sum1(1,k)*Sum5(1,k)-
           Sum3(1,k)*Sum2(1,k);
        end
    end
    plot(0:T/Nsteps:T,path);
    fprintf('The average sum is %s\n',Sum1);
    Sum4;
    S1=sum(Sum1);
    S2=sum(Sum2);
```

```
S3=sum(Sum3);
S4=sum(Sum4);
S5=sum(Sum5);
S6=sum(Sum6);
A=S1/Npaths;
B=S2/Npaths;
C=S3/Npaths;
D=S4/Npaths;
E=S5/Npaths;
F=S6/Npaths;
S_avg=[A,B,C,D,E,F];
S_imp=S_avg;

disp('The sum S1/Npaths is:')
disp(S1/Npaths)
disp('The sum S2/Npaths is:')
disp(S2/Npaths)
disp('The sum S3/Npaths is:')
disp(S3/Npaths)
disp('The sum S4/Npaths is:')
disp(S4/Npaths)
disp('The sum S5/Npaths is:')
disp(S5/Npaths)
disp('The sum S6/Npaths is:')
disp(S6/Npaths)
title('OU type process driven by Double Exponential Jump
    Process')
xlabel('Time')
xlabel('Number of steps')
```

```
ylabel('Path of Process X_t')
%%% OU_jump(.5,.5,1.1,1.1,20000,1,20,1000,1.2,1.4)
%OU_jump(.5,.5,1.1,0.9,20000,1,20,1000,1.2,1.4)
%OU_jump_Sep(.6,.4,1.2,1.6,80000,1,100,1000,1,2)
%OU_jump_Sep(.65,.35,1.25,1.65,80000,1,100,1000,1,2)
```

The following code is used to generate steps sizes of the compound Poisson process using a mixture of double exponential distribution

```
function sum= doublexp(p,q,eta,phi,Nsamples,theta,T,
   Nsteps)
r=rand(1,Nsamples);
Y=zeros(1,Nsamples);
u=rand(1,Nsamples);
sum=0;
h=T/Nsteps;
for i=1:Nsamples
    if r(i)<q
        Y(i)=(1/(phi*exp(theta*h*u(i))))*log(r(i)/q);
    elseif r(i)==q
        Y(i)=0;
    else
        Y(i)=(1/eta*exp(theta*h*u(i)))*log(p/(1-r(i)));

    end
    sum=sum+Y(i);
end
```

Appendix A

Covariance matrix A

In this section we give the expression of the covariance matrix A in Lemma 3.4.1. It is very sophisticated to express the entries of this matrix in terms of the parameters of the equation (2.1). So, we keep them as expression of the invariant probability measure of $\mathbb{X}_0$ and that of $\mathbb{X}_{kh}$. First, we compute $\sigma_{g_1g_1}$.

$$\begin{aligned}\sigma_{g_1g_1} &= \mathrm{Cov}(\mathbb{X}_0,\mathbb{X}_0) + 2\sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0,\mathbb{X}_{jh})] \\ &= \mathbb{E}(\mathbb{X}_o^2) - \mathbb{E}(\mathbb{X}_o)^2 + 2\sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_o\mathbb{X}_{jh}) - \mathbb{E}(\mathbb{X}_o)\mathbb{E}(\mathbb{X}_{jh})\Big] \\ &= \mathbb{E}(\mathbb{X}_o^2) - \mathbb{E}(\mathbb{X}_o)^2 + 2\sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_o\mathbb{X}_{jh}) - [\mathbb{E}(\mathbb{X}_o)]^2\Big],\end{aligned} \tag{0.1}$$

where we used $\mathbb{E}(\mathbb{X}_{jh}) = \mathbb{E}(\mathbb{X}_0)$. Now we compute $\sigma_{g_2g_2}$.

$$\begin{aligned}\sigma_{g_2g_2} &= \mathrm{Cov}(\mathbb{X}_0^2,\mathbb{X}_0^2) + 2\sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0^2,\mathbb{X}_{jh}^2)] \\ &= \mathbb{E}(\mathbb{X}_o^4) - \mathbb{E}(\mathbb{X}_o^2)^2 + 2\sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_o^2{\mathbb{X}_{jh}}^2) - \mathbb{E}(\mathbb{X}_o^2)^2\Big].\end{aligned} \tag{0.2}$$

Similarly, we have

$$\begin{aligned}\sigma_{g_3g_3} &= \text{Cov}(\mathbb{X}_0^3, \mathbb{X}_0^3) + 2\sum_{j=1}^{\infty}[\text{Cov}(\mathbb{X}_0^3, \mathbb{X}_{jh}^3)] \\ &= \mathbb{E}(\mathbb{X}_o^6) - \mathbb{E}(\mathbb{X}_o^3)^2 + 2\sum_{j=1}^{\infty}\left[\mathbb{E}(\mathbb{X}_o^3 {\mathbb{X}_{jh}}^3) - \mathbb{E}(\mathbb{X}_o^3)^2\right]\end{aligned} \tag{0.3}$$

and

$$\begin{aligned}\sigma_{g_4g_4} &= \text{Cov}(\mathbb{X}_0\mathbb{X}_h, \mathbb{X}_0\mathbb{X}_h) + 2\sum_{j=1}^{\infty}[\text{Cov}(\mathbb{X}_0\mathbb{X}_h, \mathbb{X}_{jh}\mathbb{X}_{(j+1)h})] \\ &= \mathbb{E}((\mathbb{X}_0\mathbb{X}_h)^2) - \mathbb{E}(\mathbb{X}_0\mathbb{X}_h)^2 \\ &\quad + 2\sum_{j=1}^{\infty}\left[\mathbb{E}(\mathbb{X}_0\mathbb{X}_h\mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) - \mathbb{E}(\mathbb{X}_0\mathbb{X}_h)\mathbb{E}(\mathbb{X}_{jh}\mathbb{X}_{(j+1)h})\right].\end{aligned} \tag{0.4}$$

$\sigma_{g_1g_2}$ is computed as follows.

$$\begin{aligned}\sigma_{g_1g_2} &= \text{Cov}(\mathbb{X}_0, \mathbb{X}_0^2) + \sum_{j=1}^{\infty}[\text{Cov}(\mathbb{X}_0, \mathbb{X}_{jh}^2) + \text{Cov}(\mathbb{X}_0^2, \mathbb{X}_{jh})] \\ &= \mathbb{E}((\mathbb{X}_0)^3) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_0^2) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0\mathbb{X}_{jh}^2) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_{jh}^2) \\ &+ \mathbb{E}(\mathbb{X}_0^2\mathbb{X}_{jh}) - \mathbb{E}(\mathbb{X}_0^2)\mathbb{E}(\mathbb{X}_{jh})\Big].\end{aligned} \tag{0.5}$$

In similar way we can get

$$\begin{aligned}\sigma_{g_1g_3} &= \text{Cov}(\mathbb{X}_0, \mathbb{X}_0^3) + \sum_{j=1}^{\infty}[\text{Cov}(\mathbb{X}_0, \mathbb{X}_{jh}^3) + \text{Cov}(\mathbb{X}_0^3, \mathbb{X}_{jh})] \\ &= \mathbb{E}((\mathbb{X}_0)^4) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_0^3) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0\mathbb{X}_{jh}^3) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_{jh}^3) \\ &+ \mathbb{E}(\mathbb{X}_0^3\mathbb{X}_{jh}) - \mathbb{E}(\mathbb{X}_0^3)\mathbb{E}(\mathbb{X}_{jh})\Big]\end{aligned} \tag{0.6}$$

and

$$\sigma_{g_1g_4} = \mathrm{Cov}(\mathbb{X}_0, \mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0, \mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) + \mathrm{Cov}(\mathbb{X}_{jh}, \mathbb{X}_0\mathbb{X}_h)]$$
$$= \mathbb{E}(\mathbb{X}_0^2\mathbb{X}_h) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0\mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) - \mathbb{E}(\mathbb{X}_0)\mathbb{E}(\mathbb{X}_{jh}\mathbb{X}_{(j+1)h})$$
$$+ \mathbb{E}(\mathbb{X}_0\mathbb{X}_h\mathbb{X}_{jh}) - \mathbb{E}(\mathbb{X}_{jh})\mathbb{E}(\mathbb{X}_0\mathbb{X}_h)\Big]. \tag{0.7}$$

$\sigma_{g_2g_3}$ is similar to $\sigma_{g_1g_2}$.

$$\sigma_{g_2g_3} = \mathrm{Cov}(\mathbb{X}_0^2, \mathbb{X}_0^3) + \sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0^2, \mathbb{X}_{jh}^3) + \mathrm{Cov}(\mathbb{X}_0^3, \mathbb{X}_{jh}^2)]$$
$$= \mathbb{E}((\mathbb{X}_0)^5) - \mathbb{E}(\mathbb{X}_0^2)\mathbb{E}(\mathbb{X}_0^3) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0^2\mathbb{X}_{jh}^3) - \mathbb{E}(\mathbb{X}_0^2)\mathbb{E}(\mathbb{X}_{jh}^3) \tag{0.8}$$
$$+ \mathbb{E}(\mathbb{X}_0^3\mathbb{X}_{jh}^2) - \mathbb{E}(\mathbb{X}_0^3)\mathbb{E}(\mathbb{X}_{jh}^2)\Big].$$

Finally, we have

$$\sigma_{g_2g_4} = \mathrm{Cov}(\mathbb{X}_0^2, \mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0^2, \mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) + \mathrm{Cov}((\mathbb{X}_{jh})^2, \mathbb{X}_0\mathbb{X}_h)]$$
$$= \mathbb{E}(\mathbb{X}_0^3\mathbb{X}_h) - \mathbb{E}(\mathbb{X}_0^2)\mathbb{E}(\mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0^2\mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) - \mathbb{E}(\mathbb{X}_0^2)\mathbb{E}((\mathbb{X}_{jh}\mathbb{X}_{(j+1)h})$$
$$+ \mathbb{E}(\mathbb{X}_0\mathbb{X}_h\mathbb{X}_{jh}^2) - \mathbb{E}(\mathbb{X}_{jh}^2)\mathbb{E}(\mathbb{X}_0\mathbb{X}_h)\Big] \tag{0.9}$$

and

$$\sigma_{g_3g_4} = \mathrm{Cov}(\mathbb{X}_0^3, \mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}[\mathrm{Cov}(\mathbb{X}_0^3, \mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) + \mathrm{Cov}((\mathbb{X}_{jh})^3, \mathbb{X}_0\mathbb{X}_h)]$$
$$= \mathbb{E}(\mathbb{X}_0^4\mathbb{X}_h) - \mathbb{E}(\mathbb{X}_0^3)\mathbb{E}(\mathbb{X}_0\mathbb{X}_h) + \sum_{j=1}^{\infty}\Big[\mathbb{E}(\mathbb{X}_0^3\mathbb{X}_{jh}\mathbb{X}_{(j+1)h}) - \mathbb{E}(\mathbb{X}_0^3)\mathbb{E}((\mathbb{X}_{jh}\mathbb{X}_{(j+1)h})$$
$$+ \mathbb{E}(\mathbb{X}_0\mathbb{X}_h\mathbb{X}_{jh}^3) - \mathbb{E}(\mathbb{X}_{jh}^3)\mathbb{E}(\mathbb{X}_0\mathbb{X}_h)\Big]. \tag{0.10}$$