University of Alberta

Translation operators on group von Neumann algebras and Banach algebras related to locally compact groups

by

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A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Mathematical and Statistical Sciences

Fall 2010
Edmonton, Alberta

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ABSTRACT

Let G be a locally compact group, G^* be the set of all extreme points of the set of normalized continuous positive definite functions of G and $a(G)$ be the closed subalgebra generated by G^* in $B(G)$. When G is abelian, G^* is the set of dirac measures of the dual group of G. The general properties of G^* are investigated in this thesis. We study the properties of $a(G)$, particularly on its spectrum.

We also define translation operators on $VN(G)$ via G^* and investigate the problem of the existence of translation means on $VN(G)$ which are not topological invariant.

Lastly, we define reflexivity of subgroups of G by using G^*, and show that a subgroup H is reflexive if and only if G had H-separation property. If G is abelian, there is correspondence between closed subgroups of G and closed subgroups of the dual group $\hat{G}$. We generalize this result to the class of groups having separation property.

ACKNOWLEDGEMENTS

I am deeply indebted to my supervisor, Dr. Anthony To-Ming Lau, not only for his immeasurable guidance but also his kind encouragement and industrious supervision in the course of this research programme. To gain techniques and insights under his supervision is undoubtedly my lifetime benefit. I would like to thank Dr. Chi-Wai Leung, the supervisor of my master program, for introducing me this research area. I also wish to thank my parents and wife, for giving me their constant love, encouragement and support. Lastly, I also wish to thank the University of Alberta for the financial support provided during this period.

To My Parents

Pui-Fong Kwok and Ting-Pong Cheng

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Chapter 1

Introduction

Let G be a locally compact group, and let $A(G)$, $B(G)$ and $VN(G)$ be the Fourier algebra, Fourier-Stieltjes algebra and group von Neumann algebra of G, respectively, as defined by Eymard[11]. If G is abelian, $A(G)$ can be identified as $L^1(\hat{G})$ via the Fourier transform, $VN(G)$ can be identified as $L^\infty(\hat{G})$ via the adjoint of Fourier transform and $B(G)$ can be identified as $M(\hat{G})$ via the Fourier-Stieltjes transform, where $\hat{G}$ is the dual group of G.

Akemann and Walter first studied G^* in [2], the set consisting of all extreme points in the set of all continuous positive definite functions(see [7], [8] for reference) of G with norm one. This object is also studied by A. T.-M. Lau in [22]. If G is amenable, it is proved that the convex hull of G^* is weak*-dense in the set of means on $UCB(\hat{G})$(=norm closure of $A(G) \cdot VN(G)$). In [29], P. F. Mah and T. Miao showed that for a [SIN]-group G, G^* and $A(G)$ are disjoint if and only if G is non-compact.

The main purpose of this thesis is to study G^* from other points of view. For a locally compact abelian group G, G^* can be viewed as the set of all dirac measures of $\hat{G}$. We define $a(G)$, the algebra generated by G^* in $B(G)$, as a non-commutative analogue of $l^1(\hat{G})$ and proved that $\sigma(a(G))$ has a natural semigroup structure. The main results are as follows:

We show that if G_1, G_2 are locally compact groups and $a(G_1)$, $a(G_2)$ are isometrically isomorphic, then the unitary parts of their spectrums are either topologically isomorphic or anti-isomorphic. It is a natural question to ask when $\sigma(a(G))$ is a group. If G is a [Moore]-group, then $a(G)$ is the Fourier algebra of G^{ap}, where G^{ap} is the almost periodic compactification of G. In this case, $\sigma(a(G))$ is just G^{ap}. We show in this thesis is that $\sigma(a(G))$ is a group only when G is a [Moore]-group. Finally, we observe that if G is a discrete group, then $l^1(\hat{G})$ characterizes G. We prove the non-commutative analogue of this phenomenon: if G is an [AR]-group, then $a(G)$ characterizes G.

The translation operators are fundamental notions in the classical theory of $L^1(G)$ and $L^\infty(G)$. Thus, it is natural for us to search for a non-commutative version of translation operators in $A(G)$ and $VN(G)$. We find that "generalized" translation operators of $A(G)$ and $VN(G)$ can be defined by using G^*. Note that if G is abelian, the generalized translation operators of $A(G)$ and $VN(G)$ are precisely the usual translation operators of $L^1(\hat{G})$ and $L^\infty(\hat{G})$ under certain identifications.

The notion of amenability of a group was formulated by von Neumann. Later, Day defined amenability of a locally compact group G by using translation invariant means on $L^\infty(G)$. As mentioned above, $VN(G)$ can be viewed as the dual object of $L^\infty(G)$. Since we have a non-commutative analogue of translation invariant means on $VN(G)$, it allows us to define translation invariant means on $VN(G)$.

Granirer[14] and Rudin[34] proved independently that if G is amenable as discrete, then G is discrete if and only if all the translation invariant means on $L^\infty(G)$ are topological invariant. However, this is no longer true in general even when G is a compact group(See [6]). As a direct consequence of Granirer-Rudin's theorem, we have the following observation: if G is abelian, then G is compact if and only if all the translation invariant means on $VN(G)$ are topological invariant. We prove that the result is not true for general locally compact groups. One of the main purposes of this thesis is to generalize this result for non-abelian groups under certain assumptions.

Let H be a closed subgroup of G, and let π be a unitary representation of G. It is natural to ask if the restriction of π to H is a direct sum of irreducible representation of H in general. Surprisingly, by making use of a result concerning translation invariant means on $VN(G)$ and Granirer-Rudin's result that mentioned above, we give a negative answer to this question.

Many classical results can be generalized in the non-commutative setting via G^* and the proofs of them need only slight modifications. The closed convex translation invariant subsets in $L^1(G)$ and $L^\infty(G)$ were studied in [21]. Under the assumption that $A(G)$ has approximate identity, we generalize most of the results in [21] in the our setting. It was proved in [26] that G is amenable if and only if every completely complemented weak*-closed translation invariant subspace of $L^\infty(G)$ is invariantly complemented.. We generalize the forward implication in our setting.

If G is abelian, by using the Pontryagin duality theorem, it is shown that there is a one-to-one correspondence between the set of all closed subgroups of G and the set of all closed subgroups of the dual group $\hat{G}$. It maps a subgroup H of G to $H^\perp$, a subgroup of $\hat{G}$, which is defined by the following:

$$H^\perp = \{\chi \in \hat{G} : \chi(x) = 1 \text{ for any } x \in H\}.$$

In the non-abelian case, there is no such an correspondence since we do not have the notion of dual group defined in a natural way.

If H is a closed subgroup of G and A is a subset of G^*, we put

$$H^{\perp} = \{g^* \in G^* : g^*(x) = 1 \text{ for any } x \in H\}$$

and

$$A_{\perp} = \{x \in G : g^*(x) = 1 \text{ for any } g^* \in A\}.$$

A subgroup H of G is called reflexive if $(H^{\perp})_{\perp} = H$ and a subset A in G^* is called reflexive if $(A_{\perp})^{\perp} = A$. Some main results are as follows: We first prove that H is reflexive if H is separating in G(see [19]). We also prove that there is a one-to-one correspondence between reflexive subgroups of G and reflexive subsets of G^*. As a consequence, we generalize the correspondence of subgroups in the abelian case under the assumption that G has separation property(see [19]). We also show that the dual space of G/N can be identified with $N^{\perp}$ if N is a closed normal subgroup of G. We also give a characterization of the dual space of a product of two groups if one of them is of abelian.

This thesis is organized as follows: in Chapter 3, we define $a(G)$ and its semigroup structure. Then, we prove the main results about $a(G)$ mentioned above; in Chapter 4, we define the translation invariant operators and means on $VN(G)$, and prove the main results about translation invariant means on $VN(G)$ which are topological invariant. We also show as an application that the infinite dimensional irreducible representations of "ax+b"-group are not completely irreducible; in Chapter 5, we characterizes all the closed subalgebras of $A(G)$ which are ideals of $A(G)$. This result actually motivated the author to study of "generalized" translation operators of $A(G)$. A couple of classical results about translation operators have been generalized in our setting in this chpater; in Chapter 6, we characterize the commutativity, compactness or discreteness of G by using properties of G^*. We also give characterizations of translation invariant elements in $VN(G)$ and $W^*(G)$ at the end of this chapter; in Chapter 7, we study the closed convex G^*-invariant subsets of $A(G)$ and $VN(G)$ and give the characterization of $WAP(\hat{G})$ and $AP(\hat{G})$. We also show that every completely complemented weak*-closed G^*-invariant subspace of $VN(G)$ is invariantly complemented whenever G is amenable; in chapter 8, we discuss the reflexivity of subsets of G and G^*. We also prove that G is abelian if and only if G^* is a semigroup under certain assumptions.

Chapter 2

Some Preliminaries

Let (X, τ) be a topological space, and let Y be a subset of X. Denote by $\overline{Y}^{\tau}$ and Y^o the closure of Y and the interior of Y, respectively.

Let E be a Banach space. Throughout this thesis, E_1 and S_E will denote the unit ball and unit sphere of E respectively. Let K be a subset of E. We denote by $\mathcal{E}(K)$ the set of all extreme points of K, and denote by $\text{co}(K)$ the algebraic convex hull of K. Let E' be the dual space of E, which contains all bounded linear functional of E.

In this thesis, all groups will be assumed to be locally compact, and G will denote a locally compact group. A *left(right) Haar measure* on G is a non-zero positive Borel regular measure μ_G on G such that μ_G is *left(right) translation invariant*. Every locally compact group possesses a left(right) Haar measures, which is unique up to multiplication by a positive constant. If λ is a Haar measure of G, then there is a continuous homomorphism $\Delta : G \to [0, \infty)$, called the modular function of G, such that $\lambda(Ex) = \Delta(x)\lambda(E)$ for any $x \in G$ and Borel set $E \subseteq G$. Let m_G be a fixed left Haar measure on G. Let $1 \leq p < \infty$ and let $\mathcal{L}^p(G)$ be the set of all p-integrable functions on G with respect to m_G. Let $f_1, f_2 \in \mathcal{L}^p(G)$. f_1 and f_2 are said to be equivalent if

$\|f_1 - f_2\|_p = 0$. Denote by $L^p(G)$ the set of all equivalent classes in $\mathcal{L}^p(G)$. Define $L^\infty(G)$ to be the set of all locally (Haar-)measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e. It is known that $L^\infty(G)$ is a Banach space with norm

$$\|f\|_\infty := \inf\{c : |f| \leq c \text{ locally a.e.}\}$$

From now on, we use the following notation without further specification. Denote by $CB(G)$ the space of bounded continuous functions on G, and by $C_0(G)$ the space of continuous functions vanishing at infinity on G.

Let $M(G)$ be the set of all complex Radon measures on G. Define a norm $\|\cdot\|_{M(G)}$ by

$$\|\mu\|_{M(G)} := |\mu|(G) \ \ (\mu \in M(G))$$

Also, define the convolution operation $*$ on $M(G)$ by

$$\int_G f(x)\mathrm{d}(\mu * \nu)(x) := \int_G (\int_G f(xy)\mathrm{d}\mu(x))\mathrm{d}\nu(x) \ \ (f \in C_c(G), \mu, \nu \in M(G))$$

and the involution $\mu \mapsto \mu^*$ on $M(G)$ by

$$\int_G f(x)\mathrm{d}\mu^*(x) := \int_G f(x^{-1})d\overline{\mu}(x) \ \ (f \in C_c(G), \mu \in M(G))$$

Then $(M(G), \|\cdot\|_{M(G)}, *)$ is a unital Banach *-algebra and the unit is given by the point mass measure at the identity, δ_e. The set $M_a(G)$ of measures in $M(G)$ which are absolutely continuous with respect to the Haar measure is a closed *-ideal in $M(G)$ identified with $L^1(G)$ via

$$\mu_f(E) := \int_E f(x)\mathrm{d}m_G(x)$$

The convolution operation $*$ on $L^1(G)$ inherited from $M(G)$ is given by

$$f * g(y) = \int_G f(yx)g(x^{-1})\mathrm{d}m_G(x) \text{ a.e.}$$

The *-operation on $L^1(G)$ inherited from $M(G)$ is given by

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})} \text{ a.e.}$$

We will call $L^1(G)$, with the above convolution product, the group algebra of G. Write $\delta_G = \{\delta_x : x \in G\}$.

Let f be a function on G and $y \in G$. We define the left and right translates of f through y by

$$L_y f(x) = f(y^{-1}x), \;\; R_y f(x) = f(xy).$$

We also write ${}_xf$ and f_x for the functions $f(x\cdot)$ and $f(\cdot x)$, respectively.

A bounded continuous function f is said to be *left*(*right*) *uniformly continuous* if $\|L_y f - f\|_\infty \to 0$(resp. $\|R_y f - f\|_\infty \to 0$) as $y \to e_G$. If f is both left and right uniformly continuous, then f is called *uniformly continuous*. Denote by $UCB(G)$ the space of uniformly continuous functions on G.

A *unitary representation* of G is a homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. Let Σ_G be the class of unitary representations of G, and let $\lambda_2 : G \longrightarrow B(L^2(G))$, $[\lambda_2(x)(f)](y) := f(x^{-1}y)$ $(x, y \in G, f \in L^2(G))$ be the *left regular representation* of G. We will also denote by $\hat{G}$ the class of all irreducible unitary representations of G. If G is abelian, we also denote the dual group of G by $\hat{G}$.

Let G be a locally compact group. For any $f \in L^1(G)$, define

$$\|f\|_{C^*(G)} := \sup\nolimits_{\pi \in \Sigma_G} \|\pi(f)\|$$

It is easily seen that $\|\cdot\|_{C^*(G)}$ is a C*-norm on $L^1(G)$. Let $C^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C^*(G)}$. Then $C^*(G)$ is called the *full group C^*-algebra* or simply the *group C^*-algebra* of G. Let $B(G) := \{x \mapsto \langle \pi(x)\xi, \eta\rangle :$

$\pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$ be the *Fourier-Stieltjes algebra* of G. $B(G)$ is a commutative Banach algebra with the pointwise multiplication and its norm is given by

$$\|u\|_{B(G)} = \sup\{\|\xi\|\|\eta\| : u(x) = \langle \pi(x)\xi, \eta \rangle, \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$$

Let $A(G) := \{x \mapsto \langle \lambda_2(x)\xi, \eta \rangle : \xi, \eta \in L^2(G)\}$ be the *Fourier algebra* of G. It is well-known that $A(G)$ is a closed ideal of $B(G)$.

Let $P(G)$ be the set of all continuous positive definite functions on G. (i.e.)

$$P(G) := \{\phi \in B(G) : \int (f^* * f)\phi \geq 0 \text{ for any } f \in L^1(G)\}$$

It can be shown that $P(G) = \{\langle \pi(\cdot)\xi, \xi \rangle : \pi \in \Sigma_G, \xi \in \mathcal{H}_\pi\}$ and $\phi(e) = \|\phi\|_{B(G)}$. See [7] for reference.

Let $VN(G)$ be the von Neumann algebra generated by the image of λ_2 in $B(L^2(G))$. It is called the *group von Neumann algebra* of G. For any $f \in L^1(G)$, define

$$\|f\|_{C_r^*} := \|\lambda_2(f)\|$$

It is easily seen that $\| \cdot \|_{C^*_{\lambda_2}(G)}$ is a C*-norm on $L^1(G)$. Let $C_r^*(G)$ be the completion of $L^1(G)$ under $\| \cdot \|_{C_r^*(G)}$ Then $C_r^*(G)$ is called the *reduced group C*-algebra* of G. It is proved by Eymard[11] that $A(G)' = VN(G)$. For $u \in A(G)$ and $T \in VN(G)$, define $u \cdot T \in VN(G)$ by $\langle u \cdot T, v \rangle = \langle T, uv \rangle$, $v \in A(G)$. Let $UCB(\hat{G})$ be the closed linear span of $A(G) \cdot VN(G)$ in $VN(G)$. The set of all T in $VN(G)$ for which the operator from $A(G)$ to $VN(G)$ given by $u \mapsto u \cdot T$ is weakly compact (compact) is denoted by $WAP(\hat{G})$ (resp. $AP(\hat{G})$), *the weakly almost periodic*(resp. *almost periodic*) *functionals in* $VN(G)$.

Suppose that π is a unitary representation of G. Let $F_\pi(G) = \text{span}\, \{x \mapsto \langle \pi(x)\xi, \eta \rangle : \xi, \eta \in \mathcal{H}_\pi\}$. $A_\pi(G)$, *the Fourier spaces associated to* π, is defined to be the closure of $F_\pi(G)$ in the Banach space $B(G)$. For any representation π of G, define $VN_\pi(G)$ the von Neumann algebra generated by $\pi(G)$(or $\pi(L^1(G))$) in $\mathcal{L}(\mathcal{H}_\pi)$. If $\pi = \lambda_2$, then $A_\pi(G) = A(G) = F_\pi(G)$ and $VN_\pi(G) = VN(G)$. For each $u \in A_\pi(G)$, there exist some nets (ξ_n) and (η_n) in $\mathcal{H}_\pi$ such that

$$u(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \eta_n \rangle \text{ and } \|u\| = \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\|.$$

See [1] and [11] for more details.

Chapter 3

Semigroup structure of the spectrum of $a(G)$

In this chapter, we will study the semigroup structure of the spectrum of $a(G)$. We start with the definition of G^*, which will play an important role throughout this thesis. Let $P_1(G) = S_{B(G)} \cap P(G)$. In the other words,

$$P_1(G) = \{\langle \pi(\cdot)\xi, \xi\rangle : \pi \in \Sigma_G, \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$$

Let $G^* = \mathcal{E}(P_1(G))$, and let $\widetilde{G}$ be the semigroup generated by G^* in $B(G)$. The sets G^* and $\widetilde{G}$ are equipped with the relative weak* topology inherited from $B(G)$. G^* will be called the *dual space* of G. We shall denote the elements in G^* by g^*, h^* or k^*.

Remarks.

(a) If G is abelian, then $G^* = \widetilde{G} = \hat{G}$.

(b) $G^* = \{x \mapsto \langle \pi(x)\xi, \xi\rangle : \pi \in \hat{G}, \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$

(c) G^* separates points on G. That is, if x and y are distinct points of G, there is an element $g^* \in G^*$ such that $g^*(x) \neq g^*(y)$.(See [12, Theorem 3.34])

(d) Actually, it is proved in [2] that the following statements are equivalent:

  (a) G is abelian.

  (b) For every $g^* \in G^*$, we have $(g^*)^{-1} \in P_1(G)$.

  (c) G^*, equipped with the pointwise multiplication, is a group.

3.1 Isomorphisms between generalized Fourier algebras

Let $a_0(G)$ be the closure of the span of G^* in $B(G)$, and let $a(G)$ be the closed subalgebra generated by $a_0(G)$ in $B(G)$. We call $a(G)$ *the little Fourier algebra* of G. Denote by $vn_0(G)$ and $vn(G)$ the dual Banach spaces of $a_0(G)$ and $a(G)$, respectively. We call $vn(G)$ the *the little von Neumann algebra* of G. Then the norm-closure of the span of $\widetilde{G}$ in $B(G)$ is $a(G)$. Recall that $\bar{\pi}$ is the contragredient of π(for details, see [12, Chapter 3]). Note that $\bar{\pi}$ is irreducible for any irreducible representation π of G. It follows that $a(G)$ is a Banach *-algebra where the involution is given by the complex conjugation. Furthermore, we can show that $a(G)$ is semisimple as G^* separates points of G.

For the definitions of direct sums and internal tensor products of unitary representations of G, we refer the reader to [12, Chapter 3 and 7].

Proposition 3.1.1. Let $\pi_a = \bigoplus_{\pi \in \hat{G}} \pi$. Then $a_0(G) = A_{\pi_a}(G)$. Hence, $vn_0(G) = VN_{\pi_a}(G)$. In particular, $vn_0(G)$ is a von Neumann algebra.

Proof. Let $\mathfrak{F}$ be the set of all unitary equivalence classes of finite direct sums of irreducible representations in G. It is clear that $span(G^*) = \{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi \in \mathfrak{F}, \xi, \eta \in \mathcal{H}_\pi\}$. Suppose that $\phi \in A_{\pi_a}(G)$ such that $\phi(x) = \langle \pi_a(x)\xi, \xi\rangle$ for some $\xi \in \mathcal{H}_{\pi_a}\}$. For any $\epsilon > 0$, there exists $\xi_0 \in \mathcal{H}_\pi$ for some $\pi \in \mathfrak{F}$ and $\|\xi - \xi_0\| < \epsilon$. For any $f \in C^*(G)$,

$$|\langle \pi_a(f)\xi, \xi\rangle - \langle \pi(f)\xi_0, \xi_0\rangle| = |\langle \pi_a(f)\xi, \xi\rangle - \langle \pi_a(f)\xi_0, \xi_0\rangle|$$

$$\leq |\langle \pi_a(f)\xi, (\xi - \xi_0)\rangle| + |\langle \pi_a(f)(\xi - \xi_0), \xi_0\rangle| \leq 2\|f\|_{C^*}\|\xi\|\epsilon$$

Therefore, $\|\langle \pi_a(\cdot)\xi, \xi\rangle - \langle \pi(\cdot)\xi_0, \xi_0\rangle\|_{B(G)} \leq \epsilon$. The results thus follows. □

Let $\pi_a^{(n)} = \otimes_{i=1}^n \pi_a$ and $\sigma = \oplus_{n=1}^\infty \pi_a^{(n)}$. It is straight forward to show that $a(G) = A_\sigma(G)$ and $vn(G) = VN_\sigma(G)$. Hence, $vn(G)$ is a von Neumann algebra.

A Banach space X has the Radon-Nikodym property(RNP) if for every bounded subset C of X and $\epsilon > 0$, there is some $x \in C$ such that x does not lie in the norm closure of co$[C \setminus (x + \{y \in X : \|y\| \leq \epsilon\})]$.

Remark. If G is a compact group, then $B(G)$ has RNP. In fact, we have $B(G)$ has RNP if and only if $B(G) = a_0(G)$(See [5, Theorem 5], [36, Theorem 4.2], [25, Theorem 4.5] and [27]).

Let $A_{\mathcal{F}}(G)$ be the $\|\cdot\|_{B(G)}$ closure of $\{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi$ is a finite dimensional representation of $G, \xi, \eta \in \mathcal{H}_\pi\}$. Let $\hat{G}_{\mathcal{F}}$ be the set of all finite dimensional irreducible representations of G, and $\pi_F = \oplus_{\pi \in \hat{G}_{\mathcal{F}}} \pi$. Then $A_{\mathcal{F}}(G) = A_{\pi_F}(G) \subseteq a_0(G)$.

A [Moore]-group is a locally compact group such that all its irreducible unitary representations are finite dimensional.

Remarks.

(a) If G is abelian, $a_0(G) = a(G) \cong l^1(\hat{G})$ and $vn_0(G) = vn(G) \cong l^\infty(\hat{G})$.

(b) If G is compact, then every representation of G is a direct sum of copies of irreducible representations, hence $a_0(G) = B(G) = a(G)$.

(c) If G is [Moore]-group, it is clear that $a_0(G) = a(G) = A_{\mathcal{F}}(G)$.

(d) More generally, if $B(G)$ has a RNP, then $a_0(G) = B(G) = a(G)$.

(e) If G is the "ax+b"-group, then $a_0(G) = A_{\mathcal{F}}(G) \oplus A(G)$, which is an algebra since $A(G)$ is an ideal in $a_0(G)$. Thus $a_0(G) = a(G)$.

Let A be a commutative Banach algebra. The spectrum of A, written as $\sigma(A)$, is the set of all non-zero multiplicative linear functional of A.

From now on, π will be a unitary representation of G such that $A_\pi(G)$ is an algebra.

If $A_\pi(G)$ is a unital algebra, then it is easy to see that

$$A_\pi(G) = A_\pi(G) \cdot A_\pi(G) = \text{norm-cl}(\text{span}(A_\pi(G) \cdot A_\pi(G)))$$

Therefore, it follows that $A_\pi(G) = A_{\pi\otimes\pi}(G)$, and hence π and $\pi \otimes \pi$ are quasi-equivalent(see [1]). By a result in [9, Chapter], it follows that there is an isomorphism

$$\Phi : VN_\pi(G) \to VN_{\pi\otimes\pi}(G) \text{ such that } \Phi(\pi(g)) = (\pi \otimes \pi)(g) \text{ for any } g \in G.$$

Moreover, we have

$$\langle u, x\rangle_{A_\pi(G),VN_\pi(G)} = \langle u, \Phi(x)\rangle_{A_{\pi\otimes\pi}(G),VN_{\pi\otimes\pi}(G)} \text{ for any } u \in A_\pi(G), x \in VN_\pi(G)$$

(See [1]). It is easy to see that the isomorphism with above properties is unique.

For any $x \in VN_\pi(G)$, $\pi \otimes \pi(x)$ is defined to be the $\Phi(x)$. It is an operator of $H_\pi \otimes H_\pi$ since it is an element of $VN_{\pi\otimes\pi}(G)$. Since $\pi \otimes \pi(x)$ and $\pi(x) \otimes \pi(x)$ are operators of $H_\pi \otimes H_\pi$, it makes sense to ask if they are equal.

The following lemma is a generalization of [37, Theorem 1(ii)].

Lemma 3.1.2. If $A_\pi(G)$ is unital, then $\sigma(A_\pi(G)) := \{x \in VN_\pi(G) \setminus \{0\} : \pi \otimes \pi(x) = \pi(x) \otimes \pi(x)\}$

Proof. Let $u_i = \langle \pi(\cdot)\xi_i, \eta_i \rangle \in A_\pi(G)$ where $i = 1, 2$ and let $f = u_1 u_2$. Then $f(x) = \langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle$ for any $x \in G$. Then we have

$$\langle f, x \rangle = \langle f, \Phi(x) \rangle = \langle f, \pi \otimes \pi(x) \rangle = \langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle.$$

If $x \in \sigma(A_\pi(G))$, then

$$\langle f, x \rangle = \langle u_1, x \rangle \langle u_2, x \rangle = \langle \pi(x)\xi_1, \eta_1 \rangle \langle \pi(x)\xi_2, \eta_2 \rangle = \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle.$$

Therefore,

$$\langle \pi \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle$$

Conversely, suppose that $x \in VN_\pi(G) \setminus \{0\}$ and $\pi(x) \otimes \pi(x) = \pi \otimes \pi(x)$. Then we have

$$\langle u_1, x \rangle \langle u_2, x \rangle = \langle \pi(x)\xi_1, \eta_1 \rangle \langle \pi(x)\xi_2, \eta_2 \rangle = \langle \pi(x) \otimes \pi(x)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle f, x \rangle.$$

So, $x \in \sigma(A_\pi(G))$. □

For any $u \in A_\pi(G)$, $T \in VN_\pi(G)$, define $T_l(u)(x) = \langle \pi(x) \cdot T, u \rangle$.

Lemma 3.1.3. We have $T_l(u)(x) = \langle T, {}_x u \rangle$. If $A_\pi(G)$ is unital, then $T_l(1)(x) \equiv \langle T, 1 \rangle$.

Proof. If $u \in A_\pi(G)$ and $u(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \eta_n \rangle$ for some $\xi_n, \eta_n \in \mathcal{H}_\pi$, then $(u \cdot \pi(x))(y) = \sum_{n=1}^{\infty} \langle \pi(y)\xi_n, \pi(x)^*\eta_n \rangle = \sum_{n=1}^{\infty} \langle \pi(xy)\xi_n, \eta_n \rangle = {}_x u(y)$ for any $x, y \in G$. □

Lemma 3.1.4. $T_l(u) \in A_\pi(G)$ for each $u \in A_\pi(G)$ and $T \in VN_\pi(G)$.

Proof. $T_l(u)(x) = \langle \pi(x) \cdot T, u \rangle = \langle \pi(x), T \cdot u \rangle = (T \cdot u)(x)$. □

Lemma 3.1.5. If $T \in \sigma(A_\pi(G))$, then $T_l : A_\pi(G) \to A_\pi(G)$ is a homomorphism.

Proof. If $u, v \in A_\pi(G)$, then

$$T_l(u \cdot v)(x) = \langle T, {}_x(uv) \rangle = \langle T, {}_x u {}_x v \rangle = \langle T, {}_x u \rangle \langle T, {}_x v \rangle = T_l(u)(x) T_l(v)(x)$$

. □

For any $S, T \in VN_\pi(G)$, define $S \circ T \in VN_\pi(G)$ by $\langle S \circ T, u \rangle = \langle S, T_l(u) \rangle$ for all $u \in A_\pi(G)$.

Proposition 3.1.6. If $S, T \in VN_\pi(G)$, then $S \circ T = S \cdot T$ and $(S \cdot T)_l(u) = T_l(S_l(u))$ for all $u \in A_\pi(G)$.

Proof. By definition, the first equality holds clearly if $S = \pi(x)$ for some $x \in G$. The rest follows form the weak* density of span$(\pi(G))$ in $VN_\pi(G)$. The second equality is straightforward. □

Given a function $u : G \to \mathbb{C}$, let $\tilde{u} : G \to \mathbb{C}$ be the function defined by $\tilde{u}(x) = u(x^{-1})$.

Proposition 3.1.7. If $\sigma(A_\pi(G)) \cup \{0\}$ is equipped with the multiplication product inherited from $VN_\pi(G)$, then it is a *-semitopological semigroup. In addition, if $A_\pi(G)$ is unital and $\sigma(A_\pi(G))$ is equipped with the multiplication product inherited from $VN_\pi(G)$, then it is a compact *-semitopological semigroup.

Proof. If $T, S \in \sigma(A_\pi(G))$ and $u, v \in A_\pi(G)$, then

$$\langle T \cdot S, uv \rangle = \langle T, S_l(uv) \rangle = \langle T, S_l(u)S_l(v) \rangle$$

$$= \langle T, S_l(u) \rangle \langle T, S_l(v) \rangle = \langle T \cdot S, u \rangle \langle T \ \cdot S, v \rangle.$$

On the other hand, we have $\langle T^*, uv \rangle = \langle T, \tilde{uv} \rangle = \langle T, \tilde{u}\tilde{v} \rangle = \langle T^*, u \rangle \langle T^*, v \rangle$, so $T^* \in \sigma(A_\pi(G))$. Suppose that $A_\pi(G)$ is unital. Now $\langle T, 1 \rangle = 1 = \langle S, 1 \rangle$, so $\langle T \cdot S, 1 \rangle = \langle T, S_l(1) \rangle = \langle T, 1 \rangle = 1$. It follows that $T \cdot S \neq 0$. Hence, $T \cdot S \in \sigma(A_\pi(G))$. Since the multiplication of a von Neumann algebra is separately weak*-continuous, we conclude that they are semitopological semigroups. □

Corollary 3.1.8. $\sigma(a(G))$ is a compact *-semitopological semigroup if it is equipped with the multiplication product inherited from $vn(G)$.

Suppose that $\phi \in l^\infty(G)$ satisfying that

$$\phi f = f \text{ for any } f \in l^1(G).$$

Then, obviously, ϕ is the constant one function. We now have the following proposition which is a non-commutative analogue of this observation:

Proposition 3.1.9. Let T be a non-zero element in $vn(G)$. Then the following statements are equivalent:

(a) $Tu = u$ for all $u \in a(G)$

(b) $T = \sigma(e)$.

Proof. (b) $\Rightarrow$ (a) is clear. Suppose that (a) holds. We have $[T_l(u)](x) = (Tu)(x) = u(x)$. For any $S \in vn(G)$, we obtain $\langle S \cdot T, u \rangle = \langle S, T_l(u) \rangle = \langle S, u \rangle$. Hence, $S \cdot T = S$ for all $S \in vn(G)$. Therefore, $T = \sigma(e)$. □

Write $\sigma_u(A_\pi(G))(\sigma_{inv}(A_\pi(G)))$ the set of all unitary(resp. invertible) elements in $\sigma(A_\pi(G))$. Clearly, $\sigma_u(A_\pi(G))$ and $\sigma_{inv}(A_\pi(G))$ are semi-topological groups if equipped with the relative weak*-topology of $VN_\pi(G)$.

Theorem 3.1.10. Let π_1 and π_2 be unitary representations of G_1 and G_2, respectively. If $A_{\pi_1}(G_1)$ and $A_{\pi_2}(G_2)$ are isometrically isomorphic, then there is a homeomorphism $\phi : \sigma(A_{\pi_1}(G_1)) \to \sigma(A_{\pi_2}(G_2))$ such that:

(a) $\phi(T^*) = \phi(T)^*$ for any $T \in \sigma(A_{\pi_1}(G_1))$;

(b) for each $T, S \in \sigma(A_{\pi_1}(G_1))$, either $\phi(T \cdot S) = \phi(T)\phi(S)$ or $\phi(T \cdot S) = \phi(S)\phi(T)$;

(c) ϕ is either a *-isomorphism or a *-anti-isomorphism from $\sigma_u(A_{\pi_1}(G_1))$ onto $\sigma_u(A_{\pi_2}(G_2))$.

Proof.

Step 1:

*We construct a Jordan *-isomorphism* Φ *between* $VN_{\pi_1}(G_1)$ *and* $VN_{\pi_2}(G_2)$.

Let $\psi : A_{\pi_2}(G_2) \to A_{\pi_1}(G_1)$ be an isometric isomorphism. It is straight forward to show that $U = \psi^*(\pi_2(e)) \in \sigma(A_{\pi_2}(G_2))$. We have $V = U^* \in \sigma(A_{\pi_2}(G_2))$ by proposition 3.1.7. By lemma 3.1.6, $V_l : A_{\pi_2}(G_2) \to A_{\pi_2}(G_2)$ is a homomorphism. Since V is unitary, it is easy to see that V_l is in fact an isometric isomorphism. It follows that $\psi \circ V_l : A_{\pi_2}(G_2) \to A_{\pi_1}(G_1)$ is an isometric isomorphism. Let $\Phi = (\psi \circ V_l)^*$. Then Φ is an isometry from $VN_{\pi_1}(G_1)$ onto $VN_{\pi_2}(G_2)$. Note that

$$\langle \Phi(\pi_1(e_1)), f \rangle = \langle \psi^*(\pi(e_1)), V_l(f) \rangle = \langle U, V_l(f) \rangle = \langle \pi_2(e), f \rangle$$

for any $f \in A_{\pi_1}(G_1)$. Therefore, Φ preserves units and hence is a Jordan *-isomorphism by [18, Theorem 7].

Step 2: Let ϕ be the restriction of Φ to $\sigma(A_{\pi_1}(G_1))$. Then ϕ is a homeomorphism from $\sigma(A_{\pi_1}(G_1))$ onto $\sigma(A_{\pi_2}(G_2))$. We show that ϕ satisfies (a) and (b):

If $TS = ST$, then (b) holds as Jordan *-isomorphisms preserve commutativity. Otherwise, we have

$$\phi(T)\phi(S) + \phi(S)\phi(T) = \phi(ST) + \phi(TS)$$

Suppose that (b) does not holds. Then $\phi(T)\phi(S)$, $\phi(S)\phi(T)$, $\phi(ST)$ and $\phi(TS)$ are pairwise distinct, hence linearly independent, in $\sigma(A_{\pi_2}(G_2))$, which leads a contradiction.

By theorem 10 in [18], there exists a central projections $z_i \in VN_{\pi_i}(G_i)$ $(i = 1, 2)$, such that $\Phi = \Phi_I + \Phi_A$ and $\Phi_I : VN(G_1)z_1 \to VN(G_2)z_2$ is a *-isomorphism and $\Phi_A : VN(G_1)(\pi_1(e) - z_1) \to VN(G_2)(\pi_2(e) - z_2)$ is a *-anti-isomorphism. For each $T \in \sigma_u(A_{\pi_1}(G_1))$, define $H_T = \{S \in \sigma_u(A_{\pi_1}(G_1)) : (ST - TS)z_1 = 0\}$; $K_T = \{S \in \sigma_u(A_{\pi_1}(G_1)) : (ST - TS)(\pi_2(e) - z_1) = 0\}$.

Step 3: We show that H_T and K_T are subgroups of $\sigma_u(A_{\pi_1}(G_1))$ and $H_T \cup K_T = \sigma_u(A_{\pi_1}(G_1))$.

If $S_1, S_2 \in H_T$ and $S \in \sigma_u(A_{\pi_1}(G_1))$, then $SS_1S_2z_1 = S_1(SS_2)z_2 = S_1(S_2Sz_2) = S_1S_2Sz_2$ and $(S_1^{-1}S - SS_1^{-1})z_2 = S_1^{-1}(S_1S - SS_1)S_1^{-1} = 0$. It follows that H_T is a subgroup of $\sigma_u(A_{\pi_1}(G_1))$. Similarly, K_T is a subgroup of $\sigma_u(A_{\pi_1}(G_1))$. Finally, if $\phi(ST) = \phi(T)\phi(S)$, then $\phi(ST - TS)z_2 = 0$(since Φ_I is a *-isomorphism) which implies that $(ST - TS)z_1 = 0$. So, $S \in H_T$. Otherwise, we have $\phi(ST) = \phi(S)\phi(T)$. It follows similarly that $S \in K_T$.

Step 4: Define $H = \cap\{T \in \sigma_u(A_{\pi_1}(G_1)) : H_T = \sigma_u(A_{\pi_1}(G_1))\}$; $K =$

$\cap\{T \in \sigma_u(A_{\pi_1}(G_1)) : K_T = \sigma_u(A_{\pi_1}(G_1))\}$. *Then either* $H = \sigma_u(A_{\pi_1}(G_1))$ *or* $K = \sigma_u(A_{\pi_1}(G_1))$.

If $S_1, S_2 \in H$, then, for any $S \in \sigma_u(A_{\pi_1}(G_1))$, we have $S_1S_2Sz = S_1(SS_2z) = (SS_1)S_2z = S(S_1S_2)z$. Thus, $H_{S_1S_2} = \sigma_u(A_{\pi_1}(G_1))$. Also, we have $(S_1^{-1}S - SS_1^{-1})z = S_1^{-1}(S_1S - SS_1)S_1^{-1} = 0$. Consequently, $H_{S_1^{-1}} = \sigma_u(A_{\pi_1}(G_1))$. The final assertion is clear since $H_T = \sigma_u(A_{\pi_1}(G_1))$ or $K_T = \sigma_u(A_{\pi_1}(G_1))$ for any $T \in \sigma_u(A_{\pi_1}(G_1)$ (as H_T and K_T are subgroups of $\sigma_u(A_{\pi_1}(G_1))$).

Step 5:

If $H = \sigma_u(A_{\pi_1}(G_1))(K = \sigma_u(A_{\pi_1}(G_1)))$*, then* ϕ *is a *-anti-isomorphism (a *-isomorphism).*

Suppose that $H = \sigma_u(A_{\pi_1}(G_1))$. We claim that $\phi(S_1S_2) = \phi(S_2)\phi(S_1)$ for all $S_1, S_2 \in \sigma_u(A_{\pi_1}(G_1))$. If not so, then $\phi(S_1S_2) = \phi(S_1)\phi(S_2)$. It follows that $(\phi(S_1)\phi(S_2) - \phi(S_2)\phi(S_1))(\pi_2(e) - \phi(z_1)) = 0$. But $S_1, S_2 \in H$ implies that $(S_1S_2 - S_2S_1)z = 0$. So, $S_1S_2 = S_2S_1$. Hence, $\phi(S_1S_2) = \phi(S_2)\phi(S_1)$. Therefore, ϕ is a *-anti-isomorphism. The other case is similar. □

Corollary 3.1.11. If $a(G_1)$ and $a(G_2)$ are isometrically isomorphic, then $\sigma_u(a(G_1))$ and $\sigma_u(a(G_2))$ are topologically isomorphic.

3.2 When is the spectrum of $a(G)$ a group?

In this section, we investigate when the spectrum of $a(G)$ is a group.

Let G be a non-[Moore]-group. Let $\hat{G}_{\mathcal{I}}$ be the set of all infinite dimensional irreducible representations of G, and $\pi_I = \oplus_{\pi\in\hat{G}_{\mathcal{I}}}\pi$. Then $\pi_a = \pi_F \oplus \pi_I$. Let $\sigma_I = \bigoplus_{n\in\mathbb{N}} \pi_F \otimes \pi_I^{\otimes n}$ where $\pi_I^{\otimes n} = \bigotimes_{i=1}^n \pi_I$. It is easy to see that $\sigma = \pi_F \oplus \sigma_I$.

Lemma 3.2.1. Let $z_F \in vn(G)$ be the central projection such that $A_{\mathcal{F}}(G) =$

$z_F \cdot a(G)$. Write $a(G) = A_{\mathcal{F}}(G) \oplus A_I(G)$, where $A_I(G) = (\sigma(e) - z_F)a(G)$. Then $A_I(G)$ is the ideal generated by $A_{\pi_I}(G)$ in $a(G)$ and $A_I(G) = A_{\sigma_I}(G)$.

Proof. Note that $B(G) = A_{\mathcal{F}}(G) \oplus A_{\mathcal{PIF}}(G)$(see [33, Section 2]). Thus, $a(G) = A_{\mathcal{F}}(G) \oplus (a(G) \cap A_{\mathcal{PIF}}(G))$. By uniqueness of the translation invariant complement of $A_{\mathcal{F}}(G)$ in $a(G)$, we have $a(G) \cap A_{\mathcal{PIF}}(G) = A_I(G)$. Since $A_{\mathcal{PIF}}(G)$ is an ideal of $B(G)$(see [33, Theorem 2.3]), it follows that $A_I(G)$ is an ideal of $a(G)$. □

We have the following proposition that gives some criteria for the equality of $a(G)$ and $a_0(G)$, which is of independent interest:

Proposition 3.2.2. The following statements are equivalent:

(a) $a_0(G) = a(G)$.

(b) $a_0(G) = A_{\pi_a \otimes \pi_a}(G)$.

(c) $a(G)$ has RNP.

(d) $A_{\pi_a \otimes \pi_a}(G)$ has RNP.

(e) $A_I(G)$ has RNP.

(f) $\pi_a \otimes \pi_a$ is completely reducible.

(g) $\pi \otimes \rho$ is completely reducible for any $\pi, \rho \in \hat{G}$.

(h) $A_{\pi_I}(G)$ is an algebra and $a_0(G)A_{\pi_I}(G) = A_{\pi_I}(G)$.

Proof. Note that $a_0(G) \subseteq A_{\pi_a \otimes \pi_a}(G) \subseteq a(G)$ and $a(G) = A_{\mathcal{F}}(G) \oplus A_I(G)$. The result follows from [5, Theorem 3]. □

Remark. It follows that [27] that if $a_0(G) = a(G)$, then $a(G)$ has the weak fixed point property for non-expansive mappings. We do not know if the converse is true(see also [25]).

Note that $\sigma(A_{\mathcal{F}}(G)) = \sigma(A(G^{ap})) \cong G^{ap}$ where G^{ap} is the almost periodic compactification of G. If G is a [Moore]-group, then $a(G) = A_{\mathcal{F}}(G) = B(G^{ap}) = A(G^{ap})$. Therefore, $\sigma(a(G)) = G^{ap}$ is a group. We will prove below that the converse of it is also true:

The following lemma is a generalization of [38, Proposition 1], and the proof of it is left to the reader.

Lemma 3.2.3. Let $s \in VN_\pi(G)$ be such that $s^2 = s$. Then the following are equivalent:

(a) $s \in \sigma(A_\pi(G))$.

(b) $s \cdot A_\pi(G)$ is an algebra and $(\pi(e) - s)A_\pi(G)$ is an ideal of $A_\pi(G)$.

(c) The map $A_\pi(G) \to s \cdot A_\pi(G)$, $f \mapsto s \cdot f$ is an endomorphism.

Lemma 3.2.4. If $A_\pi(G) = A_{\pi_1}(G) \oplus A_{\pi_2}(G)$ and $m \in \sigma(A_\pi(G))$ is invertible, then $m(A_{\pi_1}(G)) \neq 0$ and $m(A_{\pi_2}(G)) \neq 0$.

Proof. Let $z[\pi_1]$ be the support projection of π_1 in $VN_\pi(G)$. Assume that $m(A_{\pi_1}(G)) = 0$. Then $m \in A_{\pi_1}(G)^\perp = (\pi(e) - z[\pi_1])VN_\pi(G)$. So, $m = (\pi(e) - z[\pi_1])m$. Hence, $\pi(e) = z[\pi_1]$. Consequently, $A_{\pi_2}(G) = 0$ which leads a contradiction. □

Lemma 3.2.5. Let $z_F \in vn(G)$ be the central projection such that $A_{\mathcal{F}}(G) = z_F \cdot a(G)$. Then $z_F \in \sigma(a(G))$.

Proof. Since $A_I(G)$ is an ideal of $a(G)$, by lemma 3.2.3, we have $z_F \in \sigma(a(G))$.

□

Note that $a_0(G) = \oplus_1\{A_\pi(G) : \pi \in \hat{G}\} = \oplus_1\{L^1(\mathcal{H}_\pi) : \pi \in \hat{G}\}$(See [1]) where $L^1(\mathcal{H}_\pi)$ is the space of all trace-class operators on $\mathcal{H}_\pi$. Let $c_0(\hat{G}) := \oplus_0\{\mathcal{K}(\mathcal{H}_\pi) : \pi \in \hat{G}\}$. Then it is easy to see that the dual space of $c_0(\hat{G})$ is $a_0(G)$.

Lemma 3.2.6. The following assertions are equivalent:

(a) G is a [Moore]-group.

(b) $a_0(G)$ is a l^1-sum of finite-dimensional Banach spaces.

(c) $c_0(\hat{G})$ is a c_0-sum of finite-dimensional C*-algebras.

(d) Every bounded linear operator $T : c_0(\hat{G}) \to a_0(G)$ is compact.

(e) Every irreducible representation of $c_0(\hat{G})$ is finite-dimensional.

Proof. By using [24, Theorem 3.6 and Theorem 4.1], we see the equivalence of (b)-(e). It suffices to proof that (e) implies that (a). Define $\hat{\pi_0} : c_0(\hat{G}) \to B(\mathcal{H}_{\pi_0})$, $(T_\pi)_{\pi\in\hat{G}} \mapsto T_{\pi_0}$. Let $\xi, \eta \in \mathcal{H}_{\pi_0} \setminus \{0\}$. There exists $S_{\pi_0} \in \mathcal{F}(\mathcal{H}_{\pi_0})$ such that $S_{\pi_0}(\xi) = \eta$. Now, define $T_\pi = S_{\pi_0}$ if $\pi = \pi_0$ and $T_\pi = 0$ if $\pi \neq \pi_0$. Then $\hat{\pi_0}((T_\pi)_{\pi\in\hat{G}})\xi = \eta$, and hence $\hat{\pi_0}$ is irreducible. Therefore, $\mathcal{H}_{\pi_0}$ is finite-dimensional. □

Remark. A Banach space is said to have *Schur's property* if all weakly convergent sequences are norm convergent. The Banach space X is said to have the *DPP* if, for any Banach space Y, every weakly compact linear operator $u : X \to Y$ sends weakly Cauchy sequences into norm convergent sequences. Actually, by using [24, Theorem 3.6 and Theorem 4.1], we can prove that the following assertions are equivalent:

(a) G is a [Moore]-group.

(b) $a_0(G)$ has Schur's property.

(c) $a_0(G)$ has DPP.

(d) $c_0(\hat{G})$ has DPP.

(e) $ap(c_0(G)) = a_0(G)$.

Theorem 3.2.7. Let G be a locally compact group. The following statements are equivalent:

(a) G is a [Moore]-group.

(b) $\sigma(a(G))$ is a group.

(c) The only idempotent of $\sigma(a(G))$ is $\sigma(e)$.

(d) $z_F \in \sigma(a(G))$ is invertible.

(e) $a(G) = A_{\mathcal{F}}(G)$

(f) $a_0(G) = A_{\mathcal{F}}(G)$

Proof. "(a) $\Rightarrow$ (b) $\Rightarrow$ (c)" and "(b) $\Rightarrow$ (d)" are clear. Suppose that (b) holds. Then $z_F = \sigma(e)$. So, $a(G) = z_F \cdot a(G) = A_{\mathcal{F}}(G)$. On the other hand, suppose that (d) holds. Then $z_F(A_I(G)) \neq 0$ by lemma 8.1.4. It contradicts that $A_I(G) = (\sigma(e) - z_F)a(G)$. We thus get $A_I(G) = 0$, ie. $a(G) = A_{\mathcal{F}}(G)$. If $a(G) = A_{\mathcal{F}}(G)$, then we have $a_0(G) = A_{\mathcal{F}}(G)$ as $A_{\mathcal{F}}(G) \subseteq a_0(G)$. Finally, assume that (f) is true. Then G is a [Moore]-group by lemma 3.2.6. $\square$

3.3 A non-commutative analogue of Wendel's theorem for discrete groups

By the last result in the previous section, we see that $\sigma(a(G))$ is not always a group. We will now study the unitary(invertible) part of $\sigma(a(G))$. Recall that the following definitions:

A unitary representation of G is *completely reducible* if it can be written as a direct sum of irreducibles. A locally compact group G is called a *[AR]-group* if $A(G)$ has RNP. It is proved that G is an [AR]-group if and only if its left regular representation is completely reducible. (See [36] for more details.)

Theorem 3.3.1. Let G be an [AR]-group. Then $\sigma_u(a(G))$ and $\sigma_{inv}(a(G))$ are topologically isomorphic to G.

Proof. We prove this proposition for $\sigma_u(a(G))$. The case for $\sigma_{inv}(a(G))$ is similar. Define $\phi : G \to \sigma_u(a(G))$ by $x \mapsto m_x$ where $m_x(u) = u(x)$. Clearly, ϕ is continuous. Since G^* separates points of G(see remark 3.2.3), the map ϕ is injective. By assumption, $A(G) \subseteq a(G)$. Let $m \in \sigma_u(a(G))$. Then $m|_{A(G)} \neq 0$ by lemma 8.1.4. Therefore, $m|_{A(G)} \in \sigma(A(G))$. Let $u \in A(G)$ and $v \in a(G)$. Note that $A(G)$ is an ideal of $a(G)$. There exists $x_0 \in G$ such that

$$m(u)m(v) = m(uv) = u(x_0)v(x_0).$$

Pick $u_0 \in A(G)$ such that $u_0(x_0) \neq 0$. We conclude that $m(v) = v(x_0)$. Hence, ϕ is surjective. The continuity of the inverse of ϕ follows from the facts that $A(G) \subseteq a(G)$ and $\sigma(A(G))$ is topologically isomorphic to G. □

If G is a discrete group, then $l^1(G) = L^1(G)$ is a total invariant of G by Wendel's theorem(See [39]). We have the following non-commutative analogue of this observation.

Corollary 3.3.2. Let G_1 and G_2 be locally compact groups such that $A(G_1)$ and $A(G_2)$ has RNP(i.e. G_1, G_2 are [AR]-groups). The following conditions are equivalent:

(a) G_1 and G_2 are topologically isomorphic.

(b) $a(G_1)$ and $a(G_2)$ are isometrically isomorphic.

(c) $\sigma_u(G_1)$ and $\sigma_u(G_2)$ are topologically isomorphic.

(d) $\sigma_{inv}(G_1)$ and $\sigma_{inv}(G_2)$ are topologically isomorphic.

Proof. It follows form corollary 3.1.11 and theorem 3.3.1. □

Remark.

(a) The product discussed in proposition 3.1.6 is motivated by [23, Section 5].

(b) The proof of theorem 3.1.10 is a generalization of [37, Theorem 2] and the proof of it is inspired by [23] [Theorem 5.8] and [37, Theorem 2].

(c) Part of the proof of theorem 3.2.7 is inspired by the proof of [37, Lemma of Theorem 2, p. 27].

(d) The proof of theorem 3.3.1 follows an idea in [38, Theorem 2].

Chapter 4

Translation invariant means in $VN(G)$ which are not topological invariant

In this chapter, we will study translation invariant means in $VN(G)$ which are not topological invariant. We begin with the definition of translation operators on $VN(G)$. For any $g^* \in G^*$, the operator $L_{g^*} : A(G) \to A(G)$, $f \mapsto g^* f$ is called *the translation operator of* $A(G)$ *by* g^*. The Banach adjoint of L_{g^*}, $L^t_{g^*} : VN(G) \to VN(G)$, is called *the translation operator of* $VN(G)$ *by* g^*. In this case, we write $g^* \cdot T = L^t_{g^*}(T)$ for any $T \in VN(G)$. Moreover, a subset $E \subseteq A(G)(F \subseteq VN(G))$ is said to be *G^*-invariant* if $g^* E \subseteq E$ for any $g^* \in G^*$(resp. $g^* \cdot F \subseteq F$ for any $g^* \in G^*$).

Let E be a subspace of $VN(G)$. E is said to be *invariant* if E is topological invariant and G^*-invariant.

Note that $C^*_r(G)$, $AP(\hat{G})$, $WAP(\hat{G})$ and $UCB(\hat{G})$ are invariant subspace of $VN(G)$.

Write $G^*_{\mathcal{F}} = \{x \mapsto \langle \pi(x)\xi, \xi\rangle : \pi \in \hat{G}_{\mathcal{F}}, \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$.

Let E be an invariant subspace of $VN(G)$ such that it is closed under involution and contains $\lambda_2(e)$. Let m be a linear functional of E such that $m(\lambda_2(e)) = 1$. Then

(a) m is said to be *a topological invariant mean* if $m(\phi \cdot T) = m(T)$ for any $\phi \in A(G) \cap P_1(G)$, $T \in E$.

(b) m is said to be *a translation invariant mean* if $m(g^* \cdot T) = m(T)$ for any $g^* \in G^*$, $T \in E$.

(c) m is said to be *a $\mathcal{F}$-translation invariant mean* if $m(g^* \cdot T) = m(T)$ for any $g^* \in G^*_{\mathcal{F}}$, $T \in E$.

4.1 Main results

Let $IM(\hat{G})$, $FIM(\hat{G})$ and $TIM(\hat{G})$ be the sets of all translation invariant means, the sets of all $\mathcal{F}$-translation invariant means and the set of all topological invariant means on $VN(G)$, respectively. If G is abelian, it is proved by Rudin that G is compact if and only if $FIM(\hat{G}) = IM(\hat{G}) = TIM(\hat{G})$. In fact, if G is an locally compact group, then $FIM(\hat{G}) \supseteq IM(\hat{G}) \supseteq TIM(\hat{G})$. For any $m \in IM(\hat{G})$, define

$$B_m(G) = \{u \in B(G) : m(u \cdot T) = u(e)m(T) \text{ for any } T \in VN(G)\}$$

Then it is easy to see that $B_m(G)$ is a closed subalgebra of $B(G)$ containing $A_{\mathcal{F}}(G)$, $m \in IM(\hat{G})$ if and only if $a(G) \subseteq B_m(G)$ and $m \in TIM(\hat{G})$ if and only if $A(G) \subseteq B_m(G)$. Put $B_{IM}(G) = \bigcap_{m \in IM(\hat{G})} B_m(G)$. Then $B_{IM}(G)$ is a closed subalgebra of $B(G)$ containing $a(G)$ and $IM(\hat{G}) = TIM(\hat{G})$ if and only if $A(G) \subseteq B_{IM}(G)$.

Proposition 4.1.1. Let G be a locally compact group. Then the following statements hold:

(a) If $G \in [AR]$(i.e. $A(G)$ has RNP.), then $IM(\hat{G}) = TIM(\hat{G})$.

(b) If G is compact, then $FIM(\hat{G}) = TIM(\hat{G})$.

(c) If $G \in [Moore]$, then $IM(\hat{G}) = FIM(\hat{G})$.

Proof. By [5, Theorem 3], we have $A(G) \subseteq a_0(G) \subseteq B_{IM}(G)$. Hence, $IM(\hat{G}) = TIM(\hat{G})$. The rest is clear. □

Example. If G is the "ax+b" group or Fell's group, then $IM(\hat{G}) = TIM(\hat{G})$. Therefore, unlike the abelian case, there is a non-compact group G such that $IM(\hat{G}) = TIM(\hat{G})$.

If H is closed subgroup of G, then $A_{\mathcal{F}}(G)|_H \subseteq A_{\mathcal{F}}(H)$.

Let G be a locally compact group. Suppose that H is a closed subgroup of G. Let $\Psi : A(G) \to A(H)$ be the restriction map, that is, $u \mapsto u|_H$.

Lemma 4.1.2. Let $\phi \in B(H), T \in VN(H)$ and $\psi \in B(G)$ such that $\psi|_H = \phi$. Then we have $\Psi^*(\phi \cdot T) = \psi \cdot \Psi^*(T)$.

Proof. Let $\phi \in B(H)$, $T \in VN(H)$ and $\psi \in B(G)$ such that $\psi|_H = \phi$. Then, for each $u \in A(G)$, we have

$$\langle \Psi^*(\phi \cdot T), u \rangle = \langle \phi \cdot T, \Psi(u) \rangle = \langle \phi \cdot T, u|_H \rangle$$

$$= \langle T, \phi u|_H \rangle = \langle T, (\psi u)|_H \rangle = \langle T, \Psi(\psi u) \rangle = \langle \Psi^*(T), \psi u \rangle = \langle \psi \cdot \Psi^*(T), u \rangle$$

Therefore,

$$\Psi^*(\phi \cdot T) = \psi \cdot \Psi^*(T).$$

□

Theorem 4.1.3. We have $\Psi^{**}(FIM(\hat{G})) \supseteq FIM(\hat{H})$ and $\Psi^{**}(TIM(\hat{G})) = TIM(\hat{H})$.

Proof. Let $m \in FIM(\hat{H})$. Put $K = \{M \in VN(G)^*_+ : \Psi^{**}(M) = m\}$. Since $A(H) \cap P(H)$ is weak*-dense in $VN(H)^*_+$, there is a net $(m_\alpha) \subseteq A(H) \cap P(H)$ such that $m_\alpha \to^{w*} m$. Also, note that $\Psi(A(G) \cap P(G)) = A(H) \cap P(H)$. For each α, there exists $M_\alpha \in A(G) \cap P(G)$ such that $\Psi(M_\alpha) = m_\alpha$. By passing to a subnet, we may assume that $M_\alpha \to^{w*} M$ where $M \in VN(G)^*_+$. Then $m_\alpha = \Psi(M_\alpha) \to^{w*} \Psi^{**}(M)$. Therefore, $\Psi^{**}(M) = m$, whence K is non-empty. It is easy to check that K is a weak* compact convex subset of $VN(G)^*$. For any $g^* \in G^*_{\mathcal{F}}$, define $T_{g^*} : K \to K$ by $T_{g^*}(M) = g^* \cdot M$ where $\langle g^* \cdot M, T\rangle = \langle M, g^* \cdot T\rangle$. We need to show that T_{g^*} is well-defined. In fact, for any $T \in VN(H)$, we have, by using lemma 4.1.2,

$$\langle \Psi^{**}(g^* \cdot M), T\rangle = \langle g^* \cdot M, \Psi^*(T)\rangle = \langle M, g^* \cdot \Psi^*(T)\rangle$$

$$= \langle M, \Psi^*(g^*|_H \cdot T)\rangle = \langle \Psi^{**}(M), g^*|_H \cdot T\rangle = g^*(e)\langle \Psi^{**}(M), T\rangle = \langle m, T\rangle.$$

where the second last equality follows from the fact that $A_{\mathcal{F}}(G)|_H \subseteq A_{\mathcal{F}}(H)$. Thus, $\{T_{g^*} : g^* \in G^*_{\mathcal{F}}\}$ is a commuting family of weak*-weak*-continuous affine maps from K to K. Therefore, by Markov-Kakutani fixed point theorem, there is a element $M_0 \in K$ such that $M_0 = g^* \cdot M_0$. Hence, $\Psi^{**}(IM(\hat{G})) \supseteq IM(\hat{H})$. The last equality can be proved similarly. □

Put $B_{FIM}(G) = \bigcap_{m \in FIM(\hat{G})} B_m(G)$. Then $B_{FIM}(G)$ is a closed subalgebra of $B(G)$ containing $A_{\mathcal{F}}(G)$ and $FIM(\hat{G}) = TIM(\hat{G})$ if and only if $A(G) \subseteq B_{FIM}(G)$.

Corollary 4.1.4. Let G be a locally compact group and H a closed subgroup of G.

(a) $B_{FIM}(G)|_H \subseteq B_{FIM}(H)$.

(b) If $FIM(\hat{G}) = TIM(\hat{G})$, then $FIM(\hat{H}) = TIM(\hat{H})$.

(c) $\text{Card}(FIM(\hat{G}) \setminus TIM(\hat{G})) \geq \text{Card}(FIM(\hat{H}) \setminus TIM(\hat{H}))$.

Corollary 4.1.5. If G has a non-compact closed abelian subgroup, then $FIM(\hat{G}) \neq TIM(\hat{G})$.

Proof. If H is a closed non-compact abelian subgroup of G, then $FIM(\hat{H}) \neq TIM(\hat{H})$. □

Let G be a locally compact group. Then G is called a [SIN]-group if it has a base for the neighborhood system at the identity of G consisting of compact neighborhoods which are invariant under all inner automorphisms of G.

A C*-algebra A is said to be CCR if $\pi(f)$ is a compact operator for every $f \in A$ and irreducible *-representation π of A. G is called a [CCR]-group if $C^*(G)$ is CCR.

A unitary *-representation π of G is primary if the center of $C(\pi) = \{T \in B(\mathcal{H}_\pi) : T\pi(x) = \pi(x)T \text{ for any } x \in G\}$ consists of scalar multiples of I. G is said to be a [Type I]-group if every primary representation of G is a direct sum of copies of some irreducible representations.

For more results of [SIN], [CCR] and [Type I]-groups, we refer the readers to [30].

Corollary 4.1.6. Let G be a [SIN]-group and $FIM(\hat{G}) = TIM(\hat{G})$. Then every closed connected subgroup of G is compact.

Proof. Let H be a connected [Moore]-group. We have $H = V \times K$ where V is a vector group and K is a compact group(See [30, 12.6.6]). By the first statement, it follows that $IM(\hat{H}) = TIM(\hat{H})$ and V is therefore trivial. Hence, H is compact. □

If G has extension property and H is closed subgroup of G, then $H^* \subseteq G^*|_H$(See [17, Proposition 2, p. 275]). Therefore, $a_0(H) \subseteq a_0(G)|_H$.

Lemma 4.1.7. Let G be a [Moore]-group and H a closed subgroup of G. Then $a_0(H) = a_0(G)|_H$.

Proof. Every [Moore]-group is a [SIN]-group, so it has extension property. If $g^* \in G^*$, then $g^*(x) = \langle \pi(x)\epsilon, \epsilon\rangle$ where $\pi \in \hat{G}$ and $\epsilon \in \mathcal{H}_\pi, \|\epsilon\| = 1$. Since G is a [Moore]-group, it follows that $\dim(\pi|_H) = \dim(\pi) < \infty$. Therefore, $g^*|_H \in A_{\mathcal{F}}(H) \subseteq a_0(H)$. □

Lemma 4.1.8. Let G be a locally compact group and H an open subgroup. Let π be a unitary representation of G. For any $f \in L^1(H)$, define $\dot{f} \in L^1(G)$ by $\dot{f}(x) = f(x)$ if $x \in G$ and $\dot{f}(x) = 0$ if $x \neq G$. Then $\pi(\dot{f}) = \pi|_H(f)$ for any $f \in L^1(H)$ and $\|\dot{f}\|_{C^*(G)} = \|f\|_{C^*(H)}$ for any $f \in L^1(H)$. Hence, the map $L^1(H) \to L^1(G)$, $f \mapsto \dot{f}$ extends to a C*-algebra monomorphism $\Phi : C^*(H) \to C^*(G)$.

Proof. Let $\xi, \eta \in \mathcal{H}_\pi$. Then we have

$$\langle \pi(\dot{f})\xi, \eta\rangle = \int_G \dot{f}(x)\langle \pi(x)\xi, \eta\rangle dx = \int_H f(x)\langle \pi|_H(x)\xi, \eta\rangle dx = \langle \pi|_H(f)\xi, \eta\rangle$$

Therefore, $\pi(\dot{f}) = \pi|_H(f)$ for any $f \in L^1(H)$. For the last statement, note that every unitary representation of H can be induced to a unitary representation of G. Hence,

$$\|\dot{f}\|_{C^*(G)} = \sup\{\|\pi(\dot{f})\| : \pi \in \Sigma_G\}$$

$$= \sup\{\|\pi|_H(f)\| : \pi \in \Sigma_G\} = \sup\{\|\pi(f)\| : \pi \in \Sigma_H\} = \|f\|_{C^*(H)}$$

□

Proposition 4.1.9. Let G be a [CCR]-group and H an open subgroup of G. Then $a_0(H) = a_0(G)|_H$.

Proof. Since $G \in [CCR]$, we have $\pi(C^*(G)) \subseteq \mathcal{K}(\mathcal{H}_\pi)$ for any irreducible representation π. Thus, $\pi|_H(C^*(H)) = \Phi(C^*(H)) \subseteq \mathcal{K}(\mathcal{H}_\pi)$ where Φ is defined as in lemma 4.1.8. By [9, Proposition 5.4.13], $\pi|_H$ is a direct sum of irreducible representations of H. Hence, $a_0(H) \supseteq a_0(G)|_H$. □

Theorem 4.1.10. Let G be a locally compact group. Suppose that H is a closed subgroup of G. Then the following holds:

(a) If $a_0(H) \subseteq a_0(G)|_H$, then $\Psi^{**}(IM(\hat{G})) \subseteq IM(\hat{H})$.

(b) If $a_0(G)|_H \subseteq a_0(H)$, then $\Psi^{**}(IM(\hat{G})) \supseteq IM(\hat{H})$.

Proof. Let $M \in IM(\hat{G})$, $\phi \in a_0(H)$, $T \in VN(H)$ and $\psi \in a_0(G)$ such that $\psi|_H = \phi$. Then

$$\langle \Psi^{**}(M), \phi \cdot T\rangle = \langle M, \Psi^*(\phi \cdot T)\rangle$$

$$= \langle M, \psi \cdot \Psi^*(T)\rangle = \psi(e)\langle M, \Psi^*(T)\rangle = \phi(e)\langle \Psi^{**}(M), T\rangle.$$

Therefore, $\Psi^{**}(M)$ is translation invariant. Note that $\Psi^*(\lambda_H(e)) = \lambda_G(e)$. It follows that $\langle \Psi^{**}(M), \lambda_H(e)\rangle = \langle M, \lambda_G(e)\rangle = 1$. The positivity of $\Psi^{**}(M)$ is clear. Hence, $\Psi^{**}(M) \in IM(\hat{H})$. Conversely, let $m \in IM(\hat{H})$. Put $K = \{M \in VN(G)^*_+ : \Psi^{**}(M) = m\}$. Since $A(H) \cap P(H)$ is weak*-dense in $VN(H)^*_+$, there is a net $(m_\alpha) \subseteq A(H) \cap P(H)$ such that $m_\alpha \to^{w*} m$. Also, note that $\Psi(A(G) \cap P(G)) = A(H) \cap P(H)$. For each α, there exists $M_\alpha \in A(G) \cap P(G)$ such that $\Psi(M_\alpha) = m_\alpha$. By passing to a subnet, we may assume that $M_\alpha \to^{w*} M$ where $M \in VN(G)^*_+$. Then $m_\alpha = \Psi(M_\alpha) \to^{w*} \Psi^{**}(M)$. Therefore, $\Psi^{**}(M) = m$, whence K is non-empty. It is easy to check that K is a weak* compact convex subset of $VN(G)^*$. For any $g^* \in G^*$, define $T_{g^*} : K \to K$ by $T_{g^*}(M) = g^* \cdot M$ where $\langle g^* \cdot M, T\rangle = \langle M, g^* \cdot T\rangle$. We need to show that T_{g^*} is well-defined. In fact, for any $T \in VN(H)$, we have, by using lemma 4.1.2,

$$\langle \Psi^{**}(g^* \cdot M), T\rangle = \langle g^* \cdot M, \Psi^*(T)\rangle = \langle M, g^* \cdot \Psi^*(T)\rangle$$

$$= \langle M, \Psi^*(g^*|_H \cdot T)\rangle = \langle \Psi^{**}(M), g^*|_H \cdot T\rangle = g^*(e)\langle \Psi^{**}(M), T\rangle = \langle m, T\rangle$$

where the second last equality follows from the assumption that $a_0(H) = a_0(G)|_H$. Thus, $\{T_{g^*} : g^* \in G^*\}$ is a commuting family of weak*-weak*-continuous affine maps from K to K. Therefore, by Markov-Kakutani fixed point theorem, there is a element $M_0 \in K$ such that $M_0 = g^* \cdot M_0$. Hence, $\Psi^{**}(IM(\hat{G})) \supseteq IM(\hat{H})$. □

Theorem 4.1.11. Let G be a locally compact group. Suppose that H is a closed subgroup of G such that $a_0(G)|_H \subseteq a_0(H)$.

(a) $B_{IM}(G)|_H \subseteq B_{IM}(H)$.

(b) If $IM(\hat{G}) = TIM(\hat{G})$, then $IM(\hat{H}) = TIM(\hat{H})$.

(c) If $IM(\hat{G}) = FIM(\hat{G})$, then $IM(\hat{H}) = FIM(\hat{H})$.

(d) $\mathrm{Card}(IM(\hat{G}) \setminus TIM(\hat{G})) \geq \mathrm{Card}(IM(\hat{H}) \setminus TIM(\hat{H}))$.

Proof. (a): Let $u \in B_{IM}(G)$. Then $u|_H \in B(H)$. For any $m \in IM(\hat{H})$ and $S \in VN(H)$, we have

$$\langle \Psi^{**}(m), u|_H \cdot S\rangle = \langle m, \Psi^*(u|_H \cdot S)\rangle$$

$$= \langle m, u \cdot \Psi^*(S)\rangle = u(e)\langle m, \Psi^*(S)\rangle = u|_H(e)\langle \Psi^{**}(m), S\rangle$$

Therefore, $B_{IM}(G)|_H \subseteq B_{IM}(H)$.
(b): If $IM(\hat{G}) = TIM(\hat{G})$, then $A(G) \subseteq B_{IM}(G)$. Consequently, $A(H) = A(G)|_H \subseteq B_{IM}(G)|_H$. This implies $A(H) \subseteq B_{IM}(H)$. Hence, $IM(\hat{H}) = TIM(\hat{H})$.
(c),(d): Straightforward. □

Clearly, if $TIM(\hat{G}) = FIM(\hat{G})$, then $TIM(\hat{G}) = IM(\hat{G})$.

Corollary 4.1.12. Let G be a [Moore]-group and $IM(\hat{G}) = TIM(\hat{G})$. Then $IM(\hat{H}) = TIM(\hat{H})$ for any closed subgroup H of G. Consequently, every connected subgroup H of G is compact. Furthermore, if G has a non-compact abelian closed subgroup, then $IM(\hat{G}) \neq TIM(\hat{G})$.

Proof. The first statement follows from lemma 4.1.7 and theorem 4.1.11. Since H is a connected [Moore]-group, we have $H = V \times K$ where V is a vector group and K is a compact group(See [30, 12.6.6]). By the first statement, it follows that $IM(\hat{H}) = TIM(\hat{H})$ and V therefore is trivial. Hence, H is compact. The last statement follows clear from the first statement. □

Corollary 4.1.13. Let G be a [CCR]-group and $IM(\hat{G}) = TIM(\hat{G})$. Then $IM(\hat{H}) = TIM(\hat{H})$ for any open subgroup H of G. Furthermore, if G has a non-compact abelian open subgroup, then $IM(\hat{G}) \neq TIM(\hat{G})$.

G is called a *central group* if the quotient group G/Z is compact where Z is the center of G. G is called an *almost connected* group if the quotient group G/G_e is compact where G_e is the connected component of e in G. For more details, see [30, Chapter 12] for reference.

Theorem 4.1.14. Let G be a locally compact group. Suppose one of the following conditions holds:

(a) G is an almost connected [Moore]-group.

(b) G has an abelian closed subgroup of finite index.

(c) G is a central group.

(d) $G = A \times K$ where A is a locally compact abelian group and K is compact group.

Then G is compact if and only if $IM(\hat{G}) = TIM(\hat{G})$(which is equivalent to say that $FIM(\hat{G}) = TIM(\hat{G})$). Moreover, if G is non-compact, then $\mathrm{Card}(IM(\hat{G}) \setminus TIM(\hat{G})) \geq 2^c$.

Proof. If G is an almost connected [Moore]-group, then $G = V \times_\eta K$ where K is compact, V is a vector group and $\eta(K)$ is finite(See [30, 12.6.6]). If $IM(\hat{G}) = TIM(\hat{G})$, then $IM(\hat{V}) = TIM(\hat{V})$ by theorem 4.1.11, which implies that V is trivial. Consequently, G is compact. Suppose that (b) holds. Then G is a [Moore]-group. Let H be such an abelian subgroup of G. If $IM(\hat{G}) = TIM(\hat{G})$, then $IM(\hat{H}) = TIM(\hat{H})$ by theorem 4.1.11. Consequently, H is compact. Therefore, G is compact since H is of finite index. Now, assume that G is central(i.e. $G/Z(G)$ is compact). But $Z(G)$ is compact by theorem 4.1.11. Hence, G is compact. Finally, note that if G satisfies (d), then G is central. □

Corollary 4.1.15. Let G be a connected [SIN]-group. Then G is compact if and only if $IM(\hat{G}) = TIM(\hat{G})$.

Let $u : G_1 \to \mathbb{C}$ and $v : G_2 \to \mathbb{C}$ be functions. Define $u \otimes v : G_1 \times G_2 \to \mathbb{C}$ by $u \otimes v(x, y) = u(x)v(y)$.

Lemma 4.1.16. Let G_1 be a [Type 1]-group and G_2 any locally compact group. Let $G = G_1 \times G_2$. Identity G_1 as $G_1 \times \{e_2\}$ and G_1 as $\{e_1\} \times G_2$. Then $a_0(G_1) = a_0(G)|_{G_1}$ and $a_0(G_2) = a_0(G)|_{G_2}$.

Proof. Note that $\hat{G} \longrightarrow \hat{G}_1 \times \hat{G}_2$, $\pi \mapsto \pi_1 \otimes \pi_2$ is a bijection. We have

$$a_0(G) = \{(x, y) \mapsto \langle \pi_1 \otimes \pi_2(x, y)\xi, \eta\rangle : \xi, \eta \in \mathcal{H}_{\pi_1 \otimes \pi_2}, \pi_1 \in \hat{G}_1, \pi_2 \in \hat{G}_2\}$$

It follows that

$$a_0(G) = \overline{\mathrm{span}(a_0(G_1) \otimes a_0(G_2))}^{\|\cdot\|_{B(G)}}$$

□

Proposition 4.1.17. Let G_1 be a [Type 1]-group and G_2 any locally compact group. Suppose that $IM(\widehat{G_1 \times G_2}) = TIM(\widehat{G_1 \times G_2})$. Then we have $IM(\hat{G}_i) = TIM(\hat{G}_i)$ for all $i = 1, 2$.

Corollary 4.1.18. Let G_1 be a non-compact locally compact abelian group and G_2 any locally compact group. Then $IM(\widehat{G_1 \times G_2}) \neq TIM(\widehat{G_1 \times G_2})$.

4.2 Some applications

Remarks. Let G be the "ax+b"-group, $H = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a > 0\}$ and $N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\}$

(a) Since $A(G)$ has RNP(i.e. G is an [AR]-group.) and $FIM(\hat{H}) = IM(\hat{H}) \neq TIM(\hat{H})$, we have $FIM(\hat{G}) \neq IM(\hat{G}) = TIM(\hat{G})$

(b) Note that $FIM(\hat{H}) = IM(\hat{H}) \neq TIM(\hat{H})$ and $G/N \cong H$. We have $FIM(\widehat{G/N}) = IM(\widehat{G/N}) \neq TIM(\widehat{G/N})$.

Question: Let H be a closed subgroup of G, and let $\pi \in \hat{G}$. Is $\pi|_H$ completely reducible in general?

We have the following observation:

Let G be a locally compact group such that $TIM(\hat{G}) = IM(\hat{G})$. If H is closed subgroup of G such that $TIM(\hat{H}) \neq IM(\hat{H})$, then $a_0(G)|_H \subsetneq a_0(H)$. Hence, there exists a irreducible representation π of G such that $\pi|_H$ is not completely reducible.

Recall that the infinite dimensional representations of the "ax+b"-group are given by:

$$[\pi_+(a, b)g](s) = a^{\frac{1}{2}} e^{2\pi i b s} g(as) \quad (a > 0, b \in \mathbb{R}, g \in L^2((0, \infty)), ds));$$

$$[\pi_-(a,b)g](s) = a^{\frac{1}{2}}e^{2\pi ibs}g(as) \quad (a>0, b\in\mathbb{R}, g\in L^2((-\infty,0)), ds)).$$

The following gives a negative answer to our question. In its proof we will use theorem 4.1.11 and Granirer-Rudin's result.

Proposition 4.2.1. Let G be the "ax+b"-group, and H be the subgroup of G defined in the remarks. Then $a_0(G)|_H \subsetneq a_0(H)$. Moreover, $\pi_+|_H$ and $\pi_-|_H$ are not completely reducible.

Proof. Since $\hat{G}$ consists of all characters, π_+ and π_-, we conclude that $\pi_+|_H$ and $\pi_-|_H$ cannot be both written as direct sums of irreducible representations of H. Let $U : L^2((0,\infty), ds) \to L^2((-\infty,0), ds)$, $Ug = \tilde{g}$ where $\tilde{g}(x) = g(-x)$. By direct calculation, we may prove that $U\pi_+(a,b) = \pi_-(a,-b)U$ for any $a>0, b\in\mathbb{R}$. Thus $U\pi_+|_H = \pi_-|_H U$, whence they are equivalent. Therefore, both of them cannot be written as direct sums of irreducible representations of H. □

Proposition 4.2.2. Let G be a locally compact group, and Z be the center of G. Then $a_0(G)|_Z \subseteq a_0(Z)$.

Proof. If π is an irreducible representation of G, then $\pi|_Z$ is a multiple of ρ where ρ is an irreducible representation of Z. In particular, ρ is completely irreducible. □

Corollary 4.2.3. Let G be a locally compact group. If the center of G is non-compact, then $TIM(\hat{G}) \neq IM(\hat{G})$.

Proof. Suppose that $TIM(\hat{G}) = IM(\hat{G})$. Then by proposition 4.2.2 and theorem 4.1.11, $TIM(\hat{Z}) = IM(\hat{Z})$ where Z is the center of G. Therefore, by Granirer-Rudin's theorem(See [14] and [23]), we conclude that Z is compact. □

We obtain a lot of examples by using the above corollary. Say, if G is $GL_n(\mathbb{R})$ or the Heisenberg group, then $TIM(\hat{G}) \neq IM(\hat{G})$.

The following corollary should be well known. However, we can also prove it independently by just using the results proved in this section.

Corollary 4.2.4. The center of an [AR]-group is always compact. In particular, if $A(G)$ has RNP, then $A(Z)$ has RNP where Z is the center of G.

Chapter 5

General properties of translation invariant means in $VN(G)$

In this chapter we will study the non-commutative analogues of classical results about translation invariant means on $L^\infty(G)$. We start by recalling the following definition. Recall that a net (μ_α) in $M(G)$ is said to converge *strictly* to μ if

$$\|g * (\mu_\alpha - \mu)\| + \|(\mu_\alpha - \mu) * g\| \to 0 \text{ for any } g \in L^1(G)$$

Remark. $co(\delta_G)$ is strictly dense in $M(G)_1^+$, and hence $l^1(G)$ is also strictly dense in $M(G)$.

Inspired by above classical definition for $M(G)$, we may define the strict topology of $B(G)$ analogously: A net (f_α) in $B(G)$ is said to converge strictly to f if

$$\|(f_\alpha - f) \cdot g\| \to 0 \text{ for any } g \in A(G)$$

5.1 Strict topology on $B(G)$

The following lemma is an analogue of the remark above, which will be particularly useful in sequel.

Lemma 5.1.1. $co(G^*)$ is strictly dense in $P_1(G)$, and hence span(G^*) is strictly dense in $B(G)$.

Proof. Note that $co(G^*) = co(\mathcal{E}(P_1(G)))$ is w*-dense in $P_1(G)$ and the strict topology coincides with the w*-topology in $B(G)_1$ by [16, Theorem B2]. Therefore, $co(G^*)$ is strictly dense in $P_1(G)$. The rest is now trivial. $\square$

Here is an application of the previous lemma. This is an analogue of [12, Proposition 2.43], which also motivates the author to study G^*.

Proposition 5.1.2. Let I be a closed subalgebra of $A(G)$. If I is G^*-invariant, then I is an ideal of $A(G)$. Suppose, in addition, that $u \in \overline{uA(G)}$ for any $u \in A(G)$. Then I is G^*-invariant if and only if I is an ideal of $A(G)$.

Proof. Let $\phi \in A(G) \cap P_1(G)$, $f \in I$, and let (ϕ_α) be a net in $co(G^*)$ such that $\phi_\alpha \longrightarrow \phi$ in the strict topology of $B(G)$. Note that $\phi_\alpha \cdot f \in I$ for each α. Since I is norm-closed, it follows that $\phi \cdot f \in I$. Conversely, let $f \in I$ and $g^* \in G^*$. Then $g^* f \in A(G)$. By assumption, there exists a net $(e_\alpha) \subseteq A(G)$ such that $(g^* f)e_\alpha \to g^* f$. However, $(g^* f)e_\alpha = (g^* e_\alpha)f \in A(G)I \subseteq I$. Therefore, $g^* f \in I$. $\square$

The following lemma should be well-known, and we give the proof of it below for the sake of completeness.

Lemma 5.1.3. Let X be a locally compact Hausdorff space. Then $C_0(X)$, the space consisting of all continuous functions on X which vanish at infinity, has an n-dimensional ideal if and only if X has exactly n distinct discrete points. Moreover, if such an n-dimensional ideal exists, then it is the linear span of $\delta_{x_1}, \delta_{x_2}, ..., \delta_{x_n}$ where $x_1, x_2, ..., x_n \in X$.

Proof. Let $x_1, ..., x_n$ be n distinct discrete points in X. Let $I := \{f \in C_0(X) : f(x_i) \neq 0 \ \ for \ \ some \ \ i \in \{1, 2, ..., n\}\}$. Clearly I is a closed ideal in A. Since $x_1, ..., x_n$ are discrete, $\chi_{x_1}, ..., \chi_{x_n}$ are continuous functions on X and form a basis for I. Conversely, let I be an n-dimensional ideal in $C_0(X)$. Define $\Omega := \{x \in G : f(x) \neq 0 \ \ for \ \ some \ \ f \in I\}$. Clearly, $|\Omega| \geq n$. It is left to show that $|\Omega| \leq n$. Suppose that $x_1, ..., x_n, x_{n+1} \in \Omega$ are distinct. For each $i \in \{1, ..., n, n+1\}$, by Urysohn's lemma, there exist $f_i \in I$ and $g_i \in C_0(G)$ such that

$$f_i(x_i) \neq 0 \ \ and \ \ g_i(x_j) = \delta_{ij} \quad 1 \leq i, j \leq n+1$$

Let $c_i \in C$ such that $\sum_{i=1}^{n+1} c_i g_i f_i = 0$. Then for each $j \in \{1, ..., n, n+1\}$,

$$\sum_{i=1}^{n+1} c_i g_i f_i(x_j) = c_j f_j(x_j) = 0 \quad \Rightarrow \quad c_j = 0$$

Therefore, $\{g_i f_i\}_{i=1}^{n+1}$ is linear independent subset of I. This leads a contradiction.

The proof of the latter part of this lemma is clear. $\square$

The following proposition is about translation-invariant elements of $A(G)$ and $B(G)$, which will be useful in next chpater.

Proposition 5.1.4. Let G be a locally compact group. Then the following statements are equivalent:

(a) G is discrete.

(b) there exists a non-zero $f \in A(G)$ such that f is G^*-invariant.

(c) there exists a non-zero $f \in B(G)$ such that f is G^*-invariant.

Moreover, if such f exists, then $f = c\delta_e$ for some $c \in \mathbb{C}$.

Proof. "(a) $\Rightarrow$ (c)" is clear. If (c) holds, choose $u \in A(G)$ such that $uf \neq 0$. Then $\{uf\}$ is a G^*-invariant subset in $A(G)$. Now, suppose that (b) holds. Let $I = \mathbb{C}f \subseteq A(G)$. Let $h \in A(G)$ and let $\{\sum c_\alpha g_\alpha^*\}$ be a net in $span(G^*)$ which is converging to h strictly. Then

$$h \cdot f = \lim_\alpha \sum c_\alpha (g_\alpha^* \cdot f) = (\lim_\alpha \sum c_\alpha) f$$

so, I is a one dimensional ideal in $A(G)$, which implies that G is discrete.
For the last statement, without loss of generality, we may assume that $f \in A(G)$. Now, I is also an ideal in $C_0(G)$. So, $I = \langle \delta_x \rangle$ for some $x \in G$ by lemma 5.1.3. It follows that $f = c\delta_x$ for some $c \in \mathbb{C}$. Without loss of generality, assume that $c = 1$. For any $y \in G$, $\pi \in \hat{G}$, $\xi \in \mathcal{H}_\pi, \|\xi\| = 1$, we have $\langle \pi(y)\xi, \xi \rangle \delta_x(y) = \delta_x(y)$. Suppose that $x \neq e$. Pick $\pi_0 \in \hat{G}$ such that $\pi_0(x) \neq \pi_0(e)$. Thus, we have $\langle \pi_0(x)\xi, \xi \rangle = 1$. By Cauchy-Schwarz's inequality, it follows that $\pi_0(x)\xi = \xi$. Hence, $\pi_0(x)$ is the identity map, which leads a contradiction. □

Remark. By lemma 5.1.3, it is not hard to see that the following statements are equivalent:

(a) G is discrete.

(b) $A(G)$ has a non-zero finite dimensional ideal.

(c) $C_0(G)$ has a non-zero finite dimensional ideal.

5.2 General properties of translation invariant means

In this section, we discuss the general properties of translation invariant means and $\mathcal{F}$-translation invariant means. These notions were defined in the beginning of chpater 4.

Remark. In fact, $VN(G)$ always has a topological invariant mean(See [32, Theorem 4]). We also notice that the set of all translation invariant means on $VN(G)$ is a w*-compact convex subset in $VN(G)^*$ and $A(G) \cap P_1(G)$ is weak* dense in the set of all means in $VN(G)$(Refer to [15]).

Recall that in chapter 4, we have the following definition: a subspace of $VN(G)$ is said to be *invariant* if it is topological invariant and G^*-invariant.

Proposition 5.2.1. Let E be an invariant closed subspace of $VN(G)$ which is closed under involution and contains $\lambda_2(e)$. Then every topological invariant mean on E is ($\mathcal{F}$-)translation invariant. Hence, for any locally compact group, E has an ($\mathcal{F}$-)translation invariant mean. Furthermore, if G is non-discrete, then $VN(G)$ has uncountably many ($\mathcal{F}$-)translation invariant means.

Proof. Let m be a topological invariant mean on E. For any $g^* \in G^*$, $T \in E$, $\phi \in A(G) \cap P_1(G)$, we have

$$m(g^* \cdot T) = m(\phi \cdot (g^* \cdot T)) = m((\phi \cdot g^*) \cdot T) = m(T)$$

Therefore, m is translation invariant. Note that $G^*_{\mathcal{F}} \subseteq G^*$. The rest follows from the remark above. □

Lemma 5.2.2. The following statements are equivalent:

(a) G is discrete.

(b) There is a bounded linear functional on $C_r^*(G)$ which is translation invariant.

(c) There is a bounded linear functional on $C_r^*(G)$ which is topological invariant.

Proof. "(a) $\Rightarrow$ (c) $\Rightarrow$ (b)" is clear. Let $\phi \in B_r(G)$ such that $\langle \phi, T\rangle = \langle \phi, g^* \cdot T\rangle = \langle g^* \cdot \phi, T\rangle$ for any $g^* \in G^*$, $T \in C_r^*(G)$. Then $g^* \cdot \phi = \phi$. Note that $\phi \in B_r(G) \subseteq B(G)$. So, G is discrete by proposition 8.2.1. □

Proposition 5.2.3. Let G be a non-discrete locally compact group, and let M be a translation invariant mean on $VN(G)$. Then the restriction of M on $C_r^*(G)$ is always zero.

Proof. Let $m = M|_{C_r^*(G)}$. Assume that $m \neq 0$. Clearly, m is positive and translation invariant on $C_r^*(G)$. Therefore, $n = m/\|m\|$ is a translation invariant mean on $C_r^*(G)$, which contradicts to lemma 5.2.2. □

Since all topological means are translation invariant(Proposition 5.2.1), we thus provide another proof of [32, Theorem 12].

Corollary 5.2.4. Let G be a locally compact group. Then G is discrete if and only if there is a translation invariant mean on $VN(G)$ belonging to $A(G) \cap P_1(G)$(or $A(G)$).

Proof. If G is discrete, then $\delta_e \in A(G) \cap P_1(G)$. Hence, $m(T) := \langle \delta_e, T\rangle$ defines a translation invariant mean on $VN(G)$. Conversely, if there is $f \in A(G) \cap P_1(G)$ such that $\langle f, T\rangle = \langle f, g^* \cdot T\rangle$ for any $g^* \in G^*$, $T \in VN(G)$. Then $f = g^* \cdot f$. So, G is discrete. □

Theorem 5.2.5. If $A(G)$ has an approximate identity, then every translation invariant mean on $UCB(\hat{G})$ is topological invariant.

Proof. Let m be a translation invariant mean on $UCB(\hat{G})$, and let $S = u \cdot T \in UCB(\hat{G})$ where $T \in VN(G)$, $u \in A(G)$. As the functional $A(G) \longrightarrow \mathbb{C}$, $f \mapsto m(f \cdot S)$ is continuous, there exists $T_0 \in VN(G)$ such that $m(f \cdot S) = \langle T_0, f \rangle$. Since m is translation invariant, for any $g^* \in G^*$, we have

$$\langle g^* \cdot T_0, f \rangle = \langle T_0, g^* \cdot f \rangle = m(g^* \cdot (f \cdot S)) = m(f \cdot S) = \langle T, f \rangle$$

That is, $g^* \cdot T_0 = T_0$. By lemma 8.2.1, $T_0 = c\lambda_2(e)$ for some constant $c \neq 0$. It follows that $m(f \cdot S) = c$ for any $f \in A(G) \cap P_1(G)$, $S \in A(G) \cdot VN(G)$. By assumption, $A(G)$ has an approximate identity $\{e_\alpha\}$. So, we have

$$m(f \cdot S) = \lim_\alpha m((f \cdot e_\alpha) \cdot S) = \lim_\alpha m(e_\alpha \cdot S) = m(S)$$

However, $A(G) \cdot VN(G)$ is a norm-dense subset of $UCB(\hat{G})$. We hence conclude that m is topological invariant. □

Corollary 5.2.6. If G is a compact group, then every ($\mathcal{F}$-)translation invariant mean on $VN(G)$ is topological invariant.

Proof. Note that G is amenable, $G^*_{\mathcal{F}} = G^*$ and $VN(G) = UCB(\hat{G})$ under the assumption. □

Recall that $WAP(\hat{G})$ is the set of all T in $VN(G)$ for which the operator from $A(G)$ to $VN(G)$ given by $u \mapsto u \cdot T$ is weakly compact is denoted by $WAP(\hat{G})$, the weakly almost periodic functionals in $VN(G)$. It is proved by Granrier[15] that $WAP(\hat{G})$ has a unique topological invariant mean.

Proposition 5.2.7. $WAP(\hat{G})$ has a unique translation invariant mean.

Proof. The proof is the same as that of [32, Theorem 1]. □

Lemma 5.2.8. Let $\phi \in A(G) \cap P_1(G)$. If m is a topological invariant mean on $UCB(\hat{G})$, then m' is a topological invariant mean on $VN(G)$, where m' is given by $m'(T) = m(\phi \cdot T)$. Furthermore, m' is independent of the choice of ϕ.

Proof. Let $T_0 \in VN(G)$. Define $F \in A(G)^*$ by $F(\psi) = m(\psi \cdot T_0)$. Now, for any $\psi \in A(G)$, $\varphi \in A(G) \cap P_1(G)$, we have $F(\varphi \cdot \psi) = m(\varphi \cdot \psi \cdot T_0) = m(\psi \cdot T_0) = F(\psi)$. So, by proposition 6.3.4, $F(\psi) = m(\psi \cdot T_0) = \langle c\lambda_2(e), \psi \rangle = c\psi(e)$. In particular, $m(\varphi \cdot T_0) = c$ for any $\varphi \in A(G) \cap P_1(G)$. Thus, m' is independent of the choice of ϕ. It is routine to check that m' is a topological invariant mean on $VN(G)$. $\square$

Proposition 5.2.9. There is a bijection between the set of topological invariant means on $UCB(\hat{G})$ and the set of topological invariant means on $VN(G)$.

Proof. If m is a topological invariant means on $UCB(\hat{G})$, then for any $T \in UCB(\hat{G})$, $m'|_{UCB(\hat{G})}(T) = m(\phi \cdot T) = m(T)$ where $\phi \in A(G) \cap P_1(G)$. On the other hand, if m is a topological invariant means on $VN(G)$, then for any $T \in VN(G)$, $(M|_{UCB(\hat{G})})'(T) = M|_{UCB(\hat{G})}(\phi \cdot T) = M(T)$ where $\phi \in A(G) \cap P_1(G)$. $\square$

Corollary 5.2.10. Suppose that $A(G)$ has an approximate identity. Then G is discrete if and only if there exists a unique (topological)translation invariant mean on $UCB(\hat{G})$.

Proof. Note that G is discrete if and only if $VN(G)$ has a unique topological invariant mean. (See [29, Theorem 11] and [32, Corollary 4.11]). The result thus follows from the last proposition and theorem 5.2.5. $\square$

Chapter 6

Some general properties on translation operators

The purpose of this chapter is to discuss how the properties of G are related to those of G^*. We will characterize abelian, compact and discrete groups via properties of G^*.

6.1 Characterization of abelian groups

We begin with the following lemma:

Lemma 6.1.1. Let A be a C*-algebra. Then A is commutative if and only if, for any $a \in A$,

$$\|a\| = \sup\{|\langle a, f\rangle| : f \in (A^*)^1_+\}$$

Proof. Suppose that A is non-abelian, there exists $a \in A$ such that $\|a\| = 1$ and $a^2 = 0$. Then for any state f on A, $|f(a)|^2 \leq \sqrt{f(a^*a)f(aa^*)} \leq f(a^*a+aa^*)/2 \leq \|a^*a+aa^*\|/2 = \max(\|a^*a\|, \|aa^*\|)/2$ (since a^*a and aa^* are orthogonal)$= 1/2$. Thus, $|f(a)| \leq 1/\sqrt{2}$ for any state f on A. □

In fact, G is abelian only when G^* has the following extraordinary properties in the non-commutative point of view:

Theorem 6.1.2. Let G be a locally compact group. The following are equivalent:

(a) G is abelian.

(b) Given any $g^* \in G^*$, $\|g^* \cdot \phi\| = \|\phi\|$ for all $\phi \in B(G)$.

(c) Given any $g^* \in G^*$, $\|g^* \cdot \phi\| = \|\phi\|$ for all $\phi \in A(G)$.

(d) For any $g^* \in G^*$, we have $g^* \cdot (T_1 T_2) = (g^* \cdot T_1)(g^* \cdot T_2)$ for any $T_1, T_2 \in VN(G)$.

(e) For any $g^* \in G^*$, we have $\|g^* \cdot T\| = \|T\|$ for any $T \in VN(G)$.

(f) The relative topology of G^* inherited from the norm-topology of $B(G)$ is discrete.

(g) The set of all extreme points of $B(G)_1$ is $\mathbb{T}G^* = \{\lambda g^* : \lambda \in \mathbb{T}, g^* \in G^*\}$.

(h) The weak*-closed convex hull of $\mathbb{T}G^*$ is $B(G)_1$.

(i) $|\langle TT^*, g^*\rangle| = |\langle T, g^*\rangle|^2$ for any $T \in C^*(G)$, $g^* \in G^*$.

(j) $\|T\| = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}$ for any $T \in C^*(G)$.

Proof. If G is abelian, then "(a) $\Rightarrow$ (b) $\Rightarrow$ (c)" is obvious. If (c) holds, for each $g^* \in G^*$, let $L_{g^*} : A(G) \longrightarrow A(G)$, $L_{g^*}(f) = g^* \cdot f$. Then L_{g^*} is clearly a bounded multiplier on $A(G)$. By using a similar idea in the proofs of [31, Lemma 1,2], L_{g^*} is an isometric linear isomorphism(onto). Therefore, $L_{g^*}(S_{A(G)}) = S_{A(G)}$. It follows that $\|g^* \cdot T\| = \|T\|$ for any $T \in VN(G)$. Suppose that (e) holds. Since $g^* \cdot \lambda_2(x) = g^*(x)\lambda_2(x)$ for each $x \in G$, we have

$|g^*(x)| = \|g^* \cdot \lambda_2(x)\| = \|\lambda_2(x)\| = 1$. It follows that $g^* \cdot \bar{g^*} = |(g^*)^2| = 1$. So, G is abelian by the remark above this proposition. Finally, Suppose that (d) holds. For any $f_1, f_2 \in C_c(G)$, we have

$$\lambda_2(g^*(f_1 * f_2)) = g^* \cdot \lambda_2(f_1 * f_2) = g^* \cdot \lambda_2(f_1) \circ g^* \cdot \lambda_2(f_2) = \lambda_2(g^* f_1 * g^* f_2).$$

However, for any $x \in G$,

$$g^*(x) f_1 * f_2(x) = \int_G g^*(x) f_1(y) f_2(y^{-1}x) dy$$

and

$$(g^* f_1) * (g^* f_2)(x) = \int_G g^*(y) f_1(y) g^*(y^{-1}x) f_2(y^{-1}x) dy.$$

Since λ_2 is a faithful representation, it follows that

$$\langle g^*(x), f_1 L_x \breve{f_2} \rangle = \langle g^* L_x \bar{g^*}, f_1 L_x \breve{f_2} \rangle$$

for each $x \in G$ where $\langle \cdot, \cdot \rangle$ denote the dual pair of $L^\infty(G)$ and $L^1(G)$. Since g^* is continuous, $g^*(x) = g^*(y) g^*(y^{-1}x)$ for any $x, y \in G$. In particular, $g^* \cdot \bar{g^*} = 1$. Hence, by the remark again, G is abelian. Finally, if G is abelian, then $G^* = \delta_{\hat{G}}$ and $B(G) = M(\hat{G})$. Since $\|\delta_x - \delta_y\| = 2$ whenever $x, y \in \hat{G}$ and $x \neq y$, the forward direction follows. Conversely, suppose that G is not abelian. Then there exists $\pi \in \hat{G}$ such that dim $\mathcal{H}_\pi > 1$. Let $\eta_1, \eta_2 \in \mathcal{H}_\pi, \|\eta_1\| = \|\eta_2\| = 1$ such that η_1, η_2 are linear independent. Put $\epsilon_n = (\eta_1 + \eta_2/n)/\|\eta_1 + \eta_2/n\|$. Then ϵ_n and η_1 are linearly independent and $\epsilon_n \to \eta_1$. Hence, $\langle \pi(x)\eta_1, \eta_1 \rangle \neq \langle \pi(x)\epsilon_n, \epsilon_n \rangle$ for each $n \in \mathbb{N}$ and $x \in G$. However, for each $f \in C^*(G)$,

$$\|\langle \pi(f)\eta_1, \eta_1 \rangle - \langle \pi(f)\epsilon_n, \epsilon_n \rangle\|$$

$$\leq \|\langle \pi(f)(\eta_1 - \epsilon_n), \eta_1 \rangle + \langle \pi(f)(\epsilon_n - \eta_1), \epsilon_n \rangle\| \leq 2\|f\|_{C^*(G)} \|\eta_1 - \epsilon_n\| \to 0$$

Consequently, the relative topology of G^* inherited from the norm-topology of $B(G)$ is non-discrete. By Krein-Milman theorem, it is easy to see that (a)⇒(g)⇒(h)⇒(j). (a)⇒(i) is also clear. Note that $\|T\| = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}$ for any $T \in C^*(G)_{sa}$(See lemma 5.1.1 and [35, 1.5.4]). If (i) holds, then

$$\|T\| = \|TT^*\|^{1/2} = \sup\{|\langle TT^*, g^*\rangle|^{1/2} : g^* \in G^*\} = \sup\{|\langle T, g^*\rangle| : g^* \in G^*\}.$$

Finally, suppose that (j) holds. Then $\|T\| = \sup\{|\langle T, h\rangle| : h \in \mathrm{co}(G^*)\}$ for any $T \in C^*(G)$. Note that $co(G^*) = co(\mathcal{E}(P_1(G)))$ is w*-dense in $P_1(G)$. It follows that $\|T\| = \sup\{|\langle T, h\rangle| : h \in P_1(G)\}$ for any $T \in C^*(G)$. Consequently, $C^*(G)$ is commutative(by lemma 6.1.1), hence G is abelian. □

6.2 Characterization of compact groups

For any $\pi \in \hat{G}$, write $G^*_\pi = \{\langle \pi(\cdot)\xi, \xi\rangle : \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$.

The following proposition gives a characterization of compact groups by properties of G^*:

Proposition 6.2.1. Let G be a separable group. The following statements are equivalent:

(a) G is compact.

(b) The identity map $id : (G^*, w^*) \mapsto (G^*, \|\cdot\|)$ is continuous(a homeomorphism).

(c) The interior of G^*_π is non-empty for each $\pi \in \hat{G}$.

Proof. "(a) ⇒ (b)": If G is compact, then the w*-topology and the norm-topology coincides on $S_{B(G)}$([16, Corollary 2, p.463]).
"(b) ⇒ (c)": Let $g_0^* \in G^*_\pi$. By assumption, there exists a w*-open set U containing g_0^* such that

$$U \subseteq \{g^* \in G^* : \|g^* - g_0^*\| < 2\}$$

However, we have $\{g^* \in G^* : \|g^* - g_0^*\| < 2\} \subseteq G_\pi^*$(See [10, 2.12.1]). Therefore, $(G_\pi^*)^o$ is non-empty.

"(c) $\Rightarrow$ (a)": Note that the natural map $q : G^* \to \hat{G}$ is open([10, Theorem 3.4.11]) and $\{\pi\} = q((G_\pi^*)^o)$ by the definition of q and the assumption that $G_\pi^*)^o$ is non-empty. It follows that the hull-kernel topology on $\hat{G}$ is discrete, hence G is compact. □

6.3 Characterization of discrete groups

The following proposition is a consequence of theorem 8.2.1, which gives some characterizations of discrete groups.

Proposition 6.3.1. Let G be a locally compact group. Then the following statements are equivalent:

(a) G is discrete.

(b) $P_1(G)$ is weak* compact.

(c) $B_r(G) \cap P_1(G)$ is weak* compact.

Hence, if G^* is weak* compact, then G is discrete.

Proof. If G is discrete, then $P_1(G) = \{\phi \in B(G) : \phi(e) = \langle \phi, \delta_e \rangle = 1 = \|\phi\|\}$ is clearly weak* compact. Suppose that (b) holds. For each $g^* \in G^*$, define $T_{g^*} : P_1(G) \to P_1(G)$ by $T_{g^*}(\phi) = g^* \cdot \phi$. Then $\{T_{g^*} : g^* \in G^*\}$ is a commutating family of continuous affine maps on $P_1(G)$. By the Markov-Kakutani fixed point theorem, there exists $\phi_0 \in P_1(G)$ such that $g^* \cdot \phi_0 = \phi_0$. Thus, G is discrete by theorem 8.2.1. The proof of the equivalence of (a) and (c) is similar. □

It gives another proof of the following theorem which appears in [28].

Corollary 6.3.2. Let G be a locally compact group. Then the following statements are equivalent:

(a) G is discrete.

(b) $C^*(G)$ is unital.

(c) $C_r^*(G)$ is unital.

Proof. Note that if A is a unital C*-algebra, then $(A^*)_1^+$ is w*-compact. □

The following theorem characterizes all the translation-invariant elements of $W^*(G)$.

Theorem 6.3.3. Let T be a non-zero element in $W^*(G)$. Then the following statements are equivalent:

(a) $g^* \cdot T = T$ for all $g^* \in G^*$

(b) $T = c\omega(e)$ for some non-zero $c \in \mathbb{C}$.

Proof. (b) $\Rightarrow$ (a) is clear. Suppose that (a) holds. Let $g^* \in G^*$. Then $\langle T, g^* \rangle = \langle g^* \cdot T, 1 \rangle = \langle T, 1 \rangle$. If $\langle T, 1 \rangle = 0$, then $\langle T, g^* \rangle = 0$ for each $g^* \in G^*$. Hence, $T = 0$, which leads a contradiction. Therefore, we may assume that $\langle T, 1 \rangle = 1$. By the above observation, $\langle T, g^* \rangle = 1$ for all $g^* \in G^*$. Thus, $\langle T, f \rangle = 1$ for all $f \in P_1(G)$ by lemma 5.1.1. For any non-degenerate representation π of $C^*(G)$, we have $\langle \pi(T)\epsilon, \epsilon \rangle = 1$ for all $\epsilon \in \mathcal{H}_{\pi \cdot 1}$. It follows by Cauchy-Schwarz's inequality that $\pi(T) = id_\pi$. Therefore, T is the identity in $W^*(G)$. □

The following theorem gives a characterization of translation-invariant elements of $VN(G)$. For the definition of the support of an element of $VN(G)$, the basic reference is [11, Chapter 4].

Theorem 6.3.4. Let T be a non-zero element in $VN(G)$. Then the following statements are equivalent:

(a) $g^* \cdot T = T$ for all $g^* \in G^*$

(b) $\phi \cdot T = T$ for all $\phi \in B(G)_1^+$

(c) $\phi \cdot T = T$ for all $\phi \in A(G) \cap P_1(G)$

(d) $T = c\lambda_2(e)$ for some non-zero constant $c \in \mathbb{C}$.

Proof. "(a) $\Rightarrow$ (b):" For any $\phi \in P_1(G)$, there exists a net $\{e_\alpha\}$ in $co(G^*)$ such that $e_\alpha \longrightarrow \phi$ strictly. Observe that $e_\alpha \cdot T = T$. So, for any $u \in A(G)$, $\langle T, u\rangle = \langle e_\alpha \cdot T, u\rangle = \langle T, e_\alpha \cdot u\rangle \to \langle T, \phi \cdot u\rangle$. Thus, $T = \phi \cdot T$. "(b) $\Rightarrow$ (c)" is clear. Suppose that (c) holds. By [11, Proposition 4.4.8], $\text{supp}(T) = \text{supp}(\phi \cdot T) \subseteq \text{supp}(\phi) \cap \text{supp}(T)$. It follows that $\text{supp}(T) \subseteq \text{supp}(\phi)$ for any $\phi \in A(G) \cap P_1(G)$. However, for any $x \neq e \in G$, there exists $f \in A(G) \cap P_1(G)$ such that x lies outside $\text{supp}(f)$. Therefore, we have $\text{supp}(T) = \{e\}$. Hence, the result follows by [11, Theorem 4.4.9]. □

Note that $C^*(G)$ and $C_r^*(G)$ are $B(G)$-bimodules, and hence they are G^*-invariant.

Corollary 6.3.5. Let G be a locally compact group. Then the following statements are equivalent:

(a) G is discrete.

(b) There exists a non-zero $T \in C^*(G)$ such that T is G^*-invariant.

(c) There exists a non-zero $T \in C_r^*(G)$ such that T is G^*-invariant.

Moreover, if such T exists, then $T = c\delta_e$ for some $c \in \mathbb{C}$.

Proof. It follows from corollary 6.3.2, theorem 6.3.3 and theorem 6.3.4. □

We may characterize translation-invariant elements of $vn(G)$:

Proposition 6.3.6. Let T be a non-zero element in $vn(G)$. Then the following statements are equivalent:

(a) $g^* \cdot T = T$ for all $g^* \in G^*$

(b) $T = c\sigma(e)$ for some non-zero $c \in \mathbb{C}$.

Proof. (b) ⇒ (a) is clear. Suppose that (a) holds. Let $g^* \in G^*$. Then $\langle T, g^* \rangle = \langle g^* \cdot T, 1 \rangle = \langle T, 1 \rangle$. If $\langle T, 1 \rangle = 0$, then $\langle T, g^* \rangle = 0$ for each $g^* \in G^*$. Hence, $T = 0$, which leads a contradiction. Therefore, we may assume that $\langle T, 1 \rangle = 1$. By the above observation, $\langle T, g^* \rangle = 1$ for all $g^* \in G^*$. Since $g^* = \langle \pi(\cdot)\epsilon, \epsilon \rangle$ for some irreducible representation π and $\epsilon \in \mathcal{H}_\pi$, $\|\epsilon\| = 1$, we obtain $T|_{\mathcal{H}_\pi} = id_{\mathcal{H}_\pi}$. Consequently, $T|_{\mathcal{H}_{\pi_a}} = \pi_a(e)$. Note that $(g_1^* ... g_n^*) \cdot T = T$ for any $g_1^*, ..., g_n^* \in G^*$. By the similar argument, $T|_{\mathcal{H}_{\pi_a^{(n)}}} = \pi_a^{(n)}(e)$. Hence, $T = \sigma(e)$. □

Chapter 7

Closed convex G^*-invariant subsets in $A(G)$ and $VN(G)$

In this chapter, we will study the closed convex G^*-invariant subsets in $A(G)$ and $VN(G)$, and also the operators on $A(G)$ and $VN(G)$ which commute with the actions of G^*.

7.1 Some general results

Let τ denote the locally convex topology on $B(G)$ determined by the separating family of semi-norms $\{p_{f,T} : f \in A(G), T \in VN(G)\}$ where $p_{f,T}(\phi) = \langle T, f \cdot \phi \rangle$ for each $\phi \in B(G)$.

It is easy to see that the strict topology is stronger than the τ-topology on $B(G)$.

Lemma 7.1.1. For any locally compact group G, we have $P_1(G) \subseteq \overline{co(G^*)}^{(\tau)}$. Moreover, G is amenable if and only if $P_1(G) \subseteq \overline{A(G) \cap P_1(G)}^{(\tau)}$.

Proof. Since $co(G^*)$ is strictly dense in $P_1(G)$, the first part of this lemma is straightforward. Let $(e_\alpha) \subseteq A(G) \cap P_1(G)$ be a BAI in $A(G)$. For any $f \in P_1(G)$, we have $(fe_\alpha)_\alpha \subseteq A(G) \cap P_1(G)$ and

$$\langle T, (fe_\alpha)g - fg \rangle \leq \|T\| \|f\| \|e_\alpha g - g\| \to 0 \text{ for all } T \in VN(G), g \in A(G)$$

The converse follows from [16, Theorem B2]. □

Lemma 7.1.2. Let G be a locally compact group.

(a) For any $u \in A(G)$, the function $B(G) \to A(G)$, $\phi \mapsto \phi \cdot u$ is continuous when $B(G)$ has τ-topology and when $A(G)$ has the weak topology.

(b) For any $T \in VN(G)$, the function $B(G) \to VN(G)$, $\phi \mapsto \phi \cdot T$ is continuous when $B(G)$ has τ-topology and when $VN(G)$ has the weak* topology.

(c) For any $\psi \in B(G)$, the function $B(G) \to B(G)$, $\phi \mapsto \phi \cdot \psi$ is weak*-weak*-continuous.

Proof. (a) Let $\{\phi_\alpha\}$ be a net in $B(G)$ converging to some ϕ in $B(G)$ in the τ-topology. By definition,

$$\langle T, \phi_\alpha \cdot u \rangle \to \langle T, \phi \cdot u \rangle \text{ for all } T \in VN(G).u \in A(G)$$

In particular, $\phi_\alpha \cdot f \to \phi \cdot f$ weakly.

(b) Let $\{\phi_\alpha\}$ be a net in $B(G)$ converging to some ϕ in $B(G)$ in the τ-topology. By definition,

$$\langle \phi_\alpha \cdot T, u \rangle = \langle T, \phi_\alpha \cdot u \rangle \to \langle T, \phi \cdot u \rangle = \langle \phi \cdot T, u \rangle \text{ for all } T \in VN(G).u \in A(G)$$

In particular, $\phi_\alpha \cdot T \to \phi \cdot T$ in the weak* topology.

(c) Clear. Note that $C^*(G)$ is a $B(G)$-bimodule.

□

Theorem 7.1.3. Let G be a locally compact group and let K be a closed convex subset of $A(G)$.

(a) If $g^* \cdot K \subseteq K$ for each $g^* \in G^*$, then $\phi \cdot K \subseteq K$ for each $\phi \in A(G) \cap P_1(G)$.

(b) Suppose, in addition, that G is amenable. Then $g^* \cdot K \subseteq K$ for each $g^* \in G^*$ if and only if $\phi \cdot K \subseteq K$ for each $\phi \in A(G) \cap P_1(G)$.

Proof. (a) Let $\phi \in A(G) \cap P_1(G)$, $f \in K$, and let (ϕ_α) be a net in $co(G^*)$ such that $\phi_\alpha \longrightarrow^\tau \phi$. Note that $\phi_\alpha \cdot f \in K$ for each α. Since K is weakly closed, it follows from lemma 7.1.2 that $\phi \cdot f \in K$.

(b) Since G is amenable, $P_1(G) \subseteq \overline{A(G) \cap P_1(G)}^{(\tau)}$ by lemma 7.1.1. For any $g^* \in G^*$, let (ϕ_α) be a net in $A(G) \cap P_1(G)$ such that $\phi_\alpha \longrightarrow^\tau g^*$. Since $\phi_\alpha \cdot f \in K$ for each α and $f \in K$, by lemma 7.1.2 again, we have $g^* \cdot K \subseteq K$.

□

Theorem 7.1.4. Suppose that G is amenable. Let $v_0 \in A(G)$, then $\overline{co}\{g^* \cdot v_0 : g^* \in G^*\} = \{\phi \cdot v_0 : \phi \in A(G) \cap P_1(G)\}^-$.

Proof. Let $K_1 = \overline{co}\{g^* \cdot v_0 : g^* \in G^*\}$ and $K_2 = \{\phi \cdot v_0 : \phi \in (A(G) \cap P_1(G))\}^-$. Then $g^* \cdot K_1 \subseteq K_1$ for each $g^* \in G^*$. So, by theorem 7.1.3, $\phi \cdot K_1 \subseteq K_1$ for each $\phi \in A(G) \cap P_1(G)$. Therefore, $K_2 \subseteq K_1$. Conversely, let ϕ_α be a net in $A(G) \cap P_1(G)$ converging to 1. By lemma 7.1.2, $\phi_\alpha \cdot v_0 \longrightarrow v_0$ in the weak-topology of $A(G)$. It follows that $v_0 \in K_2$. By the same argument, $g^* \cdot K_2 \subseteq K_2$ for each $g^* \in G^*$. In particular, $g^* \cdot v_0 \in K_2$ for each $g^* \in G^*$. □

Theorem 7.1.5. Let A,B be a closed G^*-invariant convex subsets of $A(G)$. Suppose that Φ is an affine norm-continuous map from A into B, then the following are equivalent:

(a) $\Phi(g^* \cdot f) = g^* \cdot \Phi(f)$ for each $g^* \in G^*$, $f \in A$

(b) $\Phi(\phi \cdot f) = \phi \cdot \Phi(f)$ for each $\phi \in A(G) \cap P_1(G)$, $f \in A$

Proof. Suppose that (a) holds. Let $\phi \in A(G) \cap P_1(G)$, and $(\phi_\alpha) \in co(G^*)$ such that $\phi_\alpha \longrightarrow^\tau \phi$. By lemma 7.1.2, $\phi_\alpha \cdot f \longrightarrow \phi \cdot f$ weakly for each $f \in A$. Since A is closed and convex, and T is affine, Φ is also continuous when A, B have the respective weak topology. Thus, $\Phi(\phi \cdot f) = \lim \Phi(\phi_\alpha \cdot f) = \lim \phi_\alpha \cdot \Phi(f) = \phi \cdot \Phi(f)$. Conversely, let $g^* \in G^*$, and $(\phi_\alpha) \in A(G) \cap P_1(G)$ such that $\phi_\alpha \longrightarrow^\tau g^*$. By lemma 7.1.2, $\phi_\alpha \cdot f \longrightarrow g^* \cdot f$ weakly for each $f \in A$. Then, $\Phi(g^* \cdot f) = \lim \Phi(\phi_\alpha \cdot f) = \lim \phi_\alpha \cdot \Phi(f) = g^* \cdot \Phi(f)$. □

Theorem 7.1.6. G is discrete if and only if there exists a weakly compact, convex, G^*-invariant, non-zero subset in $A(G)$.

Proof. If G is discrete, then $K = \{\delta_e\}$ is such a subset. Conversely, suppose that G is non-discrete. Let $v_0 \neq 0$ be an element in K. Define $T : A(G) \rightarrow A(G)$, $T(u) = v_0 u$. Then T is weakly compact. In fact, let $K_1 = \overline{co}^{weak}\{g^* \cdot v_0 : g^* \in G^*\}$. By theorem 7.1.4, we have $K_1 = \{\phi \cdot v_0 : \phi \in (A(G) \cap P_1(G))\}^-$. Consider the weakly compact set $K_2 = \{\lambda k : \lambda \in [0,1], k \in K_1\}$. For each $u \in A(G)_1$, we have $v_0 u \in K_2 - K_2 + i(K_2 - K_2)$. So, $T(A(G)_1)$ is relatively weakly compact. It follows that $v_0 u = T(u) = 0$ for all $u \in A(G)$(See [22, Proposition 6.9]). Therefore, $u = 0$. □

Theorem 7.1.7. G is non-discrete if and only if every norm-compact, convex, G^*-invariant, non-zero subset of $C^*(G)$ contains zero.

Proof. If G is discrete, then $K = \{\delta_e\}$ is such a subset which does not contain zero. Conversely, for any $g^* \in G^*$, define $T_{g^*} : K \rightarrow K$, $f \mapsto g^* \cdot f$. Then each T_{g^*} is a norm-continuous affine map from K to K. Also, $\{T_{g^*} : g^* \in G^*\}$ is a commuting family. By Markov-Kakutani fixed point theorem, there exists $f_0 \in C^*(G)$ such that f_0 is G^*-invariant. If follows from corollary 6.3.5 that $f_0 = 0$ as G is not discrete. □

Theorem 7.1.8. Let B be a closed G^*-invariant subset of $A(G)$, and let Γ be a continuous affine mapping from $A(G) \cap P_1(G)$ into B. Then $\Gamma(g^* \cdot \psi) = g^* \cdot \Gamma(\psi)$ for any $g^* \in G^*$, $\psi \in A(G) \cap P_1(G)$ if and only if there exists $\phi \in B(G)$ such that $\Gamma(\psi) = \phi \cdot \psi$ for any $\psi \in A(G) \cap P_1(G)$.

Proof. Assume that $\Gamma(g^* \cdot \psi) = g^* \cdot \Gamma(f)$ for any $g^* \in G^*$, $\psi \in A(G) \cap P_1(G)$. Let (ϕ_α) be a net in $A(G) \cap P_1(G)$ such that $\phi_\alpha \longrightarrow^\tau 1$. By lemma 7.1.2, $\phi_\alpha \cdot \psi \longrightarrow \psi$ in the weak topology of $A(G)$. Thus, $\Gamma(\phi_\alpha) \cdot \psi = \Gamma(\phi_\alpha \cdot \psi) \longrightarrow \Gamma(\psi)$ weakly. On the other hand, since $\{\Gamma(\phi_\alpha)\}$ is bounded, we may assume that $\Gamma(\phi_\alpha) \longrightarrow^{w^*} \phi$ for some $\phi \in B(G)$. By lemma 7.1.2 again, $\Gamma(\phi_\alpha) \cdot \psi \longrightarrow \phi \cdot \psi$ in the weak* topology of $B(G)$. Hence, $\Gamma(\psi) = \phi \cdot \psi$ for each $\psi \in A(G) \cap P_1(G)$. The converse is trivial. □

Theorem 7.1.9. Let G be a locally compact group and let K be a weak*-closed convex subset of $VN(G)$.

(a) If $g^* \cdot K \subseteq K$ for each $g^* \in G^*$, then $\phi \cdot K \subseteq K$ for each $\phi \in A(G) \cap P_1(G)$.

(b) Suppose, in addition, that G is amenable. Then $g^* \cdot K \subseteq K$ for each $g^* \in G^*$ if and only if $\phi \cdot K \subseteq K$ for each $\phi \in A(G) \cap P_1(G)$.

Proof. By using lemma 7.1.1 and lemma 7.1.2, the proof of it is similar to that of theorem 7.1.3. □

Theorem 7.1.10. Suppose that G is amenable. If $T \in VN(G)$, then $\overline{co}^{\mathrm{w}^*}\{g^* \cdot T : g^* \in G^*\} = \{\phi \cdot T : \phi \in A(G) \cap P_1(G)\}^{-\mathrm{w}^*}$.

Proof. The proof is similar to the proof of theorem 7.1.4. □

Theorem 7.1.11. Let A be a closed G^*-invariant convex subset of $A(G)$, and B be w*-closed G^*-invariant convex subset of $VN(G)$. Suppose that Φ is an affine norm-continuous map from A into B, then the following are equivalent:

(a) $\Phi(g^* \cdot f) = g^* \cdot \Phi(f)$ for each $g^* \in G^*$, $f \in A$

(b) $\Phi(\phi \cdot f) = \phi \cdot \Phi(f)$ for each $\phi \in A(G) \cap P_1(G)$, $f \in A$

Proof. The proof is just a slight modification of the proof of theorem 7.1.5. □

Theorem 7.1.12. Let A, B be w*-closed G^*-invariant convex subsets of $VN(G)$. Suppose that Ψ is an affine w*-w*-continuous map from A into B, then the following are equivalent:

(a) $\Psi(g^* \cdot T) = g^* \cdot \Psi(T)$ for each $g^* \in G^*$, $T \in A$

(b) $\Psi(\phi \cdot T) = \phi \cdot \Psi(T)$ for each $\phi \in A(G) \cap P_1(G)$, $T \in A$

Proof. The proof is just a slight modification of the proof of theorem 7.1.5. □

7.2 Characterizations of generalized function spaces in amenable groups

Let us recall the definition of $AP(G)$ and $WAP(G)$:
For any $f \in L^\infty(G)$, write $o(f) = \{L_x f : x \in G\}$. Then

$$AP(G) := \{o(f) \text{ is compact in } L^\infty(G)\} \ ;$$

$$WAP(G) := \{o(f) \text{ is weakly compact in } L^\infty(G)\}$$

Notice that these definitions depend on the concept of translations of functions in $L^\infty(G)$. In defining the non-commutative analogue of them, namely, $AP(\hat{G})$

and $WAP(\hat{G})$, we use the notion of (weakly)compact operators instead. However, if G is amenable, we are able to characterize $AP(\hat{G})$ and $WAP(\hat{G})$ by using the concept of generalized translation operators.

For any $T \in VN(G)$, write $O(T) = \{g^* \cdot T : g^* \in G^*\}$.

Theorem 7.2.1. If G is amenable, then

$$WAP(\hat{G}) = \{T \in VN(G) : O(T) \text{ is relatively weakly compact in } VN(G)\}.$$

Proof. The set $K := \{u \cdot T : u \in A(G)_1\}$ is relatively weakly compact. Since the weak*-topology on $\overline{K}^{weak}$ is Hausdorff, the weak topology and the weak*-topology coincide on K. By using this observation and lemma 7.1.10, we have

$$O(T) \subseteq \overline{co}^{w*}(O(T)) = \{\phi \cdot T : \phi \in A(G) \cap P_1(G)\}^{-w^*} \subset \overline{K}^{w^*} = \overline{K}^{weak}$$

Therefore, $O(T)$ is relatively weakly compact. Conversely, suppose that $O(T)$ is relatively weakly compact. Therefore, $co(O(T))$ is also relatively weakly compact. By the same argument, the weak topology and the weak*-topology coincide on $K' = \overline{co(O(T))}^{weak}$. So, $K' = \overline{co}^{w*}(O(T)) = \{\phi \cdot T : \phi \in A(G) \cap P_1(G)\}^{-w^*} = \{\phi \cdot T : \phi \in A(G) \cap P_1(G)\}^{-}$. Consequently, for any $u \in A(G)_1$, we have $u \cdot T \in K' - K' + i(K' - K')$. □

By replacing "weak topology" by "norm topology" everywhere in the proof of theorem 7.2.1, one can show that:

Theorem 7.2.2. If G is amenable, then

$$AP(\hat{G}) = \{T \in VN(G) : O(T) \text{ is relatively compact in } VN(G)\}.$$

7.3 Invariantly complemented subspace of $VN(G)$

Throughout this section, we will assume that $A(G)$ has an approximate identity.

Lemma 7.3.1. For any locally compact group G, we have $\overline{\text{span}(G^*)}^{\tau} = B(G)$ and $\overline{A(G)}^{\tau} = B(G)$.

Proof. The first assertion follows from lemma 7.1.1. To prove the second assertion, let $f \in B(G)$, there exists a net $(f_\alpha) \subseteq A(G)$ such that $f_\alpha \to f$ strictly, i.e. $f_\alpha g \to fg$ for any $g \in A(G)$. But, it follows clearly that $\langle T, (f_\alpha - f), g\rangle \to 0$ for any $g \in A(G)$. □

Lemma 7.3.2. Let X be a weak*-closed subspace of $VN(G)$. Then X is G^*-invariant if and only if X is a $A(G)$-submodule of $VN(G)$.

Proof. Let $f \in A(G)$. By lemma 7.3.1, there exists a net $f_\alpha \in \text{span}(G^*)$ such that $f_\alpha \to^\tau f$. For all $g \in X$, we have $f_\alpha g \in X$ Since X is weak*-closed, it follows that $fg \in X$. Conversely, let $g^* \in G^*$. By lemma 7.3.1, there exists a net $f_\alpha \in A(G)$ such that $f_\alpha \to^\tau g^*$. Then we use the same argument. □

Definition 7.3.3. Let X be a weak*-closed G^*-invariant subspace of $VN(G)$.

(a) X is said to be *invariantly complemented* in $VN(G)$ such that $P(g^* \cdot T) = g^* \cdot T$ $(g^* \in G^*, T \in VN(G))$.

(b) X is said to be *topologically invariantly complemented* in $VN(G)$ such that $P(f \cdot T) = f \cdot T$ $(f \in A(G), T \in VN(G))$.

The notion of (topologically)invariantly complemented subspaces of $UCB(\hat{G})$ is defined similarly.

Theorem 7.3.4. Let X be a weak*-closed G^*-invariant subspace of $VN(G)$. Consider the following conditions:

(a) $X_\perp$ has a BAI.

(b) X is topologically invariantly complemented in $VN(G)$.

(c) X is invariantly complemented in $VN(G)$.

(d) $X \cap UCB(\hat{G})$ is invariantly complemented in $UCB(\hat{G})$.

(e) $X \cap UCB(\hat{G})$ is topologically invariantly complemented in $UCB(\hat{G})$.

Then we have (a)⇒(b)⇒(c)⇒(d)⇒(e). If G is amenable, then the above conditions are equivalent to each other. Moreover, G is amenable if and only if (a) is equivalent to any of (b)-(e).

Proof. (a)⇒(b): Let (v_α) be a BAI in $X_\perp$ and let N be a weak*-limit point of (v_α) in $VN(G)^*$. Without loss of generality, assume that $v_\alpha \longrightarrow^{w^*} N$. Define $P : VN(G) \to VN(G)$ by

$$\langle PT, f\rangle = \langle T, f\rangle - \lim_\alpha \langle T, v_\alpha f\rangle (\text{or } \langle T, f\rangle - \langle f \cdot T, N\rangle)$$

Then

$$
\begin{aligned}
&\langle P^2 T, f\rangle \\
&= \langle PT, f\rangle - \lim_\alpha \langle PT, v_\alpha f\rangle \\
&= \langle T, f\rangle - \lim_\alpha \langle T, v_\alpha f\rangle - \lim_\alpha(\langle T, v_\alpha f\rangle - \lim_\beta \langle T, v_\beta v_\alpha f\rangle) \\
&= \langle T, f\rangle - \langle f \cdot T, N\rangle - \langle f \cdot T, N\rangle + \langle f \cdot T, N^2\rangle \\
&= \langle T, f\rangle - \langle f \cdot T, N\rangle - \langle f \cdot T, N\rangle + \langle f \cdot T, N\rangle \\
&= \langle T, f\rangle - \langle f \cdot T, N\rangle \\
&= \langle PT, f\rangle \text{ for any } f \in A(G) \text{ and } T \in VN(G).
\end{aligned}
$$

So, P is a projection. Let $T \in X$ and $f \in X_\perp$, then $\langle PT, f\rangle = \langle T, f\rangle - \lim_\alpha \langle T, v_\alpha f\rangle = 0 - 0 = 0$. Therefore, $PT \in (X_\perp)^\perp = X$.

(b)⇒(c): Let $P : VN(G) \to X$ be a projection such that $P(f \cdot T) = f \cdot T$ $(f \in A(G), T \in VN(G))$, and let (u_α) be an approximate identity for $A(G)$. If $g^* \in G^*$, $f \in A(G)$ and $T \in VN(G)$, then we have

$\langle P(g^* \cdot T), f \rangle$

$= \lim_\alpha \langle P(g^* \cdot T), u_\alpha f \rangle = \lim_\alpha \langle u_\alpha \cdot P(g^* \cdot T), f \rangle = \lim_\alpha \langle P((u_\alpha g^*) \cdot T), f \rangle$

$= \lim_\alpha \langle (u_\alpha g^*) \cdot P(T), f \rangle = \lim_\alpha \langle g^* \cdot P(T), u_\alpha f \rangle = \langle g^* \cdot P(T), f \rangle$

Hence, $P(g^* \cdot T) = g^* \cdot P(T)$.

(c)$\Rightarrow$(d): Let $P : VN(G) \to X$ be a projection such that $P(g^* \cdot T) = g^* \cdot T$ ($g^* \in G^*$, $T \in VN(G)$). If $f \in A(G)$, then there exists a net of functions $(f_\alpha) \subseteq \text{span}(G^*)$ such that $f_\alpha \to f$ strictly by proposition 5.1.1. So, we get

$\langle P(f \cdot (u \cdot T)), v \rangle$

$= \langle P((f \cdot u) \cdot T), v \rangle = \lim_\alpha \langle P((f_\alpha \cdot u) \cdot T), v \rangle$

$= \lim_\alpha \langle f_\alpha \cdot P(u \cdot T), v \rangle = \lim_\alpha \langle P(u \cdot T), f_\alpha v \rangle$

$= \langle P(u \cdot T), fv \rangle = \langle f \cdot P(u \cdot T), v \rangle$ for any $u, v \in A(G)$ and $T \in VN(G)$.

Since $A(G) \cdot VN(G)$ is dense in $UCB(\hat{G})$, we have

$$P(f \cdot S) = f \cdot P(S) \text{ for all } f \in A(G) \text{ and } S \in UCB(\hat{G}) \qquad (*)$$

By the same argument in proving (b)$\Rightarrow$(c), we have $P(g^* \cdot S) = g^* \cdot P(S)$ for all $g^* \in G^*$ and $S \in UCB(\hat{G})$.

(d)$\Rightarrow$(e): Using the same argument in proofing (*) above.

For the last statement, we only need to show that G is amenable if and only if (a) is equivalent to (e). Suppose that (a) and (e) are equivalent. Let $X = \{0\}$, then $X \cap UCB(\hat{G}) = \{0\}$ is topologically invariantly complemented in $UCB(\hat{G})$. Therefore, $A(G) = X_\perp$ has a BAI, and hence G is amenable by Leptin's theorem. The converse follows by [Proposition 6.4] and [Proposition 7.5] of [13]. $\square$

Remark. The above proof is inspired by the proof of [4, Theorem 1].

A subspace X of $VN(G)$ is said to be *completely complemented* in $VN(G)$ if there exists a completely bounded projection from $VN(G)$ onto X (see [40] for more details).

Theorem 7.3.5. Consider the following conditions:

(a) G is amenable.

(b) A closed ideal I of $A(G)$ has a BAI if and only if $I^\perp$ is completely complemented in $VN(G)$.

(c) Every completely complemented weak*-closed G^*-invariant subspace of $VN(G)$ is topologically invariantly complemented.

(d) Every completely complemented weak*-closed G^*-invariant subspace of $VN(G)$ is invariantly complemented.

Then we have (a)⇔(b)⇒(c)⇒(d).

Proof. (a)⇒(b) follows by [40, Theorem 3] and (b)⇒(c)⇒(d) follows by theorem 7.3.4. Suppose that (b) holds. Let $I = A(G)$. Then $I^\perp = \{0\}$ is completely complemented in $VN(G)$. Thus, $A(G) = X_\perp$ has a BAI, and hence G is amenable by Leptin's theorem. □

Chapter 8

Reflexivity and duality of subgroups

We will use the following notations throughout this chapter:

$$P_0(G) := \{\phi \in P(G) : 0 \leq \phi(e) \leq 1\} \text{ and } P_1(G) := \{\phi \in P(G) : \phi(e) = 1\}.$$

Let G^* be the the set of all extreme points of $P_1(G)$. (i.e. $G^* = \mathcal{E}(P_1(G))$), equipped with the relative weak*-topology inherited from $B(G)$. It is called the *dual space* of G. Given a non-empty subset X in G, let

$$X^{\odot} := \{\phi \in P_0(G) : \phi(x) = \phi(y) \text{ for any } x, y \in X\},$$

$$X^{\circ} := \{\phi \in P_1(G) : \phi(x) = 1 \text{ for any } x \in X\},$$

$$X^{\perp} := \{g^* \in G^* : g^*(x) = 1 \text{ for any } x \in X\}.$$

It is easy to see that $X^{\circ} = P_1(G) \cap X^{\odot}$ and $X^{\perp} = G^* \cap X^{\odot}$.

8.1 Extreme points of subsets of Fourier Stieljies algebras

We quote the following classical results about positive definite functions, which is useful in the sequel.

Proposition 8.1.1. Let $\phi \in P(G)$. Then $|\phi(e)\phi(yz) - \phi(y)\phi(z)|^2 \leq (\phi(e)^2 - |\phi(y)|^2)(\phi(e)^2 - |\phi(z)|^2)$ for any $y, z \in G$.

The closed subgroup generated by X in G is denoted by $\langle X \rangle$.

Lemma 8.1.2. For any non-empty subset X in G, we have

$$X^\circ = \langle X \rangle^\circ \text{ and } X^\perp = \langle X \rangle^\perp.$$

If X contains identity, then

$$X^\odot = \langle X \rangle^\odot$$

The following lemma generalizes [12, Lemma 3.26] where the proof follows from a slight modification of the proof of [12, Lemma 3.26].

Lemma 8.1.3. Let X be a closed non-empty subset of G. Then $\mathcal{E}(X^\circ) = X^\perp$. If X contains identity, then $\mathcal{E}(X^\odot) = X^\perp \cup \{0\}$.

Proof. For the first equality, by lemma 8.4.1, it suffices to assume that X is a closed subgroup of G. Suppose that $g^* \in X^\perp$. Let $g^* = 1/2(\phi_1 + \phi_2)$ for some $\phi_1, \phi_2 \in X^\circ$. Then, we have $g^* = \phi_1 = \phi_2$ since g^* is extreme in $P^1(G)$. It follows that $g^* \in \mathcal{E}(X^\circ)$. Conversely, assume that $\phi \in \mathcal{E}(X^\circ)$. Let $\phi = 1/2(\phi_1 + \phi_2)$ for some $\phi_1, \phi_2 \in P_1(G)$. Then $1 = \phi(x) = 1/2(\phi_1(x) + \phi_2(x))$ for any $x \in X$. Since 1 is an extreme point of $\mathbb{T}$ and $\phi_1(x), \phi_2(x) \in \mathbb{T}$, we get $\phi_1(x) = \phi_2(x) = 1$ for all $x \in X$. Therefore, $\phi_1, \phi_2 \in X^\circ$. Consequently,

$\phi = \phi_1 = \phi_2$. Hence, $\phi \in G^*$. For the second equality, let $\phi_1, \phi_2 \in X^{\odot}$. Suppose that $0 = 1/2(\phi_1 + \phi_2)$. Then, we have $0 = 1/2(\phi_1(e) + \phi_2(e))$. It follows that $\phi_1 = \phi_2 = 0$ since $\phi_1(e), \phi_2(e) \geq 0$. Therefore, 0 is an extreme point. Let $g^* \in X^{\perp}$. If $g^* = 1/2(\phi_1 + \phi_2)$ for some $\phi_1, \phi_2 \in X^{\odot}$, then $1 = g^*(e) = 1/2(\phi_1(e) + \phi_2(e))$. Consequently, $\phi_1(e) = 1 = \phi_2(e)$, ie. $\phi_1, \phi_2 \in X^{\circ}$. Therefore, g^* is extreme by the first assertion. Conversely, if $\phi \in X^{\odot} \setminus (X^{\circ} \cup \{0\})$, then $\phi = \|\phi\|(\phi/\|\phi\|) + (1 - \|\phi\|)0$. So, ϕ is not an extreme point. □

Proposition 8.1.4. Let G be a locally compact group, and let X be a non-empty subset of G. Then $X^{\odot}$ is weak*-compact, and hence $co(X^{\perp})$ is weak*(strictly) dense in X°.

Proof. Let $x \in X$, $(\phi_\alpha) \subseteq X^{\odot}$ and $\phi \in B(G)$ such that $\phi_\alpha \to \phi$ in the weak*-topology. Since the weak*-topology and the topology of uniform convergence on compacta coincide on the unit ball of $B(G)$(see [16]), we have $\phi_\alpha(x) \to \phi(x)$ for any $x \in X$. If $x, y \in X$ such that $\phi_\alpha(x) = \phi_\alpha(y)$, then $\phi(x) = \phi(y)$. Also, we have

$$\|\phi_\alpha\| = \phi_\alpha(e) \to \phi(e) = \|\phi\|.$$

Hence, $\phi \in X^{\odot}$ and therefore $X^{\odot}$ is weak*-compact. With lemma 8.1.3, the rest is basically the same as that of [12, Theorem 3.27]. We will prove it here for the sake of completeness. By Krein-Milman theorem and lemma 8.1.3, for any $\phi \in X^{\circ}$, ϕ is the weak*-limit of a net of functions ϕ_α of the form $c_1 g_1^* + ... + c_n g_n^* + c_{n+1} 0$, where $g_1^*, ... g_n^* \in X^{\perp}$, $c_1, ..., c_n, c_{n+1} \geq 0$, and $\sum_j c_j = 1$. Since $\|\phi\|_\infty = 1$ and $\|\phi_\alpha\|_\infty \leq 1$, we have

$$1 = \|\phi\|_\infty \leq \underline{\lim}\|\phi_\alpha\|_\infty \leq \overline{\lim}\|\phi_\alpha\|_\infty \leq 1.$$

Put $\phi'_\alpha = \phi_\alpha/\phi_\alpha(e)$. We have $\phi'_\alpha = \sum_{i=1}^n (c_i/\phi_\alpha(e)) g_i^*$ and $\sum_{i=1}^n c_i/\phi_\alpha(e) = \phi_\alpha(e)/\phi_\alpha(e) = 1$. Thus, $\phi'_\alpha \in co(X^{\perp})$ and $\phi = \lim^{w*} \phi_\alpha$. □

8.2 Reflexivity on subgroups

Let A be a subset of G^*. Write $A_\perp := \{x \in G : g^*(x) = 1 \text{ for all } g^* \in A\}$.

The proof of the following lemma is easy and left to the reader.

Lemma 8.2.1. Let X, Y be non-empty subsets of G, A, B be non-empty subsets of G^* and P, Q be non-empty subsets of $P_1(G)$. Then we have the following inclusions:

(a) $X \subseteq (X^\perp)_\perp$ and $Y^\perp \subseteq X^\perp$ if $X \subseteq Y$.

(b) $X \subseteq (X^\circ)_\perp$ and $Y^\circ \subseteq X^\circ$ if $X \subseteq Y$.

(c) $A \subseteq (A_\perp)^\perp$ and $B_\perp \subseteq A_\perp$ if $A \subseteq B$.

(d) $P \subseteq (P_\perp)^\circ$ and $Q_\perp \subseteq P_\perp$ if $P \subseteq Q$.

Let X and A be subsets of G and G^*. respectively.

(a) X is said to be *reflexive* in G if $(X^\perp)_\perp = X$(or equivalently, $(X^\perp)_\perp \subseteq X$).

(b) A is said to be *reflexive* in G^* if $(A_\perp)^\perp = A$(or equivalently, $(A_\perp)^\perp \subseteq A$).

Lemma 8.2.2. Let A be a subset of G^*. Then $A_\perp$ is a closed reflexive subgroup in G.

Proof. Since g^* is continuous, the set $\{x \in G : g^*(x) = 1\}$ is closed in G. Therefore,

$$A_\perp = \bigcap_{g^* \in A} \{x \in G : g^*(x) = 1\}$$

is closed in G. Let $x, y \in A_\perp$ and $g^* \in A$. By lemma 8.1.1, we have $g^*(xy) = g^*(x)g^*(y) = 1$ and $g^*(x^{-1}) = \overline{g^*(x)} = 1$. Let $H = A_\perp$. Then we have $H \subseteq (H^\perp)_\perp$ by lemma 8.2.1(a). On the other hand, since $A \subseteq (A_\perp)^\perp$, we have

$$(H^\perp)_\perp = ((A_\perp)^\perp)_\perp \subseteq A_\perp = H$$

by lemma 8.2.1(c). □

Therefore, only closed subgroups can be reflexive.

Similarly, we have the following lemma:

Lemma 8.2.3. Let X be a subset of G. Then $X^{\perp}$ is a closed reflexive subset in G^*.

Corollary 8.2.4. Let H be a closed subgroup of G and $N = (H^{\perp})_{\perp}$. Then $N^{\perp} = H^{\perp}$ and N is the smallest closed subgroup in G containing H such that N is reflexive in G.

Proof. The equality that $N^{\perp} = H^{\perp}$ follows from lemma 8.2.3. Let K be a subset of G such that $H \subseteq K$ and K is reflexive. Then we have

$$N = (H^{\perp})_{\perp} \subseteq (K^{\perp})_{\perp} = K$$

□

Notation. Write

$$\mathcal{G} = \{X \subseteq G : X \text{ is reflexive.}\}$$

and

$$\mathcal{G}^* = \{B \subseteq G^* : B \text{ is reflexive.}\}$$

Remark. It is not hard to see that

$$\mathcal{G} = \{A_{\perp} : A \subseteq G^*\} = \{(X^{\perp})_{\perp} : X \subseteq G\} = \{(H^{\perp})_{\perp} : H \text{ is a subgroup of } G\}$$

and

$$\mathcal{G}^* = \{X^{\perp} : X \subseteq G\} = \{(A_{\perp})^{\perp} : A \subseteq G^*\} = \{H^{\perp} : H \text{ is a subgroup of } G\}$$

In conclusion, we have the following theorem:

Theorem 8.2.5. $\Phi : \mathcal{G} \to \mathcal{G}^*$, $\Phi(H) = H^\perp$ is a bijection, and its inverse is given by $\Psi : \mathcal{G}^* \to \mathcal{G}$, $\Psi(B) = B_\perp$.

Proof. Write $B = H^\perp$ for some closed subgroup H of G. We have

$$\Phi(\Psi(B)) = \Phi(B_\perp) = \Phi((H^\perp)_\perp) = ((H^\perp)_\perp)^\perp = H^\perp = B$$

(the second last equality follows by lemma 8.2.3). On the other hand, for any closed subgroup H of G, we have

$$\Psi(\Phi((H^\perp)_\perp)) = \Psi(((H^\perp)_\perp)^\perp) = \Psi(H^\perp) = (H^\perp)_\perp$$

(the second last equality follows again by lemma 8.2.3). □

Lemma 8.2.6. Let H be a closed subgroup of G. Then the following conditions are equivalent:

(a) H is reflexive.

(b) for any $x \in G \setminus H$, there exists an element $g^* \in H^\perp$ such that $g^*(x) \neq 1$.

Proof. If $x \in (H^\perp)_\perp$, then $g^*(x) = 1$ for any $g^* \in H^\perp$. By assumption, $x \in H$. Conversely, let $x \in G \setminus H = G \setminus (H^\perp)_\perp$. By assumption, $g^*(x) \neq 1$ for some $g^* \in H^\perp$. Therefore, H is reflexive in G. □

Remark. Let G be a locally compact group. Lau and Kanuith defined in [19] the following separation property: H is said to be separating in G if for any $x \in G \setminus H$, there exists $\phi \in H^\circ$ such that $\phi(x) \neq 1$. In our notation, it is just equivalent to say that $(H^\circ)_\perp = H$. It follows that if H is reflexive, then H is separating in G.

We are going to prove that the converse is also true.

Let τ be the topology on $B(G)$ defined as the following:
A net (u_α) in $B(G)$ is said to converge to $u \in B(G)$ in the τ-topology if

$$\langle u_\alpha \cdot T, v\rangle \to \langle u \cdot T, v\rangle \text{ for any } v \in A(G), T \in VN(G)$$

Lemma 8.2.7. Let G be a locally compact group and let X be a subset of G. Then we have $X^\circ \subseteq \overline{co(X^\perp)}^{(\tau)}$.

Proof. It follows from the fact that $co(X^\perp)$ is strictly dense in X°(Lemma 8.1.4). □

For any closed subgroup H of G, let $VN_H(G)$ be the von Neumann subalgebra of $VN(G)$ generated by $\lambda_2(H)$.

Theorem 8.2.8. Let H be a closed subgroup of G. Then the following conditions are equivalent:

(a) H is separating in G.

(b) H is reflexive in G.

(c) $VN_H(G) = \{T \in VN(G) : g^* \cdot T = T \text{ for all } g^* \in H^\perp\}$.

(d) $VN_H(G) = \{T \in VN(G) : \phi \cdot T = T \text{ for all } \phi \in H^\circ\}$.

Proof. The equivalence of (a) and (c) was shown in [19, Proposition 3.1]. (b) $\Rightarrow$ (a) is clear.
(c) $\Rightarrow$ (d). If $g^* \cdot T = T$ for all $g^* \in H^\perp$, then clearly $\phi \cdot T = T$ for all $\phi \in co(H^\perp)$. Let $\psi \in H^\circ$ and (ϕ_α) be a net in $co(H^\perp)$ converging to ψ in the strict topology. For any $u \in A(G)$, by lemma 8.2.7, we have

$$\langle T, u\rangle = \langle \phi_\alpha \cdot T, u\rangle \to \langle \psi \cdot T, u\rangle.$$

Therefore, $\psi \cdot T = T$ for all $\phi \in H^{\circ}$.

(d) $\Rightarrow$ (b). It follows in the same way as [19, Proposition 3.1]((iii) $\Rightarrow$ (i)). We will give a proof for the sake of completeness. Let $x \in (H^{\perp})_{\perp}$. Then for $u \in A(G)$ and $g^* \in H^{\perp}$,

$$\langle g^* \cdot \lambda_2(x), u\rangle = (g^* u)(x) = u(x) = \langle \lambda_2(x), u\rangle.$$

We conclude that $\lambda_2(x) \in VN_H(G)$ since $g^* \cdot \lambda_2(x) = \lambda_2(x)$. Hence, $x \in H$ by the definition of $VN_H(G)$. □

Examples.

(a) Let G be the "ax+b"-group, and $H = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a > 0\}$. From the calculation in Example 3(i) of [19], we see that $H^{\perp} = \{1\}$, and hence $(H^{\perp})_{\perp} = G$.

(b) Let G be the Heisenberg group, and $H = \{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R}\}$, and let $K = \{\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}\}$. It is known(see [20]) that if G is a connected nilpotent group and H is a closed subgroup of G, then H is reflexive if and only if G is a normal of subgroup of G. It is straightforward to check that K is the smallest normal subgroup of G which contains H as a subgroup. Therefore, by corollary 8.2.4, we have $(H^{\perp})_{\perp} = K$.

(c) Let $G = G_{3,4}(\alpha)$, $\alpha \in \mathbb{R}$, $H = \{(0, x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, x, y) : x, y \in \mathbb{R}\}$. Then $(H^{\perp})_{\perp} = K$(see Example 3(iii) of [19]).

8.3 Dual spaces of quotient and product groups

Let N be a closed normal subgroup of G, and let $\sigma : G \longrightarrow G/N$ be the canonical homomorphism. Define $j : B(G/N) \longrightarrow B(G)_N$, $j(f) = f \circ \sigma$. Then j is an isometric isomorphism. Furthermore, we have $j(P_1(G/N)) = N^\circ$(See [11, Corollary 2.26]).

Proposition 8.3.1. The restriction of j on $(G/N)^*$ maps $(G/N)^*$ onto $N^\perp$.

Proof. If $\phi(\sigma(x)) = \langle \pi(\sigma(x))\xi, \xi\rangle$ for some $\pi \in (G/N)^\wedge$, then $\pi \circ \sigma \in \hat{G}$. Therefore, we have $j(\phi) = \phi \circ \sigma \in G^*$. It follows that j maps $(G/N)^*$ into $N^\perp$. Now, let $\psi \in N^\perp \subseteq P_1(G)$. Then $j^{-1}(\psi) \in P_1(G/N)$. If $j^{-1}(\psi) = (\bar{\psi}_1 + \bar{\psi}_2)/2$ for some $\bar{\psi}_1, \bar{\psi}_2 \in P_1(G/N)$, then $\psi = (j(\bar{\psi}_1) + j(\bar{\psi}_2))/2$ where $j(\bar{\psi}_1), j(\bar{\psi}_2) \in P_1(G)$. But $\psi \in G^*$, so $\psi = j(\bar{\psi}_1)$. Therefore, $j^{-1}(\psi) = \bar{\psi}_1$ is an extreme point of $P_1(G/N)$. □

Corollary 8.3.2. If N is a closed normal subgroup of G such that $N^\perp = \{1\}$, then $G = N$.

Proof. Note that $(G/N)^*$ consists of the constant function 1 only and separates points in G/N(see the proof of [12, Theorem 3.34]). It follows that $G/N = \{e_{G/N}\}$. □

Suppose that G_1, G_2 are locally compact groups, and π_1 and π_2 are representations of G_1 and G_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We recall that the definition of the outer tensor product of π_1 and π_2 of $G_1 \times G_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by

$$(\pi_1 \otimes \pi_2)(x, y) = \pi_1(x) \otimes \pi_2(y)$$

Remark.

(a) The map $\hat{G}_1 \times \hat{G}_2 \longrightarrow (G_1 \times G_2)^\wedge$, $(\pi_1, \pi_2) \mapsto \pi_1 \otimes \pi_2$ is a bijection if either G_1 or G_2 is type 1([12, Theorem 7.25]).

(b) $\pi_1 \otimes \pi_2$ and $\rho_1 \otimes \rho_2$ are unitary equivalent if and only if π_1 is unitary equivalent to ρ_1 and π_2 is unitary equivalent to ρ_2([12, Corollary 7.22]).

Proposition 8.3.3. Let G_1 be an abelian locally compact group, and G_2 a locally compact group. Then $G_1^* \times G_2^* \longrightarrow (G_1 \times G_2)^*$, $(g_1^*, g_2^*) \mapsto g_1^* \otimes g_2^*$ is a bijection where $g_1^* \otimes g_2^*$ is given by $g_1^* \otimes g_2^*(x, y) = g_1^*(x)g_2^*(y)$.

Proof. The map is well-defined as $\pi_1 \otimes \pi_2$ is irreducible if and only if π_1 and π_2 are irreducible, see [12, Corollary 7.20]. Given any $h^* \in (G_1 \times G_2)^*$, $h(x, y) = \langle \Pi(x, y)\epsilon, \epsilon\rangle$ for some $\Pi \in (G_1 \times G_2)^\wedge$, $\epsilon \in \mathcal{H}_\Pi$. Since Π is unitary equivalent to $\pi_1 \otimes \pi_2$ for some $\pi_1 \in \hat{G}_1$, $\pi_2 \in \hat{G}_2$, we have $h(x, y) = \langle \pi_1 \otimes \pi_2(x, y)\xi, \xi\rangle$ for some $\xi \in \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}$. Since $\mathcal{H}_{\pi_1}$ is one-dimensional, we may assume that $\xi = 1 \otimes \eta$ for some $\eta \in \mathcal{H}_{\pi_2}$. Thus, $h(x, y) = \langle \pi_1 \otimes \pi_2(x, y)(1 \otimes \eta), 1 \otimes \eta\rangle = \langle \pi_1(x)(1), 1\rangle\langle \pi_2(y)\eta, \eta\rangle$. So, the map is onto. The injectivity of it follows from the fact that $\hat{G}_1$ is a group and the last remark. □

8.4 Some results on the algebraic structure of G^*

A locally compact group G is a [MAP]-group if $\hat{G}_{\mathcal{F}}$ separates points in G.

Proposition 8.4.1. Let G be a locally compact group. Then G is a [MAP]-group if and only if $(G^*_{\mathcal{F}})_\perp = \{e\}$ where e is the identity element of G.

Proof. Suppose that G is a [MAP]-group. Let x be an element of $(G^*_{\mathcal{F}})_\perp$. For any $g^* \in G^*_{\mathcal{F}}$, we have $g^*(x) = 1$. Therefore, $\langle \pi(x)\xi, \xi\rangle = 1$ for any $\pi \in \hat{G}_{\mathcal{F}}$

and $\xi \in \mathcal{H}_\pi$. Consequently, $\pi(x) = \pi(e)$ and hence $x = e$. Conversely, let x and y be elements in G such that $\pi(x) = \pi(y)$ for any $\pi \in \hat{G}_{\mathcal{F}}$. Now, we have $\pi(xy^{-1}) = 1$. Therefore, for any $g^* \in G^*_{\mathcal{F}}$, we have $g^*(xy^{-1}) = 1$. It follows that $xy^{-1} = e$. □

For the definition of amenable representations, we refer readers to [3]. Let $\hat{G}_{am}$ be the set of all amenable irreducible representations of G. Let G^*_{am} be the set $\{x \mapsto \langle \pi(x)\xi, \xi\rangle : \pi \in \hat{G}_{am}, \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$. Since every finite dimensional representation is amenable(see [3, Theorem 1.3(i)]), it follows that $G^*_{\mathcal{F}}$ is a subset of G^*_{am}.

Proposition 8.4.2. Let G be a locally compact group. Then $\hat{G}_{am}$ separates points in G if and only if $(G^*_{am})_\perp = \{e\}$ where e is the identity element of G.

Proof. The proof is similar to proposition 8.4.1 and is left to readers. □

Theorem 8.4.3. Let G be a locally compact group. Consider the following conditions:

(a) G is abelian.

(b) G^* is a group.

(c) G^* is a semigroup.

(d) G^*_{am} is a semigroup.

(e) $G^*_{\mathcal{F}}$ is a semigroup.

(f) Every finite-dimensional irreducible unitary representation of G is one-dimensional.

Then we have (a)⇔(b)⇒(c)⇒(d)⇒(e)⇒(f).

Proof. It is trivial to see that "(a) ⇒ (b) ⇒ (c)". The implication "(c) ⇒ (d)" follows from [3, Corollary 5.4(ii)]. Since tensor products of finite dimensional representations is finite dimensional, "(d) ⇒ (e)" is also clear. Suppose that $G^*_{\mathcal{F}}$ is a semigroup. Let $g^* \in G^*_{\mathcal{F}}$. Then $g^*(\cdot) = \langle \pi(\cdot)\epsilon, \epsilon \rangle$ for some $\pi \in \hat{G}_{\mathcal{F}}$ and $\epsilon \in \mathcal{H}_\pi$. Since $g^* \cdot \bar{g^*} \in G^*_{\mathcal{F}}$, we conclude that $\pi \otimes \bar{\pi}$ is irreducible. However, the trivial representation 1 is a subrepresentation of $\pi \otimes \bar{\pi}$. We hence conclude that 1 is unitarily equivalent to $\pi \otimes \bar{\pi}$. It follows that g^* is invertible, and hence π is 1-dimensional(See [2]). □

Corollary 8.4.4. If G is a [MAP]-group, then all of below conditions are equivalent to each other:

(a) G is abelian.

(b) G^* is a group.

(c) G^* is a semigroup.

(d) G^*_{am} is a semigroup.

(e) $G^*_{\mathcal{F}}$ is a semigroup.

(f) Every finite-dimensional irreducible unitary representation of G is one-dimensional.

Proof. Assume that (f) holds. Let $\chi \in G^*_{\mathcal{F}}$, $x, y \in G$. We have

$$\chi(xyx^{-1}y^{-1}) = \chi(x)\chi(x)\chi(y)\chi(x) = 1.$$

By proposition 8.4.1, it follows that $xyx^{-1}y^{-1} = e$ for any $x, y \in G$. □

Corollary 8.4.5. Suppose that G is a [CCR]-group such that $\hat{G}_{am}$ separates points in G. Then all of below conditions are actually equivalent to each other:

(a) G is abelian.

(b) G^* is a group.

(c) G^* is a semigroup.

(d) G^*_{am} is a semigroup.

(e) Every amenable irreducible unitary representation of G is one-dimensional.

Proof. Suppose that G^*_{am} is a semigroup. Let $g^* \in G^*_{am}$. Then $g^*(\cdot) = \langle \pi(\cdot)\epsilon, \epsilon \rangle$ for some $\pi \in \hat{G}_{am}$ and $\epsilon \in \mathcal{H}_\pi$. Since $g^* \cdot \bar{g^*} \in G^*_{am}$, we conclude that $\pi \otimes \bar{\pi}$ is irreducible. However, the trivial representation 1 is always weakly contained in $\pi \otimes \bar{\pi}$. Since G is a [CCR]-group, we may easily conclude that 1 is unitarily equivalent to $\pi \otimes \bar{\pi}$. It follows that g^* is invertible, and hence π is 1-dimensional(See [2]). □

Remarks.

(a) Suppose that G has the property that $\hat{G}_{am}$ separates points in G. Then G is not necessarily amenable. In fact, if G is a [MAP] group, then $\hat{G}_{am}$ separates points in G since $\hat{G}_{\mathcal{F}}$ does. Therefore, $\mathbb{F}_2$, the free group on two generators, is a [MAP]-group which is not amenable.

(b) An amenable [CCR] group is not necessarily a [MAP] group(see [30]).

Chapter 9

Open Questions

Problem 1. Is it true in general that G^* is a semigroup if and only if G is abelian?

Problem 2. Does G^*, with some extra structures on it, characterize G?

Problem 3. Is $a_0(G)$ an algebra in general?

Problem 4. Are $\sigma_u(a(G))$ and $\sigma_{inv}(a(G))$ equal in general?

Problem 5. If the embedding $\phi : G \to \sigma_u(a(G))$ defined in the proof of theorem 3.3.1 is surjective, is it necessarily true that $A(G)$ has RNP?

Problem 6. Let G be a locally compact group. Does any one of the following statements hold in general?

(a) $G \in [AR]$ if and only if $IM(\hat{G}) = TIM(\hat{G})$.

(b) G is compact if and only if $FIM(\hat{G}) = TIM(\hat{G})$.

(c) $G \in [Moore]$ if and only if $IM(\hat{G}) = FIM(\hat{G})$.

Problem 8. Is it possible to embed $UCB(\hat{G})$ as subspaces of $vn(G)$?

Problem 9. Could we characterize $UCB(\hat{G})$ as a closed subspace of $VN(G)$ via G^*?

Problem 10. Is is true that G is amenable if and only if every completely complemented weak*-closed G^*-invariant subspace of $VN(G)$ is invariantly complemented.?

Problem 11. Could we characterize closed G^*-invariant subalgebras of $C_r^*(G)$ or $C^*(G)$?

Problem 12. Is there any non-commutative analogue of "$\hat{G}/H^{\perp} \cong \hat{H}$" by using G^*?

Problem 13. If G is non-abelian, is it possible to find a non-zero element f in $A(G)$ such that $g^*f = 0$ for some $g^* \in G^*$?

Problem 14. Is it true that G is abelian if and only if $G = \{x \in G : g^*(x) \neq 0\}$ for all $g^* \in G^*$?

Appendix A

Classical objects or notions and their ”dual” versions

OBJECTS AND NOTIONS	”DUAL” OBJECTS OR NOTIONS
G	$\widehat{G}$ or G^*
$L^1(G)$	$A(G)$
$M(G)$	$B(G)$
$C_0(G)$	$C^*(G)$
$L^\infty(G)$	$VN(G)$
$UCB(G)$	$UCB(\widehat{G})$
$\Vert \cdot \Vert_\infty$	$\Vert \cdot \Vert_{C^*}$
$\Vert \cdot \Vert_{C^*}$	$\Vert \cdot \Vert_\infty$
Discrete group	Compact group
Compact group	Discrete group

Appendix B

New objects or notions and their "dual" versions

Objects or notions	When G is abelian	Definition
G^*	$\delta_{\hat{G}}$	$\mathcal{E}(P_1(G))$
$G^*_{\mathcal{F}}$	$\delta_{\hat{G}}$	$\{\pi(\cdot)\xi, \xi\rangle : \pi \in \hat{G}_{\mathcal{F}}, \xi \in \mathcal{H}_\pi, \|\xi\| = 1\}$
$a_0(G)$	$l^1(\hat{G})$	Norm closure of span(G^*)
$a(G)$	$l^1(\hat{G})$	Norm closure of $\langle G^* \rangle$
$A_{\mathcal{F}}(G)$	$l^1(\hat{G})$	Norm closure of span$(G^*_{\mathcal{F}})$
$vn_0(G)$	$l^\infty(\hat{G})$	Dual space of $a_0(G)$
$vn(G)$	$l^\infty(\hat{G})$	Dual space of $a(G)$

($\hat{G}_{\mathcal{F}}$ is the class of all finite dimensional irreducible representations and $\langle G^* \rangle$ denotes the algebra generated by G^* in $B(G)$.)

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