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THE UNIVERSITY OF ALBERTA

Color-critical Hypergraphs

BY
DONOVAN ROSS HARE

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Do Hare

Permanent Address:
11036 86 Avenue
Edmonton, Alberta
Canada T6G 0W9

Date: August 21, 1987

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Color-critical Hypergraphs** submitted by **Donovan Ross Hare** in partial fulfillment of the requirements for the degree of **Master of Science**.

H P Abbott
Supervisor

S K Sehgal

Keith Authie

Date: August 21, 1987

To my parents Carl and Clara

ABSTRACT

This thesis investigates certain coloring problems in hypergraph theory. In Chapter 1 we give an overview. This includes definitions, examples, the statements of the problems, a brief history as well as some motivational material. Also included is a section describing the new results of the thesis. Chapter 2 contains the proofs of a number of results which pertain to the problem of how few edges color-critical hypergraphs may have. In Chapter 3 we establish the existence of certain color-critical hypergraphs whose existence was not previously known. All the proofs of the thesis are constructive in nature.

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Chapter 1. Overview

This thesis investigates certain coloring problems in hypergraph theory. The intent of this chapter is to provide motivation and background material for the problems being studied. In the first section we give most of the basic definitions and we give some examples that illustrate them. We refer to these examples from time to time later in this chapter and in subsequent chapters. In section two we state the problems dealt with in the thesis and in the next section we provide a brief history. We do not give an exhaustive survey of the literature, but we try to mention all of the main developments and we discuss all of the earlier work that has some bearing on ours. In the fourth section we formulate the statements of the results that are obtained in this thesis. We explain how they relate to previously known results and how they fit into the overall picture. The proofs are given in Chapters 2 and 3. We shall need, in the later chapters, a number of results of a technical nature. We have found it convenient to collect these in one place and have put them in the fifth section of this overview. The reader may wish to omit this section on first reading and refer to it as the need arises. Many of the results contained in this thesis are obtained by exploiting a certain general construction. We describe this construction in the last section of this chapter and make some remarks as to how it will be used.

1.1 Definitions

We now give some of the basic definitions. The term being defined is shown in bold face type. As a rule, we do not explain set theortic terminology, since it is standard. We also do not explain the terminology for ordinary graphs. In this regard we follow Bollobás [B1]. A **hypergraph** is an ordered pair $(V, \mathcal{G})$, where V is a finite nonempty set whose elements are called **vertices** and $\mathcal{G}$ is a collection of nonempty subsets of V whose members are called **edges**. We shall usually assume that there are no isolated vertices; that is, we assume $V = \bigcup \mathcal{G} = \bigcup \{E : E \in \mathcal{G}\}$. Thus, when we refer to the hypergraph $\mathcal{G}$ we shall mean $(\bigcup \mathcal{G}, \mathcal{G})$. The **order** of a hypergraph is the number of its vertices, and the **size** of a hypergraph is the number of its edges.

A hypergraph $\mathcal{G}$ is n-**uniform** (or an n-**graph**) if each of its edges is an n-subset of $\bigcup \mathcal{G}$ for some $n \geq 2$; that is, if for all $F \in \mathcal{G}$, $|F| = n$ for some $n \geq 2$. Observe that a 2-graph is an ordinary graph. A hypergraph $\mathcal{G}$ is **linear** if any two of its edges have at most one vertex in common; that is, if for all $E, F \in \mathcal{G}$, $E \neq F$, we have $|E \cap F| \leq 1$. Note that a 2-graph is necessarily linear.

Let $\mathcal{G}$ be a hypergraph. A hypergraph $\mathcal{H}$ is a **subgraph** of $\mathcal{G}$ if $\mathcal{H} \subseteq \mathcal{G}$, and is a **proper subgraph** if $\mathcal{H} \subset \mathcal{G}$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is a **spanning** subgraph of $\mathcal{G}$ if $\bigcup \mathcal{H} = \bigcup \mathcal{G}$. If $E \in \mathcal{G}$, then $\mathcal{G} - E$ is the subgraph obtained from $\mathcal{G}$ by removing the edge E. If $L \subset \bigcup \mathcal{G}$, $L \neq \emptyset$, then $\mathcal{G} - L = \{E : E \in \mathcal{G}, E \cap L = \emptyset\}$; that is, $\mathcal{G} - L$ is the hypergraph consisting of all edges of $\mathcal{G}$ that do not contain any vertices of L.

If $v \in \bigcup \mathcal{G}$, we write $\mathcal{G} - v$ instead of $\mathcal{G} - \{v\}$.

Two hypergraphs $\mathcal{G}$ and $\mathcal{H}$ are **isomorphic** if there is a bijection $\theta: \bigcup \mathcal{G} \to \bigcup \mathcal{H}$ such that $\theta(E)$ is an edge of $\mathcal{H}$ if and only if E is an edge of $\mathcal{G}$; that is, θ is a bijection which perserves edges. If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic hypergraphs whose vertex sets are disjoint we say that $\mathcal{H}$ is a **copy** of $\mathcal{G}$. When we say that $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$ are copies of $\mathcal{G}$, it is understood that the vertex sets of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$ are pairwise disjoint.

An r-**coloring** of a hypergraph is an assignment of $r \geq 2$ distinct colors (denoted by $1, 2, \ldots, r$) to its vertices so that none of its edges is **monochromatic**; that is, no edge has all of its vertices assigned the same color. Equivalently, an r-coloring of $\mathcal{G}$ is a partition of $\bigcup \mathcal{G}$ into r sets $V_1, V_2, \ldots, V_r$ so that for each $E \in \mathcal{G}$, $E \not\subset V_i$ for any i. The sets $V_1, V_2, \ldots, V_r$ are called the **color classes** of the r-coloring: V_i consists of those vertices that have been assigned color i. If r is small, we frequently denote the colors by red, blue, green, etc. The definition makes sense when $r = 1$, but in that case the hypergraph has no edges.

A hypergraph is r-**colorable** if it has an r-coloring. If a hypergraph is r-colorable then it is necessarily $(r+k)$-colorable for every positive integer k. The **chromatic number** of $\mathcal{G}$ is the smallest integer r for which $\mathcal{G}$ is r-colorable. $\mathcal{G}$ is r-**chromatic** if it has chromatic number r. Moreover, $\mathcal{G}$ is r-**critical** if it is r-chromatic and all of its proper subgraphs are not. Observe that in order to show that an r-chromatic hypergraph $\mathcal{G}$ is r-critical it suffices to show that $\mathcal{G} - E$ is $(r-1)$-

colorable for each $E \in \mathcal{G}$. A hypergraph is **critical** if it is r-critical for some integer r. $\mathcal{G}$ is **r-vertex-critical** if it is r-chromatic and if $\mathcal{G} - v$ is $(r-1)$-colorable for all vertices v of $\mathcal{G}$. A hypergraph is **vertex-critical** if it is r-vertex-critical for some r. A critical hypergraph is necessarily vertex-critical but not vice versa. Every r-chromatic hypergraph has an r-critical subgraph, and every r-vertex-critical hypergraph has an r-critical spanning subgraph.

An r-critical n-uniform hypergraph of order m is called an **(m, n, r)-graph.**

We now give some examples that illustrate some of the above terms.

Example 1. *The complete r-critical n-graph.*

Let $M(n,r) = (n-1)(r-1)+1$, $n, r \geq 2$. Let V be a set of size $M(n,r)$ and let $\mathcal{G}_1$ be the n-graph whose edges are all the n-subsets of V. That $\mathcal{G}_1$ is r-colorable is clear. Partition V into r sets, $r-1$ of size $n-1$ and one set of size 1. These sets form the color classes of an r-coloring. There is no $(r-1)$-coloring of $\mathcal{G}_1$. If there were, one of the color classes, by the box principle, would contain at least n vertices and hence an edge. Thus $\mathcal{G}_1$ is r-chromatic. For any $E \in \mathcal{G}_1$ we may set up an $(r-1)$-coloring of $\mathcal{G}_1 - E$ by letting E be a color class and letting the other $r-2$ color classes be $r-2$ pairwise disjoint $(n-1)$-subsets of $V \setminus E$. It follows that $\mathcal{G}_1$ is r-critical.

Example 2. *A* $(6,3,3)$*-graph.*

Let $\mathcal{G}_2$ be the 3-graph with vertex set $V = \{1,2,3,4,5,6\}$ and edge set

$\{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,5\},$

$\{2,4,6\},\{3,4,5\},\{3,4,6\}\}$. $\mathcal{G}_2$ is an example of a $(6,3,3)$-graph. Even though we are dealing here with a small graph it may be instructive if we provide a brief argument explaining why $\mathcal{G}_2$ is 3-critical. That $\mathcal{G}_2$ is 3-colorable is clear: any partition of V into three sets of size 2 will give a 3-coloring. That $\mathcal{G}_2$ is not 2-colorable may be seen as follows: since every 4-subset of V contains an edge of $\mathcal{G}_2$, no 2-coloring can have a color class of size 4 or more. Thus, if there were a 2-coloring, each color class must have size 3. However, it is easy to check that for each 3-subset E of V one of E or $V \setminus E$ is an edge of $\mathcal{G}_2$, and we therefore have a monochromatic edge. Finally, that $\mathcal{G}_2$ is 3-critical follows from the fact that if E is an edge of $\mathcal{G}_2$, then $\mathcal{G}_2 - E$ may be 2-colored by coloring the vertices of E red and those of $V \setminus E$ blue. This works because any two edges of $\mathcal{G}_2$ have nonempty intersection. We remark that there does not exist a $(6,3,3)$-graph with fewer than 10 edges.

Example 3. *A linear* $(7,3,3)$*-graph*

Let $\mathcal{G}_3$ be the 3-graph with vertex set $V = \{1,2,3,4,5,6,7\}$ and edges $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$. This hypergraph arises from the well known Fano plane, $PG(2,2)$ (the projective geometry of dimension 2 over the field of 2 elements). The vertices of $\mathcal{G}_3$ are the points of the plane and the edges are the lines of the plane. See Figure 1.1.

With a little effort one may show the $\mathcal{G}_3$ is 3-chromatic. That it is 3-critical may be seen by noting that for any edge E we may obtain a 2-coloring of $\mathcal{G}_3 - E$ by

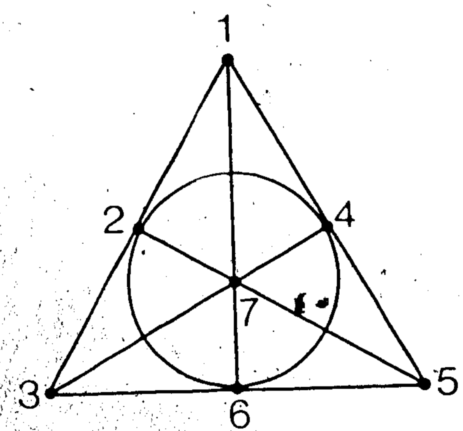


Fig. 1.1

coloring E red and $V \setminus E$ blue. A glance at Figure 1.1 makes this obvious.

We remark that linear $(m, 3, 3)$-graphs exists for $m = 7$ and $m \geq 9$. No such graph exists for $m < 7$ or $m = 8$.

1.2 Statements of Problems

Perhaps the most basic question that arises in connection with color critical hypergraphs is that of existence–for which values of m, n, r is there an (m, n, r)-graph or a linear (m, n, r)-graph? If a reasonable satisfactory answer to this question can be found, there are other questions that naturally arise. Several of these questions are extremal problems; for example, what is the largest number of edges such a graph may have, or how sparse can such a graph be; that is, how few edges can it have? There are also enumeration problems: for example, how many pairwise non-isomorphic (m,n,r)-graphs are there? One may also ask questions concerning the classification or characterization of (m,n,r)-graphs. Most of the literature in the area is devoted to the existence problem, which is not completely solved, and to the extremal problems. This thesis falls into these categories.

We now formulate the questions that we investigate in this thesis.

1(a) For given n and r, for which m do (m,n,r)-graphs exist?

(b) For those m for which (m,n,r)-graphs exist, what is the least number of edges such a graph may have? That is, what is the value of

$$E(m,n,r) = \min\{|\mathcal{G}| : \mathcal{G} \text{ is an } (m,n,r)\text{-graph }\}?$$

(c) What is the least number of edges an r-critical n-graph may have? Here there is no restriction on the number of vertices. Equivalently, what is the value of

$$E(n,r) = \min_{m} E(m,n,r)?$$

Question 2 is identically posed except that we restrict the graphs to be linear, with $E^*(m,n,r)$ and $E^*(n,r)$ defined in the natural way.

1.3 History

It is not suprising that these questions first arose for 2-graphs. Here problems 1 and 2 coincide. We now discuss some of the main developments for 2-graphs.

Until further notice, graph will mean 2-graph. While there are a few results on r-critical graphs in the early literature on graph theory, they were first studied systematically by G.A. Dirac in the 1950's (see [D1], [D2] and [D3]). It is a straightforward exercise to show that the only 3-critical graphs are the odd cycles, so that $E(2l+1,2,3) = 2l+1$, $E(2l,2,3)$ is not defined, and $E(2,3) = 3$. Dirac was hoping that there might be some reasonable characterization of the 4-critical graphs which would shed some light on the famous Four Color Problem (now a theorem). No such characterization has been found and it seems unlikely that there is any. The reader should see Chapter 11 of Ore's book [O1] for connections with the Four Color Problem and Bollobás [B2, Chapter 5] for aspects of r-critical graphs not mentioned here.

The complete graph on r vertices is r-critical, and there is clearly no such graph of smaller order. For $r \geq 4$ Dirac constructed r-critical graphs of order m for $m \geq r+2$ and showed that no such graph exists for $m = r+1$. Thus Questions 1(a) and 1(c) have been solved completely for graphs. Question 1(b), however, has turned out to be of a greater level of difficulty and a complete solution has not yet been found. We summarize a few of the key developments. We suppose $r \geq 4$ and $m \geq r+2$.

Since, in an r-critical graph, each vertex has degree at least $r-1$ we get, as was pointed out by Dirac [D2],

$$E(m,2,r) \geq \frac{m(r-1)}{2}. \tag{1}$$

The classical theorem of Brooks (see Bollobás [**B1**, page 91]) implies that

$$E(m,2,r) \geq \frac{m(r-1)}{2} + 1. \tag{2}$$

This is a slight improvement over the trivial bound given by (1), but it turns out, that (2) is actually equivalent to Brook's Theorem, thus suggesting that it may be difficult to improve (2) substantially. One of the main results obtained by Dirac [**D3**] is the following sharpening of (2):

$$E(m,2,r) \geq \frac{m}{2}(r-1) + \frac{r-3}{2}. \tag{3}$$

T. Gallai published two important papers on color critical graphs [**G1**], [**G2**]. In [**G1**] he obtained a lower bound for $E(m,2,r)$ which, if m is large compared to r, is superior to (3); namely,

$$E(m,2,r) \geq \frac{m}{2}(r-1) + \frac{m(r-3)}{2(r^2-3)}. \tag{4}$$

Observe that the second term on the right in (4) grows linearly with m, while the corresponding term in (3) depends only on r, so that for large m, (4) is a much stronger result.

G. Hajós [**H1**] described a method of constructing an r-critical graph from two smaller such graphs. From his construction it may be deduced that

$$E(m,2,r) \leq \frac{mr}{2} - \frac{m}{r-1}. \tag{5}$$

Furthermore, it follows from Hajós' construction that

$$E(m_1+m_2-1,2,r) \leq E(m_1,2,r) + E(m_2,2,r) - 1. \tag{6}$$

From (6) and a variant of Fekete's lemma on subadditive functions [**F1**], Hajós deduced that there exists a number $\alpha(2,r)$ such that, as $m \to \infty$,

$$E(m,2,r) = (\alpha(2,r) + o(1))\, m. \tag{7}$$

From (4) and (5) it follows that

$$\frac{r-1}{2} + \frac{r-3}{2(r^2-3)} \le \alpha(2,r) \le \frac{r}{2} - \frac{1}{r-1}. \tag{8}$$

No value of $\alpha(2,r)$ has been determined and, in fact, no improvement on the bounds given by (8) have been obtained, although it has been conjectured [**O1**, Chapter 11] that equality holds on the right in (8) for all r.

In summary, with regard to Question 1(b), it is known that for each $r \ge 4$, $E(m,2,r)$ grows in an essentially linear fashion with m, but the precise nature of this growth has not been determined. This completes our discussion of the case of 2-graphs.

We now discuss the case $n \ge 3$. It will be convenient if we follow roughly the chronological development. For this reason we begin with Question 1(c). P. Erdős and A. Hajnal [**E1**] raise the problem of determining the least number of edges a 3-critical n-graph may have. In our notation, they ask for the value of $E(n,3)$. They remark that the collection of all n-subsets of a set of size $2n-1$ is a 3-critical n-graph so that $E(n,3)$ exists and satisfies

$$E(n,3) \le \binom{2n-1}{n}. \tag{9}$$

In addition to the value $E(2,3) = 3$ mentioned earlier, only one value of $E(n,3)$ has been determined, namely $E(3,3) = 7$. The graph in Example 3 is a $(7,3,3)$-graph with 7 edges and one may show that there is no such graph with fewer edges. That $E(4,3) \leq 35$ follows from (9). B. Toft **[T1]** and P.D. Seymour **[S1]** independently found 3-critical 4-graphs showing that $E(4,3) \leq 23$. In the other direction P. Aizley and J.L. Selfridge **[A1]** announced that via an extensive computer search, they can show that $E(4,3) \geq 19$, but no details of this have been published.

Erdős **[E2]**, **[E3]** proved that for all $n \geq 2$,

$$2^{n-1} < E(n,3) \leq n^2 2^{n+1} . \tag{10}$$

Considerable effort has gone into the problem of improving the bounds given by (10) (see **[E4]**, **[S2]**, **[J1]**, **[E5]**, **[H2]**, for example). The best upper bound currently known is that of M. Herzog and J. Schönheim **[H2]** who proved that

$$E(n,3) \leq e^{\frac{1}{2}}(\log 2)n^2 2^n$$

and the best lower bound is that of J. Beck **[B3]** who showed that for each $\epsilon > 0$, if $n \geq n_0(\epsilon)$,

$$E(n,3) > n^{\frac{1}{3}-\epsilon} 2^n . \tag{11}$$

Beck's proof of (11) is complicated. A much simplified version can be found in the paper of J. Spencer [S3].

The hypergraph that establishes (9) is the special case $r = 3$ of the graph in

Example 1. It follows from Example 1 that $E(n,r)$ exists and satisfies

$$E(n,r) \leq \binom{(n-1)(r-1)+1}{n}. \tag{12}$$

It is an attractive conjecture of Erdős [E6, page 282] that equality holds in (12) for each fixed n, provided r is sufficiently large. The argument used by Beck to prove (11) gives, with only minor modifications in detail,

$$E(n,r) > n^{\frac{1}{3}-\epsilon}(r-1)^n \tag{13}$$

for each $\epsilon > 0$ and each $r \geq 3$, provided $n \geq n_0(\epsilon, r)$.

It should also be noted that Herzog and Schönheim [H2] proved that for all $n \geq 3$, $r \geq 3$,

$$E(n,r) \leq e^{(r-2)/(r-1)} \quad (\log r - 1)n^2(r-1)^n \tag{14}$$

but (14) is weaker that (12) for fixed n, and r large compared to n. This perhaps lends some slim support to Erdős' conjecture conerning (12).

This completes our summary of the main results concerning Question 1(c).

We turn now to questions 1(a) and 2(a). Question 1(a) has been solved completely: (m,n,r)-graphs exist if and only if $m \geq M(n,r) = (n-1)(r-1)+1$. This was shown by H.L. Abbott and D. Hanson [A2] in the case $r = 3$ and by Toft [T1] for $r \geq 4$. The hypergraphs constructed in [A2] and [T1] are not linear graphs and thus the solution to Question 1(a) sheds no light on 2(a). Erdős and Hajnal [E1] state the problem and attribute it to Gallai. They note that the Fano plane (the graph of Example 3) is a linear $(7,3,3)$-graph, but they give no other examples.

It is not immediately obvious that such graphs exist for $n \geq 4$. Proofs of the existence of such graphs were given by several authors at about the same time and by a variety of methods. Abbott [A3] showed that for each n there is at least one m for which a linear $(m, n, 3)$-graph exists and Liu [L1] showed that the argument works for $n \geq 4$ also. No specific value of m was given. The proof uses Ramsey's Theorem (see Bollobás [B1, Chapter 6]) and gives only an upper bound for m in terms of certain undetermined Ramsey Numbers. Erdős and Hajnal [E7] showed by probabilistic methods that for each pair n, r there are arbitrarily large values of m for which linear (m, n, r)-graphs exist. A constructive proof of this was given by L. Lovász [L2] and somewhat later Erdős and Lovász [E8] gave another proof using probabilistic methods which provided a much better estimate for the least value of m in question. It should be noted that in the papers [E7], [L2], [E8] the following stronger result is proved: there exist r-chromatic n-graphs containing no cycles of length $\leq l$ for any r, n, l, $r \geq 3$, $n \geq 2$, $l \geq 2$. We do not define a cycle in a hypergraph here but simply note that for $n \geq 3$, the condition that a hypergraph has no cycle of length 2 is equivalent to the condition that it is linear. Yet another proof of the existence of linear $(m, n, 3)$-graphs was given by A.W. Hales and R.I. Jewett [H3] in their paper on positional games. Imagine the game of tic-tac-toe being played on a "board" of side n in k dimensions. A result of Hales and Jewett is that if k $(= k(n))$ is sufficiently large, the game cannot end in a draw. One may think of two players alternately coloring the squares of the board red and blue.

Since one of the players has to win, there must result a monochromatic "line". Thus the hypergraph whose vertices are the squares and whose edges are the lines of the board is (at least) 3-chromatic, and therefore contains a 3-critical subgraph. Since the graph is clearly linear, this establishes the existence of a linear $(m, n, 3)$-graph for some m.

In [A4] Abbott and Liu show that for each pair n, r, $n \geq 3$, $r \geq 3$, there are only finitely many values of m for which linear (m, n, r)-graphs do not exist. In other words, for each pair n, r, there corresponds a least integer $M^*(n, r)$such that for all $m \geq M^*(n, r)$ there exists a linear (m, n, r)-graph. Only one value of $M^*(n, r)$has been determined, namely $M^*(3, 3) = 9$. In Chapter 3 of this thesis we obtain some new results concerning $M^*(4, 3)$ and $M^*(3, 4)$.

This completes our summary of the literature concerning Question 1(a) and 2(a).

As far as Question 2(c) is concerned we state only one result. Erdős and Lovász [E8] prove that

$$\frac{4^n}{256n^3} \leq E^*(n, 3) \leq 6420n^4 4^n. \tag{15}$$

The reader should note that $E^*(n, 3)$ exhibits a much faster rate of growth than $E(n, 3)$ as can be seen by comparing (15) with (10). Bounds for $E^*(n, r)$, $r \geq 4$, are also given in [E8], but they are complicated and since we de not need to refer to them we do not state them here.

We now turn to Question 1(b) and 2(b). Seymour [S4] and D.R. Woodall [W1], answering a question posed by Erdős, independently showed that any 3-critical

n-graph has at least as many edges as vertices, so that for $m \geq M(n,3)$

$$E(m,n,3) \geq m. \tag{16}$$

In his thesis [L1], Liu showed that for each $n \geq 3$,

$$E(m,n,3) = m + O(1), \quad \text{as } m \to \infty, \tag{17}$$

where the constant implied by the O-notation depends only on n. M. Burstein [B4] proved the following stronger and very striking result: for each fixed n and all sufficiently large m,

$$E(m,n,3) = m. \tag{18}$$

It is thus curious that while $E(m,4,3) = m$ for large m, we do not know the least m for which this is so, nor do we know the value of $E(9,4,3)$ (see [E4] and [A5]).

In [L1], Liu showed that the argument used by Seymour to prove (16) and the natural extension of the observation on which (1) is based may be used to show that

$$E(m,n,r) \geq \max\left\{1, \frac{r-1}{n}\right\} m. \tag{19}$$

Of course, (19) holds for $E^*(m,n,r)$ as well. In [L1] Liu also showed that for $n \geq 3$, $r \geq 3$, the limits

$$\alpha(n,r) = \lim_{m\to\infty} \frac{E(m,n,r)}{m}$$

and

$$\alpha^*(n,r) = \lim_{m\to\infty} \frac{E^*(m,n,r)}{m}$$

exist and are finite. That $\alpha(2,r)$ exists for $r \geq 4$ is the result of Hajós mentioned earlier.

It follows from (19) that

$$\alpha^*(n,r) \geq \alpha(n,r) \geq \max\left\{1, \frac{r-1}{n}\right\}. \tag{20}$$

Also, from (17) and (18) it follows that $\alpha(n,3) = 1$ for $n \geq 3$. The constructions establishing (17) and (18) do not yield linear graphs except in the case $n = 3$, where we get $\alpha^*(3,3) = 1$. No other explicit values of $\alpha(n,r)$ or $\alpha^*(n,r)$ have been previously determined. We shall obtain some new information on these limits in Chapter 2.

1.4 Results of the Thesis

Chapter 2 of this thesis gives some new information on the numbers $\alpha(n,r)$ and $\alpha^*(n,r)$. As was noted in Section 1.3, $\alpha^*(3,3) = 1$ and $\alpha(n,3) = 1$ for all $n \geq 3$. The graphs of Erdős and Lovász that establish the upper bound in (15) have $320n^4 2^n$ vertices and $6400n^4 4^n$ edges, so that the ratio of edges to vertices is about $(20)2^n$. Moreover, in their paper, they show that no 3-critical linear n-graph has fewer than $\frac{2^n}{16}$ vertices, so that in a 3-critical n-graph of smallest size the order of magnitude of the ratio of edges to vertices cannot be made appreciably smaller than 2^n. Also, if one uses this graph of smallest order in conjunction with one of the constructions in [A4] one may get an upper bound for $\alpha^*(n,3)$ which, while its precise form will be fairly complicated, is exponential in n.

One of the main results of this thesis is that for all $n \geq 3$

$$\alpha^*(n,3) = 1, \tag{21}$$

in sharp contrast to what one might have expected, given the remarks in the preceding paragraph.

We also obtain information concerning $\alpha(n,r)$ and $\alpha^*(n,r)$ for $r \geq 4$. Definitive results should not be expected here, however, since even for $n = 2$, exact results are not known (see (8)). We shall prove that, for $r \geq 3$, $n \geq 3$,

$$\alpha(n,r+1) \leq \alpha(n,r) + 1 \tag{22}$$

and

$$\alpha^*(n,r+1) \leq \alpha^*(n,r) + 1. \tag{23}$$

These inequalities, when combined with $\alpha(n,3) = \alpha^*(n,3) = 1$ give

$$\alpha(n,r) \leq r - 2 \tag{24}$$

and

$$\alpha^*(n,r) \leq r - 2. \tag{25}$$

Note that even though the gap between (20) and (24), (25) is fairly wide, the upper bounds given above are independent of n.

Denote by $T = T(n)$ the least integer for which there exists a $(T,n,3)$-graph with T edges. Such a T exists by the result (18) of Burstein. We shall prove that

$$\alpha(n,4) \leq 2 - \frac{2}{\alpha(n-1,4)T + 1}. \tag{26}$$

This, when combined with (22), shows that for $r \geq 4$, strict inequality holds in (24). We do not know whether strict inequality holds in (25) for $r \geq 4$, or whether $\alpha(n,r) < \alpha^*(n,r)$.

Observe that in the case $r = n + 1$ we get from (20) that $\alpha(n, n+1) \geq 1$. We shall prove that this can be strengthened to

$$\alpha(n, n+1) \geq 1 + \frac{n-2}{n^2} \tag{27}$$

but we can obtain no improvement over (20) for any other value of r and n.

The only value of $T(n)$ which is known is $T(3) = 7$. Thus it is not feasible to try to extract from (26) explicit upper bounds for $\alpha(n,4)$ for $n \geq 4$, but it may be worthwhile to record the results that come from (26) and (27) when $n = 3$:

$$\frac{10}{9} \leq \alpha(3,4) \leq \frac{35}{19}.$$

In Chapter 3 of this thesis we obtain some new information on the numbers $M^*(4,3)$ and $M^*(3,4)$. As was noted in Section 1.3, Abbott and Liu [**A4**] showed that $M^*(3,3) = 9$. In his thesis, Liu showed that $M^*(4,3) \leq 8928$ and $M^*(3,4) \leq 62835$. These were improved in [**A4**] to $M^*(4,3) \leq 124$ and $M^*(3,4) \leq 1399$ and in [**A6**] it was shown that $M^*(3,4) \leq 719$. We shall prove that

$$M^*(4,3) \leq 51 \tag{28}$$

and

$$M^*(3,4) \leq 100. \tag{29}$$

1.5 Other Needed Results

We describe in this section some technical results that we shall need in Chapters 2 and 3. As was mentioned in the introductory paragraph of this overview, the reader may wish to omit this section on first reading and refer to it as the need arises.

1.5.1 Block Designs

A **block design** with parameters (v, b, r, k, λ), or a (v, b, r, k, λ)**-design**, is an arrangement of v objects into b sets, called **blocks**, such that each block contains exactly k objects, each object occurs in exactly r different blocks, and every pair of objects occurs in exactly λ blocks. The parameters are not independent; simple counting arguments show that

$$vr = bk \qquad \text{and} \qquad (v-1)\lambda = (k-1)r. \tag{30}$$

The connection with our work is that any (v, b, r, k, λ)-design is a k-graph whose vertices are the objects and whose edges are the blocks, and if $\lambda = 1$, the k-graph is linear.

We shall not need any of the general theory of block designs, but we need two specific examples. As a general reference see Hall ([H4], especially the table in Appendix 1). The two examples that we need are the following:

a) A $(25, 50, 8, 4, 1)$-design.

We will use the $(25, 50, 8, 4, 1)$-design whose objects are the 25 elements of $Z_5 \times Z_5$ and whose blocks are obtained from $\{(0,0), (1,0), (0,1), (4,4)\}$ and

$\{(0,0),(2,0),(0,2),(3,3)\}$ by adding each element of $Z_5 \times Z_5$ to both (Design #22 in Hall's table). It was verified by Abbott and Liu that this design is a linear $(25,4,3)$-graph.

b) A $(31,155,15,3,1)$-design.

We will also use the $(31,155,15,3,1)$-design whose objects are the elements of Z_{31} and whose blocks are obtained by adding each element of Z_{31} to the blocks $\{0,1,18\}$, $\{0,2,5\}$, $\{0,4,10\}$, $\{0,8,20\}$, $\{0,9,16\}$ (Design #101 in Hall's table). We remark that this design is a model of $PG(4,2)$, the projective geometry of dimension 4 over the field of 2 elements. The objects are the points of the geometry and the blocks are the lines. Rosa [R1] proved that this design is 4-chromatic. It is not known whether it is critical, but Liu verified that it is vertex-critical. It therefore contains a linear $(31,3,4)$-graph.

We remark that the graphs in Examples 2 and 3 in Section 1.2 are also block designs; the $(6,3,3)$-graph is a $(6,10,5,3,2)$-design and the linear $(7,3,3)$-graph is a $(7,7,3,3,1)$-design.

1.5.2 Difference Sets

A set $D = \{d_1, d_2, \ldots, d_t\}$ of integers is a **difference set** if no positive integer has more than one representation in the form $d_j - d_i$. Let k be a integer and let $D+k = \{d_1+k, d_2+k, \ldots, d_t+k\}$. We call $D+k$ a **translate** of D. The following simple lemma explains how difference sets may be used to contruct linear graphs.

LEMMA 1.1. *Let $D = \{d_1, d_2, \ldots, d_t\}$ be a difference set. Then any two distinct translates of D have at most one element in common.*

PROOF: Let k_1 and k_2 be integers, $k_2 > k_1$. Suppose $s_1, s_2 \in (D + k_1) \cap (D + k_2)$. Then $s_1 = d_{i_1} + k_1 = d_{i_2} + k_2$, and $s_2 = d_{j_1} + k_1 = d_{j_2} + k_2$, for some $1 \le i_1, i_2, j_1, j_2 \le t$. Hence $k_2 - k_1 = d_{i_1} - d_{i_2} = d_{j_1} - d_{j_2}$. Since D is a difference set, $i_1 = j_1$ and $i_2 = j_2$ so that $s_1 = s_2$. Thus $|(D + k_1) \cap (D + k_2)| \le 1$. □

A consequence of this lemma is that any finite collection of translates of D gives the edge set of a linear graph.

1.5.3 The Erdõs - Hanani Theorem

In proving that $\alpha^*(n, r+1) \le \alpha^*(n, r) + 1$ we shall need to have at our disposal certain structures which are "almost" block designs with $\lambda = 1$. The precise result we use is the following theorem of Erdõs and Hanani [E9, Theorem 1] which we formulate as a lemma.

LEMMA 1.2. *Let S be a set of size v, and let $k \ge 2$. Let $\mathcal{B}$ be maximal collection of k-subsets of S such that no pair of elements occurs in more than one member of $\mathcal{B}$. Then*

$$\lim_{v \to \infty} \frac{|\mathcal{B}|}{v^2} = \frac{1}{k(k-1)}.$$

Note that if there were a (v, b, r, k, λ)-design, the number b of blocks would be $\frac{v(v-1)}{k(k-1)}$ by (30).

1.6 A General Construction

Most of the proofs of this thesis involve constructions of new hypergraphs from existing ones. In this section we describe a construction which will be used several times in subsequent chapters.

For $i = 1, 2, \ldots, l$, let $\mathcal{G}_i$ be a linear (m_i, n, r)-graph, E_i an edge of $\mathcal{G}_i$, and v_i a vertex of E_i. Let v be a new vertex. Let

$$E = \left(\bigcup_{i=1}^{l} E_i \setminus \{v_i\} \right) \cup \{v\}$$

and for $F \in \mathcal{G}_i$ let

$$F' = \begin{cases} (F \setminus \{v_i\}) \cup \{v\} & \text{if } v_i \in F \\ F & \text{if } v_i \notin F. \end{cases}$$

Let $\mathcal{G}$ be the hypergraph whose edges are:

i) the edge E

ii) the edges F', $F \in \mathcal{G}_i$ for some i, $1 \leq i \leq l$.

Less formally, $\mathcal{G}$ is the hypergraph obtained from the graphs $\mathcal{G}_i$ by identifying each v_i with v. The edge E is just the union of the E_i, with v_i replaced by v (see Figure 1.2).

We say that $\mathcal{G}$ is a **long edge graph** and we refer to E as the **long edge**. We shall use the notation

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{l} (\mathcal{G}_i, E_i, v_i). \tag{31}$$

The long edge graph $\mathcal{G}$ given by (31) has the properties given in the following lemma.

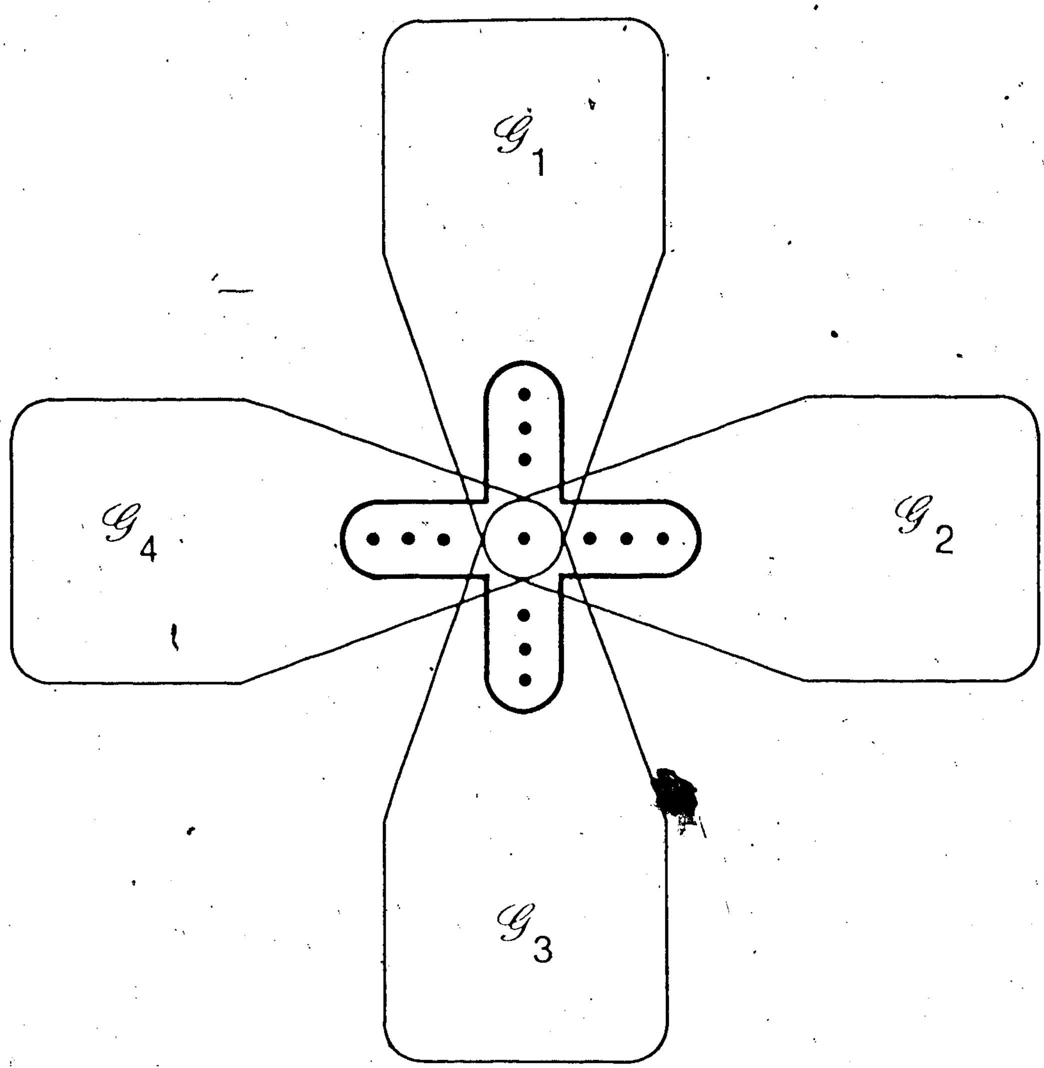


Fig. 1.2

LEMMA 1.3 (LONG EDGE).

a) the size of the long edge is $l(n-1)+1$.

b) $\mathcal{G}$ *has order* $\left(\sum_{i=1}^{l} m_i\right) - l + 1$

c) $\mathcal{G} - E$ *is n-uniform and linear*

d) $\mathcal{G} - E$ *is* $(r-1)$*-colorable and in any* $(r-1)$*-coloring of* $\mathcal{G} - E$, E *is monochromatic.*

PROOF: a), b) and c) are clear. An $(r-1)$-coloring of $\mathcal{G} \setminus E$ induces an $(r-1)$-coloring of each $\mathcal{G}_i - E_i$, $i = 1, 2, \ldots, l$, if we color vertex v_i the same color as vertex v. Since $\mathcal{G}_i$ is r-critical, E_i must be monochromatic, and since each v_i is colored the same as v, every E_i must be the same color. Thus E is monochromatic, and d) holds also. □

We now make some brief comments as to how the long edge construction will be used in the subsequent work. We hope that these comments will make the arguments seem less ad hoc, and that there is, running through them, a common theme.

In many of the proofs in Chapters 2 and 3 we shall need to construct r-chromatic graphs which contain large r-critical subgraphs. The long edge graphs, or variants of them will play a central role in these constructions. Our graphs will be constructed in stages. At the first stage we construct an r-chromatic long edge graph $\mathcal{G}$. At the second stage we delete the long edge E and add some new vertices and edges so as to get a graph $\mathcal{H}$ which contains $\mathcal{G} - E$ as a subgraph. Some of the new edges will have vertices in common with E. Our object will be to show that $\mathcal{H}$ contains a large r-critical subgraph. Note that in attempting to $(r-1)$-color $\mathcal{H}$ we are forced, by part d) of the Long Edge Lemma, to color all of the vertices of E the same. If the new edges are chosen appropriately (the way in which they are chosen will, of course, depend on the problem at hand) the fact that E is monochromatic will impose limitations as to how the colors are assigned to the rest of the graph. The goal is to show that the limitations are severe enough to rule out the possibility of any $(r-1)$-coloring of $\mathcal{H}$, but not severe enough to rule out the possibility of $(r-1)$-coloring $\mathcal{H} - F$ where F is any edge in some large subgraph $\mathcal{H}'$ of $\mathcal{H}$.

The details will of course vary from one situation to another. In fact, we shall sometimes need to show that the whole graph $\mathcal{H}$ is r-critical, in which case there has to be considerable care taken at the first stage of the construction. This is especially so in the proofs of (28) and (29).

We make a few remarks as to how the proofs are presented. It is frequently the case in combinatorial mathematics that when trying to establish the existence of a combinatorial object with certain properties the hard part is the actual finding of the object (if there is one). The verification that it has the desired properties may be much more straightforward. This seems to be the case in some of the questions dealt with in this thesis, and our presentation of the proofs of the results have been influenced by this. Our work involves the construction of hypergraphs for which certain coloring properties are claimed. In each case we give complete details of the construction, and when we state that a particular hypergraph $\mathcal{H}$ is r-critical, we exhibit an explicit $(r-1)$-coloring of $\mathcal{H} - F$ for each $F \in \mathcal{H}$. However, we sometimes omit some (or all) of the details of the verification that a certain $(r-1)$-coloring is, in fact, such, especially if the verification is straightforward or similar to one that has been described earlier in the thesis.

Chapter 2. Estimates of $\alpha^*(n,r)$ and $\alpha(n,r)$

In this chapter we present the proofs of the results concerning $\alpha^*(n,r)$ and $\alpha(n,r)$ stated in Section 1.4.

THEOREM 2.1. $\alpha^*(n,3) = 1, \qquad n \geq 3.$

PROOF: Since $\alpha^*(n,3) \geq \alpha(n,3) = 1$ it is only necessary to prove that $\alpha^*(n,3) \leq 1$. Moreover, since $\alpha^*(3,3) = 1$ (see Section 1.3) we may suppose that $n \geq 4$. Let $D = \{d_1, d_2, \ldots, d_{n-2}\}$ be a difference set such that $1 = d_1 < d_2 < \ldots < d_{n-3} < d_{n-2}$. Let $t = d_{n-2}$ and $s = t + 2$. Let $m = M^*(n,3)$ and for $i = 1, 2, \ldots, s$, let $\mathcal{G}_i$ be a linear $(m,n,3)$-graph, E_i be an edge of $\mathcal{G}_i$, v_i be a vertex of E_i, and let v be a new vertex. Let $\mathcal{G}$ be the long edge graph

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{s} (\mathcal{G}_i, E_i, v_i).$$

It is straightforward to verify, using the definition of s, that

$$|E| = s(n-1) + 1 \geq (t+1)(n-2) + \left\lfloor \frac{n}{2} \right\rfloor + t. \tag{32}$$

Let $S_1, S_2, \ldots, S_t$ be pairwise disjoint subsets of E of size $n-2$ and let $x, f_1, f_2, \ldots, f_{\lceil \frac{n}{2} \rceil}, h_1, h_2, \ldots, h_t$ be distinct elements of E not contained in any of the S_i.

Let $\mathcal{G}'$ be a copy of $\mathcal{G}$. Let $S_1', S_2', \ldots, S_{t+1}'$ be pairwise disjoint $(n-2)$-subsets of the long edge E' of $\mathcal{G}'$, and let $f_1', f_2', \ldots, f'_{\lfloor \frac{n}{2} \rfloor}, h_1', h_2', \ldots, h_t'$ be distinct elements of E' not occuring in any of the S_i'. That such sets and vertices may be chosen follows from (32).

Let k be a positive integer and let 1, 2, ..., $(k+1)t$ and $\overline{1}$, $\overline{2}$, ..., $\overline{(k+1)t}$ be new vertices. For $i = 0, 1, \ldots, kt-1$, let $\overline{D+i} = \{\overline{d_1+i}, \overline{d_2+i}, \ldots, \overline{d_{n-2}+i}\}$. For $i = 1, 2, \ldots, t$ and for $j = 1, 2, \ldots, kt-1$ put $h_{jt+i} = h_i$, $h'_{jt+i} = h'_i$.

Let $\mathcal{H}_k$ be the hypergraph whose edges are:

i) those of $\mathcal{G} - E$ and $\mathcal{G}' - E'$

ii) $F = \{f_1, f_2, \ldots, f_{\lceil \frac{n}{2} \rceil}, f'_1, f'_2, \ldots, f'_{\lfloor \frac{n}{2} \rfloor}\}$

iii) $F_1 = S_1 \cup \{x, 1\}$

iv) $F_i = S_i \cup \{\overline{i-1}, i\}$ for $i = 2, 3, \ldots, t$

v) $\overline{F}_i = S'_i \cup \{i, \overline{i}\}$ for $i = 1, 2, \ldots, t$

vi) $H_i = (\overline{D+i-1}) \cup \{h_i, t+i\}$ for $i = 1, 2, \ldots, kt$

vii) $\overline{H}_i = (D+i-1) \cup \{h'_i, \overline{t+i}\}$ for $i = 1, 2, \ldots, kt$

viii) $H^*_k = S'_{t+1} \cup \{(k+1)t-1, (k+1)t\}$.

Then $\mathcal{H}_k$ is clearly n-uniform. By appealing to Lemma 1.1 on difference sets and on recalling how t is chosen ($t = d_{n-2}$), it is straightforward to verify that $\mathcal{H}_k$ is linear.

$\mathcal{H}_k$ is 3-chromatic by the following argument. Suppose, to the contrary, that $\mathcal{H}_k$ is 2-colorable and color it red and blue. Then the sets E and E' will be monochromatic by Lemma 1.3, part d). If E and E' were assigned the same color, F would be monochromatic. Thus we may suppose, without loss of generality, that E is red and E' is blue. It follows that the sets S_i and the vertices x, h_1, h_2, ..., h_t, f_1, f_2, ..., $f_{\lceil \frac{n}{2} \rceil}$ are red, and the sets S'_i and vertices h'_1, h'_2, ..., h'_t, f'_1, f'_2, ... $f'_{\lfloor \frac{n}{2} \rfloor}$ are blue. Vertex 1 must be colored blue since otherwise F_1 would be red. This, in turn, forces $\bar{1}$ to be red since otherwise $\overline{F}_1$ would be blue. Let $j > 1$ and suppose we have proved that for $1 \leq i < j$, vertex i is colored blue and vertex $\bar{i}$ is colored red. If $j \leq t$ then the edge F_j forces vertex j to be colored blue. This, in turn, forces vertex $\bar{j}$ to be colored red since otherwise $\overline{F}_j$ would be blue. If $t+1 \leq j \leq (k+1)t$ then edge H_{j-t} forces vertex j to be colored blue, and and similiarily edge $\overline{H}_{j-t}$ ensures vertex $\bar{j}$ is colored red. It follows, by induction, that 1, 2, ..., $(k+1)t$ must be blue and the vertices $\bar{1}$, $\bar{2}$, ..., $\overline{(k+1)t}$ must be red. But then H_k^* is blue, a contradiction. Thus $\mathcal{H}_k$ is not 2-colorable. If at the last step in the above argument we were to color $(k+1)t$ green we would get a 3-coloring of $\mathcal{H}_k$. Therefore $\mathcal{H}_k$ is 3-chromatic.

$\mathcal{H}_k$ may not be 3-critical. However, if k is large $\mathcal{H}_k$ will contain a large critical subgraph. Let $\mathcal{H}_k'$ be a 3-critical subgraph of $\mathcal{H}_k$. We prove that for $j = 1, 2, \ldots,$ $kt-1$, H_j, $\overline{H}_j$ are edges of $\mathcal{H}_k'$. It will suffice to exhibit 2-colorings of $\mathcal{H}_k - H_j$ and $\mathcal{H}_k - \overline{H}_j$. The idea is to exploit the fact that in our attempt to 2-color $\mathcal{H}_k$ we were forced to color 1, 2, ..., $(k+1)t$ blue and $\bar{1}$, $\bar{2}$, ..., $\overline{(k+1)t}$ red. However, if we delete H_j or $\overline{H}_j$ we gain some flexibility and are able to complete the 2-coloring of the resulting graph. The details now follow. Let $j \in \{1, 2, \ldots, kt-1\}$. Color $\mathcal{G} - E$ and $\mathcal{G}' - E'$ red and blue so that E is red and E' is blue. Color vertices $\bar{1}$, $\bar{2}$, ..., $\overline{t+j-1}$ red and color vertices $1, 2, \ldots, t+j-1$ blue. Note that at this stage there is no monochromatic edge. The coloring of the rest of the graph will depend on whether we are considering H_j or $\overline{H}_j$.

Consider H_j. Color vertices $t+j+2l$ and $\overline{t+j+2l}$ red, $l = 0, 1, \ldots, \left\lfloor \frac{kt-j}{2} \right\rfloor$, and color the vertices $t+j+2l+1$ and $\overline{t+j+2l+1}$ blue, $l = 0, 1, \ldots, \left\lfloor \frac{kt-j-1}{2} \right\rfloor$. Then the (deleted) edge H_j is red. However, since $\{t+j-1, \overline{t+j}\} \subset \overline{H}_j$, $\overline{H}_j$ is not monochromatic. Moreover, since $\{\overline{t+l-1}, t+l\} \subset H_l$ and $\{t+l-1, \overline{t+l}\} \subset \overline{H}_l$, H_l and $\overline{H}_l$ are not monochromatic, $l = j+1, j+2, \ldots, kt$. Finally, vertex $t(k+1)-1$ is colored opposite to vertex $t(k+1)$ since one is numbered evenly and the other is numbered oddly, and hence the edge H_k^* is not monochromatic. Therefore $\mathcal{H}_k - H_j$ is 2-colorable, and hence H_j is an edge of $\mathcal{H}_k'$.

Consider $\overline{H}_j$. Color vertices $t+j+2l$ and $\overline{t+j+2l}$ blue, $l = 0, 1, \ldots, \left\lfloor \frac{kt-j}{2} \right\rfloor$, and color the vertices $t+j+2l+1$ and $\overline{t+j+2l+1}$ red, $l = 0, 1, \ldots, \left\lfloor \frac{kt-j-1}{2} \right\rfloor$. The preceding argument now applies with only obvious changes.

Let q be the number of edges of type i) to v). Note that q depends only on the graphs $\mathcal{G}_i$ and the difference set D and thus only on n. The number of vertices of $\mathcal{H}_k'$ is at least $2(kt+t-1)$, and since $\mathcal{H}_k'$ is a subgraph of $\mathcal{H}_k$ it has at most $q+2kt+1$ edges. Hence

$$\alpha^*(n,3) \leq \lim_{k\to\infty} \frac{q+2kt+1}{2(kt+t-1)} = 1.$$

□

THEOREM 2.2. $\alpha^*(n,r+1) \leq \alpha^*(n,r)+1, \qquad n,r \geq 3.$

PROOF: The argument uses the long edge construction and the theorem of Erdős and Hanani given in Section 1.5.3 as Lemma 1.2.

Let $m = M^*(n, r+1)$, $l \geq 2$, and let $\mathcal{G}$ be the long edge graph

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{l} (\mathcal{G}_i, E_i, v_i)$$

where each $\mathcal{G}_i$ is a linear $(m, n, r+1)$-graph with t edges. Let $\mathcal{B} = \{M_1, M_2, \ldots, M_p\}$ be a maximal collection of $(n-1)$-subsets of the long edge E such that every 2-subset of E is contained in at most one member of $\mathcal{B}$. $\mathcal{B}$ is a linear $(n-1)$-graph and, by the result of Erdős and Hanani (with $v = l(n-1)+1$ and $k = n-1$) we have

$$p = \frac{(n-1)l^2}{n-2}(1 + o(1)), \qquad \text{as } l \to \infty. \tag{33}$$

Choose l so large that $p \geq M^*(n, r)$. (33) ensures that this is possible. Let $\mathcal{F}$ be a linear (p, n, r)-graph with $q = E^*(p, n, r)$ edges. Let the vertex set of $\mathcal{F}$ be $\{x_1, x_2, \ldots, x_p\}$ and let $\mathcal{M} = \{M_1 \cup \{x_1\}, M_2 \cup \{x_2\}, \ldots, M_p \cup \{x_p\}\}$. Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{G} - E$, $\mathcal{F}$ and $\mathcal{M}$. Then $\mathcal{H}$ is n-uniform and linear (see Figure 2.1).

We show that $\mathcal{H}$ is $(r+1)$-chromatic. Suppose, to the contrary, that we can r-color $\mathcal{H}$ in colors 1, 2, ..., r. Then by Lemma 1.3, the long edge E is monochromatic. So, without loss of generality, suppose E is colored r. Now no vertex of $\mathcal{F}$ can be colored with color r for otherwise there would be a monochromatic edge of $\mathcal{M}$. Hence the subgraph $\mathcal{F}$ must be colored in colors 1, 2, ..., $r-1$. But $\mathcal{F}$ is r-critical, a contradiction. Hence $\mathcal{H}$ is not r-colorable. It is clearly $(r+1)$-colorable and therefore $\mathcal{H}$ is $(r+1)$-chromatic.

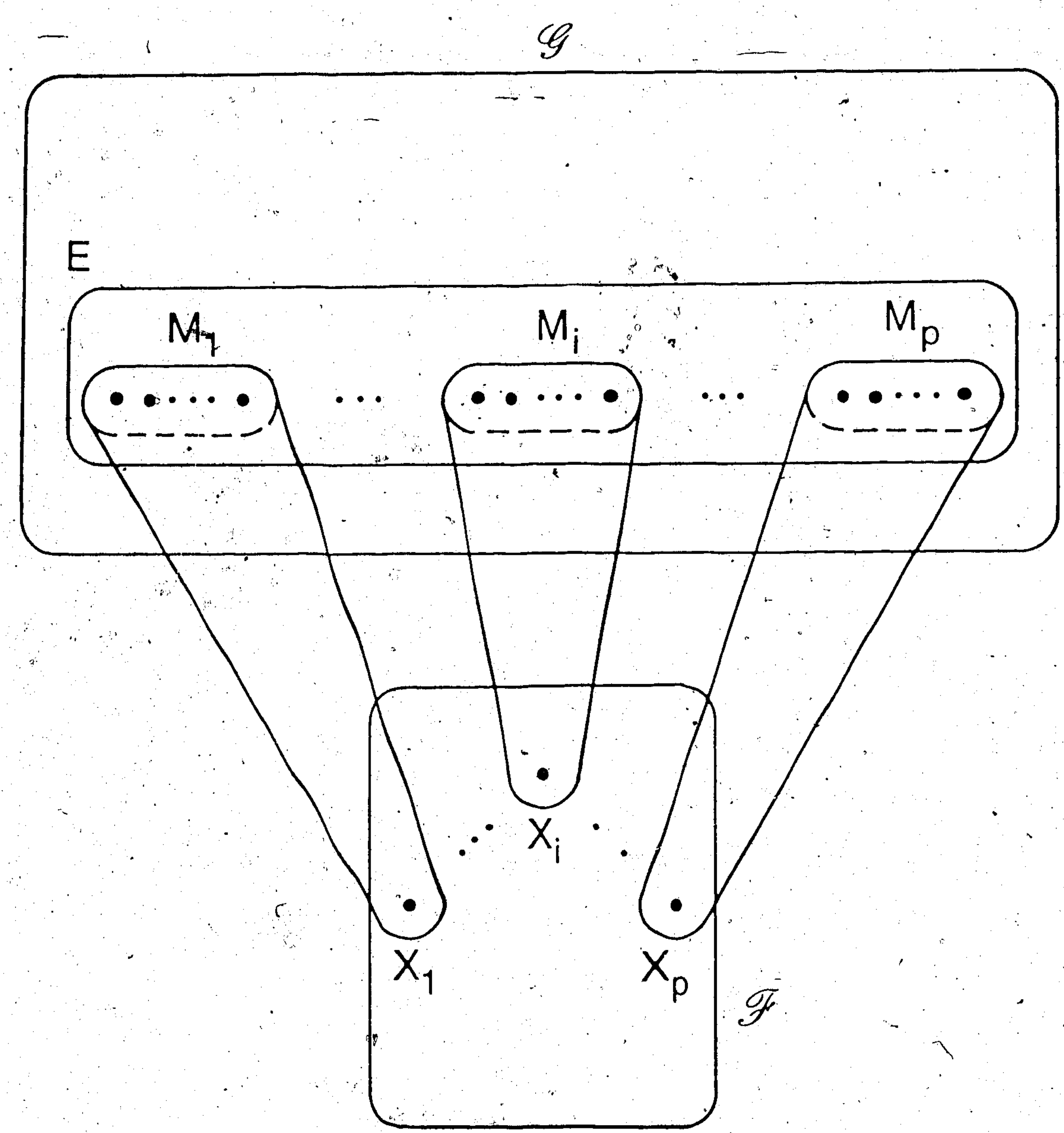


Fig. 2.1

The hypergraph $\mathcal{H}$ may not be critical, but it does contain a large $(r+1)$-critical subgraph. Let $\mathcal{H}'$ be an $(r+1)$-critical subgraph of $\mathcal{H}$. We prove that $\mathcal{M} \subset \mathcal{H}'$. In order to do this it suffices to exhibit an r-coloring of $\mathcal{H} - M$, where $M = M_j \cup \{x_j\}$ for some $j \in \{1, 2, \ldots, p\}$. Since $\mathcal{F}$ is r-critical we can $(r-1)$-color $\mathcal{F} - x_j$ in colors $1, 2, \ldots, r-1$. Color $\mathcal{G}$ in r colors $1, 2, \ldots, r$, so that the long edge E is colored r. Then v_j may also be colored r and we have an r-coloring for $\mathcal{H} - M$. Thus $\mathcal{M} \subset \mathcal{H}'$.

$\mathcal{H}'$ has at least $l(n-1)+1+p$ vertices (the number of vertices of $\mathcal{M}$) and at most $l(t-1)+q+p$ edges (the number of edges of $\mathcal{H}$). Thus

$$\begin{aligned}
\alpha^*(n, r+1) &\leq \lim_{l \to \infty} \frac{l(t-1)+q+p}{l(n-1)+1+p} \\
&= \lim_{l \to \infty} \frac{\frac{l}{p}(t-1)+\frac{q}{p}+1}{\frac{l}{p}(n-1)+\frac{1}{p}+1} \\
&= \lim_{p \to \infty} \frac{q}{p}+1 \qquad \text{, by (33)} \\
&= \alpha^*(n,r)+1 \qquad \text{, since } q = E^*(p,n,r).
\end{aligned}$$

□

COROLLARY 2.3. *For $n, r \geq 3$,*

$$\max\left\{1, \frac{r-1}{n}\right\} \leq \alpha^*(n,r) \leq r-2.$$

PROOF: The upper bound follows by induction using Theorems 2.1 and 2.2. The lower bound is from (20). □

Remark: It would be of interest to decide how $\alpha^*(n,r)$ depends upon n. In this regard it would be useful to know whether for fixed n,

$$\lim_{r\to\infty} \frac{\alpha^*(n,r)}{r} = L_n$$

exists. That $L_2 = \frac{1}{2}$ follows from (8), but we can make no progress with this problem for $n \geq 3$. Note, however, that if L_n exists it satisfies $\frac{1}{n} \leq L_n \leq 1$ by Corollary 2.3.

We now prove the analog of Theorem 2.2 for $\alpha(n,r)$.

THEOREM 2.4. $\alpha(n,r+1) \leq \alpha(n,r) + 1, \qquad n,r \geq 3.$

PROOF: Let $q = M(n-1,r)$ and let $\mathcal{G}$ be a $(q,n-1,r)$-graph with edges E_1, E_2, ..., E_p. Let $m \geq M(n,r)$ and let $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_p$ be copies of an (m,n,r)-graph with $k = E(m,n,r)$ edges. Moreover, let S be a set of new vertices of size $r(n-2)+1$. Let $\mathcal{H}$ to be the hypergraph whose edges are:

i) those of $\mathcal{F}_i$, $i = 1, 2, \ldots, p$

ii) the pm edges of the form $E_i \cup \{v\}$, where v is a vertex of $\mathcal{F}_i$, $i = 1, 2, \ldots, p$.

iii) all edges of the form $S' \cup \{g\}$ where S' is an $(n-1)$-subset of S and g is a vertex of $\mathcal{G}$.

Then $\mathcal{H}$ has order $s = pm + q + r(n-2) + 1$, size $pk + pm + q\binom{r(n-2)+1}{n-1}$, and is n-uniform (see Figure 2.2).

We show that $\mathcal{H}$ is $(r+1)$-chromatic. Suppose, to the contrary, that $\mathcal{H}$ has an r-coloring in colors 1, 2, ..., r. Then by the box principle, there is some $(n-1)$-subset S' of S of size $n-1$ which is monochromatic. Suppose S' is colored 1. Then

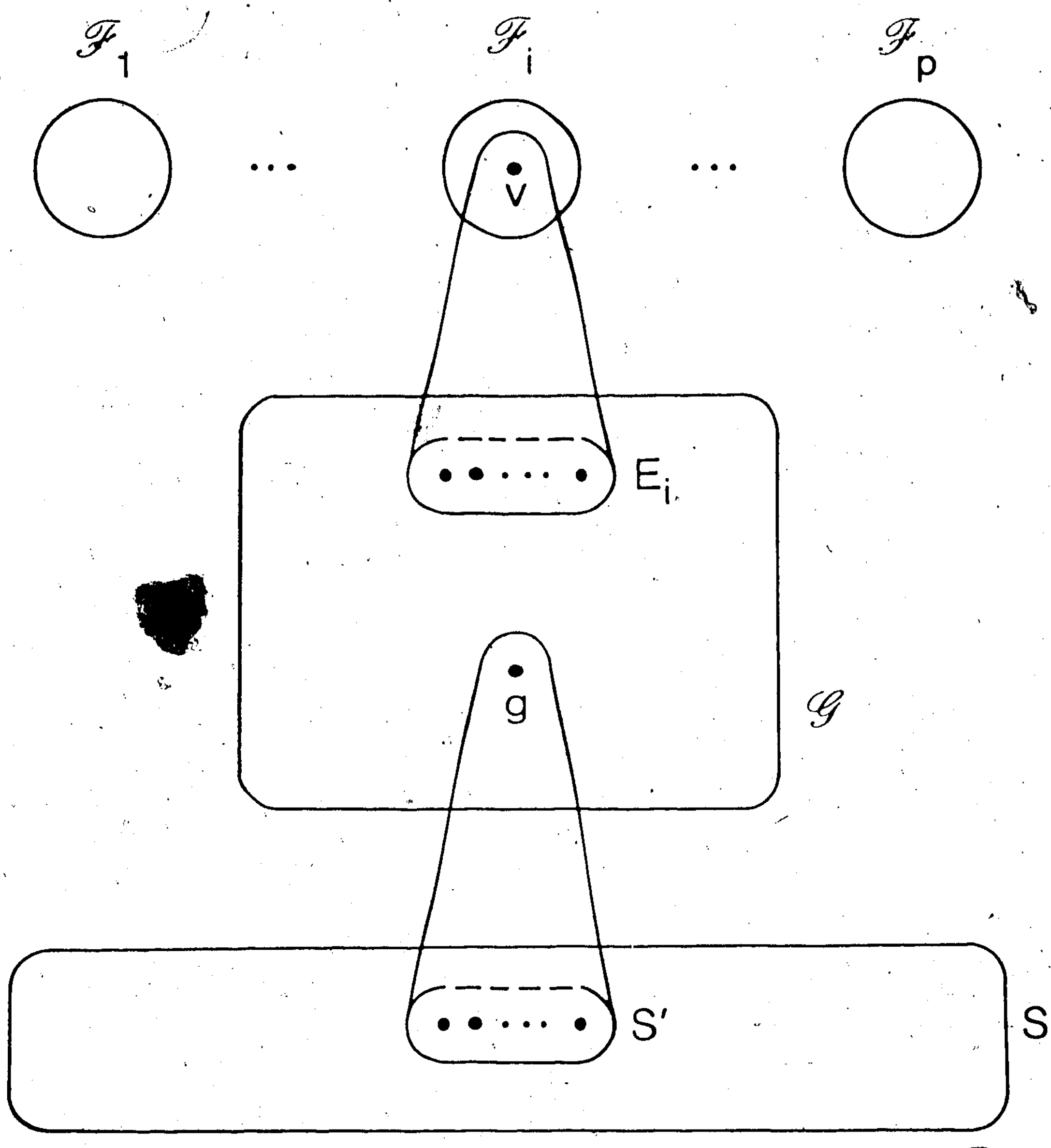


Fig. 2.2

no vertex of $\mathcal{G}$ can be colored 1 since otherwise there would be a monochromatic edge of type iii). Thus the vertices of $\mathcal{G}$ are colored in colors 2, 3, ..., r. Since $\mathcal{G}$ is r-critical, E_j is monochromatic for some $j \in \{1, 2, \ldots, p\}$. Suppose E_j is colored r. Then all the vertices of $\mathcal{F}_j$ must be colored 1, 2, ..., $r-1$, since otherwise there would be a monochromatic edge of type ii). However, $\mathcal{F}_j$ is r-critical, and thus there is a monochromatic edge of type i). This is a contradiction. Thus $\mathcal{H}$ is not r-colorable. There is, however, an $(r+1)$-coloring of $\mathcal{H}$. Color $\mathcal{G}$ and each $\mathcal{F}_i$ in colors 1, 2, ..., r. Partition S into $r+1$ sets S_1, S_2, ..., S_{r+1} each of size at most $n-2$, and color each vertex of S_i with color i. Since any $(n-1)$-subset S' of S must intersect at least two of the S_i, no edge of type iii) is monochromatic. Thus $\mathcal{H}$ is $(r+1)$-chromatic.

The following argument shows that $\mathcal{H}$ is $(r+1)$-critical. Let F be an edge of $\mathcal{H}$. We need to exhibit an r-coloring of $\mathcal{H} - F$.

Case 1 F is an edge of type i).

Then $F \in \mathcal{F}_j$ for some $j \in \{1, 2, \ldots, p\}$. Since $\mathcal{F}_j$ is r-critical, $(r-1)$-color $\mathcal{F}_j - F$ in colors 1, 2, ..., $r-1$. For $i \neq j$, r-color $\mathcal{F}_i$ in colors 1, 2, ..., r. Since $\mathcal{G}$ is r-critical, $(r-1)$-color $\mathcal{G} - E_j$ in colors 2, 3, ..., r so that edge E_j is colored r. Partition S into r sets S_1, S_2, ..., S_r, where S_1 has size $n-1$ and the rest have size $n-2$. Assign color i to each vertex of S_i. Note that edge $S_1 \cup \{g\}$ of $\mathcal{H}$ is not monochromatic for all vertices g of $\mathcal{G}$ since S_1 is colored 1 and $\mathcal{G}$ does not use 1.

Case 2 F is an edge of type ii).

Then $F = E_j \cup \{v\}$ for some $j \in \{1, 2, \ldots, p\}$ and some vertex v of $\mathcal{F}_j$. $(r-1)$-color $\mathcal{F}_j - v$ in colors 1, 2, ..., $r - 1$. For $i \neq j$, r-color $\mathcal{F}_i$, in colors 1, 2, ..., r. $(r-1)$-color $\mathcal{G} - E_j$ colors 2, 3, ..., r so that E_j is colored r. Partition S into r sets $S_1, S_2, \ldots, S_r$ where S_1 has size $n-1$ and all others have size $n-2$. Assign color i to each vertex of S_i. Finally, assign color r to vertex v.

Case 3 F is an edge of type iii).

Then $F = S' \cup \{g\}$ for some $(n-1)$-subset S' of S and some vertex g of $\mathcal{G}$. r-color $\mathcal{G}$ and each of the $\mathcal{F}_i$ in colors 1, 2, ..., r. Partition S into r sets $S_1, S_2, \ldots, S_r$ where $S_r = S'$ and all other sets in the partition have size $n-2$. Assign color i to each vertex of S_i.

Therefore $\mathcal{H}$ is $(r+1)$-critical. The ratio of the number of edges of $\mathcal{H}$ to the number of vertices is

$$\frac{pk + pm + q\binom{r(n-2)+1}{n-1}}{pm + q + r(n-2) + 1} = \frac{\frac{k}{m} + 1 + \frac{q}{pm}\binom{r(n-2)+1}{n-1}}{1 + \frac{q}{pm} + \frac{r(n-2)+1}{pm}}.$$

If we let m tend to infinity and keep in mind that n, p, q, r are fixed and $k = E(m, n, r-1)$, we get

$$\alpha(n, r+1) \leq \lim_{m \to \infty} \frac{k}{m} + 1 = \alpha(n, r) + 1.$$

□

It follows from Theorem 2.4 and $\alpha(n, 3) = 1$ that $\alpha(n, r) \leq r - 2$ for $n, r \geq 3$. We now establish a result which shows that strict inequality holds for $r \geq 4$.

THEOREM 2.5. *Let $n \geq 3$ and let T $(= T(n))$ be the least integer for which there exists a $(T, n, 3)$-graph with T edges. Then*

$$\alpha(n,4) \leq 2 - \frac{2}{\alpha(n-1,4)T+1}.$$

PROOF: Let $p \geq M(n-1,4)$ and let $\mathcal{G}$ be a $(p, n-1, 4)$-graph with $q = E(p, n-1, 4)$ edges, E_1, E_2, ..., E_q. Let $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_q$ be copies of a $(T, n, 3)$-graph with T edges. For $i = 1, 2, \ldots, q$, let $\mathcal{M}_i = \{ E_i \cup \{v\} \mid v \text{ is a vertex of } \mathcal{F}_i\}$. Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{F}_i$ and of $\mathcal{M}_i$, $i = 1, 2, \ldots, q$.

We show that $\mathcal{H}$ is 4-chromatic. Suppose we try to 3-color $\mathcal{H}$ red, blue and green. Since $\mathcal{G}$ is 4-critical some $E_j \in \mathcal{G}$ will be monochromatic. Suppose E_j is red. If some vertex of $\mathcal{F}_j$ were red, there would be a red edge in $\mathcal{M}_j$. Thus the vertices of $\mathcal{H}_j$ are colored blue and green. However, this contradicts the fact that $\mathcal{F}_j$ is 3-critical. Thus $\mathcal{H}$ is not 3-colorable. $\mathcal{H}$ is clearly 4-colorable and hence 4-chromatic.

Next we show that $\mathcal{H}$ is 4-critical. To do this we need to give a 3-coloring of $\mathcal{H} - F$ for each $F \in \mathcal{H}$.

If $F \in \mathcal{F}_j$ for some $j \in \{1, 2, \ldots, q\}$, 2-color $\mathcal{F}_j - F$ red and blue so that F is red, 3-color $\mathcal{G} - E_j$ red, blue and green so that E_j is green, and for $i \neq j$ 3-color $\mathcal{F}_i$ red, blue, and green. This is a 3-coloring of $\mathcal{H} - F$.

If $F \in \mathcal{M}_j$ for some $j \in \{1, 2, \ldots, q\}$; then $F = E_j \cup \{v\}$ for some vertex v of $\mathcal{F}_j$. 3-color $\mathcal{F}_i$ and $\mathcal{G} - E_j$ red, blue and green so that v is the only green vertex of $\mathcal{F}_j$ and E_j is green. This is a 3-coloring of $\mathcal{H} - F$.

It follows that $\mathcal{H}$ is 4-critical. Moreover, it is n-uniform and has $qT + p$ vertices and $2qT$ edges. Therefore,

$$\alpha(n,4) \le \lim_{p\to\infty} \frac{2qT}{qT+p} = \lim_{p\to\infty} 2 - \frac{2}{1+\frac{qT}{p}} = 2 - \frac{2}{1+\alpha(n-1,4)T}.$$

□

COROLLARY 2.6. *For $n \ge 3$, $r \ge 4$,*

$$\max\left\{1, \frac{r-1}{n}\right\} \le \alpha(n,4) < r-2.$$

PROOF: By induction using Theorems 2.4 and 2.5, and (20). □

Our next result deals with the problem of improving the lower bound for $\alpha(n,r)$ given by (20). We get from (20)

$$\alpha(n,r) \ge \frac{r-1}{n} \text{ if } r \ge n+1 \qquad \text{and} \qquad \alpha(n,r) \ge 1 \text{ in all cases.} \tag{34}$$

The first inequality in (34) is based on the fact that in an r-critical hypergraph each vertex is contained in at least $r-1$ edges. Thus if we are going to improve on (34) in the case $r \ge n+1$, we need to show that a vertex of a large r-critical n-graph, on the average, is contained in more than $r-1$ edges. We shall prove that this is so in the case $r = n+1$; that is, we shall prove $\alpha(n, n+1) > 1$. Our proof of this may appear to be somewhat unmotivated, so we make a few comments which make it appear less artifical. The proof is not direct. We take an $(m,n,r+1)$-graph $\mathcal{G}$ and we manufacture from it an $(n+1)$-critical 2-graph G. The manufacturing process is such that if $\mathcal{G}$ has few edges, G will also be sparse, so sparse, in fact, that Gallai's lower bound for $\alpha(2, n+1)$ given by (8) will be violated. It will then follow that $\mathcal{G}$ must have many edges. When all of this is made precise, the result of Theorem 2.7 will follow.

We point out that the idea of tackling problems on hypergraphs by reducing them to problems about 2-graphs, or conversely of obtaining results about 2-graphs from hypergraphs, is not a new one. It has been used successfully on several occasions. See [**L2**] and [**T2**] for example.

THEOREM 2.7. *For* $n \geq 3$,

$$\alpha(n, n+1) \geq 1 + \frac{n-2}{n^2}.$$

PROOF: Let $m \geq M(n, n+1)$ and let $\mathcal{G}$ be a $(m, n, n+1)$-graph with $p = E(m, n, n+1)$ edges, $E_1, E_2, \ldots, E_p$. For $i = 1, 2, \ldots, p$ let G_i be a copy of K^n, the complete 2-graph on n vertices. For each i set up a matching M_i (in the ordinary graph theoretic sense; see [**B1**, Chapter 3]) between the vertices of E_i and the of vertices of G_i. Let G to be the 2-graph whose edges are those of M_i and of G_i, $i = 1, 2, \ldots, p$. Then G has $\left(n + \binom{n}{2}\right) p$ edges and $m + np$ vertices (see Figure 2.3).

We show that G is $(n+1)$-chromatic. Suppose, to the contrary, that G has a n-coloring. Since $\mathcal{G}$ is $(n+1)$-critical, there results a monochromatic edge E_j of $\mathcal{G}$, $j \in \{1, 2, \ldots, p\}$. The color of E_j cannot be assigned to any vertex of G_j, because of the matching M_j. Thus G_j must be $(n-1)$-colored. But this is not possible

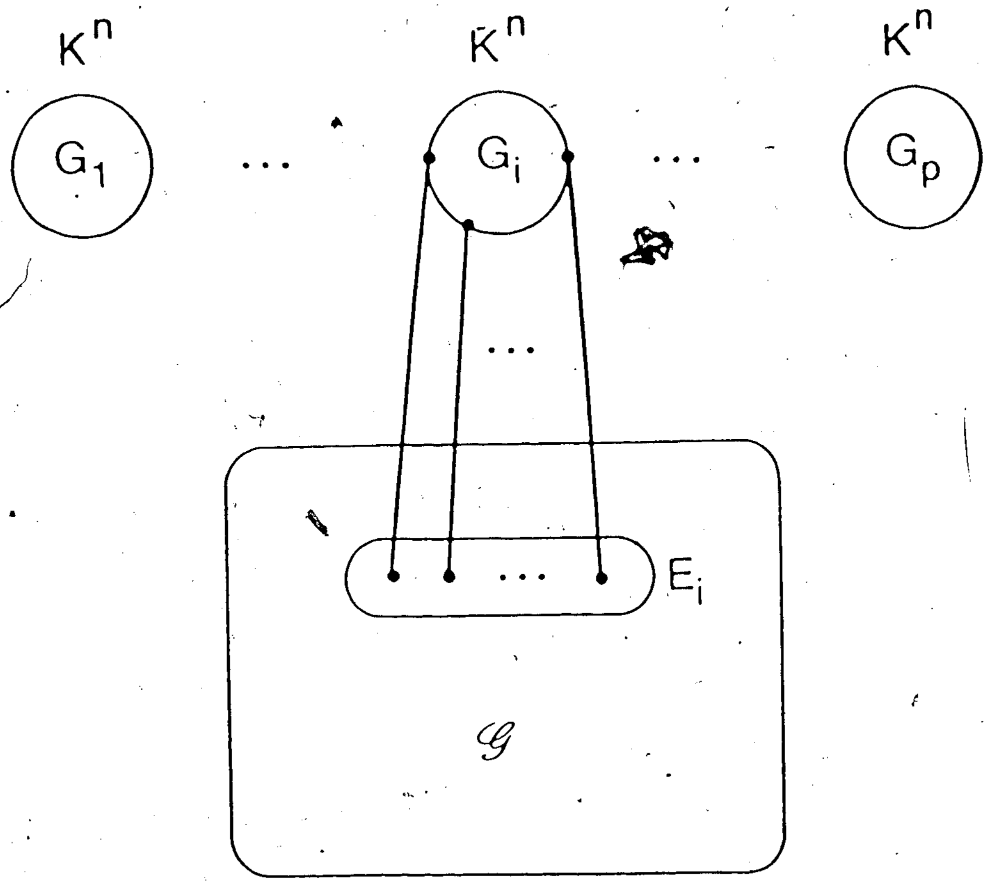


Fig. 2.3

since $G_j = K^n$, an n-critical graph. Thus G is not n-colorable. It is clear that G has an $(n+1)$-coloring and it is thus $(n+1)$-chromatic.

We next show that G is $(n+1)$-critical. Let uv be an edge of G. We need to exhibit an n-coloring of $G - uv$.

Case 1 uv is an edge of G_j for some $j \in \{1, 2, \ldots, p\}$.

$(n-1)$-color $G_j - uv$ in colors 1, 2, ..., $n-1$, and n-color $\mathcal{G} - E_j$ in colors 1, 2, ..., n so that color n is assigned to E_j. Then n-color G_i for $i \neq j$ in colors 1, 2, ..., n so that no edge of M_i is monochromatic.

Case 2 uv is and edge of M_j for some $j \in \{1, 2, \ldots, p\}$.

Without loss of generality, suppose u is a vertex of G_j and v is a vertex of E_j. $(n-1)$-color $G_j - u$ in colors 1, 2, ..., $n-1$, n-color $\mathcal{G} - E_j$ in colors 1, 2, ..., n so that color n is assigned to E_j. Assign color n to u and n-color G_i for $i \neq j$ in colors 1, 2, ..., n so that no edge of M_i is monochromatic.

G is an $(m+np, 2, n+1)$-graph, and therefore

$$\frac{E(m+np, 2, n+1)}{m+np} \leq \frac{(n + \binom{n}{2})p}{m+np} = \frac{n(n+1)}{2(\frac{m}{p} + n)}.$$

Let $m \to \infty$. This gives

$$\alpha(2, n+1) \leq \frac{n(n+1)}{2\left(\frac{1}{\alpha(n,n+1)} + n\right)}. \tag{35}$$

From the theorem of Gallai (the left inequality in (8)) we get

$$\alpha(2, n+1) \geq \frac{n}{2} + \frac{n-2}{2(n^2 + 2n - 2)} \tag{36}$$

and from (35) and (36) it follows that

$$\alpha(n, n+1) \geq 1 + \frac{n-2}{n^2}.$$

□

The above results give a bound on $\alpha(3,4)$.

COROLLARY 2.8. $\frac{10}{9} \leq \alpha(3,4) \leq \frac{35}{19}$.

PROOF: The lower bound is immediate from Theorem 2.7. To get the upper bound, observe that $T(3) \doteq 7$ and $\alpha(2,4) \leq \frac{5}{3}$ by the result of Hajós (see(8)). The upper bound then follows from Theorem 2.5. □

Chapter 3. Upper Bounds for $M^*(4,3)$ and $M^*(3,4)$

This chapter contains the proofs of the improved upper bounds for $M^*(4,3)$ and $M^*(3,4)$ stated in Section 1.4.

THEOREM 3.1. $M^*(4,3) \leq 51$.

PROOF: The proof is based on five constructions, three of which are new, and two of which are given in [A4]. The most important of these, from the point of view of this theorem, is the first one, since it establishes the existence of $(m,4,3)$-graphs for all $m \geq 55$. We point out also that it uses a variant of the ideas on which the proof of Theorem 2.1 is based.

Let $S_1 = \{ m \mid \text{there exists a linear } (m,4,3)\text{-graph}\}$.

Note: In this proof, when we say that we 2-color a hypergraph we will mean that we 2-color it red and blue. It is hoped that this assumption will facilitate the reading of the proof.

CONSTRUCTION 1. *If $m_1, m_2 \in S_1$ then $m \in S_1$ for $m \geq m_1 + m_2 + 5$.*

Let $\mathcal{G}_1$ be a linear $(m_1,4,3)$-graph and $\mathcal{G}_2$ be a linear $(m_2,4,3)$-graph. Let $E_1 = \{a,b,c,d\}$ be an edge of $\mathcal{G}_1$, $E_2 = \{e,f,g,h\}$ be an edge of $\mathcal{G}_2$. Let t be a positive integer and for $1 \leq i \leq t$ let

$$H_i = \begin{cases} \{c, i+1, i+3, i+4\}, & \text{if } i \equiv 1 \pmod 4 \\ \{h, i+1, i+3, i+4\}, & \text{if } i \equiv 2 \pmod 4 \\ \{b, i+1, i+3, i+4\}, & \text{if } i \equiv 3 \pmod 4 \\ \{g, i+1, i+3, i+4\}, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

and let

$$H_t^* = \begin{cases} \{f,h,t+2,t+4\}, & \text{if } t \equiv 1 \pmod 4 \\ \{a,d,t+2,t+4\}, & \text{if } t \equiv 2 \pmod 4 \\ \{f,g,t+2,t+4\}, & \text{if } t \equiv 3 \pmod 4 \\ \{a,d,t+2,t+4\}, & \text{if } t \equiv 0 \pmod 4. \end{cases}$$

Let $\mathcal{H}_t$ be the hypergraph whose edges are

i) those of $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$

ii) $\{c,d,e,f\}$, $\{a,b,c,1\}$, $\{e,h,1,2\}$, $\{b,d,2,3\}$ and $\{g,1,3,4\}$.

iii) H_i, $i = 1, 2, \ldots, t$

iv) H_t^*.

$\mathcal{H}_t$ is a 4-graph and it is not difficult to check, using the fact that $\{1,3,4\}$ is a difference set, that $\mathcal{H}_t$ is linear. The number of vertices of $\mathcal{H}_t$ is $m_1 + m_2 + t + 4$. We need to show that $\mathcal{H}_t$ is 3-critical for each $t \geq 1$. We do this by induction on t.

We first show that $\mathcal{H}_1$ is 3-chromatic. Suppose, to the contrary, that $\mathcal{H}_1$ is 2-colorable and 2-color it. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are 3-critical, edges E_1 and E_2 will be monochromatic, and edge $\{c,d,e,f\}$ forces them to have opposite colors. So without loss of generality, suppose E_1 is red and E_2 is blue. Then edge $\{a,b,c,1\}$ ensures vertex 1 is colored blue, and hence edge $\{e,h,1,2\}$ ensures vertex 2 is colored red. Furthermore, vertex 3 must be blue because of edge $\{b,d,2,3\}$ and hence edge $\{g,1,3,4\}$ forces vertex 4 to be red. Finally, edge H_1 ensures vertex 5 is colored blue. But then edge H_1^* is blue, a contradiction. If, however, at the last step we were to color vertex 5 green we would get a 3-coloring for $\mathcal{H}_1$. Therefore $\mathcal{H}_1$ is 3-chromatic.

We now show that $\mathcal{H}_1$ is 3-critical. Let E be an edge of $\mathcal{H}_1$. We need to exhibit a 2-coloring $\mathcal{H}_1 - E$.

Case 1 E is an edge of $\mathcal{G}_1 - E_1$.

2-color $\mathcal{G}_1 - E$ so that d is red and $\{a, b, c\}$ is not red. 2-color $\mathcal{G}_2 - E_2$ so that E_2 is blue. Color 1, 3 and 5 red, and color 2 and 4 blue.

Case 2 E is an edge of $\mathcal{G}_2 - E_2$.

2-color $\mathcal{G}_2 - E$ so that e is blue and $\{f, g, h\}$ is not blue. 2-color $\mathcal{G}_1 - E_1$ so that E_1 is red. If g is red, color 1, 3 and 4 blue, and color 2 and 5 red. If g is blue then color 1, 3 and 5 blue, and color 2 and 4 red.

Case 3 E is not an edge of $\mathcal{G}_1$ or of $\mathcal{G}_2$.

2-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$. E_1 and E_2 will be monochromatic. We may obtain a 2-coloring of $\mathcal{H}_1 - E$ by choosing the colors of E_1 and E_2 and assigning colors to 1, 2, 3, 4, 5 according to the following table.

E	red	blue
$\{c, d, e, f\}$	E_1, E_2	$1, 2, 3, 4, 5$
$\{a, b, c, 1\}$	$E_1, 1, 4, 5$	$E_2, 2, 3$
$\{e, h, 1, 2\}$	$E_1, 3, 4$	$E_2, 1, 2, 5$
$\{b, d, 2, 3\}$	$E_1, 2, 3, 4$	$E_2, 1, 5$
$\{g, 1, 3, 4\}$	$E_1, 2, 5$	$E_2, 1, 3, 4$
$H_1 = \{c, 2, 4, 5\}$	$E_1, 2, 4, 5$	$E_2, 1, 3$
$H_1^* = \{f, h, 3, 5\}$	$E_1, 2, 4$	$E_2, 1, 3, 5$

Therefore $\mathcal{H}_1$ is 3-critical. Suppose now that $t \geq 1$ and that $\mathcal{H}_t$ is 3-critical. We deduce that $\mathcal{H}_{t+1}$ is 3-critical. Note that $\mathcal{H}_{t+1} = (\mathcal{H}_t - H_t^*) \bigcup \{H_{t+1}, H_{t+1}^*\}$. We need to first show that $\mathcal{H}_{t+1}$ is 3-chromatic. Suppose, to the contrary, that $\mathcal{H}_{t+1}$ is 2-colorable and 2-color it. Then H_t^* is monochromatic by the induction hypothesis, and without loss of generality let it be red. Since $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ are subgraphs of $\mathcal{H}_t - H_t^*$ the edges E_1 and E_2 must be assigned opposite colors (the edge $\{c, d, e, f\}$ is also an edge of $\mathcal{H}_t - H_t^*$). Hence, the set $H_{t+1}^* \setminus \{t+5\}$ must be monochromatic, and since H_t^* is red $H_{t+1} \setminus \{t+5\}$ must be red. Hence vertex $t+5$ must be colored blue. Then the set $H_{t+1} \setminus \{t+3\}$ is blue, and hence vertex $t+3$ must be colored red. But since H_t^* is red and $t+4$ is only in edges H_t and H_t^* if we re-color vertex $t+4$ blue we will have a 2-coloring for $\mathcal{H}_t$, a contradiction. However, by 2-coloring $\mathcal{H}_t - H_t^*$ (this is possible by the induction hypothesis) and by coloring vertex $t+5$ green, we have a 3-coloring for $\mathcal{H}_{t+1}$. Therefore, $\mathcal{H}_{t+1}$ is 3-chromatic.

We now show that $\mathcal{H}_{t+1}$ is critical. Let E be an edge of $\mathcal{H}_{t+1}$. We need to show that $\mathcal{H}_{t+1} - E$ is 2-colorable.

<u>*Case* 1</u> $E = H_{t+1}^*$.

2-color $\mathcal{H}_t - H_t^*$ and assign to $t+5$ the color not assigned to $t+4$.

<u>*Case* 2</u> $E = H_{t+1}$.

2-color $\mathcal{H}_t - H_t^*$ and assign to $t+5$ the color not assigned to $t+3$.

<u>*Case* 3</u> $E = \{c, d, e, f\}$.

2-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that E_1 and E_2 are both red. Color 1, 2, ..., $t+5$ blue.

<u>*Case* 4</u> E is an edge of $\mathcal{G}_1 - E_1$.

2-color $\mathcal{G}_1 - E$ so that d is red and $\{a, b, c\}$ is not red. 2-color $\mathcal{G}_2 - E_2$ so that E_2 is blue. Color the odd numbered vertices red and the even numbered vertices blue.

<u>*Case* 5</u> E is an edge of $\mathcal{G}_2 - E_2$.

2-color $\mathcal{G}_2 - E$ so that f is blue and $\{e, g, h\}$ is not blue. 2-color $\mathcal{G}_1 - E_1$ so that E_1 is red. If g is red, color 1 and 3 blue, 2 red, all other odd numbered vertices red, and all other even vertices blue. If g is blue, color 1 and all even numbered vertices blue and all other odd numbered vertices red.

<u>*Case* 6</u> E is not one of the edges covered by Cases 1–5.

2-color $\mathcal{H}_t - E$. This is possible by the induction hypothesis. Since $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ are subgraphs of $\mathcal{H}_t - E$, and since $\mathcal{G}_1$ and $\mathcal{G}_2$ are 3-critical, E_1 and E_2 are monochromatic. Since $\{c, d, e, f\}$ is an edge of $\mathcal{H}_t - E$, E_1 and E_2 must have different colors. The only vertex that has not been colored is $t + 5$, [illegible] 5 is contained only in H_{t+1} and H^*_{t+1}. Assign to $t + 5$ the color not as[illegible] vertex $t + 3$. Then H^*_{t+1} is not monochromatic. If $t + 2$ and $t + 4$ are colored differently, then H_{t+1} is not monochromatic. If $t + 2$ and $t + 4$ are colored the same, then since in the 2-coloring of $\mathcal{H}_t - E$, H^*_t is not monochromatic and since $E_j \cap H^*_t$ and $E_j \cap H_{t+1}$ are non-empty for some $j \in \{1,2\}$, the edge H_{t+1} is not monochromatic in this case also. Thus we have a 2-coloring for $\mathcal{H}_{t+1} - E$.

It follows that $\mathcal{H}_t$ is 3-critical for all $t \geq 1$ and thus that $m \in S_1$ for $m \geq m_1 + m_2 + 5$.

<u>Note</u> The $(25, 50, 8, 4, 1)$-design given as example a) in Section 1.5.1 is an example of a linear $(25, 4, 3)$-graph. Thus, if we take $m_1 = m_2 = 25$ in Construction 1, we find that $m \in S_1$ for all $m \geq 55$.

CONSTRUCTION 2. *If* $m_1, m_2 \in S_1$ *then* $m_1 + m_2 + 1 \in S_1$.

This is a special case of one of the constructions given in [A4], so we do not give the details here.

<u>Note</u> If we take $m_1 = m_2 = 25$ we see that $51 \in S_1$.

CONSTRUCTION 3. *If* $m_1, m_2 \in S_1$ *then* $m_1 + m_2 + 4 \in S_1$.

This is another of the constructions given in [A4], so we do not give the details here.

<u>Note</u> If we take $m_1 = m_2 = 25$ in Construction 3 we see that $54 \in S_1$.

CONSTRUCTION 4. *If* $m_1, m_2 \in S_1$ *then* $m_1 + m_2 + 3 \in S_1$.

Let $\mathcal{G}_1$ be a linear $(m_1, 4, 3)$-graph and $\mathcal{G}_2$ be a linear $(m_2, 4, 3)$-graph. Let $E_1 = \{a, b, c, d\}$ be an edge of $\mathcal{G}_1$, $E_2 = \{e, f, g, h\}$ be an edge of $\mathcal{G}_2$, and let 1, 2, 3 be new vertices. Let $\mathcal{H}$ be the hypergraph whose edges are:

i) those of $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$

ii) $\{c, d, e, f\}$, $\{a, b, c, 1\}$, $\{e, g, h, 2\}$, $\{b, d, 2, 3\}$, $\{f, h, 1, 3\}$.

Then $\mathcal{H}$ is linear, 4-uniform and has order $m_1 + m_2 + 3$ (see Figure 3.1).

We show that $\mathcal{H}$ is 3-chromatic. Suppose, to the contrary, that $\mathcal{H}$ has a 2-coloring and 2-color it. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are 3-critical, edges E_1 and E_2 will be monochromatic, and must be assigned different colors, since otherwise edge $\{c, d, e, f\}$ would be monochromatic. Without loss of generality, suppose E_1 is red and E_2 is blue. Then, a glance at the edges in ii) reveals that no matter how the colors are assigned

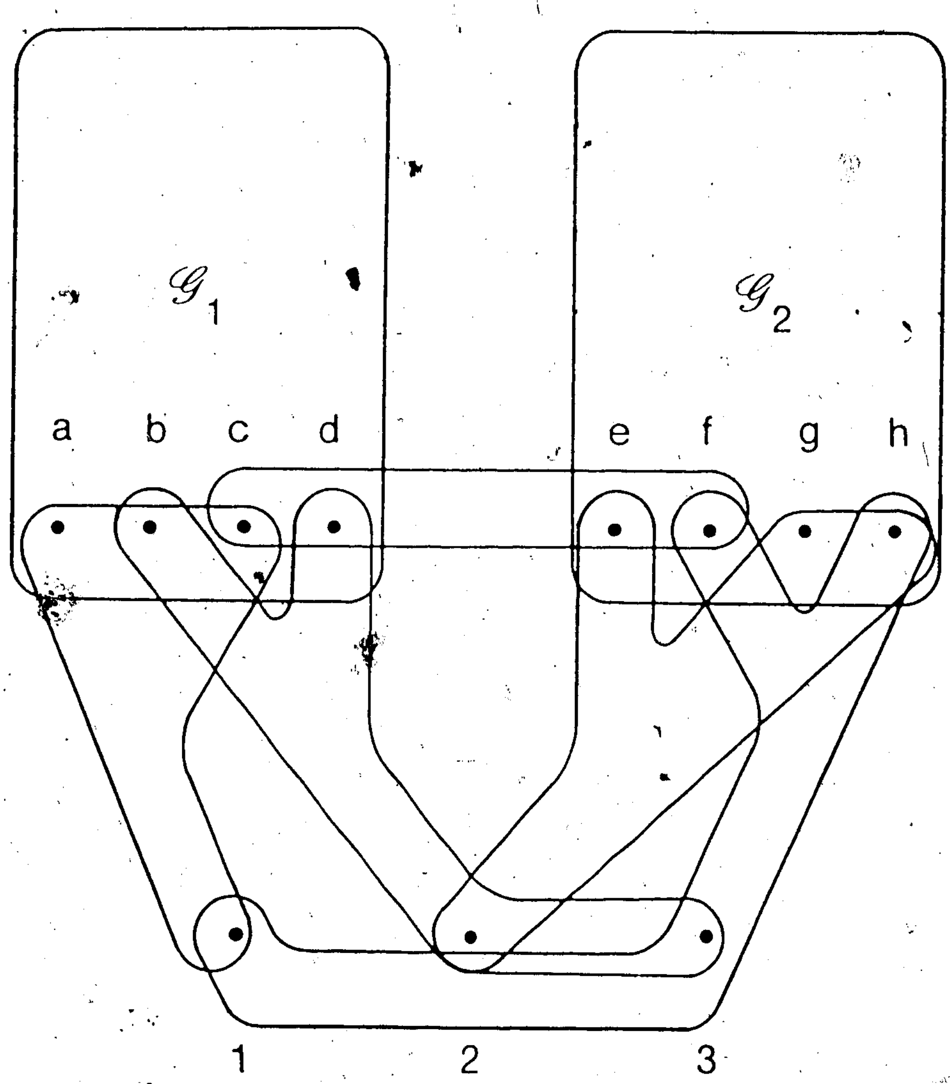


Fig. 3.1

to 1, 2 and 3, there must be a monochromatic edge. Thus $\mathcal{H}$ has no 2-coloring. $\mathcal{H}$ is clearly 3-colorable and hence 3-chromatic.

Now we show that $\mathcal{H}$ is 3-critical. Let E be an edge of $\mathcal{H}$. We need to exhibit a 2-coloring of $\mathcal{H} - E$.

<u>*Case* 1</u> E is an edge of $\mathcal{G}_1 - E_1$.

2-color $\mathcal{G}_1 - E$ so that d is red and $\{a, b, c\}$ is not red. 2-color $\mathcal{G}_2 - E_2$ so that E_2 is blue. Color 1 and 2 red and 3 blue.

<u>*Case* 2</u> E is an edge of $\mathcal{G}_2 - E_2$.

2-color $\mathcal{G}_2 - E$ so that f is blue and $\{e, g, h\}$ is not blue. 2-color $\mathcal{G}_1 - E_1$ so that E_1 is red. Color 1 and 2 blue and 3 red.

<u>*Case* 3</u> E is not an edge of $\mathcal{G}_1 - E_1$ or of $\mathcal{G}_2 - E_2$.

2-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$. E_1 and E_2 are then monochromatic. We may obtain a 2-coloring of $\mathcal{H} - E$ by choosing the colors of E_1 and E_2 and assigning colors to 1, 2, 3 according to the following table.

E	red	blue
$\{c, d, e, f\}$	E_1, E_2	$1, 2, 3$
$\{a, b, c, 1\}$	$E_1, 1, 2$	$E_2, 3$
$\{e, g, h, 2\}$	$E_1, 3$	$E_2, 1, 2$
$\{b, d, 2, 3\}$	$E_1, 2, 3$	$E_2, 1$
$\{f, h, 1, 3\}$	$E_1, 2$	$E_2, 1, 3$

Thus $\mathcal{H}$ is 3-critical. Since $\mathcal{H}$ has order $m_1 + m_2 + 3$, $m_1 + m_2 + 3 \in S_1$.

<u>Note</u> If we take $m_1 = m_2 = 25$ in Construction 4 we find that $53 \in S_1$.

CONSTRUCTION 5. $52 \in S_1$.

We shall need to establish a special property of the linear $(25,4,3)$-graph given by the $(25,50,8,4,1)$-design. Let us denote this graph by D_1. The property that we need is the following:

(P1) There exists an edge $E = \{a,b,c,d\}$ of D_1 such that for any vertex $v \neq c,d$ of D_1 there is a 2-coloring of $D_1 - v$ in which c and d are assigned the same color.

Recall that the vertices of D_1 are the elements of $Z_5 \times Z_5$ and the edges are obtained by adding each element of $Z_5 \times Z_5$ to $\{(0,0),(1,0),(0,1),(4,4)\}$ and $\{(0,0),(2,0),(0,2),(3,3)\}$. We show that (P1) holds for

$E = \{(0,0),(1,0),(0,1),(4,4)\}$ where $c = (0,0)$ and $d = (1,0)$.

Case 1 $v \in \{(0,1),(0,2),(0,4),(1,1),(1,3),(1,4),(2,0),(2,1),(2,3),(3,2),(3,3),$
$(4,0),(4,2),(4,4)\} = A$.

If $v = (2,0)$, the following is a 2-coloring of $D_1 - v$ in which $(0,0)$ and $(1,0)$ are red.

red	(0,0)	(1,0)	(0,4)	(1,1)	(1,3)	(1,4)	(2,1)	(2,4)	(3,1)	(3,2)	(3,4)	(4,1)
blue	(0,1)	(0,2)	(0,3)	(1,2)	(2,2)	(2,3)	(3,0)	(3,3)	(4,0)	(4,2)	(4,3)	(4,4)

We do not give explicit 2-colorings of $D_1 - v$ for other $v \in A$ since they may be obtained from this one via translates by appropriate elements of $Z_5 \times Z_5$. The reader should see Figure 3.2 a).

In this figure (and the three succeeding figures) the dots represent the 25 elements of $Z_5 \times Z_5$ with the standard Cartesian coordinate system ((0,0) is at lower left

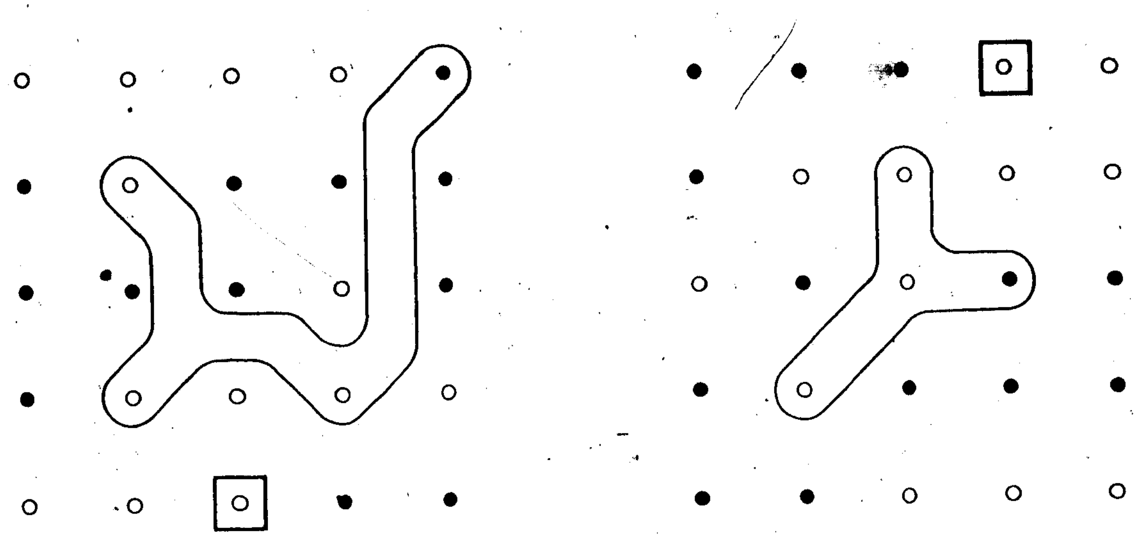


Fig. 3.2 a) Fig. 3.2 b)

corner of the figure). A white dot represents a red vertex and a black dot dot represents a blue vertex. The box indicates the deleted vertex. In Figure 3.2 a) a translate of the edge $\{(0,0),(2,0),(0,2),(3,3)\}$ is drawn and in Figure 3.2 b) one of $\{(0,0),(1,0),(0,1),(4,4)\}$ is drawn. To obtain a 2-coloring of $\mathcal{D}_1 - v'$, one simply searches a figure for any two horizontally adjacent vertices which have the same color (vertices in the same row and on the sides of the figure are also considered adjacent). One then translates the coordinate system of the figure so that these vertices become $(0,0)$ and $(1,0)$ in the new system. This is then a 2-coloring of $\mathcal{D}_1 - v'$..

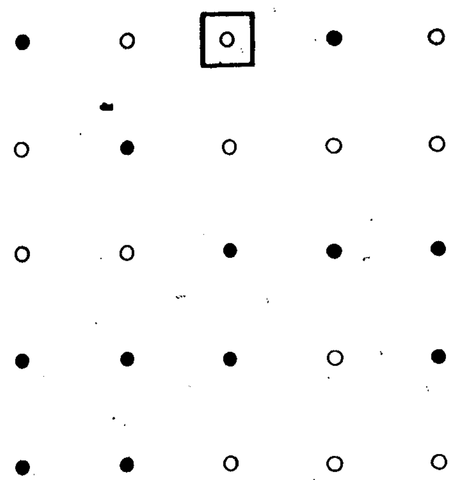


Fig. 3.3 a)

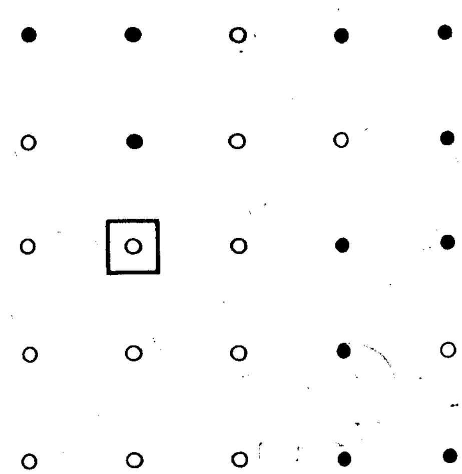


Fig. 3.3 b)

Case 2 $v \in \{(0,3),(3,0),(3,4),(4,3)\}$.

The following is a 2-coloring of $D_1 - v$ for $v = (3,4)$ (see Figure 3.2 b). The others may be obtained via suitable translations of this one.

blue	(0,0)	(1,0)	(0,1)	(0,3)	(0,4)	(1,2)	(1,4)	(2,1)	(2,4)	(3,1)	(3,2)	(4,1)	(4,2)
red	(0,2)	(1,1)	(1,3)	(2,0)	(2,2)	(2,3)	(3,0)	(3,3)	(4,0)	(4,3)	(4,4)		

Case 3 $v \in \{(2,2),(2,4),(3,1),(4,1)\}$.

The following 2-coloring of $D_1 - v$ works for $v = (2,4)$ (see Figure 3.3 a). Again, translations of this coloring work for the others.

blue	(0,0)	(1,0)	(0,1)	(0,4)	(1,1)	(1,3)	(2,1)	(2,2)	(3,2)	(3,4)	(4,1)	(4,2)
red	(0,2)	(0,3)	(1,2)	(1,4)	(2,0)	(2,3)	(3,0)	(3,1)	(3,3)	(4,0)	(4,3)	(4,4)

Case 4 $v = (1,2)$.

The following is a suitable 2-coloring of $\mathcal{D}_1 - v$.

red	(0,0) (1,0) (0,1) (0,2) (0,3) (1,1) (2,0) (2,1) (2,2) (2,3) (2,4) (3,3) (4,1)
blue	(0,4) (1,3) (1,4) (3,0) (3,1) (3,2) (3,4) (4,0) (4,2) (4,3) (4,4)

We now return to the construction of a linear $(52,4,3)$-graph. Let $\mathcal{D}_1$ be the linear $(25,4,3)$-graph and $E = \{a,b,c,d\}$ its special edge. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be copies of $\mathcal{D}_1$ and let $E_1 = \{a,b,c,d\}$ and $E_2 = \{e,f,g,h\}$ be the copies of the special edge E (with $e = (0,0)$ and $f = (1,0)$). Let x, y, 1, 2, 3, 4 be new vertices. Let $\mathcal{H}$ be the hypergraph obtained as follows: First, we identify c and e with x and d and f with y. (Here the reader may find it helpful to think of this as a variant of the long edge construction– the set $\{a,b,x,y,g,h\}$ will play the role of the long edge). $\mathcal{H}$ also contains the edges: $\{a,b,x,1\}$, $\{a,y,g,2\}$, $\{b,y,h,3\}$, $\{x,g,h,4\}$, and $\{1,2,3,4\}$. $\mathcal{H}$ is linear, 4-uniform, and has order 52. See Figure 3.4.

$\mathcal{H}$ is 3-chromatic by the following argument. Suppose, to the contrary, $\mathcal{H}$ is 2-colorable and 2-color it. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are 3-critical, E_1 and E_2 are monochromatic, and because of the identification process, $\{a,b,x,y,g,h\}$ is monochromatic. But now one of the added edges is monochromatic, a contradiction. Thus $\mathcal{H}$ has no 2-coloring. It is easily seen to be 3-colorable and is thus 3-chromatic.

We do not know whether $\mathcal{H}$ is 3-critical. We shall now show, however, that it is 3-vertex-critical and thus contains a 3-critical spanning subgraph. To do this we need to exhibit a 2-coloring of $\mathcal{H} - v$ for each vertex v of $\mathcal{H}$.

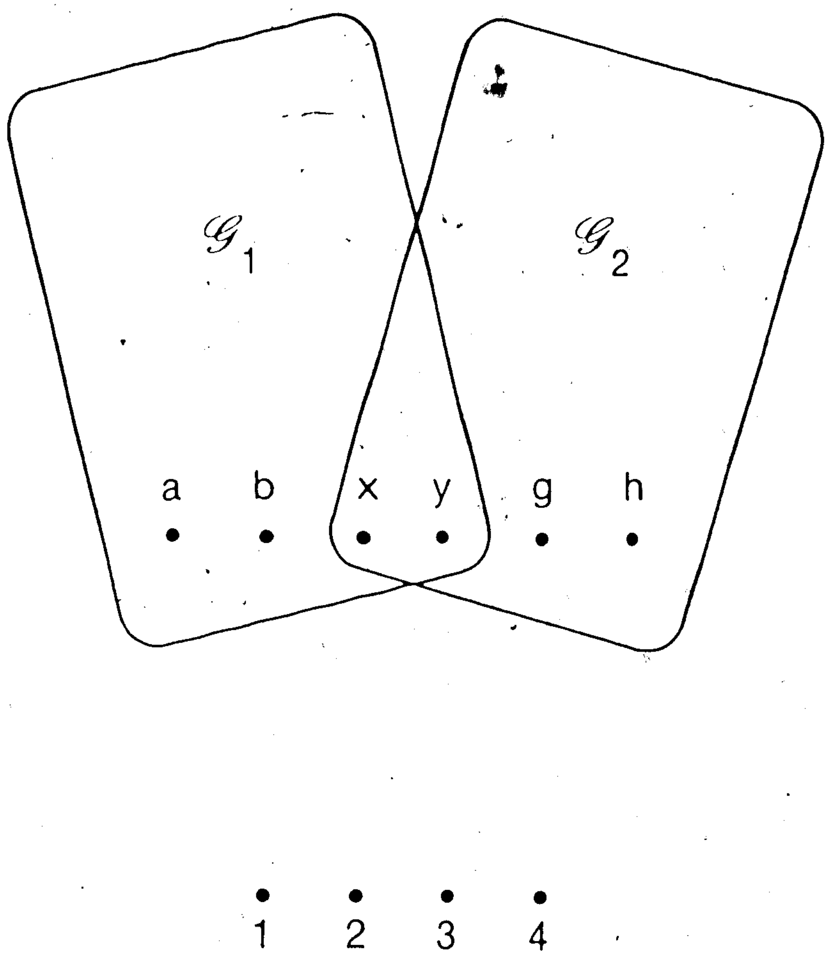


Fig. 3.4

Case 1 $v \in \{1,2,3,4\}$.

2-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that E_1 and E_2 are red. Color x and y red, and $\{1,2,3,4\} \setminus \{v\}$ blue.

Case 2 $v = x$ or y.

Suppose $v = x$. 2-color $\mathcal{G}_1 - c$ and $\mathcal{G}_2 - e$ so that vertices d and f are red. Color 1, 2 red and 3, 4 blue. This is a 2-coloring for $\mathcal{H} - x$. The case $v = y$ is similar.

Case 3 v is a vertex of $\mathcal{G}_1 - \{c,d\}$.

2-color $\mathcal{G}_1 - v$ so that c and d are red. This can be done, by (P1). 2-color $\mathcal{G}_2 - E_2$ so that E_2 is red. Color 1, x and y red and color 2, 3 and 4 blue. This induces a 2-coloring of $\mathcal{H} - v$.

Case 4 v is a vertex of $\mathcal{G}_2 - \{e,f\}$.

2-color $\mathcal{G}_2 - v$ so that e and f are red. This can be done, by (P1). 2-color $\mathcal{G}_1 - E_1$ so that E_1 is red. Color 4, x and y red and color 1, 2 and 3 blue. This induces a 2-coloring of $\mathcal{H} - v$.

Therefore $\mathcal{H}$ is 3-vertex-critical and hence contains a linear $(52,4,3)$-graph. This completes the proof of Theorem 3.1. $\square$

THEOREM 3.2. $M^*(3,4) \leq 100$.

PROOF: We give six constructions from which the theorem may be deduced. The main construction is the first one–from it alone we may deduce that $M^*(3,4) \leq 160$, a considerable improvement over the bound of 719 given in [A6]. It also gives linear $(m,3,4)$-graphs for a few values of $m < 160$. The remaining five constructions enable us to fill the interval $[100, 159]$.

Note: Unless stated otherwise, we make the assumption in this proof that 3-coloring of a hypergraph will mean a 3-coloring using the colors red, blue and green. We will also make the assumption that a 2-coloring of a hypergraph will be in colors red and blue.

CONSTRUCTION 1.

We exploit the long edge graph. Let l be a positive integer, $l \geq 2$. For $i = 1, 2, \ldots, l$, let $\mathcal{G}_i$ be a linear $(m_i, 3, 4)$-graph, E_i be an edge of $\mathcal{G}_i$, and v_i be a vertex of E_i, and let v be a new vertex. Let $\mathcal{G}$ be the long edge graph

$$\mathcal{G} = (\mathcal{G}, E, v) = \bigoplus_{i=1}^{l} (\mathcal{G}_i, E_i, v_i) \,.$$

Let t satisfy

$$t \in \begin{cases} \{7, 9, 10, 11, \ldots, \binom{2l+1}{2}\} & 2 \leq l \leq 6 \\ \{9, 10, 11, \ldots, \binom{2l+1}{2}\} & l = 7 \\ \{l+1, l+2, l+3, \ldots, \binom{2l+1}{2}\} & l \geq 8 \end{cases} \qquad (37)$$

Let $m = m_1 + m_2 + \ldots + m_l - l + 1 + t$. We show that there exists a linear $(m,3,4)$-graph.

Let $\mathcal{L}$ be a linear $(t,3,3)$-graph and label its vertices $v_1, v_2, \ldots, v_t$. Let M_1, $M_2, \ldots, M_t$ be distinct 2-subsets of the long edge E so that $\bigcup_{i=1}^{t} M_i = E$. This requires $2t \geq 2l+1$ and (since $|E| = 2l+1$) $t \leq \binom{2l+1}{2}$. Thus $l+1 \leq t \leq \binom{2l+1}{2}$, and this condition is satisfied. Of course, we cannot allow $t < 7$ or $t = 8$ in the case $l \leq 7$ since then no linear $(t,3,3)$-graph exists.

Let $\mathcal{M} = \{M_1 \cup \{v_1\}, M_2 \cup \{v_2\}, \ldots, M_t \cup \{v_t\}\}$ and let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{G} - E$, $\mathcal{L}$ and $\mathcal{M}$. $\mathcal{H}$ is 3-uniform, linear and has order m. We show that $\mathcal{H}$ is 4-critical.

We first show that $\mathcal{H}$ is 4-chromatic. Suppose, to the contrary, that $\mathcal{H}$ is 3-colorable and 3-color it. By Lemma 1.3, the long edge E is monochromatic. We may suppose it is red. No vertex of $\mathcal{L}$ can be colored red, because otherwise there would be a monochromatic edge in $\mathcal{M}$. Thus the vertices of $\mathcal{L}$ are colored blue and green. However, $\mathcal{L}$ is 3-critical, and we get a contradiction. Thus $\mathcal{H}$ is not 3-colorable. It is clear that $\mathcal{H}$ is 4-colorable and therefore 4-chromatic.

$\mathcal{H}$ is also 4-critical by the following argument. Let F be an edge of $\mathcal{H}$. We need to show that $\mathcal{H} - F$ is 3-colorable.

<u>Case 1</u> F is an edge of $\mathcal{L}$.

2-color $\mathcal{L} - F$ so that F is red. 3-color $\mathcal{G} - E$ so that E is green.

<u>Case 2</u> F is an edge of $\mathcal{M}$.

We have $F = M_j \cup \{v_j\}$ for some $j \in \{1, 2, \ldots, t\}$. 3-color $\mathcal{G} - E$ so that E is green. 3-color $\mathcal{L}$ so that v_j is the only vertex in $\mathcal{L}$ colored green.

<u>*Case 3*</u> F is an edge of $\mathcal{G} - E$.

We have $F \in \mathcal{G}_j$ for some $j \in \{1,2,\ldots,l\}$. For $i \neq j$, 3-color $\mathcal{G}_i - E_i$ so that E_i is green. 3-color $\mathcal{G}_j - F$ so that vertex v_j is green. Color v green. Since, in the 3-coloring of $\mathcal{G}_j - F$, E_j is not monochromatic, there is a $y \in E_j$ which is not green. Since $\bigcup_{i=1}^{t} M_i = E$, $y \in M_k$ for some $k \in \{1,2,\ldots,t\}$. 3-color $\mathcal{L}$ so that v_k is the only green vertex of $\mathcal{L}$. This induces a 3-coloring of $\mathcal{H} - F$.

Thus $\mathcal{H}$ is 4-critical and hence a linear $(m,3,4)$-graph exists.

<u>Note</u> Let $S_2 = \{\, m \mid$ there exists a linear $(m,3,4)$-graph$\}$. The $(31,155,15,3,1)$-design given as Example b) in Section 1.4.1 contains a linear $(31,3,4)$-graph. If we let $\mathcal{G}_i$ be a copy of this graph we find, after doing some simple calculations based on (37), that $m \in S_2$ for $m \geq 160$. Furthermore, the following numbers < 160 are in S_2:

$l = 2$: 68, 70, 71

$l = 3$: 98, 100, 101, 102, …, 112

$l = 4$: 128, 130, 131, 132, …, 157

$l = 5$: 158.

The remaining five constructions must therefore establish the existence of linear $(m,3,4)$-graphs for

$$m = 113, 114, 115, \ldots, 126, 127, 129, 159. \tag{38}$$

CONSTRUCTION 2. *A linear* $(m,3,4)$*-graph for* $m \in \{115,117,118,119,\ldots,127\}$.

Recall that the $(31,155,15,3,1)$-design may be described as follows: its objects are t[illegible]nts of Z_{31} and its blocks obtained from

$$\{0,1,18\},\{0,2,5\},\{0,4,10\},\{0,8,20\},\{0,9,16\}$$

by translating each of these by each element of Z_{31}. Denote the design by D_2. We shall need to know that D_2 has the following property:

(P2) If $F = \{a, b, c\}$ is an edge of D_2 then for any vertex v of D_2, other than b or c, there exists a 3-coloring of $D_2 - v$ in which b and c are colored the same.

In order to show this note first that since D_2 is a model of $PG(4,2)$, there is no loss of generality in assuming that $F = \{0,1,18\}$ and $a = 0$, $b = 1$, $c = 18$ (all lines are alike in a projective geometry). We now exhibit four 3-colorings of $D_2 - 0$ in which 1 and 18 are assigned the same color. Each of these 3-colorings will be 3-colorings for $D_2 - v$ in which 1 and 18 are assigned the same color by translating the color classes by $-v$ (modulo 31), for each v listed above the colorings.

For $v \in \{0,2,3,4,7,8,9,11,12,14,17,21,26\}$, the 3-coloring is:

red	1	2	3	4	5	10	15	18	24	29
blue	6	8	9	12	16	17	20	23	27	30
green	7	11	13	14	19	21	22	25	26	28

For $v \in \{5,15,20,22,23,24,25,29\}$, the 3-coloring is:

red	1	2	4	5	6	10	13	14	18	27
blue	3	8	12	17	20	21	23	25	29	30
green	7	9	11	15	16	19	22	24	26	28

For $v \in \{6,16,19\}$, the 3-coloring is:

red	1	2	4	5	8	9	10	16	18	20
blue	3	13	14	15	17	21	24	25	29	30
green	6	7	11	12	19	22	23	26	27	28

For $v \in \{10, 13, 27, 28, 30\}$, the 3-coloring has coloring classes:

red	1	4	10	13	14	18	21	25	26	28
blue	2	5	7	8	9	11	16	19	20	22
green	3	6	12	15	17	23	24	27	29	30

Thus (P2) holds.

Remark: The above colorings were obtained in the following way. We found a 3-coloring, with the help of a computer, that worked for a particular v and then took as many translations of this coloring as possible. We then chose a v not covered by these, found a 3-coloring that worked for it and then took translates of this coloring, and so on.

We now describe the construction. Let $\mathcal{D}_2$ be the graph given above, and let $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be copies of a 4-critical spanning subgraph of $\mathcal{D}_2$. Note that (P2) holds for each $\mathcal{G}_i$. For $i = 0, 1, 2$, let $E_i = \{v_1^i, v_2^i, v_3^i\}$ be the special edge of $\mathcal{G}_i$. Let z be a new vertex. Let $\mathcal{G}$ be the hypergraph whose edges are:

i) those of $\mathcal{G}_i - E_i$, $i = 0, 1, 2$

ii) $\{v_1^i, v_2^i, z\}$, $i = 0, 1, 2$.

It is straightforward to check that $\mathcal{G}$ is 3-chromatic. We make a simple but important observation. In any 3-coloring of $\mathcal{G}$, E_0, E_1 and E_2 are monochromatic and at least two of them are assigned the same color. (That E_0, E_1 and E_2 are monochromatic is clear and if no two of them have the same color, there would be a monchromatic edge of type ii)) The importance of this observation is that the m under consideration lie between those covered by the case $l = 3$ of Construction 1 (where the long edge has size 7) and the case $l = 4$ (where the long edge has size 9). Here $E_0 \cup E_1 \cup E_2$ will play the role similar to that played by the long edge in previous constructions.

We continue with the construction. For $i = 0, 1, 2$ define i' by $i' \equiv i + 1 \bmod 3$, $0 \leq i' \leq 2$. Let

$$M_1^i = \{v_1^{i'}, v_3^{i'}\}, \qquad M_2^i = \{v_2^{i'}, v_3^{i'}\}$$

and for $j = 1, 2, 3$ let

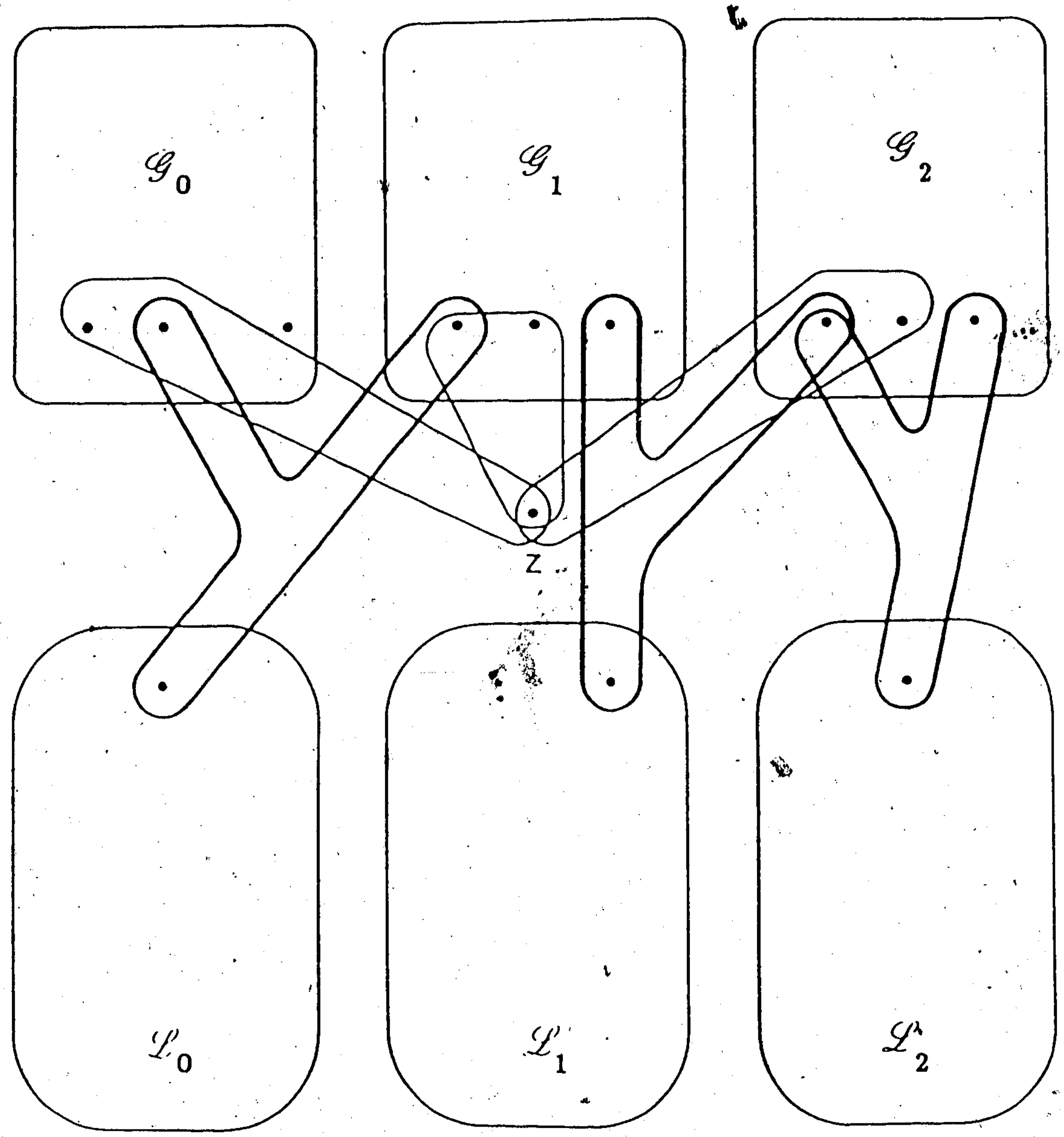
$$M_{j+2}^i = \{v_1^i, v_j^{i'}\}, \quad M_{j+5}^i = \{v_3^i, v_j^{i'}\}, \quad M_{j+8}^i = \{v_2^i, v_j^{i'}\}.$$

These sets are 2-subsets of $E_i \cup E_{i'}$. The reason for describing them in this way will become apparent later. Note that $\{v_1^i, v_2^i\}$ and $\{v_1^{i'}, v_2^{i'}\}$ are not included.

For $i = 0, 1, 2$, let $\mathcal{L}_i$ be a linear $(p_i, 3, 3)$-graph so that $p_i \in \{7, 9, 10, 11\}$, (hence $p_1 + p_2 + p_3 \in \{21, 23, 24, 25, \ldots, 33\}$). It may be checked that each $\mathcal{L}_i$ has a 3-coloring in which the color classes V_1^i, V_2^i, V_3^i satisfy $|V_1^i| \geq 3$ and $|V_2^i| \geq \max\{2, p_i - 6\}$. Let the vertices of $\mathcal{L}_i$ be labelled $l_1^i, l_2^i, \ldots, l_{p_i}^i$, and suppose that the labelling is such that $l_1^i, l_2^i \in V_2^i$ and $l_3^i, l_4^i, l_5^i \in V_1^i$ and in case $p_i \neq 7, l_{j+8}^i \in V_2^i$ for $j = 1, 2, \ldots, p_i - 8$.

For $i = 0, 1, 2$ let $\mathcal{M}_i = \left\{\{l_1^i\} \cup M_1^i, \{l_2^i\} \cup M_2^i, \ldots, \{l_{p_i}^i\} \cup M_{p_i}^i\right\}$. Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{G}$, $\mathcal{L}_i$ and $\mathcal{M}_i$, $i = 0, 1$ and 2. The reader should see Figure 3.5.

$\mathcal{H}$ is 3-uniform, linear (since $\{v_1^i, v_2^i\}$ is not a subset of any member of $\mathcal{M}_j$) and has order $94 + p_1 + p_2 + p_3$.



$\mathcal{G}_0$
$\mathcal{G}_1$
$\mathcal{G}_2$
Z
$\mathcal{L}_0$
$\mathcal{L}_1$
$\mathcal{L}_2$

Fig. 3.5

We show that $\mathcal{H}$ is 4-chromatic. Suppose that $\mathcal{H}$ is 3-colorable, and 3-color it. Then, by the observation made earlier, E_0, E_1, and E_2 are monochromatic and, at least two of them, say E_i and $E_{i'}$, have the same color, say red. Since $E_i \cup E_{i'} = \bigcup_{j=1}^{p_i} M_j^i$, no vertex of $\mathcal{L}_i$ can be colored red, since otherwise there would be a monochromatic edge in $\mathcal{M}_i$. Thus $\mathcal{L}_i$ must be 2-colored, contradicting the fact that $\mathcal{L}_i$ is 3-critical. It follows that $\mathcal{H}$ is not 3-colorable. It is clear that $\mathcal{H}$ is 4-colorable and hence 4-chromatic.

We do not know whether $\mathcal{H}$ is 4-critical. It is enough, however, to show that it is 4-vertex-critical and thus contains a 4-critical spanning subgraph. Let v be a vertex of $\mathcal{H}$. We need to show that $\mathcal{H} - v$ is 3-colorable.

<u>Case 1</u> v is a vertex of $\mathcal{G}_k$ for some $k \in \{0,1,2\}$, $v \notin \{v_2^k, v_3^k\}$.

3-color $\mathcal{G}_k - v$ so that v_1^k is red and $\{v_2^k, v_3^k\}$ is green. It is possible to do this by (P2). 3-color $\mathcal{G}_{k'} - E_{k'}$ and $\mathcal{G}_{k''} - E_{k''}$ so that $E_{k'}$ is red and $E_{k''}$ is blue (here $k'' = (k')'$). Color vertex z green. Then no edge of $\mathcal{G}$ is monochromatic. 3-color $\mathcal{L}_k$ so that V_1^k is blue, V_2^k is green and V_3^k is red. 3-color $\mathcal{L}_{k'}$ so that $V_2^{k'}$ is red, $V_1^{k'}$ is blue and $V_3^{k'}$ is green. 3-color $\mathcal{L}_{k''}$ so that $V_2^{k''}$ is red, $V_1^{k''}$ is blue and $V_3^{k''}$ is green. Then no edge of $\mathcal{M}_i$ or $\mathcal{L}_i$ is monochromatic for $i = 0, 1, 2$.

<u>Case 2</u> $v = v_2^k$ for some $k \in \{0,1,2\}$.

3-color $\mathcal{G}_k - v$, $\mathcal{G}_{k'} - E_{k'}$ and $\mathcal{G}_{k''} - E_{k''}$ so that E_k is green, $E_{k'}$ is red and $E_{k''}$ is blue. Color z green (this creates no problem since $\{v_1^k, v_2^k, z\}$ is not an edge of $\mathcal{H} - v$). [illegible] 2, we may 3-color $\mathcal{L}_i$ so that V_2^i is not assigned the same color as $E_{i'}$.

<u>*Case 3*</u> $v = v_3^k$ for some $k \in \{0, 1, 2\}$.

For $i = 0, 1, 2$, we may 3-color $\mathcal{G}_i - E_i$ so that E_k and $E_{k'}$ are red, and $E_{k''}$ is blue. Color z green. For $i = k', k''$, 3-color $\mathcal{L}_i$ so that V_2^i is not assigned the same color as $E_{i'}$. Color $\mathcal{L}_k$ so that V_1^k is blue, V_2^k is green and V_3^k is red. Note that M_6^k, M_7^k and (in case $p_k \neq 7$) M_8^k are not subsets of any edge of $\mathcal{H} - v$, and if $p_k \neq 7$, then $l_{j+8}^i \in V_2^k$, for $j = 1, 2, \ldots, p_k - 8$, and so edge $M_{j+8}^k \cup \{l_{j+8}^k\} = \{l_{j+8}^k, v_2^k, v_j^{k'}\}$ is not monochromatic.

<u>*Case 4*</u> $v = z$.

For $i = 0, 1, 2$, we may 3-color $\mathcal{G}_i - E_i$ so that E_0 is red, E_1 is blue and E_2 is green, and 3-color $\mathcal{L}_i$ so that V_2^i is not assigned the same color as E_i.

<u>*Case 5*</u> v is a vertex of $\mathcal{L}_k$ for some $k \in \{0, 1, 2\}$.

For $i = 0, 1, 2$, we may 3-color $\mathcal{G}_i - E_i$ so that E_k and $E_{k'}$ are red, and $E_{k''}$ is blue. Color z green. 2-color $\mathcal{L}_k - v$ blue and green. For $i = k', k''$, 3-color $\mathcal{L}_i$ so that V_1^i is blue, V_2^i is green and V_3^i is red.

It follows that $\mathcal{H}$ is 4-vertex-critical and thus contains a 4-critical spanning subgraph.

<u>Note</u> If we delete from the list given in (38) those values of m covered in Construction 2, we get the following values of $m \geq 100$ for which we need to establish the existence of linear $(m, 3, 4)$-graphs:

$m = 113,$ [illegible] $16, 129, 159.$ (39)

CONSTRUCTION 3. *If $m_1, m_2 \in S_2$ then $m_1 + m_2 + 11 \in S_2$.*

Let $\mathcal{G}_1$ be a linear $(m_1, 3, 4)$-graph, $\mathcal{G}_2$ be a linear $(m_2, 3, 4)$-graph, $E_1 = \{a, b, c\}$ be an edge of $\mathcal{G}_1$ and $E_2 = \{d, e, f\}$ be an edge of $\mathcal{G}_2$. Let $\mathcal{L}$ be the linear $(11, 3, 3)$-graph with edges $\{1,2,3\}$, $\{1,4,9\}$, $\{1,5,11\}$, $\{1,6,7\}$, $\{1,8,10\}$, $\{2,6,11\}$, $\{3,5,6\}$, $\{3,7,11\}$, $\{4,5,7\}$, $\{4,8,11\}$, $\{5,8,9\}$, $\{9,10,11\}$. Let $\mathcal{M}$ be the linear 3-graph with edges $\{1,a,b\}$, $\{2,a,c\}$, $\{3,b,c\}$, $\{1,d,e\}$, $\{2,d,f\}$, $\{3,e,f\}$, $\{4,a,d\}$, $\{5,a,e\}$, $\{6,a,f\}$, $\{7,b,d\}$, $\{8,b,e\}$, $\{9,b,f\}$, $\{10,c,d\}$, $\{11,c,e\}$. Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{G}_1 - E_1$, $\mathcal{G}_2 - E_2$, $\mathcal{L}$ and $\mathcal{M}$.

$\mathcal{H}$ is linear, 3-uniform and has order $m_1 + m_2 + 11$. $\mathcal{H}$ is 4-chromatic by the following argument. Suppose, to the contrary, $\mathcal{H}$ is 3-colorable and 3-color it. Then E_1 and E_2 are monochromatic. If they have different colors: E_1 red, E_2 blue, the first six edges listed in $\mathcal{M}$ force 1, 2 and 3 to be green thus giving a monochromatic edge (namely $\{1,2,3\}$) in $\mathcal{L}$. Thus E_1 and E_2 have the same color, say red. Since $\mathcal{L}$ is 3-critical some vertex of $\mathcal{L}$ must be colored red. This implies that there is a red edge in $\mathcal{M}$, a contradiction. Thus $\mathcal{H}$ is not 3-colorable. It is clearly 4-colorable and thus 4-chromatic.

We now show that $\mathcal{H}$ is 4-critical. Let E be an edge of $\mathcal{H}$. We need to show that $\mathcal{H} - E$ is 3-colorable.

Case 1 E is an edge of $\mathcal{L}$.

2-color $\mathcal{L} - E$ and 3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that E_1 and E_2 are green.

Case 2 E is an edge of $\mathcal{M}$.

Let j be the vertex of $\mathcal{L}$ contained in E. 3-color $\mathcal{L}$ so that vertex j is the only green vertex. 3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that E_1 and E_2 are green.

Case 3 E is an edge of $\mathcal{G}_1 - E_1$.

3-color $\mathcal{G}_1 - E$ and $\mathcal{G}_2 - E_2$ so that a is red and E_2 is green. If all three colors are used on E_1 let b be blue and c be green. 3-color $\mathcal{L}$ so that 4 is the only green vertex of $\mathcal{L}$. This gives a 3-coloring of $\mathcal{H} - E$. Otherwise, one color, which we may take to be green, is not used in E_1. Thus either $\{a, b\}$ is red, $\{a, c\}$ is red, or $\{b, c\}$ is blue. 2-color $\mathcal{L} - \{1, 2, 3\}$ red and green so that $\{1, 2, 3\}$ is red and re-color 1 and 2 blue. This gives a 3-coloring of $\mathcal{H} - E$.

Case 4 E is an edge of $\mathcal{G}_2 - E_2$.

Color $\mathcal{G}_2 - E$ and $\mathcal{G}_1 - E_1$ so that d is red and E_1 is green. If all three colors are used on E_2, let e be blue and f be green. 3-color $\mathcal{L}$ so that 4 is the only green vertex of $\mathcal{L}$. This is a 3-coloring of $\mathcal{H} - E$. Otherwise, one color, which we may take to be green, is not used in E_2. Thus either $\{d, e\}$ is red, $\{d, f\}$ is red, or $\{e, f\}$ is blue. 2-color $\mathcal{L} - \{1, 2, 3\}$ red and green so that $\{1, 2, 3\}$ is red and re-color 1 and 2 blue. This is a 3-coloring of $\mathcal{H} - E$.

It follows that $\mathcal{H}$ is 4-critical and hence that $m_1 + m_2 + 11 \in S_2$.

CONSTRUCTION 4. *If* $m_1, m_2 \in S_2$ *then* $m_1 + m_2 + 12 \in S_2$.

Let $\mathcal{G}_1$, $\mathcal{G}_2$, E_1, E_2 be as in Construction 3. Let $\mathcal{L}$ be the linear $(12, 3, 3)$-graph whose edges are: $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{1, 8, 9\}$, $\{2, 4, 8\}$, $\{2, 6, 9\}$, $\{2, 5, 7\}$,

$\{3,4,6\}$, $\{3,7,8\}$, $\{3,5,9\}$, $\{5,6,10\}$, $\{5,8,11\}$, $\{6,8,12\}$, $\{10,11,12\}$. Let $\mathcal{M}$ be the same as in Construction 3, but with the additional edge $\{12,c,f\}$. Let $\mathcal{H}$ be the linear 3-graph whose edges are those of $\mathcal{G}_1 - E_1$, $\mathcal{G}_2 - E_2$, $\mathcal{L}$ and $\mathcal{M}$.

The argument used in Construction 3, with no changes, shows that $\mathcal{H}$ is 4-critical. We do not give the details. Since $\mathcal{H}$ has order m_1+m_2+12 we have $m_1+m_2+12 \in S_2$. Note Construction 4, with $m_1 = m_2 = 31$ shows that $74 \in S_2$. Construction 1, with $l = 2$, $m_1 = 74$, $m_2 = 31$, and $t = 9$ or 10 shows that 113 and 114 are in S_2. Construction 3, with $m_1 = m_2 = 74$, shows that $159 \in S_2$. Thus three of values in (39) are removed, leaving just $m = 116$ and $m = 129$.

CONSTRUCTION 5. *A linear* $(116,3,4)$*-graph.*

The construction is complicated. We make a few remarks which may serve to motivate the main idea. We wish to exploit the long edge notion. The case $l = 3$ of Construction 1 gives a linear $(115,3,4)$-graph. The long edge (of size 7) is not quite long enough to cover the case $m = 116$. Our construction is given in two stages. At the first stage we construct a 3-chromatic 3-graph in which in every 3-coloring a certain set of 8 vertices is forced to be monochromatic. This set of 8 vertices will then play the role of the long edge.

We proceed to the details. For $i = 1, 2, 3$ let $\mathcal{G}_i$ be a copy of the linear $(31,3,4)$-graph, and let E_i be an edge of $\mathcal{G}_i$. For $i = 2, 3$ let v_i be a vertex of E_i. Let v be a new vertex. Let $\mathcal{G}$ be the long edge graph obtained from $\mathcal{G}_2$ and $\mathcal{G}_3$:

$$\mathcal{G} = (\mathcal{G}, E, v) = (\mathcal{G}_2, E_2, v_2) \bigoplus (\mathcal{G}_3, E_3, v_3).$$

Let $E_1 = \{a, b, c\}$ and label the vertices of E so that $E = \{d, e, f, g, h\}$ where $\{d, e\} \subset \bigcup \mathcal{G}_2$, $\{g, h\} \subset \bigcup \mathcal{G}_3$ and $f = v$. Let $\bar{1}$, $\bar{2}$ and $\bar{3}$ be new vertices and let $\mathcal{F}$ be the graph whose edges are:

i) those of $\mathcal{G}_1 - E_1$ and $\mathcal{G} - E$

ii) $\{a, b, \bar{1}\}$, $\{a, c, \bar{2}\}$, $\{b, c, \bar{3}\}$, $\{d, e, \bar{1}\}$, $\{d, f, \bar{2}\}$, $\{e, f, \bar{3}\}$

iii) $\{\bar{1}, \bar{2}, \bar{3}\}$.

$\mathcal{F}$ is 3-colorable. The key observation is that in any 3-coloring of $\mathcal{F}$ the set $\{a, b, c, d, e, f, g, h\} = E'$ is monochromatic. To see this, consider a 3-coloring of $\mathcal{F}$. By Lemma 1.3, E must be monochromatic. Also E_1 is monochromatic. If E_1 and E_2 have different colors, then these colors cannot be used to color $\bar{1}$, $\bar{2}$ or $\bar{3}$ since otherwise there would be a monochromatic edge of type iii). But then $\{\bar{1}, \bar{2}, \bar{3}\}$ is monochromatic. It follows that in any 3-coloring of $\mathcal{F}$ the set E' is monochromatic.

$\mathcal{F}$ has 95 vertices. We now construct a linear 3-graph of order 116 which contains $\mathcal{F}$ as a subgraph. Let 1, 2, ..., 21 be new vertices and let $\mathcal{L}$ be the linear $(21, 3, 3)$-graph with edges $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{1, 8, 9\}$, $\{2, 4, 8\}$, $\{2, 6, 9\}$, $\{2, 5, 7\}$, $\{3, 4, 6\}$, $\{3, 7, 8\}$, $\{3, 5, 9\}$, $\{5, 6, 10\}$, $\{5, 8, 11\}$, $\{6, 8, 12\}$, $\{10, 11, 13\}$, $\{10, 12, 14\}$, $\{11, 12, 15\}$, $\{13, 14, 16\}$, $\{13, 15, 17\}$, $\{14, 15, 18\}$, $\{16, 17, 19\}$, $\{16, 18, 20\}$, $\{17, 18, 21\}$, $\{19, 20, 21\}$. and let $\mathcal{M}$ be the linear 3-graph with edges $\{1, a, d\}$, $\{2, a, e\}$, $\{3, a, f\}$, $\{4, b, d\}$, $\{5, b, e\}$, $\{6, b, f\}$, $\{7, c, d\}$, $\{8, c, e\}$, $\{9, c, f\}$, $\{10, a, g\}$, $\{11, a, h\}$, $\{12, b, g\}$, $\{13, b, h\}$, $\{14, c, g\}$, $\{15, c, h\}$, $\{16, d, g\}$, $\{17, d, h\}$, $\{18, c, g\}$, $\{19, e, h\}$, $\{20, f, g\}$, $\{21, f, h\}$.

Let $\mathcal{H}$ be the hypergraph whose edges are those of $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{M}$. The reader should see Figure 3.6.

$\mathcal{H}$ is a linear 3-graph of order 116. We show that $\mathcal{H}$ is 4-critical.

First observe that $\mathcal{H}$ is 4-chromatic. If there were a 3-coloring, the set E' would be monochromatic. Then the color assigned to E' cannot be used to color $\mathcal{L}$ since otherwise there would be a monochromatic edge in $\mathcal{M}$. Thus $\mathcal{L}$ must be colored in 2 colors, contradicting the fact that $\mathcal{L}$ is 3-critical. Hence $\mathcal{H}$ is not 3-colorable. It is clearly 4-colorable and therefore 4-chromatic.

We now show that $\mathcal{H}$ is 4-critical. Let F be an edge of $\mathcal{H}$. We need to show that $\mathcal{H} - F$ is 3-colorable.

Case 1 F is an edge of $\mathcal{G}_1 - E_1$.

3-color $\mathcal{G}_1 - F$ so that a is red and $\{b, c\}$ is not red. 3-color $\mathcal{G} - E$ so that E is blue, and 3-color $\mathcal{L}$ so that 1 is the only blue vertex of $\mathcal{L}$. Then no edge of $\mathcal{M}$ is monochromatic since in $\{1, a, d\}$, 1 is blue and a is red and each of the remaining edges of $\mathcal{M}$ has a red or green vertex from $\mathcal{L}$ and a blue vertex from E.

Case 2 F is an edge of $\mathcal{F} - E$.

Let $F' = (F \setminus \{v\}) \cup \{v_2\}$ if $v \in F$ and set $F' = F$ otherwise. We give details for the case $F' \in \mathcal{G}_2 - E_2$. The case involving $\mathcal{G}_3 - E_3$ is similar. 3-color $\mathcal{G}_2 - F'$ so that v_2 is blue and $\{d, e\}$ is not blue. 3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_3 - E_3$ so that E_1 and E_3 are blue. Color $\bar{1}$ red and $\bar{2}$ green. Then no edge of type ii) or iii) is monochromatic. Now one of the vertices of d or e is not blue. If d is not blue, 3-color $\mathcal{L}$ so that $\bar{1}$ is

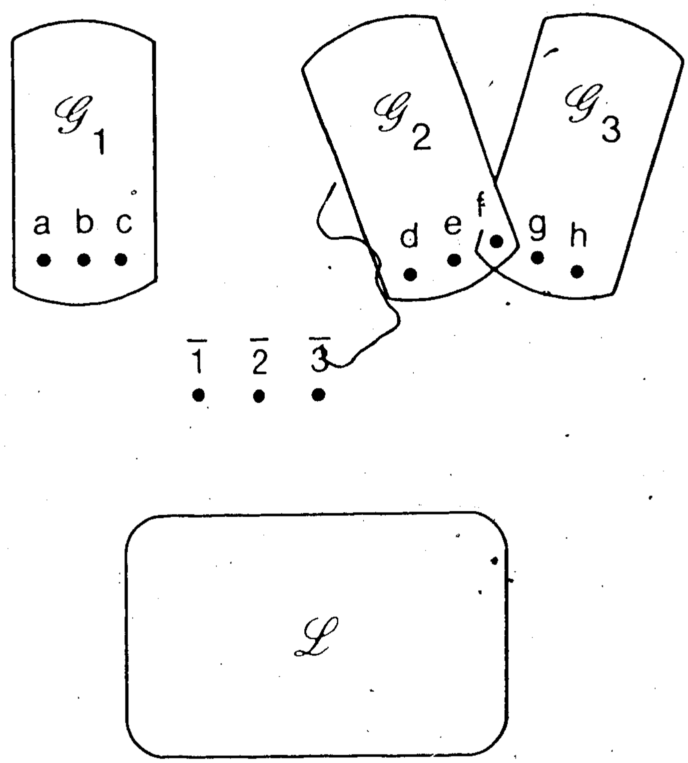

$\mathscr{G}_1$
a b c
$\mathscr{G}_2$
d e f
$\mathscr{G}_3$
g h
$\bar{1}$ $\bar{2}$ $\bar{3}$
$\mathscr{L}$

Fig. 3.6

the only blue vertex of $\mathcal{L}$. Then no edge of $\mathcal{M}$ is monochromatic since in $\{1, a, d\}$, 1 is blue and d is not and the rest of the edges of $\mathcal{M}$ have a blue vertex from $E_1 \cup E_2$ and a red or a green vertex from $\mathcal{L}$ (it is important to note that $\{d, e\}$ is not a subset of any edge of $\mathcal{M}$). If d is blue, then e is not blue and we may 3-color $\mathcal{L}$ so that 2 is the only blue vertex of $\mathcal{L}$. In both cases we get a 3-coloring of $\mathcal{H} - F$.

Case 3 F is an edge of $\mathcal{L}$.

3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G} - E$ so that E_1 and E are blue. Color $\bar{1}$ red and $\bar{2}$, $\bar{3}$ green. 2-color $\mathcal{L} - F$ red and green so that F is green.

Case 4 F is an edge of type ii) or iii).

3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G} - E$ so that E_1 is blue and E is red. If $F = \{\bar{1}, \bar{2}, \bar{3}\}$ then color F green. If F is of type ii) let $\bar{i}$ be the vertex common to F and $\{\bar{1}, \bar{2}, \bar{3}\}$. Color $\bar{i}$ so that F is monochromatic and color the other vertices of $\{\bar{1}, \bar{2}, \bar{3}\}$ green. 3-color $\mathcal{L}$ so that 1 is the only red vertex in $\mathcal{L}$. Then no edge of $\mathcal{M}$ is monochromatic: in $\{1, a, d\}$, 1 is red and a blue, and the rest of the edges from $\mathcal{M}$ have a blue or green vertex from $\mathcal{L}$, and a red vertex from E.

Case 5 F is an edge of $\mathcal{M}$.

3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G} - E$ so that E_1 and E are blue. Color $\bar{1}$ and $\bar{2}$ red, and $\bar{3}$ green. Let j be the vertex of $\mathcal{F}$ which is also a vertex of $\mathcal{L}$ and 3-color $\mathcal{L}$ so that j is the only blue vertex of $\mathcal{L}$. Then no edge of $\mathcal{M} - F$ is monochromatic since each such edge contains a red or a green vertex from $\mathcal{L}$ and a blue vertex from $E_1 \cup E$.

It follows that $\mathcal{H}$ is a 4-critical and is therefore a linear $(116, 3, 4)$-graph.

There remains only one other value of $m \geq 100$ to dispose of, namely $m = 129$ (it is the value missed by the case $l = 3$ of Construction 1 because of the lack of a linear $(8,3,3)$-graph). In our search for a linear $(129,3,4)$-graph we had to abandon the idea of using a variant of the long edge construction. 129 is near the lower end of the block of consecutive numbers covered by the case $l = 3$ of Construction 1 so that the idea used in Construction 5 does not apply. Also, it is just missed by Construction 2 since it is clear from Construction 2 that the numbers p_1, p_2, p_3 occurring there cannot exceed 11. In order for it to work here we would have to have $p_1 + p_2 + p_3 = 35$.

CONSTRUCTION 6. *A linear* $(129,3,4)$*-graph.*

We describe the construction in two steps.

The first step is to obtain a linear 4-critical graph in which all edges except one have size 3 and one edge has size 2. We refer to it as a **short edge graph**. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be copies of the linear $(31,3,4)$-graph. Let $E_1 = \{a,b,c\}$ be an edge of $\mathcal{G}_1$ and $E_2 = \{d,e,f\}$ be an edge of $\mathcal{G}_2$. Let 1, 2 and 3 be new vertices and let $\mathcal{F}$ be the hypergraph whose edges are:

i) those of $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$

ii) $\{a,b,1\}$, $\{a,c,2\}$, $\{b,c,3\}$, $\{d,e,1\}$, $\{d,f,2\}$, $\{e,f,3\}$, and $\{1,2,3\}$.

iii) $F^* = \{c,d\}$ (the short edge).

Note that $\mathcal{F} - F^*$ is 3-uniform, linear and has order 65 (see Figure 3.7).

It is easy to check that $\mathcal{F}$ is 4-chromatic. We show that it is 4-critical. Let $F \in \mathcal{F}$.

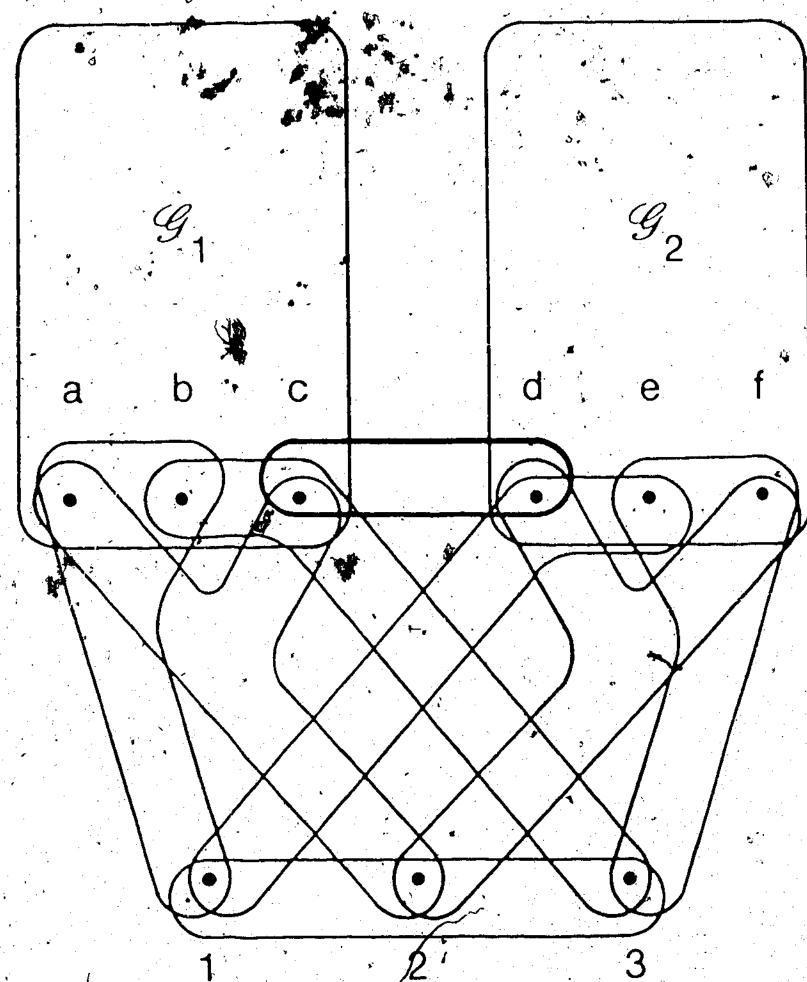


Fig. 3.7

We need to exhibit a 3-coloring of $\mathcal{F} - F$.

Case 1 F is an edge of $\mathcal{G}_1 - E_1$.

3-color $\mathcal{G}_1 - F$ so that c is red and $\{a, b\}$ is not red. 3-color $\mathcal{G}_2 - E_2$ so that E_2 is green. Color 1 red and 2 and 3 blue.

Case 2 F is an edge of $\mathcal{G}_2 - E_2$.

3-color $\mathcal{G}_1 - F$ so that d is red and $\{e, f\}$ is not red. 3-color $\mathcal{G}_1 - E_1$ so that E_1 is green. Color 1 and 2 blue and 3 red.

Case 3 F is an edge of type ii).

3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that E_1 is red and E_2 is blue. If $F = \{1, 2, 3\}$, color F green. Otherwise, $F = \{j\} \cup F'$ where $F' \subset E_1$ or $F' \subset E_2$, and $j \in \{1, 2, 3\}$. Color vertex j blue if $F' \subset E_1$ and red if $F' \subset E_2$. Color $\{1, 2, 3\} \setminus \{j\}$ green.

Case 4 $F = F^*$.

3-color $\mathcal{G}_1 - E_1$ and $\mathcal{G}_2 - E_2$ so that $E_1 \cup E_2$ is green. Color 1 red and 2 and 3 blue.

It follows that $\mathcal{F}$ is 4-critical.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be copies of the short edge graph $\mathcal{F}$ described above. Let the short edges of $\mathcal{F}_1$ and $\mathcal{F}_2$ be $F_1^* = \{u, v\}$ and $F_2^* = \{x, y\}$ respectively. Let z be a new vertex and let $\mathcal{H}$ be the hypergraph obtained from $\mathcal{F}_1$ and $\mathcal{F}_2$ by identifying v and x with z. $\mathcal{H}$ is a linear, 3-graph–in the identification process the short edges merge so as to give a single edge of size 3– and has order 129 (see Figure 3.8).

It is straightforward to check that $\mathcal{H}$ is 4-chromatic. $\mathcal{H}$ is also 4-critical. Let F be

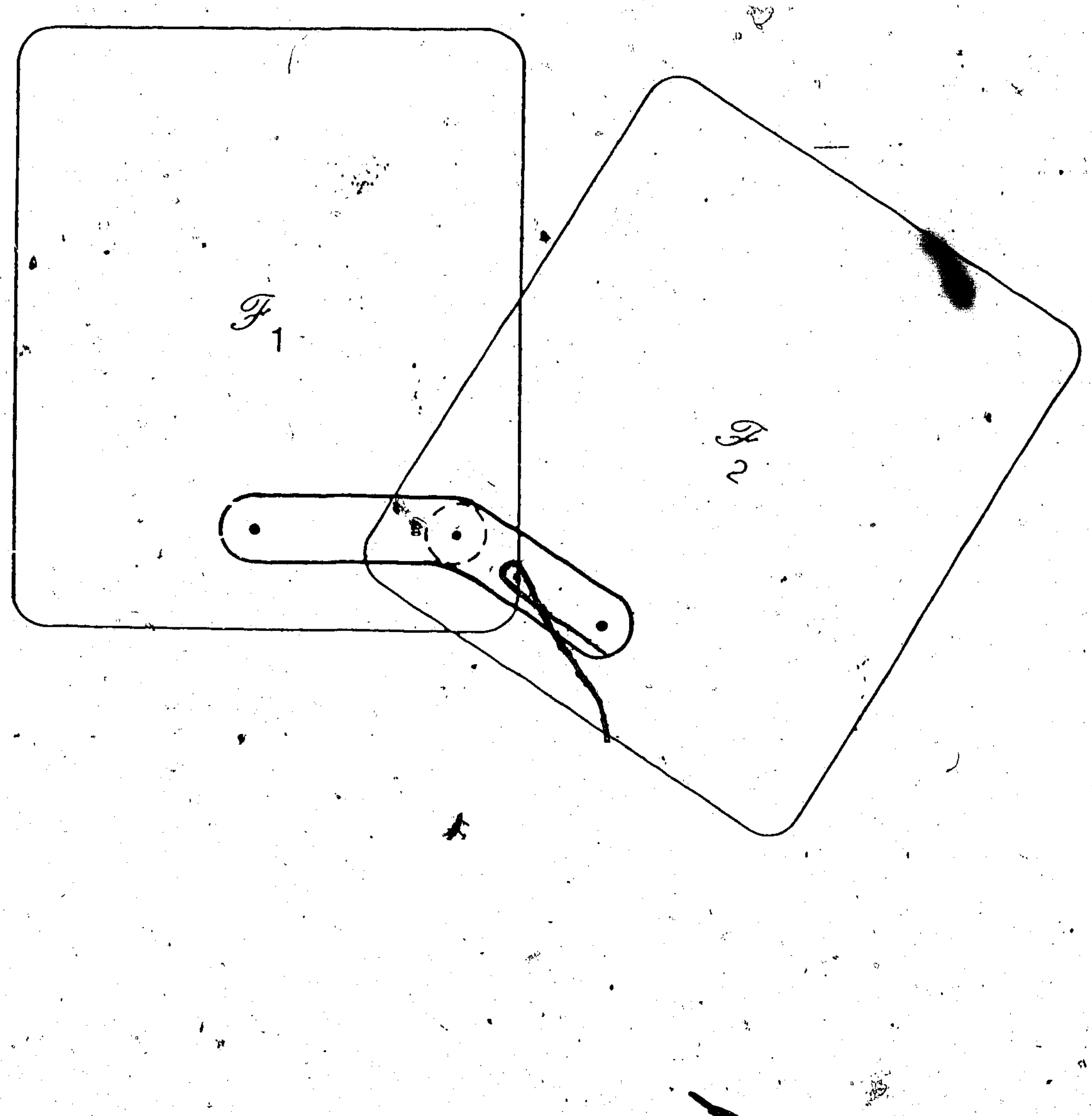


Fig. 3.8

an edge of $\mathcal{H}$. Without loss of generality we may suppose that $F \in \mathcal{F}_1 - F_1^*$. 3-color $\mathcal{F}_1 - F$ so that u is red and v is blue. 3-color $\mathcal{F}_2 - F_2^*$ so that F_2^* is blue. Color z blue. This yields a 3-coloring of $\mathcal{H} - F$. Hence $\mathcal{H}$ is 4-critical

This completes the proof of Theorem 3.2. □

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