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BY

JOHN PRICE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "BURNSIDE GROUPS THROUGH FUNCTION MODULES" submitted by JOHN PRICE in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Date Oct. 1, 1975

To Sue,

My Wife.

ABSTRACT

The purpose of this thesis is to use the new method of H. Wielandt [11], that of considering permutation groups through function modules, to prove two theorems on B-groups. Chapter I gives basic definitions on permutation groups and a summary of Wielandt's method as related to B-groups. Chapter II contains the work to find function modules invariant under the action of certain groups. Finally, in the third chapter we prove that the product of the quaternions of order eight and an abelian group of odd order is a B-group and that the group $Z_4 \times Z_2 \times Z_p \times Z_p$ is a B-group for p an odd prime where Z_n is the cyclic group of order n.

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I would like to express my grateful thanks to my supervisor, Dr. R. Bercov, for the excellent advice and help, and for the considerable time and effort that he has spent on my work while I have been completing this thesis.

I would also like to thank my wife for the part she has played in my work. Without her I could never have finished this thesis, and it is to her, and the encouragement and loving care which she has given me, that I dedicate this work.

TABLE OF CONTENTS

LIST OF DIAGRAMS

CHAPTER I

Introduction

Given a set Ω containing n elements, the set of bijective mappings from Ω to itself forms a group, known as the *symmetric group of degree* n and written S_n. The subgroups of S_n are the *permutation groups* of degree n and it is these groups and their various properties that we shall study.

§1. Basic Definitions.

Let us begin with some definitions essential to our work.

Let G be a subgroup of S_n and denote the action of $g \in G$ on $\alpha \in \Omega$ by α^g.

Definition 1.1: Define the relation $\alpha \sim \beta$, $\alpha,\beta \in \Omega$, by

$$\alpha \sim \beta \text{ if and only if } \beta \in \{\alpha^g \mid g \in G\} .$$

This is an equivalence relation which partitions Ω into equivalence classes, known as the *orbits* of G on Ω.

Definition 1.2: Let $A \subseteq \Omega$. A is a *fixed block* if and only if $A^G = A$, i.e. $\alpha^g \in A$ for all $g \in G$, $\alpha \in A$.

A ~~fixed~~ block is a union of orbits of G and an orbit is a minimal fixed block.

Definition 1.3: G is *transitive* if and only if the only fixed blocks of G on Ω are Ω and the empty set ϕ. An equivalent definition of transitivity is: for all $\alpha, \beta \in \Omega$ there exists a $g \in G$ such that $\alpha^g = \beta$. This means that there is always an element of G which enables us to "move" from one point of Ω to another.

It is this definition that we extend to define multiple transitivity.

Definition 1.4: G is r - *transitive* if given any two ordered r - tuples $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $(\beta_1, \beta_2, \ldots, \beta_r)$ with $\alpha_i, \beta_i \in \Omega$ for all i and $\alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$ for all $i \neq j$, then there exists a $g \in G$ such that

$$(\alpha_1^g, \alpha_2^g, \ldots, \alpha_r^g) = (\beta_1, \beta_2, \ldots, \beta_r) .$$

So, for example, 2-transitivity (often called double transitivity) is the ability to "move" from any ordered pair (α_1, α_2) of Ω^2 $(\alpha_1 \neq \alpha_2)$ to any other ordered pair (β_1, β_2) of Ω^2 $(\beta_1 \neq \beta_2)$.

1 - transitivity is, of course, the same as transitivity.

Definition 1.5: A *block* is a subset A of Ω such that for each $g \in G$ we have either $A^g \cap A = A$ or $A^g \cap A = \phi$.

A is a subset which is either invariant under the action of a group element or is taken to a set with no point in common with A.

Clearly, a fixed block is a block and so are Ω, ϕ, and the single point sets of Ω.

<u>Definition 1.6:</u> G is called *primitive* if the only blocks of G on Ω are Ω, ϕ and the single point sets of Ω and G is transitive on Ω.

It can be seen from the definitions that

$$G \text{ 2 - transitive} \Rightarrow G \text{ primitive} \Rightarrow G \text{ transitive.}$$

Hence primitivity is a property lying between 2 - transitivity and transitivity. Those groups which are primitive but not 2 - transitive have come under special attention under the name uni-primitive groups.

It is the relationship between 2 - transitivity and primitivity which is the underlying theme of this thesis. This will be made clear in definition 1.8.

<u>Definition 1.7:</u> Any abstract group can be realized as a permutation group in the following way.

Let G be a group. Then if $g \in G$ define g^* by

$$x^{g^*} = xg \quad \text{for all} \quad x \in G \ .$$

We thus get a permutation group $G^* = \{g^* \mid g \in G\}$ of degree $|G|$ where $\Omega = G$ and $G \cong G^*$.

G^* is known as the *right regular representation* of G. (A left regular representation is obtained by taking multiplication on the left.) For example, if we denote the cyclic group of order n by Z_n, we obtain $Z_4 \times Z_2$ as a subgroup of S_8 in this way.

The true motivating force of this thesis is the study of Burnside groups, henceforth known as B - groups, named after the English mathematician W. Burnside, and we are at last in a position to give their definition.

Definition 1.8: A group H is a B - *group* if every primitive group containing the right (or left) regular representation of H as a transitive group is 2 - transitive.

Burnside was the first to consider groups with this property and proved, using character theory, that every cyclic group whose order is of the form p^m (p prime, $m > 1$) is a B - group. Since that time a considerable collection of groups have been shown to be B - groups, though Burnside's assertion that all Abelian groups which are not elementary Abelian are B - groups has been shown to be false by . Manning using an idea of W.A. Manning. The results were obtained through either character theory or Schur Ring theory, both of these methods being used only very slightly in the work of this thesis. The results for abelian groups can be summed up in the following theorem of Bercov [1].

Theorem 1.9: Let H be abelian, P a Sylow p-subgroup of H and a an element of P of maximal order, p^{α}. Let A be the cyclic group generated by a. Then $H = A \times B \times C$ where $P = A \times B$ and C is of order prime to p.

(a) If $B \neq 1$ if of exponent $p^{\beta} < p^{\alpha}$ (and $\alpha \geq 3$ if $p = 2$) then either H is a direct product of groups of the same order greater than 2 or else H is a B - group.

(b) If $B = 1$ then H is a B - group unless $C = 1$ and $\alpha = 1$.

In this thesis we will consider the case $\alpha = 2$, $\beta = 1$, $p = 2$, as well as the non-abelian case where $H = P \times C$ with P being the quaternion group of order 8.

For completeness, we list the non-abelian B - group results known to date.

Theorem 1.10:

(a) Wielandt [9].

Every dihedral group is a B - group.

(b) Scott [8; pg. 416].

Every generalised dicyclic group (defined by $x^{2n} = 1$, $y^2 = x^n$, $y^{-1}xy = x^{-1}$) is a B - group.

(c) Nagai [5] and Nagao [6].

Every non-abelian group of order $3(2.3^a+1)$ or $3(6\ell+1)$ is a B - group, where ℓ and p are prime, $\ell > 7$, $a > 2$.

(d) Enomoto [4].

Groups of nilpotence class two which have a maximal cyclic subgroup and do not have order 16 are B - groups.

§2. The Function Space F_k.

Character theory and Schur Ring theory were the two main methods of proving permutation group results until H. Wielandt [11] introduced a new approach in 1969. To illustrate the idea with respect to B - groups he produced a rather beautiful proof of Burnside's original theorem and since then Bercov [2], using the same method has produced a new elegant proof of Wielandt's generalization of Schur's main B - group theorem. Hence the hope has risen that new results may be obtained through this new method of attack. This section will explain Wielandt's method and give the important results obtained by Wielandt that will be used in subsequent chapters.

Definition 1.11: Let G be a permutation group acting on Ω.

For a given field F we define the *function space* F_k as the space of functions from the cartesian product Ω^k to F.

Taking addition and multiplication as being pointwise, F_k is a commutative, associative algebra for which the characteristic functions of the points of Ω^k form a basis. For example, F_1, which we shall write as F from now on, has, as a basis, the n functions χ_α, $\alpha \in \Omega$ where

$$\chi_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta . \end{cases}$$

We define the action of G on F_k by

$$f^g(\alpha) = f(\alpha^{g^{-1}}) \qquad g \in G, \ f \in F_k, \ \alpha \in \Omega^k .$$

Definition 1.12: A submodule M of F_k is a G - *module* (written $M \underset{G}{\leq} F_k$) if and only if M is invariant under the action of G; i.e. $f \in M \Rightarrow f^g \in M$ for all $g \in G$.

A G - *algebra* is a G - module closed under multiplication.

For example, the set of constant functions C_k defined by

$$C_k = \{f \in F_k \mid f(\alpha) = a \text{ for all } \alpha \in \Omega^k, \text{ fixed } a \in F\}$$

is a G - module of F_k. [Write C for the set C_1.]

Now we give Wielandt's characterization of primitivity.

Theorem 1.13 [11; pg. 59]: G is primitive on Ω if and only if F is the only G - algebra which properly contains the constant functions C.

Since it is important to produce a G - algebra to invoke this characterization we now give various methods for finding G - modules. It is from these modules that we will hope to produce the algebras.

Definition 1.14: Let M, N be modules of F_k.

(a) Define MN as the smallest module which contains the set of functions

$$\{f \cdot g \mid f \in M, g \in N\}.$$

(b) Define $M:N$ as the smallest module which contains the set of functions

$$\{f \mid f \cdot N \subseteq M\}.$$

Theorem 1.15 [11; pg. 58]: Let M, N be G - modules of F_k. Then

(a) $M \cap N$, $M+N$, MN and $M:N$ are G - modules.

(b) $M:N$ is a G - algebra if $M \leq N$.

For example, for any G - module M, $M:M$ is a G - algebra and the "square" $M^2 = MM$ is a G - module.

Definition 1.16: Let A be any subset of F_k and r any natural number. Then the r - *closure* of A, written $A^{(r)}$, is defined by $f \in A^{(r)}$ if and only if for every set $\{\alpha_1,\alpha_2,\ldots,\alpha_r\}$ of r points of Ω^k, there exists a $g \in A$ such that

$$g(\alpha_i) = f(\alpha_i) \quad \text{for all} \quad i = 1,\ldots,r .$$

Note that $[A^{(r)}]^{(r)} = A^{(r)}$ so taking r - closure more than once will not produce a larger G - module.

Theorem 1.17 [11; pg. 60]: If $M \underset{G}{\leq} F_k$ then $M^{(r)} \underset{G}{\leq} F_k$ and $F_k \geq M^{(1)} \geq M^{(2)} \geq \ldots \geq M$.

Definition 1.18: Let $f \in F_k$.

(a) Define the *integral* of f, $\int f$, by

$$\int f = \sum_{\alpha\in\Omega^k} f(\alpha) .$$

(b) For any module M of F_k define $M^{\perp}$ by

$$M^{\perp} = \{f \in F_k \mid \int fm = 0 \quad \text{for all} \quad m \in M\} .$$

Theorem 1.19 [11; pg. 61]: If $M \underset{G}{\leq} F_k$ then $M \to M^{\perp}$ is an anti-isomorphism of the lattice of all G - submodules of F_k .

For example, $C^{\perp} \underset{G}{<} F$.

To end both the section and the chapter we shall give a brief summary of the tactics we shall be using in Chapter III to gain our B - group results.

Since, in the right regular representation of an abstract group H, the set Ω is in fact the group H, to show that a group H is a B - group we may assume Ω to be H and any $G \geq H$ to be acting on H rather than Ω. This makes the action of H on itself merely multiplication on the right, i.e.

$$x^h = xh \quad \text{for all} \quad x,h \in H .$$

Hence we will begin by finding all the H - modules of F and since a G - module must be an H - module if $G \geq H$ we will have pinned down the modules that could possibly be G - modules. This first step is so basic to our work that the whole of Chapter II is devoted to the calculation of the H - module structure for certain groups H.

Having found the H - modules, the H - algebras, if any, will be isolated, for they play an important role. Our initial assumption will be that G is not 2 - transitive and we will hope to show that G is therefore not primitive by finding a G - module I lying inside an H - algebra $A \neq F$ but not equal to C, so that the algebra generated by I and C will be contained in A and therefore 1.13 will give the desired result. To do this we shall consider a G - endomorphism ϕ of F, (an endomorphism ϕ with the property $\phi(\alpha^g) = [\phi(\alpha)]^g$), its image, Im ϕ, which is clearly a G - module, and certain other G - modules obtained from it by using 1.15, 1.17, 1.19 and the map

ϕ .

In practise, we will not know Im ϕ very precisely, and we will therefore have to consider a number of cases.

CHAPTER II

H - Module Structures

In this chapter we consider some simple examples of groups containing a cyclic subgroup of index 2. They are the abelian group $Z_4 \times Z_2$, the dihedral group D_4 of order 8, and the quaternion groups Q_2 and Q_4 or order 8 and 16 respectively. We also give some general results on groups of the form $A \times B$, where B is an abelian group of order prime to the order of A and where the A - module structure is known.

Throughout sections 1 to 4 of this chapter $F = \{0,1\}$.

§1. $H = Z_4 \times Z_2$.

Let $Z_4 \times Z_2$ have generators x, y such that

$$x^4 = 1 \quad , \quad y^2 = 1 \quad \text{and} \quad xy = yx \quad .$$

We may write any function

$$f : H \to F \quad \text{as} \quad (f_1, f_2)$$

where f_1 and f_2 are functions from $\langle x \rangle$ to F and

$$(f_1, f_2)(x^i y^j) = \begin{cases} f_1(x^i) & \text{if } j = 0 \\ f_2(x^i) & \text{if } j = 1 \end{cases} .$$

Thus if $f = (f_1, f_2)$ then $f^x = (f_1^x, f_2^x)$ and $f^y = (f_2, f_1)$.

Definition 2.1:

(i) Define Δf by $\Delta f = f^x + f$. Note that $\Delta f = (\Delta f_1, \Delta f_2)$.

(ii) Define the functions χ , ψ , i and 1 by $\chi = \chi_1$, $\psi = \Delta\chi = \chi_{1,x}$, $i = \Delta^2\chi = \chi_{1,x^2}$ and $1 = \Delta^3\chi = \chi_{1,x,x^2,x^3}$.

(iii) Define the *degree* of $f \in F$ as the smallest integer n such that $\Delta^n f = 0$.

Write $d(f)$ for the degree of f .

The four functions $(\chi,0)$, $(\psi,0)$, $(i,0)$, $(1,0)$ and the four functions formed by acting on them by y are linearly independent and form a basis for F ; i.e.

$$f \in F \Rightarrow f = (a_1\chi + b_1\psi + c_1 i + d_1, a_2\chi + b_2\psi + c_2 i + d_2)$$

where $a_i, b_i, c_i, d_i \in F$, $i = 1,2$.

Now we find those modules which are invariant under the action of H . In this case we need only consider the action of x and of y , or, more conveniently, the action of Δ and of y . The process of calculating these modules, even with the above notation, is a somewhat tedious process. Two observations make the operation a little less time-consuming.

Observation 2.2:

(a) Every module contains functions of the type

$$(\chi, f_2)\ ,\ (\psi, f_2)\ ,\ (i, f_2)\ \text{or}\ (1, f_2)\ .$$

This is true because of the following: If $(\chi+a_1\psi+b_1 i+c_1, f_2)$ is contained in an H - module M then

$$(\chi, f_2+a_1\Delta f_2+(a_1^2+b_1)\Delta^2 f_2+(a_1 b_1+c_1+(a_1^2+b_1)a_1)\Delta^3 f_2) \in M\ .$$

If $(\psi+b_1 i+c_1, f_2) \in M$ then

$$(\psi, f_2+b_1\Delta f_2+(b_1^2+c_1)\Delta^2 f_2) \in M\ .$$

If $(i+c_1, f_2) \in M$ then

$$(i, f_2+c_1\Delta f_2) \in M\ .$$

(b) We need only consider those modules containing functions

$$f = (f_1, f_2)\ \text{where}\ d(f_1) = d(f_2)\ \text{or}\ f_2 = 0\ .$$

This is because if $d(f_1) > d(f_2)$ then we can show that the G - module also contains a function $(f_1^*, 0)$ where $d(f_1^*) = d(f_1)$. [If $d(f_2) > d(f_1)$ then consider the function $(f_2, f_1) = f^y$.]

For example, let $f = (\chi, a_2\psi+b_2 i+c_2)$ be contained in an H - module M . Therefore

$$\Delta f = (\psi, a_2 i+b_2) \in M\ ,$$

$$\Delta^2 f = (i, a_2) \in M\ ,$$

$$\Delta^3 f = (1, 0) \in M\ ,\ \text{and}$$

$$(\Delta^3 f)^y = (0, 1) \in M\ .$$

By adding, if necessary, this last function to the 4 others we can obtain

$$(\chi, a_2\psi + b_2 i) \in M \ ,$$

$$(\psi, a_2 i) \in M \ ,$$

$$(i, 0) \in M \ , \quad \text{and hence}$$

$$(0, i) = (i, 0)^y \in M \ .$$

Using this last function enables us to obtain

$$(\chi, a_2\psi) \in M \ ,$$

$$(\psi, 0) \in M \ , \quad \text{and hence}$$

$$(0, \psi) \quad M \ .$$

Finally, if $a_2 \neq 0$, then $(\chi,\psi) + (0,\psi) = (\chi, 0) \in M$.

A similar process is used for functions of the type (ψ, f_2) , (i, f_2) , $(1, f_2)$ and observation (a) tells us that these are the only functions we need consider.

Diagram I shows the lattice of the H - modules finally calculated. Two examples follow to give an idea of how the structure was calculated.

(1) Let $f = (\chi,\chi) \in M$. $f^y = (\chi,\chi) \in M = f$.

Then $\Delta f = (\psi,\psi) \in M$, $(\Delta f)^y = (\psi,\psi) = \Delta f$.

$\Delta^2 f = (i, i) \in M$, $(\Delta^2 f)^y = \Delta^2 f$.

and $\Delta^3 f = (1, 1) \in M$, $(\Delta^3 f)^y = \Delta^3 f$.

Thus the smallest H - module containing the function (χ,χ) is a four-dimensional module with basis $\{(\chi,\chi), (\psi,\psi), (i,i), (1,1)\}$. We shall call this the module generated by (χ,χ) and write it as $\langle\chi,\chi\rangle$.

(2) Let $f = (\chi,\chi+\psi+i+1) \in M$. $f^y = (\chi+\psi+i+1,\chi)$.

Then $\Delta f = (\psi,\psi+i+1)$, $(\Delta f)^y = (\psi+i+1,\psi)$.

$\Delta^2 f = (i,i+1)$, $(\Delta^2 f)^y = (i+1,i) = \Delta^2 f + \Delta^3 f$,

and $\Delta^3 f = (1,1)$. $(\Delta^3 f)^y = (1,1) = \Delta^3 f$.

So $\langle\chi,\chi+\psi+i+1\rangle$ is a 6-dimensional module and is equal to $\langle\chi,\chi+\psi\rangle$.

To find the H - algebras we need the multiplication table which is

×	1	i	ψ	χ
1	1	i	ψ	χ
i	i	i	χ	χ
ψ	ψ	χ	ψ	χ
χ	χ	χ	χ	χ

Using this table we consider the module $\langle\chi,\chi+i\rangle$:

$(\chi,\chi+i)(\chi,\chi+i) = (\chi,\chi+i)$ $\qquad (\psi,\psi+1)(i,i) = (\chi,\chi+i)$

$(\chi,\chi+i)(\psi,\psi+1) = (\chi,\chi+\chi+\chi+i) = (\chi,\chi+i)$ $\qquad (\psi,\psi+1)(1,1) = (\psi,\psi+1)$

$(\chi,\chi+i)(i,i) = (\chi,\chi+i)$

$(\psi,\psi+1)(\psi,\psi+1) = (\psi,\psi+1)$ $\qquad (i,i)(i,i) = (i,i)$

So $<\chi,\chi+i>$ is an algebra. The complete list of algebras is:

$$F, \quad \phi,$$

$$<\chi,\chi>, \quad <\chi,\chi+i>, \quad <i,0>,$$

$$<i,i>, \quad <i,i+1>, \quad <1,0> \text{ and } <1,1>.$$

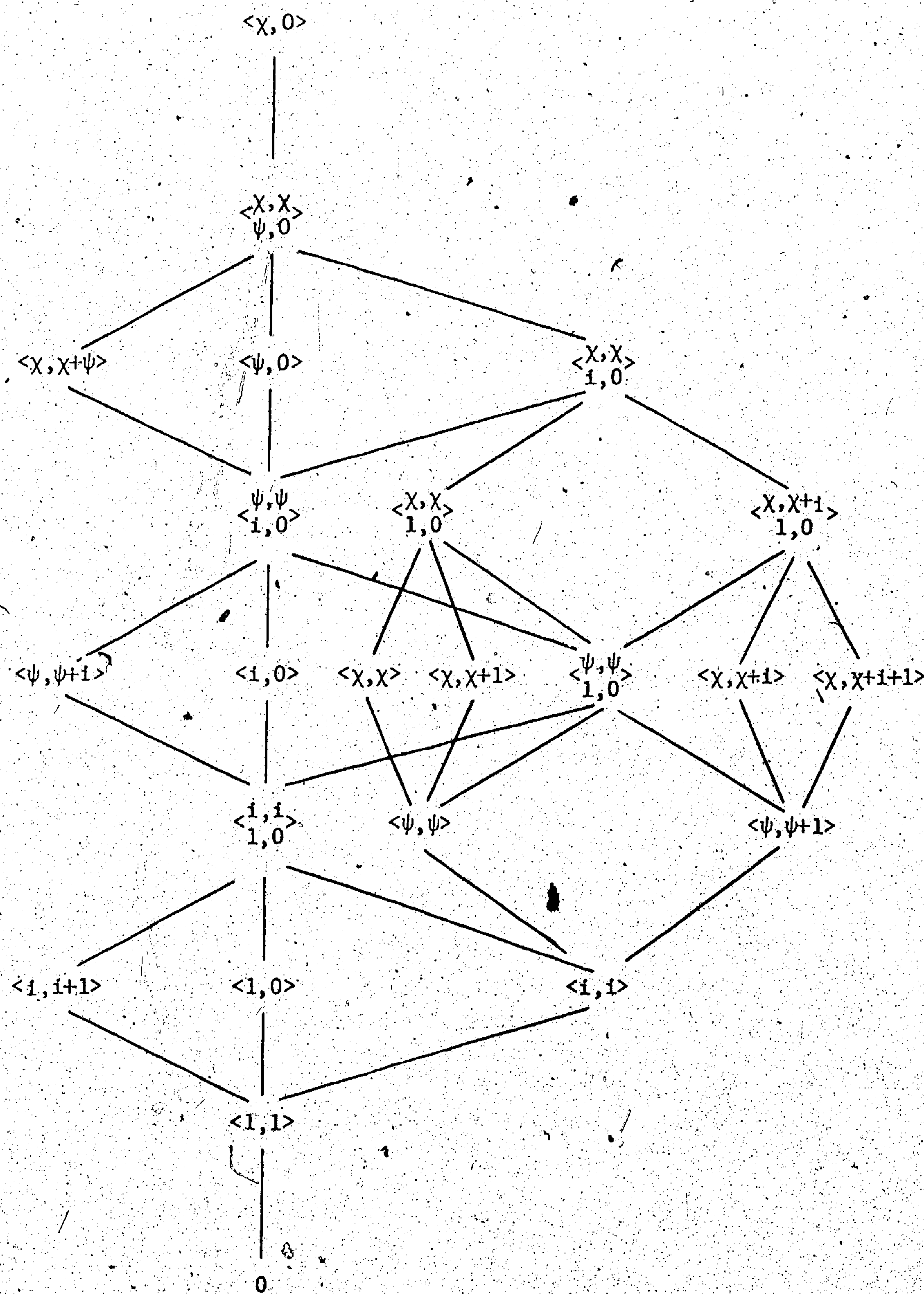


Diagram I $H = Z_4 \times Z_2$

§2. $H = D_4$.

Let D_4 have generators x , y such that $x^4 = 1$, $y^2 = 1$, $y^{-1}xy = x^{-1}$. The same notation is used as that in §1. The action of y remains the same: i.e. $f^y = (f_2, f_1)$ but the action of Δ is different.

$$\Delta f = (\Delta f_1, \Delta^* f_2) \quad \text{where} \quad \Delta^* f = f^{x^{-1}} + f \quad \text{so that}$$

$$\Delta^* \chi = \psi + i + 1 \quad , \qquad \Delta^* i = 1 \quad ,$$

$$\Delta^* \psi = i + 1 \quad , \quad \text{and} \quad \Delta^* 1 = 0 \quad .$$

The H - module structure was calculated in the same fashion as Section 1 and is shown in Diagram II. Here again are 2 examples to illustrate the calculations.

(1) Let $f = (\chi, \chi) \in M$. $f^y = f$.

Then $\Delta f = (\psi, \psi+i+1) \in M$, $(\Delta f)^y = (\psi+i+1, \psi) = \Delta f + \Delta^2 f + \Delta^3 f$.

$\Delta^2 f = (i, i) \in M$, $(\Delta^2 f)^y = \Delta^2 f$.

and $\Delta^3 f = (1,1) \in M$. $(\Delta^3 f)^y = \Delta^3 f$.

Hence $\langle\chi, \chi\rangle$ is a 4-dimensional H - module.

(2) Let $f = (\chi, \chi+\psi+i+1) \in M$. $f^y = f + \Delta f + \Delta^2 f + \Delta^3 f$.

Then $\Delta f = (\psi, \psi+1) \in M$, $(\Delta f)^y = \Delta f + \Delta^3 f$.

$\Delta^2 f = (i, i+1) \in M$, $(\Delta^2 f)^y = \Delta^2 f + \Delta^3 f$.

and $\Delta^3 f = (1,1) \in M$. $(\Delta^3 f)^y = \Delta^3 f$.

Hence $\langle \chi, \chi+\psi+i+1 \rangle$ is a 4-dimensional module.

The multiplication table is, of course, the same and the H-algebras are

$$F \ , \quad \phi \ ,$$

$$\langle \chi, \chi \rangle \ , \quad \langle \chi, \chi+i \rangle \ , \quad \langle \chi, \chi+\psi \rangle \ , \quad \langle \chi, \chi+\psi+i \rangle \ , \quad \langle i, 0 \rangle$$

$$\langle i, i \rangle \ , \quad \langle i, i+1 \rangle \ , \quad \langle 1, 0 \rangle \text{ and } \langle 1, 1 \rangle \ .$$

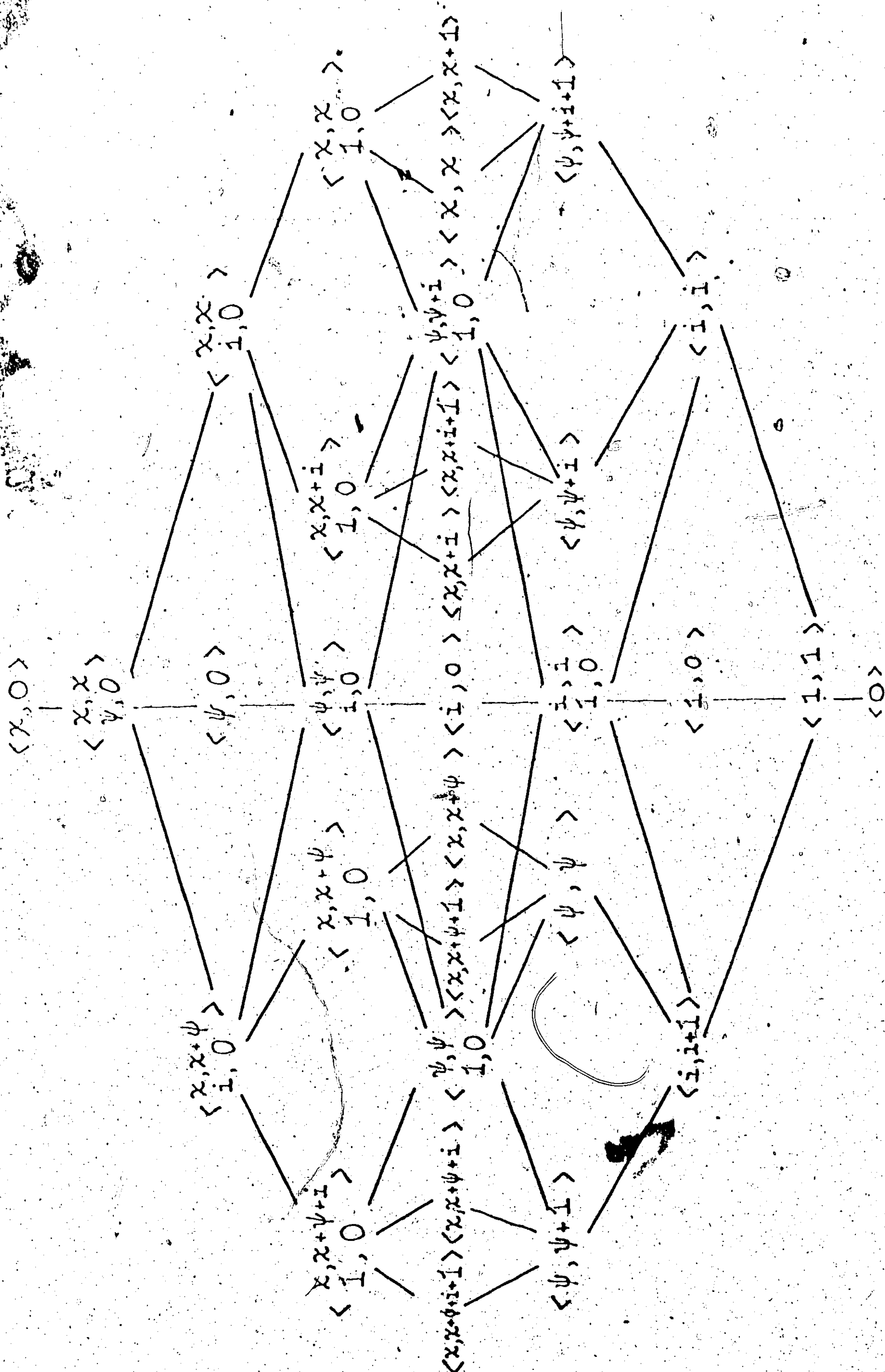


Diagram II $H = D_4$

§3. $H = Q_2$.

Let Q_2 have generators x , y such that $x^4 = 1$, $y^2 = x^2$, $y^{-1}xy = x^{-1}$. Again, with the notation of section 2, we consider the action of Δ and y .

$\Delta f = (\Delta f_1, \Delta^* f_2)$ where Δ^* is as in section 2. But now $f^y = (f_2^{x^2}, f_1)$ and

$$\chi^{x^2} = \chi + i \ , \qquad i^{x^2} = i \ ,$$

$$\psi^{x^2} = \psi + 1 \ , \quad \text{and} \quad 1^{x^2} = 1 \ .$$

Diagram III shows the H - module lattice for $H = Q_2$.

(1) Let $f = (\chi, \chi) \in M$. $\quad f^y = (\chi + i, \chi)$.

Then $\Delta f = (\psi, \psi + i + 1) \in M$, $\quad (\Delta f)^y = (\psi + i, \psi)$.

$\Delta^2 f = (i, i) \in M$, $\quad (\Delta^2 f)^y = \Delta^2 f$,

and $\Delta^3 f = (1,1) \in M$. $\quad (\Delta^3 f)^y = \Delta^3 f$.

$\langle \chi, \chi \rangle$ is a 6-dimensional module.

(2) Let $f = (\chi, \chi + \psi + i + 1) \in M$. $\quad f^y = (\chi + \psi, \chi)$.

Then $\Delta f = (\psi, \psi + 1) \in M$, $\quad (\Delta f)^y = (\psi, \psi)$.

$\Delta^2 f = (i, i+1) \in M$, $\quad (\Delta^2 f)^y = \Delta^2 f + \Delta^3 f$

and $\Delta^3 f = (1,1) \in M$. $\quad (\Delta^3 f)^y = \Delta^3 f$.

$\langle \chi, \chi + \psi + i + 1 \rangle = \langle \chi, \chi + \psi \rangle$ is a 6-dimensional module.

The H - algebras are

$$F \;,\; \phi \;,$$

$$\langle i,0\rangle \;,$$

$$\langle i,i\rangle \;,\; \langle i,i+1\rangle \;,\; \langle 1,0\rangle \text{ and } \langle 1,1\rangle \;.$$

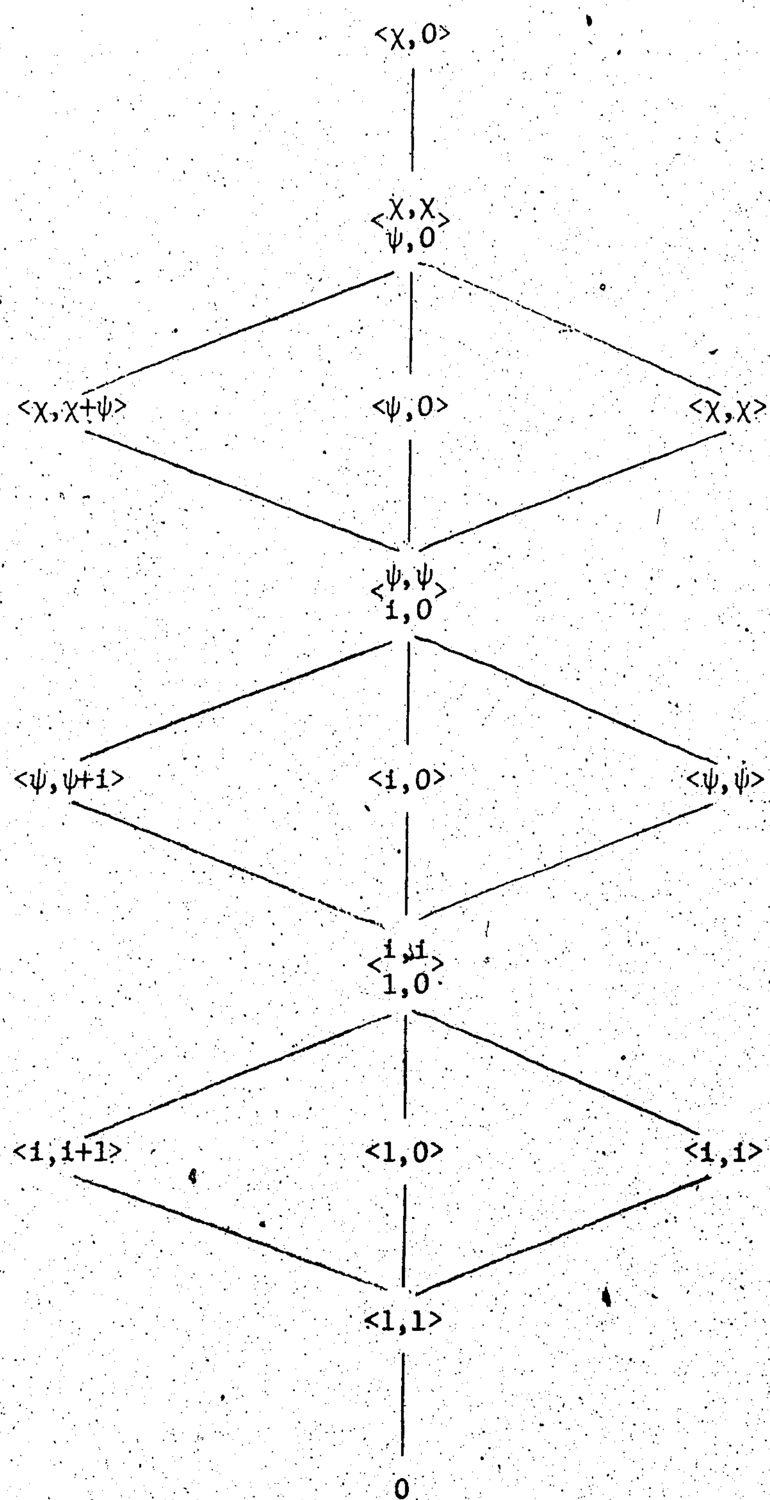


Diagram III

H = Q_2

§4. $H = Q_4$.

Let Q_4 have generators x, y such that $x^8 = 1$, $y^2 = x^4$, $y^{-1}xy = x^{-1}$. Because of the size of this group a slightly different notation will be more convenient.

Let $\chi = x_1$ and $\chi^i = \Delta^i\chi$. So χ^7 is the constant function from $\langle x\rangle$ to F. For the action of y we have $f^y = (f_2^{x^4}, f_1)$ so that,

$$(\chi)^{x^4} = \chi + \chi^4,$$

$$(\chi^1)^{x^4} = \chi^1 + \chi^5,$$

$$(\chi^2)^{x^4} = \chi^2 + \chi^6,$$

$$\vdots$$

$$(\chi^n)^{x^4} = \chi^n + \chi^{n+4}.$$

The action of Δ is $\Delta f = (\Delta f_1, \Delta^* f_2)$ where $\Delta^* f = f^{x^{-1}} + f$ so that

$$\Delta^*\chi = \chi^1 + \chi^2 + \chi^3 + \chi^4 + \chi^5 + \chi^6 + \chi^7,$$

$$\Delta^*\chi^n = \chi^{n+1} + \chi^{n+2} + \ldots .$$

The H - module structure was calculated in the usual way. For example:

Let $f = (\chi^3, \chi^3+\chi^4+\chi^7) \in M$. $f^y = (\chi^3+\chi^4, \chi^3)$.

Then $\Delta f = (\chi^4, \chi^4) \in M$, $\Delta(f^y) = (\chi^4+\chi^5, \chi^4+\chi^5+\chi^6+\chi^7)$

$\Delta^2 f = (\chi^5, \chi^5+\chi^6+\chi^7) \in M$, $= \Delta f + \Delta^2 f$.

$\Delta^3 f = (\chi^6, \chi^6) \in M$, $\Delta^2(f^y) = \Delta^2 f + \Delta^3 f$.

$$\Delta^4 f = (\chi^7,\chi^7) \in M \quad , \quad \Delta^3(f^y) = \Delta^3 f + \Delta^4 f \quad ,$$

and $$\Delta^5 f = 0 \quad . \quad \Delta^4(f^y) = \Delta^4(f) \quad .$$

Hence $<\chi^3,\chi^3+\chi^4+\chi^7>$ is a 6-dimensional module.

The multiplication table is:

$\times$	χ^7	χ^6	χ^5	χ^4	χ^3	χ^2	χ^1	χ
χ^7	χ^7	χ^6	χ^5	χ^4	χ^3	χ^2	χ^1	χ
χ^6		χ^6	χ^4	χ^4	χ^2	χ^2	χ	χ
χ^5			χ^5	χ^4	χ^1	χ	χ^1	χ
χ^4				χ^4	χ	χ	χ	χ
χ^3					χ^3	χ^2	χ^1	χ
χ^2						χ^2	χ	χ
χ^1							χ^1	χ
χ								χ

The H - module $<\chi^4,0>$ is an algebra.

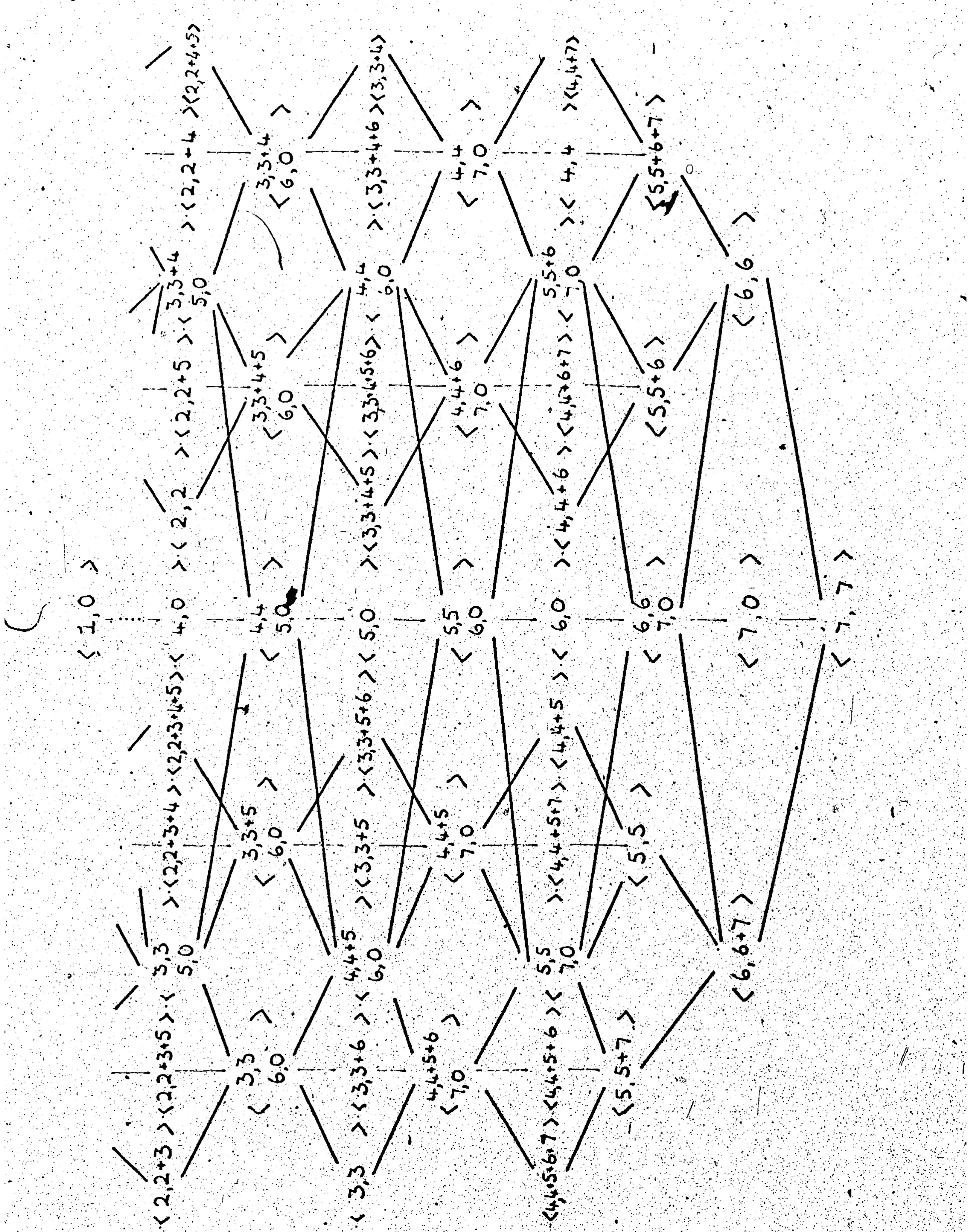


Diagram IV H = Q4

§5. $H = A \times B$.

Let $H = A \times B$ where B is abelian of order prime to the order of A and prime to the characteristic of the field F.

In this situation we are assuming we know the H module structure when $H = A$ and will find out how the "attaching" of an abelian group of order prime to the order of A affects this structure. The results are due to Bercov [2].

First we extend the field F, used when $H = A$, by the m^{th} root of unity, where $|B| = m$. This is in order that the set L of all linear characters of B takes on its values in the new field. [Note. This is our sole excursion into character theory.] The H - module structure of $H = A$ may be affected by this field extension. Each module will contain more functions but to all intents and purposes we can regardit as being unchanged, but new modules may arise, e.g. $\langle \psi, \psi + \alpha i \rangle$, $\alpha \in F$, appears when $A = Q_2$. We assume the effect of this extension is known.

If we let F_A be the set of functions which are zero outside of A then F_A is clearly A - isomorphic to the functions from A to F, so we know all the A - modules of F_A.

Define, for every $f \in F_A$, and every $\lambda \in L$,

$$f^{\lambda} = \sum_{b \in B} \lambda(b) f^{b} ,$$

so that for $a \in A$, $b \in B$ we have $f^{\lambda}(ab) = f(a)\lambda(b)$, and for $N \leq F_A$ put $N^{\lambda} = \{f^{\lambda} \mid f \in N\}$. We put $F_A{}^{\lambda} = F^{\lambda}$.

Again, F_A is clearly A - isomorphic to F^λ for all $\lambda \epsilon L$ so we know the A - modules of F^λ, namely the set

$$\{N^\lambda \mid N \underset{A}{\leq} F_A\} .$$

However,

$$(f^\lambda)^b = (\sum_{b_1 \epsilon B} \lambda(b_1) f^{b_1})^b = \sum_{b_1 \epsilon B} \lambda(b_1) f^{b_1 b}$$

$$= \lambda(b^{-1}) \sum_{b_1 \epsilon B} \lambda(b_1 b) f^{b_1 b} = \lambda(b^{-1}) f^\lambda$$

so that $N^\lambda \leq F^\lambda$ is an $A{\times}B$ - module if and only if N is an A - module of F_A .

Thus we know all the H - modules of F^λ for all $\lambda \epsilon L$. To get the H modules of F we have the following lemmas.

Lemma 2.3: F is the direct sum of the F^λ, $\lambda \epsilon L$.

Proof: Suppose the sum of the F^λ is not direct. Then for an appropriate numbering of the characters of L and for some $f_i \epsilon F_A$ and $\lambda_i \epsilon L$ we have

$$f_1^{\lambda_1} = \sum_{i>1} f_i^{\lambda_i} \neq 0 .$$

For any $a \epsilon A$ with $f_1(a) \neq 0$ and all $b \epsilon B$ we get

$$f_1^{\lambda_1}(ab) = f_1(a)\lambda_1(b) = \sum_{i>1} f_i(a)\lambda_i(b)$$

and therefore

$$\lambda_1 = \sum_{i>1} \frac{f_i(a)}{f_1(a)} \lambda_i$$

contrary to the well-known fact [7; pg. 326] that L is an independent set over $\mathcal{F}$. Since each F^λ has $\mathcal{F}$ - dimension equal to that of F_A, the direct sum has dimension m times the dimension of F_A which is the dimension of F. Thus the proof is complete.

For any H - module M, we put $M^\lambda = M \cap F^\lambda$.

Lemma 2.4: Let M be an H - module. Then M is the direct sum of the M^λ, $\lambda \in L$.

Proof: We have that the sum is direct from the previous lemma. Thus to show that M is the sum of the M^λ it suffices to show that if $f^\lambda + \sum_{\lambda_i \neq \lambda} f_i^{\lambda_i}$ belongs to M then f^λ belongs to M. If we have such a function, say g, with k λ_i's $\neq \lambda$ [without loss of generality let $i = 1,\ldots,k$] then we may write g as

$$g = f^\lambda + f_1^{\lambda_1} + \sum_{i=2}^{k} f_i^{\lambda_i} .$$

Since $\lambda_1 \neq \lambda$ there exists a $b \in B$ such that $\lambda_1(b) \neq \lambda(b)$ and so the function $\frac{1}{\lambda(b)-\lambda_1(b)} [g^{b^{-1}} - \lambda_1(b)g]$ is contained in M and is the sum of f^λ and fewer $f_i^{\lambda_i}$.

Eliminating all of the λ_i's $\neq \lambda$ in this way gives

$$f^\lambda \in M .$$

For information about multiplication we have the following lemma.

Lemma 2.5:

(a) $F^\lambda F^\mu \leq F^{\lambda\mu}$ for all $\lambda,\mu \in L$.

(b) If N is an algebra of F_A then the module M of F defined by $M = \sum_{\lambda\in L} N^\lambda$ = span N is an algebra of F .

Proof: Let $a \in A$; $b \in B$.

$$f^\lambda g^\mu(ab) = f^\lambda(ab)g^\mu(ab) = f(a)\lambda(b)g(a)\mu(b)$$

$$= fg(a)\lambda\mu(b) = (fg)^{\lambda\mu}(ab) \ .$$

This gives both (a) and (b).

Finally, for information on the action of G - endomorphisms on H - modules, where $G \geq H$, we have the following lemma.

Lemma 2.6: Let ϕ be an endomorphism on F . Then

$$\phi(F^\lambda) \leq F^\lambda \quad \text{for all} \quad \lambda \in L \ .$$

Proof: Suppose not. Then for some functions $f_i \in F_A$, $\lambda,\mu \in L$, with $\lambda \neq \mu$ and $f,g \neq 0$, we have

$$\phi(f^\lambda) = g^\mu + \sum_{\lambda_i\neq\mu} f_i^{\lambda_i} \ .$$

Choose $b \in B$ with $\lambda(b) \neq \mu(b)$. Then

$$\phi((f^{\lambda})^{b^{-1}}) = \phi(\lambda(b)f^{\lambda}) = \lambda(b)g^{\mu} + \sum_{\lambda_i \neq \mu} \lambda(b)\ f_i^{\lambda_i}$$

and

$$(\phi(f^{\lambda}))^{b^{-1}} = \mu(b)g^{\mu} + \sum_{\lambda_i \neq \mu} \lambda_i(b)\ f_i^{\lambda_i} .$$

Since ϕ is an H - map these expressions should be equal but this would contradict lemma 2.3.

§6. Perpendiculars and Kernels.

Perpendiculars: It will be useful to know explicitly the effect of taking the perpendicular of a module M, i.e. knowing $M^{\perp}$. Because of theorem 1.19 we know the dimension of $M^{\perp}$ immediately and finding the correct module does not prove to be difficult in the cases dealt with in sections 1-3. We need to find every $g \in F$ such that

$$\sum_{a \in H} f \cdot g(a) = 0 \quad \text{for all} \quad f \in M .$$

Every function in the module $\langle{}^{\chi,\chi}_{\psi,0}\rangle$ has non-zero values on an *even* number of points of H and every function not contained in $\langle{}^{\chi,\chi}_{\psi,0}\rangle$ has non-zero values on an *odd* number of points of H so to show a module N is not $M^{\perp}$ we need only find a $g \in N$ such that

$$f \cdot g \notin \langle{}^{\chi,\chi}_{\psi,0}\rangle \quad \text{for some} \quad f \in M .$$

For example, let $H = Z_4 \times Z_2$ and $M = \langle{}^{\chi,\chi}_{1,0}\rangle$. Then

$$M^{\perp} \neq \langle 1,0 \rangle \quad \text{because} \quad (\chi,\chi)\cdot(1,0) = (\chi,0)$$

and

$$M^{\perp} \neq \langle i,i+1 \rangle \quad \text{because} \quad (\chi,\chi)\cdot(i,i+1) = (\chi,0) .$$

Hence $M^{\perp} = \langle i,i \rangle$.

Instead of giving a complete list of perpendiculars, Diagrams I, II and III have been drawn so that the intuitive idea of "$\perp$" turning the lattice structure "upside down" really does work and they, therefore, contain all the information required.

In the case $H = A \times B$, where the lattice of modules is shown for $H = A$, and B is abelian and of order prime to the order of A, section 5 tells us what the lattice structure is like. Taking perpendiculars, in this case, causes the module you would expect to appear in the σ component to appear in the σ^{-1} component. Fortunately, in one case that we consider in Chapter 3 we can prove that

$$\text{if } M^{\lambda} = N^{\lambda} \text{ then } M^{\lambda^{-1}} = N^{\lambda^{-1}} \quad \text{for } M \leq F \text{ and } N \leq F_A$$

so the intuitive effect of "$\perp$" is still correct.

Kernels: In chapter 3 we shall create a G - endomorphism ϕ, and it will be useful to know the kernel of ϕ, $\ker \phi$. If we know the image of this map, $\mathrm{Im}\ \phi$, then we can use the identity

$$\mathrm{Im}\ \phi \cong F / \ker \phi$$

and because ϕ is a G - endomorphism this is a G - isomorphism.

Given $\mathrm{Im}\ \phi$, the dimension of $\ker \phi$ is easily calculated. To find the modules of this dimension which obey the above identity is not so easy, yet it happens for the group $Z_4 \times Z_2$, that the only possibility for $\ker \phi$ is $(\mathrm{Im}\ \phi)^{\perp}$. Hence again Diagram I contains the information in pictorial form.

We give two examples showing the calculation of a kernel.

Let $H = Z_4 \times Z_2$ and let $\mathrm{Im}\ \phi = \langle i, i \rangle$. Hence the dimension of $\ker \phi = 6$.

Let $\phi(\chi,0) = a(i,i) + b(1,1)$, $a, b \in F$.

Therefore $\phi(0,\chi) = a(i,i) + b(1,1)$ and so

$$\phi(\chi,\chi) = a(i,i) + b(1,1) + a(i,i) + b(1,1) = 0 .$$

Also $\phi(\psi,0) = a(1,1)$ and therefore

$$\phi(i,0) = 0$$

Thus the module $\langle{}^{\chi,\chi}_{i,0}\rangle$ is contained in $\ker \phi$ but the dimension of $\ker \phi$ is 6 . Hence $\ker \phi = \langle{}^{\chi,\chi}_{i,0}\rangle$.

Let $H = Q_2$ and $\mathrm{Im}\,\phi = \langle\psi,\psi\rangle$. Hence the dimension of $\ker \phi$ is 4 .

Let $\phi(\chi,0) = a(\psi,\psi) + b(i,i+1) + c(1,1) + d(1,0)$,

$a, b, c, d \in F$.

Therefore $\phi(\psi,0) = a(i,i+1) + b(1,1)$,

$\phi(i,0) = a(1,1)$,

$\phi(0,i) = a(1,1)$ and

$\phi(0,\psi) = a(i+1,i) + b(1,1)$.

Hence
$$\begin{aligned}\phi(\psi,\psi+i) &= a(i,i+1) + b(1,1) \\ &+ a(i+1,i) + b(1,1) \\ &+ a(1,1) \\ &= 0\end{aligned}$$

Thus $\ker \phi = \langle\psi,\psi+1\rangle$.

CHAPTER III

B - Groups

Now we are armed with the H - module structure of several groups and several lemmas from Chapter I to deal with them. We are therefore in a position to consider the possibility of some of these groups being B - groups.

We begin by constructing a G - endomorphism and give a useful lemma concerning this endomorphism.

§1. The G - Endomorphism.

We need to use a small amount of Schur Ring theory to construct this map.

We assume that $G \geq H$ is transitive but not 2-transitive on H and therefore the orbit S of $G_1 = \{g \in G \mid 1^g = 1\}$ which contains x^2 is not equal to $H/\{1\}$, [10, p. 63]. Taking $T = S$ or $T = S\cup\{1\}$ according as S is even or odd, we define a G - endomorphism ϕ by

$$\phi(\chi_1) = \chi_T \ , \ \phi(f^g) = [\phi(f)]^g \text{ and } \phi(f_1+f_2) = \phi(f_1) + \phi(f_2) \ .$$

This map is well-defined since T is invariant under G_1.

Denote the G - module $\text{Im}\ \phi$ and $\ker \phi$ by I and K respectively.

Lemma 3.1: Let $H = A\times B$, with $A = Z_4\times Z_2$ or Q_2 and B an abelian group of odd order. If I^{λ} is generated by $f^{\lambda} = \phi(\chi_1^{\lambda})$, then $I^{\lambda^{-1}}$ is generated by $g^{\lambda^{-1}} = \phi(\chi_1^{\lambda^{-1}})$ where $g(a) = f(a^{-1})$ for all $a \in A$.

<u>Proof</u>: If S is an orbit of G_1 then $S^* = \{t^{-1} \mid t \in S\}$ is also an orbit of G_1, [10, p. 57]. So we have $S = S^*$ because orbits are equal or disjoint and therefore $T = T^*$.

$$\chi_1^{\lambda} = \sum_{b \in B} \lambda(b)\chi_b \quad \text{so for } a \in A,\ c \in B$$

$$f^{\lambda}(ac) = \phi(\chi_1^{\lambda})(ac) = \phi[\sum_{b\in B} \lambda(b)\chi_b](ac) = [\sum_{b\in B} \lambda(b)\phi(\chi_b)]\ (ac)$$

$$= \sum_{b\in B} \lambda(b)\chi_{Tb}(ac) = \sum_{t\in T\cap aB} \lambda(t^{-1}ac) .$$

Hence

$$f(a) = \sum_{t\in T\cap aB} \lambda(t^{-1}a) \quad \text{for all } a \in A .$$

Thus

$$g(a) = \sum_{t\in T\cap aB} \lambda^{-1}(t^{-1}a) = \sum_{t\in T\cap aB} \lambda(a^{-1}t) = \sum_{t\in T\cap aB} \lambda(ta^{-1})$$

$$= \sum_{t^{-1}\in T\cap aB} \lambda(t^{-1}a^{-1}) = \sum_{t\in T\cap a^{-1}B} \lambda(t^{-1}a^{-1}) = f(a^{-1})$$

since if $t^{-1} \in T\cap aB$ then $t \in T\cap Ba^{-1} = T\,a^{-1}B$. Thus $g(a) = f(a^{-1})$ for all $a \in A$. □

For the following two sections we shall take F to be the required extention of the field $\{0,1\}$.

§2. $H = Q_2 \times B$.

In this section we consider the group $H = A \times B$ where $A = Q_2$ and B is an abelian group of odd order. The H - module structure comes from sections 3 and 5 of Chapter II and is shown in Diagram V.

Theorem 3.2: H is a B - group.

Proof: We assume that a group G contains the right regular representation of H as a transitive group but that G is not 2-transitive. We shall show that this implies that G is not primitive by assuming G is primitive and then proving that $I = \text{Im}\,\phi$ does not exist, thus obtaining a contradiction.

Let $D(M) = \max_{\lambda \varepsilon L} \{\text{dimension of } M^{\lambda}\}$.

(1) $I \not\subset C$.

Proof: $\phi(\chi_1)$ is 1 on T and 0 on the complement of T and so is not a constant function.

(2) $D(I) \neq 8$.

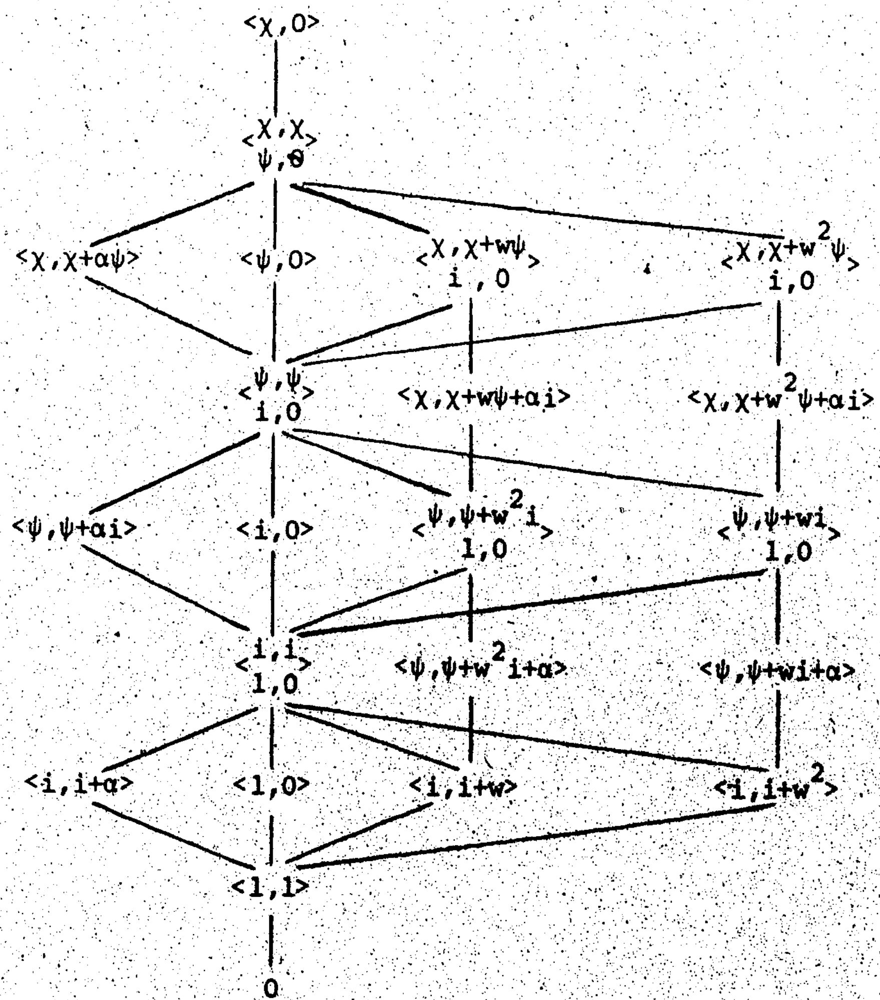
Proof: $\phi(\chi_H) = \phi(\sum_{h \varepsilon H} \chi_h) = \sum_{h \varepsilon H} \phi(\chi_1)^h = \sum_{h \varepsilon H} \chi_T^{\,h}$

$$= \sum_{h \varepsilon H} \sum_{t \varepsilon T} \chi_{th} = |T| \sum_{h \varepsilon H} \chi_h = 0 .$$

So $K^1 \neq \phi$ and hence $I^1 \neq F^1$.

Let $S = \{\lambda \varepsilon L \mid I^{\lambda} = F^{\lambda}\}$.

Since the module $\langle 1,1 \rangle^{\lambda}$ is contained in all non-zero modules of F^{λ}, if $K^{\lambda} \neq \phi$ and if $I^{\lambda} \neq \phi$ then both K^{λ} and I^{λ} contain $\langle 1,1 \rangle^{\lambda}$. This means that if we repeat the map ϕ enough times



[α ranges through F except $\alpha \neq w^n$ where $w^3 = 1$.]

<u>Diagram V</u>

$H = Q_2 \times B$

we can get the G-module $N = \sum_{\lambda \varepsilon S} F^{\lambda}$

Now $F^1 \leq N:N$ since $F^1 F^{\lambda} \leq F^{\lambda}$ for all λ [2.13] but $F^{\lambda^{-1}} \not\leq N:N$ for all $\lambda \varepsilon S$ since $F^1 \not\leq N$. Hence $N:N$ is a G-algebra [1.15] containing C but strictly less than F, contradicting 1.13. □

(3) $D(I) \not< 3$.

Proof: If $D(I) < 3$ then $I \leq$ Span $<i,0>$ and so the algebra generated by I and C is strictly greater than C but is contained in span $<i,0>$, contradicting 1.13. □

(4) $D(I) \leq 4$.

Proof: We may discount all those modules which are generated by more than one function, for I^{λ} is generated by only one function, namely $\phi(\chi_i^{\lambda})$.

If $I^{\lambda} = <\chi,\chi+\alpha\psi>^{\lambda}$ then $[\ker \phi]^{\lambda} = <i,i+\alpha>^{\lambda}$ and if $I^{\lambda} = <\psi,0>^{\lambda}$ then $[\ker \phi]^{\lambda} = <1,0>^{\lambda}$ so in both cases $\phi^3(F) \leq$ span $<i,0>$ and $D[\phi^3(F)] = 2$.

If $I^{\lambda} = <\chi,\chi+w\psi+\alpha i>^{\lambda}$ where $w^3 = 1$ then $[\ker \phi]^{\lambda}$ is of the form $<\psi,\psi+w^2 i+\beta>^{\lambda}$ for some $\beta \varepsilon F$ and so $\phi^2(F) \leq$ span $<i,0>$ and $D[\phi^2(F)] = 2$. □

(5) $D(I) \neq 3$.

[N.B. If $w \not\in F$, where $w^3 = 1$, then the only 3 dimensional module is $<{i,i \atop 1,0}>$.]

Proof: If $I^{\lambda} = \langle\psi,\psi+wi+\alpha\rangle^{\lambda}$ then $[\ker\phi]^{\lambda}$ is of the form $\langle\chi,\chi+w\psi+\beta i\rangle^{\lambda}$ for some $\beta \in F$. Therefore $I^{\lambda} \cap \ker\phi = \langle i,i+w\rangle^{\lambda}$. For those components I^{σ} with dimension less than 3

$$I^{\sigma} \cap \ker\phi = I^{\sigma},$$

so $I \cap \ker\phi \leq \text{span}\langle i,0\rangle$ and $D[I \cap \ker\phi] = 2$. □

(6) $$D(I) \neq 4 .$$

Proof: If there is no component of the form $\langle\psi,\psi+\alpha i\rangle^{\lambda}$ for some $\lambda \in L$, $\alpha \in F$ then $I \leq \text{span}\langle i,0\rangle$ so we may assume we have such a component. If $I^{\lambda} = \langle\psi,\psi+\alpha i\rangle^{\lambda}$ then $[\ker\phi]^{\lambda} = \langle\psi,\psi+(1+\alpha)i\rangle$ and so $I^{\lambda} \cap \ker\phi = \langle\begin{smallmatrix} i,i \\ 1,0 \end{smallmatrix}\rangle^{\lambda}$.

Therefore $I \cap \ker\phi \leq \text{span}\langle i,0\rangle$ and $D(I \cap \ker\phi) = 3$. □

We have now shown that I does not exist, giving the contradiction hoped for, and therefore the proof of 3.2 .

3. $H = Z_4 \times Z_2 \times B$

In this section we consider the group $H = Z_4 \times Z_2 \times B$ but restrict our studies to the case when $B = Z_p \times Z_p$ (p an odd prime). As in section 2 of this chapter we perform a case by case analysis of the possible values of I. The H-module structure is shown in diagram VI.

Lemma 3.3. Let $H = A \times B$, B abelian of odd order, $A = Z_4 \times Z_2$. If $I^{\sigma} = \langle f \rangle^{\sigma}$ then $I^{\sigma^n} = \langle f \rangle^{\sigma^n}$ for all integers n prime to the order of B.

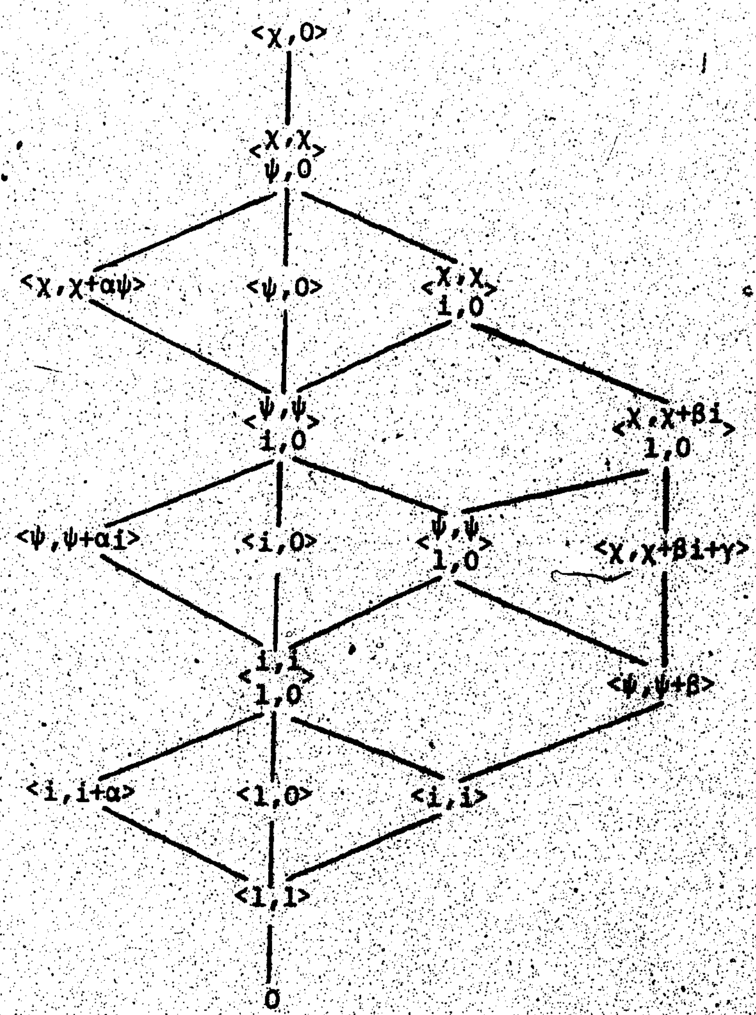
Proof: Since H is abelian, if S is an orbit of G_1 on H then $S^{(m)} = \{t^m | t \in S, m \text{ an integer}\}$ is an orbit of G_1 for all m prime to the order of H [10, p. 58]. Hence, since we chose S to contain an element of order 2, we have that $S = S^{(m)}$ and so $T = T^{(m)}$ for all m prime to the order of B.

Now, from 3.1, if $f^{\sigma} = \phi(\chi_1^{\sigma})$ and $g^{\sigma^m} = \phi(\chi_1^{\sigma^m})$ then

$$f(a) = \sum_{t \in T \cap aB} \sigma(t^{-1}a) \quad \text{and}$$

$$g(a) = \sum_{t \in T \cap aB} \sigma^m(t^{-1}a) = \sum_{t \in T \cap aB} \sigma(t^{-m}a^m) = \sum_{t^m \in T \cap a^m B} \sigma(t^{-m}a^m) = f(a^m)$$

since $T = T^{(m)}$ and if $t \in T \cap aB$ then $t^m \in T^{(m)} \cap Ba^m = T \cap a^m B$. We may choose an n such that $n \equiv m (\text{mod } |B|)$ and $m \equiv 1 (\text{mod } 4)$. Hence $g(a) = f(a)$ for all $a \in A$ and so if $I^{\sigma} = \langle f \rangle^{\sigma}$ then $I^{\sigma^n} = \langle f \rangle^{\sigma^n}$. □



[α, β and γ range through F , α ≠ 1 .]

Diagram VI

$H = Z_4 \times Z_2 \times B$

This property, that $T = T^{(m)}$ for all m prime to the order of H, also tells us that I^{λ} must be generated by a characteristic function for all $\lambda \in L$. For if $ab \in T \cap aB$ then $(ab)^m = ab^m \in T \cap aB$ for all $m \equiv 1 \pmod 4$ and $(m, |B|) = 1$. Hence, if the order of b is p, by suitable choices of m we have that $ab^n \in T \cap aB$ for all integers $n < p$. Therefore $\sum_{t \in T \cap a<b>} \lambda(t^{-1}a) = \lambda(b) + \lambda(b^2) + \ldots + \lambda(b^{p-1}) = 1$. So $\phi(\chi_1^{\lambda}) = f^{\lambda}$, where $f(a) = \sum_{t \in T \cap aB} \lambda(t^{-1}a) = \sum_{t \in T \cap a} \lambda(t^{-1}a) + \sum \sum_{t \in T \cap a<b>} \lambda(t^{-1}a)$ is a sum of ones and is therefore itself either one or zero. Thus f is a characteristic function.

If $I \not\geq C$ then we could consider the G-module $I + C$ so let us assume from now on that $I \geq C$.

<u>Lemma 3.4</u>: If $I^{\sigma} = \langle f \rangle^{\sigma}$ then $f^1 \in [I^{(4)}]^1$ where f is a characteristic function.

<u>Proof</u>: We consider every possible choice of 4 points of H and give the function from I which agrees with f^1 on these 4 points.

If f^1 has non-zero value on 4 points chosen use $(1,1)^1$.

If f^1 has non-zero value on 3 points chosen, zero on 1, use $(1,1)^1 + C_1(f^{\sigma} + (1,1)^{\sigma})$, $C_1 \in F$.

If f^1 has non-zero value on 1 point chosen, zero on 3, use $C_2 f^{\sigma}$, $C_2 \in F$.

If f^1 has non-zero value on 2 points chosen, zero on 2, then we have the following situation.

Points	We want f^1		and we have in I $f^{\sigma}+(1,1)^{\sigma}$	$(1,1)^1$	$f^{\sigma^{-1}}+(1,1)^{\sigma^{-1}}$
a_1b_1	1		0	1	0
a_2b_2	1		0	1	0
a_3b_3	0		$\sigma(b_3)$	1	$\sigma^{-1}(b_3)$
a_4b_4	0		$\sigma(b_4)$	1	$\sigma^{-1}(b_4)$

[Note that we have $f^{\sigma^{-1}}$ in I because of 3.4.)

If $\sigma(b_3) = \sigma(b_4)$ then use $(1,1)^1 + \sigma^{-1}(b_3)[f^{\sigma}+(1,1)^{\sigma}]$.

If $\sigma(b_3) \neq \sigma(b_4)$ then $\sigma^{-1}(b_3) \neq \sigma^{-1}(b_4)$ so we can use

$$(1,1)^1 + [\sigma(b_4)+\sigma^2(b_3)\sigma^{-1}(b_4)]^{-1}[f^{\sigma}+(1,1)^{\sigma}+\sigma^2(b_3)(f^{\sigma^{-1}}+(1,1)^{\sigma^{-1}})] +$$
$$[\sigma(b_3)+\sigma^2(b_4)\sigma^{-1}(b_3)]^{-1}[f^{\sigma}+(1,1)^{\sigma}+\sigma^2(b_4)(f^{\sigma^{-1}}+(1,1)^{\sigma^{-1}})] .$$

[In essence we create functions which are

0	0	
0	0	on the 4 chosen points and then add them
1	0	both to $(1,1)^1$
0	1	.]

□

We shall now give a lemma which we shall use frequently in this section.

Lemma 3.5: The 4-closure of span N is equal to span N for all $N \leq F_A$, N generated by characteristic functions, $N \neq \langle \chi_{\psi}, \chi_0 \rangle$.

Proof: For various functions f we show that if $f \notin N$ then $f^{\rho} \notin [\text{span } N]^{(4)}$ for all $\rho \in L$ by giving 4 points of H on which no function of span N a[illegible] f^{ρ}, all $\rho \in L$.

Function f		4 Points
$(\psi$ [illegible]		$1, x^2, y, x^2y$
$(i,$ [illegible]		$1, x, y, xy$
	or	$1, x, x^2y, x^3y$
	or	$1, x, xy, x^2y$
$(1,0)$		$1, y, xy, x^2y$
	or	$1, y, xy, x^3y$
	or	$1, y, x^2y, x^3y$
	or	$1, xy, x^2y, x^3y$
$(\chi, \chi+\alpha\psi+\beta i+\delta)$		$1, x, x^2, x^3$
$(\psi, \psi+\alpha i+\beta)$		$1, x, x^2, x^3$
$(i, i+\alpha)$		$1, x, x^2, x^3$

where $\alpha, \beta, \gamma\ \delta \in F$. □

Now let us consider the possible values of I

(1) (a) $I \not\leq C$,

(b) $D(I) \neq 8$,

(c) $D(I) \not\leq 2$ and

(d) $D(I) \not> 4$

Proof: Parts (a) and (b) are similar to 3.2 parts (1) and (2).

Part (c) is true because if $D(I) \leq 2$ then $I \leq \text{span} <i,0>$, contradicing 1.13.

Part (d) comes from our knowledge of $\ker \phi$.

If $D(I) = 7$ then $\phi^6(F) \leq \text{span} <i,0>$ and $D[\phi^6(F)] = 2$.

If $D(I) = 6$ then $\phi^3(F) \leq \text{span} <i,0>$ and $D[\phi^3(F)] = 2$.

If $D(I) = 5$ then $\phi^2(F) \leq \text{span} <i,0>$ and $D[\phi^2(F)] = 2$.

[In fact $D(I) = 7$ and $D(I) = 5$ may be discounted for another reason, namely that all components of this dimension are generated by more than one function, whereas I^σ is generated by one function, $\phi(\chi_1^\sigma)$ for all $\sigma \in L$.]

(2) $D(I) \neq 3$.

Proof: If $I \leq \text{span} <\chi,\chi>$, $\text{span} <\chi,\chi+i>$ or $\text{span} <i,0>$ then the algebra generated by I and C contradicts 1.13. Hence we may assume $[I^{(4)}]^1 = <{\psi,\psi \atop 1,0}>$.

We therefore have that $[I^{(4)}]^2 \leq \text{span} <{\chi,\chi \atop \psi,0}>$ and the σ component of $[I^{(4)}]^2 = <{\chi,\chi \atop \psi,0}>^\sigma$ for all $\sigma \in L$ such that dimension $I^\sigma = 3$.

By reference to the kernel of ϕ we have that $D[\phi(I^{(4)})^2] = 2$ and so $\phi[(I^{(4)})^2] \leq$ span $<i,0>$. The algebra generated by $\phi[(I^{(4)})^2]$ and C contradicts 1.13. □

(3) $I^\sigma \neq <\underset{1,0}{\psi,\psi}>^\sigma$ for all $\sigma \in L$.

<u>Proof</u>: I^σ is generated by *one* function, namely $\phi(\chi_1^\sigma)$, but $<\underset{1,0}{\psi,\psi}>^\sigma$ is a module which is generated by more than one function. □

(4) $\dot{I}^\sigma \neq <\psi,\psi+i>^\sigma$ for all $\sigma \in L$, $\sigma \neq 1$.

<u>Proof</u>: By using 3.5 we may split the situation, I containing $<\psi,\psi+i>^\sigma = \dot{I}^\sigma$ for some $\sigma \neq 1$, into 3 cases,

(a) $[I^{(4)}]^1 = <\psi,\psi+i>$,

(b) $[I^{(4)}]^1 = <\underset{1,0}{\psi,\psi}>$ and

(c) $[I^{(4)}]^1 = <\underset{i,0}{\chi,\chi}>$.

(a) $[I^{(4)}]^2 \leq$ span $<\underset{\psi,0}{\chi,\chi}>$ and the λ component of $[I^{(4)}]^2 = <\underset{\psi,0}{\chi,\chi}>^\lambda$ for all λ such that dimension $I^\lambda = 4$.

By reference to ker ϕ ,

$$D(\phi[(I^{(4)})^2]) = 3 \text{ and } \phi[(I^{(4)})^2] \leq \text{span } <i,0> .$$

The algebra generated by $\phi[(I^{(4)})^2]$ and C contradicts 1.13. □

(b) We make the following claim.

<u>Claim 3.6</u>: If $(1,i)^\sigma$, $(1,0)^\sigma$ and $(i,0)^1$ are contained in $I^{(4)}$

then $(i,0)^\sigma \in I^{(4)}$.

Proof: Using the notation of $A_1 = \langle x \rangle$, $A_2 = \langle x \rangle y$,

Number of Points from		
$A_1 \times B$	$A_2 \times B$	
4	0	use $(i,i)^\sigma$
3	1	use $(i,i)^\sigma + C_1(0,1)^\sigma$
1	3	use $(1,0)^\sigma$
2	2	then

if $(i,0)^\sigma$ is non-zero on both points of $A_1 \times B$ then use $(1,0)^\sigma$;
if $(i,0)^\sigma$ is non-zero on only one point of $A_1 \times B$ then use $C_2(i,0)^1$, $C_2 \in F$. □

This claim tells us that if $I^\sigma = \langle \psi, \psi+i \rangle$, $\sigma \neq 1$, and $[I^{(4)}]^1 = \langle {\psi,\psi \atop i,0} \rangle$ then $(i,0)^\sigma \in I^{(4)}$ and by 3.3, $I^{(4)} \leq \text{span} \langle {\psi,\psi \atop i,0} \rangle$. Hence $[I^{(4)}]^\sigma = \langle {\psi,\psi \atop i,0} \rangle$.

By considering $\ker \phi$,

$$\phi[I^{(4)}] \leq \text{span} \langle 1,1 \rangle \quad \text{and} \quad [\phi(I^{(4)})]^\sigma = \langle 1,1 \rangle^\sigma .$$

The algebra generated by $\phi[I^{(4)}]$ and C contradicts 1.13. □

(c) The claim of part (b) still holds but in this case we only know that $I^{(4)} \leq \text{span} \langle {\chi,\chi \atop i,0} \rangle$.

Thus if $I^\sigma = \langle \psi, \psi+i \rangle$ then

$$\langle {\psi,\psi \atop i,0} \rangle^\sigma \leq [I^{(4)}]^\sigma \leq \langle {\chi,\chi \atop i,0} \rangle^\sigma$$

and

$$\phi[I^{(4)}] \leq \text{span} <i,i> \quad , \quad <1,1>^{\sigma} \leq [\phi(I^{(4)})]^{\sigma} \leq <i,i>^{\sigma} \quad .$$

Again, the algebra generated by $\phi[I^{(4)}]$ and C contradicts 1.13.

This completes the proof of (4). □

(5) $I^{\sigma} \neq <\chi,\chi+1>$ or $<\chi,\chi+i+1>$ for all $\sigma \in L$, $\sigma \neq 1$.

Proof: It is sufficient to show $I^{\sigma} \neq <\chi,\chi+1>^{\sigma}$ because $<\chi,\chi+1>$ and $<\chi,\chi+i+1>$ would interchange if a different coset representative in A had been chosen, i.e. x^2y instead of y.

The proof is similar to (4) in that we deal with 3 possible cases, the final two needing a claim about 4-closure.

(a) $[I^{(4)}]^1 = <\chi,\chi+1>$:

$$[I^{(4)}]^2 \leq \text{span} <\begin{smallmatrix}\chi,\chi \\ \psi,0\end{smallmatrix}>$$

and the λ component of $[I^{(4)}]^2 = <\begin{smallmatrix}\chi,\chi \\ \psi,0\end{smallmatrix}>$ for all λ such that dimension $I^{\lambda} = 4$.

By reference to $\ker \phi$,

$$D(\phi[(I^{(4)})^2]) = 3 \quad \text{and} \quad \phi[(I^{(4)})^2] \leq \text{span} <\chi,\chi> \quad .$$

The algebra generated by $\phi[(I^{(4)})^2]$ and C gives the usual contradiction to 1.13. □

(b) $[I^{(4)}]^1 = \langle \chi,\chi_{1,0} \rangle$:

We make the following claim.

<u>Claim 3.7</u>: If $(\psi,\psi)^\sigma \in I^{(4)}$ and $(\chi,\chi)^1 \in I^{(4)}$ then $(\chi,\chi)^\sigma \in I^{(4)}$.

<u>Proof</u>:

If $(\chi,\chi)^\sigma$ has non-zero value on 4 points, use $(1,1)^\sigma$.

If $(\chi,\chi)^\sigma$ has non-zero value on 3 points, zero on 1, use $(1,1)^\sigma + C_1(\chi+1,\chi+1)^1$, $C_1 \in F$.

If $(\chi,\chi)^\sigma$ has non-zero value on 1 point, zero on 3, use $C_2(\chi,\chi)^1$, $C_2 \in F$.

If $(\chi,\chi)^\sigma$ has non-zero value of 2 points, zero on 2, then we must have that 2 points are in the set $\{1,y\}\times B$ and 2 points in the set $\{x,x^2,x^3,xy,x^2y,x^3y\} \times B$.

If 2 points are contained in $\{x^2,x^3,x^2y,x^3y\}$ use $(\psi,\psi)^\sigma$.

If 2 points are contained in $\{x,x^3,xy,x^3y\}$ use $(i,i)^\sigma$.

If 2 points are contained in $\{x,x^2,xy,x^2y\}$ use $(\psi+i+1,\psi+i+1)^\sigma$.

This covers all possibilities for the 4 chosen points. □

From 3.3 we have that $I^{(4)} \leq \text{span} \langle \chi,\chi_{1,0} \rangle$ and with this claim it means that if $I^\sigma = \langle \chi,\chi+1 \rangle$ then

$$[I^{(4)}]^\sigma = \langle \chi,\chi_{1,0} \rangle .$$

Considering $\ker \phi$,

$$\phi[I^{(4)}] \leq \text{span} <1,1> \quad \text{and} \quad (\phi[I^{(4)})^{\sigma} = <1,1>^{\sigma} .$$

The algebra generated by $\phi[I^{(4)}]$ and C gives us the contradiction. □

(c) $[I^{(4)}]^1 = <^{\chi,\chi}_{i,0}>$:

We use the claim of part (b) and 3.3 to give that if

$$I^{\sigma} = <\chi,\chi+1>^{\sigma} \quad \text{then} \quad <^{\chi,\chi}_{1,0}>^{\sigma} \leq [I^{(4)}]^{\sigma} \leq <^{\chi,\chi}_{i,0}>$$

and

$$\phi[I^{(4)}] \leq \text{span} <i,i> \quad , \quad <1,1>^{\sigma} \leq [\phi(I^{(4)})]^{\sigma} \leq <i,i>^{\sigma} .$$

$\phi[I^{(4)}]$ and C gives us the contradiction.

(6) $[I^{(4)}]^1 \neq <\chi,\chi>$ or $<\chi,\chi+i>$ or $<i,0>$.

<u>Proof:</u> If this were the case then $D(I) = 4$ but I would be contained in span $<\chi,\chi>$, span $<\chi,\chi+i>$ or span $<i,0>$ which would give us our usual contradiction. □

We are now left with four cases to deal with:

(A) $[I^{(4)}]^1 = <\chi,\chi_{i,0}>$,

(B) $[I^{(4)}]^1 = <\psi,\psi_{i,0}>$,

(C) $[I^{(4)}]^1 = <\chi,\chi_{1,0}>$ and

(D) $[I^{(4)}]^1 = <\chi,\chi+1_{1,0}>$.

Since (C) and (D) would interchange with a different choice of coset representatives in $Z_4 \times Z_2$, we shall only consider (C) .

(A) $[I^{(4)}]^1 = <\chi,\chi_{i,0}>$:

(1) If I^1 is a 4-dimensional component then

$$D[\phi(I^{(4)})] = 2 \text{ and } \phi(I^{(4)}) \leq \text{span} <i,0> .$$

The algebra generated by $\phi(I^{(4)})$ and C contradicts 1.13 .

(2) If I^1 is not a 4-dimensional component then there exists a $\sigma \neq 1$ and a $\tau \neq \sigma^n$ for all integers n such that

either (a) $I^\sigma = <\chi,\chi>$, $I^\tau = <\chi,\chi+i>$ or (b) $I^\sigma = <\chi,\chi>$, $I^\tau = <i,0>$ or (c) $I^\sigma = <\chi,\chi+i>$, $I^\tau = <i,0>$.

(a) By considering $\phi(C^\perp)$ if necessary we may assume I^1 is one of $\{<i,i> , <1,1> , 0\}$ and for $\rho \not\in <\sigma> \cup <\tau>$, I^ρ is one of $\{<i,0> , <\chi,\chi> , <\chi,\chi+i> , <\psi,\psi> , <\psi,\psi+1> , <i,i> , <i,i+1> , <1,0> , <1,1> , 0\}$.

Therefore $[I^2]^1 = <\chi,\chi_{i,0}>$, $[I^2]^\rho = <\chi,0>$ for all $\rho \not\in <\tau> \cup <\sigma>$ and if there is a $\rho \not\in \langle\sigma\rangle \cup \langle\tau\rangle$ with $I^{\rho} \neq 0$ then $\langle i,i\rangle^{1} \leq [I^{2}]^{\perp}$ $\leq$ span $\langle i,0\rangle$. If $I^{\rho} = 0$ for all $\rho \not\in \langle\sigma\rangle \cup \langle\tau\rangle$ then $\langle i,i\rangle^{1} \leq I^{2} : I^{2} < F$ contradicting 1.13.

(b) and (c) follow in a similar fashion.

(B) $[I^{(4)}]^{1} = \langle{}^{\psi,\psi}_{i,0}\rangle$:

In this situation the only I^{σ}, $\sigma \neq 1$, such that $\dim I^{\sigma} = 4$ are $I^{\sigma} = \langle i,0\rangle^{\sigma}$. If there is no $\sigma \neq 1$ such that $I^{\sigma} = \langle i,0\rangle$ then by considering the image under ϕ of $C^{\perp}$ we obtain a G - module which has been dealt with in part (2), i.e. we squared the 4-closure and the image under ϕ of the result was contained in span $\langle i,0\rangle$.

So let us assume $I^{\sigma} = \langle i,0\rangle$ for some $\sigma \neq 1$. By considering $\sigma(C^{\perp})$ of necessary we may assume I^{1} is one of $\{\langle i,0\rangle, \langle{}^{i,i}_{1,0}\rangle, \langle i,i\rangle, \langle 1,1\rangle, 0\}$. Since $[I^{(4)}]^{1} = \langle{}^{\psi,\psi}_{i,0}\rangle$ we must therefore have a $\tau \neq \sigma^{n}$ for all integers n such that

(a) $I^{\tau} = \langle\psi,\psi\rangle$ or

(b) $I^{\tau} = \langle\psi,\psi+1\rangle$.

(a) $I^{\sigma} = \langle i,0\rangle$, $I^{\tau} = \langle\psi,\psi\rangle$.

The possible values of the ρ components, $\rho \not\in \langle\tau\rangle\cup\langle\sigma\rangle$, are $\{\langle i,0\rangle, \langle\psi,\psi\rangle, \langle\psi,\psi+1\rangle, \langle i,i\rangle, \langle 1,0\rangle, \langle i,i+1\rangle, \langle 1,1\rangle, 0\}$.

Therefore

$$[I^2]^1 = \langle\chi,\chi\rangle_{1,0}\ ,\ [I^2]^\rho = \langle\chi,0\rangle \text{ for all } \rho \notin \langle\tau\rangle\cup\langle\sigma\rangle$$

and

$$[I^2]^{\tau^n} \text{ is } \textit{at least } \langle\chi,\chi\rangle\ ,\ \tau^n \in \langle\tau\rangle\ ,$$

$$[I^2]^{\sigma^n} \text{ is } \textit{at least } \langle i,0\rangle\ ,\ \sigma^n \in \langle\sigma\rangle\ .$$

If I^2 attains its minimum value then $\langle i,i\rangle^1 \leq I^2 : I^2 < F$, contradicting 1.13.

If I^2 is greater than its minimum then $[I^2]^{\tau^n} \geq \langle\chi,\chi\rangle_{1,0}^{\tau^n}$ and so $\langle i,i\rangle^1 \leq [I^2]^1 \leq \text{span}\ \langle i,0\rangle$, against 1.13.

(b) $I^\sigma = \langle i,0\rangle$, $I^\tau = \langle\psi,\psi+1\rangle$.

This is dealt with in a similar fashion to (a). □

(C) $[I^{(4)}]^1 = \langle\chi,\chi\rangle_{1,0}$:

In this case the only I^σ, $\sigma\neq 1$, such that $\dim I^\sigma = 4$ are $I^\sigma = \langle\chi,\chi\rangle$. If there is no such $\sigma \neq 1$ then by considering the image of $C^\perp$ we have a case which has been dealt with before and was refered to at the beginning on (B), i.e. $\langle\psi,\psi\rangle^1 \leq \phi[[\phi(C^\perp)^{(4)}]^2] \leq \text{span}\ \langle\chi,\chi\rangle$.

So let us assume $I^\sigma = \langle\chi,\chi\rangle$ for some $\sigma \neq 1$. By considering $\phi(C^\perp)$ if necessary we may assume I^1 is one of $\{\langle\chi,\chi\rangle, \langle\psi,\psi\rangle, \langle i,i\rangle, \langle 1,1\rangle, 0\}$. Since $[I^{(4)}]^1 = \langle\chi,\chi\rangle_{1,0}$ we must therefore have a $\tau \notin \langle\sigma\rangle$ such that

(a) $I^\tau = \langle\psi,\psi+1\rangle$ or

(b) $I^{\tau} = \langle i,i+1\rangle$ or

(c) $I^{\tau} = \langle 1,0\rangle$.

[Note: $\langle {i,i \atop 1,0} \rangle$ is not generated by one function so cannot be an I^{ρ} for any $\rho \in L$.]

(a) $I^{\sigma} = \langle \chi,\chi\rangle$, $I^{\tau} = \langle \psi,\psi+1\rangle$.

The possible values of the ρ components, $\rho \notin \langle\tau\rangle \cup \langle\sigma\rangle$, are $\{\langle\chi,\chi\rangle , \langle\psi,\psi+1\rangle , \langle\psi,\psi\rangle , \langle i,i\rangle , \langle 1,0\rangle , \langle i,i+1\rangle , \langle 1,1\rangle , 0\}$. Therefore

$$[I^2]^1 = \langle {\chi,\chi \atop 1,0} \rangle \ ; \quad [I^2]^{\rho} = \langle \chi,0\rangle \text{ for all } \rho \notin \langle\tau\rangle\cup\langle\sigma\rangle$$

and

$$[I^2]^{\tau^n} \text{ is } \textit{at least } \langle\chi,\chi+i\rangle \ , \quad \tau^n \in \langle\tau\rangle \ ,$$
$$[I^2]^{\sigma^n} \text{ is } \textit{at least } \langle\chi,\chi\rangle \ ; \quad \sigma^n \in \langle\sigma\rangle \ .$$

If I^2 attains its minimum value, then $\langle i,i\rangle \leq I^2 : I^2 < F$, contradicting 1.13.

If I^2 is greater than its minimum then $[I^2]^{\tau^n} \geq \langle {\chi,\chi \atop 1,0} \rangle^{\tau^n}$, and so $\langle i,i\rangle^{\perp} \leq [I^2]^{\perp} \leq \text{span } \langle\chi,\chi\rangle$, against 1.13.

(b) $I^{\sigma} = \langle\chi,\chi\rangle$, $I^{\tau} = \langle i,i+1\rangle$.

From part (a) we may assume that there is no $\rho \neq 1$ with $I^{\rho} = \langle\psi,\psi+1\rangle$. $[I^2]^1 = \langle {\chi,\chi \atop 1,0} \rangle$ and $[I^2]^{\rho} = \langle\chi,0\rangle$ for all $\rho \notin \langle\tau\rangle\cup\langle\sigma\rangle$. If there is a $\rho \notin \langle\tau\rangle\cup\langle\sigma\rangle$ with $(1,1)^{\rho} \in I^{\rho}$ then

$$[I^2]^{\tau^n} \text{ is } \textit{at least } \langle{\chi,\chi \atop 1,0}\rangle \ , \quad \tau^n \in \langle\tau\rangle \ ,$$

and

$$[I^2]^{\tau^n} \text{ is } \textit{at least } \langle{\chi,\chi \atop 1,0}\rangle \ , \quad \sigma^n \in \langle\sigma\rangle \ .$$

Thus $\langle\psi,\psi\rangle^1 \leq [I^2]^{\perp} \leq \text{span} \langle\chi,\chi\rangle$, which contradicts 1.13.

If there is no such ρ , then

if $I^1 = 0$, $\langle 1,1\rangle^1$, or $\langle 1,i\rangle^1$ we have

$$\langle\chi,\chi\rangle^{\sigma} \leq ([[I^2]^{\perp}]^2)^{\perp} \leq \text{span} \langle\chi,\chi\rangle \text{ , against 1.13;}$$

if $I^1 = \langle\chi,\chi\rangle$ then

$$\langle\chi,\chi\rangle^{\sigma} \leq [I^2]^{\perp} \leq \text{span} \langle\chi,\chi\rangle \quad \text{, against 1.13.}$$

(c) $I^{\sigma} = \langle\chi,\chi\rangle^{\sigma}$, $I^{\tau} = \langle 1,0\rangle$.

The arguements are analogous to part (b).

If there is a $\rho \notin \langle\tau\rangle \cup \langle\sigma\rangle$ with $(1,1)^{\rho} \in I^{\rho}$ then $\langle\psi,\psi\rangle^1 \leq [I^2]^{\perp} \leq \text{span} \langle\chi,\chi\rangle$, which contradicts 1.13.

If there is no such ρ then if $I^1 = 0$, $\langle 1,1\rangle^1$ or $\langle 1,i\rangle^1$ we have $\langle\chi,\chi\rangle^{\sigma} \leq ([[I^2]^{\perp}]^2)^{\perp} \leq \text{span} \langle\chi,\chi\rangle$, against 1.13; if $I^1 = \langle\chi,\chi\rangle$ then $\langle\chi,\chi\rangle^{\sigma} \leq [I^2]^{\perp} \leq \text{span} \langle\chi,\chi\rangle$, against 1.13.

There are now only two more cases which have not been dealt with, namely

$$\text{(a)} \quad I = \langle \chi, \chi+1 \rangle^{1} + \sum_{i=1}^{p-1} \langle \chi, \chi \rangle^{\sigma^{i}} + \sum_{i=1}^{p-1} \langle 1, 0 \rangle^{\tau^{i}} \quad \text{and}$$

$$\text{(b)} \quad I = \langle \chi, \chi+1 \rangle^{1} + \sum_{i=1}^{p-1} \langle \chi, \chi \rangle^{\sigma^{i}} + \sum_{i=1}^{p-1} \langle i, i+1 \rangle^{\tau^{i}} \ .$$

Much to the surprise of the author, these last two possibilities cannot be discounted using Wielandt's method of function modules. Using all our available methods of generating a G - module from another G - module, that is sum $(M+N)$, intersection $(M \cap N)$, multiplication (MN), division $(M:N)$, r-closure for all natural numbers r, and perpendiculars $(M^{\perp})$, each case belongs to a set of nine G - modules which is closed under all these operations yet contains no G - module contradicting 1.13. We therefore shall use Classical Schur Ring Theory to eliminate these cases.

Let us assume that $I = \text{Im}\,\phi$ is either (a) or (b).

For a subset P of H and for $b \in B$ let

$$[P_b]b = P \cap Ab \ .$$

From our knowledge of $\text{Im}\,\phi$ we shall discover the possible values of T_b for all $b \in B$.

We recall that $T = T^{*}$, so that if $x \in T_b$ then x^{-1} must be contained in T_b. Also we know $T = T^{(m)}$ for m prime to the order of B (lemma 3.4) so we have that if $T_b = N$ then $T_{b^{i}} = N$ for $N \subseteq A$ and all $i < p$.

Now consider $\ker \phi$. It tells us that

$$\text{(i)}\quad \phi\Big(\sum_{\rho\in L} (\chi,\chi)^{\rho}\Big) = \phi(\chi_{\{1,y\}}) = \sum_{i=1}^{p} (1,1)^{\tau^{i}} \quad \text{and}$$

$$\text{(ii)}\quad \phi\Big(\sum_{\rho\in L} (1,0)^{\rho}\Big) = \phi(\chi_{\{1,x,x^2,x^3\}}) = \sum_{i=1}^{p} (1,1)^{\sigma^{i}} \ .$$

So $\chi_T + \chi_{T_y} = \sum_{i=1}^{p} (1,1)^{\tau^{i}}$ and $\chi_T + \chi_{Tx} + \chi_{Tx^2} + \chi_{Tx^3} = \sum_{i=1}^{p} (1,1)^{\sigma^{i}}$.

Let $u,v \in B$ be generators of B such that $\sigma(u) = 1$ and $\tau(v) = 1$. If $b \notin \langle u\rangle \cup \langle v\rangle$ then the first equation tells us that T_b contains an even number of elements in each of the cosets of $\{1,y\}$ in A and the second that T_b contains an even number of elements in the cosets of $\langle x\rangle$ in A .

So $T_b = \phi, K, A$ or $A-K$ where $K = \{1,x^2,y,x^2y\}$. If $b \in \langle v\rangle$, $b \neq 1$ then T_b contains an odd number of elements in the cosets of $\{1,y\}$ and an even number from the cosets of $\langle x\rangle$.

Thus

$$T_b = \langle x\rangle\ ,\ A-\langle x\rangle\ ,\ \{1,x^2,xy,x^3y\}\ \text{ or }\ A-\{1,x^2,xy,x^3y\}\ .$$

If $b \in \langle u\rangle$, $b \neq 1$ then this time T_b will contain an even number of the cosets of $\{1,y\}$ and an odd number from the cosets of $\langle x\rangle$.

Hence

$$T_b = \langle y\rangle\ ,\ A-\langle y\rangle\ ,\ \{x^2,x^3y\}\ \text{ or }\ A-\{x^2,x^2y\}\ .$$

Finally, for $b = 1$ T_b contains an odd number of the cosets of $\{1,y\}$ and of $\langle x\rangle$. So

$$T_1 = \{1,xy,x^2y,x^3y\}\ ,\ \{x,x^2,x^3,y\}\ ,\ \{x^2,y,xy,x^3y\}\ \text{ or }\ \{1,x,x^3,x^2y\}\ ,$$

but by our choice of T , $x^2 \in T_1$ and so

$$T_1 = \{x,x^2,x^3,y\} \quad \text{or} \quad T_1 = \{x^2,y,xy,x^3y\}\ .$$

Hence $T = S$ and T is an orbit, not a union of orbits, of G_1 .

Classical Schur Ring Theory tells us that the coefficients of x^2 and y in T^2 should be equal in characteristic 2 <u>and</u> in characteristic 0 . The coefficients of these two elements come from $[T_b]^2$ for all $b \in B$.

If $b \notin \langle u\rangle \cup \langle v\rangle$ then coeff x^2 - coeff $y = 0$.

If $b \in \langle v\rangle$, $b \neq 1$, then coeff x^2 - coeff $y = 4$.

If $b \in \langle u\rangle$, $b \neq 1$, then coeff x^2 - coeff $y = -2$.

If $b = 1$, then coeff x^2 - coeff $y = 2$.

Thus the coefficients of x^2 is greater than that of y which is a contradiction. Hence cases (a) and (b) are impossible.

The work of section 9 is finally complete and enables us to state the following theorem.

<u>Theorem 3.9</u>: Let $H = A\times B$ where $A = Z_4\times Z_2$ and $B = Z_p\times Z_p$, p an odd prime. Then H is a B - group.

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