THE UNIVERSITY OF ALBERTA

OPTIMAL LONG-TERM OPERATION OF HYDRO-RESERVOIRS

by

SOLIMAN ABDEL-HADY SOLIMAN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

AND RESEARCH IN PARTIAL FULFILMENT OF THE

REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

SPRING 1986

ISBN 0-315-30206-2

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: SOLIMAN ABDEL-HADY SOLIMAN

TITLE OF THESIS: OPTIMAL LONG-TERM OPERATION OF HYDRO-RESERVOIRS

DEGREE: DOCTOR OF PHILOSOPHY

YEAR THIS DEGREE GRANTED: 1986

(Signed) ....

PERMANENT ADDRESS:

Department of Electrical Engineering

University of Ain Shams

Abbassia, Cairo, EGYPT

DATED: Jan. 9. 1986.

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled OPTIMAL LONG-TERM OPERATION OF HYDRO-RESERVOIRS submitted by SOLIMAN ABDEL-HADY SOLIMAN, in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering.

....
Supervisor

....

....

....

....
External Examiner

Date: Jan. 14, 1986

To my mother and the memory of my father

and

To the Spirit of my wife's mother and sister

ABSTRACT

In this thesis the long-term optimal operating problem of multireservoir power systems under different water conditions is discussed. The scheduling problems are solved by the use of functional analysis, where the minimum norm formulation is employed.

Optimal scheduling problems for series and parallel multireservoir power systems having linear storage-elevation curves and constant water conversion factors are discussed. The total benefits obtained in this case are greater than those obtained using the Decomposition Approach and Dynamic Programming, and the computing time is less by several orders of magnitude than that using the Dynamic Programming. For power systems in which the water heads vary by a considerable amount, the assumption of constant water conversion factors is not true. The optimal scheduling problems of multireservoir power systems having a linear storage-elevation curve, and a variable water conversion factor are discussed. We assume a linear relation for this variation with either the net head of the plant or the storage of the reservoir. The total benefits obtained using this linear model are better than those obtained using other approaches, and the computing time is very small compared to that of other approaches.

The assumption of linear storage-elevation curves and linear water conversion factors may yield a significant error in the storage of some reservoirs, which may be greater than the minimum natural inflow to these reservoirs. For this reason, we assume a nonlinear quadratic storage-elevation curve and a non-linear quadratic water conversion factor in modelling the system. The cost function in this

case is highly nonlinear; we define a set of pseudo-state variables to cast the problem into a quadratic problem. The total benefits obtained in this case are better than those obtained using the linear model. On the other hand, no spillage took place at all.

New optimal equations for scheduling the operation of a multireservoir power system under critical water conditions are developed. To meet all the requirements during this period, we maximize the generation from the system during this period, and we make the generation during this period uniform. The load on the system and the transmission line losses which may be included in this load are taken into account. The load in this section is equal to a certain percentage of the total generation at the end of each year of the "critical period" (a variable load). The model used in this case is a nonlinear model of the average storage; to avoid underestimation in production for rising water levels and overestimation for falling water levels, we define a set of pseudo-state variables to cast the problem into a quadratic problem. The total benefits obtained using this model are greater by a significant amount than those obtained using the Dynamic Programming and the computer time was very small compared to the computing time using the Dynamic Programming.

New optimal equations for optimizing the firm hydro energy capability from multireservoir power systems are developed under critical water conditions. The cost function in this study is a highly nonlinear function, we define a set of pseudo-state variables to cast the problem into a quadratic problem.

In this thesis the stochasticity of the river flows is taken into account. We use the average values of the random variables (the expected value for the statistically independent random variables with normal distribution), the times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason those times are not taken into account; the reservoirs water losses due to evaporation, seepage and irrigation are also neglected.

The computational aspects of the obtained scheduling equations are discussed. Practical examples are also given to illustrate the results obtained.

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my adviser, Dr. G.S. Christensen for encouragement, council and aid throughout my thesis. The theoretical principles behind this dissertation were first introduced to me by Dr. G.S. Christensen, through frequent discussions.

I wish to thank my graduate committee members, Dr. D.H. Kelly, Dr. D. Koval of the Electrical Engineering Department and Dr. M.N. Oguztoreli of the Mathematics Department, University of Alberta, for their advice and encouragement.

I owe a debt of gratitude to my wife Mrs. Laila Soliman for her encouragement, patience and understanding during this work. This work would not have been possible without the patience of our children, Rasha, Shady and Samia.

Thanks goes to my Father-in-Law Dr. Mousa S. Lasheen, former Vice-President of Al-Azhar University, Cairo, Egypt for his encouragement and moral support during my studies in Canada.

I wish to thank the National Research Council of Canada, and the Department of Electrical Engineering, the University of Alberta for the financial support during the course of this work.

Thanks also to Dr. M.S. Morsy, Dr. M.A. Abdel-Halim, of the Department of Electrical Engineering, Ain Shams University, Cairo, Egypt, and Dr. S.Y. Mansour of the Electrical Engineering Department, University of Alberta for their encouragement and moral support during this work.

Thanks also goes to Mrs. Barbara J. Peck of Canterbury Executive Services for a neatly done job in typing this thesis.

I wish to thank B.C. Hydro, Vancouver, B.C. for providing the reservoir data. Finally I wish to express my gratitude to EGYPT and CANADA for giving me the opportunity to learn and to continue my studies.

TABLE OF CONTENTS

Page

Page

Page

Page

Page

Page

Page

LIST OF TABLES

LEAF xviv OMITTED IN PAGE NUMBERING.

LEAF xxiv OMITTED IN PAGE NUMBERING.

LEAF xxix OMITTED IN PAGE NUMBERING.

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Background

The hydro optimization problem involves the planning of the usage of a limited resource over a period of time. The resource is the water available for hydro-generation. Most of the hydro-electric plants are multipurpose in nature. In such cases, it is necessary to meet certain obligations other than power generation. These may include a maximum forebay elevation not to be exceeded due to flood prospects, and a minimum plant discharge and spillage to meet irrigational and navigational commitments. Thus, the optimum operation of the hydro system depends upon the conditions which exist over the entire optimization interval (31).

Other distinctions among power systems are the number of hydro stations, their location and special operating characteristics. The problem is quite different if the hydro stations are located on the same stream or on different ones. An upstream station will highly influence the operation of the next downstream station. The latter, however, also influences the upstream plant by its effect on the tail water elevation and effective head. Close coupling of stations by such a phenomenon is a complicating factor (31).

The problem of determining the optimal long-term operation of a multireservoir power system has been the subject of numerous publications over the past forty years, and yet no completely satisfactory solution has been obtained, since in every publication the problem has been simplified in order to be solved (1).

Aggregation of the multireservoir hydroplant into a single complex equivalent reservoir and solution by stochastic dynamic programming (SDP) is one of the earlier approaches that has been used (3,4). Obviously, such a representation of the reservoir cannot take into account all local constraints on the contents of the reservoir, water flows and hydroplant generation. This method can perform satisfactorily for systems where reservoirs and inflow characteristics are sufficiently "similar" to justify aggregation into a single reservoir and hydroplant model (11).

Turgeon (1,2) has proposed two methods for the solution of the problem. The first is really an extension to the aggregation method, and it breaks the problem into a two-level problem. At the second level the problem is to determine the monthly generation of the valley. This problem is solved by Dynamic Programming. The problem at the first level is to allocate that generation to the installation, this is done by finding functions that relate the water level of each reservoir to the total amount of potential energy stored in the valley (41). The second method is the decomposition method by combining many reservoirs into one reservoir for the purpose of optimization of a multireservoir power system connected in series on a river, and using the dynamic programming for solving n-1 problems of two state-variables each. The solution obtained by this method is a function of the water content of that reservoir and the total energy content of the downstream reservoirs. The main drawback is that the approach avoids answering basic question as to how the individual reservoirs in the system are to be operated in an optimal fashion. Also the inflows to some reservoirs may be periodic in phase with the annual demand cycle, while other reservoirs have an inflow cycle which lags by a certain time (12).

Stochastic dynamic programming with successive approximation (DPSA) has been proposed to solve the problem of a parallel multireservoir hydroelectric power system (2). The successive approximation involves a "one-at-a-time" stochastic optimization of each reservoir. The major drawback of this approach is that it ignores the dependence of the operation of one reservoir on the actual energy content of other reservoirs.

In the Aggregation-Decomposition (AD) approach (2), the optimization of a system of n reservoirs is broken into n subproblems in which one reservoir is optimized knowing the total energy content of the rest of the reservoirs. For each subproblem one of the reservoir-hydroplant models is retained and the remaining n-1 are aggregated into an equivalent reservoir hydro-plant model. A comparison of the last two approaches, DPSA and AD, has been done in (2) on a simulation basis for a six complex reservoir system. The results indicate that the AD approach gives better results and the computational effort increases only linearly with the number of reservoirs. More precisely, for each new reservoir added to the system, only one additional Dynamic Programming problem of two-state variables has to be solved. The computing time for each one of the last two approaches was 150 minutes in CPU units.

In Ref. 5, linear and Dynamic Programming have been applied to the optimization of the production of hydroelectric power. the solution was obtained in two steps with linear and Dynamic Programming methods. The models that have been used are deterministic. The first step in the solution was the long-term optimization problem; this problem was solved by a linear programming (LP) method. The variation in the efficiency of the turbines, the variation of the water heads and the time delays are neglected. The second step is the optimal short-term run of the turbine-generator units, this is determined by the dynamic programming (DP). The total computation time on a typical minicomputer was 1 to 3 min. per power station.

In Ref. (42), successive Linear Programming, an optimal control algorithm, and a combination of linear programming and Dynamic Programming (LP-DP) are employed to optimize the operation of multireservoir hydro systems given a deterministic inflow forecast. The algorithm maximized the total benefits from the system (maximization of energy produced, plus the estimated value of water remaining in storage at the end of the planning period). the LP-DP algorithm is dominated: it takes longer to find a solution and produces significantly less hydropower than the other two procedures. Successive linear programming (SLP) appear to find the global maximum and is easily implemented. The optimal control algorithm could find the optimum in about one fifth the time required by SLP for small systems, but is harder to implement. The computing costs were reasonable with SLP and the optimal control algorithm and increases only as the square of the number of reservoirs in contrast to the exponential growth of Dynamic Programming.

Halliburton and Sirisena (12) compared three methods for scheduling releases from the hydrostorage reservoirs of power systems operating both hydro and thermal stations. The three methods are: stochastic dynamic programming, the open-loop feedback controller and a linear feedback law. Stochastic Dynamic Programming is capable of finding the true optimal solution. However, this method is computationally infeasible for a multireservoir model. An open loop feedback controller (OLFC) involves a deterministic algorithm using mean values of stochastic variables. Only the results of the first time interval are actually implemented. At the end of this period the algorithm is repeated using updated values. Linear decision rules have been used by water resources managers and can be adapted for the hydrothermal problem. They compared the above three methods on the basis of one reservoir system. They concluded that the OLFC solution is the best that can be obtained by using a deterministic method, but is not as good as that given using a linear feedback rule, with chance constraints.

Marino and Loaiciga (44) applied a quadratic optimization model to a large scale reservoir system to obtain operation schedules. The model used has the minimum possible dimensionality, treats spillage and penstock releases as decision variables, and takes advantage of system dependent features to reduce the size of the decision space. They compared the quadratic model with a simplified linear model.

1.2 Scope of the Thesis

In this thesis, optimum generation schedules will be developed for a hydro electric power system. The scheduling problem is solved by use of functional analysis where the minimum norm formulation is employed. The power system considered contains an arbitrary number of plants. The optimization problem is described and formulated as the optimal control of a multivariable state-space model in which the state and control variables are constrained by a set of equality and inequality constraints to satisfy the multipurpose stream use requirements. Lagrange and Kuhn-Tucker multipliers are used to adjoin these constraints to the objective function.

Chapter II is concerned with the problem of multireservoir power systems connected either in series on a river or in parallel on a multiriver system. The system has a constant water conversion factor and this conversion factor is equal to the average megawatt-hours produced in a month by an outflow of one Mm^3 ($1Mm^3=10^6m^3$). Section (2.1) is concerned with a series multireservoir power system. The storage-elevation curve used is linear. The total benefits obtained in this section is greater, by a significant amount, than that obtained by the dynamic programming for the same system. However, the computing time is less by several orders of magnitude than that using dynamic programming. Section (2.2) is concerned with a parallel multireservoir power system. The water conversion factor used is also constant and the storage-elevation curve is linear.

Chapter III is concerned with the same problem mentioned in Chapter II, but the water conversion factor assigned for each hydroplant is a variable, and this variation has a linear relation with the head in section (3.1). But in section (3.2) this variation is linear with the average storage. The problem formulated in this chapter, has a more accurate representation than that solved in Chapter II.

Chapter IV is concerned with solving the long-term optimal operating problem of a multireservoir power system with a nonlinear storage-elevation curve. Section (4.1) is concerned with a series multireservoir power system. Section (4.2) is concerned with a parallel multireservoir power system which may contain a run-of-river hydroplant. Since the run-of-river plants have little storage capacity and use water as it becomes available, the hydro generation from these plants is equal to a constant times the discharge through the turbines. The same problem mentioned in section 4.2 has been solved again in section 4.3, but the water conversion factor assigned for each hydroplant has a nonlinear relation (quadratic relation) with the average storage between two successive months not the storage of the previous month. This is done in order to avoid underestimation of production for rising water levels and overestimation for falling water levels. We also compare the two models we used in both sections 4.2 and 4.3.

There is a period in the water management during which the inflow to the sites is very low, and the reservoir should be drawn down from full to empty. This period is referred to as the "critical period", and the stream flows which occurred during the critical period are called the "critical stream flows". Chapter V is concerned with the optimal operation of a parallel multireservoir power system during this period. In this chapter we maximize the total generation from the system during this period. In section 5.1, the model used for each reservoir is a linear model of the average head. By using this model the error in the storage of some reservoirs is greater than the minimum natural inflow to the reservoir, for this reason, in section 5.2 we use a more accurate model for the reservoirs, quadratic model, from the storage because the higher terms are negligible.

Chapter VI is concerned with the maximization of production from multireservoir power systems during the "critical period", but the system in this chapter has a specified monthly generation, and this generation is equal to a certain percentage (a^k) of the total generation at the end of each year of the "critical period". In section (6.1), the model used is a nonlinear, quadratic model with the storage. In section (6.2), the model used is a quadratic model with average storage. The problem in this chapter is a highly nonlinear problem; we define a set of pseudo-state variables to reduce the problem to a quadratic problem.

Chapter VII is concerned with the maximization of the firm hydro-energy capability of a multireservoir power system with a fixed load (d^k) during the "critical period", at the same time this energy should be uniform during that period. The model used in this section for the generating function is similar to that used in section (6.2). The problem in this chapter is a highly nonlinear problem, we define a set of pseudo-state variables to cast the problem into a quadratic one.

Only the titles and the broad outline of the problems considered in this thesis are mentioned here. A more detailed description of each of the problems and its relationship to the previous work in this area will be found at each chapter.

1.3 Mathematical Background

Our object in this section is to state one important minimum norm result which plays an important part in the solution of problems treated in this work. Before we do this, a brief discussion of relevant concepts from functional analysis is given (37)

A. Norms (37)

A norm, commonly denoted by $||\cdot||$, is real-valued and positive definite. The norm satisfies the following axioms:

(1) $||x|| \geq 0$ for all x X, $||x||=0$ if and only if $x=0$

(2) $||x+y|| \leq ||x||+||y||$ for each $x,y \in X$

(3) $||\alpha x||=|\alpha|\cdot||x||$ for all scalars α and each $x \in X$.

A normed linear (vector) space X is a linear space in which every vector x has a norm (Length). The norm functional is used to define a distance and a convergence measure

$$d(x,y) = ||x-y||$$

For example let [1,K] be a closed bounded interval. The space of the discrete function x(k) on [1,K] can have one of the following norms:

$$||x||_1 = \sum_{k=1}^{K} |x(k)|$$

$$||x||_2 = [\sum_{k=1}^{K} ||x(k)||^2]^{1/2}$$

B. Inner Product (37)

A very important concept in Euclidean geometry is the concept of orthogonality of two vectors. Two vectors are orthogonal if their inner product is zero. Extension of this concept to more generalized spaces leads to powerful results. The concept of orthogonality is not present in all normed spaces and leads to the definition of an inner product defined on XxX denoted by <.>. The inner product satisfies

(1) $\langle x,y\rangle = \langle y,x\rangle$

(2) $\langle \alpha x+\beta y,z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

(3) $\langle x,x\rangle \geq 0$, $\langle x,x\rangle = 0 \leftrightarrow x=0$

(4) $\langle x,x\rangle = ||x||^2$

Hilbert Space (37,38)

A linear space X is called a Hilbert space if X is an inner product space that is complete with respect to the norm induced by the inner product.

Equivalently, a Hilbert space is a Banach space whose norm is induced by an inner product. Let us now consider some specific examples of Hilbert spaces. The space E^n is a Hilbert space with inner product as defined by

$$\langle x,y\rangle = x^T y$$

or

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$$

The space $L_2(k_o k_f)$ is a Hilbert space with inner product

$$\langle x,y\rangle = \sum_{k=k_o}^{k_f} x^T(kT)y(k\tau)$$

An extremely useful Hilbert space is used in this study. The elements of the space are vectors whose components are functions of time over the interval $[k_o k_f]$. We can define the Hilbert space $L^n_{2B}(k_o, k_f)$. The inner product in this case is given by

$$<V(k\tau),\ U(k\tau)> = \sum_{k=k_o}^{k_f} V^T(k\tau)\ B(k\tau)\ U(k\tau)$$

for every $V(k_\tau)$ and $U(k_\tau)$ in the space.

Hilbert Space of Random Variables (38)

If $P(\xi)$ is the probability that the random variable x assumes a value less than or equal to the number ξ. The expected value of a discrete random variable g denoted by μ_x, is defined by

$$E[x] = \mu_x = \sum_x g(\xi)P(\xi)$$

where $\sum_x$ means sum over all x values.

Given a finite collection of real random variables $\{x_1, x_2, \ldots, \ldots, x_{11}\}$, we define their joint probability distribution P as

$$P(\xi_1, \xi_2, \ldots, \xi_n) = \text{Prob}(x_1 \leq \xi_1, x_2 \leq \xi_2, \ldots, x_n \leq \xi_n)$$

i.e., the probability of the simultaneous occurrence of $x_i \leq \xi_i$ for all i. The expected value of any discrete function g of x_i's is defined by

$$E[g(x_1, x_2, \ldots, x_n)] = \sum_x g(\xi_1, \xi_2, \ldots, \xi_n)P(\xi_1, \ldots, \xi_n)$$

Two random variables x_i, x_j are said to be uncorrelated if

$$E(x_i x_j) = E(x_i)E(x_j)$$

With this elementary background material, we now construct a Hilbert space of random variables. Let $\{y_1, y_2, \ldots, y_m\}$ be a finite collection of random variables with $E(y_i^2)<\infty$ for each i. We define a Hilbert space H consisting of all random variables that are linear combination of the y_i's. The inner product of two elements x,y in H is defined as

$$\langle x,y\rangle = E(xy)$$

If $x= \sum \alpha_i y_i$, $y= \sum \beta_i y_i$, then

$$E(xy)=E((\sum_i \alpha_i y_i)(\sum_j \beta_j y_j))$$

The space H is a finite-dimensional Hilbert space with dimension equal to at most m. If in the Hilbert space H each random variable has an expected value equal to zero, then two vectors x,z are orthogonal if they are uncorrelated:

$$\langle x,z\rangle = E(x)E(z)=0$$

The concept of a random variable can be generalized in an important direction. An n-dimensional vector-values random variable x is an ordered collection of n scalar-valued random variables. Notationally, x is written as a column vector

$$x = \text{col.}(x_1, x_2, x_3, \ldots, x_n)$$

(the components being random variables). x in the above equation is referred to as a random vector.

A Hilbert space of random vectors can be generated from a given set of random vectors in a manner analogous to that for random variables. Suppose $\{y_1, y_2, \ldots, y_m\}$ is a collection of n-dimensional random vectors. Each element y_i has n components y_{ij}; $j=1,2,\ldots,\ldots,n$ each of which is a random variable with finite variance. We define the Hilbert space H of n-dimensional random vectors as consisting of all vectors whose components are linear combinations of the components of the y_i's. Thus an arbitrary element y in this space can be expressed as

$$y = K_1 y_1 + K_2 y_2 + \ldots. + K_n y_n$$

where the K_i's are real nxn matrices.

If x and z are elements of H, we define their inner product as

$$\langle x,z \rangle = E\left(\sum_{i=1}^{n} x_i z_i\right)$$

which is the expected value of the n-dimensional inner product. A convenient notation is

$$\langle x,z \rangle = E(x^T z)$$

The norm of an element x in the space of n-dimensional random vectors can be written as

$$||x|| = (\text{Trace } E(xx^T))^{1/2} ,$$

where

$$E(xx^T) = \begin{bmatrix} E(x_1x_1) & E(x_1x_2)\ldots\ldots E(x_1x_n) \\ E(x_2x_1) & E(x_2x_2)\ldots\ldots E(x_2x_n) \\ E(x_nx_1) & E(x_nx_2)\ldots\ldots E(x_nx_n) \end{bmatrix}$$

is the expected value of the random matrix dyad xx^T. Similarly, we have

$$\langle x,z\rangle = \text{Trace } E(xz^T).$$

A Minimum Norm Theorem (31,37,38)

With the preliminary definitions and concepts outlined above, we are now in a position to introduce one powerful result in optimization theory. Our result is a generalization of the idea that the shortest distance from a point to a line is given by the orthogonal to the line from the point.

Theorem. Let B and D be Hilbert spaces. Let T be a bounded linear transformation defined on B with values in D. Let $\hat{u}$ be a given vector in B. For each ξ in the range of T, there exists a unique element $u_\xi \in B$ that satisfies

$$\xi = Tu$$

while minimizing the objective functional

$$J(u) = ||u-\hat{u}||$$

The unique optimal $u_\xi \in B$ is given by

$$u = T^{\dagger}[\xi - T\hat{u}] + \hat{u}$$

where the pseudoinverse operator $T^{\dagger}$ is given by

$$T^{\dagger}\xi = T^{*}[TT^{*}]^{-1}\xi$$

provided that the inverse of TT^{*} exists.

We can obtain the above result by using a Lagrange multiplier argument. Here we consider an augmented objective

$$\tilde{J} = ||u-\hat{u}||^2 + \langle\lambda, (\xi - Tu)\rangle$$

where λ is a multiplier (in fact $\lambda \epsilon D$) to be determined so that the constraint $\xi = Tu$ is satisfied. By utilizing properties of the inner products we can write

$$\tilde{J} = ||u-\hat{u}-T^{*}(\lambda/2)||^2 - ||T^{*}(\lambda/2)||^2 + \langle\lambda, \xi\rangle$$

only the first norm depends explicitly on the choice of u. To minimize $\tilde{J}$ we therefore require that

$$u_\xi = \hat{u} + T^{*}(\lambda/2)$$

The vector $(\lambda/2)$ is obtained using the constraint as the solution to

$$\xi = T\hat{u} + TT^{*}(\lambda/2)$$

or

$$(\frac{\lambda}{2}) = [TT^*]^{-1}[\xi - T\hat{u}]$$

It is therefore clear that with an invertible TT^* we write

$$u_\xi = T^*[TT^*]^{-1}[\xi - T\hat{u}] + \hat{u}$$

which is the required result.

The theorem as stated is an extension of the fundamental minimum norm problem where the objective functional is

$$J(u) = ||u||$$

the optimal solution for this case is

$$u_\xi = T^*[TT^*]^{-1}\xi.$$

In applying this result to our physical problem we need to recall two important concepts from ordinary constrained optimization. These are the Lagrange multiplier rule and the Kuhn-Tucker multipliers. An augmented objective functional is formed by adding to the original functional terms corresponding to the constraints using the necessary multipliers. The object in these cases is to ensure that the augmented functional can indeed be cast as a norm in the chosen space.

CHAPTER II

A MULTIRESERVOIR POWER SYSTEM WITH A CONSTANT WATER CONVERSION FACTOR AND A LINEAR STORAGE-ELEVATION CURVE

2.1 Background

In this chapter the long-term optimal operating problem of a multireservoir power system connected either in series on a river or series-parallel on a multiriver system is considered. The system considered here is characterized by having a constant water conversion factor (MWh/MM^3), and this conversion factor is equal to the average number of megawatt-hours produced in a month by an outflow of one Mm^3.

The optimization of the monthly operating policy of a multireservoir hydroelectric power system is a stochastic nonlinear problem. For a small system this problem can be solved by dynamic programming, but for a large system another method is required. Various optimization techniques have been proposed in the past to solve this problem.

Aggregation of the multireservoir hydroplant into a single complex equivalent reservoir and solution by Stochastic Dynamic Programming (SDP) is one of the earlier approaches that has been used (3,4). Obviously, such a representation of the reservoirs cannot take into account all local constraints on the contents of the reservoirs, water flows, and hydroplants generations. This model can perform satisfactorily for systems where reservoirs and inflow characteristics are sufficiently "similar" to justify aggregation into a single reservoir and hydroplant model (11).

Turgeon (1,2) has proposed two methods for the solution of the problem. the first method (1978) is really an extension to the aggregation method, and it breaks the problem into a two-level problem. At the second level the problem is to determine the monthly generation of the valley. This problem is solved by Dynamic Programming. The problem at the first level is to allocate that generation to the installation. This is done by finding functions that relate the water level of each reservoir to the total amount of potential energy stored in the valley. The second method (1981) is the decomposition method by combining many reservoirs into one reservoir for the purpose of optimization of operation of a multireservoir power system in series on a river and using the Dynamic Programming for solving n-1 problems of two state variables each. The solution obtained by this method is a function of the water content of that reservoir and the total energy content of the downstream reservoirs. The main drawback is that the approach avoids answering the basic question as to how the individual reservoirs in the system are to be operated in an optimal fashion (15). Also, the inflows to some reservoirs may be periodic in phase with the annual demand cycle, while other reservoirs have an inflow cycle which lags by a certain time (12).

Stochastic Dynamic Programming with successive approximation (DPSA) has been proposed to solve the problem of a parallel multireservoir hydrdoelectric power system (2). The successive approximation involves a "one-at-a-time" stochastic optimization of each reservoir. The procedure is repeated over all the reservoirs until convergence is attained. The major drawback of this approach is that it ignores the dependence of the operation of one reservoir on the actual energy content of other reservoirs.

In the Aggregation-Decomposition (AD) approach (2), the optimization of a system of n reservoirs is broken into n subproblems in which one reservoir is optimized knowing the total energy content of the rest of the reservoirs. For each subproblem one of the reservoir-hydroplant models is retained and the remaining n-1 are aggregated into an equivalent reservoir hydro-plant model.

A comparison of the last two approaches, DPSA and AD, has been done in reference (2) on a simulation basis for a six complex reservoir system. The results indicate that the AD approach gives better results and the computational effort increases only linearly with the number of reservoirs. More precisely, for each new reservoir added to the system, only one additional Dynamic Programming problem of two-state variables has to be solved. The computing time for each one of the last two approaches was 150 minutes in CPU units.

Unfortunately, all the present techniques used for solving the long-term stochastic optimization problem are limited to stochastic Dynamic Programming which suffers from major problems, when it is applied to a multidimensional problem, including excessive demands on computing time and storage requirements (2).

In this chapter the optimal long-term optimization problem for series and parallel multireservoir power systems is considered. We assume that the water conversion factor assigned for each hydroplant is a constant, and this conversion factor is equal to the average number of megawatt-hours produced in a month by an outflow of one $Mm^3 (1Mm^3 = 10^6 m^3)$. Also, we assume that the reservoir's storage-elevation curve is linear.

2.2 A Series Multireservoir Power System*

2.2.1 Problem Formulation

2.2.1.1 The System Under Consideration

The system under consideration consists of n reservoirs connected in series on a river. We will number the installations from upstream to downstream (Figure 2.1), and denote by the following:

y_i^k A random variable representing the natural inflow to reservoir i in period k in Mm^3 $(1Mm^3 = 10^6 m^3)$. These are statistically independent random variables with normal distribution

x_i^k The storage of reservoir i at the end of month k in Mm^3,

$$\underline{x}_i \leq x_i^k \leq \overline{x}_i$$

where $\underline{x}_i$ and $\overline{x}_i$ are the minimum and maximum storages in Mm^3

s_i^k The spill from the reservoir i during a period k in Mm^3;

$$s_i^k \geq 0 \; ; \; s_0^k = 0$$

---

* A version of this section has been accepted for publication in the Journal of Optimization Theory and Application (JOTA), January 1985, Ref. 23.

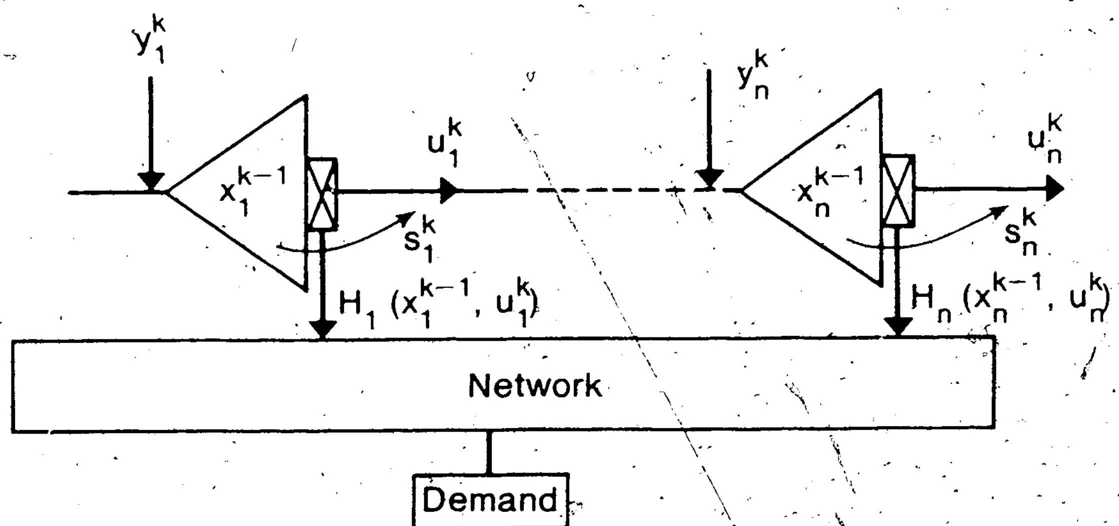

Figure 2.1 A Series Multireservoir Power System

u_i^k The discharge from reservoir i during a period k in Mm^3;

$$\underline{u}_i^k \leq u_i^k \leq \overline{u}_i^k$$

where $\underline{u}_i^k$ and $\overline{u}_i^k$ are the minimum and maximum discharge. If $u_i^k > \overline{u}_i^k$ then, $u_i^k - \overline{u}_i^k$ is discharged through the spillway

$H_i(u_i^k, x_i^{k-1})$ The generation of plant i in period k in MWh

$V_i(x_i^K)$ Value of the water remaining in reservoir i at the end of the last period studied in dollars

c^k Value in dollars of one MWh generated anywhere on the river

i Subscript denoting the installation number; i=1,...,...,n

k Superscript denoting the month; k=1,...,...,K

2.2.1.2 Statement of the Problem

The problem is to determine the optimal operating policy of n hydroelectric power plants in series on a river. In mathematical terms, the problem for the power system of Figure (2.1) is to determine the discharge u_i^k; i=1,...,...,n; k=1,...,...,K that maximizes

$$J = E\left[\sum_{i=1}^{n} V_i(x_i^K) + \sum_{i=1}^{n} \sum_{k=1}^{K} c^k H_i(u_i^k, x_i^{k-1})\right] \quad \$ \qquad (2.1)$$

Subject to the following constraints

(1) The reservoir dynamics may be adequately described by the following discrete continuity equation

$$x_i^k = x_i^{k-1} + y_i^k + u_{i-1}^k - u_i^k + s_{i-1}^k - s_i^k \qquad (2.2)$$

where

$$s_i^k = \begin{cases} (x_i^{k-1} + y_i^k + s_{i-1}^k + u_{i-1}^k - x_i^k) - \bar{u}_i^k ; \text{If} (x_i^{k-1} + y_i^k + s_{i-1}^k + u_{i-1}^k - x_i^k) > \bar{u}_i^k \\ \text{and } x_i^k > \bar{x}_i \\ 0, \text{ otherwise} \end{cases} \qquad (2.3)$$

(2) The operating reservoir constraints are

$$\underline{x}_i \leq x_i^k \leq \bar{x}_i \qquad (2.4)$$

$$\underline{u}_i^k \leq u_i^k \leq \bar{u}_i^k \qquad (2.5)$$

The first set of the inequality constraints simply states that the reservoir storage (or elevation) may not exceed a maximum level, nor be lower than a minimum level. For the maximum level this is determined by the elevation of the spillway crest or the top of the spillway gates. The minimum level may be fixed by the elevation of the lowest outlet in the dam or by conditions of operating efficiency for the turbines. The second set is determined by the discharge capacity of the power plant as well as its efficiency (32) we are given

x_i^0, $H_i(u_i^k, x_i^{k-1})$, $V_i(x_i^K)$ and parameter c^k; the symbol E in equation (2.1) stands for the expected value.

2.2.1.3 Modelling of the System

The conventional approach for obtaining the equivalent reservoir and hydroplant is based on the potential energy concept. The reservoir on a river is mathematically represented by an equivalent potential energy balance equation. The potential energy balance equation is obtained by multiplying both sides of the reservoir balance-of-water equation by the water conversion factors of at-site and downstream hydroplants (3,4). We may choose the following for the function $V_i(x_i^K)$ (1)

$$V_i(x_i^K) = \sum_{j=i}^{n} h_j x_i^K \qquad (2.6)$$

where h_j is the average water conversion factor (MWh/Mm^3) at site j. In the above equation we assumed that the cost of this energy is one dollar/MWh (The average cost during the year).

The generation of a hydroelectric plant is a non-linear function of the water discharge u_i^k and the net head, which itself is a function of the storage. In this section, we will assume a linear relation between the storage and the head (the storage-elevation curve is linear and the tailwater elevation is constant independent of the discharge). We may choose the following for the function $H_i(x_i^{k-1}, u_i^k)$ (1)

$$H_i(u_i^k, x_i^{k-1}) = a_i u_i^k + b_i u_i^k x_i^{k-1} \text{ MWh} \qquad (2.7)$$

where a_i and b_i are constants for the reservoir i. These were obtained by least square curve fitting to typical plant data available.

Now, the cost functional in equation (2.1) becomes

$$J=E[\sum_{i=1}^{n}\sum_{j=i}^{n} h_j x_i^K + \sum_{i=1}^{n}\sum_{k=1}^{K} (A_i^k u_i^k + u_i^k B_i^k x_i^{k-1})] \quad \$ \tag{2.8}$$

where

$$A_i^k = c^k a_i \tag{2.9}$$

$$B_i^k = c^k b_i \tag{2.10}$$

Subject to the following constraints

$$x_i^k = x_i^{k-1} + y_i^k + u_{i-1}^k - u_i^k + s_{i-1}^k - s_i^k \tag{2.11}$$

$$\underline{x}_i \leq x_i^k \leq \overline{x}_i \tag{2.12}$$

$$\underline{u}_i^k \leq u_i^k \leq \overline{u}_i^k \tag{2.13}$$

The problem now is that of maximizing (2.8) subject to satisfying constraints (2.11-2.13).

2.2.2 A Minimum Norm Formulation

The augmented cost functional is obtained by adjoining to the cost function (2.8) equation (2.11) via Lagrange's multiplier, λ_i^k, and the inequality constraints (2.12) and (2.13) via Kuhn-Tucker multipliers

(40). One thus obtains

$$\tilde{J}=E[\sum_{i=1}^{n}\sum_{j=i}^{n} h_j x_i^{K}+\sum_{i=1}^{n}\sum_{k=1}^{K}\{A_i^{k}u_i^{k}+u_i^{k}B_i^{k}x_i^{k-1}+\lambda_i^{k}(-x_i^{k}+x_i^{k-1}$$

$$+y_i^{k}+u_{i-1}^{k}-u_i^{k}+s_{i-1}^{k}-s_i^{k})+e_i^{k}(\underline{x}_i-x_i^{k})$$

$$+e_i^{1k}(x_i^{k}-\bar{x}_i)+f_i^{k}(\underline{u}_i^{k}-u_i^{k})+f_i^{1k}(u_i^{k}-\bar{u}_i^{k})\}] \qquad (2.14)$$

where e_i^k, e_i^{1k}, f_i^k and f_i^{1k} are Kuhn-Tucker multipliers. These are equal to zero, if the constraints are not violated and greater than zero if the constraints are violated (32).

Define the following nx1 column vector such that

$$H = \text{col.}\ (H_1,\ldots,\ldots,H_n) \qquad (2.15)$$

where

$$H_i = \sum_{j=i}^{n} h_j \qquad (2.16)$$

$$A(k) = \text{col.}\ (A_1^{k},\ldots,\ldots,A_n^{k}) \qquad (2.17)$$

$$x(k) = \text{col.}\ (x_1^{k},\ldots,\ldots,x_n^{k}) \qquad (2.18)$$

$$u(k) = \text{col.}\ (u_1^{k},\ldots,\ldots,u_n^{k}) \qquad (2.19)$$

$$y(k) = \text{col.}\ (y_1^{k},\ldots,\ldots,y_n^{k}) \qquad (2.20)$$

$$s(k) = \text{col.}\ (s_1^k, \ldots, \ldots, s_n^k) \tag{2.21}$$

$$\lambda(k) = \text{col.}\ (\lambda_1^k, \ldots, \ldots, \lambda_n^k) \tag{2.22}$$

$$\mu_i^k = e_i^{1k} - e_i^k \tag{2.23}$$

$$\mu(k) = \text{col.}\ (\mu_1^k, \ldots, \ldots, \mu_n^k) \tag{2.24}$$

$$\psi_i^k = f_i^{1k} - f_i^k \tag{2.25}$$

$$\psi(k) = \text{col.}\ (\psi_1^k, \ldots, \ldots, \psi_n^k) \tag{2.26}$$

Define the following nxn diagonal matrix

$$B(k) = \text{diag.}(B_1^k, \ldots, \ldots, B_n^k) \tag{2.27}$$

Furthermore, define the following lower triangular matrix M whose elements are given by

$$\left.\begin{array}{l} \text{(i)}\ \ m_{ii} = -1 \ ;\ i=1,\ldots,\ldots,n \\ \text{(ii)}\ m_{(j+1)j} = 1 \ ;\ j=1,\ldots,\ldots,n-1 \\ \text{(iii) the rest of the elements are equal to zero} \end{array}\right\} \tag{2.28}$$

Using all the above definitions, the augmented cost functional in equation (2.14) becomes

$$\tilde{J}=E[H^T x(K)+\sum_{k=1}^{K}\{A^T(k)u(k)+u^T(k)B(k)x(k-1)$$

$$+\lambda^T(k)(-x(k)+x(k-1)+Mu(k))+Ms(k))$$

$$+\mu^T(k)x(k)+\psi^T(k)u(k)] \qquad (2.29)$$

Employing the discrete version of integration by parts (33)*, and dropping terms explicitly independent of x(k-1) and u(k), one thus obtains

$$J=E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)+\sum_{k=1}^{K}\{u^T(k)B(k)x(k-1)$$

$$+(\lambda(k)-\lambda(k-1)+\mu(k))^T x(k-1)$$

$$+(A(k)+M^T\lambda(k)+M^T\lambda(k)+\psi(k))^T u(k))\}] \qquad (2.30)$$

---

*The discrete version of integration by parts is as follows

$$\sum_{k=1}^{K}\lambda^T(k)x(k)=\lambda^T(1)x(1)+\lambda^T(2)x(2)+\ldots+\ldots+\lambda^T(K-1)x(K-1)+\lambda^T(K)x(K)$$

If one adds $\lambda^T(0)x(0)$ and subtracts $\lambda^T(0)x(0)$ from the right hand side of the above equation, one obtains

$$\sum_{k=1}^{K}\lambda^T(k)x(k)=-\lambda^T(0)x(0)+\lambda^T(K)x(K)+(\lambda^T(0)x(0)+\ldots+\ldots+\lambda^T(K-1)x(K-1)$$

$$\sum_{k=1}^{K}\lambda^T(k)x(k)=-\lambda^T(0)x(0)+\lambda^T(K)x(K)+\sum_{k=1}^{K}\lambda^T(k-1)x(k-1)$$

It will be noticed that $\tilde{J}$ in equation (2.30) is composed of a boundary part and a discrete integral part, which are independent of each other.

If one defines the vector X(k) such that

$$X^T(k) = [x^T(k-1), \ u^T(k)] \tag{2.31}$$

Then, $\tilde{J}$ in equation (2.30) can be written as

$$\tilde{J}=E\{(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)+\sum_{k=1}^{K} \{X^T(k)L(k)X(k)+R^T(k)X(k)\}\} \tag{2.32}$$

where

$$L(k)=\begin{bmatrix} 0 & 1/2\ B(k) \\ 1/2B^T(k) & 0 \end{bmatrix}, \quad \text{is 2nx2n matrix} \tag{2.33}$$

and

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\mu(k))^T,(A(k)+M^T\mu(k)+\psi(k))^T] \tag{2.34}$$

Note that terms which do not explicitly depend on X(k) are dropped in equation (2.32).

If one defines the vector V(k) such that

$$V(k) = L^{-1}(k)R(k) \tag{2.35}$$

then, the cost function in equation (2.32) can be written in the following form by a process similar to completing the squares as

$$\tilde{J}=E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)+\sum_{k=1}^{K}\{(X(k)+1/2V(k))^T L(k)(X(k)+1/2V(k)) - 1/4\, V^T(k)L(k)V(k)\}] \tag{2.36}$$

The last term in equation (2.36) does not depend explicitly on X(k), so that it is necessary only to consider

$$\tilde{J}=E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0) + \sum_{k=1}^{K}\{(X(k)+ 1/2\ V(k))^T L(k)(X(k)+ 1/2\ V(k))\}] \tag{2.37}$$

To maximize $\tilde{J}$ in equation (2.37), one maximizes the boundary and the discrete integral parts separately (40). Hence one can write

$$\underset{[x(K),X(k)]}{\text{Max.}}\tilde{J}= \underset{x(K)}{\text{Max.}}E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)] + \underset{X(k)}{\text{Max.}}E[\sum_{k=1}^{K}(X(k)+ 1/2\ V(k))^T L(k)(X(k)+ 1/2\ V(k))^T] \tag{2.38}$$

The discrete integral part in equation (2.38) defines a norm, hence this part can be written as

$$\underset{X(k)}{\text{Max.}}J_2 = \underset{X(k)}{\text{Max.}}E[||X(k) + 1/2\ V(k)||]_{L(k)} \qquad (2.39)$$

2.2.3 The Optimal Solution

There is exactly one optimal solution to the problem formulated in equation (2.38). The boundary part in equation (2.38) is optimal when

$$E[\lambda(K)'] = H \qquad (2.40)$$

because $\delta x(K)$ is arbitrary and $x(0)$ is constant.

Since it is desired to maximize J in equation (2.8), the problem is mathematically equivalent to minimizing the norm in equation (2.39). This norm is considered to be an element of the Hilbert space because X(k) is always positive. Hence, the discrete integral part in equation (2.38) is maximized when the norm of equation (2.39) is equal to zero.

$$E[X(k) + 1/2\ V(k)] = [0] \qquad (2.41)$$

Substituting from equation (2.35) into equation (2.41) for V(k), one finds the optimal solution is given by

$$E[R(k) + 2L(k)\ X(k)] = [0] \qquad (2.42)$$

Writing equation (2.42) explicitly and adding the continuity equation (2.11), one obtains the following long-term optimal equations for a series multireservoir power system.

$$E[-x(k)+x(k-1)+y(k)+Mu(k)+Ms(k)] = [0] \tag{2.43}$$

$$E[\lambda(k)-\lambda(k-1)+\mu(k)+B(k)u(k) = [0] \tag{2.44}$$

$$E[A(k) + M^T\lambda(k)+M^T\mu(k)+\psi(k)+B(k)x(k-1)] = [0] \tag{2.45}$$

We can now state the optimal solution of equations (2.43) – (2.45) in component form

$$E[-x_i^k+x_i^{k-1}+y_i^k+u_{i-1}^k-u_i^k+s_{i-1}^k-s_i^k]=0; i=1,\ldots,\ldots,n \tag{2.46}$$

$$E[\lambda_i^k-\lambda_i^{k-1}+\mu_i^k+c^kb_iu_i^k]=0; i=1,\ldots,\ldots,n \tag{2.47}$$

$$E[c^ka_i+\lambda_{i+1}^k-\lambda_i^k+\mu_{i+1}^k-\mu_i^k+\psi_i^k+c^kb_ix_i^{k-1}]=0; i=1,\ldots,\ldots,n \tag{2.48}$$

Besides the above equations, one has the following limits on the variables (40)

$$\left.\begin{array}{l} \text{If } x_i^k < \underline{x}_i, \text{ then put } x_i^k=\underline{x}_i \\ \text{If } x_i^k > \bar{x}_i, \text{ then put } x_i^k=\bar{x}_i \\ \text{If } u_i^k < \underline{u}_i^k, \text{ then put } u_i^k=\underline{u}_i^k \\ \text{If } u_i^k > \bar{u}_i^k, \text{ then put } u_i^k=\bar{u}_i^k \end{array}\right\} \tag{2.49}$$

One also has the following exclusion equations which must be satisfied at the optimum (32)

$$\left.\begin{aligned} e_i^{k}(\underline{x}_i - x_i^{k}) &= 0 \\ e_i^{1k}(x_i^{k} - \overline{x}_i) &= 0 \\ f_i^{k}(\underline{u}_i^{k} - u_i^{k}) &= 0 \\ f_i^{1k}(u_i^{k} - \overline{u}_i^{k}) &= 0 \end{aligned}\right\} \quad (2.50)$$

Equations (2.46-2.50) with equation (2.40) completely specify the optimal solution. The following algorithm is used to solve these equations.

2.2.4 Algorithm for Solution

Assume given: the number of reservoirs (n), the expected value for the natural inflow y_i^k, the initial storage x_i^0, and the cost of energy in \$/MWh c^k.

Step 1 - Assume initial guess for the variable u(k) such that

$\underline{u}(k) \leq u^i(k) \leq \overline{u}(k)$, i=iteration number; i=0

Step 2 First assume that s(k) is equal to zero. Solve equation (2.43) forward in stages with x(0) given

Step 3 - Check the limits on x(k) If x(k) satisfies the inequality;

$\underline{x} < x(k) < \overline{x}$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 4 Calculate the new discharge u(k) from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k) - x(k-1) - y(k))]$$

Step 5 Check the limits on $u(k)$. If $u(k)$ satisfies the inequality

$$\underline{u}(k) \le u(k) \le \overline{u}(k)$$

go to Step 14, otherwise put $u(k)$ to its limits and go to Step 6

Step 6 Calculate the spill at month k from the following

$$E[s(k)]=E[[M]^{-1}(x(k)-x(k-1)-y(k))-\overline{u}(k)]$$

If $s(k) < 0$, put $s(k)=0$

Step 7 Calculate the discharge from the following equation

$$E[u(k)]=E[[M]^{-1}(x(k)-x(k-1)-y(k)-Ms(k))]]$$

Step 8 Solve equation (2.43) forward in stages with $x(0)$ given, and using the value of $s(k)$ calculated in Step 6

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4

Step 10 With $\mu(k)=0$, solve equation (2.44) backward in stages with equation (2.40) as a terminal condition

Step 11 Calculate Kuhn-Tucker multipliers for $u(k)$, $\psi(k)$, from the following equation

$$E[\psi(k)]=E[M^T B(k)u(k)-M^T\lambda(k-1)-B(k)x(k-1)-A(k)]$$

If $u(k)$ satisfies the inequality

$$\underline{u}(k) \le u(k) < \overline{u}(k)$$

put $\psi(k) = 0$

Note that, the above equation for $\psi(k)$ is obtained by multiplying equation (2.44) by $[M^T]$, and subtracting the resulting equation from equation (2.45)

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)] = E[u^i(k) + \alpha Du^i(k)]$$

where

$$E[Du(k)] = E[A(k) + M^T\lambda(k) + B(k)x(k-1) + \psi(k)]$$

and α is a scalar which is chosen with consideration given to such factors as convergence

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2.

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)] = E[B(k)u(k)-[M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)]$$

Step 15 Determine Kuhn-Tucker multipliers for x(k), $\mu(k)$, from the following equation

$$E[\mu(k)] = E[-\lambda(k)-[M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)]$$

If x(k) satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

put $\mu(k) = 0$

Step 16 Determine a new state iterate from the approximation

$$E[x^{i+1}(k)] = E[x^i(k) + \alpha Dx^i(k)]$$

where

$$E[Dx^i(k)] = E[\lambda(k) - \lambda(k-1) + \mu(k) + B(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and J in equation (2.8) is a maximum

2.2.5 Practical Example (1)

The algorithm of the last section has been used to determine the optimal monthly operation for a period of a year for the same system mentioned in Ref. (1). This system consists of four reservoirs (n=4) connected in series on a river, the characteristics of which are given in Table (2.1).

If d^k denote the number of days in month k, then the maximum and minimum releases in Mm^3 during a month k is given by:

$$\left.\begin{aligned} \bar{u}_i^k &= 0.0864 d^k (\text{maximum effective discharge in } m^3/\text{sec}) \\ \underline{u}_i^k &= 0.0864 d^k (\text{minimum effective discharge in } m^3/\text{sec}) \end{aligned}\right\} \quad (2.51)$$

where the maximum and minimum effective discharges are given in Table (2.1).

As we mentioned earlier in this chapter, the storage elevation curve is linear. In tables (2.2-2.5) we give the variation in the percentage error of the water conversion factor (Mwh/Mm^3) with the storage. We assume that the efficiency of the plant is constant. From these tables one can notice that the percentage error in the MWh/Mm^3 generated under this assumption is very small.

The MWh generated at each power house for the four reservoirs is given by:

$$
\left.
\begin{aligned}
H_1(u_1^k, x_1^{k-1}) &= 11.8u_1^k + 1.3\times10^{-3} x_1^{k-1} . u_1^k \quad \text{MWh} \\
H_2(u_2^k, x_2^{k-1}) &= 231.5u_2^k + 9.532\times10^{-3} x_2^{k-1} u_2^k \quad \text{MWh} \\
H_3(u_3^k, x_3^{k-1}) &= 251.82u_3^k + 12.667\times10^{-3} x_3^{k-1} u_3^k \quad \text{MWh} \\
H_4(u_4^k, x_4^{k-1}) &= 437.0u_4^k + 11.173\times10^{-3} x_4^{k-1} u_4^k \quad \text{MWh}
\end{aligned}
\right\} \qquad (2.52)
$$

We have simulated the monthly operation of the system for widely different water conditions. In Tables (2.6-2.11) we have reported the results obtained in two typical years. One with high flow and one with low flow. the natural inflows to the sites in the year of high flow, which we call year 1 are given in Table 2.6.

In Table 2.7 we give the optimal monthly release from each reservoir and the profits realized in year 1 for the optimal global-feedback solution. In Table 2.8 we also give the optimal storage for each reservoir during year 1.

The monthly natural inflows to the four-sites in year 2, which is the year of the low flow are given in Table 2.9. In Table 2.10 we also give the optimal monthly releases from each reservoir and the profits realized during year 2. In Table 2.11 we give the optimal storage for each reservoir during year 2. We began both years with

$$x^T(0) = [6688.5,\ 557.9,\ 48.9,\ 3347.4]\ \text{Mm}^3$$

The processing time required to determine the optimal monthly operating policy for a period of a year for the system just described was 1.2 sec. in CPU units.

Table 2.1 - Characteristics of the Installations

Site	Minimum capacity $\underline{x}_i$ Mm^3	Maximum capacity $\bar{x}_i$ Mm^3	Maximum effective discharge m^3/sec	Minimum effective discharge m^3/sec	Average monthly productibility MWh/Mm^3	Reservoir's Constants a_i MWh/Mm^3	Reservoir's Constants b_i $MWh/(Mm^3)^2$
1	0	9628	400	0	18.31	11.8	1.3×10^{-3}
2	0	570	547	0	234.36	231.5	9.532×10^{-3}
3	0	50	594	0	216.14	215.82	12.667×10^{-3}
4	0	3420	1180	0	453.44	437.00	11.173×10^{-3}

Table 2.2: Variation of MWh/Mm^3 with the storage and the percentage error for the first reservoir

Storage Mm^3	Given MWh/Mm^3	Calculated MWh/Mm^3	% Error
0	11.28	11.80	4.610
1925.6	14.41	14.30	-0.741
3851.2	17.02	16.81	-1.254
5776.8	19.54	19.31	-1.178
7702.4	21.97	21.81	-0.714
9628.0	24.32	24.32	0.000

Table 2.3: Variation of MWh/Mm^3 with the storage and the percentage error for the second reservoir

Storage Mm^3	Given MWh/Mm^3	Calculated MWh/Mm^3	% Error
0	231.53	231.500	-0.0130
114	232.68	232.587	-0.0412
228	233.81	233.673	-0.0585
342	234.89	234.760	-0.0553
456	235.94	235.850	-0.0396
570	236.96	236.933	-0.0113

Table 2.4: Variation of MWh/Mm^3 with the storage and the percentage error for the third reservoir

Storage Mm^3	Given MWh/Mm^3	Calculated MWh/Mm^3	% Error
0	215.82	215.82	0.00000
10	215.95	215.95	0.00000
20	216.08	216.07	-0.00463
30	216.20	216.20	0.00000
40	216.33	216.33	0.00000
50	216.46	216.45	-0.0031

Table 2.5: Variation of MWh/Mm^3 with the storage and the percentage error for the fourth reservoir

Storage Mm^3	Given MWh/Mm^3	Calculated MWh/Mm^3	% Error
0	431.75	437.00	1.22
684	441.99	444.64	0.60
1368	449.94	452.28	0.50
2052	456.54	459.93	0.74
2736	462.32	467.57	1.14
3420	467.52	475.21	1.65

Table 2.6: Monthly inflows to the four sites in year 1 and the cost of energy

Month k	y_1^k Mm^3	y_2^k Mm^3	y_3^k Mm^3	y_4^k Mm^3	c^k \$/MWh
1	828	380	161	1798	0.78
2	829	331	82	1201	0.93
3	578	224	41	810	1.03
4	394	146	32	486	1.34
5	265	95	18	302	1.42
6	233	82	14	258	1.34
7	193	68	6	194	1.21
8	219	127	279	1485	0.98
9	1101	614	181	3239	0.83
10	1887	781	205	2560	0.73
11	1150	491	146	1583	0.70
12	824	363	132	1454	0.76

Table 2.7: Optimal monthly releases from the reservoirs and the profits realized in year 1

Month k	u_1^k Mm^3	u_2^k Mm^3	u_3^k Mm^3	u_4^k Mm^3	Profits $
1	0	368	528	2253	990816
2	1037	1418	1499	2700	1828432
3	853	1076	1118	1928	1473793
4	1071	1465	1547	2244	2300938
5	968	1130	1098	2855	2656644
6	1071	1336	1400	3161	2785338
7	1037	1127	1133	1580	1475190
8	0	0	229	1310	609669
9	0	446	627	2326	1051100
10	0	506	711	2330	979250
11	0	491	637	2117	873319
12	0	363	495	1794	788087
Value of water remaining in the reservoir at the end of the year					10414598.
Total Profits					28227174

Table 2.8: Optimal reservoir storage for year 1

Month k	x_1^k Mm^3	x_2^k Mm^3	x_3^k Mm^3	x_4^k Mm^3
1	7517	570	50	3420
2	7309	520	50	3420
3	7034	520	50	3420
4	6357	272	0	3209
5	5654	205	50	1755
6	4816	22	0	253
7	3972	00	0	0
8	4191	127	50	404
9	5292	295	50	1943
10	7179	570	50	2884
11	8329	570	50	2987
12	9153	570	50	3142

Table 2.9: Monthly inflows to the four sites in year 2

Month k	y_1^k Mm^3	y_2^k Mm^3	y_3^k Mm^3	y_4^k Mm^3
1	568	227	29	708
2	442	173	150	772
3	460	171	22	608
4	505	187	22	601
5	324	116	22	272
6	305	113	[illegible]	307
7	208	81	13	219
8	161	110	259	1327
9	498	301	64	1539
10	642	274	51	886
11	429	169	22	605
12	527	229	75	895

Table 2.10: Monthly releases from the reservoirs and the profits realized in year 2

Month k	u_1^k Mm^3	u_2^k Mm^3	u_3^k Mm^3	u_4^k Mm^3	Profits $
1	0	215	343	878	405647
2	1037	1262	1412	2242	1573185
3	881	1052	1074	1624	1308419
4	1071	1465	1537	3107	2829756
5	968	1319	1341	2855	2759020
6	1071	1260	1299	2628	2380707
7	1037	1118	1131	1418	1383345
8	0	0	209	1470	675513
9	0	0	64	1454	541237
10	0	115	166	91	75287
11	0	169	191	114	93025
12	0	229	304	154	144909
	Value of water remaining in the reservoir at the end of the year				7165203
	Total Benefits				21335253

Table 2.11: Optimal reservoir storage in year 2

Month k	x_1^k Mm^3	x_2^k Mm^3	x_3^k Mm^3	x_4^k Mm^3
1	7257	570	50	3420
2	6662	518	50	3362
3	6240	518	50	3420
4	5674	312	0	2452
5	5030	76	0	1210
6	4246	0	0	189
7	3435	0	0	121
8	3596	110	50	187
9	4094	411	50	337
10	4736	570	50	1298
11	5160	570	50	1980
12	5687	570	50	3025

2.2.6 Discussion

From Table 2.7, it will be noticed that the total benefits from the system during the first year (wet year) using our approach is \$28,227,174 with computing time 1.2 sec. in CPU units, while for the same using the decomposition approach and the Dynamic Programming (DP) as mentioned in Ref. 1 is \$28,165,760 with computing time 12 sec. in CPU units. The increases in the total benefits during the first year is \$61,414 while the computing time is less by several orders of magnitude than the computing time required by Ref. 1.

On the other hand from Table 2.10, the total benefits from the system during the second year (dry year) using our approach is \$21,335,253 with computing time 1.2 sec. in CPU units, while for the same system as mentioned in Ref. 1 is \$21,155,030 using the decomposition approach and the Dynamic Programming with computing time 12 sec. The increase in the total benefits is \$180,223 which is less than one percent (0.85%). Although the total benefits are the same, the computing time using our approach is very small compared to the computing time using the Dynamic Programming (one tenth the computing time using DP).

2.3 A Parallel Multireservoir Power System*

In section 2.2, we proposed a new attractive and efficient approach to solve the optimal long-term operating problem of multireservoir power systems connected in series on a river. The problem is formulated as a minimum norm problem. We compared our approach with one of the well-known approaches, Dynamic Programming, through an example. The computing time using our approach was very small compared to the computing time using Dynamic Programming for the same system. At the same time the total benefits obtained using the proposed approach is greater than the total benefits obtained using the Dynamic Programming.

This section is devoted to solving the optimal long-term operating problem of a parallel multireservoir power system. In formulating the problem as a minimum norm problem, we assumed that the water conversion factor assigned for each hydroplant is a constant, and the time of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason, those times are not taken into account. Transmission line losses are also neglected.

2.3.1 Problem Formulation

2.3.1.1 The System Under Consideration (2)

The system under consideration consists of m independent rivers, with one or several reservoirs and power plants in series on each, and

---

* A version of this Section has been accepted for publication in Journal of Optimization Theory and Applications, January 1985, Ref. 24.

interconnection lines to the neighboring system through which energy may be exchanged (Figure 2.2). We will denote by the following:

y_{ij}^k A random variable representing the natural inflows to reservoir i on river j during a period k in Mm^3 ($1Mm^3 = 10^6 m^3$). These are statistically independent random variables with normal distribution

u_{ij}^k The effective discharge from reservoir i on river j in period k in Mm^3. This is the water released from the reservoir to the allied power plant to produce electricity;

$$\underline{u}_{ij}^k \le u_{ij}^k \le \overline{u}_{ij}^k$$

where $\underline{u}_{ij}^k$ and $\overline{u}_{ij}^k$ are the minimum and maximum discharges. If $u_{ij}^k > \overline{u}_{ij}^k$ and the reservoir is full, then $u_{ij}^k - \overline{u}_{ij}^k$ Mm^3 is discharged through the spillways.

x_{ij}^k The content of reservoir i on river j at the end of month k in Mm^3;

$$\underline{x}_{ij} \le x_{ij}^k \le \overline{x}_{ij}$$

where $\underline{x}_{ij}$ and $\overline{x}_{ij}$ are the minimum and maximum storages

$H_{ij}(u_{ij}^k, x_{ij}^{k-1})$ The generation of plant i on river j in period k in MWh. It is a function of the discharge u_{ij}^k and of the water head, which itself is a function of the allied reservoir storage

$V_{ij}(x_{ij}^K)$ Value in dollars of the water remaining in reservoir i of river j at the end of the last period studied

c_j^k Value in dollars of one MWh produced anywhere on the river j

s_{ij}^k The spill from reservoir i on river j in Mm^3; $s_{ij}^k \geq 0$

n_j Number of reservoirs on river j

k Superscript denoting the period, $k=1,\ldots,\ldots,K$

m Number of rivers

2.3.1.2 Statement of the Problem

The problem is to determine the monthly optimal operating policy of a multireservoir hydroelectric power system in parallel (Figure 2.2). In mathematical terms, the problem for the power system of Figure 2.2 is to determine the discharge u_{ij}^k that maximizes the total benefits from the system (benefits from hydro generation and from the amount of water left in storage at the end of the planning period). i.e. find the discharge u_{ij}^k that maximizes

$$J=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j} V_{ij}(x_{ij}^K)+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} \bar{c}_j^k H_{ij}(u_{ij}^k,x_{ij}^{k-1})] \quad (2.53)$$

Subject to satisfying the following constraints

(1) The reservoir dynamics may be described by the following discrete continuity equation

$$x_{ij}^k = x_{ij}^{k-1}+y_{ij}^k+u_{(i-1)j}^k-u_{ij}^k+s_{(i-1)j}^k-s_{ij}^k \quad (2.54)$$

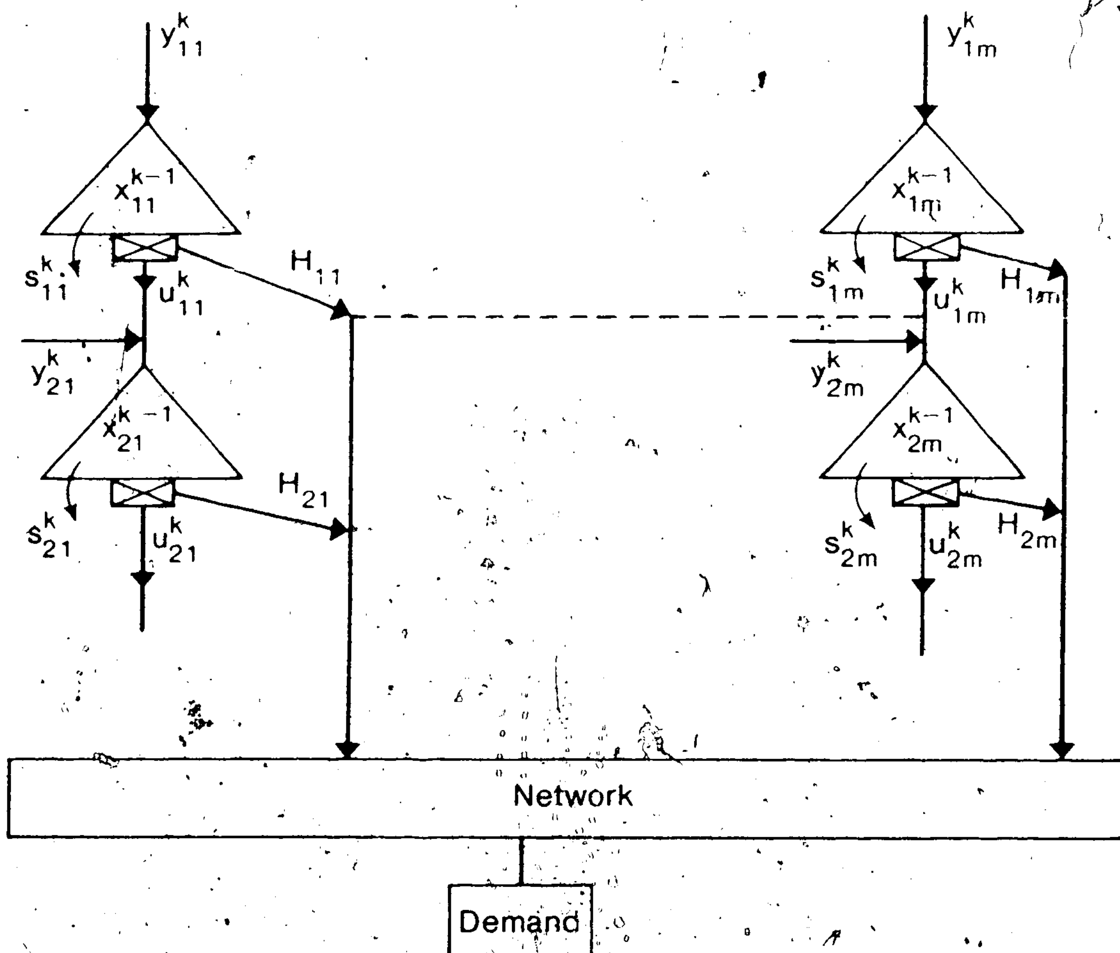

Figure 2.2 A Parallel Multireservoir Power System

(2) The operational reservoir constraints are

$$\underline{x}_{ij} \le x_{ij}^{k} \le \bar{x}_{ij} \qquad (2.55)$$

$$\underline{u}_{ij}^{k} \le u_{ij}^{k} \le \bar{u}_{ij}^{k} \qquad (2.56)$$

where

$$s_{ij}^{k} = \begin{cases} (x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-x_{ij}^{k})-\bar{u}_{ij}^{k}; \text{ If } (x_{ij}^{k-1} \\ \quad +y_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-x_{ij}^{k}) > \bar{u}_{ij}^{k} \text{ and } x_{ij}^{k} > \bar{x}_{ij} \\ 0, \text{ otherwise} \end{cases} \qquad (2.57)$$

and given x_{ij}^{0}; $i=1,\ldots,\ldots,n_j$, $j=1,\ldots,\ldots,m$, the functions $H_{ij}(u_{ij}^{k}, x_{ij}^{k-1})$ and $V_{ij}(x_{ij}^{K})$ and the parameter c_j^{k}; the symbol E stands for the expected value. The expection in equation (2.53) is taken with respect to the random variable y_{ij}^{k}.

2.3.1.3 Mathematical Model

The generation of a hydroelectric plant is a nonlinear function of water discharge u_{ij}^{k} and reservoir head, which in turn, is a function of storage. In this study we will assume that the storage-elevation curve is approximately linear. We may choose the following for the functions $H_{ij}(u_{ij}^{k},x_{ij}^{k-1})$ and $V_{ij}(x_{ij}^{K})$ (1)

$$H_{ij}(u_{ij}^{k},x_{ij}^{k-1}) = f_{ij}(x_{ij}^{k-1})\, u_{ij}^{k} \quad \text{MWh} \qquad (2.58)$$

$$V_{ij}(x_{ij}^{K}) = \sum_{v=i}^{n_j} h_{vj} x_{ij}^{K}; \; j=1,\ldots,\ldots,m \qquad (2.59)$$

where

$$f_{ij}(x_{ij}^{k-1}) = a_{ij}+b_{ij}x_{ij}^{k-1} \quad \text{MWh/Mm}^3 \qquad (2.60)$$

and a_{ij}, b_{ij} are constants, these were obtained by least square fitting to typical plant data available.

Substituting from equation (2.60) into equation (2.58), one obtains

$$H_{ij}(u_{ij}^{k}, x_{ij}^{k-1}) = a_{ij}u_{ij}^{k}+b_{ij}u_{ij}^{k}x_{ij}^{k-1} \quad \text{MWh} \qquad (2.61)$$

In equation (2.59), we modelled the amount of water stored at the last month studied by multiplying this amount of water by the water conversion factor of at-a-site and downstream reservoirs, and we assumed that the cost of this energy is one dollar/MWh (the average cost of energy during the year).

Now, the cost functional in equation (2.53) becomes

$$J=E[\sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{v=i}^{n_j} h_{vj}x_{ij}^{K} + \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} \{A_{ij}{}^{k}u_{ij}^{k} + B_{ij}{}^{k}u_{ij}^{k}x_{ij}^{k-1}\}] \qquad (2.62)$$

Subject to satisfying the following constraints

$$x_{ij}^{k} = x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k} \qquad (2.63)$$

$$\underline{x}_{ij} \le x_{ij}^{k} \le \bar{x}_{ij} \tag{2.64}$$

$$\underline{u}_{ij}^{k} \le u_{ij}^{k} \le \bar{u}_{ij}^{k} \tag{2.65}$$

where

$$\left.\begin{aligned} A_{ij}^{k} &= c_{j}^{k} a_{ij} \\ B_{ij}^{k} &= c_{j}^{k} b_{ij} \end{aligned}\right\} \tag{2.66}$$

2.3.2 A Minimum Norm Formulation

The augmented cost functional is obtained by adjoining to the cost function in equation (2.62), the equality constraints (2.63) via Lagrange's multiplier and the inequality constraints (2.64-2.65) via Kuhn-Tucker multipliers, one thus obtains

$$\tilde{J}=E\Big[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{v=i}^{n_j} h_{vj}x_{ij}^{K} + \sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}\{A_{ij}^{k}u_{ij}^{k}+B_{ij}^{k}u_{ij}^{k}x_{ij}^{k-1}$$

$$+\lambda_{ij}^{k}(-x_{ij}^{k}+x_{ij}^{k-1}+u_{(i-1)j}^{k}-u_{ij}^{k}+q_{ij}^{k})+e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})$$

$$+e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij})+f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})+f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k})\}\Big] \tag{2.67}$$

where λ_{ij}^{k} is Lagrange's and e_{ij}^{k}, e_{ij}^{lk}, f_{ij}^{k} and f_{ij}^{lk} are Kuhn-Tucker multipliers. These are equal to zero, if the constraints are not violated and greater than zero, if the constraints are violated (32).

Now, define the following column vectors such that

$$H_{ij} = \sum_{j=1}^{m} \sum_{v=i}^{n_j} h_{vj} \quad i=1,\ldots,\ldots,n_j;\; j=1,\ldots,\ldots,m \tag{2.68}$$

$$H = \text{col.}\ (H_1,\ldots,\ldots,H_m) \tag{2.69}$$

$$H_1 = \text{col.}\ (H_{11},\ldots,\ldots,H_{1m}) \tag{2.70}$$

$$H_m = \text{col.}\ (H_{1m},\ldots,\ldots,H_{n_m}) \tag{2.71}$$

$$A(k) = \text{col.}\ (A_1(k),\ldots,\ldots,A_m(k)) \tag{2.72}$$

$$A_1(k) = \text{col.}\ (A_{11}^{k},\ldots,\ldots,A_{n_1 1}^{k}) \tag{2.73}$$

$$A_m(k) = \text{col.}\ (A_{1m}^{k},\ldots,\ldots,A_{n_m m}^{k}) \tag{2.74}$$

$$u(k) = \text{col.}\ (u_1(k),\ldots,\ldots,u_m(k)) \tag{2.75}$$

$$u_1(k) = \text{col.}\ (u_{11}^{k},\ldots,\ldots,u_{n_1 1}^{k}) \tag{2.76}$$

$$u_m(k) = \text{col.}\ (u_{1m}^{k},\ldots,\ldots,u_{n_m m}^{k}) \tag{2.77}$$

$$x(k) = \text{col.}\ (x_1(k),\ldots,\ldots,x_m(k)) \tag{2.78}$$

$$x_1(k) = \text{col.}\ (x_{11}{}^k,\ldots,\ldots,x_{n_1 1}{}^k) \tag{2.79}$$

$$x_m{}^k = \text{col.}\ (x_{1m}{}^k,\ldots,\ldots,x_{n_m m}{}^k) \tag{2.80}$$

$$q_{ij}{}^k = y_{ij}{}^k + s_{(i-1)j}{}^k - s_{ij}{}^k \tag{2.81}$$

$$q(k) = \text{col.}\ (q_1(k),\ldots,\ldots,q_m(k)) \tag{2.82}$$

$$q_1(k) = \text{col.}\ (q_{11}{}^k,\ldots,\ldots,q_{n_1 1}{}^k) \tag{2.83}$$

$$q_m(k) = \text{col.}\ (q_{1m}{}^k,\ldots,\ldots,q_{n_m m}{}^k) \tag{2.84}$$

$$y(k) = \text{col.}\ (y_1(k),\ldots,\ldots,y_m(k)) \tag{2.85}$$

$$y_1(k) = \text{col.}\ (y_{11}{}^k,\ldots,\ldots,y_{n_1 1}{}^k) \tag{2.86}$$

$$y_m(k) = \text{col.}\ (y_{1m}{}^k,\ldots,\ldots,y_{n_m m}{}^k) \tag{2.87}$$

$$s(k) = \text{col.}\ (s_1(k),\ldots,\ldots,s_m(k)) \tag{2.88}$$

$$s_1(k) = \text{col.}\ (s_{11}{}^k,\ldots,\ldots,s_{n_1 1}{}^k) \tag{2.89}$$

$$s_m{}^k = \text{col.}\ (s_{1m}{}^k,\ldots,\ldots,s_{n_m m}{}^k) \tag{2.90}$$

$$\lambda(k) = \text{col.}\ (\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \qquad (2.91)$$

$$\lambda_1(k) = \text{col.}\ (\lambda_{11}{}^k,\ldots,\ldots,\lambda_{n_1 1}{}^k) \qquad (2.92)$$

$$\lambda_m(k) = \text{col.}\ (\lambda_{1m}{}^k,\ldots,\ldots,\lambda_{n_m m}{}^k) \qquad (2.93)$$

$$\mu_{ij}{}^k = e_{ij}{}^{1k} - e_{ij}{}^k \qquad (2.94)$$

$$\mu(k) = \text{col.}\ (\mu_1(k),\ldots,\ldots,\mu_m(k)) \qquad (2.95)$$

$$\mu_1(k) = \text{col.}\ (\mu_{11}{}^k,\ldots,\ldots,\mu_{n_1 1}{}^k) \qquad (2.96)$$

$$\mu_m(k) = \text{col.}\ (\mu_{1m}{}^k,\ldots,\ldots,\mu_{n_m m}{}^k) \qquad (2.97)$$

$$\psi_{ij}{}^k = f_{ij}{}^{1k} - f_{ij}{}^k \qquad (2.98)$$

$$\psi(k) = \text{col.}\ (\psi_1(k),\ldots,\ldots,\psi_m(k)) \qquad (2.99)$$

$$\psi_1(k) = \text{col.}(\psi_{11}{}^k,\ldots,\ldots,\psi_{n_1 1}{}^k) \qquad (2.100)$$

$$\psi_m(k) = \text{col.}(\psi_{1m}{}^k,\ldots,\ldots,\psi_{n_m m}{}^k) \qquad (2.101)$$

Furthermore, define the following diagonal matrices

$$B(k) = \text{diag.}\ (B_1(k),\ldots,\ldots,B_m(k)) \qquad (2.102)$$

$$B_1(k) = \text{diag.}\ (B_{11}^{\ k},\ldots,\ldots,B_{n_1 1}^{\ k}) \tag{2.103}$$

$$B_m(k) = \text{diag.}\ (B_{1m}^{\ k},\ldots,\ldots,B_{n_m m}^{\ k}) \tag{2.104}$$

$$M = \text{diag.}\ (M_1,\ldots,\ldots,M_m) \tag{2.105}$$

where the matrices $M_1,\ldots,M_m$ are lower triangular matrices whose elements are given by

$$\left.\begin{array}{ll} \text{(i)} & m_{ii} = -1;\ i=1,\ldots,\ldots,n_j;\ j=1,\ldots,\ldots,m \\ \text{(ii)} & m_{(v+1)v}=1;\ v=1,\ldots,\ldots,n_j-1;\ j=1,\ldots,\ldots,m \\ \text{(iii)} & \text{the rest of the elements are equal to zero} \end{array}\right\} \tag{2.106}$$

Now using all the above definitions, the cost functional in equation (2.67) becomes

$$\tilde{J}=E[H^T x(K)+ \sum_{k=1}^{K} \{A^T(k)u(k)+u^T(k)B(k)x(k-1)+\lambda^T(k)(-x(k)+x(k-1)+$$

$$+ q(k)+Mu(k))+\mu^T(k)(x(k-1)+Mu(k)+q(k))+\psi^T(k)u(k)\}] \tag{2.107}$$

Note that constant terms are dropped above .

Employing the discrete version of integration by parts (33), and dropping terms explicitly independent of x(k-1) and u(k), one obtains

$$\tilde{J}=E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)+\sum_{k=1}^{K}[1/2x^T(k-1)B(k)u(k) +$$

$$+ 1/2\, u^T(k)B(k)x(k-1)+(\lambda(k)-\lambda(k-1)+\mu(k))^T x(k-1) +$$

$$+ (A(k)+M^T\lambda(k)+M^T\mu(k)+\psi(k))^T u(k)]]. \qquad (2.108)$$

It will be noticed that $\tilde{J}$ in equation (2.108) is composed of a boundary part and a discrete integral part, which are independnet of each other.

If one defines the vector X(k) such that

$$X^T(k) = [x^T(k-1),\ u^T(k)] \qquad (2.109)$$

Then $\tilde{J}$ can be written as

$$\tilde{J}=E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0) +$$

$$+ \sum_{k=1}^{K} [X^T(k)L(k)X(k)+R^T(k)X(k)]] \qquad (2.110)$$

where

$$L(k) = \begin{bmatrix} 0 & 1/2B(k) \\ 1/2B(k) & 0 \end{bmatrix} \qquad (2.111)$$

and

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\mu(k))^T,\ (A(k)+M^T\lambda(k)+M^T\mu(k)+\psi(k))^T] \qquad (2.112)$$

To maximize $\tilde{J}$ in equation (2.110), one maximizes the boundary and the discrete integral parts separately (40). Hence, one can write

$$\underset{[x(K),X(k)]}{\text{Max.}}\ \tilde{J} = \underset{x(K)}{\text{Max.}}E[(H-\lambda(K))^T x(K)+\lambda^T(0)x(0)]$$

$$+ \underset{X(k)}{\text{Max.}}E[\sum_{k=1}^{K} X^T(k)L(k)X(k)+R^T(k)X(k))] \qquad (2.113)$$

If one defines the vector V(k) such that

$$V(k) = L^{-1}(k)\ R(k) \qquad (2.114)$$

then, the discrete integral part of equation (2.113) can be written in the following form by a process similar to completing the squares as:

$$\underset{X(k)}{\text{Max.}}\ J_2 = \underset{X(k)}{\text{Max.}}E\ [[\sum_{k=1}^{K} [(X(k)+ 1/2V(k))^T L(k)(X(k)+ 1/2V(k))^T$$

$$- 1/4\ V^T(k)\ L(k)V(k)]] \qquad (2.115)$$

The last term in equation (2.115) does not depend on X(k), so that it is only necessary to consider

$$\underset{X(k)}{\mathrm{Max}}.J_2 = \underset{X(k)}{\mathrm{Max}}.E\left[\sum_{k=1}^{K}(X(k)+ 1/2V(k))^T L(k)(X(k)+ 1/2V(k))\right] \tag{2.116}$$

Equation (2.116) defines a norm. Hence equation (2.116) can be written as

$$\underset{X(k)}{\mathrm{Max}}.J_2 = \underset{X(k)}{\mathrm{Max}}.E[||X(k)+ 1/2\ V(k)||]_{L(k)} \tag{2.117}$$

2.3.3 The Optimal Solution

There is exactly one optimal solution to the problem formulated in equation (2.117). This solution is clearly achieved when

$$E[X(k) + 1/2\ V(k)] = [0] \tag{2.118}$$

Substituting from equation (2.114) into equation (2.118), one finds that the optimal solution is given by

$$E[R(k) + 2L(k)\ X(k)] = [0] \tag{2.119}$$

The boundary part in equation (2.113) is optimized when

$$E[\lambda(K)] = H \tag{2.120}$$

because $\delta x(K)$ is arbitrary and $x(0)$ is constant.

Equations (2.119) and (2.120) are the optimal solution for the long-term operation.

Writing equation (2.119) explicitly and adding the continuity equation (2.63), one obtains the following set

$$E[-x(k)+x(k-1)+y(k)+Mu(k)+Ms(k)] = [0] \quad (2.121)$$

$$E[\lambda(k)-\lambda(k-1)+\mu(k)+B(k)u(k)] = [0] \quad (2.122)$$

$$E[A(k)+M^T\lambda(k)+M^T\mu(k)+\psi(k)+B(k)x(k-1)] = [0] \quad (2.123)$$

We can state the optimal equations (2.121-2.123) in component form as

$$E[-x_{ij}^{k}+x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k}] = 0;$$

$$i=1,\ldots,\ldots,n_j;\ j=1,\ldots,\ldots,m \quad (2.124)$$

$$E[\lambda_{ij}^{k}-\lambda_{ij}^{k-1}+\mu_{ij}^{k}+c_j^{k}b_{ij}u_{ij}^{k}] = 0;\ j=1,\ldots,\ldots,n_j;j=1,\ldots,m$$

(2.125)

$$E[c_j^k a_{ij} + \lambda_{(i+1)j}^k - \lambda_{ij}^k + \mu_{(i+1)j}^k - \mu_{ij}^k + \psi_{ij}^k + c_j^k b_{ij} x_{ij}^{k-1}] = 0;$$

$$i=1,\ldots,\ldots,n_j;\quad j=1,\ldots,\ldots,m \qquad (2.126)$$

Besides the above equations, one has the following Kuhn-Tucker exclusion equations which must be satisfied at the optimum

$$e_{ij}^k(\underline{x}_{ij} - x_{ij}^k) = 0 \qquad (2.127)$$

$$e_{ij}^{1k}(x_{ij}^k - \bar{x}_{ij}) = 0 \qquad (2.128)$$

$$f_{ij}^k(\underline{u}_{ij}^k - u_{ij}^k) = 0 \qquad (2.129)$$

$$f_{ij}^{1k}(u_{ij}^k - \bar{u}_{ij}^k) = 0 \qquad (2.130)$$

One also has the following limits on the variables

$$\left.\begin{array}{l}
\text{If } x_{ij}^k < \underline{x}_{ij}, \text{ then we put } x_{ij}^k = \underline{x}_{ij} \\
\text{If } x_{ij}^k > \bar{x}_{ij}, \text{ then we put } x_{ij}^k = \bar{x}_{ij} \\
\text{If } u_{ij}^k < \underline{u}_{ij}^k, \text{ then we put } u_{ij}^k = \underline{u}_{ij}^k \\
\text{If } u_{ij}^k > \bar{u}_{ij}^k, \text{ then we put } u_{ij}^k = \bar{u}_{ij}^k
\end{array}\right\} \qquad (2.131)$$

Equations (2.124 - 2.131) with equation (2.120) completely specify the optimum solution. The following algorithm is used to solve these optimality equations.

2:3.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoir on each river (n_j), the expected value for the natural inflows y_{ij}^k, the initial storage x_{ij}^0 and the cost of energy on each river (c_j^k).

Step 1 Assume initial guess for u(k) such that

$$\underline{u}(k) \leq u^i(k) \leq \overline{u}(k);\ i\text{=iteration number, } i=0$$

Step 2 Assume first that s(k) is equal to zero. Solve equation (2.121) forward in stages with x(0) given

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$\underline{x} \leq x(k) < \overline{x}$

go to Step 4, otherwise put x(k) to its limits and go to Step 10

Step 4 Calculate the new discharge, u(k), from the following

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$\underline{u}(k) < u(k) < \overline{u}(k)$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill s(k) at month k from the following equation

$$E[s(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k))-\bar{u}(k)]$$

If $s(k) < 0$, put $s(k) = 0$

Step 7 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k)-Ms(k))]$$

Step 8 Solve again equation (2.121) forward in stages with $x(0)$ given and using the value of $\hat{s}(k)$ calculated in Step 6

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality

$\underline{x} \leq x(k) \leq \bar{x}$

go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4

Step 10 With $\mu(k)=0$, solve equation (2.122) backward in stages with equation (2.120) as a terminal condition

Step 11 Calculate Kuhn-Tucker multipliers for $u(k)$, $\psi(k)$, from the following equation

$$E[\psi(k)] = E[M^T B(k)u(k)-M^T\lambda(k-1)-A(k)-B(k)x(k-1)]$$

If $u(k)$ satisfies the inequality; $\underline{u}(k) \leq u(k) \leq \bar{u}(k)$ put $\psi(k)=0$

The above equation for $\psi(k)$ is obtained by multiplying equation (2.122) by M^T, and subtracting the resulting equation from equation (2.123)

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)] = E[u^i(k)+\alpha\Delta u^i(k)] \quad ; \; i = \text{iteration number}$$

where

$$E[\Delta u(k)] = E[A(k)+M^T\lambda(k)+B(k)x(k-1)+\psi(k)]$$

and α is a positive scalar which is chosen with consideration given to such factors as convergence.

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality; $\underline{u}(k) < u^{i+1}(k) < \overline{u}(k)$ go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)] = E[B(k)u(k) - [M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)]$$

Step 15 Determine the Kuhn-Tucker multiplier for $x(k)$, $\mu(k)$, from the following

$$E[\mu(k)] = E[-\lambda(k)-[M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)]$$

If $x(k)$ satisfies the inequality

$\underline{x} < x(k) < \overline{x}$

put $\mu(k)=0$

Step 16 Determine a new state iterate from the approximation

$$E[x^{i+1}(k)] = E[x^i(k)+\alpha\Delta x^i(k)]$$

where

$$E[\Delta x(k)] = E[\lambda(k)-\lambda(k-1)+\mu(k)+B(k)u(k)]$$

Step 17 Repeat the calculate starting from Step 3. Continue until the state $x(k)$ and the control $u(k)$ do not change significantly from iteration to iteration and the cost function J in equation (2.62) is a maximum

2.3.5 Practical Example

The algorithm of the last section has been used to determine the monthly optimal operation during one year for the example mentioned in section (2.26), but with the following arrangement:

(1) On river 1, reservoirs number 1,2,3 and 4 are connected in series ($n_1=4$)

(2) On river 2, reservoirs number 1,2 and 3 are connected in series ($n_2=3$)

(3) On river 3, reservoirs number 1 and 2 are connected in series ($n_3=2$)

The natural inflows to the sites are given in Table (2.12). In Tables (2.13-2.15), we give the monthly release from each reservoir and the profits realized for the optimal global-feedback solution; we began with

$$x_1(0)=[6688.5,\ 557.9,\ 48.9,\ 3347.4]^T \quad Mm^3$$

$$x_2(0)=[6688.5,\ 557.9,\ 48.9]^T \quad Mm^3$$

and

$$x_3(0)=[6688.5,\ 557.9]^T \quad Mm^3$$

It is assumed that no correlation exists between flows of independent rivers and the times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason those times are not taken into account. Transmission line losses are also neglected.

The processing time required to determine the optimal monthly operating policy for a period of a year for the system just described was 2.9 sec. in CPU units.

Table 2.12: Monthly inflows to the reservoirs

Month	River number 1				River number 2			River number 3	
k	y_{11}^k Mm^3	y_{21}^k Mm^3	y_{31}^k Mm^3	y_{41}^k Mm^3	y_{12}^k Mm^3	y_{22}^k Mm^3	y_{32}^k Mm^3	y_{13}^k Mm^3	y_{23}^k Mm^3
1	828	380	161	1798	828	380	161	828	380
2	829	331	82	1201	829	331	82	829	331
3	578	224	41	810	578	224	41	578	224
4	394	146	32	486	394	146	32	394	146
5	265	95	18	302	265	95	18	265	95
6	233	82	14	258	233	82	14	233	82
7	193	68	6	194	193	68	6	193	68
8	219	127	279	1485	219	127	279	219	127
9	1101	614	181	3239	1101	614	181	1101	614
10	1887	781	205	2560	1887	781	205	1887	781
11	1150	491	146	1583	1150	491	146	1150	491
12	824	363	132	1454	363	132	132	824	363

Table 2.13: Optimal monthly releases from the reservoirs and the profits realized on the first river

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{31}^k Mm^3	u_{41}^k Mm^3	Profits $
1	0	368	528	2253	990,816
2	1037	1418	1499	2700	1,828,432
3	853	1077	1118	1928	1,473,793
4	1071	1465	1547	2244	2,300,938
5	968	1130	1098	2855	2,656,644
6	1071	1336	1400	3161	2,785,338
7	1037	1127	1133	1580	1,475,190
8	0	0	229	1310	609,669
9	0	446	627	2326	1,051,100
10	0	506	711	2330	979,250
11	0	491	637	2117	873,319
12	0	363	495	1794	788,087
	Value of water remaining in the reservoir at the end of the year				10,414,598
	Total profits (benefits)				28,227,174

Table 2.14: Optimal monthly releases from the reservoirs and the profits realized on the second river

Month k	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	u_{32}^{k} Mm^3	Profits $
1	0	368	528	157,061
2	1037	1393	1475	624,632
3	925	1149	1190	565,478
4	1071	1465	1547	915,163
5	968	1230	1318	863,611
6	1071	1174	1188	736,193
7	1037	1138	1144	640,632
8	0	94	324	89,924
9	0	181	362	100,008
10	0	682	887	257,640
11	0	491	637	177,949
12	0	363	495	146,793
	Value of water remaining in the reservoir at the end of the year			4,524,836
	Total benefits			9,799,920

Table 2.15: Optimal monthly releases from the reservoirs and the profits realized on the third river

Month k	u_{13}^k Mm^3	u_{23}^k Mm^3	Profits $
1	0	368	67,956
2	1037	1382	325,283
3	1028	1252	327,889
4	1071	1396	458,666
5	968	1328	469,001
6	1071	1270	422,918
7	1037	1105	331,831
8	0	83	18,906
9	0	560	107,706
10	82	826	141,206
11	0	407	66,280
12	0	202	35,923
		Value of water remaining in the reservoir at the end of the year	2,336,758
		Total profits	5,110,333

Table 2.16: Optimal total benefits from the system

River	1	2	3
Profits $	28,227,147	9,779,920	5,110,333
Total Profits	43,117,400		

2.3.6 Discussion

We have presented in this section an effecient and new approach to solve the optimal long-term operating problem for a parallel multireservoir power system. The problem is formulated as a minimum norm problem using functional analysis. In formulating the problem as a minimum norm problem, we assume that the water conversion factor (MWh/Mm^3) assigned for each hydro plant is a constant and the storage-elevation curve for each reservoir is a linear curve.

We applied our algorithm to a practical system consisting of three rivers; the first river has four series reservoirs, the second river has three series reservoirs and the third river has two series reservoirs. The computing time to get the optimal solution for a period of a year was 2.9 sec. in CPU units, which is very small compared to other techniques. (A. Turgeon 150 min. in CPU units for a system of six complex reservoirs, using the aggregation/decomposition approach).

CHAPTER III

MULTIRESERVOIR POWER SYSTEMS WITH A VARIABLE WATER CONVERSION FACTOR AND LINEAR STORAGE-ELEVATION CURVES

3.1 Background

In Chapter 2, the long-term optimal operating problem of multireservoir power systems connected either in series on a river or in parallel on a multiriver system is considered. In formulating the problem as a minimum norm problem, we assumed a constant water conversion factor assigned for each hydro-plant and this conversion factor is equal to the average number of megawatt-hours produced in a month by an outflow of one Mm^3(1). In hydro electric power systems in which the water heads vary by a considerable amount this assumption is not correct, which is the case in this chapter.

This chapter is devoted to the solution of the long-term optimal operating problem of multireservoir power systems connected either in series on a river or in parallel on a multiriver system. The system considered here is characterized by having a variable water conversion factor (variable head), and a linear storage-elevation curve.

A wide variety of techniques for solving such a problem have been reported in the literature. The model in Refs. 3 and 4 was obtained by applying curve fitting to past generation data without any assurance that an optimal, even a good, operating policy has been followed. The other models suppose that: 1) the production of a hydroplant is a constant time the discharge; 2) spillage will not occur or will occur at every site when it does; 3) the reservoirs will not become empty or will all become empty simultaneously. All these assumptions have been used to obtain exact composite models of the valleys.

In addition to being inexact, these composite models do not give any indication of how the total production of a valley should be divided among its installations. On the other hand, the assumption that the production of a hydroplant is a constant time the discharge (constant water conversion factor) has no meaning especially for the hydroplant which have a considerable water head variation. Also, the second assumption about spillage which may occur from one of the reservoirs, while it does not occur from the other is not valid. Finally, regarding the third assumption, may be one of the reservoirs becomes empty while, at the same time the other reservoirs are not empty. For all these reasons, the outflow of potential energy from the composite reservoir will not be equal to the actual number of MWh generated. Therefore, a generation function must be constructed that relates the actual generation to the MWh outflow and the energy content of the composite reservoir (1).

Stochastic Dynamic Programming is, in principle, the optimization technique most studied to solve this kind of problem. Computing time and storage requirements however, make it impractical for systems with more than three reservoirs, since they grow exponentially with the number of state variables.

3.2 A Series Multireservoir Power System* (25)

3.2.1 Problem Formulation

3.2.1.1 The System Under Study

The system under consideration consists of n reservoirs in series on a river. This system has a considerable water head variation (variable conversion factor). The reservoirs are characterized by having a linear storage-elevation. We will number the installations from upstream to downstream, and denote by the following (Figure 3.1):

y_i^k A random variable representing the natural inflow to site in in period k in Mm^3 ($1Mm^3=10^6m^3$). These are statistically independent random variables with normal distribution

h_i^k The net head of the reservoir i at the end of period k in meters. We assume that the tailwater elevation is constant (the reference elevation used is constant);

$$\underline{h}_i \leq h_i^k \leq \overline{h}_i$$

where $\underline{h}_i$ and $\overline{h}_i$ are the minimum and maximum heads respectively

---

* A version of this Section has been accepted for publication in the Journal of Optimization Theory and Applications, May 1985 (Ref. 25).

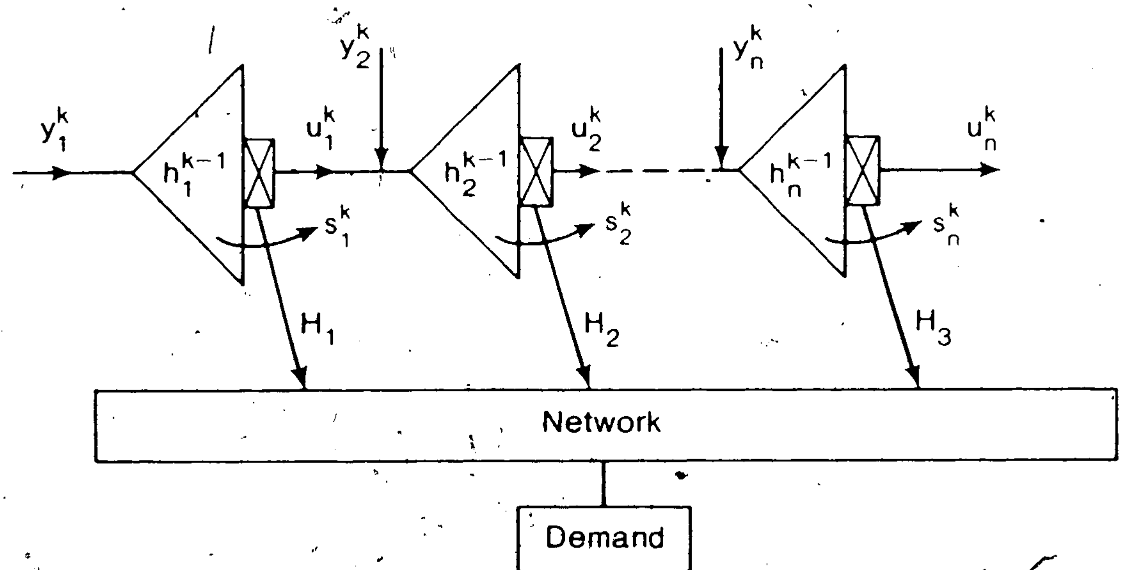

Figure 3.1 A Series Multireservoir Power System with a Variable Conversion Factor

u_i^k The discharge from reservoir i during the period k in Mm^3;

$$\underline{u}_i^k \le u_i^k \le \bar{u}_i^k \; ; \; u_0^k = 0$$

where $\underline{u}_i^k$ and $\bar{u}_i^k$ are the minimum and maximum discharges. If $u_i^k > \bar{u}_i^k$, then $u_i^k - \bar{u}_i^k$ Mm^3 is discharged through the spillways.

$H_i(u_i^k, h_i^{k-1})$ The generation of plant i in month k in MWh. It is a function of the discharge and the net head

$V_i(h_i^K)$ Value in dollars of water remaining in reservoir i at the end of the last period studied

s_i^k The spill from reservoir i during a period k in Mm^3;

$$s_i^k \ge 0 \; ; \; s_0^k = 0$$

c^k Value in dollars of one MWh produced anywhere on the river

i Subscript denoting the installation number; i=1,...,...,n

n Total number of installation

k Superscript denoting the month; k=1,...,...,K

3.2.1.2 Statement of the Problem

The long-term optimal operating problem aims to find the release u_i^k; i=1,...,...,n, k=1,...,...,K from each reservoir that maximizes the total expected benefits (benefits from the generation and benefits from the amount of water left in storage at the end of the planning horizon). In mathematical terms, the problem for the power system in Figure 3.1 is to determine the discharge u_i^k that maximizes

$$J=E[\sum_{i=1}^{n} V_i(h_i^K)+\sum_{i=1}^{n}\sum_{k=1}^{K} c^k H_i(u_i^k,h_i^{k-1})] \text{ in } \$ \quad (3.1)$$

Subject to satisfying the following constraints.

(1) The reservoir dynamics may be described by the following discrete continuity equation

$$x_i^k=x_i^{k-1}+y_i^k+u_{(i-1)}^k-u_i^k+s_{i-1}^k-s_i^k \quad (3.2)$$

(2) The storage-elevation curve may be adequately described by the following linear equation

$$x_i^k = a_i + b_i h_i^k \quad (3.3)$$

In the above equation, we assume that the tailrace water elevation is constant

(3) The operational reservoir constraints are

$$\underline{h}_i \leq h_i^k \leq \overline{h}_i \quad (3.4)$$

$$\underline{u}_i^k \leq u_i^k \leq \overline{u}_i^k \quad (3.5)$$

Substituting from equation (3.3) into equation (3.2), one obtains the following continuity equation

$$h_i^k = h_i^{k-1} + 1/b_i\, y_i^k + 1/b_i\, u_{i-1}^k - 1/b_i\, u_i^k + 1/b_i\, s_{i-1}^k - 1/b_i\, s_i^k \tag{3.6}$$

where a_i and b_i are constants for each reservoir, these were obtained by least square curve fitting to typical plant data available;

and

$$s_i^k = \begin{cases} (b_i h_i^{k-1} + u_{i-1}^k + y_i^k + s_{i-1}^k - b_i h_i^k) - u_i^k; \text{ If } (b_i h_i^{k-1} + u_{i-1}^k + y_i^k + s_{i-1}^k - b_i h_i^k) \\ > \bar{u}_i^k \text{ and } h_i^k > \bar{h}_i \\ 0, \text{ otherwise} \end{cases} \tag{3.7}$$

and for given $V_i(h_i^K)$, $H_i(u_i^k, h_i^{k-1})$, h_i^0 and the parameter c^k in \$/MWh. The symbol E in equation (3.1) stands for the expected value. The expectation in equation (3.1) is taken over the random variable [illegible]^k.

3.3 Modelling of the System

The water conversion factor of at-a-site as a function of the head is given by the following equation

$$C_i^k = d_i h_i^k \qquad i=1,\ldots,\ldots,n \tag{3.8}$$

where

$$d_i = 2.723\, \eta_i \qquad i=1,\ldots,\ldots,n$$

η_i = The efficiency of the hydropower plant. In this study we assume a constant efficiency for each hydropower plant.

To model the amount of water left in storage at the end of the planning period, we multiply this amount of water by the water conversion factor of at-a-site and downstream reservoirs; we may choose the following for the function $V_i(h_i^K)$

$$V_i(h_i^K) = \sum_{j=i}^{n} d_j h_j^K (a_i + b_i h_i^K); \quad i=1,\ldots,\ldots,n \text{ in } \$ \tag{3.9}$$

The generation of a hydroelectric power plant is a nonlinear function of the discharge and the net head. We may choose the following for the function $H_i(u_i^k, h_i^{k-1})$

$$H_i(u_i^k, h_i^{k-1}) = d_i u_i^k h_i^{k-1} \text{ MWh} \tag{3.10}$$

Now the cost function in equation (3.1) becomes

$$J = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} d_j h_j^K (a_i + b_i h_i^K) + \sum_{i=1}^{n} \sum_{k=1}^{K} c^k d_i h_i^{k-1} u_{io}^k \right] \$ \tag{3.11}$$

Subject to satisfying the following constraints

$$h_i^k = h_i^{k-1} + 1/b_i \, y_i^k + 1/b_i \, u_{i-1}^k - 1/b_i \, u_i^k + 1/b_i \, s_{i-1}^k - 1/b_i \, s_i^k \tag{3.12}$$

$$\underline{h}_i \leq h_i^k \leq \bar{h}_i \tag{3.13}$$

$$\underline{u}_i^k \leq u_i^k \leq \bar{u}_i^k \tag{3.14}$$

The problem now is that of maximizing (3.11) subject to satisfying constraints (3.12-3.14).

3.2.2 A Minimum Norm Formulation

The augmented cost functional is obtained by adjoining equation (3.12) to the cost function via Lagrange's multipliers, and the inequality constraints (3.13) and (3.14) via Kuhn-Tucker multipliers. One thus obtains

$$\tilde{J}=E[\sum_{i=1}^{n}\sum_{j=i}^{n} d_j h_j^K(a_i+b_i h_i^K)+\sum_{i=1}^{n}\sum_{k=1}^{K}\{c^k d_i h_i^{k-1} u_i^k+\lambda_i^k(-h_i^k+h_i^{k-1}$$

$$+1/b_i y_i^k+1/b_i\ u_{i-1}^k+1/b_i\ s_{i-1}^k-1/b_i u_i^k-1/b_i\ s_i^k)$$

$$+e_i^k(\underline{h}_i-h_i^k)+e_i^{1k}(h_i^k-\bar{h}_i)$$

$$+f_i^k(\underline{u}_i^k-u_i^k)+f_i^{1k}(u_i^k-\bar{u}_i^k)\}] \tag{3.15}$$

where λ_i^k is Lagrange's multiplier and e_i^k, e_i^{1k}, f_i^k and f_i^{1k} are Kuhn-Tucker multipliers, these are equal to zero, if the constraints are not violated and greater than zero if the constraints are violated (32).

Now define the following n column vectors such that

$$A = \text{col.}\ (A_1, \ldots, \ldots, \ldots, A_n) \tag{3.16}$$

where

$$A_i = d_i \sum_{j=1}^{i} a_j \tag{3.17}$$

$$h(k) = \text{col.}\ (h_1^{\,k}, \ldots, \ldots, h_n^{\,k}) \tag{3.18}$$

$$u(k) = \text{col.}\ (u_1^{\,k}, \ldots, \ldots, u_n^{\,k}) \tag{3.19}$$

$$y(k) = \text{col.}\ (y_1^{\,k}, \ldots, \ldots, y_n^{\,k}) \tag{3.20}$$

$$s(k) = \text{col.}\ (s_1^{\,k}, \ldots, \ldots, s_n^{\,k}) \tag{3.21}$$

$$\lambda(k) = \text{col.}\ (\lambda_1^{\,k}, \ldots, \ldots, \lambda_n^{\,k}) \tag{3.22}$$

$$\mu_i^{\,k} = e_i^{\,1k} - e_i^{\,k} \tag{3.23}$$

$$\mu(k) = \text{col.}\ (\mu_1^{\,k}, \ldots, \ldots, \mu_n^{\,k}) \tag{3.24}$$

$$\psi_i^{\,k} = f_i^{\,1k} - f_i^{\,k} \tag{3.25}$$

$$\psi(k) = \text{col.}\ (\psi_1^{\,k}, \ldots, \ldots, \psi_n^{\,k}) \tag{3.26}$$

Furthermore, define the following matrices

G is nxn matrix whose elements are given by:

$$\left.\begin{array}{ll}\text{(i)} & g_{ii} = b_i d_i \quad i=1,\ldots,\ldots,n \\ \text{(ii)} & g_{j(j+1)} = g_{(j+1)j} = 1/2\ b_j\ d_{(j+1)} \quad j=1,\ldots,n-1\end{array}\right\} \tag{3.27}$$

M is nxn lower triangular matrix whose elements are given by

$$\left.\begin{array}{ll}\text{(i)} & m_{ij} = -1/b_i \ ; \ i=1,\ldots,\ldots,n \\ \text{(ii)} & m_{(j+1)j} = 1/b_{j+1} ; \ j=1,\ldots,\ldots,n-1\end{array}\right\} \tag{3.28}$$

$$B = \text{diag.}\ (1/b_1,\ldots,\ldots,1/b_n) \tag{3.29}$$

$$D(k) = \text{diag.}\ (c^k d_1,\ldots,\ldots,c^k d_n) \tag{3.30}$$

Using all the above definitions, the augmented cost functional in equation (3.15) becomes

$$\tilde{J}=E[A^T h(K)+h^T(K)Gh(K)+\sum_{k=1}^{K}\{u^T(k)D(k)h(k-1)$$
$$+\lambda^T(k)(-h(k)+h(k-1)+By(k)+Mu(k)+Ms(k))$$
$$+\mu^T(k)h(k)+\psi^T(k)u(k)\}] \tag{3.31}$$

Note that constant terms are dropped in the above equation.

Employing the discrete version of integration by parts, substituting for h(k) and dropping terms explicitly independent of $h(k-1)$ and $u(k)$ (33), one thus obtains

$$\tilde{J} = E[h^T(K)Gh(K) + (A-\lambda(K))^T h(K) + \lambda^T(0)h(0)$$

$$+ \sum_{k=1}^{K} \{1/2 u^T(k)D(k)h(k-1) + 1/2\, h^T(k-1)D(k)u(k)$$

$$+ (\lambda(k) - \lambda(k-1) + \mu(k))^T h(k-1)$$

$$+ (M^T\lambda(k) + M^T\mu(k) + \nu(k))^T u(k))\}] \qquad (3.32)$$

It will be noticed that $\tilde{J}$ in equation (3.32) is composed of a boundary part and a discrete integral part, which are independent of each other. To maximize $\tilde{J}$ in equation (3.32), one can maximize the boundary and the discrete integral parts separately. If one defines the following vector such that

$$N(K) = A - \lambda(K) \qquad (3.33)$$

$$W(K) = G^{-1} N(K) \qquad (3.34)$$

$$X^T(k) = [h^T(k-1) \,;\, u^T(k)] \qquad (3.35)$$

$$L(k) = \begin{bmatrix} 0 & 1/2D(k) \\ 1/2D(k) & 0 \end{bmatrix} \tag{3.36}$$

and

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\mu(k))^T,(M^T\lambda(k)+M^T\mu(k)+\nu(k))^T] \tag{3.37}$$

Then the boundary part can be written as

$$J_1=E[(h(K)+ 1/2\ W(K))^T G(h(K)+ 1/2\ W(K))- 1/4\ W^T(K)GW(K)$$

$$+\lambda^T(0)h(0)] \tag{3.38}$$

Since, it is desired to maximize J_1 with respect to h(K); the problem is equivalent to

$$J_1'=E[(h(K)+ 1/2\ W(K))^T G(h(K)+ 1/2\ W(K))] \tag{3.39}$$

because W(K) is independent of h(K) and h(0) is constant. Equation (3.39) defines a norm, hence it can be written as:

$$\underset{h(K)}{\text{Max.}}J_1 = \underset{h(K)}{\text{Max.}}E[\,||h(K) + 1/2\ W(K)||\,]_G \tag{3.40}$$

Also, the discrete integral part in equation (3.32) can be written as

$$\max_{X(k)} J_2 = \max_{X(k)} E\left[\sum_{k=1}^{K} \left[\left(X^T(k)L(k)X(k) + R^T(k)X(k) \right) \right] \right] \tag{3.41}$$

If one defines the following vector such that

$$V(k) = L^{-1}(k)R(k) \tag{3.42}$$

then the discrete integral part in equation (3.41) can be written in the following form by a process similar to completing the squares as

$$\max_{X(k)} J_2 = \max_{X(k)} E\left[\sum_{k=1}^{K} \left[(X(k) + 1/2\ V(k))^T L(k)(X(k) + 1/2\ V(k)) - 1/4\ V^T(k)L(k)V(k) \right] \right] \tag{3.43}$$

Since it is desired to maximize J_2 with respect to $X(k)$, the problem is equivalent to

$$\max_{X(k)} J_2 = \max_{X(k)} E\left[\sum_{k=1}^{K} (X(k) + 1/2 V(k))^T L(k)(X(k) + 1/2\ V(k))\right] \tag{3.44}$$

because $V(k)$ is independent of $X(k)$. Equation (3.44) defines a norm, hence equation (3.44) can be written as

$$\underset{X(k)}{\text{Max.}}\ J_2 = \underset{X(k)}{\text{Max.}}\ E[||X(k) + 1/2\ V(k)||]_{L(k)} \tag{3.45}$$

3.2.3 The Optimal Solution

There is exactly one optimal solution to the problem formulated in equations (3.40) and (3.45). The optimal solution for the problem formulated in equation (3.40) is obtained, when the norm of that equation is equal to zero

$$E[h(K) + 1/2\ W(K)] = [0] \tag{3.46}$$

Substituting from equations (3.33) and (3.34) into equation (3.46), one finds that the optimal boundary condition is given by

$$E[\lambda(K)] = E[A + 2Gh(K)] \tag{3.47}$$

Also, the optimal solution for the problem formulated in equation (3.45) is obtained when the norm of that equation is equal to zero.

$$E[X(k) + 1/2\ V(k)] = [0] \tag{3.48}$$

Substituting from equation (3.42) into equation (3.48), one finds that the optimal condition for the discrete integral part is given by

$$E[R(k) + 2L(k)\ X(k)] = [0] \qquad (3.49)$$

Writing equation (3.49) explicitly by substituting from equations (3.35-3.37) into that equation and adding equation (3.12), one obtains the following set of optimal equations:

$$E[-h(k)+h(k-1)+By(k)+Mu(k)+Ms(k)] = [0] \qquad (3.50)$$

$$E[\lambda(k)-\lambda(k-1)+\mu(k)+D(k)u(k)] = [0] \qquad (3.51)$$

$$E[M^T\lambda(k)+M^T\mu(k)+\psi(k)+D(k)h(k-1)] = [0] \qquad (3.52)$$

We can state the optimal equations (3.47) and (3.50-3.52) in component form as

$$E[\lambda_i^K-2b_i d_i h_i^K-b_i d_{(i+1)} h_{(i+1)}^K-\sum_{j=1}^{i} d_i a_j]=0;\ i=1,\dots,n \qquad (3.53)$$

$$E[-h_i^k+h_i^{k-1}+1/b_i(y_i^k+u_{i-1}^k-u_i^k+s_{i-1}^k-s_i^k)]=0;$$

$$i=1,\dots,\dots,n,k=1,\dots,\dots,K \qquad (3.54)$$

$$E[\lambda_i^k-\lambda_i^{k-1}+(e_i^{1k}-e_i^k)+c^k d_i u_i^k]=0;\ i=1,\dots,\dots,n;$$

$$k=1,\dots,\dots,K \qquad (3.55)$$

$$E[-1/b_{i+1}\ _{i+1}^{k} - 1/b_i\ _i^{k} + 1/b_{i+1}(e_{i+1}^{1k} - e_{i+1}^{k})$$

$$- 1/b_i(e_i^{1k} - e_i^{k}) + (f_i^{1k} - f_i^{k}) + c^k d_i h_i^{k-1}];\ i=1,\ldots,\ldots,n;$$

$$k=1,\ldots,\ldots,K \qquad (3.56)$$

Besides the above equations, one has the following limits on the variables

$$\left.\begin{array}{l} \text{If } h_i^k < \underline{h}_i, \text{ then we put } h_i^k = \underline{h}_i \\ \text{If } h_i^k > \overline{h}_i, \text{ then we put } h_i^k = \overline{h}_i \\ \text{If } u_i^k < \underline{u}_i^k, \text{then we put } u_i^k = \underline{u}_i^k \\ \text{If } u_i^k > \overline{u}_i^k, \text{then we put } u_i^k = \overline{u}_i^k \end{array}\right\} \qquad (3.57)$$

One also has the following Kuhn-Tucker exclusion equations

$$e_i^k(\underline{h}_i - h_i^k) = 0 \qquad (3.58)$$

$$e_i^{1k}(h_i^k - \overline{h}_i) = 0 \qquad (3.59)$$

$$f_i^k(\underline{u}_i^k - u_i^k) = 0 \qquad (3.60)$$

$$f_i^{1k}(u_i^k - \overline{u}_i^k) = 0 \qquad (3.61)$$

Equations (3.53-3.61) or equation (3.47) with equations (3.50-3.52) and equations (3.57-3.61) completely specify the optimal solution. the following algorithm is used to solve these equations.

3.2.4 Algorithm for Solution

Assume given: the number of reservoirs (n), the expected value for the natural inflows to each site y_i^k, the initial head h_i^0, and the cost of energy c^k in \$/MWh.

Step 1 Assume initial guess for the variable $u^i(k)$, such that

$$\underline{u}(k) < u^i(k) < \overline{u}(k); \ i = \text{iteration counter}; \ i=0$$

Step 2 Assume first that s(k) is equal to zero. Solve equation (3.50) forward in stages with h(0) given

Step 3 Check the limits on h(k). If h(k) satisfies the inequality

$$\underline{h} < h(k) < \overline{h}$$

go to Step 10, otherwise put h(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(h(k)-h(k-1)-By(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \overline{u}(k)$$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill at the month k from the following equation

$$E[s(k)]=E[[M]^{-1}(h(k)-h(k-1)-By(k))-\bar{u}(k)]$$

If s(k) is less than zero, put s(k) equal to zero

Step 7. Calculate the discharge from the following equation

$$E[u(k)]=E[[M]^{-1}(h(k)-h(k-1)-By(k)-Ms(k))]$$

Step 8 Solve again equation (3.50) forward in stages with h(0) given

Step 9 Check the limits on h(k). If h(k) satisfies the inequality

$\underline{h} < h(k) < h$

go to Step 10, otherwise put h(k) to its limits and go to Step 4

Step 10 With μ(k)=0, solve equation (3.51) backward in stages with equation (3.47) as the terminal conditon

Step 11 Calculate the Kuhn-Tucker multiplier for u(k), ψ(k), from the following equation

$$E[\psi(k)]=E[M^T D(k)u(k)-M^T\lambda(k-1)-D(k)h(k-1)]$$

If u(k) satisfies the inequality

$\underline{u}(k) < u(k) < \bar{u}(k)$

put ψ(k) equal to zero

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)] = E[u^{i}(k)+\alpha Du^{i}(k)]$$

where

$$E[Du^{i}(k)] = E[M^{T}\lambda(k)+\psi(k)+D(k)h(k-1)]$$

and α is a positive scalar which is chosen with consideration given to such factors as convergence

Step 13 Check the limits of $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \bar{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)] = E[D(k)u(k)-[M^{T}]^{-1}D(k)h(k-1)]$$

Step 15 Calculate Kuhn-Tucker multipliers for h(k), $\mu(k)$, from the following equation

$$E[\mu(k)] = E[-\lambda(k)-[M^{T}]^{-1}D(k)h(k-1)]$$

If h(k) satisfies the inequality

$\underline{h} < h(k) < h$

put $\mu(k)=0$

Step 16 Determine a new state iterate from the approximation

$$E[h^{i+1}(k)] = [h^i(k)+\alpha Dh^i(k)]$$

where

$$E[Dh^i(k)] = [\lambda(k)-\lambda(k-1)+\mu(k)+D(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state h(k) and the control u(k) do not change significantly from iteration to iteration and the cost function J in equation (3.11) is a maximum

3.2.5 Practical Example

The algorithm of the last section has been used to determine the optimal monthly operation of a two series reservoir power system for a period of a year. The characteristics of the installations are given in Table 3.1.

If we let d^k denote the number of days in month k, then the maximum and minimum releases are given by

$$\left.\begin{aligned} \bar{u}_i^{\,k} &= 0.0864d^k(\text{maximum effective discharge in } m^3/\text{sec}) \\ \underline{u}_i^{\,k} &= 0.0864d^k(\text{minimum effective discharge in } m^3/\text{sec} \end{aligned}\right\} \quad (3.62)$$

where the maximum and minimum effective discharges are given in Table 3.1.

The MWh generated at each power house is given by

$$H_1(u_1^k, h_1^{k-1}) = d_1 u_1^k h_1^{k-1} \quad \text{MWh} \tag{3.63}$$

$$H_2(u_2^k, h_2^{k-1}) = d_2 u_2^k h_2^{k-1} \quad \text{MWh} \tag{3.64}$$

where d_1 and d_2 are given in Table 3.1 (we assume a constant efficiency for each reservoir).

The expected natural inflows to the sites in the year of high flow which we call year 1 and the cost of energy are given in Table (3.2). In Table (3.3), we give the optimal monthly release from each reservoir and the profits realized in year 1 for the optimal global-feedback solution. In Table (3.4) and Table (3.5) we give the optimal head and the spill for each reservoir in year 1.

We have simulated the monthly operation for widely different water conditions. In Tables (3.6-3.9), we have reported the results for year 2, which is the year of the low flow. We began both years with

$$h(0) = [184.4 \quad 128.9]^T \text{ meters}$$

In Table (3.6) we give the expected natural inflows to the sites in year 2. In Table (3.7) we give the monthly release from each reservoir and the profits realized in year 2. In Tables (3.8) and (3.9) we give the optimal head and the spill for each reservoir for the same year.

The computing time required to determine the optimal monthly operating policy for a period of a year for the system just described was 0.58 sec. in CPU units.

Table 3.1: The characteristics of the installations

Site name	Capacity of the reservoir Mm^3	Maximum effective discharge (m^3/sec)	Minimum effective discharge (m^3/sec)	Maximum head m	Minimum head m	Reservoir's constants		
						a_i	b_i	d_i
R_1	24763	1119	85	184.4	137.2	-34360.83	315.96	2.451
R_2	5304	1583	85	128.9	113.7	-4287.2	72.11	2.451

Table 3.2: The expected monthly inflows to the sites and the cost of energy in year 1

Month k	y_1^k Mm^3	y_2^k Mm^3	c^k \$/MWh
1	892	689	1.4
2	922	454	1.4
3	462	319	1.4
4	310	176	0.8
5	280	103	0.8
6	253	116	0.8
7	698	190	0.8
8	2712	1227	0.8
9	3417	1324	0.8
10	5388	2136	0.8
11	4566	1995	1.1
12	2631	1097	1.1

Table 3.3: Optimal monthly releases from the reservoirs in year 1 and the profits realized

Month k	u_1^k Mm^3	u_2^k Mm^3	Profits $
1	1515	2735	2,190,667
2	1487	1617	1,601,528
3	1021	1125	1,116,710
4	475	1120	448,575
5	399	967	370,524
6	462	700	319,871
7	2327	2562	1,395,888
8	2997	4240	1,956,095
9	2900	4103	1,891,709
10	2997	4240	1,978,378
11	2997	4240	2,922,978
12	2900	3998	2,831,306
	Value of water remaining in the reservoirs at the end of the year		19,679,840
	Total profits		38,704,069

Table 3.4: The optimal reservoir head in year 1

Month k	h_1^k m	h_2^k m
1	182.4	120.9
2	180.6	125.4
3	178.9	128.4
4	178.4	121.9
5	178.0	115.4
6	177.3	113.7
7	172.0	113.7
8	171.1	114.2
9	172.7	115.9
10	180.3	128.3
11	184.4	128.9
12	183.6	128.9

Table 3.5: Optimal spill from each reservoir in year 1

Month k	s_1^k Mm^3	s_2^k Mm^3
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	0.0	0.0
5	0.0	0.0
6	0.0	0.0
7	0.0	0.0
8	0.0	0.0
9	0.0	0.0
10	0.0	0.0
11	275.51	980.82
12	0.0	0.0

Table 3.6: The expected monthly inflows to the sites in year 2

Month k	y_1^k Mm^3	y_2^k Mm^3
1	847	358
2	371	344
3	234	261
4	227	176
5	205	113
6	266	155
7	456	204
8	2004	1123
9	3223	1508
10	3402	1312
11	2585	677
12	1305	311

Table 3.7: Optimal monthly releases from the reservoirs in year 2 and the profits realized

Month k	u_1^k Mm^3	u_2^k Mm^3	Profits $
1	847	2301	1,553,883
2	1123	969	1,088,689
3	1058	1143	1,134,114
4	228	1020	326,051
5	206	376	156,776
6	228	382	165,261
7	1058	1135	625,636
8	1064	1611	735,535
9	2900	4071	1,999,555
10	2997	4240	2,138,921
11	2110	3171	2,142,324
12	2900	2826	2,383,476
	Value of the water remaining an the end of the year		18,434,656
	Total profits		32,884,877

Table 3.8: Optimal reservoir head in year 2

Month k	h_1^k m	h_2^k m
1	184.4	113.7
2	182.0	120.6
3	179.4	123.1
4	179.4	114.5
5	179.4	113.7
6	179.5	113.7
7	177.6	115.5
8	180.6	123.5
9	181.6	128.9
10	182.9	128.9
11	184.4	123.6
12	179.4	128.9

Table 3.9: Optimal spill from each reservoir in year 2

Month k	s_1^k Mm^3	s_2^k Mm^3
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0
9	0	0
10	0	69.3
11	0	0
12	0	0

3.2.6 Discussion

In this section the optimal long-term operating problem of a multireservoir power system connected in series on a river has been discussed. The water conversion factor variation is taken into account. In this section we assume that the storage-elevation curve is linear. We report the results obtained for an actual system in operation, this system consists of two reservoirs in series.

Expressing the storage as a linear function of the head may yield a large error in calculating the storage, which for some reservoirs is greater than the minimum natural inflow to the reservoir. For example, for the reservoir R_1 the storage at the maximum head, which is 184.4m, is 23902 Mm^3 using this linear function, but from Table 3.1, the reservoir storage corresponding to this head is 24763 Mm^3. The difference between the two storages is 861Mm^3, which is greater than the minimum natural inflow to the reservoir (the minimum inflow to the reservoir as given in Table 3.2 is 253 Mm^3 at k=6).

On the other hand, the modelling of the hydroplant generation as a function of the discharge and the head of the previous month, $H_1(u_1^k, h_1^{k-1})$, may cause an underestimation in production for rising water levels and overestimation for falling water levels.

3.3 A Multireservoir Power System in Parallel*

In section 3.2, the long-term optimal operating problem for a multireservoir power system connected in series on a river has been discussed. The system described there is characterized by having a variable water conversion factor, this variation is linear with the head, and a linear storage elevation curve. We reported the results obtained for a system consisting of two reservoirs in series on a river for widely different water conditions.

This section is devoted to solve the long-term optimal operating problem for multireservoir power systems connected in parallel. The systems described here are characterized by having a variable water conversion factor (linear variation with the storage), and a linear storage-elevation curve. In the formulation of the generating function, an average of begin and end-of-time step storage is used, to avoid underestimation of production for rising water levels and overestimation for falling water levels.

3.3.1 Problem Formulation

3.3.1.1 The System Under Study

The system under consideration consists of m independent rivers, with one or several reservoirs and power plants in series, and interconnection lines to the neighboring system through which energy may be exchanged. It is assumed that no correlation exists between

---

* A version of this section has been submitted to IEEE PES, Summer Meeting, 1986.

flows of independent rivers or of different periods of time (Figure 3.2). We will denote by:

y_{ij}^{k} A random variable representing the natural inflow to reservoir i of river j in period k in Mm^3 ($1Mm^3=10^6m^3$). These are statistically independent random variables with normal distribution

x_{ij}^{k} The storage of reservoir i on river j at the end of period k, in Mm^3;

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \overline{x}_{ij}$$

u_{ij}^{k} The effective discharge from reservoir i of river j in period k in Mm^3. This is the water released from the reservoir i to the allied powerplant to produce electricity;

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \overline{u}_{ij}^{k}$$

where $\underline{u}_{ij}^{k}$ and $\overline{u}_{ij}^{k}$ are the minimum and maximum discharges. If $u_{ij}^{k} > \overline{u}_{ij}^{k}$, then $u_{ij}^{k}-\overline{u}_{ij}^{k}$ Mm^3 is discharged through the spillways. Water is spilt when the reservoir is filled to capacity, and the inflow to the reservoir exceeds $\overline{u}_{ij}^{k}$

$G_{ij}(u_{ij}^{k}, 1/2(x_{ij}^{k}+x_{ij}^{k-1})$ The generation of plant i on river j in period k in MWh. It is a function of the discharge and the average storage between two successive months

$V_{ij}(x_{ij}^{K})$ Value in dollars of the water remaining in reservoir i of river j at the end of the planning period

c_{j}^{k} Value in dollars of one MWh generated anywhere on river j

s_{ij}^{k} The spill from reservoir i on river j during period k in Mm^3;

$$s_{ij}^{k} \geq 0$$

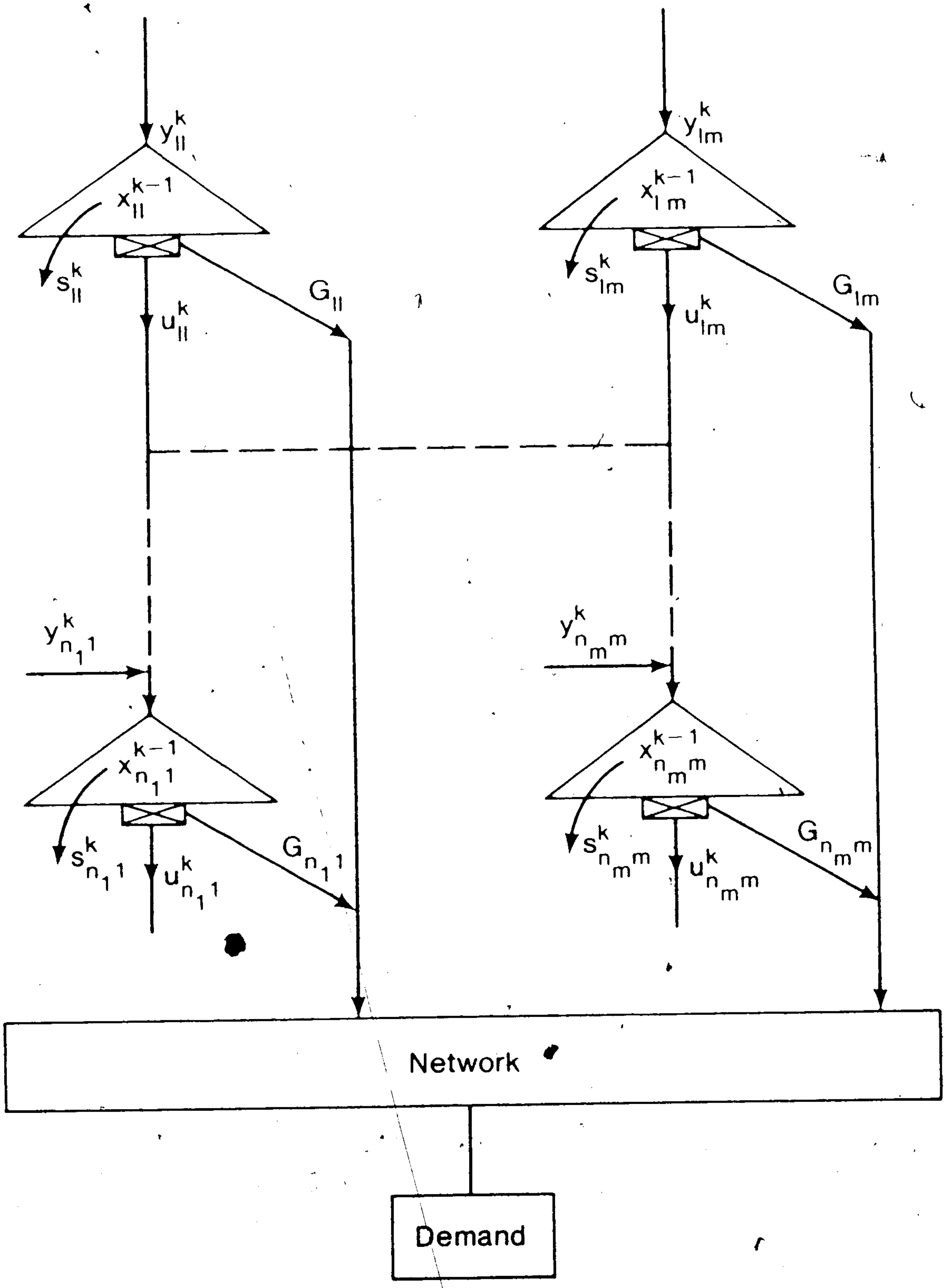

Figure 3.2 A parallel multiresevoir power system with a variable water conversion factor.

n_j The number of reservoir on river j

m Number of rivers

k Superscript denoting the period; k=1,...,...,K

3.3.1.2 Objective Function

The long-term stochastic optimization problem determines the discharge u_{ij}^k; $i=1,...,...,n_j$, $j=1,...,...m$, $k=1,...,...,K$ that maximizes the value of energy generated by a hydropower system over the planning period plus the expected future returns from water left in storage at the end of that period. In mathematical terms, the problem for the power system in Figure (3.2) is to determine the discharge u_{ij}^k that maximizes:

$$J=E[\sum_{j=1}^{m}\sum_{i=1}^{n} V_{ij}(x_{ij}^K)+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} c_j^k G_{ij}(u_{ij}^k,\ 1/2(x_{ij}^k+x_{ij}^{k-1}))] \quad \$ \tag{3.65}$$

Subject to satisfying the following constraints

(1) The reservoir dynamics may be adequately described by the following discrete continuity equation

$$x_{ij}^k=x_{ij}^{k-1}+y_{ij}^k+u_{(i-1)j}^k-u_{ij}^k+s_{(i-1)j}^k-s_{ij}^k \tag{3.66}$$

(2) Storage levels and discharge can be bounded above and below to allow for recreation and flood control objectives. These bounds are given by

$$\underline{x}_{ij} \le x_{ij}^{k} \le \bar{x}_{ij} \tag{3.67}$$

$$\underline{u}_{ij}^{k} \le u_{ij}^{k} \le u_{ij}^{k} \tag{3.68}$$

where

$$s_{ij}^{k} = \begin{cases} (x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-x_{ij}^{k})-\bar{u}_{ij}^{k}; \text{If } (x_{ij}^{k-1}+y_{ij}^{k} \\ \quad +u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-x_{ij}^{k}) \ge \bar{u}_{ij}^{k} \quad \text{and} \quad x_{ij}^{k} > \bar{x}_{ij} \\ 0, \text{ otherwise} \end{cases} \tag{3.69}$$

The initial storage x_{ij}^{0} and the expected value for the natural inflows into each stream during each month are assumed to be known.

<u>3.3.1.3 Modelling of the System</u>

To model the water left in storage at the end of the planning period, we multiply this amount of water by the water conversion factor of at-a-site and downstream reservoirs. In this section we assume that the water conversion factor at the end of the planning period has a linear relation with the storage at that period. We may choose the following for the function $V_{ij}(x_{ij}^{K})$

$$V_{ij}(x_{ij}^{K}) = \sum_{v=i}^{n_j} x_{ij}^{K}(a_{vj}+b_{vj}x_{vj}^{K}) \quad \$ \tag{3.70}$$

In the above equation; the term between brackets is the water conversion factor, MWh/Mm^3, and we assume that the cost of this energy is one dollar per MWh, (the average cost during the planning period), since no-one knows when this energy will be used in the future.

The generation of a hydroelectric plant is a nonlinear function of the water discharge u_{ij}^{k} and reservoir head, which itself is a function of the storage x_{ij}^{k}. To avoid underestimation of production for rising water levels and overestimation for falling water level, an average of begin and end-of time step storage is used. We may choose the following for the function $G_{ij}(u_{ij}^{k}, 1/2(x_{ij}^{k}+x_{ij}^{k-1}))$

$$G_{ij}(u_{ij}^{k},1/2(x_{ij}^{k}+x_{ij}^{k-1})) = a_{ij}u_{ij}^{k}+ 1/2\ b_{ij}(x_{ij}^{k}+x_{ij}^{k-1})u_{ij}^{k} \tag{3.71}$$

Substituting from equation (3.66) into equation (3.71) for x_{ij}^{k}, one obtains:

$$G_{ij}(u_{ij}^{k},1/2(x_{ij}^{k}+x_{ij}^{k-1}))=A_{ij}^{k}u_{ij}^{k}+b_{ij}u_{ij}^{k}x_{ij}^{k-1} + 1/2b_{ij}u_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k})\ \text{MWh} \tag{3.72}$$

where

$$\left.\begin{aligned} q_{ij}^{k} &= y_{ij}^{k} + s_{(i-1)j}^{k} - s_{ij}^{k} \\ A_{ij}^{k} &= a_{ij} + 1/2\ b_{ij}q_{ij}^{k} \end{aligned}\right\} \tag{3.73}$$

Now the cost functional in equation (3.65) becomes:

$$J=E\Big[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{v=i}^{n_j}\left(a_{vj}+b_{vj}x_{vj}^{K}\right)x_{ij}^{K}$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}\left\{\alpha_{ij}^{k}u_{ij}^{k}+2\beta_{ij}^{k}u_{ij}^{k}x_{ij}^{k-1}+\beta_{ij}^{k}u_{ij}^{k}\left(u_{(i-1)j}^{k}-u_{ij}^{k}\right)\right\}\Big]$$

(3.74)

Subject to satisfying the following constraints

$$x_{ij}^{k}=x_{ij}^{k-1}+q_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k} \qquad (3.75)$$

$$\underline{x}_{ij}\leq x_{ij}^{k}\leq \bar{x}_{ij} \qquad (3.76)$$

$$\underline{u}_{ij}^{k}\leq u_{ij}^{k}\leq \bar{u}_{ij}^{k} \qquad (3.77)$$

where

$$\left.\begin{aligned}\alpha_{ij}^{k}&=c_{j}^{k}A_{ij}^{k}\\ \beta_{ij}^{k}&=1/2\,c_{j}^{k}b_{ij}\end{aligned}\right\} \qquad (3.78)$$

The problem now is that of maximizing equation (3.74) subject to satisfying constraints (3.75-3.77).

3.3.2 A Minimum Norm Formulation

The augmented cost functional J is obtained by adjoining to the cost functional J in equation (3.74), the equality constraint (3.75) via Lagrange's multiplier and the inequality constraints (3.76-3.77) via Kuhn-Tucker multipliers, one thus obtains

$$J=E\{\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{v=i}^{n_j}(a_{vj}+b_{vj}x_{vj}^{K})x_{ij}^{K}+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}(\alpha_{ij}^{k}u_{ij}^{k}+2\varepsilon_{ij}^{k}u_{ij}^{k}x_{ij}^{k-1}$$

$$+\beta_{ij}^{k}u_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k})+\lambda_{ij}^{k}(-x_{ij}^{k}+x_{ij}^{k-1}+q_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k})$$

$$+e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})+e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij}^{k})+f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})$$

$$+f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k}))\} \tag{3.79}$$

where λ_{ij}^{k} is Lagrange's multiplier, this is to be determined so that the corresponding equality constraint is satisfied, and e_{ij}^{k}, e_{ij}^{1k}, f_{ij}^{k} and f_{ij}^{1k} are Kuhn-Tucker multipliers, these are equal to zero if the constraints are not violated and greater than zero if the constraints are violated (32).

Define the following column vectors such that

$$A = \text{col.}(A_1,\ldots,\ldots,\ldots,A_m) \tag{3.80}$$

$$A_1 = \text{col.}(A_{11},\ldots,\ldots,\ldots,A_{n_1 1}) \tag{3.81}$$

$$A_m = \text{col.}(A_{1m},\ldots,\ldots,\ldots,A_{n_m m}) \tag{3.82}$$

where

$$A_{ij} = \sum_{v=i}^{n_j} a_{vj}; \quad i=1,\ldots,\ldots,n_j \ ; \ j=1,\ldots,\ldots,m \tag{3.83}$$

$$u(k) = \text{col.}(u_1(k),\ldots,\ldots,u_m(k)) \tag{3.84}$$

$$u_1(k) = \text{col.}(u_{11}{}^k,\ldots,\ldots,u_{n_1 1}{}^k) \tag{3.85}$$

$$u_m(k) = \text{col.}(u_{1m}{}^k,\ldots,\ldots,u_{n_m m}{}^k) \tag{3.86}$$

$$x(k) = \text{col.}(x_1(k),\ldots,\ldots,x_m(k)) \tag{3.87}$$

$$x_1(k) = \text{col.}(x_{11}{}^k,\ldots,\ldots,x_{n_1 1}{}^k) \tag{3.88}$$

$$x_m(k) = \text{col.}\ (x_{1m}{}^k,\ldots,\ldots,x_{n_m m}{}^k) \tag{3.89}$$

$$y(k) = \text{col.}(y_1(k),\ldots,\ldots,y_m(k)) \tag{3.90}$$

$$y_1(k) = \text{col.}(y_{11}{}^k,\ldots,\ldots,y_{n_1 1}{}^k) \tag{3.91}$$

$$y_m(k) = \text{col.}(y_{1m}{}^k,\ldots,\ldots,y_{n_m m}{}^k) \tag{3.92}$$

$$s(k) = \text{col.}s_1(k),\ldots,\ldots,s_m(k)) \tag{3.93}$$

$$s_1(k) = \text{col.}(s_{11}{}^k,\ldots,\ldots,s_{n_1 1}{}^k) \tag{3.94}$$

$$s_m(k) = \text{col.}(s_{1m}{}^k,\ldots,\ldots,s_{n_m m}{}^k) \tag{3.95}$$

$$\mu_{ij}^{k} = e_{ij}^{1k} - e_{ij}^{k} \tag{3.96}$$

$$\mu(k) = \text{col.}(\mu_1(k),\ldots,\ldots,\mu_m(k)) \tag{3.97}$$

$$\mu_1(k) = \text{col.}(\mu_{11}^{k},\ldots,\ldots,\mu_{n_1 1}^{k}) \tag{3.98}$$

$$\mu_m(k) = \text{col.}(\mu_{1m}^{k},\ldots,\ldots,\mu_{n_m m}^{k}) \tag{3.99}$$

$$\psi_{ij}^{k} = f_{ij}^{1k} - f_{ij}^{k} \tag{3.100}$$

$$\psi(k) = \text{col.}(\psi_1(k),\ldots,\ldots,\psi_m(k)) \tag{3.101}$$

$$\psi_1(k) = \text{col.}(\psi_{11}^{k},\ldots,\ldots,\psi_{n_1 1}^{k}) \tag{3.102}$$

$$\psi_m(k) = \text{col.}(\psi_{1m}^{k},\ldots,\ldots,\psi_{n_m m}^{k}) \tag{3.103}$$

$$q(k) = \text{col.}(q_1(k),\ldots,\ldots,q_m(k)) \tag{3.104}$$

$$q_1(k) = \text{col.}(q_{11}^{k},\ldots,\ldots,q_{n_1 1}^{k}) \tag{3.105}$$

$$q_m(k) = \text{col.}(q_{1m}^{k},\ldots,\ldots,q_{n_m m}^{k}) \tag{3.106}$$

$$\alpha(k) = \text{col.}(\alpha_1(k),\ldots,\ldots,\alpha_m(k)) \tag{3.107}$$

$$\alpha_1(k) = \text{col.}(\alpha_{11}^{k},\ldots,\ldots,\alpha_{n_1 1}^{k}) \tag{3.108}$$

$$\alpha_m(k) = \text{col.}(\alpha_{1m}^{\;k}, \ldots, \ldots, \alpha_{n_m m}^{\;k}) \tag{3.109}$$

Furthermore, define the following matrices

$$\beta = \text{diag.}(\beta_1, \ldots, \ldots, \beta_m) \tag{3.110}$$

where the elements of any matrix $\beta_1, \ldots, \beta_m$ are given by

(1) $\beta_{ii} = b_{ij}$; $i=1,\ldots,n_j$; $j=1,\ldots,m$

(3.111)

(2) $\beta_{(v+1)v} = \beta_{v(v+1)} = 1/2\ b_{(v+1)j}$; $v=1,\ldots,n_j-1$, $j=1,\ldots,m$

$$\beta_1(k) = \text{diag.}(\beta_{11}^{\;k}, \ldots, \ldots, \beta_{n_1 1}^{\;k}) \tag{3.112}$$

$$\beta_m(k) = \text{diag.}(\beta_{1m}^{\;k}, \ldots, \ldots, \beta_{n_m m}^{\;k}) \tag{3.113}$$

$$M = \text{diag.}(M_1, \ldots, \ldots, M_m) \tag{3.114}$$

where $M_1, \ldots, \ldots, M_m$ are lower triangular matrices whose elements are given by

(1) $m_{ii} = -1$; $i=1,\ldots,\ldots, n_j$; $j=1,\ldots,\ldots,m$

(2) $m_{(v+1)v} = 1$; $v=1,\ldots,\ldots,n_{j-1}$; $j=1,\ldots,m$

(3.115)

Using all the above definitions, the augmented cost functional J in equation (3.79) becomes

$$J=E[A^T x(K)+x^T(K)\beta x(K)+\sum_{k=1}^{K}\{\alpha^T(k)u(k)+u^T(k)\xi(k)x(k-1)$$

$$+x^T(k-1)\beta(k)u(k)+u^T(k)\beta(k)Mu(k)+\lambda^T(k)(-x(k)$$

$$+x(k-1)+Mu(k))+\mu^T(k)x(k)+\psi^T(k)u(k)\}] \quad (3.116)$$

Note that constant terms are dropped in equation (3.116).

Employing the discrete version of integration by parts (33), substituting for x(k) and dropping terms explicitly independent of x(k-1) and u(k) .One obtains

$$J=E[\{x^T(K)\beta x(K)+(A-\lambda(K))^T x(K)+\lambda^T(0)x(0)\}$$

$$+\sum_{k=1}^{K}\{u^T(k)\beta(k)Mu(k)+x^T(k-1)\beta(k)u(k)+u^T(k)\beta(k)x(k-1)$$

$$+(\lambda(k)-\lambda(k-1)+\mu(k))^T x(k-1)+(\alpha(k)+M^T\lambda(k)+R^T\mu(k)+\psi(k))^T u(k)\}] \quad (3.117)$$

If one defines the following vector such that

$$N(K) = A - \lambda(K) \quad (3.118)$$

$$W(K) = \beta^{-1}N(K) \tag{3.119}$$

$$X^T(k) = [x^T(k-1),\ u^T(k)] \tag{3.120}$$

$$L(k) = \begin{bmatrix} 0 & \beta(k) \\ \beta(k) & \beta(k)M \end{bmatrix} \tag{3.121}$$

and

$$R^T(k) = [(\lambda(k)-\lambda(k-1)+\mu(k))^T, (\alpha(k)+M^T\lambda(k)+M^T\mu(k)+\psi(k))^T] \tag{3.122}$$

then the cost functional in equation (3.117) can be written as

$$\tilde{J} = E[(x(K)+ 1/2\ W(K))^T\beta(x(K)+ 1/2\ W(K)) - 1/4\ W^T(K)\beta W(K)+\lambda^T(0)x(0)$$

$$+ \sum_{k=1}^{K} \{X^T(k)L(k)X(k)+R^T(k)X(k)\}] \tag{3.123}$$

It will be noticed that $\tilde{J}$ in equation (3.123) is composed of a boundary part and a discrete integral part, which are independent of each other. $\tilde{J}$ in equation (3.123) can be written as

$$J = J_1 + J_2$$

where the boundary part is given by

$$J_1 = E[(x(K) + 1/2\ W(K))^T (x(K) + 1/2\ W(K))] \quad (3.124)$$

because W(K) is independent of x(K) and x(0) is constant, and the discrete integral part is given by

$$J_2 = E[\sum_{k=1}^{K} (X^T(k)L(k)X(k)) + R^T(k)\hat{X}(k))] \quad (3.125)$$

If one defines the vector V(k) such that

$$V(k) = L^{-1}(k)\ R(k) \quad (3.126)$$

then, equation (3.125) can be written as

$$J_2 = E[\sum_{k=1}^{K} (X(k) + 1/2\ V(k))^T L(k)(X(k) + 1/2\ V(k)) - 1/4\ V^T(k)L(k)V(k)] \quad (3.127)$$

The last term in equation (3.127) is constant independent of X(k), then we can consider only

$$J_2 = E[\sum_{k=1}^{K} (X(k) + 1/2\ V(k))^T L(k)(X(k) + 1/2\ V(k))] \quad (3.128)$$

Now, equation (3.124) defines a norm, hence one can write equation (3.124) as

$$\underset{x(K)}{\text{Max.}} J_1 = \underset{x(K)}{\text{Max.}} E\ [||x(K) + 1/2\ W(K)||]_\beta \quad (3.129)$$

Also, equation (3.128) defines a norm, hence one can write equation (3.128) as

$$\underset{X(k)}{\text{Max.}} J_2 = \underset{X(k)}{\text{Max.}} E[\,||X(k) + 1/2\ V(k)||\,]_{L(k)} \qquad (3.130)$$

3.3.3 The Optimal Solution

There is only one optimal solution to the problem formulated in equations (3.129) and (3.130). The maximum of J_1 is achieved when

$$E[x(K) + 1/2\ W(K)] = [0] \qquad (131)$$

Substituting from equations (3.118) and (3.119) into equation (3.131) one obtains the following optimal equation at the boundary

$$E[\lambda(K)] = E[A + 2Bx(K)] \qquad (3.132)$$

The above equation gives the value of Lagrange's multiplier at the end of the last period.

The maximum of J_2 in equation (3.130) is achieved when

$$E[X(k) + 1/2\ V(k)] = [0] \qquad (3.133)$$

Substituting from equation (3.126) into equation (3.133) for V(k), one obtains

$$E[R(k) + 2L(K)\ X(k)] = [0] \tag{3.134}$$

Writing eqution (3.134) explicitly by substituting from equations (3.120-3.122) and adding the continuity equation (3.75), one obtains the following set.

$$E[-x(k)+x(k-1)+y(k)+Mu(k)+Ms(k)] = [0] \tag{3.135}$$

$$E[\lambda(k)-\lambda(k-1)+\mu(k)+2\beta(k)u(k)] = [0] \tag{3.136}$$

$$E[\alpha(k)+M^T\lambda(k)+M^T\mu(k)+\psi(k)+2\beta(k)x(k-1) +$$

$$+ 2\beta(k)Mu(k)] = [0] \tag{3.137}$$

We can now state the optimal equation (3.132) and equations (3.135-3.137) in component form as follows

$$E[\lambda_{ij}^{K}] = E[\sum_{v=i}^{n_j} a_{vj}+2b_{ij}x_{ij}^{K}+\sum_{w=1}^{n_j-1} b_{(w+1)j}x_{(w+1)j}^{K}] = 0 \tag{3.138}$$

$$E[-x_{ij}^{k}+x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k}] = 0 \tag{3.139}$$

$$E[\lambda_{ij}^{k}-\lambda_{ij}^{k-1}+\mu_{ij}^{k}+c_j^{k}b_{ij}u_{ij}^{k}] = 0 \tag{3.140}$$

$$E[\alpha_{ij}^{k}+\lambda_{(i+1)j}^{k}-\lambda_{ij}^{k}+\mu_{(i+1)j}^{k}-\mu_{ij}^{k}+\psi_{ij}^{k}+c_{j}^{k}b_{ij}x_{ij}^{k-1}$$

$$+\ c_{j}^{k}b_{ij}(u_{(i-1)j}^{k}-u_{ij}^{k})] = 0 \qquad (3.141)$$

Besides the above four equations, one has the following limits on the variables

$$\left.\begin{array}{l} \text{If } x_{ij}^{k} < \underline{x}_{ij}, \text{ then we put } x_{ij}^{k} = \underline{x}_{ij} \\ \text{If } x_{ij}^{k} > \bar{x}_{ij}, \text{ then we put } x_{ij}^{k} = \bar{x}_{ij} \\ \text{If } u_{ij}^{k} < \underline{u}_{ij}^{k}, \text{ then we put } u_{ij}^{k} = \underline{u}_{ij}^{k} \\ \text{If } u_{ij}^{k} > \bar{u}_{ij}^{k}, \text{ then we put } u_{ij}^{k} = \bar{u}_{ij}^{k} \end{array}\right\} \qquad (3.142)$$

One also has the following Kuhn-Tucker exclusion equtions which must be satisfied at the optimum (40)

$$e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k}) = 0 \qquad (3.143)$$

$$e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij}) = 0 \qquad (3.144)$$

$$f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \qquad (3.145)$$

$$f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k}) = 0 \qquad (3.146)$$

Equations (3.138 - 3.146) or equation (3.132) with equations (3.135–3.137) completely specify the optimal solution. The following algorithm is used to solve these equations

3.3.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the initial storage, x(0), the expected natural inflows to the reservoirs, y(k), and the cost of energy on each river, c_j^k, in $/MWh.

Step 1 Assume initial guess of the control variable $u^i(k)$, such that

$\underline{u}(k) \le u^i(k) \le \bar{u}(k)$; i=iteration counter, i=0

Step 2 Assume first that s(k) is equal to zero. Solve equation (3.135) forward in stages with x(0) given

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$\underline{x} < x(k) < \bar{x}$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equation

$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k))]$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$\underline{u}(k) < u(k) < \bar{u}(k)$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k))-\bar{u}(k)]$$

If $s(k)<0$, put $s(k) = 0$

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-y(k)-Ms(k))]$$

Step 8 Solve again equation (3.135) forward in stages with $x(0)$ given

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4

Step 10 With $\mu(k)=0$, solve equation (3.136) backward in stages with equation (3.132) as a terminal condition

Step 11 Calculate Kuhn-Tucker multiplier for $u(k)$, $\psi(k)$, from the following equation

$$E[\psi(k)]=E[2M^T\beta(k)u(k)-M^T\lambda(k-1)-\alpha(k)-2\beta(k)x(k-1)-$$

$$- 2\beta(k)Mu(k)]$$

If $u(k)$ satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

put $\psi(k)=0$

Step 12 Determine a new control iterate from the following

$$E[u^{i+1}(k)]=[u^{i}(k)+\alpha Du^{i}(k)]$$

where

$$E[Du^{i}(k)]=E[\alpha(k)+M^{T}\lambda(k)+\psi(k)+2\beta(k)x(k-1)$$

$$+ 2\beta(k)Mu(k)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \bar{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)]=E[[M^{T}]^{-1}(2M^{T}\beta(k)u(k)-\alpha(k)-2\beta(k)x(k-1)$$

$$- 2\beta(k)Mu(k)]$$

Step 15 Determine Kuhn-Tucker multiplier for $x(k)$, $\mu(k)$, from the following equation

$$E[\mu(k)]=E[-[M^T]^{-1}(\alpha(k)+M^T\lambda(k)+2\beta(k)x(k-1)$$

$$+ 2\beta(k)Mu(k)]$$

If $x(k)$ satisfies the inequality; $\underline{x} < x(k) < \bar{x}$, put $\mu(k)=0$

Step 16 Determine a new state iterate from the following equation

$$E[x^{i+1}(k)]=E[x^i(k)+\alpha Dx^i(k)]$$

where

$$E[Dx(k)]=E[\lambda(k)-\lambda(k-1)+\mu(k)+2\beta(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and J in equation (3.74) is a maximum

3.3.5 Practical Example

The algorithm of the last section has been used to determine the optimal long-term operation of a real system in operation. The system consists of three rivers; each river has two series reservoirs. The characteristics of the installations are given in Table (3.10). The optimization is done on a monthly time basis for a period of a year. The times of water travel between upstream and downstream reservoirs are neglected, also transmission line losses are neglected.

If d^k denotes thes number of days in a month k, then the maximum and minimum discharges are given by

$$\bar{u}_{1j}^{k} = 0.0864d^{k} \text{ (maximum effective discharge)}$$

$$\underline{u}_{1j}^{k} = 0.0864d^{k} \text{ (minimum effective discharge)}$$

where the maximum and minimum effective discharges are given in Table (3.10).

The expected monthly natural inflows to the sites in the year of high flow, which we call year 1, and the cost of energy are given in Table (3.11). In Table (3.12) we give the optimal monthly release from each reservoir and the profits realized in year 1 for the optimal global-feedback solution. In Table (3.13), we give the optimal storage for each reservoir during the first year and in Table (3.14), we give the optimal spill from each reservoir during the first year.

We have simulated the monthly operation of the system for widely different water conditions. The expected monthly natural inflows to the sites in year 2 which is the year of low flow are given in Table (3.15). In Table (3.16) we give the optimal monthly release from each reservoir and the profits realized during year 2. In Tables (3.17) and (3.18), we give the optimal storage and the optimal spill for each reservoir during year 2. We started both years with the reservoirs full.

The processing time required to get the optimal solution for a period of a year for the system just described above was 1.6 sec. in CPU units, which is very small compared to what has been done so far using other techniques.

Table 3.10: Characteristic of the installation

Reservoir	Maximum Storage $\bar{x}_{ij}$ (Mm^3)	Minimum Storage $\underline{x}_{ij}$ (Mm^3)	Maximum Effective Discharge m^3/sec	Minimum Effective Discharge m^3/sec	Reservoir Constants a_{ij}	b_{ij}
R_{11}	24763	9949	1119	85	268.672	7.62×10^{-3}
R_{21}	5304	3734	1583	85	148.467	3.3×10^{-2}
R_{12}	74225	33196	1877	283	274.100	1.91×10^{-3}
R_{22}	0	0	1930	283	100.735	0
R_{13}	45264	24467	1632	283	224.061	5.621×10^{-3}
R_{23}	9175	8886	1876	283	116.500	1.892×10^{-2}

Table 3.11: The expected monthly inflows to the sites in year 1 and the cost of energy

Month k	y_{11}^k Mm^3	y_{21}^k Mm^3	y_{12}^k Mm^3	y_{22}^k Mm^3	y_{13}^k Mm^3	y_{23}^k Mm^3	c_j^k \$/MWh
1	948	326	2526	30	2799	318	1.4
2	482	189	1226	14	1632	193	1.4
3	350	148	1001	15	1380	221	1.4
4	300	113	849	8	1035	234	0.8
5	238	83	724	7	825	164	0.8
6	225	78	644	8	767	169	0.8
7	385	160	962	7	794	229	0.8
8	1388	910	4558	53	2017	373	0.8
9	4492	2143	17322	147	15509	1920	0.8
10	5028	2026	7660	76	6453	999	0.8
11	2685	963	5195	69	4953	712	1.1
12	1402	580	2349	29	3376	414	1.1

Table 3.12: Optimal releases from the reservoirs in year 1 and the profits realized

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	u_{13}^k Mm^3	u_{23}^k Mm^3	Profits $
1	2396	3255	4483	4514	4371	4689	15,441,752
2	2220	1875	4391	4405	4230	4422	13,816,672
3	1924	2117	4550	4565	4178	4399	13,337,558
4	831	1355	3344	3356	3054	3535	5,366,083
5	621	1523	2959	2966	2744	2903	4,506,654
6	688	1058	3348	3357	3079	3249	4,741,587
7	710	870	3219	3226	2983	3212	4,445,930
8	1002	1912	3361	3414	3152	3526	4,943,712
9	973	3116	3294	3441	3179	4863	6,453,121
10	2479	3897	4761	4837	4371	5025	9.092.486
11	2997	2997	4928	4996	4371	5025	12,942,779
12	2900	3480	3532	3562	4230	4643	11,800,057
Value of water remaining in the reservoirs at the end of the year							93,236,784
Total profits							200,125,205

Table 3.13: Optimal reservoir storage during year 1

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3	x_{13}^{k} Mm^3	x_{23}^{k} Mm^3
1	23315	4771	72297	44100	9132
2	21577	5304	69131	41502	9132
3	20003	5258	65582	38703	9132
4	19472	4848	63083	36683	8886
5	19080	4026	60822	34735	8886
6	18617	3734	58117	32422	8886
7	18292	3734	55858	30232	8886
8	18677	3734	57056	29097	8886
9	22196	3734	71083	41426	9122
10	24744	4342	73982	44508	9132
11	24432	5304	74248	45091	9132
12	22933	5304	73064	44236	9132

Table 3.14: Optimal spill from each reservoir in year 1

Month k	s_{11}^k Mm^3	s_{21}^k Mm^3	s_{12}^k Mm^3	s_{12}^k Mm^3	s_{13}^k Mm^3	s_{23}^k Mm^3
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	334
11	0.0	0.0	0.0	0.0	0.0	57
12	0.0	0.0	0.0	0.0	0.0	0.0

Table 3.15: The expected monthly inflows to the sites in year 2

Month k	y_{11}^k Mm^3	y_{21}^k Mm^3	y_{12}^k Mm^3	y_{22}^k Mm^3	y_{13}^k Mm^3	y_{23}^k Mm^3
1	551	220	1987	23	2260	531
2	361	285	122[illegible]	14	991	169
3	316	298	1054	15	849	84
4	273	129	744	10	720	54
5	245	10	491	6	599	72
6	265	92	563	7	579	90
7	608	251	1182	14	598	96
8	1593	925	3898	54	2336	432
9	3851	2065	7487	73	5872	668
10	4566	1630	2556	30	4179	675
11	2701	1137	1828	22	3299	865
12	2348	1080	1813	22	2664	852

Table 3.16: Optimal releases from the reservoirs in year 2 and the profits realized

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	u_{13}^k Mm^3	u_{23}^k Mm^3	Profits $
1	2997	3197	4420	4443	4371	4902	16,013,177
2	2485	2769	4369	4384	4230	4398	14,196,242
3	1296	2884	4567	4582	3448	3532	11,565,489
4	227	636	1873	1883	975	1275	2,464,047
5	205	205	1485	1491	710	780	1,464,047
6	227	227	1873	1879	1039	1129	1,965,371
7	220	220	1747	1761	1007	1102	1,873,458
8	529	1140	1903	1956	1268	1700	2,747,102
9	2900	4061	1843	1916	1667	2335	4,961,431
10	2997	4239	2033	2064	2852	3281	6,446,289
11	2997	4134	3405	3428	4195	5025	12,075,924
12	2900	3980	3317	3339	4230	4863	11,599,919
	Value of water remaining in the reservoirs at the end of the year						84,799,056
	Total profits						171,855,992

Table 3.17: Optimal reservoir storage during year 2

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3	x_{13}^{k} Mm^3	x_{23}^{k} Mm^3
1	22317	5304	71821	43561	9132
2	20192	5304	68678	40322	9132
3	19212	4014	65164	37722	9132
4	19258	3734	64035	37466	8886
5	19289	3744	63023	37334	8886
6	19327	3836	61713	36874	8886
7	19714	4086	61148	36464	8886
8	20778	4400	63143	37532	8886
9	21728	5304	68786	41736	8886
10	23297	5304	69308	43063	9132
11	23000	5304	67730	42167	9132
12	2248	5304	66226	40601	9132

Table 3.18: Optimal spill from each reservoir in year 2

Month k	s_{11}^{k} Mm^3	s_{21}^{k} Mm^3	s_{12}^{k} Mm^3	s_{22}^{k} Mm^3	s_{13}^{k} Mm^3	s_{23}^{k} Mm^3
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	388	0	0	0	0
11	0	0	0	0	0	35
12	0	0	0	0	0	218

3.3.6 Discussion

In this section functional analysis and minimum norm formulation have been used to solve the long-term optimal hydrdoelectric geeration of a multireservoir power system. The method takes into account the water conversion factor variation, and the stochasticity of the river flow. In formulating the problem as a minimum norm problem, we used an average of begin and end-of-time step storage to avoid underestimation of production for rising water levels, and overestimation for falling water levels.

Numerical results are presented for a real systesm in operation including up to six reservoirs for widely different water conditions. The proposed method is computional efficient compared to previous techniques.

We used a linear relation for the water conversion factor with the storage. This may yield a big error in the water conversion factor, for example for the reservoir R_{11} at the maximum capacity the water conversion factor using this linear relation is $447 MWh/Mm^3$, but it is actually equal to $452 MWh/Mm^3$ with percentage error equal to -1.1%. On the other hand, the water conversion factor at the minimum storage is equal to $344.5 MWh/Mm^3$ using this linear relation, but it is actually equal to $336.3 MWh/Mm^3$, with percentage error equal to 2.5%.

Also, for the reservoir R_{12}, the water conversion factor at the maximum storage using this linear model is equal to $323.5 MWh/Mm^3$, but it is actually equal to $316 MWh/Mm^3$ with percentage error equal to 2.4%, and so on for other reservoirs. In conclusion, using a linear model for the water conversion factor may cause overestimation in the total benefits from the system.

CHAPTER IV

MULTIRESERVOIR POWER SYSTEMS WITH A NONLINEAR WATER CONVERSION FACTOR AND NONLINEAR STORAGE-ELEVATION CURVE

4.1 Background

In the previous two chapters the long-term optimal operating problem for multireservoir power systems connected either in series or in parallel, and having either a constant or a variable water conversion factor has been discussed. We assumed that the storage-elevation curve for both cases was linear.

This chapter is devoted to solve the long-term optimal operating problem for multireservoir power systems connected either in series on a river or in parallel on a multiriver system. The systems described here are characterized by having a nonlinear storage-elevation curve and a nonlinear water conversion factor. We used, for both a quadratic function of the storage; the resulting problem has a highly nonlinear objective function and linear constraints. We propose a transformation such that the system equations are reduced to a linear-quadratic form. Lagrange and Kuhn-Tucker multipliers are used to adjoin the equality and the inequality constraints to the objective function.

The problem of optimizing the operation of a multireservoir power system is a difficult problem because first, it has a nonlinear objective function. Second, the production energy function of the hydroplant is a non-separable function of the discharge and the head, which itself is a function of the storage. Third, there are linear constraints on both the state (storage) and decision (release) variables. Fourth, it is a stochastic problem with respect to the river flows (41).

Over the past several years a number of methods have been developed to solve the problem. Aggregation of the multireservoir and hydrdoplant system into a single equivalent reservoir is one of the earlier approaches that has been used. Obviously, such a representation of the reservoirs cannot take into account all local constraints on the contents of the reservoir. Stochastic Dynamic Programming is used to solve the resulting one state problem (3,4). Stochastic Dynamic Programming with successive approximations was used to solve the parallel multireservoir problem (2).

The Aggregation/Decomposition method was used for the solution of two-state variable optimization subproblems. The solution obtained by this method appears questionable, because the inflows to some reservoirs may be periodic in phase with the annual demand cycle, while other reservoirs may have an inflow cycle which lags by a certain time. In this case the decomposition has no meaning (12). Also, this approach avoids answering the basic question as to how the individual reservoirs in the system are to be operated in an optimal fashion (15).

Recently (42), successive Linear Programming, an optimal control algorithm and a combination of Linear Programming and Dynamic Programming (LP-DP) are employed to optimize the operation of multireservoir hydro system given a deterministic inflow forecast. The algorithm maximizes the value of energy produced, plus the estimated value of water remaining in storage at the end of the planning period.

The LP-DP algorithm is clearly dominated: it takes longer to find a solution and produces significantly less hydropower than the other two procedures. Successive Linear Programming (SLP) appears to find the global maximum and is easily implemented. For simple systems (one reservoir or maximum two reservoirs) the optimal control algorithm finds the optimum in about one fifth the time required by SLP but is harder to implement.

In reference (5) a combination of Linear and Dynamic Programming is applied to optimize the production of hydroelectric power for a system consisting of six reservoirs. The computing time to get the optimum using this approach on a typical minicomputer is 1 to 3 minutes per power station.

In this chapter functional analysis and minimum norm forulation have been used to solve the optimal long-term operating problem of a series and a parallel multireservoir power system.

4.2 A Multireservoir Power System in Series on a River*

This section is devoted to the solution of the long-term optimal operating problem for a multireservoir power system in series on a river. The system described here is characterized by the nonlinearity of the storage-elevation curve and the stochasticity of the river flows. In formulating the problem as a minimum norm problem, we assume

---

* A version of this section has been accepted for publication in the Journal of Optimization Theory and Application (JOTA), May 1985, Ref. 27.

that the times of water travel between upstream and downstream reservoirs are shorter than a month, for this reason those times are not taken into account. Transmission line losses are also neglected.

4.2.1 Problem Formulation

4.2.1.1 The System Under Study

The system under consideration consists of n reservoirs connected in series on a river. We will number the installations from upstream to downstream, and denote by the following (Figure 4.1).

I_i^k A random variable representing the natural inflow to site i in period k in Mm^3. These are statistically independent random variables with normal distribution

u_i^k The discharge from plant i during a period k in Mm^3. This is the water released to the allied plant to produce electricity;

$$\underline{u}_i^k \leq u_i^k \leq \overline{u}_i^k \; ; \; u_0^k = 0$$

where $\underline{u}_i^k$ and $\overline{u}_i^k$ are the minimum and maximum releases. If $u_i^k > \overline{u}_i^k$, then $u_i^k - \overline{u}_i^k$ are discharged through the spillways

x_i^k The storage of the reservoir i at the end of month k in Mm^3;

$$\underline{x}_i \leq x_i^k \leq \overline{x}_i$$

where $\underline{x}_i$ and $\overline{x}_i$ are the minimum and maximum storages

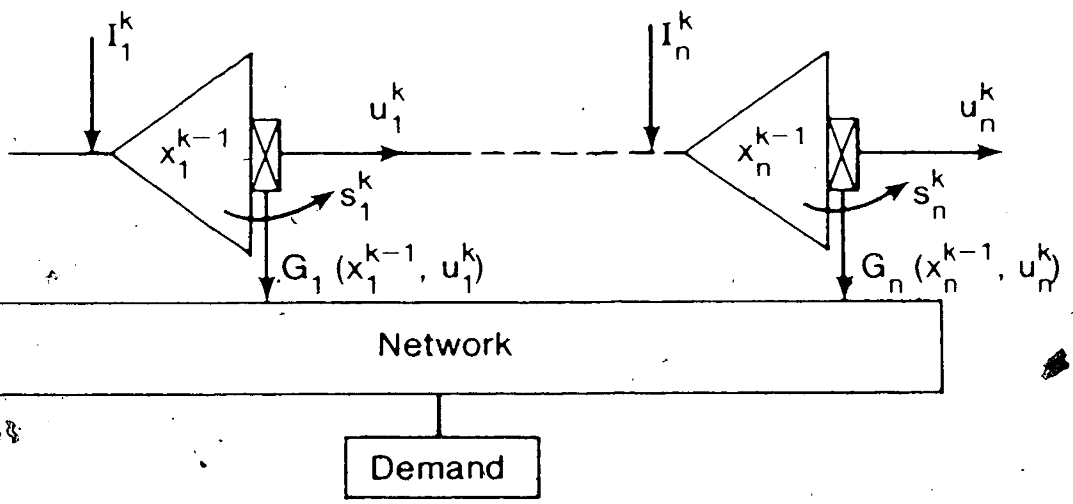

Figure 4.1 A Series Multireservoir Power System

$G_i(u_i^k, x_i^{k-1})$ The generation of plant i in month k in MWh. It is a function of the discharge and the head, which itself is a function of the storage, and this function is a nonlinear function

s_i^k The spill from reservoir i during the period k; $s_0^k=0$; $s_i^k \geq 0$. Water is spilt when the reservoir is filled to capacity and the inflow to the reservoir exceeds $\bar{u}_{ij}^k$

$V_i(x_i^K)$ Value in dollars of the water left in storage at the end of the planning period

c^k Value in dollars of one MWh produced anywhere on the river

k Superscript denoting the period; k=1,...,...,K

i Subscript denoting the reservoir number; i=1,...,...,n

4.2.1.2 The Objective Function

The problem for the power system of Figure 4.1 is to find the discharge u_i^k; i=1,...,...,n; k=1,...,...,K that maximizes the value of energy generated by the system over the planning period plus the expected future returns from water left in storage at the end of that period. In mathematical terms, the basic optimization problem for a n reservoirs system can be written as:

$$J=E\left[\sum_{i=1}^{n} V_i(x_i^K)+\sum_{i=1}^{n}\sum_{k=1}^{K} c^k G_i(u_i^k, x_i^{k-1})\right] \quad \$ \qquad (4.1)$$

Subject to the following constraints

(1) The reservoir's dynamic equation is described by the following difference equation

$$x_i^k = x_i^{k-1} + I_i^k + u_{i-1}^k - u_i^k + s_{i-1}^k - s_i^k \qquad (4.2)$$

where

$$s_i^k = \begin{cases} (x_i^{k-1} + I_i^k + u_{i-1}^k + s_{i-1}^k - x_i^k) - \bar{u}_i^k ; \text{ If } (x_i^{k-1} + I_i^k + u_{i-1}^k + s_{i-1}^k - x_i^k) > \bar{u}_i^k \text{ and } x_i^k = \bar{x}_i \\ 0, \text{ otherwise} \end{cases} \qquad (4.3)$$

(2) Storage levels and discharges can be bounded above and below to allow for recreation and flood control objectives

$$\underline{x}_i \leq x_i^k \leq \bar{x}_i \qquad (4.4)$$

$$\underline{u}_i^k \leq u_i^k \leq \bar{u}_i^k \qquad (4.5)$$

The initial storage x_i^0, the expected natural inflows into each stream during each stream I_i^k and the parameter c^k are assumed to be known.

4.2.1.3 Mathematical Model

To model the amount of water left in storage at the end of the planning period we multiply this amount by the water conversion factor of at-a-site and downstream reservoirs. In this study we assume that the water conversion factor of at-a-site is a nonlinear function of the storage, and this function is a quadratic function. Then, the value in dollars of the amount of water left in storage at the end of the planning period is given by:

$$V_i(x_i^K) = \sum_{j=1}^{n} (\alpha_j+\beta_j x_j^K+\gamma_j (x_j^K)^2)\, x_i^K; \quad i=1,\ldots,\ldots,n \tag{4.6}$$

where α_i, β_i and γ_i are constants. These were obtained by least square fitting to typical plant data available.

The generation of the hydroelectric plant in MWh is a nonlinear function of the water discharge u_i^k and the head, which itself is a function of the storage. We may choose the following $G_i(u_i^k, x_i^{k-1})$.

$$G_i(u_i^k, x_i^{k-1}) = \alpha_i u_i^k + \beta_i u_i^k x_i^{k-1} + \gamma_i u_i^k (x_i^{k-1})^2 \tag{4.7}$$

The above equation is a highly nonlinear function. If one defines the following n dimensional pseudo-state variables such that (17)

$$y_{ij}^k = (x_{ij}^k)^2 \tag{4.8}$$

Then, equation (4.7) can be written as

$$G_i(u_i^k, x_i^{k-1}) = \alpha_i u_i^k + \beta_i u_i^k x_i^{k-1} + \gamma_i u_i^k y_i^{k-1} \tag{4.9}$$

Now, the cost functional in equation (4.1) becomes

$$J=E\Big[\sum_{i=1}^{n}\sum_{j=i}^{n} x_i^K(\alpha_j+\beta_j x_j^K+\gamma_j y_j^K)+\sum_{i=1}^{n}\sum_{k=1}^{K}\{c^k_{1i}u_i^k+c^k_{2i}u_i^k x_i^{k-1}$$

$$+\gamma_i c^k u_i^k y_i^{k-1}\}\Big] \tag{4.10}$$

Subject to satisying the following constraints

$$x_i^k=x_i^{k-1}+I_i^k+u_{i-1}^k-u_i^k+s_{i-1}^k-s_i^k;\ i=1,\ldots,n;k=1,\ldots,K \tag{4.11}$$

$$y_i^k=(x_i^k)^2\ ;\ i=1,\ldots,n;k=1,\ldots,\ldots,K \tag{4.12}$$

$$\underline{x}_i \le x_i^k \le \bar{x}_i;\ i=1,\ldots,n;k=1,\ldots,K \tag{4.13}$$

$$\underline{u}_i^k \le u_i^k \le \bar{u}_i^k;\ i=1,\ldots,\ldots,n;\ k=1,\ldots,\ldots,K \tag{4.14}$$

4.2.2 A Minimum Norm Formulation

The problem now is that of maximizing (4.10) subject to satisfying constraints (4.11-4.14). We can now form an augmented cost functional $\tilde{J}$ by adjoining the equality constraints via Lagrange's multipliers, λ_i^k and μ_i^k, and the inequality constraints via Kuhn-Tucker multipliers.

$$\tilde{J}=E[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^K(\alpha_j+\beta_j x_j^K+\gamma_j y_j^K)+\sum_{i=1}^{n}\sum_{k=1}^{K}(A_i^k u_i^k$$

$$+B_i^k u_i^k x_i^{k-1}+C_i^k u_i^k y_i^{k-1}$$

$$+\lambda_i^k(-x_i^k+x_i^{k-1}+I_i^k+u_{i-1}^k-u_i^k+s_{i-1}^k-s_i^k)+\mu_i^k(-y_i^k+(x_i^k)^2)$$

$$+e_i^{1k}(\underline{x}_i-x_i^k)+e_i^k(x_i^k-\bar{x}_i)+g_i^{1k}(\underline{u}_i^k-u_i^k)+g_i^k(u_i^k-\bar{u}_i^k)\}] \qquad (4.15)$$

where e_i^{1k}, e_i^k, g_i^{1k} and g_i^k are Kuhn-Tucker multipliers. These are equal to zero if the constraints are not violated and greater than zero if the constraints are violated (32). The values of Lagrange's multipliers, λ_i^k and μ_i^k, are to be determined such that the corresponding equality constraints are satisfied.

Now, define the following nx1 column vectors

$$A = \text{col.}(A_1,\ldots,\ldots,A_n) \qquad (4.16)$$

where

$$A_i = \sum_{j=i}^{n} \alpha_j;\ i=1,\ldots,\ldots,n \qquad (4.17)$$

$$x(k) = \text{col.}(x_1^k,\ldots,\ldots,x_n^k) \qquad (4.18)$$

$$y(k) = \text{col.}(y_1^k,\ldots,\ldots,y_n^k) \qquad (4.19)$$

$$u(k) = \text{col.}(u_1^k,\ldots,\ldots,u_n^k) \qquad (4.20)$$

$$s(k) = col.(s_1^k,\ldots,\ldots,s_n^k) \tag{4.21}$$

$$I(k) = col.(I_1^k,\ldots,\ldots,I_n^k) \tag{4.22}$$

$$\lambda(k) = col.(\lambda_1^k,\ldots,\ldots,\lambda_n^k) \tag{4.23}$$

$$\mu(k) = col.(\mu_1^k,\ldots,\ldots,\mu_n^k) \tag{4.24}$$

$$A(k) = col.(A_1^k,\ldots,\ldots,A_n^k) \tag{4.25}$$

where

$$A_i^k = c^k \alpha_i \tag{4.26}$$

$$\nu_i^k = e_i^k - e_i^{lk} \tag{4.27}$$

$$\nu(k) = col.(\nu_1^k,\ldots,\ldots,\nu_n^k) \tag{4.28}$$

$$\psi_i^k = g_i^k - g_i^{lk} \tag{4.29}$$

$$\psi(k) = col.(\psi_1^k,\ldots,\ldots,\psi_n^k) \tag{4.30}$$

Furthermore, define the following nxn matrices

$$B(k) = diag.(B_1^k,\ldots,\ldots,B_n^k) \tag{4.31}$$

where

$$B_i^k = c^k \beta_i \tag{4.32}$$

$$C(k) = \text{diag.}(C_1^k, \ldots, \ldots, C_n^k) \tag{4.33}$$

where

$$C_i^k = c^k \gamma_i \tag{4.34}$$

B is nxn matrix whose elements are given by

(1) $b_{ii} = \beta_i$; ;$i=1,\ldots,n$ (4.35)

(2) $b_{i(j+1)} = b_{(j+1)i} = 1/2\ \beta_{j+1}$; $i=1,\ldots,\ldots,n; j=1,\ldots,\ldots,n-1$

C is nxn upper triangular matrix whose elements are given by

(1) $c_{ii} = \gamma_i$; $i=1,\ldots,\ldots,n$ (4.36)

(2) $c_{i(j+1)} = \gamma_{j+1}$; $i=1,\ldots,n; j=1,\ldots,n-1,\ i = j+1$

M is nxn lower triangular matrix whose elements are given by

(1) $m_{ii} = -1$; $i=1,\ldots,n$ (4.37)

(2) $m_{(j+1)i} = 1$; $i=1,\ldots,n$; $j=1,\ldots,n-1$

Using all the above definitions, the augmented cost functional in equation (4.15) becomes

$$\tilde{J}=E[A^T x(K)+x^T(K)Bx(K)+x^T(K)Cy(K)$$

$$+\sum_{k=1}^{K}\{A^T(k)u(k)+1/2\ x^T(k-1)B(k)u(k)+1/2\ u^T(k)B(k)x(k-1)$$

$$+1/2\ u^T(k)C(k)y(k-1)+1/2\ y^T(k-1)C(k)u(k)+\lambda^T(k)(-x(k)+x(k-1)$$

$$+I(k)+Mu(k)+Ms(k))+\mu^T(k)(-y(k)+x^T(k)\vec{H}x(k))$$

$$+\nu^T(k)x(k)+\psi^T(k)u(k)\}] \qquad (4.38)$$

Note that constant terms are dropped in the above equation, $\vec{H}$ in the above equation is a vector matrix in which the vector index varies from 1 to n while the dimension of $\vec{H}$ is nxn*(17).

---

*If the index is equal to 1, then the matrix $\vec{H}$ is given by:

$$\vec{H}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \text{ is nxn matrix}$$

Also, if the index is equal to 2, then the matrix $\vec{H}$ is given by:

$$\vec{H}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \text{ is nxn matrix}$$

Finally, if the index is equal to n, the matrix $\vec{H}$ is given by:

$$\vec{H}_n = \begin{bmatrix} 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 0 & \cdots\cdots & 1 \end{bmatrix} \text{ is nxn matrix}$$

Employing the discrete version of integration by parts, and dropping constant terms, one thus obtains (33)

$$\tilde{J}=E[(x^T(K)(B+\mu^T(K)\vec{H})x(K)+x^T(K)Cy(K)+(A-\lambda(K))^T x(K)$$

$$-\quad \mu^T(K)y(K)-x^T(0)\mu^T(0)\vec{H}x(0)+\lambda^T(0)x(0)+\mu^T(0)y(0))$$

$$+\sum_{k=1}^{K}(x^T(k-1)\mu^T(k-1)\vec{H}x(k-1)+ 1/2\ x^T(k-1)B(k)u(k)$$

$$+ 1/2\ u^T(k)B(k)x(k-1)+ 1/2\ u^T(k)C(k)y(k-1)+ 1/2\ y^T(k-1)C(k)u(k)$$

$$+ (\lambda(k)-\lambda(k-1)+ \nu(k))^T x(k-1)+(A(k)+M^T\lambda(k)+M^T\nu(k)$$

$$+ \psi(k))^T u(k) - \mu^T(k-1)y(k-1))] \qquad (4.39)$$

It will be noticed that $\tilde{J}$ in equation (4.39) is composed of a boundary part and a discrete integral part, which are independent of each other.

If one defines the vector Z(K) such that

$$Z^T(K) = [x^T(K),\ y^T(K)] \qquad (4.40)$$

Then, the boundary part can be written as:

$$J_1=E[Z^T(K)Q(K)Z(K) + F^T(K)Z(K)] \qquad (4.41)$$

where

$$Q(K)=\begin{bmatrix} B+\mu^T(K)H & 1/2C \\ 1/2C^T & 0 \end{bmatrix} \quad \text{is 2nx2n matrix} \qquad (4.42)$$

and

$$F^T(K)=[(A-\lambda(K))^T, \ -\mu^T(K)] \qquad (4.43)$$

Note that, since the problem is to maximize J_1 with respect to $Z(K)$ and $x(0)$, $y(0)$ are constants, we dropped constant terms in equation (4.41).

Now, define the following vector such that

$$V(K) = Q^{-1}(K)F(K) \qquad (4.44)$$

then, the boundary term can be written in the following form by a process similar to completing the squares as

$$J_1=E[(Z(K)+ 1/2\ V(K))^T Q(K)(Z(K)+ 1/2\ V(K))- 1/4\ V^T(K)Q(K)V(K)] \qquad (4.45)$$

since it is desired to maximize J_1 with respect to $Z(K)$, the problem is equivalent to

$$\underset{Z(K)}{\mathrm{Max.}}J_1=\underset{Z(K)}{\mathrm{Max.}}E[(Z(K)+1/2V(K))^T Q(K)(Z(K)+1/2V(K))] \qquad (4.46)$$

because V(K) is independent of Z(K).

Equation (4.46) defines a norm, hence this equation can be written as (38)

$$\underset{Z(K)}{\mathrm{Max.}}J_1=\underset{Z(K)}{\mathrm{Max.}}E[\,||Z(K) + 1/2\ V(K)||\,]_{Q(K)} \qquad (4.47)$$

If one defines the following vector X(k) such that

$$X^T(k) = [x^T(k-1),\ y^T(k-1),\ u^T(k)] \qquad (4.48)$$

then, the discrete integral part can be written as

$$J_2=E[\sum_{k=1}^{K}\{X^T(k)L(k)X(k) + R^T(k)X(k)\}] \qquad (4.49)$$

where

$$L(k)=\begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 1/2\ B(k) \\ 0 & 0 & 1/2\ C(k) \\ 1/2B(k) & 1/2C(k) & 0 \end{bmatrix} \text{ is 3nx3n matrix} \qquad (4.50)$$

and

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\nu(k))^T,-\mu^T(k-1),(A(k)+M^T\lambda(k)+M^T\nu(k)$$

$$+\psi(k))^T] \quad (4.51)$$

Now define the vector W(k) such that

$$W(k) = L^{-1}(k)R(k) \quad (4.52)$$

then J_2 can be written in the following form by a process similar to completing the squares as (40)

$$\underset{X(k)}{\text{Max.}}J_2=\underset{X(k)}{\text{Max.}}E[\sum_{k=1}^{K}[(X(k)+1/2W(K))^TL(k)(X(k)+1/2\,W(k))$$

$$-1/4\,W^T(k)L(k)W(k)]] \quad (4.53)$$

Since it is desired to maximize J_2 with respect to X(k), the problem is equivalent to

$$\underset{X(k)}{\text{Max.}}J_2=\underset{X(k)}{\text{Max.}}E[\sum_{k=1}^{K}[(X(k)+1/2\,W(k))^TL(k)(X(k)+1/2\,W(k))]] \quad (4.54)$$

because W(k) is independent of X(k). Equation (4.54) also defines a norm, then equation (4.54) can be written as (38)

$$\underset{X(k)}{\text{Max.}}J_2 = \underset{X(k)}{\text{Max.}}E[||X(k) + 1/2\ W(k)||]_{L(k)} \qquad (4.55)$$

4.2.3 The Optimal Solution

As has now become obvious, the problem is finally in the minimum norm form. The optimal solution is of the standard form and is obtained in a similar fashion to that outlined previously. To be specific, the optimal solution for the problem formulated in equation (4.47) is given by

$$E[Z(K) + 1/2\ V(K)] = [0] \qquad (4.56)$$

Writing equation (4.56) explicitly, one obtains the following optimality equations at the boundary:

$$E[2Bx(K)+2\mu^T(K)H\underline{x}(K)+Cy(K)+A-\lambda(K)] = [0] \qquad (4.57)$$

$$E[\mu(K) - C^T x(K)] = [0] \qquad (4.58)$$

Also, the optimal solution for the problem formulated in equation (4.55) is given by

$$E[X(k) + 1/2\ W(k)] = [0] \qquad (4.59)$$

Writing equation (4.59) explicitly and adding equations (4.11) and (4.12), one obtains the following set .

$$E[-x(k)+x(k-1)+I(k)+Mu(k)+Ms(k)] = [0] \tag{4.60}$$

$$E[-y(k)+x^T(k)Hx(k)] = [0] \tag{4.61}$$

$$E[-\mu(k-1)+C(k)u(k)] = [0] \tag{4.62}$$

$$E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^T(k-1)Hx(k-1)+B(k)u(k)] = [0] \tag{4.63}$$

$$E[A(k)+M^T\lambda(k)+M^T\lambda(k)+M^T\nu(k)+\psi(k)+B(k)x(k-1)$$

$$+C(k)y(k-1)] = [0] \tag{4.64}$$

We can now state the optimal solution of equations (4.57-4.58) and equation (4.60-4.64) in component form.

$$E[2\beta_i x_i^K+\beta_i \sum_{j=1}^{i-1} x_j^K + \sum_{j=1}^{n} \beta_j x_j^K+2\mu_i^K x_i^K + \sum_{j=i}^{n} \gamma_j y_j^K$$

$$+ \sum_{j=i}^{n} \alpha_j + \lambda_i^K] = 0 \; ; \; i=1,\ldots,n \tag{4.65}$$

$$E[\mu_i^K - \gamma_i \sum_{j=1}^{i} x_j^K] = 0 \; ; \; i=1,\ldots,n \tag{4.66}$$

$$E[-x_i^k+x_i^{k-1}+I_i^k+u_{i-1}^k-u_i^k+s_{i-1}^k-s_i^k]=0; \; i=1,\ldots,\ldots,n \tag{4.67}$$

$$E[-y_i^k + (x_i^k)^2] = 0;\ i=1,\ldots,\ldots,n \tag{4.68}$$

$$E[-\mu_i^{k-1} + \gamma_i c^k u_i^k] = 0;\ i=1,\ldots,\ldots,n \tag{4.69}$$

$$E[\lambda_i^k - \lambda_i^{k-1} + \nu_i^k + 2\mu_i^{k-1} x_i^{k-1} + \beta_i c^k u_i^k] = 0;\ i=1,\ldots,\ldots,n \tag{4.70}$$

$$E[\alpha_i c^k + \lambda_{i+1}^k - \lambda_i^k + \nu_{i+1}^k - \nu_i^k + \psi_i^k + \beta_i c^k x_i^{k-1} + \gamma_i c^k y_i^{k-1}] = 0;\ i=1,\ldots,\ldots,n \tag{4.71}$$

Besides the above equations, one has the following exclusion equations which msut be satisfied at the optimal (40).

$$e_i^{1k}(\underline{x}_i - x_i^k) = 0 \tag{4.72}$$

$$e_i^k(x_i^k - \overline{x}_i) = 0 \tag{4.73}$$

$$f_i^{1k}(\underline{u}_i^k - u_i^k) = 0 \tag{4.74}$$

$$f_i^k(u_i^k - \overline{u}_i^k) = 0 \tag{4.75}$$

One also has the following limits on the variables (40)

$$\left.\begin{array}{l} \text{If } x_i^k < \underline{x}_i, \text{ then we put } x_i^k = \underline{x}_i \\ \text{If } x_i^k > \overline{x}_i, \text{ then we put } x_i^k = \overline{x}_i \\ \\ \text{If } u_i^k < \underline{u}_i^k, \text{ then we put } u_i^k = \underline{u}_i^k \\ \text{If } u_i^k > \overline{u}_i^k, \text{ then we put } u_i^k = \overline{u}_i^k \end{array}\right\} \tag{4.76}$$

Equations (4.60-4.64) with equations (4.57) and (4.58) or equations (4.65-4.76) completely specify the optimal solution for the system. The following algorithm is used to solve these equatios.

4.2.4 Algorithm For Solution

Assume given: the number of reservoirs(n), the expected value for the natural inflows to each site I_i^k, the initial storage x_i^0, and the cost of energy c^k in $/MWh.

Step 1 : Assume initial guess for the variable u(k) such that

$$\underline{u}(k) \le u^i(k) \le \bar{u}(k) \text{ ; } i = \text{iteration number; } i = 0$$

Step 2 Assume first that s(k) is equal to zero. Solve equations (4.60-4.62) forward in stage with x(0) given

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill at a month k from the following equation

$$E[s(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k))-\bar{u}(k)]$$

If s(k) is less than zero, put s(k) = 0

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k)-Ms(k))]$$

Step 8 Solve again equations (4.60-4.62) forward in stages with x(0) given

Step 9 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put x(k) to its limtis and go to Step 4

Step 10 With $\nu(k)=0$, solve equation (4.63) backward in stages with equation (4.57) as a terminal condition

Step 11 Calculate Kuhn-Tucker multipliers for u(k), $\psi(k)$, from the following equation

$$E[\psi(k)]=E[M^T B(k)u(k)+2M^T u^T(k)C(k)Hx(k-1)$$

$$- B(k)x(k-1)-C(k)y(k-1)-M^T\lambda(k-1)-A(k)]$$

If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

put $\psi(k) = 0$

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)] = E[u^i(k)+\alpha Du^i(k)]$$

where

$$E[Du(k)] = E[A(k)+M^T\lambda(k)+\psi(k)+B(k)x(k-1)+C(k)y(k-1)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies [illegible] inequality

$$\underline{u}(k) < u^{i+1}(k) < \bar{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)] = E[B(k)u(k) - [M^T]^{-1}B(k)x(k-1) - [M^T]^{-1}C(k)y(k-1)$$

$$+ 2\mu^T(k-1)H\vec{x}(k-1) - [M^T]^{-1}A(k)]$$

Step 15 Determine Kuhn-Tucker multipliers for x(k), ν(k), from the following equation

$$E[\nu(k)] = E[-[M^T]^{-1}A(k) - \lambda(k) - [M^T]^{-1}B(k)x(k-1)$$

$$- [M^T]^{-1}C(k)y(k-1)]$$

If x(k) satisfies the inequality; $\underline{x} < x(k) < \overline{x}$, put $\nu(k) = 0$

Step 16 Determine a new state iterate from the following equation

$$E[x^{i+1}(k)] = E[x^i(k) + \alpha Dx^i(k)]$$

where

$$E[Dx(k)] = E[\lambda(k) - \lambda(k-1) + \nu(k) + 2\mu^T(k-1)H\vec{x}(k-1)$$

$$+ B(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and J in equation (4.10) is a maximum.

4.2.5 Practical Example

The algorithm of the previous section has been used to determine the monthly optimal operation for two reservoirs connected in series on a river. The characteristics of these two reservoirs are given in Table 4.1. The optimization is done on a monthly time basis for a period of a year. The times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month; for this reason, those times are not taken into account, also reservoirs water losses due to seepage, evaporation and irrigation are neglected as are transmission line losses.

The expected natural inflows to the two sites and the cost of energy are given in Table (4.2) for the wet year, which we call year 1. In Table (4.3) we give the optimal monthly releases from each reservoir and thse profits realized in year 1, for the optimal global feedback solution. In Table (4.4), we give the optimal reservoir's storage in year 1.

We tested our algorithm for widely different water conditions. In Tables ((4.5)-(4.7)), we reported the results for a dry year, which we call year 2. We started both years with reservoirs full. The computing time to get the optimal solution for the system just described was 0.58sec. in CPU units, which is less than the computing time using other approaches.

Table 4.1: Characteristics of the installations

Site name	Maximum storage ($\bar{x}$) Mm^3	Minimum storage ($\underline{x}$)	Maximum effective discharge (m^3/sec)	Minimum effective discharge (m^3/sec)	Reservoirs constant		
					α_i	β_i	γ_i
R_1	24763	9949	1119	85	212.11095	146.956×10^{-4}	-2050.31×10^{-10}
R_2	5304	3734	1583	85	117.2017	569.71×10^{-4}	-36811.989×10^{-10}

Table 4.2: The expected monthly inflows to the sites in year 1 and the cost of energy

Month k	I_1^k Mm^3	I_2^k Mm^3	c^k $/MWh
1	892	689	1.4
2	922	454	1.4
3	462	319	1.4
4	310	176	0.8
5	280	103	0.8
6	253	116	0.8
7	698	190	0.8
8	2712	1227	0.8
9	3417	1324	0.8
10	5388	2136	0.8
11	4566	1995	1.1
12	2631	1097	1.1

Table 4.3: Optimal monthly releases from the reservoirs and the profits realized in year 1

Month k	u_1^k Mm^3	u_2^k Mm^3	Profits $
1	2523	3213	3,011,408
2	2341	2796	2,686,580
3	2288	2608	2,545,721
4	1465	1898	976,044
5	1251	1226	720,880
6	1432	1467	836,165
7	1363	1506	815,377
8	1444	2721	1,142,981
9	1964	3288	1,465,964
10	1708	3844	1,538,007
11	2345	4239	2,602,937
12	2900	3997	2,822,082
Value of water remaining in storage at the end of the year			20,206,320
Total profits			41,370,466

Table 4.4: Optimal reservoir storage during year 1

Month k	x_1^k Mm^3	x_2^k Mm^3
1	23131	5304
2	21712	5304
3	19885	5304
4	18729	5304
5	17757	5176
6	16577	5257
7	15911	5304
8	17179	5304
9	18632	5304
10	22311	5304
11	24531	5304
12	24262	5304

Table 4.5: The expected monthly inflows to the sites in year 2

Month k	I_1^k Mm^3	I_2^k Mm^3
1	847	358
2	371	344
3	234	261
4	227	176
5	205	113
6	266	155
7	456	204
8	2004	1123
9	3223	1508
10	3402	1312
11	2585	677
12	1305	311

Table 4.6: Optimal monthly releases from the reservoirs and the profits realized in year 2

Month k	u_1^k Mm^3	u_2^k Mm^3	Profits $
1	2997	3355	3,372,875
2	2900	3243	3,219,276
3	1254	3085	2,109,900
4	227	227	126,752
5	205	205	115,319
6	227	236	130,417
7	220	414	169,988
8	227	227	139,342
9	220	1725	511,848
10	2997	4239	2,143,635
11	2997	3673	2,757,101
12	2900	3211	2,542,615
		Value of water remaining in storage at the end of the year	**18,668,544**
		Total profits	**35,997,612**

Table 4.7: Optimal reservoir storage during year 2

Month k	x_1^k Mm^3	x_2^k Mm^3
1	22163	5304
2	20086	5304
3	19062	3734
4	19062	3910
5	19060	4023
6	19098	4168
7	19334	4178
8	21111	5301
9	24114	5304
10	24520	5304
11	24107	5304
12	22512	5304

4.2.6 Discussion

The total expected benefits for this system in year 1 using a linear model as mentioned in Table 3.3 is $38,704,069 but from Table 4.3, the total expected benefits using a quadratic model for the same system in the same year is $41,370,466. The percentage increases in the total benefits is about 6.9% in year 1.

On the other hand, the total expected benefits in year 2 as mentioned in Table 3.7 using a linear model for this system is $32,884,877 but from Table 4.6, the total expected benefits for the same system using a quadratic model is $35,997,612. The percentage increases in the total benefits is about 9.5% in year 2.

4.3 A Parallel Multireservoir Power System*

This section is devoted to the solution of long-term optimal operation of a parallel multireservoir power system, which may contain run-of-river downstream reservoirs. The systems considered in this section are characterized by having a highly nonlinear hydroelectric generating function of the storage and the discharge, and the water conversion factor assigned for each hydroplant is a nonlinear function of the storage.

4.3.1 Problem Formulation

4.3.1.1 The System Under Study

The system under consideration consists of r rivers, each river has n_j series reservoirs; $j=1,\ldots,\ldots,r$, m_j from these reservoirs are run-of-river downstream reservoirs. More precisely there are n_j-m_j series controllable reservoirs on each river. We will number the installations from upstream to downstream and denote by the following (Figure 4.2).

I_{ij}^k A random variable representing the natural inflow to site i on river j; $i=1,\ldots,\ldots,n_j$; $j=1,\ldots,\ldots,r$ in month k; $k=1,\ldots,\ldots,K$ in Mm^3 ($1Mm^3=10^6m^3$). These are statistically independent random variables with normal distribution. It is assumed that no correlation exists between flows of independent rivers or of a different period of time.

---

* A version of this section has been accepted for publication in the Canadian Electrical Engineering Journal, June 1985, Ref. (30).

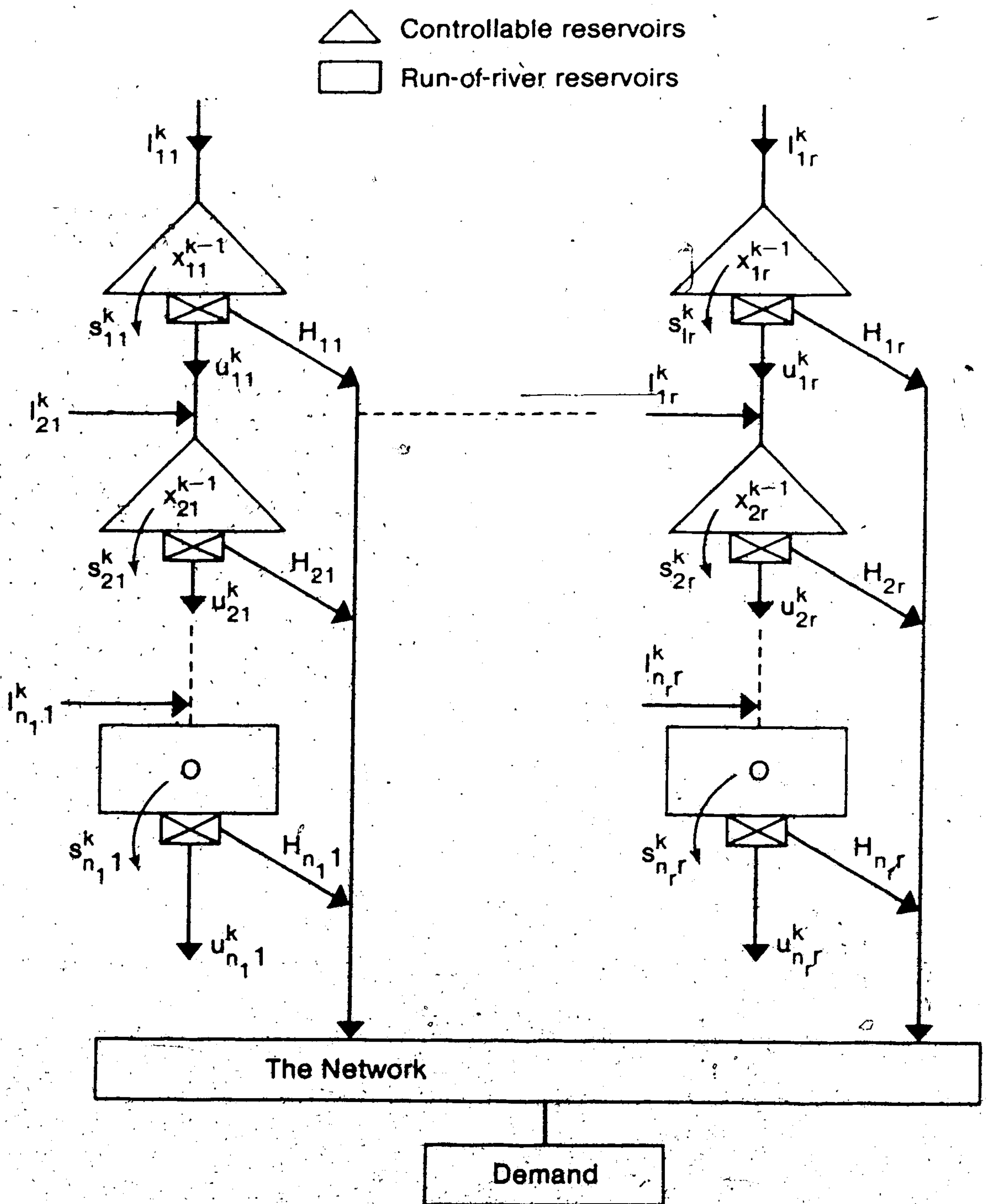

Figure 4.2 A Parallel Hydroelectric System

x_{ij}^k The storage of reservoir i on river j at the end of period k in Mm^3; $i=1,\ldots,\ldots,n_j-m_j$; $j=1,\ldots,\ldots,r$; $k=1,\ldots,\ldots,K$;

$$\underline{x}_{ij} \leq x_{ij}^k \leq \overline{x}_{ij}$$

where $\underline{x}_{ij}$ and $\overline{x}_{ij}$ are the minimum and maximum storages.

u_{ij}^k The discharge from reservoir i on river j during the period k in Mm^3;

$$\underline{u}_{ij}^k \leq u_{ij}^k \leq \overline{u}_{ij}^k \; ; \; u_{0j}^k = 0$$

where $\underline{u}_{ij}^k$ and $\overline{u}_{ij}^k$ are the minimum and maximum discharges of the allied power plant. If $u_{ij}^k > \overline{u}_{ij}^k$, then $u_{ij}^k - \overline{u}_{ij}^k$ Mm^3 of water is discharged through the spillways.

s_{ij}^k The spillage from reservoir i on river j during the period k in Mm^3; $s_{0j}^k=0$; $s_{ij}^k \geq 0$,

c_j^k Value in dollars of one MWh produced anywhere on the river j in $/MWh.

$V_{ij}(x_{ij}^K)$ Value in dollars of the water left in storage in reservoir i of river j; $i=1,\ldots,\ldots,n_j-m_j$; $j=1,\ldots,\ldots,r$, at the end of the planning period.

h_{ij}^k The water conversion factor in MWh/Mm^3 of power plant i on river j in period k. In this study it is assumed that

$$h_{ij}^k = \alpha_{ij} + \beta_{ij} x_{ij}^k + \gamma_{ij} (x_{ij}^k)^2 \; MWh/Mm^3 \; ; \qquad (4.77)$$

$i=1,\ldots,\ldots,n_j-m_j$; $j=1,\ldots,\ldots,r$ for the controllable reservoirs

and

$$h_{ij}^{k} = \alpha_{ij} \text{ MWh/Mm}^3;\ i=n_j-m_j+1,\ldots,n_j; \tag{4.78}$$

$j=1,\ldots,\ldots,r$ for run-of-river reservoirs.

$H_{ij}(u_{ij}^{k},x_{ij}^{k-1})$ The generation of plant i on river j during period k MWh. It is assumed that

$$H_{ij}(u_{ij}^{k},\ x_{ij}^{k-1}) = h_{ij}^{k-1}u_{ij}^{k};\ i=1,\ldots,\ldots,n_j-m_j \tag{4.79}$$

$$H_{ij}(u_{ij}^{k},x_{ij}^{k-1})=\alpha_{ij}u_{ij}^{k};\ i=n_j-m_j+1,\ldots,n_j \tag{4.80}$$

4.3.1.2 Objective Function

The long-term stochastic optimization problem determines the discharge u_{ij}^{k}; $i=1,\ldots,\ldots,n_j$; $j=1,\ldots,\ldots,r$ that maximizes the total expected benefits from the system while satisfying certain constraints. In mathematical terms, the problem for the power system in Figure (4.2) is to determine u_{ij}^{k} that maximizes

$$J=E\left[\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j} V_{ij}(x_{ij}^{K})+\sum_{j=1}^{r}\sum_{i=1}^{n_j}\sum_{k=1}^{K} c_j^{k}H_{ij}(u_{ij}^{k},x_{ij}^{k-1})\right]\ \$ \tag{4.81}$$

Subject to satisfying the following constraints

(i) For the controllable reservoirs

(1) The reservoir's dynamics may be described by the following difference equation

$$x_{ij}^{k}=x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k};\ ;i=1,\ldots,\ldots,$$

$$,\ n_j-m_j;\ j=1,\ldots,\ldots,r \qquad (4.82)$$

(2) The operational reservoir constraints are given by

$$\underline{x}_{ij} \le x_{ij}^{k} \le \bar{x}_{ij} \qquad (4.83)$$

$$\underline{u}_{ij}^{k} \le u_{ij}^{k} \le \bar{u}_{ij}^{k} \qquad (4.84)$$

where

$$s_{ij}^{k}=\begin{cases}(x_{ij}^{k-1}+I_{ij}^{k}+s_{(i-1)j}^{k}+u_{(i-1)j}^{k}-x_{ij}^{k})-\bar{u}_{ij}^{k}; \text{If}(x_{ij}^{k-1}+I_{ij}^{k} \\ \quad + s_{(i-1)j}^{k}+u_{(i-1)j}^{k})>\bar{u}_{ij}^{k} \quad \text{and } x_{ij}^{k}>\bar{x}_{ij} \\ 0, \text{ otherwise.}\end{cases}$$

<u>(ii) For the run-of-river downstream reservoirs</u>

(1) The reservoir's dynamics may be described by the following difference equation in which the inflow is equal to the outflow

$$0=u_{(i-1)j}^{k}+I_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k}-u_{ij}^{k};\ i=n_j-m_j+1,\ldots,\ldots$$

$$,n_j;\ j=1,\ldots,\ldots,r \qquad (4.86)$$

(2) The operational reservoir constraints are given by

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k};\ i=n_j-m_j+1,\ldots,n_j;\ j=1,\ldots,\ldots,r \tag{4.87}$$

where

$$s_{ij}^{k}=\begin{cases}(u_{(i-1)j}^{k}+I_{ij}^{k}+s_{(i-1)j}^{k})-\bar{u}_{ij}^{k}; \text{If}(u_{(i-1)j}^{k}+I_{ij}^{k}+s_{(i-1)j}^{k}) > \bar{u}_{ij}^{k} \\ 0, \text{ otherwise}\end{cases} \tag{4.88}$$

4.3.1.3 Modelling of the System

To model the amount of water left in storage (to change this amount into equivalent MWh) at the end of the planning period, we multiply this amount by the water conversion factor of at-a-site and downstream hydroplants. If x_{ij}^{K} is the amount of water left in storage at the last period studied, K, then the value in dollars of this amount is given by

$$V_{ij}(x_{ij}^{K})= \sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}(x_{vj}^{K})^2) \quad \$ \tag{4.89}$$

In the above equation we assume that the cost of energy is one dollar/MWh, which is the average cost during the planning period.

Using equations (4.79) and (4.80) for the generating functions, the cost functional in equation (4.81) becomes

$$J=E[\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}(x_{vj}^{K})^2)+\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{k=1}^{K} c_j^{k}(\alpha_{ij}$$

$$+\beta_{ij}x_{ij}^{k-1}+\gamma_{ij}(x_{ij}^{k-1})^2)u_{ij}^{k}+\sum_{j=1}^{r}\sum_{i=n_j-m_j+1}^{n_j}\sum_{k=1}^{K} c_j^{k}\alpha_{ij}u_{ij}^{k}] \tag{4.90}$$

The cost functional in equation (4.89) is a highly nonlinear functional. To cast the problem into a quadatic problem, we may define the following N, $N=\sum_{j=1}^{r}(n_j-m_j)$, pseudo-state variable as (17).

$$y_{ij}^{k}=(x_{ij}^{k})^2;\; i=1,\ldots,\ldots,n_j-m_j;\; j=1,\ldots,r \tag{4.91}$$

[illegible] the cost functional in equation (4.89) becomes

$$J=E[\sum_{j=1}^{n_j}\sum_{i=1}^{n_j-m_j}\sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}y_{vj}^{K})$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{k=1}^{K} c_j^{k}u_{ij}^{k}(\alpha_{ij}+\beta_{ij}x_{ij}^{k-1}+\gamma_{ij}y_{ij}^{k-1})$$

$$+\sum_{j=1}^{r}\sum_{i=n_j-m_j+1}^{n_j}\sum_{k=1}^{K} c_j^{k}\alpha_{ij}u_{ij}^{k}] \tag{4.92}$$

subject to satisfying the constraints (4.82-4.88) and the following constraint

$$y_{ij}^{k}=(x_{ij}^{k})^2 \tag{4.93}$$

The problem now, is that of maximizing (4.92) subject to satisfying the constraints (4.82-4.88) and (4.93).

4.3.2 A Minimum Norm Formulation

The augmented cost functional $\tilde{J}$ is obtained by adjoining to the cost function (4.92) the equality constraints via Lagrange's multipliers and the inequality constraints via Kuhn-Tucker multipliers (40). One thus obtains

$$\tilde{J}=E[\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{v=i}^{n_j-m_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}y_{vj}^{K})+\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{v=n_j-m_j+1}^{n_j}\alpha_{vj}x_{ij}^{K}$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{n_j-m_j}\sum_{k=1}^{K}\{\alpha_{ij}c_j^{k}u_{ij}^{k}+c_j^{k}u_{ij}^{k}\beta_{ij}x_{ij}^{k-1}+c_j^{k}\gamma_{ij}u_{ij}^{k}y_{ij}^{k-1}$$

$$+\lambda_{ij}^{k}(-x_{ij}^{k}+x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k})+\mu_{ij}^{k}(-y_{ij}^{k}$$

$$+(x_{ij}^{k})^2)+e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})+e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij})+f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})$$

$$+f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k})\}+\sum_{j=1}^{r}\sum_{i=n_j-m_j+1}^{n_j}\sum_{k=1}^{K}\{c_j^{k}\alpha_{ij}u_{ij}^{k}+\lambda_{ij}^{k}(u_{(i-1)j}^{k}$$

$$+I_{ij}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k})+f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})+f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k})\}] \tag{4.94}$$

where λ_{ij}^{k} and μ_{ij}^{k} are Lagrange's multipliers. These are to be determined such that the corresponding equality constraints are satisfied, and e_{ij}^{k}, e_{ij}^{1k}, f_{ij}^{k} and f_{ij}^{1k} are Kuhn-Tucker multipliers, these are equal to zero if the constraints are not violated, and greater than zero if the constraints are violated (32).

To obtain a norm related formulation, we may define the following vectors(40)

$$A_{ij} = \sum_{v=i}^{n_j} \alpha_{vj} \; ; \; i=1,\ldots,n_n-m_j \; ; \; j=1,\ldots,\ldots,r \tag{4.95}$$

$$A = \text{col.}(A_1,\ldots,\ldots,A_r) \text{ is } N\text{x}1 \; ; \; N = \sum_{j=1}^{r} (n_j-m_j) \tag{4.96}$$

$$A_1 = \text{col.}(A_{11},\ldots,\ldots,A_{(n_1-m_1)1}) \text{ is } (n_1-m_1)\text{x}1 \tag{4.97}$$

$$A_r = \text{col.}(A_{1r},\ldots,\ldots,A_{(n_r-m_r)r}) \text{ is } (n_r-m_r)\text{x}1 \tag{4.98}$$

$$x(k) = \text{col.}(x_1(k),\ldots,\ldots,x_r(k)) \text{ is } N\text{x}1 \tag{4.99}$$

$$x_1(k) = \text{col.}(x_{11}{}^k,\ldots,\ldots,x_{(n_1-m_1)_1}{}^k) \text{ is } (n_1-m_1)\text{x}1 \tag{4.100}$$

$$x_r(k) = \text{col.}(x_{1r}{}^k,\ldots,\ldots,x_{(n_r-m_r)_1}{}^k) \text{ is } (n_r-m_r)\text{x}1 \tag{4.101}$$

$$y(k) = \text{col.}(y_1(k),\ldots,\ldots,y_r(k)) \text{ is } N\text{x}1 \tag{4.102}$$

$$y_1(k) = \text{col.}(y_{11}{}^k,\ldots,\ldots,y_{(n_1-m_1)_1}{}^k) \text{ is } (n_1-m_1)\text{x}1 \tag{4.103}$$

$$y_r(k) = \text{col.}(y_{1r}{}^k,\ldots,\ldots,y_{(n_r-m_r)_r}{}^k) \text{ is } (n_r-m_r)\text{x}1 \tag{4.104}$$

$$u(k) = \text{col.}(u_1(k),\ldots,\ldots,u_r(k)) \text{ is } (\sum_{j=1}^{r} n_j)\text{x}1 \tag{4.105}$$

$$u_1(k) = \text{col.}(u_{11}{}^k,\ldots,\ldots,u_{n_1 1}{}^k) \text{ is } n_1\text{x}1 \tag{4.106}$$

$$u_r(k) = \text{col.}(u_{1r}^{k},\ldots,\ldots,u_{n_r r}^{k}) \text{ is } n_r x1 \quad (4.107)$$

$$\lambda(k) = \text{col.}(\lambda_1(k),\ldots,\ldots,\lambda_r(k)) \text{ is } (\sum_{j=1}^{r} n_j)x1 \quad (4.108)$$

$$\lambda_1(k) = \text{col.}(\lambda_{11}^{k},\ldots,\ldots,\lambda_{n_1 1}^{k}) \text{ is } n_1 x1 \quad (4.109)$$

$$\lambda_r(k) = \text{col.}(\lambda_{1r}^{k},\ldots,\ldots,\lambda_{n_r r}^{k}) \text{ is } n_r x1 \quad (4.110)$$

$$\nu_{ij}^{k} = e_{ij}^{1k} - e_{ij}^{k} \; ; \; i=1,\ldots,\ldots,n_j \; ; \; j=1,\ldots,\ldots,r \quad (4.111)$$

$$\psi_{ij}^{k} = f_{ij}^{1k} - f_{ij}^{k} \; ; \; i=1,\ldots,\ldots,n_j \; ; \; j=1,\ldots,\ldots,r \quad (4.112)$$

$$\nu(k) = \text{col.}(\nu_1(k),\ldots,\ldots,\nu_r(k)) \text{ is } (\sum_{j=1}^{r} n_j x1) \quad (4.113)$$

$$\nu_1(k) = \text{col.}(\nu_{11}^{k},\ldots,\ldots,\nu_{n_1 1}^{k}) \text{ is } n_1 x1 \quad (4.114)$$

$$\nu_r(k) = \text{col.}(\nu_{1r}^{k},\ldots,\ldots,\nu_{n_r r}) \text{ is } n_r x1 \quad (4.115)$$

$$\psi(k) = \text{col.}(\psi_1(k),\ldots,\ldots,\psi_r(k)) \text{ is } n_r x1 \quad (4.116)$$

$$\psi_1(k) = \text{col.}(\psi_{11}^{k},\ldots,\ldots,\psi_{n_1 1}^{k}) \text{ is } n_1 x1 \quad (4.117)$$

$$\psi_r(k) = \text{col.}(\psi_{1r}^{k},\ldots,\ldots,\psi_{n_r r}^{k}) \text{ is } n_r x1 \quad (4.118)$$

$$s(k) = \text{col.}(s_1(k),\ldots,\ldots,s_r(k)) \text{ is } (\sum_{j=1}^{r} n_j x1) \quad (4.119)$$

$$s_1(k) = \text{col.}(s_{11}^{\ k},\ldots,\ldots,s_{n_1 1}^{\ k}) \text{ is } n_1 x1 \tag{4.120}$$

$$s_r(k) = \text{col.}(s_{1r}^{\ k},\ldots,\ldots,s_{n_r r}^{\ k}) \text{ is } n_r x1 \tag{4.121}$$

$$A(k) = \text{col.}(A_1(k),\ldots,\ldots,A_r(k)) \text{ is } (\sum_{j=1}^{r} n_j x1) \tag{4.122}$$

$$A_1(k) = \text{col.}(c_1^{\ k}\alpha_{11},\ldots,\ldots,c_1^{\ k}\alpha_{n_1 1}) \text{ is } n_1 x1 \tag{4.123}$$

$$A_r(k) = \text{col.}(c_r^{\ k}\alpha_{1r},\ldots,\ldots,c_r^{\ k}\alpha_{n_r r}) \text{ is } n_r x1 \tag{4.124}$$

$$I(k) = \text{col.}(I_1(k),\ldots,\ldots,I_r(k)) \text{ is } (\sum_{j=1}^{r} n_j x1) \tag{4.125}$$

$$I_1(k) = \text{col.}(I_{11}^{\ k},\ldots,\ldots,I_{n_1 1}^{\ k}) \text{ is } n_1 x1 \tag{4.126}$$

$$I_r(k) = \text{col.}(I_{1r}^{\ k},\ldots,\ldots,I_{n_r r}^{\ k}) \text{ is } n_r x1 \tag{4.127}$$

Furthermore, define the following square matrices

$$B(k)=\text{diag.}(B_1(k),\ldots,\ldots,B_r(k)) \text{ is NxN matrix} \tag{4.128}$$

$$B_1(k) = \text{diag.}(c_1^{\ k}\beta_{11},\ldots,\ldots,c_1^{\ k}\beta_{(n_1-m_1)1}) \text{ is } (n_1-m_1)x(n_1-m_1) \tag{4.129}$$

$$B_r(k) = \text{diag.}(c_r^{\ k}\beta_{1r},\ldots,\ldots,c_r^{\ k}\beta_{(n_r-m_r)r}) \text{ is } (n_r-m_r)x(n_r-m_r) \tag{4.130}$$

$$M = \text{diag.}(M_1,\ldots,\ldots,M_r) \text{ is } (NxN) \tag{4.131}$$

where $M_1,\ldots,\ldots M_r$ are lower triangular matrices whose elements are given by

$$\left.\begin{array}{ll}\text{(i)} & m_{vv}=-1\ ;\ v=1,\ldots,\ldots,n_j\ ;\ j=1,\ldots,\ldots,r \\ \text{(ii)} & m_{(v+1)v}=1;\ v=1,\ldots,\ldots,n_j-1;\ j=1,\ldots,\ldots,r \\ \text{(iii)} & \text{the rest of the elements are equal to zero}\end{array}\right\} \quad (4.132)$$

$$B=\text{diag.}(B_1,\ldots,B_r)\ \text{is}\left(\sum_{j=1}^{r} n_j-m_j\right)x\left(\sum_{j=1}^{r} n_j-m_j\right)\ \text{matrix} \quad (4.133)$$

B_1 is $(n_1-m_1)x(n_1-m_1)$ square matrix whose elements are given by

$$\begin{array}{lll}\text{(i)} & b_{ii}=\beta_{i1} & ;\quad i=1,\ldots,\ldots,n_1-m_1 \\ \text{(ii)} & b_{(v+1)v}=b_{v(v+1)}=1/2\ \beta_{(v+1)1} & ;\ v=1,\ldots,\ldots,n_1-m_1-1\end{array} \quad (4.134)$$

B_r is $(n_r-m_r)x(n_r-m_r)$ square matrix whose elements are given by

$$\begin{array}{lll}\text{(i)} & b_{ii}=\beta_{ir} & ;\ i=1,\ldots,\ldots,n_r-m_r \\ \text{(ii)} & b_{(v+1)v}=b_{v(v+1)}=1/2\ \beta_{(v+1)r} & ;\ v=1,\ldots,n_r-m_r-1\end{array} \quad (4.135)$$

$$C=\text{diag.}(C_1,\ldots,\ldots,C_r)\ \text{is}\ \left(\sum_{j=1}^{r} n_j-m_j\right)\left(\sum_{j=1}^{r} n_j-m_j\right)\text{matrix} \quad (4.136)$$

C_1 is $(n_1-m_1)x(n_1-m_1)$ upper triangular matrix whose elements are given by

(i) $c_{ii} = \gamma_{i1}$; $i=1,\ldots,n_1-m_1$ (4.137)

(ii) $c_{v(v+1)} = \gamma_{(v+1)v}$; $v=1,\ldots,\ldots,n_1-m_1-1$

C_r is $(n_r-m_r)x(n_r-m_r)$ upper triangular matrix whose elements are given by

(i) $c_{ii} = \gamma_{ir}$; $i=1,\ldots,n_r-m_r$ (4.138)

(ii) $c_{v(v+1)} = \gamma_{(v+1)r}$; $v=1,\ldots,n_r-m_r-1$

Using all the above definitions, the augmented cost functional in equation (4.93) becomes

$$\tilde{J}=E[A^T x(K)+x^T(K)Bx(K)+x^T(K)Cy(K)$$

$$+\sum_{k=1}^{K}[A^T(k)u(k)+x^T(k-1)B(k)u(k)+u^T(k)C(k)y(k-1)$$

$$+\lambda^T(k)(-x(k)+x(k-1)+I(k)+Mu(k)+Ms(k))$$

$$+\mu^T(k)(-y(k)+x^T(k)\vec{H}x(k))+\nu^T(k)x(k)+\psi^T(k)u(k)\}] \quad (4.139)$$

In the above equation $\vec{H}$ is a vector matrix in which the vector index varies from 1 to N, while the matrix dimension of $\vec{H}$ is NxN (17).

Employing the discrete version of integration by parts, substituting for x(k) and dropping terms which do not depend on x(k-1), y(k-1) and u(k) (33), one obtains

$$\tilde{J}=E[x^T(K)(B+\mu^T(K)\vec{H})x(K)+x^T(K)Cy(K)+(A-\lambda(K))^T x(K)$$

$$-\mu^T(K)y(K)-x^T(0)\mu^T(0)\vec{H}x(0)+\lambda^T(0)x(0)+\mu^T(0)y(0)$$

$$+\sum_{k=1}^{K}\{x^T(k-1)\mu^T(k-1)\vec{H}x(k-1)+x^T(k-1)B(k)u(k)+u^T(k)C(k)y(k-1)$$

$$+(\lambda(k)-\lambda(k-1)+\nu(k))^T x(k-1)+(A(k)+M^T\lambda(k)+M^T\nu(k)$$

$$+\psi(k))^T u(k)-\mu^T(k-1)y(k-1)\}] \qquad (4.140)$$

It will be noticed that $\tilde{J}$ in equation (4.140) is composed of a boundary part and a discrete integral part, which are independent of each other. To maximize $\tilde{J}$ in equation (4.140), one maximizes the boundary and the discrete parts separately (40).

To formulate the boundary term as a norm, one defines the vector Z(K) such that

$$Z^T(K) = [x^T(K), y^T(K)] \qquad (4.141)$$

then, J can be written as

$$J_1=E[Z^T(K)W(K)Z(K)+Q^T(K)Z(K)] \qquad (4.142)$$

where

$$W(K) = \begin{bmatrix} B+\mu^T(K)\vec{H} & 1/2\ C \\ 1/2\ C^T & 0 \end{bmatrix} \quad \text{is 2Nx2N matrix} \tag{4.143}$$

and

$$Q^T(K)=[(A-\lambda(K))^T,\ -\ \mu^T(K)] \tag{4.144}$$

Since x(0) and y(0) are constants, they are dropped in equation (4.142).

Now define the vector V(K) such that

$$V(K) = W^{-1}(K)Q(K) \tag{4.145}$$

then the boundary term (J_1) can be written in the following form similar to completing the squares as

$$J_1=E[(Z(K)+ 1/2\ V(K))^T W(K)(Z(K)+ 1/2\ V(K))- 1/4\ V^T(K)W(K)V(K)] \tag{4.146}$$

Since it is desired to maximize J_1 with respect to Z(K), the problem is equivalent to

$$\underset{Z(K)}{\text{Max.}}J_1 = \underset{Z(K)}{\text{Max.}}E[(Z(K)+ 1/2\ V(K))^T W(K)(Z(K) + 1/2\ V(K))] \tag{4.147}$$

Equation (4.147) defines a norm, then equation (4.147) can be written as

$$\underset{Z(K)}{\mathrm{Max}}.J_1 = \underset{Z(K)}{\mathrm{Max}}.E[||Z(K) + 1/2\ V(K)||]_{W(K)} \tag{4.148}$$

To formulate the discrete integral part (J_2) as a norm, define the vector X(k) such that

$$X^T(k) = [x^T(k-1),\ y^T(k-1),\ u^T(k)] \tag{4.149}$$

then, the discrete integral part, J_2, of equation (4.140) can be written as

$$J_2 = E[\sum_{k=1}^{K} [X^T(k)L(k)X(k) + R^T(k)X(k)]] \tag{4.150}$$

where

$$L(k) = \begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 1/2B(k) \\ 0 & 0 & 1/2C(k) \\ 1/2B(k) & 1/2C(k) & 0 \end{bmatrix} \text{ is } 3N\text{x}3N \tag{4.151}$$

$$R^T(k) = [(\lambda(k)-\lambda(k-1)+\nu(k))^T, -\mu^T(k-1), (A(k)+M^T\hat{\lambda}(k)+M^T\nu(k)+\psi(k))^T] \tag{4.152}$$

Now define the following vector N(k) such that

$$N(k) = L^{-1}(k)\ R(k) \tag{4.153}$$

then, equation (4.150) can be written in the following form similar to completing the squares as

$$J_2 = E[\sum_{k=1}^{K} \{(X(k)+ 1/2N(K))^T L(k)(X(k)+ 1/2N(k))- 1/4N^T(k)L(k)N(k)\}] \tag{4.154}$$

Since it is desired to maximize J_2 with respect to $X(k)$, the problem is equivalent to

$$\underset{X(k)}{\text{Max.}} J_2 = \underset{X(k)}{\text{Max.}} E[\sum_{k=1}^{K} (X(k)+ 1/2N(k))^T L(k)(X(k)+ 1/2N(k))] \tag{4.155}$$

Equation (4.155) defines a norm, hence equation (4.155) can be written as

$$\underset{X(k)}{\text{Max.}} J_2 = \underset{X(k)}{\text{Max.}} E[||X(k) + 1/2\ N(k)||]_{L(k)} \tag{4.156}$$

4.3.3 The Optimal Solution

There is only optimum solution to the problem formulated in equations (4.148) and (4.156). The maximum of J_1 in equation (4.148) is clearly achieved when the norm of that equation is equal to zero

$$E[||Z(K) + 1/2\ V(K)||\ ] = [0] \ . \tag{4.157}$$

Substituting from equation (4.145) into equation (4.157), one finds that the optimal conditon for the boundary term is given by

$$E[Q(K) + 2W(K)Z(K)] = [0] \qquad (4.158)$$

Writing equation (4.158) explicitly, one obtains the following terminal optimal equations

$$E[A-\lambda(K)+2Bx(K)+Cy(K)+2\mu^T(K)\vec{H}x(K)] = [0] \qquad (4.159)$$

$$E[C^T x(K) - \mu(K)] = [0] \qquad (4.160)$$

Equations (4.159) and (4.160) give the values of Lagrange's multipliers, $\lambda(K)$ and $\mu(K)$, at the last period K.

The maximum of J_2 is clearly achieved for

$$E[||X(k) + 1/2\ N(k)||] = [0] \qquad (4.161)$$

Substituting from equation (4.153) into equation (4.161), one finds that the optimal solution is given by

$$E[R(k) + 2L(k)\ X(k)] = [0] \qquad (4.162)$$

Writing equation (4.162) explicitly and adding equation (4.82), one obtains

$$E[-x(k)+x(k-1)+l(k)+Mu(k)+Ms(k)]=[0] \quad (4.163)$$

$$E[-y(k)+x^T(k)\vec{H}x(k)] = [0] \quad (4.164)$$

$$E[-\mu(k-1)+C(k)u(k)] = [0] \quad (4.165)$$

$$E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^T(k-1)\vec{H}x(k-1)+B(k)u(k)] = [0] \quad (4.166)$$

$$E[A(k)+M^T\lambda(k)+M^T\nu(k)+\psi(k)+B(k)x(k-1)+C(k)y(k-1)] = [0] \quad (4.167)$$

Besides the above equations, one has the following exclusion equations which must be satisfied at the optimum (40)

$$e_{ij}^{1k}(\underline{x}_{ij}-x_{ij}^{k}) = 0 \quad (4.168)$$

$$e_{ij}^{k}(x_{ij}^{k}-\overline{x}_{ij}) = 0 \quad (4.169)$$

$$f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \quad (4.170)$$

$$f_{ij}^{1k}(u_{ij}^{k}-\overline{u}_{ij}^{k}) = 0 \quad (4.171)$$

one also has the following limits on the variables (40)

$$\left.\begin{array}{l} \text{If } x_{ij}^{k} < \underline{x}_{ij}, \text{ then put } x_{ij}^{k} = \underline{x}_{ij} \\ \text{If } x_{ij}^{k} > \overline{x}_{ij}, \text{ then put } x_{ij}^{k} = \overline{x}_{ij} \\ \\ \text{If } u_{ij}^{k} < \underline{u}_{ij}^{k}, \text{then put } u_{ij}^{k} = \underline{u}_{ij}^{k} \\ \text{If } u_{j}^{k} > \overline{u}_{ij}^{k}, \text{ then put } u_{ij}^{k} = \overline{u}_{ij}^{k} \end{array}\right\} \quad (4.172)$$

Equations (4.163-4.172) with equations (4.159) and (4.160) completely specify the optimum solution. The following algorithm is used to solve these equations.

4.3.4 Algorithm for Solution

Assume given: The number of rivers (r), the number of reservoirs on each river (n_j), the number of run-of-river reservoirs on each river (m_j), the expected monthly natural inflows I(k), the cost of energy on each river c_j^k in $/MWh and the initial storage (x_{ij}^0).

Step 1 Assume initial guess for the control variable u(k) such that

$$\underline{u}(k) \le u^{0}(k) \le \overline{u}(k) \qquad i = \text{iteration number, } i = 0$$

Step 2 Assume first that s(k) is equal to zero. Solve equations (4.163-4.165) forward in stages with x(0) given

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$\underline{x} < x(k) < \overline{x}$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equations

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$\underline{u}(k) < u(k) < \overline{u}(k)$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)]=E[[M]^{-1}(x(k)-x(k-1)-I(k))-\overline{u}(k)]$$

If $s(k) < 0$, put $s(k)=0$

Step 7 Calcualte the new discharge from the following equation

$$E[u(k)]=E[[M]^{-1}(x(k)-x(k-1)-I(k)-Ms(k))]$$

Step 8 Solve equations (4.163-4.165) forward in stages with x(0) given

Step 9 Check the limits on x(k). If x(k) satisfies the inequality

$\underline{x} < x(k) < \overline{x}$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 10 With $\nu(k)=0$, solve equation (4.166) backward in stages with equations (4.159) and (4.160) as terminal conditions

Step 11 Calculate Kuhn-Tucker multipliers for u(k), $\psi(k)$, from the following equation

$$E[\psi(k)] = E[M^T B(k)u(k) + 2M^T \mu^T(k-1)\vec{H}x(k-1) - M^T\lambda(k-1) - A(k)$$

$$- B(k)x(k-1) - C(k)y(k-1)]$$

where

$$E[y(k)] = E[x^T(k)\vec{H}x(k)]$$

If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

then put $\psi(k) = 0$

Step 12 Determine a new control iterate from the following

$$E[u^{i+1}(k)] = E[u^i(k) + \alpha Du^i(k)]$$

where

$$E[Du(k)] = E[A(k) + M^T\lambda(k) + \psi(k) + B(k)x(k-1) + C(k)y(k-1)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \bar{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)] = E[B(k)u(k)+2\mu^T(k-1)\vec{H}x(k-1)$$

$$-[M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)-[M^T]^{-1}C(k)y(k-1)]$$

where

$$E[y(k)] = E[x^T(k)\vec{H}x(k)]$$

Step 15 Determine Kuhn-Tucker multipliers for x(k), $\nu(k)$, from the following equation

$$E[\nu(k)] = E[-[M^T]^{-1}(A(k)+M^T\lambda(k)+B(k)x(k-1)+C(k)y(k-1))]$$

If x(k) satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

put $\nu(k) = 0$

Step 16 Determine a new state iterate from the following equation

$$E[x^{i+1}(k)] = E[x^i(k)+\alpha Dx^i(k)]$$

where

$$E[Dx^1(k)]=E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^T(k-1)\vec{H}x(k-1)$$

$$+ B(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and J in equation (4.92) is a maximum.

4.3.5 Practical Example

The algorithm of the last section has been used to determine the monthly long-term optimal operation of a hydroelectric power system consisting of three rivers (r=3). The first river has two series controllable reservoirs ($n_1=2$, $m_1=0$), the second river has one controllable reservoir and one run-of-river downstream reservoir ($n_2=2$, $m_2=1$), and the third river has two controllable reservoirs ($n_3=2$, $m_3=0$). The characteristic of the installations are given in Table (4.8).

The maximum and minimum discharges in Mm^3 are given by

$\bar{u}_{ij}^k=0.0864d^k$(maximum effective discharge in m^3/sec)

$\underline{u}_{ij}^k=0.0864d^k$(minimum effective discharge in m^3/sec)

where d^k the number of days in month k, and the maximum and minimum effective discharges are given in Table 4.8.

The MWh generated at each power house is given by

$$H_{ij}(u_{ij}^{k},x_{ij}^{k-1})=\alpha_{ij}u_{ij}^{k}+\beta_{ij}u_{ij}^{k}x_{ij}^{k-1}+\gamma_{ij}u_{ij}^{k}(x_{ij}^{k-1})^{2} \text{ MWh}$$

where α_{ij}, β_{ij} and γ_{ij} are given in Table (4.8). These values were obtained by least squares curve fitting to typical plant data available.

The expected value of the natural inflows to the sites for a year of high flow, we call year 1, and the cost of energy are given in Table (3.11). In Table (4.9), we give the optimal monthly releases from each reservoir and the profits realized in year 1 for the optimal global-feedback solution. In Table (4.10), we give the optimal reservoir storage in year 1.

We have simulated the operation of the system for widely different water conditions, i.e,. a year of high flow (wet year) and a year of low year (dry year). The expected value of the monthly natural inflows to the sites for the year of low flow, which we call year 2, are given in Table (3.15). We give in Table (4.11) the optimal monthly releases from each reservoir and the profits realized in year 2. In Table (4.12), we give the optimal storage in the same year.

Table 4.8: Characteristics of the installations

River	Site name	Capacity of reservoir ($\overline{x}$) Mm^3	Minimum storage ($\underline{x}$) Mm^3	Maximum effective discharge m^3/sec	Minimum effective discharge m^3/sec	Reservoir constants α_{ij} MWh/Mm^3	β_{ij} $MWh/(Mm^3)^2$	γ_{ij} $MWh(Mm^3)^2$
1	R_{11}	24763	9949	1119	85	212.11	146.96×10^{-4}	$-20503142.65\times10^{-14}$
	R_{21}	5304	3743	1583	85	117.20	569.71×10^{-4}	$-368119890.48\times10^{-14}$
2	R_{12}	74255	33196	1877	283.2	232.46	359.45×10^{-4}	$-1603544.3196\times10^{-14}$
	R_{22}	0	0	1930	283.2	10074	0	0
3	R_{13}	45672	24467	1632	283.2	176.28	105.63×10^{-4}	$-10022665.7196\times10^{-14}$
	R_{23}	9132	8886	1876.3	283.2	131.44	200.89×10^{-4}	$-34741725.6\times10^{-14}$

Table 4.9: Optimal releases from the reservoirs and the profits realized during year 1

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	u_{13}^k Mm^3	u_{23}^k Mm^3	Profits $
1	1919	3134	4088	4118	4042	4361	9,859,915
2	1837	1352	3962	3976	3929	4142	8,690,246
3	1914	1847	4094	4109	3916	4137	9,013,311
4	1690	1803	3848	3855	3698	3932	4,797,369
5	1534	1614	3459	3466	3331	3490	4,264,354
6	1670	1749	3850	3858	3623	3810	4,624,607
7	1616	1776	3721	3728	3474	3770	4,440,306
8	1671	2581	3852	3905	3514	4014	4,720,464
9	1597	3740	3727	3874	3534	4863	5,122,684
10	1723	3748	3874	3950	3600	4602	5,419,751
11	2079	3042	4163	4231	4371	5025	8,146,824
12	1858	2438	3861	3890	4230	4643	7,484,473
	Value of the water remaining in the reservoirs at the end of the year						89,414,256
	Total profits						165,998,560

Table 4.10: Optimal reservoir storage during year 1

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3	x_{13}^{k} Mm^3	x_{23}^{k} Mm^3
1	23792	4415	72692	44428	9132
2	22436	5089	69956	42121	9132
3	20872	5304	66863	39595	9132
4	19482	5304	63864	36932	9132
5	18177	5304	61103	34397	9115
6	16731	5304	57897	31540	9115
7	15499	5304	55137	28860	9048
8	15216	5304	55844	27362	8921
9	18110	5304	69438	39336	9132
10	21415	5304	73223	43189	9128
11	22021	5304	74255	43772	9132
12	21564	5304	72742	42918	9132

Table 4.11: Optimal monthly releases from the reservoir and the profits realized in year 2

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	u_{13}^{k} Mm^3	u_{23}^{k} Mm^3	Profits $
1	1688	3145	2530	2553	2260	3037	6,934,622
2	1633	1614	2450	2464	2291	2214	5,764,761
3	1675	1683	2531	2546	2358	2441	6,012,178
4	1619	1500	2469	2479	2302	2356	3,296,474
5	1471	1239	2226	2232	2089	2158	2,946,267
6	1613	1529	2470	2476	2289	2388	3,275,203
7	1513	1812	2389	2403	2225	2321	3,233,888
8	1612	2538	2470	2523	2293	2725	3,563,849
9	1551	3616	2390	2463	2221	2889	3,802,820
10	1597	3625	2473	2503	2290	2965	3,855,912
11	1620	2720	2504	2527	2314	3179	5,262,758
12	1542	2622	2424	2446	2343	3195	5,180,320
	Value of water left in storage at the end of the year						89,377,792
	Total profits						142,506,846

Table 4.12: Optimal reservoir storage during year 2

Month k	x_{11}^k Mm^3	x_{21}^k Mm^3	x_{12}^k Mm^3	x_{13}^k Mm^3	x_{23}^k Mm^3
1	23626	4047	73712	45672	8886
2	22353	4315	72487	44371	9132
3	2094	4641	71010	42862	9132
4	19648	4888	69284	41279	9132
5	18414	5129	67532	39768	9132
6	17066	5304	65624	38048	9132
7	16112	5304	64417	36421	9132
8	16092	5304	65845	36463	9132
9	18391	5304	70941	40114	9132
10	21360	5267	71024	42002	9132
11	2240	5304	70347	42987	9132
12	23245	5304	69736	43307	9132

4.3.6 Discussion

We have considered in this section a hydro-system consisting of r rivers with each river having n_j reservoirs; j=1,...,...,r connected in series. These reservoirs include both storage reservoir-plant and run-of-river plants. In formulating the problem as a minimum norm problem, Lagrange's multipliers are used for adjoining the equality constraints, while Kuhn-Tucker multipliers are employed for the inequality constraints.

It should be noted that in this section we have considered a practical example consisting of three rivers with each river having two series connected plants. The same system was solved numerically in Ref.28. However, by using the more accurate representation for the reservoirs employed in this section much better results were obtained than in Ref. 28. For example, in this case no spilling of water took place at all and the computed benefits were considerably larger than those found for the system in Ref. 28.

4.4 A Multireservoir Power System in Parallel with an Average Storage*

In section 3.3, the problem of optimal operation of a multireservoir power system in parallel is discussed. The model used for each reservoir is a linear model with the average storage. The error in the storage for this model is greater than the inflow to some reservoirs. For example, for the reservoir R_{11} the error in the storage-elevation curve at some point is 5.6% of the total capacity which is 5304 Mm^3, this error is equal to 297 Mm^3 which is too great. The total benefits obtained by using this model ar[illegible]stimated as well. In section 4.3 we solved the same problem, but the model used in that case is a nonlinear model with the storage of the previous month. This model may cause an overestimation of production for falling water levels and underestimation for rising water level.

In this section we solve the same problem for the same system. The model used in this section has a nonlinear generating function of the average storage. To cast the problem in a quadratic form, we define a set of pseudo-state variables (17).

4.4.1 The Problem Formulation

4.4.1.1 The System Under Study

The system under study consists of m independent rivers with one or several reservoirs and power plants in series on each, and interconnection lines to the neighbouring system through which energy may be exchanged. We will denote by (Figure 4.3).

---

*A version of this section has been submitted to Optimal Control and Calculus of Variation Meeting, West Germany, June 1986.

I_{ij}^k A random variable representing the natural inflow to the reservoir i on river j during a period k in Mm^3. It is assumed that no correlation exists between flows of independent rivers at different periods of time. These random variables are statistically independent with normal distribution ($1\ Mm^3=10^6 m^3$).

x_{ij}^k The storage of reservoir i on river j at the end of period k in Mm^3

u_{ij}^k The discharge from reservoir i on river j during a period k in Mm^3

s_{ij}^k The spill from reservoir i on river j during a period k in Mm^3

c_{ij}^k Value in dollars of one MWh produced anywhere on river j

$\overline{x}_{ij}$ The maximum storage of reservoir i on river j in Mm^3.

$\underline{x}_{ij}$ The minimum storage of reservoir i on river j in Mm^3

$\overline{u}_{ij}^k$ The maximum discharge through the turbines in Mm^3

$\underline{u}_{ij}^k$ The minimum discharge through the turbines in Mm^3

$G_{ij}(u_{ij}^k, 1/2(x_{ij}^k+x_{ij}^{k-1}))$ The generation of plant i on river j during a period k in MWh. It is a nonlinear function of the discharge u_{ij}^k and the average storage between two successive months

$V_{ij}(x_{ij}^K)$ Value in dollars of the water left in storage at the end of the planning horizon

n_j Number of reservoirs on river j; $i=1,\ldots,n_j$; $j=1,\ldots,\ldots,m$

m The total number of rivers

k Superscript denoting the period; $k=1,\ldots,\ldots,K$

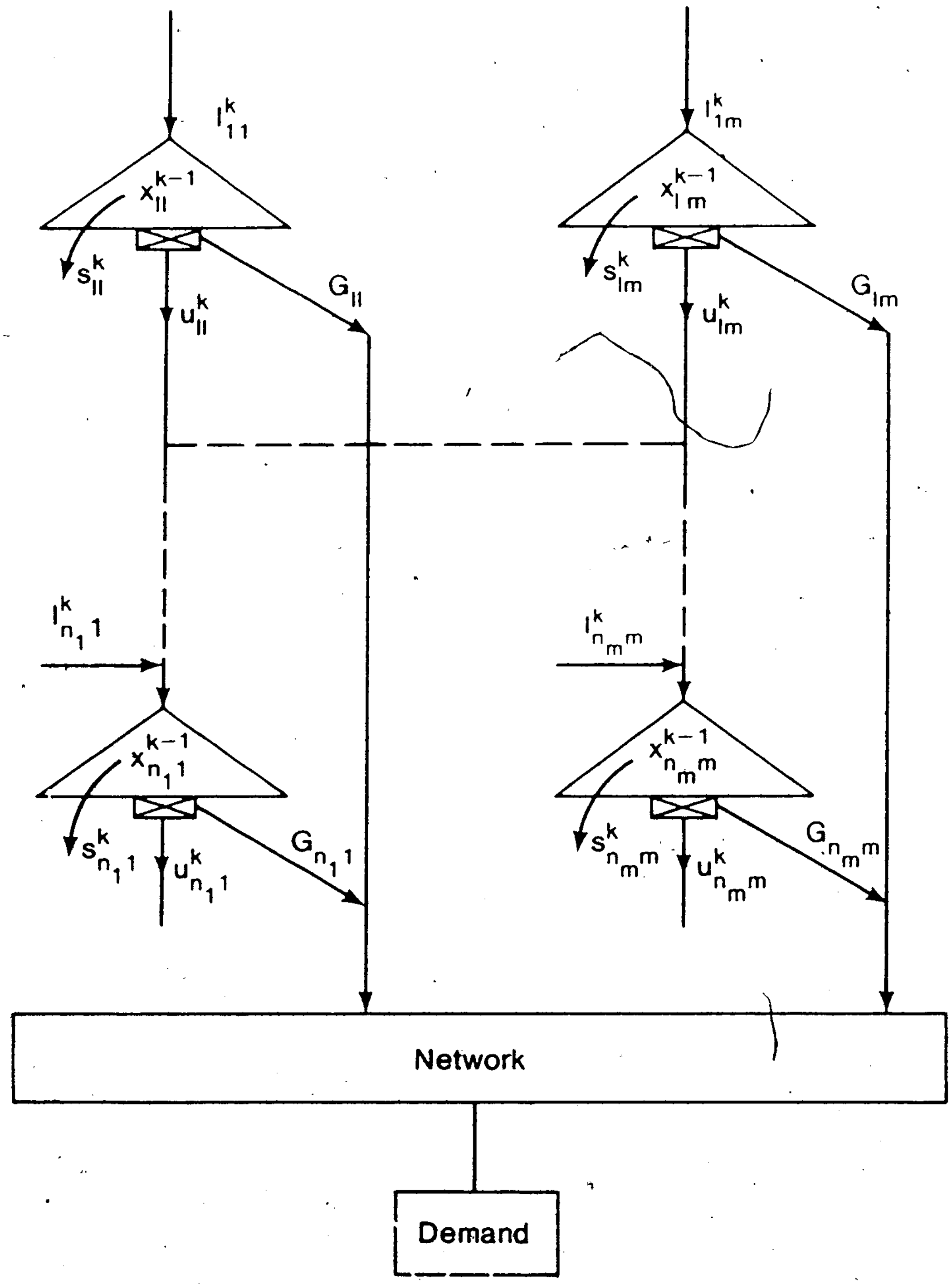

Figure 4.3 A Multiriver Hydroelectric System

4.4.1.2 The Objective Function

The long term optimal operating problem aims to find the discharge u_{ij}^{k}; $i=1,\ldots,\ldots,n_j$; $j=1,\ldots,\ldots,m$ that maximizes the total expected benefits from the system (benefits from the generation and benefits from the amount of water left in storage at the end of the planning period), while satisfying certain constraints. In mathematical terms, the problem of the power system in Figure 4.3 is to find the discharge u_{ij}^{k} that maximizes

$$J=E\left[\sum_{j=1}^{m}\sum_{i=1}^{n_j} V_{ij}(x_{ij}^{K})+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} c_j^{k} G_{ij}(u_{ij}^{k},1/2(x_{ij}^{k}+x_{ij}^{k-1}))\right] \tag{4.173}$$

Subject to satisfying the following constraints

(1) The water conservation equation (continuity equation) for each reservoir may be adequately described by the following difference equation

$$x_{ij}^{k}=x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k} \tag{4.174}$$

where

$$s_{ij}^{k}=\begin{cases}(x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-x_{ij}^{k})-\bar{u}_{ij}^{k};\text{If}(x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k} \\ +s_{(i-1)j}^{k}-x_{ij}^{k})>\bar{u}_{ij}^{k} \text{ and } x_{ij}^{k}>\bar{x}_{ij} \\ 0, \text{ otherwise}\end{cases}$$

water is spilt when the reservoir is filled to capacity, and the inflow to the reservoir exceeds $\bar{u}_{ij}^{k}$

(2) To satisfy multipurpose stream use requirements, such as flood control, irrigation, fishing and other purposes if any, the following upper and lower limits on the variables should be satisfied

(a) upper and lower bounds on the storage

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \bar{x}_{ij} \qquad (4.176)$$

(b) upper and lower bounds on the discharge

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \qquad (4.177)$$

The first set of the inequality constraints simply states that the reservoir storage may not exceed a maximum level, nor be lower than a minimum level. The maximum level is determined by the elevation of the spillway crest or the top of the spillway gates. The minimum level may be fixed by the elevation of the lowest outlet in the dam or by conditions of operating efficiency for the turbines. The second set is determined by the discharge capacity of the power plant as well as its efficiency (32).

The initial storage x_{ij}^{0} and the expected value for the natural inflows into each stream during each month are assumed to be known.

4.4.1.3 Modelling of the System

The value in dollars of water left in storage at the end of the planning period is obtained as follows

(a) Multiply the amount of water left in storage by the water conversion factor of at-a-site and downstream hydroplants. (i.e.) we convert this amount of water to the equivalent electrical energy MWh.

(b) Since no one knows when this energy will be used in the future, we assumed the value of this energy is 1$/MWh, the average value during a year . Following the above two steps we may choose the following for the function $V_{ij}(x_{ij}^{K})$.

$$V_{ij}(x_{ij}^{K}) = \sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}(x_{vj}^{K})^2) \text{ in } \$ \qquad (4.178)$$

In equation (4.178), we assumed that the water conversion factor (MWh/Mm^3) at-a-site has a quadratic relation with the storage. (The storage-elevation curve is a quadratic). The constants α_{ij}, β_{ij} and γ_{ij} were obtained by least square curve fitting to typical plant data available.

The generation of a hydroelectric plant is a nonlinear function of the water discharge u_{ij}^{k} and the reservoir head, which itself is a function of the storage. to avoid underestimation of production for rising water levels and overestimation for falling water levels, an average of begin and-end-of time step (month) storage is used. We may choose the following for the function $G_{ij}(u_{ij}^{k};\ 1/2(x_{ij}^{k}+x_{ij}^{k-1}))$

$$G_{ij}(u_{ij}^{k}, 1/2(x_{ij}^{k}+x_{ij}^{k-1})) = \alpha_{ij}u_{ij}^{k} + 1/2\,\beta_{ij}u_{ij}^{k}(x_{ij}^{k}+x_{ij}^{k-1})$$

$$+ 1/4\ \gamma_{ij}u_{ij}^{k}(x_{ij}^{k}+x_{ij}^{k-1})^2 \qquad (4.179)$$

Substituting for x_{ij}^k from equation (4.174) into equation (4.179), one obtains

$$G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1})) = b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1}$$

$$+ u_{ij}^k f_{ij}^k (u_{(i-1)j}^k - u_{ij}^k) + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2$$

$$+ 1/4\ u_{ij}^k\ \gamma_{ij}((u_{ij}^k)^2 + (u_{(i-1)j}^k)^2)$$

$$+ \gamma_{ij} u_{ij}^k x_{ij}^{k-1}(u_{(i-1)j}^k - u_{ij}^k) - 1/2\ \gamma_{ij} u_{(i-1)j}^k (u_{ij}^k)^2 \qquad (4.180)$$

where

$$q_{ij}^k = I_{ij}^k + s_{(i-1)j}^k - s_{ij}^k \qquad (4.181)$$

$$b_{ij}^k = \alpha_{ij} + 1/2\ \beta_{ij} q_{ij}^k + 1/4\ \gamma_{ij}(q_{ij}^k)^2 \qquad (4.182)$$

$$d_{ij}^k = \beta_{ij} + \gamma_{ij} q_{ij}^k \qquad (4.183)$$

$$f_{ij}^k = 1/2\ \beta_{ij} + 1/2\ \gamma_{ij} q_{ij}^k \qquad (4.184)$$

It will be noticed that the generating function in equation (4.180) is a highly nonlinear function. If one defines the following pseudo-state variables such that (17)

$$y_{ij}^{k}=(x_{ij}^{k})^2;\ i=1,\dots,n_j;j=1,\dots,m;k=1,\dots,K \tag{4.185}$$

$$z_{ij}^{k}=(u_{ij}^{k})^2;\ i=1,\dots,n_j;j=1,\dots,\dots,m;k=1,\dots,\dots,K \tag{4.186}$$

$$r_{ij}^{k-1}=u_{ij}^{k}x_{ij}^{k-1};i=1,\dots,\dots,n_j;j=1,\dots,\dots,m;k=1,\dots,\dots,K \tag{4.187}$$

Then, the function G_{ij} in equation (4.180) becomes

$$\begin{aligned}G_{ij}(u_{ij}^{k},1/2(x_{ij}^{k}+x_{ij}^{k-1}))&=b_{ij}^{k}u_{ij}^{k}+u_{ij}^{k}d_{ij}^{k}x_{ij}^{k-1}\\ &+u_{ij}^{k}f_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k})+\gamma_{ij}u_{ij}^{k}y_{ij}^{k-1}\\ &+1/4\ u_{ij}^{k}\gamma_{ij}(z_{(i-1)j}^{k}+z_{ij}^{k})\\ &+\gamma_{ij}r_{ij}^{k-1}(u_{(i-1)j}^{k}-u_{ij}^{k})-1/2\gamma_{ij}z_{ij}^{k}u_{(i-1)j}^{k}\end{aligned} \tag{4.188}$$

Now, the cost functional in equation (4.173) becomes

$$J=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}y_{vj}^{K})+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}\{c_j^k b_{ij}^k u_{ij}^k$$

$$+ c_j^k u_{ij}^k d_{ij}^k x_{ij}^{k-1}+c_j^k u_{ij}^k f_{ij}^k(u_{(i-1)j}^k-u_{ij}^k)$$

$$+ c_j^k \gamma_{ij} u_{ij}^k y_{ij}^{k-1}$$

$$+ 1/4c_j^k u_{ij}^k \gamma_{ij}(z_{(i-1)j}^k+z_{ij}^k)+c_j^k \gamma_{ij} r_{ij}^{k-1}(u_{(i-1)j}^k-u_{ij}^k)$$

$$- 1/2c_j^k \gamma_{ij} z_{ij}^k u_{(i-1)j}^k\}] \qquad (4.189)$$

Subject to satisfying the following constraints.

$$y_{ij}^k = (x_{ij}^k)^2 \qquad (4.190)$$

$$z_{ij}^k = (u_{ij}^k)^2 \qquad (4.191)$$

$$r_{ij}^{k-1} = u_{ij}^k x_{ij}^{k-1} \qquad (4.192)$$

$$x_{ij}^k = x_{ij}^{k-1}+q_{ij}^k+u_{(i-1)j}^k-u_{ij}^k \qquad (4.193)$$

$$\underline{x}_{ij} \leq x_{ij}^k \leq \overline{x}_{ij} \qquad (4.194)$$

$$\underline{u}_{ij}^k \leq u_{ij}^k \leq \overline{u}_{ij}^k \qquad (4.195)$$

Now, the problem is that of maximizing (4.189) subject to satisfying the constraints (4.190-4.195).

4.4.2 A Minimum Norm Formulation

We can form an augmented cost functional $\tilde{J}$, by adjoining to the cost function in equation (4.189) the equality constraints (4.190-4.192) via Lagrange's multipliers, and the inequality constraints (4.194-4.195) via Kuhn-Tucker multipliers (40), one thus obtains

$$\tilde{J}=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{v=i}^{n_j} x_{ij}^{K}(\alpha_{vj}+\beta_{vj}x_{vj}^{K}+\gamma_{vj}y_{vj}^{K})+\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}\{c_j^k b_{ij}^k u_{ij}^k$$

$$+ c_j^k u_{ij}^k d_{ij}^k x_{ij}^{k-1}+c_j^k u_{ij}^k f_{ij}^k(u_{(i-1)j}^k-u_{ij}^k)$$

$$+ c_j^k \gamma_{ij}u_{ij}^k y_{ij}^{k-1}+ 1/4\ c_j^k u_{ij}^k \gamma_{ij}(z_{(i-1)j}^k+z_{ij}^k)$$

$$+ c_j^k \gamma_{ij}r_{ij}^{k-1}(\hat{u}_{(i-1)j}^k-u_{ij}^k)-1/2c_j^k \gamma_{ij}z_{ij}^k u_{(i-1)j}^k$$

$$+ \mu_{ij}^k(-y_{ij}^k+(x_{ij}^k)^2)+\phi_{ij}^k(-z_{ij}^k+(u_{ij}^k)^2)+\psi_{ij}^k(-r_{ij}^{k-1}+u_{ij}^k x_{ij}^{k-1})$$

$$+ \lambda_{ij}^k(-x_{ij}^k+x_{ij}^{k-1}+q_{ij}^k+u_{(i-1)j}^k-u_{ij}^k)+e_{ij}^k(\underline{x}_{ij}-x_{ij}^k)+e_{ij}^{1k}(x_{ij}^k-\bar{x}_{ij})$$

$$+ g_{ij}^k(\underline{u}_{ij}^k-u_{ij}^k)+g_{ij}^{1k}(u_{ij}^k-\bar{u}_{ij}^k)\}] \qquad (4.196)$$

where μ_{ij}^k, ϕ_{ij}^k, ψ_{ij}^k and λ_{ij}^k are Lagrange's multipliers, they are to be determined, so that the corresponding equality constraints are satisfied and e_{ij}^k, e_{ij}^{1k}, g_{ij}^k and g_{ij}^{1k} are Kuhn-Tucker multipliers, these are equal to zero if the constraints are not violated, and greater than zero if the constraints are violated (17).

Define the following column vectors:

$$A_{ij}= \sum_{v=1}^{n_j} a_{vj} \; ; \; i=1,\ldots,\ldots,n_j; \; ;j=1,\ldots,\ldots,m \qquad (4.197)$$

$$A=\text{col.}(A_1,\ldots,\ldots,\ldots,A_m) \qquad (4.198)$$

$$A_1=\text{col.}(A_{11},\ldots,\ldots,\ldots,A_{n_1 1}) \qquad (4.199)$$

$$A_m=\text{col.}(A_{1m},\ldots,\ldots,\ldots,A_{n_m m}) \qquad (4.200)$$

$$b(k)=\text{col.}(b_1(k),\ldots,\ldots,b_m(k)) \qquad (4.201)$$

$$b_1(k)=\text{col.}(c_1{}^k b_{11}{}^k,\ldots,\ldots,c_1{}^k b_{n_1 1}{}^k) \qquad (4.202)$$

$$b_m(k)=\text{col.}(c_m{}^k b_{1m}{}^k,\ldots,\ldots,c_m{}^k b_{n_m m}{}^k) \qquad (4.203)$$

$$x(k)=\text{col.}(x_1(k),\ldots,\ldots,x_m(k)) \qquad (4.204)$$

$$x_1(k)=\text{col.}(x_{11}{}^k,\ldots,\ldots,x_{n_1 1}{}^k) \qquad (4.205)$$

$$x_m(k)=\text{col.}(x_{1m}{}^k,\ldots,\ldots,x_{n_m m}{}^k) \qquad (4.206)$$

$$y(k)=\text{col.}(y_1(k),\ldots,\ldots,y_m(k)) \qquad (4.207)$$

$$y_1(k)=\text{col.}(y_1{}^k,\ldots,\ldots,y_{n_1 1}{}^k) \qquad (4.208)$$

$$y_m(k)=\text{col.}(y_{1m}{}^k,\ldots,\ldots,y_{n_m m}{}^k) \tag{4.209}$$

$$u(k)=\text{col.}(u_1(k),\ldots,\ldots,u_m(k)) \tag{4.210}$$

$$u_1(k)=\text{col.}(u_{11}{}^k,\ldots,\ldots,u_{n_1 1}{}^k) \tag{4.211}$$

$$u_m(k)=\text{col.}(u_{1m}{}^k,\ldots,\ldots,u_{n_m m}{}^k) \tag{4.212}$$

$$z(k)=\text{col.}(z_1(k),\ldots,\ldots,z_m(k)) \tag{4.213}$$

$$z_1(k)=\text{col.}(z_{11}{}^k,\ldots,\ldots,z_{n_1 1}{}^k) \tag{4.214}$$

$$z_m(k)=\text{col.}(z_{1m}{}^k,\ldots,\ldots,z_{n_m m}{}^k) \tag{4.215}$$

$$r(k)=\text{col.}(r_1(k),\ldots,\ldots,r_m(k)) \tag{4.216}$$

$$r_1(k)=\text{col.}(r_{11}{}^k,\ldots,\ldots,r_{n_1 1}{}^k) \tag{4.217}$$

$$r_m(k)=\text{col.}(r_{1m}{}^k,\ldots,\ldots,r_{n_m m}{}^k) \tag{4.218}$$

$$\mu(k)=\text{col.}(\mu_1(k),\ldots,\ldots,\mu_m(k) \tag{4.219}$$

$$\mu_1(k)=\text{col.}(\mu_{11}{}^k,\ldots,\ldots,\mu_{n_1 1}{}^k) \tag{4.220}$$

$$\mu_m(k)=\text{col.}(\mu_{1m}{}^k,\ldots,\ldots,\mu_{n_m m}{}^k) \tag{4.221}$$

$$\phi(k)=col.(\phi_1(k),\ldots,\ldots,\phi_m(k)) \tag{4.222}$$

$$\phi_1(k)=col.(\phi_{11}{}^k,\ldots,\ldots,\phi_{n_1 1}{}^k) \tag{4.223}$$

$$\phi_m(k)=col.(\phi_{1m}{}^k,\ldots,\ldots,\phi_{n_m m}{}^k) \tag{4.224}$$

$$\psi(k)=col.(\psi_1(k),\ldots,\ldots,\psi_m(k)) \tag{4.225}$$

$$\psi_1(k)=col.(\psi_{11}{}^k,\ldots,\ldots,\psi_{n_1 1}{}^k) \tag{4.226}$$

$$\psi_m(k)=col.(\psi_{1m}{}^k,\ldots,\ldots,\psi_{n_m m}{}^k) \tag{4.227}$$

$$\lambda(k)=col.(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \tag{4.228}$$

$$\lambda_1(k)=col.(\lambda_{11}{}^k,\ldots,\ldots,\lambda_{n_1 1}{}^k) \tag{4.229}$$

$$\lambda_m(k)=col.(\lambda_{1m}{}^k,\ldots,\ldots,\lambda_{n_m m}{}^k) \tag{4.230}$$

$$q(k)=col.(q_1(k),\ldots,\ldots,q_m(k) \tag{4.231}$$

$$q_1(k)=col.(q_{11}{}^k,\ldots,\ldots,q_{n_1 1}{}^k) \tag{4.232}$$

$$q_m(k)=col.(q_{1m}{}^k,\ldots,\ldots,q_{n_m m}{}^k) \tag{4.233}$$

$$s(k)=col.(s_1(k),\ldots,\ldots,s_m(k) \tag{4.234}$$

$$s_1(k)=\text{col.}(s_{11}^{\ k},\ldots,\ldots,s_{n_1 1}^{\ k}) \tag{4.235}$$

$$s_m(k)=\text{col.}(s_{1m}^{\ k},\ldots,\ldots,s_{n_m m}^{\ k}) \tag{4.236}$$

$$\nu_{ij}^{\ k}=e_{ij}^{\ 1k}-e_{ij}^{\ k} \tag{4.237}$$

$$\nu(k)=\text{col.}(\nu_1(k),\ldots,\ldots,\nu_m(k)) \tag{4.238}$$

$$\nu_1(k)=\text{col.}(\nu_{11}^{\ k},\ldots,\ldots,\nu_{n_1 1}^{\ k}) \tag{4.239}$$

$$\nu_m(k)=\text{col.}(\nu_{1m}^{\ k},\ldots,\ldots,\nu_{n_m m}^{\ k}) \tag{4.240}$$

$$\sigma_{ij}^{\ k}=g_{ij}^{\ 1k}-g_{ij}^{\ k} \tag{4.241}$$

$$\sigma(k)=\text{col.}(\sigma_1(k),\ldots,\ldots,\sigma_m(k)) \tag{4.242}$$

$$\sigma_1(k)=\text{col.}(\sigma_{11}^{\ k},\ldots,\ldots,\sigma_{n_1 1}^{\ k}) \tag{4.243}$$

$$\sigma_m(k)=\text{col.}(\sigma_{1m}^{\ k},\ldots,\ldots,\sigma_{n_m m}^{\ k}) \tag{4.244}$$

Furthermore, define the following matrices

$$B=\text{diag.}(B_1,\ldots,\ldots,B_m) \tag{4.245}$$

B_1 is $n_1 x n_1$ matrix whose elements are given by

(i) $b_{ii}=\beta_{i1}$; $i=1,\ldots,\ldots,n_1$

(ii) $b_{(v+1)i}=b_{i(v+1)}=1/2\beta_{(v+1)1}$; $v=1,\ldots,\ldots,n_1-1$ (4.246)

B_m is $n_m x n_m$ matrix whose elements are given by

(i) $b_{ii}=\beta_{im}$ $i=1,\ldots,n_m$

(ii) $b_{(v+1)i}=b_{i(v+1)}=1/2\ \beta_{(v+1)m}$; $v=1,\ldots,\ldots,n_m-1$ (2.247)

$C = \text{diag.}(C_1,\ldots,\ldots,C_m)$ (4.248)

C_1 is upper triangular matrix whose elements are given by

(i) $c_{ii}=\gamma_{i1}$; $i=1,\ldots,\ldots,n_1$

(ii) $c_{(v+1)}=\gamma_{(v+1)1}$; $v=1,\ldots,\ldots,n_1-1$ (4.249)

C_m is upper triangular matrix whose elements are given by

(i) $c_{ii}=\gamma_{im}$; $i=1,\ldots,n_m$

(ii) $c_{i(v+1)}=\gamma_{(v+1)m}$; $v=1,\ldots,n_m-1$ (4.250)

$d(k)=\text{diag.}(d_1(k),\ldots,\ldots,d_m(k))$ (4.251)

$$d_1(k)=\text{diag.}(c_1^{\,k} d_{11}^{\,k},\ldots,\ldots,c_1^{\,k} d_{n_1 1}^{\,k}) \tag{4.252}$$

$$d_m(k)=\text{diag.}(c_m^{\,k} d_{1m}^{\,k},\ldots,\ldots,c_m^{\,k} d_{n_m m}^{\,k}) \tag{4.253}$$

$$f(k)=\text{diag.}(f_1(k),\ldots,\ldots,f_m(k)) \tag{4.254}$$

$$f_1(k)=\text{diag.}(c_1^{\,k} f_{11}^{\,k},\ldots,\ldots,c_1^{\,k} f_{n_1 1}^{\,k}) \tag{4.255}$$

$$f_m(k)=\text{diag.}(c_m^{\,k} f_{1m}^{\,k},\ldots,\ldots,c_m^{\,k} f_{n_m m}^{\,k}) \tag{4.256}$$

$$C(k)=\text{diag.}(C_1(k),\ldots,\ldots,C_m(k)) \tag{4.257}$$

$$C_1(k)=\text{diag.}(c_1^{\,k}\gamma_{11},\ldots,\ldots,c_1^{\,k}\gamma_{n_1 1}) \tag{4.258}$$

$$C_m(k)=\text{diag.}(c_m^{\,k}\gamma_{1m},\ldots,\ldots,c_m^{\,k}\gamma_{n_m m}) \tag{4.259}$$

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \tag{4.260}$$

where any matrix $M_1,\ldots,\ldots,M_m$ is a lower triangular matrix, whose elements are given by

$$\left.\begin{array}{ll}
\text{(i)} & m_{ii}=-1\ ;\ i=1,\ldots,\ldots,n_j \\
\text{(ii)} & m_{(v+1)v}=1\ ;\ v=1,\ldots,n_j-1 \\
\text{(iii)} & \text{the rest of the elements are equal to zero}
\end{array}\right\} \tag{4.261}$$

$N = \text{diag.}(N_1, \ldots, \ldots, \ldots, N_m)$ (4.262)

where any matrix $N_1, \ldots, \ldots, N_m$ is a lower triangular matrix whose elements are given by

$$\left.\begin{array}{ll} \text{(i)} & n_{ii} = 1 \;;\; i = 1, \ldots, \ldots, n_j \\ \text{(ii)} & n_{(v+1)v} = 1 \;;\; v = 1, \ldots, \ldots, n_j - 1 \\ \text{(iii)} & \text{the rest of thes elements are equal to zero} \end{array}\right\} \quad (4.263)$$

$L = \text{diag.}(L_1, \ldots, \ldots, \ldots, L_m)$ (4.264)

where any matrix $L_1, \ldots, \ldots, L_m$ is a lower triangular matrix whose elements are given by

$$\left.\begin{array}{ll} \text{(i)} & \ell_{(v+1)v} = 1 \;;\; v = 1, \ldots, n_j - 1 \\ \text{(ii)} & \text{the rest of the elements are equal to zero} \end{array}\right\} \quad (4.265)$$

Using all the above definitions, the augmented cost functional in equation (4.196) becomes

$$\tilde{J}=E[A^T x(K)+x^T(K)Bx(K)+x^T(K)Cy(K)+\sum_{k=1}^{K} \{b^T(k)u(k)$$

$$+ u^T(k)d(k)x(k-1)+u^T(k)f(k)Mu(k)+u^T(k)C(k)y(k-1)$$

$$+ 1/4\ u^T(k)C(k)Nz(k)+r^T(k-1)C(k)Mu(k)-\ 1/2\ z^T(k)C(k)Lu(k)$$

$$+ \mu^T(k)(-y(k)+x^T(k)\vec{H}x(k))+\phi^T(k)(-z(k)+u^T(k)\vec{H}u(k))$$

$$+ \psi^T(k)(-r(k-1)+u^T(k)\vec{H}x(k-1))+\lambda^T(k)(-x(k)+x(k-1)+q(k)$$

$$+ Mu(k))+\nu^T(k)(x(k-1)+q(k)+Mu(k))+\sigma^T(k)u(k)\}] \qquad (4.266)$$

In equation (4.266) . $\vec{H}$ is a vector matrix in which the vector index varies from 1 to $\sum_{j=1}^{m} n_j$, while the dimension of $\vec{H}$ is $(\sum_{j=1}^{m} n_j)x(\sum_{j=1}^{m} n_j)$ (17).

Employing the discrete version of integration by parts (33), and dropping the constant terms, one obtains

$$\tilde{J}=E[x^T(K)(B+\mu^T(K)\vec{H})x(K)+x^T(K)Cy(K)+(A-\lambda(K))^Tx(K)$$

$$-\mu^T(K)y(K)+x^T(0)\mu^T(0)Hx(0)+\lambda^T(0)x(0)+\mu^T(0)y(0)$$

$$+\sum_{k=1}^{K}\{x^T(k-1)\mu^T(k-1)\vec{H}x(k-1)+u^T(k)d(k)x(k-1)+u^T(k)f(k)Mu(k)$$

$$+u^T(k)C(k)y(k-1)+1/4\ u^T(k)C(k)Nz(k)+r^T(k-1)C(k)Mu(k)$$

$$-1/2\ z^T(k)C(k)Lu(k)-\mu^T(k-1)y(k-1)-\phi^T(k)z(k)+u^T(k)\phi^T(k)\vec{H}u(k)$$

$$-\psi^T(k)r(k-1)+u^T(k)\psi^T(k)\vec{H}x(k-1)+(\lambda(k)-\lambda(k-1)+\nu(k))^Tx(k-1)$$

$$+(b(k)+M^T\lambda(k)+M^T\nu(k)+\sigma(k))^Tu(k)\}] \qquad (4.267)$$

If one defines the following vectors such that (32)

$$Z^T(K)=[x^T(K),\ y^T(K)] \qquad (4.268)$$

$$W(K)=\begin{bmatrix} B+\mu^T(K)\vec{H} & 1/2\ C \\ 1/2C^T & 0 \end{bmatrix} \qquad (4.269)$$

$$Q^T(K)=[(A-\lambda(K))^T,\ -\mu^T(K)] \qquad (4.270)$$

$$X^T(k)=[x^T(k-1),\ y^T(k-1),\ z^T(k),\ r^T(k-1),\ u^T(k)] \qquad (4.271)$$

$$L(k)=\begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 0 & 0 & 1/2(d(k)+\psi^T(k)\vec{H}) \\ 0 & 0 & 0 & 0 & 1/2\ C(k) \\ 0 & 0 & 0 & 0 & (1/8N^TC(k)-1/4C(k)L) \\ 0 & 0 & 0 & 0 & 1/2\ C(k)M \\ 1/2(d(k)+\psi^T(k)\vec{H}) & 1/2C(K) & (1/8C(k)N^T-1/4L^TC(k)) & 1/2M^TC(k) & (f(k)M+\Phi^T(k)\vec{H}\ ) \end{bmatrix} \quad (4.272)$$

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\nu(k))^T, -\mu^T(k-1),\ -\phi^T(k), -\psi^T(k), (b(k)+M^T\lambda(k)+M^T\nu(k) + \sigma(k))^T] \quad (4.273)$$

Then, the augmented cost functional in equation (4.267) can be written as:

$$J=E[\{Z^T(K)W(K)Z(K)+Q^T(K)Z(K)+x^T(0)\mu^T(0)\vec{H}x(0) + \lambda^T(0)x(0)+\mu^T(0)y(0)\}+\sum_{k=1}^{K}\{X^T(k)L(k)X(k)+R^T(k)X(k)\}] \quad (4.274)$$

Furthermore, if one defines the following vectors such that

$$N(K)=W^{-1}(K)Q(K) \tag{4.275}$$

$$V(k)=L^{-1}(k)R(k) \tag{4.276}$$

Then, by using the process of competing the squares equation (4.274) can be written as:

$$\tilde{J}=E[\{(Z(K)+1/2N(K))^{T}W(K)(Z(K)+1/2N(K))$$

$$-1/4N^{T}(K)W(K)N(K)+x^{T}(0)\mu^{T}(0)\vec{H}x(0)+\lambda^{T}(0)x(0)$$

$$+\mu^{T}(0)y(0)\}+\sum_{k=1}^{K}\{(X(k)+1/2V(k))^{T}L(k)(X(k)+1/2V(k))$$

$$-1/4V^{T}(k)L(k)V(k)\}] \tag{4.277}$$

Equation (4.277) can be written as

$$\tilde{J}=E[\{(Z(K)+1/2N(K))^{T}W(K)(Z(K)+1/2N(K))\}$$

$$+\sum_{k=1}^{K}\{(X(k)+1/2V(k))^{T}L(k)(X(k)+1/2V(k))\}] \tag{4.278}$$

because $x(0)$, $y(0)$ are constants and $N(K)$, $V(k)$ are independent of $Z(K)$ and $X(k)$ respectively.

It will be noticed that $\tilde{J}$ in equation (4.278) is composed of a boundary part and a discrete integral part, which are independent of each other. To maximize $\tilde{J}$ in equation (4.278), one maximizes the boundary part and the discrete part separately (40). $\tilde{J}$ in equation (4.278) can be written as

$$\tilde{J} = J_1 + J_2 \tag{4.279}$$

with

$$J_1 = E[(Z(K)+1/2N(K))^T W(K)(Z(K)+1/2N(K))] \tag{4.280}$$

$$J_2 = E[\sum_{k=1}^{K} \{(X(k)+1/2V(k))^T L(k)(X(k)+1/2V(k))\}] \tag{4.281}$$

J_1 in equation (4.280) defines a norm, hence one can write equation (4.280) as:

$$J_1 = E[||Z(K)+1/2N(K)||]_{W(K)} \tag{4.282}$$

Also J_2 in equation (4.281) defines a norm, hence J_2 can be written as:

$$J_2 = E[||X(k) + 1/2\ V(k)||]_{L(k)} \tag{4.283}$$

4.4.3 The Optimal Solution

The boundary term in equation (4.282) is optimized, when the norm in that equation is equal to zero

$$E[Z(K) + 1/2\ N(K)] = [0] \tag{4.284}$$

Substituting from equation (4.275) into equation (4.284) one obtains:

$$E[2W(K)Z(K)+Q(K)] = [0] \qquad (4.285)$$

Writing equation (4.285) explicitly by substituting from equations (4.268-4.270) into equation (4.285) one obtains:

$$E[2Bx(K)+2\mu^T(K)H\vec{x}(K)+Cy(K)+A-\lambda(K)] = [0] \qquad (4.286)$$

$$E[C^T x(K)-\mu(K)] = [0] \qquad (4.287)$$

Equations (4.286) and (4.287) give the values of Lagrange's multipliers, $\lambda(K)$, $\mu(K)$, at the end of the planning horizon.

The discrete integral part in equation (4.283) is optimized, when the norm in that equation is equal to zero (40)

$$E[X(k) + 1/2\ V(k)] = [0] \qquad (4.288)$$

By substituting from equation (4.276) into equation (4.288) one obtains the following equation of optimality.

$$E[2L(k)X(k) + R(k)] = [0] \qquad (4.289)$$

Writing equation (4.289) explicitly, by substituting from equations (4.271-4.273) into equation (4.289), and by adding the equality constraints (4.190-4.193), one obtains the following set:

$$E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^T(k-1)\vec{H}x(k-1)+d(k)u(k)+\psi^T(k)\vec{H}u(k)=[0] \tag{4.290}$$

$$E[-\mu(k-1)+C(k)u(k)] = [0] \tag{4.291}$$

$$E[-\phi(k)+ 1/4\ N^T C(k)u(k)- 1/2\ C(k)Lu(k)] = [0] \tag{4.292}$$

$$E[-\psi(k) + C(k)\ Mu(k)] = [0] \tag{4.293}$$

$$E[\delta(k)+M^T\lambda(k)+M^T\nu(k)+\sigma(k)+d(k)x(k-1)+\psi^T(k)\vec{H}x(k-1)$$

$$+ C(k)y(k-1)+ 1/4\ C(k)N^T z(k)- 1/4\ L^T C(k)z(k)+M^T C(k)r(k-1)$$

$$+ 2f(k)Mu(k)+2\phi^T(k)\vec{H}u(k)] = [0] \tag{4.294}$$

$$E[-x(k)+x(k-1)+q(k)+Mu(k)] = [0] \tag{4.295}$$

$$E[-y(k)+x^T(k)\vec{H}x(k)] = [0] \tag{4.296}$$

$$E[-z(k)+u^T(k)\vec{H}u(k)] = [0] \tag{4.297}$$

$$E[-r(k-1)+u^T(k)\vec{H}x(k-1)] = [0] \tag{4.298}$$

Besides the above equations, one has the following limits on the variables (40).

$$\left.\begin{array}{l} \text{If } x(k) < \underline{x}, \text{ then we put } x(k) = \underline{x} \\ \text{If } x(k) > \overline{x}, \text{ then we put } x(k) = \overline{x} \\ \\ \text{If } u(k) < \underline{u}(k), \text{ then we put } u(k) = \underline{u}(k) \\ \text{If } u(k) > \overline{u}(k), \text{ then we put } u(k) = \overline{u}(k) \end{array}\right\} \quad (4.299)$$

One also has the following exclusion equation which must be satisfied at the optimum (40)

$$\left.\begin{array}{ll} e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})=0; & e_{ij}^{1k}(x_{ij}^{k}-\overline{x}_{ij})=0 \\ g_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})=0; & g_{ij}^{1k}(u_{ij}^{k}-\overline{u}_{ij}^{k})=0 \end{array}\right\} \quad (4.300)$$

Equations (4.290-4.300) with equations (4.286) and (4.287) completely specify the optimal long-term operation of the system. The following algorithm is used to solve these equations.

4.4.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the natural inflows to each site, and the cost of energy on each valley c_j^k in \$/MWh. The following steps are used to solve the optimal equations.

Step 1 Assume initial guess for the variable u(k) such that

$\underline{u}(k) \le u^{i}(k) \le \overline{u}(k)$; i = iteration counter; i=0

Step 2 Assume first that s(k) is equal to zero, solve equations (4.291-4.293) with equations (4.295-4.298) forward in stages with x(0) given

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$\underline{x} < x(k) < \overline{x}$

go to Step 10, otherwise put x(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(-x(k-1)-I(k)+x(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$\underline{u}(k) < u(k) < \overline{u}(k)$

go to Step 14, otherwise put u(k) to its limits ad go to Step 6

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)] = E[(M)^{-1}(x(k)-x(k-1)-I(k))-\overline{u}(k)]$$

If s(k) is less than zero, put s(k) =0, otherwise go to Step 7

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[(M)^{-1}(x(k)-x(k-1)-I(k)-Ms(k))]$$

and go to Step 8

Step 8 Solve again, equations (4.291-4.293) with equations (4.295-4.298) forward in stages with $x(0)$ given, but $s(k)$ has the value obtained from Step 7

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4

Step 10 With $\nu(k)=0$, solve equation (4.290) backward in stages with equations (4.286) and (4.287) as the terminal conditions

Step 11 Calculate Kuhn-Tucker multipliers for $u(k)$, $\sigma(k)$, from the following equation:

$$E[\sigma(k)]=E[2M^T\mu^T(k-1)\vec{H}x(k-1)+M^Td(k)u(k)+M^T\psi^T(k)\vec{H}u(k)$$

$$-b(k)-M^T\lambda(k-1)-d(k)x(k-1)-\psi^T(k)\vec{H}x(k-1)-C(k)y(k-1)$$

$$-1/4C(k)N^Tz(k)+1/4L^TC(k)z(k)-M^TC(k)r(k-1)$$

$$-2f(k)Mu(k)-2\phi^T(k)\vec{H}u(k)]$$

If $u(k)$ satisfies the inequality

$$\underline{u}(k) < u(k) < \overline{u}(k)$$

put $\sigma(k)=0$

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)]=[u^{i}(k)+\alpha D^{i}u(k)]$$

where

$$E[Du(k)]=E[b(k)+M^{T}\lambda(k)+\ (k)+d(k)x(k-1)+\psi^{T}(k)Hx(k-1)$$

$$+\ C(k)y(k-1)+1/4C(k)N^{T}z(k)-1/4L^{T}C(k)z(k)$$

$$+\ M^{T}C(k)r(k-1)+2f(k)Mu(k)+2\phi^{T}(k)Hu(k)]$$

and α is chosen with consideration to such factors as convergence

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 6

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)]=E[2\mu^T(k-1)\vec{H}x(k-1)+d(k)u(k)+\psi^T(k)\vec{H}u(k)$$

$$-[M^T]^{-1}b(k)-[M^T]^{-1}d(k)x(k-1)$$

$$-[M^T]^{-1}\psi^T(k)\vec{H}x(k-1)-[M^T]^{-1}C(k)y(k-1)$$

$$-1/4[M^T]^{-1}C(k)N^Tz(k)+1/4[M^T]^{-1}L^TC(k)z(k)$$

$$+C(k)r(k-1)-2[M^T]^{-1}f(k)Mu(k)-2[M^T]^{-1}\phi^T(k)\vec{H}u(k)]$$

Step 15 Determine Kuhn-Tucker multipliers for x(k), $\nu(k)$, from the following equation

$$E[\nu(k)]=[[M^T]^{-1}(-b(k)-M^T\lambda(k)-d(k)x(k-1)$$

$$-\psi^T(k)\vec{H}x(k-1)-C(k)y(k-1)-1/4C(k)N^Tz(k)$$

$$+1/4\,L^TC(k)z(k)-C(k)r(k-1)-2f(k)Mu(k)$$

$$-2\phi^T(k)\vec{H}u(k)]$$

If x(k) satisfies the inequality

$\underline{x} < x(k) < \overline{x}$

put $\nu(k)=0$

Step 16 Determine a new state iterate from the following

$$E[x^{i+1}(k)]=E[x^{i}(k)+\alpha D^{i}x(k)]$$

where

$$E[Dx(k)]=E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^{T}(k-1)\vec{H}x(k-1)$$

$$+ d(k)u(k)+\psi^{T}(k)\vec{H}u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and the cost function in equation (4.189) is a maximum.

4.4.5 Practical Example

Again, the algorithm of the last section has been used to solve the same example mentioned earlier in both sections 3.3 and 4.3 using the model of this section. The characteristics of the installation are given in Table 4.13.

The expected natural inflows to the sites for a year of high flow, which we call year 1, and the cost of energy in \$/MWh for that year are given in Table 4.14. In Table 4.15, we give the optimal monthly releases from each reservoir and the profits realized at the end of year 1 for the optimal global-feedback solution. In Table 4.16 we give the optimal storage schedule for each reservoir during year 1.

We have simulated the monthly operation of the system for widely different water conditions. The monthly natural inflows to the sites in year 2, which is the year of low flow, are given in Table 4.17. In Table 4.18 we give the optimal monthly releases from each reservoir and the profits realized at the end of year 2. In Table 4.18 we give the optimal storage for each reservoir during year 2. We started both years with the reservoirs full.

The computing time to get the optimal solution for a period of a year was 3.5 sec. in CPU units, which is very small compared to what has been done so far using other approaches.

Table 4.13: The characteristics of the installations

River	Site name	Capacity of the reservoir Mm^3	Minimum storage Mm^3	Max.effective discharge m^3/sec	Min.effective discharge m^3/sec	Reservoir constants		
						α_{ij}	β_{ij}	γ_{ij}
1	R_{11}	24763	9949	1119	85	212.11	146.96×10^{-4}	$-20503142.65\times10^{-14}$
	R_{21}	5304	3734	1583	85	117.20	569.71×10^{-4}	$-368119890.49\times10^{-14}$
2	R_{12}	74255	33196	1877	283	232.46	359.45×10^{-4}	$-1603544.32\times10^{-14}$
	R_{22}	0	0	1930	283	100.74	0	0
3	R_{13}	45672	24467	1632	283	176.28	105.626×10^{-4}	$-10022665.72\times10^{-14}$
	R_{23}	9132	8886	1876	283	131.44	200.897×10^{-4}	$-34741725.6\times10^{-14}$

Table 4.14: The expected natural inflows to the sites in year 1 and the cost of energy

Month k	I_{11}^k Mm^3	I_{21}^k Mm^3	I_{12}^k Mm^3	I_{22}^k Mm^3	I_{13}^k Mm^3	I_{23}^k Mm^3	c^k \$/MWh
1	948	236	2526	30	2799	318	1.4
2	482	189	1226	15	1632	192	1.4
3	350	148	1001	15	1380	221	1.4
4	300	113	849	8	1035	234	0.8
5	229	80	699	7	796	158	0.8
6	225	78	644	8	766	170	0.8
7	385	160	962	7	794	229	0.8
8	1388	910	4588	53	2017	374	0.8
9	4492	2143	17322	147	15509	1920	0.8
10	5028	2025	7660	76	7453	999	0.8
11	2685	963	5195	68	4953	711	1.1
12	1402	580	2349	29	3376	413	1.1

Table 4.15: The optimal releases from the turbines and the profits realized in year 1

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	u_{13}^k Mm^3	u_{23}^k Mm^3	Profits $
1	1649	3451	3821	3851	3641	4010	9898372
2	1589	1453	3703	3718	3538	3730	8685940
3	1552	1578	3839	3855	3601	3871	8970313
4	1318	401	3690	3698	3294	3725	4554491
5	1177	1257	3304	3311	2931	3089	4214638
6	1322	1400	3695	3703	3236	3406	4658819
7	1293	1453	3567	3574	3092	3321	4491206
8	1387	2298	3699	3752	3185	3559	4886224
9	1318	3461	3853	3730	3150	4824	5598097
10	1637	3662	4540	4616	4371	5025	6627686
11	2266	3229	5027	5169	4371	5025	9571348
12	2706	3286	3416	3445	4029	4442	8345312
Value of water left in storage at the end of the year							99,334,960
Total benefits from the system							179,837,406

Table 4.16: Optimal reservoir storage during year 1

Month k	x_{11}^k Mm^3	x_{21}^k Mm^3	x_{12}^k Mm^3	x_{13}^k Mm^3	x_{23}^k Mm^3
1	24062	3828	72959	44779	9132
2	22955	4152	70481	42873	9132
3	21752	4274	67642	40652	9083
4	20734	5304	64801	38393	8886
5	19786	5304	62196	36258	8886
6	18689	5304	59145	33788	8886
7	17780	5304	56539	31489	8886
8	17780	5304	57397	30320	8886
9	20953	5304	71135	42678	9132
10	24345	5304	74255	45672	9132
11	24763	5304	74255	45672	9132
12	23458	5304	73187	450189	9132

Table 4.17: The expected natural inflows to the sites in year 2

Month k	I_{11}^k Mm^3	I_{21}^k Mm^3	I_{12}^k Mm^3	I_{22}^k Mm^3	I_{13}^k Mm^3	I_{23}^k Mm^3
1	551	200	1987	23	2260	531
2	361	285	1226	15	991	169
3	316	298	1054	15	849	83
4	273	128	744	10	720	54
5	237	10	474	6	578	70
6	265	92	563	7	579	90
7	608	251	1182	15	598	95
8	1593	925	3898	53	2336	432
9	3851	2065	7487	73	5872	668
10	4566	1631	2556	50	4179	675
11	2701	1137	1828	23	3299	865
12	2348	1080	1813	22	2644	851

Table 4.18: The optimal releases from the turbines and the profits realized in year 2

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	u_{13}^{k} Mm^3	u_{23}^{k} Mm^3	Profits $
1	2997	3197	2426	2449	2260	2037	8143159
2	2900	3185	2335	2350	1711	1880	7073434
3	2997	3576	2491	2506	1957	2041	7588778
4	2300	2148	1888	1898	922	975	2764157
5	2214	2223	1508	1515	685	754	2399639
6	2310	2401	1902	1909	1026	1117	2843036
7	1704	1955	1779	1794	968	1064	2439814
8	1592	2518	1922	1975	1389	1821	2978143
9	1423	3488	2358	2432	3861	4283	4949586
10	2174	3805	2555	2586	4178	4854	5618122
11	2997	4134	1827	1850	4371	5025	7907461
12	2900	3980	2115	2137	4230	4863	7929455
Value of the water left in storage at the end of the year							90,212,080
Total benefits during year 2							152,846,864

Table 4.19: Optimal reservoir storage during year 2

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3	x_{13}^{k} Mm^3	x_{23}^{k} Mm^3
1	22316	5304	73816	45672	8886
2	19777	5304	72706	44951	8886
3	17096	5023	71269	43843	8886
4	15068	5304	70124	43641	8886
5	13091	5304	69089	43534	8886
6	11045	5304	67749	43085	8886
7	9949	5304	67151	42715	8886
8	9949	5304	69127	43661	8886
9	12376	5304	74255	45672	9132
10	14768	5304	74255	45672	9132
11	14472	5304	74255	44600	9132
12	13918	5304	73053	43034	9132

4.4.6 Discussion

It will be noticed from Table 4.15 that the total benefits during year 1 using this model is $179,837,406. However, by using the model mentioned in section 4.3, the total benefits from the same system during the same year is $165,998,560. Hence, by using a more accurate model the total benefits increased by 8.4 percent.

On the other hand, the total benefits during year 2 for the same system using this model as given in Table 4.18 is $152,846,864. But the total benefits from the same system using the model of section 4.3, during year 2 is $142,506,846. The total benefits increased by 7.3 percent.

From the above results, we conclude that this model is the more accurate one for calculating the total expected benefits from a multireservoir hydroelectric power system, for optimal long-term regulation, and by using this model we avoid the disadvantages of the other models explained in the previous chapters.

CHAPTER V

MODELLING AND OPTIMIZATION OF A MULTIRESERVOIR POWER SYSTEM FOR CRITICAL WATER CONDITIONS

5.1 Background

The period in which reservoirs are drawn down from full to empty is referred to as the "critical period", and the stream flows which occur during the critical period are called "critical period stream flow". The duration of the critical period is determined by the amount of reservoir storage in the hydroelectric system, the amount of energy support available from thermal, gas turbine plants and possible purchase and depends on how these resources are committed to support the hydroelectric system.

The basic requirement for the critical period is that the generation during this period should be equal during each year of the critical period and should supply the required load on the system, and at the same time the reservoirs are drawn down empty at the end of the critical period.

Over the past several years a number of methods have been developed to solve the problem. These methods are limited to either Dynamic Programming, Linear Programming or a combination of them, but they suffer from major problems, when they are applied to the multi-dimensional systems including excessive demands on computing time and storage requirement (1,2).

In Ref. (44), the maximization of energy capability in terms of NonLinear Programming methods has been done for a large scale system of what is believed to be one of the largest hydroelectric Nonlinear

Programming problems attempted. In maximizing the energy capability during the critical period, the problem is formulated as a general nonlinear problem. Next, to account for certain soft constraints for which constraints parameter settings may eliminate the feasible region, the objective function is modified by adding penalty functions. This results in a program with a nonlinear objective function, linear constraints equations, and simple bounds. A solution method is presented employing the Conjugate Gradient Method (45).

In this chapter, the maximization of the production of hydroelectric power for a multireservoir power system has been done. The first section of this chapter is devoted to the maximization of the generation from the system using a linear model of the discharge and the average head between two successive months, and at the same time the generation during the critical period should be equal during each year. In the second section, we maximize the generation for a multireservoir power system; the model used is a quadratic model of the discharge and the storage, and also the generation during the critical period should be equal during each year.

The optimization is done on a monthly time basis for a period of a year, the times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason those times are not taken into account. Transmission line losses are also neglected.

5.2 A Linear Model for the Reservoir*

In this section, we maximize the generation from a multireservoir power system. The model used here is a linear model of the discharge and the average head, in order to avoid overestimation for falling water levels and under estimation for rising water levels. Water head variation, and stochasticity of the river flows are taken into account. According to the data available here, the tail water elevation and the efficiency of the plant are assumed be constant.

5.2.1 Problem Formulation

5.2.1.1 The System Under Study

The system under consideration consists of m independent rivers with one or several reservoirs and power plants in series on each, and interconnection lines to the neighboring system through which energy may be exchanged (Figure 5.1), we will denote by the following:

y_{ij}^k A random variable representing the natural inflow to the reservoir i of river j in period k in Mm^3. These are statistically independent random variables with normal distribution. It is assumed that no correlation exists between flows of independent rivers of different periods of time.

---

* A version of this section has been accepted for publication in the Journal of Optimization Theory and Applications (JOTA), May 1985. (Ref. 26).

h_{ij}^k The net head of the reservoir i on river j at the end of period k in meters;

$$\underline{h}_{ij} \leq h_{ij}^k \leq \overline{h}_{ij}$$

where $\underline{h}_{ij}$ and $\overline{h}_{ij}$ are the minimum and maximum heads respectively.

u_{ij}^k The effective discharge from reservoir i on river j in period k. This is the water released from the reservoir to the allied power plant to produce electricity, in Mm^3;

$$\underline{u}_{ij}^k \leq u_{ij}^k \leq \overline{u}_{ij}^k$$

where $\underline{u}_{ij}^k$ and $\overline{u}_{ij}^k$ are the minimum and maximum discharges respectively; $u_{0j}^k = 0$.

If $u_{ij}^k > \overline{u}_{ij}^k$, and the reservoir is full, $h_{ij}^k = \overline{h}_{ij}$, then $u_{ij}^k - \overline{u}_{ij}^k$ is discharged through the spillways.

s_{ij}^k The spill from reservoir i of river j during the period k in Mm^3;

$$s_{ij}^k \geq 0; \; s_{0j}^k = 0$$

$G_{ij}(u_{ij}^k, 1/2(h_{ij}^k + h_{ij}^{k-1}))$ The generation of plant i on river j in period k in MWh. It is a function of the water discharges and the average head between two successive months, k-1, k.

c_j^k Value in dollars of one MWh generated anywhere on the river j.

n_j The number of reservoirs on each river.

k Superscript denoting the month; $k=1,\ldots,K$.

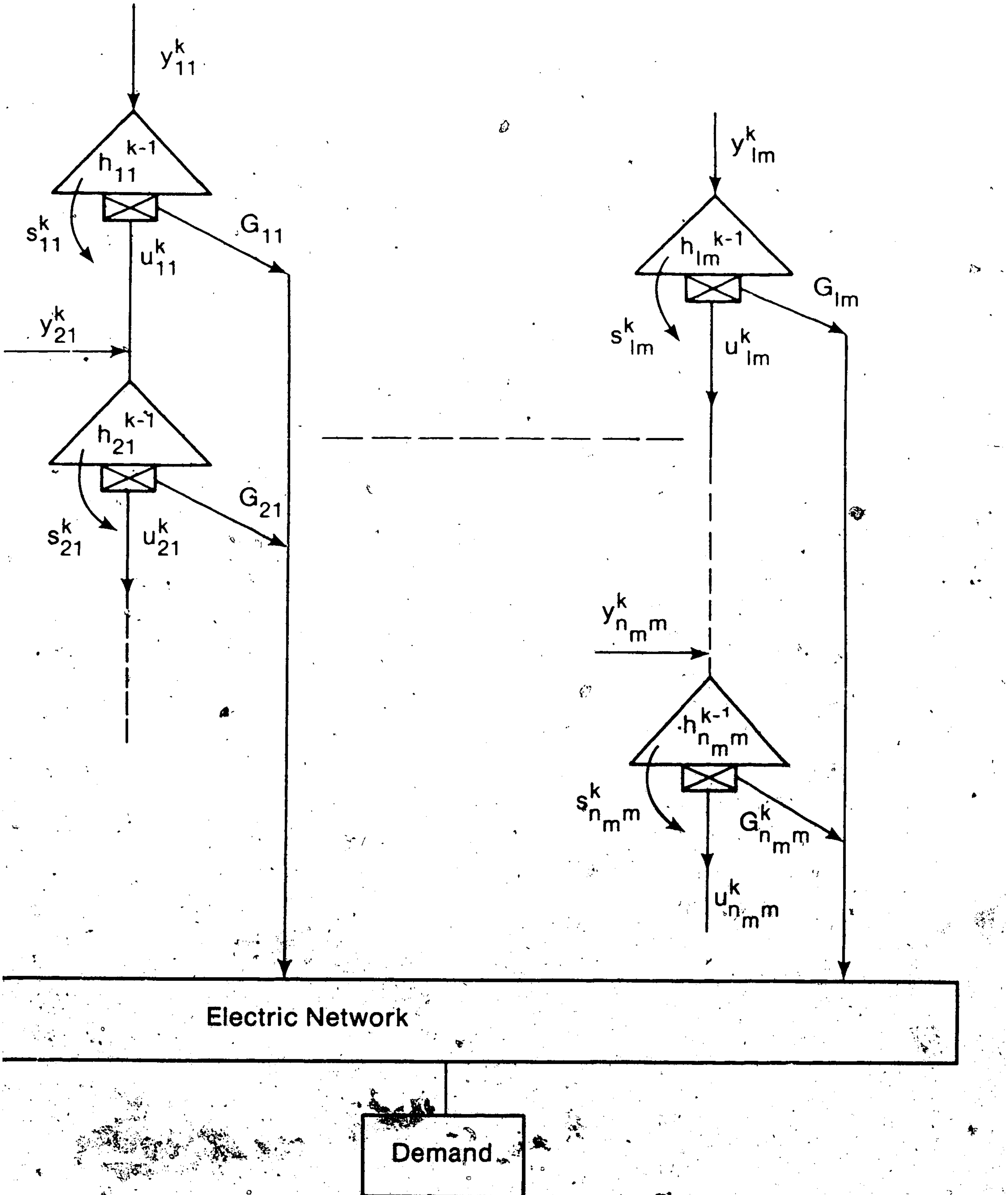

Figure 5.1 A variable head multireservoir power system

5.2.1.2 Objective Function

The optimization objective function is to find the discharge u_{ij}^{k}, that maximizes the total generating benefits from the system during the "critical period", with acceptable equality in this generation during each year of the critical period. This objective function can be expressed as

$$\text{Maximum}\{J=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} c_j^{k} G_{ij}(u_{ij}^{k}, 1/2(h_{ij}^{k}+h_{ij}^{k-1}))]\} \qquad (5.1)$$

In order to be realizable, and also to satisfy multipurpose stream use requirements, the storage management schedule must satisfy the water conservation equations, and the inequality constraints as follows:

(1) The water conservation equation for each reservoir may be adequately described by the following difference equation

$$x_{ij}^{k}=x_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k} \qquad (5.2)$$

where

x_{ij}^{k} is the storage of reservoir i of river j at the end of period k in Mm^3. In this Section, we assumed that the relation between the storage and the net head is a linear relation and this relation is given by

$$x_{ij}^{k} = a_{ij} + b_{ij}h_{ij}^{k} \qquad (5.3)$$

where a_{ij} and b_{ij} are constants. These were obtained by least square curve fitting to typical data available. In this section the variables are the head and the discharge, substituting from equation (5.3) into equation (5.2), one obtains the following water conservation equation as a function of the head.

$$h_{ij}^{k}=h_{ij}^{k-1}+1/b_{ij}\, y_{ij}^{k}+1/b_{ij}\, u_{(i-1)j}^{k}-1/b_{ij}^{k}+u_{ij}^{k}$$

$$+1/b_{ij}\, s_{(i-1)j}^{k}-1/b_{ij}\, s_{ij}^{k} \text{ and } h_{ij}^{k}>\bar{h}_{ij} \qquad (5.4)$$

where

$$s_{ij}^{k}=\begin{cases}(b_{ij}h_{ij}^{k-1}+y_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-b_{ij}h_{ij}^{k})-\bar{u}_{ij}^{k};\text{If}(b_{ij}h_{ij}^{k-1} \\ +y_{ij}^{k}+u_{(i-1)j}^{k}+s_{(i-1)j}^{k}-b_{ij}h_{ij}^{k})>\bar{u}_{ij}^{k} \\ 0, \text{ otherwise}\end{cases} \qquad (5.5)$$

(2) To satisfy multipurpose stream use requirements, such as flood control, irrigation, fishing and other purposes if any, the following upper and lower limits on the variables should be satisfied.

(a) upper and lower bounds on the head.

$$\underline{h}_{ij}\leq h_{ij}^{k}\leq \bar{h}_{ij} \qquad (5.6)$$

(b) upper ad lower bounds on the discharge

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \overline{u}_{ij}^{k} \qquad (5.7)$$

The initial head h_{ij}^{0}, the natural inflows into each stream during each month y_{ij}^{k} and the cost of energy c_j^{k} are assumed to be known. The expectation in equation (5.1) is taken over the equivalent reservoir random inflows.

5.2.1.3 Modelling of the System

The generation of a hydroelectric plant is a non-linear function of the water discharge u_{ij}^{k} and the net head. To avoid underestimation of production for rising water levels and overestimation for falling water levels an average of begin and end-of-time step head has been used. We may choose the following for the function $G_{ij}(u_{ij}^{k}, 1/2(h_{ij}^{k}+h_{ij}^{k-1}))$

$$G_{ij}(u_{ij}^{k}, 1/2(h_{ij}^{k}+h_{ij}^{k-1}))=1/2\, u_{ij}^{k} d_{ij}(h_{ij}^{k}+h_{ij}^{k-1}) \text{ MWh} \qquad (5.8)$$

where

$$d_{ij} = 2.723\, \eta_{ij} \qquad (5.9)$$

η_{ij} = efficiency of plant i on river j

Now, the cost functional in equation (5.1) becomes

$$J=E[\sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} 1/2\, d_{ij}u_{ij}^{k}(h_{ij}^{k}+h_{ij}^{k-1})] \qquad (5.10)$$

Subject to satisfying the constraints (5.4–5.7)

5.2.2 A Minimum Norm Formulation

The problem is now that of maximizing (5.10) subject to satisfying the constraints (5.4–5.7). We can form an augmented cost functional by adjoining to the cost functional in equation (5.10), the equality constraints via Lagrange's multipliers, λ_{ij}^{k}, and the inequality constraints via Kuhn-Tucker multipliers. One thus obtains:

$$\tilde{J}=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}[1/2c_j^{k}d_{ij}u_{ij}^{k}(h_{ij}^{k}+h_{ij}^{k-1})+\lambda_{ij}^{k}(-h_{ij}^{k}+h_{ij}^{k-1}+1/b_{ij}\,y_{ij}^{k}$$

$$+1/b_{ij}\,u_{(i-1)j}^{k}-1/b_{ij}\,u_{ij}^{k}+1/b_{ij}\,s_{(i-1)j}^{k}-1/b_{ij}\,s_{ij}^{k})$$

$$+e_{ij}^{k}(\underline{h}_{ij}-h_{ij}^{k})+e_{ij}^{1k}(h_{ij}^{k}-\bar{h}_{ij})$$

$$+g_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})+g_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k})]] \qquad (5.11)$$

where λ_{ij}^{k} is to be determined such that the corresponding equality constraints must be satisfied, and e_{ij}^{1k}, e_{ij}^{k}, g_{ij}^{k} ad g_{ij}^{1k} are equal to zero, if the constraints are not violated and greater than zero, if the constraints are violated (32).

Now, define the following vectors such that

$$h(k)=\text{col.}(h_1(k),\ldots,\ldots,h_m(k)) \qquad (5.12)$$

$$h_1(k)=\text{col.}(h_{11}^{k},\ldots,\ldots,h_{n_1 1}^{k}) \qquad (5.13)$$

$$h_m(k)=\text{col.}(h_{1m}{}^k,\ldots,\ldots,h_{n_m m}{}^k) \tag{5.14}$$

$$u(k)=\text{col.}(u_1(k),\ldots,\ldots,u_m(k)) \tag{5.15}$$

$$u_1(k)=\text{col.}(u_{11}{}^k,\ldots,\ldots,u_{n_1 1}{}^k) \tag{5.16}$$

$$u_m(k)=\text{col.}(u_{1m}{}^k,\ldots,\ldots,u_{n_m m}{}^k) \tag{5.17}$$

$$\lambda(k)=\text{col.}(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \tag{5.18}$$

$$\lambda_1(k)=\text{col.}(\lambda_{11}{}^k,\ldots,\ldots,\lambda_{n_1 1}{}^k) \tag{5.19}$$

$$\lambda_m(k)=\text{col.}(\lambda_{1m}{}^k,\ldots,\ldots,\lambda_{n_m m}{}^k) \tag{5.20}$$

$$y(k)=\text{col.}(y_1(k),\ldots,\ldots,y_m(k)) \tag{5.21}$$

$$y_1(k)=\text{col.}(y_{11}{}^k,\ldots,\ldots,y_{n_1 1}{}^k) \tag{5.22}$$

$$y_m(k)=\text{col.}(y_{1m}{}^k,\ldots,\ldots,y_{n_m m}{}^k) \tag{5.23}$$

$$s(k)=\text{col.}(s_1(k),\ldots,\ldots,s_m(k)) \tag{5.24}$$

$$s_1(k)=\text{col.}(s_{11}{}^k,\ldots,\ldots,s_{n_1 1}{}^k) \tag{5.25}$$

$$s_m(k)=\text{col.}(s_{1m}^{\ k},\ldots,\ldots,s_{n_m m}^{\ k}) \qquad (5.26)$$

$$\psi_{ij}^{\ k} = e_{ij}^{\ lk} - e_{ij}^{\ k} \qquad (5.27)$$

$$\psi(k)=\text{col.}(\psi_1(k),\ldots,\ldots,\psi_m(k)) \qquad (5.28)$$

$$\psi_1(k)=\text{col.}(\psi_{11}^{\ k},\ldots,\ldots,\psi_{n_1 1}^{\ k}) \qquad (5.29)$$

$$\psi_m(k)=\text{col.}(\psi_{1m}^{\ k},\ldots,\ldots,\psi_{n_m m}^{\ k}) \qquad (5.30)$$

$$\mu_{ij}^{\ k} = g_{ij}^{\ lk} - g_{ij}^{\ k} \qquad (5.31)$$

$$\mu(k)=\text{col.}(\mu_1(k),\ldots,\ldots,\mu_m(k)) \qquad (5.32)$$

$$\mu_1(k)=\text{col.}(\mu_{11}^{\ k},\ldots,\ldots,\mu_{n_1 1}^{\ k}) \qquad (5.33)$$

$$\mu_m(k)=\text{col.}(\mu_{1m}^{\ k},\ldots,\ldots,\mu_{n_m m}^{\ k}) \qquad (5.34)$$

Furthermore, define the following diagonal matrices

$$D(k)=\text{diag.}(D_1(k),\ldots,\ldots,D_m(k)) \qquad (5.35)$$

$$D_1(k)=\text{diag.}(c_1^{\ k}d_{11},\ldots,\ldots,c_1^{\ k}d_{n_1 1}) \qquad (5.36)$$

$$D_m(k)=\text{diag.}(c_m^{\ k}d_{1m},\ldots,\ldots,c_m^{\ k}d_{n_m m}) \qquad (5.37)$$

$$B=\text{diag.}(B_1,\ldots,\ldots,B_m) \tag{5.38}$$

$$B_1=\text{diag.}(1/b_{11},\ldots,\ldots,1/b_{n_1 1}) \tag{5.39}$$

$$B_m=\text{diag.}(1/b_{1m},\ldots,\ldots,1/b_{n_m m}) \tag{5.40}$$

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \tag{5.41}$$

where the matrices $M_1,\ldots,M_m$ are lower triangular matrices, whose elements are given by

$$\left.\begin{array}{ll} \text{(i)} & m_{ij}=-1/b_{ij};\ i=1,\ldots,\ldots,n_{ij};\ j=1,\ldots,m \\ \text{(ii)} & m_{(\omega+1)}=1/b_{(\omega+1)j};\ \omega=1,\ldots,n_j-1;\ j=1,\ldots,\ldots,m \end{array}\right\} \tag{5.42}$$

Using all the above definitions, the augmented cost functional in equation (5.11) becomes:

$$\tilde{J}=E[\sum_{k=1}^{K}\{1/2\ h^T(k-1)D(k)u(k)+1/2\ u^T(k)D(k)h(k)+\lambda^T(k)(-h(k)$$

$$+\ h(k-1)+By(k)+Ms(k)+Mu(k))+\psi^T(k)h(k)+\mu^T(k)u(k)\}] \tag{5.43}$$

Note that constant terms are dropped in the above equation.

Employing the discrete version of integration by parts, and substituting for h(k), one obtains (33)

$$J=E[-\lambda^T(K)h(K)+\lambda^T(0)h(0)+\sum_{k=1}^{K}\{1/2\ h^T(k-1)D(k)u(k)$$

$$+\ 1/2\ u^T(k)D(k)h(k-1)+\ 1/2\ u^T(k)D(k)Mu(k)$$

$$+(\lambda(k)-\lambda(k-1)+\psi(k))^T h(k-1)+(M^T\lambda(k)+M^T\psi(k)+\mu(k)+1/2D(k)By(k)$$

$$+\ 1/2D(k)Ms(k))u(k)\}] \tag{5.44}$$

Note that terms explicitly independent of h(k-1) and u (k) are dropped in the above equation.

It will be noticed that J in equation (5.44) is composed of a boundary part and a discrete integral part, which are indpendent of each other. If one defines the following such that

$$X^T(k)=[h^T(k-1),\ u^T(k)] \tag{5.45}$$

$$L(k)=\begin{bmatrix} 0 & 1/2D(k) \\ 1/2D(k) & 1/2D(k)M \end{bmatrix} \tag{5.46}$$

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\psi(k))^T,(M^T\lambda(k)+M^T\psi(k)+\mu(k)+1/2D(k)By(k)\ +$$

$$+\ 1/2D(k)Ms(k))] \tag{5.47}$$

then, the cost functional in equation (5.44) can be written as

$$J=E[-\lambda^T(K)h(K)+\lambda^T(0)h(0)+\sum_{k=1}^{K}\{X^T(k)L(k)X(k)+R^T(k)X(k)\}] \quad (5.48)$$

To maximize J in equation (5.48), one maximizes the boundary part and the discrete part separately. If one defines the following vector such that

$$V(k) = L^{-1}(k)R(k) \quad (5.49)$$

then, the discrete integral part can be written in the following form by a process similar to completing the squares as (33)

$$J_2=E[\sum_{k=1}^{K}(X(k)+1/2V(k))^TL(k)(X(k)+1/2V(k))-1/4V^T(k)L(k)V(k)] \quad (5.50)$$

Since it is desired to maximize J_2 with respect to X(k), the problem is equivalent to

$$\underset{X(k)}{Max.}J_2=\underset{X(k)}{Max.}E[\sum_{k=1}^{K}(X(k)+1/2V(k))^TL(k)(X(k)+1/2V(k))] \quad (5.51)$$

because V(k) is independent of X(k) Equation (5.51) defines a norm, hence equation (5.51) can be written as

$$\underset{X(k)}{Max.}J_2=\underset{X(K)}{Max.}E[\,||X(k)+1/2V(k)||\,]_{L(k)} \quad (5.52)$$

5.2.3 The Optimal Solution

The maximum of J_2 is achieved when the norm in equation (5.52) is equal to zero

$$E[X(k)+1/2V(k)] = [0] \tag{5.53}$$

Substituting from equation (5.49) into equation (5.53), one obtains the following optimality condition

$$E[R(k)+2L(k)X(k)] = [0] \tag{5.54}$$

The boundary term in equation (5.48) is maximized when

$$E[\lambda(K)] = [0] \tag{5.55}$$

because h(0) is constant and $\delta h(K)$ is arbitrary.

Writing equation (5.54) explicitly by substituting from equations (5.45-5.47) into this equation, one obtains the following equations for long-term optimal operation under critical water conditions

$$E[-h(k)+h(k-1)+By(k)+Ms(k)+Mu(k)] = [0] \tag{5.56}$$

$$E[\lambda(k)-\lambda(k-1)+\psi(k)+D(k)u(k)] = [0] \tag{5.57}$$

$$E[M^T\lambda(k)+M^T\psi(k)+\mu(k)-1/2D(k)By(k)-1/2D(k)Ms(k)+D(k)h(k)] = [0] \tag{5.58}$$

We can now state the optimal solution of equations (5.56-5.58) in component form

$$E\{-h_{ij}^k+h_{ij}^{k-1}+1/b_{ij}y_{ij}^k+1/b_{ij}s_{(i-1)j}^k-1/b_{ij}s_{ij}^k-1/b_{ij}u_{ij}^k$$

$$+ 1/b_{ij}u_{(i-1)j}^k] = 0 \tag{5.59}$$

$$E[\lambda_{ij}^k-\lambda_{ij}^{k-1}+\psi_{ij}^k+c_j^k d_{ij}u_{ij}^k] = 0 \tag{5.60}$$

$$E[1/b_{(i+1)j}\lambda_{(i+1)j}^k-1/b_{ij}\lambda_{ij}^k+1/b_{(i+1)j}\psi_{(i+1)j}^k+1/b_{ij}\psi_{ij}^k+\mu_{ij}^k$$

$$-1/2c_j^k d_{ij}/b_{ij}\; y_{ij}^k-1/2\; c_j^k d_{ij}/b_{ij}\; (s_{(i-1)j}^k-s_{ij}^k)$$

$$+c_j^k d_{ij}h_{ij}^k] \tag{5.61}$$

Besides the above equations, one has the following limits on the variable (40).

$$\left.\begin{array}{l} \text{If } h_{ij}^k < \underline{h}_{ij}, \text{ then we put } h_{ij}^k = \underline{h}_{ij} \\ \text{If } h_{ij}^k > \overline{h}_{ij}, \text{ then we put } h_{ij}^k = \overline{h}_{ij} \\ \\ \text{If } u_{ij}^k < \underline{u}_{ij}^k, \text{ then we put } u_{ij}^k = \underline{u}_{ij}^k \\ \text{If } u_{ij}^k > \overline{u}_{ij}^k, \text{ then we put } u_{ij}^k = \overline{u}_{ij}^k \end{array}\right\} \tag{5.62}$$

One also has the following exclusion equations which must be satisfied at the optimum (40).

$$e_{ij}^{k}(\underline{h}_{ij}-h_{ij}^{k}) = 0 \qquad (5.63)$$

$$e_{ij}^{1k}(h_{ij}^{k}-\overline{h}_{ij}) = 0 \qquad (5.64)$$

$$g_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \qquad (5.65)$$

$$g_{ij}^{1k}(u_{ij}^{k}-\overline{u}_{ij}^{k}) = 0 \qquad (5.66)$$

Equations (5.56-5.66) with equation (5.55) completely specify the optimum solution. The following algorithm is used to solve these equations.

5.2.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the expected value for the natural inflows during the critical period (y_{ij}^{k}), the initial head h_{ij}^{0} and the cost of energy on each river c_j^{k} in $/MWh.

Step 1 Assume initial guess for the variable u(k) such that

$$\underline{u}(k) \leq u^{i}(k) \leq \overline{u}(k) \quad i = \text{iteration number; } i=0$$

Step 2 Assume first that s(k) is equal to zero, solve the following stochastic difference equation

$$E[-h(k)+h(k-1)+By(k)+Mu(k)] = [0]$$

forward in stages with h(0) given.

Step 3 Check the limits on h(k). If h(k) satisfies the inequality

$$\underline{h} < h(k) < \overline{h}$$

go to Step 10, otherwise put h(k) to its limits and go to Step 4

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(h(k)-h(k-1)-By(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \overline{u}(k)$$

go to Step 14, otherwise put u(k) to its limits and go to Step 6

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)] = E[[M]^{-1}(h(k)-h(k-1)-By(k))-u(k)]$$

If s(k) is less than zero, put s(k)=0, otherwise go to Step 7

Step 7 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(h(k)-h(k-1)-By(k)-Ms(k)]$$

Step 8 Solve again the following equation forward in stages

$$E[h(k)] = E[h(k-1)+By(k)+Mu(k)+Ms(k)]$$

with h(0) given

Step 9 Check the limits on h(k). If h(k) satisfies the inequality

$$\underline{h} < h(k) < \overline{h}$$

go to Step 10, otherwise put h(k) to its limits and go to Step 4

Step 10 With $\psi(k)=0$, solve the following stochastic difference equation

$$E[\lambda(k-1)] = [\lambda(k)+D(k)u(k)]$$

backward in stages with the terminal condition

$$E[\lambda(K)] = 0$$

Step 11 Calculate the Kuhn-Tucker multipliers for u(k), $\mu(k)$, from the following equation

$$E[\mu(k)] = E[M^T D(k)u(k) - D(k)h(k) - M^T\lambda(k-1)$$

$$+ 1/2\ D(k)By(k) + 1/2\ D(k)Ms(k)]$$

If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \overline{u}(k)$$

put $\mu(k) = 0$

Step 12 Determine a new control iterate from the following approximation

$$E[u^{i+1}(k)] = E[u^i(k) + \alpha Du^i(k)]$$

where

$$E[Du(k)] = E[M^T\lambda(k) + \mu(k) + D(k)h(k) - 1/2D(k)By(k)$$

$$- 1/2\ D(k)Ms(k)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence.

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 6

Step 14 Solve the following difference equation

$$E[\lambda(k-1)]=E[D(k)u(k)-1/2[M^T]^{-1}D(k)By(k)$$

$$\pm 1/2[M^T]^{-1}D(k)Ms(k)-[M^T]^{-1}D(k)h(k)]$$

forward in stages

Step 15 Calculate the Kuhn-Tucker multiplier for $h(k)$, $\psi(k)$, from the following equation

$$E[\psi(k)]=E[-\lambda(k)+1/2[M^T]^{-1}D(k)By(k)+1/2[M^T]^{-1}D(k)Bs(k)$$

$$-[M^T]^{-1}D(k)h(k)]$$

If $\underline{h}<h(k)<\overline{h}$, put $\psi(k)=0$

Step 16 Determine a new state iterate from the following approximation

$$E[h^{i+1}(k)]=E[h^i(k)+\alpha Dh^i(k)]$$

where

$$E[Dh(k)]=[\lambda(k)-\lambda(k-1)+\psi(k)+D(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state $h(k)$ and the control $u(k)$ do not change sigificantly from iteration to iteration and J in equation (5.10) is a maximum and equal for each year of the critical period.

5.2.5 Practical Example

The previous algorithm has been used to optimize the generation from a hydroelectric power system consisting of two rivers, each river has two series reservoirs, for the 43 months critical period. The characteristics of the installations are given in Table (5.1).

The minimum and maximum releases in Mm^3 can be obtained from the effective minimum and maximum discharges through the turbines as:

$$\underline{u}_{ij}^{k} = 0.0864d^{k} \text{ (minimum effective discharge in } m^3/\text{sec)}$$
$$\bar{u}_{ij}^{k} = 0.0864d^{k} \text{ (maximum effective discharge in } m^3/\text{sec)}$$

where d^k is the number of days in month k, and the maximum and minimum effective discharges are given in Table (5.1).

In Tables (5.3-5.5), we give for each site, the storage-head characteristics, and the percentage error in the storage for using a linear function between the storage and the head.

The expected natural inflows to the sites during the critical period are given in Table (5.2). In Tables (5.6-5.9) we give the optimal monthly releases from each reservoir and the profits realized from each year during the critical period. It will be noticed that the benefits during each year of the critical period are almost equal.

In Tables (5.10-5.13) we give the optimal head for each reservoir during the critical period. We started the critical period with the reservoirs full and we end up the critical period with empty reservoirs.

The computing time to get the optimal long-term operation for the critical period, which is 43 months, was about 4.41 sec. in CPU units, which is very small compared to what has been done so far using other approaches.

Table 5.1: Characteristics of the installations

River	Reservoir	Capacity	Maximum effective discharge	Minimum effective discharge	Maximum head	Minimum head	Reservoirs constants	
j	R_{ij}	Mm^3	m^3/sec	m^3/sec	m	m	a_{ij}	b_{ij}
1	R_{11}	24763	1119	84.95	184.4	137.2	-34360.8	315.95
	R_{21}	5304	1583	84.95	128.9	113.7	-4287.2	72.11
2	R_{12}	74255	1877	283.20	168.0	136.0	-141291.2	1267.6
	R_{22}	0	1930	283.20	41.1	41.1	0	0

Table 5.2: The expected monthly inflows to the sites during the critical period

Month k	y_{11}^k Mm^3	y_{21}^k Mm^3	y_{12}^k Mm^3	y_{22}^k Mm^3
1	796	372	1085	23
2	369	184	910	7
3	288	140	645	8
4	207	74	781	8
5	190	81	452	6
6	313	70	485	6
7	947	521	866	7
8	1456	849	3898	53
9	2833	1307	9175	73
10	4611	1714	4877	61
11	3148	895	1798	23
12	1285	426	1585	22
13	811	387	1320	15
14	363	183	1299	15
15	219	97	872	8
16	188	67	880	8
17	184	102	493	6
18	213	76	473	6
19	411	288	1740	22
20	1798	1024	4103	53
21	3428	1666	7120	88
22	2950	1168	3989	53
23	2700	834	2184	23
24	1798	683	1549	22
25	918	493	2124	23
26	545	321	1431	22
27	293	225	1024	15
28	227	193	727	9
29	190	185	647	8
30	236	146	497	6
31	241	157	559	7
32	1623	956	4194	53
33	3346	1637	7560	73
34	3572	1229	3534	46
35	2526	789	1737	23
36	1270	433	1262	15
37	658	275	1380	15
38	340	162	837	7
39	224	161	689	8
40	149	105	465	6
41	149	84	351	5
42	234	80	411	5
43	465	291	837	7

Table 5.3: Head-storage characteristic and the percentage error in the storage for reservoir R_{11}

Head m	Storage Mm^3	% Error in the storage
137.10	9904.5	-3.80
140.20	10545.8	-2.40
143.26	11205.9	-1.20
149.35	12614.5	0.86
152.40	13383.2	1.65
156.97	14643.0	2.40
158.50	15096.4	2.50
161.54	16062.8	2.50
164.59	17090.4	2.23
167.64	18158.4	1.81
170.69	19267.5	1.25
172.82	20068.1	0.71
175.26	21007.4	0.06
177.69	21973.1	-0.77
179.83	22839.5	-1.55
182.88	24111.7	-2.80
184.40	24763.3	-3.50

Table 5.4: Head-storage characteristic and the percentage error in the storage for reservoir R_{21}

Head m	Storage Mm^3	% Error in the storage
74.1	1233.6	-3.35
104.6	2960.5	5.60
113.7	3762.3	2.80
119.8	4317.5	0.64
124.4	4687.4	-0.08
128.9	5304.2	-5.60

Table 5.5: Head-storage characteristic and the percentage error in the storage for reservoir R_{12}

Head m	Storage Mm^3	% Error in the storage
135.94	33010.9	-2.67
134.47	34403.1	-1.90
138.99	35839.3	-1.30
140.52	37324.4	-0.70
142.04	38711.7	-0.10
143.56	40441.5	-0.33
145.09	42068.6	0.75
146.61	43744.5	1.10
148.13	45471.9	1.35
149.66	47255.6	1.56
151.18	49097.9	1.68
152.71	51006.4	1.72
154.23	52985.7	1.65
155.75	55040.9	1.48
157.28	57174.5	1.20
158.80	59388.8	0.83
161.24	63095.5	0.00
163.37	66496.4	-0.95
165.51	70036.8	-2.10
167.94	74254.9	-3.60

Table 5.6: Optimal releases from the reservoirs and the profits realized during the first year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits \$
1	1138	1509	2382	2405	2,211,074
2	896	1080	2323	2330	1,927,593
3	917	1057	2386	2394	1,950,518
4	919	993	2387	2395	1,919,313
5	866	946	2205	2211	1,777,283
6	923	993	2389	2396	1,895,785
7	909	1430	2330	2337	1,989,025
8	937	1786	2395	2448	2,152,412
9	2900	4103	4153	4226	4,624,473
10	2997	4239	4876	4937	5,111,436
11	2997	4042	2403	2426	3,805,175
12	1731	2006	2345	2367	2,573,737
The total benefits from the generation					31,937,824

Table 5.7: Optimal releases from the reservoirs and the profits realized during the second year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits $
1	1312	1699	1783	1798	2,005,966
2	1312	1494	1761	1776	1,921,542
3	1397	1495	1792	1800	1,959,549
4	1364	1431	1796	1803	1,911,283
5	1306	1408	1718	1724	1,825,635
6	1198	1274	1800	1806	1,766,817
7	1142	1430	1778	1800	1,774,213
8	1566	2591	1816	1869	2,329,294
9	2900	4103	4865	4953	4,865,134
10	2997	4165	4994	5047	4,994,847
11	2793	3831	2686	2709	3,633,917
12	2216	3079	2054	2076	2,831,200
The total benefits from the generation					31,819,397

Table 5.8 Optimal releases from the reservoirs and the profits realized during the third year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits $
1	1899	2007	2373	2396	2,533,056
2	1931	2253	2351	2373	2,604,947
3	1921	2146	2390	2406	2,558,283
4	1925	2119	2396	2405	2,521,139
5	1278	1463	2304	2312	2,020,492
6	236	381	2403	2409	1,362,521
7	241	398	2374	2381	1,346,895
8	1623	2579	2420	2473	2,527,237
9	2046	3683	3865	3938	3,744,704
10	1995	3224	5022	5067	4,173,851
11	2997	3786	2725	2748	3,589,547
12	2487	2920	2257	2271	2,885,410
The total benefits from the generation					31,868,082

Table 5.9: Optimal releases from the reservoirs and the profits realized during the rest of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits $
1	489	1138	5027	5043	2,952,121
2	473	1101	4865	4873	2,808,546
3	489	908	5027	5036	2,778,431
4	489	595	5027	5033	2,639,836
5	425	509	4541	4546	2,329,853
6	234	313	4465	4470	2,139,283
7	465	756	837	844	731,082
The total benefits from the generation					16,379,152

Table 5.10: Optimal head during the first year of the critical period

Month k	h_{11}^{k} m	h_{21}^{k} m	h_{12}^{k} m	h_{22}^{k} m
1	183.3	128.9	167.5	41.1
2	181.7	128.9	166.5	41.1
3	179.7	128.9	165.1	41.1
4	177.4	128.9	163.8	41.1
5	175.3	128.9	162.4	41.1
6	173.3	128.9	160.9	41.1
7	173.5	128.9	159.8	41.1
8	175.1	128.9	160.9	41.1
9	174.9	128.9	164.9	41.1
10	180.0	128.9	164.9	41.1
11	180.5	126.8	164.4	41.1
12	179.1	128.9	163.8	41.1

Table 5.11: Optimal head during the second year of the critical period

Month k	h_{11}^{k} m	h_{21}^{k} m	h_{12}^{k} m	h_{22}^{k} m
1	177.5	128.9	163.5	41.1
2	174.5	128.9	163.1	41.1
3	170.7	128.9	162.4	41.1
4	167.0	128.9	161.6	41.1
5	163.5	128.9	160.7	41.1
6	160.4	128.9	159.6	41.1
7	158.0	128.9	159.6	41.1
8	158.8	128.9	161.4	41.1
9	160.4	128.9	163.2	41.1
10	160.3	128.9	162.4	41.1
11	160.	126.1	162.0	41.1
12	158.7	123.6	161.6	41.1

Table 5.12: Optimal head during the third year of the critical period

Month k	h_{11}^{k} m	h_{21}^{k} m	h_{12}^{k} m	h_{22}^{k} m
1	155.6	128.9	161.4	41.1
2	151.2	128.9	160.7	41.1
3	146.0	128.9	159.6	41.1
4	140.7	128.9	158.3	41.1
5	137.2	128.9	156.9	41.1
6	137.2	128.9	155.5	41.1
7	137.2	128.9	154.0	41.1
8	137.2	128.9	155.4	41.1
9	141.3	128.9	158.3	41.1
10	146.3	128.9	157.2	41.1
11	144.8	128.9	156.4	41.1
12	141.0	128.9	155.6	41.1

Table 5.13: Optimal head during the rest of the critical period

Month k	h_{11}^{k} m	h_{21}^{k} m	h_{12}^{k} m	h_{22}^{k} m
1	140.4	123.7	152.7	41.1
2	140.0	117.3	149.5	41.1
3	139.2	113.7	146.1	41.1
4	138.1	113.7	142.5	41.1
5	137.2	113.7	139.2	41.1
6	137.2	113.7	136.0	41.1
7	137.2	113.7	136.0	41.1

5.2.6 Discussion

In this Section we maximized the generation from a hydroelectric power system, the model used for each reservoir is a linear model of storage and the head. For this model the error in the storage for some reservoirs is greater than the natural inflow to this reservoir during a certain month, for example the percentage error in the storage of the reservoir R_{21} at the maximum head is (-5.6%) from the maximum storage which is 5304Mm^3. This error is about 297Mm^3, while the minimum inflow to this reservoir is 67Mm^3 during the fourth month of the second year of the critical period.

Hence, by using this model which may not be accurate, the total benefits from the system during the last seven months of the critical period is small. However, the total benefits during the critical period using this model is $112,004,460 with an average $31,257,057 which are greater than the total benefits and the average obtained for the same system using the other approaches.

5.3 A Nonlinear Model for the Reservoir*

In section 5.2 we maximized the generation from a hydroelectric power system under critical water condition. The model used for each reservoir was a linear model, this model is not accurate, since in some cases, the error in the storage for some reservoirs is greater than the minimum natural inflow to the reservoir. As a result the total generation during the last couple of months of the critical period is small.

In this section we maximize the generation from the same system, but the model use in this case is a nonlinear model of storage (quadratic function). To overcome these nonlinearities during the formulation of the problem as a minimum norm problem, we introduce a set of pseudo-state variables to cast the problem into a quadratic form. A set of discrete optimizing equations is obtained, these equations are solved forward and backward in time. In the solution of these equations it was found that it is convenient to let the final states be free and assign values to them to get uniform generation during each year of the critical period (9).

5.3.1 The Problem Formulation

5.3.1.1 The System Under Study

The system under consideration consists of m independent rivers

---

* A version of this Section has been accepted for publication in the Journal of Optimization Theory and Application (JOTA), May 1985 (Ref. 27).

with one or several reservoirs and power plants in series on each, and interconnection lines to the neighbouring system through which energy may be exchanged (Figure 5.2). We will denote by the following

I_{ij}^{k} A random variable representing the natural inflow to the site i of river j during a period k in Mm^3. These are statistically independent random variables with normal distribution. It is assumed that no correlation exists between flows of independent rivers at different periods of time.

x_{ij}^{k} The storage of reservoir i on river j at the end of period k in Mm^3;

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \overline{x}_{ij}$$

where $\underline{x}_{ij}$ and $\overline{x}_{ij}$ are the minimum and maximum storages.

u_{ij}^{k} The discharge from reservoir i on river j in month k in Mm^3;

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \overline{u}_{ij}^{k} \; ; \; u_{0j}^{k} = 0$$

where $\overline{u}_{ij}^{k}$ and $\underline{u}_{ij}^{k}$ are the maximum and minimum releases from the turbines. If $u_{ij}^{k} > \overline{u}_{ij}^{k}$, then $u_{ij}^{k} - \overline{u}_{ij}^{k}$ Mm^3 is discharged through the spillways.

s_{ij}^{k} The spill from reservoir i of river j in month k in Mm^3;

$$s_{ij}^{k} \geq 0; \; s_{0j}^{k} = 0$$

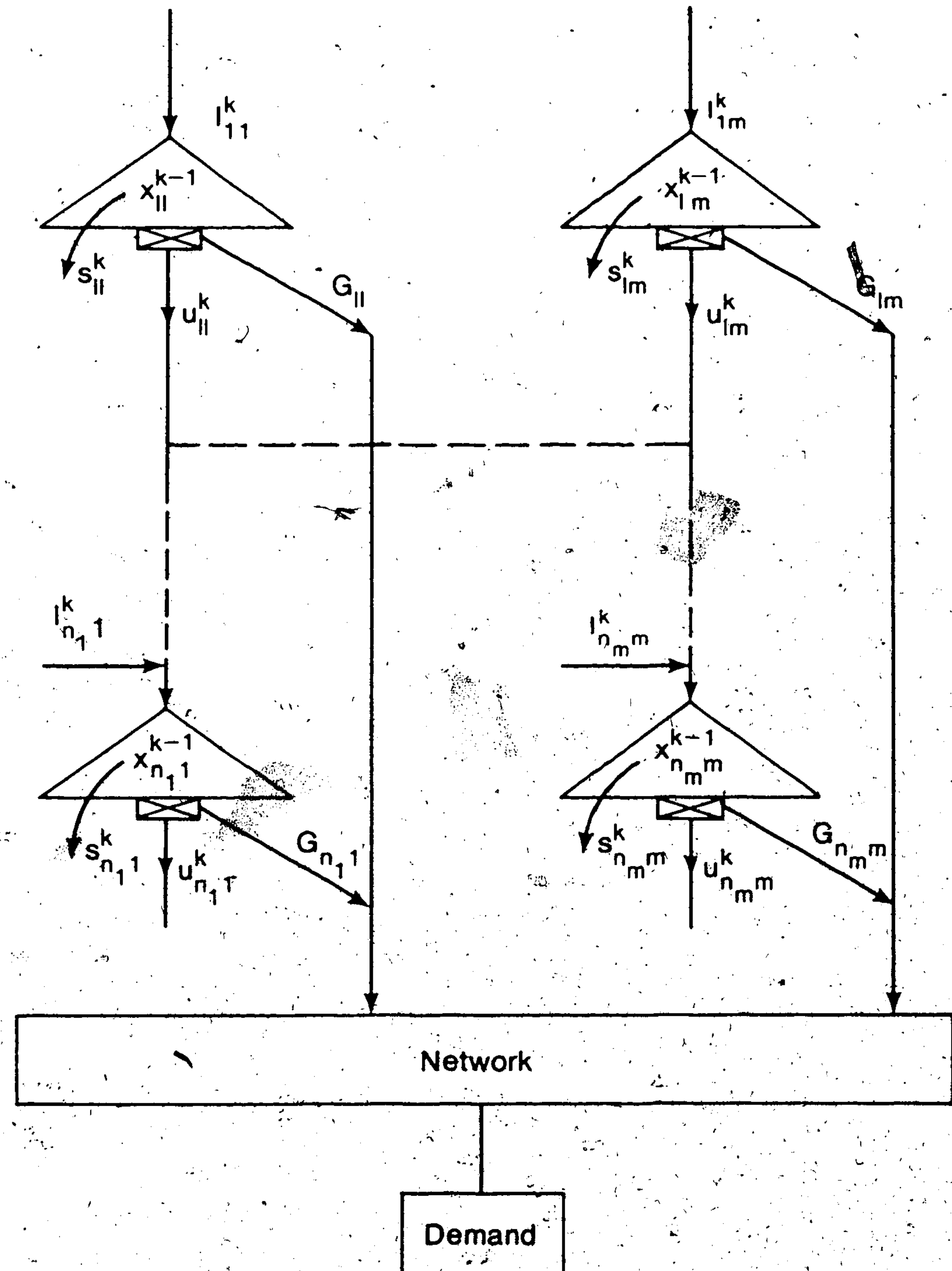

Figure 5.2 The Power System

$G_{ij}(u_{ij}^{k}, x_{ij}^{k-1})$ The generation of plant i on river j in month k in MWh. It is a function of the discharge u_{ij}^{k} and the water head which itself is a function of the reservoir content.

c_{j}^{k} The value in dollars of one MWh produced anywhere on the river j in month k.

k Superscript denoting the month; $k=1,\ldots,K$.

i Subscript denoting the reservoir's number; $i=1,\ldots,n_j$.

j Subscript denoting the river's number; $j=1,\ldots,m$.

5.3.1.2 The Objective Function

Consider the electric system shown in Figure 5.2, with m rivers and each river has n_j series reservoirs. The problem is to find the discharge, u_{ij}^{k}, as a function of time under the following conditions.

1. The total hydroelectric generation from the system over the optimization interval is maximum.
2. The generation during each year of the critical period is equal.
3. The water conservation equation for each reservoir is adequately described by the following difference equation

$$x_{ij}^{k}=x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k} \quad (5.61)$$

where the spill at any month k is given by

$$s_{ij}^{k} = \begin{cases} (x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) - \bar{u}_{ij}^{k}; \text{If } (x_{ij}^{k-1} \\ + I_{ij}^{k} + u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) > \bar{u}_{ij}^{k} \text{ and } x_{ij}^{k} \\ > \bar{x}_{ij} \\ 0, \text{ otherwise} \end{cases} \tag{5.62}$$

4. To satisfy the multipurpose stream use requirements the following operational constraints should be satisfied

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \bar{x}_{ij}^{k}$$

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \tag{5.63}$$

In mathematical terms the objective function for the hydroelectric system in Figure (5.2) can be written as:

$$\text{Max.}[J = E[\sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} c_j^{k} G_{ij}(u_{ij}^{k}, x_{ij}^{k-1})]] \tag{6.64}$$

Subject to satisfying the following constraints

$$x_{ij}^{k} = x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} - u_{ij}^{k} + s_{(i-1)j}^{k} - s_{ij}^{k} \tag{5.65}$$

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \bar{x}_{ij} \tag{5.66}$$

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \tag{5.67}$$

The initial storage x_{ij}^{0}, the expected value for the natural inflows into each stream during each month, and the cost of energy on each river are assumed to be known.

5.3.1.3 Modelling of the System

The generation of a hydroelectric plant is a nonlinear function of water discharge u_{ij}^k and of the water head which itself is a function of the reservoir content. It is assumed that the tail-race elevation at any of the hydro-plants does not change with the water discharge. We may choose the following for the function $G_{ij}(u_{ij}^k, x_{ij}^{k-1})$.

$$G_{ij}(u_{ij}^k, x_{ij}^{k-1}) = \alpha_{ij} u_{ij}^k + \beta_{ij} u_{ij}^k x_{ij}^{k-1} + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2 \quad (5.68)$$

where α_{ij}, β_{ij} and γ_{ij} are constants. These are obtained by least square curve fitting to typical data available.

Now, the cost funtional in equation (5.64) becomes

$$J = E[\sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} c_j^k \alpha_{ij} u_{ij}^k + c_j^k \beta_{ij} u_{ij}^k x_{ij}^{k-1} + c_j^k \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2] \quad (5.69)$$

It will be noticed that the cost functional in equation (5.69) is a highly nonlinear function. If one defines the following n_j pseudo-state variables such that (17)

$$y_{ij}^k = (x_{ij}^k)^2 \quad i=1,\ldots,n_j;\ j=1,\ldots,m \quad (5.70)$$

then the cost functional in equation (5.69) becomes

$$J=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} c_j^k\alpha_{ij}u_{ij}^k+c_j^k\beta_{ij}u_{ij}^k x_{ij}^{k-1}+c_j^k\gamma_{ij}u_{ij}^k y_{ij}^{k-1}] \quad (5.71)$$

Subject to satisfying the constraints (5.65-5.67) and (5.70).

5.3.2 A Minimum Norm Formulation

The problem now is that of maximizing (5.71) subject to satisfying the constraints (5.65-5.67) and (5.70). We can form an augmented cost functional by adjoining to the cost function (5.71) the equality constraints (5.65) and (5.70) via Lagrange's multipliers λ_{ij}^k, μ_{ij}^k and the inequality constraints (5.66) and (5.67) via Kuhn-Tucker multipliers (40). One thus obtains:

$$\begin{aligned}\tilde{J}=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} \{ & c_j^k\alpha_{ij}u_{ij}^k+c_j^k\beta_{ij}u_{ij}^k x_{ij}^{k-1}+c_j^k\gamma_{ij}u_{ij}^k y_{ij}^{k-1} \\ & + \lambda_{ij}^k(-x_{ij}^k+x_{ij}^{k-1}+I_{ij}^k+u_{(i-1)j}^k-u_{ij}^k+s_{(i-1)j}^k-s_{ij}^k) \\ & + \mu_{ij}^k(-y_{ij}^k+(x_{ij}^k)^2) \\ & + e_{ij}^k(\underline{x}_{ij}-x_{ij}^k)+e_{ij}^{1k}(x_{ij}^k-\bar{x}_{ij})+f_{ij}^k(\underline{u}_{ij}^k-u_{ij}^k) \\ & + f_{ij}^{1k}(u_{ij}^k-\bar{u}_{ij}^k)\}]\end{aligned} \quad (5.72)$$

where λ_{ij}^k and μ_{ij}^k are to be determined such that the corresponding equality constraints are satisfied, and e_{ij}^{k}, e_{ij}^{lk}, f_{ij}^{k} and f_{ij}^{lk} are Kuhn-Tucker multipliers. These are equal to zero, if the constraints are not violated and greater than zero, if the constraints are violated (32).

Define the following column vectors such that (32)

$$u(k)=col.(u_1(k),\ldots,\ldots,u_m(k)) \tag{5.73}$$

$$u_1(k)=col.(u_{11}^{k},\ldots,\ldots,u_{n_1 1}^{k}) \tag{5.74}$$

$$u_m(k)=col.(u_{1m}^{k},\ldots,\ldots,u_{n_m m}^{k}) \tag{5.75}$$

$$x(k)=col.(x_1(k),\ldots,\ldots,x_m(k)) \tag{5.76}$$

$$x_1(k)=col.(x_{11}^{k},\ldots,\ldots,x_{n_1 1}^{k}) \tag{5.77}$$

$$x_m(k)=col.(x_{1m}^{k},\ldots,\ldots,x_{n_m m}^{k}) \tag{5.78}$$

$$y(k)=col.(y_1(k),\ldots,\ldots,y_m(k)) \tag{5.79}$$

$$y_1(k)=col.(y_{11}^{k},\ldots,\ldots,y_{n_1 1}^{k}) \tag{5.80}$$

$$y_m(k)=col.(y_{1m}^{k},\ldots,\ldots,y_{n_m m}^{k}) \tag{5.81}$$

$$s(k)=col.(s_1(k),\ldots,\ldots,s_m(k)) \tag{5.82}$$

$$s_1(k)=col.(s_{11}^k,\ldots,\ldots,s_{n_1 1}^k) \quad (5.83)$$

$$s_m(k)=col.(s_{1m}^k,\ldots,\ldots,s_{n_m m}^k) \quad (5.84)$$

$$I(k)=col.(I_1(k),\ldots,\ldots,I_m(k)) \quad (5.85)$$

$$I_1(k)=col.(I_{11}^k,\ldots,\ldots,I_{n_1 1}^k) \quad (5.86)$$

$$I_m(k)=col.(I_{1m}^k,\ldots,\ldots,I_{n_m m}^k) \quad (5.87)$$

$$\lambda(k)=col.(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \quad (5.88)$$

$$\lambda_1(k)=col.(\lambda_{11}^k,\ldots,\ldots,\lambda_{n_1 1}^k) \quad (5.89)$$

$$\lambda_m(k)=col.(\lambda_{1m}^k,\ldots,\ldots,\lambda_{n_m m}^k) \quad (5.90)$$

$$\mu(k)=col.(\mu_1(k),\ldots,\ldots,\mu_m(k)) \quad (5.91)$$

$$\mu_1(k)=col.(\mu_{11}^k,\ldots,\ldots,\mu_{n_1 1}^k) \quad (5.92)$$

$$\mu_m(k)=col.(\mu_{1m}^k,\ldots,\ldots,\mu_{n_m m}^k) \quad (5.93)$$

$$\nu_{ij}^k=e_{ij}^{1k}-e_{ij}^k \quad (5.94)$$

$$\nu(k)=col.(\nu_1(k),\ldots,\ldots,\nu_m(k)) \quad (5.95)$$

$$\nu_1(k)=\text{col.}(\nu_{11}{}^k,\ldots,\ldots,\nu_{n_1 1}{}^k) \tag{5.96}$$

$$\nu_m(k)=\text{col.}(\nu_{1m}{}^k,\ldots,\ldots,\nu_{n_m m}{}^k) \tag{5.97}$$

$$\psi_{ij}{}^k=f_{ij}{}^{1k}-f_{ij}{}^k \tag{5.98}$$

$$\psi(k)=\text{col.}(\psi_1(k),\ldots,\ldots,\psi_m(k)) \tag{5.99}$$

$$\psi_1(k)=\text{col.}(\psi_{11}{}^k,\ldots,\ldots,\psi_{n_1 1}{}^k) \tag{5.100}$$

$$\psi_m(k)=\text{col.}(\psi_{1m}{}^k,\ldots,\ldots,\psi_{n_m m}{}^k) \tag{5.101}$$

$$A(k)=\text{col.}(A_1(k),\ldots,\ldots,A_m(k)) \tag{5.102}$$

$$A_1(k)=\text{col}(c_1{}^k\alpha_{11},\ldots,\ldots,c_1{}^k\alpha_{n_1 1}) \tag{5.103}$$

$$A_m(k)=\text{col.}(c_m{}^k\alpha_{1m},\ldots,\ldots,c_m{}^k\alpha_{n_m m}) \tag{5.104}$$

Furthermore, define the following NxN diagonal matrices, $N=\sum_{j=1}^{m} n_j$

$$B(k)=\text{diag.}(B_1(k),\ldots,\ldots,B_m(k)) \tag{5.105}$$

$$B_1(k)=\text{diag.}(c_1{}^k\beta_{11},\ldots,\ldots,c_1{}^k\beta_{n_1 1}) \tag{5.106}$$

$$B_m(k)=\text{diag.}(c_m{}^k\beta_{1m},\ldots,\ldots,c_m{}^k\beta_{n_m m}) \tag{5.107}$$

$$C(k)=\text{daig.}(C_1(k),\ldots,\ldots,C_m(k)) \quad (5.108)$$

$$C_1(k)=\text{diag.}(c_1{}^k\gamma_{11},\ldots,\ldots,c_1{}^k\gamma_{n_1 1}) \quad (5.109)$$

$$C_m(k)=\text{diag.}(c_m{}^k\gamma_{1m},\ldots,\ldots,c_m{}^k\gamma_{n_m m}) \quad (5.110)$$

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \quad (5.111)$$

where the matrices $M_1,\ldots,M_m$ are lower triangular matrices whose elements are given by:

$$\left.\begin{array}{lll} \text{(i)} & m_{ii}=-1 & ;\ i=1,\ldots,\ldots,n_j \\ \text{(ii)} & m_{(v+1)v}=1 & ;\ v=1,\ldots,\ldots,n_j-1 \end{array}\right\} \quad (5.112)$$

Using all the above definitions, the augmented cost functional, $\tilde{J}$, in equation (5.72) becomes:

$$\tilde{J}=E[\sum_{k=1}^{K}\{A^T(k)u(k)+1/2x^T(k-1)B(k)u(k)+1/2x^T(k-1)B(k)u(k)$$

$$+\ 1/2\ u^T(k)C(k)y(k-1)+1/2y^T(k-1)C(k)u(k)-\mu^T(k)y(k)$$

$$+\ x^T(k)\mu^T(k)\vec{H}x(k)+\lambda^T(k)(-x(k)+x(k-1)+I(k)$$

$$+\ Mu(k)+Ms(k))+\nu^T(k)x(k)+\psi^T(k)u(k)\}] \quad (5.113)$$

In the above equation $\vec{H}$ is a vector matrix in which the vector index varies from 1 to N, while the dimension of the matrix $\vec{H}$ is NxN; $N = \sum_{j=1}^{m} n_j$.

Employing the discrete version of integration by parts and dropping terms which do not depend on x(k-1), y(k-1) and u(k). One obtains:

$$\tilde{J} = E[\{x^T(K)\mu^T(K)\vec{H}x(K) - \lambda^T(K)x(K) - \mu^T(K)y(K)$$

$$+ x^T(0)\mu^T(0)\vec{H}x(0) + \lambda^T(0)x(0) - \mu^T(0)y(0)\}$$

$$+ \sum_{k=1}^{K} \{x^T(k-1)\mu^T(k-1)\vec{H}x(k-1) + 1/2x^T(k-1)B(k)u(k)$$

$$+ 1/2\, u^T(k)B(k)x(k-1) + 1/2u^T(k)C(k)y(k-1) + 1/2y^T(k-1)C(k)u(k)$$

$$+ (\lambda(k) - \lambda(k-1) + \nu(k))^T x(k-1) + (A(k) + M^T\lambda(k) + M^T\nu(k)$$

$$+ \psi(k))^T u(k) - \mu^T(k-1)y(k-1)\}] \qquad (5.114)$$

It will be noticed that $\tilde{J}$ in equation (5.114) is composed of a boundary part and a discrete integral part, which are independent of each other. If one defines the following vectors such that:

$$X^T(k) = [x^T(k-1),\ y^T(k-1),\ u^T(k)] \qquad (5.115)$$

$$L(k)=\begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 1/2B(k) \\ 0 & 0 & 1/2C(k) \\ 1/2B(k) & 1/2C(k) & 0 \end{bmatrix} \tag{5.116}$$

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\nu(k))^T, -\mu^T(k-1), (A(k)+M^T\lambda(k)+M^T\nu(k)+\psi(k))^T] \tag{5.117}$$

Then, $\tilde{J}$ in equation (5.114) can be written as:

$$J=E[\{x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)+x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+\lambda^T(0)x(0)-\mu^T(0)y(0)\}+\sum_{k=1}^{K}\{X^T(k)L(k)X(k)+R^T(k)X(k)\}]. \tag{5.118}$$

To maximize $\tilde{J}$ in equation (5.118), one can maximize the boundary and the discrete integral parts separately (40). If one defines the vector V(k) such that

$$V(k) = L^{-1}(k)R(k) \tag{5.119}$$

then, the discrete integral part can be written as

$$J_2=E[\sum_{k=1}^{K}(X(k)+1/2V(k))^T L(k)(X(k)+1/2V(k))-1/4V^T(k)L(k)V(k)] \tag{5.120}$$

The last term in equation (5.120) does not depend explicitly on X(k). Thus one needs only to consider maximizing

$$J_2 = E\left[\sum_{k=1}^{K} (X(k)+1/2V(k))^T L(k)(X(k)+1/2V(k)) \right] \quad (5.121)$$

Equation (5.121) defines a norm, hence equation (5.121) can be written as

$$\underset{X(k)}{\text{Max.}} J_2 = \underset{X(k)}{\text{Max.}} E[\,||X(k)+1/2V(k)||\,]_{L(k)} \quad (5.122)$$

5.3.3 The Optimal Solution

The optimal solution to the problem formulated in equation (5.122) is given by

$$E[X(k)+1/2V(k)] = [0] \quad (5.123)$$

Substituting from equation (5.119) into equation (5.123), one obtains thes following equation, which must be satisfied at the optimum

$$E[R(k)+2L(k)X(k)] = [0] \quad (5.124)$$

The boundary term in equation (5.118) is optimized when

$$E[\mu(K)] = [0] \quad (5.125)$$

$$E[\lambda(K)] = [0] \quad (5.126)$$

Writing equation (5.124) explicitly and adding the water conservation equation (5.65) and equation (5.70), one obtains the following set

$$E[-x(k)+x(k-1)+I(k)+Mu(k)+Ms(k)] = [0] \quad (5.127)$$

$$E[-y(k)+x^T(k)\vec{H}x(k)] = [0] \quad (5.128)$$

$$E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^T(k-1)\vec{H}x(k-1)+B(k)u(k)] = [0] \quad (5.129)$$

$$E[-\mu(k-1)+2C(k)u(k)] = [0] \quad (5.130)$$

$$E[A(k)+M^T\lambda(k)+M^T\nu(k)+\psi(k)+B(k)x(k-1)+C(k)y(k-1)] = [0] \quad (5.131)$$

We can now state the optimal solution of equations (5.127-5.131) in component form

$$E[-x_{ij}^{k}+x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k}]=0 \quad (5.132)$$

$$E[-y_{ij}^{k} \qquad +(x_{ij}^{k})^2 \;]=0 \quad (5.133)$$

$$E[\lambda_{ij}^{k}-\lambda_{ij}^{k-1}+\nu_{ij}^{k}+2\mu_{ij}^{k-1}x_{ij}^{k-1}+\beta_{ij}c_j^{k}u_{ij}^{k}]=0 \quad (5.134)$$

$$E[-\mu_{ij}^{k-1}+2\gamma_{ij}c_j^{k}u_{ij}^{k}]=0 \quad (5.135)$$

$$E[\alpha_{ij}c_j^k+\lambda_{(i+1)j}^k-\lambda_{ij}^k-\nu_{ij}^k+\nu_{(i+1)j}^k+\psi_{ij}^k+\beta_{ij}c_j^k x_{ij}^{k-1}$$

$$+\gamma_{ij}c_j^k y_{ij}^{k-1}]=0 \qquad (5.136)$$

One has the following Kuhn-Tucker exclusion equations which must be satisfied at the optimum (40).

$$e_{ij}^k(\underline{x}_{ij}-x_{ij}^k)=0 \qquad (5.137)$$

$$e_{iji}^{1k}(x_{ij}^k-\overline{x}_{ij})=0 \qquad (5.138)$$

$$f_{ij}^k(\underline{u}_{ij}^k-u_{ij}^k)=0 \qquad (5.139)$$

$$f_{ij}^{1k}(u_{ij}^k-\overline{u}_{ij}^k)=0 \qquad (5.140)$$

One also has the following limits on the variables

$$\left.\begin{array}{ll} \text{If } x_{ij}^k < \underline{x}_{ij}, & \text{then we put } x_{ij}^k=\underline{x}_{ij} \\ \text{If } x_{ij}^k > \overline{x}_{ij}, & \text{then we put } x_{ij}^k=\overline{x}_{ij} \\ & \\ \text{If } u_{ij}^k < \underline{u}_{ij}^k, & \text{then we put } u_{ij}^k=\underline{u}_{ij}^k \\ \text{If } u_{ij}^k > \overline{u}_{ij}^k, & \text{then we put } u_{ij}^k=\overline{u}_{ij}^k \end{array}\right\} \qquad (5.141)$$

Equations (5.132-5.141) with equations (5.125) and (5.126) completely specify the optimal solution. The following algorithm is used to solve these equations.

5.3.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the expected value for the natural inflows, the initial storage x_{ij}^0 and the cost of energy on each river c_j^k in $/MWh. The following steps are used to solve the above equations. In the solution of these equations we used the expected value for the random variables.

Step 1 Assume initial guess for the variable u(k) such that

$$\underline{u}(k) \leq u^i(k) \leq \overline{u}(k); \; i = \text{iteration number}; \; i = 0$$

Step 2 Assume first that s(k) is equal to zero, solve equations (5.127), (5.128) and (5.130) forward in stages with x(0) given.

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

go to Step 10, otherwise put x(k) to its limits and go to Step 4.

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k) - x(k-1) - I(k))]$$

Step 5 Check the limits on $u(k)$. If $u(k)$ satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

go to Step 14, otherwise put $u(k)$ to its limits and go to Step 6.

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)] = E[[M]^{-1}(x(k) - x(k-1) - I(k)) - \bar{u}(k)]$$

If $s(k) < 0$ put $s(k) = 0$

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k) - x(k-1) - I(k) - Ms(k))]$$

Step 8 Solve again equations (5.127), (5.128) and (5.130) forward in stages with $x(0)$ given.

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4.

Step 10 With $\nu(k)=0$, solve equation (5.129) backward in stages with equation (5.126) as the terminal conditions.

Step 11 Calculate the Kuhn-Tucker multipliers for u(k), $\psi(k)$, from the following equation

$$E[\Psi(k)]=E[M^T B(k)u(k)+2M^T\mu^T(k-1)\vec{H}x(k-1)$$

$$- B(k)x(k-1)-C(k)y(k-1)-A(k)-M^T\lambda(k-1)]$$

If u(k) satisfies the inequality

$$\underline{u}(k)<u(k)<\bar{u}(k)$$

put $\psi(k)=0$

Step 12 Determine a new control from the following equation

$$E[u^{i+1}(k)]=E[u^i(k)+\alpha Du^i(k)]$$

where

$$E[Du(k)]=E[A(k)+M^T\lambda(k)+B(k)x(k-1)+C(k)y(k-1)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence.

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k)<u^{i+1}(k)<\overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 2.

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)]=E[B(k)u(k)-[M^T]^{-1}B(k)x(k-1)-[M^T]^{-1}C(k)y(k-1)$$

$$+ 2\mu^T(k-1)\vec{H}x(k-1)-[M^T]^{-1}A(k)]$$

Step 15 Determine Kuhn-Tucker multipliers for $x(k)$, $\nu(k)$, from the following equation

$$E[\nu(k)]=E[-\lambda(k)-[M^T]^{-1}A(k)-[M^T]^{-1}B(k)x(k-1)$$

$$-[M^T]^{-1}C(k)y(k-1)]$$

If $x(k)$ satisfies the inequality

$$\underline{x}<x(k)<\overline{x}$$

then put $\nu(k)=0$

Step 16 Determine a new state iterate from the following

$$E[x^{i+1}(k)]=E[x^{i}(k)+\alpha Dx^{i}(k)]$$

where

$$E[Dx^{i}(k)]=E[\lambda(k)-\lambda(k-1)+2\mu^{T}(k-1)H\vec{x}(k-1)$$

$$+ B(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. Continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration, and J in equation (5.71) is maximum and equal during each year of the critical period.

5.3.5 Practical Example

The algorithm of the last section has been used to solve the same example mentioned in section (5.2.5), but the model used is a quadratic model. The characteristics of the installations are given in Table (5.14). In Tables (5.15-5.17) we give the variation of the water conversion factors (MWh/Mm^3) with the storage and the percentage error in those factors using a quadratic model. It will be noticed from these tables that the quadratic model for the reservoir is adequate, since the maximum error is very small compared to the linear model used in section (5.2.5).

The expected natural inflows to the sites are given in Table (5.2) for the same critical period (43 months).

In Tables (5.18-5.21) we give the optimal releases from each reservoir and the profits realized during each year of the critical period. It will be noticed from these tables that the total benefits during each year of the critical period are equal.

In Tables (5.22-5.25) we give the optimal storage for each reservoir during each year of the critical period. It will be noticed that we started with the reservoirs full, and we end up the critical period with the reservoirs at the minimum storage. The computing time to get the optimal long-term operation during the critical period for 43 months was 4.84 sec. in CPU units, which is very small compared to what has been done so far using other approaches.

Table 5.14: Characteristics of the installations

Site name	Capacity of the reservoirs ($\bar{x}$) Mm^3	Minimum storage ($\underline{x}$) Mm^3	Maximum effective discharge m^3/sec	Minimum effective discharge m^3/sec	Reservoirs constants α MWh/Mm^3	β $MWh/(Mm^3)^2$	γ $MWh/(Mm^3)^3$
R_{11}	24763	9949	1119	85.0	212.110947	146.956×10^{-4}	$-20503142.65\times10^{-14}$
R_{21}	5304	3734	1583	65.0	117.2016907	569.71×10^{-4}	$-368119890.482\times10^{-14}$
R_{12}	74255	33195	1877	283.2	232.45593262	359.4496×10^{-4}	$-1603544.3196\times10^{-14}$
R_{22}	0	0	1930.3	283.2	100.736	0	0

Table 5.15: Variation of MWh/M^3 with the storage and the percentage error for the reservoir R_{11}

Given Storage	MWh/Mm^3 Given	MWh/Mm^3 Calculated	% Error
9904.50000	336.04004	337.55005	0.44935
10545.80078	343.52002	344.28540	0.22281
11205.89844	350.98999	351.04199	0.01482
12614.50000	365.87988	364.86279	-0.27798
13383.19922	373.37988	372.06177	-0.35302
14643.00000	384.64990	383.33643	-0.34147
15096.39844	388.33008	387.23462	-0.28209
15062.80078	395.67993	395.26270	-0.10545
17090.39844	403.27002	403.37891	0.02700
18158.39844	410.62012	411.35522	0.17902
19267.50000	418.21997	419.14331	0.22078
20068.07031	423.36011	424.45166	0.25783
21007.39844	429.48999	430.34473	0.19901
21973.10156	435.37012	436.02637	0.15073
22839.50000	440.51001	440.79785	0.06534
24111.69922	448.11011	447.24707	-0.19259
24763.30078	451.78003	450.29297	-0.32916

Table 5.16: Variation of MWh/Mm^3 with the storage and the percentage error for the reservoir R_{21}

Given Storage	MWh/Mm^3 Given	MWh/Mm^3 Calculated	% Error
1233.60010	181.62000	181.87917	0.14270
2960.50000	256.37012	253.60017	-1.08045
3762.30005	278.67993	279.43628	0.27140
4317.50000	293.62988	294.55322	0.31446
4687.39844	304.89990	303.36499	-0.50341
5304.19922	315.92993	315.81812	-0.03539

Table 5.17: Variation of MWh/Mm^3 with the storage and the percentage error for the reservoir R_{12}

Given Storage	MWh/Mm^3 Given	MWh/Mm^3 Calculated	% Error
33020.89844	333.10010	333.63916	0.16183
34403.10156	336.80005	3337.13843	0.10047
35839.30078	340.50000	340.68311	0.05378
37324.39844	344.30005	344.27881	-0.00617
38711.69922	348.00000	347.57397	-0.12242
40441.50000	351.69995	351.59619	-0.02950
42068.60156	355.50000	355.29199	-0.05851
43744.50000	359.19995	359.01001	-0.05288
45471.89844	362.89990	362.74780	-0.04191
47255.60156	366.69995	366.50708	-0.05260
49097.89844	370.39990	370.28271	-0.03164
51006.39844	374.19995	374.07935	-0.03223
52985.69922	377.89990	377.89331	-0.00174
55040.89844	381.60010	381.72046	0.03154
57174.50000	385.30005	385.55054	0.06501
59388.80078	389.10010	389.37109	0.06965
63095.5000	395.10010	395.41455	0.07959
66496.37500	400.30005	400.57178	0.06788
70036.81250	405.50000	405.54639	0.01144

Table 5.18: Optimal monthly releases from the turbines and the profits realized during the first year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits $
1	1527	1899	1805	1827	2,226,489
2	1479	1662	1988	1995	2,218,467
3	1485	1626	2039	2047	2,225,034
4	1460	1534	2039	2047	2,171,806
5	1345	1426	1892	1898	1,998,280
6	1402	1517	2041	2047	2,113,491
7	1340	1817	1992	2000	2,141,509
8	1341	2191	2043	2096	2,284,959
9	1320	2626	4865	4938	3,864,054
10	2997	4239	4876	4937	5,125,145
11	2997	4207	1797	1820	3,550,995
12	1284	1710	1585	1607	1,905,646
The total benefits from the generation during the first year					31,825,875

Table 5.19: Optimal monthly releases from the turbines and the profits realized during the second year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits $
1	1699	2051	2646	2661	2,737,787
2	1624	2027	2578	2593	2,650,984
3	1625	2122	2560	2657	2,689,055
4	1627	1690	2651	2658	2,519,825
5	1503	1266	2442	2449	2,213,943
6	1630	1364	2653	2659	2,385,625
7	1644	1715	2584	2606	2,447,415
8	1636	2660	2656	2709	2,756,799
9	1609	3275	2592	2680	2,931,874
10	1678	2846	2672	2725	2,903,740
11	1699	2533	2677	2700	2,835,652
12	1674	2357	2610	2632	2,747,839
The total benefits from the generation during the second year					31,830,538

Table 5.20: Optimal monthly releases from the turbines and the profits realized during the third year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits $
1	1946	2439	2949	2972	3,052,808
2	1867	2246	2872	2894	2,902,146
3	1876	2352	2954	2969	2,950,186
4	1830	2393	2956	2965	2,898,514
5	1320	1889	2716	2725	2,390,010
6	235	889	2958	2964	1,807,796
7	240	397	2879	2885	1,608,273
8	1623	2317	2960	3013	2,642,784
9	1731	2314	2887	2960	2,669,109
10	1813	2787	2979	3025	2,998,333
11	1882	2671	2985	3007	3,034,301
12	1819	2341	2907	2922	2,871,918
	The total benefits from the generation during the third year				31,826,178

Table 5.21: Optimal monthly releases from the turbines and the profits realized during the rest of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits $
1	1259	1445	5027	4872	3,406,992
2	1212	1364	4865	4872	3,243,030
3	1180	1340	5027	5035	3,257,447
4	1128	1386	5027	5033	3,197,263
5	215	1318	4540	4545	2,576,443
6	233	712	3890	3896	2,038,664
7	465	755	836	844	738,421
	The total benefits from the generation				18,524,463

Table 5.22: Optimal reservoir storage during the first year of the critical period

Month k	x_{11}^k Mm^3	x_{21}^k Mm^3	x_{12}^k Mm^3
1	24032	5304	74255
2	22922	5304	73177
3	21724	5304	71782
4	20471	5304	70524
5	19316	5304	69084
6	18227	5206	67527
7	17833	5304	66400
8	17947	5304	68255
9	19460	5304	72564
10	21074	5304	72564
11	21225	4988	72564
12	21225	4988	72564

Table 5.23: Optimal reservoir storage during the second year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	20322	5022	71220
2	19050	4802	69940
3	17654	4402	68163
4	16214	4406	66392
5	14895	4745	64441
6	13477	5087	62261
7	12243	5304	61416
8	12404	5304	62863
9	14222	5304	67390
10	15494	5304	68707
11	16495	5304	68214
12	16619	5304	67153

Table 5.24: Optimal reservoir storage during the third year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	15590	5304	66327
2	14267	5247	64866
3	12683	4996	62955
4	11080	4626	60725
5	9949	4242	58656
6	9949	7324	56194
7	9949	3734	53874
8	9949	3995	55108
9	11564	5049	59780
10	13323	5304	60335
11	13965	5304	59086
12	13415	5215	57441

Table 5.25: Optimal reservoir storage during the rest of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	12814	5304	53794
2	11951	5304	49766
3	10995	5304	45428
4	10015	5152	40865
5	9949	4133	36675
6	9949	3734	33196
7	9949	3734	33196

5.3.6 Discussion

In this section we have used nonlinear storage curves opposed to the linear model used in section (5.2) for the same four reservoirs. It will be noticed that the total benefits using the nonlinear model is \$114,007,054 but in the case of linear model as mentioned in section (5.2), the total benefits is \$112,004,460, the difference is about \$2,002,594 which is a considerable amount. On the other hand the present method allows us to have the continuity equation as a linear equation, this would not have been the case if we instead had represented the storage as a quadratic function of the head

It will be noticed also that the total benefits during the last seven months for the critical period, using the nonlinear model, is better than those obtained using a linear model, and the total generation during each year is almost equal.

CHAPTER VI

OPTIMIZATION OF OPERATION OF MULTIRESEVOIR POWER SYSTEMS WITH A SPECIFIED MONTHLY GENERATION FOR CRITICAL WATER CONDITIONS

6.1 Background

In chapter 5, we maximized the generation from a hydroelectric power system during the critical water conditions. In this maximization, the required load on the system did not account for, that load may higher in winter than in summer. This chapter is devoted to the solution of the long-term optimal operating problem of a multireservoir power system with specified monthly generation, and this generation should supply the required load on the system during the critical period, and at the same time the total generation during each year of the critical period should be equal and maximum. To meet all these requirements, we maximize the generation from the system during each year of the critical period taking into account this specified load in the system. In other words, we minimize the difference between the monthly generation and the monthly specified load on the system, which is equal to the monthly percentage load required multiplied by the total generation at the end of the year.

The optimal control problem is formulated by constructing a cost functional which penalizes state (storage) and control (release) variables. The cost functional is augmented by using Lagrange and Kuhn-Tucker multipliers to adjoin the equality and inequality constraints on the system. The resulting cost functional is minimized by using the minimum norm formulation of functional analysis. A set of discrete optimizing equations is obtained, these equations are solved forward and backward in time. In the solution of these equations it was found that it is convenient to let the final states be free and assign values to them to get uniform generation during each year of the critical period (9).

The optimization is done on a monthly time basis for a period of a year. The times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason those times are not taken into account. Transmission line losses are included in the specified monthly load.

This chapter consists of two sections. The first section is devoted to solve the problem, but the model used for the water conversion factor is a quadratic function of the storage of the previous month. This assumption may cause an overestimation of production for falling water levels and underestimation for rising water levels. For this reason the second section in this chapter is devoted to solve the same problem, but the model used for the water conversion factor is a quadratic function from the average storage of two successive months.

6.2 The Water Conversion Factor is a Function from the Previous Monthly Storage*

In this section the problem of long-term optimal operation of

---

* A version of this Section has been to the Canadian Elecrical Engineering Journal, August 1985.

multireservoir power systems with specified monthly generation has been solved. The model used for the hydropower generation is a nonlinear model, and the water conversion factor assigned for each hydroplant is a nonlinear function of the previous monthly storage.

6.2.1 Problem Formulation

6.2.1.1 The System Under Study

The system under consideration consists of m independent rivers in parallel with one or several reservoirs and power plants in series on each (Figure 6.1). We will denote by:

I_{ij}^{k} A random variable representing the natural inflow to the reservoir i on river j during a period k in Mm^3. These are statistically independent random variables with normal distribution. It is assumed that no correlation exists between flows of independent rivers at different periods of time.

x_{ij}^{k} The storage of reservoir i on river j at the end of period k in Mm^3;

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \overline{x}_{ij}$$

where $\underline{x}_{ij}$ and $\overline{x}_{ij}$ are the minimum and maximum storages respectively.

u_{ij}^{k} The discharge from reservoir i on river j during a period k in Mm^3;

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \overline{u}_{ij}^{k} \; ; \; u_{0j}^{k} = 0$$

where $\underline{u}_{ij}^{k}$ and $\overline{u}_{ij}^{k}$ are the minimum and maximum discharges respectively. They are given by

$$\underline{u}_{ij}^{k} = 0.0864d^{k} \text{ (minimum effective discharge in } m^3/\text{sec)}$$

$$\overline{u}_{ij}^{k} = 0.0864d^{k} \text{ (maximum effective discharge in } m^3/\text{sec)}$$

If $u_{ij}^{k} > \overline{u}_{ij}^{k}$, then $u_{ij}^{k} - \overline{u}_{ij}^{k}$ Mm^3 is discharged through the spillways.

d^k The number of days in a month k.

s_{ij}^{k} The spill from reservoir i on river j during a period k in Mm^3;

$$s_{ij}^{k} \geq 0 \; ; \; s_{0j}^{k} = 0$$

water is spilt when the reservoir is filled to capacity, and the inflow to the reservoir exceeds $\overline{u}_{ij}^{k}$.

$G_{ij}(u_{ij}^{k}, x_{ij}^{k-1})$ The generation of plant i on river j during a period k in MWh. It is a nonlinear function of the storage and the discharge.

a^k The ratio between the monthly generation and the total yearly generation. In other words, it is the specified load during a month k. This ratio may include the transmission line losses.

n_j Number of reservoirs on river j; j=1,...,..,m.

m The total number of rivers.

k Superscript denoting the period, k=1,...,...,K.

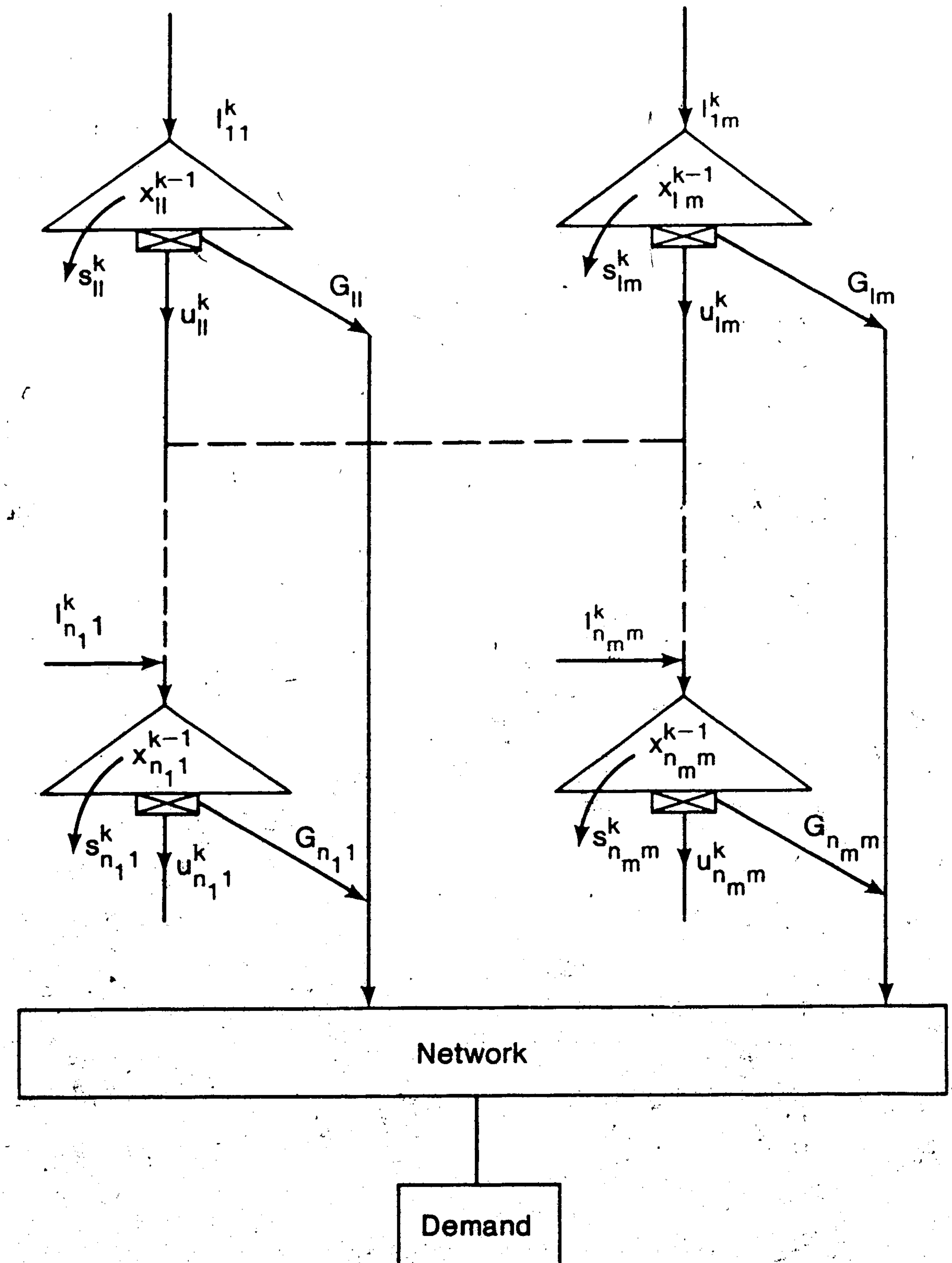

Figure 6.1 The Hydroelectric Power Systems

6.2.1.2 The Objective Function

Given the system shown in Figure 6.1, the problem is to find the discharge u_{ij}^{k} as a function of the time over the optimization interval under the following conditions:

(1) The total generation from the system over the optimization interval is a maximum

(2) The total system generation should match the load on the system.

(3) The difference between the montly generation and the specified load should be minimum.

(4) The water conservation equation (continuity equation) for each reservoir may be adequately described by the following difference equation

$$x_{ij}^{k} = x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} - u_{ij}^{k} + s_{(i-1)j}^{k} - s_{ij}^{k} \qquad (6.1)$$

where

$$s_{ij}^{k} = \begin{cases} (x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) - \bar{u}_{ij}^{k} ; \text{If } x_{ij}^{k} > \bar{x}_{ij} \\ \text{and} (x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) > \bar{u}_{ij}^{k} \\ 0, \text{ otherwise} \end{cases} \qquad (6.2)$$

(5) To satisfy multipurpose stream use requirements, such as flood control, irrigation, fishing and other purposes if any, the

following upper and lower limits on the variables should be satisfied.

(a) upper and lower bounds on the storage

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \overline{x}_{ij} \tag{6.3}$$

(b) upper and lower bounds on the discharge

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \overline{u}_{ij}^{k} \tag{6.4}$$

In mathematical terms, the object of the optimizing computation is to find the discharge u_{ij}^{k} that minimizes

$$J^{k}=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j} G_{ij}(u_{ij}^{k},x_{ij}^{k-1})-a^{k}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} G_{ij}(u_{ij}^{k},x_{ij}^{k-1})] \tag{6.5}$$

subject to satisfying the constraints (6.1-6.4).

6.2.1.3 Modelling of the Systesm

There are different approaches to model the hydroelectric power generating function. For the power systems in which the water head variation is small, we can consider the generation from these systems is equal to a constant time the discharge, and this constant is equal to the water conversion factor (41) and this conversion factor is equal to the average number of MWh generated in a month by an outflow of one Mm^3. But for the power systems in which the water heads vary with a considerable amount, this assumption is not true, and the water conversion factor varies with the head which itself is a function of the storage. In this section we assume that the water conversion factor has a quadratic relation with the storage. For this reason the generation of a hydroplant is a nonlinear function of the discharge and the storage. We may choose the following for the function $G_{ij}(u_{ij}^k, x_{ij}^{k-1})$

$$G_{ij}(u_{ij}^k, x_{ij}^{k-1}) = \alpha_{ij} u_{ij}^k + \beta_{ij} u_{ij}^k x_{ij}^{k-1} + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2 \qquad (6.5)$$

where α_{ij}, β_{ij} and γ_{ij} are constants. These were obtained by least square curve fitting to typical plant data available.

Now, the cost functional in equation (6.5) becomes

$$J^k = E[\sum_{j=1}^{m} \sum_{i=1}^{n_j} (\alpha_{ij} u_{ij}^k + \beta_{ij} u_{ij}^k x_{ij}^{k-1} + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2,$$
$$-a^k \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} (\alpha_{ij} u_{ij}^k + \beta_{ij} u_{ij}^k x_{ij}^{k-1} + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2)] \qquad (6.7)$$

Subject to satisfying the constraints (6.1-6.4).

The cost functional in equation (6.7) is a highly nonlinear function. If one introduces the following nonlinear transformation such that

$$y_{ij}^k = (x_{ij}^k)^2 \qquad (6.8)$$

Then, the cost functional in equation (6.7) becomes

$$J^k = E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}(\alpha_{ij}u_{ij}^k+\beta_{ij}u_{ij}^k x_{ij}^{k-1}+\gamma_{ij}u_{ij}^k y_{ij}^{k-1})-a^k\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}(\alpha_{ij}u_{ij}^k$$

$$+\beta_{ij}u_{ij}^k x_{ij}^{k-1}+\gamma_{ij}u_{ij}^k y_{ij}^{k-1})] \tag{6.9}$$

Subject to satisfying the following constraints

$$x_{ij}^k = x_{ij}^{k-1}+I_{ij}^k+u_{(i-1)j}^k-u_{ij}^k+s_{(i-1)j}^k-s_{ij}^k \tag{6.10}$$

$$y_{ij}^k = (x_{ij}^k)^2 \tag{6.11}$$

$$\underline{x}_{ij} \leq x_{ij}^k \leq \bar{x}_{ij} \tag{6.12}$$

$$\underline{u}_{ij}^k \leq u_{ij}^k \leq \bar{u}_{ij}^k \tag{6.13}$$

Now the problem is that of minimizing equation (6.9) subject to satifying the constraints (6.10–6.13). The initial storage x_{ij}^0, the percentage load a^k, and the natural inflows into each stream during each period are assumed to be known. The symbol E in equation (6.9) stands for the expected value, this expectation is taken over the random variable I_{ij}^k.

6.2.2 A Minimum Norm Formulation

We can now form an augmented cost function by adjoining to the cost function in equation (6.9) the equality constraints (6.10) and

(6.11) via Lagrange's multipliers and the inequality constraints (6.12) and (6.13) via Kuhn-Tucker multipliers, one thus obtains (40)

$$\tilde{J}=E[\sum_{j=1}^{m}\sum_{i=1}^{n_j}(\alpha_{ij}u_{ij}^{k}+\beta_{ij}u_{ij}^{k}x_{ij}^{k-1}+\gamma_{ij}u_{ij}^{k}y_{ij}^{k-1})$$

$$-a^{k}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K}\{\alpha_{ij}u_{ij}^{k}+\beta_{ij}u_{ij}^{k}x_{ij}^{k-1}$$

$$+\gamma_{ij}u_{ij}^{k}y_{ij}^{k-1}+\lambda_{ij}^{k}(-x_{ij}^{k}+x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k})$$

$$+\mu_{ij}^{k}(-{}_{ij}^{K}+(x_{ij}^{k})^{2})+e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})+e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij})+f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})$$

$$+f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k})\}] \qquad (6.14)$$

where λ_{ij}^{k}, μ_{ij}^{k} are Lagrange's multipliers, they are to be determined so that the corresponding equality constraints are satisfied; and e_{ij}^{k}, e_{ij}^{1k}, f_{ij}^{k} and f_{ij}^{1k} are Kuhn-Tucker multipliers. They are equal to zero, if the constraints are not violated and greater than zero if the constraints are violated.

To formulate the augmented cost functional in equation (6.14) as a norm, we may define the following vectors (32)

$$A(k)=\text{col.}(A_1(k),\ldots,\ldots,A_m(k)) \qquad (6.15)$$

$$A_{12}(k)=\text{col.}(\alpha_{11}/a^{k},\ldots,\ldots,\alpha_{n_1 1}/a^{k}) \qquad (6.16)$$

$$A_m(k)=\text{col.}(\alpha_{1m}/a^k,\ldots,\ldots,\alpha_{n_m m}/a^k) \tag{6.17}$$

$$A=\text{col.}(A_1,\ldots,\ldots,A_m) \tag{6.18}$$

$$A_1=\text{col}(\alpha_{11},\ldots,\ldots,\alpha_{n_1 1}) \tag{6.19}$$

$$A_m=\text{col.}(\alpha_{1m},\ldots,\ldots,\alpha_{n_m m}) \tag{6.20}$$

$$u(k)=\text{col.}(u_1(k),\ldots,\ldots,u_m(k)) \tag{6.21}$$

$$u_1(k)=\text{col.}(u_{11}{}^k,\ldots,\ldots,u_{n_1 1}{}^k) \tag{6.22}$$

$$u_m(k)=\text{col.}(u_{1m}{}^k,\ldots,\ldots,u_{n_m m}{}^k) \tag{6.23}$$

$$x(k)=\text{col.}(x_1(k),\ldots,\ldots,x_m(k)) \tag{6.24}$$

$$x_1(k)=\text{col.}(x_{11}{}^k,\ldots,\ldots,x_{n_1 1}{}^k) \tag{6.25}$$

$$x_m(k)=\text{col.}(x_{1m}{}^k,\ldots,\ldots,x_{n_m m}{}^k) \tag{6.26}$$

$$y(k)=\text{col.}(y_1(k),\ldots,\ldots,y_m(k)) \tag{6.27}$$

$$y_1(k)=\text{col.}(y_{11}{}^k,\ldots,\ldots,y_{n_1 1}{}^k) \tag{6.28}$$

$$y_m(k)=\text{col.}(y_{1m}{}^k,\ldots,\ldots,y_{n_m m}{}^k) \tag{6.29}$$

$$I(k)=\text{col.}(I_1(k),\ldots,\ldots,I_m(k)) \tag{6.30}$$

$$I_1(k)=\text{col.}(I_{11}{}^k,\ldots,\ldots,I_{n_1 1}{}^k) \tag{6.31}$$

$$I_m(k)=\text{col.}(I_{1m}{}^k,\ldots,\ldots,I_{n_m m}{}^k) \tag{6.32}$$

$$s(k)=\text{col.}(s_1(k),\ldots,\ldots,s_m(k)) \tag{6.33}$$

$$s_1(k)=\text{col.}(s_{11}{}^k,\ldots,\ldots,s_{n_1 1}{}^k) \tag{6.34}$$

$$s_m(k)=\text{col.}(s_{1m}{}^k,\ldots,\ldots,s_{n_m m}{}^k) \tag{6.35}$$

$$\lambda(k)=\text{col.}(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \tag{6.36}$$

$$\lambda_1(k)=\text{col.}(\lambda_{11}{}^k,\ldots,\ldots,\lambda_{n_1 1}{}^k) \tag{6.37}$$

$$\lambda_m(k)=\text{col.}(\lambda_{1m}{}^k,\ldots,\ldots,\lambda_{n_m m}{}^k) \tag{6.38}$$

$$\mu(k)=\text{col.}(\mu_1(k),\ldots,\ldots,\mu_m(k)) \tag{6.39}$$

$$\mu_1(k)=\text{col.}(\mu_{11}{}^k,\ldots,\ldots,\mu_{n_1 1}{}^k) \tag{6.40}$$

$$\mu_m(k)=\text{col.}(\mu_{1m}{}^k,\ldots,\ldots,\mu_{n_m m}{}^k) \tag{6.41}$$

$$\nu_{ij}{}^k = e_{ij}{}^{1k} - e_{ij}{}^k \tag{6.42}$$

$$\nu(k)=\text{col.}(\nu_1(k),\ldots,\ldots,\nu_m(k)) \tag{6.43}$$

$$\nu_1(k)=\text{col.}(\nu_{11}{}^k,\ldots,\ldots,\nu_{n_1 1}{}^k) \tag{6.44}$$

$$\nu_m(k)=\text{col.}(\nu_{1m}{}^k,\ldots,\ldots,\nu_{n_m m}{}^k) \tag{6.45}$$

$$\psi_{ij}{}^k = f_{ij}{}^{\prime k} - f_{ij}{}^k \tag{6.46}$$

$$\psi(k)=\text{col.}(\psi_1(k),\ldots,\ldots,\psi_m(k)) \tag{6.47}$$

$$\psi_1(k)=\text{col.}(\psi_{11}{}^k,\ldots,\ldots,\psi_{n_1 1}{}^k) \tag{6.48}$$

$$\psi_m(k)=\text{col.}(\psi_{1m}{}^k,\ldots,\ldots,\psi_{n_m m}{}^k) \tag{6.49}$$

Furthermore, define the following diagonal matrices

$$B(k)=\text{diag.}(B_1(k),\ldots,\ldots,B_m(k)) \tag{6.50}$$

$$B_1(k)=\text{diag.}(\beta_{11}/a^k,\ldots,\ldots,\beta_{n_1 1}/a^k) \tag{6.51}$$

$$B_m(k)=\text{diag.}(\beta_{1m}/a^k,\ldots,\ldots,\beta_{n_m m}/a^k) \tag{6.52}$$

$$C(k)=\text{diag.}(C_1(k),\ldots,\ldots,C_m(k)) \tag{6,53}$$

$$C_1(k)=\text{diag.}(\gamma_{11}/a^k,\ldots,\ldots,\gamma_{n_1 1}/a^k) \tag{6.54}$$

$$C_m(k)=\text{diag.}(\gamma_{1m}/a^k,\ldots,\ldots,\gamma_{n_m m}/a^k) \quad (6.55)$$

$$B=\text{diag.}(B_1,\ldots,\ldots,B_m) \quad (6.56)$$

$$B_1=\text{diag.}(\beta_{11},\ldots,\ldots,\beta_{n_1 1}) \quad (6.57)$$

$$B_m=\text{diag.}(\beta_{1m},\ldots,\ldots,\beta_{n_m m}) \quad (6.58)$$

$$C=\text{diag.}(C_1,\ldots,\ldots,C_m) \quad (6.59)$$

$$C_1=\text{diag.}(\gamma_{11},\ldots,\ldots,\gamma_{n_1 1}) \quad (6.60)$$

$$C_m=\text{diag.}(\gamma_{1m},\ldots,\ldots,\gamma_{n_m m}) \quad (6.61)$$

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \quad (6.62)$$

where the matrices $M_1,\ldots,\ldots,M_m$ are lower triangular matrices, whose elements are given by:

$$\left.\begin{array}{lll} \text{(i)} & m_{ii}=-1 & ;\ i=1,\ldots,\ldots,n_j;\quad j=1,\ldots,\ldots,m \\ \text{(ii)} & m_{(v+1)v}=1 & ;\ v=1,\ldots,\ldots,n_j-1;\ j=1,\ldots,\ldots,m \end{array}\right\} \quad (6.63)$$

Using all the above definitions, the cost functional in equation (6.14) can be written as:

$$\tilde{J}=E[A^T(k)u(k)+u^T(k)B(k)x(k-1)+u^T(k)C(k)y(k-1)$$

$$-\sum_{k=1}^{K}\{A^T u(k)+u^T(k)Bx(k-1)+u^T(k)Cy(k-1)+\lambda^T(k)(-x(k)$$

$$+x(k-1)+I(k)+Mu(k)+Ms(k))+\mu^T(k)(-y(k)+x^T(k)\vec{H}x(k))$$

$$+\nu^T(k)(x(k-1)+I(k)+Mu(k)+Ms(k))+\psi^T(k)u(k)\}] \qquad (6.64)$$

In the above equation $\vec{H}$ is a vector matrix in which the vector index varies from 1 to N, while the matrix dimension of $\vec{H}$ is NxN, $N=\sum_{j=1}^{m} n_j$, (17).

Employing the discrete version of integration by parts (33), and dropping terms which do not depend on x(k-1), y(k-1) and u(k). One obtains:

$$\tilde{J}=E[\{x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)-x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+\lambda^T(0)x(0)+\mu^T(0)y(0)\}+\{A^T(k)u(k)+u^T(k)B(k)x(k-1)$$

$$+u^T(k)C(k)y(k-1)-\sum_{k=1}^{K}\{x^T(k-1)\mu^T(k-1)\vec{H}x(k-1)+u^T(k)Bx(k-1)$$

$$+u^T(k)Cy(k-1)+(\lambda(k)-\lambda(k-1)+\nu(k))^T x(k-1)-\mu^T(k-1)y(k-1)$$

$$+(M^T\lambda(k)+M^T\nu(k)+\psi(k)+A)^T u(k)\}\}] \qquad (6.65)$$

If one defines the following vectors such that:

$$X^T(k)=[x^T(k-1),\ y^T(k-1),\ u^T(k)] \text{ is } 1\text{x}3N \text{ vector} \qquad (6.66)$$

$$W(k)=\begin{bmatrix} 0 & 0 & 1/2B(k) \\ 0 & 0 & 1/2C(k) \\ 1/2B(k) & 1/2C(k) & 0 \end{bmatrix} \text{ is 3Nx3N matrix} \quad (6.67)$$

$$L(k)=\begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 1/2B \\ 0 & 0 & 1/2C \\ 1/2B & 1/2C & 0 \end{bmatrix} \text{ is 3Nx3N matrix} \quad (6.68)$$

$$Q^T(k)=[0, \quad 0, \quad A^T(k)] \text{ is 1x3N vector} \quad (6.69)$$

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+\nu(k))^T, -\mu^T(k-1), (A+M^T\lambda(k)+M^T\nu(k)$$

$$+ \psi(k))^T] \quad (6.70)$$

Then, the cost functional in equation (6.65) can be written as:

$$\tilde{J}=E[\{x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)+x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+ \lambda^T(0)x(0)+\mu^T(0)y(0)\}+\{X^T(k)W(k)X(k)+Q^T(k)X(k)$$

$$- \sum_{k=1}^{K} \{X^T(k)L(k)X(k)+R^T(k)X(k)\}\}] \quad (6.71)$$

$\tilde{J}$ in equation (6.71) can be written as two independent terms as:

$$J_1=[x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)+x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+\lambda^T(0)x(0)+\mu^T(0)y(0)] \tag{6.72}$$

$$J_2=E[X^T(k)W(k)X(k)+Q^T(k)X(k)-\sum_{k=1}^{K}\{X^T(k)L(k)X(k)$$

$$+R^T(k)X(k)\}] \tag{6.73}$$

If one defines the following vectors such that

$$F(k) = W(k) - KL(k) \tag{6.74}$$

$$P(k) = Q(k) - KR(k) \tag{6.75}$$

then equation (6.73) can be written as:

$$J_2=E[\sum_{k=1}^{K} X^T(k)F(k)X(k)+P^T(k)X(k)] \tag{6.76}$$

Now define the following vector such that

$$V(k) = F^{-1}(k)P(k) \tag{6.77}$$

then, equation (6.76) can be written in the following form by a process similar to completing the squares as:

$$J_2=E[\sum_{k=1}^{K} \{(X(k)+1/2V(k))^TF(k)(X(k)+1/2V(k))-1/4V^T(k)F(k)V(k)\}] \tag{6.78}$$

Since the term, $1/4V^T(k)F(k)V(k)$ of equation (6.78) does not depend on $X(k)$, equation (6.78) can be reduced to

$$\underset{X(k)}{\text{Min.}}J_2 = \underset{X(k)}{\text{Min.}}E\left[\sum_{k=1}^{K} (X(k)+1/2V(k))^T F(k)(X(k)+1/2V(k))\right] \qquad (6.79)$$

Equation (6.79) defines a norm, then one can write equation (6.79) as:

$$\underset{X(k)}{\text{Min.}}J_2 = \underset{X(k)}{\text{Min}}\left[\,||X(k)+1/2V(k)||\,\right]_{F(k)} \qquad (6.80)$$

6.2.3 The Optimal Solution

To minimize $\tilde{J}$ in equation (6.71), one minimizes J_1 and J_2 separately. The minimum of J_1 is clearly achieved for:

$$E[\lambda(K)] = [0] \qquad (6.81)$$

$$E[\mu(K)] = [0] \qquad (6.82)$$

because $\delta x(K)$ and $\delta y(K)$ are arbitrary and $x(0)$ and $y(0)$ are constant.

Equations (6.81) and (6.82) give the values of Lagrange's multipliers at the last period studied.

Equation (6.80) is in the norm form, the minimum of this equation is achieved for:

$$E[X(k)+1/2V(k)] = [0] \tag{6.83}$$

Substituting from equation (6.77) into equation (6.83), one obtains the following optimal equation

$$E[P(k)+2F(k)X(k)] = [0] \tag{6.84}$$

Substituting from equations (6.74) and (6.75) into equation (6.84), one obtains:

$$E[(R(k)-(1/K)Q(k))+(L(k)-(1/K)W(k))X(k)] = [0] \tag{6.85}$$

Writing equation (6.85) explicitly, and adding the reservoir's dynamic equations (6.9) and (6.10). One obtains the following equation for the optimal long-term operation.

$$E[-\mathbf{x}(k)+\mathbf{x}(k-1)+I(k)+Mu(k)+Ms(k)] = [0] \tag{6.86}$$

$$E[-y(k)+\mathbf{x}^T(k)\vec{H}\mathbf{x}(k)] = [0] \tag{6.87}$$

$$E[\lambda(k)-\lambda(k-1)+2\mu^T(k-1)\vec{H}\mathbf{x}(k-1)+\nu(k)+\beta(k)u(k)] = [0] \tag{6.88}$$

$$E[-\mu(k-1)+\Gamma(k)u(k)] = [0] \tag{6.89}$$

$$E[\alpha(k)+M^T\lambda(k)+M^T\nu(k)+\psi(k)+\beta(k)\mathbf{x}(k-1)+\Gamma(k)y(k-1)] = [0] \tag{6.90}$$

where

$$\left.\begin{aligned} \alpha(k) &= A-(1/K)A(k) \\ \beta(k) &= B-(1/K)B(k) \\ \Gamma(k) &= C-(1/K)C(k) \end{aligned}\right\} \qquad (6.91)$$

We can now state the optimal equations (6.86-6.90) in component form:

$$E[-x_{ij}^{k}+x_{ij}^{k-1}+I_{ij}^{k}+u_{(i-1)j}^{k}-u_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k}] = 0 \quad (6.92)$$

$$E[-y_{ij}^{k} + (x_{ij}^{k})^{2}] = 0 \qquad (6.93)$$

$$E[\lambda_{ij}^{k}-\lambda_{ij}^{k-1}+2\mu_{ij}^{k-1}x_{ij}^{k-1}+\nu_{ij}^{k}+\beta_{ij}^{k}u_{ij}^{k}] = 0 \qquad (6.94)$$

$$E[-\mu_{ij}^{k-1}+\Gamma_{ij}^{k}u_{ij}^{k}] = 0 \qquad (6.95)$$

$$E[\alpha_{ij}^{k}+\lambda_{(i+1)j}^{k}-\lambda_{ij}^{k}+\nu_{(i+1)j}^{k}-\nu_{ij}^{k}+\psi_{ij}^{k}+\beta_{ij}^{k-1}x_{ij}^{k-1}+\Gamma_{ij}^{k}y_{ij}^{k-1}] = 0 \qquad (6.96)$$

where

$$\left.\begin{aligned} \alpha_{ij}^{k} &= \alpha_{ij} - (\alpha_{ij}/Ka^{k}) \\ \beta_{ij}^{k} &= \beta_{ij} - (\beta_{ij}/Ka^{k}) \\ \Gamma_{ij}^{k} &= \gamma_{ij} - (\gamma_{ij}/Ka^{k}) \end{aligned}\right\} \qquad (6.97)$$

Besides the above equations, one has the following Kuhn-Tucker exclusion equations, which must be satisfied at the optimum (40).

$$e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k}) = 0 \qquad (6.98)$$

$$e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij}) = 0 \qquad (6.99)$$

$$f_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \qquad (6.100)$$

$$f_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k}) = 0 \qquad (6.101)$$

One also has the following limits on the variable (40).

$$\left.\begin{aligned} &\text{If } x_{ij}^{k} < \underline{x}_{ij}, \quad \text{then we put } x_{ij}^{k} = \underline{x}_{ij} \\ &\text{If } x_{ij}^{k} > \bar{x}_{ij}, \quad \text{then we put } x_{ij}^{k} = \bar{x}_{ij} \\ &\text{If } u_{ij}^{k} < \underline{u}_{ij}^{k}, \quad \text{then we put } u_{ij}^{k} = \underline{u}_{ij}^{k} \\ &\text{If } u_{ij}^{k} > \bar{u}_{ij}^{k}, \quad \text{then we put } u_{ij}^{k} = \bar{u}_{ij}^{k} \end{aligned}\right\} \qquad (6.102)$$

Equations (6.92-6.102) with equations (6.81) and (6.82) completely specify the optimal solution for the system. The following algorithm is used to solve these equations.

6.2.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the expected value for the natural inflows, the required monthly generation, a^k, (this may include the system transmission losses) and the initial storage, x_{ij}^0. The following steps are used to solve the optimal equations.

Step 1 Assume initial guess for the variable u(k) such that

$$\underline{u}(k) \leq u^i(k) \leq \overline{u}(k); \quad i = \text{iteration counter}; \ i=0$$

Step 2 Assume first that s(k) is equal to zero. Solve equations (6.86), (6.87) and (6.89) forward in stages with x(0) given.

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

go to Step 10, otherwise put x(k) to its limits and go to Step 4.

Step 4 Calculate the new discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-I(k)-x(k-1))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

go to Step 14, otherwise put u(k) to its limits and go to Step 6.

Step 6 Calculate the spill at month k from the following equation

$$E[s(k)]=E[[M]^{-1}(x(k)-x(k-1)-I(k))-\bar{u}(k)]$$

If s(k) is less than zero, put s(k) equal to zero and go to Step 7.

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k)-Ms(k))]$$

Step 8 Solve again equations (6.86), (6.87) and (6.89) forward in stages with x(0) given, but s(k) has the value obtained from Step 6.

Step 9 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put x(k) to its limits and go to Step 4.

Step 10 With $\nu(k)=0$, solve equation (6.88) backward in stages with equation (6.81) as the terminal condition.

Step 11 Calculate Kuhn-Tucker multipliers for $u(k)$, $\psi(k)$, from the following equation

$$E[\psi(k)]=[M^T\beta(k)u(k)+2M^Tu^T(k)\Gamma(k)H\vec{x}(k-1)$$

$$-\beta(k)x(k-1)-\Gamma(k)y(k-1)-M^T\lambda(k-1)-\alpha(k)]$$

If $u(k)$ satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

put $\psi(k) = 0$.

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)] = E[u^i(k) - \alpha Du(k)]$$

where

$$E[Du(k)] = E[\alpha(k)+M^T\lambda(k)+\psi(k)+\beta(k)x(k-1)+\Gamma(k)y(k-1)]$$

and α is chosen with consideration to such factors as convergence.

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k) < u^{i+1}(k) < \overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 6.

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)]=E[\beta(k)u(k)-[M^T]^{-1}\beta(k)x(k-1)-[M^T]^{-1}\Gamma(k)y(k-1)$$

$$+ 2u^T(k)\Gamma(k)\vec{H}x(k-1)-[M^T]^{-1}\alpha\ (k)]\ .$$

Step 15 Determine Kuhn-Tucker multiplier for x(k), $\nu(k)$, from the following equation

$$E[\nu(k)]=E[-[M^T]^{-1}(\alpha(k)+\beta(k)x(k-1)+\Gamma(k)y(k-1)-\lambda(k))]$$

If x(k) satisfies the inequality

$$\underline{x} < x(k) < \overline{x}$$

put $\nu(k) = 0$

Step 16 Determine a new state iterate from the following

$$E[x^{i+1}(k)] = E[x^{i}(k)-\alpha Dx(k)]$$

where

$$E[Dx(k)] = E[\lambda(k)-\lambda(k-1)+\nu(k)+2\mu^{T}(k-1)H\vec{x}(k-1)$$

$$+ \beta(k)u(k)]$$

Step 17 Repeat the calculation starting from Step 3. If the solution does not converge to the optimal solution, assume another initial guess for u(k) ad repeat the calculation starting from Step 2. But if the solution converges to the optimal solution continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and the cost function in equation (6.8) is a minimum.

The above algorithm is summarized in the main flow chart given in Figure (6.2).

6.2.5 Practical Example

The algorithm of the last section has been used to solve the same example mentioned earlier in section 5.2.5. The optimization is done on a monthly time basis for a period of a year, the times of water travel between upstream and downstream reservoirs are assumed to be shorter than a month, for this reason those times are not taken into

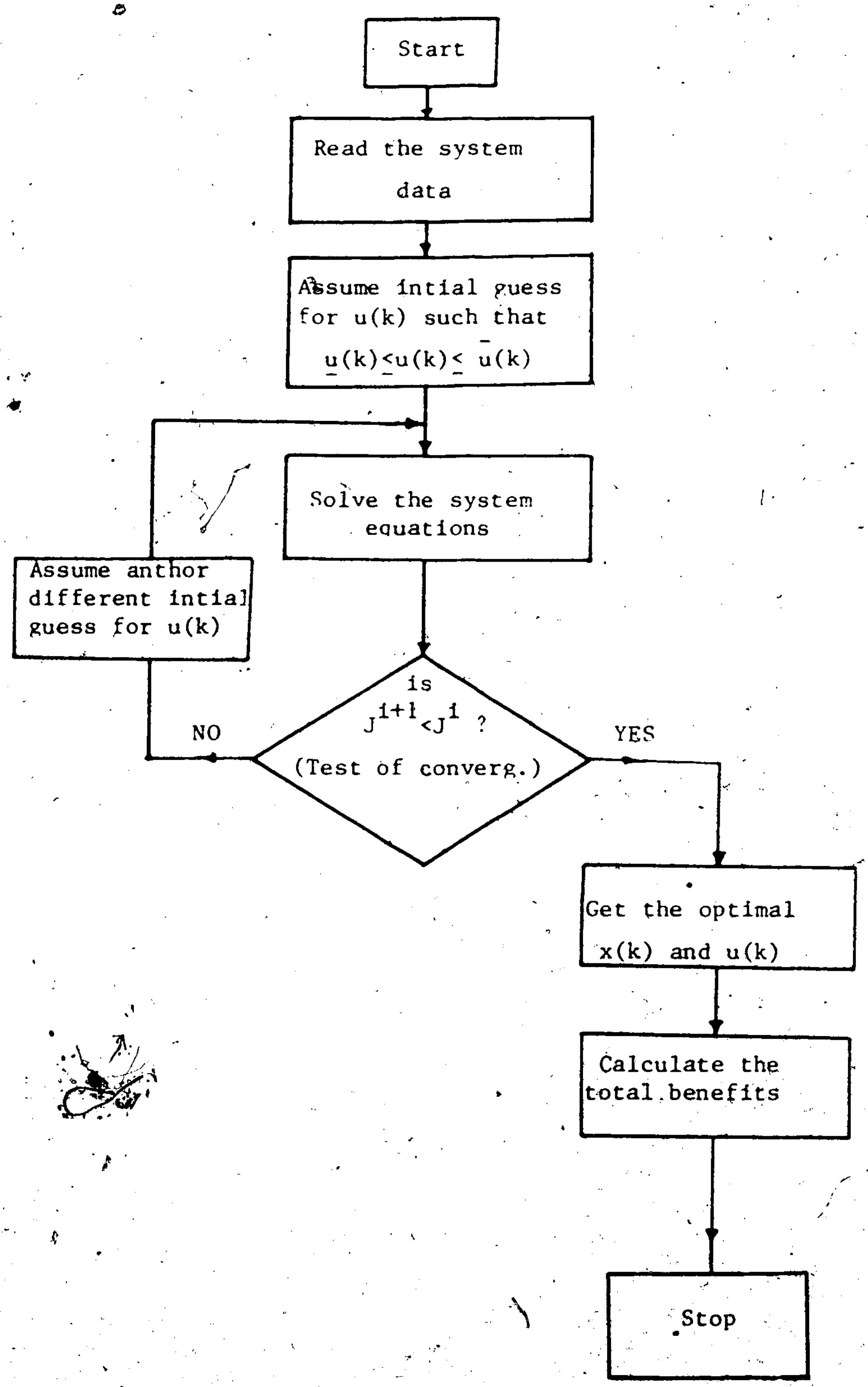

Figure 6.2: The main flow chart

account. As we mentioned earlier, in the solution of the optimal equations, it was found that it is convenient to let the final states be free, and assign values to them to get equal generation during each year of the critical period.

The characteristics of the installations are given in Table (5.14), the required monthly percentage load on the system during one year of the critical period is given in Table (6.1). The expected natural inflows to the sites during the critical period are given in table (5.2).

In Tables (6.2-6.5), we give the optimal releases from the reservoirs, the percentage monthly generation and the total benefits during each year of the critical period. In Tables (6.6-6.9), we give the optimal reservoir storage during each year of the critical period. We started the critical period with the reservoirs full, and end up the critical period with the reservoirs empty.

In Tables (6.10-6.13) we give the value of the cost function J^k at the optimum, and the percentage error in the calculated load. This percentage error is given by:

% Error=(J^k/(total benefits at the end of each year))x100 where J^k is given from equation (6.7).

Table 6.1: The percentage monthly load on the system during one year of the critical period

Month	k	% Load
Oct	1	8.34
Nov	2	8.87
Dec	3	9.39
Jan	4	9.77
Feb	5	8.75
Mar	6	9.04
Apr	7	7.99
May	8	7.74
Jun	9	7.49
Jul	10	7.43
Aug	11	7.64
Sep	12	7.55

Table 6.2: Optimal releases from the turbines, the total benefits and the calculated percentage load during the first year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	% load Calculated
1	1387	2131	2588	2611	2643421	8.341
2	1522	2341	2715	2722	2810554	8.868
3	1682	2384	2968	2977	2975314	7.388
4	1884	1959	3378	3386	3096023	9.769
5	1785	1790	2982	2988	2772653	8.749
6	1892	1860	3089	3095	2864938	9.040
7	1516	1897	2766	2773	2532372	7.991
8	1484	1923	2624	2677	2453483	7.742
9	1490	1954	2386	2460	2374029	7.491
10	1175	2889	1876	1936	2354331	7.429
11	1339	2234	2211	2234	2421756	7.642
12	1304	2081	2293	2315	2392756	7.550
	Total benefits from the generation				31,691,630	100.00

Table 6.3: Optimal releases from the turbines the total benefits and the calculated percentage load during the second year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits MWh	% Load Calculated
1	1384	2307	2566	2582	2643591	8.340
2	1501	2429	2795	2710	2811846	8.870
3	2091	2188	2879	2887	2976766	9.931
4	1966	2032	3373	3381	3096872	7.769
5	1842	1820	3024	3030	2774041	8.751
6	2116	2006	2947	2953	2865604	9.040
7	1770	1780	2741	2763	2533057	7.991
8	1600	1992	2599	2652	2453208	7.739
9	1388	2703	2066	2154	2374121	7.489
10	1353	2521	2055	2108	2355275	7.430
11	1461	2295	2202	2224	2421737	7.640
12	1408	2091	2288	2310	2393342	7.550
Total benefits from the generation					31,699,460	100.00

Table 6.4: Optimal releases from the turbines, the total benefits, and the calculated percentage load during the third year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	% Load Calculated
1	1420	2238	2681	2703	2644533	8.342
2	1548	2437	2830	2852	2812179	8.871
3	1707	2609	3031	3046	2976800	9.390
4	2019	2213	3407	3416	3096678	9.768
5	1861	1988	3096	3104	2773646	8.749
6	1337	1403	4079	4085	2865097	9.038
7	322	356	4808	4815	2533360	7.991
8	1524	1986	2865	2918	2454439	7.742
9	1510	2331	2414	2487	2374313	7.489
10	1346	2574	2157	2202	2344441	7.430
11	1439	2227	2381	2404	2421835	7.639
12	1422	2092	2404	2419	2393476	7.550
Total benefits from the generation					31701907	100.00

Table 6.5: Optimal releases from the turbines, the profits realized and the calculated percentage load during the rest of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	% Load Calculated
1	991	1766	3143	3158	2480291	8.346
2	1343	2094	3044	3052	2634390	8.865
3	1600	1840	3432	3440	2789622	9.387
4	1315	1379	4253	4259	2902518	9.767
5	1364	1759	3606	3611	2677991	9.012
6	233	324	5027	5032	2493403	8.39
7	465	755	4702	4709	2490285	8.38
Total benefits from the generation					18,469,500	100.00

Table 6.6: Optimal reservoir storage during the first year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	24172	4932	73471
2	23019	4296	71666
3	21625	3734	69343
4	19947	3734	66745
5	18352	3810	64215
6	16772	3911	61610
7	16203	4051	59710
8	16175	4461	60984
9	17517	5304	67772
10	20953	5304	70772
11	22760	5304	70358
12	22740	5016	69650

Table 6.7: Optimal reservoir storage during the second year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	22168	4479	68403
2	21029	3734	65906
3	19156	3734	64898
4	17378	3734	62404
5	15719	3858	59872
6	13815	4044	57398
7	12456	4322	56396
8	12653	4654	57899
9	14692	5304	62953
10	16289	5304	64886
11	17528	5304	64868
12	17918	5304	64128

Table 6.8: Optimal reservoir storage during the third year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	17415	4979	63571
2	16411	4411	62171
3	14997	3734	60164
4	13203	3734	57483
5	11532	3791	55034
6	10430	3871	51450
7	10348	3995	47201
8	10447	4488	48529
9	12283	5304	53675
10	14508	5304	55052
11	15595	5304	54407
12	15442	5067	53265

Table 6.9: Optimal reservoir storage during the rest of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	15109	4567	51502
2	14106	3978	49295
3	12730	3899	46552
4	11564	3940	42764
5	10349	3734	39509
6	10350	3734	34893
7	10350	3734	33195

Table 6.10: The perentage calculated load, the percentage error in that load and the value of J^k at the optimum for the first year of the critical period

Month k	% Calculated Load	% Error σ	J^k MWh
1	8.34107	0.00108	342.00000
2	8.86844	-0.00155	-490.00000
3	9.38833	-0.00166	-527.00000
4	9.76921	-0.000077	-245.00000
5	8.74885	-0.00114	-362.00000
6	9.04005	0.00005	17.00000
7	7.99066	0.00067	213.00000
8	7.74174	0.00174	553.00000
9	7.49103	0.00103	328.00000
10	7.42887	-0.00111	-353.00000
11	7.64163	0.00163	518.00000
12	7.55012	0.00013	41.00000

Table 6.11: The percentage calculated load, the percentage error in that load and the value of J^k at the optimum for the second year of the critical period

Month k	% Calculated Load	% Error	J^k MWh
1	8.34187	0.00188	596.
2	8.87069	0.00070	223.
3	9.38997	-0.00002	-7.
4	9.76811	-0.00188	-596.
5	8.74914	-0.0085	-269.
6	9.03761	-0.00238	-754.
7	7.99119	0.00120	379.
8	7.74224	0.00225	713.
9	7.48950	-0.00050	-158.
10	7.43031	0.00032	102.
11	7.63940	-0.00060	-189.
12	7.54994	-0.00005	-16.

Table 6.12: The percentage calculated load, the percentage error in that load and the value of J^k at the optimum for the third year of the critical period

Month k	% Calculated Load	% Error	J^k MWh
1	8.33954	-0.00045	-142.00000
2	8.87033	0.00034	107.00000
3	9.39059	0.00060	189.00000
4	9.76947	-0.00051	-162.00000
5	8.75106	0.00107	340.00000
6	9.03991	-0.00008	-26.00000
7	7.99085	0.00086	272.00000
8	7.73895	-0.00103	-328.00000
9	7.48947	-0.00053	-167.00000
10	7.43002	0.00003	8.00000
11	7.63968	-0.00032	-100.00000
12	7.55010	0.00011	35.00000

Table 6.13: The percentage calculated load, the percentage error in that load and the value of J^k at the optimum for the rest of the critical period

Month k	% Calculated Load	% Error	J^k MWh
1	8.3460	0.0060	1108.0
2	8.8650	-0.0050	-923.0
3	9.3870	-0.0030	-554.0
4	9.7670	-0.0030	-554.0
5	9.0120	0.2620	48390.0
6	8.3900	-0.6500	-120052.0
7	8.3800	0.3900	72031.0

6.2.6 Discussion

From Tables (6.2-6.5), the total benefits from the system during the critical period with this model and taking the load on the system into account is 113,562,500 MWh with yearly average equal to 31,691,860 MWh. It will be noticed from these tables that the total benefits from generation during each year of the critical period are almost equal.

From the Tables (6.10-6.13), it will be noticed that, at the optimum the percentage error in the calculated load is almost zero and at this point the calculated load is equal to the required load on the system.

In this section, the model used for the generating function of each reservoir is a nonlinear model of the discharge and the storage of the previous month. This model may cause an overestimation of production for falling the water levels and underestimation for rising water levels. To avoid this an average of begin-and-end of time step (month) storage is used which is explained in detail in the next section.

6.3 The Water Conversion Factor is a Function of the Average Storage*

In section 6.2, the problem of long-term optimal operation of a hydroelectric power system under critical water conditions has been solved. The model used for each hydroplant was a nonlinear model. It was a function of the discharge and the storage of the previous month, $G_{ij}(u_{ij}^{k}, x_{ij}^{k-1})$. This model may cause an underestimation for rising water levels and overestimation for falling water levels. This section is an extension to the previous section, the model used here for the same system is a nonlinear model, it is a funciton of the discharge through the turbines and the average monthly storage between two successive months k-1 and k to avoid an underestimation for rising water levels and overestimation for falling water levels in the hydroelectric production.

6.3.1 Problem Formulation

6.3.1.1 The Objective Function

Given the hydroelectric system in Figure 6.1, the problem is to find the discharge u_{ij}^{k} as a function of time over the critical period under the following conditions:

(1) The total generation from the system over the optimization interval is a maximum.

(2) The total system generation should match the load on the system.

(3) The difference between the monthly generation and the specified monthly load should be minimum.

---

* A version of this Section has been submitted to IEEE Transactions on A.C. December 1985

(40) The water conservation equation (continuity equation) for each reservoir may be adequately described by the following difference equation:

$$x_{ij}^{k} = x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} - u_{ij}^{k} + s_{(i-1)j}^{k} - s_{ij}^{k} \tag{6.92}$$

where

$$s_{ij}^{k} = \begin{cases} (x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) - \bar{u}_{ij}^{k}; \text{If} (x_{ij}^{k} + I_{ij}^{k} + \\ u_{(i-1)j}^{k} + s_{(i-1)j}^{k} - x_{ij}^{k}) > \bar{u}_{ij}^{k} \text{ and } x_{ij}^{k} > \bar{x}_{ij} \\ 0, \text{ otherwise} \end{cases} \tag{5.93}$$

(5) To satisfy the multipurpose stream use requirements, such as flood control, irrigation, fishing and other purposes, if any, the following upper and lower limits on the variables should be satisfied:

(a) upper and lower bounds on the storage

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \bar{x}_{ij} \tag{6.94}$$

(b) upper and lower bounds on the discharge

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \tag{6.95}$$

In mathematical terms, the object of the optimizing computation is to find the discharge u_{ij}^k that minimizes

$$J^k = E\left[\sum_{j=1}^{m} \sum_{i=1}^{n_j} G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1})) - \right.$$

$$\left. - a^k \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1})) \right] \qquad (6.96)$$

subject to satisfying the constraints (6.29–6.95).

The initial storage x_{ij}^0, the percentage load a^k, and the expected value for the natural inflows into each stream during each month are assumed to be known.

6.3.1.2 Modelling of the System

The generation of a hydroplant is a nonlinear function of the discharge u_{ij}^k and the net head, which itself is a nonlinear function of the storage. To avoid overestimation of production for falling water levels and underestimation for rising water levels, an average of begin and end-of-time step storage has been used. We may choose the following for the function $G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1}))$

$$G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1})) = \alpha_{ij} u_{ij}^k + 1/2\beta_{ij} u_{ij}^k (x_{ij}^k + x_{ij}^{k-1})$$

$$+ 1/4\gamma_{ij} u_{ij}^k (x_{ij}^k + x_{ij}^{k-1})^2 \qquad (6.97)$$

Substituting from equation (6.92) into equation (6.97) for x_{ij}^k, one obtains

$$G_{ij}(u_{ij}^k, 1/2(x_{ij}^k + x_{ij}^{k-1})) = b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1} + u_{ij}^k f_{ij}^k (u_{(i-1)j}^k$$

$$- u_{ij}^k) + \gamma_{ij} u_{ij}^k (x_{ij}^{k-1})^2$$

$$+ 1/4 \gamma_{ij} u_{ij}^k ((u_{ij}^k)^2 + (u_{(i-1)j}^k)^2)$$

$$+ \gamma_{ij} u_{ij}^k x_{ij}^{k-1} (u_{(i-1)j}^k - u_{ij}^k)$$

$$+ 1/2 \gamma_{ij} u_{(i-1)j}^k (u_{ij}^k)^2 \tag{6.98}$$

where

$$q_{ij}^k = \dot{I}_{ij}^k + s_{(i-1)j}^k - s_{ij}^k \tag{6.99}$$

$$b_{ij}^k = \alpha_{ij} + 1/2\ \beta_{ij} q_{ij}^k + 1/4\ \gamma_{ij} (q_{ij}^k)^2 \tag{6.100}$$

$$d_{ij}^k = \beta_{ij} + \gamma_{ij} q_{ij}^k \tag{6.101}$$

$$f_{ij}^k = 1/2\ \beta_{ij} + 1/2\ \gamma_{ij} q_{ij}^k \tag{6.102}$$

and α_{ij}, β_{ij} and γ_{ij} are constants, these were obtained by least square curve fitting to typical plant data available.

It will be noticed that the generating function in equation (6.98) is a highly nonlinear function. If one defines the following n_j pseudo-state variables such that (17)

$$y_{ij}^{k} = (x_{ij}^{k})^2 \tag{6.103}$$

$$z_{ij}^{k} = (u_{ij}^{k})^2 \tag{6.104}$$

$$r_{ij}^{k-1} = u_{ij}^{k} x_{ij}^{k-1} \tag{6.105}$$

Then, the function G_{ij} in equation (6.98) becomes:

$$\begin{aligned} G_{ij}(u_{ij}^{k}, 1/2(x_{ij}^{k}+x_{ij}^{k-1})) = {} & b_{ij}^{k} u_{ij}^{k} + u_{ij}^{k} d_{ij}^{k} x_{ij}^{k-1} \\ & + u_{ij}^{k} f_{ij}^{k} (u_{(i-1)j}^{k} - u_{ij}^{k}) \\ & + \gamma_{ij} u_{ij}^{k} y_{ij}^{k-1} + 1/4 u_{ij}^{k} \gamma_{ij} (z_{(i-1)j}^{k} + z_{ij}^{k}) \\ & + \gamma_{ij} r_{ij}^{k-1} (u_{(i-1)j}^{k} - u_{ij}^{k}) \\ & - 1/2 \, \gamma_{ij} z_{ij}^{k} u_{(i-1)j}^{k} \end{aligned} \tag{6.106}$$

Now, the cost functional in equation (6.96) becomes:

$$J^k = E\Big[\sum_{j=1}^{m}\sum_{i=1}^{n_j} \{b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1} + u_{ij}^k f_{ij}^k (u_{(i-1)j}^k - u_{ij}^k)$$

$$+ \gamma_{ij} u_{ij}^k y_{ij}^{k-1} + 1/4\, u_{ij}^k \gamma_{ij} (z_{(i-1)j}^k + z_{ij}^k)$$

$$+ \gamma_{ij} r_{ij}^{k-1} (u_{(i-1)j}^k - u_{ij}^k) - 1/2\, \gamma_{ij} z_{ij}^k u_{(i-1)j}^k\}$$

$$- a^k \sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{k=1}^{K} \{b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1}$$

$$+ u_{ij}^k f_{ij}^k (u_{(i-1)j}^k - u_{ij}^k) + \gamma_{ij} u_{ij}^k y_{ij}^{k-1}$$

$$+ 1/4\, u_{ij}^k \gamma_{ij} (z_{(i-1)j}^k + z_{ij}^k) + \gamma_{ij} r_{ij}^{k-1} (u_{(i-1)j}^k - u_{ij}^k)$$

$$- 1/2\, \gamma_{ij} z_{ij}^k u_{(i-1)j}^k\}\Big] \qquad (6.107)$$

Subject to satisfying the following constraints

$$y_{ij}^k = (x_{ij}^k)^2 \qquad (6.108)$$

$$z_{ij}^k = (u_{ij}^k)^2 \qquad (6.109)$$

$$r_{ij}^{k-1} = u_{ij}^k x_{ij}^{k-1} \qquad (6.110)$$

$$x_{ij}^k = x_{ij}^{k-1} + q_{ij}^k + u_{(i-1)j}^k - u_{ij}^k \qquad (6.111)$$

$$\underline{x}_{ij} \le x_{ij}^k \le \overline{x}_{ij} \qquad (6.112)$$

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \tag{6.113}$$

6.3.2 A Minimum Norm Formulation

We can now form an augmented cost functional by adjoining to the cost function in equation (6.107), the equality constraints (6.108-6.111) via Lagrange's multipliers, and the inequality constraints via Kuhn-Tucker multipliers, one thus obtains:

$$
\begin{aligned}
J^k = E\Big[\sum_{j=1}^{m} \sum_{i=1}^{n_j} & \{ b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1} + u_{ij}^k f_{ij}^k (u_{(i-1)j}^k - u_{ij}^k) \\
& + \gamma_{ij} u_{ij}^k y_{ij}^{k-1} + 1/4\, u_{ij}^k \gamma_{ij} (z_{(i-1)j}^k + z_{ij}^k) \\
& + \gamma_{ij} r_{ij}^{k-1} (u_{(i-1)j}^k - u_{ij}^k) - 1/2\, \gamma_{ij} z_{ij}^k u_{(i-1)j}^k \} \\
& - a^k \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{k=1}^{K} \{ b_{ij}^k u_{ij}^k + u_{ij}^k d_{ij}^k x_{ij}^{k-1} \\
& + u_{ij}^k f_{ij}^k (u_{(i-1)j}^k - u_{ij}^k) + \gamma_{ij} u_{ij}^k y_{ij}^{k-1} \\
& + 1/4\, u_{ij}^k \gamma_{ij} (z_{(i-1)j}^k + z_{ij}^k) + \gamma_{ij} r_{ij}^{k-1} (u_{(i-1)j}^k - u_{ij}^k) \\
& - 1/2\, \gamma_{ij} z_{ij}^k u_{(i-1)j}^k + \mu_{ij}^k (-y_{ij}^k + (x_{ij}^k)^2 \\
& + \phi_{ij}^k (-z_{ij}^k + (u_{ij}^k)^2 + \psi_{ij}^k (-r_{ij}^k + u_{ij}^k x_{ij}^{k-1}) \\
& + \lambda_{ij}^k (-x_{ij}^k + x_{ij}^{k-1} + q_{ij}^k + u_{(i-1)j}^k - u_{ij}^k) + e_{ij}^k (\underline{x}_{ij} - x_{ij}^k) \\
& + e_{ij}^{1k} (x_{ij}^k - \bar{x}_{ij}) + g_{ij}^k (\underline{u}_{ij}^k - u_{ij}^k) + g_{ij}^{1k} (u_{ij}^k - \bar{u}_{ij}^k) \} \Big]
\end{aligned}
\tag{6.114}
$$

To formulate the problem as a norm, we may define the following vectors. Define the following column vectors:

$$B(k)=\text{col.}(B_1(k),\ldots,\ldots,B_m(k)) \tag{6.115}$$

$$B_1(k)=\text{col.}(b_{11}{}^k/a^k,\ldots,\ldots,b_{n_1 1}{}^k/a^k) \tag{6.116}$$

$$B_m(k)=\text{col.}(b_{1m}{}^k/a^k,\ldots,\ldots,b_{n_m m}{}^k/a^k) \tag{6.117}$$

$$b(k)=\text{col.}(b_1(k),\ldots,\ldots,b_m(k)) \tag{6.118}$$

$$b_1(k)=\text{col.}(b_{11}{}^k,\ldots,\ldots,b_{n_1 1}{}^k) \tag{6.119}$$

$$b_m(k)=\text{col.}(b_{1m}{}^k,\ldots,\ldots,b_{n_m m}{}^k) \tag{6.120}$$

$$u(k)=\text{col.}(u_1(k),\ldots,\ldots,u_m(k)) \tag{6.121}$$

$$u_m(k)=\text{col.}(u_{1m}{}^k,\ldots,\ldots,u_{n_m m}{}^k) \tag{6.123}$$

$$x(k)=\text{col.}(x_1(k),\ldots,\ldots,x_m(k)) \tag{6.124}$$

$$x_1(k)=\text{col.}(x_{11}{}^k,\ldots,\ldots,x_{n_1 1}{}^k) \tag{6.125}$$

$$x_m(k)=\text{col.}(x_{1m}{}^k,\ldots,\ldots,x_{n_m m}{}^k) \tag{6.126}$$

$$y(k)=\text{col.}(y_1(k),\ldots,\ldots,y_m(k)) \tag{6.127}$$

$$y_1(k)=\text{col.}(y_{11}^k,\ldots,\ldots,y_{n_1 1}^k) \qquad (6.128)$$

$$y_m(k)=\text{col.}(y_{1m}^k,\ldots,\ldots,y_{n_m m}^k) \qquad (6.129)$$

$$z(k)=\text{col.}(z_1(k),\ldots,\ldots,z_m(k)) \qquad (6.130)$$

$$z_1(k)=\text{col.}(z_{11}^k,\ldots,\ldots,z_{n_1 1}^k) \qquad (6.131)$$

$$z_m(k)=\text{col.}(z_{1m}^k,\ldots,\ldots,z_{n_m m}^k) \qquad (6.132)$$

$$r(k)=\text{col.}(r_1(k),\ldots,\ldots,r_m(k)) \qquad (6.133)$$

$$r_1(k)=\text{col.}(r_{11}^k,\ldots,\ldots,r_{n_1 1}^k) \qquad (6.134)$$

$$r_m(k)=\text{col.}(r_{1m}^k,\ldots,\ldots,r_{n_m m}^k) \qquad (6.135)$$

$$\mu(k)=\text{col.}(\mu_1(k)\ldots,\ldots,\mu_m(k)) \qquad (6.136)$$

$$\mu_1(k)=\text{col.}(\mu_{11}^k,\ldots,\ldots,\mu_{n_1 1}^k) \qquad (6.137)$$

$$\mu_m(k)=\text{col.}(\mu_{1m}^k,\ldots,\ldots,\mu_{n_m m}^k) \qquad (6.138)$$

$$\phi(k)=\text{col.}(\phi_1(k),\ldots,\ldots,\phi_m(k)) \qquad (6.139)$$

$$\phi_1(k)=\text{col.}(\phi_{11}^k,\ldots,\ldots,\phi_{n_1 1}^k) \qquad (6.140)$$

$$\phi_m(k)=col.(\phi_{1m}^k,\ldots,\ldots,\phi_{n_m m}^k) \quad (6.141)$$

$$\lambda(k)=col.(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \quad (6.142)$$

$$\lambda_1(k)=col.(\lambda_{11}^k,\ldots,\ldots,\lambda_{n_1 1}^k) \quad (6.143)$$

$$\lambda_m(k)=col.(\lambda_{1m}^k,\ldots,\ldots,\lambda_{n_m m}^k) \quad (6.144)$$

$$\nu_{ij}^k=e_{ij}^{1k}-e_{ij}^k \quad (6.145)$$

$$\nu(k)=col.(\nu_1(k),\ldots,\ldots,\nu_m(k)) \quad (6.146)$$

$$\nu_1(k)=col.(\nu_{11}^k,\ldots,\ldots,\nu_{n_1 1}^k) \quad (6.147)$$

$$\nu_m(k)=col.(\nu_{1m}^k,\ldots,\ldots,\nu_{n_m m}^k) \quad (6.148)$$

$$\sigma_{ij}^k = g_{ij}^{1k} - g_{ij}^k \quad (6.149)$$

$$\sigma(k)=col.(\sigma_1(k),\ldots,\ldots,\sigma_m(k)) \quad (6.150)$$

$$\sigma_1(k)=col.(\sigma_{11}^k,\ldots,\ldots,\sigma_{n_1 1}^k) \quad (6.151)$$

$$\sigma_m(k)=col.(\sigma_{1m}^k,\ldots,\ldots,\sigma_{n_m m}^k) \quad (6.152)$$

Furthermore, define the following diagonal matrices:

$$d(k)=\text{diag.}(d_1(k),\ldots,\ldots,d_m(k)) \tag{6.153}$$

$$d_1(k)=\text{diag.}(d_{11}{}^k,\ldots,\ldots,d_{n_1 1}{}^k) \tag{6.154}$$

$$d_m(k)=\text{diag.}(d_{1m}{}^k,\ldots,\ldots,d_{n_m m}{}^k) \tag{6.155}$$

$$D(k)=\text{diag.}(D_1(k),\ldots,\ldots,D_m(k)) \tag{6.156}$$

$$D_1(k)=\text{diag.}(d_{11}{}^k/a^k,\ldots,\ldots,d_{n_1 1}{}^k/a^k) \tag{6.157}$$

$$D_m(k)=\text{diag.}(d_{1m}{}^k/a^k,\ldots,\ldots,d_{n_m m}{}^k/a^k) \tag{6.158}$$

$$f(k)=\text{diag.}(f_1(k),\ldots,\ldots,f_m(k)) \tag{6.159}$$

$$f_1(k)=\text{diag.}(f_{11}{}^k,\ldots,\ldots,f_{n_1 1}{}^k) \tag{6.160}$$

$$f_m(k)=\text{diag.}(f_{1m}{}^k,\ldots,\ldots,f_{n_m m}{}^k) \tag{6.161}$$

$$F(k)=\text{diag.}(F_1(k),\ldots,\ldots,F_m(k)) \tag{6.162}$$

$$F_1(k)=\text{diag.}(f_{11}{}^k/a^k,\ldots,\ldots,f_{n_1 1}{}^k/a^k) \tag{6.163}$$

$$F_m(k)=\text{diag.}(f_{1m}{}^k/a^k,\ldots,\ldots,f_{n_m m}{}^k/a^k) \tag{6.164}$$

$$C(k)=\text{diag.}(C_1(k),\ldots,\ldots,C_m(k)) \tag{6.165}$$

$$C_1(k)=\text{diag.}(\gamma_{11}/a^k,\ldots,\ldots,\gamma_{n_1 1}/a^k) \tag{6.166}$$

$$C_m(k)=\text{diag.}(\gamma_{1m}/a^k,\ldots,\ldots,\gamma_{n_m m}/a^k) \tag{6.167}$$

$$C=\text{diag.}(C_1,\ldots,\ldots,C_m) \tag{6.168}$$

$$C_1=\text{diag.}(\gamma_{11},\ldots,\ldots,\gamma_{n_1 1}) \tag{6.169}$$

$$C_m=\text{diag.}(\gamma_{1m},\ldots,\ldots,\gamma_{n_m m}) \tag{6.170}$$

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \tag{6.171}$$

where the matrices $M_1,\ldots,M_m$ are lower triangular matrices whose elements are given by:

$$\left.\begin{array}{ll}(1)\ m_{ii}=-1 & ;\ i=1,\ldots,\ldots,n_j\ ;\ j=1,\ldots,\ldots,m\\ (2)\ m_{(v+1)v}=1 & ;\ v=1,\ldots,\ldots,n_j-1\\ (3)\ \text{the rest of the elements are equal to zero} & \end{array}\right\} \tag{6.172}$$

$$N=\text{diag.}(N_1,\ldots,\ldots,N_m) \tag{6.173}$$

where the matrices $N_1,\ldots,\ldots,N_m$ are lower triangular matrices whose elements are given by:

$$\left.\begin{array}{ll}(1) & n_{ii}=1 \quad ; \; i=1,\ldots,\ldots,n_j \\ (2) & n_{(v+1)v}=1 \quad ; \; v=1,\ldots,\ldots,n_j-1 \\ (3) & \text{the rest of the elements are equal to zero}\end{array}\right\} \tag{6.174}$$

$$L=\text{diag.}(L_1,\ldots,\ldots,L_m) \tag{6.175}$$

where $L_1,\ldots,L_m$ are lower triangular matrices whose elements are given by:

$$\left.\begin{array}{ll}(1) & \ell_{(v+1)v}=1; \; v=1,\ldots,\ldots,n_j-1 \; ; \; j=1,\ldots,m \\ (2) & \text{the rest of the elements are equal to zero}\end{array}\right\} \tag{6.176}$$

Using all the above definitions, the cost functional in equation (6.114) can be written as:

$$\begin{aligned}J^k=E[\{&B^T(k)u(k)+u^T(k)D(k)x(k-1)+u^T(k)F(k)Mu(k) \\ &+u^T(k)C(k)y(k-1)+1/4\;u^T(k)C(k)Nz(k)+r^T(k-1)C(k)Mu(k) \\ &-1/2\;z^T(k-1)C(k)Lu(k)\}-\sum_{k=1}^{K}\{b^T(k)u(k)+u^T(k)d(k)x(k-1) \\ &+u^T(k)f(k)Mu(k)+u^T(k)Cy(k-1)+1/4\;u^T(k)CNz(k)+r^T(k-1)CMu(k) \\ &-1/2\;z^T(k)CLu(k)+\mu^T(k)(-y(k)+x^T(k)\vec{H}x(k))+\phi^T(k)(-z(k)+u^T(k)\vec{H}u(k))\end{aligned}$$

$$+ \psi^T(k)(-r(k-1)+x^T(k-1)\vec{H}u(k))+\lambda^T(k)(-x(k)+x(k-1)+q(k)+Mu(k))$$

$$+ \nu^T(k)(x(k-1)+q(k)+Mu(k))+\sigma^T(k)u(k)\}] \qquad (6.177)$$

In the above equation $\vec{H}$ is a vector matrix in which the vector index varies from 1 to $\sum_{j=1}^{m} n_j$, while the dimension of $\vec{H}$ is $(\sum_{j=1}^{m} n_j x \sum_{j=1}^{m} n_j)$ (17).

Employing the discrete version of integration by part (33) and dropping constant terms, one obtains:

$$J^k = E[x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)-x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+ \lambda^T(0)x(0)+\mu^T(0)y(0)+\{B^T(k)u(k)+u^T(k)D(k)x(k-1)$$

$$+ u^T(k)F(k)Mu(k)+u^T(k)C(k)y(k-1)+ 1/4\ u^T(k)C(k)Nz(k)$$

$$+ r^T(k-1)C(k)Mu(k)- 1/2\ z^T(k)C(k)Lu(k)\}- \sum_{k=1}^{K} \{b^T(k)u(k)$$

$$+ u^T(k)d(k)x(k-1)+u^T(k)f(k)Mu(k)+u^T(k)Cy(k-1)+ 1/4\ u^T(k)CNz(k)$$

$$+ (\lambda(k)-\lambda(k-1)+\nu(k))^T x(k-1)+(M^T\lambda(k)+M^T\nu(k)+\sigma(k))^T u(k)$$

$$+r^T(k-1)CMu(k)-1/2z^T(k)CLu(k)-\mu^T(k-1)y(k-1)+x^T(k-1)\mu^T(k-1)\vec{H}x(k-1)$$

$$- \phi^T(k)z(k)+u^T(k)\phi^T(k)\vec{H}u(k)-\psi^T(k)r(k-1)+x^T(k-1)\psi^T(k)\vec{H}u(k)\}]$$

$$(6.178)$$

If one defines the following such that

$$X^T(k) = [x^T(k-1),\ y^T(k-1),\ u^T(k),\ z^T(k),\ r^T(k-1)] \tag{6.179}$$

$$L(k) = \begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 1/2(d(k)+\psi^T(k)\vec{H}) & 0 & 0 \\ 0 & 0 & 1/2C & 0 & 0 \\ 1/2(d(k)+\psi^T(k)\vec{H}) & 1/2C & (f(k)M+\phi^T(k)M) & (1/8CN-1/4L^TC) & 1/2M^TC \\ 0 & 0 & (1/8N^TC-1/4CL) & 0 & 0 \\ 0 & 0 & 1/2CM & 0 & 0 \end{bmatrix}$$

(6.180)

$$W(k) = \begin{bmatrix} 0 & 0 & 1/2D(k) & 0 & 0 \\ 0 & 0 & 1/2C(k) & 0 & 0 \\ 1/2D(k) & 1/2C(k) & F(k)M & (1/8C(k)N-1/4L^TC(k) & 1/2M^TC(k) \\ 0 & 0 & (1/8N^TC(k)-1/4C(k)L) & 0 & 0 \\ 0 & 0 & 1/2C(k)M & 0 & 0 \end{bmatrix}$$

(6.181)

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+ (k)^T, -\mu^T(k-1), (b(k)+M^T\lambda(k)+\sigma(k)+M^T\nu(k))^T,$$

$$-\phi^T(k), -\psi^T(k)] \qquad (6.182)$$

$$Q^T(k) = [0, 0, B^T(k), 0, 0] \qquad (6.183)$$

then, the cost functional in equation (6.177) can be written as:

$$J^k=E[\{x^T(K)\mu^T(K)\vec{H}x(K)-\mu^T(K)x(K)-\lambda^T(K)y(K)-x^T(0)\mu^T(0)\vec{H}x(0)$$

$$+\lambda^T(0)x(0)+\mu^T(0)y(0)\} + \{X^T(k)W(k)X(k)+Q^T(k)X(k)$$

$$-\sum_{k=1}^{K}(X^T(k)L(k)X(k)+R^T(k)X(k))] \qquad (6.184)$$

Since $x(K)$ and $y(K)$ are arbitrary and $x(0)$, $y(0)$ are constant, then the cost functional in equation (6.184) can be written as:

$$J^k=E[\{x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)-\mu^T(K)y(K)\}$$

$$+\{X^T(k)W(k)X(k)+Q^T(k)X(k)-\sum_{k=1}^{K}(X^T(k)L(k)X(k)+R^T(k)X(k))\}]$$

(6.185)

It will be noticed that J^k in equation (6.185) is composed of a boundary part and a discrete integral part, which are independent of each other. The cost functional in equation (6.185) can be written as:

$$J^k = J_1^K + J_2^k \qquad (6.186)$$

where

$$J_1^K = E[x^T(K)\mu^T(K)\vec{H}x(K) - \lambda^T(K)x(K) - \mu^T(K)y(K)] \qquad (6.187)$$

$$J_2^k = E[X^T(k)W(k)X(k) + Q^T(k)X(k) - \sum_{k=1}^{K} (X^T(k)L(k)X(k) + R^T(k)X(k))] \qquad (6.188)$$

If one defines the following vectors such that:

$$A(k) = W(k) - KL(k) \qquad (6.189)$$

$$P(k) = Q(k) - KR(k) \qquad (6.190)$$

Then, equation (6.188) can be written as:

$$J_2 = E[\sum_{k=1}^{K} \{X^T(k)A(k)X(k) + P^T(k)X(k)\}] \qquad (6.191)$$

Now, define the vector V(k) such that

$$V(k) = A^{-1}(k)P(k) \qquad (6.192)$$

then equation (6.191) can be written in the following form by a process similar to completing the squares as

$$J_2 = E\left[\sum_{k=1}^{K} \{(X(k)+1/2\ V(k))^T A(k)(X(k)+ 1/2\ V(k)) - 1/4\ V^T(k)A(k)V(k)\}\right]$$ (6.193)

Since it is desired to minimize J_2 with respect to $X(k)$, the problem is equivalent to:

$$J_2 = E\left[\sum_{k=1}^{K} \{(X(k)+ 1/2\ V(k))^T A(k)(X(k)+ 1/2\ V(k))\}\right] \quad (6.194)$$

because $V(k)$ is independent of $X(k)$. Equation (6.194) defines a norm, hence equation (6.144) becomes:

$$\underset{X(k)}{\mathrm{Min.}} J_2 = \underset{X(k)}{\mathrm{Min}} E[\,||X(k) + 1/2\ V(k)||\,]_{A(k)} \quad (6.195)$$

6.3.3 The Optimal Solution

To minimize J^k in equation (6.186), one minimizes each term in equation (6.186) separately. The minimum of $J_1{}^K$ is clearly achieved when

$$E[\lambda(K)] = [0] \quad (6.196)$$

$$E[\mu(K)] = [0] \quad (6.197)$$

because $\delta x(K)$ and $\delta y(K)$ are arbitrary.

The minimum of $J_2{}^k$ in equation (6.195) is achieved when the norm is equal to zero

$$E[X(k) + 1/2\ V(k)] = [0] \tag{6.198}$$

Substituting from equations (6.189), (6.190) and equation (6.192) into equation (6.198), one obtains the following optimal equation.

$$E[(L(k)-1/K\ W(k))X(k)+R(k)-1/K\ Q(k)] = [0] \tag{6.199}$$

Writing equation (6.199) explicitly and adding equations (6.108-6.111), one obtains the following equations:

$$E[-x(k)+x(k-1)+I(k)+Mu(k)+Ms(k)] = [0] \tag{6.200}$$

$$E[-y(k)+x^T(k)\vec{H}x(k)] = [0] \tag{6.201}$$

$$E[-z(k)+u^T(k)\vec{H}u(k)] = [0] \tag{6.202}$$

$$E[-r(k-1)+u^T(k)\vec{H}x(k-1)] = [0] \tag{6.203}$$

$$E[\lambda(k)-\lambda(k-1)+2\mu^T(k-1)\vec{H}x(k-1)+(\Delta(k)+\psi^T(k)\vec{H})u(k)$$

$$+ \nu(k)] = [0] \tag{6.204}$$

$$E[\Gamma(k)u(k) - \mu(k-1)] = [0] \tag{6.205}$$

$$E[\beta(k)+M^T\lambda(k)+M^T\nu(k)+\sigma(k)+(\Delta(k)+\psi^T(k)\vec{H})x(k-1)$$

$$+\ \Gamma(k)y(k-1)+2(\theta(k)M+\phi^T(k)\vec{H})u(k)$$

$$+\ (1/4\Gamma(k)N-\ 1/2\ L^T\ \Gamma(k))z(k)] = [0] \qquad (6.206)$$

$$E[-\phi(k)+\ 1/4\ N^T\Gamma(k)u(k)-\ 1/2\ \Gamma(k)Lu(k)] = [0] \qquad (6.207)$$

$$E[-\psi(k)+\Gamma(k)Mu(k)] = [0] \qquad (6.208)$$

where

$$\begin{aligned} \beta(k) &= b(k) - 1/K\ B(k) \\ \Delta(k) &= d(k) - 1/K\ D(k) \\ \Gamma(k) &= C - 1/K\ C(k) \\ \theta(k) &= f(k) - 1/K\ F(k) \end{aligned} \qquad (6.209)$$

We can state the optimal equations (6.200-6.208) in component form as:

$$E[-x_{ij}^{\ k}+x_{ij}^{\ k-1}+I_{ij}^{\ k}+u_{(i-1)j}^{\ k}-u_{ij}^{\ k}+s_{(i-1)j}^{\ k}-s_{ij}^{\ k}] = 0 \qquad (6.210)$$

$$E[-y_{ij}^{\ k}+(x_{ij}^{\ k})^2] = 0 \qquad (6.211)$$

$$E[-z_{ij}^{k}+(u_{ij}^{k})^2] = 0 \tag{6.212}$$

$$E[-r_{ij}^{k-1}+u_{ij}^{k}x_{ij}^{k-1}] = 0 \tag{6.213}$$

$$E[\lambda_{ij}^{k}-\lambda_{ij}^{k-1}+2\mu_{ij}^{k-1}x_{ij}^{k-1}+\Delta_{ij}^{k}u_{ij}^{k}+\psi_{ij}^{k}u_{ij}^{k}+\nu_{ij}^{k}] = 0 \tag{6.214}$$

$$E[\Gamma_{ij}^{k}u_{ij}^{k}-\mu_{ij}^{k-1}] = 0 \tag{6.215}$$

$$E[\beta_{ij}^{k}+\lambda_{(i+1)j}^{k}-\lambda_{ij}^{k}+\nu_{(i+1)j}^{k}-\nu_{ij}^{k}+\sigma_{ij}^{k}+\Delta_{ij}^{k}x_{ij}^{k-1}+\psi_{ij}^{k}x_{ij}^{k-1}$$

$$+\Gamma_{ij}^{k}y_{ij}^{k-1}+2\phi_{ij}^{k}u_{ij}^{k}+2\theta_{ij}^{k}u_{(i-1)j}^{k}-2\theta_{ij}^{k}u_{ij}^{k}+1/4\ \Gamma_{ij}^{k}u_{ij}^{k}$$

$$+1/4\ \Gamma_{ij}^{k}u_{(i-1)j}^{k}-1/2\ \Gamma_{(i+1)j}^{k}z_{(i+1)j}^{k}] = 0 \tag{6.216}$$

$$E[-\phi_{ij}^{k}+1/4\ \Gamma_{ij}^{k}u_{ij}^{k}+1/4\ \Gamma_{(i+1)j}^{k}u_{(i+1)j}^{k}-1/2\ \Gamma_{ij}^{k}u_{(i-1)j}^{k}] = 0 \tag{6.217}$$

$$E[-\psi_{ij}^{k}+\Gamma_{ij}^{k}u_{(i-1)j}^{k}-\Gamma_{ij}^{k}u_{ij}^{k}] = 0 \tag{6.218}$$

where

$$\left.\begin{aligned}
\beta_{ij}^{k} &= b_{ij}^{k}-(b_{ij}^{k}/Ka^{k})\\
\Delta_{ij}^{k} &= d_{ij}^{k}-(d_{ij}^{k}/Ka^{k})\\
\Gamma_{ij}^{k} &= \gamma_{ij}-(\gamma_{ij}/Ka^{k})\\
\theta_{ij}^{k} &= f_{ij}^{k}-(f_{ij}^{k}/Ka^{k})
\end{aligned}\right\} \tag{6.219}$$

Besides the above equations, one has the following Kuhn-Tucker exclusion equations which must be satisfied at the optimum (40)

$$e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k}) = 0 \tag{6.220}$$

$$e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij}) = 0 \tag{6.221}$$

$$g_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \tag{6.222}$$

$$g_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k}) = 0 \tag{6.223}$$

One also has the following limits on the variable (40)

$$\left.\begin{array}{l} \text{If } x_{ij}^{k} < \underline{x}_{ij}, \text{ then we put } x_{ij}^{k}=\underline{x}_{ij} \\ \text{If } x_{ij}^{k} > \bar{x}_{ij}, \text{ then we put } x_{ij}^{k}=\bar{x}_{ij} \\ \\ \text{If } u_{ij}^{k} < \underline{u}_{ij}^{k}, \text{ then we put } u_{ij}^{k}=\underline{u}_{ij}^{k} \\ \text{If } u_{ij}^{k} > \bar{u}_{ij}^{k}, \text{ then we put } u_{ij}^{k}=\bar{u}_{ij}^{k} \end{array}\right\} \tag{6.224}$$

Equations (6.210-6.224) with equations (6.196) and (6.197) completely specify the optimal solution. The following algorithm is used to solve these equations.

6.3.4 Algorithm for Solution

Assume given: The number of rivers (m), the number of reservoirs on each river (n_j), the expected value for the natural inflows, the initial storage x_{ij}^{0}, and the percentage load on the system during each month a^k. The following steps are used to solve the optimal system equations.

Step 1 Assume initial guess for the variable u(k) such that

$$\underline{u}(k) \leq u^i(k) \leq \bar{u}(k) \text{ ; i=iteration number; i=0}$$

Step 2 Assume first that s(k) is equal to zero. Solve equations (6.200–6.203), (6.205), (6.207) and (6.208) forward in stages with x(0) given.

Step 3 Check the limits on x(k). If x(k) satisfies the inequality

$$\underline{x} < x(k) < \bar{x}$$

go to Step 10, otherwise put x(k) to its limits and go to Step 4.

Step 4 Calculate the new dischage from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k))]$$

Step 5 Check the limits on u(k). If u(k) satisfies the inequality

$$\underline{u}(k) < u(k) < \bar{u}(k)$$

go to Step 14, otherwise put u(k) to its limits and go to Step 6.

Step 6 Calculate the spill at month k from the following

$$E[s(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k))-\bar{u}(k)]$$

If $s(k) < 0$, put $s(k) = 0$.

Step 7 Calculate the discharge from the following equation

$$E[u(k)] = E[[M]^{-1}(x(k)-x(k-1)-I(k)-Ms(k))]$$

and go to Step 8.

Step 8 Solve again equations (6.200–6.203) with equations (6.205), (6.207) and (6.208) forward in stages with $x(0)$ given, but $s(k)$ has the value obtained from Step 6.

Step 9 Check the limits on $x(k)$. If $x(k)$ satisfies the inequality $\underline{x} < x(k) < \bar{x}$, go to Step 10, otherwise put $x(k)$ to its limits and go to Step 4.

Step 10 With $\nu(k)=0$, solve equation (6.204) backward in stages with equations (6.196) and (6.197) as the terminal conditions and go to Step 11.

Step 11 Calculate Kuhn-Tucker multipliers for $u(k)$, $\sigma(k)$, from the following equation

$$E[\sigma(k)]=E[2M^T\mu^T(k-1)\vec{H}x(k-1)+(M^T\Delta(k)+M^T\psi^T(k)\vec{H})u(k)$$

$$-\beta(k)-M^T\lambda(k-1)-(\Delta(k)+\psi^T(k)\vec{H})x(k-1)-\Gamma(k)y(k-1)$$

$$- 2(\theta(k)M+\phi^T(k)\vec{H})u(k)-(1/4\ \Gamma(k)N-\ 1/2\ L^T\Gamma(k))x(k)]$$

If $u(k)$ satisfies the inequality, $\underline{u}(k)<u(k)<\overline{u}(k)$, put $\sigma(k)=0$.

Step 12 Determine a new control iterate from the following equation

$$E[u^{i+1}(k)]=E[u^i(k)-\alpha D^i u(k)]$$

where

$$E[Du(k)]=E[\beta(k)+M^T\lambda(k)+\sigma(k)+(\Delta(k)+\psi^T(k)\vec{H})x(k-1)\ +$$

$$+\ \Gamma(k)y(k-1)+2(\theta(k)M+\phi^T(k)\vec{H})u(k)\ +$$

$$+\ (1/4\ \Gamma(k)N-\ 1/2\ L^T\Gamma(k))z(k)]$$

and α is a positive scalar which is chosen with consideration to such factors as convergence.

Step 13 Check the limits on $u^{i+1}(k)$. If $u^{i+1}(k)$ satisfies the inequality

$$\underline{u}(k)<u^{i+1}(k)<\overline{u}(k)$$

go to Step 14, otherwise put $u^{i+1}(k)$ to its limits and go to Step 6.

Step 14 Solve the following equation forward in stages

$$E[\lambda(k-1)]=E[2\mu^T(k-1)\vec{H}x(k-1)+(\Delta(k)+\psi^T(k)\vec{H})u(k)$$

$$-[M^T]^{-1}\beta(k)-([M^T]^{-1}\Delta(k)+[M^T]^{-1}\psi^T(k)\vec{H})x(k-1)$$

$$-[M^T]^{-1}\Gamma(k)y(k-1)-2([M^T]^{-1}\theta(k)M$$

$$+[M^T]^{-1}\phi(k)\vec{H})u(k)$$

$$-(1/4[M^T]^{-1}\Gamma(k)N-1/2[M^T]^{-1}L^T\Gamma(k))z(k)]$$

Step 15 Determine Kuhn-Tucker multiplier for x(k), $\nu(k)$, from the following equation

$$E[\nu(k)]=E[-[M^T]^{-1}\{\beta(k)+M^T\lambda(k)$$

$$+(\Delta(k)+\psi^T(k)\vec{H})x(k-1)+\Gamma(k)y(k-1)+2(\theta(k)M+\psi^T(k)\vec{H})\cdot$$

$$\cdot u(k)+(1/4\ \Gamma(k)N-1/2\ L^T\Gamma(k))z(k)\}]$$

If x(k) satisfies the inequality, $\underline{x}<x(k)<\overline{x}$, put $\nu(k)=0$

Step 16 Determine a new state iterate from the following equation

$$E[x^{i+1}(k)]=E[x^i(k)-\alpha D^i x(k)]$$

where

$$E[Dx(k)] = E[\lambda(k) - \lambda(k-1) + 2\mu^T(k-1)\vec{H}x(k-1)$$

$$+ (\Delta(k) + \psi^T(k)\vec{H})u(k) + \nu(k)]$$

Step 17 Repeat the calculation starting from Step 3. If the solution does not converge to the optimal solution, assume another different initial guess for u(k) and repeat the calculation starting from Step 2. But if the solution converges to the optimal continue until the state x(k) and the control u(k) do not change significantly from iteration to iteration and J^k in equation (6.107) is a minimum.

6.3.5 Two Practical Examples

The algorithm of the last section has been used to solve two practical examples. The first example, the same example mentioned in section (6.2.5), is solved using this model. The second example is the same as the first example except that the operation of the reservoirs R_{11} and R_{21} are constrained by a certain treaty contract and for this example, the maintainance schedule for the turbines is taken into account. As a matter of fact, we found that the discharge from the turbines is less than the maximum discharge from the turbines during the maintainance period.

The results for the first example are reported in Tables (6.14-6.21). In Tables (6.14-6.17) we give the optimal monthly releases from the reservoirs, the profits realized and the calculated percentage load on the system during each year of the critical period. It will be noticed from these tables that the calculated percentage load is equal to the required percentage load on the system (a^k). In Tables (6.18-6.21) we give the optimal reservoir storage during the critical period.

In Tables (6.22-6.25) we give the percentage load, the difference between the required load on the system and the calculated load and the value of J^k (the cost function). It will be noticed from these tables that as this difference approaches zero, the calculated load is equal to the required load (a^k).

The total benefits obtained from the system using this model during the critical period is 112,499,920 MWh and the yearly average is 31,395,327 MWh

The computing time to get the optimal solution during the critical period (43 months) was 6.85 sec. in CPU units.

Table 6.14: Optimal releases from the turbines, the profits realized and the percentage load required during the first year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits MWh	%Load Cal.	%Load Given
1	1470	2117	2494	2517	2619563	8.340	8.34
2	2038	2040	2404	2411	2786490	8.871	8.87
3	1454	2552	2992	3001	2949777	9.391	9.39
4	1842	2424	3107	3114	3068649	9.769	9.77
5	1764	1837	2957	2963	2748784	8.751	8.75
6	1849	1906	3091	3097	2839532	9.040	9.04
7	1573	2064	2614	2622	2510435	7.992	7.99
8	1498	2079	2488	2541	2430745	7.738	7.74
9	1420	1966	2355	2428	2352717	7.490	7.49
10	1232	2455	2042	2103	2333654	7.429	7.43
11	1314	2209	2192	2215	2399492	7.639	7.64
12	1364	1906	2275	2297	2371382	7.549	7.55
Total benefits from the generation during the first year of the critical period					31411168	100.00	100.00

Table 6.15: Optimal releases from the turbines, profits realized and the percentage load required during the second year of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits MWh	%Load Cal.	%Load Given
1	1491	2150	2529	2544	2619398	8.339	8.34
2	2077	2078	2438	2453	2785707	8.869	8.87
3	1475	2594	3041	3049	2949907	9.391	9.39
4	1910	2318	3221	3229	3069785	9.773	9.77
5	1793	1888	3007	3013	2748036	8.749	8.75
6	1872	1935	3179	3185	2839552	9.040	9.04
7	1653	1913	2751	2773	2509918	7.991	7.99
8	1529	2126	2542	2595	2431018	7.739	7.74
9	1435	2006	2382	2471	2352861	7.491	7.49
10	1267	2435	2101	2154	2333571	7.429	7.43
11	1367	2201	2268	2291	2399486	7.639	7.64
12	1370	2053	2288	2310	2371875	7.551	7.55
Total benefits from the generation during the second year of the critical period					31411056	100.00	100.00

Table 6.16: Optimal releases from the turbines, profits realized and the percentage load required during the third year of the critical period

Month k	u_{11}^{k} Mm^3	u_{21}^{k} Mm^3	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	%Load Cal.	%Load Given
1	1529	2184	2579	2602	2619493	8.339	8.34
2	2049	2251	2487	2509	2786245	8.870	8.87
3	1513	2630	3091	3106	2949134	9.389	9.39
4	1879	2698	3173	3182	3068526	9.769	9.77
5	1830	2023	3077	3085	2740968	8.752	8.75
6	1771	1909	3461	3467	2840465	7.043	9.04
7	255	390	4796	4803	2509759	7.990	7.99
8	1583	2203	2633	2686	2432202	7.743	7.44
9	1488	2066	2473	2546	2353636	7.493	7.49
10	1324	2407	2199	2245	2332939	7.427	7.43
11	1410	2199	2355	2378	2399289	7.638	7.64
12	1436	2012	2399	2414	2370568	7.547	7.55
Total benefits from the generation during the third year of the critical period					31411152	100.00	100.00

Table 6.17: Optimal releases from the turbines, profits realized and the percentage load required during the rest of the critical period

Month k	u_{11}^k Mm^3	u_{21}^k Mm^3	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits MWh	%Load Cal.	%Load Given
1	779	1441	3519	3534	2466724	8.393	8.34
2	1180	2203	3099	3106	2602812	8.856	8.87
3	1444	1785	3579	3587	2756584	7.379	9.39
4	1207	1313	4373	4379	2868556	9.769	9.77
5	1624	1608	3314	3319	2559364	8.710	8.75
6	683	862	4826	4831	2675327	9.102	9.04
7	465	755	4410	4417	2337184	7.956	7.99
Total benefits from the generation during the rest of the critical period					18266544	62.165	62.15

Table 6.18: Optimal reservoir storage during the first year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	24088.39844	5026.95313	73565.43750
2	22419.17969	5199.93359	72071.12500
3	21253.19141	4241.70703	69723.81250
4	19618.23438	3734.00000	67397.68750
5	18044.49609	3741.81177	64892.28906
6	16507.83984	3754.94678	62285.46094
7	15881.36719	3784.47144	60536.80469
8	15838.98438	4052.69971	61946.90234
9	17252.08591	4813.08203	68766.25000
10	20630.50781	5304.19922	71600.43750
11	22463.90234	5304.19922	71205.0000
12	22384.19922	5187.05859	70514.93750

Table 6.19: Optimal reservoir storage during the second year of the critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	21704.51172	4914.93750	69304.75000
2	19989.33203	5097.31250	68165.18750
3	18733.31641	4075.27490	65995.68750
4	17010.77344	3734.00000	63653.91797
5	15400.62500	3741.52344	61138.93750
6	13741.71094	3754.38501	58433.14844
7	12498.86328	3782.31079	57420.90625
8	12766.90234	4209.58203	58981.62109
9	14759.15625	5304.19922	63718.12891
10	15442.28906	5304.19922	65605.62500
11	17775.22656	5304.19922	65521.16406
12	18202.51953	5304.19922	6481.41797

Table 6.20: Optimal reservoir storage during the third year of the critical period

Month k	x_{11}^k Mm^3	x_{21}^k Mm^3	x_{12}^k Mm^3
1	17590.51563	5142.03516	64325.31484
2	16085.42969	5261.57031	63269.33594
3	14865.00781	4368.90625	61201.72656
4	13212.67578	3742.86377	58754.78125
5	11572.34766	3734.00000	56324.73438
6	10036.80859	3741.15430	53360.37891
7	10021.75781	3763.70752	49123.04688
8	10061.26953	4099.75000	50683.39844
9	11919.75781	5158.27344	55770.05859
10	14167.24219	5304.19922	57104.83203
11	15282.39453	5304.19922	65485.76563
12	15115.46094	5161.42969	55348.51953

Table 6.21: Optimal reservoir storage during the rest of critical period

Month k	x_{11}^{k} Mm^3	x_{21}^{k} Mm^3	x_{12}^{k} Mm^3
1	14993.76953	4774.42578	53209.72656
2	14153.11328	3913.73535	50947.33203
3	12932.23438	3734.00000	48057.27344
4	11873.80078	3734.00000	44148.40234
5	10398.88672	3833.39600	41184.32813
6	9949.00000	3734.00000	36769.14453
7	9949.00000	3734.00000	33195.50000

Table 6.22: The percentage calculated load, the percentage error and the value of J^k at the optimum for the first year of the critical period

Month k	% Calculated load	% Error	J^k MWh
1	8.33959	-0.00040	-126.00000
2	8.87101	0.00103	322.00000
3	9.39085	0.00086	270.00000
4	9.76929	-0.00070	-220.00000
5	8.75097	0.00098	308.00000
6	9.03988	-0.00011	-36.00000
7	7.99217	0.00218	684.00000
8	7.73847	-0.00152	-478.00000
9	7.49006	0.00007	22.00000
10	7.42937	-0.00061	-193.00000
11	7.63897	-0.00102	-320.00000
12	7.54948	-0.00051	-159.00000

Table 6.23: The percentage calculated load, the percentage error and the value of J^k at the optimum for the second year of the critical period

Month k	% Calculated load	% Error	J^k MWh
1	8.33910	0.00090	-282.00000
2	8.86855	-0.00144	-451.00000
3	9.39130	0.00131	410.00000
4	9.77294	0.00295	927.00000
5	8.74862	-0.00137	430.00000
6	9.03997	-0.00002	-6.00000
7	7.99055	0.00056	176.00000
8	7.73936	-0.00062	-196.00000
9	7.49055	0.00055	174.00000
10	7.42913	-0.00085	-260.00000
11	7.63989	-0.00101	-318.00000
12	7.55108	0.00109	342.00000

Table 6.24: The percentage calculated load, the percentage error and the value of J^k at the optimum for the third year of the critical period

Month k	% Calculated Load	% Error	J^k MWh
1	8.33937	0.00062	195.00000
2	8.87024	0.00025	78.00000
3	9.38881	-0.00118	-372.00000
4	9.76890	-0.00109	341.00000
5	8.75156	0.00157	493.00000
6	9.04285	0.00286	898.00000
7	7.99002	0.00003	10.00000
8	7.74311	0.00312	980.00000
9	7.49299	0.00300	942.00000
10	7.42710	-0.00289	-907.00000
11	7.63833	-0.00166	-522.00000
12	7.54690	-0.00309	-972.00000

Table 6.25: The percentage calculated load, the percentage error and the value of J^k at the optimum for rest of the critical period

Month k	% Calculated Load	% Error	J^k MWh
1	8.39277	0.052767	15356.0000
2	8.85600	-0.014000	-3823.0000
3	9.37900	-0.011000	-3489.0000
4	9.76900	-0.001000	-2943.0000
5	8.71000	-0.040000	-10737.0000
6	9.10200	+0.062000	17546.0000
7	7.95600	-0.034000	-11892.0000

The results for the second example are reported in Tables (6.26-6.33), as we mentioned earlier the operation of the first river which contains reservoirs R_{11} and R_{21} is constrained by what is called "treaty contract". The operation of the two reservoirs are fixed according to this contract. In this example we optimized the operation of the other two reservoirs in such a way that the total benefits from the system are maximum. In Tables (6.26-6.29) we give the optimal releases from the reservoirs, the profits realized and the percentage calculated load during the critical period. In Tables (6.30-6.33) we give the optimal reservoir storage during the critical period. It will be noticed from these tables that we started the critical period full but we did not end up empty to satisfy the treaty requirements.

The total benefits from the system under the treaty constraints is 111,946,688 Wh with yearly average 31,240,932 Wh. If we compare these results with that obtained in the first example, one can notice that due to the treaty constraints the total benefits from the system is decreased by 553,232 Wh and the yearly average is decreased by 154,395

Table 6.26: Optimal releases from the turbines, the profits realized and the percentage load required during the first year of the critical period under treaty constraints

Month k	u_{11}^{k} Mm^3 Treaty	u_{21}^{k} Mm^3 Treaty	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	% Calculated load
1	1046	1417	3262	3285	2610259.000	8.314
2	1123	1307	3622	3629	2782248.000	8.862
3	2123	2264	2520	2528	2942984.000	9.374
4	2123	2198	2856	2864	3063773.000	9.758
5	1575	1656	3075	3081	2740524.000	8.729
6	1289	1359	3747	3753	2838911.000	9.042
7	1101	1622	3095	3102	2498796.000	7.959
8	758	1607	3223	3276	2420443.000	7.709
9	734	2040	2765	2838	2346145.000	7.473
10	758	2472	2376	2437	2326424.000	7.410
11	2333	3228	773	796	2463758.000	7.847
12	1277	1703	2440	2462	2362353.000	7.524
	Total benefits from the generation				31,396,560	100.00

Table 6.27: Optimal releases from the turbines, the profits realized and the percentage load required during the second year of the critical period under treaty constraints

Month k	u_{11}^k Mm^3 Treaty	u_{21}^k Mm^3 Treaty	u_{12}^k Mm^3	u_{22}^k Mm^3	Profits MWh	% Calculated load
1	1061	1448	3307	3322	2618955.000	8.341
2	1117	1300	3712	3727	2788479.000	8.881
3	2123	2220	2616	2624	2952204.000	9.403
4	2123	2190	2932	2940	3068909.000	9.774
5	1575	1677	3143	3149	2747244.000	8.750
6	1289	1365	3828	3834	2836162.000	9.033
7	1101	1388	3333	3355	2502941.000	7.972
8	758	1782	3198	3251	2425440.000	7.725
9	734	2400	2614	2702	2354555.000	7.499
10	758	1926	2803	2856	2329661.000	7.420
11	758	1592	3145	3168	2399335.000	7.642
12	2202	2884	959	981	2374175.000	7.562
	Total benefits from the generation				31,398,000	100.00

Table 6.28: Optimal releases from the turbines, the profits realized and the percentage load required during the third year of the critical period under treaty constraints

Month k	u_{11}^{k} Mm^3 Treaty	u_{21}^{k} Mm^3 Treaty	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	%Calculated load
1	2047	2540	1818	1841	2615178.000	8.329
2	2348	2670	1841	1863	2782111.000	8.861
3	2578	2803	1967	1982	2952002.000	9.402
4	2578	2772	2314	2323	3065924.000	9.765
5	1027	1212	4033	4041	2755350.000	8.776
6	1137	1283	4159	4165	2846396.000	9.066
7	917	1074	3834	3841	2505565.000	7.980
8	758	1714	3378	3431	2427422.000	7.731
9	734	2370	2740	2813	2353722.000	7.497
10	758	1987	2863	2909	2329193.000	7.419
11	758	1547	3280	3303	2397031.000	7.635
12	734	1167	3509	3524	2367041.000	7.539
	Total benefits from the generation				31,396,560	100.00

Table 6.29: Optimal releases from the turbines, the profits realized and the percentage load required during the rest of the critical period under treaty constraints

Month k	u_{11}^{k} Mm^3 Treaty	u_{21}^{k} Mm^3 Treaty	u_{12}^{k} Mm^3	u_{22}^{k} Mm^3	Profits MWh	% Calculated load
1	758	1033	3644	3659	2384338.000	8.346
2	771	934	4067	4074	2532555.000	8.865
3	2578	2739	1538	1546	2682864.000	9.392
4	2578	2685	1909	1915	2791420.000	9.771
5	1195	1481	3419	3424	2498024.000	8.744
6	1137	1738	3589	3594	2584342.000	9.046
7	1895	3002	1546	1553	2281606.000	7.986
	Total benefits from the generation				17,755,248	

Table 6.30: Optimal reservoir storage during the first year of the critical period

Month k	x_{11}^k Mm^3 Treaty	x_{21}^k Mm^3 Treaty	x_{12}^k Mm^3
1	24515.37500	5304.19531	72797.37500
2	23768.09766	5304.19141	70085.37500
3	21938.64063	5304.18750	68210.25000
4	20027.11719	5304.18359	66134.50000
5	18646.25781	5304.17969	63511.50781
6	17673.72656	5304.17188	60248.59375
7	17509.31250	5304.16406	58019.44141
8	18175.85547	5304.15016	58694.83594
9	20224.75000	5304.14844	67604.25000
10	24005.89844	5304.14844	67604.25000
11	24762.43850	5304.14453	68628.75000
12	24762.46094	5304.13672	67773.56250

Table 6.31: Optimal reservoir storage during the second year of the critical period

Month k	x_{11}^{k} Mm^3 Treaty	x_{21}^{k} Mm^3 Treaty	x_{12}^{k} Mm^3
1	24515.32422	5304.12891	65785.81250
2	23768.05078	5304.12891	63372.82813
3	21869.89453	5304.12109	61628.68359
4	19939.67578	5304.11719	59576.31641
5	18552.18750	5304.11328	56925.60938
6	17479.72656	5304.10938	53570.74219
7	16779.70313	5304.10156	51977.06641
8	17787.04688	5304.09766	52882.38281
9	20429.27734	5304.09375	57387.73047
10	22547.67578	5304.08984	58573.84766
11	24430.99219	5304.08594	57612.40234
12	24021.14063	5304.08203	58201.31250

Table 6.32: Optimal reservoir storage during the third year of the critical period

Month k	x_{11}^{k} Mm^3 Treaty	x_{21}^{k} Mm^3 Treaty	x_{22}^{k} Mm^3
1	22894.12109	5304.09766	58506.28516
2	21097.67969	5304.09375	58096.28125
3	18818.30469	5304.08594	57152.41016
4	16471.94531	5304.07813	55564.65625
5	15638.91406	5304.07031	52178.94531
6	14741.10938	5304.06641	48515.87109
7	14054.51172	5304.06250	45240.39453
8	14886.92969	5304.06250	46056.53125
9	17447.17188	5304.05859	50876.67969
10	20185.80469	5304.05469	51547.24609
11	21893.62891	5304.05078	50003.31641
12	22423.92188	5304.04688	47756.53516

Table 6.33: Optimal reservoir storage during the rest of the critical period

Month k	x_{11}^{k} Mm^3 Treaty	x_{21}^{k} Mm^3 Treaty	x_{12}^{k} Mm^3
1	22326.64063	5304.04297	45492.12109
2	21902.94922	5303.59766	42261.60156
3	19554.84766	5303.58984	41412.67578
4	17131.37500	5302.13281	39967.72656
5	16090.36719	5099.46875	36899.23438
6	15190.44531	4578.75391	33720.78125
7	13750.87109	3762.21875	33010.80078

6.4 General Discussions

In this chapter the problem of long-term optimal operation of a multireservoir power system with specified monthly generation under critical water conditions has been treated. The model used for each reservoir in the first section of this chapter is a nonlinear model of the discharge and the storage of the previous month, this model may cause an overestimation of production for rising water levels and underestimation for falling water levels. To avoid these under and overestimations of production, an average of begin-and-end-of time step (month) storage is used, this is explained in detail in the second section of this chapter. The total benefits obtained in the second section is less than the total benefits obtained in the first Section.

According to the specifications of the source which provided us with the reservoir data, the operation of the first river, which contains reservoirs R_{11} and R_{21}, is constrained by what is called "treaty contract". We fixed the operation of this river according to the treaty operation, and optimized the operation of the second river, so that the total benefits from the system during the critical period is maximum. Also, we took into account the maintainance schedule of the reservoirs. The effect of this schedule is that the maximum discharge from the turbines is decreased during the period of maintainance. As a matter of fact, we found that the maintainance schedule has no effect in the optimal system operation, since the discharge from the turbines at the optimal operation is less than the maximum discharge during the maintainance period.

In Table (6.34) we give the optimal benefits from the system under difference modes of operation.

Table 6.34: Comparision between different modes of operation during the critical period

The model used	Total benefits MWh	Yearly average MWh
1. Nonlinear model of the storage of the previous month	113,562,500	31,691,860
2. Nonlinear model of the average storage	112,499,920	31,395,327
3. Nonlinear model of the average storage and treaty constraints	111,946,688	31,240,932

CHAPTER VII

OPTIMIZATION OF THE FIRM HYDRO ENERGY CAPABILITY UNDER CRITICAL WATER CONDITIONS

7.1 Background

In chapter 6, we maximized the generation from hydro electric multireservoir power systems having a specified monthly load under critical water conditions. This load is equal to a certain percentage of the total generation at the end of each year of the critical period. The GWh for this load during each month varies according to the total generation at the end of the optimization period. We found that our approach was efficient in calculating the hydro energy generation during the critical period, and the computing time was very small compared to what has been done so far using other approaches.

This chapter is devoted to the maximization of the firm hydro capability from multireservoir power systems under critical water conditions. In this chapter the load on the system is fixed, i.e. there is no relation between that load and the total generation at the end of optimization interval, and at the same time this firm energy should be uniform during each year of the critical period.

Solutions to this maximization problem have been determined in the past by skilled engineers using digital computer simulation programs and "cut and try" methods wherein reservoir storage-management schedules are successfully modified until it is believed that maximum energy capability for the system (total generation averaged over all months in the adverse streamflow period) with system constraints satisfied has been achieved. This approach has become cumbersome with present day requirements to deal with a large number of reservoirs and with a long period (45).

In Ref. (45), the maximization of the energy capability for a large scale hydroelectric system using the Nonlinear Programming approach has been done, the computing time for such systems was very large (41 minutes in CPU units), they considered this time as a reasonable computing time of what is believed to be one of the largest hydroelectric Nonlinear Programming problems ever attempted.

In this chapter, functional analysis and minimum norm formulation have been used to maximize the surplus energy from a hydroelectric power system under critical water conditions. The optimal control problem is formulated by constructing a cost funcion in which we maximize the surplus energy (the difference between the total generation during a certain month and the load on the system during that month), and at the same time this surplus energy should be uniform over the planning period. The cost functional is augmented by using Lagrange and Kuhn-Tucker multipliers to adjoin the equality and inequality constraints. The model used here is a nonlinear model, it is a function of the discharges through the turbines and the average monthly storage between two successive months, to avoid an underestimation for rising water level and overestimation for falling water levels in the hydroelectric generation.

7.2 Problem Formulation

7.2.1 The Objective Function

Given the hydroelectric system in Figure 6.1, the problem for this system is to find the discharge u_{ij}^{k}; $i=1,\dots,n$: $j=1,\dots,m$;

$k=1,\ldots,K$ as a function of time over the critical period under the following conditions.

(1) The total generation from the system over the optimization interval is a maximum.

(2) The system expected surplus energy over load for each time interval should be a maximum and at the same time uniform.

(3) The water conservation equation for each reservoir may be adequately described by the continuity-type equation

$$x_{ij}^{k} = x_{ij}^{k-1} + I_{ij}^{k} + u_{(i-1)j}^{k} - u_{ij}^{k} + s_{(i-1)j}^{k} - s_{ij}^{k} \qquad (7.1)$$

(4) In order to be realizable and also to satisfy multipurpose stream use requirements, the following inequality constraints must be satisfied

a) upper and lower bounds on reservoir plant outflows

$$\underline{u}_{ij}^{k} \leq u_{ij}^{k} \leq \bar{u}_{ij}^{k} \qquad (7.2)$$

b) upper and lower bounds on reservoir contents

$$\underline{x}_{ij} \leq x_{ij}^{k} \leq \bar{x}_{ij} \qquad (7.3)$$

In mathematical terms, the object of the optimizing computation is to find the discharge u_{ij}^{k} that maximizes

$$J=E[\sum_{k=1}^{K}\{\sum_{j=1}^{m}\sum_{i=1}^{n_j} G_{ij}(u_{ij}^k,\ 1/2(x_{ij}^k+x_{ij}^{k-1}))-d^k\}^2]\ \text{GWh} \quad (7.4)$$

Subject to satisfying the constraints (7.1-7.3).

The initial storage x_{ij}^0, the expected value for the natural inflows into each stream during each month, and the value of the load d^k are assumed to be known.

7.2.2 Modelling of the System

The generation of a hydroelectric plant is a nonlinear function of the discharge u_{ij}^k and the reservoir water head, which itself is a function of the storage. To avoid underestimation of production for rising water levels and overestimation for falling water levels, an average of begin and end-of-time step (month) storage is used, hence the generation of plant i on river j may be choosen as:

$$G_{ij}(u_{ij}^k,1/2(x_{ij}^k+x_{ij}^{k-1}))=\alpha_{ij}u_{ij}^k+1/2\ \beta_{ij}u_{ij}^k(x_{ij}^k+x_{ij}^{k-1})$$
$$+\ 1/4\ \gamma_{ij}(x_{ij}^k+x_{ij}^{k-1})^2 \quad (7.5)$$

where α_{ij}, β_{ij} and γ_{ij} are constants. These were obtained by least square curve fitting to typical plant data available.

Substituting from equation (7.1) into equation (7.5) for x_{ij}^k. One obtains

$$G_{ij}(u_{ij}^k,1/2(x_{ij}^k+x_{ij}^{k-1}))=b_{ij}^k u_{ij}^k+d_{ij}^k r_{ij}^k+f_{ij}^k\omega_{ij}^k+\gamma_{ij}g_{ij}^k$$
$$+\ 1/4\ \gamma_{ij}t_{ij}^k+\gamma_{ij}n_{ij}^k-1/4\ \gamma_{ij}m_{ij}^k \quad (7.6)$$

where

$$q_{ij}^{k}=I_{ij}^{k}+s_{(i-1)j}^{k}-s_{ij}^{k} \tag{7.7}$$

$$b_{ij}^{k}=\alpha_{ij}+ 1/2\ \beta_{ij}q_{ij}^{k}+ 1/4\ \gamma_{ij}(q_{ij}^{k})^2 \tag{7.8}$$

$$d_{ij}^{k}=\beta_{ij}+\gamma_{ij}q_{ij}^{k} \tag{7.9}$$

$$f_{ij}^{k}=1/2\ \beta_{ij}+1/2\ \gamma_{ij}q_{ij}^{k} \tag{7.10}$$

$$y_{ij}^{k}=(x_{ij}^{k})^2 \tag{7.11}$$

$$z_{ij}^{k}=(u_{ij}^{k})^2 \tag{7.12}$$

$$r_{ij}^{k}=u_{ij}^{k}x_{ij}^{k-1} \tag{7.13}$$

$$\omega_{ij}^{k}=u_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k}) \tag{7.14}$$

$$g_{ij}^{k}=u_{ij}^{k}y_{ij}^{k-1} \tag{7.15}$$

$$t_{ij}^{k}=u_{ij}^{k}(z_{(i-1)j}^{k}+z_{ij}^{k}) \tag{7.16}$$

$$n_{ij}^{k}=r_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k}) \tag{7.17}$$

$$m_{ij}^{k}=z_{ij}^{k}u_{(i-1)j}^{k} \tag{6.18}$$

Now, the cost functional in equation (7.4) becomes:

$$J=E[\sum_{k=1}^{K} \{\sum_{i=1}^{n_j} \sum_{j=1}^{m} (b_{ij}^k u_{ij}^k+d_{ij}^k r_{ij}^k+f_{ij}^k \omega_{ij}^k+\gamma_{ij} g_{ij}^k$$

$$1/4\, \gamma_{ij} t_{ij}^k+\gamma_{ij} n_{ij}^k-1/4\gamma_{ij} m_{ij}^k-d^k)^2\}] \tag{7.19}$$

Subject to satisfying the following constraints:

$$x_{ij}^k=x_{ij}^{k-1}+q_{ij}^k+u_{(i-1)j}^k-u_{ij}^k \tag{7.20}$$

$$y_{ij}^k=(x_{ij}^k)^2 \tag{7.21}$$

$$z_{ij}^k=(u_{ij}^k)^2 \tag{7.22}$$

$$r_{ij}^k=u_{ij}^k x_{ij}^{k-1} \tag{7.23}$$

$$\omega_{ij}^k=u_{ij}^k(u_{(i-1)j}^k-u_{ij}^k) \tag{7.24}$$

$$g_{ij}^k=u_{ij}^k y_{ij}^{k-1} \tag{7.25}$$

$$t_{ij}^k=u_{ij}^k(z_{(i-1)j}^k+z_{ij}^k) \tag{7.26}$$

$$n_{ij}^k=r_{ij}^k(u_{(i-1)j}^k-u_{ij}^k \tag{7.27}$$

$$m_{ij}^k = z_{ij}^k (u_{(i-1)j}^k \qquad (7.28)$$

$$\underline{x}_{ij} \le x_{ij}^k \le \overline{x}_{ij} \qquad (7.29)$$

$$\underline{u}_{ij}^k \le u_{ij}^k \le \overline{u}_{ij}^k \qquad (7.30)$$

The problem now is that of maximizing (7.19) subject to satisfying the constraints (7.20-7.30).

7.3 A Minimum Norm Formulation

The augmented cost functional, $\tilde{J}$, is obtained by adjoining to the cost functional the equality constraints (7.20-7.28) via Lagrange's multipliers and the inequality constraints (7.29) and (7.30) via Kuhn-Tucker multipliers. One thus obtains (48):

$$\tilde{J} = E[\sum_{k=1}^{K} \{\sum_{j=1}^{m} \sum_{i=1}^{n_j} (b_{ij}^k u_{ij}^k + d_{ij}^k r_{ij}^k + f_{ij}^k \omega_{ij}^k + \gamma_{ij} g_{ij}^k$$

$$+ 1/4\, \gamma_{ij} t_{ij}^k + \gamma_{ij} n_{ij}^k - 1/4\, \gamma_{ij} m_{ij}^k - d^k)^2$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{n_j} \{\lambda_{ij}^k (-x_{ij}^k + x_{ij}^{k-1} + q_{ij}^k + u_{(i-1)j}^k - u_{ij}^k)$$

$$+ \mu_{ij}^k (-y_{ij}^k + (x_{ij}^k)^2) + \phi_{ij}^k (-z_{ij}^k + (u_{ij}^k)^2)$$

$$+ \psi_{ij}^k (-r_{ij}^k + u_{ij}^k x_{ij}^{k-1}) + \theta_{ij}^k (-\omega_{ij}^k + u_{ij}^k (u_{(i-1)j}^k - u_{ij}^k))$$

$$+ \delta_{ij}^k (-g_{ij}^k + u_{ij}^k y_{ij}^{k-1}) + \Omega_{ij}^k (-t_{ij}^k + u_{ij}^k (z_{(i-1)j}^k + z_{ij}^k))$$

$$+ \Gamma_{ij}^{k}(-n_{ij}^{k}+r_{ij}^{k}(u_{(i-1)j}^{k}-u_{ij}^{k}))+\nu_{ij}^{k}(-m_{ij}^{k}+z_{ij}^{k}u_{(i-1)j}^{k})$$

$$+ e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k})+e_{ij}^{1k}(x_{ij}^{k}-\bar{x}_{ij})+\ell_{ij}^{k}(\underline{u}_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k})$$

$$+ \ell_{ij}^{1k}(u_{ij}^{k}-\bar{\mu}_{ij}^{k})]] \qquad (7.31)$$

where λ_{ij}^{k}, μ_{ij}^{k}, ϕ_{ij}^{k}, ψ_{ij}^{k}, θ_{ij}^{k}, δ_{ij}^{k}, Ω_{ij}^{k}, Γ_{ij}^{k} and ν_{ij}^{k} are Lagrange's multipliers. These are to be determined such that the corresponding equality constraints must be satisfied, and e_{ij}^{k}, e_{ij}^{1k}, ℓ_{ij}^{k} and ℓ_{ij}^{1k} are Kuhn-Tucker multipliers. They are equal to zero if the constraints are not violated and greater than zero if the constraints are violated (32).

Now define the following column vectors such that (32)

$$b(k)=col.(b_1(k),\ldots,\ldots,b_m(k)) \qquad (7.32)$$

$$b_1(k)=col.(b_{11}^{k},\ldots,\ldots,b_{n_1 1}^{k}) \qquad (7.33)$$

$$b_m(k)=col.(b_{1m}^{k},\ldots,\ldots,b_{n_m m}^{k}) \qquad (7.34)$$

$$d(k)=col.(d_1(k),\ldots,\ldots,d_m(k)) \qquad (7.35)$$

$$d_1(k)=col.(d_{11}^{k},\ldots,\ldots,d_{n_1 1}^{k}) \qquad (7.36)$$

$$d_m(k)=col.(d_{1m}^{k},\ldots,\ldots,d_{n_m m}^{k}) \qquad (7.37)$$

$$f(k)=col.(f_1(k),...,...,f_m(k)) \qquad (7.38)$$

$$f_1(k)=col.(f_{11}{}^k,...,...,f_{n_1 1}{}^k) \qquad (7.39)$$

$$f_m(k)=col.(f_{1m}{}^k,...,...,f_{n_m m}{}^k) \qquad (7.40)$$

$$\gamma=col.(\gamma_1,...,...,\gamma_m) \qquad (7.41)$$

$$\gamma_1=col.(\gamma_{11},...,...,\gamma_{n_1 1}) \qquad (7.42)$$

$$\gamma_m=col.(\gamma_{1m},...,...,\gamma_{n_m m}) \qquad (7.43)$$

$$y(k)=col.(y_1(k),...,...,y_m(k)) \qquad (7.44)$$

$$y_1(k)=col.(y_{11}{}^k,...,...,y_{n_1 1}{}^k) \qquad (7.45)$$

$$y_m(k)=col.(y_{1m}{}^k,...,...,y_{n_m m}{}^k) \qquad (7.46)$$

$$r(k)=col.(r_1(k),...,...,r_m(k)) \qquad (7.47)$$

$$r_1(k)=col.(r_{11}{}^k,...,...,r_{n_1 1}{}^k) \qquad (7.48)$$

$$r_m(k)=col.(r_{1m}{}^k,...,...,r_{n_m m}{}^k) \qquad (7.49)$$

$$\omega(k)=col.(\omega_1(k),...,...,\omega_m(k)) \qquad (7.50)$$

$$\omega_1(k)=\text{col.}(\omega_{11}{}^k,\ldots,\ldots,\omega_{n_1 1}{}^k) \tag{7.51}$$

$$\omega_m(k)=\text{col.}(\omega_{1m}{}^k,\ldots,\ldots,\omega_{n_m m}{}^k) \tag{7.52}$$

$$g(k)=\text{col.}(g_1(k),\ldots,\ldots,g_m(k)) \tag{7.53}$$

$$g_1(k)=\text{col.}(g_{11}{}^k,\ldots,\ldots,g_{n_1 1}{}^k) \tag{7.54}$$

$$g_m(k)=\text{col.}(g_{1m}{}^k,\ldots,\ldots,g_{n_m m}{}^k) \tag{7.55}$$

$$t(k)=\text{col.}(t_1(k),\ldots,\ldots,t_m(k)) \tag{7.56}$$

$$t_1(k)=\text{col.}(t_{11}{}^k,\ldots,\ldots,t_{n_1 1}{}^k) \tag{7.57}$$

$$t_m(k)=\text{col.}(t_{1m}{}^k,\ldots,\ldots,t_{n_m m}{}^k) \tag{7.58}$$

$$n(k)=\text{col.}(n_1(k),\ldots,\ldots,n_m(k)) \tag{7.59}$$

$$n_1(k)=\text{col.}(n_{11}{}^k,\ldots,\ldots,n_{n_1 1}{}^k) \tag{7.60}$$

$$n_m(k)=\text{col.}(n_{1m}{}^k,\ldots,\ldots,n_{n_m m}{}^k) \tag{7.61}$$

$$m(k)=\text{col.}(m_1(k),\ldots,\ldots,m_m(k)) \tag{7.62}$$

$$m_1(k)=\text{col.}(m_{11}{}^k,\ldots,\ldots,m_{n_1 1}{}^k) \tag{7.63}$$

$$m_m(k)=col.(m_{1m}^{k},\ldots,\ldots,m_{n_m m}^{k}) \quad (7.64)$$

$$\lambda(k)=col.(\lambda_1(k),\ldots,\ldots,\lambda_m(k)) \quad (7.65)$$

$$\lambda_1(k)=col.(\lambda_{11}^{k},\ldots,\ldots,\lambda_{n_1 1}^{k}) \quad (7.66)$$

$$\lambda_m(k)=col.(\lambda_{1m}^{k},\ldots,\ldots,\lambda_{n_m m}^{k}) \quad (7.67)$$

$$x(k)=col.(x_1(k),\ldots,\ldots,x_m(k)) \quad (7.68)$$

$$x_1(k)=col.(x_{11}^{k},\ldots,\ldots,x_{n_1 1}^{k}) \quad (7.69)$$

$$x_m(k)=col.(x_{1m}^{k},\ldots,\ldots,x_{n_m m}^{k}) \quad (7.70)$$

$$q(k)=col.(q_1(k),\ldots,\ldots,q_m(k)) \quad (7.71)$$

$$q_1(k)=col.(q_{11}^{k},\ldots,\ldots,q_{n_1 1}^{k}) \quad (7.72)$$

$$q_m(k)=col.(q_{1m}^{k},\ldots,\ldots,q_{n_m m}^{k}) \quad (7.73)$$

$$I(k)=col.(I_1(k),\ldots,\ldots,I_m(k)) \quad (7.74)$$

$$I_1(k)=col.(I_{11}^{k},\ldots,\ldots,I_{n_1 1}^{k}) \quad (7.75)$$

$$I_m(k)=col.(I_{1m}^{k},\ldots,\ldots,I_{n_m m}^{k}) \quad (7.76)$$

$$s(k)=col.(s_1(k),\ldots,\ldots,s_m(k)) \tag{7.77}$$

$$s_1(k)=col.(s_{11}^k,\ldots,\ldots,s_{n_1 1}^k) \tag{7.78}$$

$$s_m(k)=col.(s_{1m}^k,\ldots,\ldots,s_{n_m m}^k) \tag{7.79}$$

$$u(k)=col.(u_1(k),\ldots,\ldots,u_m(k)) \tag{7.80}$$

$$u_1(k)=col.(u_{11}^k,\ldots,\ldots,u_{n_1 1}^k) \tag{7.81}$$

$$u_m(k)=col.(u_{1m}^k,\ldots,\ldots,u_{n_m m}^k) \tag{7.82}$$

$$\mu(k)=col.(\mu_1(k),\ldots,\ldots,\mu_m(k)) \tag{7.83}$$

$$\mu_1(k)=col.(\mu_{11}^k,\ldots,\ldots,\mu_{n_1 1}^k) \tag{7.84}$$

$$\mu_m(k)=col.(\mu_{1m}^k,\ldots,\ldots,\mu_{n_m m}^k) \tag{7.85}$$

$$\phi(k)=col.(\phi_1(k),\ldots,\ldots,\phi_m(k)) \tag{7.86}$$

$$\phi_{11}(k)=col.(\phi_{11}^k,\ldots,\ldots,\phi_{n_1 1}^k) \tag{7.87}$$

$$\phi_m(k)=col.(\phi_{1m}^k,\ldots,\ldots,\phi_{n_m m}^k) \tag{7.88}$$

$$z(k)=col.(z_1(k),\ldots,\ldots,z_m(k)) \tag{7.89}$$

$$z_1(k)=\text{col.}(z_{11}^{k},\ldots,\ldots,z_{n_1 1}^{k}) \quad (7.90)$$

$$z_m(k)=\text{col.}(z_{1m}^{k},\ldots,\ldots,z_{n_m m}^{k}) \quad (7.91)$$

$$\psi(k)=\text{col.}(\psi_{1}(k)\ldots,\ldots,\psi_m(k)) \quad (7.92)$$

$$\psi_1(k)=\text{col.}(\psi_{11}^{k},\ldots,\ldots,\psi_{n_1 1}^{k}) \quad (7.93)$$

$$\psi_m(k)=\text{col.}(\psi_{1m}^{k},\ldots,\ldots,\psi_{n_m m}^{k}) \quad (7.94)$$

$$\theta(k)=\text{col.}(\theta_1(k),\ldots,\ldots,\theta_m(k) \quad (7.95)$$

$$\theta_1(k)=\text{col.}(\theta_{11}^{k},\ldots,\ldots,\theta_{n_1 1}^{k}) \quad (7.96)$$

$$\theta_m(k)=\text{col.}(\theta_{1m}^{k},\ldots,\ldots,\theta_{n_m m}^{k}) \quad (7.97)$$

$$\delta(k)=\text{col.}(\delta_1(k),\ldots,\ldots,\delta_m(k) \quad (7.98)$$

$$\delta_1(k)=\text{col.}(\delta_{11}^{k},\ldots,\ldots,\delta_{n_1 1}^{k}) \quad (7.99)$$

$$\delta_m(k)=\text{col.}(\delta_{1m}^{k},\ldots,\ldots,\delta_{n_m m}^{k}) \quad (7.100)$$

$$\Omega(k)=\text{col.}(\Omega_1(k),\ldots,\ldots,\Omega_m(k)) \quad (7.101)$$

$$\Omega_1(k)=\text{col.}(\Omega_{11}^{k},\ldots,\ldots,\Omega_{n_1 1}^{k}) \quad (7.102)$$

$$\Omega_m(k)=\text{col.}(\Omega_{1m}^{\ k},\ldots,\ldots,\Omega_{n_m m}^{\ k}) \quad (7.103)$$

$$\Gamma(k)=\text{col.}(\Gamma_1(k),\ldots,\ldots,\Gamma_m(k)) \quad (7.104)$$

$$\Gamma_1(k)=\text{col.}(\Gamma_{11}^{\ k},\ldots,\ldots,\Gamma_{n_1 1}^{\ k}) \quad (7.105)$$

$$\Gamma_m(k)=\text{col.}(\Gamma_{1m}^{\ k},\ldots,\ldots,\Gamma_{n_m m}^{\ k}) \quad (7.106)$$

$$\nu(k)=\text{col.}(\nu_1(k),\ldots,\ldots,\nu_m(k)) \quad (7.107)$$

$$\nu_1(k)=\text{col.}(\nu_{11}^{\ k},\ldots,\ldots,\nu_{n_1 1}^{\ k}) \quad (7.108)$$

$$\nu_m(k)=\text{col.}(\nu_{1m}^{\ k},\ldots,\ldots,\nu_{n_m m}^{\ k}) \quad (7.109)$$

$$h_{ij}^{\ k}=e_{ij}^{\ 1k}-e_{ij}^{\ k}$$

$$(7.110)$$

$$\xi_{ij}^{\ k}=\ell_{ij}^{\ 1k}-\ell_{ij}^{\ k}$$

$$h(k)=\text{col.}(h_1(k),\ldots,\ldots,h_m(k)) \quad (7.111)$$

$$h_1(k)=\text{col.}(h_{11}^{\ k},\ldots,\ldots,h_{n_1 1}^{\ k}) \quad (7.112)$$

$$h_m(k)=\text{col.}(h_{1m}^{\ k},\ldots,\ldots,h_{n_m m}^{\ k}) \quad (7.113)$$

$$\xi(k)=\text{col.}(\xi_1(k),\ldots,\ldots,\xi_m(k)) \quad (7.114)$$

$$\xi_1(k)=\text{col.}(\xi_{11}^{\;k},\ldots,\ldots,{}_{n_1}\xi_1^{\;k}) \tag{7.115}$$

$$\xi_m(k)=\text{col.}(\xi_{1m}^{\;k},\ldots,\ldots,\xi_{n_m m}^{\;k}) \tag{7.116}$$

Using the above definitions the cost functional in equation (7.31) becomes:

$$\tilde{J}=E[\sum_{k=1}^{K}\{u^T(k)b(k)b^T(k)u(k)+r^T(k)d(k)d^T(k)r(k)+\omega^T(k)f(k)f^T(k)\omega(k)$$

$$+g^T(k)\gamma\gamma^Tg(k)+1/16\;t^T(k)\gamma\gamma^Tt(k)+n^T(k)\gamma\gamma^Tn(k)+1/16\;m^T(k)\gamma\gamma^Tm(k)$$

$$+2u^T(k)b(k)d^T(k)r(k)+2u^T(k)b(k)f^T(k)\omega(k)+2u^T(k)b(k)\gamma^Tg(k)$$

$$+2/4\;u^T(k)b(k)\gamma^Tt(k)+2u^T(k)b(k)\gamma^Tn(k)-2/4\;u^T(k)b(k)\gamma^Tm(k)$$

$$+2r^T(k)d(k)f^T(k)\omega(k)+2r^T(k)d(k)\gamma^Tg(k)+2/4\;r^T(k)d(k)\gamma^Tt(k)$$

$$+2r^T(k)d(k)\gamma^Tn(k)-2/4\;r^T(k)d(k)\gamma^Tm(k)+2\omega^T(k)f(k)\gamma^Tg(k)$$

$$+2/4\;\omega^T(k)f(k)\gamma^Tt(k)+2\omega^T(k)f(k)\gamma^Tn(k)-2/4\;\omega^T(k)f(k)\gamma^Tm(k)$$

$$+2/4\;g^T(k)\gamma\gamma^Tt(k)+2g^T(k)\gamma\gamma^Tn(k)-2/4\;g^T(k)\gamma\gamma^Tm(k)$$

$$+2/4\;t^T(k)\gamma\gamma^Tn(k)-2/16\;t^T(k)\gamma\gamma^Tm(k)-2/4\;n^T(k)\gamma\gamma^Tm(k)$$

$$+x^T(k)\mu^T(k)\vec{H}x(k)+u^T(k)\phi^T(k)\vec{H}u(k)+u^T(k)\psi^T(k)\vec{H}x(k-1)$$

$$+ u^T(k)\phi^T(k)\vec{A}u(k)+u^T(k)\delta^T(k)\vec{H}y(k-1)+u^T(k)\Omega^T(k)\vec{L}z(k)$$

$$+ r^T(k)\Gamma^T(k)\vec{F}u(k)+z^T(k)\gamma^T(k)\vec{Q}u(k)$$

$$+(\xi^T(k)+\lambda^T(k)M-2d^k b^T(k)+h^T(k)M)u(k)+(-\phi^T(k)z(k)$$

$$+ (-2d^k d^T(k)-\psi^T(k))r(k)+(-2d^k f^T(k)-\theta^T(k))\omega(k)$$

$$+ (-2d^k \gamma^T-\delta^T(k))g(k)+(-1/2\ d^k \gamma^T-\Omega^T(k))t(k)$$

$$+ (-2d^k \gamma^T-\Gamma^T(k))n(k)+(-1/2\ d^k \gamma^T-\nu^T(k))m(k)$$

$$+ (-\mu^T(k)y(k)) + \lambda^T(k)(-x(k)+x(k-1))+h^T(k)x(k-1))] \tag{7.117}$$

Note that constant terms are dropped in the above equation.

Employing the discrete version of integration by parts, one obtains the following equation for the augmented cost functional.

$$J=E[x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)+\mu^T(K)y(K)$$

$$- x^T(0)\mu^T(0)\vec{H}x(0)-\mu^T(0)y(0)+\lambda^T(0)x(0)$$

$$+ \sum_{k=1}^{K} \{X^T(k)L(k)X(k)+R^T(k)X(k)\}] \tag{7.118}$$

where

$$X^T(k)=[x^T(k-1),y^T(k-1),z^T(k),r^T(k),\omega^T(k),g^T(k),t^T(k),n^T(k),$$

$$,m^T(k),u^T(k)] \tag{7.119}$$

$$R^T(k)=[(\lambda(k)-\lambda(k-1)+h(k))^T,-\mu^T(k-1),-\phi^T(k),(-\psi(k)-2d^k d(k))^T,$$

$$,(-\theta(k)-2d^k f(k))^T,(-2d^k\gamma-\gamma(k))^T,(-\Omega(k)-1/2\ d^k\gamma)^T,$$

$$,(-\Gamma(k)-2d^k\gamma)^T,(-\nu(k)-1/2\ d^k\gamma)^T,(\xi(k)+M^T\lambda(k)-2d^k b(k)+M^T h(k))^T] \tag{7.120}$$

where

$$M=\text{diag.}(M_1,\ldots,\ldots,M_m) \tag{7.121}$$

and $M_1,\ldots,M_m$ are lower triangular matrices whose elements are given by:

$$\left.\begin{array}{lll} \text{(i)}\ m_{ii}=-1 & ;\ i=1,\ldots,n_j; & j=1,\ldots,m \\ \text{(ii)}\ m_{(v+1)v}=1 & ;\ v=1,\ldots,n_j-1; & j=1,\ldots,m \end{array}\right\} \tag{7.122}$$

and

$$
L(k) = \begin{bmatrix} L_{11}(k) & L_{21}(k) \\ & \\ L_{21}(k) & L_{22}(k) \end{bmatrix} \tag{7.123}
$$

where

$$
L_{11}(k) = \begin{bmatrix} \mu^T(k-1)\vec{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d(k)d^T(k) & d(k)f^T(k) \\ 0 & 0 & 0 & f(k)d^T(k) & f(k)f^T(k) \end{bmatrix} \tag{7.124}
$$

$$
L_{21}(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/2\psi^T(k)\vec{H} \\ 0 & 0 & 0 & 0 & 1/2\delta^T(k)\vec{H} \\ 0 & 0 & 0 & 0 & 1/2\nu^T(k)\vec{Q}+1/2\Omega^T(k)\vec{L} \\ d(k)\gamma^T & 1/4d(k)\gamma^T & d(k)\gamma^T & -1/4d(k)\gamma^T & d(k)b^T(k)+1/2\Gamma^T(k)\vec{F} \\ f(k)\gamma^T & 1/4f(k)\gamma^T & f(k)\gamma^T & -1/4f(k)\gamma^T & f(k)b^T(k) \end{bmatrix}
$$

(7.125)

$$L_{12}(k) = \begin{bmatrix} 0 & 0 & 0 & \gamma d^T(k) & \gamma f^T(k) \\ 0 & 0 & 0 & 1/4\gamma d^T(k) & 1/4\gamma f^T(k) \\ 0 & 0 & 0 & \gamma d^T(k) & \gamma f^T(k) \\ 0 & 0 & 0 & -1/4\gamma d^T(k) & -1/4\gamma f^T(k) \\ 1/2\psi^T(k)\vec{H} & 1/2\delta^T(k)\vec{H} & 1/2\nu^T(k)\vec{Q}+ & b(k)d^T(k)+ & b(k)f^T(k) \\ & & +1/2\Omega^T(k)\vec{L} & +1/2\Gamma^T(k)\vec{F} & \end{bmatrix}$$

(7.126)

$$L_{22}(k) = \begin{bmatrix} \gamma\gamma^T & 1/4\gamma\gamma^T & \gamma\gamma^T & -1/4\gamma\gamma^T & \gamma b^T(k) \\ 1/4\gamma\gamma^T & 1/16\gamma\gamma^T & 1/4\gamma\gamma^T & -1/16\gamma\gamma^T & 1/4\gamma b^T(k) \\ \gamma\gamma^T & 1/4\gamma\gamma^T & \gamma\gamma^T & -1/4\gamma\gamma^T & \gamma b^T(k) \\ -1/4\gamma\gamma^T & -1/16\gamma\gamma^T & -1/4\gamma\gamma^T & 1/16\gamma\gamma^T & -1/4\gamma b^T(k) \\ b(k)\gamma^T & 1/4b(k)\gamma^T & b(k)\gamma^T & -1/4b(k)\gamma^T & b(k)b^T(k)+\theta^T(k)\vec{A}+ \\ & & & & +\theta^T(k)\vec{H} \end{bmatrix}$$

(7.127)

It will be noticed that, $\tilde{J}$, in equation (7.118) is composed of a boundary part and a discrete integral part which are independent of each other. To maximize $\tilde{J}$ in equation (7.118) one maximizes the boundary and the discrete integral parts separately (40).

$$\underset{[x(K),y(K),X(k)]}{\text{Max.}\tilde{J}} = \underset{[x(K),y(K)]}{\text{Max.E}} \; [x^T(K)\mu^T(K)\vec{H}x(K)-\lambda^T(K)x(K)+\mu^T(K)y(K)$$

$$- x^T(0)\mu^T(0)\vec{H}(x(0)-\mu^T(0)y(0)+\lambda^T(0)x(0)]$$

$$+ \underset{X(k)}{\text{Max}}[\sum_{k=1}^{K} (X^T(k)L(k)X(k)+R^T(k)X(k)]$$

If one defines the vector V(k) such that:

$$V(k) = L^{-1}(k)R(k) \qquad (7,128)$$

then, the discrete integral part of equation (7.123) can be written as:

$$\underset{X(k)}{\text{Max.}J_2} = \underset{X(k)}{\text{Max}}[\sum_{k=1}^{K} \{(X(k)+ 1/2V(k))^T L(k)(X(k)+1/2\, V(k))$$

$$- 1/4\, V^T(k)L(k)V(k)\}] \qquad (7,129)$$

Since it is desired to maximize J_2 with respect to X(k), the problem is equivalent to:

$$\underset{X(k)}{\text{Max}.}J_2 = \underset{X(k)}{\text{Max}}\left[\sum_{k=1}^{K} \{(X(k)+ 1/2\ V(k))^T L(k)(X(k)+ 1/2\ V(k))\}\right] \tag{7.130}$$

because V(k) is independent of X(k). Equation (7.126) defines a norm, one can write equation (7.126) as:

$$\underset{X(k)}{\text{Max}.}J_2 = \underset{X(k)}{\text{Max}.}E[\,||X(k)+ 1/2\ V(k)||\,]_{L(k)} \tag{7.131}$$

7.4 The Optimal Solution

The condition of optimality for the problem formulated in equation (7.127) is that the norm of this equation should be equal to zero

$$E[X(k) + 1/2\ V(k)] = [0] \tag{7.132}$$

Substituting from equation (7.124) into equation (7.118), one obtains the following optimal equation

$$E[\bar{R}(k)+2L(k)X(k)] = [0] \tag{7.133}$$

The boundary part in eqution (7.123) is optimized when

$$E[\lambda(K)] = [0] \tag{7.134}$$

$$E[\mu(K)] = [0] \tag{7.135}$$

Equations (7.134) and (7.135) give the values of Lagrange's multipliers at the last period studied.

Writing equation (7.133) explicitly and adding equations (7.20-7.28), one obtains thes following optimal equations.

$$E[\lambda(k)-\lambda(k-1)+2\mu^T(k-1)\vec{H}x(k-1)+\psi^T(k)\vec{H}u(k)+h(k)] = [0] \tag{7.136}$$

$$E[-\mu(k-1)+\delta^T(k)\vec{H}d(k)] = [0] \tag{7.137}$$

$$E[-\phi(k)+\nu^T(k)\vec{Q}u(k)+\Omega^T(k)\vec{L}u(k)] = [0] \tag{7.138}$$

$$E[-\psi(k)-2d^k d(k)+2d(k)d^T(k)r(k)+2d(k)f^T(k)\omega(k)+2d(k)\gamma^T g(k)$$

$$+ 1/2\ d(k)\gamma^T t(k)+2d(k)\gamma^T n(k)- 1/2\ d(k)\gamma^T m(k)+2d(k)b^T(k)u(k)$$

$$+ \Gamma^T(k)\vec{F}u(k)] = [0] \tag{7.139}$$

$$E[-\theta(k)-2d^k f(k)+2f(k)d^T(k)r(k)+2f(k)f^T(k)\omega(k)+2f(k)\gamma^T g(k)$$

$$+ 1/2\ f(k)\gamma^T t(k)+2f(k)\gamma^T n(k)-1/2\ f(k)\gamma^T m(k)+2f(k)b^T(k)u(k)]=[0] \tag{7.140}$$

$$E[-\delta(k)-2d^{k}\gamma+2\gamma d^{T}(k)r(k)+2\gamma f^{T}(k)\omega(k)+2\gamma\gamma^{T}g(k)+1/2\ \gamma\gamma^{T}t(k)$$

$$+2\gamma\gamma^{T}n(k)-1/2\gamma\gamma^{T}m(k)+2\gamma b^{T}(k)u(k)]=[0] \qquad (7.141)$$

$$E[-\Omega(k)-1/2\ d^{k}\gamma+1/2\gamma d^{T}(k)r(k)+1/2\gamma\ f^{T}(k)\omega(k)+1/2\gamma\gamma^{T}g(k)+1/8\gamma\gamma^{T}t(k)$$

$$+1/2\gamma\gamma^{T}n(k)-1/8\ \gamma\gamma^{T}m(k)+1/2\ \gamma b^{T}(k)u(k)]=[0] \qquad (7.142)$$

$$E[-\Gamma(k)-2d^{k}\gamma+2\gamma d^{T}(k)r(k)+2\gamma f^{T}(k)\omega(k)+2\gamma\gamma^{T}g(k)+1/2\ \gamma\gamma^{T}t(k)$$

$$+2\gamma\gamma^{T}n(k)-1/2\ \gamma\gamma^{T}m(k)+2\gamma b^{T}(k)u(k)]=[0] \qquad (7.143)$$

$$E[-\nu(k)-1/2\ d^{k}\gamma-1/2\ \gamma d^{T}(k)r(k)-1/2\ \gamma f^{T}(k)\omega(k)-1/2\ \gamma\gamma^{T}g(k)-1/8\ \gamma\gamma^{T}t(k)$$

$$-1/2\ \gamma\gamma^{T}n(k)+1/8\ \gamma\gamma^{T}m(k)-1/2\gamma\ b^{T}(k)u(k)]=[0] \qquad (7.144)$$

$$E[\xi(k)+M^{T}\lambda(k)-2d^{k}b(k)+M^{T}h(k)+\psi^{T}(k)\vec{H}x(k-1)+\delta^{T}(k)\vec{H}y(k-1)$$

$$+\nu^{T}(k)\vec{Q}z(k)+\Omega^{T}(k)\vec{L}z(k)+(2b(k)d^{T}(k)+\Gamma^{T}(k)\vec{F})r(k)$$

$$+\ 2b(k)f^{T}(k)\omega(k)+2b(k)\gamma^{T}g(k)+1/2\ b(k)\gamma^{T}t(k)+2b(k)\gamma^{T}n(k)$$

$$-1/2\ b(k)\gamma^{T}m(k)+2b(k)b^{T}(k)u(k)+2\theta^{T}(k)\vec{A}u(k)+2\phi^{T}(k)\vec{H}u(k)]=[0]$$

$$(7.145)$$

$$E[-x(k)+x(k-1)+q(k)+Mu(k)] = [0] \tag{7.146}$$

$$E[-y(k)+x^T(k)Hx(k)] = [0] \tag{7.147}$$

$$E[-z(k)+u^T(k)\vec{H}u(k)] = [0] \tag{7.148}$$

$$E[-r(k)+u^T(k)\vec{H}x(k-1)] = [0] \tag{7.149}$$

$$E[-\omega(k)+u^T(k)\vec{A}u(k)] = [0] \tag{7.150}$$

$$E[-g(k)+u^T(k)\vec{H}y(k-1)] = [0] \tag{7.151}$$

$$E[-t(k)+u^T(k)\vec{L}z(k)] = [0] \tag{7.152}$$

$$E[-n(k)+r^T(k)\vec{F}u(k)] = [0] \tag{7.153}$$

$$E[-m(k)+z^T(k)\vec{Q}u(k)] = [0] \tag{7.154}$$

According to Kuhn-Tucker theory, the following exclusion equations must be satisfied at the optimum (32), (48)

$$e_{ij}^{k}(\underline{x}_{ij}-x_{ij}^{k}) = 0 \tag{7.155}$$

$$e_{ij}^{1k}(x_{ij}^{k}-x_{ij}) = 0 \tag{7.156}$$

$$\xi_{ij}^{k}(\underline{u}_{ij}^{k}-u_{ij}^{k}) = 0 \tag{7.157}$$

$$\xi_{ij}^{1k}(u_{ij}^{k}-\bar{u}_{ij}^{k}) = 0 \qquad (7.158)$$

One also has the following limits on the variables

$$\left.\begin{array}{ll} \text{If } x_{ij}^{k} < \underline{x}_{ij}, & \text{then we put } x_{ij}^{k}=\underline{x}_{ij} \\ \text{If } x_{ij}^{k} > \underline{x}_{ij}, & \text{then we put } x_{ij}^{k}=\bar{x}_{ij} \\ & \\ \text{If } u_{ij}^{k}<\underline{u}_{ij}^{k}, & \text{then we put } u_{ij}^{k}=\underline{u}_{ij}^{k} \\ \text{If } u_{ij}^{k}>u_{ij}^{k}, & \text{then we put } u_{ij}^{k}=\bar{u}_{ij}^{k} \end{array}\right\} \qquad (7.159)$$

Equations (7.136-7.159) with equations (7.134) and (7.135) completely specify the optimal solution for the system. Figure 7.1 gives the main flow charge to solve these equations.

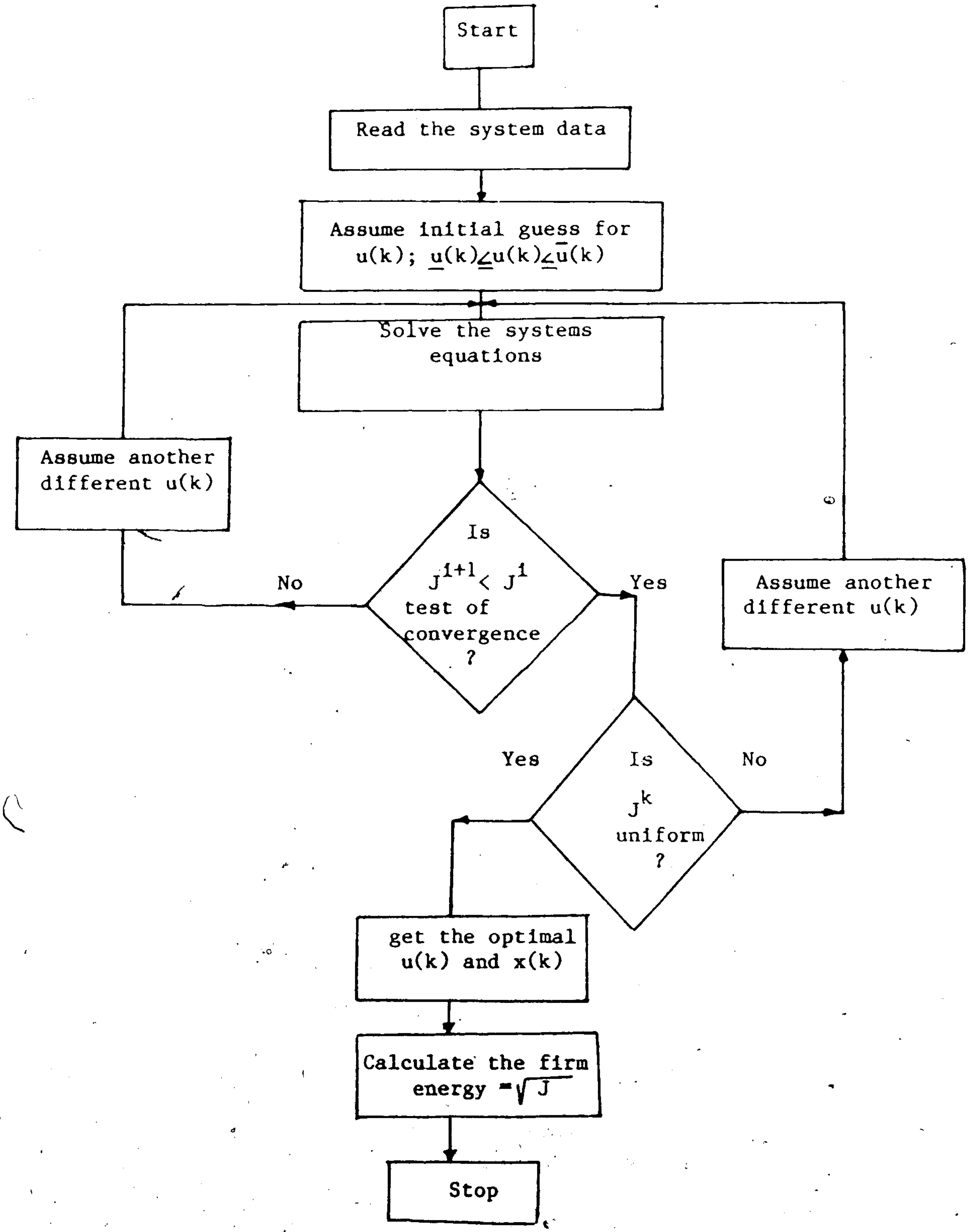

Figure 7.1: The main flow chart.

CHAPTER VIII

CONCLUDING REMARKS

8.1 Conclusions

In this thesis a functional analytic optimization technique is applied to problems of optimal long-term operation of multireservoir hydroelectric power systems. Here, the minimum norm formulation is employed to find the optimum discharge that maximizes the total expected benefits from the systems under different water conditions (the benefits from the generation and the water left in storage at the end of the planning period). This investigation shows how the powerful minimum norm formulation can be applied to complex problems of high dimension.

In chapter 2, the optimal long-term operating problem of multireservoir power systems having a linear storage-elevation curve and a constant water conversion factor is considered. The model used for each reservoir (generating function) is a nonlinear stochastic model of the storage and the discharge, and the equations obtained are stochastic discrete equations. It should be noted that the results obtained in this chapter correspond very closely to those obtained using Dynamic Programming. However, the CPU time required to find the solution by the method presented in this is less by several orders of magnitude than the CPU time required by Dynamic Programming. Furthermore, no approximations are necessary in using the present method.

For the hydroelectric power systems in which the water heads vary by a considerable amount, the assumption of constant water conversion factors is not true. In chapter 3 the optimal scheduling problems of multireservoir power systems having a linear storage-elevation curve and a variable water conversion factor is considered. The model used for this water conversion factor is a linear model of the average storage to avoid underestimation in production for rising water levels and overestimation for falling water levels, and the generating function for each reservoir is a nonlinear function of the storage and the discharge. Numerical results are presented for a real system in operation including up to six resrvoirs for widely different water conditions. The proposed method is computationaly efficient compared to previous techniques.

The assumption of linear storage-elevation curves and linear water conversion factors may yield a significant error in the storage of some reservoirs, which may be greater than the minimum natural inflow to these reservoirs. In chapter 4, the optimal scheduling problems of multireservoir power systems having a nonlinear storage elevation curve and nonlinear water conversion factors is considered. The nonlinearity in the cost function in this chapter is reduced to a quadratic form by introducing a set of pseudo-state variables. The total benefits obtained using this model is greater, by a significant amount, than those obtained using a linear model for the reservoir.

In chapter 5 new optimal equations for scheduling the operation of a multireservoir power sysem under critical water conditions are developed. To meet all the requirements during this period, we maximize the generation from the system during this period. In the first part of this chapter the model used for the reservoir is a nonlinear model of the average head and the discharge and the water conversion factor in this case is proportional to the head. But in the second part we have used nonlinear storage curves opposed to the linear models used in the first part. It will be noticed that the total benefits have increased considerably due to the more exact model used here. The present model in this section allows us to have the continuity equation as a linear equation, this would not have been the case if we instead had represented the storage as a quadratic function of the head.

The generation given for the example is essentially constant during the critical period.

In chapter 5, we maximize the generation from the system during the critical water conditions. In this maximization, the required load on the system was not accounted for. In chapter 6 the solution of the long-term optimal operating problem of a multireservoir power system with specified monthly generation is considered. In this chapter, we minimize the difference between the monthly generation and the monthly specified load on the system. New optimal equations for scheduling the operation of the system are developed. The model used for each reservoir is a nonlinear stochastic model, we define a set of pseudo-state variables to cast the problem into a quadratic problem.

In chapter 7, new optimal equations for optimizing the firm hydro energy capability from multireservoir power systems are developed under critical water conditions. The model used in this study is a nonlinear model of the discharge and the average storage, we define a set of pseudo-state variables to overcome these nonlinearities in the cost functional.

An important aspect of the problems considered in this thesis is the stochasticity of the river flows. We used the average values for the random variables. Furthermore, the solution for the Kuhn-Tucker multipliers when the variables violate the constraints make the problem converge to the optimal solution in a few iterations.

8.2 Suggestions for Further Research

The minimum norm formulation employed in this investigation has demonstrated the capability of solving compex power system scheduling problems. For example, the long-term optimal scheduling for realistic systems having thermal, hydro and nuclear power plants may be possible to solve using the same technique.

It may be possible to solve the short-term optimal operating problem for multireservoir power systems (hydroelectric system only) using the results obtained from the long-term optimal operation for the same system. In this study the losses of the system minimized using the exact load flow model for the network under the constraint of the availability of the amount of water from the long-term study. In the short-term study the time delay of water flow may be taken into account.

REFERENCES

1. A. Turgeon, "A Decomposition Method for the Long-Term Scheduling of Reservoirs in Series", Water Resources Research, Vol. 17, No. 6, 1981.

2. A. Turgeon, "Optimal Operation of Multireservoir Power Systems With Stochastic Inflows", Water Resources Research, Vol. 16, No. 2, 1980.

3. N.V. Arvanitidies and J. Rosing, "Composite Representation of a Multireservoir Hydroelectric Power System", IEEE Transactions, Vol. PAS-89, No. 2, 1971.

4. N.V. Arvanitidies and J. Rosing, "Optimal Operation of Multireservoir Systems Using a Composite Representation", IEEE Transcations, Vol. PAS-89, No. 2, 1971.

5. S. Olcer, C. Harsa, and A. Roch, "Application of Linear and Dynamic Programming to the Optimization of the Production of Hydroelectric Power", Optimal Control Application and Methods, Vol. 6, 43-56, 1985.

6. Y. Ikura, and G. Gross, "Efficient Large-Scale Hydro System Scheduling With Spill Conditions", IEEE 84 WM 006-3.

7. A. Turgeon, "Selection of Installed Capacities for Hydroelectric Power Plants in Series", IEEE 84 WM 245-7.

8. L. Hano, Y. Tamura and S. Narita, An Application of the Maximum Principle to the Most Economical Operation of Power Systems", IEEE Transactions on PAS Vol. PAS-85, No. 5, 1966.

9. C. Lyra, H. Tavares and S. Soares, "Modelling and Optimization of Hydrothermal Generation Scheduling", IEEE Transactions, Vol. PAS-103, No. 8, 1984.

10. S. Soares, C. Lyra and H. Tavares, "Optimal Generation Scheduling of Hydrothermal Power Systems", IEEE Transactions on PAS, Vol. PAS-99, No. 3, 1980.

11. H. Duran, C. Pusch, J. Diaz and G. Sanchez, "Optimal Operation of Multireservoir Systems Using an Aggregation-Decomposition Approach", IEEE Transactions on PAS, Vol. PAS-104, No. 8, 1985.

12. T.S. Halliburton and H.R. Sirisena, "Development of a Stochastic Optimization for Multireservoir Scheduling", IEEE Transactions on AC, Vol. AC-29, No. 1, 1984.

13. H.R. Sirisena and T.S. Halliburton, "Long-Term Optimization of Hydrothermal Power Systems by Generalized Conjugate Gradient Methods", Optimal Control Application & Methods, Vol. 2, 351-364, 1981.

14. D. Sjetvgren and T.S. Dillon, "Seasonal Planning Approach to the Hydrothermal System Based on the Network Concept Used at the Swedish State Power Board", IEEE, 83, WM 074-2.

15. T.S. Dillon, R.W. Martin and D. Sjelvgren, "Stochastic Optimization and Modelling of Large Hydrothermal Systems for Long-Term Regulation", Electrical Power & Energy System, Vol. 2, No. 1, 1980.

16. S. Narita, et.al., "An Application of the Maximum Principle to the Most Economic Power System Operations", Elect.Eng.Jap. Vol. 67, 1967.

17. A. Shamaly, G.S. Christensen and M.E. El-Hawary, "A Transformation for Necessary Optimality Conditions for Systems with Polynomial Nonlinearities", IEEE Transactions on AC, Vol. AC-24, No. 6, 1979.

18. V.T. Sherkat, R. Campo, K. Maslehi and E.O. Lo, "Stochastic Long-Term Hydrothermal Optimization for a Multireservoir System", IEEE Transactions on PAS, Vol. PAS-108, No. 8, 1985.

19. S.K. Agarwal, "Optimal Stochastic Scheduling of Hydrothermal Systems", Proc. IEE, Vol. 120, No. 6, 1973.

20. T.S. Halliburton, and H.R. Sirisena, "Long-Term Optimal Operation of a Power System", Proc. IEE, Vol. 129, No. 5, Pt. C, 1982.

21. A. Shamaly, G.S. Christensen and M.E. El-Hawary, "Realistic Feedback Control of Turboalternator Using Functional Analysis", JOTA, Vol. 35, No. 2, 1981.

22. S.A. Soliman, G.S. Christensen and M.A. Abdel-Halim, "Optimal Operation of Multireservoir Power Systems Using Functional Analysis", In Press. Journal of Optimization Theory and Applications, January, 1985.

23. G.S. Christensen and S.A. Soliman, "Long-Term Optimal Operation of a Parallel Multireservoir Power System Using Functional Analysis", In Press, Journal of Optimization Theory and Applications, January, 1985.

24. S.A. Soliman and G.S. Christensen, "Modelling and Optimization of Series Reservoirs for Long-Term Regulation With Variable Head Using Functional Analysis", In Press, Journal of Optimization Theory and Applications, May, 1985.

25. G.S. Christensen and S.A. Soliman, "Optimal Long-Term Operation of a Multireservoir Power System for Critical Water Conditions Using Functional Analysis", In Press, Journal of Optimization Theory and Applications, May, 1985.

26. S.A. Soliman and G.S. Christensen, "Optimization of Reservoirs in Series on a River With Nonlinear Storage Curve for Long-Term Regulation by Using the Minimum Norm Formulation of Functional Analysis", In Press, Journal of Optimization Theory and Applications, May, 1985.

27. G.S. Christensen and S.A. Soliman, "Modelling and Optimization of Parallel Reservoirs Having Nonlinear Storage Curves Under Critical Water Conditions for Long-Term Regulation Using Functional Analysis", In Press, Journal of Optimization Theory and Applications, May, 1985.

28. S.A. Soliman and G.S. Christensen, "Application of Functional Analysis to Optimization of a Variable Head Multireesrvoir Power System for Long-Term Regulation", Accepted for publication in Water Resources Research, January 1986.

29. G.S. Christensen and S.A. Soliman, "A New Analytical Approach for Long-Term Optimal Operation of a Parallel Multireservoir Power System Based on Functional Analysis", In Press, Canadian Electrical Engineering Journal, July, 1985.

30. M.E. El-Hawary, "Application of Functional Analysis Optimization Techniques to the Economic Scheduling of Hydro-Thermal Electric Power Systems", Ph.D. Thesis, The University of Alberta, Edmonton, Alberta, Canada, 1972.

31. M.E. El-Hawary and G.S. Christensen, "Optimal Economic Operationn of Electric Power Systems", Academic Press, New York, N.Y., 1979.

32. A. Sage, "Optimal Systems Control", Prentice-Hall, Englewood Cliffs, New Jersey, 1968.

33. A. Sage and J.L. Melsa, "System Identification", Academic Press, New York, New York, 1971.

34. D.E. Kirk, "Optimal Control Theory; An Introduction", Prentice-Hall Inc., 1970.

35. R. Fletcher, "Practical Methods of Optimization", Vol. 2, Constrained Optimization, John Wiley & Sons, 1980.

36. R.L. Fox, "Optimization Methods for Engineering Design", Addison Wesley Company, 1971.

37. W.A. Porter, "Modern Foundations of System Engineering", New York, MacMillan, 1966.

38. D.G. Luenberger, "Optimization by Vector Space Methods", Wiley, New York, 1969.

39. A. Shamaly, G.S. Christensen and M.E. El-Hawary, "Optimal Control of Large Turboalternator", Journal of Optimization Theory and Applications, Vol. 34, No. 1, 1981.

40. A. Turgeon, "A Decomposition/Projection Method for the Multireservoir Operating Problem", Paper presented at the Joint National TIMS/ORSA Meeting, Los Angeles, Calif., November 1978.

41. J.C. Grygier and J.R. Stedinger, "Algorithms for Optimization Hydropower System Operation", Water Resources Research, Vol. 21, No. 1, January, 1985.

42. M.A. Marino and H.A. Loaiciga, "Dynamic Model for Multireservoir Operation", Water Resources Research, Vol. 21, No. 5, 1985.

43. M.A. Marino and H.A. Loaiciga, "Quadratic Model for Reservoir Management: Application to the Central Valley Project", Water Resources Research, Vol. 21, No. 5, 1985.

44. R.H. Hicks, C.R. Gagnon and S.L.S. Jacoby, "Large Scale, Nonlinear Optimization of Energy Capability for the Pacific Northwest Hydroelectric System", IEEE Press, WM, T74, 110-3.

45. G.S. Christensen and S.A. Soliman, "Long-Term Operation of Series Parallel Reservoirs for Critical Period With Specified Monthly Generation and Average Monthly Storage", submitted to IEEE, Transaction on A.C., November, 1985.

46. G.S. Christensen and S.A. Soliman, "On the Application of Functional Analysis to the Optimization of the Production of Hydroelectric Power", Submitted to IEEE, Summer Meeting, July, 1985.

47. J. Peschon, D.S. Piercy and W.F. Tinney, "Optimization Control of Reactive Power Flow", IEEE Transactions on PAS, Vol. PAS-87, 1968.

48. W.B. Davenport, "Probability and Random Processes", McGraw-Hill Book Company, New York, 1970.

49. R.F. Curtain and A.J. Pritchard, "Functional Analysis in Modern Applied Mathematics", Academic Press, New York, New York, 1977.

50. V.L. Yorre and S.G. Kapoor, "A Multi-Stage Quadratic Programming Optimization Technique for Optimal Management of Large Hydro-Power Operation", Engineering Optimization Journal, Vol. 8, PP.

103-118, 1985.

51. S.A. Soliman and G.S. Christensen, "Long-Term Optimal Operation of Series-Parallel Reservoirs for Critical Period With Specified Monthly Generation", Submitted to Canadian Electrical Engineering Journal, September, 1985.

52. M.E. El-Hawary and G.S. Christensen, "Optimal Active-Reactive Hydro-Thermal Schedules Using Functional Analysis", Optimal Control Application & Methods, Vol. 1, 1980.

53. C.M. Shen and M.A. Laughton, "Determination of Optimum Power-System Operating Conditions Under Constraints", Proc. IEE, Vol. 116, No. 2, 1969.

54. V.P. Lukie, "Optimal Operating Policy for Energy Storage", IEEE Transactions on PAS, Vol. PAS-101, No. 9, 1982.

55. Y.N. Oh, "An Application of the Discrete Maximum Principle to the Most Economical Power-System Operation", Elect.Eng.Jap., 87, 17-29, 1967.

56. F.J. Rees and R.E. Larson, "Computer-Aided Dispatching and Operation Planning for An Electric Utility With Multiple Types of Generation", IEEE Transactions on PAS, Vol. PAS-90, No. 2, 1971.

57. M.V.F. Pereira and L.M.V.G. Pinto, "Stochastic Optimization of Multireservoir Hydroelectric System: A Decomposition Approach", Water Resources Research Vol. 21, No. 6, June 1985.

58. S.A. Soliman and G.S. Christensen, "Optimization of the Production of Hydroelectric Power Systems With a Variable Head", In Press, Journal of Optimization Theory and Applications, November, 1985.

59. S.A. Soliman and G.S. Christensen, "A Minimum Norm Approach to Optimization of Production of Multireservoir Power Systems with Specified Monthly Generation", In Press, Journal of Optimization Theory and Applications, October, 1985.

60. S.A. Soliman and G.S. Christensen, "A New Approach for Optimizing Hydropower System Operation with a Quadratic Model", Submitted to Optimal Control and Calculus of Variation Meeting", West Germany, June 1986.

61. S.A. Soliman and G.S. Christensen, "Discrete Stochastic Optimal Long-Term Scheduling of Hydro-Thermal Power Systems", Fourteenth IASTED International Conference, Vancouver, Canada, June 4-6, 1986.

62. S.A. Soliman and G.S. Christensen, "Discrete Stochasticc Optimal Long-Term Scheduling of Series Reservoirs", Fourteenth IASTED International Conference, Vancouver, Canada, June 4-6, 1986.

VITA

NAME: Soliman Abdel-Hady Soliman

PLACE OF BIRTH: Cairo, EGYPT

YEAR OF BIRTH: 1950

POST-SECONDARY EDUCATION:

1. B.Sc. in Elect.Eng., Ain Shams University Cairo, Egypt, 1973 (Very Good (83.45%) Top of the Class).

2. M.Sc. in Elect.Eng., Ain Shams University Cairo, Egypt, 1977.

HONOURS AND AWARDS:

1. Egyptian Scholarship 1972-1973.

2. University of Alberta 1984-1986.

RELATED WORK EXPERIENCE:

1. Demonstrator, Faculty of Engineering, Ain Shams University, Elect.Eng. Dept., 1973-1977.

2. Assistant Lecturer, Elect.Eng. Dept., Ain Shams University, 1977-1983.

3. Teaching Assistant, Elect.Eng. Dept., University of Saskatchewan, Saskatoon, Canada, 1983.

4. Teaching Assistant, Elect.Eng. Dept., University of Alberta, Edmonton, Alberta, Canada, 1984-1986.

PUBLICATIONS

1. S.A. Soliman, G.S. Christensen and M.A. Abd-Elhalim, "Optimal Operation of Multireservoir Power Systems Using Functional Analysis", In Press, Journal of Optimization Theory and Applications, January 1985.

2. G.S. Christensen and S.A. Soliman, "Long-Term Optimal Operation of a Parallel Multireservoir Power System Using Functional Analysis", In Press, Journal of Optimization Theory and Applications (JOTA), January 1985.

3. S.A. Soliman and G.S. Christensen, "Modelling and Optimization of Series Reservoirs for Long-Term Regulation with Variable Head Using Functional Analysis", In Press, Journal of Optimization Theory and Applications (JOTA), May 1985.

4. S.A. Soliman and G.S. Christensen, "Optimization of the Production of Hydroelectric Power Systems With a Variable Head", In Press, Journal of Optimization Theory and Applications, November, 1985.

5. S.A. Soliman and G.S. Christensen, "Optimization of Reservoirs in Series on a River with a Nonlinear Storage Curve for Long-Term Regulation by Using the Minimum Norm Formulation of Functional Analysis", In Press, Journal of Optimization Theory and Applications, May 1985.

6. G.S. Christensen and S.A. Soliman, "New Analytical Approach for Long-Term Optimal Operation of a Parallel Multireservoir Power System Based on Functional Analysis", In Press, Canadian Electrical Engineering Journal, July 1985.

7. G.S. Christensen and S.A. Soliman, "Optimal Long-Term Operation of a Multireservoir Power System for Critical Water Conditions Using Functional Analysis", In Press, JOTA, May 1985.

8. G.S. Christensen and S.A. Soliman, "Modelling and Optimization of Parallel Reservoirs Having Nonlinear Storage Curves Under Critical Water Conditions for Long-Term Regulation Using Functional Analysis", In Press, JOTA, May 1985.

9. G.S. Christensen and S.A. Soliman, "On the Application of Functional Analysis to the Optimization of the Production of Hydroelectric Power", Submitted to IEEE, Summer Meeting, 1986.

10. G.S. Christensen and S.A. Soliman, "Long-Term Optimal Operation of Series-Parallel Reservoirs for Critical Period with Specified Monthly Generation and Average Monthly Storage", Submitted to IEEE Transaction on A.C., November, 1985.

11. S.A. Soliman and G.S. Christensen, "Long Term Optimal Operation of Series-Parallel Reservoirs for Critical Period with Specified Monthly Generation", Submitted to Canadian Electrical Engineering Journal, September 1985.

12. S.A. Soliman and G.S. Christensen, "Application of Functional Analysis to Optimization of a Variable Head Multireservoir Power System for Long-Term Regulation", Accepted for publication in Water Resources Research, January, 1986.

13. S.A. Soliman and G.S. Christensen, "A New Approach for Optimizing Hydropower System Operation with a Quadratic Model", Submitted to Optimal Control and Calculus of Variation Meeting, West Germany, June 1986.

14. S.A. Soliman and G.S. Christensen, "A Minimum Norm Approach to Optimization of Production of Multireservoir Power Systems with Specified Monthly Generation", In press JOTA, January 1986

15. S.A. Soliman and G.S. Christensen, "Discrete Stochastic Optimal Long-Term Scheduling of Hydro-Thermal Power Systems", Fourteenth IASTED International Conference, Vancouver, Canada, June 4-6, 1986.

16. S.A. Soliman and G.S. Christensen, "Discrete Stochastic Optimal Long-Term Scheduling of Series Reservoirs", Fourteenth IASTED International Conference, Vancouver, Canada, June 4-6, 1986.

17. S.A. Soliman, M.S. Morsy and M.A. Badr, "A Constant Frequency Generator for Wind Energy; Part I: Steady State Analysis", Fourteenth IASTED International Conference, Vancouver, Canada, June 4-6, 1986.