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Characterizations of ℓ_p^n and Constructions of Banach Spaces with No Unconditional Basis

BY

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ABSTRACT

The first part of the paper establishes some characterizations of ℓ_p^n spaces in terms of p-summing or p-nuclear norms of the identity operator on the given space E. In particular, for an n-dimensional Banach space E, $1 \leq p < 2$, E is isometric to ℓ_p^n if and only if $\pi_p(E^*) \geq n^{\frac{1}{p}}$ and E^* has cotype p' with the constant one. Furthermore, ℓ_p^n spaces are characterized by inequalities for p-summing norms of operators related to the ellipsoid of maximal volume contained in the unit ball of E.

The aim of the second note is to show a rather general construction of Banach spaces with no unconditional basis. As a corollary, for example, one obtains a weak Hilbert space with no unconditional basis. The novelty of the preset general approach consists of the use of a tree of partitions. It is strong enough to construct spaces which contain subspaces with no unconditional basis.

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NOTATION

A_p	$\equiv$	$(E\|\gamma_1\|^p)^{1/p}$
$d(X,Y)$	$\equiv$	the Banach-Mazur distance
d_E	$\equiv$	$d(E, \ell_2^{\|E\|})$
$\|S\|$	$\equiv$	the cardinality of the set S
X_δ	$\equiv$	the version of the 2-convexified Tsirelson space
$C_q(X)$	$\equiv$	the cotype constant
$D(X_1 \oplus \cdots \oplus X_m)$	$\equiv$	the diagonal subspace
γ_i	$\equiv$	the standard Gaussian random variables
$E\xi$	$\equiv$	the expected value of ξ
$\mathrm{span}\,(S)$	$\equiv$	the linear span of S
$\overline{\mathrm{span}}\,(S)$	$\equiv$	the norm closed linear span of S
$\pi_p(S)$	$\equiv$	the p-summing norm of S
$\nu_p(S)$	$\equiv$	the p-nuclear norm of S
r_i	$\equiv$	the Rademacher functions
$\mathrm{unc}\,(X)$	$\equiv$	the unconditional constant of X
$\mathrm{unc}\,(Z_k)_k$	$\equiv$	the unconditional constant of $(Z_k)_k$

INTRODUCTION

In this dissertation we discuss two separate topics from the theory of Banach spaces. One comes from the local theory of finite dimensional spaces, and the other concerns certain structure properties of infinite dimensional Banach spaces.

The first topic, studied in Chapters 1 and 2, is concerned with relationships between geometric properties of finite dimensional normed spaces and operator ideal properties of certain natural operators acting on them. We use techniques of vector valued random variables and of p-summing operators; both of them are natural in this context.

The second topic, described in Chapter 3, is devoted to rather general constructions of infinite dimensional Banach spaces with no unconditional basis. In fact, we identify several large classes of Banach spaces such that for a space in one of these classes we show how to construct a subspace with no unconditional basis. It is clear that the construction is more difficult if a space is close, in a sense, to a Hilbert space. One of classes we identified contains spaces which are very close indeed to ℓ_2; in particular, it contains the best known examples of so-called weak Hilbert spaces, 2-convexified Tsirelson space and its dual.

Now we pass to more detailed description of a background material and of the results of the thesis.

In Chapter 1 we establish characterizations of ℓ_p^n-spaces in terms of ideal norms of certain natural operators related to an n-dimensional Banach space E. These characterizations generalize several known results for ℓ_1^n and ℓ_∞^n ([D], [FJ], [G1], [N], [PT]). Some characterizations are given by conditions on p-summing and p'-nuclear norms of the identity operator on E, combined with assumption on the cotype of the space. Others involve operators determined by the John's ellipsoid of maximal volume contained in the unit ball of E. In particular we show that some inequalities for these norms characterize ℓ_p^n.

In arguments, the most important results are established in Proposition 2.1 and Theorem 3.1. In the first, we obtain an upper estimate of the p-summing norm by the p-th moment of a related vector valued Gaussian random variable. In the second, we prove that if $1 \le p < 2$ and E is an n-dimensional Banach space such that $\pi_p(id : E \to E) \ge n^{\frac{1}{p}}$, then there exist vectors $e_1, \ldots, e_n$ in E such that, for every sequence of scalars $a_1, \ldots, a_n$ one has

$$\max_{i=1,\ldots,n} |a_i| \le \left\| \sum_{i=1}^{n} a_i e_i \right\| \le \left(\sum_{i=1}^{n} |a_i|^{p'} \right)^{1/p'},$$

(here $1/p + 1/p' = 1$). It turns out that the vectors $e_1, \ldots, e_n$ are the contact points of the unit ball B_E of E with the John's ellipsoid.

In Chapter 2 we present some consequences of our results from Chapter 1 for subspaces of L_p. We also prove that if $2 < p < \infty$, then an n-dimensional subspace of L_p with the maximal Euclidean distance is isometric to ℓ_p^n. This complements a result obtained in [BT] for $1 < p < 2$. The proof is based on a well-known result of D. R. Lewis [L] which establishes the existence of good bases in finite dimensional subspaces of L_p.

Let us describe the content and a background of the next part of the thesis which is presented in Chapter 3.

A useful and widely studied class of Banach spaces consists of spaces with an unconditional basis. The existence of such a basis gives many information on the structure of the space. However, there are spaces which do not satisfy such a nice property. To construct or to show the existence of such spaces is not very easy. Difficulties are caused by the fact that "to be unconditional" is not a local property; that is, it can not be described in terms of finite dimensional subspaces. However, there are some techniques which help to justify whether a given Banach space has or does not have an unconditional basis.

The literature on this particular problem is very extensive and varies. Some new parameters were introduced and some new arguments were presented. Equally important constants which are very closely related to unconditionality are: the *Gordon-Lewis constant* $g\ell(X)$ and the *local unconditional structure constant* $l.u.st(X)$.

The first one which was introduced by Y. Gordon and D. R. Lewis (1974) relates the existence of an unconditional structure in a Banach space X to L_1-factorizations of 1-summing operators from X into a Hilbert space. This parameter became an indispensable tool in the study of unconditional structures of Banach spaces. Y. Gordon and D. R. Lewis [GL] gave the first examples of Banach spaces without G-L property. In particular, they studied the Schatten classes C_p^n of operators acting in ℓ_2^n obtained the estimate $g\ell(C_p^n) \geq cn^{|\frac{1}{p}-\frac{1}{2}|}$, valid for all $1 \leq p \leq \infty$, where $c > 0$ is a universal constant. Later T. Figiel, S. Kwapien and A. Pelczynski in 1977 [FKP] and then T. Figiel and W. B. Johnson in 1980 [FJ], by using a random technique, have shown that there exist finite-dimensional spaces for which $g\ell(X)$ and $\mathrm{unc}(X)$ have indeed the largest possible order. It follows that L_p-spaces, for $p > 2$, contain subspaces without G-L property; hence without unconditional basis.

The second parameter is a localization of the notion on unconditional basis, and more generally, of a Banach lattice. The definition of $l.u.st(X)$ (see [GL], c.f. e.g. [T-J]) which is used in studing Banach spaces in the present time allows us to look at the problem locally. In fact, it differs from the definition first introduced, and it is more suitable in the context.

In this dissertation, we do not discuss properties $g\ell$ and $l.u.st.$ Our approach is based on techniques first introduced by W. B. Johnson, J. Lindenstrauss and G. Schechtman [JLS] for studying the Kalton-Peck space [KP]. These techniques were refined by T. Ketonen [K] and later generalized further by A. Borzyszkowki [B]. The Kalton-Peck space was the first example of a Banach space without an unconditional basis which is an unconditional sum of two dimensional subspaces. Every space constructed in this dissertation has such a property. The essential idea, contained in above papers, is summarized in Proposition 2.1 (Chapter 3), which is a version of Proposition A in [B] and Lemma 3.1 in [K]. We give the proof of Proposition 2.1 as it contains a necessary estimation and it is slightly different and shorter than those which are presented in [B] and [K].

The novelty of a more general situation discussed here consists of the use of a familty of partitions of a basis. This concept appears in all our constructions. It is strong enough to give a general procedure of the construction of subspaces in many Banach spaces, including well-known examples of weak Hilbert spaces, which have no unconditional basis (Theorem 2.2, Propositions 4.2 and 4.3, Theorem 5.1 in Chapter 3).

CHAPTER 1

CHARACTERIZATIONS OF ℓ_p^n-SPACES IN TERMS OF OPERATOR IDEALS

1.1. Notations. Let $(E, \|\cdot\|)$ be a finite dimensional Banach space over either $\mathbb{R}$ or $\mathbb{C}$, and let $\|\cdot\|_2$ denote the Euclidean norm on E induced by the ellipsoid of maximal volume contained in the unit ball of E. Let $\langle\cdot,\cdot\rangle$ denote the induced inner product, let $\|\cdot\|_*$ be the norm on E, dual to the original norm $\|\cdot\|$, and let $i_{2E} : (E, \|\cdot\|_2) \to (E, \|\cdot\|)$ and $i_{E2} = (i_{2E})^{-1}$ be the formal identity operators.

Let $1 \le p < \infty$, and let X and Y be Banach spaces. For an operator $S : X \to Y$ set $\pi_p(S) = \inf c$, where the infimum is taken over all constants c such that

$$(1.1) \qquad \Big(\sum_j \|Sx_j\|^p\Big)^{1/p} \le c \sup_{\substack{x^* \in X^* \\ \|x^*\| \le 1}} \Big(\sum_j |\langle x_j, x^*\rangle|^p\Big)^{1/p}$$

for all finite sequences (x_j) in X; if no such a c exists, then $\pi_p(S) = \infty$. If $\pi_p(S) < \infty$, then S is said to be p-summing and $\pi_p(S)$ is called the p-summing norm of S. For $p = \infty$ put $\pi_\infty(S) = \|S\|$.

Next, an operator $S : X \to Y$ is called p-nuclear if S can be written in the form

$$S = \sum_j x_j^* \otimes y_j,$$

where (x_j^*) in X^* and (y_j) in Y satisfy $N_p((x_j^*),(y_j)) < \infty$. Here

$$N_p\big((x_j^*),(y_j)\big) = \Big(\sum_j \|x_j^*\|\Big)\Big(\sup_j \|y_j\|\Big) \quad \text{for } p = 1,$$

$$N_p\big((x_j^*),(y_j)\big) = \Big(\sum_j \|x_j^*\|^p\Big)^{\frac{1}{p}} \sup_{\substack{y^* \in Y^* \\ \|y^*\| \le 1}} \Big(\sum_j |\langle y_j, y^*\rangle^{p'}\Big)^{\frac{1}{p'}} \quad \text{for } 1 < p < \infty,$$

$$N_p\big((x_j^*),(y_j)\big) = \Big(\sup_j \|x_j^*\|\Big) \sup_{\substack{y^* \in Y^* \\ \|y^*\| \le 1}} \Big(\sum_j |\langle y_j, y^*\rangle|\Big) \quad \text{for } p = \infty.$$

Each such representation of S is called a p-nuclear representation. The p-nuclear norm, $\nu_p(S)$, is denoted by

$$\text{(1.2)} \qquad \nu_p(S) = \inf N_p\big((x_j^*),(y_j)\big),$$

where the infimum is taken over all p-nuclear representations of S.

Let $1 \le p < \infty$. It is well known, that the p-summing norm π_p and the p'-nuclear norm $\nu_{p'}$ are in trace duality. It means that if u is an operator $(u : X \to Y)$, then

$$\text{(1.3)} \qquad \pi_p(u) = \sup\{|\operatorname{trace} u\,\omega| \mid \omega : Y \to X,\ \nu_{p'}(\omega) \le 1\}$$

(cf. e.g. [T-J], p. 52).

Now, let us state a fundamental characterization of p-summing operators given by Pietsch in 1967 [Pch] (cf. also, [T-J], p. 47).

Theorem 1.1. *Let $1 \le p < \infty$. Let X and Y be Banach spaces. Let $K = \{x^* \in X^* \mid \|x^*\| \le 1\}$, equipped with the compact topology $\sigma(X^*, X)$.*

(i) Let $u : X \to Y$ *be a* p*-summing operator; let* $C = \pi_p(u)$*. There exists a probability measure* μ *on* K *such that*

$$(1.4) \qquad \|ux\| \leq C\Big(\int_K |\langle x, \xi\rangle|^p \, d\mu(\xi)\Big)^{\frac{1}{p}} \qquad \text{for } x \in X.$$

(ii) Conversely, any operator $u : X \to Y$ *satisfying (1.4) with some constant* C *is* p*-summing and* $\pi_p(u) \leq C$*.*

The next proposition establishes a factorization of p-nuclear operators (cf. [T-J], p. 41).

PROPOSITION 1.2. *Let* $1 \leq p \leq \infty$*. Let* X *and* Y *be Banach spaces. An operator* $u : X \to Y$ *is* p*-nuclear if and only if* u *has a factorization* $u = v_2 \Delta v_1$ *:*

$$(1.5) \qquad X \xrightarrow[v_1]{} \ell_\infty \xrightarrow[\Delta]{} \ell_p \xrightarrow[v_2]{} Y,$$

where Δ *is a diagonal compact operator. Moreover,*

$$\nu_p(u) = \inf \|v_2\| \, \|\Delta\| \, \|v_1\|,$$

where the infimum is taken over all factorizations (1.5).

Sometimes, in the case when E is an n-dimensional Banach space, we will use the following abbreviations:

$$\pi_p(E) = \pi_p(id : E \to E),$$

$$\nu_p(E) = \nu_p(id : E \to E).$$

For a real valued random variable ξ on a probability space (Ω, P), $\mathsf{E}\xi$ denotes the expected value of ξ.

Let $\gamma_1, \dots, \gamma_n$ denote real or complex standard Gaussian random variables on (Ω, P). For $s \geq 1$ set $A_s = (\mathsf{E}|\gamma_1|^s)^{1/s}$. For any orthonormal basis (e_i) in ℓ_2^n, let $\mathbf{X}$ denote the ℓ_2^n-valued random variable defined by

$$\mathbf{X} = \sum_{i=1}^{n} \gamma_i e_i. \tag{1.6}$$

Notice that the distribution of $\mathbf{X}$ does not depend on a choice of the basis (e_i).

Finally, for isomorphic Banach spaces X and Y, the quantity $d(X,Y) = \inf \|T\|\,\|T^{-1}\|$, with the infimum taken over all isomorphisms from X onto Y, is called the Banach-Mazur distance.

In the case when F is an n-dimensional Banach space, we will use the following abbreviation: $d_F = d(F, \ell_2^n)$.

1.2. Preliminaries on p-summing norms. We start by stating a simple observation which will be often used throughout the chapter. It follows directly from the definition (1.1) of the p-summing norms (cf. e.g., [T-J]) or from Theorem 1.1.

PROPOSITION 2.1. *Let $1 \leq p < \infty$. Let T be an operator between two Banach spaces X and Y.*

(i) Suppose that there are functionals $x_1^*, x_2^*, \dots \in X^*$ such that

$$\|Tx\|^p \leq \sum_j |\langle x_j^*, x\rangle|^p \qquad \text{for all} \quad x \in X.$$

Then $\pi_p(T) \leq \left(\sum_j \|x_j^*\|^p\right)^{1/p}$.

(ii) Let ξ *be a random variable on a probability space* (Ω, P) *with values in* $(X^*, \sigma(X^*, X))$. *Suppose that* $\|Tx\|^p \leq \mathsf{E}|\langle \xi, x\rangle|^p$ *for all* $x \in X$. *Then* $\pi_p(T) \leq (\mathsf{E}\|\xi\|^p)^{1/p}$.

Recall that X is the ℓ_2^n-valued random variable defined in (1.6). It is easy to calculate that

$$\|x\|_2 = A_s^{-1}\big(\mathsf{E}|\langle \mathsf{X}, x\rangle|^s\big)^{1/s} \qquad \text{for} \quad s \geq 1. \tag{2.1}$$

Now, let us give some simple conclusions from Proposition 2.1 which we will need further.

PROPOSITION 2.2. *The following equalities are true.*

(i) $\pi_p(id : \ell_2^n \to \ell_2^n) = A_p^{-1}(\mathsf{E}\|\mathsf{X}\|_2^p)^{1/p}, \qquad 1 \leq p \leq \infty,$

(ii) $\pi_p(id : \ell_{p'}^n \to \ell_2^n) = n^{1/p}, \qquad 1 \leq p \leq 2,$

(iii) $\pi_p(id : \ell_2^n \to \ell_p^n) = n^{1/p}, \qquad 1 \leq p \leq \infty,$

(iv) $\pi_{p'}(id : \ell_p^n \to \ell_2^n) = n^{1/p'}, \qquad 1 \leq p \leq 2,$

(v) $\pi_p(id : \ell_{p'}^n \to \ell_\infty^n) = n^{1/p}, \qquad 1 \leq p \leq \infty.$

Equality (i) was proved, in a slightly different formulation by D. J. H. Garling [G2] (cf. also, [T-J], p. 60).

Other equalities are well-known to specialists. For sake of the completeness, we give the proof.

PROOF:(i) The upper estimate follows from (2.1) and Proposition 2.1 (ii). For the lower estimate, it is enough to observe that

$$(\mathsf{E}\|\mathsf{X}\|_2^p)^{1/p} = A_p^{-1}(\mathsf{E}\|\mathsf{X}\|_2^p)^{1/p} \cdot \sup_{\|x^*\|_2=1} (\mathsf{E}|\langle x^*, \mathsf{X}\rangle|^p)^{1/p}.$$

(ii) For fixed $x^* \in \ell_{p'}^n$ one has

$$\|id(x^*)\|_2 \leq \left(\sum_{i=1}^{n} |\langle x^*, e_i\rangle|^p\right)^{1/p}.$$

By using Proposition 2.1 (i), we conclude that $\pi_p(id : \ell_{p'}^n \to \ell_2^n) \leq n^{1/p}$.

On the other hand, we have

$$\left(\sum_{i=1}^{n} \|id(e_i)\|_2^p\right)^{1/p} = n^{1/p} \sup_{\|y\|_p=1} \left(\sum_{i=1}^{n} |\langle y, e_i\rangle|^p\right)^{1/p}$$

which concludes the proof of (ii).

(iii) Fix $x \in \ell_2^n$. Then $\|id(x)\|_p = (\sum_{i=1}^{n} |\langle x, e_i\rangle|^p)^{\frac{1}{p}}$; hence

$$\pi_p(id : \ell_2^n \longrightarrow \ell_p^n) \leq n^{1/p}.$$

Next, observe that

$$(E\|X\|_p^p)^{1/p} = n^{1/p} \sup_{\|y\|_2=1} (E|\langle y, X\rangle^p)^{1/p}.$$

(iv) Fix $x \in \ell_p^n$. Then

$$\|id(x)\|_2 = \Big(\sum_{\overline{\epsilon_i}=\underbrace{(\pm 1, \ldots, \pm 1)}_{n \text{ times}}} \frac{1}{2^n}|\langle x, \overline{\epsilon}_i\rangle|^2\Big)^{1/2}$$

$$\leq \Big(\sum_{\overline{\epsilon_i}=(\pm 1, \ldots, \pm 1)} \frac{1}{2^n}|\langle x, \overline{\epsilon_i}\rangle|^{p'}\Big)^{1/p'}.$$

Again, by Proposition 2.1 (i), we obtain

$$\pi_{p'}(id : \ell_p \to \ell_2) \leq \Big(\frac{1}{2^n} \sum_{\overline{\epsilon_i}=(\pm 1, \ldots, \pm 1)} \|\overline{\epsilon_i}\|_{p'}^{p'}\Big)^{1/p'} = n^{1/p'}.$$

Conversely,

$$\Big(\sum_{i=1}^{n}\|e_i\|_2^{p'}\Big)^{1/p'} = n^{1/p'} \sup_{\|y\|_{p'}=1} \Big(\sum_{i=1}^{n}|\langle y, e_i\rangle|^{p'}\Big)^{1/p'}.$$

(v) Similarly as before, for $x \in \ell_{p'}^n$, one has

$$\|id(x)\|_\infty \le \|x\|_p = \Big(\sum_{i=1}^{n}|\langle x, e_i\rangle|^p\Big)^{1/p};$$

thus

$$\pi_p\big(id : \ell_{p'}^n \longrightarrow \ell_\infty^n\big) \le n^{1/p}.$$

The opposite inequality follows from the equation:

$$\Big(\sum_{i=1}^{n}\|id(e_i)\|_\infty^p\Big)^{1/p} = n^{1/p} \sup_{\|x\|_p=1} \big(\sum_{i=1}^{n}|\langle x, e_i\rangle|^p\big)^{1/p}.$$

□

As an interesting consequence, we get an isometric characterization of ℓ_2^n as follows.

COROLLARY 2.3. *Let $1 \le p < \infty$. An n-dimensional Banach space E is isometric to ℓ_2^n if and only if*

$$\pi_p\big((i_{2E})^*\big) = \pi_p(id : \ell_2^n \to \ell_2^n).$$

PROOF:By (2.1) and Proposition 2.1 (ii), we obtain

$$\pi_p\big((i_{2E})^*\big) \le A_p^{-1}(\mathsf{E}\|\mathsf{X}\|^p)^{1/p}.$$

Since $\|x\| \le \|x\|_2$ for every $x \in E$, by Proposition 2.2 (i), it follows that

$$A_p^{-1}(\mathbb{E}\|\mathsf{X}\|^p)^{1/p} \le A_p^{-1}(E\|\mathsf{X}\|_2^p)^{1/p}$$

$$= \pi_p(id : \ell_2^n \to \ell_2^n) = \pi_p\big((i_{2E})^*\big).$$

Combining the two estimates, we see that $\|\mathsf{X}(\omega)\| = \|\mathsf{X}(\omega)\|_2$ almost everywhere. Hence, by the continuity, $\|\mathsf{X}(\omega)\| = \|\mathsf{X}(\omega)\|_2$ for every $\omega \in \Omega$, completing the proof. □

REMARK. For an n-dimensional Banach space E one has $\pi_2\big((i_{2E})^*\big) \le \sqrt{n}\,\|(i_{2E})^*\| = \sqrt{n}$. Corollary 2.3 says in particular that if the 2-summing norm of the operator $(i_{2E})^*$ is maximal, then E is isometric to ℓ_2^n.

1.3. Characterizations of ℓ_p^n in terms of ideal norms of the identity operator. In this section, we present characterizations of ℓ_p^n in terms of p'-summing and p-nuclear norms of the identity operator on the space.

The definition of type p and cotype q constants, T_p' and C_q', respectively, used here, differ from the usual ones by replacing the L_2-Rademacher averages by the L_p- and L_q-averages respectively (cf. e.g. [T-J], p. 14). Namely, for $1 \le p \le 2 \le q < \infty$, we define T_p', C_q' to be the smallest constants such that, for arbitrary vectors $x_1, \ldots, x_n$ in a given Banach space X, one has

$$\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|^p dt\Big)^{1/p} \le T_p'\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p},$$

$$C_q'\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|^q dt\Big)^{1/q} \ge \Big(\sum_{j=1}^n \|x_j\|^q\Big)^{1/q}.$$

The main result of the section states:

THEOREM 3.1. *Let E be an n-dimensional Banach space. Let $1 \leq p < 2$. The following are equivalent:*

(i) $\pi_p(E) \geq n^{1/p}$,

(ii) there exist vectors $e_1, \dots, e_n \in E$ *such that, for every choice of scalars* $a_1, \dots, a_n$, *one has*

$$\max_{i=1,\dots,n} |a_i| \leq \|\sum_{i=1}^{n} a_i e_i\| \leq \Big(\sum_{i=1}^{n} |a_i|^{p'}\Big)^{1/p'},$$

(iii) $\nu_{p'}(E) \leq n^{1/p'}$.

Furthermore, E is isometric to $\ell_{p'}^n$ if and only if E satisfies one of the above conditions, and $C'_{p'}(E) = 1$.

For $p = 1$, implication (i) $\Longrightarrow$ (ii) was proved in [D] and [G1]; implication (iii) $\Longrightarrow$ (ii) is the isometric version of a classical P_λ problem, proved by Nachbin [N].

The proof of the theorem is based on several results of independent interest. Proposition 3.2 below is crucial for further investigation. It involves the operator i_{E2} associated to the ellipsoid of maximal volume. Case $p = 1$ was proved in [FJ] (cf. also, [T-J], p. 266).

PROPOSITION 3.2. *Let $1 \leq p < 2$. Let E be an n-dimensional Banach space such that*

$$\pi_p(i_{E2}) \geq n^{1/p}.$$

Then there exists an orthonormal basis $(e_j)_{j=1}^n$ in $(E, \|\cdot\|_2)$ such that $\|e_j\| = \|e_j\|_* = \|e_j\|_2 = 1$, $j = 1, \dots, n$.

PROOF OF PROPOSITION 3.2: By the well-known John's result (cf., e.g., [T-J], p. 118), there exist a positive integer N, vectors $x_1,\dots,x_N$ in E and positive scalars $c_1,\dots,c_N$ such that $\|x_j\| = \|x_j\|_* = 1$ $(j = 1,\dots,N)$, $\sum_{j=1}^{N} c_j = n$ and $x = \sum_{i=1}^{N} c_i\langle x, x_j\rangle x_j$ for $x \in E$.

We need the following Lemmas.

LEMMA 3.3. *Assume that $x_1,\dots,x_N$ and $c_1,\dots,c_N$ are as above. Let $M \subset \{1,\dots,N\}$ be a subset such that $\sum_{j\in M} c_j = m$, for some positive integer, and that*

$$\langle x_s, x_j\rangle = 0 \qquad \text{for} \quad s \notin M,\ j \in M.$$

Let $F_M = \operatorname{span}(x_j)_{j\in M}$ and let $P : (E, \|\cdot\|_2) \to (E, \|\cdot\|_2)$ be the orthogonal projection onto F_M. If $\pi_p(Pi_{E2}) \geq m^{1/p}$, then there is a subset $J \subset M$ with $|J| = m$ such that $\langle x_i, x_j\rangle = 0$ for $i \neq j$, $i, j \in J$.

Obviously, Proposition 3.2 follows from Lemma 3.3 applied for $M = \{1,\dots,N\}$ and $m = n$.

LEMMA 3.4. *Let $1 \leq p \leq 2$. Let X be a Banach space, let H be a Hilbert space, and let $T : X \to H$ be a linear operator. Suppose that A and B are orthogonal subspaces of H, and that P, Q are orthogonal projections on A and B, respectively. Then*

$$\pi_p((P+Q)T) \leq \left(\pi_p^p(PT) + \pi_p^p(QT)\right)^{1/p}.$$

PROOF OF LEMMA 3.4: Fix $x_1,\dots,x_n \in X$. We have

$$\left(\sum_{i=1}^{n} \|(P+Q)Tx_i\|^p\right)^{1/p} = \left(\sum_{i=1}^{n} \left(\|PTx_i\|^2 + \|QTx_i\|^2\right)^{p/2}\right)^{1/p}$$

$$\le \Big(\sum_{i=1}^{n} \|PTx_i\|^p + \|QTx_i\|^p \Big)^{1/p}$$

$$\le \left(\pi_p^p(PT) + \pi_p^p(QT)\right)^{1/p} \cdot \sup_{\substack{\|x^*\|=1 \\ x^* \in X^*}} \Big(\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p \Big)^{1/p}.$$

Using the definition of p-summing norm (see (1.1)), we conclude the required inequality. □

PROOF OF LEMMA 3.3: Proceeding by induction, we assume that $m > 1$ and that the Lemma is true for $m-1$. Pick a vector $y \in F_M$ such that $a = \sum_{j \in M} c_j |\langle y, x_j \rangle|^2$ is a maximal subject to $\sum_{j \in M} c_j |\langle y, x_j \rangle|^p = 1$.

Since $\|y\|_2^2 = \sum_{j=1}^{N} c_j |\langle y, x_j \rangle|^2 \le \sum_{j=1}^{N} c_j |\langle y, x_j \rangle|^p \|y\|_2^{2-p}$, it follows that $a \le 1$. On the other hand, for every $x \in E$ one has

$$\|Px\|_2 = \Big(\sum_{j \in M} c_j |\langle Px, x_j \rangle|^2\Big)^{1/2} \le a^{1/2} \Big(\sum_{j \in M} c_j |\langle Px, x_j \rangle|^p\Big)^{1/p}$$

$$\le a^{1/2} \Big(\sum_{j \in M} c_j |\langle x, x_j \rangle|^p\Big)^{1/p};$$

hence

$$m^{1/p} \le \pi_p(Pi_{E2}) \le a^{1/2} \Big(\sum_{j \in M} c_j \Big)^{1/p} = a^{1/2} m^{1/p} \qquad \text{and} \qquad a = 1.$$

Next, since $\big(\sum_{j \in M} c_j |\langle y, x_j \rangle|^2\big)^{1/2} = \big(\sum_{j \in M} c_j\, |\langle y, x_j \rangle|^p\big)^{1/p} = 1$ and $|\langle y, x_j \rangle| \le 1$, it follows that there exists a subset $K \subset M$ such that

$$|\langle y, x_j \rangle| = \begin{cases} 1 & \text{for } \ s \in K \\ 0 & \text{for } \ s \in (1, \ldots, N) \setminus K. \end{cases}$$

Let $k_0 \in K$. Then for every $k \in K$, $x_k = \epsilon_k x_{k_0}$ with $|\epsilon_k| = 1$. Indeed, for $k \in K$ and $k \neq k_0$, define a functional φ on E as follows

$$\varphi = \frac{1}{2}\big(\overline{\langle y, x_k\rangle}\, x_k + \overline{\langle y, x_{k_0}\rangle}\, x_{k_0}\big).$$

Then

$$1 = \langle y, \varphi\rangle \leq \|y\|_2\, \|\varphi\|_2 \leq \frac{1}{2}\big(\|x_k\|_2 + \|x_{k_0}\|_2\big) = 1.$$

This implies that

$$\|\overline{\langle y, x_k\rangle}\, x_k + \overline{\langle y, x_{k_0}\rangle}\, x_{k_0}\|_2 = \|\overline{\langle y, x_k\rangle}\, x_k\|_2 + \|\overline{\langle y, x_{k_0}\rangle} x_{k_0}\|_2.$$

Hence $\overline{\langle y, x_{k_0}\rangle}\, x_{k_0} = \overline{\langle y, x_k\rangle}\, x_k$ and

$$x_k = \varepsilon_k\, x_{k_0} \qquad \text{with} \qquad |\varepsilon_k| = 1.$$

We may assume that $y = x_{k_0}$.

Put $M_1 = M \setminus K$. Then $\langle y, x_i\rangle = 0$ for $i \in M_1$. In addition, $\langle x_s, x_k\rangle = 0$ for $s \in M_1$, $k \notin M_1$, and $\sum\limits_{i \in M_1} c_i = m - 1$.

Finally, if $Q :$ $\|_2) \to (E, \|\ \|_2)$ is the orthogonal projection onto $F_{M_1} = \operatorname{span}(x_i)_{i \in M_1}$, then

$$\pi_p(Q i_{E2}) \geq (m-1)^{1/p}.$$

Indeed, for every $x \in E$ one has

$$\|(P-Q)x\|_2 = \Big(\sum_{j=1}^{N} c_j |\langle (P-Q)x, x_j\rangle|^2\Big)^{1/2}$$

$$= \Big(\sum_{j \in M} c_j |\langle (P-Q)x, x_j \rangle|^2 \Big)^{1/2}$$

$$\leq a^{1/2} \Big(\sum_{j \in M} c_j |\langle (P-Q)x, x_j \rangle|^p \Big)^{1/p}$$

$$\leq \Big(\sum_{j \in K} c_j |\langle x, x_j \rangle|^p \Big)^{1/p}.$$

By applying Proposition 2.1(i), we get

$$\pi_p((P-Q)i_{E2}) \leq \Big(\sum_{j \in K} c_j \Big)^{1/p} = 1.$$

Moreover, by Lemma 3.4, we have

$$\pi_p(Pi_{E2}) \leq \big(\pi_p((P-Q)i_{E2})^p + \pi_p(Qi_{E2})^p\big)^{1/p};$$

which shows the required inequality.

The inductive hypothesis applied to the subset M_1 and the projection Q yields: there is a subset $J_0 \subset M_1$ with $|J_0| = m-1$ such that $\langle x_j, x_i \rangle = \delta_{ij}$, $i, j \in J_0$. Then $J_0 \cup \{k_0\}$ obviously satisfies the condition of Lemma 3.3.

□

In order to prove the next proposition we require the following lemma.

LEMMA 3.5. *Let* $1 \leq p \leq q \leq \infty$ *and let* $(E, \|\cdot\|)$ *be a normed space. Then for every choice of vectors* $x_1, \ldots, x_n \in X$ *the following inequality holds*

$$\Big(\sum_{i=1}^n \|x_i\|^q \Big)^{1/q} \leq \Big(\alpha \sum_{i=1}^n \|x_i\|^p + \beta \|\sum_{i=1}^n x_i\|^p \Big)^{1/p}, \tag{3.1}$$

where $\alpha = 2^{p/q-1}$ *and* $\beta = 1 - \alpha$.

PROOF:The lemma is obvious for $n = 1$. Proceeding by induction assume that the lemma is true for $n-1$. Without loss of generality, we may assume that $1 < p < q < \infty$, $\sum_{i=2}^{n} \|x_i\|^q = 1$ and $0 < \|x_1\| \le \cdots \le \|x_n\|$. To prove (3.1) it is enough to check the following stronger inequality:

$$(3.2) \qquad \Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \le \Big(\alpha \sum_{i=1}^{n} \|x_i\|^p + \beta \Big| \|x_1\| - \|\sum_{i=2}^{n} x_i\| \Big|^p\Big)^{1/p}.$$

Next, let us introduce the following notations:

$$(w)^s = \operatorname{sign}(w) \cdot |w|^s \qquad \text{for} \quad s > 1, \ w \in \mathbf{R};$$

$$A = \sum_{i=2}^{n} \|x_i\|^p;$$

$$a = \|\sum_{i=2}^{n} x_i\|;$$

$$\|x_1\| = t \in [0,1].$$

Observe that, in the above terms, the following formulas are true:

$$\frac{d}{dt}|w|^p = p(w)^{p-1} \qquad \text{and} \qquad w(w)^{p-1} = |w|^p.$$

Finally, we can rewrite the inequality (3.2) in the following way:

$$(3.3) \qquad 0 \le f(t) = \alpha(t^p + A) + \beta|t - a|^p - (t^q + 1)^{p/q}, \quad \text{where} \ \ t \in [0,1].$$

To prove (3.3) observe that $f(0) \ge 0$ (by inductive hypothesis) and

$$\begin{aligned} f(1) &= \alpha(1 + A) + \beta|1 - a|^p - (1+1)^{p/q} \\ &\ge 2\alpha - 2^{p/q} = 2 \cdot 2^{p/q-1} - 2^{p/q} = 0 \end{aligned}$$

(since $A \geq 1$).

Let us suppose the contrary. There exists $t \in (0,1)$ such that $f'(t) = 0$ and $f(t) < 0$. Then

$$\begin{aligned}
0 &= p^{-1}(t-a) \cdot f'(t) \\
&= (t-a)[\alpha t^{p-1} + \beta(t-a)^{p-1} - t^{q-1}(t^q+1)^{p/q-1}] \\
&= \alpha(t-a)t^{p-1} - \alpha(t^p + A) + f(t) + (t^q+1)^{p/q} - t^{q-1}(t-a)(t^q+1)^{p/q-1} \\
&< -\alpha(at^{p-1} + A) + (t^q+1)^{p/q-1}[1 + t^{q-1}a] \\
&\leq (t^q+1)^{p/q-1}[-at^{p-1} - A + 1 + t^{q-1}a] \qquad (\text{since } (t^q+1)^{p/q-1} < \alpha) \\
&\leq (t^q+1)^{p/q-1}a(t^{q-1} - t^{p-1}).
\end{aligned}$$

Summarize, $0 < (t^q+1)^{p/q-1}a(t^{q-1} - t^{p-1})$ which gives $t^{q-1} > t^{p-1}$ and $p > q$. This is contradictory to the assumption and completes the proof of the lemma.

□

PROPOSITION 3.6. *Let $(E, \|\cdot\|)$ be an n-dimensional Banach space and let $1 \leq p < q \leq \infty$. Suppose that there exist vectors, $e_1, \ldots, e_n \in E$ and $e_1^*, \ldots, e_n^* \in E^*$, such that $\langle e_j^*, e_i \rangle = \delta_{ij}$ and $\|e_i\| = \|e_i^*\| = 1$, $i = 1, \ldots, n$. Consider on E the ℓ_q^n norm, say $\|\cdot\|_q$, induced by the basis $(e_i)_{i=1}^n$. Let i_{Eq} denote the formal identity operator from $(E, \|\cdot\|)$ to $(E, \|\cdot\|_q)$.*

If $\pi_p(i_{Eq}) \geq n^{1/p}$, then for every $a_1, \ldots, a_n \in \mathbb{C}$ one has

$$\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^n a_i e_i^*\Big\|_*.$$

PROOF: We suppose that $q < \infty$. In the case $q = \infty$, the proof is similar. First, we will show that

$$\pi_p(Ti_{Eq}) = \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p}, \qquad \text{where} \quad T = \sum_{i=1}^{n} a_i e_i^* \otimes e_i. \tag{3.4}$$

Fix $x \in E$. Then $\|Ti_{Eq}x\|_q \le (\sum_{i=1}^{n} |a_i|^p |\langle x, e_i^*\rangle|^p)^{1/p}$; hence

$$\pi_p(Ti_{Eq}) \le \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p}. \tag{3.5}$$

To see opposite inequality, choose $g_i \in \mathbb{C}$ $(i = 1, \dots, n)$ such that

$$\max_i |a_i| = (|a_i|^p + |g_i|^p)^{1/p}.$$

Define an operator

$$S : E \to E, \qquad S = \sum_{i=1}^{n} g_i e_i^* \otimes e_i.$$

Then for every $x = (x_1, \dots, x_n) \in \ell_q^n$ one has

$$\begin{aligned}
\max |a_i|^p \|x\|_q^p &= \Big(\sum_{i=1}^{n} (|a_i x_i|^p + |g_i x_i|^p)^{q/p}\Big)^{p/q} \\
&\le \Big(\sum_{i=1}^{n} |a_i x_i|^{p\cdot\frac{q}{p}}\Big)^{p/q} + \Big(\sum_{i=1}^{n} |g_i x_i|^{p\cdot\frac{q}{p}}\Big)^{p/q} \\
&= \|Tx\|_q^p + \|Sx\|_q^p .
\end{aligned}$$

Hence

$$\max |a_i| \, \|x\|_q \le \big(\|Tx\|_q^p + \|Sx\|_q^p\big)^{1/p}. \tag{3.6}$$

Next, using definition (1.1) and (3.6), we obtain

$$\max\ |a_i|\pi_p(i_{Eq}) \leq \left(\pi_p^p(Ti_{Eq}) + \pi_p^p(Si_{Eq})\right)^{1/p}.$$

By above estimation, and by (3.5) for T and S, respectively, we have the following:

$$n^{1/p}\max\ |a_i| \leq \max|a_i|\pi_p(i_{Eq}) \leq [\pi_p^p(Ti_{Eq}) + \pi_p^p(Si_{Eq})]^{1/p}$$

$$\leq \left[\sum_{i=1}^{n}|a_i|^p + \sum_{i=1}^{n}|g_i|^p\right]^{1/p} = n^{1/p}\max\ |a_i|;$$

thus (3.4) holds as required.

By using Lemma 3.5, it follows that

$$\|Tx\|_q = \left(\sum_{i=1}^{n}|\langle Tx, e_i^*\rangle|^q\right)^{1/q}$$

$$\leq \left(\alpha\sum_{i=1}^{n}|\langle Tx, e_i^*\rangle|^p + \beta\left|\sum_{i=1}^{n}\langle Tx, e_i^*\rangle\right|^p\right)^{1/p}$$

$$= \left(\alpha\sum_{i=1}^{n}|a_i|^p|\langle x, e_i^*\rangle|^p + \beta\left|\langle x, \sum_{i=1}^{n}a_ie_i^*\rangle\right|^p\right)^{1/p}.$$

Finally, the condition (3.4) and Proposition 2.1(i) show

$$\left(\alpha\sum_{i=1}^{n}|a_i|^p + \beta\left\|\sum_{i=1}^{n}a_ie_i^*\right\|_*^p\right)^{1/p} \geq \pi_p(Ti_{Eq}) = \left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}$$

completing the proof. □

Now, we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1: The fact that (i) implies (ii) follows from: Proposition 3.6 for $q = 2$, Proposition 3.2, and the inequality

$$\pi_p(i_{E2}) \geq \pi_p(E) \geq n^{1/p}.$$

Next, condition (ii) implies that the following factorization holds

$$E \xrightarrow[V_1]{} \ell_\infty^n \xrightarrow[\Delta]{} \ell_{p'}^n \xrightarrow[V_2]{} E$$

with $\nu_{p'}(E) \leq \|V_1\| \, \|\Delta\| \, \|V_2\| \leq n^{1/r}$ (see Proposition 1.2). Finally, since

$$n = \text{trace}(id : E \to E) \leq \pi_p(E)\nu_{p'}(E),$$

it follows that (iii) implies (i). Note that π_p and $\nu_{p'}$ are in trace duality (see (1.3)).

Before we pass to the second part of the theorem, observe that

$$C'_{p'}(E) = 1 \qquad \text{iff} \qquad T'_p(E^*) = 1.$$

This can be checked directly for two vectors, and by induction for more vectors. Suppose that $\pi_p(E) \geq n^{1/p}$; so, $\pi_p(i_{E2}) \geq n^{1/p}$. By using Proposition 2.1(ii) and (2.1), we obtain

$$\begin{aligned} n^{1/p} &\leq \pi_p(i_{E2}) \leq A_p^{-1}(\mathsf{E}\|\mathsf{X}\|_*^p)^{1/p} \\ &= A_p^{-1}\left(\mathsf{E}\int_0^1 \Big\|\sum_{i=1}^n r_i(t)\gamma_i e_i\Big\|_*^p \, dt\right)^{1/p} \\ &\leq A_p^{-1}\left(\mathsf{E}\sum_{i=1}^n |\gamma_i|^p \|e_i\|_*^p\right)^{1/p} = n^{1/p}. \end{aligned}$$

Therefore, $A_p^{-1}(\mathsf{E}\|\mathsf{X}\|_p^p)^{1/p} = n^{1/p} = A_p^{-1}(\mathsf{E}\|\mathsf{X}\|_*^p)^{1/p}$. Since $\|x\|_p \le \|x\|_*$ for every $x \in E$, it follows that $\|\cdot\|_p = \|\cdot\|_*$, as in the proof of Corollary 2.3. □

1.4. The ellipsoid of maximal volume and other characterizations of ℓ_p^n spaces. In this section, we give some characterizations of ℓ_p^n space in terms of p-summing norms of an operator associated with the ellipsoid of maximal volume contained in the unit ball of E.

Before we start, let us introduce some new notations. Let $i_{E\infty} : (E, \|\cdot\|) \to (E, \|\cdot\|_\infty)$ denote the formal identity operator, where the norm $\|\cdot\|_\infty$ is given by a fixed Auerbach system on E. Similarly, we define $i_{E^*\infty} : (E^*, \|\cdot\|_*) \to (E, \|\cdot\|_\infty)$. Finally, let

$$i_{E^*2} = (i_{2E})^* \qquad \text{and} \qquad i_{2E^*} = (i_{E2})^*.$$

THEOREM 4.1. *Let $1 \le p < 2$. Let E be an n-dimensional linear space. The following are equivalent:*

(i) E^* *is isometric to* ℓ_p^n,

(ii) $\pi_p(i_{E2}) \ge n^{1/p}$ and $\pi_{p'}(i_{E^*2}) \ge n^{1/p'}$,

(iii) $\pi_p(i_{E2}) \ge n^{1/p}$ and $\pi_p(i_{2E^*}) \le n^{1/p}$.

Moreover, for $1 \le p < \infty$, condition (i) is equivalent to

(iv) $\pi_p(i_{E\infty}) \ge n^{1/p}$ and $\pi_{p'}(i_{E^*\infty}) \ge n^{1/p'}$.

PROOF:By Proposition 2.2, we see that the condition (i) implies (ii), (iii) and (iv).

First, suppose that

$$\pi_p(i_{E2}) \ge n^{1/p}.$$

By Proposition 3.2 and Proposition 3.6, we conclude that

$$\|x\| \leq \|x\|_{p'} \qquad \text{for} \quad x \in E. \tag{4.1}$$

Now, assume that

$$\pi_{p'}(i_{E^*2}) \geq n^{1/p'}.$$

Applying (4.1) and Proposition 2.1(ii) to (2.1) for $s = p'$, it follows that

$$n^{1/p'} \leq \pi_{p'}(i_{E^*2}) \leq A_{p'}^{-1}(\mathsf{E}\|\mathsf{X}\|^{p'})^{1/p'}$$

$$\leq A_{p'}^{-1}(\mathsf{E}\|\mathsf{X}\|_{p'}^{p'})^{1/p'} = n^{1/p'}.$$

Hence $\|\cdot\|_{p'} = \|\cdot\|$, as in the proof of Corollary 2.3.

Next, let us suppose that (iii) holds. Again, by (2.1) for $s = p$, we obtain

$$n^{1/p} \leq \pi_p(i_{E2}) \leq A_p^{-1}(\mathsf{E}\,\|\mathsf{X}\|_*^p)^{1/p}. \tag{4.2}$$

By using Theorem 1.1, one can find a probability measure μ on $S_2^{n-1} = \{x : \|x\|_2 = 1\}$ such that

$$\|x^*\|_* \leq n^{1/p}\Big(\int_{S_2^{n-1}} |\langle y, x^*\rangle|^p d\mu(y)\Big)^{1/p} \qquad \text{for} \quad x^* \in E^*.$$

By (4.2) and the above inequality, it follows that

$$n^{1/p} \leq A_p^{-1}(\mathsf{E}\,\|\mathsf{X}\|_*^p)^{1/p} \leq n^{1/p}A_p^{-1}\Big(\mathsf{E}\int_{S_2^{n-1}} |\langle y, \mathsf{X}\rangle|^p d\mu(y)\Big)^{1/p} = n^{1/p}.$$

Therefore, $\mathsf{E}\,\|\mathsf{X}\|_*^p = \mathsf{E}\,\|\mathsf{X}\|_p^p$ and $\|\cdot\|_* = \|\cdot\|_p$, as before.

Finally, by using Proposition 3.6 for $q = \infty$, we conclude from (iv) that

$$\|x^*\|_p \leq \|x^*\|_* \qquad \text{for} \qquad x^* \in E^*;$$

hence

$$\|x\|_{p'} \leq \|x\| \qquad \text{for} \qquad x \in E.$$

This implies that E is isometric to $\ell_{p'}^n$, completing the proof. □

CHAPTER 2

FINITE DIMENSIONAL SUBSPACES OF L_p

In this chapter, we apply Theorem 3.1 from Chapter 1 to subspaces of L_p. We also get a characterization of n-dimensional subspaces of L_p with the maximal Euclidean distance.

COROLLARY 1.1. *Let E be an n-dimensional subspace of $L_p(\Omega,\mu)$. Then E is isometric to ℓ_p^n if and only if $\pi_{p'}(E) \geq n^{1/p'}$ for $2 < p < \infty$ or $\pi_p(E^*) \geq n^{1/p}$ for $1 \leq p < 2$.*

The corollary follows immediately from Theorem 3.1 and the fact that $T_p'(L_p(\Omega,\mu)) = 1$.

PROPOSITION 1.2. *Fix n and $2 < p < \infty$. Then any n-dimensional subspace E of $L_p(\Omega,\mu)$ whose Euclidean distance is maximal, i.e., $d(E,\ell_2^n) = n^{1/2-1/p}$, is isometric to ℓ_p^n.*

For $1 < p < 2$, an analogous result was proved in [BT].

The proof of Proposition 1.2 is based on well-known result of D. R. Lewis [L] which states:

Proposition 1.3. *Fix n and $1 < p < \infty$. Then for any n-dimensional subspace E of $L_p(\Omega,\mu)$ there exist vectors $f_1,\dots,f_n \in E$ such that*

$$\int f_i\overline{f_j}F^{p-2}d\mu = \delta_{ij}, \qquad \text{where} \quad F = \Big(\sum_{i=1}^n |f_i|^2\Big)^{1/2}. \tag{1.1}$$

Proof of Proposition 1.2: Fix an arbitrary n-dimensional subspace E of $L_p(\Omega,\mu)$. First, we follow Lewis' argument from [L]. Observe that (1.1) implies the following

$$\int \Big|\sum_{i=1}^n a_i f_i\Big|^2 F^{p-2}\,d\mu = \sum_{i,j=1}^n a_i a_j \int f_i \overline{f}_j\, F^{p-2}\,d\mu = \sum_{i,j=1}^n |a_i|^2,$$

and

$$n = \int \sum_{i=1}^n f_i \overline{f}_i\, F^{p-2}\,d\mu = \int F^p\,d\mu.$$

Summarizing,

$$\int \Big|\sum_{i=1}^n a_i f_i\Big|^2 F^{p-2}\,d\mu = \sum_{i=1}^n |a_i|^2, \tag{1.2}$$

$$\|F\|_p = n^{1/p}. \tag{1.3}$$

Define an operator $T : E \to L_2(\Omega,\mu)$ by $Tf = fF^{\frac{p-2}{2}}$, for $f \in E$. By using Hölder's inequality for $\frac{p}{2}$, we obtain

$$\|Tf\|_2^2 \le \|f\|_p^2\, \|F\|_p^{p-2}.$$

Indeed,

$$\begin{aligned}\|Tf\|_2^2 &= \int f^2 F^{p-2}\, d\mu \\ &\le \left(\int |f|^p\right)^{2/p} \left(\int F^{(p-2)\cdot\frac{p}{p-2}}\right)^{\frac{p-2}{p}} \\ &= \|f\|_p^2\, \|F\|_p^{p-2}.\end{aligned}$$

Thus, by (1.3), $\|T\| \le n^{1/2-1/p}$.

On the other hand, by (1.2) and Cauchy-Schwarz inequality, we see that for every $h = \sum_{i=1}^n a_i f_i \in E$ one has

$$\text{(1.4)}\qquad \begin{aligned}\|h\|_p^p &= \int \Big|\sum_{i=1}^n a_i f_i\Big|^2 \Big|\sum_{i=1}^n a_i f_i\Big|^{p-2} d\mu \\ &\le \int \Big|\sum_{i=1}^n a_i f_i\Big|^2 \left[\left(\sum_{i=1}^n |a_i|^2\right)^{1/2} F\right]^{p-2} d\mu \\ &= \left(\sum_{i=1}^n |a_i|^2\right)^{\frac{p-2}{2}} \int \Big|\sum_{i=1}^n a_i f_i\Big|^2 F^{p-2} d\mu \\ &= \left(\sum_{i=1}^n |a_i|^2\right)^{p/2} = \left[\int \Big|\sum_{i=1}^n a_i f_i\Big|^2 F^{p-2} d\mu\right]^{p/2} = \|Th\|_2^p.\end{aligned}$$

Thus $\|T^{-1}\| \le 1$, and so,

$$\text{(1.5)}\qquad d_E \le n^{1/2-1/p}.$$

Now, we proceed by induction in n. Assume that the proposition is valid for $(n-1)$-dimensional subspaces.

Let $E \subset L_p(\Omega, \mu)$, $\dim E = n$, $d_E = n^{1/2-1/p}$. Then $\|T^{-1}\| = 1$. Fix $h \in E$ such that $\|h\|_p = \|Th\|_2 = 1$ and $h = \sum_{i=1}^n a_i f_i$ for some scalars $a_1, \dots, a_n$, where $f_1, \dots, f_n$ are as in Proposition 1.3. Since all the inequalities in (1.4) become equalities, it follows that $|h| = F$ *a.e.* in the support A of h.

Moreover, there exists a functional ϕ such that $f_i = \phi a_i$ *a.e.*

Since the f_i's are linearly independent, there exists $i_0 \in \{1, \dots, n\}$ such that $a_i = \delta_{ii_0}$. Without loss of generality, assume that $i_0 = 1$. Therefore, $|h| = |f_1|$ *a.e.* and $f_2 = f_3 = \cdots = f_n = 0$ *a.e.* on A. Next, observe that for any $f \in E$, the restriction $f \cdot \chi_A$ of f to A belongs to the one-dimensional subspace $[h]$ of $L_p(\Omega, \mu)$ generated by h.

Summarizing, $E = [h] \oplus_p E_1$ where

$$E_1 = \{f \in E : f(w) = 0 \ \ a.e. \text{ on } A\}.$$

We shall show that

$$d_E \leq \left(1 + d_{E_1}^{\frac{1}{1/2-1/p}}\right)^{1/2-1/p}. \tag{1.6}$$

Observe that, by (1.5) for the space E_1 and above, we obtain that $d_{E_1} = (n-1)^{1/2-1/p}$.

By (1.6) and by using the inductive hypothesis, we conclude the proof. To see (1.6) fix $\varepsilon > 0$. Choose an operator $H : E_1 \to \ell_2^{n-1}$ such that $\|H^{-1}\| = 1$ and $\|H\| \leq d_{E_1} + \varepsilon$. Consider the following diagram

$$\begin{array}{ccc} [h] \oplus_p & E_1 & = E \\ \downarrow S & \downarrow H & \\ [e_1] \oplus_2 & \ell_2^{n-1} & = \ell_2^n, \end{array}$$

where $S : [h] \to [e_1]$ is a linear operator defined by $S(h) = e_1$.

Pick $g \in E$ such that $\|g\|_p = 1$. Choose $f \in E_1$ and scalars λ, ξ such that $\|f\|_p = 1$ and $g = \lambda h + \xi f$. Since h and f have disjoint supports, it follows that $\lambda^p + \xi^p = 1$. Moreover,

$$\|S \oplus H(g)\|_2 = \|\lambda e_1 + \xi H(f)\|_2 = (\lambda^2 + \xi^2 \|H(f)\|_2^2)^{1/2}$$

$$\leq (\lambda^2 + \xi^2 \|H\|^2)^{1/2}$$

$$\leq (\lambda^p + \xi^p)^{1/p} \left(1 + \|H\|^{2(\frac{p}{2})'}\right)^{\frac{1}{(\frac{p}{2})' \cdot 2}} .$$

Since $(\frac{p}{2})' = \frac{p}{p-2}$ and $\|H\| \leq d_{E_1} + \varepsilon$, we have

$$\|S \oplus H\| \leq \left(1 + (d_{E_1} + \varepsilon)^{\frac{1}{1/2 - 1/p}}\right)^{1/p - 1/p} . \tag{1.7}$$

On the other hand, by similar calculation as above, we have

$$\|(S \oplus T)^{-1}\| \leq \sup_{\lambda^2+\xi^2=1} \left(\lambda^p + \xi^p \|H^{-1}\|^p\right)^{1/p}$$

$$= \sup_{\lambda^2+\xi^2=1} (\lambda^p + \xi^p)^{1/p} \leq \sup_{\lambda^2+\xi^2=1} (\lambda^2 + \xi^2)^{1/2} = 1.$$

Combining the last estimation with (1.7), we obtain

$$d_E \leq \|S \oplus T\| \, \|(S \oplus T)^{-1}\| \leq \left[1 + (d_{E_1} + \varepsilon)^{\frac{1}{1/2 - 1/p}}\right]^{1/2 - 1/p} .$$

Since $\varepsilon > 0$ is arbitrary, it shows (1.6). $\square$

CHAPTER 3

GENERAL CONSTRUCTIONS OF SPACES WITH NO UNCONDITIONAL BASIS

3.1. Notations. The aim of this chapter is to show a rather general construction of Banach spaces with no unconditional basis. The idea of the construction is based on some techniques first introduced by W. B. Johnson, J. Lindenstrauss and G. Schechtman [JLS] for studying the Kalton-Peck space [KP]. These techniques were refined by T. Ketonen [K] and later generalized further by A. Borzyszkowski [B]. The Kalton-Peck space was the first example of a Banach space without an unconditional basis which is an unconditional sum (even a symmetric sum) of two dimensional subspaces.

The standard notations from the Banach theory used throughout Chapter 3 can be found, e.g., in [P] and [T-J].

Let us recall some fundamental definitions. A basis (e_i) in a Banach space X is called unconditional if there exists a constant K such that, for every $x = \sum_i a_i e_i$ in X one has

$$\Big\| \sum_i \varepsilon_i a_i e_i \Big\| \le K\|x\| \qquad \text{for all } \ \varepsilon_i = \pm 1, \ i = 1, 2, \ldots .$$

The smallest K is called the unconditional basic constant and is denoted by $\mathrm{unc}\,((e_i))$. For a Banach space X with an unconditional basis, we set $\mathrm{unc}\,(X) = \inf\{\mathrm{unc}\,((e_i)) \mid (e_i) \text{ is an unconditional basis in } X\}$.

If X has no unconditional basis, we set $\operatorname{unc}(X) = \infty$. A basis (e_i) is called 1-unconditional if $\operatorname{unc}((e_i)) = 1$.

If $(Z_k)_{k=1}^{\infty}$ is a family of finite dimensional subspaces of a Banach space X, then the unconditional constant of $(Z_k)_{k=1}^{\infty}$, denoted by $\operatorname{unc}(Z_k)_{k=1}^{\infty}$, is the infimum of numbers $K > 0$ such that, for all finite sequences of vectors $(x_k)_k$, with $x_k \in Z_k$, and for all choices of signs (ε_k), the following holds

$$\Big\|\sum_k \varepsilon_k x_k\Big\| \le K\Big\|\sum_k x_k\Big\|.$$

Moreover, if $\overline{\operatorname{span}}(Z_k)_k = X$, then we say that X has finite dimensional decomposition. If $\operatorname{unc}(Z_k)_k$ is finite (equal to one), we call $(Z_k)_{k=1}^{\infty}$ an unconditional (1-unconditional) decomposition.

If A is a subset of the positive integers, we denote $\overline{\operatorname{span}}(\{e_i\}_{i\in A})$ in X by $X|_A$.

DEFINITION 1.1. A Banach space X with a normalized 1-unconditional basis (e_i) is said to satisfy an *upper*, respectively, *lower 2-estimate* if there exists a constant $K < \infty$ such that, for every choice of scalars (a_i), and every $n = 1, 2, 3, \ldots,$ one has

$$(*) \qquad \Big\|\sum_{i=1}^{n} a_i e_i\Big\| \le K\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2},$$

respectively,

$$(**) \qquad \Big\|\sum_{i=1}^{n} a_i e_i\Big\| \ge K^{-1}\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2}.$$

The smallest constant K satisfying $(*)$ or $(**)$ is called the *upper*, respectively, *lower 2-estimate constant* of X.

DEFINITION 1.2. Let X be a Banach space with a normalized 1-unconditional basis. We say that X *contains* ℓ_2^n *'s uniformly on subsequences* if there exists a constant $K < \infty$ such that, for every subsequence (e_n) of the basis, and every k, there exists a subsequence $n_1, \dots, n_k$ of indexes such that

$$K^{-1}\Big(\sum_{i=1}^{k} |a_i|^2\Big)^{1/2} \le \Big\| \sum_{i=1}^{k} a_i e_{n_i} \Big\| \le K\Big(\sum_{i=1}^{k} |a_i|^2\Big)^{1/2}$$

for all scalars a_i.

Let X and Y be two Banach spaces with a normalized 1-unconditional basis (e_k). The natural identity operator acting between X and Y we denote by id.

3.2. Spaces with no unconditional basis and with a finite dimensional decomposition. The following result is essential in the construction of Banach spaces with no unconditional basis. T. Ketonen [K] and A. Borzyszkowski [B] used this approach for the ℓ_p-space, $1 \le p < 2$. Our argument is slightly different and shorter than those which are presented in [B] and [K]. Therefore, for the sake of completeness and to obtain some estimations, we will give the proof.

PROPOSITION 2.1. *Let* $(Y, \|\cdot\|)$ *be a Banach space of finite cotype* q, *and let* $C_q(Y)$ *be the cotype constant of* Y. *Suppose that* $\mathrm{unc}(Y) < \infty$, *and that* Y *has a 1-unconditional decomposition* $(Z_k)_k$. *Assume that* $\dim Z_k = 2$ *for each* k. *Then there exist an operator* $T : Y \to Y$ *and an increasing function* $f_0 : [1, \infty) \to [1, \infty)$ *such that*

(i) $T(Z_k) \subset Z_k$ *for each* k,

(ii) $\|T\| \le f_0(\mathrm{unc}(Y) \cdot C_q(Y))$,

(iii) $\|(T - \lambda id)|_{Z_k}\| \ge \frac{1}{8}$ *for each* $\lambda \in \mathbf{R}$, $k = 1, 2, 3, \dots$.

Before we start the proof, we recall some necessary information.

Let X be a Banach space with a 1-unconditional basis (e_k). Let $x_1, \dots, x_n$ be a finite sequence in X. Let $(x_{ik})_k$ be the coordinates of x_i so that $x_i = \sum_k x_{ik} e_k$. We denote by

$$\Big(\sum_i |x_i|^2\Big)^{1/2}$$

the element of X defined as follows

$$\Big(\sum_i |x_i|^2\Big)^{1/2} = \sum_k \Big(\sum_{i=1}^n |x_{ik}|^2\Big)^{1/2} e_k.$$

Recall (see [LT] Theorem 1.d.6 and Corollary 1.f.9) that if X is of cotype $q < \infty$, then there exists a constant M such that

$$E\Big\|\sum_i r_i x_i\Big\| \le M\Big\|\Big(\sum_i |x_i|^2\Big)^{1/2}\Big\| \tag{2.1}$$

for all finite sequences (x_i) in X.

Note that M in (2.1) depends only on the cotype constant, and in fact, it can be considered as an increasing function of $C_q(X)$.

Recall that in finite-dimensional spaces a series $\sum\limits_{i=1}^{\infty} x_i$ converges unconditionally if and only if it converges absolutely, i.e. $\sum\limits_{i=1}^{\infty} \|x_i\| < \infty$. More precisely, if X is an n-dimensional Banach space, then

$$\sup_{\varepsilon_i=\pm 1} \Big\|\sum_i \varepsilon_i x_i\Big\| \ge \frac{1}{n}\sum_i \|x_i\| \qquad \text{for all } x_i \in X,\ i = 1, 2, \dots. \tag{2.2}$$

Indeed, let $(e_k, e_k^*)_{k=1}^n$ be an Auerbach system for the space X. Then

$$\sum_i \|x_i\| = \sum_i \Big\|\sum_{k=1}^n e_k^*(x_i) e_k\Big\| \le \sum_{k=1}^n \sum_i |e_k^*(x_i)|$$

$$= \sum_{k=1}^{n} e_k^* \Big(\sum_i \operatorname{sign}(e_k^*(x_i)) x_i \Big)$$

$$\le n \sup_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i x_i \Big\|.$$

PROOF OF PROPOSITION 2.1: Let (e_i) be an unconditional basis for Y. Denote $D = \operatorname{unc}((e_i))$. We introduce on Y the equivalent norm $|||\cdot|||$ to $\|\cdot\|$ as follows:

$$\Big|\Big|\Big| \sum_i a_i e_i \Big|\Big|\Big| = \sup_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i a_i e_i \Big\|.$$

Then $\|\cdot\| \le |||\cdot||| \le D\|\cdot\|$, and (e_i) becomes a normalized 1-unconditional basis for $(Y, |||\cdot|||)$.

Let P_k be the natural projection from Y onto Z_k, and let $\Lambda_{\bar\varepsilon}$ be an operator defined as follows: $\Lambda_{\bar\varepsilon} : Y \to Y$, $\Lambda_{\bar\varepsilon} = \sum_i \varepsilon_i e_i^* \otimes e_i$, where (e_i^*) is the biorthogonal system associated with the basis (e_i), $\bar\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$, $\varepsilon_i = \pm 1$.

For every k pick $\bar\varepsilon_k = (\pm 1, \pm 1, \dots)$ such that

$$\frac{3}{4} \sup_{\bar\varepsilon} \inf_{\lambda \in \mathbf{R}} \|P_k \Lambda_{\bar\varepsilon} P_k - \lambda id\| \le \inf_{\lambda \in \mathbf{R}} \|P_k \Lambda_{\bar\varepsilon_k} P_k - \lambda id\|. \tag{2.3}$$

Define $T = \sum_k P_k \Lambda_{\bar\varepsilon_k} P_k$.

Observe that (i) follows directly from the definition of T. To see (ii) fix an x in Y such that $\|x\| = 1$. By (2.1) and since $\operatorname{unc}(Z_k)_k = 1$, we have

$$\|Tx\| = \Big\| E\Big(\sum_k r_k P_k\Big)\Big(\sum_k r_k \Lambda_{\bar\varepsilon_k} P_k(x)\Big) \Big\|$$

$$\le \max_{\varepsilon_k = \pm 1} \Big\| \sum_k \varepsilon_k P_k \Big\| \, E \Big|\Big|\Big| \sum_k r_k \Lambda_{\bar\varepsilon_k} P_k(x) \Big|\Big|\Big|$$

$$\le M \Big|\Big|\Big| \Big(\sum_k |\Lambda_{\bar\varepsilon_k} P_k(x)|^2 \Big)^{1/2} \Big|\Big|\Big|$$

$$= M\Big|\Big|\Big|\Big(\sum_k |P_k(x)|^2\Big)^{1/2}\Big|\Big|\Big|$$

$$= M\Big|\Big|\Big|\Big(E\Big|\sum_k r_k P_k(x)\Big|^2\Big)^{1/2}\Big|\Big|\Big|$$

$$\leq \sqrt{2}M\Big|\Big|\Big|E\Big|\sum_k r_k P_k(x)\Big|\Big|\Big|\Big|$$

$$\leq \sqrt{2}ME\Big|\Big|\Big|\sum_k r_k P_k(x)\Big|\Big|\Big|$$

$$\leq \sqrt{2}MDE\Big\|\sum_k r_k P_k(x)\Big\| = \sqrt{2}MD\|x\|.$$

Note that the constant M depends on $C_q(Y, ||| \cdot |||)$, and by renorming on $DC_q(Y, \|\cdot\|)$.

In order to prove (iii), we consider a 3-dimensional space B/B_0, where B is a space of linear operators on Z_k and $B_0 = \text{span}(id)$. For $S \in B$ let $\widetilde{S}$ be the image of S under the quotient map.

Fix k. Define $R_i = P_k(e_i^* \otimes e_i)P_k$. Observe that

$$\|\widetilde{R}_i\| = \inf_{\lambda\in\mathbf{R}} \|R_i - \lambda id\| \geq \frac{1}{2}\|R_i\| \tag{2.4}$$

This is trivially true for $|\lambda| < \|R_i\|/2$, and in the case $|\lambda| \geq \|R_i\|/2$ it is sufficient to notice that, since $R_i(Z_k)$ is one-dimensional, there is an $x \in \ker R_i$ such that $\|x\| = 1$. By (2.3), (2.4) and (2.2) for $n = 3$, we have

$$\inf_{\lambda\in\mathbf{R}} \|(T - \lambda id)|_{Z_k}\| \geq \frac{3}{4} \sup_{\bar{\varepsilon}} \inf_{\lambda\in\mathbf{R}} \|P_k \Lambda_{\bar{\varepsilon}} P_k - \lambda id\|$$

$$= \frac{3}{4} \sup_{\varepsilon_i=\pm1} \Big\|\sum_i \varepsilon_i \widetilde{R}_i\Big\| \geq \frac{3}{4}\cdot\frac{1}{3}\sum_i \|\widetilde{R}_i\|$$

$$\geq \frac{1}{8}\sum_i \|R_i\| \geq \frac{1}{8}\Big\|\sum_i P_k(e_i^* \otimes e_i)P_k\Big\| = \frac{1}{8}\|id\|. \qquad \square$$

The next theorem illustrates the use of Proposition 2.1. The idea of the proof appears later in a more general setting, in the proof of Proposition 4.3. The class of spaces presented in this theorem is very large. In particular, as we will see in Section 5, it allows us to construct a weak Hilbert space with no unconditional basis.

THEOREM 2.2. *Let $\alpha \geq 1$. Let $X_1, \ldots, X_4$ be Banach spaces of finite cotype with a normalized 1-unconditional basis. Suppose that $\|id : X_{i+1} \to X_i\| < \alpha$, and that no subsequence of X_i is equivalent to this in X_{i+1}, $i = 1, 2, 3$. Then there exists a subspace Y in $X_1 \oplus \cdots \oplus X_4$ such that* $\mathrm{unc}(Y) = \infty$.

PROOF:Let $(e_{i,k})_k$ be a normalized 1-unconditional basis in X_i, $i = 1, 2, 3, 4$. Let $(A_m)_m$ and $(A_{m,n})_{m,n}$ be two partitions of the positive integers such that:

$$|A_m| = \infty, \quad |A_{m,n}| = \infty, \quad m, n = 1, 2, 3, \ldots, \tag{2.5}$$

and

$$A_m = \bigcup_{n=1}^{\infty} A_{m,n}. \tag{2.6}$$

For $k \in A_{m,n}$ define

$$x_k = e_{1,k} + e_{2,k} + 2^{-m} e_{3,k},$$

$$y_k = \qquad e_{2,k} \qquad + 2^{-(m+n)} e_{4,k}.$$

Put $Z_k = \mathrm{span}(x_k, y_k)$, $Y = \overline{\mathrm{span}}(Z_k)_k$. We will show that $\mathrm{unc}(Y) = \infty$. Assume on the contrary that $\mathrm{unc}(Y) < \infty$. Observe that $\mathrm{unc}(Z_k)_k = 1$; and since

$$(2.7)\qquad \max(|t|,|w|) \le \|tx_k + wy_k\| \le 3(|t|+|w|),$$

it follows that $x_1, y_1, x_2, y_2, \dots$ is a basic sequence.

By Proposition 2.1 there exists an operator $T : Y \to Y$ such that (i), (ii) and (iii) are satisfied. Let $\begin{pmatrix} a'_k & b_k \\ c_k & d_k \end{pmatrix}$ be the matrix of $T|_{Z_k}$ with respect to (x_k, y_k). Observe that sequences (a'_k), (b_k), (c_k), (d_k) are bounded. Put $a_k = a'_k - d_k$. Let $T_1 : Y \to Y$ be an operator such that $\begin{pmatrix} a_k & b_k \\ c_k & 0 \end{pmatrix}$ is the matrix of $T_1|_{Z_k}$ with respect to (x_k, y_k). We will show that T_1 is bounded, and that for every k one has

$$(2.8)\qquad \max(|a_k|,|b_k|,|c_k|) > \eta_0,$$

where η_0 is an absolute constant.

Indeed, let $S : Y \to Y$ be an operator such that $\begin{pmatrix} d_k & 0 \\ 0 & d_k \end{pmatrix}$ is the matrix of $S|_{Z_k}$ with respect to (x_k, y_k). Since $\mathrm{unc}(Z_k)_k = 1$, it follows that S is bounded; hence $T_1 = T - S$ is bounded. Condition (2.8) follows from Proposition 2.1(iii) and (2.7).

Fix $A_{m,n}$. By (2.5) and since no subsequence of $(e_{4,k})$ is equivalent to this of $(e_{3,k})$, it follows that there exists a sequence of scalars $(\beta_k)_{k\in A_{m,n}}$ such that $\sum \beta_k e_{3,k}$ converges, but $\sum \beta_k e_{4,k}$ does not converge.

We have $\|\sum \beta_k x_k\| < \infty$; hence $\|T_1(\sum \beta_k x_k)\| = \|\sum \beta_k(a_k x_k + c_k y_k)\| < \infty$. It follows that $\sum c_k \beta_k e_{4,k}$ converges; so, $\inf_{k\in A_{m,n}} |c_k| = 0$.

For every m, n pick $k_{m,n} \in A_{m,n}$ such that

$$(2.9)\qquad \sum_{m,n} |c_{k_{m,n}}| < \eta_0.$$

Let K be the set of such k 's.

Define an operator $T_2 : \overline{\text{span}}\,(Z_k)_{k\in K} \to \overline{\text{span}}\,(Z_k)_{k\in K}$ such that $\begin{pmatrix} a_k & b_k \\ 0 & 0 \end{pmatrix}$ is the matrix of $T_2|_{Z_k}$ with respect to (x_k, y_k). By (2.7) and (2.9), T_2 is bounded.

Fix A_m. Consider $A_m \cap K$. For $k \in A_m \cap K$ define n_k such that $k \in A_{m,n_k}$. It follows from (2.6) that $\lim\limits_{\substack{k\to\infty \\ k\in A_m\cap K}} n_k = \infty$. As before, choose $(\beta_k)_{k\in A_m\cap K}$ such that $\sum \beta_k e_{2,k}$ converges, but $\sum \beta_k e_{3,k}$ does not converge.

We have $\|\sum \beta_k y_k\| \le \|\sum \beta_k e_{2,k}\| + \sum\limits_{k\in A_m\cap K} 2^{-(m+n_k)}|\beta_k| < \infty$; hence

$$\left\|T_2(\sum \beta_k y_k)\right\| = \left\|\sum \beta_k b_k x_k\right\| < \infty.$$

It follows that $\sum \beta_k b_k e_{3,k}$ converges; so, $\inf_{k\in A_m\cap K} |b_k| = 0$.

For every m pick $k_m \in A_m \cap K$ such that

$$\sum_m |b_{k_m}| < \eta_0. \tag{2.10}$$

Let K' be the set of such k 's.

Define an operator $T_3 : \overline{\text{span}}\,(Z_k)_{k\in K'} \to \overline{\text{span}}\,(Z_k)_{k\in K'}$ such that $\begin{pmatrix} a_k & 0 \\ 0 & 0 \end{pmatrix}$ is the matrix of $T_3|_{Z_k}$ with respect to (x_k, y_k). As before $\|T_3\|$ is bounded, and by (2.8), (2.9) and (2.10), $|a_k| > \eta_0$ for $k \in K'$.

For $k \in K'$ define m_k such that $k \in A_{m_k}$. Then $\lim\limits_{\substack{k\to\infty \\ k\in K'}} m_k = \infty$. Choose $(\beta_k)_k \in K'$ such that $\sum \beta_k e_{1,k}$ converges, but $\sum \beta_k e_{2,k}$ does not converge. We have

$$\left\|\sum \beta_k (x_k - y_k)\right\| \le \left\|\sum \beta_k e_{1,k}\right\| + 2\sum_{k\in K'} 2^{-m_k}|\beta_k| < \infty;$$

thus

$$\infty > \left\| T_3\left(\sum \beta_k(x_k - y_k)\right)\right\| = \left\|\sum \beta_k a_k x_k\right\| \geq \inf_{k\in K'} |a_k| \cdot \left\|\sum \beta_k e_{2,k}\right\|$$
$$> \eta_0 \left\|\sum \beta_k e_{2,k}\right\| = \infty.$$

This contradicts the fact that $\mathrm{unc}(Y) < \infty$. Hence $\mathrm{unc}(Y) = \infty$. □

3.3. Preliminary quantitative results. Let us introduce some new notations important for the further argument.

Let X be a Banach space with a normalized 1-unconditional basis (e_k). Let $(X_i)_{i=1}^m$ be a sequence of subspaces in X with the basis $(e_{i,k})_k$, $i = 1, \ldots, m$, respectively. Assume that $(e_{i,k})_k$, $i = 1, \ldots, m$, are disjoint subsequences of (e_k). By $D(X_1 \oplus \cdots \oplus X_m)$ we mean the diagonal space in $X_1 \oplus \cdots \oplus X_m$ (i.e., $\overline{\mathrm{span}}(e_{1,k} + \cdots + e_{m,k})_k$) with a normalized 1-unconditional basis $\big((e_{1,k} + \cdots + e_{m,k})/\|e_{1,k} + \cdots + e_{m,k}\|\big)_k$.

Suppose that I is any subset of the positive integers $\mathbb{N}$, and that Δ_1, Δ_2 are any two partitions of I. We write $\Delta_1 \succ \Delta_2$ if for every $A \in \Delta_2$, there exists a family (B_n) in Δ_1 such that $A = \bigcup_n B_n$.

In the construction, we will consider only four partitions. If necessary, members of Δ are numerated and denoted by $\Delta = (A_n)$.

Let X and Y be two Banach spaces with a normalized 1-unconditional basis $(e_k)_{k\in I}$. Let (Δ_i) be a family of partitions of I such that $\Delta_1 \succ \Delta_2 \succ \cdots$. For every $A \in \Delta_i$, $i = 2, 3, \ldots$, we define

$$d_A(X, Y) = \inf \|id : X|_{(k_n)} \to Y|_{(k_n)}\|, \tag{3.1}$$

where the infimum is taken over all sequences (k_n) in A such that $k_n \in B_n$, $B_n \in \Delta_{i-1}$ and $A = \bigcup_n B_n$.

The above parameter is crucial for the construction of Banach spaces with no unconditional basis. It will be used for the lower estimate of the norm of the operator T from Proposition 2.1; and hence, for the lower estimate of $\mathrm{unc}(Y)$ as well. Therefore, in all situations considered in here, we try to get d_A sufficiently large.

The following lemma illustrates that, in some cases, we can obtain $d_A(X,Y)$ as large as we wish.

LEMMA 3.1. *Let X_1 and X_2 be two Banach spaces with a normalized 1-unconditional basis. Let $\Delta = (A_n)$ be a partition of $\mathbb{N}$ such that every A_n is finite. Suppose that for every infinite subset A of the positive integers the following holds*

$$\|id : X_2|_A \to X_1|_A\| = \infty.$$

Then for every $K \geq 1$, there exists $m \in \mathbb{N}$ such that, for every sequence $(k_n)_{n=1}^m$, $k_n \in A_n$, one has

$$\|id : X_2|_{(k_n)_{n=1}^m} \to X_1|_{(k_n)_{n=1}^m}\| \geq K.$$

In particular we have:

COROLLARY 3.2. *Let X be a Banach space with a normalized 1-unconditional basis. Let $\Delta = (A_n)$ be a partition of $\mathbb{N}$ such that every A_n is finite. Suppose that no subsequence of the basis in X satisfies an upper (respectively lower)*

2-estimate. Then for every $K \geq 1$, *there exists* $m \in \mathbb{N}$ *such that, for every sequence* $(k_n)_{n=1}^m$, $k_n \in A_n$, *one has*

$$\|id : \ell_2^m \to X|_{(k_n)_{n=1}^m}\| \geq K$$

$$(\textit{respectively } \|id : X|_{(k_n)_{n=1}^m} \to \ell_2^m\| \geq K).$$

Proof of Lemma 3.1: Assume on the contrary that there exists a constant $K \geq 1$ such that, for every $m = 1, 2, 3, \ldots$, there exists a sequence $(k_n)_{n=1}^m$, $k_n \in A_n$, such that

$$\|id : X_2|_{(k_n)_{n=1}^m} \to X_1|_{(k_n)_{n=1}^m}\| < K.$$

By the standard Cantor diagonal process, we can extract an increasing sequence $(k'_n)_{n=1}^\infty$, $k'_n \in A_n$ (note that each A_n is finite) such that $\|id : X_2|_{(k'_n)_{n=1}^\infty} \to X_1|_{(k'_n)_{n=1}^\infty}\| \leq K$. This is impossible by our assumption. □

As an immediate consequence, from Corollary 3.2, we obtain:

Corollary 3.3. *Let* X *be a Banach space with a normalized 1-unconditional basis. Suppose that* X *contains* ℓ_2^n *'s uniformly on subsequences, and that for every infinite subset* A *of the positive integers one has*

$$\|id : \ell_2 \to X|_A\| = \infty$$

(i.e., no subsequence of the basis satisfies an upper 2-estimate).

Then for any choice of positive numbers $(M_{i,n})_n$, $i = 2, 3, 4$, *there exist four disjoint subsequences* $(e_{i,k})_k$ *of the basis and four partitions* $\Delta_i = (A_{i,n})_n$

of $\mathbb{N}$, $i = 1, 2, 3, 4$, such that, setting $X_i = \text{span}(e_{i,k})_k$, the following conditions are satisfied:

(i) $\Delta_1 \succ \cdots \succ \Delta_4$,

(ii) there exists a constant $\alpha \geq 1$ such that, for every $1 \leq i \leq j \leq 4$, and for every $A \in \Delta_i$, one has

$$\|id : X_i|_A \to X_j|_A\| < \alpha,$$

(iii) for every $A_{i,n} \in \Delta_i$, $i = 2, 3, 4$, one has

$$d_{A_{i,n}}(X_i, X_{i-1}) > M_{i,n}.$$

COROLLARY 3.4. *Let X be a Banach space with a normalized 1-unconditional basis. Suppose that X contains ℓ_2^n's uniformly on subsequences, and that for any basic sequence (f_k) of the form: $f_k = e'_{1,k}$, $f_k = e'_{1,k} + e'_{2,k}$ or $f_k = e'_{1,k} + e'_{2,k} + e'_{3,k}$, where $(e'_{1,k})$, $(e'_{2,k})$ and $(e'_{3,k})$ are disjoint subsequences of the basis, the following holds*

$$\|id : \overline{\text{span}}(f_k) \to \ell_2\| = \infty$$

(i.e., (f_k) does not satisfy a lower 2-estimate).

Then for any choice of positive numbers $(M_{i,n})_n$, $i = 2, 3, 4$, there exist four disjoint subsequences $(e_{i,k})_k$ of the basis and four partitions $\Delta_i = (A_{i,n})_n$ of $\mathbb{N}$, $i = 1, 2, 3, 4$, such that, setting $X_i = \text{span}(e_{i,k})_k$, the following conditions are satisfied:

(i) $\Delta_1 \succ \cdots \succ \Delta_4$,

(ii) there exists a constant $\alpha \geq 1$ *such that, for every* $1 \leq i \leq j \leq 4$, *and for every* $A \in \Delta_i$, *one has*

$$\|id : X_i|_A \to X_j|_A\| < \alpha,$$

(iii) for every $A_{i,n} \in \Delta_i$, $i = 2, 3, 4$, *one has*

$$d_{A_{i,n}}(D(X_1 \oplus \cdots \oplus X_{i-1}), X_i) > M_{i,n}.$$

PROOF OF COROLLARIES 3.3 AND 3.4: By induction, we can choose $\Delta_1 \succ \cdots \succ \Delta_4$ and $X_1, \ldots, X_4$ in such a way, that for every $A \in \Delta_i$, $i = 1, 2, 3, 4$, the space $X_i|_A$ is uniformly "close" to $\ell_2^{|A|}$; hence (ii) holds. Condition (iii) follows immediately from the main assumption and Corollary 3.2. □

3.4. Main estimates and quantitative results. In this section, we establish, in Propositions 4.2 and 4.3, the general construction of Banach spaces with no unconditional basis. This result is the most important in this part of the thesis. The argument is very similar to that one which was presented in the proof of Theorem 2.2. As a result we will obtain the following:

THEOREM 4.1. *Let* X *be a Banach space of finite cotype with a normalized 1-unconditional basis. Suppose that* X *contains* ℓ_2^n *'s uniformly on subsequences, and that one of the following conditions is satisfied:*

(i) no subsequence of the basis satisfies an upper 2-estimate,

or

(ii) no basic sequence of the form: $f_k = e'_{1,k}$, $f_k = e'_{1,k} + e'_{2,k}$ *or* $f_k = e'_{1,k} + e'_{2,k} + e'_{3,k}$, *where* $(e'_{1,k})$, $(e'_{2,k})$ *and* $(e'_{3,k})$ *are disjoint supported subsequences of the basis, satisfies a lower 2-estimate.*

Then there exists a subspace Y *in* X *such that* $\mathrm{unc}(Y) = \infty$.

In the next two propositions, by " f_0 " we denote the function established in Proposition 2.1(ii).

PROPOSITION 4.2. *Let* X *be a Banach space of finite cotype* q, *with the cotype constant* $C_q(X)$. *Suppose that* X *has a normalized 1-unconditional basis. Let* $(e_{i,k})_k$, $i = 1,2,3,4$, *be disjoint subsequences of the basis. Let* $\Delta_1, \ldots, \Delta_4$ *be partitions of* $\mathbb{N}$ *such that* $\Delta_1 \succ \cdots \succ \Delta_4$. *Let* $A_0 \in \Delta_4$, $\alpha \geq 1$, $\eta \geq 1$.

Setting $X_i = span(e_{i,k})_k$, *assume that*

$$(4.1) \qquad d_A(D(X_1 \oplus \cdots \oplus X_{i-1}), X_i) > \eta \qquad \text{for all } A \in \Delta_i \cap \{A_0\},\ i = 2,3,4,$$

and

$$(4.2) \qquad \|id : X_i|_A \to X_j|_A\| < \alpha \qquad \text{for all } A \in \Delta_i \cap \{A_0\},\ 1 \leq i \leq j \leq 4.$$

Then there exists a subspace Y *in* $X_1 \oplus \cdots \oplus X_4$ *such that*

$$\mathrm{unc}(Y) \geq C_q(X)^{-1} f_0^{-1}(10^{-4}\alpha^{-1}\eta^{\frac{1}{8}}).$$

PROPOSITION 4.3. *Let* X *be a Banach space of finite cotype* q, *with the cotype constant* $C_q(X)$. *Suppose that* X *has a normalized 1-unconditional basis. Let* $(e_{i,k})_k$, $i = 1, \ldots, 4$, *be disjoint subsequences of the basis. Let* $\Delta_1, \ldots, \Delta_4$ *be partitions of* $\mathbb{N}$ *such that* $\Delta_1 \succ \cdots \succ \Delta_4$. *Let* $A_0 \in \Delta_4$, $\alpha \geq 1$, $\eta \geq 1$.

Setting $X_i = span(e_{i,k})_k$, *assume that*

$$(4.3) \qquad d_A(X_i, X_{i-1}) > \eta \qquad \text{for all } A \in \Delta_i \cap \{A_0\},\ i = 2,3,4,$$

$$(4.4) \qquad \|id : X_i|_A \to X_j|_A\| < \alpha \qquad \text{for all } A \in \Delta_i \cap \{A_0\},\ 1 \le i \le j \le 4,$$

$$(4.5) \qquad \sum_{B \in \Delta_2} d_B(X_2, X_1)^{-1/2} + \sum_{B \in \Delta_3} d_B(X_3, X_2)^{-1/2} < 1.$$

Then there exists a subspace Y *in* $X_1 \oplus \cdots \oplus X_4$ *such that*

$$unc(Y) \ge C_q(X)^{-1} f_0^{-1}(10^{-6}\alpha^{-1}\eta^{1/2}).$$

Assuming the truth of Propositions 4.2 and 4.3, we can easily prove Theorem 4.1. It follows immediately from Corollaries 3.3 and 3.4. Notice that, by a suitable choice of $(M_{i,n})$, and then by changing A_0 in Δ_4, we can obtain the lower estimations in (4.1) and (4.3) as large as we wish.

Before we pass to the proof of Propositions 4.2 and 4.3, we discuss a general idea of the use of Proposition 2.1.

We define

$$(4.6) \qquad x_k = \alpha_{1,k}e_{1,k} + \cdots + \alpha_{4,k}e_{4,k}, \qquad y_k = \beta_{1,k}e_{1,k} + \cdots + \beta_{4,k}e_{4,k},$$

where $(\alpha_{i,k})_k$, $(\beta_{i,k})_k$, $i = 1, \ldots, 4$, are any scalars such that

$$(4.7) \qquad \max(|t|, |w|) \le \|tx_k + wy_k\| \le 3(|t| + |w|).$$

Put $Z_k = \text{span}(x_k, y_k)$ and $Y = \overline{\text{span}}(Z_k)_k$. Observe that for $k \ne k'$ the spaces Z_k and $Z_{k'}$ have disjoint supports; hence $\text{unc}(Z_k)_k = 1$, and by (4.7), it follows that $x_1, y_1, x_2, y_2, \ldots$ is a basic sequence.

Suppose that $\text{unc}(Y) < \infty$. By Proposition 2.1, there exists an operator $T : Y \to Y$ such that conditions (i), (ii), (iii) are satisfied.

Let

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

be the matrix of $T|_{Z_k}$ with respect to the basis (x_k, y_k) (i.e., $T(\alpha x_k + \beta y_k) = (\alpha a_k + \beta b_k)x_k + (\alpha c_k + \beta d_k)y_k$).

For simplicity, we introduce the following notations. Let $S : Z_k \to Z_k$ be an operator, and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix of S with respect to (x_k, y_k). By $|||S|||$ we mean the ℓ_∞^4 norm of S, i.e. $|||S||| = \max\{|a|, |b|, |c|, |d|\}$. The norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent; in fact, $\frac{\|S\|}{12} \le |||S||| \le 3\|S\|$. This is a consequence of (4.7) and the fact that Z_k is two-dimensional. Hence, by Proposition 2.1(iii), we obtain

$$10^{-2} \le \max(|a_k - \lambda|, |d_k - \lambda|, |c_k|, |b_k|) \qquad \text{for all } \lambda \in \mathbb{R}. \tag{4.8}$$

In the proof of Proposition 4.2 (4.3) we will choose scalars $(\alpha_{i,k})_k$, $(\beta_{i,k})_k$ in (4.6) such that (4.7) and (4.8) are satisfied.

Proof of Proposition 4.2: For $\varepsilon = \eta^{-1/3}$, we have

$$d_A(D(X_1 \oplus \cdots \oplus X_{i-1}), X_i) > \varepsilon^{-3} \quad \text{for all } A \in \Delta_i \cap \{A_0\},\ i = 2, 3, 4. \tag{4.9}$$

Define x_k, y_k in (4.6) as follows:

$$\begin{aligned} x_k &= e_{1,k} + \varepsilon e_{3,k} + \varepsilon^2 e_{4,k}, \\ y_k &= e_{2,k} + \varepsilon^2 e_{4,k}. \end{aligned}$$

We will show that

$$\|T\| \geq 10^{-4}\alpha^{-1}\varepsilon^{-1}. \tag{4.10}$$

Pick $B \in \Delta_2 \cap \{A_0\}$. By (4.9), there exists a sequence of scalars $(\beta_k)_{k\in B}$ such that $\|\sum \beta_k e_{1,k}\| = 1$ and $\|\sum \beta_k e_{2,k}\| = \varepsilon^{-1}$ (by continuality, we may suppose that). By (4.2), we have

$$\begin{aligned}\Big\|\sum \beta_k x_k\Big\| &\leq \Big\|\sum \beta_k e_{1,k}\Big\| + \varepsilon\Big\|\sum \beta_k e_{3,k}\Big\| + \varepsilon^2\Big\|\sum \beta_k e_{4,k}\Big\| \\ &\leq 1 + \varepsilon\alpha\Big\|\sum \beta_k e_{2,k}\Big\| + \varepsilon^2\alpha\Big\|\sum \beta_k e_{2,k}\Big\| < 3\alpha,\end{aligned}$$

and

$$\begin{aligned}3\alpha\|T\| &> \Big\|T\Big(\sum \beta_k x_k\Big)\Big\| = \Big\|\sum \beta_k(a_k x_k + c_k y_k)\Big\| \\ &\geq \Big\|\sum \beta_k c_k e_{2,k}\Big\| \geq \inf_{k\in B}|c_k|\Big\|\sum \beta_k e_{2,k}\Big\| = \varepsilon^{-1}\inf_{k\in B}|c_k|.\end{aligned}$$

For every $B \in \Delta_2 \cap \{A_0\}$ pick $k \in B$ such that $|c_k| < 3\varepsilon\alpha\|T\|$. Let K be the set of such k 's. Suppose that $|c_k| < \frac{1}{400}$, for all $k \in K$ (if not, then (4.10) holds).

Pick $C \in \Delta_3 \cap \{A_0\}$. By (4.9), there exists $(\beta_k)_{k\in C\cap K}$ such that

$$\Big\|\sum \beta_k\Big(\frac{e_{1,k} + e_{2,k}}{\|e_{1,k} + e_{2,k}\|}\Big)\Big\| = 1 \qquad \text{and} \qquad \Big\|\sum \beta_k e_{3,k}\Big\| = \varepsilon^{-2}.$$

As before, by (4.2), one has

$$\begin{aligned}\Big\|\sum \beta_k y_k\Big\| &\leq \Big\|\sum \beta_k e_{2,k}\Big\| + \varepsilon^2\Big\|\sum \beta_k e_{4,k}\Big\| \\ &\leq 2\Big\|\sum \beta_k\Big(\frac{e_{1,k} + e_{2,k}}{\|e_{1,k} + e_{2,k}\|}\Big)\Big\| + \varepsilon^2\alpha\Big\|\sum \beta_k e_{3,k}\Big\| \leq 3\alpha,\end{aligned}$$

and

$$3\alpha\|T\| > \left\|T\left(\sum \beta_k y_k\right)\right\| = \left\|\sum \beta_k(b_k x_k + d_k y_k)\right\|$$
$$\geq \varepsilon\left\|\sum \beta_k b_k e_{3,k}\right\| \geq \varepsilon^{-1} \inf_{k\in K\cap C} |b_k|.$$

For every $C \in \Delta_3 \cap \{A_0\}$ pick $k \in C \cap K$ such that $|b_k| < 3\varepsilon\alpha\|T\|$ and $|b_k| < \frac{1}{400}$ (if not, then (4.10) holds as required). Let K' be the set of such k 's.

Finally, choose $(\beta_k)_{k\in A_0\cap K'}$ such that

$$\left\|\sum \beta_k e_{4,k}\right\| = \varepsilon^{-3} \qquad \text{and} \qquad \left\|\sum \beta_k\left(\frac{e_{1,k}+e_{2,k}+e_{3,k}}{\|e_{1,k}+e_{2,k}+e_{3,k}\|}\right)\right\| < 1.$$

Thus we see that

$$\left\|\sum \beta_k(x_k - y_k)\right\| < \left\|\sum \beta_k(e_{1,k}+e_{2,k}+e_{3,k})\right\| < 3.$$

By (4.8), and since $|c_k| < \frac{1}{400}$, $|b_k| < \frac{1}{400}$ for $k \in K'$, it follows that $|a_k - d_k| > \frac{1}{100}$ for $k \in K'$. Hence

$$3\|T\| > \left\|T\left(\sum \beta_k(x_k - y_k)\right)\right\|$$
$$= \left\|\sum \beta_k[(a_k - b_k)x_k + (c_k - d_k)y_k]\right\|$$
$$\geq \varepsilon^2\left\|\sum \beta_k[(a_k - b_k) + (c_k - d_k)]e_{4,k}\right\|$$
$$\geq \varepsilon^{-1} \inf_{k\in K'\cap A_0}\left(|a_k - d_k| - |c_k| - |b_k|\right)$$
$$\geq \varepsilon^{-1}\left(\frac{1}{100} - \frac{1}{400} - \frac{1}{400}\right) = \frac{\varepsilon^{-1}}{200}.$$

Summarizing, $\|T\| > \frac{1}{1200\alpha\varepsilon} \geq 10^{-4}\alpha^{-1}\eta^{\frac{1}{3}}$, and by Proposition 2.1, we obtain that $\operatorname{unc}(Y) \geq C_q(X)^{-1} \cdot f_0^{-1}(10^{-4}\alpha^{-1}\eta^{\frac{1}{3}})$ as required. □

PROOF OF PROPOSITION 4.3: Define x_k, y_k in (4.6) as follows: for $k \in B \cap C \cap A_0$, where $B \in \Delta_2$, $C \in \Delta_3$, put

$$x_k = \qquad d_C(X_3, X_2)^{-\frac{1}{2}} e_{2,k} + e_{3,k} + e_{4,k},$$

$$y_k = d_B(X_2, X_1)^{-\frac{1}{2}} e_{1,k} \qquad + e_{3,k}.$$

The result will follow from Proposition 2.1 and the following esimation:

$$\|T\| \geq 10^{-6} \alpha^{-1} \eta^{\frac{1}{2}}. \tag{4.11}$$

Fix $B \in \Delta_2 \cap \{A_0\}$. In particular, $\|id : X_2|_B \to X_1|_B\| \geq d_B(X_2, X_1)$. Choose $(\beta_k)_{k\in B}$ such that $\|\sum \beta_k e_{2,k}\| = 1$ and $\|\sum \beta_k e_{1,k}\| \geq d_B(X_2, X_1)$. By using (4.4), we see that $\|\sum \beta_k x_k\| < 3\alpha$ and

$$3\alpha\|T\| > \left\|T\left(\sum \beta_k x_k\right)\right\| = \left\|\sum \beta_k (a_k x_k + c_k y_k)\right\|$$
$$\geq d_B(X_2, X_1)^{-\frac{1}{2}} \left\|\sum \beta_k c_k e_{1,k}\right\| \geq d_B(X_2, X_1)^{\frac{1}{2}} \inf_{k\in B} |c_k|.$$

For every $B \in \Delta_2 \cap \{A_0\}$ pick $k \in B$ such that

$$|c_k| \leq 3\alpha d_B(X_2, X_1)^{-\frac{1}{2}} \|T\|. \tag{4.12}$$

Let K be the set of such k 's. We numerate members of $\Delta_2 \cap \{A_0\} = (B_k)_{k\in K}$ such that $k \in B_k$.

Suppose that $|c_k| < 10^{-2}$ for $k \in K$ (if not, then (4.11) holds by (4.3)).

Define $T_1 : \overline{\text{span}}\,(Z_k)_{k\in K} \to \overline{\text{span}}\,(Z_k)_{k\in K}$ such that

$$\begin{pmatrix} a_k & b_k \\ 0 & d_k \end{pmatrix}$$

is the matrix of $T_1|_{Z_k}$ with respect to (x_k, y_k). By (4.12) and (4.5), we see that

$$\|T_1\| \leq \|T\| + \sum_{k\in K} \left\| \begin{pmatrix} 0 & 0 \\ c_k & 0 \end{pmatrix} \right\| \leq \|T\| + 12 \sum_{k\in K} \left|\left|\left| \begin{pmatrix} 0 & 0 \\ c_k & 0 \end{pmatrix} \right|\right|\right|$$
$$\leq \|T\| + 36\alpha\|T\| \sum_{k\in K} d_{B_k}(X_2, X_1)^{-\frac{1}{2}} \leq 40\alpha\|T\|.$$

Hence

$$\|T_1\| \leq 40\alpha\|T\|. \tag{4.13}$$

Next, for fixed $C \in \Delta_3 \cap \{A_0\}$, we have

$$\left\| id : X_3|_{C\cap K} \to X_2|_{C\cap K} \right\| \geq d_C(X_3, X_2).$$

Choose $(\beta_k)_{k\in C\cap K}$ such that $\|\sum \beta_k e_{3,k}\| = 1$ and $\|\sum \beta_k e_{2,k}\| \geq d_C(X_3, X_2)$. Thus, by (4.5), one has

$$\left\| \sum \beta_k y_k \right\| \leq \sum_{k\in C\cap K} d_{B_k}(X_2, X_1)^{-\frac{1}{2}} |\beta_k| + \left\| \sum_k \beta_k e_{3,k} \right\| < 2;$$

hence, by (4.13),

$$80\alpha\|T\| \geq 2\|T_1\| \geq \left\| T_1\left(\sum \beta_k y_k\right) \right\| = \left\| \sum \beta_k (b_k x_k + d_k y_k) \right\|$$
$$\geq d_C(X_3, X_2)^{-\frac{1}{2}} \left\| \sum \beta_k b_k e_{2,k} \right\| \geq d_C(X_3, X_2)^{\frac{1}{2}} \inf_{k\in C\cap K} |b_k|.$$

For every $C \in \Delta_3 \cap \{A_0\}$ pick $k \in C \cap K$ (say $C = C_k$) such that

$$|b_k| \leq 80\alpha\|T\| d_{C_k}(X_3, X_2)^{-\frac{1}{2}}. \tag{4.14}$$

Let K' be the set of such k 's. As before, we may suppose that $|b_k| < 10^{-2}$ for $k \in K'$. Define $T_2 : \overline{\text{span}}\,(Z_k)_{k\in K'} \to \overline{\text{span}}\,(Z_k)_{k\in K'}$ such that

$$\begin{pmatrix} a_k & 0 \\ 0 & d_k \end{pmatrix}$$

is the matrix of $T_2|_{Z_k}$ with respect to (x_k, y_k). By (4.13), (4.14) and (4.5), we have

$$\begin{aligned} \|T_2\| &\le \|T_1\| + 12 \sum_{k\in K'} \left\| \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix} \right\| \\ &\le 40\alpha\|T\| + 12\cdot 80\cdot \alpha\|T\| \sum_{k\in K'} d_{C_k}(X_3, X_2)^{-\frac{1}{2}} \\ &< 10^3\alpha\|T\|; \end{aligned}$$

hence,

$$\|T_2\| < 10^3\alpha\|T\|. \tag{4.15}$$

By (4.8), and since $|c_k| < 10^{-2}$, $|b_k| < 10^{-2}$ for $k \in K'$, it follows that

$$|a_k - d_k| \ge 10^{-2} \qquad \text{for } k \in K'. \tag{4.16}$$

Observe that $\|id : X_4|_{A_0\cap K'} \to X_3|_{A_0\cap K'}\| \ge d_{A_0}(X_4, X_3)$. Choose $(\beta_k)_{k\in A_0\cap K'}$ such that $\|\sum \beta_k e_{4,k}\| = 1$ and $\|\sum \beta_k e_{3,k}\| \ge d_{A_0}(X_4, X_3)$. Again, by (4.5), we have

$$\begin{aligned} \left\|\sum \beta_k(x_k - y_k)\right\| \le &\sum_{k\in A_0\cap K'} d_{B_k}(X_2, X_1)^{-\frac{1}{2}}|\beta_k| \\ &+ \sum_{k\in A_0\cap K'} d_{C_k}(X_3, X_2)^{-\frac{1}{2}}|\beta_k| + \left\|\sum \beta_k e_{4,k}\right\| < 2; \end{aligned}$$

hence by (4.15) and (4.16), we see that

$$\begin{aligned}2\cdot 10^3\alpha\|T\| &\geq \Big\|T_2\Big(\sum\beta_k(x_k-y_k)\Big)\Big\| \\ &= \Big\|\sum\beta_k(a_kx_k-d_ky_k)\Big\| \geq \Big\|\sum\beta_k(a_k-d_k)e_{3,k}\Big\| \\ &\geq 10^{-2}\Big\|\sum\beta_k e_{3,k}\Big\| \geq 10^{-2}d_{A_0}(X_4,X_3).\end{aligned}$$

Finally, the last inequality shows (4.11), and by using Proposition 2.1(ii), we obtain the required estimation. □

3.5. Examples of spaces with no unconditional basis. In this section we shall present some specific examples of Banach spaces with no unconditional basis which are relatively "close" to ℓ_2. It is known that ℓ_p, for $p\neq 2$, contains subspaces with no unconditional basis; however, for various other spaces such examples were unknown.

As an illustration we will show that there exists a weak Hilbert space with no unconditional basis, thus answering a question raised some years ago by several authors (c.f. e.g. [CS], [P]).

Next, we use some notations and facts contained in [CS] and [P].

Recall that a Banach space X is a weak Hilbert space if for every $0<\delta<1$, there is a constant C_δ with the following property: every finite dimensional subspace $E\subset X$ contains a subspace $F\subset E$ with $\dim F\geq\delta\dim E$ such that $d(F,\ell_2^{|\dim F|})\leq C_\delta$ and there is a projection $P:X\to F$ with $\|P\|\leq C_\delta$.

Let X_δ be a version of the 2-convexified Tsirelson space, presented in [CS] and in [P] (Chapter 15). Then X_δ (respectively X_δ^*) is a weak Hilbert space (actually, X_δ is of type 2 and any cotype $q>2$). The standard unit vector basis is normalized and 1-unconditional in X_δ (in X_δ^* respectively). Moreover, X_δ

and X_δ^* contain ℓ_2^n 's uniformly on subsequences and do not contain subsequences equivalent to the unit vector basis of ℓ_2. In addition, in [J] we find that X_δ does not contain any isomorphic copy of ℓ_2.

Let us state the main theorem of this section.

THEOREM 5.1. *The space X_δ (respectively X_δ^*) contains a subspace with a basis, but not unconditional basis.*

PROOF:First consider X_δ^*. Then X_δ^* is of cotype 2, and since no subsequence of the basis in X_δ^* is equivalent to ℓ_2, it follows that no subsequence of the basis satisfies an upper 2-estimate. By Theorem 4.1(i), and since X_δ^* contains ℓ_2^n 's uniformly on subsequences, there exists a subspace Y in X_δ^* such that $\mathrm{unc}(Y) = \infty$. In fact, it follows from the construction of Y, that Y has a basis $x_1, y_1, x_2, y_2, \ldots$.

Next consider X_δ. Then X_δ is of type 2. Since X_δ does not contain any isomorphic copy to ℓ_2, it follows that condition (ii) in Theorem 4.1 is satisfied; hence, there exists a subspace in Y in X_δ such that $\mathrm{unc}(Y) = \infty$. □

REMARK 5.2. The existence of a weak Hilbert space with no unconditional basis follows also from Theorem 2.2. It is a consequence of the fact (cf. [C]) that the spaces X_δ are all mutually non-isomorphic (i.e. if $\delta \neq \delta'$, then X_δ and $X_{\delta'}$ have non-isomorphic infinite dimensional subspaces) and the observation that a direct sum of finitely many copies of weak Hilbert spaces is a weak Hilbert space.

REMARK 5.3. Recall that the class of weak Hilbert spaces can be characterized by a certain linear behaviour of various functions associated with the finite-dimensional structure of Banach spaces. Such functions are, for instance, the codimension of a nicely complemented Euclidean subspace in any n-dimensional subspace of X (cf. e.g. [P]), or, as shown in [JP], the uniformity function of the uniform approximation property of X. It is well known ([P]) that for examples related to the Tsirelson space, these functions have much slower growth than linear, and the same has been recently shown in [NT] for an arbitrary weak Hilbert space with an unconditional basis. Obviously, for the weak Hilbert spaces without unconditional basis, constructed by using Theorem 4.1, the first mentioned function, as well as many other functions discussed in [P], which do not necessarily characterize weak Hilbert spaces, has growth as slow as in the Tsirelson space.

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