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A STUDY ON CERTAIN HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS OF MANGERON

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies & Research for acceptance, a thesis entitled A STUDY ON CERTAIN HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS OF MANGERON submitted by SUNDARAM EASWARAN in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Supervisor

W. Allegretto

Z. Ditzian

External Examiner

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ABSTRACT

This thesis is devoted to the study of certain non-classical partial differential equations referred to by many authors as Mangeron's equations or polyvibrating equations of Mangeron; specifically, we investigate here the partial differential equation

$$(1) \qquad Lu = \frac{\partial^2}{\partial x \partial y}\left(\theta \frac{\partial^2 u}{\partial x \partial y}\right) + pu = f$$

subject to various types of boundary conditions on the boundary ∂R of the rectangle $R : \{a \leq x \leq b \; ; \; c \leq y \leq d\}$, where $\theta(x,y)$, $p(x,y)$ and $f(x,y)$ are functions defined on R with certain specified properties. Differential equations of the above type have applications in many problems of mathematical physics and in some areas of multidimensional interpolation theory.

In the present work, using the tools of functional analysis, we establish a general abstract background for the theory of polyvibrating equations and unify several known results with our new ones. The underlying connection between all the results obtained is the positive definiteness of polyvibrating operators subject to certain restrictions, which shall be made explicit in the course of the thesis. We achieve our objective, essentially, in three steps.

First, we consider an associated variational problem which consists of minimizing a functional of the form

$$\int_a^b \int_c^d f(x,y,u,u_{xy})\, dxdy \tag{2}$$

over a specified class of functions. Necessary and sufficient conditions, analogous to fixed end point problems of the calculus of variations, are derived in the second chapter.

The second step is devoted to the study of the existence of a solution in a Hilbert space $V_0^{(1)}$ consisting of functions which are absolutely continuous in the sense of Vitali and satisfy certain additional properties. The Hilbert space $V_0^{(1)}$ seems to be considered here for the first time. We generalize the concept of eigenfunctions and eigenvalues, and show that, for the associated Sturm-Liouville problem, the eigenfunctions form a complete set in $V_0^{(1)}$. We consider eigenvalue problems involving natural boundary conditions in Chapter IV and prove a comparison theorem. Chapter V deals with partial differential equations of the type (1) involving mixed boundary conditions. We construct Green's functions for polyvibrating operators and it is also shown that in certain cases the Green's function is positive, a property which is very important in the study of the oscillatory nature of eigenfunctions.

Lastly we generalize many results concerning Equation 1 to higher order polyvibrating equations. For this purpose we introduce certain new Hilbert spaces $V_0^{(n)}$ $(n = 1,2,\cdots)$, which are subspaces of $V_0^{(1)}$. We conclude the thesis with the study of the existence of a solution to certain integro-partial differential equations and with an up to date bibliography.

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CHAPTER I

Scope of the Thesis

§1. Introduction.

Many properties of the mixed derivative u_{xy} of a function $u(x,y)$ of two real variables x,y are extensions of properties of the ordinary first derivative of a function of one variable. For example, if $v(t)$ denotes a function defined on $[a,b]$ such that $v'(t)$ exists and is continuous in (a,b) , then

$$(1) \qquad v(t) = v(a) + \int_a^t v'(\xi)d\xi \quad .$$

This shows that the function value $v(t)$ can be recovered provided we are given the derivative in the interval $a < t < b$ and the initial value of the function at $t = a$. This characterestic property of $\frac{dv}{dt}$ extends to the function of two variables as follows. If $u(x,y)$ is a continuous function defined on a rectangle $R:\{a \leq x \leq b;\ c \leq y \leq d\}$ such that the derivative $u_{xy}(x,y)$ exists in the interior of R , then

$$(2) \qquad u(x,y) = u(a,y) + u(x,c) - u(a,c) + \int_a^x\int_c^y u_{\xi\eta}d\xi d\eta \quad .$$

Taylor's formula for functions of a single variable can also be extended as follows:

Theorem I. 1.1 [13]. *If $v(x)$ is a function that is continuous together with its first $(n+1)$ derivatives on an interval containing a and x , then the value of the function at x is given by*

$$(3) \qquad v(x) = v(a) + v'(a)(x-a) + \cdots + \frac{v^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{v^{(n)}(a)}{n!}(x-a)^n + R_n(x,a) \quad ,$$

where

$$R_n(x,a) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \quad .$$

The equation (*I.1.3*) can also be written in the form

$$(4) \qquad v(x) = v(a) + \int_a^x v'(a)d\xi + \int_a^x \frac{(x-\xi)}{1!} v''(a)d\xi + \cdots + \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!} v^{(n)}(a)d\xi + \int_a^x \frac{(x-\xi)^n}{n!} v^{n+1}(\xi)d\xi \quad .$$

This form suggests the extension, as given by M. Picone [1].

Theorem I. 1.2 (M. Picone [1]). *Let* $u(x,y)$ *be a real valued function of two variables such that it is continuous together with the partial derivatives*

$$(5) \qquad \frac{\partial^{h+k} u}{\partial x^h \partial y^k} \qquad (0 \le h \le n+1 \ , \ 0 \le k \le n+1)$$

in $a \le x \le b$ *and* $c \le y \le d$. *Let*

$$(6) \qquad u^{(k)}(x,y) = \frac{\partial^{2k} u(x,y)}{\partial x^k \partial y^k} \qquad (0 \le k \le n+1)$$

and

$$(7) \qquad u_k(x,y) = u^{(k)}(a,y) + u^{(k)}(x,c) - u^{(k)}(a,c)$$

for $0 \le k \le n+1$. *Then* $u(x,y)$ *can be written in the form*

$$(8) \qquad u(x,y) = u_0(x,y) + \int_a^x\int_c^y u_1(\xi,\eta)d\xi d\eta + \cdots$$

$$+ \int_a^x\int_c^y \frac{(x-\xi)^k(y-\eta)^k}{(k!)^2} u_{k+1}(\xi,\eta)d\xi d\eta + \cdots$$

$$+ \int_a^x\int_c^y \frac{(x-\xi)^{n-1}(y-\eta)^{n-1}}{[(n-1)!]^2} u_n(\xi,\eta)d\xi d\eta + \int_a^x\int_c^y \frac{(x-\xi)^n(y-\eta)^n}{(n!)^2} u^{(n+1)}(\xi,\eta)d\xi$$

Another interesting similarity between the ordinary derivative for functions of one variable and the mixed derivative $u_{xy}(x,y)$ is to be found in the theory of probability. Specifically, if $v(x)$ denotes the probability that the random variable R takes on a value less than or equal to x and if $v(x)$ is differentiable then $\frac{dv}{dx}$ is the probability density function. Similarly, if $u(x_1,x_2)$ denotes the probability that the random variables R_1,R_2 take values less than or equal to x_1 and x_2 , respectively, and if the mixed derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ exists, then it is the joint probability density function of this random process.

If a function $v(t)$ is absolutely continuous in (a,b) then its derivative exists almost everywhere in (a,b) . To show how this property is extended, we need the following definition of absolute continuity of a function of two variables given by Vitali:

Definition I. 1.3 [14]. *A function* $u(x,y)$ *defined on the rectangle* $R:\{a \le x \le b \; ; \; c \le y \le d\}$ *is said to be absolutely continuous in the sense of Vitali in* R , *if given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for any finite or infinite set of nonoverlapping subrectangles* $\{R_i\}$ *of* R

$$meas\ (\cup_i R_i) < \delta \Rightarrow \sum_i |F_u(R_i)| < \varepsilon$$

where $F_u(R_i)$ *denotes the following double difference*

$$F_u(R_i) = u(a_i,c_i) - u(a_i,d_i) - u(b_i,c_i) + u(b_i,d_i)$$

for the rectangle

$$R_i : \{a_i \leq x \leq b_i \ ; \ c_i \leq y \leq d_i\} \quad .$$

Extensive work was done in the beginning of this century on this type of absolute continuity by G.H. Hardy, M. Krause and W. Young. Hardy and Krause have defined the concept of bounded variation for functions of two variables and used it in the study of expansion of functions of two variables in terms of their double Fourier Series. A detailed account can be found in the book by E.W. Hobson [*14*].

Thus it is quite natural to expect that there should be similarities between the ordinary second derivative of a function of one variable and the fourth order partial derivative $\frac{\partial^4 u}{\partial x^2 \partial y^2}$ of a function $u(x,y)$ of two variables. This idea has been used by D. Mangeron [*2*] in his habilitation thesis, at the suggestion of M. Picone. D. Mangeron specifically considered the eigenvalue problem for the equation

$$(9) \qquad \frac{\partial^4 u}{\partial x^2 \partial y^2} = \lambda\, A(x,y)u$$

subject to the boundary conditions

(9)* $$u(c,y) = u(b,y) = u(x,c) = u(x,d) = 0$$

where $A(x,y)$ is a positive continuous function defined on the rectangle $R:\{a \leq x \leq b \ ; \ c \leq y \leq d\}$. He extended many properties which hold for the following simple eigenvalue problem for the ordinary differential system

$$\frac{d^2v}{dx^2} = -\lambda \ p(x)u$$

$$v(a) = v(b) = 0$$

D. Mangeron [2] showed that the eigenvalue problem (*I.1.9*) - (*I.1.9*)* is equivalent to the minimization of the following double integral

(10) $$\int_a^b\int_c^d u_{xy}^2 \ dxdy$$

over the class of $C^2(R^2)$ functions satisfying the conditions

(11) $$\begin{cases} \text{(i)} \quad u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0 \\ \text{(ii)} \quad \int_a^b\int_c^d A(x,y)u^2 \ dxdy = 1 \ . \end{cases}$$

Noticing the similarity between the problem of minimization of (*I.1.10*) and simple integral problems of the calculus of variations, M. Salvadori [39] has considered the problem of minimization of integrals of the type

(12) $$\int_a^b\int_c^d f(x,y,u;u_{xy})dxdy$$

over a specified class of functions and has generalized may of the results in the calculus of variations which hold for the following simple integral

(13) $$\int_a^b f(x,y,y')dx \quad .$$

Further, Tonelli [3] has given many criteria which ensure the existence of an absolute minimum for functionals of the type (*I.1.13*). These results have been generalized to the problem (*I.1.12*) by G. Stampacchia [*37*]. In this connection it is important to mention a lemma of M. Mason [*24*], which seems to have been missed by the above mentioned authors. The article by A. Huke [*15*] gives an excellent account of the fundamental lemmas of the calculus of variations, and this is our source for the lemma of Mason. This lemma will be given in Chapter II.

F. Maneresi [*20*] has considered the following Sturm Liouville problem

(14) $$(\theta\, u_{xy})_{xy} + pu = \lambda\, qu$$

(14)* $$u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0$$

where $\theta(x,y)$ is positive continuous function in R such that $\theta_x, \theta_y, \theta_{xy}$ are all continuous and p,q are nonnegative continuous functions defined on R . The problem of F. Maneresi is quite easily seen to be very similar to the following Sturm-Liouville problem for an ordinary self adjoint differential system

(15) $$\begin{cases} (r(x)y')' + s(x)y = \mu\, t(x)y \\ y(a) = y(b) = 0 \end{cases}$$

where $r(x)$, $s(x)$, and $t(x)$ are sufficiently smooth in $[a,b]$. We mention in what follows some work done in this direction. D. Mangeron and L.E. Krivosein have considered in a series of papers [22] integro-partial differential equations of the type

$$\frac{\partial^{2n} u}{\partial x^n \partial y^n} = f(x,y) + \lambda \int_a^b \int_c^d K(x,y;\xi,\eta)\, u(\xi,\eta) d\xi d\eta$$

$$u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0$$

$$\frac{\partial^{2i} u}{\partial x^i \partial y^i}(a,y) = \frac{\partial^{2i} u}{\partial x^i \partial y^i}(x,c) = 0 \qquad (i = 1,2,\cdots n-1)$$

for $c \leq y \leq d$ and $a \leq x \leq b$, respectively. They have considered the problem both when $K(x,y;\xi,\eta)$ is a Fredholm or Volterra type of kernel. In this connection we should also mention a series of papers published by D. Mangeron and M.N. Oguztoreli [23] where the authors have considered the following partial differential difference equation

$$\frac{\partial^2}{\partial x \partial y} u(x,y;\alpha) = u(x,y;\alpha+1) \quad .$$

This work extends the work of F. Truesdell [39] on the ordinary differential-difference equation

$$\frac{dy}{dx}(x,\alpha) = y(x,\alpha+1) \quad .$$

If we make a transformation of variables

$$x = \xi + \eta \quad ; \quad y = \xi - \eta$$

then the partial differential operator $\dfrac{\partial^2 u}{\partial x \partial y}$ is transformed into the partial differential operator $(\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial^2 u}{\partial\eta^2})$, and $\frac{\partial^{2n} u}{\partial x^n \partial y^n}$ goes over to $(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2})^{(n)} u$. It is well known that the partial differential equation

$$\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial^2 u}{\partial\eta^2} = a^2 u$$

represents the equation of a vibrating string. Thus rightly D. Mangeron has called the operator $\frac{\partial^{2n}}{\partial x^n \partial y^n}$ a polyvibrating operator of order n for the same reason as in the case of polyharmonic operators. I.N. Vekua [*40*] has considered the solution of polyharmonic equations

$$\Delta^{(n)} u = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^{(n)} u = f \qquad (16)$$

by making a transformation of the type

$$z = x + iy \qquad \bar{z} = x - iy$$

and then the equation (*I.1.16*) is equivalent to the following partial differential equation

$$\frac{\partial^{2n} u}{\partial z^n \partial \bar{z}^n} = f(\frac{z+\bar{z}}{2} \;,\; \frac{z-\bar{z}}{2i}) \;.$$

We have also to mention the application of these types of operators in approximation theory for functions of two variables: in [*2*] it has been shown that a two dimensional spline is a function which minimizes the integral

$$\int_a^b\int_c^d \{\frac{\partial^4 u}{\partial x^2 \partial y^2}\}^2 \, dxdy \qquad (17)$$

where $u(x,y)$ is assumed to satisfy certain differentiability conditions. G. Birkhoff and W. Gordon [7] have used these ideas very recently in their paper on the Draftsmen's equation and related problems. In this connection we should mention that many European Mathematicians have also considered such problems. (cf: M. Picone [1,2], D.V. Ionescu [16]).

It is well known [30] that certain plane problems in elasticity theoryare equivalent to the minimization of the quadratic form

$$(18) \qquad \Phi[u] = \iint_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^2 dxdy$$

over the space of functions u satisfying the boundary conditions

$$(19) \qquad u\Big|_{\partial R} = 0 \quad ; \quad \frac{du}{dn}\Big|_{\partial R} = 0 \quad .$$

It can easily be seen, by virtue of the boundary conditions, that there exists constants c_1 and c_2 such that

$$(20) \qquad c_1 \iint_R u_{xy}^2 \, dxdy \leq \Phi[u] \leq c_2 \iint u_{xyxy}^2 \, dxdy$$

The above inequality has been used by N. Aronszajn and W.F. Donoghue [4] to find upper and lower bounds for the functional $\Phi[u]$. This has been done by solving the eigenvalue problems

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \lambda \, u \quad , \quad \frac{\partial^8 u}{\partial x^4 \partial y^4} = \mu \, u$$

subject to the boundary conditions (I.1.19). Partial differential equations of this type also occur in some problems of chemistry. M.N. Oguztoreli

[4] has considered the above type of problems in connection with his study on distributed parameter control systems.

§2. Description of the Work Done in This Thesis.

The fundamental problem considered in this thesis is to investigate systematically certain boundary value and mixed problems associated with a class of polyvibrating equations of Mangeron, to unify the results obtained up to date with our new ones, and to give a solid foundation for the theory of polyvibrating equations.

In Chapter II we first give a simple proof of a lemma of Mason for continuous functions $u(x,y)$ whose partial derivatives u_x , u_y , u_{xy} have discontinuities only along lines parallel to the coordinate axes. This proof of Mason's lemma is then used to prove some theorems of M. Salvadori to the above class of functions. In this way we prove certain results in one dimensional problems in the calculus of variations (*I.1.13*), where the admissible functions are continuous functions whose derivatives are piecewise continuous in a specified domain of the real line. Necessary conditions and sufficient conditions for the existence of an extremal are established and the results obtained in this chapter are applied to establish the existence and uniqueness of the solution of the Mangeron equation (*I.1.14*) subject to certain boundary conditions.

In Chapter III we show that the functions defined on R which are absolutely continuous in the sense of Vitali and which vanish on the boundary of R , form a Hilbert space $V_0^{(1)}$, with respect to the inner product

(1) $$((u,v)) = \int_a^b \int_c^d u_{xy}\, v_{xy}\, dxdy \quad .$$

We have also shown, using a method due to N. Aronszan and Donoghue[4], that this Hilbert space is the completion of $C_0^\infty(R)$, functions which are infinitely differentiable in R and with compact support in Int (R) , with respect to the norm defined by (I.2.1). This leads us to the consideration of the problem of existence and uniqueness of a generalized solution to the following Manaresi type system:

(2) $$(\theta\, u_{xy})_{xy} + pu = f$$

(3) $$u(a,y) = u(x,c) = u(b,y) = u(x,d) = 0$$

where $\theta(x,y)$, $p(x,y)$ are essentially bounded functions and $f(x,y) \varepsilon L_2(R)$. We also assume the existence of a positive constant θ_0 such that $\theta(x,y) \geq \theta_0$. Further, we consider generalized eigenfunctions and eigenvalues for the partial differential equation (I.2.2) subject to the boundary conditions (I.2.3). In this analysis we make use of an idea due to E.M. Landesman and A.C. Lazer [19].

In Chapter IV we give some "fundamental" inequalities connecting the function $u(x,y)$ and its mixed derivative $u_{xy}(x,y)$ where $u(x,y)$ is subjected to certain boundary conditions. One of these is an extension of Poincare's inequality [1]. This inequality will then be used to investigate the existence of a solution of the simple boundary value problem

(4) $$\frac{\partial^4 u}{\partial x^2 \partial y^2} = f(x,y) \qquad f \varepsilon L_2(R)$$

$$(5) \qquad u_{xy}(a,y) = u_{xy}(b,y) = u_{xy}(x,c) = u_{xy}(x,d) = 0 .$$

It will also be shown that the boundary conditions ($I.2.5$) are of unstable type. This shows the peculiarity of this type of problems, which are parabolic rather than hyperbolic.

In Chapter V we consider the classical operator ($I.2.2$), i.e., we assume that θ , p , f are continuous functions in R such that $\theta(x,y) > 0$, $p(x,y) \geq 0$ and $\theta_x(x,y)$, $\theta_y(x,y)$ and $\theta_{xy}(x,y)$ are all continuous. The boundary conditions subject to which the solutions are sought are of the form

$$(6) \qquad \begin{cases} \alpha_i \, u(x_i,y) + (-1)^i u_x(x_i,y) = 0 \\ \\ \beta_j \, u(x,y_j) + (-1)^j u_y(x,y_j) = 0 \end{cases}$$

where $x_1 < x_2$, $y_1 < y_2$ are given, and α_i , β_j are nonnegative constants such that at least one of the products $\{\alpha_i \beta_j | i,j=1,2\}$ is not equal to zero. Subject to these conditions we will show the positive definiteness of the partial differential operator in ($I.1.14$) and also show that the operator has a countable sequence of eigenvalues tending to infinity. The interesting fact about this type of boundary conditions is that when $\alpha_i = \beta_j = \infty$, they reduce to

$$u(x_i,y) = u(x,y_i) = 0 \qquad (i=1,2)$$

for $y_1 \leq y \leq y_2$ and $x_1 \leq x \leq x_2$. Thus this interpretation gives us a comparison between eigenvalues of (*I.1.14*) subject to two different types of boundary conditions.

In Chapter VI we investigate the concept of Green's functions for polyvibrating operators. These will be found explicitly by use of some functional analytic results. We will also study the variation diminishing property of the Green's functions in certain simple cases. The proof follows closely to the one given by R. Bellman [5] for ordinary differential equations.

In Chapter VII we give generalizations of the results obtained in previous chapters to higher order polyvibrating operators of the form

$$L_1\, u = \sum_{i=0}^{n} \frac{\partial^{2i}}{\partial x^i \partial y^i} \left(\theta_i \frac{\partial^{2i} u}{\partial x^i \partial y^i}\right)$$

$$L_2\, u = \frac{\partial^2}{\partial x \partial y} \left[\theta \frac{\partial^2}{\partial x \partial y} \left(\theta_1 \frac{\partial^2}{\partial x \partial y}\, \theta\, \frac{\partial^2 u}{\partial x \partial y}\right)\right]$$

where the θ_i's are functions with properties to be specified. Particularly of interest is the question of the positive definiteness of these operators. These properties help us in defining Hilbert spaces $V_0^{(n)}$, which are subspaces of the Hilbert space $V_0^{(1)}$ introduced in Chapter III.

We conclude this thesis with an up to date bibliography and with an indication of many open problems which could be pursued in future.

CHAPTER II

Variational Problems Associated With Polyvibrating Equations

§1. Introduction.

In this chapter we consider the problem of finding a function $\overset{\circ}{u}(x,y)$, belonging to a class of functions to be specified, which minimizes the integral

$$(1) \qquad J[u] = \int_a^b \int_c^d f(x,y,u,u_{xy})dxdy$$

and investigate the properties of this minimizing function. It is assumed that f is a real valued function of class $C'(R^4)$.

The above mentioned problem was originally suggested by M. Picone [21] and has been extensively studied by D. Mangeron [21], M. Salvadori [34] and by G. Stampacchia [37]. Our presentation in this chapter closely follows that of M. Salvadori, with the addition of some new results. In Section 2 we begin with some notation and definitions. Sections 3 and 4 deal with a lemma of M. Mason [24] and its extensions. First and second variations of $J[u]$ and necessary conditions for the existence of a minimizing function are given in Sections 5 and 6. Sufficient conditions for the functional $J[u]$ to have an absolute minimum are dealt with in Section 7. In Section 8 we use the results of Section 7 to show the existence and uniqueness of solutions of a boundary value problem for Mangeron's equations (*I.2.14*).

§2. Notation and Definitions.

Let R be the rectangle $\{a \leq x \leq b \; ; \; c \leq y \leq d\}$ in the two dimensional euclidean plane and ∂R be its boundary. By $U^{(1)}$ we shall denote the class of all continuous functions $u(x,y)$ defined on R such that the partial derivatives $u_x(x,y)$, $u_y(x,y)$ and $u_{xy}(x,y)$ have discontinuities only along a finite number of lines parallel to the axes of coordinates. $U^{(2)}$ will denote the functions u such that u_{xy} belongs to $U^{(1)}$. Similarly $U^{(n)}$ will denote the class of functions $u(x,y) \in U^{(n)}$ such that $\dfrac{\partial^{2n-2} u}{\partial x^{n-1} \partial y^{n-1}} \in U^{(1)}$. The space of functions $u(x,y) \in {}^{(n)}$ such that $u = g$ (given function) on ∂R will be denoted by $U_g^{(n)}$.

By $\Gamma^{(1)}$ we shall specify the subclass of functions u in $U^{(1)}$ such that all the partial derivatives u_x , u_y and $u_{xy}(x,y)$ are continuous in R . Inductively $\Gamma^{(n)}$ will represent the class of functions such that $\dfrac{\partial^{2n-2} u}{\partial x^{n-1} \partial y^{n-1}}$ belongs to $\Gamma^{(1)}$. Further $\Gamma_g^{(n)}$ will denote the functions u in $\Gamma^{(n)}$ such that $u = g$ on the boundary of R . Observe that $\Gamma^{(n)} \subset U^{(n)}$.

§3. Mason's Lemma.

In this section we state and prove the following lemma due to M. Mason [24], which plays an important role in the calculus of variations.

Lemma II.3.1. *If* $F(x,y)$ *is a continuous function defined on* R *such that*

$$(1) \qquad \int_a^b \int_c^d F(x,y)\, U_{xy}\, dx dy = 0$$

for all $u \in U_0^{(1)}$, *then*

$$(2) \qquad F(x,y) = A(x) + B(y)$$

where A *and* B *depend only on* F.

Proof: Let $(\bar{x},\bar{y}) \in R$. Choose $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \frac{\bar{x}-a}{2}$, $0 < \varepsilon_2 < \frac{\bar{y}-c}{2}$ and consider the functions $z_1(x)$ and $z_2(y)$ defined as follows:

$$z_1(x) = \begin{cases} \frac{1}{\varepsilon_1}(x-a) & a \le x \le a+\varepsilon_1 \\ 1 & a+\varepsilon_1 \le x \le \bar{x}-\varepsilon_1 \\ \frac{1}{\varepsilon_1}(\bar{x}-x) & \bar{x}-\varepsilon_1 \le x \le \bar{x} \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_2(y) = \begin{cases} \frac{1}{\varepsilon_2}(y-c) & c \le y \le c+\varepsilon_2 \\ 1 & c+\varepsilon_2 \le y \le \bar{y}-\varepsilon_2 \\ \frac{1}{\varepsilon_2}(\bar{y}-y) & \bar{y}-\varepsilon_2 \le y \le \bar{y} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly the function $u(x,y) = z_1(x)z_2(y)$ belongs to $U_0^{(1)}$ and by virtue of condition $(II.4.1)$, we have

$$\int_a^b\int_c^d F(x,y)u_{xy}\,dydx = \frac{1}{\varepsilon_1\varepsilon_2}\Big[\int_a^{a+\varepsilon_1}\int_c^{c+\varepsilon_2} F(x,y)\,dydx$$

$$(3) \qquad - \int_{\bar{x}-\varepsilon_1}^{\bar{x}}\int_c^{c+\varepsilon_2} F(x,y)\,dydx + \int_{\bar{x}-\varepsilon_1}^{\bar{x}}\int_{\bar{y}-\varepsilon_2}^{\bar{y}} F(x,y)\,dydx$$

$$- \int_a^{a+\varepsilon_1}\int_{\bar{y}-\varepsilon_2}^{\bar{y}} F(x,y)dydx\Big] = 0 \quad .$$

Since $F(x,y)$ is continuous, letting $\varepsilon_1,\varepsilon_2$ tend to zero, we obtain

$$F(\bar{x},\bar{y}) - F(a,\bar{y}) - F(\bar{x},c) + F(a,c) = 0$$

which proves the assertion of the lemma.

We now recall the following definition of the concept of quasi-monotonicity of a function of two variables.

Definition [14]. *A function* $F(x,y)$ *defined on* R *is said to be quasi-monotone if and only if the inequalities* $x_1 \geq x_2$, $y_1 \geq y_2$ *imply the inequality*

$$(4) \qquad F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1) \geq 0 \quad .$$

The following result is of particular interest in the theory of quasi-monotone functions.

Corollary: *If* $\overset{+}{\Gamma}{}_o^{(1)}$ *denotes the set of nonnegative functions belonging to* $\Gamma_o^{(1)}$ *, then a function* $F(x,y)$ *defined on* R *is quasi-monotone if and only if*

$$\int_a^b \int_c^d F(x,y)\; u_{xy}(x,y)\; dy dx \geq 0$$

for all functions $u \in \overset{+}{\Gamma}{}_o^{(1)}$.

Proof: The 'if' part follows from the lemma (*II.4.1*) by replacing (a,c) and $(\overline{x},\overline{y})$ by (x_1,y_1) and (x_2,y_2) respectively. In order to prove the converse assertion, we consider the function $q(x,y)$ defined below which is a continuous extension of $F(x,y)$.

$$q(x,y) = \begin{cases} F(a,y) & x < a \quad ; \; c \leq y \leq d \\ F(b,y) & x > b \quad ; \; c \leq y \leq d \\ F(x,d) & a \leq x \leq b \; ; \; y > d \\ F(x,c) & a \leq x \leq b \; ; \; y < c \\ F(x,y) & a \leq x \leq b \; ; \; c \leq y \leq d \\ F(a,d) & x < a \quad ; \; y > d \\ F(b,d) & x > b \quad ; \; y > d \\ F(b,c) & x > b \quad ; \; y < c \\ F(a,c) & x < a \quad ; \; y < c \end{cases}$$

Clearly $q(x,y)$ is quasi-monotone in R . Put

(5) $$q(x,y:\alpha,\beta) = \frac{1}{4\alpha\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} q(x+\xi,y+\eta)\; d\eta d\xi$$

$$= \frac{1}{4\alpha\beta}\int_{x-\alpha}^{x+\alpha}\int_{y-\beta}^{y+\beta} q(\xi,\eta)\; d\eta d\xi \; .$$

Then we have

(6) $$\frac{\partial q}{\partial x\partial y}(x,y;\alpha,\beta) = \frac{1}{4\alpha\beta}[q(x+\alpha,y+\beta) - q(x-\alpha,y+\beta) -$$

$$- q(x+\alpha,y-\beta) + q(x-\alpha,y-\beta)]$$

which is nonnegative, since $q(x,y)$ is quasi-monotone on R. Thus if $u \in \overset{+}{U}_o^{(1)}$ we have

$$(7) \qquad \int_a^b\int_c^d q_{xy}(x,y;\alpha,\beta)\ u\ dxdy = \int_a^b\int_c^d q(x,y;\alpha,\beta)\ u_{xy}\ dxdy \geq 0$$

because $u(x,y)$ vanishes on the boundary ∂R. Taking the limit as α,β tends to zero, we obtain $q(x,y;\alpha,\beta) \to q(x,y)$ and consequently

$$\int_a^b\int_c^d F(x,y)\ u_{xy}\ dxdy \geq 0 \quad .$$

This completes the proof of the second part of the corollary.

§5. The First and Second Variations of J.

As usual we denote the partial derivatives of $f(x,y,u,v)$ by f_x, f_y, f_{xy}, f_{yy} ··· etc. Let $U_g^{(1)}$ be the class of admissible functions and let $u \in U_g^{(1)}$ and $w \in U_o^{(1)}$. Choose $\delta > 0$ such that the function $u + \varepsilon w$ is admissible if ε is on the range $-\delta \leq \varepsilon \leq \delta$. Clearly the function

$$(1) \qquad F(\varepsilon) = J[u + \varepsilon w] = \int_a^b\int_c^d f(x,y,u+\varepsilon w,u_{xy}+\varepsilon w_{xy})\ dxdy$$

is in the class $C''(-\delta,\delta)$. The derivative $F'(o)$ of F at $\varepsilon = 0$, which is called the first variation of J at $u(x,y)$, is denoted by $J'(u,w)$. By differentiating (*II.5.1*) with respect to ε at $\varepsilon = 0$, we obtain

$$(2) \qquad J'[u,w] = \int_a^b\int_c^d [f_u\, w(x,y) + f_v\, w_{xy}(x,y)]\; dxdy$$

where the arguments in the partial derivatives of f are (x,y,u,u_{xy}) . The second variation of J along u is denoted by the symbol $J''(u,w)$. It can easily be shown that

$$(3) \qquad J''(u,w) = \int_a^b\int_c^d 2\,W(x,y,u,u_{xy})\; dxdy$$

where

$$2\,W(x,y,u,u_{xy}) = f_{u\,u}w^2 + 2\,f_{uv}ww_{xy} + f_{vv}w^2{}_{xy}$$

and the arguments of the derivatives of f are (x,y,u,u_{xy}) .

Lemma II.5.1. *Given a function* $u \in U_g^{(1)}$ *there is a unique* $z \in U_o^{(1)}$ *such that*

$$(4) \qquad J'(u,w) = ((w,z))$$

for all $w \in U_o^{(1)}$ *, where*

$$(5) \qquad ((w,z)) = \int_a^b\int_c^d w_{xy}\; z_{xy}\; dxdy \quad .$$

The function $z(x,y)$ *is defined by the relationship*

$$(6) \qquad z(a,y) = z(x,c) = z(a,c) = 0$$

and

$$(7) \quad z_{xy}(x,y) = f_v(x,y,u,u_{xy}) + \int_a^x\int_c^y f_u(\xi,\eta,u,u_{\xi\eta})d\xi d\eta - A(x) - B(y) + C$$

where

$$A(x) = \frac{1}{d-c}\int_c^d \chi(x,y)\ dy$$

$$B(y) = \frac{1}{(b-a)}\int_a^b \chi(x,y)\ dx$$

$$C = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \chi(x,y)\ dxdy$$

and

$$\chi(x,y) = f_v(x,y,u,u_{xy}) + \int_a^x\int_c^y f_u(\xi,\eta,u,u_{\xi\eta})\ d\xi d\eta \quad .$$

Proof: First we establish (*II.5.4*) . Let $w \in U_o^{(1)}$. Then, since

$$w(a,y) = w(b,y) = w(x,c) = w(x,d) = 0$$

we obtain

$$(8) \qquad \int_a^b\int_c^d w_{xy}(x,y)\ [\int_a^x\int_c^y f_u d\xi d\eta - A(x) - B(y) + C]\ dxdy \ =$$

$$= \int_a^b \int_c^d w(x,y) \; f_u(x,y,u,u_{xy}) \; dxdy \; .$$

The left hand side of (*I.5.8*) is equal to

$$\text{(9)} \qquad \int_a^b \int_c^d w_{xy} [-f_v + z_{xy}] \; dxdy \; ,$$

by virtue of (*I.5.7*). Hence, combining (*I.5.8*) and (*I.5.9*), the result follows. We now show uniqueness of function $z(x,y)$. If there were two functions z_1 and z_2 in $U_o^{(1)}$ such that

$$((w,z_1)) = ((w,z_2)) \quad ,$$

then we would have

$$((w,z_1 - z_2)) = 0 \quad .$$

Hence, taking $w = z_1 - z_2$, we obtain

$$\frac{\partial^2}{\partial x \partial y} (z_1 - z_2) = 0$$

except perhaps at the discontinuities of $\frac{\partial^2}{\partial x \partial y} (z_1 - z_2)$ and therefore

$$z_1(x,y) - z_2(x,y) = C(x) + \mathcal{D}(y)$$

C , $\mathcal{D}$ are <u>arbitrary</u> <u>functions</u> of x,y . But the boundary conditions

($II.5.6$) imply that

$$z_1(x,y) \equiv z_2(x,y)$$

and this completes the proof.

Theorem II.5.2. *If a function* $\overset{o}{u}(x,y) \in U_g^{(1)}$ *minimizes* $J[u]$ *, then*

$$\text{(i)} \qquad J'(\overset{o}{u},w) = 0 \quad \text{and} \quad J''(\overset{o}{u},w) \geq 0$$

for all $w \in U_o^{(1)}$ *;*

$$\text{(ii)} \qquad J'(\overset{o}{u},w) = 0 \ \text{ holds for all } \ w \in U_o^{(1)}$$

if and only if

$$\overset{o}{f}_v(x,y) + \int_a^x \int_c^y \overset{o}{f}_u(\xi,\eta)\, d\xi d\eta = A(x) + B(y) - C$$

where $A(x)$ *,* $B(y)$ *and* C *are as defined in lemma I.5.1.* $\overset{o}{f}_v(x,y)$ *and* $\overset{o}{f}_u(x,y)$ *denote* $f_v(x,y,\overset{o}{u},\overset{o}{u}_{xy})$ *and* $f_u(x,y,\overset{o}{u},\overset{o}{u}_{xy})$ *respectively.*

Proof: Suppose that $\overset{o}{u}(x,y)$ minimizes $J[u]$ in $U_g^{(1)}$. Then the function $F(\varepsilon)$ in ($II.5.1$) with $u = \overset{o}{u}(x,y)$ has a minimum at $\varepsilon = 0$. Hence it follows that

$$(10) \qquad F'(0) = J'(\overset{o}{u},w) = 0 \quad \text{and} \quad J''(\overset{o}{u},w) \geq 0 \ .$$

This proves the assertion (*i*).

The proof of part (*ii*) follows from (*II.5.10*) and from lemma (*II.4.1*), the extension of Mason's lemma.

§6. <u>Weierstrass' Necessary Condition</u>.

Let $E(x,y,u,u_{xy},v)$ be a function defined by the equality

$$(1) \qquad E(x,y,u,u_{xy},v) = f(x,y,u,v) - f(x,y,u,u_{xy}) - (v-u_{xy})\, f_v(x,y,u,u_{xy}) \;.$$

<u>*Theorem II.6.1.*</u> *Suppose the function* $\overset{\circ}{u}(x,y) \in U_g^{(1)}$ *minimizes the functional* $J[u]$ *. Then*

$$(2) \qquad E(x,y,u,u_{xy},v) \geq 0$$

holds for all (x,y,u,u_{xy},v) *such that* (x,y,u,u_{xy}) *are evaluated along* $\overset{\circ}{u}(x,y)$ *and* (x,y,u,v) *is a 4-vector.*

<u>Proof</u>: Let $\overset{\circ}{u}(x,y) \in U_g^{(1)}$. Consider a point $(\overline{x},\overline{y}) \in R$ which does not lie on any line of discontinuity of $\overset{\circ}{u}_x$, $\overset{\circ}{u}_y$, $\overset{\circ}{u}_{xy}$. Define

$$\overline{u} = \overset{\circ}{u}(\overline{x},\overline{y}) \quad , \quad \overline{v} = \overset{\circ}{u}_{xy}(\overline{x},\overline{y}) \quad .$$

Choose v and $\delta_o > 0$ such that $(\overline{x},\overline{y},\overline{u},v)$ is admissible and

$$\bar{x} + \sqrt{\delta} < b \quad , \quad \bar{y} + \sqrt{\delta} < d$$

for $0 \leq \delta \leq \delta_0$. Let $0 < \varepsilon < 1$. Consider the auxiliary function

$$\phi(x,y) = \begin{cases} (x-\bar{x})(y-\bar{y})(v-\bar{v}) & \bar{x}<x<\bar{x}+\sqrt{\varepsilon\delta} \;\; ; \;\; \bar{y}<y<\bar{y}+\sqrt{\varepsilon\delta} \\ \sqrt{\frac{\varepsilon}{1-\varepsilon}}\,(x-\bar{x})(\bar{y}+\delta-y)(v-\bar{v}) & \bar{x}<x<\bar{x}+\sqrt{\varepsilon\delta} \;\; ; \;\; \bar{y}+\sqrt{\varepsilon\delta}<y<\bar{y}+\sqrt{\delta} \\ \sqrt{\frac{\varepsilon}{1-\varepsilon}}\,(y-\bar{y})(\bar{x}+\delta-x)(v-\bar{v}) & \bar{x}+\sqrt{\varepsilon\delta}<x<\bar{x}+\sqrt{\delta} \;\; ; \;\; \bar{y}<y<\bar{y}+\sqrt{\varepsilon\delta} \\ \frac{\varepsilon}{1-\varepsilon}\,(\bar{y}+\sqrt{\delta}-y)(\bar{x}+\sqrt{\delta}-x)(v-\bar{v}) & \bar{x}+\sqrt{\varepsilon\delta}<x<\bar{x}+\sqrt{\delta} \;\; ; \;\; \bar{y}+\sqrt{\varepsilon\delta}<y<\bar{y}+\sqrt{\delta} \\ 0 & \text{otherwise} \end{cases}$$

Put

$$u(x,y;\varepsilon,\delta) = \mathring{u}(x,y) + \phi(x,y) \quad .$$

Then, we can easily verify that $u(x,y;\varepsilon,\delta)$ is admissible, and since $\mathring{u}(x,y)$ minimi es $J[u]$ we have

$$(3) \qquad J[u(x,y;\varepsilon,\delta)] - J[\mathring{u}(x,y)] = G_1 + G_2 + G_3 + G_4 \geq 0$$

where

$$G_1 = \int_{\bar{x}}^{\bar{x}+\sqrt{\varepsilon\delta}} \int_{\bar{y}}^{\bar{y}+\sqrt{\varepsilon\delta}} \{f(x,y,u(x,y,\varepsilon,\delta),\overset{\circ}{u}_{xy}+(v-\bar{v})) - f(x,y,\overset{\circ}{u},\overset{\circ}{u}_{xy})\}dxdy$$

$$G_2 = \int_{\bar{x}}^{\bar{x}+\sqrt{\delta}} \int_{\bar{y}+\sqrt{\varepsilon\delta}}^{\bar{y}+\sqrt{\delta}} \{f(x,y,u(x,y,\varepsilon,\delta),\overset{\circ}{u}_{xy} - \sqrt{\frac{\varepsilon}{1-\varepsilon}}\,(v-\bar{v})) - f(x,y,\overset{\circ}{u},\overset{\circ}{u}_{xy})\}dxdy$$

$$G_3 = \int_{\bar{x}+\sqrt{\varepsilon\delta}}^{\bar{x}+\sqrt{\delta}} \int_{\bar{y}}^{\bar{y}+\sqrt{\delta}} \{f(x,y,u(x,y,\varepsilon,\delta),\overset{\circ}{u}_{xy} - \sqrt{\frac{\varepsilon}{1-\varepsilon}}\,(v-\bar{v}) - f(x,y,\overset{\circ}{u},\overset{\circ}{u}_{xy})\}\ dxdy$$

$$G_4 = \int_{\bar{x}+\sqrt{\varepsilon\delta}}^{\bar{x}+\sqrt{\delta}} \int_{\bar{y}+\sqrt{\varepsilon\delta}}^{\bar{y}+\sqrt{\delta}} \{f(x,y,u(x,y,\varepsilon,\delta),\overset{\circ}{u}_{xy} + \frac{\varepsilon}{1-\varepsilon}\,(v-\bar{v})) - f(x,y,\overset{\circ}{u},\overset{\circ}{u}_{xy})\}dxdy\ .$$

By virtue of (*II.6.3*), we have

$$\lim_{\delta\to 0} \frac{J[u(x,y;\varepsilon,\delta)] - J[\overset{\circ}{u}(x,y)]}{\varepsilon\delta} = f(\bar{x},\bar{y},\bar{u},v) - f(\bar{x},\bar{y},\bar{u},\bar{v}) \quad + $$
$$+\,2\,\frac{f(\bar{x},\bar{y},\bar{u},\bar{v}-\eta_1(v-\bar{v})) - f(\bar{x},\bar{y},\bar{u},\bar{v})}{\eta_1}$$
$$+\ \frac{f(\bar{x},\bar{y},\bar{u},\bar{v}+\eta_2(v-\bar{v})) - f(\bar{x},\bar{y},\bar{u},\bar{v})}{\eta_2}$$

where

$$\eta_1^{\,2} = \eta_2 = \frac{\varepsilon}{1-\varepsilon} \quad .$$

Letting $\varepsilon \to 0$, which implies $\eta_1 \to 0$ and $\eta_2 \to 0$, we obtain

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to 0}} \frac{J[u(x,y;\varepsilon,\delta)] - J[\overset{o}{u}(x,y)]}{\varepsilon\delta} = E(\overline{x},\overline{y},\overline{u},\overline{v},v) \geq 0$$

which completes the proof of the theorem.

§7. Sufficient Conditions for the Existence of an Absolute Minimum.

In the previous two sections we have assumed the existence of a minimum of the functional J in a specified class of functions and investigated the properties of the corresponding minimizing function. In this section we investigate the conditions which ensure the existence of an absolute minimum of a functional $I[u]$. As is well known this type of theorem was studied by Tonelli [*3*] in the case of simple integral problems of the calculus of variations. A particular case of our problem has been studied by G. Stampacchia [*37*].

Specifically we consider the problem of the existence of an absolute minimum of the following general Bolza type functional

$$(1) \qquad I[u] = \Phi[u(a,y) \ , \ u(x,c)] + \int_a^b\int_c^d f(x,y,u,u_{xy})\,dx\,dy \ .$$

The functional $I[u]$ is defined on the class V of functions $u(x,y)$ which are absolutely continuous in the sense of Vitali in R (cf: Definition *I.1.3*) and which are such that

$$(2) \qquad u(b,y) = g_1(y) \quad ; \quad u(x,d) = f_1(x)$$

where $f_1(x)$ and $g_1(y)$ are absolutely continuous functions defined $a \leq x \leq b$ and $c \leq y \leq d$ respectively. Clearly V is an infinite set.

All the integrations considered in this section are in the sense of Lebesgue. The function $f(x,y,u,v)$ is of $C^1(G \times R)$, where G is a closed set in three dimensional space. The function $\Phi(x,y)$ is assumed to have a finite lower bound. We also assume that there exists constants $\alpha > 0$, β , and $p > 1$ for which

$$(3) \qquad f(x,y,u,v) \geq \alpha |v|^p + \beta \quad .$$

First we have the following observations. The assumptions concerning the function Φ and the condition (*II.7.3*) for arbitrary $u \in V$ gives us that

$$(4) \qquad I[u] \geq \beta(b-a)(d-c) + \alpha \int_a^b \int_c^d |u_{xy}|^p \, dxdy + \gamma$$

where

$$\gamma = min \ \Phi \quad .$$

Hence we have

$$\mu = \inf_{u \in V} I[u] \geq \beta(b-a)(d-c) + \gamma \ .$$

This ensures the existence of a minimizing sequence, which we shall designate by $\{u_n\}$. The sequence $\{u_n\}$ can be chosen in such a way that

$$I[u_n] \leq I[u_1] = A \qquad (n=1,2,3,\cdots) \quad .$$

In this case the relation (*II.7.4*) yields the inequality

$$\alpha \int_a^b \int_c^d \left|\frac{\partial^2 u_n}{\partial x \partial y}\right|^p dxdy \leq A - \beta(b-a)(d-c) - \gamma \quad .$$

So, there exists a positive constant B such that

$$(5) \qquad \int_a^b \int_c^d \left|\frac{\partial^2 u_n}{\partial x \partial y}\right|^p dxdy \leq B^p \qquad (n=1,2,3,\cdots) \quad .$$

The main objective of this section is to prove the following theorem whose analogue for simple integral problems is well known [3].

Theorem I.7.1. *Let V be the class of functions defined as above and*

(i) $f(x,y,u,v)$, in addition to the above assumptions, is continuously differentiable twice with respect to v and $f_v(x,y,u,v)$ is a nondecreasing function of v for $(x,y,u) \in G$ and all v .

(ii) The continuous function $\Phi(u(\xi,y),u(x,\eta))$ has a finite lower bound when $\xi = a$ and $\eta = c$.

(iii) There exists constants $\alpha > 0$, β *and* $p > 1$ *such that*

$$f(x,y,u,v) \geq \alpha\ |v|^p + \beta \quad .$$

Then the functional $I[u]$ *has an absolute minimum* μ *on* V *and there exists a function* $u \in V$ *such that*

$$I[u] = \mu \quad . \tag{6}$$

Proof: The proof will be established as a combination of the following lemmas.

Lemma II.7.2. If $\{u_n\}$ *is a sequence of functions belonging to* V *such that*

$$\int_a^b \int_c^d \left|\frac{\partial^2 u_n}{\partial x \partial y}\right|^p dxdy \leq B^p \tag{7}$$

then we can select a subsequence of $\{u_n\}$ *converging uniformly to a function* $u(x,y)$ *belonging to* V *and such that*

$$\int_a^b \int_c^d \left|\frac{\partial^2 u}{\partial x \partial y}\right|^p dxdy \leq B^p \quad . \tag{8}$$

Proof: Consider the following relations

$$|u_n(x+h,y+k) - u_n(x,y+k) - u(x+h,y) + u_n(x,y)| =$$

$$= |\int_x^{x+h}\int_y^{y+k} \frac{\partial^2 u}{\partial\xi\partial\eta}\, d\xi d\eta| \leq \int_x^{x+h}\int_y^{y+k} |\frac{\partial^2 u_n}{\partial\xi\partial\eta}|\, d\xi d\eta$$

for $a \leq x < x+h \leq b$ and $c \leq y < y+k \leq d$. Applying Holder's inequality, we obtain the inequality

$$(9) \qquad |u_n(x+h,y+k) - u_n(x,y+k) - u_n(x+h,y) + u_n(x,y)| \leq$$

$$\leq (hk)^{\frac{1}{q}} \{\int_x^{x+h}\int_y^{y+k} |\frac{\partial^2 u_n}{\partial x\partial y}|^p\, dxdy\}^{\frac{1}{p}} ,$$

$\frac{1}{p} + \frac{1}{q} = 1$. Hence

$$|u_n(x,y) - u_n(b,y) - u_n(x,d) + u_n(b,d)| \leq (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} B .$$

Further we have

$$(u_n(x'',y'') - u_n(x',y')) = (u_n(b,y'') - u_n(b,y')) + (u_n(x'',d) - u_n(x',d))$$

$$+ \int_{x''}^{b}\int_{y''}^{d} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta - \int_{x'}^{b}\int_{y'}^{d} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta$$

$$= (g_1(y'') - g_1(y')) + (f_1(x'') - f_1(x'))$$

$$- \int_{x'}^{x''}\int_{y'}^{y''} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta - \int_{x''}^{b}\int_{y'}^{y''} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta - \int_{x'}^{x''}\int_{y''}^{d} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta .$$

Therefore

$$|u_n(x'',y'') - u_n(x',y')| \leq |g_1(y'') - g_1(y')| + |f_1(x'') - f_1(x')|$$

$$+ B\{[(x''-x')(y''-y')]^{\frac{1}{q}} + [(b-x'')(y''-y')]^{\frac{1}{q}} + [(x''-x')(d-y'')]^{\frac{1}{q}}\}$$

by virtue of Equation (*II.7.7*) and (*II.7.9*) . Since $g_1(y)$ and $f_1(x)$ are uniformly continuous on $c \leq y \leq d$ and $a \leq x \leq b$, respectively, $\{u_n\}$ is a uniformly bounded equicontinuous family of functions; thus we can select a uniformly convergent subsequence by the Arzela-Ascoli theorem. Without loss of generality we can assume that $\{u_n\}$ itself converges uniformly to a continuous function, say, $u(x,y)$. This proves the first part of the lemma. To prove the second part of the lemma we have to show that $u(x,y)$ is absolutely continuous on R and satisfies (*II.7.8*). To do so we observe that for $a \leq x' < x'' \leq b$ and $c \leq y' < y'' \leq d$,

$$|u_n(x'',y'') - u_n(x',y'') - u_n(x'',y') + u_n(x',y')| \leq$$

$$\leq (x''-x')^{\frac{1}{q}} (y''-y')^{\frac{1}{q}} [\int_{x'}^{x''}\int_{y'}^{y''} |\frac{\partial^2 u_n}{\partial x \partial y}|^p]^{\frac{1}{p}}$$

by virtue of Equation (*II.7.9*). Let $\{R_i\}_{i=1}^m$ be a set of nonoverlapping subrectangles in R with vertices (x_i',y_i') and (x_i'',y_i'') as the opposite corners. Using the notation

$$F\, u_n(R_i) = u_n(x_i'',y_i'') - u_n(x_i',y_i'') - u_n(x_i'',y_i') + u_n(x_i',y_i')$$

we can write that

$$\sum_{i=1}^{m} |F\ u_n(R_i)| \leq \sum_{i=1}^{m} |(x''_i - x'_i)^{\frac{1}{q}} (y''_i - y'_i)^{\frac{1}{q}}| \ [\int_{x'_i}^{x''_i}\int_{y'_i}^{y''_i} |\frac{\partial^2 u_n}{\partial x \partial y}|^p]^{\frac{1}{p}} \ .$$

Then Hölder's inequality for sums [5] yields

$$\sum_{i=1}^{m} |F_{u_n}(R_i)| \leq [\sum_{i=1}^{m} (x''-x')(y''-y')]^{\frac{1}{q}} \ [\sum_{i=1}^{m} \int_{x'_i}^{x''_i}\int_{y'_i}^{y''_i} |\frac{\partial^2 u_n}{\partial x \partial y}|^p \ dx dy]^{\frac{1}{p}}$$

$$\leq B[\sum_{i=1}^{m} (x''_i - x'_i)(y''_i - y'_i)]^{\frac{1}{q}} \ .$$

Taking limit as $n \to \infty$ and using the uniform convergence of $\{u_n\}$ to $u(x,y)$, we establish the absolute continuity of $u(x,y)$ in the sense of Vitali in R .

Further it follows from Equation (*II.7.7*), that the inequality

$$\{\int_a^b\int_c^d |\frac{\partial^2 u_n}{\partial x \partial y}|^p\}^{\frac{1}{p}} \leq B$$

holds for all n , and we have

$$|u_n(x+h,y+k) - u_n(x,y+k) - u_n(x+h,y) + u_n(x,y)| \leq$$

$$\leq (hk)^{\frac{1}{q}} \{\int_x^{x+h}\int_y^{y+k} |\frac{\partial^2 u_n}{\partial \xi \partial \eta}|^p \ d\xi d\eta\}^{\frac{1}{p}}$$

for $a \leq x < x+h \leq b$ and $c \leq y < y+k \leq d$, which implies the inequality

$$\left|\frac{1}{hk}\,[u_n(x+h,y+k) - u_n(x,y+k) - u_n(x+h,y) + u_n(x,y)]\right|^p \le$$

$$\le \frac{1}{hk}\int_x^{x+h}\int_y^{y+k}\left|\frac{\partial^2 u_n}{\partial\xi\partial\eta}\right|^p d\xi d\eta \le \frac{1}{hk}\int_0^h\int_0^k \left|\frac{\partial^2 u_n}{\partial\xi\partial\eta}(x+\xi,y+\eta)\right|^p d\xi d\eta \;.$$

Integrating both sides of the above inequality over the rectangle $\{a \le x \le \bar{x}\;;\; c \le y \le \bar{y}\}$ with $a \le \bar{x} \le b$; $c \le \bar{y} \le d$, and using the uniform convergence of the sequence $\{u_n\}$ to the function u , we obtain

$$\int_a^{\bar{x}}\int_c^{\bar{y}} \left[\frac{u(x+h,y+k) - u(x,y+k) - u(x+h,y) + u(x,y)}{hk}\right]^p dxdy \le B^p \;.$$

Since $u(x,y)$ is absolutely continuous in the sense of Vitali on R , the integrand tends to $u_{xy}(x,y)$ a.e. Hence, by an obvious extension of the Lebesgue dominated convergence theorem to functions of two variables, we can easily show that $u_{xy}(x,y)$ satisfies Equation *(II.7.8)*. This completes the proof of the lemma *(I.7.2)*.

The next lemma gives us further information about the convergence of of the sequence $\{\frac{\partial^2 u_n}{\partial x\partial y}\}$.

Lemma II.7.3. *Let* $\{u_n\}$ *and* u *be as in Lemma I.7.2. Then the sequence* $\{\frac{\partial^2 u_n}{\partial x\partial y}\}$ *converges weakly to* $\frac{\partial^2 u}{\partial x\partial y}$ *in* $L_p(R)$.

Proof: We have to show that the relationship

$$\lim_{n\to\infty} \int_a^b\int_c^d \frac{\partial^2 u_n}{\partial x\partial y}\chi(x,y)\ dxdy = \int_a^b\int_c^d \frac{\partial^2 u}{\partial x\partial y}\chi(x,y)\ dxdy$$

holds for any $\chi \in L_q(R)$. Given $\chi \in L_q(R)$ and $\varepsilon > 0$, we can find a polynomial $P_N(x,y)$ of degree $N(\varepsilon)$ such that

$$\int_a^b\int_c^d |\chi(x,y) - P_N(x,y)|^q\ dxdy \leq \varepsilon^q .$$

Further we have

$$(10) \qquad \{ \int_a^b\int_c^d \left|\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\right|^p\ dxdy\}^{\frac{1}{p}} \leq 2\ B$$

by virtue of *(II.7.7)* and *(II.7.8)*, and

$$\left|\int_a^b\int_c^d \left(\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\right)\ (\chi(x,y) - P_N(x,y))\, dxdy\right| \leq$$

$$\leq \left[\int_a^b\int_c^d \left|\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\right|^p\ dxdy\right]^{\frac{1}{p}} \left[\int_a^b\int_c^d |\chi(x,y) - P_N(x,y)|^q\ dxdy\right]^{\frac{1}{q}}$$

$$\leq 2\ B\,\varepsilon$$

by virtue of Hölder's inequality. Hence

$$\overline{\lim_{n\to\infty}}\ \left|\int_a^b\int_c^d \left\{\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\right\}\ \chi(x,y)\ dxdy\right.$$

$$\leq 2\ B\,\varepsilon + \overline{\lim_{n\to\infty}}\ \left|\int_a^b\int_c^d \left\{\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\right\}\ P_N(x,y)\ dxdy\right| .$$

Therefore it is sufficient to show that the equality

$$\lim_{n\to\infty} \int_a^b\int_c^d \{\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}\} \, p(x,y) \, dxdy = 0$$

holds for any arbitrary polynomial $p(x,y)$. To do so, consider the identity

$$(u_n-u) \frac{\partial^2 p}{\partial x\partial y} - p \frac{\partial^2}{\partial x\partial y} (u_n-u)$$

$$= \frac{\partial}{\partial y} [(u_n-u) \frac{\partial p}{\partial x}] + \frac{\partial}{\partial x} [(u_n-u) \frac{\partial p}{\partial y}] = \frac{\partial^2}{\partial x\partial y} [(u_n-u)p]$$

which yields

$$\int_a^b\int_c^d [\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u}{\partial x\partial y}] \, p \, dxdy = \int_a^b\int_c^d (u_n-u) \frac{\partial^2 p}{\partial x\partial y} \, dxdy$$

$$- \int_a^b [(u_n-u) \frac{\partial p}{\partial y} (b,y) - (u_n-u) \frac{\partial p}{\partial x} (a,y)] \, dy$$

$$- \int_a^b [(u_n-u) \frac{\partial p}{\partial y} (x,d) - (u_n-u) \frac{\partial p}{\partial y} (x,c)] \, dx$$

$$+ F_{(u_n-u)p}[R] \quad .$$

Then, the result follows from the uniform convergence of the sequence $\{u_n\}$ to the function u .

Lemma I.7.5. *Let* $f(x,y,u,v)$ *be a function such that,* $f_v(x,y,u,v)$ *is a nondecreasing function in* v *for* $(x,y,u) \varepsilon G$, $-\infty < v < \infty$. *Let* $\{u_n\}$ *and* u *be as above. Then*

$$\mu = \lim_{n\to\infty} I[u_n] \geq I[u]. \tag{11}$$

Proof: We shall give a proof which resembles the well known proof in the case of simple integral problems. (cf: N.I. Akhiezer [3]).

Let us denote by E_M the set of points in R at which

$$|u_{xy}(x,y)| \leq M \quad . \tag{12}$$

Consider the identity

$$f(x,y,u_n,u_{n_{xy}})-\beta = \{f(x,y,u,u_{xy})-\beta\} + \{f(x,y,u_n,u_{n_{xy}}) - f(x,y,u_n,u_{xy})\} + \{f(x,y,u_n,u_{xy}) - f(x,y,u,u_{xy})\} \tag{13}$$

where β is the constant which appears in the formula (*II.7.3*). Because $u_n(x,y)$ converges uniformly to $u(x,y)$, we can find $N(\varepsilon)$ such that for arbitrary $\varepsilon > 0$ and for $n > N(\varepsilon)$

$$|f(x,y,u_n,u_{xy}) - f(x,y,u,u_{xy})| < \varepsilon$$

at every point of E_M . Hence the relation (*II.7.12*) implies that for $n > N(\varepsilon)$

$$(14) \qquad \iint_{E_M} \{f(x,y,u_n,u_{n_{xy}})-\beta\}dxdy \geq \iint_{E_M} \{f(x,y,u,u_{xy})-\beta\}dxdy +$$

$$+ \iint_{E_M} \{f(x,y,u_n,u_{n_{xy}}) - f(x,y,u_n,u_{xy})\}dxdy - \varepsilon(b-a)(d-c) .$$

On the other hand, the monotonicity of $f_v(x,y,u,v)$ yields the inequality

$$(15) \qquad f(x,y,u_n,u_{n_{xy}}) - f(x,y,u_n,u_{xy}) \geq (u_{n_{xy}}-u_{xy})\, f_v(x,y,u_n,u_{xy}) .$$

Because $f(x,y,u_n,u_{n_{xy}})-\beta \geq 0$, inequality (*II.7.13*) implies that

$$(16) \qquad \int_a^b\int_c^d\{f(x,y,u_n,u_{n_{xy}})-\beta\}dxdy \geq \iint_{E_M} \{f(x,y,u,u_{xy})-\beta\}dxdy +$$

$$+ \iint_{E_M} (u_{n_{xy}}-u_{xy})\, f_v(x,y,u_n,u_{xy})dxdy - \varepsilon(b-a)(d-c)$$

$$= \iint_{E_M} \{f(x,y,u,u_{xy})-\beta\}dxdy + \iint_{E_M} (u_{n_{xy}}-u_{xy})[f_v(x,y,u_n,u_{xy}) -$$

$$- f_v(x,y,u,u_{xy})\}dxdy + \iint_{E_M} (u_{n_{xy}}-u_{xy})\, f_v(x,y,u,u_{xy})dxdy -$$

$$- \varepsilon(b-a)(d-c) .$$

But due to the weak convergence of $\{\frac{\partial^2 u_n}{\partial x \partial y}\}$ to $\frac{\partial^2 u}{\partial x \partial y}$ in $L_p(R)$ we can see that the second integral in the right hand side of the equality in (*II.7.16*) tends to zero. On the other hand, with the aid of Holders inequality and formula (*II.7.10*), we can write

$$| \iint_{E_M} (u_{n_{xy}} - u_{xy}) \{f_v(x,y,u_n,u_{xy}) - f_v(x,y,u,u_{xy})\} \, dxdy$$

$$\leq 2B\{ \iint_{E_M} |f_v(x,y,u_n,u_{xy}) - f_v(x,y,u,u_{xy})|^q dxdy \}^{1/q} .$$

The difference

$$f_v(x,y,u_n,u_{xy}) - f_v(x,y,u,u_{xy})$$

converges uniformly to zero on the set E_M as $n \to \infty$. Thus

$$\lim_{n\to\infty} \iint_{E_M} (u_{n_{xy}} - u_{xy}) \{f_v(x,y,u_n,u_{xy}) - f_v(x,y,u,u_{xy})\} dxdy = 0 .$$

After passage to the limit, inequality (*II.7.16*) becomes

$$\lim_{n\to\infty} \int_a^b \int_c^d \{f(x,y,u_n,u_{n_{xy}}) - \beta\} \, dxdy$$

$$\geq \iint_{E_M} \{f(x,y,u,u_{xy}) - \beta\} dxdy - \varepsilon(b-a)(d-c)$$

or

$$\mu \geq \Phi(u(x,c),u(a,y)) + \iint_{E_M} \{f(x,y,u,u_{xy})-\beta\} + (\beta-\varepsilon)(b-a)(d-c) \quad .$$

Hence by increasing M to infinity we obtain

$$\mu \geq \Phi(u(x,c),u(a,y)) + \int_a^b\int_c^d \{f(x,y,u,u_{xy})-\beta\}dydx + (\beta-\varepsilon)(b-a)(d-c)$$

or

$$J[u] \leq \mu + \varepsilon(b-a)(d-c) \quad .$$

Because $\varepsilon > 0$ is arbitrary, the proof of lemma I.7.5 is completed.

§8. Solution of a Boundary Value Problem for a Fourth Order Polyvibrating Equation.

In this section, as an application of the results established in the previous sections, we shall prove the existence and uniqueness of the solutions of a polyvibrating equation of the fourth order subject to certain boundary conditions. More specifically we shall investigate the absolutely continuous solutions of the polyvibrating equation

$$(1) \qquad \frac{\partial^2}{\partial x \partial y} (\theta(x,y) \frac{\partial^2 u}{\partial x \partial y}) + p(x,y)u = g(x,y)$$

in R , subject to the boundary conditions

$$(2)\qquad \begin{cases} u(a,y) = Y_1(y) & u(b,y) = Y_2(y) \qquad (c \le y \le d) \\ \\ u(x,c) = X_1(x) & u(x,d) = X_2(x) \qquad (a \le x \le b) \end{cases} .$$

We assume that $X_i(x)$ and $Y_i(y)$ are absolutely continuous functions such that

$$(3)\qquad Y_1(d) = X_2(a)\ ,\ X_1(a) = Y_1(c)\ ,\ X_2(b) = Y_2(d)\ ,\ X_1(b) = Y_2(c)\ .$$

Further, $\theta(x,y) > 0$, $p(x,y) \ge 0$ and $g(x,y)$ are assumed to be continuous functions defined on R . It can easily be seen that the equation $(II.8.1)$ is the Euler-Lagrange equation of the quadratic functional

$$(4)\qquad I[u] = \int_a^b\int_c^d [\theta\ u_{xy}^2 + p(x,y)u^2 - 2gu]\ dydx$$

over the class $\overline{V}$ of the functions absolutely continuous in the sense of Vitali in R and satisfying the boundary condition $(II.8.2)$. In this case we have

$$f(x,y,u,v) = \theta(x,y)v^2 + p(x,y)u^2 - 2g(x,y)u\ .$$

First we shall show that to find a minimum for the functional $I[u]$ on $\overline{V}$, we can restrict ourselves to the functions in $\overline{V}$ such that

$$(5)\qquad -m \le u \le m\ .$$

For this purpose let us take a function u_o which satisfies the boundary condition (*II.8.2*). For example, let

$$u_o(x,y) = [X_1(x) + Y_1(y) - X_1(a)]$$

$$+ \frac{y-c}{d-c} [X_2(x) - X_2(a) - X_1(x) + X_1(a)]$$

$$+ \frac{(x-a)}{b-a} [Y_2(y) - Y_2(c) - Y_1(y) + Y_1(c)]$$

$$- \frac{(x-a)(y-c)}{(b-a)(d-c)} [Y_2(d) - Y_2(c) - Y_1(d) + Y_1(d)] \quad .$$

Put $I[u_o] = A$. To find the minimum of the functional we can restrict ourselves to the admissible functions $u \in \overline{V}$ for which

$$I[u] \leq A \, .$$

Since $p(x,y) \geq 0$

$$\int_a^b \int_c^d \theta \, u_{xy}^2 \, dxdy \leq A + 2 \int_a^b \int_c^d g \, u \, dxdy \quad .$$

Hence

$$(6) \qquad \theta_{min} \int_a^b \int_c^d u_{xy}^2 \, dxdy \leq A + 2 \, |u|_{max} \int_a^b \int_c^d g(x,y) \, dxdy \quad .$$

which implies the inequality

$$(7) \qquad \int_a^b\int_c^d u_{xy}^2 \, dxdy \leq A_1 + A_2 \, |u|_{max}$$

where A_1 , A_2 are well defined constants. On the other hand, the equality

$$u(x,y) = X_1(x) + Y_1(y) - X_1(a) + \int_a^x\int_c^y u_{xy} \, dxdy$$

yields the inequality

$$(8) \quad |u|_{max} \leq max|X_1(x) + Y_1(y) - X_1(a)| + \sqrt{(b-a)(d-c)} \sqrt{\int_a^b\int_c^d u_{xy}^2 \, dxdy}$$

Comparing the relations (*II.8.6*) and (*II.8.7*) we obtain

$$(9) \qquad |u|_{max} \leq M_1 + \sqrt{(b-a)(d-c)} \, \{A_1 + A_2 \, |u|_{max}\}^{\frac{1}{2}} \quad .$$

Therefore we obtain

$$|u|_{max} \leq m$$

where m is the constant whose value can be easily found by solving the quadratic equation with respect to $|u|_{max}$. Thus the function $u = u(x,y)$ satisfies the inequalities (*II.8.5*). Clearly we have

$$f(x,y,u,v) \geq \alpha \, v^2 + \beta$$

in G , where

$$\alpha = \theta_{min} \qquad \beta = - 2m|g|_{max} .$$

Hence the condition (*II.7.3*) of theorem II.7.1 is satisfied at each point of the region G for arbitrary but finite v . The monotonicity condition for f is also satisfied because of the equalities

$$f_v(x,y,u,v) = 2\ \theta(x,y)v \quad ; \quad f_{vv}(x,y,u,v) = \theta(x,y) > 0$$

Hence by Theorem II.7.1 there exists a function $u(x,y)$, absolutely continuous in the sense of Vitali in R, with its mixed derivative $u_{xy}(x,y) \in L_2(R)$, for which the functional $I[u]$ assumes its absolute minimum on $\overline{V}$ and such that it satisfies the equation (*II.8.1*) a.e.. This proves the existence of a solution to the boundary value problem (*II.8.1*) - (*II.8.2*) .

To show the uniqueness of the solution we proceed as follows. If there were two solutions u_1 , u_2 in $\overline{V}$ then the function $v = u_1 - u_2$, would satisfy the homogeneous equation

$$(10) \qquad \frac{\partial^2}{\partial x \partial y}\left(\theta(x,y)\frac{\partial^2 v}{\partial x \partial y}\right) + p(x,y)v = 0$$

and the homogeneous boundary condition

$$V\Big|_{\partial R} = 0 \quad .$$

Multiplying the expression (*II.8.10*) by v , and integrating by parts, we find

$$\int_a^b\int_c^d [\theta\ v_{xy}^2 + pv^2]\ dxdy = 0$$

which is possible if and only if $v(x,y) \equiv 0$ by virtue of our assumptions about the functions θ and p . This proves the uniqueness of the solution.

CHAPTER III

Existence Theorems for Polyvibrating Equations Using a Hilbert Space Approach

§0. Introduction.

In this chapter we generalize the work of F. Manaresi [20]. F. Manaresi has considered the following Sturm-Liouville problem

$$\frac{\partial^2}{\partial x \partial y}\left(\theta(x,y)\frac{\partial^2 u}{\partial x \partial y}\right) + p(x,y)u = \lambda\, u \,. \tag{1}$$

$$u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0 \tag{2}$$

where $\theta(x,y) > 0$ is a continuous function defined on R such that $\theta_x(x,y)$, $\theta_y(x,y)$, $\theta_{xy}(x,y)$ are all continuous in R, while $p(x,y) \geq 0$ is a continuous function defined on R . In this chapter we consider the same problem but the assumptions on $\theta(x,y)$ are considerably weakened. There $\theta(x,y)$, $p(x,y)$ are measurable functions belonging to $L_2(R)$ and $p(x,y)$ is a positive function. We assume there exists a constant θ_o such that

$$\theta(x,y) \geq \theta_o > 0 \,. \tag{3}$$

Due to the assumptions on $\theta(x,y)$ the partial differential operator (*III.0.1*) should be first considered as a formal partial differential operator. Thus the partial differential equation is a differential equation in a generalized sense which will be made explicit in the following section. The motivation for this comes from the similar work done in the case of elliptic partial differential operators of the second order. In this regard we refer to the book by S. Agmon [1]. In Section 1 we state without proofs some theorems from functional analysis. Next we prove the properties of a suitable Hilbert space $V_0^{(1)}$, in which our problem will have a solution. Sections 3, 4 and 5 are concerned with the characterisation of the Hilbert space $V_0^{(1)}$. Sections 6 and 7 are devoted to the study of a generalized Sturm-Liouville problem.

§1. Positive Definite Operators.

In this section we state without proofs some properties of positive definite operators. The proofs can be found in the book of S.G. Mikhlin [26].

Let H be a Hilbert space over the reals and let A be a symmetric linear transformation defined on a linear subspace M which is dense in H. The inner product in H will be denoted by the usual notation (u,v) for u, $v \in H$.

Definition III.1.1. A *is called strictly positive definite on* M *if and only if there exists a positive constant* γ*, such that*

$$(1) \qquad (A\,u,u) \geq \gamma^2 \, \|u\|^2 \quad , \quad u \in M .$$

Then, following K. Friedrichs [10], we define on the subset M, a new inner product by

(2) $$[u,v] = (A\,u,v) \quad , \quad u\,,\,v \in M\,.$$

With this inner product, M becomes an inner product space which, when completed in the usual way, yields a Hilbert space H_A. Then we have the following theorem:

Theorem III.1.2. *The Hilbert space H_A can be identified with a subspace of H :*

(3) $$M \subset H_A \subset H \quad .$$

On the basis of this theorem it is easy to show that

(4) $$[u,v] = (A\,u,v) \quad , \quad u \in M \quad , \quad v \in H_A \quad .$$

(5) $$|||u|||^2 = [u,u] \geq \gamma^2\ ||u||^2 \quad , \quad u \in H_A \quad .$$

Let f be any element in H and consider the linear functional

(6) $$F_f(u) = (u,f) \quad , \quad u \in H_A \quad .$$

Since

(7) $$|F_f(u)| = (u,f) \leq ||u||\ ||f|| \leq \gamma^{-1}\ |||u|||\ ||f||\ .$$

$F_f(u)$ is a bounded linear functional on H_A . By the Riesz representation theorem there exists a unique element $u_f \in H_A$ such that

$$(8) \qquad F_f(u) = (u,f) = [u,u_f] \quad , \quad u \in H_A \; .$$

<u>*Theorem (III.1.3).*</u> *Let* A *be a positive definite operator. If the equation*

$$(9) \qquad A\,u = f \quad , \quad f \in H$$

has a solution, then the functional

$$(10) \qquad F[u] = (A\,u,u) - (u,f) - (f,u)$$

assumes its minimum value for this solution. Conversely an element which minimizes the functional (III.1.10) satisfies the equation (III.1.9).

The basic variational problem, generally speaking, consists of finding an element belonging to M for which the functional (*III.1.10*) attains its minimum on M . In general this problem does not have a solution in M . In order that the problem become solvable we modify it somewhat. First of all if $u \in M$ then

$$(11) \qquad (A\,u,u) = [u,u] \quad .$$

Further, by our previous remark, if f is a fixed element of H and u is an arbitrary element of H_A , (u,f) is a bounded linear functional on H_A . Thus there exists $u_f \in H_A$ such that

$$(A\,u,u) = (u,f) = [u,u_f] \quad \text{for all} \quad u \in H_A \ . \tag{12}$$

We now have

$$F[u] = [u,u] - [u,u_f] - [u_f,u] \quad \text{for all} \quad u \in H_A \ . \tag{13}$$

Formula (*III.1.13*) was established for $u \in M$, but its right hand side is meaningful on all of H_A . Using (*III.1.13*) we extend $F[u]$ on all of H_A and we seek a minimum on H_A . As a matter of fact

$$F[u] = [u-u_f,u-u_f] - [u_f,u_f] \tag{14}$$

and from (*III.1.14*) it is clear that $F[u]$ assumes its minimum on H_A for $u = u_f$. Thus it is clear that if $Au = f$, then

$$(A\,u,u) = (f,u) = (A\,u_f,u) \qquad u \in H_A \ , \ u_f \in H_A \ . \tag{15}$$

Hence we have the following definition.

Definition III.1.2. *Given* $f \in H$, $u_f \in H_A$ *will be a weak solution or a generalized solution of the equation*

(9) $$A\,u = f$$

if the relationship

(16) $$(A\,u_f, u) = (f, u)$$

holds for all $u \in H_A$ *and* $f \in H$.

Theorem III.1.3. *Let* A *be a positive definite operator on a Hilbert space and*

(17) $$m = inf \frac{(A u, u)}{(u, u)} \quad .$$

Let (u_n) *be a normalized minimizing sequence. If* $\{u_n\}$ *contains a convergent subsequence then* m *is an eigenvalue of* A .

Theorem III.1.4. *Suppose that elements* $\{\phi_n\}$ *belong to the domain of definition of the operator* A *, and that the sequence* $\{A\phi_n\}$ *is complete in* H *. Then the sequence* $\{\phi_n\}$ *is complete in* H_A .

Theorem III.1.5. *Let the positive definite operator* A *be such that every bounded set in* H_A *is compact in* H *. Then*

(i) A *has a countable set of eigenvalues tending to infinity;*

(ii) the sequence of eigenvectors is complete in H_A *as well as in* H .

§2. Absolutely Continuous Functions.

It is well known that absolutely continuous functions of a single variable, defined over a specified interval of the real line, form a Hilbert space when a suitable inner product is chosen. In this section we deal with the extension of this fact to functions of several variables. We will be mainly concerned here with the definition of absolute continuity, given by Vitali, for functions of several variables (cf: Definition I.1.3). We establish our results in the case of two variables, their extensions to more than two variables are obvious.

Definition III.2.1 [14]. *A function* $u(x,y)$ *defined on the rectangle* $R : \{a \leq x \leq b \, ; \, c \leq y \leq d\}$ *is said to be absolutely continuous in the sense of Vitali, if for any* $\varepsilon > 0$ *, there exists a* $\delta > 0$ *such that for any finite or infinite set of nonoverlapping subrectangles* $\{R_i\}_{i \geq 1}$ *of* R

$$(1) \qquad meas\,(\bigcup_{i \geq 1} R_i) < \delta \Rightarrow \sum_{i \geq 1} |F_u(R_i)| < \varepsilon$$

where $F_u(R_i)$ *denotes the following double difference*

$$(2) \qquad F_u(R_i) = u(a_i,c_i) - u(a_i,d_i) - u(b_i,c_i) + u(b_i,d_i)$$

for the rectangle $R_i : \{a_i \leq x \leq b_i \, ; \, c_i \leq y \leq d_i\}$.

It is known that every definition of absolute continuity has associated with it a special derivative. This is the case for Definition III.2.1.

Definition III.2.2. *A function $u(x,y)$ defined on R is said to be differentiable at $(x_0,y_0) \in R$ in the sense of Picone* [5] *whenever the limit*

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{u(x_0+h,y_0+k) - u(x_0,y_0+k) - u(x_0+h,y_0) + u(x_0,y_0)}{h\,k} \tag{3}$$

exists.

We call this limit the generalized derivative or hyperbolic derivative of $u(x,y)$ in the sense of Picone at (x_0,y_0) and denote it by the symbol $\frac{\partial^2 u}{\partial x \partial y}(x_0,y_0)$. Now, it is clear that if $u(x,y) = f_1(x) + f_2(y)$ where neither $f_1(x)$ nor $f_2(y)$ is differentiable, then $u(x,y)$ does not have any derivative in the classical sense, but the generalized Picone derivative does exist and is equal to zero. We now state the following theorems whose proofs can be found in the book of E.W. Hobson [14]:

Theorem III.2.3. *If a function defined on R is absolutely continuous in R in the sense of Vitali then it has a generalized derivative in the sense of Picone almost everywhere in R .*

Theorem III.2.4. *If $u(x,y)$ is absolutely continuous in the sense of Vitali in R , then*

(4) $$\int_a^x\int_c^y \frac{\partial^2 u}{\partial\xi\partial\eta}\, d\xi d\eta = u(x,y) - u(a,y) - u(x,c) + u(a,c) \quad .$$

<u>*Theorem III.2.5.*</u> *If $f(x,y)$ is a summable function in R and if*

(5) $$u(x,y) = \int_a^x\int_c^y f(\xi,\eta)\, d\xi d\eta$$

then $u(x,y)$ is absolutely continuous in the sense of Vitali and hence the generalized derivative in the sense of Picone exists almost everywhere in R and we have

(6) $$\frac{\partial^2 u}{\partial x\partial y} = f(x,y)$$

almost everywhere in R .

<u>*Theorem III.2.6.*</u> *A necessary and sufficient condition that a function defined on R be absolutely continuous in the sense of Vitali is that it be the indefinite integral of a function summable in R .*

Now, let V denote the class of all absolutely continuous functions (in the sense of Vitali) defined on R . Let $V_0^{(1)}$ denote the functions in V which vanish on the boundary of the rectangle R and such that their generalized derivative belongs to $L_2(R)$, the Hilbert space of all square integrable functions defined on R , i.e.,

(7) $$V_0^{(1)} = \{u : u \text{ is A.C.}, u_{xy} \in L_2(R) ; u|_{\partial R} = 0\} .$$

<u>*Theorem III.2.7.*</u> $V_0^{(1)}$ *is a Hilbert space with respect to the inner product*

(8) $$((u,v)) = \int_a^b \int_c^d u_{xy} v_{xy} \, dxdy \qquad u , v \in V_0^{(1)}$$

and the norm

(9) $$|||u|||^2 = \int_a^b \int_c^d u_{xy}^2 \, dxdy \qquad u \in V_0^{(1)} .$$

<u>Proof</u>: The properties of the norm are verified as follows. First we show that $|||u||| = 0$ implies $u \equiv 0$ a.e. in R. Since $u(x,y)$ vanishes on the boundary of R, by Theorem III.2.4 we have

(10) $$u(x,y) = \int_a^x \int_c^y u_{\xi\eta} \, d\xi d\eta .$$

Squaring both sides of (*III.2.10*) and applying Cauchy-Schwarz inequality to the right hand side yields

(11) $$u^2(x,y) \leq (x-a)(y-c) \int_a^b \int_c^d u_{xy}^2 \, dxdy .$$

Integrating either side of (*III.2.11*), we obtain

(12) $$\int_a^b\int_c^d u^2(x,y)\ dxdy \leq \frac{(b-a)^2(d-c)^2}{4}\ |||u|||^2\quad .$$

But since $|||u||| = 0$, we have

(13) $$\int_a^b\int_c^d u^2(x,y)\ dxdy = 0$$

which in tern implies $u = 0$ a.e.

The triangle inequality and symmetry property of the norm are proved as usual. Thus it remains to show that the space $V_o^{(1)}$ is complete with respect to the norm (*III.2.9*). To do so we proceed as follows. Let $\{u_n\}$ be a Cauchy sequence with respect to the norm (*III.2.9*). Then we have, since each $\{u_n\}$ vanishes on the boundary of R ,

(14) $$u_n(x,y) - u_m(x,y) = \int_a^x\int_c^y \frac{\partial^2(u_n-u_m)}{\partial\xi\partial\eta}\ d\xi d\eta$$

and therefore

(15) $$\begin{aligned} |u_n(x,y) - u_m(x,y)| &\leq \int_a^x\int_c^y \left|\frac{\partial^2(u_n-u_m)}{\partial\xi\partial\eta}\right|\ d\xi d\eta \\ &\leq \sqrt{(x-a)(y-c)}\ \sqrt{\int_a^b\int_c^d \left[\frac{\partial^2(u_n-u_m)}{\partial x\partial y}\right]^2\ dxdy} \\ &\leq (b-a)(d-c)\ |||u_n-u_m||| \end{aligned}$$

by virtue of the Cauchy-Schwarz inequality. Further since

$$u_n(x'',y'') - u_n(x',y') = \int_a^{x''}\int_c^{y''} \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta - \int_a^{x'}\int_c^{y'} \frac{\partial^2 u_n}{\partial\xi\partial\eta}$$

for $a \leq x' < x'' \leq b$; $c \leq y' < y'' \leq d$ we have

$$|u_n(x'',y'') - u_n(x',y')| = \left|\int_a^{x'}\int_{y'}^{y''} \frac{\partial^2 u_n}{\partial x\partial y}\, dxdy + \int_{x'}^{x''}\int_c^{y'} \frac{\partial^2 u_n}{\partial x\partial y}\, dxdy\right.$$

$$\left. + \int_{x'}^{x''}\int_{y'}^{y''} \frac{\partial^2 u_n}{\partial x\partial y}\, dxdy\right| \leq \int_a^{x'}\int_{y'}^{y''} \left|\frac{\partial^2 u_n}{\partial x\partial y}\right| dxdy$$

$$+ \int_{x'}^{x''}\int_c^{y'} \left|\frac{\partial^2 u_n}{\partial x\partial y}\right| dxdy + \int_{x'}^{x''}\int_{y'}^{y''} \left|\frac{\partial^2 u_n}{\partial x\partial y}\right| dxdy$$

$$\leq [\sqrt{(x'-a)(y''-y')} + \sqrt{(x''-x')(y'-c)}$$

$$+ \sqrt{(x''-x')(y''-y')}]\ |||u_n|||$$

which proves the equicontinuity of the family $\{u_n\}$.

This shows that the sequence $\{u_n(x,y)\}$ converges uniformly to a function, say, $\overset{\circ}{u}(x,y)$. Completeness will follow if we can show that $\overset{\circ}{u}(x,y)$ is absolutely continuous in R in the sense of Vitali. To do so, first we observe that $\{\frac{\partial^2 u_n}{\partial x\partial y}\}$ is a Cauchy sequence in $L_2(R)$. Due to the completeness of $L_2(R)$, there exists a function $g \in L_2(R)$.

$$\text{(16)} \qquad \lim_{n\to\infty} \int_a^b\int_c^d \left|\frac{\partial^2 u_n}{\partial x\partial y} - g\right|^2 dxdy = 0 \quad .$$

On the other hand, we have

$$u_n(x,y) = \int_a^x \int_c^y \frac{\partial^2 u_n}{\partial\xi\partial\eta}\, d\xi d\eta \tag{17}$$

by virtue of (*III.2.4*). Hence, passing to the limit in (*III.2.16*) as $n \to \infty$ and using Equation (*III.2.17*) and the uniform convergence of $u_n(x,y)$, we obtain

$$\overset{\circ}{u}(x,y) = \int_a^x \int_c^y g(\xi,\eta)\, d\xi d\eta\ . \tag{18}$$

Thus $\overset{\circ}{u}(x,y)$ is the indefinite integral of a summable function $g(x,y)$ and hence by Theorem III.2.5, $\overset{\circ}{u}(x,y)$ is absolutely continuous in the sense of Vitali in R. Further we can easily prove that $\overset{\circ}{u}\big|_{\partial R} = 0$. Thus $V_0^{(1)}$ is a Hilbert space.

§3. Fundamental Properties of the Hilbert Space $V_0^{(1)}$.

The Hilbert space $V_0^{(1)}$ has many similarities with the Sobolev spaces usually considered in the theory of partial differential equations. In fact, it is known that the Sobolev space $H_0^{(1)}$ is the completion of $C_0^\infty(R)$, the space of infinitely differentiable functions with compact supports in R, with respect to the Dirichlet norm. A similar property is also true for $V_0^{(1)}$. This fact will be stated and proved in this section.

<u>*Theorem III.3.1.*</u> *$V_0^{(1)}$ is the completion of $C_0^\infty(R)$ equipped with the inner product (III.2.4) and the norm (III.2.5).*

Proof: For the sake of simplicity we will study the problem in the rectangle

(1) $$\mathcal{D} = \{(x,y) \; ; -\alpha \leq x \leq \alpha \; ; \; -\beta \leq y \leq \beta\} \quad .$$

We can easily show that the transformation

(2) $$\begin{cases} \xi = (\frac{x}{\alpha} + 1)(\frac{b-a}{2}) + a \\ \eta = (\frac{y}{\beta} + 1)(\frac{d-c}{2}) + c \end{cases}$$

takes $\mathcal{D}$ into R and changes the norm involved only by a multiplicative constant.

Let T be the linear transformation defined on $L_2(\mathcal{D})$ by the equation

(3) $$T\, f(x,y) = \int_{-\alpha}^{x}\int_{-\beta}^{y} f(\xi,\eta)\, d\xi d\eta \quad .$$

T is a completely continuous transformation of $L_2(\mathcal{D})$ into itself, and, in particular, for any continuous $f(x,y) \in L_2(\mathcal{D})$ we have

(4) $$f(x,y) = \frac{\partial^2}{\partial x \partial y}(T\, f) \quad .$$

It is clear that for any $f \in L_2(\mathcal{D})$, $T\, f$ vanishes for $x = -\alpha$ and

$y = -\beta$ but $T\,f$ need not vanish on other parts of the boundary. Let M denote the subspace of $L_2(\mathcal{D})$, consisting of elements f for which Tf vanishes on all of the boundary of $\mathcal{D}$. Then we have

$$(5) \qquad \int_{-\beta}^{\beta}\int_{-\alpha}^{x} f(\xi,\eta)\ d\xi d\eta = 0 \quad ; \quad \int_{-\beta}^{y}\int_{-\alpha}^{\alpha} f(\xi,\eta)\ d\xi d\eta = 0$$

for any f in M and for all $(x,y) \in \mathcal{D}$. This may be written as

$$(6) \qquad \int_{\mathcal{D}} f(\xi,\eta)\ \chi(\xi,\eta)\ d\xi d\eta = 0$$

where $\chi(\xi,\eta)$ is the characteristic function of rectangles of the form $-\alpha \le \xi \le x$; $-\beta \le \eta \le \beta$ or of the form $-\alpha \le \xi \le \alpha$; $-\beta \le \eta \le y$. It is also evident that any $g \in L_2(\mathcal{D})$ which is orthogonal to the characteristic functions of such rectangles or strips has the property that $T\,g$ vanishes on the boundary, i.e. it is in M. Thus M is a closed subspace of $L_2(\mathcal{D})$ such that $T(M)$ contains $C_o^\infty(\mathcal{D})$.

Given an f in M, let $f_r(x,y)$ $(0 < r < 1)$ be a function defined on the rectangle $\mathcal{D}$ as follows:

$$(7) \qquad f_r(x,y) = \begin{cases} f(\frac{x}{r}\,,\,\frac{y}{r}) & |x| \le r\,\alpha\ ;\ \ |y| \le r\,\beta \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $f_r(x,y)$ is in $L_2(\mathcal{D})$. In fact we show that $f_r(x,y)$

is in M. Since, if χ is the characteristic function of a strip

$$(8) \qquad -\alpha \le \xi \le x \quad ; \quad -\beta \le y \le \beta \ ,$$

we have

$$(9) \qquad \int_D f_r(\xi,\eta)\ \chi(\xi,\eta)\ d\xi d\eta = \int_{-\alpha r}^{x}\int_{-\beta r}^{\beta r} f_r(\xi,\eta)\ d\xi d\eta$$

$$= \int_{-\alpha r}^{min(x,\alpha r)}\int_{-\beta r}^{\beta r} f(\frac{\xi}{r}\ ,\ \frac{\eta}{r})\ d\xi d\eta$$

$$= \int_{-\alpha}^{min(\frac{x}{r}\ ,\ \alpha)}\int_{-\beta}^{\beta} f(\xi,\eta)\ r^2\ d\xi d\eta$$

$$= r^2 \int_D f\ \chi'\ d\xi d\eta = 0$$

χ' being the characteristic function of the strip

$$-\alpha \le x \le min(\frac{x}{r}\ ,\ \alpha) \quad ; \quad -\beta \le y \le \beta \ .$$

Similarly for strips $-\alpha \le \xi \le \alpha$; $-\beta \le \eta \le \beta$. The functions f_r converge to f in the $L_2(D)$ topology as r tends to 1. Given any f_r in M we form its regularization $f_{r,t}$ of radius t as follows.

$$(10) \qquad f_{r,t}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_r(\xi,\eta)\ K_t(x-\xi,y-\eta)\ d\xi d\eta \ .$$

$K_t(x,y)$ is a function defined on the plane vanishing outside a circle of radius t about the origin and which is given in that circle by

$$K_t(x,y) = \frac{k}{t^2} \exp\left(\frac{-t^2}{t^2-x^2-y^2}\right) \qquad (11)$$

Here the number k is a normalizing factor. It is well known that [31] the regularization of a function in $L_2(D)$ is a $C_o^\infty(D)$ function which converges to that function in the $L_2(D)$ topology as the radius t converges to zero. In our case the regularization $f_{r,t}$ of f_r is to be taken with radius $t < \frac{\alpha}{4}(1-r)$ and is a function which vanishes outside a closed subset of the interior of D. Since f_r was in M, the regularization also belongs to M. We have, writing χ for the characteristic function of a strip

$$\int_D f_{r,t}\, \chi \, dxdy = \int_D \chi(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_r(\xi,\eta)\, K_t(x-\xi,y-\eta)\, d\xi d\eta\, dxdy \qquad (12)$$

$$= \int_D \chi(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_r(x-\xi,y-\eta)\, K_t(\xi,\eta)\, d\xi d\eta\, dxdy$$

$$= \int_{-t}^{t}\int_{-t}^{t} K_t(\xi,\eta) \left\{ \int_D f_r(x-\xi,y-\eta)\, \chi(x,y)\, dxdy \right\} d\xi d\eta \quad .$$

Since $|\xi|$, $|\eta|$ are sufficiently small, being less than t , the translation $f_r(x-\xi,y-\eta)$ is a function vanishing outside a closed rectangle in the interior of D and is in M . Hence the integral above is zero. Thus for any f in M we can construct a C_o^∞ function $f_{r,t}$ in M which vanishes outside of a rectangle completely interior to D and which is as close to f in $L_2(D)$ norm as we like. Hence $T f_{r,t}$ is also in C_o^∞ and vanishes outside the same rectangle as $f_{r,t}$. $T f_{r,t}$ is in

$C_o^\infty(\mathcal{D})$ and approximates Tf in the norm (*III.2.5*). Thus we have shown that $T(M)$ is the completion of $C_o^\infty(\mathcal{D})$ with respect to the norm (*III.2.5*)

Let us note that in the above proof we have been guided by a method of Aronszajn and Donoghue [4].

<u>*Theorem III.3.2.*</u> *The Hilbert space* $V_o^{(1)}$ *is a subspace of the Sobolev space* $H_o^{(1)}$.

<u>Proof</u>: We have the following relation

$$u_x(x,y) = \int_c^y u_{x\eta}(x,\eta)\, d\eta$$

for any function $u \in C_o^\infty(R)$. Squaring and applying Cauchy-Schwarz inequality we obtain

$$u_x^2(x,y) \le (y-c) \int_c^d u_{xy}^2(x,y)\, dy \quad . \tag{13}$$

Integration of both sides yields

$$\int_a^b\int_c^d u_x^2(x,y)\, dxdy \le \frac{(d-c)^2}{2} \int_a^b\int_c^d u_{xy}^2(x,y)\, dydx \ . \tag{14}$$

Similarly we obtain

$$\int_a^b\int_c^d u_y^2(x,y)\, dxdy \le \frac{(b-a)^2}{2} \int_a^b\int_c^d u_{xy}^2(x,y)\, dxdy \ . \tag{15}$$

Addition of the inequalities (*III.3.14*) and (*III.3.15*) side by side yields

$$\int_a^b\int_c^d [u_x^2(x,y) + u_y^2(x,y)]\ dxdy \le m \int_a^b\int_c^d u_{xy}^2\ dxdy \tag{16}$$

where $m = max\ \{\frac{(b-a)^2}{2}, \frac{(d-c)^2}{2}\}$. Thus the Dirichlet norm on $C_o^\infty(R)$ is dominated by the norm (*III.2.5*) and this gives the conclusion of the theorem.

§4. A Boundary Value Problem of Mangeron.

In what follows we use the above characterization of $V_o^{(1)}$ to show the existence and uniqueness of the generalized solution to a simple boundary value problem due to D. Mangeron. Let $L_2(R)$ denote the Hilbert space of all square integrable real valued functions defined on R . Let $C_o^\infty(R)$ have the usual meaning. It is well known that $C_o^\infty(R)$ is dense in $L_2(R)$. Recall that in $L_2(R)$ the scalar product and the norm are defined by

$$(u,v) = \int_a^b\int_c^d uv\ \ dxdy \quad ; \quad ||u||^2 = \int_a^b\int_c^d |u|^2\ \ dxdy\ . \tag{1}$$

We consider the boundary value problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = f(x,y) \tag{2}$$

$$u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0 \tag{3}$$

where $f(x,y)$ is a function belonging to $L_2(R)$.

Using the same notations introduced in the preceding sections, we take $H = L_2(R)$ and $M = C_o^\infty(R)$ and $A = \frac{\partial^4}{\partial x^2 \partial y^2}$. The operator A is well defined on $C_o^\infty(R)$. The symmetry of A is proved as follows. Integrating over R both sides of the identity

$$(4) \qquad u\, v_{xyxy} = [u\, v_{xy}]_{xy} - [u_x v_{xy}]_y - [u_y v_{xy}]_x + u_{xy} v_{xy}$$

where $u(x,y)$ and $v(x,y)$ are any two functions belonging to $C_o^\infty(R)$, we find

$$(5) \qquad \int_a^b\int_c^d u\, v_{xyxy}\, dxdy = [u\, v_{xy}]_{a,c}^{b,d} - \int_a^b [u_x v_{xy}(x,d) - u_x v_{xy}(x,c)]\, dx$$

$$- \int_c^d [u_y v_{xy}(b,y) - u_y v_{xy}(a,y)]\, dy$$

$$+ \int_a^b\int_c^d u_{xy}\, v_{xy}\, dxdy.$$

Since $u(x,y)$ satisfies the boundary conditions (*III.4.3*) we have

$$(6) \qquad (Av,u) = \int_a^b\int_c^d u\, v_{xyxy} dxdy = \int_a^b\int_c^d u_{xy} v_{xy}\, dxdy \quad ,$$

which proves the symmetry of A . We now introduce the new norm defined on $C_o^\infty(R)$ by

$$(7) \qquad (A\,u,u) = \int_a^b\int_c^d uu_{xyxy}\,dxdy = \int_a^b\int_c^d u_{xy}^2\,dxdy.$$

The positive definiteness of A can be proved as follows: Since

$$(8) \qquad u(x,y) = \int_a^x\int_c^y u_{\xi\eta}\,d\xi d\eta \quad ,$$

squaring and applying Cauchy-Schwarz inequality to the right hand side, we obtain

$$(9) \qquad u^2(x,y) \leq (x-a)(y-c)\int_a^b\int_c^d u_{xy}^2\,dxdy \quad .$$

Integrating both sides we obtain

$$(10) \qquad \int_a^b\int_c^d u^2(x,y)\,dxdy \leq \frac{(b-a)^2(d-c)^2}{4}\int_a^b\int_c^d u_{xy}^2\,dxdy \quad .$$

Thus

$$(11) \qquad (A\,u,u) = |||u|||^2 \geq \frac{4}{(b-a)^2(d-c)^2}||u||^2$$

which proves the positive definiteness of A on $C_o^\infty(R)$ with $\gamma = \frac{2}{(b-a)(d-c)}$. In section 3 we have shown that the completion of $C_o^\infty(R)$ with respect to the norm (*III.2.5*) is $V_o^{(1)}$. Hence, by virtue of the results of Section 1, we have the following:

Theorem III.4.1. *The boundary value problem (III.4.2) and (III.4.3) has a generalized solution in* $V_o^{(1)}$ *and this solution is unique.*

§5. Characterization of the Solutions.

In this section, using the method of reproducing kernels due to N. Aronszajn [17] we characterize the subspace of $V_o^{(1)}$ which contains the solution to our problem. We begin with the

Definition III.5.1 [19]. *A functional completion* $\overline{M}$ *of an incomplete function space* M *is defined to be a completion by adjunction of point functions with the property that for each* (x,y) *there exists a number, call it* $M_{x,y}$ *such that, if the sequence* $\{u_n\}$ *is convergent in* M *, and there exists a* $u \in \overline{M}$ *for which* $\lim_{n\to\infty} ||u_n-u|| = 0$ *, then*

$$(1) \qquad |u_n(x,y) - u(x,y)| \leq M_{x,y} \; ||u_n-u||$$

for each (x,y) *in the domain of definition of functions in* $\overline{M}$ *.*

As was done in Section 3 choose $M = C_o^\infty(R)$ and define on $C_o^\infty(R)$ the inner product

$$(III.2.4) \qquad ((u,v)) = \int_a^b\int_c^d u_{xy}\, v_{xy}\, dxdy, |||u|||^2 = \int_a^b\int_c^d u_{xy}^2\, dxdy\ .$$

D. Mangeron [2] has shown that the function

$$(b-a)(d-c)\ G(x,y;\xi,\eta) = \begin{cases} (b-x)(d-y)(\xi-a)(\eta-c) & \xi \leq x\ ;\quad \eta \leq y \\ (b-x)(y-c)(\xi-a)(d-\eta) & \xi \leq x\ ;\quad \eta \geq y \\ (x-a)(d-y)(b-\xi)(\eta-c) & \xi \geq x\ ;\quad \eta \leq y \\ (x-a)(y-c)(b-\xi)(d-\eta) & \xi \geq x\ ;\quad \eta \geq y \end{cases}$$

is the Green's function of the operator $A = \dfrac{\partial^4}{\partial x^2 \partial y^2}$ for the boundary conditions (*III.4.3*). Since the Green's function $G(x,y;\xi,\eta)$ is a reproducing kernel, i.e.

$$(2) \qquad u(x,y) = \int_a^b \int_c^d G(x,y;\xi,\eta)\ \frac{\partial^4 u}{\partial \xi^2 \partial \eta^2}\ d\xi d\eta\ ,\quad u \in C_o^\infty(R) \quad ,$$

thus for any fixed (x,y) we have, by the Cauchy-Schwarz inequality.

$$(3) \qquad |u(x,y)| \leq |||G_{x,y}|||\quad |||u|||$$

where $|||G_{x,y}|||$ is the norm of $G(x,y;\xi,\eta)$ for fixed (x,y) defined by (*III.2.9*). In this case it should be noted that discontinuities of G should be taken into account and also

$$(4) \qquad |||G_{x,y}|||^2 = (G_{\xi\eta}(x,y;\xi,\eta),\ G_{\xi\eta}(x,y;\xi,\eta))$$

$$= G(x,y;x,y)$$

which can be easily verified. So that

$$(5) \qquad |u(x,y)| \leq \sqrt{G(x,y;x,y)} \; |||u||| \; .$$

Thus $[G(x,y;x,y)]^{\frac{1}{2}}$ can be taken as the number $M_{x,y}$ appearing in Definitions III.5.1. Thus we can make a functional completion of $C_o^\infty(R)$ as follows: Let $\{u_n\}$ be a Cauchy sequence in $C_o^\infty(R)$ with respect to (*III.2.5*). We have

$$0 \leq \lim_{n\to\infty} |u_n(x,y) - u_m(x,y)| \leq \sqrt{G(x,y;x,y)} \lim_{n\to\infty} |||u_n - u_m||| \; .$$

so that $\{u_n(x,y)\}$ is a Cauchy sequence of real numbers, which, therefore converges to, say, a function $u(x,y)$. The desired functional completion $\overline{C}_o^\infty(R)$ consists of all functions $u(x,y)$ defined in this way and it has the following properties:

(i) normwise convergence in $\overline{C}_o^\infty(R)$, implies pointwise convergence everywhere in R .

(ii) $C_o^\infty(R)$ is dense in $\overline{C}_o^\infty(R)$.

(iii) $\overline{C}_o^\infty(R)$ is complete.

(iv) For $(x,y) \in R$, the function $G(x,y;\xi,\eta)$ as a function of (ξ,η) is in $\overline{C}_o^\infty(R)$ and

$$(6) \qquad ((G(x,y;\xi,\eta) \; , \; u(\xi,\eta))) = u(x,y)$$

for all $u \in \overline{C}_o^\infty$. Thus, $G(x,y;\xi,\eta)$ is a reproducing kernel for the complete space $\overline{C}_o^\infty(R)$. For the functions $u \in C_o^\infty(R)$ we have

$$(7) \qquad \int_a^b\int_c^d G(x,y;\xi,\eta)\,\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}\,d\xi d\eta = u(x,y) \quad .$$

Let us now consider the space $\overline{\overline{C}}_o^\infty(R)$ of all functions of the form

$$(8) \qquad Gw = \int_a^b\int_c^d G(x,y;\xi,\eta)\,w(\xi,\eta)\,d\xi d\eta \quad , \quad w \in L_2(R)$$

where $w(x,y)$ is required to belong only to $L_2(R)$. Since

$$(9) \qquad \int_a^b\int_c^d\int_a^b\int_c^d G^2(x,y;\xi,\eta)\,dxdy\;d\xi d\eta < \infty$$

all functions of the form Gw belong to $L_2(R)$ and since $C_o^\infty(R)$ is dense in $L_2(R)$ we can easily show that $C_o^\infty(R)$ is also dense in $\overline{\overline{C}}_o^\infty(R)$. Hence $\overline{C}_o^\infty(R)$ is isomorphic to $\overline{\overline{C}}_o^\infty(R)$, since all completions are isometrically isomorphic to each other. Thus we can put

$$(10) \qquad \overline{C}_o^\infty(R) = \{\int_a^b\int_c^d G(x,y;\xi,\eta)\,w(\xi,\eta)\,d\xi d\eta \quad ; \quad w \in L_2(R)\} \quad .$$

By differentiation with respect to the parameter under the integral signs, we obtain

$$(11)\qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(b-a)(d-c)} \Big[\int_a^x\int_c^y (\xi-a)(\eta-c)w \, d\xi d\eta - \int_x^b\int_c^y (b-\xi)(\eta-c)w \, d\xi d\eta$$

$$- \int_a^x\int_y^d (\xi-a)(d-\eta)w \, d\xi d\eta + \int_x^b\int_y^d (b-\xi)(d-\eta)w \, d\xi d\eta \Big]$$

if

$$(12)\qquad u(x,y) = \int_a^b\int_c^d G(x,y;\xi,\eta)\, w(\xi,\eta)\, d\xi d\eta \quad .$$

Since the integrals are all continuous functions, $u_{xy}(x,y)$ is continuous. But then we have

$$(13)\qquad u_{xyxy} = \left[\frac{(x-a)(y-c) + (b-x)(y-c) + (x-a)(d-y) + (b-x)(d-y)}{(b-a)\ (d-c)}\right]w$$

$$= w(x,y)$$

almost everywhere in R and if $w(x,y)$ is continuous then $u_{xyxy}(x,y)$ exists everywhere. Thus we see that if in the boundary value problem (*III.4.2*) and (*III.4.3*) $f(x,y)$ is continuous then the generalized solution is also a classical solution. Incidentally we have also shown that a subspace $V_o^{(1)}$ is isometrically isomorphic to the function space $\overline{C}_o^{\infty}(R)$

§6. A More General Boundary Value Problem.

In this section, to generalize the results of the problems in the previous sections, we deal with the problem of the existence and uniqueness of a weak solution to the following boundary value problem due to D. Mangeron [2]. The classical case has been treated by F. Manaresi [1]. Our approach will be parallel to the one due to E.M. Landesman and A.C. Lazer [19]. Specifically, we consider the partial differential equation

$$(1) \qquad A\,u = \frac{\partial^2}{\partial x \partial y}\left(\theta(x,y)\,\frac{\partial^2 u}{\partial x \partial y}\right) + q(x,y)u = f(x,y)$$

for $a < x < b$ and $c < y < d$, subject to the boundary conditions

$$(2) \qquad u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0$$

where $\theta(x,y)$ and $q(x,y)$ are nonnegative measurable bounded functions defined on R such that

$$(3) \qquad 0 < \theta_0 \leq \theta(x,y)$$

and $f(x,y) \in L_2(R)$. A is considered as a differential operator in the formal sense, since the differentiability conditions on $\theta(x,y)$ are dropped.

We consider the inner product on $C_0^\infty(R)$ defined by

$$(4) \qquad \langle u,v \rangle = \int_a^b \int_c^d [\theta\, u_{xy} v_{xy} + q\, uv]\; dy dx \qquad u,v \in C_0^\infty(R) .$$

Lemma III.6.1. The operator A *defined on* $C_0^\infty(R)$ *is positive definite on* $C_0^\infty(R)$ *with respect to* $\langle u,u\rangle$.

Proof: Since $u \in C_0^\infty(R)$ we have

$$(5) \qquad u(x,y) = \int_a^x\int_c^y u_{\xi\eta}\, d\xi d\eta$$

$$= \int_a^x\int_c^y \frac{1}{\sqrt{\theta(\xi,\eta)}} \sqrt{\theta(\xi,\eta)}\; u_{\xi\eta}\, d\xi d\eta \;.$$

Hence squaring and applying the Cauchy-Schwarz inequality, we obtain

$$(6) \qquad u^2(x,y) \le \int_a^x\int_c^y \frac{1}{\theta(\xi,\eta)}\, d\xi d\eta \int_a^x\int_c^y \theta\, u_{\xi\eta}^2\, d\xi\, d\eta$$

$$\le \int_a^b\int_c^d \frac{1}{\theta(\xi,\eta)}\, d\xi d\eta \int_a^b\int_c^d \theta\, u_{\xi\eta}^2\, d\xi\, d\eta \quad .$$

Integration of both sides of the above inequality yields the inequality

$$(7) \qquad \int_a^b\int_c^d u^2\, dxdy \le (b-a)(d-c) \int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy \int_a^b\int_c^d \theta\, u_{xy}^2\, dxdy \quad .$$

From (*III.6.6*) we obtain

$$(8) \qquad \int_a^b\int_c^d u^2\, dxdy \le (b-a)(d-c)\int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy \;\; \langle u,u\rangle \quad .$$

Hence

$$(9) \qquad (A\,u,u) = \int_a^b\int_c^d [\theta(x,y)u_{xy}^2 + q(x,y)u^2]\,dxdy = \langle u,u\rangle$$

for all functions u in $C_0^\infty(R)$, which proves the positive definiteness of the operator A on $C_0^\infty(R)$.

Theorem III.6.2. *The norms defined by (III.6.4) and (III.2.5) are equivalent.*

Proof: We have to show that there exist constants m and M such that

$$(10) \qquad m\int_a^b\int_c^d u_{xy}^2\,dxdy \le \langle u,u\rangle \le M\int_a^b\int_c^d u_{xy}^2\,dxdy \quad .$$

The first inequality is readily obtained, since

$$(11) \qquad \theta_0\int_a^b\int_c^d u_{xy}^2\,dxdy \le \int_a^b\int_c^d (\theta\,u_{xy}^2 + q\,u^2)\,dxdy \quad .$$

To prove the second inequality, we observe that

$$(12) \qquad \int_a^b\int_c^d (\theta\,u_{xy}^2 + qu^2)\,dxdy \le \int_a^b\int_c^d \theta\,u_{xy}^2 + q_1\int_a^b\int_c^d u^2\,dxdy$$

where

$$q_1 = \sup_{(x,y)\in R}[q(x,y)] \quad .$$

Hence, combining (*III.6.8*) and (*III.6.12*), we infer that

$$\langle u,u\rangle \leq [(b-a)(d-c)\ q_1 \int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy + 1] \int_a^b\int_c^d \theta\, u_{xy}^2\, dxdy.$$

On the other hand, by our assumption on $\theta(x,y)$, we have

$$\sup_{(x,y)\in R} \theta(x,y) < \infty$$

and our theorem is thus proved.

Therefore, the completion of $C_0^{\infty}(R)$ with respect to $\langle u,u\rangle$ gives us the Hilbert Space $V_0^{(1)}$. As before, the following definition connects the operator A and the quadratic form $\langle u,u\rangle$:

<u>*Definition III.6.3.*</u> *A generalized (weak) solution of the boundary value problem (III.6.1) and (III.6.2) is a member* $u \in V_0^{(1)}$ *such that*

$$\langle v,u\rangle = (v,f)$$

for all $v \in V_0^{(1)}$.

Combining the above theorems with theorems from Section 1, we have the following theorem:

<u>*Theorem III.6.4.*</u> *The boundary value problem (III.6.1) and (III.6.2) has a unique generalized solution in* $V_0^{(1)}$.

At this point, we wish to mention that in the case when $\theta(x,y)$ is sufficiently differentiable and $p(x,y)$ and $f(x,y)$ are continuous, F. Manaresi [1] has shown that there exists a classical solution to the boundary value problem (III.6.1) and (III.6.2), by using the method of successive approximations.

§7. Generalized Eigenfunctions and Weak Eigenvalues.

In this section, following the general spirit of the work of E.M. Landesman and A.C. Lazer [19], we extend the concept of eigenfunctions and eigenvalues of the following generalized Sturm-Liouville problem of D. Mangeron [2]. Specifically we consider the following problem

(1) $$A\, u = \frac{\partial^2}{\partial x \partial y}\left(\theta(x,y)\, \frac{\partial^2 u}{\partial x \partial y}\right) + q(x,y)u = \lambda\, p(x,y)u$$

(2) $$u(a,y) = u(x,c) = u(b,y) = u(x,d) = 0$$

where $\theta(x,y)$ and $q(x,y)$ satisfy the conditions specified in Section 6, and $p(x,y)$ is a measureable function defined on R satisfying the conditions

(3) $$\delta \leq p(x,y) \leq \Delta$$

for all $(x,y) \in R$ where δ , Δ are constants. Let us define the following inner product on $L_2(R)$

$$(4) \qquad (u,v)_p = \int_a^b \int_c^d p \; uv \; dxdy \qquad u,v \in L_2(R) \quad .$$

Clearly, due to the assumptions on $p(x,y)$, the inner product (*III.7.4*) defines an equivalent norm on $L_2(R)$, which induces the same topology on $L_2(R)$ as (u,v) does.

Let us define a linear functional on $V_o^{(1)}$ by

$$(5) \qquad L_{p,w}(\phi) = (\phi,w)_p = \int_a^b \int_c^d p \, \phi \, w \; dxdy$$

where $w \in L_2(R)$. Then, by virtue of (*III.7.3*), it can easily be seen that $L_{p,w}$ is a bounded linear functional on $V_o^{(1)}$. Since $V_o^{(1)}$ is a Hilbert Space, as shown in Section 2 there exists a unique $T_p(w) \in V_o^{(1)}$ such that

$$(6) \qquad \langle \phi, T_p(w) \rangle = L_{p,w}(\phi) = (\phi,w)p$$

for all $\phi \in V_o^{(1)}$. This defines a linear map

$$(7) \qquad T_p : L_2(R) \to V_o^{(1)}$$

but since $V_o^{(1)} \subset L_2(R)$, we may consider T_p as a linear map from $L_2(R)$ into $L_2(R)$. Since

$$(III.6.8) \qquad \int_a^b \int_c^d u^2 \; dxdy \leq (b-a)(d-c) \int_a^b \int_c^d \frac{1}{\theta(x,y)} \; dxdy \; \langle u,u \rangle$$

and since $p(x,y)$ satisfies the condition (*III.7.3*) it follows that T_p is continuous and maps bounded subsets of $L_2(R)$ into bounded sets of $V_o^{(1)}$. Thus by Rellich's selection principle [*1*] T_p is completely continuous. Moreover T_p is symmetric with respect to $(\, , \,)_p$ as the inner product. For if $u,v \in L_2(R)$ we have

$$(8) \qquad (T_p u,v) = \langle T_p u \, , \, T_p v\rangle$$

$$= \langle T_p v \, , \, T_p u\rangle$$

$$= (T_p v \, , \, u)_p$$

$$= (u \, , T_p v)_p \quad .$$

If for some $u \in L_2(R)$, $T_p u = 0$, then

$$(9) \qquad (\phi,u)_p = 0$$

for all $\phi \in C_o^\infty(R)$ and since $C_o^\infty(R)$ is dense in $L_2(R)$ we have $u \equiv 0$. Thus, if $u \in L_2(R)$, it follows from (*III.7.6*) , letting $\phi = T_p u \in V_o^{(1)}$ that

$$(10) \qquad (T_p u,u)_p = \langle T_p u,T_p u\rangle > 0$$

which is positive for $u \neq 0$. Hence T_p is positive and symmetric. Now applying the results of §93 and §94 of [*33*] about positive symmetric operators we infer the existence of a sequence $\{\phi_k\}_1^\infty$ in $L_2(R)$ such that

$$(11) \qquad \begin{cases} \phi_k = \lambda_k \, T_p \, \phi_k \\ (\phi_k, \phi_j)_p = \delta_{kj} \end{cases}$$

where δ_{kj} is the Kronecker delta and

$$(12) \qquad T_p \, u = \sum_{k=1}^{\infty} \frac{(u, \phi_k)_p}{\lambda_k} \, \phi_k$$

for all $u \in L_2(R)$. Moreover the sequence $\{\lambda_k\}$ has no finite cluster point and so we may assume

$$(12)' \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \quad .$$

Using (*III.7.6*) and (*III.7.11*) we obtain

$$\langle w, \phi_k \rangle = (w, \lambda_k \phi_k)_p = (w, \lambda_k p \phi_k)$$

for $w \in V_o^{(1)}$. Hence for each $k = 1,2,3,\cdots,\phi_k$ is a nontrivial weak solution to the boundary value problem (*III.7.1*) and (*III.7.2*). We therefore call λ_k a weak eigenvalue corresponding to p and denote it by $\lambda_k(p)$. It can easily be seen that

(13) $$(u,u)_p = \sum_{k=1}^{\infty} (u,\phi_k)_p^2$$

for all $u \varepsilon L_2(R)$.

We can now write a similar identity involving $V_o^{(1)}$. Using (III.7.6) and (III.7.11) we have

(14) $$\begin{aligned} <\phi_k, \phi_j> &= <\lambda_k(p) T_p \phi_k, \phi_k> \\ &= \lambda_k(p) \; (\phi_k, \phi_j)_p \\ &= \lambda_k(p) \; \delta_{kj} \end{aligned}$$

which shows that the sequence $\{\frac{1}{\sqrt{\lambda_k(p)}} \phi_k\}_1^{\infty}$ in $V_o^{(1)}$, is orthonormal with respect to the inner product $< , >$. If for some $w \varepsilon V_o^{(1)}$

(15) $$<\phi_k, w> = 0$$

for all k , then by (III.7.6) and (III.7.11)

$$(\phi_k, w)_p = 0$$

for all k. Hence $w \equiv 0$ a.e.. Thus the sequence $\{\frac{1}{\sqrt{\lambda_k(p)}} \phi_k\}$ is complete in $V_o^{(1)}$ and Parseval's formula yields

(16) $$\langle u,u\rangle = \sum_{k=1}^{\infty} \langle \frac{1}{\sqrt{\lambda_k(p)}} \phi_k, u\rangle^2 .$$

Using (*III.7.12*) and (*III.7.13*) we can write (*III.7.16*) as

(17) $$\langle u,u\rangle = \sum_{k=1}^{\infty} \lambda_k(p)\ (u,\phi_k)_p^2 .$$

The identities (*III.7.16*) and (*III.7.17*) together with (*III.7.12'*) now yield the following variational characterization of weak eigenvalues in terms of the inner product $\langle\ ,\ \rangle$ and $(\ ,\ )_p$.

(18) $$\begin{cases} \lambda_1(p) = \min \{\langle u,u\rangle \ , \quad u \in V_o^{(1)} \ ; \quad (u,u)_p = 1 \} \\[2ex] \lambda_{k+1}(p) = \min \left\{ \begin{matrix} \langle u,u\rangle \ , & u \in V_o^{(1)} \ ; & (u,u)_p = 1 \\ (u,\phi_j)_p = 0 & & j = 1,2,3,\cdots,k \end{matrix} \right\} \end{cases}$$

Note that Courant's Min-Max principle can also be extended very easily. In this connection we have the following:

<u>*Theorem III.7.1.*</u> *If for* $v_1, v_2, v_3, \cdots, v_k$ *in* $L_2(R)$, *we define*

$$\mu_k(p)(v_1,v_2,\cdots,v_k) = \inf \left\{ \begin{matrix} \langle\theta,\theta\rangle \ , & \theta \in V_o^{(1)} \ , & (\theta,\theta)_p = 1 \\ (\theta,v_j)_p = 0 & & j = 1,2,\cdots,k-1 \end{matrix} \right\}$$

then

$$\lambda_{k+1} = \sup \left\{ \begin{array}{l} \mu_k(p) \ (v_1, v_2, \cdots, v_k) \ \big| \ v_j \in L_2(R) \\ \qquad\qquad j = 1, 2, \cdots, k \end{array} \right. .$$

CHAPTER IV

Fundamental Inequalities and Natural Boundary Conditions

§1. Inequalities.

Various types of inequalities have been proved for functions of one variable, relating them with their derivatives. In this connection, we refer to the excellent books by E.F. Beckenbach and R. Bellman [5] and D.S. Mitrinovic [27]. In this chapter, we give analogous inequalities for functions of two variables $u(x,y)$ defined on R, relating them with their mixed partial derivative $u_{xy}(x,y)$, Assuming its existence in the classical sense. The extension of some of these inequalities has been considered by D. Mangeron [12].

<u>Theorem IV.1.1.</u> *Let* $u(x,y) \in \Gamma^{(1)}$ *be such that* $u(a,y) = u(x,c) = u(a,c) = 0$. *Then*

$$\int_a^b \int_c^d u^2 \, dx dy \leq \frac{(b-a)^2 (d-c)^2}{4} \int_a^b \int_c^d u_{xy}^2 \, dx dy \, . \tag{1}$$

<u>Proof</u>: Since $u(a,y) = u(x,c) = u(a,c) = 0$, we have

$$u(x,y) = \int_a^x \int_c^y u_{\xi\eta} \, d\xi d\eta \tag{2}$$

which implies

(3) $$u^2(x,y) = \left(\int_a^x\int_c^y u_{\xi\eta}\, d\xi d\eta\right)^2$$

and utilising the Cauchy-Schwarz inequality we obtain

(4) $$u^2(x,y) \leq (x-a)(y-c) \int_a^x\int_c^y u_{\xi\eta}^2\, d\xi d\eta$$

$$\leq (x-a)(y-c) \int_a^b\int_c^d u_{\xi\eta}^2\, d\xi d\eta \quad .$$

Integrating either side of the inequality (*IV.1.4*) we obtain (*IV.1.1*).

Corollary IV.1.2. If $u(x,y) \in \Gamma^{(1)}$ and $u(a,y) = u(x,c) = u(a,c) = 0$, then

(5) $$\int_a^b\int_c^d |u\ u_{xy}|\ dxdy \leq \frac{(b-a)(d-c)}{2} \int_a^b\int_c^d u_{xy}^2\ dxdy \ .$$

Proof: Applying the Cauchy-Schwarz inequality to the left hand side we obtain

(6) $$\int_a^b\int_c^d |u\ u_{xy}|\ dxdy \leq \sqrt{\int_a^b\int_c^d u^2\ dxdy}\ \sqrt{\int_a^b\int_c^d u_{xy}^2\ dxdy}$$

and combining (*IV.1.6*) and (*IV.1.1*) we obtain (*IV.1.5*).

Theorem IV.1.3. If $u(x,y) \in \Gamma^1$ *, then*

$$(7) \qquad 4(b-a)(d-c) \int_a^b\int_c^d u^2\, dxdy + 4\left[\int_a^b\int_c^d u\, dxdy\right]^2$$

$$\leq (b-a)^3(d-c)^3 \int_a^b\int_c^d u_{xy}^2\, dxdy + 4(b-a)\int_a^b dx\left[\int_c^d u(x,y)dy\right]^2$$

$$+ 4(d-c)\int_c^d dy\left[\int_a^b u(x,y)dx\right]^2 \quad .$$

Proof: Let $a < x_1 < x_2 < b \;;\; c < y_1 < y_2 < d$. Then

$$(8) \quad [u(x_2,y_2) - u(x_2,y_1) - u(x_1,y_2) + u(x_1,y_1)]^2 = \left[\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}\, dxdy\right]^2$$

which on applying the Cauchy-Schwarz inequality yields

$$(9) \qquad \left[\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}\, dxdy\right]^2 \leq (x_2-x_1)(y_2-y_1)\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2\, dxdy$$

$$\leq (b-a)(d-c)\int_a^b\int_c^d u_{xy}^2\, dxdy \quad .$$

Expanding the term

$$[u(x_2,y_2) - u(x_1,y_2) - u(x_2,y_1) + u(x_1,y_1)]^2$$

in (*IV.1.8*) and integrating the resulting inequality in the 4-dimensional rectangle $\{a \leq x_1 \leq b \;;\; a \leq x_2 \leq b \;;\; c \leq y_1 \leq d \;;\; c \leq y_2 \leq d\}$ (*IV.1.7*) follows.

Corollary IV.1.4. If $u \in \Gamma^{(2)}$ is such that

$$u(a,y) = u(b,y) = u(x,c) = u(x,d) = 0 \quad ,$$

then

$$(10) \qquad \int_a^b \int_c^d u^2 \, dxdy \leq \frac{(b-a)^4(d-c)^4}{16} \int_a^b \int_c^d (u_{xyxy})^2 \, dxdy \; .$$

Proof: In the inequality IV.1.7, replace u by u_{xy} and since

$$(11) \qquad \int_c^d u_{xy} \, dy = u_x(x,d) - u_x(x,c) = 0$$

and also

$$(12) \qquad \int_a^b u_{xy} \, dx = u_y(b,y) - u_y(a,y) = 0$$

we have

$$(13) \qquad \int_a^b \int_c^d u_{xy}^2 \, dxdy \leq \frac{(b-a)^2(d-c)^2}{4} \int_a^b \int_c^d u_{xyxy}^2 \, dxdy \quad .$$

Combining this with (*IV.1.1*) we have

$$(14) \qquad \int_a^b \int_c^d u^2 dxdy \leq \frac{(b-a)^4(d-c)^4}{16} \int_a^b \int_c^d u_{xyxy}^2 \, dxdy \quad .$$

Theorem IV.1.5. *If* $u(x,y) \in \Gamma^{1}([0,2\pi]\times[0,2\pi])$ *is such that*

$$(15)\quad \begin{cases} (i) & u(o,y) = u(2\pi,y) \quad ; \quad u(x,o) = u(x,2\pi) \\ (ii) & \int_0^{2\pi} u(x,y)dx = 0 \quad ; \quad \int_0^{2\pi} u(x,y)dy = 0 \quad . \end{cases}$$

Then

$$(16)\qquad \int_0^{2\pi}\int_0^{2\pi} u^2 \, dxdy \leq \int_0^{2\pi}\int_0^{2\pi} u_{xy}^2 \, dxdy \quad .$$

Proof: Applying Wirtinger's inequality [5] to $u(0,y)$ as a function of x alone we have,

$$(17)\qquad \int_0^{2\pi} u^2(x,y)dx \leq \int_0^{2\pi} u^2_{\ x}(x,y)dx$$

which implies

$$(18)\qquad \int_0^{2\pi}\int_0^{2\pi} u^2(x,y)dxdy \leq \int_0^{2\pi}\int_0^{2\pi} u_x^2(x,y)dxdy \quad .$$

But since

$$(19)\qquad u_x(x,o) = u_x(x,2\pi) \quad \text{and} \quad \int_0^{2\pi} u_x(x,y)dy = 0$$

a repeated application of Wirtinger's inequality yields the result.

Remark 1: Since $u(x,y)$ is periodic and satisfies the condition (*IV.1.15(i)*), it has been shown by M. Picone [6] that $u(x,y)$ can be expressed as Fourier series

$$(20) \qquad u(x,y) = \sum_{i,j=1} (a_{ij}\ Cos\ ix\ Cos\ jy + b_{ij}\ Sin\ jy\ Cos\ ix + c_{ij}\ Sin\ ix\ Cos\ jy + d_{ij}\ Sin\ ix\ Sin\ jy) .$$

Using this expansion, it can easily be shown that the equality sign in the inequality (*IV.1.16*) holds if and only if $u(x,y)$ is of the form

$$(21) \qquad u(x,y) = a_{11}\ Cos\ x\ Cos\ y + b_{11}\ Cos\ x\ Sin\ y + c_{11}\ Cos\ y\ Sin\ x + d_{11}\ Sin\ x\ Sin\ y .$$

Remark 2: If in Theorem IV.1.5, $u(x,y)$ satisfies the condition (*IV.1.15(ii)*) only, then the inequality of Theorem IV.1.3 gives that

$$(22) \qquad \int_0^{2\pi}\int_0^{2\pi} u^2\ dxdy \leq \frac{\pi^4}{4} \int_0^{2\pi}\int_0^{2\pi} u_{xy}^2\ dxdy .$$

(II) If in the theorem (*IV.1.5*), $u(x,y)$ satisfies the condition (*IV.1.15(i)*) only, then Corollary IV.1.4 gives that

$$(23) \qquad \int_0^{2\pi}\int_0^{2\pi} u^2(x,y)\,dxdy \le \frac{\pi^8}{16}\int_0^{2\pi}\int_0^{2\pi} u_{xyxy}^2\,dxdy \quad .$$

Theorem IV.1.6. *Let* $p(x,y)$ *be a bounded positive function defined on* R . *Then*

$$(24) \qquad \lambda \int_a^b\int_c^d p\, u^2\,dxdy \le \int_a^b\int_c^d u_{xy}^2\,dxdy$$

where λ *is the smallest eigenvalue of the problem*

$$(25) \qquad \begin{cases} (i) & \dfrac{\partial^4 u}{\partial x^2 \partial y^2} = \lambda\, p(x,y)u \\ (ii) & u(x,c) = u(x,d) = u(a,y) = u(b,y) = 0 \quad . \end{cases}$$

Proof: Consider the problem of minimizing the functional

$$(26) \qquad \int_a^b\int_c^d u_{xy}^2\,dxdy \qquad u \in \Gamma_o^{(2)}$$

subject to the condition

$$(27) \qquad \int_a^b\int_c^d p\, u^2\,dxdy = 1$$

in the class of functions vanishing on the boundary of R .

Using the method of Lagrange multipliers, the problem is equivalent to minimizing the quadratic functional

(28) $$J[u] = \int_a^b \int_c^d (u_{xy}^2 - \lambda\, pu^2)\,dxdy$$

over all functions $u(x,y)$ for which the integral exists and the functions vanish on the boundary of R. The corresponding Euler equation is

(29) $$\frac{\partial^4 u}{\partial x^2 \partial y^2} - \lambda\, p(x,y)u = 0 \quad .$$

If $u(x,y)$ satisfies the equation and the boudary conditions then

(30) $$0 = \int_a^b \int_c^d u[\frac{\partial^4 u}{\partial x^2 \partial y^2} - \lambda\, p(x,y)u]dxdy = \int_a^b \int_c^d u_{xy}^2 \; dxdy - \lambda$$

Thus the minimum value of the functional (*IV.1.26*) is the smallest eigenvalue of the Sturm Liouville problem (*IV.1.25*).

§2. Natural Boundary Conditions.

It is known that for the operator $-\frac{d^2u}{dx^2}$ $(a < x < b)$ the conditions $u(a) = 0$, $u(b) = 0$ are principal and the conditions $u'(a) = 0$ and $u'(b) = 0$ are natural. If the operator A has the form

$$A\,u = \sum_{k=0}^{s} \sum_{\substack{i_1,i_2,\cdots,i_k=1 \\ j_1,j_2,\cdots,j_k=1}} \frac{\partial^k}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \left\{ A^{j_1 j_2 \cdots j_k}(x_1,x_2,\cdots,x_n) \frac{\partial^k u}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}} \right\}$$

so that its order equals $2s$, let this operator be positive and bounded below for a set of functions which satisfy certain boundary conditions.

Then the natural boundary conditions will be the homogeneous ones into which there enter derivatives of u of order s and higher, and the principal ones will be – those which contain derivatives of u up to the order $s-1$. First we have the following theorem.

<u>*Theorem IV.2.1.*</u> *If* $u(x,y) \in \Gamma^{(1)}$ *is such that* $u(a,y) = u(x,c) = u(a,c) = 0$ *and* $u(x,y)$ *minimizes the functional*

$$\int_a^b \int_c^d u_{xy}^2 \, dx dy \tag{1}$$

subject to the conditions

$$\int_a^b \int_c^d u^2 \, dx dy = 1 \tag{2}$$

then

$$u_{xy}(x,d) = u_{xy}(b,y) = u_{xy}(b,d) = 0 \tag{3}$$

and

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} - \lambda \, u(x,y) = 0 \tag{4}$$

where λ *is the minimum eigenvalue.*

<u>Proof:</u> First of all Theorem IV.1.1 shows that the minimum exists. By the method of Lagrange multipliers, the first variation of the resulting

functional is equal to zero. That is

$$(5) \qquad \int_a^b\int_c^d (u_{xy}v_{xy} - \lambda\, uv)\, dxdy = 0$$

for all admissible $v(x,y)$. Also as a consequence of Mason's lemma (cf: §II.3) we can easily show that $u(x,y)$ satisfies $(IV.2.4)$. We now have the following identity

$$(6) \qquad u_{xyxy}v = [u_{xy}v]_{xy} - [u_{xy}v_x]_y - [u_{xy}v_y]_x + u_{xy}v_{xy} \quad .$$

Integrating the above identity we obtain

$$(7) \qquad \int_c^d\int_a^b [u_{xy}v_{xy} - \lambda\, uv]\, dxdy = \int_c^d\int_a^b (u_{xyxy}-\lambda u)v\, dxdy - u_{xy}(b,d)\; v(b,d)$$

$$+ \int_a^b u_{xy}(x,d)\; v_x(x,d)\, dx + \int_c^d u_{xy}(b,y) v_y(b,y)\, dy$$

$$= 0 \quad .$$

But due to the arbitrariness of v, we can first choose $v(x,y)$ such that $v(x,y)$ vanishes on the boundary of R. Then we get the Euler equation. Thus

$$(8) \qquad -u_{xy}(b,d)v(b,d) + \int_a^b u_{xy}(x,d)v_x(x,d)\, dx + \int_c^d u_{xy}v_y(b,y)\, dy = 0$$

for all $v(x,y)$ such that $v(a,y) = v(x,c) = 0$. Now let us choose v as follows:

(9) $$v(x,y) = f(x)g(y) \qquad f \in C^1(a,b) \quad , \quad g \in C^1(c,d)$$

$$\begin{cases} f(a) = f(b) = 0 \qquad f(x) \not\equiv 0 \ . \\ g(c) = 0 \quad , \quad g(y) \not\equiv 0 \quad \text{and} \quad g(d) \neq 0 \ . \end{cases}$$

Substituting this value of $v(x,y)$ in Equation (*IV.2.8*) we get

$$g(d)\int_a^b u_{xy}(x,d)\ f'(x)dx = 0$$

for all $f(x) \in C^1(a,b)$ with $f(a) = f(b) = 0$. Hence by the fundamental lemma of the calculus of variations [11] we have

(10) $$u_{xy}(x,d) = c_1 \ ,$$

(11) $$u_{xy}(b,y) = c_2$$

c_1 and c_2 being constants. But Equations (*IV.2.10*) and (*IV.2.11*) are compatible iff $c_1 = c_2$. Substituting this back into Equation (*IV.2.8*) we find

$$-c_1\ v(b,d) + c_1\ v(b,d) + c_1\ v(b,d) = c_1\ v(b,d) = 0 \quad .$$

But, since $v(x,y)$ can be chosen so that $v(b,d) \neq 0$, we have $c_1 = 0$. This completes the proof of the theorem.

§3. A Simple Boundary Value Problem Involving Natural Boundary Conditions.

In this section we investigate the boundary value problem

(1) $$\frac{\partial^4 u}{\partial x^2 \partial y^2} = f(x,y) \qquad \begin{pmatrix} a < x < b \\ c < y < d \end{pmatrix}$$

(2) $$u_{xy}(b,y) = u_{xy}(a,y) = u_{xy}(x,c) = u_{xy}(x,d) = 0$$

$f(x,y)$ being a continuous function. Clearly the solvability of the boundary value problem (*IV.3.1*) and (*IV.3.2*) can be replaced by the problem of minimizing the functional

(3) $$I[u] = \int_a^b \int_c^d (u_{xy}^2 - 2\, uf)\, dxdy$$

over the above specified class of functions. However the operator

(4) $$A\, u = \frac{\partial^4 u}{\partial x^2 \partial y^2}$$

is not positive definite on the class of functions satisfying the boundary conditions (*IV.3.2*), since in this case

(5) $$(Au,u) = \int_a^b \int_c^d u\, u_{xyxy}\, dxdy = \int_a^b \int_c^d u_{xy}^2\, dxdy = 0$$

$$(10) \qquad (Au,u) = |||u|||^2 = \int_a^b \int_c^d u_{xy}^2 \, dxdy \qquad u \in M_o$$

and as before, in this completed space $\overline{M}_o$ we can show that there exists a solution.

In the case of $V_o^{(1)}$ we found that the functions in the completed space satisfy the same boundary conditions as the original class of functions. But this need not be the case here. Let $u(x,y) \in C^4(R^2)$ be any function which satisfies $(IV.3.9)$ but not $(IV.3.8)$. Then since the functions

$$\{\sin \frac{k\pi(x-a)}{(b-a)} \sin \frac{\ell\pi(y-c)}{(d-c)}\}_{k,\ell}$$

form a complete orthogonal system in R with respect to the $L_2(R)$ norm, we can expand $u_{xy}(x,y)$ in terms of its Fourier Series, i.e.

$$(12) \qquad u_{xy}(x,y) = \sum_{k,\ell=1}^{\infty} a_{k,\ell} \sin \frac{k\pi(x-a)}{(b-a)} \sin \frac{\ell\pi(y-c)}{(d-c)}$$

where

$$(13) \qquad a_{k\ell} = \int_a^b \int_c^d u_{xy}(x,y) \sin \frac{k\pi(x-a)}{(b-a)} \sin \frac{\ell\pi(y-c)}{(d-c)} \, dxdy$$

and the series converges in the $L_2(R)$ norm. Integrating the series $(IV.3.12)$, we obtain

$$(14)\quad u(x,y) = f(x) + g(y) + \sum_{k,\ell=1}^{\infty} a_{k,\ell} \frac{(b-a)(d-c)}{k\,\ell\,\pi^2} \cos\frac{k\pi(x-a)}{(b-a)} \cos\frac{\ell\pi(y-c)}{(d-c)}$$

where $f(x)$ and $g(y)$ are arbitrary continuously differentiable functions in the respective domains. Since $u(x,y)$ satisfies the conditions $(IV.3.9)$, we find that $f(x) + g(y) \equiv 0$. For the conditions $(IV.3.9)$ imply

$$f(x)(d-c) + \int_c^d g(y)dy = 0$$

$$g(y)(b-a) + \int_a^b f(x)dx = 0 \quad .$$

Thus

$$f(x) + g(y) = -\frac{1}{(b-a)}\int_a^b f(x)dx - \frac{1}{(d-c)}\int_c^d g(y)dy$$

$$= \text{constant}$$

$$= c_1 \quad .$$

But since

$$\int_a^b\int_c^d u(x,y)dxdy = 0$$

implies that $c_1 = 0$.

$$(15) \qquad u(x,y) = \sum_{k,\ell=1}^{\infty} a_{k,\ell} \frac{(b-a)(d-c)}{k\ell\pi^2} \cos\frac{k\pi(x-a)}{(b-a)} \cos\frac{\ell\pi(y-c)}{(d-c)} .$$

We now put

$$u_n(x,y) = \sum_{k,\ell=1}^{n} a_{k,\ell} \frac{(b-a)(d-c)}{k\ell\pi^2} \cos\frac{k\pi(x-a)}{(b-a)} \cos\frac{\ell\pi(y-c)}{(d-c)} .$$

Clearly the functions $\{u_n(x,y)\}$, belong to M_o for all n. From the expansion (*IV.3.12*) we find

$$\lim_{\substack{n\to\infty \\ n\to\infty}} \left|\left|\frac{\partial^2 u_n}{\partial x\partial y} - \frac{\partial^2 u_m}{\partial x \partial y}\right|\right| = \lim_{\substack{n\to\infty \\ n\to\infty}} |||u_n - u_m||| = 0 .$$

By definition there exists a function $\overline{u}(x,y) \in \overline{M}_o$ such that $\{u_n(x,y)\}$ converges both in $L_2(R)$ norm and the norm (*IV.3.10*). But then

$$u(x,y) = \overline{u}(x,y) ,$$

since $u_n(x,y) \to u(x,y)$ in $L_2(R)$ norm. Thus $u(x,y) \in \overline{M}_o$. This function, however does not satisfy the boundary conditions by our assumptions.

§4. A Comparison Theorem for Eigenvalues.

In this section we prove a comparison theorem for eigenvalues of a partial differential equation of D. Mangeron involving natural boundary conditions. This is an extension of a theorem of Z. Nehari [28] for ordinary differential equations. The second theorem proved in this section is an extension of an inequality of Liapunov for ordinary differential equations.

<u>*Theorem IV.4.1.*</u> *Let* $p_1(x,y)$, $p_2(x,y)$ *be positive continuous functions defined on* R *such that*

$$\int_a^x \int_c^y p_1(\xi,\eta)\,d\xi d\eta \leq \int_a^x \int_c^y p_2(\xi,\eta)\,d\xi d\eta \quad .$$

Let λ_1 *and* λ_2 *respectively be the least eigenvalues of the equations*

$$\text{(1)} \qquad \frac{\partial^4 u_i}{\partial x^2 \partial y^2} = \lambda_i p_i(x,y) u_i \qquad (i=1,2)$$

$$\text{(2)} \qquad \frac{\partial^2}{\partial x \partial y} u_i(a,y) = \frac{\partial^2}{\partial x \partial y} u_i(x,c) = 0$$

$$\text{(3)} \qquad u_i(b,y) = u_i(x,d) = 0 \quad .$$

Then we have $\lambda_2 \leq \lambda_1$ *, where the equality sign holds when* $p_1(x,y) = p_2(x,y)$.

Before we give the proof of the theorem we have the following lemma:

<u>*Lemma IV.4.2*</u> . *If* $\phi(x,y)$, $\psi(x,y)$ *are functions belonging to* $\Gamma^{(1)}$ *such that*

$$\text{(4)} \qquad \phi(a,y) = \phi(x,c) = 0 \quad ; \quad \psi(x,d) = \psi(b,y) = 0 \quad ,$$

then

$$\int_a^b \int_c^d \phi \, \psi_{xy} \, dxdy = \int_a^b \int_c^d \phi_{xy} \, \psi \, dxdy \quad . \tag{5}$$

Proof: Integrating the identity

$$\phi \, \psi_{xy} - \phi_{xy} \, \psi = [\phi \, \psi_x]_y - [\phi_y \, \psi]_x \tag{6}$$

and making use of the boundary conditions (*IV.4.4*) we find

$$\begin{aligned} (7) \quad & \int_a^b \int_c^d [\phi\psi_{xy} - \phi_{xy}\psi] dxdy \\ &= \int_a^b [\phi\psi_x(x,d) - \phi\psi_x(x,c)] dx + \int_c^d [\phi_y\psi(b,y) - \phi_y\psi(a,y)] dy \\ &= 0 \end{aligned}$$

and the lemma follows.

Proof of Theorem IV.4.1. It can easily be seen using the identity (*IV.2.6*), that

$$\int_a^b \int_c^d [\frac{\partial^2 u_1}{\partial x \partial y}]^2 dxdy = \lambda_1 \int_a^b \int_c^d p_1 \, u_1^2 \quad dxdy. \tag{8}$$

Now in Lemma IV.4.2, we take

$$(9) \qquad \psi(x,y) = u_1^2(x,y) \quad ; \quad \phi(x,y) = \int_a^x\int_c^y p_1(\xi,\eta)\,d\xi d\eta$$

which yields

$$(10) \qquad \int_a^b\int_c^d \left[\frac{\partial^2 u_1}{\partial x\partial y}\right]^2 dxdy = \lambda_1 \int_a^b\int_c^d p_1\, u_1^2\, dxdy$$

$$= \lambda_1 \int_a^b\int_c^d \left[\int_a^x\int_c^y p_1 d\xi d\eta\right] \frac{\partial^2 u_1}{\partial x\partial y}^2\, dxdy.$$

Since $u_1(x,y)$ is the eigenfunction corresponding to the least eigenvalue λ_1, we have $u_1(x,y) \geq 0$ by R. Jentzsch's theorem [17]. Further by virtue of boundary conditions (*IV.4.2*) and (*IV.4.3*) we have

$$(11) \qquad \frac{\partial^2 u_1}{\partial x\partial y}(x,y) = \lambda_1 \int_a^x\int_c^y p_1(\xi,\eta)\, u_1(\xi,\eta)\,d\xi d\eta \geq 0 \quad .$$

Thus we have

$$(12) \qquad u_{1x}(x,d) - u_{1x}(x,y) = \int_y^d u_{1x\eta}(x,\eta)\,d\eta \geq 0 \quad .$$

Since $u_x(x,d) = 0$, we see that $u_{1x}(x,y) \leq 0$. By a similar argument $u_{1y}(x,y) \leq 0$. Thus

$$(13) \qquad \frac{\partial^2(u_1^2)}{\partial x\partial y} = 2u_1 \frac{\partial^2 u_1}{\partial x\partial y} + 2\,\frac{\partial u_1}{\partial x}\,\frac{\partial u_1}{\partial y} \geq 0 \ .$$

Hence

$$(14) \qquad \int_a^b\int_c^d \left[\frac{\partial^2 u_1}{\partial x\partial y}\right]^2 dxdy = \lambda_1 \int_a^b\int_c^d \left(\int_a^x\int_c^y p_1\, d\xi d\eta\right) \frac{\partial^2 u_1}{\partial x\partial y}^2 \, dxdy$$

$$< \lambda_1 \int_a^b\int_c^d \left(\int_a^x\int_c^y p_2 d\xi d\eta\right) \left(\frac{\partial^2 u_1}{\partial x\partial y}^2\right) dxdy$$

$$= \lambda_1 \int_a^b\int_c^d p_2(x,y) u_1^2 \, dxdy \quad .$$

Thus for the solution $x_1(x,y)$ satisfying the boundary conditions (*IV.4.2*) and (*IV.4.3*)

$$(15) \qquad \int_a^b\int_c^d \left[\frac{\partial^2 u_1}{\partial x\partial y}\right]^2 dxdy < \lambda_1 \int_a^b\int_c^d p_2\, u_1^2 \, dxdy \quad .$$

But by virtue of Theorem IV.1.6, we have

$$(16) \qquad \lambda_2 \int_a^b\int_c^d p_2\, u_1^2\, dxdy \le \int_a^b\int_c^d \left[\frac{\partial^2 u_1}{\partial x\partial y}\right]^2 dxdy \le \lambda_1 \int_a^b\int_c^d p_2\, u_1^2\, dxdy \quad .$$

This yields the assertion of the theorem.

Theorem IV.4.3. *Let* $\theta(x,y) > 0$ *be such that* $\theta_x(x,y)$, $\theta_y(x,y)$ *and* $\theta_{xy}(x,y)$ *are continuous functions in* R *and let* $p(x,y)$ *be a continuous function defined on* R . *If the boundary value problem*

$$(17) \qquad (\theta\, u_{xy})_{xy} - p(x,y)u = 0$$

$$(18) \qquad u(a,y) = u(x,c) = u_{xy}(b,y) = u_{xy}(x,d) = 0$$

has a non-trivial continuous solution, then

$$\int_a^b\int_c^d \frac{dxdy}{\theta(x,y)} \int_a^b\int_c^d p_+(x,y)\,dxdy \geq 1 \tag{19}$$

where

$$p_+(x,y) = max\ \{0\ ,\ p(x,y)\}\ .$$

Proof: First of all we observe that because of our assumption of the existence of a nontrivial solution, we have

$$\int_a^b\int_c^d \theta\ u_{xy}^2\ dxdy = \int_a^b\int_c^d p\ u^2\ dxdy > 0 \quad . \tag{20}$$

Also

$$u(x,y) = \int_a^x\int_c^y u_{\xi\eta}\ d\xi d\eta \tag{21}$$

$$= \int_a^x\int_c^y \frac{1}{\sqrt{\theta(\xi,\eta)}}\ \sqrt{\theta(\xi,\eta)}\ u_{\xi\eta}\ d\xi d\eta$$

squaring and applying the Cauchy-Schwarz inequality, we obtain

$$u^2(x,y) \leq \int_a^x\int_c^y \frac{1}{\theta(\xi,\eta)}\ d\xi d\eta \int_a^x\int_c^y \theta\ (\xi,\eta)\ u_{\xi\eta}^2\ \ d\xi d\eta \tag{22}$$

$$\leq \int_a^b\int_c^d \frac{1}{\theta(x,y)}\ dxdy \int_a^b\int_c^d \theta\ u_{xy}^2\ dxdy$$

$$= \int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy \int_a^b\int_c^d p\ u^2\, dxdy \quad .$$

Hence

$$(23) \qquad \max_{(x,y)\varepsilon R} u^2(x,y) \leq \int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy \int_a^b\int_c^d p_+(x,y)\, dxdy \max_{(x,y)\varepsilon R} u^2(x,y) \ .$$

Since $u(x,y)$ is nontrivial, we have

$$(24) \qquad 1 \leq \int_a^b\int_c^d \frac{1}{\theta(x,y)}\, dxdy \int_a^b\int_c^d p_+(x,y)\, dxdy$$

which completes the proof of the theorem.

Theorem (*IV.4.3*) is an extension of a result of D.F. St. Mary [*36*], who has proved it for the case of ordinary differential equations.

CHAPTER V

Problems With Mixed Boundary Conditions

§1. Introduction.

In this chapter we consider the Sturm-Liouville problems of the type (*III.6.1*) subject to mixed boundary conditions. We have not been able to find problems of this type discussed in the literature. Throughout this chapter, for the sake of convenience, R will denote the rectangle

$$\{x_1 \leq x \leq x_2 \quad ; \quad y_1 \leq y \leq y_2\} \quad .$$

Specifically we consider the following boundary value problem

$$L\,u = \frac{\partial^2}{\partial x \partial y}(\theta\,\frac{\partial^2 u}{\partial x \partial y}) + p(x,y)u = f(x,y) \quad . \tag{1}$$

Subject to the homogenous boundary conditions of the form

$$\alpha_i\,u(x_i,y) + (-1)^i u_x(x_i,y) = 0 \tag{2}$$

for $y_1 \leq y \leq y_2$ and $i = 1,2$,

$$\beta_j\,u(x,y_j) + (-1)^j u_y(x,y_j) = 0 \tag{3}$$

for $x_1 \leq x \leq x_2$ and $j = 1,2$, and the compatibility conditions

$$u_x(x_1,y_1)u_x(x_2,y_2)u_y(x_1,y_2)u_y(x_2,y_1) =$$

$$u_x(x_1,y_2)u_x(x_1,y_2)u_y(x_1,y_1)u_y(x_2,y_2) \quad .$$

Let $\theta(x,y) > 0$, $p(x,y) \geq 0$ be continuous functions defined on R . We assume that θ_x , θ_y , θ_{xy} are all continuous functions defined on R . It is clear that there exists a positive constant θ_o such that $\theta(x,y) \geq \theta_o$. Let $\Gamma^{(2)}_{\alpha,\beta}$ denote the class of functions in $\Gamma^{(2)}$ which satisfy the boundary conditions (V.1.2.3). Here α_i and β_j $(i,j=1,2)$ are nonnegative constants such that at least one of the products $\{\alpha_i\beta_j \mid i,j = 1,2\}$ does not vanish.

§2. Symmetry and Positive Definiteness of L in $\Gamma^{(2)}_{\alpha,\beta}$.

In this section we deal with the symmetry and positive definiteness of the operator L in $\Gamma^{(2)}_{\alpha,\beta}$. First we prove the following lemmas.

Lemma V.1.1. The operator L is symmetric on $\Gamma^{(2)}_{\alpha,\beta}$.

Proof: We have to show that if $u,v \in \Gamma^{(2)}_{\alpha,\beta}$ then

$$(1) \qquad (Lu,v) = (u,Lv)$$

where (,) denotes the inner product in $L_2(R)$.

Integrating the identity

$$(2) \qquad v(\theta u_{xy})_{xy} = [v\,\theta\, u_{xy}]_{xy} - [v_x\,\theta\, u_{xy}]_y - [v_y\,\theta\, u_{xy}]_x + \theta\, u_{xy} v_{xy}$$

we have

$$(3) \qquad \int_{x_1}^{x_2}\int_{y_1}^{y_2} v\; Lu\; dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} [v(\theta\, u_{xy})_{xy} + puv]\, dxdy \quad .$$

Substituting ($V.2.2$) into ($V.2.3$) and integrating we find

$$(4) \qquad \int_{x_1}^{x_2}\int_{y_1}^{y_2} v\; Lu\; dxdy = [v\,\theta\, u_{xy}]_{x_1,y_1}^{x_2,y_2} - \int_{x_1}^{x_2} [v_x\;\theta\; u_{xy}(x,y_2) - v_x\;\theta\; u_{xy}(x,y_1)]\, dx - \int_{y_1}^{y_2} [v_y\theta\; u_{xy}(x_2,y) - v_y\;\theta\; u_{xy}(x_1,y)]\, dy + \int_{x_1}^{x_2}\int_{y_1}^{y_2} (\theta\; u_{xy}v_{xy} + p\,u\,v)\, dxdy \quad .$$

Further, by the boundary conditions ($V.1.2$-3) we have

$$(5) \qquad \begin{cases} \alpha_i\beta_j\; u(x_i,y_j) = (-1)^{i+j} u_{xy}(x_i,y_j) \\ u_{xy}\;(x,y_j) = (-1)^{j+1}\;\beta_j\; u_x(x,y_j) \\ u_{xy}\;(x_i,y) = (-1)^{i+1}\;\alpha_i\; u_y(x_i,y) \end{cases} \quad .$$

Substituting these values in (*V.1.4*) we obtain

$$(6)\qquad (Lu,v) = \sum_{i,j=1}^{2} \alpha_i\beta_j\ \theta(x_i,y_i)\ u(x_i,y_j)\ v(x_i,y_j)$$

$$+ \int_{x_1}^{x_2} \sum_{j=1}^{2} \beta_j\ \theta(x,y_j)\ u_x(x,y_j)\ v_x(x,y_j)\,dx$$

$$+ \int_{y_1}^{y_2} \sum_{i=1}^{2} \alpha_i\ \theta(x_i,y)\ u_y(x_i,y)\ v_y(x_i,y)\,dy$$

$$+ \int_{x_1}^{x_2}\int_{y_1}^{y_2} [\theta\ u_{xy}v_{xy} + puv]\,dx\,dy \quad .$$

Since the expression on the right hand side of (*V.2.6*) is symmetric in u and v, the proof is completed.

<u>*Theorem V.2.2.*</u> *Under the assumption on* $\{\alpha_i\beta_j\}$ *the operator is positive definite on* $\Gamma^{(2)}_{\alpha,\beta}$.

<u>Proof:</u> We have to show that there exists a constant γ such that

$$(Lu,u) \geq \gamma^2(u,u)$$

for all $u \in \Gamma^{(2)}_{\alpha,\beta}$. First of all, we have from (*V.1.6*) that

$$(7)\quad (Lu,u) = \sum_{i,j=1}^{2} \alpha_i\beta_j\ \theta(x_i,y_j)u^2(x_i,y_j) + \int_{x_1}^{x_2} \sum_{j=1}^{2} \beta_j\ \theta(x,y_j)u_x^2(x,y_j)\,dx$$

$$+ \int_{y_1}^{y_2} \sum_{i=1}^{2} \alpha_i\ \theta(x_i,y)u_y^2(x_i,y) + \int_{x_1}^{x_2}\int_{y_1}^{y_2} (\theta\ u_{xy}^2 + pu^2)\,dx\,dy \quad .$$

For simplicity let us assume that $\alpha_1\beta_1 \neq 0$. Then from (V.1.7), removing suitable terms, we obtain

$$(8) \qquad (Lu,u) \geq \int_{x_1}^{x_2}\int_{y_1}^{y_2} \theta\, u_{xy}^2 + \alpha_1\beta_1\ \theta(x_1,y_1)u^2(x_1,y_1) + \\ + \int_{x_1}^{x_2} \beta_1\ \theta(x,y_1)u^2(x,y_1)\,dx + \int_{y_1}^{y_2} \alpha_1\ \theta(x_1,y)u_y^2(x_1,y)\,dy.$$

By virtue of our assumptions on $\theta(x,y)$, we may write

$$(9) \qquad (Lu,u) \geq m\{\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2\ dxdy + u^2(x_1,y_1) + \int_{x_1}^{x_2} u_x^2(x,y_1)\,dy + \\ + \int_{y_1}^{y_2} u_y^2(x_1,y)\,dy\}$$

where

$$m = \theta_0\delta \quad \text{and} \quad \delta = \min\{1,\alpha_1,\beta_1,\alpha_1\beta_1\} \quad .$$

Further we have

$$u(x,y) = u(x_1,y) + u(x,y_1) - u(x_1,y_1) + \int_{x_1}^{x}\int_{y_1}^{y} u_{\xi\eta}\ d\eta d\xi \quad .$$

Using the inequality

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$$

which holds for any four real numbers a , b , c and d, we have

$$(10)\quad u^2(x,y) \leq 4\ \{u^2(x_1,y) + u^2(x,y_1) + u^2(x_1,y_1) + [\int_{x_1}^{x}\int_{y_1}^{y} u_{\xi\eta}\ d\eta d\xi]^2\}$$

By the Cauchy-Schwarz inequality applied to the integral in (V.1.10) we can write

$$(11)\qquad u^2(x,y) \leq 4\{u^2(x_1,y) + u^2(x,y_1) + u^2(x_1,y_1) + (x-x_1)(y-y_1) \int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2\ dxdy\}$$

and integrating (V.1.11) over the rectangle R , we obtain

$$(12)\quad \int_{x_1}^{x_2}\int_{y_1}^{y_2} u^2(x,y)dxdy \leq 4\{(x_2-x_1)(y_2-y_1)u^2(x_1,y_1) + (x_2-x_1)\int_{y_1}^{y_2} u^2(x_1,y)dy + (y_2-y_1)\int_{x_1}^{x_2} u^2(x,y_1)dx + \frac{(x_2-x_1)^2(y_2-y_1)^2}{4}\int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2\ dxdy$$

Further we have the relationship

$$u(x,y_1) = u(x_1,y_1) + \int_{x_1}^{x} u_\xi(\xi,y_1)d\xi$$

hence

$$u^2(x,y_1) \le 2u^2(x_1,y_1) + 2(x-x_1) \int_{x_1}^{x_2} u_x^2(x,y_1)\,dx$$

$$(13) \qquad \int_{x_1}^{x_2} u^2(x,y_1)\,dx \le 2(x_2-x_1)u^2(x_1,y_1) + (x_2-x_1)^2 \int_{x_1}^{x_2} u_x^2(x,y_1)\,dx \ .$$

By a similar argument, we can show that

$$(14) \qquad \int_{y_1}^{y_2} u^2(x_1,y)\,dy \le 2(y_2-y_1)u^2(x_1,y_1) + (y_2-y_1)^2 \int_{y_1}^{y_2} u_y^2(x_1,y)\,dy \ .$$

Multiplying (*V.1.13*) by $4(y_2-y_1)$ and (*V.1.14*) by $4(x_2-x_1)$ and adding

$$(15) \qquad 4(y_2-y_1) \int_{x_1}^{x_2} u^2(x,y_1)\,dx + 4(x_2-x_1) \int_{y_1}^{y_2} u^2(x_1,y)\,dy$$

$$\le 16(x_2-x_1)(y_2-y_1)u^2(x_1,y_1) + 4(x_2-x_1)^2(y_2-y_1) \int_{x_1}^{x_2} u_x^2(x,y_1)\,dx$$

$$+ 4(y_2-y_1)^2(x_2-x_1) \int_{y_1}^{y_2} u_y^2(x_1,y)\,dy \ .$$

Combining (*V.1.12*) and (*V.1.15*) we obtain

$$(16) \qquad \int_{x_1}^{x_2}\int_{y_1}^{y_2} u^2(x,y)\,dxdy \le 20(x_2-x_1)(y_2-y_1)u^2(x_1,y_1)$$

$$+ 4(x_2-x_1)^2(y_2-y_1) \int_{x_1}^{x_2} u_x^2(x,y_1)\,dx$$

$$+ 4(y_2-y_1)^2(x_2-x_1) \int_{y_1}^{y_2} u_y^2(x_1,y)\,dy$$

$$+ \int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2 \ dxdy \ .$$

Thus

$$(17) \qquad \int_{x_1}^{x_2}\int_{y_1}^{y_2} u^2(x,y)\,dxdy \leq C\{u^2(x_1,y_1) + \int_{x_1}^{x_2} u_x^2(x,y_1)\,dx + \int_{y_1}^{y_2} u_y^2(x_1,y)\,dy + \int_{x_1}^{x_2}\int_{y_1}^{y_2} u_{xy}^2\,dxdy\}$$

where C is the maximum of the coefficients in (*V.1.16*). Combining (*V.2.9*) and (*V.2.17*) we get the positive definiteness of the operator L.

§3. Eigenvalues of L.

In this section we investigate the eigenvalues of L in the completion of $\Gamma^{(2)}_{\alpha,\beta}$ and show the completeness of eigenfunctions in $L_2(R)$. We note that using the theorem of K. Friedrichs mentioned in [*IV.1*], the space $\Gamma^{(2)}_{\alpha,\beta}$ can be completed with respect to the norm $(Lu,u)^{1/2}$. In this completed space we can show that there is a unique solution to the boundary value probem (*V.1.1,2,3*). Now we have the following theorem:

Theorem V.3.1. *If* $\{u_n\}$ *is a sequence of functions from* $\Gamma^{(2)}_{\alpha,\beta}$ *such that*

$$(1) \qquad (Lu_n,u_n) \leq M$$

where M *is a constant, then we can extract a uniformly convergent subsequence.*

Proof: For the sake of convenience let us assume that $\alpha_1\beta_1 \neq 0$. Then (V.2. 1) yields

$$(2)\quad \begin{cases} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left(\frac{\partial^2 u_n}{\partial x \partial y}\right)^2 dx dy \leq \frac{M}{\theta_o} \quad , \quad \int_{x_1}^{x_2} \left[\frac{\partial u_n}{\partial x}(x,y_1)\right]^2 dx \leq \frac{M}{\theta_o \beta_1} \\ \int_{y_1}^{y} \left[\frac{\partial u_n}{\partial y}(x_1,y)\right]^2 dy \leq \frac{M}{\theta_o \alpha_1} \quad \text{and} \quad u_n^2(x_1,y_1) \leq \frac{M}{\theta_o \alpha_1 \beta_1} \end{cases}$$

. Thus by the Bolzano-Weierstrass Theorem we can choose a subsequence which is again denoted by $\{u_n(x_1,y_1)\}$ such that $\{u_n^2(x_1,y_1)\}$ converges. Further we have

$$u_n(x'',y_1) - u_n(x',y_1) = \int_{x'}^{x''} \frac{\partial u_n}{\partial x}(x,y_1)dx$$

for $a \leq x' < x'' \leq b$, which implies

$$|u_n(x'',y_1) - u_n(x',y_1)| \leq \int_{x'}^{x''} \left|\frac{\partial u_n}{\partial x}(x,y_1)\right| dx \ .$$

An application of the Cauchy-Scwarz inequality to the integral on the right hand side of the above inequality yields

$$(3)\qquad |u_n(x'',y_1) - u_n(x',y_1)| \leq \sqrt{(x''-x')}\sqrt{\frac{M}{\theta_o\beta_1}} \quad .$$

Similarly, we obtain

(4) $$|u_n(x_1,y'') - u_n(x_1,y')| \leq \sqrt{y''-y'}\ \sqrt{\frac{M}{\theta_o \alpha_1}} \quad .$$

The inequalities ($V.3.3$) and ($V.3.4$) show the equicontinuity and uniform boundedness of the families $\{u_n(x,y_1)\}$ and $\{u_n(x_1,y)\}$ respectively. Thus by the Arzela-Ascoli Theorem we can choose a uniformly convergent subsequence (where we keep the same indices for the sake of simplicity). Now we exhibit the uniform convergence of this subsequence $\{u_n(x,y)\}$. We have

(5) $$u_n(x'',y'') - u_n(x',y') = u_n(x'',y_1) - u_n(x',y_1) +$$
$$+ u_n(x_1,y'') - u_n(x_1,y')$$
$$+ \int_{x_1}^{x''}\int_{y_1}^{y''} \frac{\partial^2 u_n}{\partial x \partial y}\, dxdy - \int_{x_1}^{x'}\int_{y_1}^{y'} \frac{\partial^2 u_n}{\partial x \partial y}\, dxdy$$

and therefore

(6) $$|u_n(x'',y'') - u_n(x',y')| \leq |u_n(x'',y_1) - u_n(x',y_1)|$$
$$+ |u_n(x_1,y'') - u_n(x_1,y')|$$
$$\int_{x'}^{x''}\int_{y'}^{y'} |\frac{\partial^2 u_n}{\partial x \partial y}|\, dxdy + \int_{x'}^{x''}\int_{y'}^{y''} |\frac{\partial^2 u_n}{\partial x \partial y}|\, dxdy + \int_{x_1}^{x'}\int_{y'}^{y''} |\frac{\partial^2 u_n}{\partial x \partial y}|\, dxdy \quad .$$

Applying Cauchy-Schwarz inequality to the integrals in ($V.1.23$) we find

$$(7) \qquad |u_n(x'',y'') - u_n(x',y')| \le |u_n(x'',y_1) - u_n(x',y_1)|$$

$$+ |u_n(x_1,y'') - u_n(x_1,y')|$$

$$k \{ \sqrt{(x''-x')(y''-y')} + \sqrt{(x_2-x_1)(y''-y')} + \sqrt{(y_2-y_1)(x''-x')} \} .$$

where k is a constant depending on M, θ_o, α_1 and β_1. Since $\{u_n(x,y_1)\}$ and $\{u_n(x_1,y)\}$ are equicontinuous and uniformly bounded, $\{u_n(x,y)\}$ is equicontinuous and uniformly bounded by virtue of (*V.3.7*). Then Arzela-Ascoli Theorem yields the existence of a uniformly convergent subsequence.

Combining Theorem V.1.3 and Theorem III.1.5, we obtain the following theorem:

Theorem V.3.2. *The operator* L *subject to the boundary conditions (V.1.2-3) has a countable set of eigenvalues tending to infinity and the eigenfunctions form a complete set with respect to both the* $L_2(R)$ *norm and the norm*

$$|||u|||^2 = (Lu,u) .$$

§4. Comparison Theorems for Eigenvalues.

We state and prove the following comparison theorem for eigenvalues of the operators L and $L^{\#}$. A similar theorem has been proved, for ordinary differential equations by K. Kreith [*18*].

Theorem V.1.5. *If* $\theta^*(x,y)$ *and* $p^*(x,y)$ *are functions such that* $\theta \leq \theta^*$ *and* $p \leq p^*$ *and* γ_i*'s and* δ_i*'s are constants satisfying*

$$\alpha_i < \gamma_i \qquad \beta_i < \delta_i$$

then the eigenvalues of L^* *subject to the boundary conditions*

$$(1) \quad \gamma_i\, u(x_i,y) + (-1)^i\, u_x(x_i,y) = 0 \;; \quad \delta_j\, u(x,y_j) + (-1)^j\, u_y(x,y_j) = 0$$

where $L^{\#}$ *is defined by*

$$(2) \qquad L^{\#}u = \frac{\partial^2}{\partial x \partial y}\left(\theta^*(x,y)\, \frac{\partial^2 u}{\partial x \partial y}\right) + p^*(x,y)\, u$$

majorize those of L *subject to (V.1.2-3).*

Proof: Indeed, from our hypothesis it can easily be seen that

$$(Lu,u) \leq (L^{\#}u,u)$$

for all admissible functions and the result follows.

Corollary: Eigenvalues of the problem (*V.1.1*) - (*V.1.2*) are majorized by those of the problem

$$\begin{cases} \dfrac{\partial^2}{\partial x \partial y} \left(\theta(x,y) \dfrac{\partial^2 y}{\partial x \partial y}\right) + pu = \lambda\, u \\ u(x_i,y) = u(x,y_i) = 0 \qquad\qquad (i=1,2) \end{cases}$$

for $y_1 \leq y \leq y_2$ and $x_1 \leq x \leq x_2$ respectively.

Proof: The assertion follows by taking $\theta^* = \theta$, $p^* = p$ and $\gamma_i = \delta_i = \infty$ in Theorem V.1.5.

CHAPTER VI

Green's Functions of Polyvibrating Operators

§1. Introdution.

It is well known that finding Green's functions of differential operators is equivalent to finding the inverse operators to these differential operators. In practice it is easy to prove the existence of inverse operators using functional analysis as a tool. Also once we find the Green's function it is easy to give an explicit representation of the solution. But the actual construction of the Green's functions for many partial differential operators seems to be quite difficult. In this chapter, we give explicit representations of the Green's functions for certain of the polyvibrating operators of D. Mangeron [2]. We quote below two theorems from functional analysis on which our representation is based.

Theorem VI.1.1 [25]. *If* A *is a completely continuous, symmetric, positive, transformation between two Hilbert spaces, then*

(i) All the eigenvalues of A *are real and different from zero. Each is of finite multiplicity and they are either finite or denumerably infinite in numbers tending to zero.*

(ii) Every element of the form Au *can be developed in terms of the orthonormal system* $\{\phi_i\}$ *of corresponding eigenfunctions*

(1) $$Au = \sum_{i=1}^{\infty} \mu_i (Au, \phi_i)\phi_i \quad .$$

Theorem VI.1.2. *If A is a symmetric, positive bounded below operator with a discrete spectrum then its inverse operator $G = A^{-1}$ is completely continuous and is symmetric.*

Combining Theorems (*VI.1.1*) and VI.1.2, we infer that the inverse operator $G = A^{-1}$ of a positive bounded below operator is given by

(2) $$G\ u = \sum_{i=1}^{\infty} \frac{(u, \phi_i)}{\lambda_i}\ \phi_i$$

where ϕ_i's are the eigenfunctions of A and λ_i are the corresponding eigenvalues. Thus

$$\begin{aligned} G\ u &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\ \phi_k(x,y) \int_a^b \int_c^d u(\xi,\eta)\ \phi_k(\xi,\eta)\,d\xi d\eta \\ &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_a^b \int_c^d u(\xi,\eta)\ \phi_k(x,y)\ \phi_k(\xi,\eta)\,d\xi d\eta \\ &= \sum_{k=1}^{\infty} \int_a^b \int_c^d u(\xi,\eta)\ \frac{\phi_k(x,y)\phi_k(\xi,\eta)}{\lambda_k}\ d\xi d\eta \\ &= \int_a^b \int_c^d u(\xi,\eta)\ \Big(\sum_{k=1}^{\infty} \frac{\phi_k(\xi,\eta)\phi_k(x,y)}{\lambda_k}\Big) d\xi d\eta \quad . \end{aligned}$$

The interchange of summation signs and integration has to be justified every time. Thus the Green's function $G(x,y;\xi,\eta)$ is given by

$$(3) \qquad G(x,y;\xi,\eta) = \sum_{k=1}^{\infty} \frac{\phi_k(x,y)\ \phi_k(\xi,\eta)}{\lambda_k} \ .$$

§2. Green's Functions.

Though in many simple cases the Green's functions could be found by elementary techniques, the above approach gives us a unified treatment of Green's functions. In §V.2 it has been proved that under suitable conditions on $\theta(x,y)$ and $p(x,y)$, the partial differential operator L,

$$(1) \qquad L\,u = \frac{\partial^2}{\partial x \partial y}\left(\theta \frac{\partial^2 u}{x\ y}\right) + p(x,y)\ u$$

subject to the boundary conditions

$$\begin{cases} \alpha_i\ u(x_i,y) + (-1)^i\ u_x(x_i,y) = 0 & c \leq y \leq d \\ \beta_i\ u(x,y_i) + (-1)^i\ u_y(x,y_i) = 0 & a \leq y \leq d \end{cases}$$

and the compatibility conditions (C), is positive definite provided at least one of the terms $\{\alpha_i \beta_j\}$ $(i = 1,2\ ;\ j = 1,2)$ is different from zero, α_i's and β_j's being nonnegative. It has also been shown in Theorem (V.2.4) that the operator L, subject to the boundary conditions

$(V.1.2)$ has a discrete spectrum $\{\lambda_i\}$ tending to infinity. Thus we can use the theorems VI.2.1 and VI.2.2 to give an explicit representation for the Green's functions. We consider several cases as follows:

Case (i). $\theta(x,y) \equiv 1$, $p(x,y) \equiv 0$, $\alpha_i = \beta_j = \infty$ $(i , j = 1,2)$.

It is clear that in this case the operator L reduces to

$$L\, u = \frac{\partial^4 u}{\partial x^2 \partial y^2} \tag{3}$$

and the boundary conditions $(VI.2.3)$ can be identified as

$$u(x_i,y) = 0 \quad ; \quad u(x,y_i) = 0 \qquad (i = 1,2)$$

for $y_1 \leq y \leq y_2$ and $x_1 \leq x \leq x_2$ respectively.

The eigenfunctions are given by

$$\{\sin \frac{k\pi(x-x_1)}{(x_2-x_1)} \sin \frac{\ell\pi(y-y_1)}{(y_2-y_1)}\} \qquad k,\ell = 1,2,\cdots$$

and the eigenvalues are $k^2\ell^2\pi^4 \,/\, (x_2-x_1)^2(y_2-y_1)^2$. Hence by Theorem VI.2.1 the Green's function is given by

$$G(x,y;\xi,\eta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \{\frac{(x_2-x_1)^2(y_2-y_1)^2}{k^2\ell^2} \sin \frac{k\pi(\xi - x_1)}{(x_2-x_1)}$$
$$\sin \frac{k\pi(x-x_1)}{(x_2-x_1)} \sin \frac{\ell\pi(y-y_1)}{y_2-y_1} \sin \frac{\ell\pi(\eta-y_1)}{(y_2-y_1)}\}$$

D. Mangeron [2] originally obtained the representation of $G(x,y;\xi,\eta)$ as given by

$$(x_2-x_1)(y_2-y_1)G(x,y;\xi,\eta) = \begin{cases} (x_2-x)(y_2-y)(\xi-x_1)(\eta-y_1) & \xi\le x;\eta\le y \\ (x_2-x)(y-y_1)(\xi-x_1)(y_2-\eta) & \xi\le x;\eta\ge y \\ (x-x_1)(y_2-y)(x_2-\xi)(\eta-y_1) & \xi\ge x;\eta\le y \\ (x-x_1)(y-y_1)(x_2-\xi)(y_2-\eta) & \xi\ge x;\eta\ge y \end{cases}$$

Case (ii). $\theta(x,y) \equiv 1$, $p(x,y) = 0$, $\alpha_2 = 0$, $\beta_2 = 0$.

The operator L reduces to (VI.2.3), but the boundary conditions reduce to

$$u(x_1,y) = u(x,y_1) = 0 \quad ; \quad u_{xy}(x_2,y) = u_{xy}(x,y_2) = 0 \quad .$$

The eigenfunctions are given by

$$\{\sin \left(\frac{2n+1}{2}\right) \frac{\pi(x-x_1)}{(x_2-x_1)} \sin \frac{(2m+1)\pi}{2} \frac{(y-y_1)}{y_2-y_1}\}_{m,n=1,2,\cdots}$$

and the corresponding eigenvalues being

$$\frac{(2m+1)^2(2n+1)^2 \pi^4}{16(x_2-x_1)^2(y_2-y_1)^2} \quad .$$

Thus once again the Green's function $G(x,y;\xi,\eta)$ is given by

$$G(x,y;\xi,\eta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \{ \frac{16(x_2-x_1)^2(y_2-y_1)^2}{(2k+1)^2(2\ell+1)\ \pi^4} \sin \frac{(2k+1)\pi(x-x_1)}{2(x_2-x_1)}$$

$$\sin \frac{(2k+1)\pi(\xi-x_1)}{2(x_2-x_1)} \sin \frac{(2\ell+1)\pi}{2(y_2-y_1)} (\eta-y_1)$$

$$\sin \frac{(2\ell+1)}{2(y_1-y_1)} (y-y_1) \} \ .$$

An alternate representation of this function can be obtained and has the form

$$G(x,y;\xi,\eta) = \begin{cases} (x_1-\xi)(y_1-\eta) & \xi \le x \ ; \ \eta \le y \\ (x_1-\xi)(y_1-y) & \xi \le x \ ; \ \eta \ge y \\ (x_1-x)(y_1-\eta) & \xi \ge x \ ; \ \eta \le y \\ (x_1-x)(y_1-y) & \xi \ge x \ ; \ \eta \ge y \ . \end{cases}$$

Thus the above examples show that, finding the explicit representations of Green's functions is equivalent to finding th e eigenfunctions of polyvibrating operators subject to suitable boundary conditions.

Following M.N. Oguztoreli [29] we consider the following problem:

Thus once again the Green's function $G(x,y;\xi,\eta)$ is given by

$$G(x,y;\xi,\eta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \{ \frac{16(x_2-x_1)^2(y_2-y_1)^2}{(2k+1)^2(2\ell+1)\ \pi^4} \sin \frac{(2k+1)\pi(x-x_1)}{2(x_2-x_1)}$$

$$\sin \frac{(2k+1)\pi(\xi-x_1)}{2(x_2-x_1)} \sin \frac{(2\ell+1)\pi}{2(y_2-y_1)} (\eta-y_1)$$

$$\sin \frac{(2\ell+1)}{2(y_1-y_1)} (y-y_1) \} \ .$$

An alternate representation of this function can be obtained and has the form

$$G(x,y;\xi,\eta) = \begin{cases} (x_1-\xi)(y_1-\eta) & \xi \leq x \ ; \ \eta \leq y \\ (x_1-\xi)(y_1-y) & \xi \leq x \ ; \ \eta \geq y \\ (x_1-x)(y_1-\eta) & \xi \geq x \ ; \ \eta \leq y \\ (x_1-x)(y_1-y) & \xi \geq x \ ; \ \eta \geq y \ . \end{cases}$$

Thus the above examples show that, finding the explicit representations of Green's functions is equivalent to finding th e eigenfunctions of polyvibrating operators subject to suitable boundary conditions.

Following M.N. Oguztoreli [29] we consider the following problem:

(4) $$\frac{\partial^2}{\partial x \partial y} (p(x)\ q(y) \frac{\partial^2 u}{\partial x \partial y}) = \lambda\ r(x) s(y)\ u$$

for $x_1 < x < x_2$; $y_1 < y < y_2$ subject to the boundary conditions

(5) $$u(x_1,y) = u(x_2,y) = 0$$

for $y_1 \leq y \leq y_2$ and

(6) $$u(x,y_1) = u(x,y_2) = 0$$

for $x_1 \leq x \leq x_2$, where λ is a parameter. We assume that $p(x)$ and $q(y)$ are continuously differentiable positive functions defined on $a \leq x \leq b$ and $c \leq y \leq d$ respectively. We wish to determine values of the parameter λ in such a way that the boundary value problem (V.2.4) and (V.2.5-6) will admit a solution in the rectangle R . This can easily be done as follows by a separation of variables. For, let

$$u(x,y) = X(x)\ Y(y)\ .$$

Then the problem is equivalent to the following two Sturm-Liouville problems of ordinary differential equations.

(7) $$\begin{cases} \frac{d}{dx} (p(x) \frac{dX}{dx}) = \mu\ r(x) X \\ X(x_1) = X(x_2) = 0 \end{cases}$$

and

$$(8) \qquad \begin{cases} \frac{d}{dy}\left(q(y)\,\frac{dY}{dy}\right) = \nu\, s(y)Y \\ Y(y_1) = Y(y_2) = 0 \end{cases}$$

where

$$\lambda = \mu\nu \quad .$$

Let μ_1 , μ_2 , μ_3 , $\cdots$ denote the eigenvalues of the problem (*VI.2.7*) and X_1 , X_2 , X_3 , $\cdots$ be the corresponding eigenfunctions. Similarly let ν_1 , ν_2 , ν_3 , $\cdots$ denote the eigenvalues of (*VI.2.8*) and Y_1 , Y_2 , Y_3 , $\cdots$ the correspesponding eigenfunctions. Existence of these eigenfunctions can easily be shown using methods of differential equations. The conditions on $p(x)$ and $q(y)$ assure that these eigenfunctions form a complete system of orthogonal functions with weights $r(x)$ and $s(y)$, respectively. Hence

$$u_{m,n} = X_m(x)\; Y_n(y)$$

form a complete system of orthogonal functions with weight $r(x)s(y)$.

CHAPTER VII

Generalizations to Higher Order Operators

§1. Introduction.

In this chapter we generalize some of the results obtained in the previous chapters to the partial differential operators of higher order. In our study we shall restrict ourselves to the class of functions $\Gamma^{(n)}$ defined in Chapter II.

In section 2, starting with certain higher order problems of the calculus of variations, we obtain the Hilbert spaces $V_o^{(n)}$ which are subspaces of the Hilbert space $V_o^{(1)}$ introduced in Chapter III. In the same section we prove the positive definiteness of certain higher order polyvibrating operators. In section 3 we use these results to demonstrate the existence and uniqueness of the solutions to certain integro-partial differential equations of polyvibrating type.

In this chapter $u_{x^n y^n}$ will denote the partial derivative $\frac{\partial^{2n} u}{\partial x^n \partial y^n}$, which is the *nth order Picone derivative of the function* $u \equiv u(x,y)$.

§2. Minima of Integrals Involving Higher Order Picone Derivatives.

In this section we investigate the properties of a function $\overset{o}{u}(x,y)$ which minimizes a functional of the form

$$(1) \qquad I_1[u] = \int_a^b \int_c^d f(x,y,u,u_{xy},\cdots,u_{x^n y^n})\,dxdy$$

over the class of functions in $\Gamma^{(n)}$ such that

$$(2) \qquad \frac{\partial^{k+\ell}u}{\partial x^k \partial y^\ell}(a,y) = A_{k\ell}(y) \quad ; \quad \frac{\partial^{k+\ell}u}{\partial x^k \partial y^\ell}(b,y) = B_{k,\ell}(y)$$

for

$$c \le y \le d \;,\; k = 0,1,2,\cdots,n-1 \quad , \quad \ell = 0,1,2,\cdots,n-1$$

and

$$(3) \qquad \frac{\partial^{k+\ell}u}{\partial x^k \partial y^\ell}(x,c) = C_{k,\ell}(x) \quad ; \quad \frac{\partial^{k+\ell}u}{\partial x^k \partial y^\ell}(x,d) = D_{k,\ell}(x)$$

for

$$a \le x \le b \;,\; k = 0,1,2,\cdots,n-1 \;,\; \ell = 0,1,2,\cdots,n-1$$

where $A_{k\ell}$, $B_{k\ell}$, $C_{k\ell}$ and $D_{k\ell}$ are certain known functions, which are sufficiently smooth and compatible with each other. We assume that $f(x,y,u_0,u_1,\cdots,u_n)$ is a function ,twice continuously differentiable with respect to all its arguments.

It can easily be seen, using techniques of Chapter II, that the first variation of the functional $I_1[u]$ is given by

(4) $$I_1'[u,\delta] = \int_a^b\int_c^d \sum_{i=0}^{n} f_{u_i}(x,y,u,u_{xy},\cdots,u_{x^n y^n}) \frac{\partial^{2i}\delta}{\partial x^i \partial y^i}\, dxdy$$

where $\delta(x,y)$ is a function belonging to $\Gamma^{(n)}$ such that

(5) $$\frac{\partial^{k+\ell}\delta}{\partial x^k \partial y^\ell}(a,y) = \frac{\partial^{k+\ell}\delta}{\partial x^k \partial y^\ell}(b,y) = 0$$

for $c \leq y \leq d$ and $k = 0,1,2,\cdots,n-1$, $\ell = 0,1,2,\cdots,n-1$, and

(6) $$\frac{\partial^{k+\ell}\delta}{\partial x^k \partial y^\ell}(x,c) = \frac{\partial^{k+\ell}\delta}{\partial x^k \partial y^\ell}(x,d) = 0$$

for $a \leq x \leq b$ and $k = 0,1,2,\cdots,n-1$, $\ell = 0,1,2,\cdots,n-1$.

Thus if $\overset{\circ}{u}(x,y) \in \Gamma^{(n)}$ satisfies the Equations (*VII.2.2-3*) and minimizes $I_1[u]$, we have

(7) $$I_1'[\overset{\circ}{u},\delta] = 0$$

for all $\delta(x,y) \in \Gamma^{(n)}$ satisfying Equations (*VII.2.5-6*). An integration by parts, yields

(8) $$\int_a^b\int_c^d \{\sum_{k=0}^{n} \int_a^x\int_c^y \frac{(x-\xi)^{n-k-1}(y-\eta)^{n-k-1}}{[(n-k-1)!]^2} \overset{\circ}{f}_{u_k}(\xi,\eta)d\xi d\eta\} \frac{\partial^{2n}\delta}{\partial x^n \partial y^n}\, dxdy = 0$$

for all $\delta(x,y) \in \Gamma^{(n)}$ and satisfying Equations (*VII.2.5-6*) where

$$\overset{\circ}{f}_{u_k}(x,y) = \frac{\partial f}{\partial u_k}(x,y,\overset{\circ}{u},\overset{\circ}{u}_{xy},\cdots,\overset{\circ}{u}_{x^n y^n}) \quad .$$

By Mason's lemma II.2.1, we have

$$\sum_{k=0}^{n} \int_a^x\int_c^y \frac{(x-\xi)^{n-k-1}(y-\eta)^{n-k-1}}{[(n-k-1)!]^2} \overset{o}{f}_{u_k}(\xi,\eta)\,d\xi d\eta = \sum_{k=0}^{n-1} x^k Y_k + y^k X_k$$

where $Y_k(y)$ and $X_k(x)$ are functions depending only on $\overset{o}{f}$ and its partial derivatives $\overset{o}{f}_{u_i}$. Since we have assumed that $f(x,y,u_o,u_1,\cdots,u_n)$ is sufficiently differentiable, differentiating either side of the above relation, we see that $\overset{o}{u}(x,y)$ is a solution of the partial differential equation

$$(9) \qquad \sum_{i=0}^{n} \frac{\partial^{2i}}{\partial x^i \partial y^i} [f_{u_i}(x,y,\overset{o}{u},\overset{o}{u}_{xy},\cdots,\overset{o}{u}_{x^n y^n})] = 0 \quad .$$

Equation (*VII.2. 9*) is a $4\ n^{th}$ order nonlinear polyvibrating partial differential equation. If we assume

$$f(x,y,u_o,u_1,\cdots,u_n) = \sum_{i=0}^{n} \theta_i\, u_i^2$$

where $\theta_i(x,y)$ are functions belonging to $\Gamma^{(i)}$, then we obtain the following linear polyvibrating equation

$$(10) \qquad L_1 u = \sum_{i=0}^{n} \frac{\partial^{2i}}{\partial x^i \partial y^i} (\theta_i(x,y) \frac{\partial^{2i} u}{\partial x^i \partial y^i}) = 0 \quad .$$

We will discuss this equation in the following section.

§3. Self Adjoint $4n^{th}$ Order Polyvibrating Equations.

Motivated by the results of Section 2, we consider the following boundary value problem

$$(1) \qquad L_1 u = \sum_{i=0}^{n} \frac{\partial^{2i}}{\partial x^i \partial y^i} \left(\theta_i(x,y) \frac{\partial^{2i} u}{\partial x^i \partial y^i}\right) = f(x,y)$$

$$(2) \qquad \frac{\partial^{k+\ell} u}{\partial x^k \partial y^\ell}(a,y) = \frac{\partial^{k+\ell} u}{\partial x^k \partial y^\ell}(b,y) = 0$$

for $c \leq y \leq d$ and $k = 0,1,2,\cdots,n-1$, $\ell = 0,1,2,\cdots,n-1$

$$(3) \qquad \frac{\partial^{k+\ell} u}{\partial x^k \partial y^\ell}(x,c) = \frac{\partial^{k+\ell} u}{\partial x^k \partial y^\ell}(x,d) = 0$$

for $a \leq x \leq b$ and $k = 0,1,2,\cdots,n-1$, $\ell = 0,1,2,\cdots,n-1$. We also suppose that $\theta_i(x,y) \in \Gamma^{(i)}$ are all positive functions defined over R for $0 \leq i \leq n$ such that there exists a constant θ_o for which

$$(4) \qquad \theta_n(x,y) \geq \theta_o > 0$$

and $f(x,y) \in L_2(R)$. First we prove the following.

Theorem VII.3.1. *The partial differential operator L_1 defined by (VII.3.1) is symmetric and positive definite with respect to the inner product of $L_2(R)$ provided Equations (VII.3.2.-3) are satisfied.*

Proof: Let us note that for any two functions u, v vanishing on the boundary of the rectangle R , we have the following formula of integration by parts

(5) $$\int_a^b \int_c^d u \, v_{xy} \, dxdy = \int_a^b \int_c^d v \, u_{xy} \, dxdy \quad .$$

Then, clearly, if u and v satisfy the boundary conditions (*VII.3.2.-3*), we have

(6) $$\int_a^b \int_c^d u \, L_1 v \, dxdy = \int_a^b \int_c^d \sum_{i=0}^{n} \theta_i(x,y) \frac{\partial^{2i} u}{\partial x^i \partial y^i} \frac{\partial^{2i} v}{\partial x^i \partial y^i}$$

which proves the symmetry of L_1 . Hence

(7) $$(L_1 u, u) = \int_a^b \int_c^d \sum_{i=0}^{n} \theta_i \, u^2_{x^i y^i} \, dxdy \quad .$$

By virtue of our assumptions on $\theta_i(x,y)$ we can write

(8) $$(L_1 u, u) \geq \int_a^b \int_c^d \theta_n(x,y) \, u^2_{x^n y^n} \, dxdy$$

$$\geq \theta_0 \int_a^b \int_c^d u^2_{x^n y^n} \, dxdy \quad .$$

Further, since

$$\frac{\partial^{2i} u}{\partial x^i \partial y^i} (a,y) = \frac{\partial^{2i} u}{\partial x^i \partial y^i} (x,c) = 0$$

we have

$$\int_a^b\int_c^d u^2_{x^i y^i}\, dxdy \leq \frac{(b-a)^2(d-c)^2}{4} \int_a^b\int_c^d u^2_{x^{i+1}y^{i+1}}\, dxdy \tag{9}$$

$i = 0,1,2,\cdots,n-1$, as a consequence of the Theorem IV.1-1. Using (VII.3.9) n times,

$$\int_a^b\int_c^d u^2\, dxdy \leq \frac{(b-a)^{2n}(d-c)^{2n}}{2^{2n}} \int_a^b\int_c^d u^2_{x^n y^n}\, dxdy \quad , \tag{10}$$

which proves the positive definiteness of L_1 .

Now we introduce the space $V_0^{(n)}$ of functions satisfying the following properties: $u \in V_0^{(n)}$ if and only if

(i) u , u_{xy} , $u_{x^2y^2}$, $\cdots$, $u_{x^{n-1},y^{n-1}}$ *are all defined on* R *and are absolutely continuous in the sense of Vitali in* R ;

(ii) $u_{x^n y^n}$ *belongs to* $L_2(R)$;

(iii) $u(x,y)$ *satisfies the boundary conditions* (VII.3.2) *and* (VII.3.3).

Then we have the following theorem:

<u>*Theorem VII.3.2.*</u> $V_0^{(n)}$ *forms a Hilbert space with respect to the inner product defined by*

(11) $$((u,v))_n = \int_a^b \int_c^d u_{x^n y^n} \, v_{x^n y^n} \; dxdy$$

and the corresponding norm defined by

(12) $$|||u|||_n^2 = \int_a^b \int_c^d u^2_{x^n y^n} \; dxdy \quad .$$

<u>Proof</u>: First we show that $|||u|||_n = 0$ implies $u = 0$ a.e. in R . This is easily shown since an extension of inequality (*III.2.12*) as in Theorem VII.3.1 yields

(13) $$\int_a^b \int_c^d u^2 \; dxdy \leq \frac{(b-a)^{2n}(d-c)^{2n}}{2^{2n}} \; |||u|||_n^2 = 0 \quad .$$

Since $u \in L_2(R)$, we have $u = 0$ a.e. in R . Other properties can easily be verified. Thus we have only to show the completeness $V_o^{(n)}$. To do this let $\{u_n\}$ be a Cauchy sequence with respect to the norm $|||\cdot|||_n$. This clearly implies that $\{\frac{\partial^{2n} u}{\partial x^n \partial y^n}\}$ is a Cauchy sequence in $L_2(R)$. By virtue of completeness of $L_2(R)$ there exists a function $g \in L_2(R)$ such that

(14) $$\lim_{n \to \infty} \int_a^b \int_c^d \left|\frac{\partial^{2n} u}{\partial x^n \partial y^n} - g\right|^2 \; dxdy = 0 \quad .$$

But since norm convergence in $L_2(R)$ implies weak convergence, we have

$$(15) \qquad \lim_{m\to\infty} \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}} = \lim_{m\to\infty} \int_a^x\int_c^y \frac{\partial^{2n} u_m}{\partial x^m \partial y^m}\, dxdy = \int_a^x\int_c^y g(x,y)\ dxdy$$

which shows the absolute continuity of the limit

$$\lim_{m\to\infty} \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}} .$$

Further since $\{\frac{\partial^{2n} u_m}{\partial x^n \partial y^n}\}$ is a Cauchy sequence in $L_2(R)$, we can show that

(i) $\{\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\}$ converges uniformly,

(ii) $\{\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\}$ is a Cauchy sequence in $L_2(R)$.

The proof of (i) follows from the relationship

$$(16) \qquad \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\ (x,y) = \int_a^x\int_c^y \frac{\partial^{2n} u_m}{\partial \xi^n \partial \eta^n}\ d\xi d\eta$$

which implies the inequality

$$(17) \qquad \sup_{(x,y)\in R} \left|\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\right| \le \int_a^x\int_c^y \left|\frac{\partial^{2n} u_m}{\partial \xi^n \partial \eta^n}\right|\ d\xi d\eta \quad .$$

To prove (ii) let us note that

$$\sup_{(x,y)\in R} \left|\frac{\partial^{2n-2} u_m}{\partial x^{m-1}\partial y^{m-1}}\right| \leq \frac{(b-a)^2(d-c)^2}{4} \quad |||u_m|||_n^2$$

by virtue of Cauchy-Schwarz inequality. Thus $\{\frac{\partial^{2n-2} u_m}{\partial x^{m-1}\partial y^{m-1}}\}$ is a uniformly bounded family since the sequence $\{\frac{\partial^{2n} u_n}{\partial x^n \partial y^n}\}$ is uniformly bounded because it is a Cauchy sequence in $L_2(R)$. As in §1.7, we can easily show the equicontinuity of the family $\{\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\}$. Thus by the theorem of Arzela-Ascoli we can extract a uniformly convergent subsequence from $\{\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\}$. We will denote this subsequence also by $\{\frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\}$ for the sake of convenience. Using this uniformly converging subsequence we see that

$$(18) \qquad \lim_{m\to\infty} \int_a^x\int_c^y \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\, dxdy = \int_a^x\int_c^y \left[\lim_{n\to\infty} \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}}\right] dxdy$$

which yields the absolute continuity of

$$(*) \qquad \lim_{m\to\infty} \frac{\partial^{2n-2} u_m}{\partial x^{n-1}\partial y^{n-1}} \quad .$$

By similar reasoning we can demonstrate that the sequences

$$(**) \qquad \{u_m\} , \{\frac{\partial^2 u_m}{\partial x \partial y}\} , \cdots , \{\frac{\partial^{2n-4} u_m}{\partial x^{n-1} \partial y^{n-1}}\}$$

converge uniformly to functions u , u_{xy} , $u_{x^2y^2}$, $\cdots$, $u_{x^{n-1}y^{n-1}}$, all of which are absolutely continuous in the sense of Vitali in R . Further it can easily be seen that $u_{x^n y^n}$ belongs to $L_2(R)$. The boundary conditions are easily seen to be satisfied by u by virtue of the uniform convergence.

Incidentally, we have shown that $V_o^{(n)}$ is a subspace of the space $V_o^{(1)}$ of chapter III for all n .

__Theorem VII.3.3.__ If $\{u_m\}$ is a sequence such that $u_m \in V_o^{(n)}$ and

$$(19) \qquad (L_1 u_m, u_m) \leq M$$

where M is a constant, then $\{u_m\}$ contains a uniformly convergent subsequence.

__Proof:__ By virtue of Equation (*VII.3.19*) we have

$$\int_a^b\int_c^d [\frac{\partial^{2n} u_m}{\partial x^n \partial y^n}]^2 dxdy \leq \frac{M}{\theta_o} .$$

Thus combining the above inequality with the inequality (*VII.3.9*) we have

$$(20) \qquad \int_a^b\int_c^d [\frac{\partial^2 u_m}{\partial x\partial y}]^2 dxdy \leq \frac{(b-a)^{2n-2}(d-c)^{2n-2}}{2^{2n-2}\ \theta_o} M \quad .$$

But then

$$(21) \qquad \sup_{(x,y)\varepsilon R} |u_m(x,y)| \leq (b-a)(d-c) \sqrt{\int_a^b\int_c^d [\frac{\partial^2 u_m}{\partial x\partial y}]^2\ dxdy}$$

$$\leq (b-a)(d-c) \quad \{\frac{(b-a)^{2n-2}(d-c)^{2n-2}}{2^{2n-2}\ \theta_o} M\}^{\frac{1}{2}} \quad .$$

Hence $\{u_n\}$ is uniformly bounded. The equicontinuity of this family can be shown as was done in the proof of the Lemma II.7. Thus we can extract a uniformly convergent subsequence by Arzela-Ascoli Theorem.

Consider now $L_1 u = \lambda u$, where λ is a parameter. Then combining the above theorem with the Theorem III.1.5 , we immediately see the existence of a countable sequence of eigenvalues λ_n tending to infinity, corresponding to the infinite sequence of eigenfunctions $\{u_n\}$ satisfying $L_1 u_n = \lambda_n u_n$ and the boundary conditions (*VII.3.2-3*).

As another generalization of the operator L of chapter V, we give the following result about the positive definiteness of the polyvibrating operator L_2 defined by the equation

$$(22) \qquad L_2 u = \frac{\partial^2}{\partial x\partial y}\ [\theta\ \frac{\partial^2}{\partial x\partial y}\ \theta_1 (\frac{\partial^2}{\partial x\partial y}\ \theta\ \frac{\partial^2 u}{\partial x\partial y})] \quad .$$

Theorem VII.3.4. *The operator* $L_2 u$ *defined by (VII.3.22) is positive definite and symmetric if*

(i) $\theta(x,y) > \theta_o$

(ii) $\theta_1(x,y) > \theta_1 > 0$ *and* $\theta(x,y)$, $\theta_1(x,y)$ *are sufficiently smooth in* R ,

over the space of functions $u(x,y) \in \Gamma^{(4)}$ *satisying the boundary conditions*

(23) $$u\Big|_{\partial R} = 0 \quad ; \quad u_{xy}\Big|_{\partial R} = 0 \quad .$$

Proof: First of all using integration by parts we find

(24) $$(L_2 u,v) = \int_a^b\int_c^d \theta_1(x,y) \left[\frac{\partial^2}{\partial x\partial y} \left(\theta \frac{\partial^2 u}{\partial x\partial y}\right)\right] \left[\frac{\partial^2}{\partial x\partial y} \left(\theta \frac{\partial^2 v}{\partial x\partial y}\right)\right] dxdy$$

if u and v satisfy the boundary conditions (*VII.3.23*) which shows the symmetry of L_2 . Thus we have,

$$(L_2 u,u) = \int_a^b\int_c^d \theta_1(x,y) \left[\frac{\partial^2}{\partial x\partial y}\left(\theta \frac{\partial^2 u}{\partial x\partial y}\right)\right]^2 dxdy \quad .$$

Also

(25) $$\theta\, u_{xy}(x,y) = \int_a^x\int_c^y \frac{\partial^2}{\partial\xi\partial\eta} \left(\theta \frac{\partial^2 u}{\partial\xi\partial\eta}\right) d\xi d\eta \quad .$$

Since $u_{xy}(a,y) = u_{xy}(x,c) = 0$ and by the Cauchy-Schwarz inequality we have

$$(26)\qquad \int_a^b\int_c^d \theta^2\, u_{xy}^2\, dxdy \le \frac{(b-a)^2(d-c)^2}{4}\int_a^b\int_c^d \Big[\frac{\partial^2}{\partial x\partial y}\Big(\theta\,\frac{\partial^2 u}{\partial x\partial y}\Big)\Big]^2\, dxdy\ .$$

Thus

$$(27)\qquad \int_a^b\int_c^d \theta^2\, u_{xy}^2\, dxdy \le \frac{(b-a)^2(d-c)^2}{4\theta_1}\int_a^b\int_c^d \theta_1(x,y)\Big[\frac{\partial^2}{\partial x\partial y}\Big(\theta\,\frac{\partial^2 u}{\partial x\partial y}\Big)\Big]^2\, dxdy$$

since $\theta_1(x,y)/\theta_1 > 1$. Combining this with (*V.1.1*) we obtain the inequality

$$(28)\qquad \int_a^b\int_c^d u^2\, dxdy \le \frac{(b-a)^2(d-c)^2}{4\theta_o^{\,2}}\int_a^b\int_c^d \theta^2\, u_{xy}^2\, dxdy \le \frac{(b-a)^4(d-c)^4}{16\theta_o^{\,2}\,\theta_1}(L_2u,u)\ .$$

This proves the positive definiteness of the operator L_2 in $\Gamma^{(4)}$ subject to the boundary conditions (*VII.3.23*).

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