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THE UNIVERSITY OF ALBERTA

PAIRED COMPARISON OF TIME SERIES UNDER A TYPE OF DEPENDENCE

by

ARIYAWANSA PERERA ILLEPERUMA

A thesis

submitted to the Faculty of Graduate Studies and Research in partial fulfilment for the requirements for the degree of MASTER OF SCIENCE.

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submitted by Ariyawansa Perera Illeperuma in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The two sample t-test based on the paired differences may be used to compare the means of two normal populations provided the observations are independent not only among the two populations but also within the populations. This study was undertaken to examine the effect on the test when one of the samples consists of observations from a stationary linear Gaussian Markov process. An asymptotic expression for the probability density function of the $F = T^2$ statistic is derived. Moreover, the effect of the dependence on the significance levels of the test is examined using some numerical results.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Let $X(t) : t \in \mathcal{T}$ and $Y(t) : t \in \mathcal{T}$ be two independent time series and $(x_1, y_1) = (X(t_1), Y(t_1)), (x_2, y_2) = (X(t_2), Y(t_2)), \ldots, (x_n, y_n) = (X(t_n), Y(t_n))$ be a paired sample of observations obtained at time points $t_1 < t_2 < \ldots < t_n$. If $\mu(t)$ and $\nu(t)$ are the expected values $X(t)$ and $Y(t)$ respectively, then at time $t \in \mathcal{T}$ we may write

$$X(t) = \mu(t) + \varepsilon(t)$$

and

$$Y(t) = \nu(t) + \eta(t)$$

where $\varepsilon(t)$ and $\eta(t)$ are random error terms.

Suppose we are interested in testing the null hypothesis

$$H_0 : \quad \mu(t) = \nu(t)$$

against the alternate hypothesis,

$$H_A : \quad \mu(t) \neq \nu(t),$$

for all $t \in \mathcal{T}$.

Consider a discrete set of time points $t_1 < t_2 < \ldots < t_n$ and define $\varepsilon_i = \varepsilon(t_i)$ and $\eta_i = \eta(t_i)$, $i = 1, 2, \ldots, n$. If it can be assumed that $(\varepsilon_i, \eta_i); i = 1, 2, \ldots, n$ are independent identically distributed bivariate normal random variables with mean (0,0), we can use the following test statistic:

or, equivalently,

$$F = T^2,$$

(where $\bar{d} = \frac{1}{n}\sum_1 d_i$, $S_d^2 = \frac{1}{(n-1)}\sum(d_i - \bar{d})^2$ and $d_i = x_i - y_i$; $i = 1,2,\ldots,n$), since under the null hypothesis T has a Student's t-distribution with n-1 degrees of freedom and F has an F-distribution with 1 and (n-1) degrees of freedom.

However when the independence of observation cannot be assumed the t-test or the F-test may be invalid since the distributions of these statistics become unknown. Chapter 2 surveys the literature dealing with this problem for certain types of dependence.

In this thesis we consider a special case in which $\{\varepsilon_i\}$; $i = 1,2,\ldots,n$ are observations of a stationary linear Markov process, i.e.

$$\varepsilon_i = \rho\varepsilon_{i-1} + e_i \quad ,$$

and e_i's are i.i.d. N(0,1) random variables, while η_i; $i = 1,2,\ldots,n$ are independent observations from a normal process, that is the η_i's are i.i.d. N(0,1) random variables. In Chapter 3 it is shown that the probability density function (p.d.f.) of F for large n is approximately given by,

$$g(F) = \frac{(n-1)[1+(1-\rho)^2]\Gamma(\frac{n}{2})\left[(1-\rho^2)(2+\rho^2-2\rho\beta)\right]^{\frac{1}{2}}(1-\beta)}{\sqrt{\pi F}\,\Gamma(\frac{n-1}{2})(1-\rho\beta)}$$

$$\times\left(\frac{n\beta}{A}\right)^{\frac{3}{2}}\left(\frac{\rho^2\beta-\rho\beta^2+2\beta-\rho}{\rho}\right)^{\frac{n-3}{2}} ,$$

where

$$\beta = \frac{B - \sqrt{B^2-4A^2}}{2A} ,$$

$$A = n\rho\left[(n-1)[1+(1-\rho)^2] + (1-\rho^2)F\right] ,$$

and,

$$B = n\left\{\left[1 + (1 - \rho)^2\right](1 + \rho^2)(n - 1 + F) - 2\rho F\right\}$$

Furthermore it is shown that as ρ tends to zero $g(F)$ tends to the exact density function of F for $\rho=0$, which is the F-distribution with degrees of freedom 1 and (n-1).

Finally, using the renormalized $g(F)$, critical points to reject the null hypothesis at 0.05 level of significance are calculated and plotted against ρ for several values of n.

CHAPTER 2

LITERATURE REVIEW

The T-statistic is used for testing hypotheses concerning the mean of a normal population having unknown variance. One of the major assumptions behind this test is that the observations drawn from the population are statistically independent. If it is not insensitive to the condition of dependence then the inference drawn on the basis of a t-test with a sample of dependent observations will be incorrect. Thus, considering the robustness of the t-test, it is of considerable importance to examine the sensitivity of the distribution of the T-statistic to the violation of the independence assumption.

Several authors have studied this problem under different types of dependence and in this chapter we summarize the outcome of their work.

Gastwirth and Rubin(1971) have compared the effect of the dependence on the asymptotic levels of the sign, Wilcoxon and t tests when observations are from a stationary stochastic process. For any sequence of statistics T_n with means $E[T_n]$ and variance $V(T_n)$ such that

$$\frac{T_n - E(T_n)}{V(T_n)^{1/2}}$$

is asymptotically an N(0,1) random variable, the critical point K_α corresponding a right-tailed test of size α is determined by the relation

$$1 - \Phi(K_\alpha) = \alpha$$

where $\Phi(x)$ is the cumulative distribution function (c.d.f.) of the standard normal distribution. When the observations are from a stationary process, the variance of the statistic T_n differs from its value in the case of independent identically distributed (i.i.d.) observations with the same marginal distribution. Suppose we denote the asymptotic variance of T_n by D, its value in the case of i.i.d. observations by V and the ratio D/V by τ. If we use the same critical value K_α for the dependent data then the level of the test is approximately,

$$1 - \Phi(K_\alpha \tau^{-1/2}) = \alpha .$$

Deriving the variances of the statistics correspond to the sign, Wilcoxon and t tests, Gastwirth and Rubin examined the behaviour of the asymptotic levels of these tests when the observations are from:

(i) a completely regular Gaussian process whose autocorrelation satisfy

$$\sum \rho_k < \infty \text{ and } \rho_k \geq 0 \text{ for all k,}$$

and

(ii) a first order autoregressive Gaussian process with a negative autocorrelation coefficient ρ.

They showed that for both cases the level of the sign test changes less than the level of the Wilcoxon test which changes less than the level of the t-test. Furthermore, they have proven that for case (i) the levels of all three tests exceed the corresponding values for independent data while in case (ii) they are less than the levels for independent data.

Scheffé(1959) studied the model where $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$ has a multivariate normal distribution with mean $\mu = (\mu, \mu, \ldots, \mu)'$ and the covariance matrix Σ where

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & 0 & 0 & \dots & 0 & 0 & 0 \\ \rho & 1 & \rho & 0 & \dots & 0 & 0 & 0 \\ 0 & \rho & 1 & \rho & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \rho & 1 & \rho \\ 0 & 0 & 0 & 0 & \dots & 0 & \rho & 1 \end{pmatrix}_{n \times n}$$

He showed that a necessary and sufficient condition for Σ to be positive definite is,

$$|\rho| < \left[2cos(\frac{\pi}{n+1})\right]^{-1},$$

and, under these conditions,

$$E(\bar{X}) = \mu \quad ,$$
$$V(\bar{X}) = \frac{\sigma^2}{n}\left[1 + 2\rho\left(1 - \frac{1}{n}\right)\right] \quad ,$$
$$E(S^2) = \sigma^2\left(1 - \frac{2\rho}{n}\right) \quad ,$$

where $\bar{X}$ and S^2 are the sample mean and sample variance respectively. Then the T-statistic is asymptotically $N(0, 1+2\rho)$ and the asymptotic level of significance of the right-tailed test of $H_0 : \mu = 0$ is

$$1 - \Phi(K_\alpha(1 + 2\rho)^{-\frac{1}{2}})$$

A more general form of the Scheffé's model was studied by Albers(1978). His model has the observations $X_1, X_2, \ldots, X_n$ which are normally distributed with mean μ as taken by Scheffé(1959), but m-dependent, that is, they have zero serial correlation coefficients of lag k if $k > m$ for some positive integer m. Thus the covariance matrix Σ has elements

$$\sigma_{ij} = cov(X_i, X_j) = \begin{cases} \sigma^2 \rho_{|i-j|}, & \text{if } 1 \leq |i-j| \leq m; \\ 0, & \text{if } |i-j| > m; \end{cases}$$

$$i, j = 1, 2, \ldots, n \quad ,$$

where ρ_k, $k = 1, 2, \ldots, m$ are constants such that Σ is positive definite. He showed that if

$$1 + 2\sum_{k=1}^{m} \rho_k > 0$$

then the T-statistic is asymptotically

$$N\left(0, 1 + 2\sum_{k=1}^{m} \rho_k (1 - \frac{k}{n})\right) \quad ,$$

and the size of the right-tailed t-test of $H_0 : \mu = 0$ is

$$1 - \Phi\left(K_\alpha \left[1 + 2\sum_{k=1}^{m} \rho_k \left(1 - \frac{k}{n}\right)\right]^{-\frac{1}{2}}\right) + o(1).$$

Albers(1978) also studied the model in which the $\{X_i\}$ are from a stationary autoregressive process of order m, i.e.,

$$\text{(2.1)} \qquad \sum_{k=1}^{m} a_k (X_{i-k} - \mu) = Z_i, \qquad i = m + 1, \ldots, n \quad ,$$

where $a_0 = 1, a_1, a_2, \ldots, a_m$ are constants and $Z_{m+1}, \ldots, Z_n$ are i.i.d. $N(0, \tau^2)$ random variables. Moreover, the a_k are such that all roots of the equation

$$\sum_{k=1}^{m} a_k w^{m-k} = 0$$

lie inside the unit circle which is necessary and sufficient for the existence of a stationary solution of (2.1). Under these conditions he showed that T is asymptotically

$$N\left(0, \sum_{k=0}^{m} a_k \rho_k / (\sum_{k=0}^{m} a_k)^2 + O(\frac{m}{n})\right).$$

For each model, Albers proposed a robust modification of the t-test. He demonstrated that under independence both of the proposed tests would require, asymptotically, mK_α^2 additional observations to obtain the power of the null t-test.

In studying the effect of dependent observations on the level of the t-test, Hotelling(1961) defined the statistic

$$R_n = \frac{n^{n/2}|\Sigma|^{-1/2}}{(\sum\sum \lambda_{ij})^{n/2}} ,$$

where Σ is the covariance matrix of the n-dimensional random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ having the multivariate normal distribution with mean $\mathbf{0} = (0, 0, \ldots, 0)'$, and λ_{ij} is the ij^{th} element of Σ^{-1}. For large values of t he showed that R_n approximates the ratio of the probability that T-statistic exceeds t under dependence to that under independence.

For a model which has the correlation between i^{th} and j^{th} observations $\rho^{|i-j|}$ for some $|\rho| < 1$, Hotelling found R_n to be

$$R_n = \frac{(1-\rho^2)^{1/2}}{(1-\rho)^n[1+2\rho(1-\rho)^{-1}n^{-1}]^{n/2}} ,$$

while for a model with observations, each with correlation ρ with each of the others and all with unit variances,

$$R_n = \left[1 + \frac{n\rho}{1-\rho}\right]^{-1/2} .$$

Motivated by the work of Hotelling, Ali(1973) considered the distribution of the T-statistic when $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is $N(\mathbf{0}, \Sigma)$ with

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}_{n\times n}$$

and the correlation coefficient ρ satisfying

$$\frac{-1}{n-1} < \rho < 1 \quad .$$

He derived the density of T to be

$$g(t) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2}) \left\{\pi(n-1)\left(1+\frac{n\rho}{1-\rho}\right)\right\}^{1/2}}$$

$$\times \frac{1}{\left\{\left(1+\frac{n\rho}{1-\rho}\right)^{-1} \frac{t^2}{n-1} + 1\right\}^{n/2}} \quad ,$$

which is symmetric and agrees with the Student's t-distribution on $(n-1)$ degrees of freedom when $\rho = 0$. Using several numerical results Ali demonstrated that if $\rho > 0$ $(\rho < 0)$ the level of the t-test for dependent data is higher (lower) than the level of independent data.

Babb(1982) examined the distribution of $F = T^2$ statistic when the observations are sampled from a stationary first order autoregressive Gaussian process. i.e.,$\{X_i\}$ have the relationship

$$X_i = \rho X_{i-1} + e_i, \qquad i = 0, \pm 1, \pm 2, \ldots$$

where $|\rho| < 1$ and e_i's are i.i.d. $N(0,1)$ random error terms. For a sample of size $n = 2$, Babb derived the exact probability density of F to be

$$\phi(F) = \frac{\sqrt{(1-\rho^2)}}{\pi[1+\rho+(1-\rho)F]\sqrt{F}}; \quad 0 < F.$$

He also showed that for large n, the probability density of F may be approximated by

$$g(F) = \sqrt{\frac{n}{n-1}} \frac{K_n \beta_1^{n/2}(1-\rho\beta_1)}{\sqrt{F}(1-\rho^2\beta_1)} \left\{1 + O(n^{-1})\right\}; \quad 0 < F$$

with

$$\beta_1 = \frac{(1+\rho^2) + F(1-\rho)^2/(n-1) - \sqrt{[(1+\rho^2) + F(1-\rho)^2/(n-1)]^2 - 4\rho^2}}{2\rho^2}$$

and

$$K_n = \frac{(1-\rho)\Gamma(\frac{n}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n-1}{2})} \left[F\left(\frac{3}{2}, \frac{n-1}{2}; \frac{n}{2}; \rho^2\right) - \frac{\rho(n-1)}{n} F\left(\frac{3}{2}, \frac{n+1}{2}; \frac{n+2}{2}; \rho^2\right) \right.$$
$$- \frac{\rho^2(n-1)(n+1)}{n(n+2)} F\left(\frac{3}{2}, \frac{n+3}{2}; \frac{n+4}{2}; \rho^2\right)$$
$$\left. + \frac{\rho^3(n-1)(n+2)(n+3)}{n(n+2)(n+4)} F\left(\frac{3}{2}, \frac{n+5}{2}; \frac{n+6}{2}; \rho^2\right) \right]^{-1}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. He also showed that $g(F)$ approaches to the F-distribution with 1 and $(n-1)$ degrees of freedom as ρ tends to zero.

A different type of violation of the independence assumption has been studied by Cressie(1980). He considered the observations $X_1, X_2, \ldots, X_n$ as a sequence of martingale differences which are not in general independent, but are uncorrelated. Then by using a result of Aldous and Eagleson(1978) he showed that as n tends to ∞ the T-statistic converges in distribution to the standard normal random variable. Thus asymptotically, the t-test is insensitive to the martingale type of dependence; problems arise when correlations within observations are introduced.

CHAPTER 3

AN ASYMPTOTIC APPROXIMATION FOR THE P.D.F. OF THE F-STATISTIC

We wish to find the probability density function of the statistic,

$$F = \frac{n\bar{d}^2}{s^2}$$

where * $\bar{d} = \frac{\sum d_i}{n}$, $s^2 = \frac{\sum (d_i - \bar{d})^2}{n-1}$, $d_i = \varepsilon_i - \eta_i$, $\varepsilon_i = \rho\varepsilon_{i-1} + e_i$ and $e_1, e_2, \ldots, e_n$, $\eta_1, \eta_2, \ldots, \eta_n$ are i.i.d. N(0,1) random variables.

We may write,

$$F = \frac{(n-1)r}{(n-r)} \quad ,$$

where

$$r = \frac{(\Sigma d_i)^2}{\Sigma d_i^2} \; .$$

If g and h denote the probability density functions of F and r respectively then we have the relationships

$$\text{(3.0.1)} \qquad g(F) = h(r)\left|\frac{dr}{dF}\right| = \frac{n(n-1)}{(F+n-1)^2} h\left(\frac{nF}{F+n-1}\right) \quad ; \quad F > 0$$

and

$$\text{(3.0.2)} \qquad h(r) = g(F)\left|\frac{dF}{dr}\right| = \frac{n(n-1)}{(n-r)^2} g\left(\frac{(n-1)r}{n-r}\right) \quad ; \quad 0 < r \le n.$$

In this chapter we shall find an asymptotic approximation for h(r) and transform it using (3.0.1).

---

* All summations in this chapter are from $i = 1$ to $i = n$ unless otherwise indicated.

Note that,

$$r = \frac{(\sum d_i)^2}{\sum d_i^2} = \frac{c}{c_0}$$

where $c = (\sum d_i)^2$ and $c_0 = \sum d_i^2$.

This is done by applying the Daniel's (1956) version of Cramer-Geary inversion formula (see Appendix II) which provides a technique for determining the p.d.f. of a ratio from the joint moment generating function of its numerator and the denominator.

3.1 The Moment Generating Function of the Joint Distribution of c_0 and c

Since we assume that the η_i's are i.i.d. N(0,1) random variables, the joint distribution $\eta_1, \eta_2, \ldots, \eta_n$ is given by,

(3.1.1) $$dF_1(\eta_1, \eta_2, \ldots, \eta_n) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\eta'\eta} d\eta_1, d\eta_2, \ldots, d\eta_n$$

where

$$\eta = (\eta_1, \eta_2, \ldots, \eta_n)'$$

Also $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is a sample of observations from the first order stationary linear autoregressive Gaussian process

$$\varepsilon_i = \rho\varepsilon_{i-1} + e_i$$

where the e_i's are i.i.d. N(0,1) random variables. Following the method given in Fuller (1976), it can be shown that,

$$Var(\varepsilon_i) = \frac{1}{1-\rho^2}$$

and

$$Cov(\varepsilon_{i-k}, \varepsilon_i) = \frac{\rho^{|k|}}{1-\rho^2}$$

for any integer k.Thus $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)'$ has a multivariate normal distribution with density

(3.1.2) $$dF_2 = (2\pi)^{-\frac{n}{2}} |\Sigma_\varepsilon|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(\varepsilon'\Sigma_\varepsilon^{-1}\varepsilon)\right\}$$

where

$$\Sigma_\varepsilon = \begin{pmatrix} \frac{1}{(1-\rho^2)} & \frac{\rho}{(1-\rho^2)} & \cdots & \frac{\rho^{n-1}}{(1-\rho^2)} \\ \frac{\rho}{(1-\rho^2)} & \frac{\rho^2}{(1-\rho^2)} & \cdots & \frac{\rho^{n-2}}{(1-\rho^2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho^{n-1}}{(1-\rho^2)} & \frac{\rho^{n-2}}{(1-\rho^2)} & \cdots & \frac{1}{(1-\rho^2)} \end{pmatrix}_{n\times n}$$

As the ε's are independent of the η's, we can combine (3.1.1) and (3.1.2) to obtain the joint distribution of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \eta_1, \eta_2, \ldots, \eta_n)$,

$$dF = (2\pi)^{-n}|\Sigma|^{-\frac{1}{2}} exp\{-\frac{1}{2}\theta'\Sigma^{-1}\theta\} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n d\eta_1 d\eta_2 \ldots d\eta_n \quad ,$$

where

$$\theta = (\varepsilon', \eta')'$$

and

$$\Sigma = \begin{pmatrix} \Sigma_\varepsilon & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix}_{2n\times 2n}$$

Here $\mathbf{I}_{n\times n}$ and $\mathbf{0}_{n\times n}$ are the n×n identity matrix and zero matrix respectively.

Also,

$$|\Sigma| = |\Sigma_\varepsilon| = \frac{1}{(1-\rho^2)} \quad ,$$

and

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_\varepsilon^{-1} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix}$$

Thus the joint distribution of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \eta_1, \eta_2, \ldots, \eta_n)$ is,

$$dF = (2\pi)^{-n} \left(\frac{1}{1-\rho^2}\right)^{1/2} exp\left\{-\frac{1}{2}\theta'\Sigma^{-1}\theta\right\} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n d\eta_1 d\eta_2 \ldots d\eta_n \quad ,$$

and the moment generating function of the joint distribution of c_0 and c is

$$(3.1.3)\quad M(T_0,T) = E[e^{c_0T_0+cT}]$$
$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} exp\left\{T_0c_0 + Tc - \frac{1}{2}\theta'\Sigma^{-1}\theta\right\}$$
$$d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n d\eta_1 d\eta_2 \ldots d\eta_n \quad ,$$

where c_0 and c are given by

$$c_0 = \sum d_i^2 = \sum(\varepsilon_i - \eta_i)^2$$
$$= \sum \varepsilon_i^2 - 2\sum \varepsilon_i\eta_i + \sum \eta_i^2 \quad ,$$

and

$$c = \left(\sum d_i\right)^2$$
$$= \left(\sum \varepsilon_i - \sum \eta_i\right)^2$$
$$= \left(\sum \varepsilon_i\right)^2 - 2\left(\sum \varepsilon_i\right)\left(\sum \eta_i\right) + \left(\sum \eta_i\right)^2$$

Taking $\mathbf{1}_{n\times n}$ to be the n×n matrix of which each element is 1, we can write,

$$\left(\sum \varepsilon_i\right)^2 = \theta'\begin{pmatrix}\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\end{pmatrix}\theta \quad ,$$

$$\left(\sum \eta_i\right)^2 = \theta\begin{pmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n}\end{pmatrix}\theta' \quad ,$$

$$2\left(\sum \varepsilon_i\right)\left(\sum \eta_i\right) = \theta'\begin{pmatrix}\mathbf{0}_{n\times n} & \mathbf{1}_{n\times n}\\ \mathbf{1}_{n\times n} & \mathbf{0}_{n\times n}\end{pmatrix}\theta$$

Thus,

$$c = \theta' \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \theta - \theta' \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \theta + \theta' \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \end{pmatrix} \theta$$

$$= \theta' \begin{pmatrix} \mathbf{1}_{n\times n} & -\mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \end{pmatrix} \theta \ .$$

Also,

$$\sum \varepsilon_i^2 = \theta' \begin{pmatrix} \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \theta \ ,$$

$$\sum \eta_i^2 = \theta' \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix} \theta \ ,$$

$$2\sum \varepsilon_i \eta_i = \theta' \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \theta \ ,$$

and hence,

$$c_0 = \theta' \begin{pmatrix} \mathbf{I}_{n\times n} & -\mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix} \theta \ .$$

Therefore,

$$T_0 c_0 + Tc - \frac{1}{2}\theta' \mathbf{\Sigma}^{-1} \theta = \frac{1}{2}\theta' \mathbf{H} \theta \ ,$$

where,

$$\mathbf{H} = -2T_0 \begin{pmatrix} \mathbf{I}_{n\times n} & -\mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix} - 2T \begin{pmatrix} \mathbf{1}_{n\times n} & -\mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{1}_{n\times n} \end{pmatrix} + \begin{pmatrix} \mathbf{\Sigma}_\varepsilon^{-1} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{pmatrix}$$

Substituting this result in (3.1.3) we get,

$$M(T_0, T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\left\{-\frac{1}{2}\theta'\mathbf{H}\theta\right\} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n d\eta_1 d\eta_2 \ldots d\eta_n \quad .$$

It follows from the theory of multivariate normal distribution that,

(3.1.4) $$M(T_0, T) = (1-\rho^2)^{1/2} |\mathbf{H}|^{-1/2}$$

since

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} exp\left\{-\frac{1}{2}\theta'\mathbf{H}\theta\right\} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n d\eta_1 d\eta_2 \ldots d\eta_n = |\mathbf{H}|^{-\frac{1}{2}} \quad .$$

3.2 An Asymptotic Approximation for $|\mathbf{H}|$

The $(2n \times 2n)$ matrix $\mathbf{H}$ can be partitioned into four $(n \times n)$ matrices as,

$$\mathbf{H} = \begin{pmatrix} \mathbf{R}_1 - 2T\mathbf{1}_{n\times n} & \mathbf{R}_3 + 2T\mathbf{1}_{n\times n} \\ \mathbf{R}_3 + 2T\mathbf{1}_{n\times n} & \mathbf{R}_2 - 2T\mathbf{1}_{n\times n} \end{pmatrix} ,$$

where

$$\mathbf{R}_1 = \begin{pmatrix} 1-2T_0 & -\rho & 0 & 0 & \dots & 0 & 0 \\ -\rho & x & -\rho & 0 & \dots & 0 & 0 \\ 0 & -\rho & x & -\rho & \dots & 0 & 0 \\ 0 & 0 & -\rho & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x & -\rho \\ 0 & 0 & 0 & 0 & \dots & -\rho & 1-2T_0 \end{pmatrix}_{n\times n} ,$$

$$\mathbf{R}_2 = (1-2T_0)\mathbf{I}_{n\times n} ,$$

$$\mathbf{R}_3 = 2T_0\mathbf{I}_{n\times n} ,$$

and

$$x = 1 + \rho^2 - 2T_0 .$$

Defining $\mathbf{1}_n$ to be the n-dimensional column vector with each element 1,

$$\text{i.e., } \mathbf{1}_n' = (1, 1, \dots, 1)_{1\times n} ,$$

the determinant $|\mathbf{H}|$ may be written as the $(2n+2)\times(2n+2)$ bordered determinant

$|\mathbf{C}_1|$,

$$|\mathbf{H}| = |\mathbf{C}_1| = \left|\begin{pmatrix} 1 & -2T\mathbf{1}'_n & 2T\mathbf{1}'_n & 0 \\ \mathbf{0}_n & \mathbf{R}_1 - 2T\mathbf{1}_{n\times n} & \mathbf{R}_3 + 2T\mathbf{1}_{n\times n} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{R}_3 + 2T\mathbf{1}_{n\times n} & \mathbf{R}_2 - 2T\mathbf{1}_{n\times n} & \mathbf{0}_n \\ 0 & 2T\mathbf{1}'_n & -2T\mathbf{1}'_n & 1 \end{pmatrix}\right| ,$$

where $\mathbf{0}_n$ is the n-dimensional column vector of which each element is zero.

Pre-multiplication of $\mathbf{C}_1$ by the $(2n+2)\times(2n+2)$ partitioned matrix,

$$\mathbf{I}_1 = \begin{pmatrix} 1 & \mathbf{0}'_n & \mathbf{0}'_n & 0 \\ -\mathbf{1}_n & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & -\mathbf{1}_n \\ 0 & \mathbf{0}'_n & \mathbf{0}'_n & 1 \end{pmatrix}$$

yields,

$$|\mathbf{C}_2| = |\mathbf{I}_1\mathbf{C}_1| = \begin{pmatrix} 1 & -2T\mathbf{1}'_n & 2T\mathbf{1}'_n & 0 \\ -\mathbf{1}_n & \mathbf{R}_1 & \mathbf{R}_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{R}_3 & \mathbf{R}_2 & -\mathbf{1}_n \\ 0 & 2T\mathbf{1}'_n & -2T\mathbf{1}'_n & 1 \end{pmatrix}$$

Since $|\mathbf{I}_1|=1$ we have,

$$|\mathbf{C}_2| = |\mathbf{I}_1\mathbf{C}_1| = |\mathbf{I}_1||\mathbf{C}_1| = |\mathbf{C}_1| = |\mathbf{H}| \quad .$$

Now define $\mathbf{I}_2$ to be the partitioned matrix,

$$\mathbf{I}_2 = \begin{pmatrix} 1 & \mathbf{0}'_n & \mathbf{0}'_n & 0 \\ \mathbf{0}_n & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_n \\ 1 & \mathbf{0}'_n & \mathbf{0}'_n & 1 \end{pmatrix}_{(2n+2)\times(2n+2)}$$

Then $|\mathbf{I}_2|=1$ and hence,

$$|\mathbf{C}_3| = |\mathbf{I}_2\mathbf{C}_2| = \begin{vmatrix} 1 & -2T\mathbf{1}_n' & 2T\mathbf{1}_n' & 0 \\ -\mathbf{1}_n & \mathbf{R}_1 & \mathbf{R}_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{R}_3 & \mathbf{R}_2 & -\mathbf{1}_n \\ 1 & \mathbf{0}_n' & \mathbf{0}_n' & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\delta & -\delta & -\delta & \cdots & -\delta & -\delta & \delta & \delta & \delta & \cdots & \delta & \delta & 0 \\ -1 & \alpha & -\rho & 0 & \cdots & 0 & 0 & \gamma & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -\rho & \beta & -\rho & \cdots & 0 & 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & -\rho & \beta & \cdots & 0 & 0 & 0 & 0 & \gamma & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \beta & -\rho & 0 & 0 & 0 & \cdots & \gamma & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & -\rho & \alpha & 0 & 0 & 0 & \cdots & 0 & \gamma & 0 \\ 0 & \gamma & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \gamma & \cdots & 0 & 0 & 0 & 0 & \alpha & \cdots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma & 0 & 0 & 0 & 0 & \cdots & \alpha & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma & 0 & 0 & 0 & \cdots & 0 & \alpha & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix} ,$$

where

$$\alpha = 1 - 2T_0 \ ,$$

$$\beta = 1 + \rho^2 - 2T_0 \ ,$$

$$\gamma = 2T_0 \ ,$$

and

$$\delta = 2T \ .$$

To produce more zero elements in the first row, define λ_1 and λ_2 to satisfy the two simultaneous equations (see Routledge(1972))

$$-2T + \lambda_1(1 + \rho^2 - 2T_0 - 2\rho) + 2T_0\lambda_2 = 0 \ ,$$

and

$$2T + \lambda_2(1 - 2T_0 - 2\rho) + 2T_0\lambda_1 = 0 \quad .$$

Then

(3.2.1) $$\lambda_1 = \frac{-2T}{4T_0^2 - (1 - 2T_0)[(1-\rho)^2 - 2T_0]} \quad ,$$

and

(3.2.2) $$\lambda_2 = \frac{2T(1-\rho)^2}{4T_0^2 - (1 - 2T_0)[(1-\rho)^2 - 2T_0]} \quad .$$

Now define $\mathbf{I}_3$ to be the $(2n+2)\times(2n+2)$ partitioned matrix of the form:

$$\mathbf{I}_3 = \begin{pmatrix} 1 & \lambda_1\mathbf{1}'_n & \lambda_2\mathbf{1}'_n & 0 \\ \mathbf{0}_n & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_n \\ 0 & \mathbf{0}'_n & \mathbf{0}'_n & 1 \end{pmatrix} .$$

Since $|\mathbf{I}_3|=1$, premultiplying $\mathbf{C}_3$ by $\mathbf{I}_3$ yields,

$$|\mathbf{H}| = |\mathbf{C}_3| = |\mathbf{I}_3\mathbf{C}_3| = \left| \begin{pmatrix} y_1 & \mathbf{y} & \mathbf{0}'_n & y_3 \\ -\mathbf{1}_n & \mathbf{R}_1 & \mathbf{R}_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{R}_3 & \mathbf{R}_2 & -\mathbf{1}_n \\ 1 & \mathbf{0}'_n & \mathbf{0}'_n & 1 \end{pmatrix} \right| ,$$

where,

$$\mathbf{y} = (y_2, 0, 0, \ldots, 0, y_2),$$

$$y_1 = 1 - n\lambda_1 \quad ,$$

$$y_2 = -2T + (1 - 2T_0 - \rho)\lambda_1 + 2T_0\lambda_2 = \rho(1-\rho)\lambda_1 \quad ,$$

$$y_3 = -n\lambda_2 \quad .$$

Since $|\rho| < 1$ and considering the order of y_2 relative to y_1 and y_3, we can ignore y_2 for large n, and hence,

(3.2.3) $$|\mathbf{H}| \sim [1 + n(\lambda_2 - \lambda_1)]|\mathbf{A}| \quad ,$$

where,

$$\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_3 \\ \\ \mathbf{R}_3 & \mathbf{R}_2 \end{pmatrix} .$$

It follows from (3.2.1) and (3.2.2) that,

$$\lambda_2 - \lambda_1 = \frac{2T[1 + (1 - \rho)^2]}{4T_0^2 - (1 - 2T_0)[(1 - \rho)^2 - 2T_0]}$$

$$= \frac{2T[1 + (1 - \rho)^2]}{2T_0 - (1 - 2T_0)(1 - \rho)^2} .$$

Substituting th : (I.6) from Appendix I in (3.2.3) gives,

(3.2.4) $$|\mathbf{H}| \sim \left\{1 + \frac{2nT[1 + (1 - \rho)^2]}{2T_0 - (1 - \rho)^2(1 - 2T_0)}\right\} \times \frac{\rho^n(1 - 2T_0)^n[z^2(a - z)^2 - z^{2n-2}(1 - az)^2]}{(1 - z^2)z^n} ,$$

where,

$$a = \frac{1 - 4T_0}{\rho(1 - 2T_0)} ,$$

$$b = a + \rho \quad ,$$

and

$$z = \frac{b + \sqrt{b^2 - 4}}{2}$$

is the root of $r^2 - br + 1 = 0$ with $|z| < 1$.

3.3 An Asymptotic Approximation for the P.D.F. of r as an Integral

As obtained at the end of the last section z and z^{-1} are the roots of

$$r^2 - br + 1 = 0.$$

Therefore,

(3.3.1)
$$\begin{aligned} z + \frac{1}{z} &= b = \rho + a \\ &= \rho + \frac{1 - 4T_0}{\rho(1 - 2T_0)} \\ &= \rho + \frac{2}{\rho} - \frac{1}{\rho(1 - 2T_0)} \end{aligned}$$

Thus,

$$\frac{1}{\rho(1 - 2T_0)} = \frac{\rho^2 z + 2z - \rho z^2 - \rho}{\rho z},$$

and

(3.3.2)
$$(1 - 2T_0) = \frac{z}{\rho^2 z - \rho z^2 + 2z - \rho}.$$

Also,

$$\begin{aligned} 2T_0 - (1-\rho)^2(1 - 2T_0) &= 1 - (1 - 2T_0) - (1-\rho)^2(1 - 2T_0) \\ &= 1 - [1 + (1-\rho)^2](1 - 2T_0). \end{aligned}$$

Substituting (3.3.2) for $(1 - 2T_0)$ yields,

(3.3.3)
$$2T_0 - (1-\rho)^2(1 - 2T_0) = \frac{-\rho(1 - z)^2}{\rho^2 z - \rho z^2 + 2z - \rho}.$$

Setting $T_0 = u - rT$ in (3.3.2) we get,

$$\begin{aligned} 1 - 2u + 2rT &= \frac{z}{\rho^2 z - \rho z^2 + 2z - \rho} \quad , \\ 2rT &= 2u + \frac{-\rho^2 z + \rho z^2 - z + \rho}{\rho^2 z - \rho z^2 + 2z - \rho} \\ &= 2u + \frac{(\rho - z)(1 - \rho z)}{\rho^2 z - \rho z^2 + 2z - \rho} \end{aligned}$$

Then

$$2T = \frac{2u}{r} + \frac{(\rho - z)(1 - \rho z)}{r(\rho^2 z - \rho z^2 + 2z - \rho)} \tag{3.3.4}$$

Equations (3.3.3) and (3.3.4) can be combined to obtain

$$\begin{aligned} &\frac{2nT[1 + (1-\rho)^2]}{2T_0 - (1-\rho)^2(1 - 2T_0)} \\ &= -n[1 + (1-\rho)^2] \left\{ \frac{2u}{r} + \frac{(\rho - z)(1 - \rho z)}{r(\rho^2 z - \rho z^2 + 2z - \rho)} \right\} \frac{(\rho^2 z - \rho z^2 + 2z - \rho)}{\rho(1-z)^2} \\ &= \frac{-n[1 + (1-\rho)^2]}{r\rho(1-z)^2} \left\{ 2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1 - \rho z) \right\} \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} &1 + \frac{2nT[1 + (1-\rho)^2]}{2T_0 - (1-\rho)^2(1 - 2T_0)} \\ &= \frac{r\rho(1-z)^2 - n[1 + (1-\rho)^2][2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1 - \rho z)]}{r\rho(1-z)^2} \quad . \end{aligned} \tag{3.3.5}$$

From (3.3.1) we can derive,

$$z(a - z) = 1 - \rho z \quad , \tag{3.3.6}$$

and

$$1 - az = z(\rho - z) \tag{3.3.7}$$

Substituting (3.3.2), (3.3.5) through (3.3.7) in (3.2.4) yields,

$$|\mathbf{H}| \sim \frac{r\rho(1-z)^2 - n[1+(1-\rho)^2][2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1-\rho z)]}{r\rho(1-z)^2}$$

$$\times \frac{\rho^n z^n[(1-\rho z)^2 - z^{2n}(\rho - z)^2]}{(\rho^2 z - \rho z^2 + 2z - \rho)^n (1 - z^2) z^n}$$

$$= \{r\rho(1-z)^2 - n[1+(1-\rho)^2][2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1-\rho z)]\}$$

$$\times \frac{\rho^{n-1}[(1-\rho z)^2 - z^{2n}(\rho - z)^2]}{r(\rho^2 z - \rho z^2 + 2z - \rho)^n (1-z)^2 (1-z^2)}$$

Since $|z| < 1$, we can ignore the z^{2n} terms, and hence

$$|\mathbf{H}| \sim \{r\rho(1-z)^2 - n[1+(1-\rho)^2][2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1-\rho z)]\}$$

$$\times \frac{\rho^{n-1}(1-\rho z)^2}{r(\rho^2 z - \rho z^2 + 2z - \rho)^n (1-z)^2 (1-z^2)}$$

Substituting this value in (3.1.4) we get,

$$M(u - rT \qquad \frac{r^{\frac{1}{2}}(\rho^2 z - \rho z^2 + 2z - \rho)^{\frac{n}{2}}(1-z)(1-z^2)^{\frac{1}{2}}}{\rho^{\frac{n-1}{2}}\psi(u,z)(1-\rho z)} \tag{3.3.8}$$

where

$$\psi(u,z) = \{r\rho(1-z)^2 - n[1+(1-\rho)^2][2u(\rho^2 z - \rho z^2 + 2z - \rho) + (\rho - z)(1-\rho z)]\}^{\frac{1}{2}}$$

Differentiating (3.3.4) with respect to z yields,

$$\frac{\partial T}{\partial z} = \frac{-\rho(1-z^2)}{2r(\rho^2 z - \rho z^2 + 2z - \rho)^2} ,$$

and with (3.3.8) we can write

$$M(u - rT, T)\frac{\partial T}{\partial z} \sim \frac{-(1-\rho^2)^{\frac{1}{2}}(\rho^2 z - \rho z^2 + 2z - \rho)^{\frac{n}{2}-2}(1-z)(1-z^2)^{\frac{3}{2}}}{2r^{\frac{1}{2}}\rho^{\frac{n-3}{2}}\psi(u,z)(1-\rho z)}$$

Therefore,

$$\frac{\partial}{\partial u}\left[M(u - rT, T)\tfrac{\partial T}{\partial z}\right]\Big|_{u=0}$$

$$\sim \frac{-n(1-\rho^2)^{\frac{1}{2}}[1+(1-\rho)^2](\rho^2 z - \rho z^2 + 2z - \rho)^{\frac{n-2}{2}}(1-z)(1-z^2)^{\frac{3}{2}}}{2r^{\frac{1}{2}}\rho^{\frac{n-3}{2}}\left\{r\rho(1-z)^2 - n[1+(1-\rho)^2](\rho-z)(1-\rho z)]\right\}^{\frac{3}{2}}(1-\rho z)}$$

Apply this result in the Cramer-Geary inversion formula, equation II.4 in Appendix II and we get an asymptotic approximation for $h(r)$ given by,

$$\text{(3.3.9)} \qquad h(r) \sim \frac{-n[1+(1-\rho)^2]}{4\pi i}\left\{\frac{1-\rho^2}{r\rho^{n-3}}\right\}^{\frac{1}{2}}\int_{\Gamma_z}\Phi(z)\,dz ,$$

where

$$\text{(3.3.10)} \qquad \Phi(z) = \frac{(\rho^2 z - \rho z^2 + 2z - \rho)^{\frac{n-2}{2}}(1-z)(1-z^2)^{\frac{3}{2}}}{\{r\rho(1-z)^2 - n[1+(1-\rho)^2](\rho-z)(1-\rho z)\}^{\frac{3}{2}}(1-\rho z)} ,$$

and Γ_z is the path of the integration which will be analyzed in the next section.

3.4 The Path of Integration

The equation (3.3.1) can be written as

$$z + \frac{1}{z} = \rho + \frac{2(1-2T_0)-1}{\rho(1-2T_0)}$$
$$= \frac{\rho^2+2}{\rho} - \frac{1}{\rho(1-2T_0)}$$

Setting $T_0 = u - rT$ with $u = 0$ we have,

(3.4.1)
$$z + \frac{1}{z} = \frac{\rho^2+2}{\rho} - \frac{1}{\rho(1+2rT)}$$
$$= \frac{\rho^2+2}{\rho} - \frac{1}{v} \qquad (v = \rho(1+2rT)) \quad ,$$
$$= \frac{\rho^2+2}{\rho} + 2S \qquad (S = -\frac{1}{2v}) \quad ,$$
$$= w \qquad (w = \frac{\rho^2+2}{\rho} + 2S) \quad .$$

Therefore (3.4.1) consists of the elementary mappings:

$$A: \quad v = \rho(1+2rT) \quad ,$$
$$B: \quad S = -\frac{1}{2v} \quad ,$$
$$C: \quad w = \frac{2+\rho^2}{\rho} + 2S \quad ,$$
$$D: \quad z + \frac{1}{z} = w \quad .$$

A: maps the T-plane cut along the interval from

$$\frac{-(1+\rho)^2}{2r[1+(1+\rho)^2]} \quad \text{to} \quad \frac{-(1-\rho)^2}{2r[1+(1-\rho)^2]}$$

onto the v-plane cut along the interval from

$$\frac{\rho}{1+(1+\rho)^2} \quad \text{to} \quad \frac{\rho}{1+(1-\rho)^2}$$

B: maps the v-plane cut along the interval from

$$\frac{\rho}{1+(1+\rho)^2} \quad \text{to} \quad \frac{\rho}{1+(1-\rho)^2}$$

onto the S-plane cut along the interval from

$$\frac{-[1+(1+\rho)^2]}{2\rho} \quad \text{to} \quad \frac{-[1+(1-\rho)^2]}{2\rho}$$

C: maps the S-plane cut along the real interval from

$$\frac{-[1+(1+\rho)^2]}{2\rho} \quad \text{to} \quad \frac{-[1+(1-\rho)^2]}{2\rho}$$

onto the w-plane cut along the real interval from -2 to 2.

D: is the Joukowski Transformation which is discussed in Babb (1982). By his argument it is shown that D maps the w-plane cut along the real interval from -2 to 2 into the interior of the unit circle $|z| = 1$.

Our particular interest is Γ_z, the path of integration in the z-plane, which is the transformed path of the imaginary axis of T-plane transformed by (3.4.1). From equation (3.4.1) we can write,

(3.4.2) $$T = \frac{(\rho - z)(1-\rho z)}{2r(\rho^2 z - \rho z^2 + 2z - \rho)}$$

Setting $z = x + iy$ and solving the equation

$$Re(T) = 0 \quad ,$$

it can be shown that the point $z = (x + iy)$ on the path satisfy the conditions

(i) $\rho^2 y^4 + By^2 + C = 0$;
where $B = 2\rho^2 x^2 - (2\rho^3 + 3\rho)x + \rho^4 + \rho^2 + 2$,

$$C = [\rho(x^2 + 1) - (\rho^2 + 2)x][\rho(x^2 + 1) - (\rho^2 + 1)x] \quad ,$$

(ii) $|x| < 1$,

(iii) x,y are real.

Therefore, the path is symmetric about the x-axis. To obtain the end points of the path of integration in the z-plane, note that from (3.4.1),

$$z + \frac{1}{z} \to \frac{2 + \rho^2}{\rho} \quad \text{as} \quad T \to \pm i\infty.$$

But, if

$$z + \frac{1}{z} = \frac{2 + \rho^2}{\rho}$$

then

$$\rho z^2 - (2 + \rho^2)z + \rho = 0 \quad ,$$

and

$$z = \frac{(2 + \rho^2) \pm \sqrt{(2 + \rho^2)^2 - 4\rho^2}}{2\rho}$$

$$= \frac{(2 + \rho^2) \pm \sqrt{4 + \rho^4}}{2\rho} \quad .$$

The solution in the interior of $|z| = 1$ is

$$z = \frac{(2 + \rho^2) - \sqrt{4 + \rho^4}}{2\rho} \quad .$$

Also from (3.4.1) we have

$$z + \frac{1}{z} = \rho + \frac{1}{\rho} \quad \text{if } T=0,$$

and the solution inside the unit circle is $z = \rho$.

Thus for positive (negative) ρ, as T traverses the imaginary axis from $-i\infty$ to 0, z moves clockwise along a path in quadrant i (iii) from $(\beta_0, 0)$ where

$$\beta_0 = \frac{2 + \rho^2 - \sqrt{4 + \rho^4}}{2\rho}$$

to $(\rho, 0)$, and as T goes along the imaginary axis from 0 to $+i\infty$, z travels clockwise in quadrant iv (ii) from $(\rho, 0)$ to $(\beta_0, 0)$.

Consider the equation,

$$\text{(3.4.3)} \qquad r\rho(1-z)^2 - n[1 + (1-\rho)^2](\rho - z)(1 - \rho z) = 0$$

which is a quadratic equation of the form

$$Az^2 - Bz + A = 0$$

where

$$A = n\rho[1 + (1-\rho)^2] - r\rho$$

$$B = n[1 + (1-\rho)^2](1 + \rho^2) - 2r\rho$$

Therefore (3.4.3) has two roots β and β^{-1} where

$$\beta = \frac{B - \sqrt{B^2 - 4A^2}}{2A}$$

If $r = 0$ then,

$$A = n\rho[1 + (1 - \rho)^2] \quad ,$$

$$B = n[1 + (1 - \rho)^2](1 + \rho^2) \quad ,$$

$$B^2 - 4A^2 = n^2[1 + (1 - \rho)^2]^2[(1 + \rho^2)^2 - 4\rho^2] \quad ,$$

$$= n^2[1 + (1 - \rho)^2]^2(1 - \rho^2)^2 \quad ,$$

$$B - \sqrt{B^2 - 4A^2} = n[1 + (1 - \rho)^2][(1 + \rho^2) - (1 - \rho^2)] \quad ,$$

$$= 2n[1 + (1 - \rho)^2]\rho^2 = 2\rho A$$

and hence

$$\beta = \rho \quad .$$

If $r = n$ then,

$$A = n\rho(1 - \rho)^2 \quad ,$$

$$B = n[1 + (1 - \rho)^2](1 + \rho^2) - 2n\rho$$

$$= n(1 - \rho)^2(2 + \rho^2) \quad ,$$

$$B^2 - 4A^2 = n^2(1 - \rho)^4[(2 + \rho^2)^2 - 4\rho^2]$$

$$= n^2(1 - \rho)^4(4 + \rho^4)$$

and

$$\beta = \frac{B - \sqrt{B^2 - 4A^2}}{2A} = \frac{n(1 - \rho)^2[2 + \rho^2 - \sqrt{4 + \rho^4}]}{2n\rho(1 - \rho)^2}$$

$$= \frac{2 + \rho^2 - \sqrt{4 + \rho^4}}{2\rho} = \beta_0 \quad .$$

Thus if $r = 0$, $\beta = \rho$ while if $r = n$, $\beta = \beta_0$.

Now consider,

$$\frac{d\beta}{dr} = \frac{1}{2A}\left\{\frac{dB}{dr} - \frac{[2B\frac{dB}{dr} - 8A\frac{dA}{dr}]}{2\sqrt{B^2 - 4A^2}}\right\} - \frac{[B - \sqrt{B^2 - 4A^2}]}{2A^2}\frac{dA}{dr}$$

Since $\frac{dB}{dr} = -2\rho$ and $\frac{dA}{dr} = -\rho$ we have,

$$\frac{d\beta}{dr} = \frac{1}{2A}\left\{-2\rho - \frac{[-4\rho B + 8\rho A]}{2\sqrt{B^2 - 4A^2}}\right\} + \frac{2\rho[B - \sqrt{B^2 - 4A^2}]}{4A^2}$$
$$= \frac{-\rho}{A}\left\{1 + \frac{(2A - B)}{\sqrt{B^2 - 4A^2}}\right\} + \frac{\rho}{A}\frac{[B - \sqrt{B^2 - 4A^2}]}{2A}$$
$$= \frac{-\rho}{A}\left\{1 + \frac{(2A - B)}{\sqrt{B^2 - 4A^2}} - \frac{[B - \sqrt{B^2 - 4A^2}]}{2A}\right\}$$
$$= \frac{-\rho(B - 2A)[B - \sqrt{B^2 - 4A^2}]}{2A^2\sqrt{B^2 - 4A^2}}$$

$$(B - 2A) = n[1 + (1 - \rho)^2](1 + \rho^2) - 2r\rho - 2n\rho[1 + (1 - \rho)^2] + 2r\rho$$
$$= n[1 + (1 - \rho)^2][1 + \rho^2 - 2\rho]$$
$$= n[1 + (1 - \rho)^2](1 - \rho)^2$$

Therefore,

$$\frac{d\beta}{dr} = \frac{-\rho n[1 + (1 - \rho)^2](1 - \rho)^2[B - \sqrt{B^2 - 4A^2}]}{2A^2\sqrt{B^2 - 4A^2}} \quad \begin{cases} < 0, & \text{if } \rho > 0; \\ > 0, & \text{if } \rho < 0. \end{cases}$$

Thus for $|\rho| < 1$, β is a strictly decreasing ($\rho > 0$) or a strictly increasing ($\rho < 0$) function of r ($0 \le r \le n$) and hence,

(3.4.4) $$min(\rho, \beta_0) \le \beta \le max(\rho, \beta_0)$$

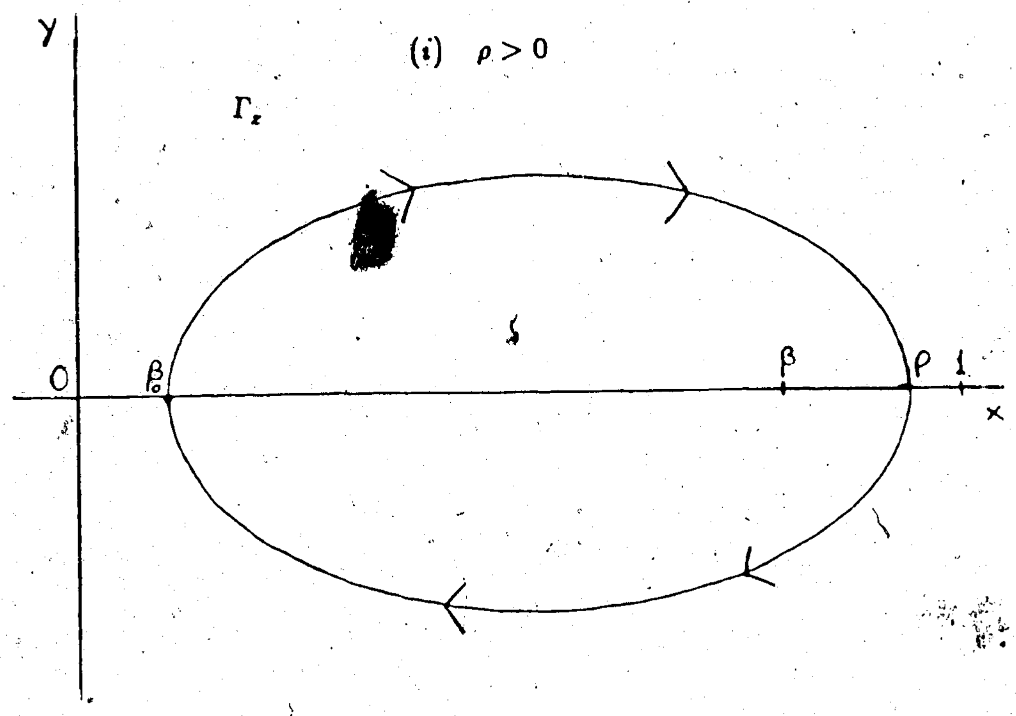
so that β (a branch point singularity) lies inside the contour of integration.

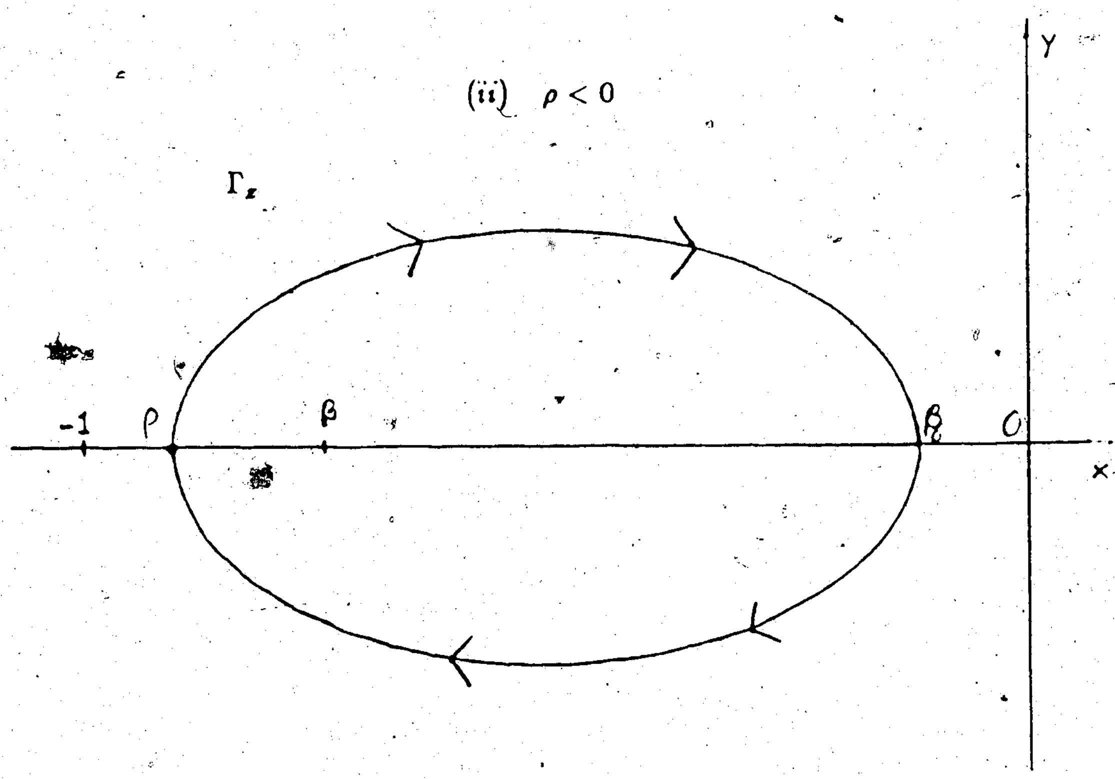
Thus we can write,

(3.4.5) $$r\rho(1 - z)^2 - n[1 + (1 - \rho)^2](\rho - z)(1 - \rho z)$$

$$= \{n\rho[1 + (1 - \rho)^2] - r\rho\}(z - \beta)(1 - \beta z)/\beta$$

Fig 3.4.1 The Path of Integration on the z-plane





Also $\rho^2 z - \rho z^2 + 2z - \rho$ can be factorized as

$$\rho^2 z - \rho z^2 + 2z - \rho = \rho(z - \beta_0)(1 - z\beta_0)/\beta_0 \tag{3.4.6}$$

and hence $\Phi(z)$ can be written as,

$$\Phi(z) = \frac{\rho^{\frac{n-2}{2}}(1-z)(1-z^2)^{\frac{3}{2}}\{(z-\beta_0)(1-\beta_0 z)\}^{\frac{n-2}{2}}\beta^{\frac{3}{2}}}{\{[n\rho[1+(1-\rho)^2]-r\rho](z-\beta)(1-\beta z)\}^{\frac{3}{2}}(1-\rho z)\beta_0^{\frac{n-2}{2}}},$$

which has branch points at $z = -1,\ 1,\ \beta,\ \frac{1}{\beta}$ and, if n is odd, at $z = \beta_0$ and $z = \frac{1}{\beta_0}$.

Let,

$$(z - \beta) = R(\theta, \beta)e^{i(\theta + 2k\pi)} \quad ,$$

where $R(\theta,\beta)$ is the length of the vector with initial point at $z = \beta$, making an angle θ with the positive direction of the real axis and terminating on the contour of integration. Then we have

$$(z-\beta)^{\frac{-3}{2}} = [R(\theta,\beta)]^{\frac{-3}{2}} e^{i(\theta+2k\pi)} .$$

which gives two solutions,

$$(z-\beta)^{-\frac{3}{2}} = [R(\theta,\beta)]^{-\frac{3}{2}} e^{-(\frac{3\theta}{2})i} \qquad \text{(Principle determination)},$$

and

$$(z-\beta)^{-\frac{3}{2}} = [R(\theta,\beta)]^{-\frac{3}{2}} e^{-(\frac{3\theta}{2}+3\pi)i} \qquad \text{(Secondary determination)}.$$

To make $(z-\beta)^{-\frac{3}{2}}$ single valued we take it to be the principle determination and cut the z-plane (assuming $\rho > 0$) from $-\infty$ to β so that we are unable to cross the cut and reach the secondary determination.

Similarly to ensure that $(z-\beta_0)^{\frac{n-2}{2}}$ is single valued we take the principle determination

$$(z-\beta_0)^{\frac{n-2}{2}} = [R(\theta,\beta_0)]^{\frac{n-2}{2}} e^{\frac{(n-2)\theta}{2}i} \ .$$

Note that in cutting the z-plane from $-\infty$ to β , it is automatically cut from $-\infty$ to β_0, as required. Also to make $(1+z)^{\frac{3}{2}}$ single valued we take the principle determination

$$(1+z)^{\frac{3}{2}} = [R(\theta,-1)]^{\frac{3}{2}} e^{\frac{3\theta}{2}} \quad ,$$

and the z-plane is already cut from $-\infty$ to -1.

Finally we wish to be able to cross the X-axis between β and ρ. Since $0 < \beta_0 < \beta < \rho < 1$, we cut the z-plane from 1 to ∞ and take the secondary determinations of $(z-1)^{\frac{3}{2}}$, $(1-\beta z)^{-\frac{3}{2}}$ and $(1-\beta_0 z)^{\frac{n-2}{2}}$.

Taking

$$(1-\beta_0 z)^{\frac{n-2}{2}} = \left[R(\theta,\frac{1}{\beta_0})\right]^{\frac{n-2}{2}} e^{i[\frac{n-2}{2}(\theta+2\pi)]}$$

$$(1-\beta z)^{\frac{-3}{2}} = \left[R(\theta,\frac{1}{\beta})\right]^{-\frac{3}{2}} e^{i[\frac{3\theta}{2}+3\pi]}$$

and

$$(1-z)^{\frac{3}{2}} = [R(\theta,1)]^{\frac{3}{2}} e^{i[\frac{3\theta}{2}+3\pi]}$$

we make $(1-\beta_0 z)^{\frac{n-2}{2}}$, $(1-\beta z)^{-\frac{3}{2}}$ and $(1-z)^{\frac{3}{2}}$ single valued and avoid the singularity at $z=\frac{1}{\rho}$.

3.5 An Asymptotic Evaluation for the Integral

The equation (3.3.9) can be written as,

$$h(r) = \frac{n[1+(1-\rho)^2](1-\rho^2)^{\frac{1}{2}}}{4\pi r^{\frac{1}{2}}\rho^{\frac{n-3}{2}}}\left(\frac{\beta}{A}\right)^{\frac{3}{2}} I \quad , \tag{3.5.1}$$

where

$$I = \int_{\Gamma_z} \frac{f(z)}{(z-\beta)^{\frac{3}{2}}}dz \quad , \tag{3.5.2}$$

$$A = n\rho[1+(1-\rho)^2] - r\rho \quad ,$$

and

$$f(z) = \frac{(1-z)(1-z^2)^{\frac{3}{2}}(\rho^2 z - \rho z^2 + 2z - \rho)^{\frac{n-2}{2}}}{(1-\rho z)(\beta z - 1)^{\frac{3}{2}}} \quad . \tag{3.5.3}$$

Since we wish to apply the integration formula developed in Babb (1982) to evaluate I, it is required that the path of integration should go through the origin. i.e. $\beta_0 = 0$. Therefore we make the linear transformation,

$$v = \frac{2\rho}{\sqrt{4+\rho^4}}(z-\beta_0) \tag{3.5.4}$$

i.e.

$$v = \frac{\sqrt{4+\rho^4} - (2+\rho^2-2\rho z)}{\sqrt{4+\rho^4}} \quad .$$

Then

$$2 - v = \frac{\sqrt{4+\rho^4} + (2+\rho^2-2\rho z)}{\sqrt{4+\rho^4}} \quad ,$$

and

$$v(2-v) = \frac{(4+\rho^4) - (2+\rho^2-2\rho z)^2}{(4+\rho^4)}$$

$$= \frac{4\rho}{(4+\rho^4)}(\rho^2 z - \rho z^2 + 2z - \rho) \quad ,$$

or

$$\rho^2 z - \rho z^2 + 2z - \rho = \frac{4+\rho^4}{4\rho} v(2-v) \quad .$$

Also from (3.5.4) we obtain,

$$z = \beta_0 + bv \quad ,$$

where, as defined before,

$$\beta_0 = \frac{2+\rho^2-\sqrt{4+\rho^4}}{2\rho} \quad ,$$

and where

$$b = \frac{\sqrt{4+\rho^4}}{2\rho} \quad .$$

Then

$$dz = bdv \quad ,$$

and

$$f(z) = l(v)\{\rho b^2 v(2-v)\}^{\frac{n-2}{2}} \quad ;$$

where

$$l(v) = \frac{(1-\beta_0-bv)[1-(\beta_0+bv)^2]^{\frac{3}{2}}}{(1-\rho\beta_0-\rho bv)(\beta bv+\beta\beta_0-1)^{\frac{3}{2}}} \quad .$$

Also,

$$z - \beta = b(v-\beta_2) \quad ,$$

where

$$\beta_2 = \frac{\beta-\beta_0}{b} \quad .$$

Thus

$$I = \int_{\Gamma_z} \frac{f(z)}{(z-\beta)^{\frac{3}{2}}} dz$$
$$= \int_{\Gamma_v} \frac{l(v) v^{\frac{n-2}{2}} (2-v)^{\frac{n-2}{2}} \rho^{\frac{n-2}{2}} t^{n-2}}{b^{\frac{3}{2}} (v-\beta_2)^{\frac{3}{2}}} b\, dv$$

or

(3.5.5) $$I = b^{n-\frac{5}{2}} \rho^{\frac{n-2}{2}} I_1 \quad ,$$

where

$$I_1 = \int_{\Gamma_v} \frac{l(v) v^{\frac{n-2}{2}} (2-v)^{\frac{n-2}{2}}}{(v-\beta_2)^{\frac{3}{2}}} dv$$

and Γ_v is the image in the v-plane of Γ_z in the z-plane transformed by the linear transformation (3.5.4) and is illustrated in Figure 3.5.1.

I_1 can be written as,

$$I_1 = \int_{\Gamma_v} \frac{g(v)}{(v-\beta_2)^{\frac{3}{2}}} dv \ ,$$

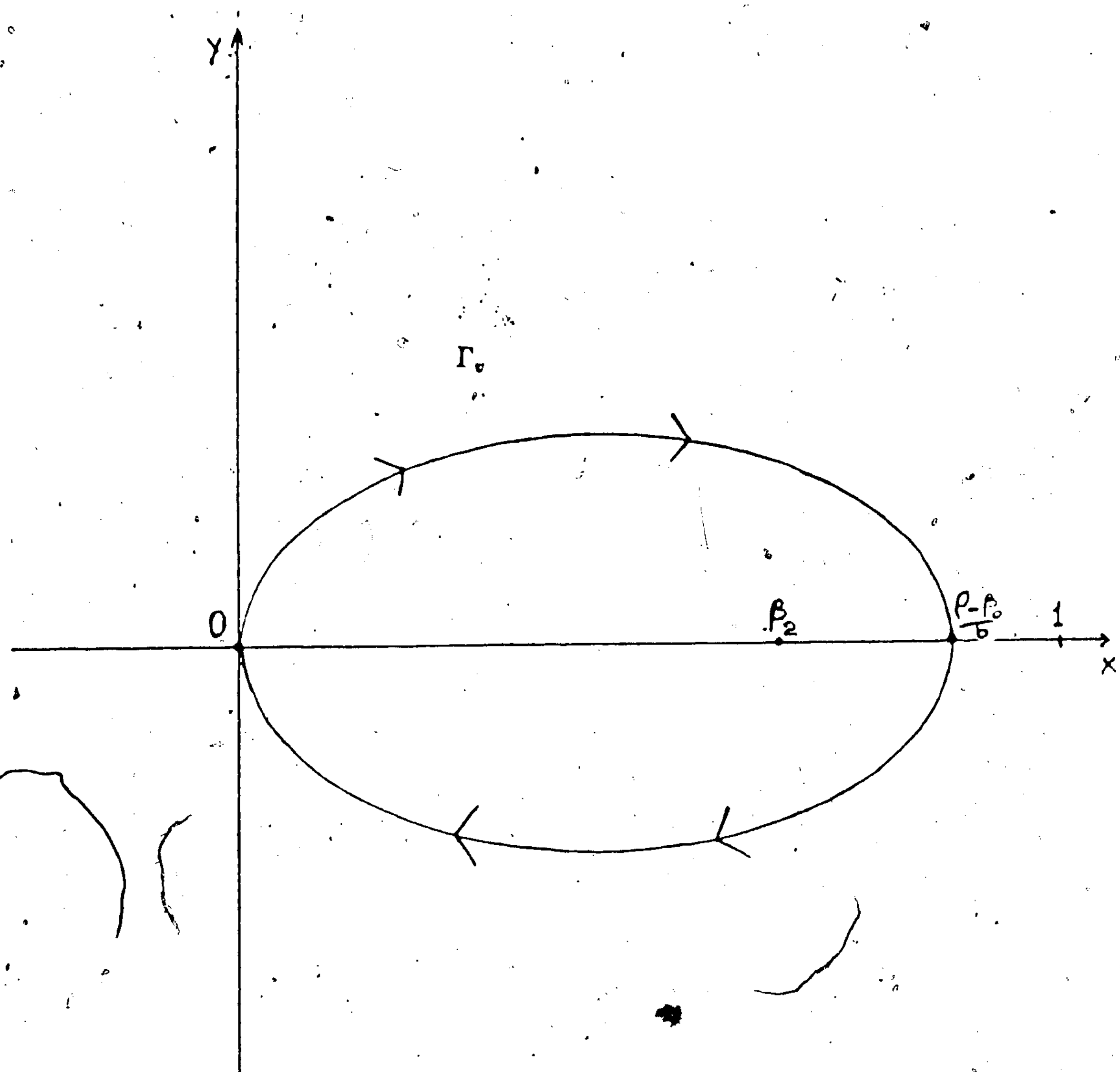
where

$$g(v) = l(v) v^{\frac{n-2}{2}} (2-v)^{\frac{n-2}{2}} \ .$$

Now applying the following integration formula developed by Babb (1982):

$$\int_{\Gamma_v} \frac{g(v)}{(v-\beta_2)^{\frac{3}{2}}} dv = \frac{-4ig(\beta_2)}{\sqrt{\beta_2}} + \frac{2i}{\beta_2} \int_0^1 y^{-\frac{3}{2}} [g(\beta_2 - \beta_2 y) - g(\beta_2)] dy$$

Fig 3.5.1 The path of Integration on the v-plane



we can write,

$$I_1 = \frac{-4il(\beta_2)}{\sqrt{\beta_2}}\beta_2^{\frac{n-2}{2}}(2-\beta_2)^{\frac{n-2}{2}} + \frac{2i}{\sqrt{\beta_2}}\int_0^1 y^{-3/2}[l(\beta_2-\beta_2 y)\beta_2^{\frac{n-2}{2}}(1-y)^{\frac{n-2}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}} - l(\beta_2)\beta_2^{\frac{n-2}{2}}(2-\beta_2)^{\frac{n-2}{2}}]dy$$

$$= \frac{-4il(\beta_2)}{\sqrt{\beta_2}}\beta_2^{\frac{n-2}{2}}(2-\beta_2)^{\frac{n-2}{2}} + 2i\beta_2^{\frac{n-3}{2}}\int_0^1 y^{-\frac{3}{2}}\left[l(\beta_2-\beta_2 y)(1-y)^{\frac{n-2}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}} - l(\beta_2)(2-\beta_2)^{\frac{n-2}{2}}\right]dy,$$

i.e.

$$I_1 = -4il(\beta_2)\beta_2^{\frac{n-3}{2}}(2-\beta_2)^{\frac{n-2}{2}} + 2i\beta_2^{\frac{n-3}{2}} I_2 \quad , \tag{3.5.6}$$

where

$$I_2 = \int_0^1 y^{-\frac{3}{2}}\left[l(\beta_2-\beta_2 y)(1-y)^{\frac{n-2}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}} - l(\beta_2)(2-\beta_2)^{\frac{n-2}{2}}\right]dy,$$

$$= \int_0^1 y^{-\frac{3}{2}}\left[\lambda(y)(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}}(1-y)^{\frac{n-2}{2}} - \lambda(0)(2-\beta_2)^{\frac{n-2}{2}}\right]dy,$$

and where $\lambda(y) = l(\beta_2-\beta_2 y)$ and hence $\lambda(0) = l(\beta_2)$. Expanding $\lambda(y)$ in Maclaurin's series we have,

$$\lambda(y) = \lambda(0) + \sum_{k=1}^{\infty}\frac{\lambda^k(0)}{k!}y^k$$

and hence,

$$I_2 = \int_0^1 y^{-\frac{3}{2}}\left\{\lambda(0)(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}}(1-y)^{\frac{n-2}{2}} - \lambda(0)(2-\beta_2)^{\frac{n-2}{2}} + \sum_{k=1}^{\infty}\frac{\lambda^k(0)}{k!}y^k(1-y)^{\frac{n-2}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}}\right\}dy \quad ,$$

or

$$I_2 = \sum_{k=0}^{\infty} \frac{\lambda^k(0)}{k!} I_k \quad , \tag{3.5.7}$$

where

$$I_0 = \int_0^1 y^{-\frac{3}{2}} \left\{ (2 - \beta_2 + \beta_2 y)^{\frac{n-2}{2}} (1-y)^{\frac{n-2}{2}} - (2-\beta_2)^{\frac{n-2}{2}} \right\} dy \quad ,$$

and for k=1,2,3,....

$$I_k = \int_0^1 y^{k-\frac{3}{2}} (2 - \beta_2 + \beta_2 y)^{\frac{n-2}{2}} (1-y)^{\frac{n-2}{2}} dy \quad .$$

To integrate I_0 by parts, let

$$u = (2 - \beta_2 + \beta_2 y)^{\frac{n-2}{2}} (1-y)^{\frac{n-2}{2}} - (2-\beta_2)^{\frac{n-2}{2}} \quad \text{and} \quad dw = y^{-\frac{3}{2}} dy .$$

Then

$$du = -(n-2)(2 - \beta_2 + \beta_2 y)^{\frac{n-4}{2}} (1-y)^{\frac{n-4}{2}} (1 - \beta_2(1-y)) dy \quad ; \quad w = -2y^{-\frac{1}{2}}$$

and hence

$$\begin{aligned} I_0 = &-2 \lim_{\varepsilon \to 0} \left\{ \frac{(2 - \beta_2 + \beta_2 y)^{\frac{n-2}{2}} (1-y)^{\frac{n-2}{2}} - (2-\beta_2)^{\frac{n-2}{2}}}{y^{\frac{1}{2}}} \right\} \Bigg|_{\varepsilon}^{1} \\ &- 2(n-2) \int_0^1 y^{-\frac{1}{2}} (2 - \beta_2 + \beta_2 y)^{\frac{n-4}{2}} (1-y)^{\frac{n-4}{2}} (1 - \beta_2(1-y)) dy \end{aligned}$$

$$\begin{aligned} = 2(2-\beta_2)^{\frac{n-2}{2}} &+ 2 \lim_{\varepsilon \to 0} \left\{ \frac{(2 - \beta_2 + \beta_2 \varepsilon)^{\frac{n-2}{2}} (1-\varepsilon)^{\frac{n-2}{2}} - (2-\beta_2)^{\frac{n-2}{2}}}{\varepsilon^{\frac{1}{2}}} \right\} \\ &- 2(n-2) \int_0^1 y^{-\frac{1}{2}} (2 - \beta_2 + \beta_2 y)^{\frac{n-4}{2}} (1-y)^{\frac{n-4}{2}} (1 - \beta_2(1-y)) dy \quad . \end{aligned}$$

Applying the L'Hospital's rule we can show that

$$\lim_{\varepsilon\to 0}\left\{\frac{(2-\beta_2+\beta_2\varepsilon)^{\frac{n-2}{2}}(1-\varepsilon)^{\frac{n-2}{2}}-(2-\beta_2)^{\frac{n-2}{2}}}{\varepsilon^{\frac{1}{2}}}\right\}=0,$$

and therefore,

$$I_0 = 2(2-\beta_2)^{\frac{n-2}{2}} - 2(n-2)\{(1-\beta_2)J_1+\beta_2 J_2\}, \tag{3.5.8}$$

where for k=1,2,3,.....

$$J_k = \int_0^1 y^{k-\frac{3}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-4}{2}}(1-y)^{\frac{n-4}{2}}\,dy \ .$$

Also,

$$\begin{aligned}
I_k &= \int_0^1 y^{k-\frac{3}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-2}{2}}(1-y)^{\frac{n-2}{2}}\,dy \\
&= \int_0^1 y^{k-\frac{3}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-4}{2}}(1-y)^{\frac{n-4}{2}}\left((2-\beta_2)-2(1-\beta_2)y-\beta_2 y^2\right)dy \\
&= (2-\beta_2)J_k - 2(1-\beta_2)J_{k+1} - \beta_2 J_{k+2}.
\end{aligned}$$

Now consider,

$$J_k = \int_0^1 y^{k-\frac{3}{2}}(1-y)^{\frac{n-4}{2}}(2-\beta_2+\beta_2 y)^{\frac{n-4}{2}}\,dy \ .$$

Setting $x = 1-y$ we get

$$\begin{aligned}
J_k &= \int_0^1 x^{\frac{n-4}{2}}(1-x)^{k-\frac{3}{2}}(2-\beta_2 x)^{\frac{n-4}{2}}\,dx \\
&= 2^{\frac{n-4}{2}}\int_0^1 x^{\frac{n-4}{2}}(1-x)^{k-\frac{3}{2}}\left(1-\frac{\beta_2}{2}x\right)^{\frac{n-4}{2}}dx \\
&= 2^{\frac{n-4}{2}}\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n+2k-3}{2})}F\left(-\frac{n-4}{2},\frac{n-2}{2};\frac{n+2k-3}{2};\frac{\beta_2}{2}\right),
\end{aligned}$$

where F is the hypergeometric function (see Gradshteyn and Ryzhik (1976)) which is defined by

$$F(a,b;c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}$$

$$+ \ldots + \frac{a(a+1)\ldots(a+j)b(b+1)\ldots(b+j)}{c(c+1)\ldots(c+j)}\frac{z^{j+1}}{(j+1)!} + \ldots .$$

Using the relationship

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$$

we can write,

$$J_k = \frac{1}{2^{\frac{2k-1}{2}}}\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n+2k-3}{2})}(2-\beta_2)^{\frac{n+2k-5}{2}}F\left(\frac{2n+2k-7}{2},\frac{2k-1}{2};\frac{n+2k-3}{2};\frac{\beta_2}{2}\right).$$

Now consider the (j+1)th term t_{j+1} of the hypergeometric series

$$F\left(\frac{2n+2k-7}{2},\frac{2k-1}{2};\frac{n+2k-3}{2};\frac{\beta_2}{2}\right),$$

$$t_{j+1} = \frac{(\frac{2n+2k-7}{2})(\frac{2n+2k-5}{2})\ldots(\frac{2n+2k+2j-7}{2})(\frac{2k-1}{2})(\frac{2k+1}{2})\ldots(\frac{2k+2j-1}{2})}{(\frac{n+2k-3}{2})(\frac{n+2k-1}{2})\ldots(\frac{n+2k+2j-3}{2})(j+1)!}\left(\frac{\beta_2}{2}\right)^{j+1}$$

$$= \left(\frac{n+\frac{2k-7}{2}}{n+2k-3}\right)\left(\frac{n+\frac{2k-5}{2}}{n+2k-1}\right)\ldots\left(\frac{n+\frac{2k+2j-7}{2}}{n+2k+2j-3}\right)$$

$$\times\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\ldots\left(k+\frac{2j-1}{2}\right)\frac{\beta^{j+1}}{(j+1)!}.$$

Since $\frac{n+a}{n+b} = 1 + \frac{a-b}{n+b} = 1 + O(n^{-1})$, we can write

$$t_{j+1} = \{1+O(n^{-1})\}\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\ldots\left(k+\frac{2j-1}{2}\right)\frac{\beta^{j+1}}{(j+1)!}.$$

Therefore,

$$F\left(\frac{2n+2k-7}{2}, \frac{2k-1}{2}; \frac{n+2k-3}{2}; \frac{\beta_2}{2}\right)$$

$$= \left\{1 + \sum_{j=0}^{\infty} \left(k - \frac{1}{2}\right)(k+1/2)\ldots\left(k + \frac{2j-1}{2}\right)\cdot \frac{\beta_2^{j+1}}{(j+1)!}\right\}\{1 + O(n^{-1})\}$$

$$= \left\{1 + \sum_{j=0}^{\infty} \frac{\Gamma(k+\frac{1}{2}+j)}{\Gamma(k-\frac{1}{2})\Gamma(j+2)}\beta_2^{j+1}\right\}\{1+O(n^{-1})\}$$

$$= (1-\beta_2)^{-(k-\frac{1}{2})}\{1+O(n^{-1})\} \quad .$$

Thus,

$$J_k = \frac{1}{2^{k-\frac{1}{2}}} \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n+2k-3}{2})} \frac{(2-\beta_2)^{\frac{n+2k-5}{2}}}{(1-\beta_2)^{k-\frac{1}{2}}}\{1+O(n^{-1})\} \quad .$$

or

$$\text{(3.5.9)} \qquad J_k = \Gamma\left(\frac{n-2}{2}\right)\sqrt{2(1-\beta_2)}(2-\beta_2)^{\frac{n-5}{2}} \frac{\Gamma(k-\frac{1}{2})}{\Gamma(\frac{n-3}{2}+k)}$$

$$\times \left\{\frac{2-\beta_2}{2(1-\beta_2)}\right\}^k \{1+O(n^{-1})\} \quad ,$$

and hence

$$J_{k+1} \sim \frac{(2-\beta_2)}{2(1-\beta_2)} \frac{(k-\frac{1}{2})}{(\frac{n-3}{2}+k)} J_k$$

$$= \frac{(2-\beta_2)}{2(1-\beta_2)} \frac{(2k-1)}{(n+2k-3)} J_k \quad .$$

Thus,

$$J_2 \sim \frac{(2-\beta_2)}{2(1-\beta_2)(n-1)} J_1 \quad ,$$

$$J_3 \sim \left\{\frac{(2-\beta_2)}{2(1-\beta_2)}\right\}^2 \frac{1.3}{(n-1)(n+1)} J_1 \quad ,$$

$$\vdots$$

$$J_{k+1} \sim \left\{\frac{2-\beta_2}{2(1-\beta_2)}\right\}^k \frac{1.3\ldots(2k-1)}{(n-1)(n+1)\ldots(n+2k-3)} J_1 \quad ,$$

and hence, to the order considered, all $J_k; k = 2, 3, \ldots$ may be neglected relative to J_1.

Now from (3.5.9) we have

$$
\begin{aligned}
J_1 &= \Gamma\left(\frac{n-2}{2}\right)\sqrt{2(1-\beta_2)}(2-\beta_2)^{\frac{n-5}{2}}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}\left\{\frac{2-\beta_2}{2(1-\beta_2)}\right\}\{1+O(n^{-1})\} \\
&= \sqrt{\pi}\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\frac{(2-\beta_2)^{\frac{n-3}{2}}}{\sqrt{2(1-\beta_2)}}\{1+O(n^{-1})\} \quad .
\end{aligned}
$$

Therefore, (3.5.8) becomes

$$
\begin{aligned}
I_0 &\sim 2(2-\beta_2)^{\frac{n-2}{2}} - 2(n-2)(1-\beta_2)\sqrt{\pi}\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\frac{(2-\beta_2)^{\frac{n-3}{2}}}{\sqrt{2(1-\beta_2)}} \\
&= 2(2-\beta_2)^{\frac{n-2}{2}} - \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}(2-\beta_2)^{\frac{n-3}{2}}2\sqrt{2(1-\beta_2)} \quad ,
\end{aligned}
$$

and, since $I_k; k = 1, 2, \ldots$ may be neglected relative to I_0,

$$
\begin{aligned}
I_2 &\sim 2\lambda(0)(2-\beta_2)^{\frac{n-2}{2}} - \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\lambda(0)(2-\beta_2)^{\frac{n-3}{2}}2\sqrt{2(1-\beta_2)} \quad , \\
&= 2l(\beta_2)(2-\beta_2)^{\frac{n-2}{2}} - \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}l(\beta_2)(2-\beta_2)^{\frac{n-3}{2}}2\sqrt{2(1-\beta_2)} \quad .
\end{aligned}
$$

Substituting this in (3.5.6) we get,

$$
I_1 \sim -i\,4\,\sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}l(\beta_2)(2-\beta_2)^{\frac{n-3}{2}}\sqrt{2(1-\beta_2)}\beta_2^{\frac{n-3}{2}} \quad ,
$$

and

$$
I \sim -i\,4\left\{\frac{\sqrt{4+\rho^4}}{2\rho}\right\}^{n-\frac{5}{2}}\rho^{\frac{n-2}{2}}\sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}l(\beta_2)(2-\beta_2)^{\frac{n-3}{2}}\sqrt{2(1-\beta_2)}\beta_2^{\frac{n-3}{2}} \quad .
$$

Now

$$l(\beta_2) = \frac{(1-\beta_0 - b\beta_2)[1-(\beta_0 + b\beta_2)^2]^{\frac{3}{2}}}{[1-\rho(\beta_0 + b\beta_2)][\beta(\beta_0 + b\beta_2) - 1]^{\frac{3}{2}}} ,$$

and

$$\beta_0 + b\beta_2 = \beta_0 + b\left(\frac{\beta - \beta_0}{b}\right) = \beta .$$

Therefore,

$$\begin{aligned} l(\beta_2) &= \frac{(1-\beta)(1-\beta^2)^{\frac{3}{2}}}{(1-\rho\beta)(\beta^2-1)^{\frac{3}{2}}} \\ &= i\frac{1-\beta}{1-\rho\beta} , \end{aligned}$$

and hence

$$I \sim 4\left\{\frac{\sqrt{4+\rho^4}}{2\rho}\right\}^{n-\frac{5}{2}} \rho^{\frac{n-2}{2}} \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\frac{(1-\beta)}{(1-\rho\beta)}(2-\beta_2)^{\frac{n-3}{2}}\sqrt{2(1-\beta_2)}\beta_2^{\frac{n-3}{2}}$$

Substituting this in (3.5.1) we get,

(3.5.10)

$$\begin{aligned} h(r) &\sim \frac{n[1+(1-\rho)^2]\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}(1-\rho^2)^{\frac{1}{2}}\left(\frac{\sqrt{4+\rho^4}}{4}\right)^{\frac{2n-5}{4}}\frac{(1-\beta)}{(1-\rho\beta)} \\ &\quad \times\left(\frac{1-\beta_2}{r}\right)^{\frac{1}{2}}\left(\frac{\beta}{A}\right)^{\frac{3}{2}}\left(\frac{\beta_2(2-\beta_2)}{\rho^2}\right)^{\frac{n-3}{2}} \\ &= \frac{n[1+(1-\rho)^2]\Gamma(\frac{n}{2})[(1-\rho^2)(2+\rho^2-2\rho\beta)]^{\frac{1}{2}}}{\sqrt{\pi r}\Gamma(\frac{n-1}{2})}\frac{(1-\beta)}{(1-\rho\beta)} \\ &\quad \times\left(\frac{\beta}{A}\right)^{\frac{3}{2}}\left(\frac{\rho^2\beta-\rho\beta^2+2\beta-\rho}{\rho}\right)^{\frac{n-3}{2}} , \end{aligned}$$

where,

$$\begin{aligned} \beta &= \frac{B-\sqrt{B^2-4A^2}}{2A} , \\ A &= \{n[1+(1-\rho)^2]-r\}\rho , \\ B &= n[1+(1-\rho)^2](1+\rho^2)-2\rho r . \end{aligned}$$

Now applying (3.0.1) we obtain an asymptotic approximation for the distribution of F,

$$(3.5.11) \quad g(F) = \frac{(n-1)[1+(1-\rho)^2]\Gamma(\frac{n}{2})\left[(1-\rho^2)(2+\rho^2-2\rho\beta)\right]^{\frac{1}{2}}(1-\beta)}{\sqrt{\pi F}\,\Gamma(\frac{n-1}{2})(1-\rho\beta)}$$

$$\times\left(\frac{n\beta}{A}\right)^{\frac{3}{2}}\left(\frac{\rho^2\beta-\rho\beta^2+2\beta-\rho}{\rho}\right)^{\frac{n-3}{2}},$$

where,

$$\beta = \frac{B-\sqrt{B^2-4A^2}}{2A},$$

$$A = \{n(n-1)[1+(1-\rho)^2]+n(1-\rho)^2F\}\rho,$$

$$B = n[1+(1-\rho)^2](1+\rho^2)(n-1+F)-2n\rho F.$$

3.6 The Limiting Distribution of r as $\rho \to 0$

If $\rho = 0$, not only the two samples but also the observations within both samples are independent of each other. Therefore the F-statistic has the F-distribution with 1 and n-1 degrees of freedom and hence the probability density function of F is given by,

$$g(F) = \begin{cases} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}\left(\frac{1}{n-1}\right)^{\frac{1}{2}} F^{-\frac{1}{2}}\left(1+\frac{1}{n-1}F\right)^{-\frac{n}{2}}; & \text{F} > 0 \\ 0; & \text{otherwise.} \end{cases}$$

Applying (3.0.2) we obtain the distribution of r to be,

$$h(r) = \frac{n(n-1)}{(n-r)^2}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}\left(\frac{1}{n-1}\right)^{\frac{1}{2}}\left[\frac{(n-1)r}{n-r}\right]^{-\frac{1}{2}}\left(1+\frac{r}{n-r}\right)^{-\frac{n}{2}}$$

or

(3.6.1) $$h(r) = \begin{cases} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\frac{1}{\sqrt{n\pi r}}\left(\frac{n-r}{n}\right)^{\frac{n-3}{2}}; & 0 < \ldots \\ 0; & \text{otherwise} \end{cases}$$

We have that,

$$\beta = \frac{B - \sqrt{B^2 - 4A^2}}{2A},$$

where

$$A = n\rho[1 + (1-\rho)^2] - r\rho,$$

$$B = n[1 + (1-\rho)^2](1+\rho^2) - 2r\rho.$$

Applying L'Hospital's rule we can show that, as $\rho \longrightarrow 0$

$$\beta \longrightarrow 0$$

and

$$\frac{\beta}{\rho} \longrightarrow \frac{2n-r}{2n}$$

Then,

$$\frac{\beta}{A} = \frac{\beta}{\rho[n[1+(1-\rho)^2]-r]} \longrightarrow \frac{2n-r}{2n}\frac{1}{2n-r} = \frac{1}{2n}$$

Also,

$$\begin{aligned}\frac{\rho^2\beta-\beta^2\rho+2\beta-\rho}{\rho} &= \rho\beta-\beta^2+2\beta/\rho-1\\ &\longrightarrow \frac{2(2n-r)}{2n}-1 = \frac{n-r}{n}\end{aligned}$$

Thus $h(r)$ given by (3.5.10) has the limiting value

$$\begin{aligned}\lim_{\rho\longrightarrow 0} h(r) &= \frac{2n\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi r}}\left(\frac{1}{2n}\right)^{\frac{3}{2}}\left(\frac{n-r}{n}\right)^{\frac{n-3}{2}}\\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n\pi r}}\left(\frac{n-r}{n}\right)^{\frac{n-3}{2}}\end{aligned}$$

which is exactly the same as (3.6.1).

Therefore the approximate probability density function of r derived in section 3.5 tends to the exact probability density function as ρ tends to zero.

3.7 The Effect of the Dependence on the Level of Significance

In this section we examine the sensitivity of the size of the test to the degree of dependence by computing the tail area under the approximate probability density function of F given by the equation (3.5.11) for several values of ρ and different sample sizes. The critical value of F to reject H_0 is selected so that the size of the test under the assumption of independence is 0.05. A computer program written in FORTRAN-77 (see Appendix III) is used for the calculations and the results are summarized in Table 3.7.1 and 3.7.2.

The results in Table 3.7.1 clearly indicate that the test is sensitive to the dependence. For positively serially correlated $\{X_i\}$ the size of the test is increased while it is decreased for negatively correlated observations. That is, positive (negative) serial correlations seem to lead to the T-statistic being heavy (light)-tailed. Furthermore, Figure 3.7.1 indicates a much higher sensitivity to positive correlations than to negative ones.

It can also be seen from Table 3.7.1 that increasing the sample size does not improve the robustness of the test by reducing the sensitivity to dependence. Critical F values to reject H_0 with a 0.05 level of significance under the approximate distribution of F are given in Table 3.7.2 and plotted against ρ in Figure 3.7.2 for samples of size 21, 41 and 121.

Table 3.7.1 Area Above the 5% Critical Value.

ρ \ n-1	20	40	120
-0.60	0.0114	0.0098	0.0085
-0.50	0.0157	0.0144	0.0133
-0.40	0.0206	0.0196	0.0187
-0.30	0.0262	0.0254	0.0247
-0.20	0.0326	0.0321	0.0317
-0.10	0.0404	0.0402	0.0399
0.0	0.0500	0.0500	0.0500
0.10	0.0626	0.0629	0.0630
0.20	0.0793	0.0800	0.0801
0.30	0.1025	0.1033	0.1034
0.40	0.1353	0.1360	0.1358
0.50	0.1827	0.1826	0.1814
0.60	0.2520	0.2494	0.2460

Table 3.7.2 5% Critical Values for F.

ρ \ n-1	20	40	120
-0.60	2.4081	2.2292	2.1432
-0.50	2.6920	2.5213	2.4203
-0.40	2.9831	2.8176	2.6988
-0.30	3.2819	3.1179	2.9784
-0.20	3.5884	3.3885	3.2281
-0.10	3.9432	3.7315	3.5416
0.0	4.3500	4.0800	3.9200
0.10	4.8351	4.5147	4.3450
0.20	5.5475	5.1379	4.8957
0.30	6.4136	5.9653	5.69[illegible]9
0.40	7.8286	7.2154	6.7673
0.50	9.9692	9.0629	8.5654
0.60	13.8782	12.2429	11.3603

Figure 3.7.1 The Effect of ρ on the Size of the Test

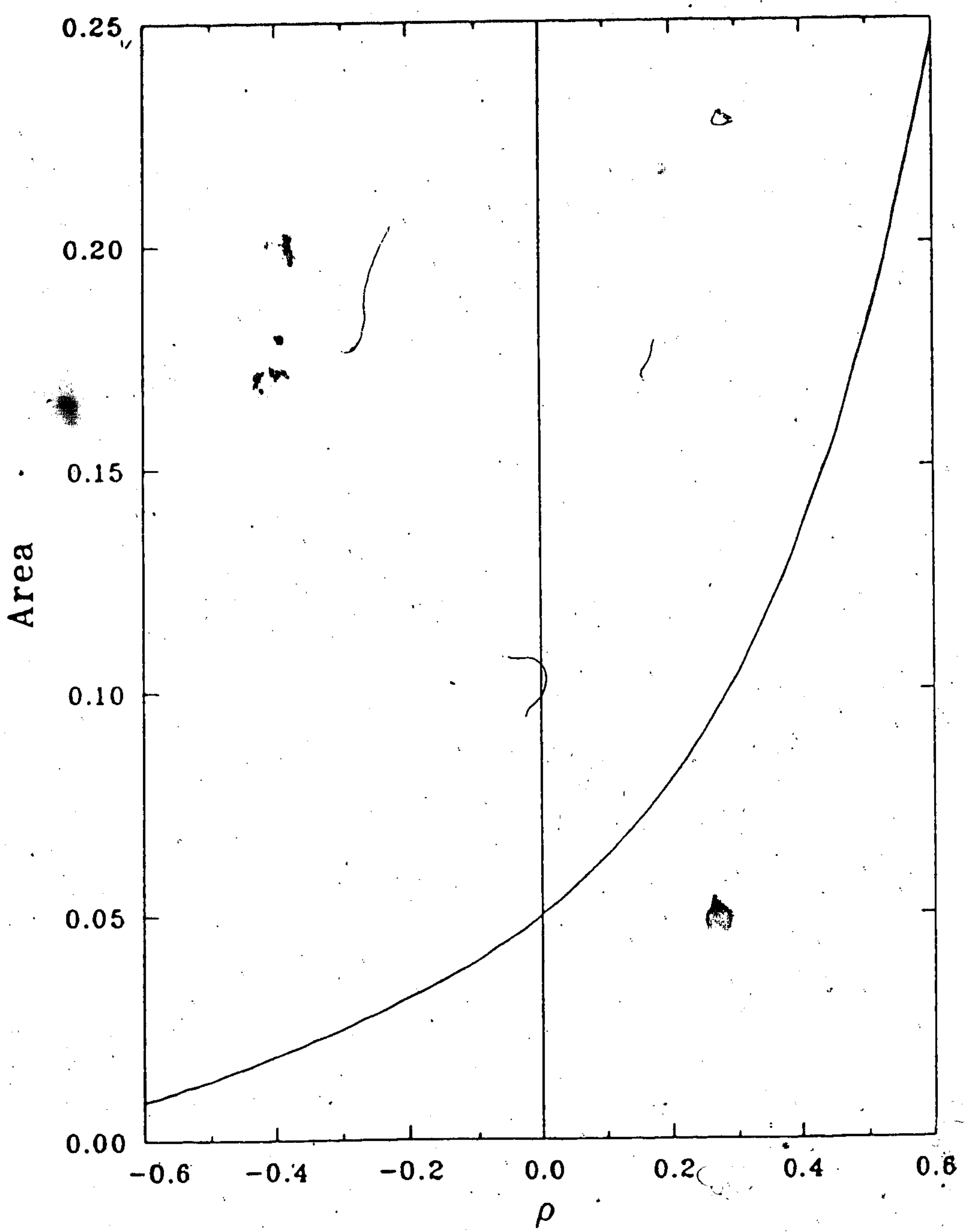
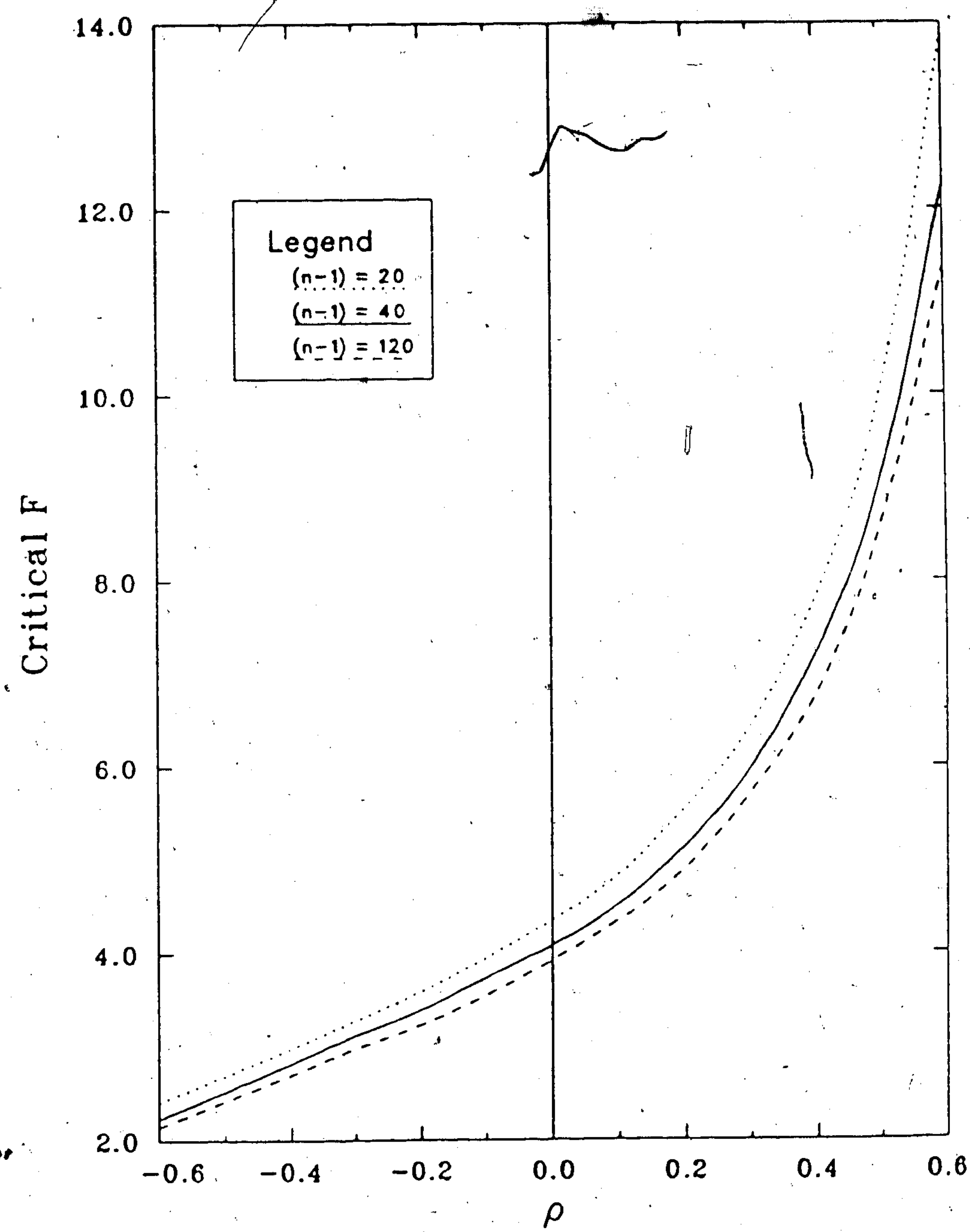


Fig. 3.7.2 5% Critical Values for F



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APPENDIX I

EVALUATION OF $|A|$

We want to evaluate the $2n \times 2n$ determinant,

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_3 & \mathbf{R}_2 \end{vmatrix} \quad ,$$

where

$$\mathbf{R}_1 = \begin{pmatrix} 1-2T_0 & -\rho & 0 & 0 & \dots & 0 & 0 \\ -\rho & x & -\rho & 0 & \dots & 0 & 0 \\ 0 & -\rho & x & -\rho & \dots & 0 & 0 \\ 0 & 0 & -\rho & x & \dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x & -\rho \\ 0 & 0 & 0 & 0 & \dots & -\rho & 1-2T_0 \end{pmatrix}_{n\times n} ,$$

$$x = 1 + \rho^2 - 2T_0 \quad ,$$

$$\mathbf{R}_2 = (1 - 2T_0)\mathbf{I}_{n\times n} \quad ,$$

$$\mathbf{R}_3 = 2T_0\mathbf{I}_{n\times n} \quad .$$

Since $\mathbf{R}_1$ and $\mathbf{R}_2$ are non-singular we can write,

$$\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_3 \\ \mathbf{R}_3 & \mathbf{R}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_2 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{R}_3 \\ \mathbf{R}_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_2 \end{pmatrix} \left\{ \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} + \begin{pmatrix} \mathbf{R}_1^{-1} & 0 \\ 0 & \mathbf{R}_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{R}_3 \\ \mathbf{R}_3 & 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{R}_1^{-1}\mathbf{R}_3 \\ \mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{I} \end{pmatrix}$$

Thus,

$$(I.1) \qquad |\mathbf{A}| = |\mathbf{R}_1||\mathbf{R}_2| \begin{vmatrix} \mathbf{I} & \mathbf{R}_1^{-1}\mathbf{R}_3 \\ \mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{I} \end{vmatrix}.$$

Applying the theory of partition matrices, (Aitken(1959)), we can write,

$$\begin{vmatrix} \mathbf{I} & \mathbf{R}_1^{-1}\mathbf{R}_3 \\ \mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{I} \end{vmatrix} = \frac{1}{|\mathbf{R}_2^{-1}\mathbf{R}_3|} \begin{vmatrix} \mathbf{I} & \mathbf{R}_1^{-1}\mathbf{R}_3 \\ \mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{R}_2^{-1}\mathbf{R}_3 \end{vmatrix}$$

$$= \frac{1}{|\mathbf{R}_2^{-1}\mathbf{R}_3|} \begin{vmatrix} \mathbf{I} & \mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{R}_2^{-1}\mathbf{R}_3 \\ \mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{R}_2^{-1}\mathbf{R}_3 \end{vmatrix}$$

$$= \frac{1}{|\mathbf{R}_2^{-1}\mathbf{R}_3|} \begin{vmatrix} \mathbf{I} - \mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{R}_2^{-1}\mathbf{R}_3 & \mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{R}_2^{-1}\mathbf{R}_3 \\ 0 & \mathbf{R}_2^{-1}\mathbf{R}_3 \end{vmatrix}$$

$$= |\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{R}_3\mathbf{R}_2^{-1}\mathbf{R}_3|$$

$$= |\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{R}_3^2\mathbf{R}_2^{-1}| \quad \text{, since } \mathbf{R}_3 \text{ is diagonal.}$$

Therefore, from (I.1) we get

$$|\mathbf{A}| = |\mathbf{R}_1\mathbf{R}_2||\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{R}_3^2\mathbf{R}_2^{-1}|$$
$$= |\mathbf{R}_1\mathbf{R}_2 - \mathbf{R}_3^2|$$
$$= |(1-2T_0)\mathbf{R}_1 - 4T_0^2\mathbf{I}|$$

or,

$$|\mathbf{A}| = \rho^n(1-2T_0)^n A_n \qquad (I.2)$$

where A_n is the $n \times n$ determinant,

$$A_n = \begin{vmatrix} a & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & b & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & b & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}$$

with

$$a = \frac{1-4T_0}{\rho(1-2T_0)}$$

and

$$b = \rho + a$$

In evaluating this determinant we follow Dixon(1944). Let $A_j^*(j < n)$ be the leading principle minor of order j of the determinant A_n. Then,

$$A_1^* = a, \quad A_2^* = ab - 1 \quad \text{etc.}$$

In general,

$$A_j^* = bA_{j-1}^* - A_{j-2}^* \quad ,$$

or

$$(I.3) \qquad A_{j+2}^* - bA_{j+1}^* + A_j^* = 0 \quad .$$

Let E be the forward operator of the calculus of finite difference defined by,

$$E(f(j)) = f(j+1) \quad .$$

Then (I.3) can be written as

$$(I.4) \qquad (E^2 - bE + 1)A_j^* = 0 \quad .$$

The auxiliary equation is

$$r^2 - br + 1 = 0$$

which has two roots

$$z = \frac{b + \sqrt{b^2 - 4}}{2} \quad \text{and} \quad \frac{1}{z} = \frac{b - \sqrt{b^2 - 4}}{2} \quad ,$$

and hence

$$z + \frac{1}{z} = b \quad .$$

Thus the general solution of (I.4) is

$$(I.5) \qquad A_j^* = K_1 z^j + \frac{K_2}{z^j} \quad ,$$

and, applying the conditions

$$A_1^* = a \quad A_2^* = ab - 1 \quad ,$$

we get

$$a = K_1 z + \frac{K_2}{z} \quad \text{and} \quad ab - 1 = K_1 z^2 + \frac{K_2}{z^2}$$

whose solutions for K_1 and K_2 are

$$K_1 = \frac{1 - az}{1 - z^2} \quad \text{and} \quad K_2 = \frac{z(a-z)}{1 - z^2} \quad .$$

Thus from (I.5) we get,

$$A_j^* = \frac{(1-az)}{(1-z^2)} z^j + \frac{z(a-z)}{(1-z^2)} \frac{1}{z^j} \quad ,(j < n).$$

But

$$A_n = aA_{n-1}^* - A_{n-2}^*$$

so that

$$\begin{aligned} A_n &= \frac{a(1-az)}{(1-z^2)} z^{n-1} + \frac{az(a-z)}{(1-z^2)} \frac{1}{z^{n-1}} - \frac{(1-az)}{(1-z^2)} z^{n-2} - \frac{z(a-z)}{(1-z^2)} \frac{1}{z^{n-2}} \\ &= \frac{1}{(1-z^2)z^n} \{a(1-az)z^{2n-1} + az^2(a-z) - (1-az)z^{2n-2} - (a-z)z^3\} \\ &= \frac{1}{(1-z^2)z^n} \{z^2(a^2 - az - az + z^2) - z^{2n-2}(1 - az - az + a^2 z^2)\} \\ &= \frac{1}{(1-z^2)z^n} \{z^2(a-z)^2 - z^{2n-2}(1-az)^2\} . \end{aligned}$$

Finally from (I.2) we get

$$(I.6) \qquad |\mathbf{A}| = \frac{\rho^n (1 - 2T_0)^n}{(1-z^2)z^n} \{z^2(a-z)^2 - z^{2n-2}(1-az)^2\} \quad ,$$

where

$$z = \frac{b + \sqrt{b^2 - 4}}{2} \text{ and } z + \frac{1}{z} = b \quad ,$$

$$a = \frac{1 - 4T_0}{\rho(1 - 2T_0)} \quad ,$$

and

$$b = \rho + a \quad .$$

APPENDIX II

A METHOD OF INVERSION TO DETERMINE THE DISTRIBUTION OF A RATIO FROM THE JOINT M.G.F. OF ITS NUMERATOR AND ITS DENOMINATOR

The following result which is known as the Cramer-Geary inversion formula is a form of Geary's (1944) extension of Cramer's theorem further extended ~~by Daniels (1956)~~.

Let the statistics c and c_0 where c_0 is almost surely positive have the joint probability density function $f(c_0, c)$. We wish to find the distribution of

$$r = c/c_0 \quad .$$

The Jacobian of the transformation

$$c = rc_0, \quad c_0 = c_0 \text{ is}$$

$$\left|\frac{\partial(c_0, c)}{\partial(c_0, r)}\right| = \begin{vmatrix} 1 & 0 \\ r & c_0 \end{vmatrix} = c_0$$

and the joint probability distribution function of the distribution of c_0 and r is

$$g(r, c_0) = c_0 f(c_0, rc_0) \quad .$$

The density of r can now be obtained as a marginal p.d.f. by integrating over c_0 to give

$$(II.1) \qquad h(r) = \int_0^\infty c_0 f(c_0, rc_0)\, dc_0 \quad .$$

Let $M(T_0,T)$ be the joint moment generating function of c_0 and c.

$$M(T_0,T) = E\left(e^{-(T_0c_0+Tc)}\right) .$$

Then by the Fourier inversion theorem

$$f(c_0,c) = \frac{1}{(2\pi i)^2}\int\int M(T_0,T)e^{-T_0c_0-Tc}\,dT_0\,dT \quad ,$$

where integration is along the imaginary axes in the T_0 and T planes from -i∞ to i∞ or along any allowable deformation of these paths. Thus,

$$f(c_0,rc_0) = \frac{1}{(2\pi i)^2}\int\int M(T_0,T)e^{-(T_0+rT)c_0}\,dT_0\,dT \quad .$$

Consider the transformation

$$u = T_0 + rT \qquad T = T$$

with Jacobian,

$$\left|\frac{\partial(T_0,T)}{\partial(u,T)}\right| = \begin{vmatrix} 1 & -r \\ 0 & 1 \end{vmatrix} = 1$$

Then $T_0 = u - rT$ and

$$f(c_0,rc_0) = \frac{1}{(2\pi i)^2}\int\int M(u-rT,T)e^{-uc_0}\,du\,dT$$

the inner integration being along the imaginary axis in the u-plane or along any allowable deformation of it. Then,

$$\int_0^\infty f(c_0,rc_0)e^{uc_0}dc_0 = \int_0^\infty\left(\frac{1}{(2\pi i)^2}\int\int M(u-rT,T)e^{-uc_0}du\,dT\right)e^{uc_0}dc_0$$
$$= \frac{1}{(2\pi i)}\int\left[\int_0^\infty\left(\frac{1}{(2\pi i)}\int M(u-rT,T)e^{-uc_0}du\right)e^{uc_0}dc_0\right]dT.$$

But by the Fourier Inversion Theorem,

$$\int_0^{\infty} \left(\frac{1}{(2\pi i)} \int M(u - rT, T) e^{-uc_0} \, du \right) e^{uc_0} \, dc_0 = M(u - rT, T) .$$

Thus,

$$\int_0^{\infty} f(c_0, rc_0) e^{uc_0} dc_0 = \frac{1}{(2\pi i)} \int M(u - rT, T) dT$$

Differentiating under the integral signs with respect to u, we get

$$(II.2) \qquad \int_0^{\infty} f(c_0, rc_0) c_0 e^{uc_0} dc_0 = \frac{1}{(2\pi i)} \int \frac{\partial}{\partial u} M(u - rT, T) dT$$

and setting u=0, we get

$$\int_0^{\infty} f(c_0, rc_0) c_0 \, dc_0 = \frac{1}{(2\pi i)} \int \frac{\partial}{\partial u} M(u - rT, T)|_{u=0} dT$$

Thus from (II.1) we have

$$(II.3) \qquad h(r) = \frac{1}{(2\pi i)} \int \frac{\partial}{\partial u} M(u - rT, T)|_{u=0} \, dT .$$

If we wish to transform from T to some other variable z by

$$T = T(z, u) \quad ,$$

then

$$M(u - rT, T) \, dT = M(u - T(z, u), T(z, u)) \frac{\partial}{\partial z} T(z, u) \, dz .$$

Substituting this in (II.2) gives,

$$\int_0^{\infty} f(c_0, rc_0) e^{uc_0} dc_0 = \frac{1}{(2\pi i)} \int M(u - rT(z, u), T(z, u)) \frac{\partial}{\partial z}(z, u) \, dz ,$$

integration being along the transformed contour in the z-plane. Finally differentiating with respect to u and setting u=0 we obtain

$$(II.4) \qquad h(r) = \frac{1}{(2\pi i)} \int \frac{\partial}{\partial u} \left\{ M(u - T(z,u), T(z,u)) \frac{\partial}{\partial z} T(z,u) \right\} \Big|_{u=0} dz$$

APPENDIX III

A PROGRAM TO COMPUTE THE SIGNIFICANCE LEVELS AND CRITICAL VALUES

This program which is written in FORTRAN-77 computes:

(i) the size of the test for given critical F, ρ and n,

and

(ii) the 5% critical F-value for given ρ and n

under the approximate distribution of F derived in this thesis. To compute the area under under the density curve the program calls the IMSL subroutine DCADRE. Since

$$F_c < F < \infty \Longleftrightarrow \frac{nF_c}{n + F_c - 1} < r < n \quad ,$$

considering the simplicity of computation, the tail area is calculated using the density function $h(r)$ of r given by the equation (3.5.10). Also, to improve the accuracy the result is renormalized by dividing by the whole area under $h(r)$.

The program handles multiple input lines. Each input line consisted of:

(i) N - the sample size,

(ii) RHOMIN and RHOMAX - the minimum amd maximum values of ρ to be used for the calculations,

(iii) RHOINC - the step in which ρ should be incremented for computations

and

(iv) 5% critical F under the independence assumption. i.e., $F_{1,n-1,0.05}$.

The program outputs the tail area above $F_{1,n-1,0.05}$ and the 5% critical F for the values of ρ from RHOMIN to RHOMAX with increments RHOINC for given N.

```
      REAL DCADRE,F,X0,X1,X2,RHO,RHOMIN,RHOMAX,RHOINC,FVALUE
      INTEGER N,NPLOTS
      EXTERNAL F
      COMMON FN,RHO
      PI=3.141592654
      AERR=0.0
      RERR=1.0E-4
      READ(5,*)NPLOTS,ERRB
      ARUB=0.05+ERRB
      ARLB=0.05-ERRB
      DO 999 I=1,NPLOTS
        READ(5,*)N,RHOMIN,RHOMAX,RHOINC,FVALUE
        WRITE(7,12)I,(N-1)
        WRITE(7,13)
        WRITE(7,14)
        WRITE(7,15)
        FN=FLOAT(N)
        X0=10.0**(-7)
        X1=FN*FVALUE/(FN-1.0+FVALUE)
        RHO=RHOMIN
88      IF (RHO.NE.0.0) THEN
          TAREA=DCADRE(F,X0,FN,AERR,RERR,ERROR,IER)
          AREA=DCADRE(F,X1,FN,AERR,RERR,ERROR,IER)/TAREA
        ELSE
          AREA=0.05
        ENDIF
        WRITE(7,16)RHO,AREA
        RHO=RHO+RHOINC
        RHO=ANINT(RHO*100.0)/100.0
        IF (RHO.LE.RHOMAX) GO TO 88
        WRITE(7,11)
        WRITE(7,17)
        RHO=RHOMIN
70      X1=FN*FVALUE/(FN-1.0+FVALUE)
        XH=FN
        XL=0.0
        IF (RHO.NE.0.0) THEN
72        TAREA=DCADRE(F,X0,FN,AERR,RERR,ERROR,IER)
          AREA=DCADRE(F,X1,FN,AERR,RERR,ERROR,IER)/TAREA
          IF (AREA .GE. ARUB) THEN
            XL=X1
            X1=(X1+XH)/2.0
            GO TO 72
          ELSEIF (AREA .LE. ARLB) THEN
            XH=X1
            X1=(X1+XL)/2.0
            GO TO 72
          ENDIF
```

```
        FV=(FN-1.0)*X1/(FN-X1)
      ELSE
        FV=FVALUE
      ENDIF
      WRITE(7,16)RHO,FV
      RHO=RHO+RHOINC
      RHO=ANINT(RHO*100.0)/100.0
      IF (RHO.LE.RHOMAX) GO TO 70
      WRITE(7,11)
999 CONTINUE
    STOP
 11 FORMAT(//1X)
 12 FORMAT(//15X,'Plot #:',I2,'   (n-1)=',I4)
 13 FORMAT(15X,'____________________'//)
 14 FORMAT(5X,'Rho',7X,'Area Above the Null')
 15 FORMAT(15X,'5% Critical Value'/)
 16 FORMAT(F8.2,F15.4)
 17 FORMAT(5X,'Rho',7X,'5% Critical Value'/)
    END
    REAL FUNCTION F(R)
    COMMON FN,RHO
    Y1=(1.0+(1.0-RHO)**2)
    A=RHO*(FN*Y1-R)
    B=FN*Y1*(1.0+RHO**2)-2.0*RHO*R
    DEL=SQRT(B*B-4.0*A*A)
    BETA=(B-DEL)/(2.0*A)
    RB=RHO*BETA
    X1=(2.0+RHO**2-2.0*RB)/R
    X2=(1.0-BETA)/(1.0-RB)
    X3=BETA/(RHO*(FN*Y1-R))
    X4=(RB*(RHO-BETA)+2.0*BETA-RHO)/RHO
    F=SQRT(X1) * X2 * X3**(3.0/2.0) * X4**((FN-3.0)/2.0)
    RETURN
    END
```