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FIELD THEORY OF HYDRODYNAMICS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled FIELD THEORY OF HYDRO-DYNAMICS, submitted by Manu B. Paranjape, in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

We consider the field theory of barotropic, inviscid hydrodynamics. Introducing the Clebsch transformation, we are able to obtain the classical hydrodynamic equations from a Lagrangian, through a variational principle. Using the canonical quantization procedure, we quantize the hydrodynamic field equations. We find conserved currents from invariant transformations and the Noether theorem. We can obtain the results of previous authors concerning the spectrum of elementary excitations. Finally, we observe the spontaneous breakdown of symmetry in hydrodynamics, and determine the origin of the resulting Goldstone boson.

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CHAPTER I

INTRODUCTION

The theory of fluids can be formulated from two general viewpoints; the microscopic and the macroscopic viewpoints. The microscopic theory attempts to explain the behaviour of the fluid from the dynamics and interactions of the individual constituent particles, quantum or classical, that make up the fluid. Although such a description can yield, if exactly solved, the complete description of the fluid, it will always be specialized to the particular system under consideration. One must specify the dynamics and interactions of the constituent particles completely. Independently of the nature of the constituents, the fluid must satisfy the macroscopic balance equations; and the two lowest order of these are commonly called the hydrodynamic equations. The relationship between equilibrium statistical mechanics and thermodynamics, is the most obvious example of this situation. Any microscopic theory must not contradict the results of a purely macroscopic theory.

This thesis project is on the hydrodynamic equations. We will study these equations from the field theoretical viewpoint, both classical and quantum. These equations are usually the two lowest order macroscopic balance equations, representing mass conservation, and the conservation of the three components of momentum. Thus we have four equations between the five field quantities, the density, the three components of velocity, and the pressure. To close this system of equations, we must externally put in a constitutive relation. For this entire thesis, we will assume that pressure is a known function of density alone. This is the condition of barotropy. Even though s[illegible] condition severely restricts the applicability of the th[illegible] without such a simplification, the field theory of the equations is almost impossible. We still have some freedom under this constraint, since the functional dependence of pressure on density is not specified.

We will consider the hydrodynamic equations as field equations, and the hydrodynamic variables as field quantities. The equations of hydrodynamics have two different forms, the Euler equations and the Lagrange equations. The Lagrange equations consider the motion of infinitesimal mass elements, and give the behaviour of the system as a function of initial coordinates and time. Such a representation, though equivalent to the Euler representation, is not amenable to the field theoretical view, as it contains reference to initial coordinates. The Euler equations are relations between the density, velocity and pressure, considered as functions of space and time coordinates, so are exactly in the form required for a field theory. We will consider the field theory of hydrodynamics in the Euler formalism.

We will first consider the classical field theory of hydrodynamics. We will then quantize under the procedure of canonical quantization. We can familiarize ourselves with the quantum field theoretical technique through studying this system as a specific example. Using the Noether theorem, we can construct many conserved currents due to invariant transformations of the equations. Studying the generators of the transformations, and the transformation properties of field quantities, we find some cases of spontaneously broken symmetry. Thus we have a chance to study the technique of spontaneous symmetry breakdown. Studying the spectral representations of the two point functions associated with the spontaneous symmetry breakdown, we can establish the origin of the Goldstone boson. Finally, we close with a discussion of the transverse excited mode, and Kubo type expressions for transport coefficients such as viscosity and future problems.

CHAPTER II

CLASSICAL FIELD THEORY OF HYDRODYNAMICS

The Hydrodynamic Equations

The usual equations of hydrodynamics, in the Euler formalism, are expressions of the conservation of mass and conservation of momentum. They are derived from the straightforward applications of these principles, to a continuous medium, where $\rho(\vec{x},t)$ and $\vec{V}(\vec{x},t)$ are the density and velocity of the medium at position $\vec{x}$ and time t. The conservation of mass is in the form of a continuity equation,

$$\frac{\partial\rho}{\partial t} + \partial_i(\rho V_i) = 0 \quad . \qquad (2.1)$$

The equation resulting from conservation of momentum is

$$\frac{\partial V_i}{\partial t} + (V_j\partial_j)V_i = -\frac{1}{\rho}\partial_i p(\rho) \quad , \quad i = 1,2,3. \qquad (2.2)$$

$p(\rho)$ is the pressure, where we have assumed it is a function of ρ alone, but the explicit dependence is not specified, and is to be supplied externally.

To obtain these equations from a Lagrangian one must modify them somewhat by integrating the equation of conservation of momentum, and introducing new hydrodynamic variables. The first successful demonstration of a Lagrangian which gave the equations of hydrodynamics, under the constraint of barotropy, was by Bateman[1]. Following the work of Hill[2], Clebsch[3] and Lamb[4], Bateman supplied the Lagrangian for a barotropic, inviscid fluid. Crucial to this Lagrangian, is the Clebsch transformation of hydrodynamic variables.

The Clebsch transformation is given by

$$V_i = -\partial_i \phi(\vec{x},t) + \lambda(\vec{x},t)\,\partial_i\,\psi(\vec{x},t) \quad . \tag{2.3}$$

A totally arbitrary velocity may be expressed in terms of the Clebsch potentials $\phi(\vec{x},t)$, $\lambda(\vec{x},t)$, and $\psi(\vec{x},t)$, as long as the lines of the curl of that velocity are integrable as the intersection of two surfaces. So if

$$\vec{\omega} = \vec{\nabla} \times \vec{V} \tag{2.4}$$

and the lines of $\vec{\omega}$ are integrated as the intersection of $\alpha(\vec{x},t)$ = constant and $\beta(\vec{x},t)$ = constant, we have

$$\vec{\omega} = P(\vec{x},t)(\vec{\nabla}\alpha \times \vec{\nabla}\beta) \quad . \tag{2.5}$$

But

$$\vec{\nabla}\cdot\vec{\omega} = \vec{\nabla}P\cdot(\vec{\nabla}\alpha \times \vec{\nabla}\beta) = 0 \quad . \tag{2.6}$$

implies

$$\frac{\partial(P,\alpha,\beta)}{\partial(x,y,z)} = 0 \tag{2.7}$$

which is just the condition

$$\dot{P} = P(\alpha,\beta) \quad . \tag{2.8}$$

If λ and ψ are functions of α and β, that is

$$\lambda = \lambda(\alpha,\beta) \tag{2.9a}$$

$$\psi = \psi(\alpha,\beta) \quad , \tag{2.9b}$$

then

$$\vec{\nabla}\lambda \times \vec{\nabla}\psi = \frac{\partial(\lambda,\psi)}{\partial(\alpha,\beta)} (\vec{\nabla}\alpha \times \vec{\nabla}\beta) \quad . \tag{2.10}$$

So we need to solve

$$P(\alpha,\beta) = \frac{\partial(\lambda,\psi)}{\partial(\alpha,\beta)} \tag{2.11}$$

which has an infinity of solutions, to guarantee that we can find λ and ψ so that

$$\vec{\omega} = \vec{\nabla}\lambda \times \vec{\nabla}\psi \quad . \tag{2.12}$$

Now it is an easy matter to find ϕ, since

$$-\vec{\nabla}\phi = \vec{V} - \lambda\vec{\nabla}\psi \tag{2.13}$$

and we know this equation is consistent since the curl of both sides vanishes. So taking the divergence of both sides, we simply have a Poisson equation for ϕ, and

$$\phi(\vec{x},t) = \int d^3x' \left[G(x,x')\vec{\nabla}'\cdot\left(\lambda(x',t)\vec{\nabla}'\psi(x',t) - \vec{V}(x',t)\right)\right] \tag{2.14}$$

where G(x,x') is the Green's function for the Poisson equation.

Then replacing the Clebsch potentials in the velocity, in the momentum conservation equation, we get

$$\frac{\partial}{\partial t}(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi) + \left((-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot\vec{\nabla}\right)(-\nabla\vec{\phi} + \lambda\vec{\nabla}\psi) = -\frac{1}{\rho}\vec{\nabla}\left[p(\rho)\right] \, . \quad (2.15)$$

Simplifying

$$\vec{\nabla}(-\dot{\phi} + \lambda\dot{\psi}) - (\vec{\nabla}\lambda)\dot{\psi} + \dot{\lambda}\vec{\nabla}\psi + \frac{1}{2}\vec{\nabla}(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)^2$$

$$- (-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi) \times \left(\vec{\nabla} \times (\lambda\vec{\nabla}\psi)\right) = -\vec{\nabla}\left[\int_{\rho_o}^{\rho}(\frac{1}{\rho}\frac{\partial p}{\partial \rho})\, d\rho\right] \quad (2.16)$$

where

$$(\vec{A}\cdot\vec{\nabla})\vec{A} = \frac{1}{2}\vec{\nabla}(\vec{A}\cdot\vec{A}) - \vec{A}\times(\vec{\nabla}\times\vec{A}). \quad (2.17)$$

was used, and ρ_o is a space and time independent constant. So finally we get

$$\vec{\nabla}\{\dot{\phi} - \lambda\dot{\psi} - \frac{1}{2}(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi) - \int_{\rho_o}^{\rho}\frac{1}{\rho}\frac{\partial p}{\partial \rho}\, d\rho\}$$

$$= -\vec{\nabla}\lambda\dot{\psi} + \dot{\lambda}\vec{\nabla}\psi - (-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\times(\vec{\nabla}\lambda\times\vec{\nabla}\psi) \, . \quad (2.18)$$

Now the right hand side of this expression is equal to

$$\text{R.H.S.} = -\vec{\nabla}\lambda\dot{\psi} + \dot{\lambda}\vec{\nabla}\psi - \left(\vec{\nabla}\lambda(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot\vec{\nabla}\psi - \vec{\nabla}\psi(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot\vec{\nabla}\lambda\right)$$

$$= -\vec{\nabla}\lambda\left(\dot{\psi} + (-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot\vec{\nabla}\psi\right) + \vec{\nabla}\psi\left(\dot{\lambda} + (-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)\cdot\vec{\nabla}\lambda\right)$$

$$= -\vec{\nabla}\lambda(\frac{D\psi}{Dt}) + \vec{\nabla}\psi(\frac{D\lambda}{Dt}) \quad (2.19)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} \qquad (2.20)$$

the Lagrange derivative or derivative following the motion. Now it can be proved that the flux of the curl of the velocity through any surface which moves with the fluid is constant[5]. Thus since the curl of the velocity was given by the intersection of two surfaces if we allow these surfaces to move with the liquid, they will continue to give the curl of the velocity for all time. Thus we can put the constraint equations, on the Clebsch potentials which give $\vec{\omega}$,

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + \vec{V}\cdot\vec{\nabla}\psi = 0 \qquad (2.21a)$$

$$\frac{D\lambda}{Dt} = \frac{\partial \lambda}{\partial t} + \vec{V}\cdot\vec{\nabla}\lambda = 0 \; . \qquad (2.21b)$$

Thus the right hand side of equation (18) is zero. Then we can integrate the left hand side of equation (18), which will give

$$\dot{\phi} - \lambda\dot{\psi} - \frac{1}{2}(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)^2 - \int_{\rho_o}^{\rho} \left(\frac{1}{\rho}\frac{\partial p}{\partial \rho}\right) d\rho = 0 \qquad (2.22)$$

where the arbitrary constant of integration is absorbed into ϕ by putting $\phi \rightarrow \phi + ct$, where c is the constant of integration, since this still leaves the velocity invariant. We will call this equation the Bernoulli equation.

Lagrangian and Hamiltonian Formalism

Now we have four field equations, namely the continuity equation, the Bernoulli equation, and the two constraint equations (2.21a) and (2.21b). We wish to obtain these hydrodynamic equations through a Lagrangian and variational principle. The Lagrangian originally given by Bateman[6] is

$$L = \int d^3x \ \mathcal{L}(x) = \int d^3x \ \{\rho(\dot{\phi} - \lambda\dot{\psi} - \tfrac{1}{2}(\vec{V})^2 - \omega(\rho))\} \quad (2.23)$$

where

$$\omega(\rho) = \int_{\rho_o}^{\rho} \left(\frac{p(\rho) - p_o}{\rho^2}\right) d\rho \quad . \quad (2.24)$$

and

$$p_o = p(\rho_o) \quad (2.25)$$

$\vec{V}$ in the above Lagrangian is to be considered short hand for $-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi$. Varying with ϕ gives the continuity equation and varying with ρ gives the Bernoulli equation. The two constraint equations come from varying with respect to λ and ψ.

If we use the Bernoulli equation we can replace in the expression for the Lagrangian density, giving

$$\mathcal{L}(x) = \rho \left\{ \int_{\rho_o}^{\rho} \left(\frac{1}{\rho}\frac{\partial p}{\partial \rho}\right) d\rho - \int_{\rho_o}^{\rho} \left(\frac{p(\rho) - p(\rho_o)}{\rho^2}\right) d\rho \right\} \quad (2.26)$$

integrating the second integral by parts, we get the simple expression

$$\mathcal{L}(x) = p(\rho) - p(\rho_o) \quad . \tag{2.27}$$

This is very useful, from the physical point of view, as it gives the meaning that the Lagrangian is the pressure fluctuation. Through the variational principle we are minimizing the pressure fluctuation.

Considering ρ and λ as canonical coordinates has its drawbacks, since the momenta conjugate to these variables vanish identically. On the other hand considering ϕ and ψ as canonical coordinates, and the Lagrangian as having been obtained from a Hamiltonian via a Legendre transformation, with the explicit dependence of the canonically conjugate momenta, on the time derivatives of the canonical coordinates, not yet replaced, is a much more consistent view, since the conjugate momenta are given by

$$\pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \rho . \tag{2.28}$$

$$\pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = - \rho \lambda \tag{2.29}$$

which can be used to just eliminate ρ and λ. Then defining the Hamiltonian in the usual way, we get

$$H = \int d^3x \, \mathcal{H}(x) = \int d^3x \, \left(\rho\dot{\phi} + (-\rho\lambda)\dot{\psi} - \mathcal{L}\right)$$

$$= \int d^3x \left(\frac{1}{2} \rho (\vec{V})^2 + \rho\omega(\rho)\right) \tag{2.30}$$

where we write

$$\vec{V} = - \vec{\nabla}\phi - \frac{\pi}{\rho} \vec{\nabla}\psi \tag{2.31}$$

and

$$\pi \equiv \pi_\psi = - \rho\lambda . \tag{2.32}$$

The Hamiltonian density is positive definite if $\rho\omega(\rho)$ is positive definite. From the form of $\omega(\rho)$, this can be ensured by restricting $p(\rho)$ to be a monotone increasing function of ρ over the positive real axis, however this is not a necessary restriction, although it is physically reasonable.

So written in terms of the canonical coordinates and conjugate momenta, the Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2} \rho\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \vec{\nabla}\phi\cdot\pi\vec{\nabla}\psi + \frac{1}{2} \frac{\pi^2}{\rho} \vec{\nabla}\psi\cdot\vec{\nabla}\psi + \rho \int_{\rho_o}^{\rho} \left(\frac{p(\rho) - p_o}{\rho^2}\right) d\rho \tag{2.33}$$

The canonical equations yield the four hydrodynamic equations

$$\dot{\phi} = \frac{\delta\mathcal{H}}{\delta\rho} = - \frac{\pi}{\rho} \dot{\psi} + \frac{1}{2} (\vec{V})^2 + \int_{\rho_o}^{\rho} (\frac{1}{\rho} \frac{\partial p}{\partial\rho}) d\rho \tag{2.34a}$$

$$\dot{\rho} = - \frac{\delta\mathcal{H}}{\delta\phi} = - \vec{\nabla}\cdot(\rho\vec{V}) \tag{2.34b}$$

$$\dot{\psi} = \frac{\delta\mathcal{H}}{\delta\pi} = - \vec{V} \cdot \vec{\nabla}\psi \tag{2.34c}$$

$$\dot{\pi} = - \frac{\delta\mathcal{H}}{\delta\psi} = - \vec{\nabla}\cdot(\pi\vec{V}) \tag{2.34d}$$

where

$$\frac{\delta}{\delta f(x)} = \frac{\partial}{\partial f} - \partial_i \frac{\partial}{\partial \partial_i f} . \qquad (2.35)$$

The independent variables then are ϕ and ψ with canonically conjugate momenta ρ and π respectively. Then this system is consistent, there is no difficulty when going to the quantum theory with vanishing conjugate momenta.

Conserved Currents

Now since the equations for ρ and π appear as conservation laws one would expect these to arise as Noether currents from invariant transformations. Using the form of the Noether theorem which uses just the Hamiltonian formulation; we have the Lagrangian, written in terms of the canonical coordinates and momenta as independent variables, is invariant under the transformation

$$\phi \to \phi' = \phi + \phi_o \qquad (2.36)$$

where ϕ_o is a constant. The conserved Noether current then is

$$\left.\begin{aligned} N_o &= -\rho(\delta\phi) = -\rho\phi_o \\ N_i &= \frac{\partial \mathcal{H}}{\partial \partial_i \phi}\delta\phi = -\rho V_i \phi_o \end{aligned}\right\} \qquad (2.37)$$

and

$$\partial_\mu N_\mu = 0 \qquad (2.38)$$

gives the continuity equation

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \quad . \tag{2.39}$$

Furthermore, the transformation

$$\psi \rightarrow \psi' = \psi + \psi_o \tag{2.40}$$

where ψ_o is a constant, gives the Noether current

$$\left.\begin{aligned} N_o &= -\pi\psi_o \\ N_i &= -\pi V_i\psi_o \end{aligned}\right\} \tag{2.41}$$

and the equation

$$\dot{\pi} + \vec{\nabla}\cdot(\pi\vec{V}) = 0 \tag{2.42}$$

results from the conservation equation.

The equation for ψ also can be obtained as a continuity equation, since

$$\frac{\partial}{\partial t}(\rho\psi) + \vec{\nabla}\cdot(\rho\psi\vec{V}) = 0 \quad . \tag{2.43}$$

This equation can be obtained by considering the transformation

$$\left.\begin{aligned} \phi &\rightarrow \phi' = \phi + \varepsilon\psi \\ \pi &\rightarrow \pi' = \pi + \varepsilon\rho \end{aligned}\right\} \tag{2.44}$$

where ε is an infinitesimal parameter. Then the Noether current obtained is,

$$\left.\begin{aligned} N_o &= -\rho\,\varepsilon\,\psi \\ N_i &= -\rho V_i \varepsilon \psi \; ; \end{aligned}\right\} \qquad (2.45)$$

and the corresponding conservation equation leads to

$$\dot{\psi} + \vec{V}\cdot\vec{\nabla}\psi = 0 \quad . \qquad (2.46)$$

One transformation under which the Lagrangian fails to show invariance is the scale transformation. One would expect any theory to be invariant under a scale transformation, since physically the theory should be invariant of the choice of unit dimension taken. However due to the existence of constants in the theory, which are not dimensionless, explicitly ρ_o, the Lagrangian fails to be invariant under the theory. The infinitesimal scale transformation is

$$\left.\begin{aligned} &x_\mu \rightarrow x'_\mu = (1 + \varepsilon)\, x_\mu \\ &\varepsilon \text{ an infinitesimal constant} \end{aligned}\right\} \qquad (2.47)$$

we must impose the transformation of the field variables as to their dimension, in units $\hbar = c = 1$, so

$$\left.\begin{aligned}
\phi(\vec{x},t) &\to \phi'(\vec{x}',t') = (1+\varepsilon)\phi(\vec{x},t) \\
\psi(\vec{x},t) &\to \psi'(\vec{x}',t') = (1+\varepsilon)\psi(\vec{x},t) \\
\rho(\vec{x},t) &\to \rho'(\vec{x}',t') = (1-4\varepsilon)\rho(\vec{x},t) \\
\pi(\vec{x},t) &\to \pi'(\vec{x}',t') = (1-4\varepsilon)\pi(\vec{x},t)
\end{aligned}\right\} \qquad (2.48)$$

Then, the Lagrangian is transformed as the

$$d^4x\ \mathcal{L}(\vec{x},t) \to d^4x'\mathcal{L}(\vec{x}',t') = d^4x(1+4\varepsilon)\rho(\vec{x},t)(1-4\varepsilon)$$

$$\times\ \{\dot{\phi}(\vec{x},t) + \frac{\pi(\vec{x},t)}{\rho(\vec{x},t)}\ \dot{\psi}(\vec{x},t) - \frac{1}{2}\vec{V}^2(\vec{x},t) - \omega((1-4\varepsilon)\rho(\vec{x},t))\}$$

$$= d^4x\left(\mathcal{L}(\vec{x},t) + 4\varepsilon\ \frac{\partial\omega(\rho)}{\partial\rho}\ \rho^2(\vec{x},t)\right)$$

$$= d^4x\left(\mathcal{L}(\vec{x},t) + 4\varepsilon(p(\rho) - p(\rho_o))\right) \qquad (2.49)$$

We can see the scale deficiency[7] is given by

$$d^4x'\ \mathcal{L}(\vec{x}',t') - d^4x\ \mathcal{L}(\vec{x},t) = d^4x\left(4\varepsilon(p(\rho)-p(\rho_o))\right). \qquad (2.50)$$

Now using the equation of motion, we have shown $\mathcal{L}(x) = p(\rho(x)) - p(\rho_o)$. Thus

$$d^4x'\ \mathcal{L}(\vec{x}',t') - d^4x\mathcal{L}(\vec{x},t) = d^4x\ (4\varepsilon\ \mathcal{L}(\vec{x},t)). \qquad (2.51)$$

The Noether current is defined as[8]

$$\left.\begin{aligned} S_o &= -\left(\rho(\phi - x_\nu \partial_\nu \phi) + \pi(\psi - x_\nu \partial_\nu \psi)\right) - t\,\mathcal{L}(\phi,\psi,\rho,\pi) \\ S_i &= -\left(\rho V_i(\phi - x_\nu \partial_\nu \phi) + \rho V_i \frac{\pi}{\rho}(\psi - x_\nu \partial_\nu \psi)\right) - x_i\,\mathcal{L}(\phi,\psi,\rho,\pi) \end{aligned}\right\} \quad (2.52)$$

and the conservation equation is modified by the scale deficiency,

$$\partial_\mu S_\mu = -4\left(p(\rho) - p(\rho_o)\right) = -4\,\mathcal{L}(\vec{x},t) \quad . \qquad (2.53)$$

CHAPTER III

QUANTUM FIELD THEORY OF HYDRODYNAMICS

Transition to the Quantum Viewpoint

In the previous section, we have obtained a classical Lagrangian and Hamiltonian formulation of the hydrodynamic equations. In as much as these equations truly represent the classical behaviour of physical system, we can formulate the quantum behaviour of the system through canonical quantization. Canonical quantization is not the only way of formulating the quantum behaviour of a hydrodynamic system, for example, Landau[9] formulated quantum hydrodynamics through an application of quantum mechanics to the equations of hydrodynamics and reinterpreting the variables of density, velocity, etc., in terms of particle quantum mechanical operators. However, although there is no proof that canonical quantization will actually describe the quantum behaviour of a system, it will give a logical and consistent method for quantization of the system.

In the quantum formulation, we wish to reinterpret the classical field variables as quantum field theoretical operators. In doing so with a classical system, there are certain ambiguities which surface[10]. If the classical equations contain products of field variables, then from the quantum viewpoint, we do not know which order to take the product. There is no rule which satisfactorily resolves this ambiguity. In certain cases we can apply general rules to guide us in choosing the appropriate order of factors. For example, we can insist that the Hamiltonian be hermitean and lead to the correct equations of motion. However, in most cases, we must compromise between symmetrization of the classical expressions, and increasing complexity with symmetrization. Another aspect of the classical equations are analytic and sometimes non-analytic (at certain points) functions of the field variables which may occur. In such cases, we must use the Taylor expansion of these functions, and then formally associate the quantum field operator with the Taylor expansion. For points of singularity, we simply assume neither the classical or quantum expression have meaning at such points.

The first classical expression we will consider is the velocity. Classically,

$$\vec{V} = -\vec{\nabla}\phi - \frac{\pi}{\rho}\vec{\nabla}\psi \quad . \tag{3.1}$$

Now the variables ϕ, ρ, ψ, and π can straightforwardly be interpreted as quantum mechanical operators. Then the function $1/\rho$ has a Taylor expansion at every point except $\rho = 0$, thus it will be interpreted in terms of its Taylor expansion about the equilibrium density ρ_o, taken as a c-number. So we take

$$\frac{1}{\rho} = \frac{1}{\rho_o}\left[1 + \left(\frac{\rho_o - \rho}{\rho_o}\right) + \left(\frac{\rho_o - \rho}{\rho_o}\right)^2 + \dots\right] \qquad (3.2)$$

and $(\rho_o - \rho)$ will be a q-number. Since we are going to use canonical quantization, ρ will commute with all expressions except those containing ϕ, thus we will expect $1/\rho$ to commute with the other variables in the second term. Thus we put forward the velocity in terms of quantum field operators,

$$\vec{V} = -\vec{\nabla}\phi - \frac{1}{\rho}\frac{1}{2}\left(\pi(\vec{\nabla}\psi) + (\vec{\nabla}\psi)\pi\right) \qquad . \qquad (3.3)$$

This will of course reduce to the classical expression if we allow the operators to commute.

Next we wish to consider the Hamiltonian. The second term in the Hamiltonian density is a function of ρ only so we will assume that we stay only in its domain of analyticity, and use its Taylor expansion for the quantum viewpoint. The first term in the Hamiltonian density, which requires modification, is $\frac{1}{2}\rho(\vec{V})^2$. As all the fields are real, the condition of hermiticity implies the expression be symmetric in terms of real fields. We could take $\frac{1}{2}\left(\frac{1}{2}(\rho(\vec{V})^2 + (\vec{V})^2\rho)\right)$; however this kind of term unduly complicates the equations of motion. It is better to take $\frac{1}{2}\,\vec{V}\cdot\rho\vec{V}$, as this is symmetric, and leads to natural generalizations of the classical equation. Thus we will take as the quantum field theoretical Hamiltonian

$$H = \int d^3x \, \mathcal{H}(x) = \int d^3x \left(\frac{1}{2} \vec{V} \cdot \rho \cdot \vec{V} + \rho\omega(\rho) \right) , \qquad (3.4)$$

where $\vec{V}$ is taken as the quantum velocity defined above.

Canonical Equations and Quantization

We find upon variation with the canonical variables, the quantum field theoretical equations,

$$\dot{\phi} = \frac{1}{2} (\vec{V})^2 + \frac{1}{2} \left\{ \frac{1}{2\rho} \left(\pi(\vec{\nabla}\psi) + (\vec{\nabla}\psi)\pi \right) \cdot \vec{V} + \vec{V} \cdot \frac{1}{2\rho} \left(\pi(\vec{\nabla}\psi) + (\vec{\nabla}\psi)\pi \right) \right\}$$

$$+ \int_{\rho_0}^{\rho} \frac{dp(\rho)}{\rho} \qquad (3.5a)$$

$$\dot{\rho} = - \frac{1}{2} \vec{\nabla} \cdot (\rho\vec{V} + \vec{V}\rho) \qquad (3.5b)$$

$$\dot{\psi} = - \frac{1}{2} \left((\vec{\nabla}\psi) \cdot \vec{V} + \vec{V} \cdot (\vec{\nabla}\psi) \right) \qquad (3.5c)$$

$$\dot{\pi} = - \frac{1}{2} \vec{\nabla} \cdot (\pi\vec{V} + \vec{V}\pi) \quad . \qquad (3.5d)$$

Again, these equations will reduce to the classical expressions if we allow the variables to commute.

To formally complete the quantization of the hydrodynamic equations, we assume the canonical commutation relations, at equal time

$$[\phi(\vec{x},t), \rho(\vec{x}',t)] = i\delta(\vec{x} - \vec{x}') \qquad (3.6a)$$

$$[\psi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x} - \vec{x}') \qquad (3.6b)$$

and all other commutators zero.

We note that since ϕ, ρ, ψ and π are real, there is no ambiguity with the statistics of the corresponding quanta; we must use commutators, not anti-commutators, and the particles will obey Bose statistics.

Physical Commutators

With these canonical commutators, we can derive certain commutation relations between operators representing more physical quantities. These commutators were first derived by Landau, using his form of quantum hydrodynamics. These commutators are useful in further work. Firstly,

$$[\rho(\vec{x},t), V_i(\vec{x}',t)] = -i\partial_i\left(\delta(\vec{x}-\vec{x}')\right) \tag{3.7}$$

which is easily derived, and second,

$$[V_i(\vec{x},t), V_j(\vec{x}',t)] = \frac{i\delta(\vec{x}-\vec{x}')}{\rho(x)}\,\varepsilon_{ijk}\,\omega_k \tag{3.8}$$

where

$$\vec{\omega} = \vec{\nabla} \times \vec{V}\ , \tag{3.9}$$

which requires considerable calculation.

Conserved Currents

We can obtain conserved quantities via the Noether theorem, similarly to how we obtained these in the classical case. The obvious invariances, omitted for the classical formulation, space-time translation, space rotation, will be discussed now, and can easily be applied to the classical case. First we will redefine the Lagrangian, as a function of the canonical coordinates, their time derivatives, and conjugate momenta, by a symmetrized Legendre transformation of the Hamiltonian,

$$L = \int d^3x\, \mathcal{L}(x) = \int d^3x \left(\frac{1}{2}(\dot{\phi}\rho + \rho\dot{\phi}) + \frac{1}{2}(\dot{\psi}\pi + \pi\dot{\psi}) - \mathcal{H}(x)\right)$$

$$= \int d^3x \left(\frac{1}{2}(\dot{\phi}\rho + \rho\dot{\phi}) + \frac{1}{2}(\dot{\psi}\pi + \pi\dot{\psi}) - \frac{1}{2}\vec{V}\cdot\rho\vec{V} - \rho\omega(\rho)\right) \quad . \tag{3.10}$$

Now this Lagrangian is obviously invariant under space-time translation

$$x_\mu \to x'_\mu = x_\mu + \varepsilon_\mu \tag{3.11}$$

where ε_μ are infinitesimal parameters. The conserved Noether current is the energy momentum tensor, $T_{\mu\nu}$, and is given by

$$\left.\begin{aligned} T_{o\nu} &= \frac{1}{2}(\pi_\alpha \partial_\nu \phi_\alpha + \partial_\nu \phi_\alpha \pi_\alpha) - \delta_{o\nu}\left(\frac{1}{2}\dot{\phi}_\alpha \pi_\alpha + \pi_\alpha \dot{\phi}_\alpha - \mathcal{H}\right) \\ T_{i\nu} &= -\frac{1}{2}\left(\frac{\partial \mathcal{H}}{\partial \partial_i \phi_\alpha}\partial_\nu \phi_\alpha + \partial_\nu \phi_\alpha \frac{\partial \mathcal{H}}{\partial \partial_i \phi_\alpha}\right) - \delta_{i\nu}\left(\frac{1}{2}(\dot{\phi}_\alpha \pi_\alpha + \pi_\alpha \dot{\phi}_\alpha) - \mathcal{H}\right) \end{aligned}\right\} \tag{3.12}$$

where the sum over α is over all canonical coordinates. Explicitly

$$T_{oo} = \mathcal{H} \tag{3.13a}$$

$$T_{oi} = -\frac{1}{2}(\rho V_i + V_i\rho) \quad . \tag{3.13b}$$

$$T_{io} = \frac{1}{2}\Big(\frac{1}{2}(\rho V_i+V_i\rho)\dot{\phi}+\dot{\phi}\frac{1}{2}(\rho V_i+V_i\rho)+\frac{1}{2}(\pi V_i+V_i\pi)\dot{\psi}+\dot{\psi}\frac{1}{2}(\pi V_i+V_i\pi)\Big) \tag{3.13c}$$

$$T_{ij} = \frac{1}{2}\Big(\frac{1}{2}(\rho V_i+V_i\rho)\partial_j\phi+(\partial_j\phi)\frac{1}{2}(\rho V_i+V_i\rho)+\frac{1}{2}(\pi V_i+V_i\pi)(\partial_j\psi)$$
$$+ (\partial_j\psi)\frac{1}{2}(\pi V_i+V_i\pi)\Big) - \delta_{ij}\Big(\frac{1}{2}(\dot{\phi}\rho+\rho\dot{\phi})+\frac{1}{2}(\pi\dot{\psi}+\dot{\psi}\pi) - \mathcal{H}\Big) . \tag{3.13d}$$

If we replace the expression for V_i in T_{ij}, we can show that T_{ij} is symmetric in i and j. We can show,

$$T_{ij} = -\frac{1}{4}\Bigg[\Big\{\{\rho,\partial_i\phi\},\partial_j\phi\Big\} + \Big\{\{\pi,\partial_i\psi\},\partial_j\phi\Big\} + \Big\{\{\pi,\partial_i\phi\},\partial_j\psi\Big\}$$
$$+ \frac{1}{2\rho}\Big\{\{\pi,\{\pi,\partial_i\psi\}\},\partial_j\psi\Big\}\Bigg] + \delta_{ij}\mathcal{L} \quad . \tag{3.14}$$

Now observing each term in this expansion and noting which variables commute, we can see the expression is symmetric in i and j. The conservation law associated with this Noether current is

$$\partial_\mu T_{\mu\nu} = 0 \quad . \tag{3.15}$$

We also expect the theory to be invariant under space rotation,

$$\left.\begin{aligned} x_i \to x_i' &= x_i + \varepsilon_{ij} x_j \\ t \to t' &= t \\ \text{and} \qquad \varepsilon_{ij} &= -\varepsilon_{ij} \end{aligned}\right\} \text{Space rotation .} \qquad (3.16)$$

ε_{ij} is infinitesimal and antisymmetric. Now the conserved current is given by

$$M_{ij\mu} = x_i T_{\mu j} - x_j T_{\mu i} \quad , \qquad (3.17)$$

where $T_{\mu j}$ is the energy-momentum tensor above. The conservation equation is

$$\partial_\mu M_{ij\mu} = 0 \quad . \qquad (3.18)$$

One other transformation under which we expect the Lagrangian to be invariant is the Galilei transformation. This is since the classical hydrodynamic equations were derived from the principle of conservation of mass and momentum, these are both Galilei invariant concepts. We get the hint of the transformation properties of the fields from knowing how the Schrödinger field is transformed under the Galilei transformation, and knowing the Schrödinger field is equivalent to an irrotational hydrodynamic velocity field where the "pressure" term may depend on the density and its derivatives. Thus, for the Galilei transformation,

$$\left.\begin{aligned} & x_i \to x_i' = x_i + v_i t \\ & t \to t' = t \\ & v_i \text{ constant velocity of Galilei frame} \end{aligned}\right\} \quad (3.19)$$

if we impose

$$\phi(\vec{x},t) \to \phi'(\vec{x}',t') = \phi(\vec{x},t) - v_i x_i - \frac{1}{2} v_i v_i t$$

$$\psi(\vec{x},t), \psi(\vec{x},t), \text{ and } \rho(\vec{x},t) \quad \text{unchanged,}$$

we can show that the Lagrangian is invariant. So we have a conserved Noether current, for each independent direction v_i,

$$\left.\begin{aligned} N_{oi}(\vec{x},t) &= x_i \rho(\vec{x},t) - t\frac{1}{2}\left(\rho(\vec{x},t)V_i(\vec{x},t)+V_i(\vec{x},t)\rho(\vec{x},t)\right) \\ N_{ji}(\vec{x},t) &= x_i \frac{1}{2}\left(\rho(\vec{x},t)V_j(\vec{x},t)+V_j(\vec{x},t)\rho(\vec{x},t)\right) + t\,T_{ji}(\vec{x},t) \end{aligned}\right\} (3.20)$$

and the expression for T_{ij} was found previously. The current conservation law is of course,

$$\partial_\mu N_{\mu i} = 0 \; . \quad (3.21)$$

We find the Galilei transformation important to spontaneous symmetry breakdown.

Spectrum of Elementary Excitations

The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\,\vec{V}\cdot\rho\cdot\vec{V} + \rho\omega(\rho)$$

$$= \frac{1}{2}\left(-\vec{\nabla}\phi - \frac{1}{2\rho}\,(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi)\right)\rho\left(-\vec{\nabla}\phi - \frac{1}{2\rho}\,(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi)\right) + \rho\omega(\rho) \qquad (3.22)$$

$$\mathcal{H} = \frac{1}{2}\Big(\vec{\nabla}\phi\cdot\rho\vec{\nabla}\phi + \vec{\nabla}\phi\cdot(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi) + \frac{1}{4\rho}(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi)\cdot(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi)\Big)$$

$$+ \rho\int_{\rho_o}^{\rho}\left(\frac{p(\rho) - p_o}{\rho^2}\right)d\rho \qquad (3.23)$$

Now in as much ρ is approximately ρ_o the equilibrium density, and all ρ dependence may be expanded, even in terms of operators, about ρ_o; we can write $\rho = \rho_o + \delta\rho$

$$\mathcal{H} = \frac{1}{2}\,\rho_o\big((\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi)\big) + \frac{1}{2\rho_o}\left(\frac{\partial p}{\partial\rho}\,(\rho_o)\right)(\delta\rho)^2 + (\vec{\nabla}\phi)\cdot(\delta\rho)\cdot(\vec{\nabla}\phi)$$

$$+ (\vec{\nabla}\phi)\cdot\big(\pi(\vec{\nabla}\psi)+(\vec{\nabla}\psi)\pi\big) + \frac{1}{6}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\,\frac{\partial p}{\partial\rho}\right)\Big|_{\rho=\rho_o}\right](\delta\rho)^3$$

$$+ \frac{1}{4\rho_o}\,\big(\pi(\vec{\nabla}\psi)+(\vec{\nabla}\psi)\pi\big)\cdot\big(\pi(\vec{\nabla}\psi)+(\vec{\nabla}\psi)\pi\big) + \frac{1}{24}\left[\frac{\partial^2}{\partial\rho^2}\left(\frac{1}{\rho}\,\frac{\partial p}{\partial\rho}\right)\Big|_{\rho=\rho_o}\right](\delta\rho)^4$$

$$+ \;\ldots\ldots\;\text{higher order terms} \qquad (3.24)$$

Such a decomposition of the Hamiltonian was first given by Landau and Khalatnikov[11]. Now we can designate the various terms in the Hamiltonian, the second order terms are

$$\mathcal{H}^2 = \frac{1}{2}\rho_o \vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}\left(\frac{1}{\rho_o}\frac{\partial p}{\partial \rho}(\rho_o)\right)(\delta\rho)^2 \qquad (3.25$$

which is the contribution to the phonon or longitudinal wave spectrum;

$$\mathcal{H}^3 = \frac{1}{2}\vec{\nabla}\phi\cdot\delta\rho\cdot\vec{\nabla}\phi + \vec{\nabla}\phi\cdot(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi) + \frac{1}{6}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial p}{\partial\rho}\right)\right]_{\rho_o}(\delta\rho)^3 \qquad (3.26)$$

are third order terms, which contribute to phonon-phonon and phonon-roton terms, where roton identifies transverse excitations.

$$\mathcal{H}^4 = \frac{1}{4\rho_o}(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi)\cdot(\pi\vec{\nabla}\psi+\vec{\nabla}\psi\pi) + \frac{1}{24}\left[\frac{\partial^2}{\partial\rho^2}\left(\frac{1}{\rho}\frac{\partial p}{\partial\rho}\right)\right]_{\rho_o}(\delta\rho)^4 \qquad (3.27)$$

Here we identify the first term as the contribution of the roton energy, the second as some higher order process..

Now we can obviously not totally diagonalize this Hamiltonian, so we must intuitively pick out terms of importance. We can make the assumption that terms of higher order are of less importance.

Then to lowest order, that is to second order,

$$\mathcal{H} = \mathcal{H}^2 = \frac{1}{2}\rho_o\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}\left(\frac{1}{\rho_o}\frac{\partial p}{\partial\rho}(\rho_o)\right)(\delta\rho)^2 \qquad (3.28)$$

Now this can be totally diagonalized, since ϕ and ρ are conjugate variables, the solution is simply the harmonic oscillator spectrum.

We can define annihilation and creation operators as

$$a_k = \frac{1}{\sqrt{2c}} \left(\sqrt{\frac{c^2}{\rho_o \sqrt{k \cdot k}}}\, p_k - i \sqrt{\rho_o \sqrt{k \cdot k}}\, q_k \right) \tag{3.30a}$$

$$a_k^{\dagger} = \frac{1}{\sqrt{2c}} \left(\sqrt{\frac{c^2}{\rho_o \sqrt{k \cdot k}}}\, p_k^{\dagger} + i \sqrt{\rho_o \sqrt{k \cdot k}}\, q_k \right) \tag{3.30b}$$

where

$$[a_k, a_{k'}^{\dagger}] = \delta_{kk'} \qquad [a_k, a_{k'}] = 0 = [a_k^{\dagger}, a_{k'}^{\dagger}] \ , \tag{3.31}$$

and

$$c^2 = \frac{\partial p}{\partial \rho} (\rho_o) \ , \tag{3.32}$$

and

$$\delta\rho = \frac{1}{V^{1/2}} \sum_k p_k \, e^{i\vec{k}\cdot\vec{x}} \qquad p_{-k} = p_k^{\dagger} \tag{3.33}$$

$$\phi = \frac{1}{V^{1/2}} \sum_k q_k \, e^{i\vec{k}\cdot\vec{x}} \qquad q_{-k} = q_k^{\dagger} \ , \tag{3.34}$$

and we have chosen box normalization of volume V. Then

$$E = \sqrt{\frac{\partial p}{\partial \rho}(\rho_o)} \sum_k \sqrt{k \cdot k}\, (a_k^{\dagger} a_k + \tfrac{1}{2}) \ , \tag{3.35}$$

so $\omega(k) = \sqrt{\frac{\partial p}{\partial \rho}(\rho_o)}\ \sqrt{k \cdot k}$ is the energy spectrum, which is the standard spectrum of harmonic excitations.

Now if we look at the roton energy term, we have

$$\mathcal{H}^{\text{roton}} = \frac{1}{4\rho_o} (\pi\vec{\nabla}\psi + \vec{\nabla}\psi\pi)\cdot(\pi\vec{\nabla}\psi + \vec{\nabla}\psi\pi) \quad . \tag{3.36}$$

This cannot be diagonalized easily without further transformations at the classical level, so following Ziman[12], define

$$\psi = \psi_1/\psi_2 \quad , \quad \pi = \frac{1}{2}\psi_2^2 \tag{3.37}$$

$$\text{or} \quad \psi_1 = \psi\sqrt{2\pi} \quad , \quad \psi_2 = \sqrt{2\pi} \quad . \tag{3.38}$$

This transformation can be demonstrated as canonical, since

$$\{\psi_1(\vec{x},t),\psi_2(\vec{x}',t)\}_{P.B.} = \delta(\vec{x}-\vec{x}') \quad . \tag{3.39}$$

Then

$$\pi\vec{\nabla}\psi = \frac{1}{2}(\psi_2\vec{\nabla}\psi_1 - \psi_1\vec{\nabla}\psi_2) \quad . \tag{3.40}$$

Further define

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \tag{3.41a}$$

$$\Psi^* = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \quad . \tag{3.41b}$$

Then

$$\pi\vec{\nabla}\psi = \frac{i}{2}(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) \tag{3.42}$$

So

$$\vec{V} = -\vec{\nabla}\phi - \frac{i}{2\rho}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) \tag{3.43}$$

which is Hermitean already. That is, we do not need to symmetrize between π and $\vec{\nabla}\psi$ for a quantum mechanical velocity. Now the roton hamiltonian is

$$\mathcal{H}_{roton} = -\frac{1}{8\rho_o}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)^2 \ . \tag{3.44}$$

With

$$\psi = \frac{1}{V^{\frac{1}{2}}}\sum_k d_k \, e^{i\vec{k}\cdot\vec{r}} \tag{3.45a}$$

$$\psi^* = \frac{1}{V^{\frac{1}{2}}}\sum_k d_k^\dagger \, e^{-i\vec{k}\cdot\vec{r}} \tag{3.45b}$$

we have the commutators

$$[d_k, d_{k'}^\dagger] = \delta_{kk'} \quad , \quad [d_k, d_{k'}] = 0 = [d_k^\dagger, d_{k'}^\dagger] \tag{3.46}$$

which gives

$$H_{roton} = \int d^3x \, \mathcal{H}_{roton} = \frac{1}{8\rho_o V}\left\{\sum_{k,\ell,m,n}{}' (k+n)\cdot(\ell+m)\,\delta_{k+\ell,m+n} d_m^\dagger d_n^\dagger d_k d_\ell \right.$$

$$\left. + \sum_{k,\ell}(k\cdot k + \ell\cdot\ell)\, d_k^\dagger d_k \right\} \ . \tag{3.47}$$

Now this expression is inherently divergent for any excited state, thus we must introduce some way of restricting the values attainable by the wave vector, a cut off.

Physically, a very large wave vector is meaningless if the medium is not a continuum, as is the case with any real liquid. A way of introducing such a cut off, analogous to the Debye theory of specific heats, we insist for N particles that we require only 6N dynamical variables to specify the state of the system. The momenta of the atoms come from ϕ, Ψ, and Ψ^*, thus if we allot 3N dynamical variables to these three potentials, we can have the 3N spatial coordinates of the system depending only on ρ. This provides a $k_{max} = \frac{2\pi n_{max}}{(V)^{1/3}}$ where $\frac{4\pi}{3} n_{max}^3 = 3N$, counting the points in wave number space inside a sphere of radius n_{max}.

Then integrating inside the sphere we get

$$\sum_{\ell} (\ell \cdot \ell) = 4\pi \left\{ \int_0^{k_{max}} \ell^4 d\ell \right\} = \frac{4\pi}{5} \left(\frac{2\pi}{V_o^{1/3}}\right)^2 n_{max}^5 \qquad (3.48)$$

or the energy of a one roton state of wave number k is given by

$$E_k = \frac{3}{8} \frac{k^2}{m} + (7.1) \rho_o^{2/3} m^{-5/3} \qquad (3.49)$$

where m is the mass of the constituent atom, given by

$$m = \frac{\rho_o V}{N} \qquad (3.50)$$

Now this is exactly the spectrum postulated by Landau[13], since the momentum of a one roton state is given by

$$\langle 1_k | \int d^3x \; - \frac{i}{2} \; (\psi^* \vec{\nabla} \psi \; - \; \psi \vec{\nabla} \psi^*) | 1_k \rangle \; = \; \vec{k} \quad . \tag{3.51}$$

so

$$E(\vec{p}) = \frac{\vec{p} \cdot \vec{p}}{2\mu} + E_o \quad , \quad \mu = \frac{4m}{3} \tag{3.52}$$

where $E_o \simeq (7°K) \cdot (k_B)$, which is in good agreement with specific heat data.

This above reduction of the Hamiltonian into phonon, roton, and interaction terms, and subsequent expressions for the energy spectrum cannot be considered completely satisfactory. Primarily because we have just selected certain special terms from the Hamiltonian, even to the extent of rejecting terms of lower order in the field variables, to get the phonon and roton spectrum alone. However, we should realize that at extremely low states of excitation, the system will essentially consist of non-interacting quantum excitations, thus for low enough temperatures the phonon and roton energy spectrum as calculated above, must be applicable.

CHAPTER IV

SPONTANEOUS BREAKDOWN OF SYMMETRY IN HYDRODYNAMICS

Generators of Invariant Transformation

From the classical and quantum theory of the hydrodynamic fields, we have seen that the hydrodynamic equations are invariant under a number of continuous transformations. Using the infinitesimal versions of these transformations, we have obtained the conservation laws associated with these symmetries of the equations. But, we have not yet discussed the generators of the transformations.

The generator of a transformation can be constructed through considering the transformation properties of the Lagrangian. For an invariant transformation, the generator can be constructed from the Noether current. One should stress that this is not always true for a non-invariant transformation. For a non-invariant transformation, the generator may be totally unrelated to the Noether current.

For an invariant transformation, the generator is related to the Noether current by

$$G(t) = \int d^3x \, N_o(\vec{x},t) \quad . \tag{4.1}$$

The Lie derivative of the field variable $\delta^L\phi_\alpha$, defined as

$$\varepsilon\delta^L\phi_\alpha(\vec{x},t) = \phi'_\alpha(\vec{x},t) - \phi_\alpha(\vec{x},t) \quad , \tag{4.2}$$

is given by the generator,

$$\delta^L \phi_\alpha(\vec{x},t) = i[\phi_\alpha(\vec{x},t), G(t)] \quad . \tag{4.3}$$

The overall internal consistency of the quantization is checked by this relation, since the left hand side is purely a transformation property, whereas the right hand side is fundamentally related to the canonical commutation relations, and the transformation properties of the Lagrangian. We can show the generator is independent of time, for an invariant transformation, since

$$\frac{d}{dt} G(t) = \int d^3x \left(\partial_t N_o(\vec{x},t) \right) = -\int d^3x \left(\partial_i N_i(\vec{x},t) \right) = 0 \tag{4.4}$$

where we assume the integration of the divergence vanishes when changed to a surface integral. Therefore from now on we will write just G for the generator of an invariant transformation.

Spontaneous Breakdown of Symmetry and the Goldstone Theorem

The Goldstone theorem, first proved by Goldstone, Salam, and Weinberg[14], concerns relativistic quantum field theory. It states that if the vacuum expectation value of the Lie derivative of a field variable, under an invariant transformation, does not vanish, then there exists a massless particle.

We can show for the non-relativistic theory also that such a particle must exist for a spontaneously broken symmetry.

The generator of an invariant transformation is important for the spontaneous breakdown of symmetry. Suppose the vacuum expectation value of the Lie derivative of a field variable does not vanish. Then,

$$- i\langle 0|\delta^L\phi_\alpha(\vec{x},t)|0\rangle = \langle 0|[\phi_\alpha(\vec{x},t),G]|0\rangle \neq 0 \ . \tag{4.5}$$

Now if we use the Ward-Takahashi identity[15], we have

$$-i\delta^L\phi_\alpha(\vec{x},t) = -\int d^4x'\left\{\partial'_\mu T\left(\phi_\alpha(\vec{x},t),N_\mu(\vec{x}',t')\right) - T\left(\phi_\alpha(\vec{x},t),\partial'_\mu N_\mu(\vec{x}',t')\right)\right\} \tag{4.6}$$

where

$$T\left(\phi(\vec{x},t),\Psi(\vec{x}',t')\right)= \theta(t-t')\phi(\vec{x},t)\Psi(\vec{x}',t') + \theta(t'-t)\Psi(\vec{x}',t')\phi(\vec{x},t) \tag{4.7}$$

for any Heisenberg boson operators $\phi(\vec{x},t)$ and $\Psi(\vec{x}',t')$. The second term vanishes due to $\partial'_\mu N_\mu(\vec{x}',t') = 0$. So,

$$- i\langle 0|\delta^L\phi_\alpha(\vec{x},t)|0\rangle = -\int d^4x'\left\{\partial'_\mu\langle 0|T\left(\phi_\alpha(\vec{x},t),N_\mu(\vec{x}',t')\right)|0\rangle\right\}. \tag{4.8}$$

Now we must use the spectral representation of the vacuum expectation value of the time ordered product of the two field operators. This is equivalent to assuming that the Fourier decomposition of this two point function exists. So,

$$<0|T\left(\phi_\alpha(\vec{x},t),N_\mu(\vec{x}',t')\right)|0> = \frac{i}{(2\pi)^4}\int d^4p\, e^{ip(x-x')}\, G_\mu(\vec{p},p_o) \quad . \tag{4.9}$$

$G_\mu(\vec{p},p_o)$ is given by

$$G_\mu(\vec{p},p_o) = (2\pi)^3\int_0^\infty dv\left\{\frac{\sigma^1_\mu(v,\vec{p})}{(p_o-v+i\delta)} - \frac{\sigma^2_\mu(v,\vec{p})}{(p_o+v-i\delta)}\right\} \tag{4.10}$$

where

$$\sigma^1_\mu(v,\vec{p}) = \sum_\lambda <0|\phi_\alpha(0)|\vec{p},v,\lambda><\vec{p},v,\lambda|N_\mu(0)|0> \tag{4.11a}$$

$$\sigma^2_\mu(v,\vec{p}) = \sum_\lambda <0|N_\mu(0)|-\vec{p},v,\lambda><-\vec{p},v,\lambda|\phi_\alpha(0)|0> \quad , \tag{4.11b}$$

and $\phi_\alpha(0)$ and $N_\mu(0)$ are related to the Heisenberg fields by,

$$\phi_\alpha(\vec{x},t) = e^{-iPx}\,\phi(0)\,e^{iPx}$$

$$N_\mu(\vec{x},t) = e^{-iPx}\,N_\mu(0)\,e^{iPx}$$

where P is the energy momentum four vector operator.
The index λ denotes all other observables characterizing the basic states. Taking the derivative with respect to x'_μ, will bring down a $-ip_\mu$, and integrating over the primed spatial coordinates will cause a four dimensional delta function in p_μ. That is,

$$i\langle 0|\delta^L\phi_\alpha(\vec{x},t)|0\rangle = \int d^4p\,\delta^3(p)\delta(p_o)\,p_\mu\,G_\mu(\vec{p},p_o)\,. \qquad (4.12)$$

We have assumed the left hand side does not vanish, thus the right hand side, which is just $p_\mu G_\mu(\vec{p},p_o)$ evaluated at $p_\mu = 0$, must not vanish. Now if we look at the same quantity, in a rotated coordinate system, and if we assume rotational invariance, the result $p'_\mu G'_\mu(\vec{p}',p'_o)$ must be invariant. Therefore we may conclude

$$p_\mu G_\mu(\vec{p},p_o) = (\vec{p})^2 g_1\left((\vec{p})^2,p_o\right) - p_o^2 g_2\left((\vec{p})^2,p_o\right)\,. \qquad (4.13)$$

Then if this is not to vanish, at $p_\mu = 0$, then either g_1 must have a pole singularity of the form $1/(p)^2$ at $p_o = 0$, or g_2 must have a pole singularity of the form $1/p_o^2$ at $\vec{p} = 0$, or a combination of these. We cannot get the exact dispersion relation from this alone, however we can see the energy must vanish for zero momentum, the spectrum is gapless.

Now if we look at the original condition, that the vacuum expectation value of the Lie derivative of the field variable does not vanish, we can see this implies that the vacuum is not invariant under the transformation; the vacuum is degenerate, the symmetry is broken. Since G is the generator of the transformation, we can define the unitary operator for an infinitesimal transformation,

$$U = 1 + i\varepsilon G \qquad (4.14)$$

where ε is an infinitesimal parameter. Then

$$\phi'_\alpha(\vec{x},t) = \phi_\alpha(\vec{x},t) + \varepsilon\delta^L\phi_\alpha(\vec{x},t) = U^\dagger\phi_\alpha(\vec{x},t)\,U \tag{4.15}$$

using the relation

$$\delta^L\phi_\alpha(\vec{x},t) = i[\phi_\alpha(\vec{x},t),G] \quad . \tag{4.16}$$

But now if

$$\langle 0|[\phi_\alpha(\vec{x},t),G]|0\rangle \neq 0 \quad , \tag{4.17}$$

this implies,

$$G|0\rangle \neq 0 \quad , \tag{4.18}$$

hence

$$U^\dagger|0\rangle = |0\rangle - i\varepsilon G|0\rangle \neq |0\rangle \quad . \tag{4.19}$$

Thus obviously, the vacuum is not invariant under the transformation, the symmetry is broken. However, $U^\dagger|0\rangle$ must behave as the vacuum in the transformed system. Thus we can say the vacuum is degenerate.

Examples of Spontaneous Symmetry Breakdown in Hydrodynamics

We can find examples of spontaneous symmetry breaking in hydrodynamics. Since we can formulate the field theory so that the vacuum expectation value corresponds to the temperature equal to zero, equilibrium state of the system[16], we must be able to solve the equations so that certain expectation values are determined. We must expect that the density, energy density, and pressure be non-vanishing for a system at equilibrium, even at zero temperature. Therefore we will assume

$$<0|\rho|0> = \rho_o \tag{4.20}$$

$$<0|T_{oo}|0> = \varepsilon_o \tag{4.21}$$

$$<0|T_{ij}|0> = -p_o \delta_{ij} \quad . \tag{4.22}$$

These relations are useful later. First consider the transformation discussed in the classical theory chapter,

$$\left.\begin{array}{l} \phi(\vec{x},t) \rightarrow \phi'(\vec{x}',t') = \phi(\vec{x},t) + \phi_o \\ \phi_o \quad \text{a constant} \quad . \end{array}\right\} \tag{4.23}$$

In the quantum field theory, this will of course be an invariant transformation, but the conservation equation will be

$$\dot{\rho} + \vec{\nabla}\cdot\left(\tfrac{1}{2}(\rho\vec{V} + \vec{V}\rho)\right) = 0 \ , \tag{4.24}$$

the continuity equation in the quantum field theory. The generator of this transformation will be

$$G_{\phi_o} = \int d^3x\left(\rho(\vec{x},t)\right) \quad . \tag{4.25}$$

Now if we wish to look for symmetry breaking, we consider

$$\delta^L \phi(\vec{x},t) = i[\phi(\vec{x},t), G_{\phi_o}] = i\,i = -1 \quad . \tag{4.26}$$

Therefore

$$\langle 0|\delta^L \phi(\vec{x},t)|0\rangle = -1 \neq 0 \quad . \tag{4.27}$$

This is exactly the required relation for spontaneously broken symmetry. Thus we know there exists a gapless boson which appears through a pole singularity at zero energy and momentum, in the spectral representation of the two point function, $\langle 0|T(\phi(\vec{x},t), N_\mu^{\phi_o}(\vec{x}',t'))|0\rangle$, where $N_\mu^{\phi_o}$ is the Noether current associated with the transformation. Similarly we can look at the other transformation considered in the classical field theory,

$$\left.\begin{array}{l} \psi(\vec{x},t) \to \psi(\vec{x}',t') = \psi(\vec{x},t) + \psi_o \\[2ex] \psi_o \text{ a constant} \quad . \end{array}\right\} \tag{4.28}$$

This will of course be an invariant transformation in the quantum field theory, with the current conservation equation modified to

$$\dot{\pi} + \vec{\nabla}\cdot\frac{1}{2}(\pi\vec{V} + \vec{V}\pi) = 0 \quad . \tag{4.29}$$

The generator of this transformation is

$$G_{\psi_o} = \int d^3x \left(\pi(\vec{x},t)\right) \quad . \tag{4.30}$$

Now if we consider

$$\delta^L\psi(\vec{x},t) = i[\psi(\vec{x},t), G_{\psi_o}] = i\,i = -1 \; , \tag{4.31}$$

exactly as in the previous case,

$$<0|\delta^L\psi|0> = -1 \neq 0 \;, \tag{4.32}$$

and we have another spontaneously broken symmetry, which results in the existence of another gapless boson, which appears as a pole singularity at zero energy and momentum in the spectral representation of $<0|T\left(\phi(\vec{x},t), N_\mu^{\psi_o}(\vec{x}',t')\right)|0>$, where $N_\mu^{\psi_o}$ is the Noether current associated with the transformation.

These two broken symmetries have arisen from the structure of the equations and the canonical quantization conditions that we impose. They can only lead to mathematical identities necessary for the internal consistency of the theory. However, there should be some broken symmetries which we can postulate on physical grounds. In the case of a crystal, the periodic lattice structure indicates a broken space-translational symmetry. Or, in the case of a ferromagnet the aligned spins in a ferromagnetic state seem to indicate a broken spin-rotational symmetry. What can we expect for the system of hydrodynamic equations, which describe a continuous media? Considering a real gas or liquid which behaves approximately hydrodynamically, we know there is some property of the substance which is not Galilei invariant, since the velocity of sound changes between different Galilei frames of reference. And as we have shown, the equations of hydrodynamics are Galilei invariant. This seems like an ideal place to look for a broken symmetry. Indeed, if we calculate the Lie derivative of the matter current and the energy current, under Galilei transformation, we find these vacuum expectation values do not vanish.

We have calculated the Noether current in the quantum field theory chapter, thus we can construct the generator of the transformation

$$G_i = \int d^3x \; N_{oi}(\vec{x},t)$$

$$= \int d^3x \left[x_i \rho(\vec{x},t) - t\frac{1}{2}\left(\rho(\vec{x},t)V_i(\vec{x},t) + V_i(\vec{x},t)\rho(\vec{x},t)\right)\right]. \quad (4.33)$$

Then using the commutation relations found in the last chapter, we can evaluate the Lie derivative of the matter current,

$$-i\delta_i^L J_j(\vec{x},t) = [J_j(\vec{x},t), G_i] = -i\delta_{ij}\rho(\vec{x},t) + it\partial_i J_j(\vec{x},t) \quad (4.34)$$

where

$$J_j(\vec{x},t) = \frac{1}{2}\left(\rho(\vec{x},t)V_i(\vec{x},t) + V_i(\vec{x},t)\rho(\vec{x},t)\right) \quad . \qquad (4.35)$$

Now taking the vacuum expectation value, we can see,

$$<0|\delta_i^L J_j(\vec{x},t)|0> = \delta_{ij}<0|\rho(\vec{x},t)|0> - t<0|\partial_i J_j(\vec{x},t)|0> . \qquad (4.36)$$

The first term is $\delta_{ij}\rho_o$. Since we may take the derivative outside of the vacuum expectation value, and the vacuum expectation value is space translationally invariant, the second term vanishes. So,

$$<0|\delta_i^L J_j(\vec{x},t)|0> = \delta_{ij}\rho_o \neq 0 \quad . \qquad (4.37)$$

Therefore we can see the Galilei symmetry is spontaneously broken. This relation gives us information about the longitudinal excitations which exists as a result of the Goldstone theorem. These of course must be gapless, and correspond to the phonons predicted by any linearized theory of the hydrodynamic equations.

We can also gain information about the spectral functions by considering the variation of the energy current T_{jo} under the Galilei transformation.

$$\begin{aligned} -i\delta_i^L T_{jo}(\vec{x},t) &= [T_{jo}(\vec{x},t),G_i] \\ &= -i\delta_{ij}T_{oo}(\vec{x},t) + iT_{ij}(\vec{x},t) + it\partial_i T_{jo}(\vec{x},t), \qquad (4.38) \end{aligned}$$

so we can see

$$<0|\delta_i^L T_{jo}(\vec{x},t)|0> = \delta_{ij}<0|T_{oo}(\vec{x},t)|0> - <0|T_{ij}(\vec{x},t)|0> - t<0|\partial_i T_{jo}(\vec{x},t)|0> \quad . \tag{4.39}$$

The last term vanishes, for the same reason as before, that the vacuum expectation value is space translationally invariant. The other two terms have the values assumed at the beginning of this section, thus,

$$<0|\delta_i^L T_{jo}(\vec{x},t)|0> = \delta_{ij}(\varepsilon_o + p_o) \neq 0 \quad . \tag{4.40}$$

The angular momentum generator density

$$M_{ijo}(\vec{x},t) = x_i T_{oj}(\vec{x},t) - x_j T_{oi}(\vec{x},t) \tag{4.41}$$

also yields a Goldstone type commutator under Galilei transformation, but this is due to the relation of M_{ijo} to the matter current, and leads to no new information.

Spectral Representations

We take the following spectral representations:

$$<0|\rho(\vec{x},t)\rho(\vec{x}',t')|0> = \frac{\rho_o}{(2\pi)^3}\int d^4k e^{ik(x-x')}\left(\int_0^\infty d\omega\left(S(k,\omega)\,\delta(k_o-\omega)\right)\right), \tag{4.42}$$

$$<0|J_j(\vec{x},t)\rho(\vec{x}',t')|0> = \frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')} \times$$

$$\left[\int_0^\infty d\omega\left(k_j\sigma_1(k,\omega)\delta(k_o-\omega)\right)\right] , \qquad (4.43)$$

$$<0|J_j(\vec{x},t)J_i(\vec{x}',t')|0> = \frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')} \times$$

$$\left[\int_0^\infty d\omega\left\{\left(\left(\delta_{ij} - \frac{k_ik_j}{(\vec{k})^2}\right)\sigma_T(\vec{k},\omega) + \frac{k_ik_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)\right)\delta(k_o-\omega)\right\}\right] \qquad (4.44)$$

Now since ρ is a real field we get from taking the complex conjugate of the first expression,

$$\int_0^\infty d\omega\left(S^*(-\vec{k},\omega)\delta(-k_o-\omega)\right) = \int_0^\infty d\omega\left(S(-\vec{k},\omega)\delta(-k_o-\omega)\right) \qquad (4.45)$$

since the complex conjugate simply reverses to order of the field. Or integrating and changing the sign of the variables,

$$S^*(k,\omega) = S(k,\omega) \quad . \qquad (4.46)$$

Furthermore, if we assume space reflection invariance, we get

$$S(\vec{k},\omega) = S(-\vec{k},\omega) \quad . \qquad (4.47)$$

The two point function $<0|J_j(\vec{x},t)\rho(\vec{x}',t')|0>$ will change sign under space reflection, if we assume the matter current reverses sign under the same. Since the spectral representation contains a factor k_j outside the spectral function $\sigma_1(\vec{k},\omega)$, the spectral function satisfies

$$\sigma_1(\vec{k},\omega) = \sigma_1(-\vec{k},\omega) \quad . \tag{4.48}$$

If we assume time reversal invariance, we can show,

$$\sigma_1(\vec{k},\omega) = \sigma_1^*(-\vec{k},\omega) \tag{4.49}$$

the calculation is in the appendix. Then these two relations imply,

$$\sigma_1(\vec{k},\omega) = \sigma_1^*(\vec{k},\omega) \quad . \tag{4.50}$$

The two point function $<0|J_j(\vec{x},t)J_i(\vec{x}',t')|0>$ has a contribution from the transverse spectrum σ_T, and the longitudinal spectrum, σ_L. Taking the divergence of the expression with either index, separates out the longitudinal part. We show in the appendix, assuming invariance under space reflection implies,

$$\sigma_T(\vec{k},\omega) = \sigma_T(-\vec{k},\omega) \tag{4.51a}$$

$$\sigma_L(\vec{k},\omega) = \sigma_L(-\vec{k},\omega) \quad , \tag{4.51b}$$

and assuming invariance under time reversal gives,

$$\sigma_T(\vec{k},\omega) = \sigma_T^*(-\vec{k},\omega) \tag{4.52a}$$

$$\sigma_L(\vec{k},\omega) = \sigma_L^*(-\vec{k},\omega) \quad . \tag{4.52b}$$

Combining these two results yields

$$\sigma_T(\vec{k},\omega) = \sigma_T^{\star}(\vec{k},\omega) \qquad (4.53a)$$

$$\sigma_L(\vec{k},\omega) = \sigma_L^{\star}(\vec{k},\omega) \qquad (4.53b)$$

This ensures

$$\langle 0|[J_i(\vec{x},t), J_j(\vec{x}', t)]|0\rangle = 0 \ . \qquad (4.54)$$

Using the conservation equations we can get relations between the spectral functions. Take the continuity equation

$$\dot{\rho}(x,t) + \partial_i \frac{1}{2}\left(\rho(\vec{x},t)V_i(\vec{x},t) + V_i(\vec{x},t)\rho(\vec{x},t)\right) = 0 \ . \qquad (4.55)$$

Multiplying by $\rho(\vec{x}',t')$ and taking the vacuum expectation value leads to,

$$\frac{\partial}{\partial t}\langle 0|\rho(\vec{x},t)\rho(\vec{x}',t')|0\rangle + \partial_i\langle 0|J_i(\vec{x},t)\rho(\vec{x}',t')|0\rangle = 0 \qquad (4.56)$$

and using the spectral representations, we get

$$\rho_o \omega S(\vec{k},\omega) = k^2\sigma_1(\vec{k},\omega) \ . \qquad (4.57)$$

Multiplying by $J_j(\vec{x}',t')$, taking vacuum expectation values, and using the spectral representations yields

$$\omega\sigma_1(\vec{k},\omega) = \sigma_L(\vec{k},\omega) \qquad (4.58)$$

for $\vec{k} \neq 0$. This shows that σ_1 is related only to the longitudinal spectrum.

Now if we look at the Goldstone type commutators, we get

$$<0|-i\delta_i^L J_j(\vec{x},t)|0> = <0|[J_j(\vec{x},t),G_i]|0> = \int d^3x'\Big\{<0|[J_j(\vec{x},t),$$

$$x_i'\rho(\vec{x}',t)]|0> + <0|[J_j(\vec{x},t),-t\,J_i(\vec{x}',t)]|0>\Big\} \quad . \qquad (4.59)$$

We have shown,

$$<0|[J_j(\vec{x},t),J_i(\vec{x}',t)]|0> = 0 \qquad (4.60)$$

thus

$$<0|-i\delta_i^L J_j(\vec{x},t)|0> = \int d^3x'\{<0|[J_j(\vec{x},t),\rho(\vec{x}',t)]|0>x_i'\} . \quad (4.61)$$

With the spectral functions, we get

$$-i\delta_{ij}\rho_o = \int d^3x' \left(\frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(\int_0^\infty d\omega\sigma_1(\vec{k},\omega)\,2k_j\right) x_i'\right) . \qquad (4.62)$$

Using integration by parts, and then integrating over x' and k, we get

$$-i\delta_{ij}\rho_o = -2i\delta_{ij}\int_0^\infty d\omega\sigma_1(\vec{o},\omega) \quad . \qquad (4.63)$$

Therefore

$$\int_0^\infty d\omega\sigma_1(\vec{o},\omega) = \rho_o/2 \quad . \tag{4.64}$$

This shows the origin of the Goldstone boson is in the longitudinal spectrum.

Sum Rule

Using the first spectral function, we can derive a sum rule,

$$<0|[\rho(\vec{x},t),\dot{\rho}(\vec{x}',t)]|0> = \frac{i2\rho_o}{(2\pi)^3}\int d^3k\, e^{ik\cdot(\vec{x}-\vec{x}')}\int_0^\infty d\omega\, \omega S(\vec{k},\omega) \quad . \tag{4.65}$$

But

$$[\rho(\vec{x},t),\dot{\rho}(\vec{x}',t)] = -\partial_i'\left[\rho(\vec{x},t),\tfrac{1}{2}(\rho(\vec{x}',t)V_i(\vec{x}',t)+V_i(\vec{x}',t)\rho(\vec{x}',t))\right]$$

$$= -\partial_i'\left(\rho(\vec{x}',t)\left(-\partial_i'(-i\delta(\vec{x}-\vec{x}'))\right)\right)$$

$$= i\left((-\partial_i'\rho(\vec{x}',t))\partial_i'\delta(\vec{x}-\vec{x}') - \rho(\vec{x}',t)\partial_i'\partial_i'\delta(\vec{x}-\vec{x}')\right) . \tag{4.66}$$

Thus

$$<0|[\rho(\vec{x},t),\dot{\rho}(\vec{x}',t)]|0> = -i\rho_o\nabla^2\delta(\vec{x}-\vec{x}')$$

$$= -i\rho_o\frac{1}{(2\pi)^3}\int d^3k\, e^{ik\cdot(\vec{x}-\vec{x}')}(-k^2) \quad . \tag{4.67}$$

Therefore,

$$\int_0^\infty d\omega\, \omega S(\vec{k},\omega) = \frac{(\vec{k})^2}{2} . \tag{4.68}$$

Then following Takahashi[17], if we assume,

$$S(\vec{k},\omega) = Z(\vec{k})\delta(\omega-\omega(\vec{k})) + S_c(\vec{k},\omega) \tag{4.69}$$

with $\lim_{\vec{k}\to 0} \omega(\vec{k}) = 0$, and $S_c(\vec{k},\omega)$ the contribution of the continuum, we get

$$Z(\vec{k})\omega(\vec{k}) + \int_0^\infty d\omega\ \omega S_c(\vec{k},\omega) = \frac{(\vec{k})^2}{2} \tag{4.70}$$

or

$$\omega(\vec{k}) = \frac{(\vec{k})^2}{2Z(\vec{k})} - \frac{1}{Z(\vec{k})}\int_0^\infty d\omega\ \omega S_c(\vec{k},\omega) \tag{4.71}$$

which has as its first term, the Feynman expression, except for a factor of 1/m which is missing because of the difference in the definition of $S(\vec{k},\omega)$. The second term is the contribution of the continuum, which always lowers the spectrum.

Asymptotic Field

The Ward-Takahashi identity reduced to,

$$i\langle 0|\delta_i^L J_j(\vec{x},t)|0\rangle = \int d^4x' \partial_\mu' \langle 0|T\left(J_j(\vec{x},t),N_\mu(\vec{x}',t')\right)|0\rangle , \tag{4.72}$$

therefore the vacuum expectation value, differentiation and subsequent integration singles out the contribution of the Goldstone particle. Thus if we put

$$\partial'_\mu \langle 0|T\left(J_j(\vec{x},t), N_\mu(\vec{x}',t')\right)|0\rangle = \eta_B \Lambda(\partial')\langle 0|T\left(J_j(\vec{x},t), B(\vec{x}',t')\right)|0\rangle$$

$$+ \text{ (term vanishing when integrated)}, \qquad (4.73)$$

then $B(\vec{x}',t')$ is the interpolating field of the massless boson, and $\Lambda(\partial')$ is the equation satisfied by the asymptotic field. $\Lambda(\partial')$ is of the form

$$\Lambda(\partial') = \frac{\partial^2}{\partial t'^2} - C_s^2(\nabla'^2)\,\nabla'^2 \quad , \qquad (4.74)$$

since the equation must describe a gapless boson, and this equation will give a dispersion relation such that the frequency vanishes for vanishing wave vector. As shown by Takahashi[18], for the Nambu-Jona-Lasinio model, the asymptotic field of the Goldstone boson carries the original transformation. We have not done this for this model, but one should be able to do so following the same method.

Deviation from Ideal Gas Law

We have assumed

$$\langle 0|T_{oo}(\vec{x},t)|0\rangle = \varepsilon_o = \langle 0|\left(\tfrac{1}{2}V\cdot\rho V + \rho\omega(\rho)\right)|0\rangle \ . \qquad (4.75)$$

Now

$$\rho = \rho_o + \delta\rho \quad , \qquad (4.76)$$

where $\delta\rho$ is a first order quantity, with

$$\langle 0|\delta\rho|0\rangle = 0 \quad . \tag{4.77}$$

Then if we also consider $\vec{V}$ a first order quantity with

$$\langle 0|\vec{V}|0\rangle = 0 \ , \tag{4.78}$$

the energy density is a second order quantity given by

$$\varepsilon_o = \frac{1}{2}\rho_o\langle 0|(\vec{V})^2|0\rangle + \left(\frac{1}{\rho_o}\frac{\partial p}{\partial\rho}(\rho_o)\right)\langle 0|(\delta\rho)^2|0\rangle \quad . \tag{4.79}$$

Now looking at

$$\langle 0|T_{ij}|0\rangle = -p_o\delta_{ij} \tag{4.80}$$

and putting in the explicit form of T_{ij}, using the above consideration, we find

$$\langle 0|T_{ij}|0\rangle = -\left(\rho_o\langle 0|V_iV_j|0\rangle + \delta_{ij}\langle 0|\mathcal{L}|0\rangle\right) . \tag{4.81}$$

Now assuming the correlations between different components of velocity are of higher order, we get,

$$\langle 0|T_{ij}|0\rangle = -p_o\delta_{ij} = -\delta_{ij}\left(\rho_o\frac{1}{3}\langle 0|(\vec{V})^2|0\rangle + \langle 0|\mathcal{L}|0\rangle\right) . \tag{4.82}$$

Comparing the expressions for the equilibrium energy density and pressure,

$$p_o = \frac{2}{3}\,\varepsilon_o + \left[<0|\mathcal{L}|0> - \frac{2}{3}\left(\frac{1}{\rho_o}\,\frac{\partial p}{\partial \rho}(\rho_o)\right) <0|(\delta\rho)^2|0> \right]. \qquad (4.83)$$

Therefore the system deviates from the ideal gas behaviour by the vacuum expectation value of the Lagrangian, and an amount proportional to the mean square density fluctuation, which must correspond to a virial expansion. This deviation allows us to hope that the theory can include information about transport coefficients, such as viscosity, arising out of quantum corrections, even though the original classical equations describe an inviscid fluid.

A generalization of the theory to finite temperatures will allow the use of the Kubo[19,20,21] type formulas for the viscosity. For example, the coefficients of viscosity η and ζ may be obtained from

$$\eta\left(\delta_{ij} - \frac{1}{3}\,\frac{k_i k_j}{(\vec{k})^2}\right) + \zeta\,\frac{k_i k_j}{(\vec{k})^2} = \frac{\beta}{4}\,\lim_{\omega\to 0}\,\lim_{\vec{k}\to 0} \int d^4x\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \left(\sum_{n,m} \frac{k_n k_m}{(\vec{k})^2} \times \right.$$

$$\left. <0|\left(T_{im}(\vec{x},t)T_{jn}(\vec{o},o) + T_{jn}(\vec{o},o)T_{im}(\vec{x},t)\right)|0> \right) \qquad (4.84)$$

where η is the shear viscosity, and ζ the bulk viscosity.

CHAPTER V

SUMMARY, CONCLUSIONS AND FUTURE OUTLOOK

Classical hydrodynamics describes the motion of a massive continuous medium, whose dynamics are governed by Newton's laws, and the principle of conservation of mass. The state of the system is specified by the hydrodynamic variables, the density, velocity, and pressure; and since it is a continuous medium, it is meaningful to speak of these variables as space and time dependent functions. In the Euler formalism of hydrodynamics, we can consider the hydrodynamics variables as fields, and the hydrodynamic equations as field equations.

The hydrodynamic equations for an inviscid fluid,

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\,\vec{V} = -\frac{1}{\rho}\vec{\nabla}p \quad , \tag{5.1}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \quad , \tag{5.2}$$

express the conservation of momentum and the conservation of mass; four equations among five variables. To close this system of equations we assume the condition of barotropy, that the pressure is a function of density alone. Such a condition severely restricts the applicability of our theory, however we do leave some freedom, since we do not specify the functional dependence of the pressure on the density, except to the extent that it be analytic. The condition of barotropy allows us to write the right hand side of equation (5.1) as the gradient of a function of ρ.

To obtain the equations of barotropic inviscid hydrodynamics from a Lagrangian, through a variational principle, we modify the equations with the Clebsch transformation,

$$\vec{V} = - \vec{\nabla}\phi + \lambda\vec{\nabla}\psi \quad . \tag{5.3}$$

Using the theorem, which applies to an inviscid barotropic fluid, that the flux of the curl of the velocity, through any surface which moves with the fluid, does not change in time; we get two equations, on the variables λ and ψ, which describe the curl of the velocity, that their derivatives moving with the fluid vanish. Then equation (5.1), with the Clebsch variables, may be integrated over the spatial coordinates to give the Bernoulli equation. These three equations and the continuity equation are obtained from a Lagrangian,

$$\mathcal{L}(x) = \rho\left\{\dot{\phi} - \lambda\dot{\psi} - \frac{1}{2}(-\vec{\nabla}\phi + \lambda\vec{\nabla}\psi)^2 - \int_{\rho_o}^{\rho} \left(\frac{p(\rho)-p(\rho_o)}{\rho^2}\right) d\rho\right\} \tag{5.4}$$

by varying with respect to λ, ψ, ρ, and ϕ.

Considering ρ and λ as canonical coördinates is not advantageous, since their canonically conjugate momenta vanish identically. This is incompatiable with canonical quantization, since this procedure assumes the commutator between a canonical coordinate, and its conjugate momentum, is non-vanishing. As we use canonical quantization, we interpret this Lagrangian as having been obtained from a Hamiltonian, with the dependence of the conjugate momenta on the canonical coordinates, not yet replaced. With this view, ϕ and ψ are the canonical coordinates, with conjugate momenta ρ and $\pi \equiv -\rho\lambda$, respectively. Using the Lagrangian, and Noether's theorem, we show that the continuity equation, and the equations for π and ψ, may be obtained from transformations of the field variables which leave the Lagrangian invariant.

The Hamiltonian is seen to be positive definite if $p(\rho)$ is a monotone increasing function of ρ, although this condition is not necessary, it is physically reasonable. The field equations now follow from the Hamiltonian, varying with ϕ, ψ, ρ and π.

We next proceed to the quantum field theory of hydrodynamics. This involves reinterpreting the hydrodynamic field variables as quantum field theoretical operators, which do not necessarily commute. Then, the classical expression for the velocity is ambiguous, since it contains the product of non-commuting operators, and we do not know which order to take in the quantum theory. We resolve this by taking the symmetrized expression for the velocity,

$$\vec{V} = - \vec{\nabla}\phi - \frac{1}{2\rho} (\pi\vec{\nabla}\psi + \vec{\nabla}\psi\pi) \quad . \qquad (5.5)$$

We do not have to symmetrize with respect to ρ, since we know that ρ will commute with the other variables π and ψ, and their derivatives. With this expression for the velocity, we can determine the Hamiltonian, which should be hermitean, and yield the correct equations of motion. The equations of motion themselves contain products of various non-commuting operators, and the Hamiltonian

$$\mathcal{H}(x) = \frac{1}{2}\vec{V}\cdot\rho\vec{V} + \rho\omega(\rho) \tag{5.6}$$

provides us with a hermitean form, which generalizes the equations of motion, so that they are no longer ambiguous as to order of operators, with the least amount of complexity. These equations of motion are obtained through the usual canonical variational procedure, but when we complete the quantization by assuming the canonical commutation relations,

$$[\phi(\vec{x},t),\rho(\vec{x}\,',t)] = i\delta(\vec{x}-\vec{x}\,') \tag{5.7a}$$

$$[\psi(x,t),\pi(x',t)] = i\delta(\vec{x}-\vec{x}\,') \tag{5.7b}$$

and all other commutators zero,

the equations follow from the Heisenberg equation,

$$i\cdot\dot{\phi}_\alpha = [\phi_\alpha, H] \tag{5.8}$$

where ϕ_α stands for ϕ, ψ, ρ, and π.

With the quantized system, we may use the Lagrangian, and the Noether theorem to obtain conservation laws from invariant transformations. It is here we obtain the conserved currents from space and time translation, which yields the energy momentum tensor, and the orbital angular momentum, from space rotation. These conserved currents may be taken directly into the classical theory, and simplify considerably when we allow the operators to commute. One other transformation we consider is the Galilei transformation. We show the Lagrangian is invariant for a finite velocity in the Galilei transformation. Then taking the infinitesimal limit by dropping second order terms, we use the Noether theorem to obtain the conserved current. Again this current may be taken classically.

Next we consider the spontaneous breakdown of symmetry. The spontaneous breakdown of symmetry occurs when there is a transformation which leaves the Lagrangian invariant, but the vacuum is not invariant under the transformation. If the vacuum is not invar[illegible] [illegible] the transformation, the generator of the trans[illegible]ation does n[illegible]t annihilate the vacuum. If the vacuum exp[illegible]tation value of the Lie derivative of a field variable i[illegible] [illegible]ot zero, the generator [illegible]f the transformation does not annihilate the vacuum, sin[illegible]

$$-i\langle 0|\varepsilon\delta^L\phi_\alpha|0\rangle = \langle 0|[\phi_\alpha,G]|0\rangle \quad 0 \qquad (5.9)$$

where G is the generator of t[illegible] transformation, ϕ_α a field variable. Now using the [illegible]d-Takahashi identity and the spectral representation, we show that there must exist a gapless boson, the Goldstone boson, if the symmetry is spontaneously broken.

We show three examples of the spontaneous breakdown of symmetry in hydrodynamics. The two field translation invariances of ϕ and ψ, are spontaneously broken. More importantly, the Galilei transformation is spontaneously broken, when we assume certain physical values for the vacuum expectation values of ρ, T_{oo}, and T_{ij}. The non-vanishing Lie derivative of the matter current under the transformation gives the Goldstone boson, which has its origin in the longitudinal spectrum.

We use the spectral representations of the two point functions $<0|J_j(\vec{x},t)\rho(\vec{x}',t')|0>$ and $<0|J_j(\vec{x},t)J_i(\vec{x}',t')|0>$ and assuming space reflection and time reversal invariance to obtain information about the spectral functions, to show the origin of the Goldstone boson in the longitudinal spectrum. We do not prove this, but we should be able to show that the asymptotic field of the Goldstone boson carries the original Galilei transformation.

The spectral function of $<0|\rho(\vec{x},t)\rho(\vec{x}',t')|0>$ satisfies a sum rule, which we can show. This spectral function is directly related to the dynamic structure factor. Using the sum rule, and assuming a form for the dynamic structure factor, we obtain an expression for the frequency spectrum, which agrees qualitatively with the Feynman expression, except for the factor of 1/m, and a contribution from the continuum, which lowers the spectrum.

Finally we show the system deviates from ideal gas behaviour characterized by the relation,

$$p_o = \frac{2}{3}\,\varepsilon_o \ . \tag{5.10}$$

Thus there exists a virial expansion, and we can hope that the transverse spectrum exists as a result of quantum corrections, even though the original classical equations describe an inviscid fluid.

We may conclude from the field theory of hydrodynamics, that the hydrodynamic equations describing a barotropic inviscid fluid may be consistently formulated as field equations, obtainable from a Lagrangian and Hamiltonian, through a variational principle. Invariant transformations and the Noether theorem readily allow us to obtain conserved currents. Following the usual canonical quantization procedure, yields the quantum field theory of the hydrodynamic equations in a logical and consistent manner. We are faced with the ambiguity of order of operators to be taken, when interpreting a classical product of non-commuting variables in the quantum theory. However, we can suitably generalize the classical expressions by symmetrizing the ambigious products, and we can obtain these generalized equations consistently from a hermitean Hamiltonian.

Finally the spontaneous breakdown of Galilei invariance gives us a Goldstone boson, which originates in the longitudinal spectrum.

Future work in the field theory of hydrodynamics should involve three major directions; complete examination of the transverse excited mode, interacting field, and superfluid. The transverse excitations should contribute to the transport coefficients, which may be obtained from the Kubo type formulas. There is always the problem of time reversal asymmetry, how can a quantum field theory, which is assumed time reversal invariant, predict macroscopically observed coefficients which occur in inherently time reversal non-invariant expressions? A detailed examination of the linear response theory would be necessary to answer the question.

The most useful interaction to consider would be the electromagnetic interaction. This may well be important to hydrodynamic plasmas.

Finally the successful application of quantum hydrodynamics to superfluids would be tremendously important. At low enough temperatures, the thermal wavelength of the particles easily extends over many atomic distances. Considering such medium as a hydrodynamic system is plausible. However it is not obvious how one would have to modify the equations, to incorporate the role of statistics in superfluidity.

APPENDIX

Parity

The hydrodynamic equations and commutation relations are invariant under space reflection,

$$\left.\begin{array}{l} \vec{x} \to \vec{x}' = -\vec{x} \\ \\ t \to t' = t \end{array}\right\} \text{ space reflection} \tag{A.1}$$

if we impose

$$\phi_\alpha(\vec{x},t) \to \phi_\alpha^P(\vec{x}',t;) = \phi_\alpha(\vec{x},t) \tag{A.2}$$

where $\phi_\alpha(\vec{x},t)$ stands for $\phi(\vec{x},t)$, $\psi(\vec{x},t)$, $\rho(\vec{x},t)$ and $\pi(\vec{x},t)$. The equations are invariant, is easily seen, since the derivatives transform as

$$\left.\begin{array}{l} \partial_i \to \partial_i' = -\partial_i \\ \\ \partial_t \to \partial_t' = \partial_t \end{array}\right\} \tag{A.3}$$

Thus,

$$\begin{aligned} V_i(\vec{x},t) \to V_i^P(\vec{x}',t') &= -\partial_i'\phi^P(\vec{x}',t') - \frac{1}{2\rho^P(\vec{x}',t')} \times \\ &\quad \left\{\pi^P(\vec{x}',t')\left(\partial_i'\psi^P(\vec{x}',t')\right) + \left(\partial_i'\psi^P(\vec{x}',t')\right)\pi^P(\vec{x}',t')\right\} \\ &= -\left\{-\partial_i\phi(\vec{x},t) - \frac{1}{2\rho(\vec{x},t)}\left(\pi(\vec{x},t)\partial_i\psi(\vec{x},t) + \partial_i\psi(\vec{x},t)\pi(\vec{x},t)\right)\right\} \\ &= -V_i(\vec{x},t) \quad . \end{aligned} \tag{A.4}$$

Then if we inspect the hydrodynamic equations, we see that the only combinations that involve V_i or ∂_i always have a product of these two quantities, thus the hydrodynamic equations are invariant under space reflection. Now since the hydrodynamic equations are invariant, we may quantize the field in the space reflected world, giving commutation relations

$$[\phi^P(\vec{x}',t_x'),\rho^P(\vec{y}',t_y')]_{t_x'=t_y'} = i\delta(\vec{x}'-\vec{y}') \tag{A.5a}$$

$$[\psi^P(\vec{x}',t_x'),\pi^P(\vec{y}',t_y')]_{t_x'=t_y'} = i\delta(\vec{x}'-\vec{y}') \tag{A.5b}$$

and all other commutators zero.

But these must be valid for all values of $\vec{x}'$, $\vec{y}'$, so we may vary $\vec{x}'$ to $-\vec{x}' = \vec{x}$. The right hand sides are unchanged, and $t_x' = t_x$, $t_y' = t_y$, thus

$$[\phi^P(\vec{x},t_x),\rho^P(\vec{y},t_y)]_{t_x=t_y} = i\delta(\vec{x}-\vec{y}) = [\phi(\vec{x},t_x),\rho(\vec{y},t_y)]_{t_x=t_y} \tag{A.6a}$$

$$[\psi^P(\vec{x},t_x),\pi^P(\vec{y},t_y)]_{t_x=t_y} = i\delta(x-y) = [\phi(\vec{x},t_x),\rho(\vec{y},t_y)]_{t_x=t_y} \tag{A.6b}$$

all all other commutators are zero.

Therefore the commutation relations are invariant, which implies the existence of a unitary transformation G_P such that

$$\phi_\alpha^P(\vec{x},t) = G_P^{-1}\phi_\alpha(\vec{x},t)\,G_P = \phi_\alpha(-\vec{x},t) \tag{A.7}$$

where $\phi_\alpha(\vec{x},t)$ stands for $\phi(\vec{x},t)$, $\psi(\vec{x},t)$, $\rho(\vec{x},t)$ and $\pi(\vec{x},t)$, and the vacuum is invariant under G_P.

Then,

$$\begin{aligned}
<0|J_j(\vec{x},t)\rho(\vec{x}',t')|0> &= <0|G_P^{-1}J_j(\vec{x},t)G_PG_P^{-1}\rho(\vec{x}',t')G_P|0> \\
&= <0|J_j^P(\vec{x},t)\rho^P(\vec{x}',t')|0> \\
&= <0|\tfrac{1}{2}\left(\rho^P(\vec{x},t)V_j^P(\vec{x},t) + V_j^P(\vec{x},t)\rho^P(\vec{x},t)\right)\rho^P(\vec{x}',t')|0> \\
&= -<0|J_j(-\vec{x},t)\rho(-\vec{x}',t')|0> \quad .
\end{aligned} \tag{A.8}$$

So

$$\frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')}\int_0^\infty d\omega k_j\sigma_1(\vec{k},\omega)\delta(k_o-\omega) = -\frac{1}{(2\pi)^3}\int d^4k\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \times e^{-ik_o(t-t')}\int_0^\infty k_j\sigma_1(\vec{k},\omega)\delta(k_o-\omega) \quad . \tag{A.9}$$

Equating Fourier coefficients after changing $\vec{k}\to-\vec{k}$ in the right hand side, we get

$$\sigma_1(\vec{k},\omega) = \sigma_1(-\vec{k},\omega) \quad . \tag{A.10}$$

The third spectral function gives

$$<0|J_j(\vec{x},t)J_i(\vec{x}',t')|0> = <0|G_P^{-1}J_j(\vec{x},t)G_PG_P^{-1}J_i(\vec{x}',t')G_P|0>$$

$$= <0|J_j^P(\vec{x},t)J_i^P(\vec{x}',t')|0>$$

$$= <0|J_j(-\vec{x},t)J_i(-\vec{x}',t')|0> \quad . \tag{A.11}$$

Replacing with the spectral representation,

$$\frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')}\int_0^\infty d\omega \left\{(\delta_{ij} - \frac{k_ik_j}{(\vec{k})^2})\sigma_T(\vec{k},\omega) + \frac{k_ik_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)\right\}\delta(k_o-\omega)$$

$$= \frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')}\int_0^\infty d\omega\left\{(\delta_{ij}-\frac{k_ik_j}{(\vec{k})^2})\sigma_T(-\vec{k},\omega)+\frac{k_ik_j}{(\vec{k})^2}\sigma_L(-\vec{k},\omega)\right\}\delta(k_o-\omega) \tag{A.12}$$

where $\vec{k}\to-\vec{k}$ has already been done in the right hand side. Equating Fourier coefficients, then multiplying by k_i and summing over i, gives

$$\sigma_L(\vec{k},\omega) = \sigma_L(-\vec{k},\omega) \quad . \tag{A.13}$$

Then equality of Fourier components implies

$$(\delta_{ij} - \frac{k_ik_j}{(\vec{k})^2})\sigma_L(\vec{k},\omega) = (\delta_{ij} - \frac{k_ik_j}{(\vec{k})^2})\sigma_T(-\vec{k},\omega) \quad , \tag{A.14}$$

as the second term cancels from both sides. Taking the trace of this, over i and j, gives

$$\sigma_T(\vec{k},\omega) = \sigma_T(-\vec{k},\omega) \quad . \tag{A.15}$$

Time Reversal

Time reversal is the transformation

$$\left.\begin{array}{l} \vec{x} \to \vec{x}' = \vec{x} \\ \\ t \to t' = -t \end{array}\right\} \text{ time reversal ,} \tag{A.16}$$

and the hydrodynamic equations are invariant, if we impose

$$\phi(\vec{x},t) \to \phi^R(\vec{x}',t') = -\phi^T(\vec{x},t) \quad . \tag{A.17a}$$

$$\psi(\vec{x},t) \to \psi^R(\vec{x}',t') = -\psi^T(\vec{x},t) \tag{A.17b}$$

$$\rho(\vec{x},t) \to \rho^R(\vec{x}',t') = \rho^T(\vec{x},t) \tag{A.17c}$$

$$\pi(\vec{x},t) \to \pi^R(\vec{x}',t') = \pi^T(\vec{x},t) \quad . \tag{A.17d}$$

where T stands for the transpose. This is readily seen if we notice

$$\left.\begin{array}{l} \partial_i \to \partial_i' = \partial_i \\ \\ \partial_t \to \partial_t' = -\partial_t \end{array}\right\} \tag{A.18}$$

and

$$V_i(\vec{x},t) \to V_i^R(\vec{x}',t') = -\partial_i'\phi^R(\vec{x}',t') - \frac{1}{2\rho^R(\vec{x}',t')} \times$$

$$\left[\pi^R(\vec{x}',t')\left(\partial_i'\psi^R(\vec{x}',t')\right) + \left(\partial_i'\psi^R(\vec{x}',t')\right)\pi^R(\vec{x}',t')\right]$$

$$= -\left[-\partial_i\phi^T(\vec{x},t) - \frac{1}{2\rho^T(\vec{x},t)}\left(\pi^T(\vec{x},t)\left(\partial_i\psi^T(\vec{x},t)\right) + \partial_i\psi^T(\vec{x},t)\pi^T(\vec{x},t)\right)\right]$$

$$= -V_i^T(\vec{x},t) \quad . \tag{A.19}$$

The Bernoulli equation

$$\dot{\phi} = \frac{1}{2}\vec{V}^2 + \frac{1}{2}\left[\frac{1}{2\rho}(\pi\vec{\nabla}\psi + \vec{\nabla}\psi\pi)\cdot\vec{V} + \vec{V}\cdot\frac{1}{2\rho}\left(\pi(\vec{\nabla}\psi) + (\vec{\nabla}\psi)\pi\right)\right] + \int_{\rho_o}^{\rho}\frac{dp(\rho)}{\rho} \tag{A.20}$$

taking the transpose,

$$\partial_t(-\phi^T) = \frac{1}{2}(-\vec{V}^T)^2 + \frac{1}{2}\left[\frac{1}{2\rho^T}\left(\pi^T(\vec{\nabla}(-\psi^T)) + (\vec{\nabla}(-\psi^T))\pi^T\right)\cdot(-\vec{V}^T) + (-\vec{V}^T)\cdot\right.$$

$$\left.\cdot\frac{1}{2\rho^T}\left(\pi^T(\vec{\nabla}(-\psi^T)) + (\vec{\nabla}(-\psi^T))\pi^T\right)\right] + \int_{\rho_o}^{\rho^T}\frac{dp(\rho^T)}{\rho^T} \tag{A.21}$$

where minus signs have been inserted. Now replacing

$$\vec{x} \to \vec{x}' = \vec{x} \qquad \partial_i \to \partial_i' = \partial_i$$

$$t \to t' = -t \qquad \partial_t \to \partial_t' = -\partial_t \tag{A.22}$$

and using the imposed transformation properties of the field, we find

$$\partial_t' \dot{S}^R(\vec{x}',t') = \frac{1}{2}\left(V^R(\vec{x}',t')\right)^2 + \frac{1}{2}\left\{\frac{1}{2\rho^R(\vec{x}',t')} \times \right.$$

$$\left(\pi^R(\vec{x}',t')\left(\vec{\nabla}'\psi^R(\vec{x}',t')\right) + \left(\vec{\nabla}'\psi^R(\vec{x}',t')\right)\pi^R(\vec{x}',t')\right)\cdot\vec{V}^R(\vec{x}',t')$$

$$+ \vec{V}^R(\vec{x}',t')\cdot\frac{1}{2\rho^R(\vec{x}',t')}\left(\pi^R(\vec{x}',t')\left(\vec{\nabla}'\psi^R(\vec{x}',t')\right)\right.$$

$$\left.\left. +\left(\vec{\nabla}'\psi^R(\vec{x}',t')\right)\pi^R(\vec{x}',t')\right)\right\} + \int_{\rho_0}^{\rho^R(\vec{x}',t')} \frac{d\rho(\rho^R)}{(\rho^R)^2} \; . \qquad \text{(A.23)}$$

Therefore the Bernoulli equation is invariant under time reversal. The remaining equations can be verified to be invariant in just as straightforward a manner. Then we may quantize the system in the time reversed coordinate system, thus we get the commutation relations

$$[\phi^R(\vec{x}',t_x'),\rho^R(\vec{y}',t_y')]_{t_x'=t_y'} = i\delta(\vec{x}'-\vec{y}') \qquad \text{(A.24a)}$$

$$[\psi^R(\vec{x}',t_x'),\pi^R(\vec{y}',t_y')]_{t_x'=t_y'} = i\delta(\vec{x}'-\vec{y}') \qquad \text{(A.24b)}$$

all other commutators zero.

Now these are valid for all times $t_x' = t_y' = t'$, so if we vary $t' \to -t' = t$, and replace $\vec{x}'$, $\vec{y}'$ with $\vec{x}$, $\vec{y}$, since these are equal, we obtain

$$[\phi^R(\vec{x},t_x),\rho^R(\vec{y},t_y)]_{t_x=t_y} = i\delta(\vec{x}-\vec{y}) \qquad \text{(A.25a)}$$

$$[\psi^R(\vec{x},t_x),\pi^R(\vec{y},t_y)]_{t_x=t_y} = i\delta(\vec{x}-\vec{y}) \quad . \qquad \text{(A.25b)}$$

Then

$$[\phi^R(\vec{x},t_x),\rho^R(\vec{y},t_y)]_{t_x=t_y} = i\delta(\vec{x}-\vec{y}) = [\phi(\vec{x},t_x),\rho(\vec{y},t_y)]_{t_x=t_y} \qquad \text{(A.26a)}$$

$$[\psi^R(\vec{x},t_x),\rho^R(\vec{y},t_y)]_{t_x=t_y} = i\delta(\vec{x}-\vec{y}) = [\psi(\vec{x},t_x),\rho(\vec{y},t_y)]_{t_x=t_y} \qquad \text{(A.26b)}$$

the equal time commutation relations are invariant. So there exists a unitary transformation G_R, such that

$$\phi^R(\vec{x},t) = G_R^{-1}\phi(\vec{x},t)G_R = -\phi^T(\vec{x},-t) \qquad \text{(A.27a)}$$

$$\psi^R(\vec{x},t) = G_R^{-1}\psi(\vec{x},t)G_R = -\psi^T(\vec{x},-t) \qquad \text{(A.27b)}$$

$$\rho^R(\vec{x},t) = G_R^{-1}\rho(\vec{x},t)G_R = \rho^T(\vec{x},-t) \qquad \text{(A.27c)}$$

$$\pi^R(\vec{x},t) = G_R^{-1}\pi(\vec{x},t)G_R = \pi^T(\vec{x},-t) \qquad \text{(A.27d)}$$

and $G_R|0\rangle = |0\rangle$.

Now looking at the two point function,

$$<0|J_j(\vec{x},t)\rho(\vec{x}',t')|0> = <0|G_R^{-1}(\vec{x},t)G_R G_R^{-1}\rho(\vec{x}',t')G_R|0>$$

$$= <0|J_j^R(\vec{x},t)\rho^R(\vec{x}',t')|0>$$

$$= -<0|J_j^T(\vec{x},-t)\rho^T(\vec{x}',-t')|0>$$

$$= -<0|\left(\rho(\vec{x}',-t')J_j(\vec{x},-t)\right)^T|0>$$

$$= -<0|\rho(\vec{x}',-t')J_j(\vec{x},-t)|0>$$

$$= -<0|J_j(\vec{x},-t)\,\rho(\vec{x}',-t')|0>^* \quad . \qquad (A.28)$$

Replacing with the spectral representation,

$$\frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')}\int_0^\infty d\omega k_j \sigma_1(\vec{k},\omega)\delta(k_o-\omega)$$

$$= -\frac{1}{(2\pi)^3}\int d^4k\; e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\; e^{-ik_o(t-t')}\int_0^\infty d\omega k_j \sigma_1^*(\vec{k},\omega)\delta(k_o-\omega) \quad . \qquad (A.29)$$

Changing $\vec{k} \to -\vec{k}$ in the right hand side, and equating Fourier coefficients, we get

$$[illegible](\vec{k},\omega) = \sigma_1^*(-\vec{k},\omega) \quad . \qquad (A.30)$$

Now [illegible]g the result from space reflection invariance, we obtain

$$\sigma_1'(\vec{k},\omega) = \sigma_1^{\star}(\vec{k},\omega) \ . \tag{A.31}$$

The third spectral function

$$\begin{aligned}
<0|J_j(\vec{x},t)J_i(\vec{x}',t')|0> &= <0|G_R^{-1}J_j(\vec{x},t)G_RG_R^{-1}J_i(\vec{x}',t')G_R|0> \\
&= <0|J_j^R(\vec{x},t)J_i^R(\vec{x}',t')|0> \\
&= <0|J_j^T(\vec{x},-t)J_i^T(\vec{x}',-t')|0> \\
&= <0|\left(J_i(\vec{x}',-t')J_j(\vec{x},-t)\right)^T|0> \\
&= <0|J_i(\vec{x}',-t')J_j(\vec{x},-t)|0> \\
&= <0|J_j(\vec{x},-t)J_i(\vec{x}',-t')|0>^{\star} \ .
\end{aligned} \tag{A.32}$$

Using the spectral representations,

$$\frac{1}{(2\pi)^3}\int d^4k\, e^{ik(x-x')}\int_0^{\infty} d\omega \left\{(\delta_{ij}-\frac{k_ik_j}{(\vec{k})^2})\sigma_T(\vec{k},\omega) + \frac{k_ik_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)\right\}\delta(k_o-\omega)$$

$$= \frac{1}{(2\pi)^3}\int d^4k\, e^{-i\vec{k}\cdot\vec{x}-\vec{x}'}\, e^{-ik_o(t-t')}\int_0^{\infty} d\omega \ \times$$

$$\left\{(\delta_{ij}-\frac{k_ik_j}{(\vec{k})^2})\sigma_T^{\star}(\vec{k},\omega) + \frac{k_ik_j}{(\vec{k})^2}\sigma_L^{\star}(\vec{k},\omega)\right\}\delta(k_o-\omega) \ . \tag{A.33}$$

Changing $\vec{k} \to -\vec{k}$ on the right hand side, then equating Fourier coefficients,

$$(\delta_{ij} - \frac{k_i k_j}{(\vec{k})^2})\sigma_T(\vec{k},\omega) + \frac{k_i k_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)$$

$$= (\delta_{ij} - \frac{k_i k_j}{(\vec{k})^2})\sigma_T^*(-\vec{k},\omega) + \frac{k_i k_j}{(\vec{k})^2}\sigma_L^*(-\vec{k},\omega) \quad , \tag{A.34}$$

multiplying by k_i and summing over i, the longitudinal part separates, giving

$$\sigma_L(\vec{k},\omega) = \sigma_L^*(-\vec{k},\omega) \quad , \tag{A.35}$$

then this drops out of the equation, and taking the trace yields,

$$\sigma_T(\vec{k},\omega) = \sigma_T^*(-\vec{k},\omega) \quad . \tag{A.36}$$

Now applying the results of space reflection invariance, we get

$$\sigma_L(\vec{k},\omega) = \sigma_L^*(\vec{k},\omega) \tag{A.37a}$$

$$\sigma_T(\vec{k},\omega) = \sigma_T^*(\vec{k},\omega) \quad . \tag{A.37b}$$

Now we can easily show

$$<0|[J_j(\vec{x},t),J_i(\vec{x}',t')]|0>_{t=t'} = <0|J_j(\vec{x},t)J_i(\vec{x}',t)|0> - <0|J_j(\vec{x},t)J_i(\vec{x}',t)|0>^*$$

$$= \frac{1}{(2\pi)^3}\int d^4k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int_0^\infty d\omega\left\{(\delta_{ij}-\frac{k_i k_j}{(\vec{k})^2})\sigma_T(\vec{k},\omega)+\frac{k_i k_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)\right\}\delta(k_o-\omega)$$

$$- \frac{1}{(2\pi)^3}\int d^4k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int_0^\infty d\omega\left\{(\delta_{ij}-\frac{k_i k_j}{(\vec{k})^2})\sigma_T^*(\vec{k},\omega)+\frac{k_i k_j}{(\vec{k})^2}\sigma_L^*(\vec{k},\omega)\right\}\delta(k_o-\omega). \tag{A.38}$$

changing $\vec{k} \to -\vec{k}$ in the second term,

$$= \frac{1}{(2\pi)^3}\int d^4k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_0^\infty d\omega \Big\{ (\delta_{ij} - \frac{k_i k_j}{(\vec{k})^2})\sigma_T(\vec{k},\omega) + \frac{k_i k_j}{(\vec{k})^2}\sigma_L(\vec{k},\omega)$$

$$- (\delta_{ij} - \frac{k_i k_j}{(\vec{k})^2})\sigma_T^\star(-\vec{k},\omega) + \frac{k_i k_j}{(\vec{k})^2}\sigma_L^\star(-\vec{k},\omega) \Big\} \delta(k_o - \omega) \qquad (A.39)$$

this obviously vanishes from the above results.

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