University of Alberta

BANACH-MAZUR DISTANCE AND RANDOM CONVEX BODIES

by

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To Brett, Chris and Shane.

ABSTRACT

In 2001, Gluskin, Litvak and Tomczak-Jaegermann, using probabilistic methods inspired by some earlier work of Gluskin's, provided an example of a convex body lacking symmetric projections. We revisit this example and give a different proof of its existence. The argument presented here makes use of a probabilistic decoupling technique due to Szarek and Tomczak-Jaegermann.

Additionally, we discuss a classical estimate due to John on the Banach-Mazur distance between an arbitrary n-dimensional Banach space and the Hilbert space ℓ_2^n. It is shown that an alternate proof of this estimate follows from a recent improvement of Kwapien's theorem.

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Chapter 1

Introduction

The theory of finite-dimensional normed spaces, or equivalently symmetric convex bodies, has witnessed a tremendous amount of activity in the last several decades. New techniques, drawing from different areas of mathematics, have been successfully applied to many difficult and long-standing open problems. This theory falls under what is now termed as *Asymptotic Geometric Analysis*, a central aspect of which is the study of certain numerical invariants that depend on dimension and the characteristic behavior of these invariants that appears as the dimension tends to infinity. One such invariant is the classical notion of Banach-Mazur distance.

The study of Banach-Mazur distance essentially began in 1948 with an estimate by John on the distance between an arbitrary symmetric convex body in $\mathbb{R}^n$ and the Euclidean ball. John proved that the distance is at most $\sqrt{n}$. The first observation in this thesis is that an alternate proof of John's estimate follows from a recent characterization of Banach-Mazur distance due to Efraim.

An immediate consequence of John's estimate is an upper bound on just how large the Banach-Mazur distance between two symmetric convex bodies in $\mathbb{R}^n$ can be: any two such bodies have a distance of at most n. The problem of actually finding examples of bodies that exhibit this maximal distance proved to be rather difficult. It was Gluskin, in 1981, who finally proved that such bodies exist. His work is important not only because it solved a previously intractable problem but also provided a fundamentally new approach to distance investigations. Gluskin was the first to introduce random bodies and, in addition to the new class of bodies, provided a set of far-reaching probabilistic methods and tools. His work has heavily influenced the development of the theory of symmetric convex bodies and a considerable amount of activity by researchers such as Mankiewicz, Szarek and Tomczak-Jaegermann.

Recent research has examined the case of convex bodies that are not symmetric. Many questions about the similarities and differences between the symmetric and non-symmetric cases have been answered in recent years. One such question is the main topic considered in this thesis. It has been conjectured that given a non-symmetric convex body in $\mathbb{R}^n$, there is always a proportional rank projection of this body which is almost symmetric. In 2001, Gluskin, Litvak and Tomczak-Jaegermann, using probabilistic techniques, provided an example that disproves the conjecture. The approach of the proof uses some basic geometric observations about non-symmetric bodies but then invokes the same methods as in the symmetric case.

The main result of this thesis is an alternate proof of the existence of said example. The approach of our proof is the same save for one key ingredient: rather than working directly with some unpleasant dependent conditions, as in the original proof, the argument presented here makes use of a recent probabilistic decoupling technique introduced by Szarek and Tomczak-Jaegermann for extracting independent behavior from dependent events.

Chapter 2

Preliminaries

2.1 Banach Spaces and Banach-Mazur Distance

We briefly recall some basic concepts from functional analysis. See, e.g., [5] for more detailed background information.

A **Banach space** is a vector space X equipped with a norm $\|\cdot\|$ such that it is complete in the metric induced by the norm. Although some definitions and background results mentioned in this thesis are stated for arbitrary Banach spaces, our results deal only with finite-dimensional Banach spaces over the field of real numbers. This means that for us a finite-dimensional Banach space is simply $\mathbb{R}^n$ equipped with a norm $\|\cdot\|$.

A **Hilbert space** is a Banach space $(X, \|\cdot\|)$ equipped with an inner-product $\langle \cdot, \cdot \rangle$ such that $\|x\| = \sqrt{\langle x, x \rangle}$ for every $x \in X$.

Example 2.1. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and for $1 \leq p \leq \infty$ let

$$\|x\|_p = \begin{cases} \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty \\ \max\limits_{i \leq n} |x_i| & \text{for } p = \infty. \end{cases}$$

Set $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$. Then ℓ_p^n is Banach space for each $1 \leq p \leq \infty$ and is a Hilbert space only for $p = 2$. □

A linear operator T between Banach spaces X and Y is **bounded** if there is a constant A such that $\|Tx\| \leq A\|x\|$ for each $x \in X$. The smallest such constant A is said to be the operator norm of T and is denoted $\|T\|$, i.e., $\|T\| = \sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$. The set of all bounded linear operators from X into Y is denoted by $\mathcal{B}(X,Y)$, which is itself a Banach space when equipped with the operator norm. $T \in \mathcal{B}(X,Y)$ is an **isomorphism** if there is an element $T^{-1} \in \mathcal{B}(Y,X)$ such that $TT^{-1} = T^{-1}T = I$. $T \in \mathcal{B}(X,Y)$ is an **isometric isomorphism** if it is an isomorphism that preserves norms, i.e., $\|Tx\| = \|x\|$ for every $x \in X$.

Banach spaces X and Y are (**isometrically**) **isomorphic** if there is some (isometric) isomorphism T mapping X onto Y.

In the case when $\dim X = \dim Y = n$, the set of all isomorphisms from X to Y can be identified with the set $\mathbb{GL}_n$ of invertible $n \times n$ matrices with real entries.

The **Banach-Mazur distance** between isomorphic Banach spaces X and Y is defined by

$$d(X,Y) := \inf\{\|T\|\|T^{-1}\| | T : X \to Y \text{ is an isomorphism}\}. \tag{2.1}$$

If X and Y are not isomorphic we set $d(X,Y) = \infty$. Note that $d(X,Y) \geq 1$ and $d(\cdot,\cdot)$ satisfies a multiplicative triangle inequality, that is to say, $d(X,Y) \leq d(X,Z)d(Z,Y)$ for any Banach spaces X, Y and Z.

If we restrict ourselves to the finite-dimensional case then Banach-Mazur distance is particularly useful since any two spaces of the same dimension are isomorphic. In this case, the infimum in (2.1) is actually attained and thus $d(X,Y) = 1$ if and only if X is isometric to Y. If we denote the closed unit ball in a Banach space Z by B_Z, i.e., $B_Z = \{z \in Z : \|z\| \leq 1\}$, then $d(X,Y)$ is the smallest positive number d such that there exists an isomorphism $T : X \to Y$ satisfying

$$B_Y \subset T[B_X] \subset dB_Y. \tag{2.2}$$

For more background information on Banach-Mazur distance, refer to [17].

2.2 Convex Geometry in $\mathbb{R}^n$

In the present section we fix our notation and terminology and recall some basic notions of convex geometry in $\mathbb{R}^n$. We assume that $\mathbb{R}^n$ is equipped with the canonical Euclidean inner product, which we denote by $\langle \cdot, \cdot \rangle$, as well as the induced norm, which we will denote by $|\cdot|$. As a word of caution, $|\cdot|$ is used elsewhere in this thesis to denote both the cardinality of a finite set and the absolute value of a scalar. The standard unit vector basis for $\mathbb{R}^n$ is denoted by $(e_i)_{i=1}^n$.

For $B \subset \mathbb{R}^n$, the **convex hull** of B is the collection of all convex combinations of elements in B, i.e., the collection of all elements of the form $\sum_{i=1}^m \lambda_i x_i$, where $m \in \mathbb{N}$, $x_1, \ldots, x_m \in B$ and the λ_i's are non-negative scalars such that $\sum_{i=1}^m \lambda_i = 1$. The **absolute convex hull** of B, denoted $\mathrm{absconv} B$, is the convex hull of $B \cup (-B)$, where $-B := \{-b : b \in B\}$. It can be shown that $x \in \mathrm{absconv} B$ if and only if $x = \sum_{i=1}^m \lambda_i x_i$, where $m \in \mathbb{N}$, $x_1, \ldots, x_m \in B$ and the λ_i's are scalars such that $\sum_{i=1}^m |\lambda_i| \leq 1$.

Recall the classical theorem of Caratheodory, the proof of which can be found, e.g,, in [9].

Theorem 2.2. *If B is a subset of $\mathbb{R}^n$ and if $x \in \mathrm{conv}\, B$ then there exists $x_1, \ldots, x_{n+1} \in B$ such that $x \in \mathrm{conv}\, \{x_1, \ldots, x_{n+1}\}$.*

Convex Bodies

A subset $K \subset \mathbb{R}^n$ is a **convex body** if it is compact, convex and has non-empty interior. Throughout this chapter, K, L and M denote convex bodies in $\mathbb{R}^n$.

The **Minkowski sum** of K and L is the set $K + L := \{x + y : x \in K, y \in L\}$. The translate of K by $a \in \mathbb{R}^n$ is the set $K_a := K - a := K + \{-a\}$ and for $\alpha \in \mathbb{R}$, we let $\alpha K := \{\alpha x : x \in K\}$. The **polar** of K is the set $K^\circ := \{y \in \mathbb{R}^n : \langle y, x \rangle \leq 1 \text{ for every } x \in K\}$.

The **Minkowski functional**, or **gauge functional**, of K, denoted $\|\cdot\|_K$, is defined by $\|x\|_K := \inf\{\lambda > 0 : x \in \lambda K\}$ for $x \in \mathbb{R}^n$. Here and throughout this thesis we use the convention that $\inf \emptyset = \infty$.

A convex body K is **centrally symmetric** if $-x \in K$ whenever $x \in K$. Thus centrally symmetric convex bodies are naturally centered at the origin. The set of all centrally symmetric convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{CS}^n$. If $K \in \mathcal{CS}^n$ then $\|\cdot\|_K$ defines a norm on $\mathbb{R}^n$. Conversely, if $X = (\mathbb{R}^n, \|\cdot\|)$ is a Banach space then the closed unit ball B_X is a centrally symmetric convex body in $\mathbb{R}^n$ and $\|\cdot\|_{B_X} = \|\cdot\|$. Thus there is a natural one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{centrally symmetric} \\ \text{convex bodies in } \mathbb{R}^n \end{array} \right\} \iff \left\{ \begin{array}{c} n\text{-dimensional} \\ \text{real Banach spaces} \end{array} \right\}. \tag{2.3}$$

If K is not centrally symmetric then $\|\cdot\|_K$ is not a norm. If, however, we assume that 0 is an interior point of K then $\|\cdot\|_K$ is a positively homogenous sublinear functional on $\mathbb{R}^n$.

The **geometric distance** $\widetilde{d}$ between K and L is defined by

$$\widetilde{d}(K, L) := \inf\{\alpha\beta : \alpha > 0, \beta > 0, (1/\beta)L \subset K \subset \alpha L\}.$$

Clearly $\widetilde{d}(K, L) \geq 1$ and $\widetilde{d}(K, L) = \widetilde{d}(L, K)$. Geometric distance also satisfies a multiplicative triangle inequality, i.e., $\widetilde{d}(K, L) \leq \widetilde{d}(K, M)\widetilde{d}(M, L)$.

Example 2.3. Let $B_p^n := B_{\ell_p^n}$. The smallest value of β for which $(1/\beta)B_2^n$ can be inscribed in B_∞^n is 1 and the smallest value of α for which αB_2^n circumscribes B_∞^n is $\sqrt{n}$. Thus the geometric distance between the Euclidean ball B_2^n and the cube B_∞^n is equal to $\sqrt{n}$. □

Given the correspondence in (2.3) we automatically have a notion of Banach-Mazur distance for centrally symmetric convex bodies in $\mathbb{R}^n$. By (2.2), the Banach-Mazur distance between *centrally symmetric* convex bodies K and L may be defined explicitly by

$$d(K, L) := \inf\{\widetilde{d}(T[K], L) : T \in \mathbb{GL}_n\}.$$

In the case of convex bodies that aren't centrally symmetric, 0 may not be the natural center for measuring geometric distance. When defining Banach-Mazur distance for bodies that are not necessarily centrally symmetric we need to consider the choice of position, i.e., we need to consider all possible centers in each body. Thus the Banach-Mazur distance between *arbitrary* convex bodies K and L is defined by

$$d(K, L) := \inf\{\tilde{d}(T[K_x], L_y) : x \in K, y \in L, T \in \mathbb{GL}_n\}. \tag{2.4}$$

In fact, it is convenient to use the equivalent definition

$$d(K, L) = \inf\{\tilde{d}(T[K_x], L_y) : x, y \in \mathbb{R}^n, T \in \mathbb{GL}_n\}. \tag{2.5}$$

To see that these are equivalent, suppose that $x, y \in \mathbb{R}^n, \lambda > 0, T \in \mathbb{GL}_n$ are such that $L_y \subset T[K_x] \subset \lambda L_y$. If $\lambda = 1$ then $L_y = T[K_x]$ and hence each body can be shifted to contain the origin. If $\lambda > 1$ then $L_y \subset \lambda L_y$, which implies that $0 \in L_y$. Indeed, by compactness there exists $z \in L_y$ with minimum distance to the origin. Then $(1/\lambda)z \in L_y$ and if $z \neq 0$ we have $|(1/\lambda)z| < |z|$, a contradiction. This means that $y \in L$ and $x \in K$.

If K and L are centrally symmetric then the infimum in (2.4) is attained at $x = y = 0$ and thus this definition is an extension of the notion of Banach-Mazur distance to arbitrary convex bodies.

A Measure of Asymmetry for Convex Bodies

In this section we discuss a notion of asymmetry for convex bodies that are not necessarily centrally symmetric. We follow the presentation in [8].

One natural measure of asymmetry for a convex body K is the quantity

$$\begin{aligned} \bar{\delta}(K) &:= \inf\{d(K, L) : L \in \mathcal{CS}^n\} \\ &= \inf\{\tilde{d}(T[K_x], L_y) : x, y \in \mathbb{R}^n, T \in \mathbb{GL}_n, L \in \mathcal{CS}^n\}. \end{aligned}$$

It will be convenient to consider a slight modification of this definition. Define the **asymmetry constant** of K by

$$\delta(K) := \inf\{\widetilde{d}(T[K_x], L) : x \in \mathbb{R}^n, T \in \mathbb{GL}_n, L \in \mathcal{CS}^n\},$$

which simply reduces to

$$\delta(K) = \inf\{\widetilde{d}(K_x, L) : x \in K, L \in \mathcal{CS}^n\}.$$

It can be shown that $\overline{\delta}(K) \leq \delta(K) \leq 2\overline{\delta}(K)$.

Although the asymmetry constant of K is defined using arbitrary bodies of $\mathcal{CS}^n$, one can use certain centrally symmetric convex bodies defined in terms of K and the center of symmetry only, as the following observation illustrates.

Lemma 2.4. *If K is a convex body in $\mathbb{R}^n$ then*

$$\begin{aligned}\delta(K) &= \inf\{\widetilde{d}(K_a, K_a \cap (-K_a)) : a \in K\} \\ &= \inf\{\widetilde{d}(K_a, \operatorname{conv}(K_a \cup (-K_a))) : a \in K\}.\end{aligned}$$

Proof: Let $A \geq 1$, $a \in K$ and $L \in \mathcal{CS}^n$ be such that $K_a \subset L \subset AK_a$. Then $-K_a \subset L \subset -AK_a$ so that

$$K_a \cap (-K_a) \subset \operatorname{conv}(K_a \cup (-K_a)) \subset L \subset A(K_a \cap (-K_a)) \subset A\operatorname{conv}(K_a \cup (-K_a)),$$

which implies the required identities. □

For $A \geq 1$, we say that K is A**-symmetric** if $\delta(K) \leq A$. More precisely, we say that K is A-symmetric with respect to (the center) $a \in K$ if $\widetilde{d}(K_a, L) \leq A$ for some $L \in \mathcal{CS}^n$.

Lemma 2.5. *Let K be a convex body in $\mathbb{R}^n$ and let $a \in K$. The following are equivalent:*

(a) *K is A-symmetric with respect to a.*

(b) $-K_a \subset AK_a$.

(c) $\|-x\|_{K_a} \leq A\|x\|_{K_a}$ *for every* $x \in \mathbb{R}^n$.

Proof: This is a trivial consequence of the preceding lemma and the following properties of gauge functionals corresponding to any convex bodies K and L.

i. $L \subset K$ if and only if $\|x\|_K \leq \|x\|_L$ for every $x \in \mathbb{R}^n$.

ii. If $\alpha > 0$ then $\|x\|_{\alpha K} = \frac{1}{\alpha}\|x\|_K$ for every $x \in \mathbb{R}^n$. □

The asymmetry constant of a simplex will be computed at the end of the next section.

2.3 John's Theorem and Some Consequences

In 1948, John proved that any convex body K in $\mathbb{R}^n$ contains a unique ellipsoid (affine image of B_2^n) $\mathcal{E}$ of maximal volume and, moreover, $K \subset n\mathcal{E}$. In addition, John showed that if K is centrally symmetric then $K \subset \sqrt{n}\mathcal{E}$. Thus John's theorem gives an upper bound of $\sqrt{n}$ for the Banach-Mazur distance between an arbitrary n-dimensional Banach space and ℓ_2^n. In the next chapter we provide an alternate proof of this estimate. We'll state John's theorem in terms of his characterization of the maximal volume ellipsoid since it will be useful in exploring some examples that illuminate the concepts presented so far. The following formulation is from [1].

Theorem 2.6. *B_2^n is the ellipsoid of maximal volume contained in the convex body $K \subset \mathbb{R}^n$ if and only if $B_2^n \subset K$ and, for some $m \geq n$, there are Euclidean unit vectors $(u_i)_{i=1}^m$, on the boundary of K, and positive numbers $(c_i)_{i=1}^m$ for which*

(a) $\sum\limits_{i=1}^{m} c_i u_i = 0$ *and*

(b) $x = \sum\limits_{i=1}^{m} c_i \langle u_i, x\rangle u_i$ *for all* $x \in \mathbb{R}^n$.

In this case, one has $K \subset nB_2^n$. If K is centrally symmetric then $K \subset \sqrt{n}B_2^n$.

Remark 2.7. There is an analogous statement for the minimal volume ellipsoid containing a convex body K for which the conditions (a) and (b) hold; one should simply reverse the inclusion $B_2^n \subset K$.

Corollary 2.8. *If X is an n-dimensional Banach space then $d(X, \ell_2^n) \leq \sqrt{n}$.*

The following example, from [2], illustrates the utility of John's theorem.

Example 2.9. We saw in Example 2.3 that the geometric distance between the Euclidean ball B_2^n and the cube B_∞^n is equal to $\sqrt{n}$; the Banach-Mazur distance is no smaller. To see this, let $\mathcal{E}_{\text{max}}$ be the ellipsoid of maximal volume contained in the cube and let $\mathcal{E}_{\text{min}}$ be the ellipsoid of minimal volume containing the cube. Then since $\pm e_i$ $(i = 1, \ldots, n)$ provide the decomposition in (a) and (b), we have $\mathcal{E}_{\text{max}} = B_2^n$. A similar decomposition for the minimal volume ellipsoid, one which uses each vertex of the cube, implies that $\mathcal{E}_{\text{min}} = \sqrt{n} B_2^n$. Thus if $T \in \mathbb{GL}_n$ is such that

$$TB_2^n \subset B_\infty^n \subset dTB_2^n$$

then

$$d^n \operatorname{vol}(TB_2^n) = \operatorname{vol}(dTB_2^n) \geq \operatorname{vol}(\mathcal{E}_{\text{min}}) = (\sqrt{n})^n \operatorname{vol}(\mathcal{E}_{\text{max}}) \geq (\sqrt{n})^n \operatorname{vol}(TB_2^n).$$

Thus $d \geq \sqrt{n}$ and hence $d(B_2^n, B_\infty^n) = \sqrt{n}$. □

Next we'll compute the asymmetry constant of a simplex. We acknowledge Dr. A. Litvak for showing us this argument. Similar conditions will have to be enforced when finding a lower bound for the asymmetry constant for (projections of) the convex body considered in chapter 5.

Example 2.10. Let S be a simplex in $\mathbb{R}^n$, i.e., the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. We will show that $\delta(S) = n$. Notice that for any $T \in \mathbb{GL}_n$ and for any $z \in \mathbb{R}^n$ we have $\delta(S) = \delta(T[S_z])$. It will be convenient to have one representation of a simplex to show the upper bound $\delta(S) \leq n$

and another representation to show the lower bound $\delta(S) \geq n$. The upper bound is immediate: if B_2^n is the maximal volume ellipsoid contained in S then $B_2^n \subset S \subset nB_2^n$ giving us $\delta(S) \leq n$. To show the lower bound, assume that B_2^n is the ellipsoid of minimal volume containing $S = \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$. Let $(u_i)_{i=1}^m$ and $(c_i)_{i=1}^m$ be given by John's theorem. Assume that the u_i's are distinct. Since each $u_i \in \partial S$, and since $S \subset B_2^n$ it follows that each u_i must be a vertex of S. But the u_i's were assumed to be distinct so we have $m \leq n+1$. If $m = n$ then $\sum_{i=1}^n c_i u_i = 0$ and, in particular, $(u_i)_{i=1}^n$ is linearly dependent, a contradiction. Hence $m = n+1$, $|v_i| = 1$ for each $i \in \{1, \ldots, n+1\}$ and we have

(a) $\sum_{i=1}^{n+1} c_i v_i = 0$ and

(b) $x = \sum_{i=1}^{n+1} c_i \langle v_i, x \rangle v_i$ for every $x \in \mathbb{R}^n$.

We will first show that $c_1 = \ldots = c_{n+1} = \frac{n}{n+1}$. It follows from (a) that $c_i = \sum_{j \neq i} c_j \langle v_j, -v_i \rangle$ for each $i \in \{1, \ldots, n+1\}$. Condition (b) implies that $1 = \langle v_i, v_i \rangle = \sum_{j=1}^{n+1} c_j \langle v_j, v_i \rangle^2$ which gives $1 - c_i = \sum_{j \neq i} c_j \langle v_j, v_i \rangle^2$ for each $i \in \{1, \ldots, n+1\}$. Also by (b), we have $n = \operatorname{trace}(I_n) = \sum_{j=1}^{n+1} c_j \langle v_j, v_j \rangle = \sum_{j=1}^{n+1} c_j$.

Using the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned}
n &= \sum_{i=1}^{n+1} \sum_{j \neq i} c_j \langle v_j, -v_i \rangle \\
&\leq \sum_{i=1}^{n+1} \sqrt{\sum_{j \neq i} c_j} \sqrt{\sum_{j \neq i} c_j \langle v_j, v_i \rangle^2} \\
&= \sum_{i=1}^{n+1} \sqrt{n - c_i} \sqrt{1 - c_i} \\
&\leq \sqrt{\sum_{i=1}^{n+1} (n - c_i)} \sqrt{\sum_{i=1}^{n+1} (1 - c_i)} \\
&= n.
\end{aligned}$$

Therefore equality holds in each inequality above and hence there exists α

such that for any $i \in \{1, \ldots, n+1\}$, the equality $\sqrt{n - c_i} = \alpha\sqrt{1 - c_i}$ holds. This implies that $c_1 = \ldots = c_{n+1} = \frac{n}{n+1}$.

Now, by Lemma 2.5, S is A-symmetric with respect to $a \in S$ if and only if

$$\begin{aligned} -S_a \subset AS_a \iff & -(v_i - a) \subset AS_a \text{ for each } i \in \{1, \ldots, n+1\} \\ \iff & \|-(v_i - a)\|_{S_a} \leq A \text{ for each } i \in \{1, \ldots, n+1\}. \end{aligned}$$

Thus to show that $\delta(S) \geq n$ it is sufficient to prove that for any $a \in S$, there is at least one $i \leq n+1$ such that

$$\|-(v_i - a)\|_{S_a} \geq n. \tag{2.6}$$

Fix $i \in \{1, \ldots, n+1\}$. Condition (a) implies that $-\frac{1}{n}v_i = \sum_{j \neq i} \frac{1}{n} v_j$, i.e., $-\frac{1}{n}v_i$ lies on a face of S. Thus $\|\frac{-1}{n}v_i\|_S = 1$, that is, $\|-v_i\|_S = n$. Thus we have shown that (2.6) is true for $a = 0$. An elementary argument shows that (2.6) is true for an arbitrary center $a \in S$. Thus we obtain the lower estimate $\delta(S) \geq n$. $\square$

2.4 Volumes of Convex Bodies

A key argument in chapter 5 involves a comparison of volumes. In particular, we will need the volume formulas

$$\operatorname{vol}(B_1^n) = \frac{2^n}{n!}$$

and

$$\operatorname{vol}(B_2^n) = \frac{\pi^{n/2}}{\Gamma(1 + n/2)},$$

where Γ denotes the Gamma function. A useful tool for calculations is Stirling's approximation: $n! \approx \sqrt{2\pi n}(n/e)^n$, where the notation $f(n) \approx g(n)$ means $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = 1$. In particular, there are absolute constants $C_i > 0, D_i > 0$

$(i = 1, 2)$ such that

$$(C_1/n)^n \leq \mathrm{vol}\,(B_1^n) \leq (C_2/n)^n$$

and

$$(D_1/n)^{\frac{n}{2}} \leq \mathrm{vol}\,(B_2^n) \leq (D_2/n)^{\frac{n}{2}}.$$

Recall the classical inequality due to Santaló [13].

Theorem 2.11. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Then*

$$\left(\frac{\mathrm{vol}\,(K)\,\mathrm{vol}\,(K^\circ)}{\mathrm{vol}\,(B_2^n)^2}\right)^{\frac{1}{n}} \leq 1.$$

The following is a result due to Ball and Pajor [3] and Gluskin [7].

Theorem 2.12. *Let $1 \leq n \leq m$ and let $x_1, \ldots, x_m \in \mathbb{R}^n$ be such that $x_i \neq 0$ for some i. Then*

$$\mathrm{vol}\,(\{x \in \mathbb{R}^n : |\langle x, x_i\rangle| \leq 1 \textit{ for every } i\})^{\frac{1}{n}} \geq \frac{2}{\alpha\sqrt{e}\sqrt{1+\log(m/n)}},$$

where $\alpha := \max\limits_{i \leq m} |x_i|$.

A consequence of this theorem is that the volume of sets of the form $K = \mathrm{absconv}\,\{x_1, \ldots, x_m\}$ is comparable, up to a logarithmic factor, with the volume of $B_1^n = \mathrm{absconv}\,\{e_1, \ldots, e_n\}$.

Corollary 2.13. *Let $1 \leq n \leq m$ and let $x_1, \ldots, x_m \in \mathbb{R}^n$. Then*

$$\mathrm{vol}\,(\mathrm{absconv}\,\{(x_i)_{i=1}^m\}) \leq \mathrm{vol}\left(3\alpha\sqrt{1+\log(m/n)}B_1^n\right),$$

where $\alpha := \max\limits_{i \leq m} |x_i|$.

Proof: Set $K := \mathrm{absconv}\,\{(x_i)_{i=1}^m\}$ so that

$$\begin{aligned} K^\circ &= \{x \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1 \text{ for all } y \in K\} \\ &= \{x \in \mathbb{R}^n : |\langle x, x_i\rangle| \leq 1 \text{ for all } i \leq m\}. \end{aligned}$$

Theorem 2.11 and Theorem 2.12 imply that

$$\begin{aligned}
\mathrm{vol}\,(K) &\le \frac{\mathrm{vol}\,(B_2^n)^2}{\mathrm{vol}\,(K^\circ)} \\
&\le \left((\alpha\sqrt{e}/2)\sqrt{1+\log(m/n)}\right)^n \frac{\mathrm{vol}\,(B_2^n)^2}{\mathrm{vol}\,(B_1^n)} \mathrm{vol}\,(B_1^n) \\
&\le \pi^n \left((\alpha\sqrt{e}/2)\sqrt{1+\log(m/n)}\right)^n \mathrm{vol}\,(B_1^n) \\
&\le \mathrm{vol}\left(3\alpha\sqrt{1+\log(m/n)}B_1^n\right).
\end{aligned}$$

□

2.5 Probabilistic Tools

Let $(\Omega, \mathbb{P})$ be a probability space. If h is a random variable defined on Ω and B is a subset of its range, we will use the notation $\mathbb{P}(\{\omega \in \Omega : h(\omega) \in B\})$ and $\mathbb{P}(\{h \in B\})$ interchangeably.

Of particular interest to us are Gaussian random variables with $N(0,1)$ distribution, i.e., those random variables $\gamma : \Omega \longrightarrow \mathbb{R}$ satisfying

$$\mathbb{P}(\gamma \in B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-\frac{x^2}{2}} dx$$

for any Borel set $B \subseteq \mathbb{R}$.

In the n-dimensional case, if γ_i $(i = 1, \ldots, n)$ are independent Gaussian random variables with $N(0,1)$ distribution then the random variable $h : \Omega \longrightarrow \mathbb{R}^n$ defined by $h = (\gamma_1, \ldots, \gamma_n)$ satisfies

$$\mathbb{P}(\{h \in B\}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_B e^{-\frac{|x|^2}{2}} dx_1 \ldots dx_n$$

for any Borel set $B \subset \mathbb{R}^n$. In this case, we say that h is a **Gaussian vector.**

A crucial property of Gaussian vectors is rotational invariance: if $U = (u_{ij})_{i,j=1}^n$ is a an orthogonal matrix then the random variable Uh has the same distribution as h, meaning that the equality

$$\mathbb{P}(\{h \in B\}) = \mathbb{P}(\{Uh \in B\})$$

is satisfied for any Borel set $B \subset \mathbb{R}^n$.

If h is a Gaussian vector, we say that $g = (1/\sqrt{n})h$ is a **normalized Gaussian vector**. This choice of normalization yields that the expected value of $|g|^2$ is equal to 1. The density of a normalized Gaussian vector is given by $(n/2\pi)^{n/2}e^{-n|x|^2/2}$. The following properties appear as part of Fact 1 in [12] and will be used in chapter 5.

Theorem 2.14. *Let $g : \Omega \to \mathbb{R}^n$ be a normalized Gaussian vector. Then*

i. *for every r-dimensional subspace $E \subset \mathbb{R}^n$, $\sqrt{\frac{n}{r}}P_E g$ is a normalized Gaussian vector in E, where P_E denotes the orthogonal projection onto E.*

ii. *$\mathbb{P}\{|g| \in [1/2, 2]\} \geq 1 - e^{-cn}$ for some absolute constant $c > 0$.*

iii. *for every Borel set $B \subset \mathbb{R}^n$ we have*

$$\mathbb{P}\{\omega \in \Omega : g(\omega) \in B\} \leq e^{n/2} \frac{\operatorname{vol}(B)}{\operatorname{vol}(B_2^n)}.$$

2.6 Gaussian Type and Cotype

Gaussian type and cotype arise in Banach-Mazur distance investigations and in this thesis will be used exactly for this purpose. See, e.g., [17] for a more detailed discussion.

Let X be a Banach space. Let $(\gamma_i)_{i=1}^{\infty}$ be a sequence of independent Gaussian random variables with $N(0,1)$ distribution defined on some probability space $(\Omega, \mathbb{P})$. For each positive integer k, define the Gaussian type 2 constant, denoted by $\alpha_2^k(X)$, as the smallest number C satisfying

$$\left(\int_{\Omega} \|\sum_{i=1}^{k} \gamma_i(\omega) x_i\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}} \leq C \left(\sum_{i=1}^{k} \|x_i\|^2\right)^{\frac{1}{2}}$$

for all $x_1, \ldots, x_k \in X$. For each positive integer k, define the Gaussian cotype 2 constant, denoted by $\beta_2^k(X)$, as the smallest number C satisfying

$$\left(\sum_{i=1}^{k} \|x_i\|^2\right)^{\frac{1}{2}} \leq C \left(\int_\Omega \|\sum_{i=1}^{k} \gamma_i(\omega) x_i\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}}$$

for all $x_1, \ldots, x_k \in X$.

Let $\alpha_2(X) = \sup_k \alpha_2^k(X)$ and $\beta_2(X) = \sup_k \beta_2^k(X)$. We say that X has **Gaussian type** 2 if $\alpha_2(X) < \infty$ and that X has **Gaussian cotype** 2 if $\beta_2(X) < \infty$.

2.7 ε-Nets in Banach Spaces

In chapter 5 we employ a standard approximation technique which uses the notion of an ε-net.

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space, Y a subset of X and $\varepsilon > 0$. A subset $\mathcal{N}$ of Y is an ε-**net** for Y if for any $y \in Y$ there exists $x \in \mathcal{N}$ such that $\|x - y\| \leq \varepsilon$.

The following estimate, while not optimal, is sufficient for our purposes. The proof can be found, for instance, in [12].

Lemma 2.15. *There exists an ε-net $\mathcal{N}$ for B_X such that $|\mathcal{N}| \leq (1 + 2/\varepsilon)^n$.*

Lemma 2.16. *If $\mathcal{N}$ is a ε-net for B_X and $U \subset B_X$ then there is a $(2\varepsilon) -$ net $\mathcal{M}$ for U such that $|\mathcal{M}| \leq |\mathcal{N}|$.*

Proof: For $x \in \mathcal{N}$ let $B(x, \varepsilon) := \{y \in X : \|x - y\| \leq \varepsilon\}$. For each $x \in \mathcal{N}$ choose $y_x \in B(x, \varepsilon) \cap U$ if possible. Let $\mathcal{M} = \{y_x\}$. Then for any $u \in U$ there is an $x \in \mathcal{N}$ with $\|u - x\| \leq \varepsilon$ and hence $B(x, \varepsilon) \cap U \neq \emptyset$. Thus there exists $y_x \in \mathcal{M}$ such that $\|x - y_x\| \leq \varepsilon$ and hence $\|u - y_x\| \leq 2\varepsilon$. $\square$

Chapter 3

The Distance to a Hilbert Space

The purpose of the present chapter is to provide another proof of the classical estimate by John on the distance of an arbitrary n-dimensional Banach space to n-dimensional Euclidean space, which we have stated as Corollary 2.8. We'll first need to develop the necessary terminology and machinery.

Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$. We say that T **factors through a Hilbert space** if there is a Hilbert space H and linear operators $R \in \mathcal{B}(X,H)$, $S \in \mathcal{B}(H,Y)$ such that $T = SR$. Let

$$\gamma_2(T) := \inf\{\|R\|\|S\| : R \in \mathcal{B}(X,H), S \in \mathcal{B}(H,Y), T = SR\}.$$

The following is a result by Lindenstrauss and Pelczyński [11]. The proof can also be found in [17] (Proposition 13.11).

Theorem 3.1. *Let X and Y be Banach spaces and let $T \in \mathfrak{B}(X,Y)$. Then T factors through a Hilbert space if and only if there exists C such that for any $k \in \mathbb{N}$, for any $x_1, \ldots, x_k \in X$, for any orthogonal matrix $U = (u_{ij})_{i,j=1}^k$ we have*

$$\left(\sum_{i=1}^k \|\sum_{j=1}^k u_{ij} T x_j\|^2\right)^{\frac{1}{2}} \le C \left(\sum_{i=1}^k \|x_i\|^2\right)^{\frac{1}{2}}.$$

Moreover, the smallest such C coincides with $\gamma_2(T)$.

The notion of factoring through a Hilbert space is particularly useful when

$\dim X = n$ since the identity map $I : X \to X$ factors through the Hilbert space ℓ_2^n and $d(X, \ell_2^n) = \gamma_2(I)$.

Another theorem that provides an upper bound on the distance between an arbitrary Banach space and a Hilbert space is the classical theorem due to Kwapien [10].

Theorem 3.2. *A Banach space X is of Gaussian type 2 and Gaussian cotype 2 if and only if it is isomorphic to a Hilbert space H. In this case, $d(X, H) \leq \alpha_2(X)\beta_2(X)$.*

If $X = (\mathbb{R}^n, \|\cdot\|)$ is an arbitrary Banach space, it can be shown (see, e.g., [17] Proposition 12.3) that $\alpha_2(X) \leq \sqrt{n}$ and $\beta_2(X) \leq \sqrt{n}$ and hence Kwapien's theorem gives the upper estimate $d(X, \ell_2^n) \leq n$. So without any a priori information on the space X, Theorem 3.2 does not provide a useful estimate.

A recent characterization of $d(X, \ell_2^n)$, due to Efraim [6], improves on the theorem of Kwapien. The improvement is achieved by revising the definitions of the Gaussian type 2 and Gaussian cotype 2 constants.

To state the theorem we will need to introduce some notation. Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space. For a positive integer k, let $X^k = \bigoplus_{i=1}^k X$. For $\overrightarrow{x} = (x_1, \ldots, x_k) \in X^k$ define $\|\overrightarrow{x}\|_2 = \left(\sum_{i=1}^k \|x_i\|^2\right)^{1/2}$. Let $\mathcal{O}(k)$ be the set of orthogonal $k \times k$ matrices. $\mathcal{O}(k)$ acts on the set X^k in a natural way: for $U = (u_{ij})_{i,j=1}^k \in \mathcal{O}(k)$ and $\overrightarrow{x} \in X^k$ let

$$U\overrightarrow{x} = \left(\sum_{j=1}^k u_{ij} x_j\right)_{i=1}^k.$$

For $\overrightarrow{x} \in X^k$, the **orbit** of $\overrightarrow{x}$ is the set $\{U\overrightarrow{x} : U \in \mathcal{O}(k)\}$.

Let $(\gamma_i)_{i=1}^\infty$ be a sequence of Gaussian random variables with $N(0,1)$ distribution defined on a probability space $(\Omega, \mathbb{P})$. For each positive integer k and for each $\overrightarrow{x} \in X^k$ we will denote by $\alpha_2^{(k)}(\overrightarrow{x})$, the smallest number C satisfying

$$\left(\int_\Omega \|\sum_{i=1}^k \gamma_i(\omega) y_i\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}} \leq C \left(\sum_{i=1}^k \|y_i\|^2\right)^{\frac{1}{2}}$$

for any $\overrightarrow{y} \in \mathcal{O}(\overrightarrow{x})$. For each positive integer k and for each $\overrightarrow{x} \in X^k$ we will denote by $\beta_2^{(k)}(\overrightarrow{x})$, the smallest number C satisfying

$$\left(\sum_{i=1}^{k} \|y_i\|^2\right)^{\frac{1}{2}} \leq C \left(\int_\Omega \|\sum_{i=1}^{k} \gamma_i(\omega) y_i\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}}$$

for any $\overrightarrow{y} \in \mathcal{O}(\overrightarrow{x})$.

The main result in [6] is the following theorem.

Theorem 3.3. *Let X be an n-dimensional Banach space. Then*

$$d(X, \ell_2^n) = \sup_{k \in \mathbb{N}} \sup_{\overrightarrow{x} \in X^k} \alpha_2^{(k)}(\overrightarrow{x}) \beta_2^{(k)}(\overrightarrow{x}) \leq 18 \sup_{\overrightarrow{x} \in X^n} \alpha_2^{(n)}(\overrightarrow{x}) \beta_2^{(n)}(\overrightarrow{x}).$$

Before sketching the proof, we will comment on the potential importance of this theorem. It is believed that this characterization may be applied to a long-standing open problem concerned with improving the estimate of Corollary 2.8 for some particular well-known Banach spaces (Banach spaces of type p, $1 < p < 2$). While we were not able to produce any results on this particular problem, an important first step is recovering the classical estimate of John.

Sketch of Proof: For $\overrightarrow{x} \in X^k$ let

$$l(\overrightarrow{x}) = \left(\int_\Omega \|\sum_{i=1}^{k} \gamma_i(\omega) x_i\|^2 d\mathbb{P}(\omega)\right)^{\frac{1}{2}}.$$

Rotational invariance of the Gaussian distribution implies that $l(\overrightarrow{x}) = l(\overrightarrow{y})$ for any $\overrightarrow{y} \in \mathcal{O}(\overrightarrow{x})$. For $\overrightarrow{x} \neq 0$ let

$$\overline{\alpha_2^{(k)}}(\overrightarrow{x}) = \frac{l(\overrightarrow{x})}{\|\overrightarrow{x}\|_2}$$

and set $\overline{\alpha_2^{(k)}}(0) = 0$. Fix a positive integer k and $\overrightarrow{x} \in X^k \setminus \{0\}$. Then

$$\overline{\alpha_2^{(k)}}(\overrightarrow{x}) \beta_2^{(k)}(\overrightarrow{x}) = \frac{l(\overrightarrow{x})}{\|\overrightarrow{x}\|_2} \sup_{\overrightarrow{y} \in \mathcal{O}(\overrightarrow{x})} \frac{\|\overrightarrow{y}\|_2}{l(\overrightarrow{y})} = \sup_{\overrightarrow{y} \in \mathcal{O}(\overrightarrow{x})} \frac{\|\overrightarrow{y}\|_2}{\|\overrightarrow{x}\|_2}. \tag{3.1}$$

In our notation, Theorem 3.1 and the comments following it imply that

$$d(X, \ell_2^n) = \sup_{k\in\mathbb{N}} \sup_{\substack{\overrightarrow{x}\in X^k \\ x\neq 0}} \sup_{\overrightarrow{y}\in\mathcal{O}(\overrightarrow{x})} \frac{\|\overrightarrow{y}\|_2}{\|\overrightarrow{x}\|_2},$$

which, given (3.1), yields

$$d(X, \ell_2^n) = \sup_{k} \sup_{\substack{\overrightarrow{x}\in X^k \\ x\neq 0}} \overline{\alpha_2^{(k)}}(\overrightarrow{x})\beta_2^{(k)}(\overrightarrow{x}).$$

Finally, it can be shown that

$$\sup_{\overrightarrow{x}\in X^k} \overline{\alpha_2^{(k)}}(\overrightarrow{x})\beta_2^{(k)}(\overrightarrow{x}) = \sup_{\overrightarrow{x}\in X^k} \alpha_2^{(k)}(\overrightarrow{x})\beta_2^{(k)}(\overrightarrow{x}).$$

We will not prove the inequality in Theorem 3.3. Let us simply mention that it is based on the following theorem by Tomczak-Jaegermann [16].

Theorem 3.4. *Let X be an n-dimensional Banach space. Then for every $k \geq n$ we have*

$$\alpha_n(X) \leq \alpha_k(X) \leq \sqrt{2\pi}\alpha_n(X)$$

and

$$\beta_n(X) \leq \beta_k(X) \leq 2\beta_n(X).$$

□

Corollary 2.8, up to an absolute constant, is now a trivial consequence.

Corollary 3.5. *Let X be an n-dimensional Banach space. Then*

$$d(X, \ell_2^n) \leq 18\sqrt{n}.$$

Proof: Notice that for $U \in \mathcal{O}(n)$ and $\overrightarrow{x} \in X^n$ we have

$$\begin{aligned}
\|U\overrightarrow{x}\|_2 &= \left(\sum_{i=1}^{n}\|\sum_{j=1}^{n} u_{ij}x_j\|_2^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\|u_{ij}x_j\|\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|u_{ij}|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}\|x_j\|^2\right)^{\frac{1}{2}} \\
&= \sqrt{n}\|\overrightarrow{x}\|_2
\end{aligned}$$

Thus Theorem 3.3 and (3.1) imply that

$$d(X, \ell_2^n) \leq 18 \sup_{\overrightarrow{x} \in X^n} \overline{\alpha_2^{(n)}}(\overrightarrow{x})\beta_2^{(n)}(\overrightarrow{x}) \leq 18 \sup_{\substack{\overrightarrow{x} \in X^n \\ \overrightarrow{x} \neq 0}} \sup_{U \in \mathcal{O}(n)} \frac{\|U\overrightarrow{x}\|_2}{\|\overrightarrow{x}\|_2} \leq 18\sqrt{n}.$$

$\square$

Chapter 4

Decoupling Weakly Dependent Events

The main argument of this thesis invokes a clever probabilistic technique introduced by Szarek and Tomczak-Jaegermann, a specific example of which is presented in [15] (Proposition 3.1). Since the generalized form [14] of the argument is unpublished, we record its statement and proof in this chapter. The statement is somewhat technical so we include the following description also taken from [14].

Suppose we have a collection of sets defined in terms of independent coordinates in a "local" way, i.e., while membership in each of the sets may depend on many or even all coordinates, it may be verified by looking at just a few coordinates at a time. Given this, the probability of the intersection of these sets can be "almost" estimated as if they were independent; more precisely, not by a product of their probabilities but by a homogeneous polynomial in their probabilities, the degree of which is high and the number of terms of which is controlled.

In our case, we are concerned with the events of some independent Gaussian vectors belonging to the convex hull of many other Gaussian vectors. Caratheodory's theorem, which we have stated as Theorem 2.2, provides the criterion for determining membership in terms of fewer coordinates.

Let $(\Omega, \mathbb{P})$ be a probability space. For $m \in \mathbb{N}$ we use the notation $[m]$ to denote the set $\{1, \dots, m\}$.

Proposition 4.1. *Let $1 \leq k \leq m/2$. Consider a family of events $\{\Omega_{i,I} : i \in [m], I \subset [m]\}$ satisfying the following conditions:*

(i) *for any $i \in [m]$ and $I \subset [m]$ we have*

$$\Omega_{i,I} = \bigcup_{\substack{I' \subset I \\ |I'| \leq k}} \Omega_{i,I'}.$$

(ii) *for any $I, J \subset [m]$ with $I \cap J = \emptyset$ the events $\{\Omega_{i,I} : i \in J\}$ are independent.*

Then for any $l \leq m/(2k+1)$, we have

$$\mathbb{P}\left(\bigcap_{i=1}^{m} \Omega_{i,\{i\}^{\complement}}\right) \leq \sum_{\substack{J \subset [m] \\ |J| = l}} \prod_{i \in J} \mathbb{P}(\Omega_{i,\{i\}^{\complement}}).$$

Proof: The proposition is based on the following lemma.

Lemma 4.2. *Let $I_1, \dots, I_m$ be subsets of $[m]$ such that $i \notin I_i$ and $|I_i| \leq k$ for all $i \in [m]$. Then there exists $J \subset [m]$ with $|J| \geq m/(2k+1)$ such that*

$$J \cap \bigcup_{i \in J} I_i = \emptyset.$$

Assume for the moment that the lemma is true. Observe that condition (i) gives us

$$I_1 \subset I_2 \subset [m] \implies \Omega_{i,I_1} \subset \Omega_{i,I_2} \tag{4.1}$$

for any $i \in [m]$.

Let $l \leq m/(2k+1)$ and set $\mathcal{J} := \{J \subset [m] : |J| = l\}$. We will prove the following inclusion:

$$\bigcap_{i=1}^{m} \Omega_{i,\{i\}^{\complement}} \subset \bigcup_{J \in \mathcal{J}} \bigcap_{i \in J} \Omega_{i,J^{\complement}}. \tag{4.2}$$

Let $\omega \in \bigcap_{i=1}^{m} \Omega_{i,\{i\}^{\complement}}$. Using (i), for each $i \in [m]$ there exists $I_i \not\ni i$ (that may depend on ω) with $|I_i| \leq k$ such that $w \in \bigcap_{i=1}^{m} \Omega_{i,I_i}$. By Lemma 4.2, there is a set J with $|J| \geq m/(2k+1)$ such that $I_i \subseteq J^{\complement}$ for $i \in J$. Then (4.1) implies that $\Omega_{i,I_i} \subset \Omega_{i,J^{\complement}}$ for each $i \in J$ and hence we obtain $\omega \in \bigcap_{i \in J} \Omega_{i,J^{\complement}}$. The set J also depends on ω but, in any case, we have proven (4.2).

By (ii), the family $\{\Omega_{i,J^{\complement}} : i \in J\}$ is independent, hence $\mathbb{P}\left(\bigcap_{i \in J} \Omega_{i,J^{\complement}}\right) = \prod_{i \in J} \mathbb{P}(\Omega_{i,J^{\complement}})$ and, using (4.1), we have $\Omega_{i,J^{\complement}} \subset \Omega_{i,\{i\}^{\complement}}$ $(i \in J)$, from which the conclusion follows. □

The inclusion (4.2) is crucial to the above argument. It says that while the events $\Omega_{i,\{i\}^{\complement}}$ may be dependent, the assumptions (i) and (ii) guarantee that the dependency is no worse than considering a (albeit much larger) collection of independent events.

Lastly, we address the proof of Lemma 4.2. This follows easily from the next theorem, a result by Ball [4] (Theorem 1.3′).

Theorem 4.3. *Let m be a positive integer and let $A = (a_{ij})_{i,j=1}^{m}$ be a matrix with real entries satisfying*

(i) $a_{ij} \geq 0$ *for all* $i, j \in [m]$ *and* $a_{ii} = 0$ *for all* $i \in [m]$.

(ii) $\sum_{j=1}^{m} a_{ij} \leq 1$ *for all* $i \in [m]$.

Then for each integer t there is a partition $\{J_s\}_{s=1}^{t}$ of $[m]$ into t mutually disjoint subsets such that

$$\sum_{j \in J_s} a_{ij} \leq \frac{2}{t}, \quad i \in J_s$$

for all $s \in [t]$.

Proof of Lemma 4.2: Let $A = (a_{ij})_{i,j}^{m}$ be defined by $a_{ij} = 1/k$ if $j \in I_i$ and $a_{ij} = 0$ otherwise. Then A satisfies the assumptions of Theorem 4.3. Thus for

$t = 2k + 1$ there is a partition $\{J_s\}_{s=1}^t$ of $\{1, \ldots, m\}$ such that

$$\sum_{j \in J_s} a_{ij} \le \frac{2}{2k+1} < \frac{1}{k}, \quad i \in J_s$$

for all $s \in [t]$. In particular, $a_{ij} = 0$ whenever $i, j \in J_s$ and $s \in [t]$. But for some s we have $|J_s| \ge \frac{m}{2k+1}$, which is what we wanted. $\square$

Chapter 5

A Convex Body Without Symmetric Projections

Some recent developments in the theory of convex bodies led to the following conjecture:

Conjecture 5.1. *For every convex body $K \subset \mathbb{R}^n$ there is a projection P with* $\operatorname{rank} P \approx n/2$ *such that*

$$\delta(PK) \leq C,$$

where C is an absolute constant.

The following theorem due to Gluskin, Litvak and Tomczak-Jaegermann [8] settles the conjecture.

Theorem 5.2. *There is an absolute constant $c > 0$ such that for any positive integer n, there exists a convex body $K \subset \mathbb{R}^n$ satisfying*

$$\delta(PK) \geq \frac{r}{c\sqrt{n \log n}},$$

for all projections P of rank $r > c\sqrt{n \log n}$.

In the present chapter we provide an alternate proof of this theorem. The approach is the same as that of the original argument but the proof of our Proposition 5.3 uses the technique in chapter 4.

Before proving the theorem we will say a few words about the general approach of the proof. Let $m \geq n$. Our convex body K will be of the form $K := \operatorname{conv}\{(g_i)_{i=1}^m\}$, where $g_1, \dots, g_m$ are points lying inside $2B_2^n$. If $r \leq n$ and P is any projection of rank r, then for $A_r = r/(c\sqrt{n\log n})$, Lemma 2.5 implies that PK is A_r-symmetric with respect to $a \in PK$ if and only if

$$-(Pg_i - a) \in A_r \operatorname{conv}\{(Pg_j - a)_{j\leq m}\} \qquad \text{for all } i \leq m,$$

which implies that

$$-(Pg_i - a) \in A_r \operatorname{conv}\{(Pg_j - a)_{j\neq i}, 0\} \qquad \text{for all } i \leq m,$$

or equivalently

$$-Pg_i \in A_r \operatorname{conv}\{(Pg_j - (1 + 1/A_r)Pa)_{j\neq i}, -(1/A_r)a\} \tag{5.1}$$

for each $i \leq m$ (note that $Pa = a$ since $a \in PK$).

Thus to prove the theorem, it is sufficient to select $g_1, \dots, g_m$ so that for any $r > c\sqrt{n\log n}$, for any projection P of rank r and any translate $a \in 2B_2^n$, (5.1) fails for at least one $i \leq m$.

Rather than considering each projection P and each translate a we will consider a suitable ε-net of projections and an ε-net of translates and violate a slightly stronger condition than that of (5.1) on these nets. It is formalized in Proposition 5.3. The theorem will then follow from a simple approximation argument.

Proof: It is sufficient to prove the theorem for orthogonal projections only. Indeed, let P be any projection and let Q be the orthogonal projection with the same kernel as P. Then Q and P have the same rank and since $\operatorname{Range}(I-P) \subset \ker P = \ker Q$, we have $Q(I-P) = 0$, i.e., $Q = QP$. Thus if PK is A-symmetric then so is $Q[PK] = QK$.

Let $m = 10n^3$, $\varepsilon = 1/(3\sqrt{n})$ and $A_r = r/(c\sqrt{n\log n})$ with $c > 0$ an absolute constant to be determined later.

Let $\mathcal{M}$ be an ε-net for $2B_2^n$ and let $\mathcal{N}_r$ be an ε-net (with respect to the operator norm) of rank r orthogonal projections. By section 2.7 we can assume that $|\mathcal{N}_r| \leq (6/\varepsilon)^{n^2}$ and $|\mathcal{M}| \leq (6/\varepsilon)^n$.

Proposition 5.3. *There exist vectors $g_1, \dots, g_m \in 2B_2^n$ such that $\forall r$ satisfying $A_r \geq 1$, $\forall Q \in \mathcal{N}_r, \forall b \in \mathcal{M}$ there is at least one $i \leq m$ such that*

$$-Qg_i \notin A_r \operatorname{conv}\left\{(Qg_j - (1+1/A_r)Qb)_{j \neq i}, -(1/A_r)Qb\right\} + 13A_r\varepsilon QB_2^n.$$

Proof of Proposition 5.3: Let $g_1, \dots, g_m$ be independent normalized Gaussian random vectors defined on a probability space $(\Omega, \mathbb{P})$ as in section 2.5. We are particularly interested in a subset $\Omega^0 \subset \Omega$, with probability exponentially close to 1, on which $|g_j|_{\Omega^0}| \in [1/2, 2]$ for each $1 \leq j \leq m$. By Theorem 2.14.ii, for each $1 \leq j \leq m$, there exists a subset $\Omega_j^0 \subset \Omega$ such that $|g_j|_{\Omega_j^0}| \in [1/2, 2]$ and $\mathbb{P}(\Omega_j^0) \geq 1 - e^{-d'n}$ for some absolute constant $d' > 0$. Thus setting $\Omega^0 := \bigcap_{j=1}^m \Omega_j^0$ and observing that

$$\mathbb{P}(\Omega \backslash \Omega^0) = \mathbb{P}\Big(\bigcup_{j=1}^m \Omega \backslash \Omega_j^0\Big) \leq m \max_j \mathbb{P}(\Omega \backslash \Omega_j^0) \leq me^{-d'n} \leq e^{-dn} \tag{5.2}$$

for some absolute constant $d > 0$, yields the desired set.

Let $r_0 := c\sqrt{n \log n}$. For any integer $r > r_0$, $Q \in \mathcal{N}_r, b \in \mathcal{M}, I \subset \{1, \dots, m\}$ and $\omega \in \Omega$, let

$$\mathfrak{A}_I(r, Q, b, \omega) := A_r \operatorname{conv}\{(Qh_j(\omega))_{j \in I}, -(1/A_r)Qb\} + 13A_r\varepsilon QB_2^n$$

and

$$\mathfrak{A}_I^{\text{abs}}(r, Q, b, \omega) := A_r \operatorname{absconv}\{(Qh_j(\omega))_{j \in I}, -(1/A_r)Qb\} + 13A_r\varepsilon QB_2^n,$$

where we have set $h_j := g_j - (1 + 1/A_r)b$.

As in chapter 4 we will use the notation $[m]$ to denote the set $\{1, \dots, m\}$.

The notation $\cdot^{\complement}$ denotes the complement of a set with respect to $[m]$. We will show that there is an absolute constant $\xi > 0$ such that

$$\mathbb{P}\left(\bigcap_{r>r_0}\bigcap_{Q\in\mathcal{N}_r}\bigcap_{b\in\mathcal{M}}\bigcup_{i\leq m}\{\omega : -Qg_i(\omega) \notin \mathfrak{A}_{\{i\}^{\complement}}(r,Q,b,\omega)\}\right) > 1 - e^{-\xi n^3} \tag{5.3}$$

by estimating the probability of the complement of the set appearing in (5.3). Note that to prove the proposition it is enough to show that the probability in (5.3) is positive.

Fix $r > r_0$, $Q \in \mathcal{N}_r$ and $b \in \mathcal{M}$. Write $\mathfrak{A}_I(\omega) = \mathfrak{A}_I(r,Q,b,\omega)$ and $\mathfrak{A}_I^{\mathsf{abs}}(\omega) = \mathfrak{A}_I^{\mathsf{abs}}(r,Q,b,\omega)$. For $i \in [m]$ and $I \subset [m]$ set

$$\Omega_{i,I} := \{\omega : -Qg_i(\omega) \in \mathfrak{A}_I(\omega)\}.$$

Consider the family of sets $\{\Omega_{i,I} : i \in [m], I \subset [m]\}$ in the context (and notation) of Proposition 4.1. The first step is to prove that (i) holds for $k = r+1$, i.e., for any $i \in [m]$ and any $I \subset [m]$ we have

$$\Omega_{i,I} = \bigcup_{\substack{I'\subset I\\ |I'|\leq r+1}} \Omega_{i,I'}. \tag{5.4}$$

To see that (5.4) holds, let $i \in [m]$ and $I \subset [m]$. Fix $\omega \in \Omega_{i,I}$. Then $\exists z_i \in 13A_r\varepsilon Q B_2^n$ such that

$$-Qg_i(\omega) - z_i \in A_r \operatorname{conv}\left\{(Qh_j(\omega))_{j\in I}, -(1/A_r)Qb\right\}.$$

By Caratheodory's theorem, there exists $I' = I'(\omega) \subset I$ with $|I'| \leq r+1$ such that

$$-Qg_i(\omega) - z_i \in A_r \operatorname{conv}\left\{(Qh_j(\omega))_{j\in I'}, -(1/A_r)Qb\right\}.$$

In other words, $\omega \in \Omega_{i,I'}$ and hence we have

$$\Omega_{i,I} \subset \bigcup_{\substack{I'\subset I\\ |I'|\leq r+1}} \Omega_{i,I'}.$$

Conversely, if $I' \subset I$ and $|I'| \leq r+1$ then clearly $\omega \in \Omega_{i,I'}$ implies that $\omega \in \Omega_{i,I}$.

Now if I and J are disjoint subsets of $[m]$, the independence of the events $\{\Omega_{i,I} : i \in J\}$ follows from the independence of the collections $(g_i)_{i\in I}$ and $(g_j)_{j\in J}$.

Having proven that both of the hypotheses in Proposition 4.1 are satisfied, we may apply it for $l = \lceil m/(5r) \rceil$ and $\mathcal{J} := \{J \subset [m] : |J| = l\}$. (Here we use the notation $\lceil x \rceil$ to denote the smallest integer larger than x.)

The remainder of the proof is devoted to estimating the probability of the events $\Omega_{i,\{i\}^{\complement}}$ $(i \in [m])$. We will use Theorem 2.14.iii and therefore need a volume estimate on the sets $\mathfrak{A}_{\{i\}^{\complement}}(\omega)$. In fact, an estimate on the volume of $\mathfrak{A}^{\mathrm{abs}}_{\{i\}^{\complement}}(\omega)$ will be sufficient. The next lemma shows that the majority of these sets (meaning $\omega \in \Omega^0$) have volume comparable to that of B_1^r.

Claim 5.4. *For any $i \in [m]$ and for any $\omega \in \Omega^0$, we have*

$$\mathrm{vol}\left(\mathfrak{A}^{\mathrm{abs}}_{\{i\}^{\complement}}(\omega)\right) \leq (72A_r\sqrt{\log n})^r \,\mathrm{vol}\,(B_1^r)\,.$$

Proof: Let $i \in [m]$ and let $\omega \in \Omega^0$. Write $g_j = g_j(\omega)$, $h_j = h_j(\omega)$ and $\mathfrak{A}^{\mathrm{abs}}_{\{i\}^{\complement}} = \mathfrak{A}^{\mathrm{abs}}_{\{i\}^{\complement}}(\omega)$. Then $|Qh_j| \leq |Qg_j| + |(1+1/A_r)Qb| \leq 6$ and $|-(1/A_r)Qb| \leq 2$. Observe that $(1/\sqrt{n})B_2^n \subset B_1^n = \mathrm{absconv}\{e_i\}_{i=1}^n$ which implies that

$$13A_r\varepsilon QB_2^n \subset A_r\,\mathrm{absconv}\,\{(Qe_i')_{i=1}^n\}\,,$$

where $e_i' := 6e_i$. Setting

$$\mathfrak{B} := A_r\,\mathrm{absconv}\,\{(Qh_j)_{j\neq i}, -(1/A_r)Qb, (Qe_i')_{i=1}^n\}$$

gives us

$$\mathfrak{A}^{\mathrm{abs}}_{\{i\}^{\complement}} = A_r\,\mathrm{absconv}\,\{(Qh_j)_{j\neq i}, -(1/A_r)Qb\} + 13A_r\varepsilon QB_2^n \subset 2\mathfrak{B}.$$

Since $\mathfrak{B}$ is the absolute convex hull of $p := (m-1)+n+1$ points all lying inside $6B_2^r$ and since $p/r \leq 2m$, Corollary 2.13 implies that

$$\begin{aligned} \mathrm{vol}\,(2\mathfrak{B}) &\leq \mathrm{vol}\left(36A_r\sqrt{1+\log(2m)}B_1^r\right) \\ &\leq \mathrm{vol}\left(72A_r\sqrt{\log n}B_1^r\right), \end{aligned}$$

which proves our claim. □

Claim 5.5. *There is an absolute constant $\zeta > 0$ for which the estimate*

$$\mathbb{P}\left(\Omega_{i,\{i\}^{\complement}}\right) \leq e^{-\zeta r}$$

is satisfied for any $i \in [m]$.

Proof: Fix $i \in [m]$. Denote by $\mathbb{P} \times \mathbb{P}$ the product measure on $\Omega \times \Omega$. Set

$$\Delta := \{(\omega, \widetilde{\omega}) \in \Omega \times \Omega : -Qg_i(\omega) \in \mathfrak{A}^{\mathsf{abs}}_{\{i\}^{\complement}}(\widetilde{\omega})\}.$$

Observe that

$$\begin{aligned} \mathbb{P}(\Omega_{i,\{i\}^{\complement}}) &\leq \mathbb{P}\{\omega \in \Omega : -Qg_i(\omega) \in \mathfrak{A}^{\mathsf{abs}}_{\{i\}^{\complement}}(\omega)\} \\ &= \mathbb{P} \times \mathbb{P}(\Delta), \end{aligned} \tag{5.5}$$

since the vector g_i is independent of the collection $(g_j)_{j\neq i}$. But (5.5) is in turn equal to

$$\begin{aligned} \int_\Omega \int_\Omega \chi_\Delta(\omega, \widetilde{\omega}) d\mathbb{P}(\omega) d\mathbb{P}(\widetilde{\omega}) &= \int_{\Omega\setminus\Omega^0} \int_\Omega \chi_\Delta(\omega, \widetilde{\omega}) d\mathbb{P}(\omega) d\mathbb{P}(\widetilde{\omega}) \\ &\quad + \int_{\Omega^0} \int_\Omega \chi_\Delta(\omega, \widetilde{\omega}) d\mathbb{P}(\omega) d\mathbb{P}(\widetilde{\omega}), \end{aligned}$$

where Ω^0 is the set satisfying (5.2). Observe first that

$$\int_{\Omega\setminus\Omega^0} \int_\Omega \chi_\Delta(\omega, \widetilde{\omega}) d\mathbb{P}(\omega) d\mathbb{P}(\widetilde{\omega}) \leq \int_{\Omega\setminus\Omega^0} d\mathbb{P}(\widetilde{\omega}) \leq e^{-dn}.$$

Recall that $A_r = r/(c\sqrt{n \log n})$. By Theorem 2.14.iii and Claim 5.4, for any fixed $\widetilde{\omega} \in \Omega^0$, we have

$$\begin{aligned}
\int_\Omega \chi_\Delta(\omega, \widetilde{\omega}) d\mathbb{P}(\omega) &= \mathbb{P}(\{\omega \in \Omega : -Qg_i(\omega) \in \mathfrak{A}^{\text{abs}}_{\{i\}^{\mathsf{C}}}(\widetilde{\omega})\}) \\
&= \mathbb{P}\left(\{\omega \in \Omega : -\sqrt{n/r}Qg_i(\omega) \in \sqrt{n/r}\mathfrak{A}^{\text{abs}}_{\{i\}^{\mathsf{C}}}(\widetilde{\omega})\}\right) \\
&\leq e^{\frac{r}{2}} \frac{\text{vol}\left(\sqrt{n/r}\mathfrak{A}^{\text{abs}}_{\{i\}^{\mathsf{C}}}(\widetilde{\omega})\right)}{\text{vol}\,(B_2^r)} \\
&\leq \left(\sqrt{\frac{ne}{r}}\right)^r (72A_r\sqrt{\log n})^r \left(\sqrt{\frac{2e}{\pi r}}\right)^r \\
&\leq (150/c)^r.
\end{aligned}$$

Thus we obtain

$$\mathbb{P}(\Omega_{i,\{i\}^{\mathsf{C}}}) \leq (150/c)^r + e^{-dn} \leq e^{-\zeta r} \tag{5.6}$$

for an appropriate choice of absolute constants c and $\zeta > 0$. As i was arbitrary, (5.6) holds for every $i \in [m]$. $\square$

Observe now that for any $1 \leq k \leq m$, Stirling's formula implies the estimate $\binom{m}{k} \leq (em/k)^k$, hence $|\mathcal{J}| = \binom{m}{\lceil m/(5r)\rceil} \leq \exp(\lceil m/(5r)\rceil \log(5er))$. Thus Proposition 4.1 yields

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^m \Omega_{i,\{i\}^{\mathsf{C}}}\right) &\leq \sum_{J\subset\mathcal{J}} \prod_{i\in J} \mathbb{P}(\Omega_{i,\{i\}^{\mathsf{C}}}) \\
&\leq |\mathcal{J}| \max_{J\in\mathcal{J}} (\mathbb{P}(\Omega_{i,\{i\}^{\mathsf{C}}}))^{|J|} \\
&\leq \exp(\lceil m/(5r)\rceil \log(5er)) \exp(-\zeta r \lceil m/(5r)\rceil) \\
&\leq e^{-\eta m},
\end{aligned} \tag{5.7}$$

where $\eta > 0$ is an absolute constant. Note that up until this point the rank r, projection Q and translate b were fixed but the calculations so far clearly do not depend on their particular values. Thus the estimate (5.7) holds for any r, Q and b.

To finish the proof of Proposition 5.3, observe that

$$\begin{aligned}\mathbb{P}\left(\bigcup_{r>r_0}\bigcup_{Q\in\mathcal{N}_r}\bigcup_{b\in\mathcal{M}}\bigcap_{i\le m}\Omega_{i,\{i\}^{\mathsf{c}}}\right) &\le n|\mathcal{N}_n||\mathcal{M}|\max_{r,Q,b}\mathbb{P}\left(\bigcap_{i\le m}\Omega_{i,\{i\}^{\mathsf{c}}}\right)\\ &\le n(18\sqrt{n})^{n^2}(18\sqrt{n})^n e^{-\eta m}\\ &\le e^{-\xi n^3}\end{aligned}$$

for some absolute constant $\xi > 0$. This proves the proposition. $\square$

Now take $g_1, \ldots, g_m$ given by Proposition 5.3 and set $K := \operatorname{conv}\{(g_i)_{i=1}^m\}$. Let $c\sqrt{n\log n} \le r \le n$ and let P be an orthogonal projection of rank r. Then PK is A_r-symmetric with respect to $a \in PK$ if and only if

$$-(Pg_i - a) \in A_r \operatorname{conv}\{(Pg_j - a)_{j\le m}\} \qquad \text{for all } i \le m$$

which implies that

$$-(Pg_i - a) \in A_r \operatorname{conv}\{(Pg_j - a)_{j\ne i}, 0\} \qquad \text{for all } i \le m$$

or equivalently

$$-Pg_i \in A_r \operatorname{conv}\{(Pu_j)_{j\ne i}, -(1/A_r)a\} \qquad \text{for all } i \le m \tag{5.8}$$

where we have set $u_j := g_j - (1 + 1/A_r)a$ (note that $Pa = a$ since $a \in PK$).

Choose $Q \in \mathcal{N}_r$ such that $\|Q-P\| \le \varepsilon$ and note that $\max_i |(Q-P)g_i| \le 2\varepsilon$. Since $P\mathcal{M}$ is an ε-net for PK there is a $b \in \mathcal{M}$ such that $|Pb - a| \le \varepsilon$ and hence $|Qb - a| \le |Qb - Pb| + |Pb - a| \le 3\varepsilon$. Fix $i \le m$. By (5.8), there are non-negative scalars $\alpha_1, \ldots, \alpha_m$ with $\sum_{j=1}^m \alpha_j = 1$ such that

$$-Pg_i = A_r \sum_{j\ne i} \alpha_j Pu_j - \alpha_i a.$$

Setting $v_j := g_j - (1 + 1/A_r)b$, we obtain

$$
\begin{aligned}
&\operatorname{dist}\left(-Qg_i, A_r \operatorname{conv}\{(Qv_j)_{j\neq i}, -(1/A_r)Qb\}\right) \\
&\leq \ \left|-Qg_i - \Big(A_r \sum_{j\neq i} \alpha_j Q v_j - \alpha_i Q b\Big)\right| \\
&\leq \ |Pg_i - Qg_i| + |A_r \sum_{j\neq i} \alpha_j (Pu_j - Qv_j)| + |\alpha_i (Qb - a)| \\
&\leq \ 2\varepsilon + 8\varepsilon A_r + 3\varepsilon \\
&\leq \ 13 A_r \varepsilon.
\end{aligned}
$$

Thus if PK is A_r-symmetric with respect to a then there are $Q \in \mathcal{N}_r$ and $b \in \mathcal{M}$ such that

$$\operatorname{dist}(-Qg_i, A_r \operatorname{conv}\{(Qv_j)_{j\neq i}, -(1/A_r)Qb\}) \leq 13 A_r \varepsilon$$

for all $i \leq m$. This contradicts Proposition 5.3 and hence proves the theorem.

□

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