University of Alberta

A Multiple-Location Model for Natural Gas Forward Curves

By

John Charles Buffington ©

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Name of Author: John Charles Buffington

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92 Chaparral Rd SE
Calgary, Alberta, Canada
T2X 3J8

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University of Alberta

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled A Multiple-Location Model for Natural Gas Forward Curves submitted by John Charles Buffington in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Finance.

Robert Elliott

Dilip Madan

David Colwell

Steven Beveridge

Abel Cadenillas

July 7, 1999

For Catherine

Abstract

Natural Gas is commodity like few others. Excluding electricity, it is the most volatile commodity traded. The price of gas is dependent primarily on weather, with local price shocks felt at other geographic locations to the extent that locations are connected by pipelines with spare capacity.

This paper takes a new approach to modelling natural gas. Instead of modelling the commodity at one location, an approach is developed whereby the natural connections between locations are incorporated.

Furthermore, as gas prices can exhibit both contango and backwardation, a stochastic convenience yield is included in the model as well as stochastic interest rates.

This term structure approach is not unknown in financial modelling; however, incorporating multiple risk factors that correspond to various locations is a new perspective. This paper also empirically tests the data from gas forward prices at Chicago, NYMEX and AECO to understand the statistically properties at each location and to ensure the proposed model is robust enough to include these properties.

This thesis also investigates the time series property of the difference of two locations (the basis) and notes that these empirical properties are consistent with the model properties.

Finally, this paper derives closed-form option solutions for call options of forward contracts and call options on forward basis. The options are calibrated and compared to other models. The thesis concludes with directions for future research.

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Chapter One: Background of Derivative Securities

Introduction

Asset prices appear to change randomly over small time intervals. Bachelier was one of the first to note this phenomena at the turn of the century. His model was developed several decades after the botanist Robert Brown observed that microscopic particles in a liquid seemed to zigzag randomly. Initially, it was thought that pollen grains were alive; we now know the random motion is due to a continuous buffeting at the molecular level. It is a source of pride among financial economists that Bachelier's work preceded, by five years, that of Einstein who described the diffusion of heat.[1].

Brownian Motion and Wiener Processes

Intuitively and non-rigorously a change in a Brownian motion path z is simply expressed as:

1.1 $$\Delta z = \varepsilon\sqrt{\Delta t}.$$

Here ε is a random variable with a N(0,1) distribution. Brownian Motion has applications in biology, physics, and in heat transfer problems.

It is generally assumed in asset markets, that discounted asset price changes are martingales. A martingale is a process whose expected future value conditional on past history is just its present value. Brownian motion itself is a martingale and the elimination of arbitrage gives rise to martingales. A central role is played by Itô processes x(t). We shall consider processes whose increments are of the form:

1.2 $$\Delta x = a(x,t)\Delta t + b(x,t)\Delta z.$$

The usual process used to model stock prices has the property that:

1.3 $$\frac{dS}{S} = \mu\,dt + \sigma\,dZ.$$

[1] As noted in the Preface of Itô and McKean

Note that the relative price change $\frac{dS}{S}$ is the sum of a deterministic increment μdt and a Brownian motion increment σdz This eliminates distortions due to the magnitude of S. This model implies that:

1.4 $$\text{Log}\, S(t) \sim N\left(\text{Log}\, S(0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right).$$

The Approach of Black & Scholes

Overview and Explanation

The pricing of options on securities prior to 1973 was, in general, poorly done. While certain models, such as Samuelson's warrant pricing model, were used they were fraught with limitations. For example, they ignored higher moments and imposed structure on the risk preferences of the investors

In their famous paper, Fischer Black and Nobel Laureate Myron Scholes derived "a new method to determine the value of derivatives [which] stands out among the foremost contributions to economic sciences over the last 25 years."[2]

Their key insight, in retrospect, was quite simple. Fundamentally, they observed that changes in a stock S, are subject to randomness due to the dz term, and because of Itô's lemma, the change in any function f(S), such as an option, must also be subject to the same source of randomness.

By considering a portfolio with Δf(S)/ΔS of the stock and a short position of 1 option valued at f(S), a portfolio is created in which losses in either asset are instantaneously and perfectly offset by gains in the other. Since f(S) is not linear, this portfolio is subject to continuous re-balancing as S changes in value. The returns in this portfolio are instantaneously risk-free so an investor following this strategy should earn the risk-free (T-bill) rate r, otherwise there would be arbitrage opportunities.

By considering the portfolio value at inception, and by substitution of results obtained by using Itô's lemma, it is possible to derive the familiar Black–Scholes differential equation:

[2] http://www.nobel.se/announcement-97/economy97.html. This quotation refers to the work of Nobel Laureate Robert C Merton as well.

1.5
$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

This is simply a variant of the well-known heat equation from partial differential equations. By clever substitution, Black and Scholes were able to solve this equation and by imposing terminal boundary conditions on f(S)—e.g. when f is a long call, f(S,T) = max ((S(T)-K),0)—they derived a closed-form solution to this equation. The solutions give the present value of European puts and calls.

One remarkable aspect of the solution is that the expected drift of the underlying process, μ, appears nowhere in the formula. The solution is in terms of the risk-free rate r.

Limitations of the Method of Black & Scholes

Cox, Ross and Rubenstein have shown that the Black-Scholes formula can be derived as the continuous-time limit of a binomial process. The formula can also be obtained by invoking a change of measure to a risk-neutral space, (see for example, Baxter and Rennie chapter 3), converting the claim to a process, and using an equivalent martingale to avoid solving the equation directly.

Fischer Black noted this model is unrealistic on several accounts. It supposes the volatility is known and constant, that the short-term rate is constant, there is unlimited lending or borrowing at a the same rate, that there are no transaction costs, no taxes, and no dividends and that no exogenous factors such as take-overs are considered.

Term Structure Models

While the log-normal model for stock prices is standard, there is no universally accepted form of dynamics for interest rate processes. In the area of interest rate modelling, the following approaches have been proposed:

Rendleman and Bartter

The approach of Rendleman and Bartter below is to model the short rate r by:

1.6
$$dr/r = Mdt + Sdz.$$

This model has the unfortunate shortcoming of entirely ignoring mean reversion, which is empirically questionable.

Vasicek

Vasicek proposed the following model to incorporate mean reversion.

1.7 $$dr = a(b-r)dt + \sigma dz.$$

Consequently r reverts to its long-term mean b at a rate a. This equation is the well-known Ornstein-Uhlenbeck model from physics where a particle under Brownian motion is slowed by friction. To use this model requires the estimation of parameters a, b and σ. A major shortcoming is that r can be negative with positive probability.

Cox, Ingersol, & Ross

Cox, Ingersol and Ross's model eliminates the problem of negative r by supposing r is given by the dynamics:

1.8 $$dr = a(b - r)dt + \sqrt{r}dz.$$

This implies r reverts to its mean as in Vasicek's model, but because r is a Bessel process[3], it remains positive.

The three previous models have one source of uncertainty, dz and may or may not fit the term structure of interest rates at time zero. In contrast, no-arbitrage models, which we now discuss, have the advantage that—at least at time zero—they force the model term structure to align with the market term structure.

No-Arbitrage Models

Heath, Jarrow, & Morton

Heath, Jarrow, and Morton (HJM) took the approach, which has been widely cited, of specifying all volatilities of all instantaneous future interest rates.

Recall that for a price process, P(t,T), of zero-coupon bonds it is straightforward to derive the yields:

[3] See Karlin and Taylor p. 367-368 for a definition of the Bessel process. It has this name presumably because the modified Bessel function appears in the transition p.d.f.

1.9 $$R(t,T) = \frac{\log P(t,T))}{T-t}.$$

The instantaneous or short rate r(t) is then just R(t,t) so that:

1.10 $$r(t) = -\frac{\partial}{\partial T}\log P(t,t).$$

By considering the forward rate of instantaneous borrowing, we see that:

1.11 $$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

What is needed is a process to specify how the f(t,T) will evolve over time. The single factor HJM approach is to write the equation for f(t,T) in differential form as:

1.12 $$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dz.$$

Here T is fixed. The key insight provided in Heath, Jarrow, and Morton's paper is that the drift and the volatility are related. This can be shown by invoking Itô's lemma, and we discover that:

1.13 $$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,\tau)d\tau.$$

Consequently, all that is needed to model the f(t,T) under the HJM assumptions is $\sigma(t,T)$, which then provides the instantaneous drift, $\alpha(t,T)$. There are some technical conditions which must be placed on $\alpha(t,T)$, but overall this model provides very useful machinery. Complete knowledge of the r(t) is sufficient to determine the initial term structure, f(t,T) and how it might evolve over time.

Ho & Lee

Six years prior to the HJM paper, Ho & Lee (HL) utilised a binomial tree based on two variables (volatility and the market price of risk) to develop a no-arbitrage term structure model. It is possible to write their model in continuous time as:

1.14 $$dr(t) = \theta(t)dt + \sigma dz.$$

with $\theta(t)$ being bounded and σ being constant. There is a closed form solution to pricing puts and calls on discount bonds by this model. (See Jamshidian). Baxter and Rennie show that this model, in HJM terms, is alternately posed as:

1.15 $$df(t,T) = \sigma^2 (T-t)\, dt + \sigma\, dz.$$

Thus this model can be properly viewed as a special case of HJM.

Hull & White

Hull & White, in 1990, extended the HL model by adding a mean-reverting coefficient to the dt term. This model,

1.16 $$dr(t) = (\theta(t)-\alpha(t)\, r(t))\, dt + \sigma(t)\, dz$$

is a hybrid between the HL and Vasicek's model. Following the approach of HJM, it is possible to show:

1.17 $$\sigma(t,T) = \sigma(t)\beta(t,T)$$

where

$$\beta(t,T) = \exp\left(-\int_t^T \alpha(s)\, ds\right)$$

and

1.18 $$f(0,T) = r(0)\beta(0,T) + \int_0^T \theta(s)\,\beta(s,T)ds - \int_0^T \sigma^2(s)\,\beta(s,T)\left(\int_s^T \beta(s,u)du\right)ds.$$

Thus, deterministic functions of time for $\theta(t)$, $\alpha(t)$ and $\sigma(t)$ are sufficient to model the current forward rate for any maturity.

There is undeniably a richness of models, described here rather briefly, that have been developed in an attempt to explain how interest rates evolve over time, given the current information set and a willingness to impose some structure on the stochastic processes.

This richness, however, has not generally trickled down to the world of non-interest rate-related commodities. Some work has been undertaken in recent years, which we shall touch upon in the following sections.

Energy Models

Simple Cost of Carry Model and Convenience Yields

Commodities differ from equities in several aspects. A commodity is generally sold on an exchange or in the over-the-counter (OTC) market for delivery in a specific time period, whereas equities are assumed to be infinitely-lived. There are two general types of commodity[4] markets; cash-and carry markets and price discovery markets.

As the name would imply, in the cash-and-carry market, it is physically possible to offset positions in the futures contract with purchases of the underlying commodity, which is then stored and later delivered upon the expiry of the contract. Gold is one such commodity.

In the price discovery market, this is simply impossible due to either the unstorability of the underlying asset—for example electricity—or the current non-existence of the underlying asset—for example a contract on next year's canola, which may not even be planted at the time the futures contract is written.

The focus of this dissertation is to consider natural gas, which is traded in a cash-and-carry market. As it is possible to purchase cash-and-carry assets physically on any day, there is a straightforward no-arbitrage relationship, which must be maintained. If it were possible to purchase an asset at time t for a price S, and costlessly store the physical asset till time T, then the relationship between the forward price and the spot price is:

1.19 $$F(S,t,T) = S \exp(r\,(T-t)).$$

This is known as the Hotelling Principle and simply indicates that, under the conditions of perfect competition, the price of an exhaustible resource should increase over time at rate r. The rationale is the following arbitrage argument.

If F were greater than S exp(r (T-t)) then arbitrageurs could sell a contract forward and receive F at time T. They then borrow S to buy the commodity in the cash market today, store it till T, and pay off the future value of the loan, S exp(r (T-t)) and cover the short futures position with physical inventory. F is received, S exp(r (T-t)) is paid out, and so a risk-free profit of F(S,t,T)-S exp(r (T-t)) was made.

If F were lower than S exp(r (T-t)), then arbitrageurs would sell "synthetic storage" by selling gas to a counter-party today to receive an immediate

[4] The distinction between forward and futures contracts is unimportant at this juncture.

cashflow of S. The gas will be "stored" and delivered to the counter-party at time T. The arbitrageur then buys a forward contract, insuring a cash outflow of F at time T and invests the S received today at a rate r. At time T, the invested S has grown to S exp(r (T-t)) and the payment of F is due. The physical delivery of gas from the futures covers the prior physical obligation of "synthetically stored" gas and a risk-free profit of S exp(r (T-t))-F is locked in.

If r is positive, then F>S, $\forall$. Even if cost of carry is permitted, contango is structural, yet we observe to be untrue. There are instances of backwardation in the gas market. Convenience yield—see Working (1948) and Brennan & Schwartz (1985)—has been proposed as a "plug" variable which would represent the utility value that would accrue to the owner of the cash commodity, but not the owner of the futures contract. If this convenience yield were high enough, i.e. if there were great value in holding the spot commodity, then the price of the spot commodity could rise considerably above the price proposed by a simple cost of carry model.

If we include cost of carry (excluding financing charges) and convenience yield as a some function $\delta(t)$, the spot forward relationship could be modelled as:

1.20 $$F(S,t,T) = S \exp(r\,(T-t)-\delta(t)\,(T-t)).$$

If r is positive, then F>S, F<S, or F=S depending on the value of $\delta(t)$.

We now describe some previously used models for commodity prices.

Gibson and Schwartz

This model used two factors, the spot price of oil and the instantaneous convenience yield, to price financial and real assets contingent of future prices of oil. These factors are modelled in the familiar way:

1.21 $$dS/S = \mu dt + \sigma_1 dz_1$$

1.22 $$d\delta = k(\alpha-\delta)dt + \sigma_2 dz_2$$

and where:

1.23 $$dz_1 \cdot dz_2 = \rho dt.$$

By stating a relationship between the spot and the futures price via Itô's lemma, the value of a future claim contingent on future spot prices could be derived. Using proxies for the unobservable S[5] and δ, and employing seemingly unrelated regression, parameters k, α, σ_1, σ_2, ρ were estimated.

For non-traded assets, such as convenience yield δ, the conversion from the original measure to a risk-neutral measure requires the estimation of the market price of risk λ.

Given all needed parameter estimates, the model was tested against out-of-sample data. Two significant results were noted. First, the λ was not stationary over time and better out-of-sample results were obtained when its estimate was updated. Second, the model mis-pricing is an increasing function of future contract maturity, even when λ is allowed to change over time.

Amin, Ng, & Pirrong

This is an extension of the HJM method, but applied to energy derivatives. They write the spot-futures relationship as:

1.24
$$F(t,T) = S(t)\exp\left[\int_t^T r(u) - \delta(t,u)\,du\right].$$

At time zero, we know F for all possible T's, S, and r, so we can back out the term structure of δ. Making the heroic assumption that the convenience yield structure is deterministic, we can model possible paths of the forward curve. Modelling the process S will provide $F(t,T)$, $\forall$ t,T.

Risk neutral valuation of options follows immediately. By introducing stochastic convenience yields, they reduce the problem to a similar form as HJM, depending only on the σ. They note that their framework is similar to Black.

While this model makes progress in modelling the forward curves and pricing options, its assumptions of the constant spot volatility and deterministic r make this model an excellent starting point for others.

Schwartz (1997)

In this recent paper, Schwartz takes the Gibson & Schwartz model and introduces a third stochastic process for interest rates:

[5] The definition of S's observability depends on whether S is defined as the cash price or the front-month contract.

1.25 $$dr = a(m-r)dt + \sigma_3 dz_3$$

where:

1.26 $$dz_1 \cdot dz_2 = \rho_1 dt,$$
$$dz_2 \cdot dz_3 = \rho_2 dt,$$
$$dz_1 \cdot dz_3 = \rho_3 dt.$$

Again there are variables which are not directly observable. In this paper, this challenge is overcome by using Kalman filters to indirectly estimate the S and the δ. Schwartz then applies the modelled forward curves of this model, the Gibson and Schwartz model, and a single factor model to estimate the value of future cash flows. See Dixit and Pindyck or Trigeorgis for an explanation of the real option approach to capital budgeting. Schwartz demonstrates how the future cash flow estimates for crude and copper projects provided by four different approaches—the three above plus the standard Discounted Cash Flow—are considerably different. At a minimum, this provides a strong incentive to ensure that the best possible model is being used to model the curves. Expensive errors are possible under model mis-specification.

Miltersen & Schwartz

As explained by Miltersen & Schwartz in their introduction:

> *"In a seminal paper, [HJM] develop a no-arbitrage model of the stochastic movements of the term structure. ... The model takes as given the initial forward ... curve and derives the drift of the risk neutral forward ... process consistent with no arbitrage. Amin, Ng, and Pirrong ... develop similar models for the term structure of commodity futures prices.*
>
> *A different approach ... is presented by Gibson & Schwartz. They develop a two factor model [.] Schwartz (1997) extends this model by introducing a third stochastic factor [.]*
>
> *In this paper, we develop a model that generalises and combines the two approaches by using all the information in the initial term structure of both interest rates and commodity futures prices.*

The form of their model for forward prices is:

1.27
$$G(t,T) = G(0,T) + \int_0^t G(u,T)\left(-\int_u^T \mu_\varepsilon(u,s)\,ds + \left\| \int_u^T \sigma_f(u,s)\,ds \right\|^2 + \frac{1}{2}\left\| \int_u^T \sigma_\varepsilon(u,s)\,ds \right\|^2 - \left(\int_u^T \sigma_f(u,s)\,ds \right) \times \left(\int_u^T \sigma_\varepsilon(u,s)\,ds \right) + \sigma_s(u) \times \left(\int_u^T \sigma_f(u,s) - \sigma_\varepsilon(u,s)\,ds \right) \right) du + \int_0^t G(u,T)\left(\sigma_s(u) + \int_u^T \sigma_f(u,s) - \sigma_\varepsilon(u,s)\,ds \right) dz(u)$$

where the spot price, the convenience yield (ε), and the forward interest rate (f) all have dynamics determined by separate stochastic processes, but with the same Brownian Motion term. Again, similarly to the HJM approach, the drift of the convenience yield can be expressed in risk-neutral space as a product of various volatilities.

Chapter Two, The Natural Gas Industry

Introduction

Exploration and Development

The natural gas industry, hereafter referred to as the gas industry, was historically segmented into two components; upstream—exploration and production—and downstream—distribution and sales. Vertically integrated companies would focus on both streams, while producers would only focus on upstream activities.

The primary objective of the production company is to undertake exploration and production activities, in the most economic manner, ensuring that their variable costs are fully covered. A secondary, and not trivial objective, was to ensure that once the production was on line—Proved Producing[6]—that they receive economic rents in excess of the variable costs. Since the owner-managers of the "juniors" are often compensated by stock options, they have a vested interest to ensure that not only is gas produced as cheaply as possible, but that it is sold for the maximum available price.

Since security analysts follow the downstream results of the upstream companies very closely, producers are further motivated to make sure that they are always beating the industry average for revenue per gigajoule[7] (GJ) of gas sold. If they are above their peer group in income, then their share prices, should theoretically reflect this superior performance. As such, most producers will not simply sell all their gas forward at a fixed price. There is a well-known industry mantra: "when prices are low, they will go higher and when prices are high, they are going higher still."

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[6] There is a hierarchy of well types, that are risk-weighted by financial analysts in estimating the volumetric reserves of a production company's wells. The hierarchy is 1) Proved Producing, 2) Proved Undeveloped, and 3) Probable Reserves. This weighting is used in conjunction with the Reserve Life Index—Remaining Reserves/Annual Production—to give a very rough estimate of how much future cash flow from gas production can be expected for a particular company, if their future exploration efforts are unsuccessful.

[7] A joule is the amount of energy (or work) equal to the force of 1 Newton when the point at which the force is applied is moved 1 metre. Gas in Canada is sold by the GJ, which is a billion joules. To help understand these units, an average home in Alberta would burn about 150 GJ of gas per year for central and water heating. Gas in the USA is sold by the British thermal unit or Btu, which is the amount of energy required to heat one pound of water from 60°F to 61°F at one atmosphere. Gas prices in the USA are generally for one million Btu's or one mmBtu. For comparison, 1 mmBtu = 1.0545 GJ. An amusing side note is that although gas is traded on an energy-content basis, it is shipped on the pipelines on a volumetric basis. Customers can choose to pay on a GJ, an mmBtu or on a 10^3m^3 basis.

Once the gas has been found, the challenge is to get it from the field to the burner tip of the customer. This involves a series of processes outlined below.

Physical Nature of Natural Gas

Natural gas is better named natural gases. Its principal components are methane (CH_4), ethane (C_2H_6), propane (C_3H_8) and butane (C_4H_{10}). It may also contain heavier gases as well as water, carbon dioxide, and hydrogen sulphide. The gas is trapped in pockets of porous rocks underground, that the producer hopes to detect with hi-tech methods such as seismic measurements. When a test well is drilled into an area that has a lot of underground gas, there must be a pressure differential for the gas to flow to the surface. A producer would typically collect the gas from several wells and process it to remove the non-gas substances noted above. These substances are harmful to pipelines, and so only processed gas is acceptable. The system of lines joining various wells to the processing plant and then to the major pipeline is called a "gathering system." These gathering systems are paid for by the upstream companies.

History of US Government Policies

The upstream natural gas business has generally been ignored by the regulators. It was always felt that the barriers to entry for producers were minimal and that the real source of anti-competitive behaviour would only occur in the downstream sector, where there were natural monopolies. As such, this half of the industry has a long history of government regulation.

The first instance of regulation in the USA occurred with the Natural Gas Act of 1938. The Federal Power Commission (FPC) was created to design regulations that would protect the public interest. It set up the mechanism by which the transmission rates for major pipelines would be charged. Like most monopolies, a rate base was established and a reasonable profit was permitted. The FPC had jurisdiction over interstate pipes.

In 1954, the US Supreme Court (The Phillips Decision) ruled that the FPC should also regulate the upstream industry, as the downstream was facing an unmatched asset-liability profile. Most producers took the rational step and simply sold on the intrastate market, where the regulators did not have the ability to set well-head prices. Ultimately, this system broke down.

In the Natural Gas Policy Act of 1978, the US government changed policy direction, and the Federal Energy Regulatory Commission (FERC) was created. The mandate was to increase interstate supply and reform the upstream price controls. Eventually, producers were allowed to buy capacity on pipelines—previously the unique bailiwick of the gas transportation companies—enabling producers to sell directly to end-users. A variety of transportation tariffs were designed, depending on how firm of a commitment the shipper required.

In 1991, FERC issued a Notice of Proposed Rule making, which paved the way to unbundle how gas was sold. The three costs of production, transportation, and distribution were no longer required to be homogeneous across all price-points. This deregulation has led to the current situation in the USA where producers, end-users, and any intermediaries can purchase gas and pipe capacity to [hopefully] sell at a profit. This was formalised under FERC Order 636 of 1992.

History of Canadian Government Policies

The deregulation of the Canadian natural gas industry took a markedly different course than that of its US counterpart. Until the 1970's, gas was sold by producers to the buyers on a negotiated agreement basis. Producers would try to extract rents, and buyers would try to lock in long-term fixed price deals, regardless of the upstream economics.

In the early 1970's the Alberta government, with a growing appetite for royalties introduced the Arbitration Act of 1973 to create a mechanism whereby either buyer or seller could initiate arbitration, whereas arbitration had previously been a possibility with mutual consent. The act stipulated that the "commodity value of gas"—read energy value relative to crude—must be considered in the price setting during arbitration. This made sense for the province as oil prices were rising due to the first energy shock, and royalties, as a fixed percentage of the sales price of gas would also rise if gas were tied, even loosely, to oil.

The Canadian federal government, never one to be usurped by a province, passed the Petroleum Administration Act of 1973 which had the objective of achieving a uniform price for gas used in Canada outside its province of production. A price was eventually set based on a Toronto price that was determined as a percentage of the energy-equivalent crude price.

The National Energy Policy of 1980 abandoned this goal, and instead opted to price natural gas so as to make it more attractive to end-users. This policy was—mercifully—short-lived and on October 31, 1985, The Agreement On Natural Gas Markets And Prices came into effect. Under this agreement, prices for gas were to be established by agreement between producers and consumers of gas. Furthermore, export prices[8], which had been regulated after 1977 and tied to crude prices, were now free to be set by negotiation.

The market was now effectively price-deregulated from the government. Now the challenge is to find a market-clearing price for agents who have competing objectives; Producers want short-term floating prices and buyers want long-term fixed prices. This has created the need for market intermediaries. It should be pointed out that it was only the price for the commodity that was deregulated. The cost of transportation—inter and intra-provincial—was still set by regulatory bodies.

The Physical Market

Supply Fundamentals

Gas is found underground in reservoirs that must be discovered, tapped into, brought to the surface and processed for shipping. Based on these four activities, gas can vary in price from region to region and the extraction costs can vary from producer to producer depending on operating efficiency and how well they manage their respective balance sheets.

It is thus realistic, that a gas purchaser in upstate New York, could call several producers in west Texas and similarly call several producers in Alberta, and receive several different price quotes for gas at the "well head".

As expectations about future prices increase, producers are motivated to explore and bring gas on-line. As prices actually fall, marginal wells are likely to be "shut-in", or taken out of service. It would be fair to assert that the supply curve of natural gas in any basin, is upward sloping. In explaining backwardation, Litzenberger and Rabinowitz show that for crude extraction, the reserve is characterised as a real call option where the extraction cost is the exercise price of the call. When the uncertainty of the futures price goes up, i.e. higher forecast volatility, the value of the call increases and the crude stays in the ground.

Demand Fundamentals

Natural gas users can be usefully partitioned into three categories: Residential, Industrial Users, and Utilities. Each of these users has different demand characteristics.

[8] It is probably self-evident, that given the nature of gas, the only significant export market for Canadian gas, to date, is the USA.

Residential customers use natural gas to heat their homes. They are generally price insensitive (they are price takers), mostly because the price of gas seems to be quasi-fixed to them, with their Local Distribution Company absorbing the price risk[9], and they are generally not sophisticated, nor are they able to change energy sources. Their demand profile is driven by the weather. When it is cold, these customers consume more gas.

Industrial users use gas for heating, co-generation—such as steam production—and for feedstock into their final products. Depending on the nature of the business, some end-user gas needs can be reduced when the cash price is too high.

Utilities use natural gas in the generation of electricity. While the gas is more expensive than coal, gas plants are cheaper to run. For example, utilities such as the Edmonton Power, have some gas generation that is only run on days when they need peak electrical capacity on cold winter days. In the deep south, the peak loads occur in the summer due to air conditioning.

It is not that the demand curve is downward sloping, and it is arguable that it is vertical due to the price in-elasticity of demand. It is the differential nature of how price effects supply and demand that creates the need for storage. If prices are very low, some users or intermediaries will buy gas and put it in storage, anticipating higher prices in the future which will cover the storage costs. If prices are very high, users will try to take gas out of storage, avoiding paying the weather premium in the cash market. The challenge is to try to make price forecasts, and volume forecasts that allow for dynamic optimisation in consumption. This is tough given that weather is the biggest source of demand uncertainty.

Transportation & Storage.

Storage, as noted above is simply a time spread, where it is believed that it is cheaper to buy and hold gas than to buy the futures contract and take later delivery. The total benefit to the marginal storer must include the convenience yield outlined in Chapter One. Storage is generally in underground caverns and is purchased for fixed time periods, with rules as to balances and cycling of gas.

Transportation, in contrast is a geographic spread. Unlike the crude market, gas is perceived to be fungible (at a particular location). The gas put in a pipeline at one point is deemed to instantaneously arrive at the destination point. If a trader thinks the current price between two points is too small, he

[9] Except when they make errors and they try to pass on excess gas costs to the residential customers. See for example the actions of Canadian Western Natural Gas and a request to the AEUB to increase the gas cost to customers, since they were under-hedged in the winter of 1996-1997.

can buy forward delivered gas at one point, buy transportation for the same time period, and wait. If the difference increases, he can then sell the gas forward at the second point, filling it with the gas and the transportation, which is cheaper than the short contract. The true (regulated) cost of transportation cannot vary much from the price differences, because of the arbitrage outlined in Chapter One.

Transportation also leads to an interesting market, the day market or cash market. Gas will only flow if there is a pressure gradient on a pipeline, and the pipeline owner makes money, and assures the integrity of the system if the pipeline operates within certain tolerances. There are different levels of transportation that are available to shippers of gas, from Firm to Interruptible, with differential tariffs. The pipeline cannot sell all capacity at fixed price, anymore than airlines can. There are always players willing to pay "student stand-by" rates if someone does not use their allocation.

For a variety of reasons it is not feasible to structure contracts so that all capacity is fully sold at all times. When capacity is released back to the pipeline (normally a customer only has to balance on a monthly basis, and can under-utilise capacity on any day), the pipeline scrambles to find "day gas" or gas in the cash market. It will use this gas to fill—the term in the industry is injecting—the pipe. Gas traders who have physical gas will try to transact to make a profit in the day market. Similarly, if there is too much gas nominated, the pipeline will curtail—the industry term here is draught—and shippers will need to find someone to take their physical gas. Since all nominations are firm by 10 am of prior day to shipping, there is a flurry of activity in the day market every morning. Hence the day market for physical gas.

Depending on whether the pipeline is injecting or draughting, and largely depending on the weather at a particular point on a pipeline, the day gas can be quite volatile. It is an article of faith in the gas industry that the day price mean reverts

The Derivatives Gas Market

The day market, or cash market is the only place where natural gas can be purchased for immediate use. As such, it is the market that commands the highest convenience yield. There is a secondary market of natural gas derivatives which is quite liquid, and which provides a variety of exchange-traded and over-the-counter securities. These products can be settled physically, with a delivery or gas or financially, with a cash settlement. These derivative markets are sometimes called "physical" gas and "paper" gas markets.

Futures

A natural gas futures contract is a derivative product that is traded on an exchange and is a claim on a future delivery of gas. The most common futures contract is the New York Mercantile Exchange (NYMEX or the "merc") contract that is for a volume of 10,000 mmBtu to be delivered at Sabine Pipe Line Company's Henry Hub in Erath, Louisiana. This contract is traded up until the third business day prior to the first calendar day of the delivery month, at which time the price of the contract for the month is deemed to be fixed. Contracts are traded for the next calendar month delivery and prices are quoted for 30 consecutive months.

The "merc" does not arrange for the delivery of the physical gas, it acts as a clearing house for buyers and sellers. It also sets the commodity standards and requires that market participants post margin as security on the contract. Because of the margin requirements, this market can be highly leveraged and market participants may include both hedgers and speculators.

A long NYMEX gas position can be closed by a corresponding short NYMEX position, thus locking in or "monetising"[10] any changes in the value of the long contract since it was purchased. Contracts are marked-to-market daily, and closed out with a new contract issued by the "merc". This is to minimise the financial risk of default to one day's change in price. It is estimated that well in excess of ninety percent of the open interest in gas contracts are closed out before physical delivery.

Forwards

A natural gas forward contract is a derivative product that is not traded on an exchange and is a claim on a future delivery of gas. It is custom designed between the buyer and the seller and unlike the futures contract, has no margin requirement. There is also no marking-to-market by the counter-parties, although this may be done internally to track the relative profitability of positions. Thus risk of default would involve more than a single day loss, and since there is no margin posted for forward contracts, there is a greater risk of a greater loss due to non-delivery of the gas by the counter-party.

Under the condition of constant interest rates, forwards and futures are identically priced [see Cox, Ingersoll & Ross 1981]. The difference under stochastic interest rates is due to the on-going refinancing costs of the futures contract due to (positive or negative) margin calls. There is re-investment or re-financing risk.

[10] We recognise that this use of the word "monetise" is at variance with the more common macroeconomic meaning of "printing" money to pay off government debt. We would prefer the term "realise", but we will bow to industry practices and use it in the gas industry sense.

Forward contracts tend to be in round volumes, e.g. 5,000 GJ per day for an entire month on gas traded in Alberta and are traded at dozens of points in North America. If there is an inlet valve into a pipeline, where the physical gas can be metered as it flows in, then a forward contract can theoretically be sold there. In the over-the-counter market, it is common to have forward curves providing price indications for up to sixty months forward, although open interest quickly drops off past one year.

Indices

Gas producers, who are systematically long gas, are expected to be writers of futures and forward contracts. This would neutralise any price movements away from the contract price at the time of inception. Natural gas users, are systematically short gas, are intuitively buyers of futures and forward contracts. This is not always the case.

As previously noted, producers try to avoid locking in fixed prices, because of company by company financial comparisons. They prefer to have the potential for some upside participation in gas prices. One mechanism to provide this floating price exposure, is the setting of gas indices every month. The price of a forward or futures contract can change in value every day. If we average the prices at which contracts were transacted for a specific period—e.g. the last three days before the contract expires—we can derive a floating price for a particular month. This price is unknowable until the averaging period has commenced. Once established, this price is fixed and is called the index price for a particular pricing point for the duration of this month.

Thus a producer may wish to sell gas on a floating price, or index basis—for example Alberta Index for July +3¢—hoping that the price of the near contract will increase (from the current price) over the last three days.

End users, especially industrial customers who are trying to beat budget, but for compensation reasons, are risk averse to large upside moves, may lock in some physical supply of gas with fixed price contracts and purchase the remainder on a floating basis.

Swaps

A swap is a financial contract that is written on paper gas. As in interest rates and currency markets, a swap represents an obligation for two parties to exchange cash flows on some notional principal value of the underlying commodity.

For example, a gas buyer in Ontario may wish to pay Alberta index price + 50¢ for gas that is delivered in Ontario. The counter-party selling this gas is buying Alberta gas, paying to ship it to Ontario, and is thus facing Ontario index price exposure. If the counter-party could find an intermediary to do a location swap (the intermediary pays Alberta index, and receives Ontario Index + 49¢) they would immediately do this transaction and monetise the penny. If the volume were for 10,000 GJ per day for a month, this is a quick profit of $3,000.

Location swaps can also be done on fixed-for-floating basis, or a fixed-for-fixed basis across two locations Another type of swap is a simple fixed-for-floating swap at the same point. In all these transactions, the physical gas is unaffected; The swap is a financial transaction that can be layered on a physical deal, or simply entered into on a speculative basis, with a view that the differential will change.

Basis

Basis[11], in the natural gas market, is the difference in the price of gas at two delivery points. The standard reference in the USA for a basis differential is NYMEX. For example, if the May Henry Hub price is $2.25 and the May Chicago price is $2.45, then the actual basis differential for May Chicago is plus $0.20 to Henry. The standard reference in Canada is the price at the AECO storage facility.

As well as the current basis differential for the near contract, which is a known number today, there is a forward basis differential, which is a market equilibrium forecast of what the basis differential will be in the future.

The basis between any two points is dynamic, but cannot vary too much. Since I can purchase transportation from the pipeline owners, and transportation is bought and sold among market participants, arbitrage arguments never let the basis meander very much from the purchased price of transport.

Options

Options in the gas industry are similar to options in equities, except for the variety of underlying assets. Options can be written on futures, on forwards, on location swaps (called swaptions), on fixed-for-floating swaps, on basis, on

[11] I recognise that "basis" for academics is the difference between the spot price and the futures price at a particular index. As with the term "monetise", I shall use the gas industry definition of "basis" as the difference in price between any two points.

volumetric variances (sometimes called swing options), and on storage. Given the complex nature of the underlying assets, the derivation of accurate option pricing formulas is still largely unsolved for this industry.

Difference Between Crude and Natural Gas

Gas and crude are quite different. It is possible to extract several different grades of crude from the ground. Because each grade has a different economic value, ownership of oil in a pipeline must be distinguishable. Crude can take about 2 months to go from Alberta to Ontario, and batches of distinct grades are separated by "plugs" of crude. The value of the crude in the pipeline can fluctuate with market conditions as the commodity is flowing east.

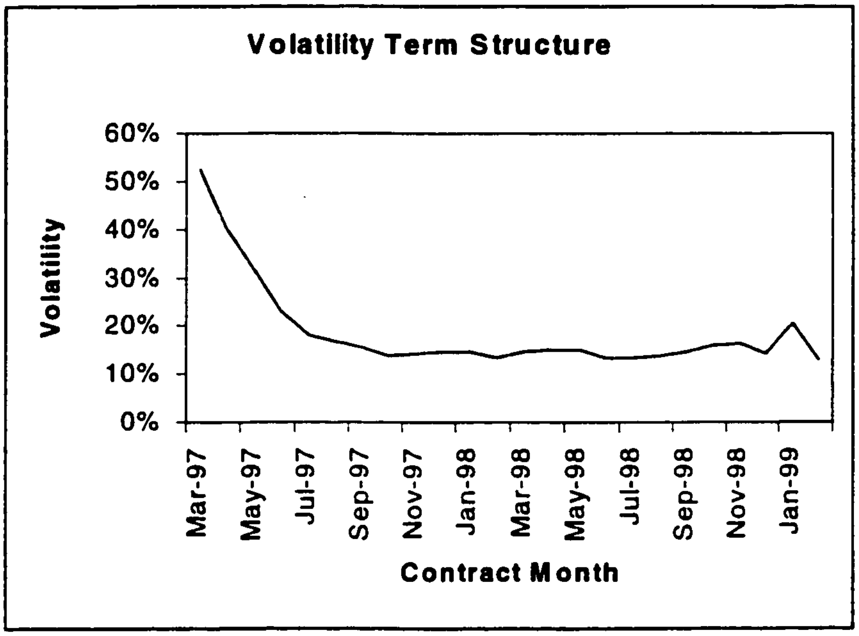
Because natural gas is a true fungible commodity, the owner is indifferent as to which molecules are received at the other end of the pipeline. Since the pipe is always full, delivery is deemed to be instantaneous. There is virtually no "storage" in the pipeline as with crude.

Volatility Term Structure

The typical assumption in the approach of Black and Scholes is that volatility is constant for the term of the option. This is not true in the natural gas market. If we consider the volatility of the Mar 97 through Feb 99 NYMEX contracts, we detect a clear term structure. The volatility is estimated from 18 daily observations and then annualised using 252 trading days.

The monotonic term structure is clear, and is also a necessary (but not sufficient) condition for mean reversion in the underlying price series. Any gas model must be consistent with this observed volatility term structure.

Graph 2.1 Volatility Term Structure



Backwardation and Contango

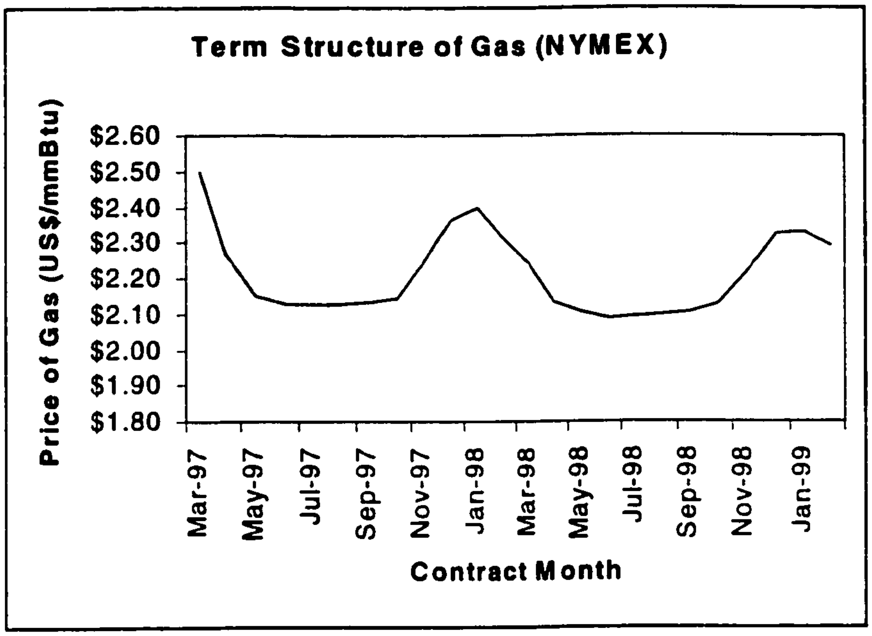
Most cash-and-carry markets exhibit long-term contango that is longer-dated contracts are priced higher than short-dated contracts. This is due to a straightforward arbitrage argument of carrying costs.

Natural gas sometimes exhibits contango and sometimes the opposite structure known as backwardation.

This makes sense when convenience yield is considered. The owner of a physical commodity—such as gas—used for consumption experiences a benefit for holding the physical stock on hand. This benefit is captured by the convenience yield, which is stochastic, and can cause a contango portion of the curve to backwardate over very short time periods. Convenience yield is less pronounced for investment commodities such as gold, which lead to an almost permanent state of contango.

Due to the nature of the commodity, there is the possibility of a seasonal component to the determination of the shape of the forward curve.

Graph 2.2 Term Structure of Gas



Stochastic Basis

Basis, as defined above, is the difference in prices between two locations. Due to changing convenience yields at one or both locations, the price differences can significantly vary over time. This is graphically illustrated below for a particular NYMEX contract.

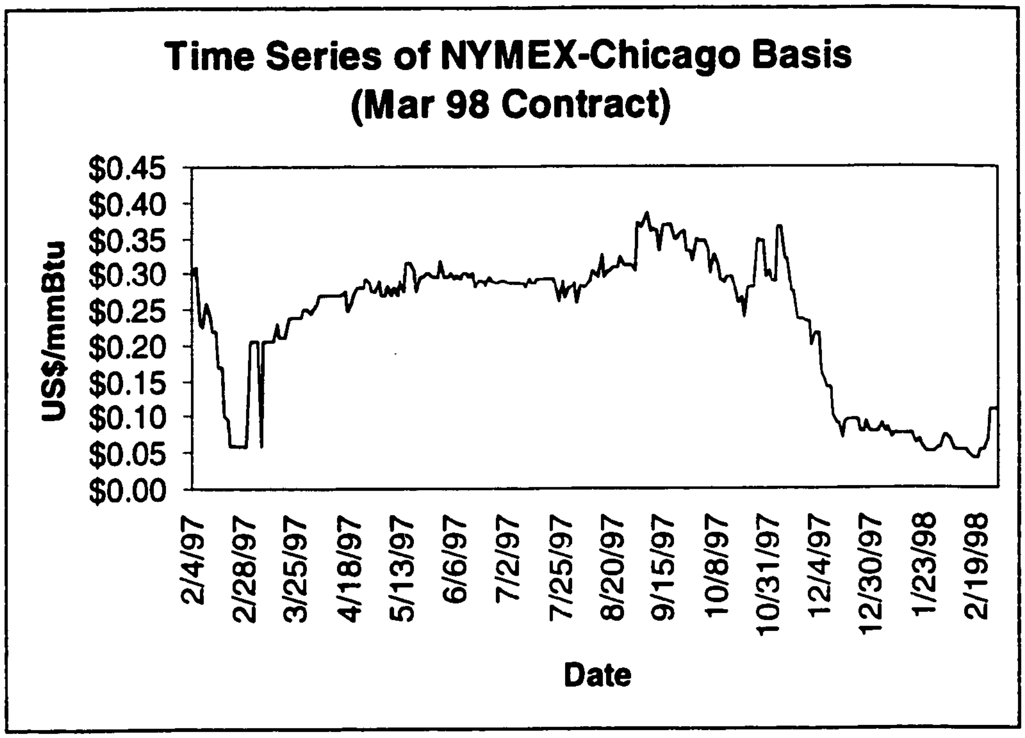
Basis is often assumed to be normally distributed see for example [Kaminski] or [Ammirati et al] for the derivation and closed-form solution of this approach. This approach ignores the multiple factors at play at both locations, as well as any correlation structure between the risk factors.

If price processes at both locations are lognormal, then the difference should not simply be assumed to be normal. Visual inspection of the graphs of the basis prices versus a true normal seems to show "fat tails". We demonstrate in a later section that normality is probably an adequate approximation to the difference of two log-normals.

We ran some time series analysis for the NYMEX-Chicago basis, and obtained the results in Appendix A. It appears the basis is best described as an AR(2) process. The derived gas model must be able to consistently capture this structure, in order to remain arbitrage-free.

We further demonstrate in a subsequent section how the AR(2) in price gives rise to a Gaussian distribution.

Graph 2.3 Time Series of NYMEX-Chicago Basis



Chapter Three: Model Outline

Introduction

The natural gas industry, as noted in Chapter Two is not a single market with one forward curve. In order for a model to be more representative of how futures prices evolve, it must consider the multiple price point nature of this market. Issues that must be addressed include: correlation across locations, the role of basis in pricing, stochastic convenience yields and cost of carry.

A three-factor model will be outlined in this chapter. It may be helpful to think of these factors as being representative of the primary sources of uncertainty at three pricing locations such as the NYMEX prices at Henry Hub, the Citygate prices at Chicago and the Nova Inventory Transfer (NIT) prices in Alberta. A pipeline directly relates the first two, while NIT and NYMEX are not directly connected.

Three state variables will be used for each geographic location in the model: spot prices, convenience yields, and interest rates. It is assumed, for model simplification, that all are subject to the same Brownian Motions at any pricing location, albeit in different manners. Market completeness dictates that we have at least as many discounted traded assets as Brownian Motions.

The objective of this model is to incorporate both the information available at time 0 from the term structure of the gas prices in the spirit of HJM, and how the state variables evolve over time in the spirit of Schwartz (1997). This framework is readily adapted to a multiple Brownian Motion model.

Model Set-up

Description of Gas Assets

Consider an economy with three locations at which forward contracts for natural gas are quoted. We index these assets with subscripts 1, 2, and 3. Let these assets assume the following form: a primary asset, a secondary asset and a tertiary asset. These assets correspond to various geographic locations.

Notation

We introduce the following notation:

$\delta_i(t,T)$ the continuously compounded forward convenience yield for time T as of time t,
$f(t,T)$ the continuously compounded forward interest rates for time T as of time t,
$F_i(t,T)$ the forward commodity price for time T as of time t,
$S_i(t)$ the spot price of the gas price at time t,
$\Gamma_{ij}(t,T)$ is the basis between two assets i and j for time T as of time t.

Instantaneous rates for these variables can be derived by allowing $t \rightarrow T$.

Separate locations will be indicated by the appropriate subscript i = 1, 2, or 3.

Expected Spot Prices

Each spot price should converge from $S_i(t)$ to $S_i(T)$ over some time interval [t,T]. To avoid arbitrage the current spot price should equal the discounted forward spot price incorporating instantaneous interest rates and convenience yields. To simplify matters, we shall assume that there is only one interest rate process, although it is subject to several Brownian Motions.

We note that:

3.1
$$S_i(t) = E\left[S_i(T) \exp\left(-\int_t^T r(u) + \delta_i(u) du \right) \middle| \Im(t) \right]$$

where $\delta_i(t)$ is the spot convenience yield (convenience yield and storage costs including transportation, insurance, and injection/withdrawal fees) for asset i at time t, and $E[\ \dots\ |\Im(t)]$ is the conditional expectation under an equivalent martingale measure conditioned on the date t information set $\Im(t)$. Furthermore $r(t)$ is shorthand for $f(t,t)$.

Expected Forward Prices

The expected forward price at any location i determined at time t, for the asset that is delivered at time T can be determined as:

3.2 $$F_i(t,T) = \frac{E\left[S_i(T)\exp\left(-\int_t^T r(u)du\right) \mid \Im(t)\right]}{P(t,T)}.$$

P(t,T) is the price of a discount bond maturing at T valued at time t and is also subject to an expectation relationship.

Similarly, the futures price i determined at time t for the asset that is delivered at time T can be written as:

3.3 $$G_i(t,T) = E\left[S_i(T) \mid \Im(t)\right].$$

Miltersen and Schwartz show the relationship between forwards and futures is given by:

3.4 $$G_i(t,T) = F_i(t,T) - \frac{S_i(t)}{P(t,T)}\mathrm{Cov}\left(\exp\left(-\int_t^T r(s)ds\right), \frac{S_i(T)}{S_i(t)} \mid \Im(t)\right).$$

It is becoming increasingly common, in light of the trading losses experienced by energy trading houses in the summer of 1998 at Cinergy, to require the posting of collateral or margin in the over-the-counter market. Correspondingly, this analysis will be done with the assumption that all prices are future and not forward prices. While this may be a bit pre-emptory, it removes a non-material source of analytic complexity. We will use the terms forward and future somewhat interchangeably.

Expected Discount Bond Prices

For all locations, we expect the following relationship:

3.5 $$P(t,T) = E\left[\exp\left(-\int_t^T r(u)du\right) \mid \Im(t)\right].$$

Ostensibly, we would expect that the interest rates at Chicago and NYMEX would be the same, while the NIT rates may be different. It is not believed that using different processes for estimating the zero coupon bonds will be material.

We recall that from Chapter One that we can extract the (non-traded) forward interest rates from the (traded) discount bond price, without any loss of information via the relationship:

3.6 $$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

Evolution of Discount Bond Prices

To employ the HJM approach, we now must specify the forward nature of the various state variables. To eliminate the expectation operator, we substitute forward for instantaneous interest rates and convenience yields.

For example, we define f(t,u) such that the discount bond price is given as:

3.7 $$P(t,T) = E\left[\exp\left(-\int_t^T r(u)du\right) \mid \Im(t)\right] = \exp\left(-\int_t^T f(t,u)du\right).$$

Evolution of Spot Prices

We suppose the spot prices have dynamics:

3.8 $$S_i(t) = S_i(0) + \int_0^t S_i(u)\mu_{Si}(u)du + \int_0^t S_i(u)\sigma_{Si}^1(u)dz_1(u) + \int_0^t S_i(u)\sigma_{Si}^2(u)dz_2(u) + \int_0^t S_i(u)\sigma_{Si}^3(u)dz_3(u)$$

for i =1, 2, 3.

Note the presence of three Brownian Motions for each asset, and that the same Brownian Motions can potentially effect the processes at each of i =1, 2, and 3. Note that so far, we have not indicated any structure on the coefficients of the Brownian Motion terms. This will be done later as we more fully specify the model. As a matter of notational clarity, we note that the superscripts above indicate an association of the coefficient with a particular Brownian Motion, and are not exponents.

Evolution of Forward Interest Rates

From the forward price of discount bonds, we may extract the forward interest rates. These may be modelled as:

3.9
$$f(t,s) = f(0,s) + \int_0^t \mu_f(u,s)du + \int_0^t \sigma_f^1(u,s)dz_1(u) + \int_0^t \sigma_f^2(u,s)dz_2(u) + \int_0^t \sigma_f^3(u,s)dz_3(u).$$

Evolution of Forward Convenience Yields

We define the continuously compound future convenience yields $\varepsilon_i(t,s)$ such that the futures price follows this relationship:

3.10
$$G_i(t,T) = S_i(t)\exp\left(\int_t^T (f(t,s) - \varepsilon_i(t,s))ds\right).$$

We may now model the relationship for convenience yields.

3.11
$$\varepsilon_i(t,s) = \varepsilon_i(0,s) + \int_0^t \mu_{\varepsilon i}(u,s)du + \int_0^t \sigma_{\varepsilon i}^1(u,s)dz_1(u) + \int_0^t \sigma_{\varepsilon i}^2(u,s)dz_2(u) + \int_0^t \sigma_{\varepsilon i}^3(u,s)dz_3(u).$$

Note that we have taken the same Brownian Motions at all locations to be the source of randomness for convenience yields, interest rates and spot prices.

The Model of Futures Prices

The Model

With the variables of interest—spot prices, convenience yields, interest rates and basis—and their processes specified, we can jointly model the futures price of gas prices.

3.12
$$G_i(t,T) = S_i(t)\exp\left(\int_t^T (f(t,s) - \varepsilon_i(t,s))ds\right).$$

It will simplify the notation if we define:

3.13 $$Y_i(t,T) = \exp\left(\int_t^T \left(f(t,s) - \varepsilon_i(t,s)\right)ds\right).$$

So,

3.14 $$G_i(t,T) = S_i(t) \cdot Y_i(t,T).$$

Observables

The observables at t=0 are:

P(0,t), the bond prices for all maturities t<T,

$S_1(0)$, $S_2(0)$, $S_3(0)$, the spot prices,

$G_1(0,t)$, $G_2(0,t)$, $G_2(0,t)$, the future curves for all maturities t<T (and by implication the basis relationships between pairs of forward curves),

$\Im(0)$, the information set including the history of the P's the S's and the G's until time 0.

We can calculate the term structure of future convenience yields:

$\varepsilon_1(0,t)$, $\varepsilon_2(0,t)$, $\varepsilon_3(0,t)$ for all maturities t<T.

Chapter Four, Mathematical Derivation of the Model

Introduction

As is usual in commodity price models, we shall assume we are working under a risk-neutral equivalent martingale measure. In the HJM model, a relation is shown to exist between drift and volatility the dynamics for the price process. We shall establish a similar relation (between the drift of the convenience yield and the volatilities of other processes) for the parameters of our model and prove the following result:

Theorem 4.1

In order that $G_l(t,T)$ is a martingale, then the following condition on the convenience yield drift must hold:

4.1
$$\mu_{\varepsilon i}(t,T)=\sum_{j=1}^{3}\left[\sigma_f^j(t,T)-\sigma_{\varepsilon i}^j(t,T)\right]\cdot\left[\sigma_{S_i(t)}^j+\int_t^T \sigma_f^j(t,s)-\sigma_{\varepsilon i}^j(t,s)ds\right].$$
$$+\sum_{j=1}^{3}\int_t^T \sigma_f^j(t,s)ds\cdot\sigma_f^j(t,T).$$

This proof of this theorem follows.

Some Relationships of Note

We know from Chapter Three that:

4.2
$$P(t,T)=E\left[\exp\left(-\int_t^T r(u)du\right)\mid \Im(t)\right]=\exp\left(-\int_t^T f(t,u)du\right)$$

This can easily be used to determine r_t by taking the partial derivative of P(t,T) with respect to T[12]. Thus we have:

4.3
$$f(t,t) = r_t.$$

Similarly, we can establish the relationships among the future, forward and spot convenience yields as:

4.4
$$\delta_i(t,t) = \varepsilon_i(t,t) = \delta_i(t)$$

[12] See Baxter and Rennie p. 132 for the derivation of this relationship.

for i = 1, 2, 3.

Multivariate Result of HJM

The results of Heath Jarrow and Morton, in a multiple Brownian motion world impose structure on the drift of the forward rate processes[13]. The rate in their model follows the process:

$$d_t f(t,T) = \sum_{i=1}^{n} \sigma_i(t,T)\, d\tilde{Z}_i(t) - \sum_{i=1}^{n} \sigma_i(t,T)\, \Sigma_i(t,T)\, dt$$

where

$$\Sigma_i(t,T) = -\int_t^T \sigma_i(t,u)\, du\,.$$

This demonstrates the volatilities in the HJM model determine the drift.

Multivariate Restriction for the Drift of the Convenience Yield

Our goal is to outline restrictions, similar to those noted by HJM, which are placed on each μ_ε under an equivalent martingale measure.

In our last chapter, we specified processes for interest rates and convenience yields. These processes were:

4.5 $$f(t,T) = f(0,T) + \int_0^t \mu_f(u,T)du + \int_0^t \sigma_f^1(u,T)dz_1(u) + \int_0^t \sigma_f^2(u,T)dz_2(u) + \int_0^t \sigma_f^3(u,T)dz_3(u)$$

and

4.6 $$\varepsilon_i(t,T) = \varepsilon_i(0,T) + \int_0^t \mu_{\varepsilon i}(u,T)du + \int_0^t \sigma_{\varepsilon i}^1(u,T)dz_1(u) + \int_0^t \sigma_{\varepsilon i}^2(u,T)dz_2(u) + \int_0^t \sigma_{\varepsilon i}^3(u,T)dz_3(u)\,.$$

We had also previously defined:

[13] See Baxter and Rennie p. 158-161 for the derivation of this relationship.

4.7 $$Y_i(t,T) = \exp\left(\int_t^T (f(t,s) - \varepsilon_i(t,s))ds\right).$$

We now define:

4.8 $$l_i(t) = \log(Y_i(t,T)).$$

It follows that:

4.9 $$\begin{aligned} l_i(t) = &\int_t^T f(0,u)\,du + \int_0^t\int_t^T \mu_f(v,u)\,du\,dv + \int_0^t\int_t^T \sigma_f^1(v,u)\,du\,dz_1(v) \\ &+ \int_0^t\int_t^T \sigma_f^2(v,u)\,du\,dz_2(v) + \int_0^t\int_t^T \sigma_f^3(v,u)\,du\,dz_3(v) \\ &- \int_t^T \varepsilon_i(0,u)\,du - \int_0^t\int_t^T \mu_{i\varepsilon}(v,u)\,du\,dv - \int_0^t\int_t^T \sigma_{i\varepsilon}^1(v,u)\,du\,dz_1(v) \\ &- \int_0^t\int_t^T \sigma_{i\varepsilon}^2(v,u)\,du\,dz_2(v) - \int_0^t\int_t^T \sigma_{i\varepsilon}^3(v,u)\,du\,dz_3(v). \end{aligned}$$

We can apply Fubini's standard and stochastic theorems to rewrite this as:

4.10 $$\begin{aligned} l_i(t) = &\int_0^T f(0,u)\,du + \int_0^t\int_v^T \mu_f(v,u)\,du\,dv + \int_0^t\int_v^T \sigma_f^1(v,u)\,du\,dz_1(v) \\ &+ \int_0^t\int_v^T \sigma_f^2(v,u)\,du\,dz_2(v) + \int_0^t\int_v^T \sigma_f^3(v,u)\,du\,dz_3(v) \\ &- \int_0^t f(0,u)\,du - \int_0^t\int_v^t \mu_f(v,u)\,du\,dv - \int_0^t\int_v^t \sigma_f^1(v,u)\,du\,dz_1(v) \\ &- \int_0^t\int_v^t \sigma_f^2(v,u)\,du\,dz_2(v) + \int_0^t\int_v^t \sigma_f^3(v,u)\,du\,dz_3(v) \\ &- \int_0^T \varepsilon_i(0,u)\,du - \int_0^t\int_v^T \mu_{i\varepsilon}(v,u)\,du\,dv - \int_0^t\int_v^T \sigma_{i\varepsilon}^1(v,u)\,du\,dz_1(v) \\ &- \int_0^t\int_v^T \sigma_{i\varepsilon}^2(v,u)\,du\,dz_2(v) - \int_0^t\int_v^T \sigma_{i\varepsilon}^3(v,u)\,du\,dz_3(v) \\ &+ \int_0^t \varepsilon_i(0,u)\,du + \int_0^t\int_v^t \mu_{i\varepsilon}(v,u)\,du\,dv + \int_0^t\int_v^t \sigma_{i\varepsilon}^1(v,u)\,du\,dz_1(v) \end{aligned}$$

$$+\int_0^t\int_v^t \sigma_{i\varepsilon}^2(v,u)\,du\,dz_2(v)+\int_0^t\int_v^t \sigma_{i\varepsilon}^3(v,u)\,du\,dz_3(v).$$

Re-writing the expression for forward interest rates, we have:

4.11
$$f(u,u)=f(0,u)+\int_0^u \mu_f(v,u)dv+\int_0^u \sigma_f^1(v,u)dz_1(v)$$
$$+\int_0^u \sigma_f^2(v,u)dz_2(v)+\int_0^u \sigma_f^3(v,u)dz_3(v).$$

The forward convenience yields are given by:

4.12
$$\varepsilon_i(u,u)=\varepsilon_i(0,u)+\int_0^u \mu_{\varepsilon i}(v,u)dv+\int_0^u \sigma_{\varepsilon i}^1(v,u)dz_1(v)$$
$$+\int_0^u \sigma_{\varepsilon i}^2(v,u)dz_2(v)+\int_0^u \sigma_{\varepsilon i}^3(v,u)dz_3(v).$$

We then obtain the following integrals:

4.13
$$\int_0^t f(u,u)du=\int_0^t f(0,u)du+\int_0^t\int_v^t \mu_f(v,u)dudv+\int_0^t\int_v^t \sigma_f^1(v,u)dudz_1(v)$$
$$+\int_0^t\int_v^t \sigma_f^2(v,u)dudz_2(v)+\int_0^t\int_v^t \sigma_f^3(v,u)dudz_3(v)$$

and

4.14
$$\int_0^t \varepsilon_i(u,u)du=\int_0^t \varepsilon_i(0,u)du+\int_0^t\int_v^t \mu_{\varepsilon i}(v,u)dudv+\int_0^t\int_v^t \sigma_{\varepsilon i}^1(v,u)dudz_1(v)$$
$$+\int_0^t\int_v^t \sigma_{\varepsilon i}^2(v,u)dudz_2(v)+\int_0^t\int_v^t \sigma_{\varepsilon i}^3(v,u)dudz_3(v).$$

Note that:

4.15
$$l_i(0)=\log(Y_i(0,T))$$
$$=\int_0^T f(0,u)du-\int_0^T \varepsilon_i(0,u)du.$$

We are now in a position to make substitutions into the expression for $l_i(t)$.

4.16
$$
\begin{aligned}
l_i(t) = {} & l_i(0) - \int_0^t f(u,u)du + \int_0^t \varepsilon_i(u,u)du \\
& + \int_0^t\int_u^T \mu_f(u,v)dvdu + \int_0^t\int_u^T \sigma_f^1(u,v)dvdz_1(u) + \int_0^t\int_u^T \sigma_f^2(u,v)dvdz_2(u) + \int_0^t\int_u^T \sigma_f^3(u,v)dvdz_3(u) \\
& - \int_0^t\int_u^T \mu_\delta(u,v)dvdu - \int_0^t\int_u^T \sigma_\delta^1(u,v)dvdz_1(u) - \int_0^t\int_u^T \sigma_\delta^2(u,v)dvdz_2(u) - \int_0^t\int_u^T \sigma_\delta^3(u,v)dvdz_3(u)\,.
\end{aligned}
$$

To simplify the notation, introduce the following variables:

4.17
$$\mu_f^*(t,T) = \int_t^T \mu_f(t,u)du$$

4.18
$$\sigma_{fi}^{j*}(t,T) = \int_t^T \sigma_{fi}^j(t,u)du \;\forall\; j = 1, 2, 3$$

4.19
$$\mu_\varepsilon^*(t,T) = \int_t^T \mu_\varepsilon(t,u)du$$

and

4.20
$$\sigma_\delta^{j*}(t,T) = \int_t^T \sigma_\delta^j(t,u)du \;\forall\; j = 1, 2, 3\,.$$

We now rewrite 4.16 as:

4.21
$$
\begin{aligned}
l_i(t) = {} & l_i(0) - \int_0^t f(u,u)du + \int_0^t \varepsilon_i(u,u)du \\
& + \int_0^t \mu_f^*(u,T)du + \int_0^t \sigma_f^{1*}(u,T)dz_1(u) + \int_0^t \sigma_f^{2*}(u,T)dz_2(u) + \int_0^t \sigma_f^{3*}(u,T)dz_3(u) \\
& - \int_0^t \mu_\delta^*(u,T)du - \int_0^t \sigma_\delta^{1*}(u,T)dz_1(u) - \int_0^t \sigma_\delta^{2*}(u,T)dz_2(u) - \int_0^t \sigma_\delta^{3*}(u,T)dz_3(u)\,.
\end{aligned}
$$

Recall that in 4.8 above, we noted that:

$$l_i(t) = \log(Y_i(t,T))$$

Consider a function $Y_i(t,T)$ with the following dynamics:

4.22
$$\frac{dY_i(t,T)}{Y_i(t,T)} = a \cdot dt + b \cdot dz_1(t) + c \cdot dz_2(t) + d \cdot dz_3(t)\,.$$

A straightforward application of Itô's lemma shows that:

4.23 $$dl_i(t) = \left(a - \frac{b^2}{2} - \frac{c^2}{2} - \frac{d^2}{2}\right)dt + b \cdot dz_1(t) + c \cdot dz_2(t) + d \cdot dz_3(t).$$

But 4.23 is simply a restating of 4.21 in differential form. We are thus able to determine the values of a, b, c, and d and substitute these expressions back into 4.22. The following result is obtained:

4.24 $$\frac{dY_i(t,T)}{Y_i(t,T)} = \left[-f_i(t,t) + \varepsilon_i(t,t) + \int_t^T \mu_f(t,s)ds - \int_t^T \mu_{\varepsilon}(t,s)ds\right]dt$$

$$+\sum_{j=1}^{3}\left[\frac{1}{2}\left\|\int_t^T \sigma_f^j(t,s)ds\right\|^2 + \frac{1}{2}\left\|\int_t^T \sigma_{\varepsilon}^j(t,s)ds\right\|^2 + \left[\int_t^T \sigma_f^j(t,s)ds\right]\cdot\left[\int_t^T \sigma_{\varepsilon}^j(t,s)ds\right]\right]dt$$

$$+\sum_{j=1}^{3}\left[\left[\int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{\varepsilon}^j(t,s)ds\right]dz_j(t)\right].$$

We now have expressions for $dS_i(t)$ and $dY_i(t,T)$ in terms of the same Brownian Motions.

Since $G_i(t,T) = S_i(t)\ Y_i(t,T)$, we can use Ito's lemma to determine the differential $dG_i(t,T)$.

We have expressions of the following forms:

4.25 $$dS_i(t) = aS_i(t)dt + bS_i(t)dz_1(t) + cS_i(t)dz_2(t) + dS_i(t)dz_3(t)$$

and

4.26 $$dY_i(t,T) = eY_i(t,T)dt + fY_i(t,T)dz_1(t) + gY_i(t,T)dz_2(t) + hY_i(t,T)dz_3(t).$$

Therefore:

4.27 $$dG_i(t,T) = (a+e+bf+cg+dh)\ G_i(t,T)dt + (b+f)\ G_i(t,T)dz_1(t) + (c+g)\ G_i(t,T)dz_2(t) + (d+h)\ G_i(t,T)dz_3(t).$$

Relying on the previous results, we can make substitutions for a, b, c, d, e, f, g, h in 4.27. This gives the following expression:

4.28 $$\frac{dG_i(t,T)}{G_i(t,T)} = \left[\int_t^T \mu_f(t,s)ds - \int_t^T \mu_{\varepsilon}(t,s)ds\right]dt$$

$$+\sum_{j=1}^{3}\left[\frac{1}{2}\left\|\int_t^T \sigma_f^j(t,s)ds\right\|^2+\frac{1}{2}\left\|\int_t^T \sigma_\delta^j(t,s)ds\right\|^2+\left[\int_t^T \sigma_f^j(t,s)ds\right]\cdot\left[\int_t^T \sigma_\delta^j(t,s)ds\right]\right]dt$$

$$+\sum_{j=1}^{3}\left[\sigma_{Si(t)}^j\left[\int_t^T \sigma_f^j(t,s)ds-\int_t^T \sigma_\delta^j(t,s)ds\right]\right]dt$$

$$+\sum_{j=1}^{3}\left[\left[\sigma_{S_i(t)}^j+\int_t^T \sigma_f^j(t,s)ds-\int_t^T \sigma_\delta^j(t,s)ds\right]dz_j(t)\right].$$

Eliminating the Convenience Yield Drift

Our objective is to derive an expression for the drift of the convenience yield under an equivalent martingale measure. We must first eliminate the drift of the forward interest rate.

We recall that:

4.29 $$\int_t^T \mu_f(t,s)ds=\sum_{j=1}^{3}\left[\int_t^T\left[\sigma_f^j(t,s)\cdot\int_t^s \sigma_f^j(t,v)dv\right]ds\right].$$

Define:

4.30 $$H^j(s)=\int_t^s \sigma_f^j(t,v)dv.$$

It follows that:

4.31 $$dH^j(s)=\sigma_f^j(t,s)\,ds.$$

We can rewrite 4.29 as:

4.32 $$\int_t^T \mu_f(t,s)ds=\sum_{j=1}^{3}\left[\int_t^T H^j(s)\,dH^j(s)\right].$$

Thus:

4.33 $$\int_t^T \mu_f(t,s)ds=\frac{1}{2}\sum_{j=1}^{3}\left[H^j(s)\right]^2\Big|_t^T.$$

We can now substitute the relationship in 4.30 and evaluate the integral to obtain:

4.34 $$\int_t^T \mu_f(t,s)ds = \frac{1}{2}\sum_{j=1}^{3}\left[\int_t^s \sigma_f^j(t,v)dv\right]^2.$$

We can make a substitution of 4.34 into 4.28 to derive the final expression for $G_i(t,T)$:

4.35 $$\frac{dG_i(t,T)}{G_i(t,T)} = \left[-\int_t^T \mu_{di}(t,s)ds + \sum_{j=1}^{3}\left\|\int_t^T \sigma_f^j(t,s)ds\right\|^2\right]dt$$
$$+\sum_{j=1}^{3}\left[\frac{1}{2}\left\|\int_t^T \sigma_{di}^j(t,s)ds\right\|^2 + \left[\int_t^T \sigma_f^j(t,s)ds\right]\cdot\left[\int_t^T \sigma_{di}^j(t,s)ds\right]\right]dt$$
$$+\sum_{j=1}^{3}\left[\sigma_{Si(t)}^j\left[\int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{di}^j(t,s)ds\right]\right]dt$$
$$+\sum_{j=1}^{3}\left[\left[\sigma_{S_i(t)}^j + \int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{di}^j(t,s)ds\right]dz_j(t)\right].$$

In order for this expression to be a martingale, the coefficients of the dt term must be equal to zero; the process must exhibit zero drift. Mathematically, this is equivalent to noting:

4.36 $$\int_t^T \mu_{di}(t,s)ds = \sum_{j=1}^{3}\left\|\int_t^T \sigma_f^j(t,s)ds\right\|^2$$
$$+\sum_{j=1}^{3}\left[\frac{1}{2}\left\|\int_t^T \sigma_{di}^j(t,s)ds\right\|^2 + \left[\int_t^T \sigma_f^j(t,s)ds\right]\cdot\left[\int_t^T \sigma_{di}^j(t,s)ds\right]\right]$$
$$+\sum_{j=1}^{3}\left[\sigma_{Si(t)}^j\left[\int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{di}^j(t,s)ds\right]\right].$$

We differentiate 4.36 term by term with respect to T to give:

4.37 $$\frac{d\left[\int_t^T \mu_{di}(t,s)ds\right]}{dT} = \mu_{di}(t,T).$$

4.38 $$\frac{d\left[\sum_{j=1}^{3}\left\|\int_t^T \sigma_f^j(t,s)ds\right\|^2\right]}{dT} = 2\cdot\sum_{j=1}^{3}\int_t^T \sigma_f^j(t,s)ds\cdot\sigma_f^j(t,T).$$

4.39 $$\frac{d\left[\sum_{j=1}^{3}\frac{1}{2}\left\|\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\right\|^2\right]}{dT}=2\cdot\frac{1}{2}\cdot\sum_{j=1}^{3}\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\cdot\sigma_{\varepsilon i}^{j}(t,T).$$

4.40 $$\frac{d\left[\sum_{j=1}^{3}\left[-\int_t^T \sigma_f^{j}(t,s)ds\cdot\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\right]\right]}{dT}$$
$$=\sum_{j=1}^{3}\int_t^T \sigma_f^{j}(t,s)ds\cdot\sigma_{\varepsilon i}^{j}(t,T)-\sum_{j=1}^{3}\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\cdot\sigma_f^{j}(t,T).$$

4.41 $$\frac{d\left[\sum_{j=1}^{3}\left[\sigma_{S_i(t)}^{j}\left[\int_t^T \sigma_f^{j}(t,s)ds-\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\right]\right]\right]}{dT}$$
$$=\sum_{j=1}^{3}\left[\sigma_{S_i(t)}^{j}\cdot\left(\sigma_f^{j}(t,T)-\sigma_{\varepsilon i}^{j}(t,T)\right)\right].$$

Substituting 4.37 through 4.41 into 4.36 and simplifying gives us the desired restriction on the drift:

4.42 $$\mu_{\varepsilon i}(t,T)=\sum_{j=1}^{3}\left[\sigma_f^{j}(t,T)-\sigma_{\varepsilon i}^{j}(t,T)\right]\cdot\left[\sigma_{S_i(t)}^{j}+\int_t^T \sigma_f^{j}(t,s)-\sigma_{\varepsilon i}^{j}(t,s)ds\right]$$
$$+\sum_{j=1}^{3}\int_t^T \sigma_f^{j}(t,s)ds\cdot\sigma_f^{j}(t,T).$$

And we have proved the result in Theorem 4.1

Thus when we estimate the convenience yield drift as per the relationship in 4.42, we are assured the process in 4.35 is the martingale:

4.43 $$\frac{dG_i(t,T)}{G_i(t,T)}=\sum_{j=1}^{3}\left[\left[\sigma_{S_i(t)}^{j}+\int_t^T \sigma_f^{j}(t,s)ds-\int_t^T \sigma_{\varepsilon i}^{j}(t,s)ds\right]dz_j(t)\right]$$

Martingale Approach for Pricing Derivatives

Introduction

With the processes now defined, and the technical restrictions in place, our objective is to price derivative securities, some of which were outlined in Chapter Two.

We will attempt to follow the martingale methodology suggested by [Baxter and Rennie] and [Elliott and Kopp] to price derivative securities (i.e. options) based on the forward prices. This requires that we evaluate $E(t) = E[P(t,T)^{-1}X|\Im(t)]$, $\forall$ i, where P(t,T) is the price of a discount bond, X is the claim at time T, E is the expectation under the measure Q, and $\Im(t)$ is the history to time t.

To hedge claims, it is necessary to find a previsible process $\phi(t)$, such that $dE(t) = \phi(t)\, dW(t)$.

Determining the Martingales

In this model, we have assumed that we are working under an equivalent martingale measure.

To explain Martingales, it is first required we briefly outline the axiomatic approach to probability as first expressed by Kolmogorov.

Gödel, in his famous incompleteness theorem[14] implies that prior attempts to define probabilities fail because of the inclusion of language such as "likelihood" in defining probability. Kolmogorov set up an axiomatic framework in which probabilities are defined using measure theory with respect to a set theory structure called a sigma field.

[14] In 1931, the Czech-born mathematician Kurt Gödel demonstrated that within any given branch of mathematics, there would always be some propositions that couldn't be proven either true or false using the rules and axioms ... of that mathematical branch itself. You might be able to prove every conceivable statement about numbers within a system by going outside the system in order to come up with new rules an axioms, but by doing so you'll only create a larger system with its own unprovable statements. The implication is that all logical system of any complexity are, by definition, incomplete; each of them contains, at any given time, more true statements than it can possibly prove according to its own defining set of rules.

Sigma Fields

A sigma field (commonly designated as $\Im$) is a collection of subsets of a set (commonly designated as Ω) which satisfies the criteria of closure under complements, and countable unions and intersections. The map of an event A, which is a set in a sigma field $\Im$, to the real interval [0,1], is called a probability measure and is commonly designated as P.

The ordered triple $(\Omega, \Im, P)$ is called a probability space.

Formal Definition of a Martingale

Let $\{X_t, t \geq 0\}$ be a set of random variables indexed by $t \geq 0$ defined on a probability space $(\Omega, \Im, P)$ and let $\{\Im_t, t \geq 0\}$ be a family of sigma fields such that if $s \leq t$ then $\Im_s \subset \Im_t$. The process $\{X_t, t \geq 0\}$ is a martingale if it satisfies the three conditions below:

(1) X_t is measurable with respect to $\Im_t$;
(2) $E(|X_t|) < \infty$;
(3) $E(|X_t \mid \Im_s) = X_s$ almost surely.

This means that a random variable is a martingale if its expected value at a future time, given "the history so far", $(\Im_t)$, is simply its value today.

Change of Probability Measures

Consider a probability space $(\Omega, \Im, P)$ and suppose Q is a second probability measure defined on $\Im$. Q is said to be equivalent to P if whenever $A \in \Im$, and $P(A) = 0$, then $Q(A) = 0$, and conversely. In this case Q can be defined by a density function, sometimes called the Radon-Nikodym derivative $\frac{dQ}{dP}$, so that for any $B \in \Im$, $Q(B) = \int_B \frac{dQ}{dP} dP$.

For clarity, expectations taken with respect to different measures will be denoted, when appropriate, with a subscript. Thus:

$$E_Q(X_t) = E_P\left(\frac{dQ}{dP} X_t\right)$$

Girsanov Theorem

We now cite one version of the well-known Girsanov[15] Theorem:

Theorem:

If z(t) is a P-Brownian motion, and $\gamma(t)$ is an $\Im$-previsible process satisfying the condition $E_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma^2(t)dt\right)\right] < \infty$, then one can define a measure Q by setting:

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma(t)dz(t) - \frac{1}{2}\int_0^T \gamma^2(t)dt\right).$$

Girsanov's theorem then states that:

$$\tilde{z}(t) = z(t) + \int_0^t \gamma(s)ds$$

is a Q-Brownian Motion.

This is simply another way of noting that a P-Brownian Motion is a Q-Brownian Motion with drift $\gamma(t)$ at time t. Since a Brownian Motion can be thought of a drift-less martingale, we have a mechanism to convert drifting processes to martingales.

Martingale Equivalence

We are now able to collect the ideas presented above and explain how they apply to pricing claims. One version of the Martingale Representation Theorem[16] is now stated:

Suppose B(t) is a Q-Brownian motion and $\{\Im_t\}$ is the family of σ-fields generated by B(t). Then if N(t) is any other $\{\Im_t, Q\}$ martingale, there exists an $\Im$-previsible process ϕ, such that $\int_0^T \phi^2(t)dt < \infty$ (a.s.) and N(t) can be written as:

[15] Baxter and Rennie, p. 74
[16] Baxter and Rennie, p. 78

$$N(t) = N(0) + \int_0^t \phi(s)dB(s)$$

For our purposes, if we have one martingale representing the asset path, and a second representing a contingent claim on that asset, then we can find a function which relates the two. Non-technically, we will be able to synthesise the claim by setting up a self-financing portfolio that always contains ϕ units of the underlying asset.

Girsanov's theorem is important because this assures us that we can take non-martingale processes and by invoking a change of measure, we can remove the drift term. It is possible to show by iterated expectations[17] that any contingent claim can be expressed as a martingale.

[17] For example see Ingersoll p. 18

Chapter Five: Option Price Derivation

Introduction

The objective of this chapter is to take the martingale processes of Chapter Four:

5.1 $$\frac{dG_i(t,T)}{G_i(t,T)} = \sum_{j=1}^{3}\left[\left[\sigma^j_{S_i(t)} + \int_t^T \sigma^j_f(t,s)ds - \int_t^T \sigma^j_{\delta i}(t,s)ds\right]dz_j(t)\right]$$

for i = 1,2,3

and price two types of options. These are options on the contract and options on the location difference of two contracts, the basis.

Call Options

The simplest derivative on the contract is a plain vanilla call option that has a maturity date of t for the T contract. This has a value at time 0 given by:

5.2 $$C_i = E\left[\exp\left(-\int_0^t f(s,s)\,ds\right)(G_i(t,T) - K)^+\right].$$

This section will use the notation suggested by Miltersen and Schwartz. Recall that:

5.3 $$f(s,s) = f(0,s) + \int_0^t \mu_f(u,s)\,du + \sum_{j=1}^{3}\int_0^t \sigma^j_f(u,s)\,dz_j(u).$$

And thus:

5.4 $$\int_0^t f(s,s)\,ds = \int_0^t f(0,s)\,ds + \int_0^t\int_u^t \mu_f(u,s)ds\,du + \sum_{j=1}^{3}\int_0^t\left(\int_u^t \sigma^j_f(u,s)\,ds\right)dz_j(u)$$

and:

5.5
$$\exp\left(-\int_0^t f(s,s)\,ds\right)$$
$$= \exp\left(-\int_0^t f(0,s)ds \;-\int_0^t\int_u^t \mu_f(u,s)ds\,du - \sum_{j=1}^{3}\int_0^t\left(\int_u^t \sigma_f^j(u,s)\,ds\right)dz_j(u)\right)$$

We can rewrite this as:

5.6
$$\exp\left(-\int_0^t f(s,s)\,ds\right) = A\exp(-X)$$

where:

5.7
$$X = \sum_{j=1}^{3}\int_0^t\left(\int_u^t \sigma_f^j(u,s)\,ds\right)dz_j(u)$$

and A is the $\Im_0$-measurable expression:

5.8
$$A = \exp\left(-\int_0^t f(0,s)ds \;-\int_0^t\int_u^t \mu_f(u,s)ds\,du\right).$$

For future reference, we note that f(s,s) = r(s), so:

$$E\left[A\exp(-X)\right] = E\left[\exp\left(-\int_0^t r(s)\,ds\right)\right] = P(0,T).$$

For ease of notation, define:

5.9
$$\int_u^t \sigma_f^j(u,s)\,ds \equiv \sigma_{P(t)}^j(u).$$

In the previous chapter, we showed that with the condition of zero drift, that

5.10
$$\frac{dG_i(t,T)}{G_i(t,T)} = \sum_{j=1}^{3}\left[\left[\sigma_{S_i(t)}^j + \int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{di}^j(t,s)ds\right]dz_j(t)\right].$$

For i = 1, 2, 3

For further ease of notation, we define:

5.11 $$\sigma^{j}_{GTi}(t) \equiv \sigma^{j}_{S_i(t)} + \int_t^T \sigma^{j}_{f}(t,s)ds - \int_t^T \sigma^{j}_{a}(t,s)ds .$$

We can re-write 5.10 as:

5.12 $$\frac{dG_i(t,T)}{G_i(t,T)} = \sum_{j=1}^{3}\left[\sigma^{j}_{GTi}(t)dz_j(t)\right].$$

We can re-write this expression with an integral sign as follows.

5.13 $$\int_0^t \frac{dG_i(u,T)}{G_i(u,T)} = \sum_{j=1}^{3}\int_0^t\left[\sigma^{j}_{GTi}(u)dz_j(u)\right].$$

That is:

5.14 $$G_i(t,T) = G_i(0,T)\cdot\exp\left(\sum_{j=1}^{3}\int_0^t \sigma^{j}_{GTi}(u)dz_j(u) - \frac{1}{2}\sum_{j=1}^{3}\int_0^t \left(\sigma^{j}_{GTi}(u)\right)^2 du\right).$$

This can be written in a simpler form as:

5.15 $$G_i(t,T) = B\cdot\exp(Z_i)$$

where

$$B = G_i(0,T)\cdot\exp\left(-\frac{1}{2}\sum_{j=1}^{3}\int_0^t \left(\sigma^{j}_{GTi}(u)\right)^2 du\right)$$

and

$$Z_i = \sum_{j=1}^{3}\int_0^t \sigma^{j}_{GTi}(u)dz_j(u).$$

Since all 3 Brownian Motions are independent, we know that:

5.16 $$\sigma_X^2 = \sum_{j=1}^{3}\int_0^t \left\|\sigma^{j}_{Pt}(u)\right\|^2 du$$

5.17 $$\sigma_{Z_i}^2 = \sum_{j=1}^{3}\int_0^t \left\|\sigma^{j}_{GTi}(u)\right\|^2 du$$

and

5.18 $$\sigma_{XZ_i} = -\sum_{j=1}^{3} \int_0^t \sigma_{Pt}^{j}(u) \cdot \sigma_{GTi}^{j}(u)\, du .$$

Substituting 5.6 and 5.15 into expression 5.2 gives:

$$C_i = E\left[A \exp(-X)(B\exp(Z_i) - K)^{+}\right]$$
$$= A \cdot E\left[e^{-X}(B\exp(Z_i) - K)^{+}\right] .$$

Our focus in valuing the option is on the expression $(B\exp(Z_i) - K)^{+}$. We can use iterated expectations to note that:

5.19 $$C_i = A \cdot E\left[E[\exp(-X) \mid Z_i] \cdot (B\exp(Z_i) - K)^{+}\right].$$

Recall that if two variables, Z_i and X are bi-variate normal, then the conditional density of X given Z_i = z, is given by $N(\mu_{X|Z_i}, \sigma^2_{X|Z_i})$ where $\mu_{X|Z_i} = \mu_X + \rho \frac{\sigma_X}{\sigma_{Z_i}}$, and $\sigma^2_{X|Z_i} = \sigma^2_X \cdot \left(1 - \frac{\sigma^2_{XZ_i}}{\sigma^2_X \sigma^2_{Z_i}}\right)$.

We recall that for any function of X, $E[u(x) \mid z] = \int_{-\infty}^{\infty} u(x) \cdot f(x \mid z)\, dx$.

Write:

5.20 $$\sigma^2 = \sigma^2_X \cdot \left(1 - \frac{\sigma^2_{XZ}}{\sigma^2_X \sigma^2_Z}\right)$$

and

5.21 $$\mu = z \frac{\sigma_{XZ}}{\sigma^2_Z} .$$

Using the conditional distribution of bi-variate normal random variables, we note:

$$E[\exp(-X) \mid Z_i = z] = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-X)\, \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$= \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{(x-\mu)^2 + 2x\sigma^2}{\sigma}\right)\right) dx.$$

Completing the square in the exponent, we obtain:

5.22
$$E[\exp(-X) \mid Z_i = z]$$
$$= \exp\left(-\mu + \frac{1}{2}\sigma^2\right) \cdot \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x-(\mu+\sigma^2)}{\sigma}\right)^2\right) dx$$

The integral now is simply the evaluation of a normal $N(\mu + \sigma^2, \sigma^2)$ distribution function, so:

5.23
$$E[\exp(-X) \mid Z_i = z] = \exp\left(-\mu + \frac{1}{2}\sigma^2\right).$$

Substituting in the expressions from 5.20 and 5.21, we obtain:

5.24
$$E[\exp(-X) \mid Z_i = z] = \exp\left(-Z_i \frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2} + \frac{1}{2}\sigma_X^2 \cdot \left(1 - \frac{\sigma_{XZ_i}^2}{\sigma_X^2 \sigma_{Z_i}^2}\right)\right).$$

This permits us to rewrite 5.19 as:

5.25
$$C_i = A \cdot \exp\left(\frac{1}{2}\sigma_X^2 \cdot \left(1 - \frac{\sigma_{XZ_i}^2}{\sigma_X^2 \sigma_{Z_i}^2}\right)\right) E\left[\exp\left(-Z_i \frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right) \cdot (B\exp(Z_i) - K)^+\right]$$

The notation $(B\exp(Z_i) - K)^+$ implies that we wish to consider only values for which the difference is positive; that is, we consider $Z_i > \log\frac{K}{B}$. We replace the notation above with the indicator function, $1_{\left\{Z > \log\frac{K}{B}\right\}}$ which is a binary payoff, giving a value of 1 when $Z_i > \log\frac{K}{B}$, and 0 otherwise. Thus 5.25 can be expressed as:

5.26
$$C_i = A \cdot B \cdot \exp\left(\frac{1}{2}\sigma_X^2 \cdot \left(1 - \frac{\sigma_{XZ_i}^2}{\sigma_X^2 \sigma_{Z_i}^2}\right)\right) E\left[1_{\left\{Z_i > \log\frac{K}{B}\right\}} \exp\left(Z_i \cdot \left(1 - \frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)\right]$$

$$-A\cdot K\cdot\exp\left(\frac{1}{2}\sigma_X^2\cdot\left(1-\frac{\sigma_{XZ_i}^2}{\sigma_X^2\sigma_{Z_i}^2}\right)\right)E\left[1_{\{Z_i>\log\frac{K}{B}\}}\exp\left(-Z_i\cdot\left(\frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)\right].$$

We now eliminate the two expectation operators. For the first:

5.27 $$E\left[1_{\{Z_i>\log\frac{K}{B}\}}\exp\left(Z_i\cdot\left(1-\frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)\right]$$

$$=\frac{1}{\sigma_{Z_i}\sqrt{2\pi}}\int_{\log\frac{K}{B}}^{\infty}\exp\left(y\cdot\left(1-\frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)\cdot\exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{Z_i}}\right)^2\right)dy.$$

Let

$$v=\left(\frac{y+\sigma_{XZ_i}-\sigma_{Z_i}^2}{\sigma_{Z_i}}\right).$$

Then 5.27 becomes:

5.28 $$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\frac{B}{K}+\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}}\exp\left((v\sigma_{Z_i}+\sigma_{Z_i}^2-\sigma_{XZ_i})\cdot\left(1-\frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)$$

$$\bullet\exp\left(-\frac{1}{2}\cdot\left(\frac{v\sigma_{Z_i}+\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}\right)^2\right)dv.$$

A little algebra can simplify this to:

5.29 $$\exp\left(\frac{1}{2}\cdot\left(\frac{(\sigma_{Z_i}^2-\sigma_{XZ_i})^2}{\sigma_{Z_i}^2}\right)\right)\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\frac{B}{K}+\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}}\exp\left(-\frac{1}{2}\cdot(v)^2\right)dv;$$

Which is simply:

5.30 $$\exp\left(\frac{1}{2}\cdot\left(\frac{(\sigma_{Z_i}^2-\sigma_{XZ_i})^2}{\sigma_{Z_i}^2}\right)\right)\cdot N\left(\frac{\log\frac{B}{K}+\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}\right).$$

Where N(.) is the N(0,1) distribution function.

A similar argument shows:

5.31
$$E\left[1_{\left\{Z_i>\log\frac{K}{B}\right\}}\exp\left(-Z_i\cdot\left(\frac{\sigma_{XZ_i}}{\sigma_{Z_i}^2}\right)\right)\right]=\exp\left(\frac{\sigma_{XZ_i}^2}{2\sigma_{Z_i}^2}\right)\cdot N\left(\frac{\log\frac{B}{K}-\sigma_{XZ_i}}{\sigma_{Z_i}}\right).$$

If we substitute 5.30 and 5.31 in 5.26 we obtain:

5.32
$$\begin{aligned}C_i &= A\cdot B\cdot\exp\left(\frac{1}{2}\sigma_X^2\cdot\left(1-\frac{\sigma_{XZ_i}^2}{\sigma_X^2\sigma_{Z_i}^2}\right)\right)\\ &\bullet\exp\left(\frac{1}{2}\cdot\left(\frac{\left(\sigma_{Z_i}^2-\sigma_{XZ_i}\right)^2}{\sigma_{Z_i}^2}\right)\right)\cdot N\left(\frac{\log\frac{B}{K}+\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}\right)\\ &-A\cdot K\cdot\exp\left(\frac{1}{2}\sigma_X^2\cdot\left(1-\frac{\sigma_{XZ_i}^2}{\sigma_X^2\sigma_{Z_i}^2}\right)\right)\cdot\exp\left(\frac{\sigma_{XZ_i}^2}{2\sigma_{Z_i}^2}\right)\cdot N\left(\frac{\log\frac{B}{K}-\sigma_{XZ_i}}{\sigma_{Z_i}}\right).\end{aligned}$$

In order to simplify 5.32, note that:

$$K\cdot A\cdot\exp\left(\frac{1}{2}\sigma_X^2\cdot\left(1-\frac{\sigma_{XZ_i}^2}{\sigma_X^2\sigma_{Z_i}^2}\right)\right)\cdot\exp\left(\frac{\sigma_{XZ_i}^2}{2\sigma_{Z_i}^2}\right)=K\cdot A\cdot\exp\left(\frac{1}{2}\sigma_X^2\right).$$

Since $X\approx N(0,\sigma_X)$, it is easy to demonstrate by introducing an integral and completing the square in the exponent that:

$$\begin{aligned}K\cdot A\cdot\exp\left(\frac{1}{2}\sigma_X^2\right)&=K\cdot A\cdot E[\exp(-X)]\\ &=K\cdot E[A\cdot\exp(-X)].\end{aligned}$$

Thus:

5.33
$$K\cdot A\cdot\exp\left(\frac{1}{2}\sigma_X^2\cdot\left(1-\frac{\sigma_{XZ_i}^2}{\sigma_X^2\sigma_{Z_i}^2}\right)\right)\cdot\exp\left(\frac{\sigma_{XZ_i}^2}{2\sigma_{Z_i}^2}\right)=K\cdot P(0,t).$$

Simplification of the first term of 5.32 yields:

$$A \cdot B \cdot \exp\left(\frac{1}{2}\sigma_X^2 \cdot \left(1 - \frac{\sigma_{XZ_i}^2}{\sigma_X^2 \sigma_{Z_i}^2}\right)\right) \cdot \exp\left(\frac{1}{2} \cdot \left(\frac{\left(\sigma_{Z_i}^2 - \sigma_{XZ_i}\right)^2}{\sigma_{Z_i}^2}\right)\right)$$

$$= A \cdot B \cdot \exp\left(\frac{1}{2}\left(\sigma_X^2 + \sigma_{Z_i}^2 - 2\sigma_{XZ_i}\right)\right).$$

Since X and Z_i are jointly normal, we can also note that:

$$A \cdot B \cdot \exp\left(\frac{1}{2}\left(\sigma_X^2 + \sigma_{Z_i}^2 - 2\sigma_{XZ_i}\right)\right) = A \cdot B \cdot E[\exp(-X + Z_i)]$$

and:

5.34 $$E[A\exp(-X) \cdot B \cdot \exp(Z_i)] = E\left[\exp\left(-\int_0^t f(s,s)\, ds\right) G_i(t,T)\right].$$

Combining 5.34 and 5.33 gives:

$$B \cdot \exp\left(\frac{1}{2}\sigma_Z^2 - \sigma_{XZ_i}\right) = \frac{E\left[\exp\left(-\int_0^t f(s,s)\, ds\right) G_i(t,T)\right]}{P(0,t)}$$

allowing us to note that:

5.35 $$\log\frac{B}{K} + \frac{1}{2}\sigma_{Z_i}^2 - \sigma_{XZ_i} = \log\left(\frac{E\left[\exp\left(-\int_0^t f(s,s)\, ds\right) G_i(t,T)\right]}{P(0,t) \cdot K}\right).$$

Above, we noted that:

5.36 $$E\left[\exp\left(-\int_0^t f(s,s)\, ds\right) G_i(t,T)\right] = A \cdot B \cdot \exp\left(\frac{1}{2}\left(\sigma_X^2 + \sigma_{Z_i}^2 - 2\sigma_{XZ_i}\right)\right)$$

which can be rewritten as:

$$P(0,t) \cdot E[G(t,T)] \cdot \exp(-\sigma_{XZ_i})$$

$$= A\,E[\exp(-X)] \cdot B\,E[\cdot \exp(Z_i)] \cdot \exp(-\sigma_{XZ_i})$$

which further simplifies to:

$$P(0,t)\cdot E[G(t,T)]\cdot \exp(-\sigma_{XZ_i}).$$

But as G(t,T) is a martingale, we note this expression is simply:

$$P(0,t)\cdot G(0,T)\cdot \exp(-\sigma_{XZ_i}).$$

To recapitulate, we see that:

5.37 $$E\left[\exp\left(-\int_0^t f(s,s)\,ds\right)G_i(t,T)\right] = P(0,t)\cdot G(0,T)\cdot \exp(-\sigma_{XZ_i}).$$

We can substitute the previous results (5.33 - 5.37) into 5.32 to provide a closed-form solution on the call option.

5.38 $$C_i = P(0,t)\left(G(0,T)\exp(-\sigma_{XZ_i})N\left(\frac{\log\frac{G(0,T)}{K}+\frac{1}{2}\sigma_{Z_i}^2-\sigma_{XZ_i}}{\sigma_{Z_i}}\right) - K\cdot N\left(\frac{\log\frac{G(0,T)}{K}-\sigma_{XZ_i}-\frac{1}{2}\sigma_{Z_i}^2}{\sigma_{Z_i}}\right)\right).$$

Recall that $\sigma_{XZ_i} = -\sum_{j=1}^{3}\int_0^t \sigma_{Pt}^j(u)\cdot\sigma_{GTi}^j(u)\,du$ as per 5.18. We are able to price options with this closed form solution and this model is calibrated in Chapter Six.

Options on Location Spreads

We are interested in the pricing of the option on the basis, or an option on a location spread. Mathematically this is

$$C_{i,h} = E\left[\exp\left(-\int_0^t f(s,s)\,ds\right)(G_i(t,T)-G_h(t,T)-K)^+\right]$$

The next few sections are background material to assist us in this endeavour.

Basis is Normal

This section is to show that the returns on the basis between two separate locations is approximated by a Normal distribution.

Recall from Chapter Four that we derived:

5.39
$$\frac{dG_i(t,T)}{G_i(t,T)} = \left[-\int_t^T \mu_{ei}(t,s)ds + \sum_{j=1}^{3} \left\| \int_t^T \sigma_f^j(t,s)ds \right\|^2 \right] dt$$
$$+ \sum_{j=1}^{3} \left[\frac{1}{2} \left\| \int_t^T \sigma_{ei}^j(t,s)ds \right\|^2 + \left[\int_t^T \sigma_f^j(t,s)ds \right] \cdot \left[\int_t^T \sigma_{ei}^j(t,s)ds \right] \right] dt$$
$$+ \sum_{j=1}^{3} \left[\sigma_{Si(t)}^j \left[\int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{ei}^j(t,s)ds \right] \right] dt$$
$$+ \sum_{j=1}^{3} \left[\left[\sigma_{S_i(t)}^j + \int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{ei}^j(t,s)ds \right] dz_j(t) \right]$$

In the previous chapter, we showed under Theorem 4.1 that:

$$\frac{dG_i(t,T)}{G_i(t,T)} = \sum_{j=1}^{3} \left[\left[\sigma_{S_i(t)}^j + \int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{ei}^j(t,s)ds \right] dz_j(t) \right].$$

As this is somewhat cumbersome, we introduce some simplifying notation and rewrite this as:

5.40
$$\frac{dG_i(t)}{G_i(t)} = \sigma_i^1 dZ_1(t) + \sigma_i^2 dZ_2(t) + \sigma_i^3 dZ_3(t)$$

This is similar to the Doléans exponential of Brownian motion. The solution is:

5.41
$$G_i(t) = G_i(0) \exp\left(\int_0^t \left(-\frac{1}{2}(\sigma_i^1(s))^2 - \frac{1}{2}(\sigma_i^2(s))^2 - \frac{1}{2}(\sigma_i^3(s))^2 \right) ds \right.$$

$$+\int_0^t \sigma_i^1(s)dZ_1(s)+\int_0^t \sigma_i^2(s)dZ_2(s)+\int_0^t \sigma_i^3(s)dZ_3(s)\Bigg)$$

Note that for any stochastic integral

5.42 $$I^j(0,t)=\int_0^t \phi(s)\,dZ_j(s)$$

the following relationship is true (under integrability conditions):

5.43 $$\left(\int_0^t \phi(s)dZ_j(s)\right)^2 = 2\int_0^t I^j(0,s)\,\phi(s)\,dZ_j(s)+\int_0^t \phi(s)^2 ds$$

In particular, define:

5.44 $$I_i^j(0,t)=\int_0^t \sigma_i^j(s)\,dZ_j(s)$$
$$\forall\ j=1,2,3$$

Recalling the Taylor series expansion of exp(y):

$$\exp(y)=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\dots$$

Including terms of o(dt), and using 5.43, $G_i(t)$ can be written as a Taylor's series as follows:

5.45 $$G_i(t)\cong G_i(0)\left(1+\left(\int_0^t\left(-\frac{1}{2}\sum_{j=1}^{3}\left(\sigma_i^j(s)\right)^2\right)ds\right.\right.$$
$$+\sum_{j=1}^{3}\int_0^t \sigma_i^j(s)dZ_j(s)+\frac{1}{2}\sum_{j=1}^{3}\int_0^t\left(\sigma_i^j(s)\right)^2 ds$$
$$\left.+\frac{2}{2}\sum_{j=1}^{3}\int_0^t I_i^j(0,t)\,\sigma_i^j(s)\,dZ_j(s)\right)$$
$$+\,o(dt)\,.$$

Thus:

5.46 $$E\left[G_i(t)\right]\cong G_i(0)\,.$$

It follows immediately that the expectation of the basis (defined as $\Gamma_{12}(t)=G_1(t)-G_2(t)$) between location 1 and 2 is given by:

5.47 $$E\left[\Gamma_{12}(t)\right]\equiv G_1(0)-G_2(0).$$

We also see that:

5.48 $$\begin{aligned}\Gamma_{12}(t)\equiv G_1(0)\Bigg(1+\Bigg(&\int_0^t\left(-\frac{1}{2}\sum_{j=1}^{3}\left(\sigma_1^j(s)\right)^2\right)ds\\&+\sum_{j=1}^{3}\int_0^t\sigma_1^j(s)dZ_j(s)+\frac{1}{2}\sum_{j=1}^{3}\int_0^t\left(\sigma_1^j(s)\right)^2ds\\&+\sum_{j=1}^{3}\int_0^t I_1^j(0,s)\sigma_1^j(s)dZ_j(s)\Bigg)\\-G_2(0)\Bigg(1+\Bigg(&\int_0^t\left(-\frac{1}{2}\sum_{j=1}^{3}\left(\sigma_2^j(s)\right)^2\right)ds\\&+\sum_{j=1}^{3}\int_0^t\sigma_2^j(s)dZ_j(s)+\frac{1}{2}\sum_{j=1}^{3}\int_0^t\left(\sigma_2^j(s)\right)^2ds\\&+\sum_{j=1}^{3}\int_0^t I_2^j(0,s)\sigma_2^j(s)dZ_j(s)\Bigg)\end{aligned}$$

To obtain the variance of the basis, recall:

$$\mathrm{Var}\left[\Gamma_{12}(t)\right]=E\left[\left(\Gamma_{12}(t)-E\left[\Gamma_{12}(t)\right]\right)^2\right].$$

It can be shown for the notation used in 5.44 that:

5.49 $$I_1^j(0,t)I_2^j(0,t)=\int_0^t I_1^j(0,s)\sigma_2^j(s)dZ_j(s)+\int_0^t I_2^j(0,s)\sigma_1^j(s)dZ_j(s)+\int_0^t\sigma_1^j(s)\sigma_2^j(s)ds.$$

Cancelling common terms of 5.47 and 5.48, squaring the difference, making the substitution of 5.49 and then taking expectations gives the following:

5.50 $$\begin{aligned}\mathrm{Var}\left[\Gamma_{12}(t)\right]=\sum_{j=1}^{3}\int_0^t\Bigg(&G_1(0)^2\sigma_1^j(s)^2\left(1+\int_0^s\sigma_1^j(u)^2\,du\right)+G_2(0)^2\sigma_2^j(s)^2\left(1+\int_0^s\sigma_2^j(u)^2\,du\right)\\&-2G_1(0)\sigma_1^j(s)G_2(0)\sigma_2^j(s)\left(1+\int_0^s\sigma_1^j(u)\sigma_2^j(u)\,du\right)\Bigg)ds.\end{aligned}$$

Thus we have the dynamics for the basis. To summarise, we have shown that $\Gamma_{12}(t)$ is approximately normal with mean and variance determined in 5.47 and 5.50 respectively.

A Continuous-time Process for AR(2)

We have previously noted that the statistical properties of the basis are those of an AR(2) process. Furthermore, in the previous section, we showed that the returns on the basis between two separate locations are approximated by a Normal distribution. In this section, we wish to show how to obtain a continuous-time analogue for the discrete-time process. The notation used in this section will be for a general process.

A continuous-time AR(2) process would be formally described by the following dynamics:

5.51 $$\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+\delta x_t=\gamma\frac{dZ}{dt},$$

where $\frac{dZ}{dt}$ is "white noise".

The objective is to interpret and solve this system of differential equations x. Recall that similar ordinary differential equations are solved by finding a general solution (x_c) to the homogeneous equation (the case where $\gamma = 0$) and a particular solution (x_p), where $\gamma \neq 0$. The final solution consists of linear combinations of x_p and x_c.

By inspection, we can see that for suitable values of a, the function $x = \exp(at)$ is a solution of the homogeneous equation. This is easy to verify as:

5.52 $$\frac{d^2x}{dt^2}=a^2\exp(at)$$

5.53 $$\frac{dx}{dt}=a\exp(at),$$

the exponential function is a solution if:

5.54 $$\exp(at)\left(a^2+\beta a+\delta\right)=0.$$

As this equation is a quadratic, there are two possible solutions.

Recall that we can also rewrite the homogeneous case of the above system in matrix notation. Thus introducing $y = \frac{dx}{dt}$, we have $\frac{dy}{dt} + \beta y + \delta x_t = 0$, or equivalently:

5.55
$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\delta & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}dt.$$

Define:

5.56
$$A = \begin{pmatrix} 0 & 1 \\ -\delta & -\beta \end{pmatrix}$$

This vector notation is required to write the second order stochastic differential equation in a less formal way as:

$$d\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}dt + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}dZ$$

Recall, the eigenvalues of A are solutions of:

5.57
$$|A - \bar{\lambda} I| = \left|\begin{pmatrix} -\lambda & 1 \\ -\delta & -\beta - \lambda \end{pmatrix}\right| = 0$$

which in this case, leads to the characteristic equation:

5.58
$$\lambda^2 + \beta\lambda + \delta = 0$$

with two distinct solutions, λ_1 and λ_2, each with a corresponding eigen vector. This is identical to the required condition above on the parameter "a" to guarantee that $x = \exp(at)$ is a solution to the differential equation.

We have two possible solutions to the second order differential equation above. Linear combinations of these two solutions also solve the homogeneous equation.

By the method of Variation of Parameters [see for example Zill section 4.5] once we have a general solution of the homogeneous equation, we can obtain a particular solution. This involves estimating the Wronskian, invoking Cramer's rule and integrating out some coefficients.

For a generic differential equation

$$\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+\delta x_t=\gamma dZ(t)$$

There are two general solutions, $x_1(t)=\exp(\lambda_1 t), x_2(t)=\exp(\lambda_2 t)$, each corresponding to roots of 5.58.

It is possible to determine two functions $u_1(t)$, and $u_2(t)$ such that a particular solution can be written as:

5.59
$$x_p(t)=u_1(t)x_1(t)+u_2(t)x_2(t)$$

It is easy to show that the $u_1(t)$, and $u_2(t)$ are determined by the following rules:

5.60
$$du_1(t)=\frac{\begin{vmatrix}0 & x_2(t)\\ \gamma dZ(t) & x_2'(t)dt\end{vmatrix}}{\begin{vmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{vmatrix}}$$

5.61
$$du_2(t)=\frac{\begin{vmatrix}x_1(t) & 0\\ x_1'(t)dt & \gamma dZ(t)\end{vmatrix}}{\begin{vmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{vmatrix}}$$

Integrating out these functions, taking determinants, and rearranging terms gives the following particular solution of the "second order stochastic differential equation":

5.62
$$x_p(t)=\frac{\exp(\lambda_1 t)}{(\lambda_2-\lambda_1)}\int_0^t\frac{-\gamma\exp(\lambda_2 u)}{\exp((\lambda_2+\lambda_1)u)}dZ(u)+\frac{\exp(\lambda_2 t)}{(\lambda_2-\lambda_1)}\int_0^t\frac{-\gamma\exp(\lambda_1 u)}{\exp((\lambda_2+\lambda_1)u)}dZ(u)$$

which reduces to:

5.63
$$x_p(t)=\frac{\exp(\lambda_1 t)}{(\lambda_2-\lambda_1)}\int_0^t-\gamma\exp(-\lambda_1 u)dZ(u)+\frac{\exp(\lambda_2 t)}{(\lambda_2-\lambda_1)}\int_0^t\gamma\exp(-\lambda_2 u)dZ(u)$$

Collecting both the general and particular solutions shows that the general solution to the second order stochastic differential equation above is:

5.64 $$x(t) = a\exp(\lambda_1 t) + b\exp(\lambda_2 t) + \frac{\exp(\lambda_1 t)}{(\lambda_2 - \lambda_1)}\int_0^t -\gamma \exp(-\lambda_1 u)dZ(u) + \frac{\exp(\lambda_2 t)}{(\lambda_2 - \lambda_1)}\int_0^t \gamma \exp(-\lambda_2 u)dZ(u).$$

It is illustrative to note the following statistical properties: $x(t)$ is normal and

5.65 $$E[x(t)] = a\exp(\lambda_1 t) + b\exp(\lambda_2 t)$$

5.66 $$Var[x(t)] = E\left[(x(t) - E[x(t)])^2\right] = E\left[x_p(t)^2\right]$$

5.67 $$= \frac{\exp(2\lambda_1 t)}{(\lambda_2 - \lambda_1)^2}\int_0^t \gamma^2 \exp(-2\lambda_1 u)du + \frac{\exp(2\lambda_2 t)}{(\lambda_2 - \lambda_1)^2}\int_0^t \gamma^2 \exp(-2\lambda_2 u)du$$

$$+ \frac{2\exp(\lambda_1 t)\exp(\lambda_2 t)}{(\lambda_2 - \lambda_1)^2}\int_0^t \gamma^2 \exp(-(\lambda_1 + \lambda_2)u)du.$$

Thus the solution to the second order differential equation, which describes the continuous-time AR(2) process, is itself Gaussian, as we required.

To recapitulate, we have shown the basis is both Normal (analytically) and AR(2) (empirically). We have also shown these results are consistent with each other, and with the proposed model at each of the endpoints of the basis relationship.

Model for Basis Options

We now wish to derive an explicit solution for an option on the basis.

$$C_{12}(t) = E\left[\exp\left(-\int_0^t f(s,s)\,ds\right)(\Gamma_{12}(t) - K)^+ \mid \Im(0)\right]$$

To simplify this process, we assume with little loss of value that f(s,s) = r(s) =r, a constant interest rate.

Then the call on the basis is:

5.68 $$C_{12}(t) = \exp(-rt)E\left[(\Gamma_{12}(t) - K)^+ \mid \Im(0)\right]$$

$$= \exp(-rt)E\left[(\Gamma_{12}(t)-K)\left(1_{\{\Gamma_{12}(t)>K\}}\right) \mid \Im(0)\right]$$

where $1_{\{\Gamma_{12}(t>K\}}$ is the indicator function.

This expression has two terms:

5.69 $$\exp(-rt)E\left[(\Gamma_{12}(t))\left(1_{\{\Gamma_{12}(t)>K\}}\right) \mid \Im(0)\right]$$

and

5.70 $$\exp(-rt)KE\left[\left(1_{\{\Gamma_{12}(t)>K\}}\right) \mid \Im(0)\right].$$

To simplify the notation, let us call the mean established in 5.47 A(t) and the variance established in 5.50 B(t), thus $\Gamma_{12}(t) \cong N\left(A(t), \sqrt{B(t)}\right)$.

Thus we can expand the expectation in expression 5.69 as:

$$\frac{1}{\sqrt{B(t)}\sqrt{2\pi}}\int_K^{\infty} \Gamma \exp\left(-\frac{1}{2}\left(\frac{\Gamma - A(t)}{\sqrt{B(t)}}\right)^2\right)d\Gamma.$$

We let:

$$\xi = \frac{\Gamma - A(t)}{\sqrt{B(t)}}.$$

So we now obtain:

$$\frac{\sqrt{B(t)}}{\sqrt{B(t)}\sqrt{2\pi}}\int_{\sqrt{B(t)}K+A(t)}^{\infty}\left(\sqrt{B(t)}K + A(t)\right)\exp\left(\frac{-\xi^2}{2}\right)d\xi$$

which simply integrates out to:

5.71 $$\frac{\sqrt{B(t)}}{\sqrt{2\pi}}\exp\left(-\frac{\left(\sqrt{B(t)} + A(t)\right)^2}{2B(t)}\right) + A(t)N\left(-\sqrt{B(t)} - A(t)\right).$$

To evaluate the expectation in 5.70, we introduce the Gaussian density function and we note that:

$$E\left[\left(1_{\{\Gamma_{12}(t)>K\}}\right)\right]$$

$$=\frac{1}{\sqrt{B(t)}\sqrt{2\pi}}\int_{K}^{\infty}\exp\left(-\frac{1}{2}\left(\frac{\Gamma-A(t)}{\sqrt{B(t)}}\right)^{2}\right)d\Gamma.$$

Introduce a change of variables:

$$\xi=\frac{\Gamma-A(t)}{\sqrt{B(t)}}.$$

So we now have:

$$\frac{\sqrt{B(t)}}{\sqrt{B(t)}\sqrt{2\pi}}\int_{\sqrt{B(t)}K+A(t)}^{\infty}\exp\left(\frac{-\xi^{2}}{2}\right)d\xi$$

which simplifies to:

5.72 $$1-N\left(\sqrt{B(t)}\,K+A(t)\right)=N\left(-\sqrt{B(t)}\,K-A(t)\right).$$

Collecting terms, we can now write the closed-form solution of the option on the basis as:

5.73 $$\exp(-rt)\left(\begin{array}{l}\frac{\sqrt{B(t)}}{\sqrt{2\pi}}\exp\left(-\frac{\left(\sqrt{B(t)}+A(t)\right)^{2}}{2B(t)}\right)\\ +A(t)N\left(-\sqrt{B(t)}-A(t)\right)-K\left(N\left(-\sqrt{B(t)}\,K-A(t)\right)\right)\end{array}\right)$$

where:

5.74 $$\mathrm{Var}[\Gamma_{12}(t)]=\sum_{j=1}^{3}\int_{0}^{t}\left(G_1(0)^2\sigma_1^j(s)^2\left(1+\int_0^s\sigma_1^j(u)\,du\right)+G_2(0)^2\sigma_2^j(s)^2\left(1+\int_0^s\sigma_2^j(u)\,du\right)\right.$$
$$\left.-2G_1(0)^2\sigma_{12}^j(s)^2G_2(0)^2\sigma_2^j(s)^2\left(1+\int_0^s\sigma_1^j(u)\sigma_2^j(u)\,du\right)\right)ds\cdot$$

and

$$A(t)=E\left[\Gamma_{12}(t)\right]\cong G_1(0)-G_2(0).$$

Chapter Six: Calibration

Model Calibration Procedures

With the forward curve model derivation completed, and with closed-form solutions for call options and basis options, based on this forward curve model, we are in a position to apply data to estimate parameter values.

We shall compare the call options derived under our model with the prices derived using Black's 76 model and with the Miltersen & Schwartz model. We shall compare the prices derived under our model for the basis with the standard basis model mentioned by Kaminski.

We have data for several forward contracts locations (AECO, Chicago, NYMEX) for several forward months. This data was observed starting in Feb 4 1997 and continues through to February 27 1998. There is data for 60 forward contracts (new contracts are created as the prompt month rolls) but we restrict our attention to the nearest 18 gas contracts (the contracts for Apr 97 delivery through to the contract for Sept 98 delivery).

We obtained the data from Engage Energy US LP, and this data consisted of mid-market exchange data for NYMEX and mid-market broker quotes for the other locations. This data was used by Engage for internal mark-to-market accounting and Value at Risk estimation; it is updated daily and is deemed to be as accurate as is reasonably commercially feasible.

Calibration of Call Option Model

We consider the call option model in 5.38

$$C_i = P(0,t)\left(G(0,T)\exp(-\sigma_{XZ_i})N\left(\frac{\log\frac{G(0,T)}{K} + \frac{1}{2}\sigma_{Z_i}^2 - \sigma_{XZ_i}}{\sigma_{Z_i}} \right) - K \cdot N\left(\frac{\log\frac{G(0,T)}{K} - \sigma_{XZ_i} - \frac{1}{2}\sigma_{Z_i}^2}{\sigma_{Z_i}} \right)\right).$$

Given the forward curves for the three locations, and making the required substitutions, we are able to price 18 options at each of three locations. These prices are shown in the next section.

Recall that we had made the definitions:

$$\sigma_{XZ_i} = -\sum_{j=1}^{3}\int_0^t \sigma_{Pt}^j(u)\cdot\sigma_{GTi}^j(u)\,du,$$

$$\int_u^t \sigma_f^j(u,s)\,ds \equiv \sigma_{P(t)}^j(u),$$

and

$$\sigma_{GTi}^j(t) \equiv \sigma_{S_i(t)}^j + \int_t^T \sigma_f^j(t,s)ds - \int_t^T \sigma_{\varepsilon i}^j(t,s)ds.$$

We had also determined that:

$$\sigma_{Z_i}^2 = \sum_{j=1}^{3}\int_0^t \left\|\sigma_{GTi}^j(u)\right\|^2 du.$$

With knowledge of the preceding model parameters, we are able to obtain prices for call options on specific assets.

Results

Prior to comparing the results of the respective call models, it is helpful to note the term structure of volatility in the following table, where these estimates are derived in the usual manner from the prior 20 days data[18]. The key idea in this table is that the volatility drops off significantly for contracts beyond a few months.

The second point to note is that the option prices under this model depend significantly on the dynamics of the convenience yield, which we have specified in Chapter 3 as:

$$\varepsilon_i(t,s) = \varepsilon_i(0,s) + \int_0^t \mu_{\varepsilon i}(u,s)du + \int_0^t \sigma_{\varepsilon i}^1(u,s)dz_1(u)$$
$$+ \int_0^t \sigma_{\varepsilon i}^2(u,s)dz_2(u) + \int_0^t \sigma_{\varepsilon i}^3(u,s)dz_3(u)$$

where the convenience yield is the "plug" variable that is directly observable from the following relationship:

[18] The risk in using too many observations in volatility estimation under conditions of a monotonically increasing term structure of volatility, is built-in bias to underestimating this non-stationary parameter.

$$G_i(t,T) = S_i(t)\exp\left(\int_t^T (f(t,s) - \varepsilon_i(t,s))ds\right).$$

Recall that due to Theorem 4.1, our choice of the drift of this convenience yield process is limited under the assumption of prices being obtained under an equivalent martingale measure.

As noted by Gibson and Schwartz, and reiterated herein, there is no obvious canonical choice for the spot asset. If we use the day gas prices as a proxy, we are at serious risk of having a spot process that is de-coupled from term gas as the pipeline balancing constraints that impact on the convenience yield of day gas are not the factors that would influence gas prices down the curve. We opt instead to use the prompt contract (in this case the March 1997 delivery contract) to proxy the spot asset.

We are immediately troubled as we back out the annualised convenience yield as shown in the relationship above for each of the three locations.

As shown in the accompanying table 6.1, these convenience yields are large and highly volatile. Furthermore, as these yields are estimated over the data as presented in February 1997 calendar time, we are immediately concerned that the convenience yield volatility estimation will be a process that is void of meaning.

Convenience yield is used to capture changes in the forward curve relative to some numeraire spot contract, but when the changes are of a nature presented in the table, it is a priori apparent that the usefulness of this concept may be limited. It is possible that this variable is simply too volatile to assist us in our forward curve modelling.

With this lurking concern, we used our model to price out the options noted above.

Table 6.1 Annualised Convenience Yields

	NYMEX Max ε	NYMEX Min ε	Chicago Max ε	Chicago Min ε	AECO Max ε	AECO Min ε
Apr-97	172%	-2%	269%	12%	350%	67%
May-97	120%	-12%	169%	-4%	220%	43%
Jun-97	84%	-13%	116%	-8%	170%	31%
Jul-97	63%	-12%	87%	-8%	144%	26%
Aug-97	50%	-12%	70%	-8%	115%	18%
Sep-97	41%	-12%	57%	-9%	82%	9%
Oct-97	35%	-12%	49%	-9%	65%	4%
Nov-97	23%	-20%	21%	-19%	52%	-1%
Dec-97	14%	-25%	13%	-25%	35%	-15%
Jan-98	11%	-25%	10%	-24%	29%	-16%
Feb-98	14%	-18%	13%	-17%	29%	-11%
Mar-98	16%	-13%	15%	-12%	34%	-2%
Apr-98	19%	-6%	27%	-4%	34%	-4%
May-98	19%	-4%	26%	-3%	34%	-1%
Jun-98	18%	-4%	25%	-2%	32%	0%
Jul-98	17%	-4%	23%	-2%	30%	0%
Aug-98	16%	-3%	21%	-2%	28%	0%
Sep-98	15%	-3%	20%	-2%	26%	0%

Table 6.2 Contract Volatility

	NYMEX	AECO	CHICAGO
Mar-97	52%	86%	49%
Apr-97	40%	55%	40%
May-97	31%	51%	31%
Jun-97	23%	41%	23%
Jul-97	18%	36%	18%
Aug-97	17%	33%	16%
Sep-97	16%	41%	15%
Oct-97	14%	42%	14%
Nov-97	14%	39%	19%
Dec-97	15%	34%	18%
Jan-98	14%	35%	18%
Feb-98	14%	32%	18%
Mar-98	15%	36%	18%
Apr-98	15%	28%	15%
May-98	15%	27%	15%
Jun-98	14%	27%	13%
Jul-98	14%	25%	13%
Aug-98	14%	24%	13%
Sep-98	15%	25%	14%
Oct-98	16%	26%	15%
Nov-98	16%	35%	24%
Dec-98	14%	34%	22%
Jan-99	21%	33%	29%
Feb-99	13%	30%	19%

The easiest comparison of the prices obtained under our model and Blacks 76, and is through the use of graphs. Although the values are discrete, we have shown them as continuous for ease of presentation.

For the NYMEX values, the comparison is shown in Graph 6.1a.

It would appear that our concerns of the excess volatility of the convenience yields as well as their inter-temporal volatility have "swamped" the impact of the other variable in our option pricing formula. To verify that this is a data and not a model issue, we offer Graph 6.1b, which includes the option prices under the Miltersen and Schwartz model, which are similarly impacted. As Miltersen and Schwartz reported no such results on copper futures in their paper, we are left to conclude that there is something is the gas data that makes this model produce these curious results. We are not immediately sure how to modify the model to capture this phenomena.

We would like to compare our model with the actual options being traded and with the prices obtained under Black's 76 model. Unfortunately, we are unable to find option quotes for the forward contracts as of Feb 27, 1997. Most sources of data tend to overwrite the historical prices on a continuing basis. We are able however, to approximate what we believe would be reasonable prices as of this valuation date.

It is well known that options traders treat options as a volatility play, and try to extract value from the market by overselling the volatility when writing options and underpaying volatility when purchasing options. As the presented results from Black's 76 model are dependant on historical volatility, and not the implied volatility which is reflective of actual transactions, it is possible that the prices under the Black's model are understated in graph 6.1a.

To verify this, we went to Bloomberg and noted that for options on summer gas there is a volatility gap of between 10 and 15% between the historical volatility (valued over 30 days) and the option-implied volatility. However, we also noted that volumes in options tends to be thin and we are not sure that much comfort can be placed on these implied volatilities as the markets do not appear to be clearing at these levels.

We add one line to chart 6.1b to show the effect of adding 15% down the volatility curve for the options priced under Black's 75 model. We call these prices the "Market" prices. As the implied volatilities in Bloomberg are derived using Black's 76 model, we believe this is as good of a proxy that we can obtain for what the actual prices would have been in February of 1997. Due to the term structure of volatility, we believe that this approach may overstate the value of the longer-term options. We are only able to try this approach to the NYMEX options as there is no service comparable to Bloomberg for the

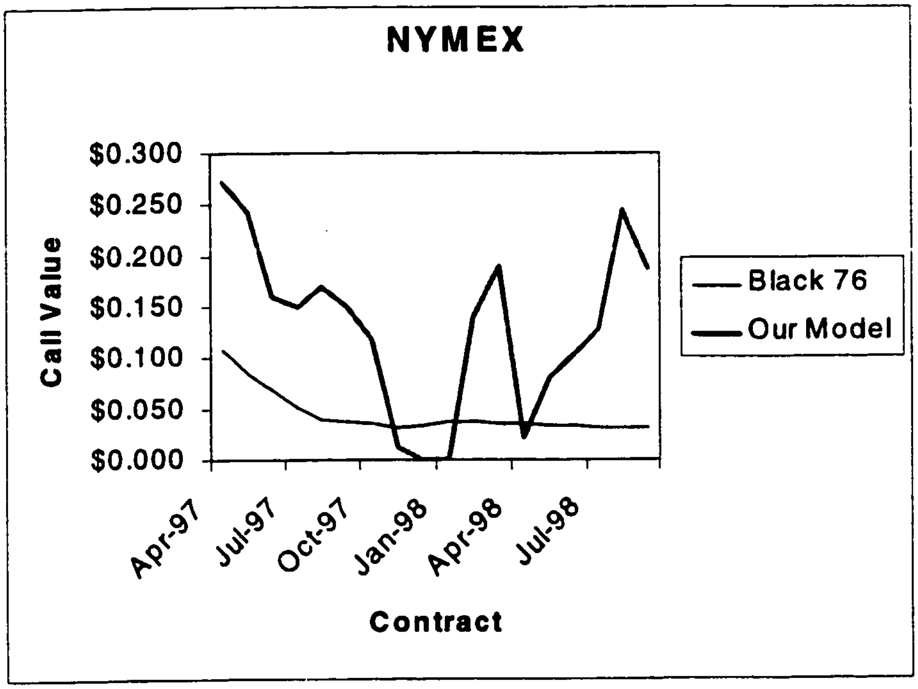
OTC markets (Chicago and AECO) and any volatility premium above historical would be a pure guess.

Similar results are obtained for Chicago and AECO options as demonstrated in Graphs 6.2a, 6.2b, 6.3a and 6.3b.

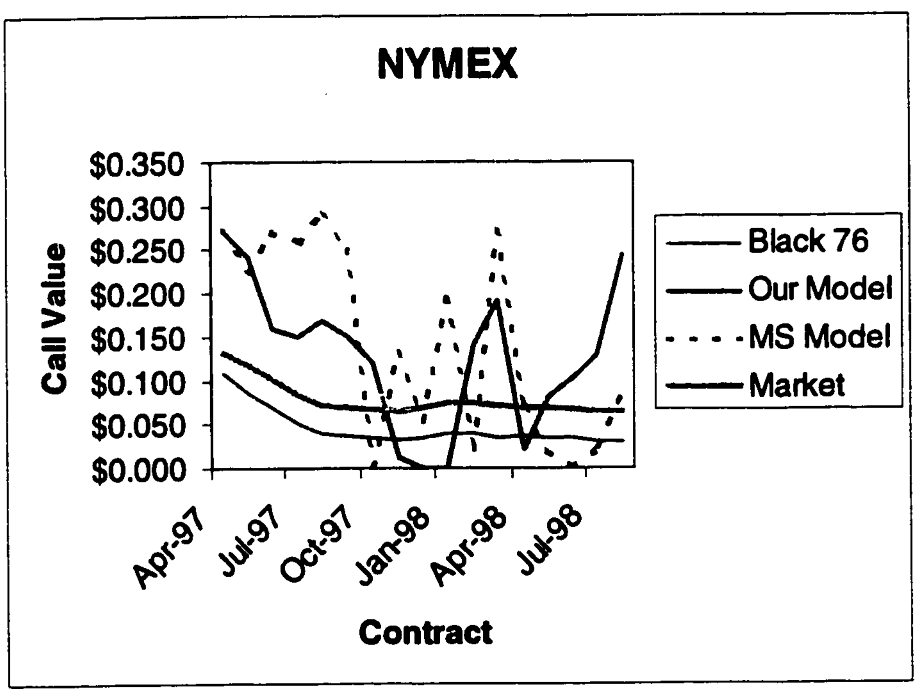
Note how the proposed model gives higher values for the most months, but occasionally drops off to values below those suggested by Black 76. The prices in all instances (including both our model and the MS model) appear to be "whipsawed" by excess convenience yield volatility. Even when compared to the "Market" prices, our model seems to be off the mark, albeit less than for prices based on historical volatility only.

To summarise, our model is certainly capable of pricing options, but the prices derived under this, and the MS model, fail to pass the test of economic reasonableness. Modifications are probably required around the estimation of the convenience yields.

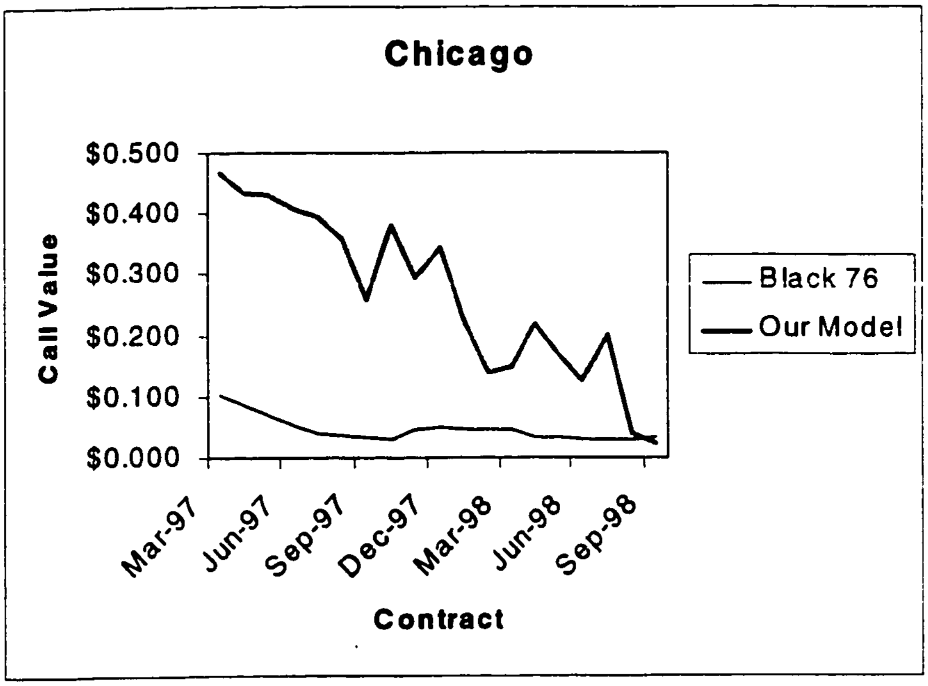
Graph 6.1a NYMEX Option Prices



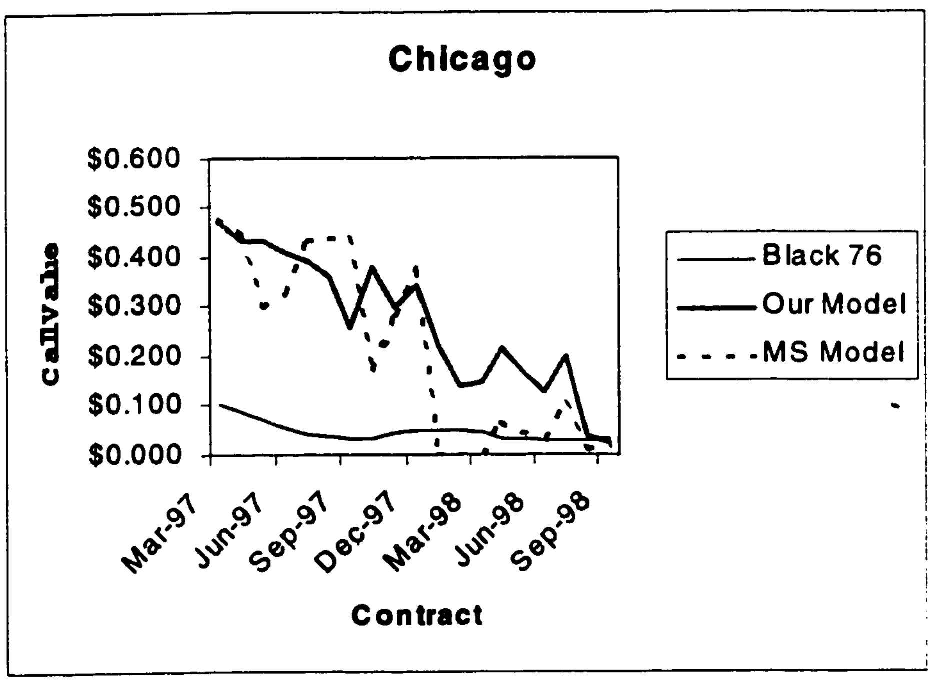
Graph 6.1b NYMEX Option Prices



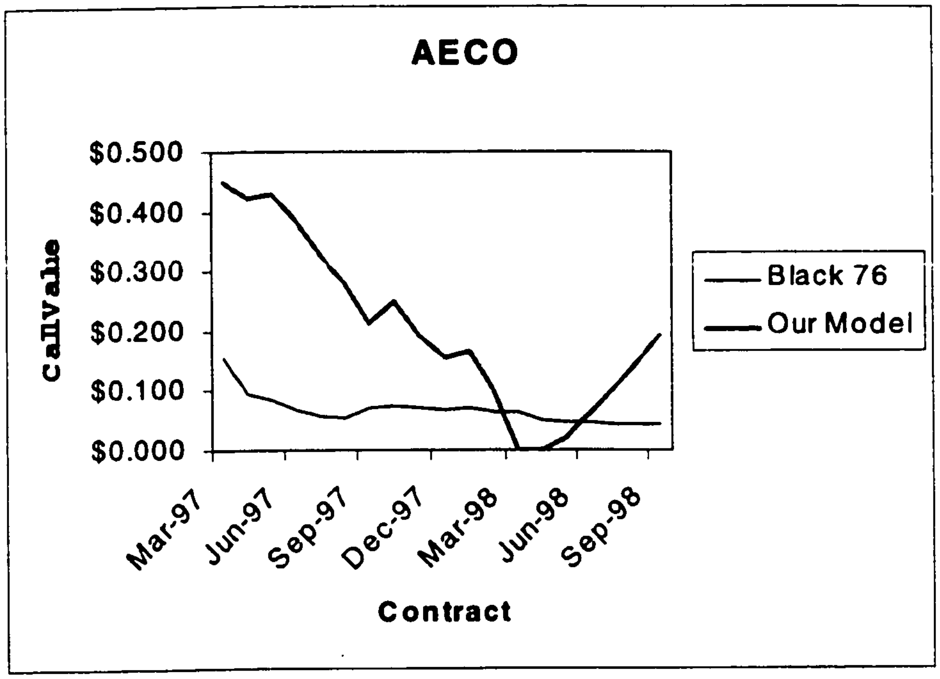
Graph 6.2a Chicago Option Prices



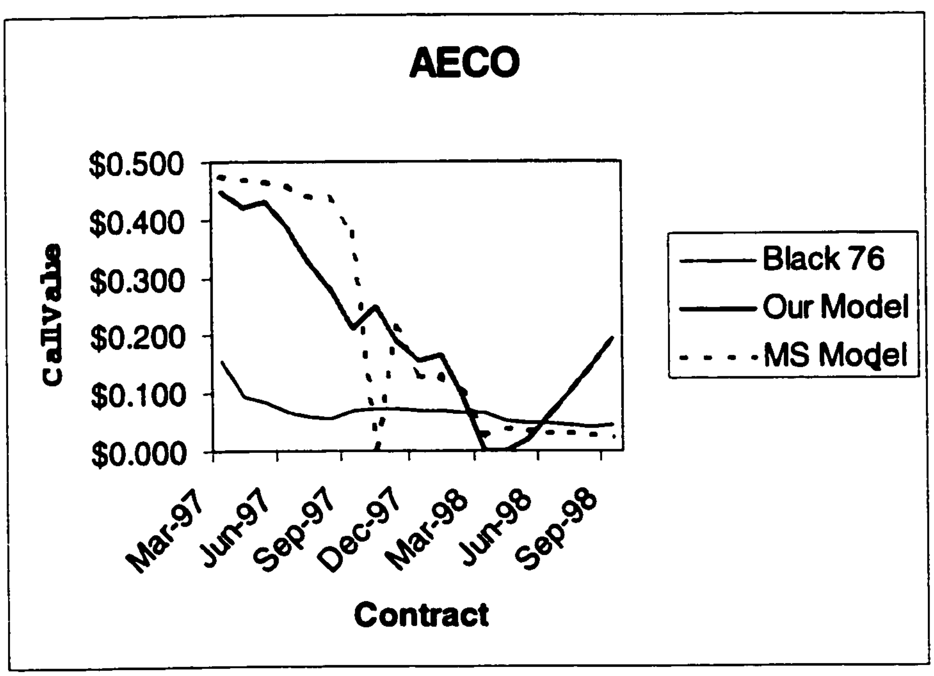
Graph 6.2b Chicago Option Prices



Graph 6.3a AECO Option Prices



Graph 6.3b AECO Option Prices



Calibration of Basis Option Model

Recall that in 5.73 we derived the following formula for call options on basis:

$$C_{12}(t) = \exp(-rt)\left(\begin{array}{l}\dfrac{\sqrt{B(t)}}{\sqrt{2\pi}}\exp\left(-\dfrac{\left(\sqrt{B(t)}+A(t)\right)^2}{2B(t)}\right)\\ +A(t)N\left(-\sqrt{B(t)}-A(t)\right)-K\left(N\left(-\sqrt{B(t)}\,K-A(t)\right)\right)\end{array}\right)$$

where:

$$A(t) \equiv G_1(0) - G_2(0)$$

and

$$B(t)^2 = \sum_{j=1}^{3}\int\left(G_1(0)^2\sigma_1(s)^2\left(1+\int_0^s \sigma_1(u)\,du\right)+G_2(0)^2\sigma_2(s)^2\left(1+\int_0^s \sigma_2(u)\,du\right)\right.$$
$$\left.-2G_1(0)^2\sigma_1(s)^2G_2(0)^2\sigma_2(s)^2\left(1+\int_0^s \sigma_1(u)\sigma_2(u)\,du\right)\right)ds\,.$$

Further recall that we had previously replaced

$$\sum_{j=1}^{3}\left[\left[\sigma^j_{S_i(t)} + \int_t^T \sigma^j_f(t,s)ds - \int_t^T \sigma^j_\delta(t,s)ds\right]dz_j(t)\right]$$

with:

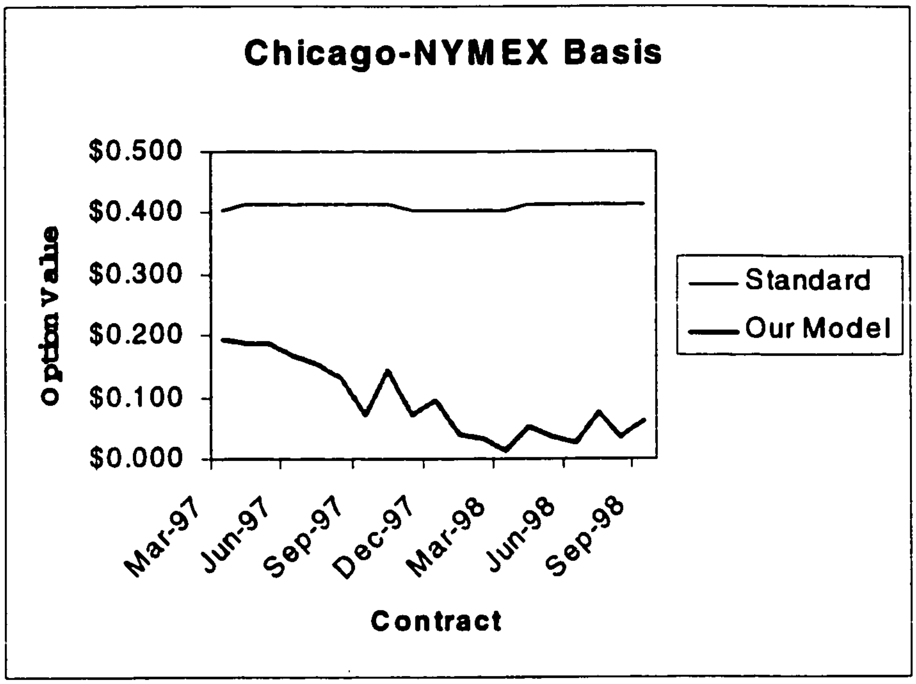
$$\sigma_i^1 dz_1(t) + \sigma_i^2 dz_2(t) + \sigma_i^3 dz_3(t)\,.$$

Thus with knowledge of $\sigma^j_{S_i(t)}$, $\int_t^T \sigma^j_f(t,s)ds$, and $\int_t^T \sigma^j_\delta(t,s)ds$, we can make the appropriate substitutions in the valuation of the basis call option.

Using the closed-form solution for this basis call option shown above, we are able to price out 18 basis options on the Chicago-NYMEX basis. Again we are concerned that the excess volatility of the convenience yield process may provide spurious results.

The following graph demonstrates the difference between the standard approach to pricing basis options and the approach developed in our model. The differences are quite apparent.

Graph 6.4 NYMEX-Chicago Basis Option Prices



Our model seems to have more of a term structure, and appears to increase with winter contracts, whereas the standard approach prices options more cheaply in the winter months, a counterintuitive result. Our model is also better are capturing the economic value of what option on the basis would be priced at in the market, in the absence of any particular model.

We would hypothesise that it appears as if the excess convenience yield has somehow netted out in the pricing of these options, providing a more reasonable result.

Commentary on Model Comparisons

All options priced in the preceding models were for a tenor of 31 days and were all for at-the-money strikes. It appears that three features are immediately noted:

1) our model is arguably too sensitive to convenience yield and generally over-prices the options on futures,

2) our model does a better job of capturing the seasonality for options on the basis, and

3) our model does a better job at pricing basis options than the standard approach which generates prices that are patently wrong. This is due to the magnitude of the volatility estimates for the standard basis model.

Is the model as now formulated ready to be employed by practitioners? We do not believe so. While it provides some superior aspects over the traditional option models (notably inclusion of parameters that we remain convinced are required to model the complexity of gas prices) it is troubling that it seems too sensitive on the pricing of the futures options. We would suggest that this model is possibly better than the standard approach for basis options, and with some modifications of convenience yield estimation may also replace Black's 76 model for futures.

Chapter Seven: Conclusions

Our Research Objective

The objective of this thesis was to:

1) derive a model that would explain the complexities of forward curves in natural gas.

2) show that this model would be consistent with the empirical data on forward curve movements and

3) use this model to derive closed-form option price models.

Our Results

We believe that we have partially met our objectives. The difficulty with natural gas modelling is that it does not easily fit into any previously derived models, as outlined in Chapter One. In Chapter Two we noted that an adequate model must take into account:

1) Volatility Term Structure,

2) Backwardation and Contango, and

3) Stochastic Basis.

On a theoretical level, our model provides dynamics that incorporate these properties and also relate prices at separate locations and thus capture the effect of transportation in these quasi-local markets.

Furthermore, we were able to show mathematically, as opposed to assert, that basis is approximately normal, and that this statistical property is consistent with our posterior beliefs, with our proposed model, and with the empirical time series results showing basis is characterised as an AR(2) process.

We were able to derive closed-form solutions, and although we are not entirely happy with the pricing results that have emerged, we are encouraged by aspects such as capture of seasonality that is a clear benefit of this model over existing ones.

Direction for Further Research

While we are pleased with the complexities this model attempts to capture, we are cognisant that financial modelling is destined to be an unfinished project.

We have attempted to use location as a proxy for the risk factors. Is this the best approach or should we undertake principal component or factor analysis in the determination of multiple risk factors? Perhaps this would help us obtain a better spanning set of factors. This model is easily extended to multiple dimensions, although we have limited our focus to three locational factors. Nothing in the model forces the factors to coincide with the locations; perhaps this could be changed to the model's benefit.

Financial models once employed largely for capital budgeting, and since 1973 used extensively in option pricing, are now being utilised in more and more creative ways. Two such uses are in the estimation of mark-to-market accounting profits on a go-forward basis and in the practice of Value-at-Risk estimation. In both processes, most energy firms use simple diffusion models.

The existence of a better forward curve model, which captures the unique pricing characteristics of energy, could be used to provide more meaningful VaR estimates, which in turn leads to more appropriate trading limits and decisions for trading purposes.

An improved model could also be used to augment capital budgeting decisions, especially those where there is a component of real optionality.

Additionally, having a model that recognises convenience yield may one day be used to synthetically replicate convenience yield, much as covariance contracting can synthetically replicate the covariance between assets.

Finally, additional research could be done on the "Greeks" in order to see if there is a better way to hedge options priced off this model. It is also foreseeable that the same approach undertaken in this model could be used to better model more complex cross-commodity instruments such as spark spreads or crack spreads.

We believe this work will make a contribution to the literature in the area of commodity forward-curve term structure modelling, and it raises additional, interesting questions for future research.

Appendix A

Non-Seasonal ARIMA Model that describe the behaviour of the price series of the NYMEX-Chicago Basis.

	Model
Mar-97	AR(1)
Apr-97	AR(1)
May-97	AR(1)
Jun-97	AR(2)
Jul-97	AR(2)
Aug-97	AR(2)
Sep-97	AR(2)
Oct-97	AR(2)
Nov-97	AR(2)
Dec-97	AR(2)
Jan-98	AR(2)
Feb-98	AR(2)
Mar-98	AR(2)
Apr-98	AR(2)
May-98	AR(2)
Jun-98	AR(2)
Jul-98	AR(2)
Aug-98	AR(2)
Sep-98	AR(2)
Oct-98	AR(2)
Nov-98	AR(2)
Dec-98	AR(2)
Jan-99	AR(2)
Feb-99	AR(2)

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