The following paper was published in:

Relevance Logics and other Tools for Reasoning. Essays in Honor of J. Michael Dunn, Bimbó, K. (ed.), (Tributes, vol. 46), College Publications, London, UK, 2022, pp. 89–127.

MODALITIES IN LATTICE-R

Katalin Bimbó and J. Michael Dunn

ABSTRACT. This paper considers modalities added to the relevance logic LR (lattice-R), which is R with the distributivity of conjunction and disjunction omitted. First, the modalities are *defined* from the Ackermann constants and the lattice connectives. Then, we introduce modalities as *primitives* equipped with some fairly usual properties. We also consider some other logics in the neighborhood. For each logic, including classical linear logic, we prove *decidability*. Lincoln, Mitchell, Scedrov, and Shankar (1992) claimed to have proved classical linear logic undecidable. We examine their work and find that their paper does not contain a proof of the admissibility of the cut rule, which would be essential for their claims to hold. Furthermore, according to their interpretation of proofs in linear logic, computations that lead to a dead-end state are not considered, unlike computations from inaccessible states that are included. The same problem with the *direction of a proof* vs the *direction of a computation* appears in all other publications that claim undecidability, including Kanovich (2016).

Keywords. Decidability, Linear logic, Modal logic, Relevance logic, Sequent calculuses

INTRODUCTION

Modality in reasoning has intrigued thinkers for millennia — at least since the time of Aristotle. Logically valid reasoning itself is often characterized in modal terms by saying that a conclusion is true necessarily, provided the premises are true. Thus it is not by chance that an attempt that aimed at tightening the connections between the notions of logical consequence and implication led to the invention of modern modal logics in the work of Clarence I. Lewis.

The *logic of entailment*, E gives a certain modal character to provable entailments. A usual definition of "$\mathcal{A}$ is necessary" in some relevance logics is by the formula $(\mathcal{A} \to \mathcal{A}) \to \mathcal{A}$. However, there are other ways to think about modality in relevance logics. In this paper, we look at an alternative definition of necessity and possibility that involves $\boldsymbol{t}$ and $\boldsymbol{f}$, then we consider $\Box$ and $\Diamond$ as primitives.

In order to narrow our considerations, we start with the logic called *lattice*-R, which is denoted by LR. This logic was derived from the logic of *relevant implication* R by omitting the assumption that conjunction and disjunction distribute over each other; it was created by Meyer [36]. The distributivity principle does not appear to be problematic from the point of view that motivates the family of relevance logics, hence,

2020 *Mathematics Subject Classification.* Primary: 03B47, Secondary: 03F52, 03B45, 68T17.

Bimbó, Katalin, (ed.), *Relevance Logics and other Tools for Reasoning. Essays in Honor of J. Michael Dunn*, (Tributes, vol. 46), College Publications, London, UK, 2022, pp. 89–127.

we might wonder why to consider lattice-R at all. Lattice-R has a straightforward sequent calculus formalization that goes back to Meyer's thesis, and it was hoped way back in the 1960s, that the *decidability* of lattice-R would be a stepping stone to the decidability of R (and that of E, T, etc.). Accordingly, we will explore the question of decidability in the context of modalities and also for logics neighboring R.

Section 1 introduces lattice-R in the way it was originally defined; then we throw in some constants. We give a sequent calculus (*L*LR) and an axiomatic (*H*LR) formulation.[1] Next, in Section 2, we take up the idea of defined modalities within *L*LR$^{\boldsymbol{c}}$, that is, lattice-R with zero-ary constants. Section 3 gives a somewhat detailed proof that *L*LR$^{\boldsymbol{c}}$ is decidable. The argument is along the standard Curry–Kripke lines, which had been successfully applied to some other logics. The next section adds $\Diamond$ and $\Box$ as new unary connectives to *L*LR$^{\boldsymbol{c}}$. We prove that the resulting logic is decidable. In Section 5, we consider a series of logics obtained by variations on the structural rules — whether they are absent, modalized or included. Then in Section 6, we give a direct and quite detailed proof of the decidability of (classical propositional) linear logic. Finally, in Section 7, we briefly outline the argument in Lincoln et al. [35], from which they conclude a theorem that conflicts with our decidability result about linear logic in the previous sections. We pinpoint some gaps in their proof of the cut elimination theorem, and we conclude with a different interpretation of *L*CLL proofs, which dissolves the appearance of a contradiction between our result and those in [35], Kanovich [28; 27] and Forster and Larchey-Wendling [21].

1. Lattice-R with Constants

The *relevant endeavor* can be quickly motivated by the desire to avoid having theorems like $\mathcal{A} \to (\mathcal{B} \to \mathcal{A})$, where $\to$ is some sort of implication. Roughly speaking, $\mathcal{B}$ gets into the theorem, although it may be completely unrelated to $\mathcal{A}$. Somewhat less obviously, $((\mathcal{A} \to \mathcal{B}) \to \mathcal{A}) \to \mathcal{A}$ is also an unwelcome theorem. It is easy to verify that the proof of these formulas in a sequent calculus for classical logic, such as Gentzen's *L*K, requires the use of some of the thinning rules. Well, then it is plain sailing to drop those rules and to see what results.

The *language* of *L*LR$^{\boldsymbol{c}}$ contains a denumerable stock of propositional variables together with a handful of logical constants.[2] The latter category is divided into three subcategories by the arity of the connectives: 0-ary, 1-ary and 2-ary. The zero-ary connectives are $\boldsymbol{t}$ ("real truth"), $\boldsymbol{f}$ ("real falsity"), $\boldsymbol{T}$ ("triviality") and $\boldsymbol{F}$ ("absurdity"). The only unary connective is $\sim$ ("De Morgan negation"). There are five binary connectives, namely, $\wedge$ ("conjunction"), $\vee$ ("disjunction"), $\circ$ ("fusion"), $\to$ ("implication" or "entailment") and $+$ ("fission"). The set of well-formed formulas is inductively defined from the base set, which comprises the propositional variables and the four zero-ary connectives, by the rest of the connectives. $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ are meta-variables that range over well-formed formulas (wff's, for short).

---

[1] We would like to forewarn the reader that L in LR stands for "lattice" and not for "logistic" as in many labels for sequent calculuses, including the original *L*K and *L*J (where we italicize *L*). Then, *L*LR is a sequent calculus formulation of LR, and so on.

[2] As we already hinted at, the superscript $\boldsymbol{c}$ in the label for the system indicates that four *zero-ary constants* are included.

Multisets constitute a datatype between sequences and sets. In a multiset, an object may have more than one occurrence, and the number of occurrences matters, but the order (of listing) of occurrences is unimportant. Here we always deal with *finite* multisets, that is, with multisets of finitely many objects, each with finitely many occurrences; we will simply talk about multisets. $\alpha,\beta,\gamma,\ldots$ are meta-variables for multisets of wff's including the empty multiset.

Definition 1. The axioms and rules of the *sequent calculus* $L\mathrm{LR}^{\boldsymbol{c}}$ are as follows.

$$\alpha;\boldsymbol{F}\vdash\beta \;\; \boldsymbol{F}\vdash \qquad \mathcal{A}\vdash\mathcal{A} \;\; \mathrm{id} \qquad \alpha\vdash\boldsymbol{T};\beta \;\; \vdash\boldsymbol{T}$$

$$\boldsymbol{f}\vdash \;\; \boldsymbol{f}\vdash \qquad \frac{\alpha\vdash\beta}{\alpha\vdash\boldsymbol{f};\beta}\;\vdash\boldsymbol{f} \qquad \frac{\alpha\vdash\beta}{\alpha;\boldsymbol{t}\vdash\beta}\;\boldsymbol{t}\vdash \qquad \vdash\boldsymbol{t} \;\; \vdash\boldsymbol{t}$$

$$\frac{\alpha;\mathcal{A}\vdash\beta}{\alpha;\mathcal{A}\wedge\mathcal{B}\vdash\beta}\;\wedge\vdash_1 \qquad \frac{\alpha;\mathcal{B}\vdash\beta}{\alpha;\mathcal{A}\wedge\mathcal{B}\vdash\beta}\;\wedge\vdash_2 \qquad \frac{\alpha\vdash\mathcal{A};\beta \qquad \alpha\vdash\mathcal{B};\beta}{\alpha\vdash\mathcal{A}\wedge\mathcal{B};\beta}\;\vdash\wedge$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \qquad \alpha;\mathcal{B}\vdash\beta}{\alpha;\mathcal{A}\vee\mathcal{B}\vdash\beta}\;\vee\vdash \qquad \frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash\mathcal{A}\vee\mathcal{B};\beta}\;\vdash\vee_1 \qquad \frac{\alpha\vdash\mathcal{B};\beta}{\alpha\vdash\mathcal{A}\vee\mathcal{B};\beta}\;\vdash\vee_2$$

$$\frac{\alpha\vdash\mathcal{A};\beta}{\alpha;{\sim}\mathcal{A}\vdash\beta}\;{\sim}\vdash \qquad \frac{\alpha;\mathcal{A}\vdash\beta}{\alpha\vdash{\sim}\mathcal{A};\beta}\;\vdash{\sim}$$

$$\frac{\alpha\vdash\mathcal{A};\beta \qquad \gamma;\mathcal{B}\vdash\delta}{\alpha;\gamma;\mathcal{A}\to\mathcal{B}\vdash\beta;\delta}\;\to\vdash \qquad \frac{\alpha;\mathcal{A}\vdash\mathcal{B};\beta}{\alpha\vdash\mathcal{A}\to\mathcal{B};\beta}\;\vdash\to$$

$$\frac{\alpha;\mathcal{A};\mathcal{B}\vdash\beta}{\alpha;\mathcal{A}\circ\mathcal{B}\vdash\beta}\;\circ\vdash \qquad \frac{\alpha\vdash\mathcal{A};\beta \qquad \gamma\vdash\mathcal{B};\delta}{\alpha;\gamma\vdash\mathcal{A}\circ\mathcal{B};\beta;\delta}\;\vdash\circ$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \qquad \gamma;\mathcal{B}\vdash\delta}{\alpha;\gamma;\mathcal{A}+\mathcal{B}\vdash\beta;\delta}\;+\vdash \qquad \frac{\alpha\vdash\mathcal{A};\mathcal{B};\beta}{\alpha\vdash\mathcal{A}+\mathcal{B};\beta}\;\vdash+$$

$$\frac{\alpha;\mathcal{A};\mathcal{A}\vdash\beta}{\alpha;\mathcal{A}\vdash\beta}\;W\vdash \qquad \frac{\alpha\vdash\mathcal{A};\mathcal{A};\beta}{\alpha\vdash\mathcal{A};\beta}\;\vdash W$$

The notion of a *proof* in $L\mathrm{LR}^{\boldsymbol{c}}$ is as usual in sequent calculuses. $\mathcal{A}$ is a *theorem* of $L\mathrm{LR}^{\boldsymbol{c}}$ iff $\vdash\mathcal{A}$ is a provable sequent.

The original lattice-R does not include the constants, that is, it comprises the axiom (id) and the rules save $(\boldsymbol{t}\vdash)$ and $(\vdash\boldsymbol{f})$. The last two rules, which are called *contraction*, are the only structural rules. Other commonly considered structural rules such as exchange and associativity are inherent in the datatype in the antecedent and succedent, whereas thinning is discarded both on the left and on the right — except for their special instances with $\boldsymbol{t}$ and $\boldsymbol{f}$.

The above sequent calculus is a sensible and well-behaved sequent calculus in light of the following theorem, which involves the single cut rule.

$$\frac{\alpha\vdash\mathcal{C};\beta \qquad \gamma;\mathcal{C}\vdash\delta}{\alpha;\gamma\vdash\beta;\delta}\;\text{single cut}$$

Theorem 2. (Cut theorem for $L\mathrm{LR}^{\boldsymbol{c}}$) *The cut rule is* admissible *in* $L\mathrm{LR}^{\boldsymbol{c}}$.

Proof. The cut rule formulated above is a version of the single cut rule. There are various ways to prove this rule admissible; one of them is by a triple induction on the degree of the cut formula, on the contraction measure of the cut and on the rank of the cut. We do not include the details here.[3] Here is a sample step, in which the degree of the cut formula $\mathcal{A}+\mathcal{B}$ is reduced.

$$\dfrac{\dfrac{\vdots \atop \alpha\vdash\mathcal{A};\mathcal{B};\beta}{\alpha\vdash\mathcal{A}+\mathcal{B};\beta} \qquad \dfrac{\vdots \atop \gamma;\mathcal{A}\vdash\delta \qquad \vdots \atop \varepsilon;\mathcal{B}\vdash\eta}{\gamma;\varepsilon;\mathcal{A}+\mathcal{B}\vdash\delta;\eta}}{\alpha;\gamma;\varepsilon\vdash\beta;\delta;\eta}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\vdots \atop \alpha\vdash\mathcal{A};\mathcal{B};\beta \qquad \vdots \atop \gamma;\mathcal{A}\vdash\delta}{\alpha;\gamma\vdash\mathcal{B};\beta;\delta} \qquad \vdots \atop \varepsilon;\mathcal{B}\vdash\eta}{\alpha;\gamma;\varepsilon\vdash\beta;\delta;\eta} \qquad \triangleleft$$

The proof of the cut theorem also establishes that the addition of the zero-ary constants (one by one, or all at once) is *conservative* over the original LR.

Lattice-R can be defined by an axiom system too. We denote the Hilbert-style system by HLR$^{\boldsymbol{c}}$. This calculus comprises the axiom schemas (A1)–(A17) and the rules (R1)–(R3). (Outside parentheses are omitted from wff's, as before.)

(A1) $\mathcal{A}\to\mathcal{A}$
(A2) $(\mathcal{A}\to\mathcal{B})\to((\mathcal{C}\to\mathcal{A})\to(\mathcal{C}\to\mathcal{B}))$
(A3) $(\mathcal{A}\to(\mathcal{B}\to\mathcal{C}))\to(\mathcal{B}\to(\mathcal{A}\to\mathcal{C}))$
(A4) $(\mathcal{A}\to(\mathcal{A}\to\mathcal{B}))\to(\mathcal{A}\to\mathcal{B})$
(A4–5) $(\mathcal{A}\wedge\mathcal{B})\to\mathcal{A}, \qquad (\mathcal{A}\wedge\mathcal{B})\to\mathcal{B}$
(A7) $((\mathcal{C}\to\mathcal{A})\wedge(\mathcal{C}\to\mathcal{B}))\to(\mathcal{C}\to(\mathcal{A}\wedge\mathcal{B}))$
(A8–9) $\mathcal{A}\to(\mathcal{A}\vee\mathcal{B}), \qquad \mathcal{A}\to(\mathcal{B}\vee\mathcal{A})$
(A10) $((\mathcal{A}\to\mathcal{C})\wedge(\mathcal{B}\to\mathcal{C}))\to((\mathcal{A}\vee\mathcal{B})\to\mathcal{C})$
(A11–2) $(\sim\mathcal{A}\to\mathcal{B})\to(\sim\mathcal{B}\to\mathcal{A}), \qquad \mathcal{A}\to\sim\sim\mathcal{A}$
(A13-4) $\boldsymbol{t}, \qquad (\boldsymbol{t}\to\sim\boldsymbol{f})\wedge(\boldsymbol{f}\to\sim\boldsymbol{t})$
(A15) $(\boldsymbol{F}\to\mathcal{A})\wedge(\mathcal{A}\to\boldsymbol{T})$
(A16) $((\mathcal{A}\circ\mathcal{B})\to\sim(\mathcal{A}\to\sim\mathcal{B}))\wedge(\sim(\mathcal{A}\to\sim\mathcal{B})\to(\mathcal{A}\circ\mathcal{B}))$
(A17) $((\mathcal{A}+\mathcal{B})\to(\sim\mathcal{A}\to\mathcal{B}))\wedge((\sim\mathcal{A}\to\mathcal{B})\to(\mathcal{A}+\mathcal{B}))$
(R1) $\mathcal{A}\to\mathcal{B}$ and $\mathcal{A}$ imply $\mathcal{B}$
(R2) $\mathcal{A}$ and $\mathcal{B}$ imply $\mathcal{A}\wedge\mathcal{B}$
(R3) $\vdash\mathcal{A}$ implies $\vdash\boldsymbol{t}\to\mathcal{A}$

The notion of a *proof* is the usual one for axiom systems, and the formulas occurring in a proof are called *theorems*.

The axiom system HLR$^{\boldsymbol{c}}$ is equivalent to LLR$^{\boldsymbol{c}}$ in the sense that the two calculuses have the same set of theorems, as we state in the following theorem. (We leave the proof, which is completely routine, to the reader.)

Theorem 3. *$\mathcal{A}$ is a* theorem *of HLR$^{\boldsymbol{c}}$ iff it is a* theorem *of LLR$^{\boldsymbol{c}}$.*

[3]Some details of a similar proof may be found in Bimbó [8, §2].

2. MODALITIES IN LLR$^{\boldsymbol{c}}$ DEFINED FROM $\boldsymbol{t}$ AND $\boldsymbol{f}$

The symbols $\Diamond$ and $\Box$ usually stand for *unary modalities*, which are read as "diamond" and "box," or in alethic modal logics, as "possibility" and "necessity." The presence of $\boldsymbol{t}$ and $\boldsymbol{f}$ in LR$^{\boldsymbol{c}}$ allows us to define surrogate unary connectives.

Definition 4. $\Box\mathcal{A}$ is $\boldsymbol{t}\wedge\mathcal{A}$, and $\Diamond\mathcal{A}$ is $\mathcal{A}\vee\boldsymbol{f}$.

Of course, the above definition in itself is nothing more than looking at formulas with a squint. However, $\Box$ and $\Diamond$ turn out to have certain properties that are reminiscent of properties the modalities often have. The notation that we introduced was intended to prefigure this.

Lemma 5. *The formulas in* (1)–(4) *are* theorems *of* LLR$^{\boldsymbol{c}}$*, and by* (5)*, necessitation is an* admissible rule *in* LLR$^{\boldsymbol{c}}$.

(1) $\Box(\mathcal{A}\to\mathcal{B})\to(\Box\mathcal{A}\to\Box\mathcal{B})$

(2) $\Box\mathcal{A}\to\mathcal{A}$

(3) $\Box\mathcal{A}\to\Box\Box\mathcal{A}$

(4) $(\Diamond\mathcal{A}\to{\sim}\Box{\sim}\mathcal{A})\wedge({\sim}\Box{\sim}\mathcal{A}\to\Diamond\mathcal{A})$

(5) *If* $\vdash\mathcal{A}$*, then* $\vdash\Box\mathcal{A}$.

Proof. The proofs of the corresponding formulas are straightforward, once the defined symbols are rewritten with the primitive connectives. For instance, (1) turns into the formula $(\boldsymbol{t}\wedge(\mathcal{A}\to\mathcal{B}))\to((\boldsymbol{t}\wedge\mathcal{A})\to(\boldsymbol{t}\wedge\mathcal{B}))$. (We omit the rest of the details.) $\triangleleft$

The formulas in (1)–(3) resemble some well-known axioms from (normal) modal logics, when $\Box$ is viewed as $\Box$, $\wedge$ as $\wedge$, and $\to$ is taken to be $\supset$ (i.e., classical conditional). In particular, (1) looks like (K), (2) looks like (T) and (3) looks like (4).[4] It may be tempting, at first sight, to conjecture that we have found S4 in LLR$^{\boldsymbol{c}}$. However, we should not forget that $\sim$ is not an orthonegation, and $\wedge$ and $\vee$ are not related to each other or to $\to$ in the way conjunction and disjunction are linked to $\supset$ (and $\neg$, orthonegation). We find another logic hidden within LLR$^{\boldsymbol{c}}$ though.

Linear logic, as defined in Girard [23], is sometimes called *classical linear logic*, because it shares more features with classical logic than with intuitionist logic.[5] We denote this logic by CLL. Linear logic without the modalities is called *multiplicative–additive linear logic* (or MALL). Linear logic was first defined as a one-sided sequent calculus. However, all fragments of classical linear logic that contain the negation connective may be defined equivalently as two-sided sequent calculuses.[6]

Classical linear logic can be (and has been) formulated in various ways, as in Avron [4] and [44], for instance. For our goals in this paper, it is convenient to rely on a sequent calculus formulation. Moreover, we will assume that sequents are defined as before, that is, they comprise a pair of multisets of wff's. The language of propositional CLL contains several connectives, and [23] uses unconventional notation to denote them. A translation that turns a symbol into a symbol that looks the same in another language is very manageable; hence, we list Girard's symbols together with his names for the connectives, but we immediately give our preferred notation that

[4] This is not a typo; (4) is the standard label for the characteristic axiom of the system S4.

[5] See, e.g., Troelstra [44], where *intuitionist linear logic* is introduced too.

[6] See Bimbó [9] for a comprehensive treatment of sequent calculuses — including calculuses for classical linear logic.

induces an identity translation between languages of logics. (In Sections 6 and 7, we turn back to using Girard's notation to facilitate comparisons.)

The zero-ary connectives are **1** (*one*, $\boldsymbol{t}$), $\bot$ (*bottom*, $\boldsymbol{f}$), $\top$ (*top*, $\boldsymbol{T}$) and **0** (*null*, $\boldsymbol{F}$). The unary connectives are ${}^{\bot}$ (*nil*, $\sim$), ! (*of course*, $\Diamond$ or $\Box$) and ? (*why not*, $\Box$ or $\Diamond$). The binary connectives are $\&$ (*with*, $\wedge$), $\oplus$ (*plus*, $\vee$), $\otimes$ (*times*, $\circ$), $⅋$ (*par*, $+$) and $\multimap$ (*entail*, $\to$).

For the so-called exponentials (! and ?), we listed both modalities. The first modality is motivated by relational semantics, whereas the second one is based on similarities of sequent calculus rules for the punctuation marks and for modalities. For the sake of translating and comparing sequent calculuses in this paper, we use the second variant. The issue is that when the Church constants ($\top$ and **0**) are not definable from negation using the lattice operations, Kripke's rules for the modalities (or their adaptations for ! and ?) do not provide both (dual) additivity and (dual) normality for either of the two monotone operations.

In a *two-sided sequent calculus* for classical linear logic, which we denote by *L*CLL, the connective rules for the connectives that have an alter ego in *L*LR$^{\boldsymbol{c}}$ are exactly as in *L*LR$^{\boldsymbol{c}}$. (Hence, we do no repeat those rules; rather, we simply assume that *L*CLL is formulated with standard vocabulary.) The contraction rules $(W\vdash)$ and $(\vdash W)$ are absent from *L*CLL. However, the rules below allow the introduction of ! and ? on the right- and left-hand sides of the turnstile, and they recuperate the effect of some of the contractions and thinnings in a traceable way.

Definition 6. The eight rules that involve the exponential connectives are the following. $!\alpha$ and $?\alpha$ are multisets in which the main connective of each formula is, respectively, ! and ?.

$$\frac{\alpha;\mathcal{A}\vdash\beta}{\alpha;\,!\mathcal{A}\vdash\beta}\;{!\vdash}\qquad\frac{!\alpha\vdash\mathcal{A};\,?\beta}{!\alpha\vdash\,!\mathcal{A};\,?\beta}\;{\vdash!}\qquad\frac{!\alpha;\mathcal{A}\vdash\,?\beta}{!\alpha;\,?\mathcal{A}\vdash\,?\beta}\;{?\vdash}\qquad\frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash\,?\mathcal{A};\beta}\;{\vdash?}$$

$$\frac{\alpha;\,!\mathcal{A};\,!\mathcal{A}\vdash\beta}{\alpha;\,!\mathcal{A}\vdash\beta}\;{!W\vdash}\qquad\frac{\alpha\vdash\beta}{\alpha;\,!\mathcal{A}\vdash\beta}\;{!K\vdash}\qquad\frac{\alpha\vdash\beta}{\alpha\vdash\,?\mathcal{A};\beta}\;{\vdash?K}\qquad\frac{\alpha\vdash\,?\mathcal{A};\,?\mathcal{A};\beta}{\alpha\vdash\,?\mathcal{A}}\;{\vdash?W}$$

If we simply omit the $(W\vdash)$ and $(\vdash W)$ rules from *L*LR$^{\boldsymbol{c}}$, then we obtain *L*MALL, a sequent calculus formalization of MALL.

Our goal now is to establish that the defined modalities in *L*LR$^{\boldsymbol{c}}$ behave sufficiently similarly to the exponentials (i.e., the modalities) of *L*CLL. Moreover, the proof of the next theorem provides a translation of wff's of CLL into LR$^{\boldsymbol{c}}$, which is of special philosophical interest, given that classical linear logic's constructive character is primarily manifest via the translation of intuitionist logic into CLL. In a similar sense, *L*LR$^{\boldsymbol{c}}$ is *linear* and *constructive*.

Theorem 7. (From *L*CLL to *L*LR$^{\boldsymbol{c}}$) *If $\mathcal{A}$ is a theorem of* *L*CLL, *then $\tau(\mathcal{A})$ is a theorem of* *L*LR$^{\boldsymbol{c}}$, *where τ is defined inductively by* (1)–(6).

(1) $\tau(p)$ *is* p, *when* p *is a propositional variable;*
(2) $\tau(\boldsymbol{c})$ *is* $\boldsymbol{c}$, *where* $\boldsymbol{c}$ *is a zero-ary constant;*
(3–5) $\tau(\,!\mathcal{A})$ *is* $\boldsymbol{t}\wedge\tau(\mathcal{A})$*;* $\tau(\,?\mathcal{A})$ *is* $\tau(\mathcal{A})\vee\boldsymbol{f}$*;* $\tau(\mathcal{A}^{\bot})$ *is* $\sim\tau(\mathcal{A})$*;*
(6) $\tau(\mathcal{A}\circledast\mathcal{B})$ *is* $\tau(\mathcal{A})\circledast\tau(\mathcal{B})$, *where* $\circledast$ *is a binary connective.*

Proof. First, we note that τ is well-defined in the sense that it is applicable to any wff of *L*CLL, and it results in a unique wff of *L*LR$^{\boldsymbol{c}}$.

The proof is by induction on χ, the height of a proof tree with root $\vdash \mathcal{A}$. We prove that if $\alpha \vdash \beta$ is provable in *L*CLL, then $\tau(\alpha) \vdash \tau(\beta)$ is provable in *L*LR$^{\boldsymbol{c}}$. (τ is applied piece-wise to a multiset, and the translation of the empty multiset is itself.)

1. If $\chi = 1$, then the proof is an instance of an axiom. We note that τ is independent of the location of a formula within a sequent. Therefore, $\tau(\mathcal{A}) \vdash \tau(\mathcal{A})$ yields $\mathcal{B} \vdash \mathcal{B}$, where $\mathcal{B}$ may be $\mathcal{A}$ or may be a different formula than $\mathcal{A}$ (if there are occurrences of $!$ or $?$ in $\mathcal{A}$). Either way, $\mathcal{B} \vdash \mathcal{B}$ is an instance of (id) in *L*LR$^{\boldsymbol{c}}$.

If the axiom is one of those that involve a zero-ary constant, then the claim is obviously true too.

2. If $\chi > 1$, then $\alpha \vdash \beta$ is by a rule.

2.1. The non-modal connective rules of *L*CLL turn into identical rules in *L*LR$^{\boldsymbol{c}}$; furthermore, the latter rules are insensitive to the concrete shape of the parametric or subaltern wff's.[7] As an example, we consider the $(\vdash \wedge)$ rule. α is α, whereas β is $\mathcal{A} \wedge \mathcal{B}; \gamma$. On the left, we have the proof segment in *L*CLL, on the right, we have the resulting proof segment in *L*LR$^{\boldsymbol{c}}$. The upper sequents are given by the hypotheses of the induction that we indicate by "i.h."

$$\dfrac{\begin{array}{c}\vdots\\ \alpha \vdash \mathcal{A};\gamma\end{array} \qquad \begin{array}{c}\vdots\\ \alpha \vdash \mathcal{B};\gamma\end{array}}{\alpha \vdash \mathcal{A} \wedge \mathcal{B};\gamma} \qquad \overset{\text{i.h.}}{\rightsquigarrow} \qquad \dfrac{\begin{array}{c}\vdots\\ \tau(\alpha) \vdash \tau(\mathcal{A});\tau(\gamma)\end{array} \qquad \begin{array}{c}\vdots\\ \tau(\alpha) \vdash \tau(\mathcal{B});\tau(\gamma)\end{array}}{\tau(\alpha) \vdash \tau(\mathcal{A}) \wedge \tau(\mathcal{B});\tau(\gamma)}$$

By clause (6), $\tau(\mathcal{A} \wedge \mathcal{B})$ is $\tau(\mathcal{A}) \wedge \tau(\mathcal{B})$. The other cases for the rules for non-modal connectives has the same general structure, and we omit including their details here.

2.2. The last rule may be a modal connective rule. *L*CLL has the same pleasing symmetry as *L*K, the original sequent calculus for classical logic has; hence, we consider in some detail the cases for $(\,!\vdash)$ and $(\vdash\,!)$, but leave the details of the dual cases (i.e., of $(\,?\vdash)$ and $(\vdash\,?)$) to the reader.

We have the following subtrees.

$$\dfrac{\begin{array}{c}\vdots\\ \gamma;\mathcal{A} \vdash \beta\end{array}}{\gamma;\,!\mathcal{A} \vdash \beta} \qquad \overset{\text{i.h.}}{\rightsquigarrow} \qquad \dfrac{\begin{array}{c}\vdots\\ \tau(\gamma);\tau(\mathcal{A}) \vdash \tau(\beta)\end{array}}{\tau(\gamma);\boldsymbol{t} \wedge \tau(\mathcal{A}) \vdash \tau(\beta)}$$

By (3), we know that $\tau(\,!\mathcal{A})$ is $\boldsymbol{t} \wedge \tau(\mathcal{A})$, as needed.

If the sequent $\alpha \vdash \beta$ is $!\gamma \vdash\, !\mathcal{A};\, ?\delta$ by $(\vdash\,!)$, then we have the following chunks of proofs.

$$\dfrac{\begin{array}{c}\vdots\\ !\gamma \vdash \mathcal{A};\,?\delta\end{array}}{!\gamma \vdash\, !\mathcal{A};\,?\delta} \qquad \overset{\text{i.h.}}{\rightsquigarrow} \qquad \dfrac{\begin{array}{c}\vdots\\ \tau(\,!\gamma) \vdash \tau(\mathcal{A});\tau(\,?\delta)\end{array} \qquad \dfrac{\vdash \boldsymbol{t}}{\tau(\,!\gamma) \vdash \boldsymbol{t};\tau(\,?\delta)}}{\tau(\,!\gamma) \vdash \boldsymbol{t} \wedge \tau(\mathcal{A});\tau(\,?\delta)}$$

The thicker line indicates possibly several applications of rules — depending on the number of wff's in $!\gamma$ and $?\delta$. For any wff $!\mathcal{B}$ in $!\gamma$, its translation is $\boldsymbol{t} \wedge \tau(\mathcal{B})$, whereas, for any wff $?\mathcal{C}$ in $?\delta$, its translation is $\tau(\mathcal{C}) \vee \boldsymbol{f}$. Each $\boldsymbol{t} \wedge \tau(\mathcal{B})$ can be obtained by $(\boldsymbol{t}\vdash)$ and $(\wedge\vdash)$; analogously, $\tau(\mathcal{C}) \vee \boldsymbol{f}$ may be gotten by $(\vdash\boldsymbol{f})$ and $(\vdash\vee)$. The last step above is justified by $(\vdash\wedge)$.

[7]These terms have their usual meanings following Curry [16].

2.3. There are four modalized structural rules in *L*CLL. First of all, the modalized contraction rules are special instances of their regular counterparts in *L*LR$^{\boldsymbol{c}}$. That is, the claim is obviously true when the last rule is $(\,!W\vdash)$ or $(\vdash\,?W)$.

If the last rule applied in the *L*CLL proof is $(\,!K\vdash)$, then α is $\gamma;\,!\mathcal{A}$, and we have the following.

$$\dfrac{\stackrel{\vdots}{\gamma\vdash\beta}}{\gamma;\,!\mathcal{A}\vdash\beta} \quad \stackrel{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\dfrac{\stackrel{\vdots}{\tau(\gamma)\vdash\tau(\beta)}}{\tau(\gamma);\boldsymbol{t}\vdash\tau(\beta)}}{\tau(\gamma);\boldsymbol{t}\wedge\tau(\mathcal{A})\vdash\tau(\beta)}$$

The rules applied in *L*LR$^{\boldsymbol{c}}$ are $(\boldsymbol{t}\vdash)$ and $(\wedge\vdash)$. The latter rule is applicable with an arbitrary $\tau(\mathcal{A})$, and $\tau(\,!\mathcal{A})$ is $\boldsymbol{t}\wedge\tau(\mathcal{A})$ by clause (3). ◁

The theorem provides a way to test wff's of *L*CLL for *non-provability*, because if their translation is not provable in *L*LR$^{\boldsymbol{c}}$, then the starting formula is not provable in *L*CLL. Of course, we are using the fact, which is well known to relevance logicians, that *L*LR is decidable. We provide some details of the proof for *L*LR$^{\boldsymbol{c}}$ in Section 3. Provability is easily decidable if a wff of CLL does not contain occurrences of ! or ?, because *L*LRW$^{\boldsymbol{c}}$'s decidability is an immediate consequence of the cut theorem. (*L*LRW$^{\boldsymbol{c}}$ is *L*LR$^{\boldsymbol{c}}$ without the $(\vdash W)$ or $(W\vdash)$ rules.)

An insight that we attribute to Kripke [30] is that, in relevance logics, a wff has to be introduced by a connective rule in order to be contracted. Once stated, the truth of this observation is obvious. However, a profound consequence, as Kripke realized, is that the contraction rules can be eliminated if operational rules *permit* some contraction but require none. Relying on the same observation, the amount of the permitted contractions in each operational rule may be minimized. The insight that we attribute to Dunn, is that it is sufficient to allow a formula to be contracted if it could not have been contracted in the premises.

In order to motivate the introduction of heap numbers (in Definition 8 below), we illustrate how we use heap numbers extracted from irredundant proofs in one calculus to bound the number of permitted contractions in another — but related — calculus.[8]

If we assume the usual definition of a subformula, then we may note the obvious fact that every formula has at least one subformula, but "often" it has more. Furthermore, if we count distinct occurrences of a subformula separately, then we find that some formulas have even more subformulas (in the sense of subformula occurrences). Then it is obvious that permitting as many contractions on a larger formula as we performed on some of its proper subformulas will produce at least as many or more occurrences of subformulas. Let us consider a small proof in [*L*LR$^{\boldsymbol{c}}$] (cf. Definition 11).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A}\vdash\mathcal{A}\quad\mathcal{B}\vdash\mathcal{B}}{\mathcal{A},\mathcal{B}\vdash\mathcal{A}\circ\mathcal{B}}{\scriptstyle[\vdash\circ]}\qquad\dfrac{\mathcal{A}\vdash\mathcal{A}\quad\mathcal{B}\vdash\mathcal{B}}{\mathcal{A},\mathcal{B}\vdash\mathcal{A}\circ\mathcal{B}}{\scriptstyle[\vdash\circ]}}{\mathcal{A},\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}{\scriptstyle[\vdash\circ]}}{\mathcal{A}\wedge\mathcal{B},\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}{\scriptstyle[\wedge\vdash]}}{\mathcal{A}\wedge\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}{\scriptstyle[\wedge\vdash]}}{\boldsymbol{t}\wedge(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}{\scriptstyle[\wedge\vdash]}$$

[8] The term "irredundant" is used in its standard sense in relevance logic; see Dunn [18, §3.6].

This proof is irredundant — with the third application of $[\vdash\circ]$ containing contractions of $\mathcal{A}$ and $\mathcal{B}$, and the second application of $[\wedge\vdash]$ containing a contraction of $\mathcal{A}\wedge\mathcal{B}$. Each of those formulas are subformulas of $\,!(\mathcal{A}\wedge\mathcal{B})$ (which translates via τ into $\boldsymbol{t}\wedge(\mathcal{A}\wedge\mathcal{B})$). Thus the heap number of $\,!(\mathcal{A}\wedge\mathcal{B})$ is at least 3. (We have not generated all the irredundant proofs here, but having this proof we know that the heap number cannot be less than 3.) The following proof in [*L*CLL] uses three contractions as part of applications of the $[\,!\vdash]$ rule. This proof also happens to be irredundant, however, that is an accidental feature. In the proof search that uses the heap number as an upper bound on contractions, we do not require the resulting proof to be irredundant. Accordingly, applications of the $(\,!\vdash)$ and $(\,!W\vdash)$ rules may be separated without loss of generality.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A}\vdash\mathcal{A}\quad\mathcal{B}\vdash\mathcal{B}}{\mathcal{A},\mathcal{B}\vdash\mathcal{A}\circ\mathcal{B}}\qquad\dfrac{\mathcal{A}\vdash\mathcal{A}\quad\mathcal{B}\vdash\mathcal{B}}{\mathcal{A},\mathcal{B}\vdash\mathcal{A}\circ\mathcal{B}}}{\mathcal{A},\mathcal{B},\mathcal{A},\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{\mathcal{A},\mathcal{B},\mathcal{A},\mathcal{A}\wedge\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{\mathcal{A},\mathcal{B},\mathcal{A}\wedge\mathcal{B},\mathcal{A}\wedge\mathcal{B}\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{\mathcal{A},\mathcal{B},\mathcal{A}\wedge\mathcal{B},\,!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{\mathcal{A},\mathcal{B},\,!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}\,[\,!\vdash]}{\mathcal{A},\mathcal{A}\wedge\mathcal{B},\,!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{\mathcal{A},\,!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}\,[\,!\vdash]}{\mathcal{A}\wedge\mathcal{B},\,!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}}{!(\mathcal{A}\wedge\mathcal{B})\vdash(\mathcal{A}\circ\mathcal{B})\circ(\mathcal{A}\circ\mathcal{B})}\,[\,!\vdash]$$

This time, we only labeled the steps that involve a contraction.

As already hinted at by the illustration, we will rely on the theorem (proved in the next section) that *L*LR^{e} is decidable. This result is a small extension of the decidability of *L*LR originally proved in [36].[9]

It may be helpful to note that the decidability proof using a proof-search tree with the sequent calculus [*L*LR^{e}] provides *all the irredundant proofs* of a provable sequent — unlike the example above where we only presented one irredundant proof.

Now we turn to the definition of heap numbers. Our definition uses the notion of "ancestors," which is essentially, Curry's notion (see [16, p. 199]), with some obvious modifications that are due to our calculuses being based on multisets. We briefly explain the notion ancestors in the paragraph after the definition.

---

[9]We thank Alasdair Urquhart for calling to our attention (in December 2016) the preprint paper Roorda [39], which claimed to have proved the decidability of classical linear logic. We did not know of Roorda's paper until well after we had our own proof, but his strategy is remarkably similar to ours. He uses the method of Kripke to construct a finite proof-search tree, but the problem seems to be that there is no guarantee that his tree will contain a proof of the candidate theorem if there is one. We provide such a guarantee via our *heap number* in Definition 8. As Urquhart pointed out to us, Roorda's proof does not appear in his subsequent Ph.D. thesis [40]. In fact, on p. 12 of his thesis, he mentions [35] and repeats their claim that CLL is undecidable. So, he apparently came to consider their proof to be correct and his own earlier proof to be mistaken.

Definition 8. (Heap number) Let $\vdash \mathcal{A}$ be a provable sequent. The *heap number of* $\mathcal{B}$ (where $\mathcal{B}$ is a subformula of $\mathcal{A}$) is the maximum of the total number of contractions on the ancestors of $\mathcal{B}$ in any irredundant proof of the sequent.

Given a proof, $\mathcal{B}$ may be parametric in an application of a rule, in which case, it has immediate parametric ancestors in the upper sequent. If $\mathcal{B}$ is the principal formula of a rule then it typically has subalterns in the upper sequent. (Since the calculuses from which we calculate the heap numbers have no explicit contraction rules, only the thinning rules have no subalterns in the upper sequent.) We call immediate parametric ancestors and subalterns immediate ancestors. If contraction is built into a rule, then the principal formula may encompass contractions of parametric formulas, in which case all the affected immediate parametric ancestors as well as the subalterns are immediate ancestors of the principal formula. $\mathcal{C}$ is an ancestor of $\mathcal{B}$ when $\mathcal{C}$ is in the reflexive transitive closure of the *immediate ancestor of* $\mathcal{B}$ relation.

Lemma 9. *Let* $\vdash \mathcal{A}$ *be a provable sequent of* LCLL*. The heap numbers of the subformulas of* $\mathcal{A}$ *(obtained from proofs in* [LLR$^{\boldsymbol{c}}$]*) are sufficiently large as bounds on the number of contractions on each formula to construct a proof of* $\vdash \mathcal{A}$ *in* LCLL *(or* $[\![L\mathrm{CLL}]\!]$*).*

Proof. It is sufficient to consider the right-handed sequent calculus for CLL. Hence, the only contraction rule is $(\vdash ?W)$. There are two rules (beyond those for zeroary constants) that can introduce several formulas into the sequent, namely, $(\vdash \circ)$ and $(\vdash ?K)$. Clearly, a formula introduced by the latter does not need to be considered, because the rule has no subaltern. (This means that if $\mathcal{A} \vee \boldsymbol{f}$ resulted from $(\vdash \boldsymbol{f})$ in [LLR$^{\boldsymbol{c}}$], then $?\mathcal{A}$ can be introduced by $(\vdash ?K)$ in LCLL, if needed.) If the immediate subformula of the contracted $?\mathcal{A}$ was introduced by $(\vdash \circ)$, then the proof-search might have contained contractions on $?\mathcal{A}$ or on its subformulas. Let us assume that $n-1$ contractions on $?\mathcal{A}$ are not sufficient for the proof of the sequent in LCLL, and the proof search in [LLR$^{\boldsymbol{c}}$] provided a heap number $\leq n-1$. Then there are formulas in the sequent that cannot be contracted in LCLL, which require at least n contractions on $?\mathcal{A}$. However, if a formula cannot be contracted in LCLL, then it remains in the sequent; hence, the sequent at the beginning of the proof search in [LLR$^{\boldsymbol{c}}$] contains any such formula. That is, contractions on those formulas cannot reduce the number of required contractions on $?\mathcal{A}$ and its subformulas in [LLR$^{\boldsymbol{c}}$]. $\lhd$

Theorem 10. *Classical linear logic* (CLL) *is decidable.*

Proof. Given a wff $\mathcal{A}$ of CLL, τ yields a wff of LLR$^{\boldsymbol{c}}$. It is decidable whether $\tau(\mathcal{A})$ is a theorem of LLR$^{\boldsymbol{c}}$; if it is not, then $\mathcal{A}$ is not a theorem of CLL. If $\tau(\mathcal{A})$ is a theorem of CLL, then we can identify all the subformulas of $\tau(\mathcal{A})$, which result from the translation of an exponential subformula; we call them exceptional. The decision procedure for $\tau(\mathcal{A})$ in [LLR$^{\boldsymbol{c}}$] produces all the irredundant proofs, hence, we can calculate the heap number for each exceptional formula. Then we search for a proof of $\tau(\mathcal{A})$ in a *restricted version* of [LLR$^{\boldsymbol{c}}$] by building a proof search tree in which contractions on exceptional formulas are limited by their heap number.

The restrictions on the rules of [LLR$^{\boldsymbol{c}}$] are the following.

1. No contraction is permitted in $[\vdash \wedge]$, $[\vee \vdash]$, $[\sim \vdash]$, $[\vdash \sim]$, $[\circ \vdash]$, $[\vdash +]$, $[\vdash \rightarrow]$.

2. A contraction is permitted in $[\wedge_1 \vdash]$ and $[\wedge_2 \vdash]$ when $\mathcal{A} \wedge \mathcal{B}$ is an exceptional wff. Dually, a contraction is permitted in $[\vdash \vee_1]$ and $[\vdash \vee_2]$, if $\mathcal{A} \vee \mathcal{B}$ is an exceptional wff.
3. A contraction is permitted in $\alpha;\gamma$ in $[\vdash \circ]$, $[+ \vdash]$ and $[\rightarrow\vdash]$ if an exceptional formula $\boldsymbol{t} \wedge \mathcal{A}$ occurs both in α and γ. Dually, a contraction is permitted in $\beta;\delta$ in an application of the same rules if an exceptional formula $\mathcal{A} \vee \boldsymbol{f}$ occurs both in β and δ. (The principal formulas of these rules cannot be contracted.)

The above restrictions match exactly the restricted contraction rules in CLL (see Definition 6), whereas the heap numbers provide the upper bounds on the number contractions. Therefore, the proof-search tree is finite. Since $L\mathrm{LR}^{\boldsymbol{c}}$ and LCLL coincide on the exponential-free fragment of CLL, and we allowed heap-number-many contractions on exceptional formulas, if $\mathcal{A}$ is a theorem of CLL, then the proof-search tree will contain a proof of $\tau(\mathcal{A})$.

Once we have proofs in the proof-search tree for the formula $\mathcal{A}$, we also check that any applications of $[\vdash \wedge]$ and $[\vee \vdash]$ with principal formulas that are translations of a !'d or ?'d formula satisfy the side conditions in Kripke's rules (i.e., of the $(\vdash\, !)$ and $(\,?\vdash)$ rules). $\triangleleft$

The theorem contradicts Theorem 3.7 in [35], which states the undecidability of classical linear logic, and Corollaries 5.5 and 5.7 in [28], which state the undecidability of two Horn-fragments of linear logic. We believe that those papers do not contain proofs of the undecidability of CLL, and will provide our argument for this in Section 7. We also give another proof of the decidability of CLL below; that proof also uses in an essential way the Curry–Kripke strategy.

3. Lattice-R's Decidability

This section is a rather detailed presentation of the decidability of $L\mathrm{LR}^{\boldsymbol{c}}$. The decidability of LR was proved by Meyer in 1966 [36]. The addition of the zero-ary constants is not a huge extension of that result, and it has a certain resemblance to the extension of the decidability result for $\mathrm{R}_{\rightarrow}$ proved by Kripke in 1959 [30] to a proof of the decidability of $\mathrm{R}^{\boldsymbol{t}}_{\rightarrow}$ in Bimbó and Dunn [13].

A core idea is to define a *contraction-free sequent calculus* that allows the proof of the same sequents as $L\mathrm{LR}^{\boldsymbol{c}}$ does. This sequent calculus must be orderly, that is, the *cut theorem* has to hold for it, and cut-free proofs must have the *subformula property*. Then, we build a *proof-search tree*, which can be shown to be *finite*.

Definition 11. The *sequent calculus* $[L\mathrm{LR}^{\boldsymbol{c}}]$ is defined by the following axioms and rules. (Sequents are as before, and the bracket notation is explained below. This use of brackets motivates the label $[L\mathrm{LR}^{\boldsymbol{c}}]$ for the calculus.)

$$\alpha;\boldsymbol{F}\vdash\beta \;\; {\scriptstyle \boldsymbol{F}\vdash} \qquad \mathcal{A}\vdash\mathcal{A} \;\; {\scriptstyle \mathrm{id}} \qquad \alpha\vdash\boldsymbol{T};\beta \;\; {\scriptstyle \vdash\boldsymbol{T}}$$

$$\boldsymbol{f}\vdash \;\; {\scriptstyle \boldsymbol{f}\vdash} \qquad \frac{\alpha\vdash\beta}{\alpha\vdash\boldsymbol{f};\beta} \;{\scriptstyle \vdash\boldsymbol{f}} \qquad \frac{\alpha\vdash\beta}{\alpha;\boldsymbol{t}\vdash\beta} \;{\scriptstyle \boldsymbol{t}\vdash} \qquad \vdash\boldsymbol{t} \;\; {\scriptstyle \vdash\boldsymbol{t}}$$

$$\frac{\alpha;\mathcal{A}\vdash\beta}{[\alpha;\mathcal{A}\wedge\mathcal{B}]\vdash\beta} \;{\scriptstyle [\wedge\vdash_1]} \qquad \frac{\alpha;\mathcal{B}\vdash\beta}{[\alpha;\mathcal{A}\wedge\mathcal{B}]\vdash\beta} \;{\scriptstyle [\wedge\vdash_2]} \qquad \frac{\alpha\vdash\mathcal{A};\beta \qquad \alpha\vdash\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\wedge\mathcal{B};\beta]} \;{\scriptstyle [\vdash\wedge]}$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \qquad \alpha;\mathcal{B}\vdash\beta}{[\alpha;\mathcal{A}\vee\mathcal{B}]\vdash\beta}\ [\vee\vdash] \qquad \frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash[\mathcal{A}\vee\mathcal{B};\beta]}\ [\vdash\vee_1] \qquad \frac{\alpha\vdash\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\vee\mathcal{B};\beta]}\ [\vdash\vee_2]$$

$$\frac{\alpha\vdash\mathcal{A};\beta}{[\alpha;\sim\mathcal{A}]\vdash\beta}\ [\sim\vdash] \qquad \frac{\alpha;\mathcal{A}\vdash\beta}{\alpha\vdash[\sim\mathcal{A};\beta]}\ [\vdash\sim]$$

$$\frac{\alpha;\mathcal{A};\mathcal{B}\vdash\beta}{[\alpha;\mathcal{A}\circ\mathcal{B}]\vdash\beta}\ [\circ\vdash] \qquad \frac{\alpha\vdash\mathcal{A};\beta \qquad \gamma\vdash\mathcal{B};\delta}{[\alpha;\gamma]\vdash[\mathcal{A}\circ\mathcal{B};\beta;\delta]}\ [\vdash\circ]$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \qquad \gamma;\mathcal{B}\vdash\delta}{[\alpha;\gamma;\mathcal{A}+\mathcal{B}]\vdash[\beta;\delta]}\ [+\vdash] \qquad \frac{\alpha\vdash\mathcal{A};\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}+\mathcal{B};\beta]}\ [\vdash+]$$

$$\frac{\alpha\vdash\mathcal{A};\beta \qquad \gamma;\mathcal{B}\vdash\delta}{[\alpha;\gamma;\mathcal{A}\to\mathcal{B}]\vdash[\beta;\delta]}\ [\to\vdash] \qquad \frac{\alpha;\mathcal{A}\vdash\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\to\mathcal{B};\beta]}\ [\vdash\to]$$

Bracketing happens in *three kinds* of situations.

1. A parametric multiset is joined with the principal wff of a rule.
2. Two parametric multisets are joined with the principal wff of a rule.
3. Two parametric multisets are joined (without the addition of the principal wff of a rule).

Situations of type 1 occur in all the $\wedge$, $\vee$ and $\sim$ rules, as well as in $[\circ\vdash]$, $[\vdash+]$ and $[\vdash\to]$. Situations of type 2 and 3 occur in the rules $[\vdash\circ]$, $[+\vdash]$ and $[\to\vdash]$ — on one or another side of the turnstile.

Definition 12. The bracketing indicates the following potential contractions — without a total loss of a wff, of course — in the respective multisets. None of the contractions is mandatory, that is, any rule can be applied without contraction, if desired.

1. The principal wff may be contracted once, if it already occurs in the parametric multiset.
2. The principal wff may be contracted once or twice, if it already occurs in one or both parametric multisets, respectively. A parametric wff may be contracted once, if it occurs in both parametric multisets.
3. A wff may be contracted once, if it already occurs in both parametric multisets.

Theorem 13. (Cut theorem for [LLR^c]) *The cut rule is* admissible *in* [LLR^c].

Proof. The proof can be carried out more or less along similar lines as the proof of Theorem 2. However, instead of dealing with the contraction rules separately, we have to verify that all the contractions, which could be carried out in a given proof, can be carried out in the transformed proof too. As a sample transformation, we give a transformation where ρ, the rank is minimal and the cut formula is a conjunction.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \alpha\vdash\mathcal{A};\beta\end{matrix} \qquad \begin{matrix}\vdots\\ \alpha\vdash\mathcal{B};\beta\end{matrix}}{\alpha\vdash\mathcal{A}\wedge\mathcal{B};\beta} \qquad \dfrac{\begin{matrix}\vdots\\ \gamma;\mathcal{A}\vdash\delta\end{matrix}}{\gamma;\mathcal{A}\wedge\mathcal{B}\vdash\delta}}{[\alpha;\gamma]\vdash[\beta;\delta]} \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ \alpha\vdash\mathcal{A};\beta\end{matrix} \qquad \begin{matrix}\vdots\\ \gamma;\mathcal{A}\vdash\delta\end{matrix}}{[\alpha;\gamma]\vdash[\beta;\delta]}$$

Since $\mathcal{A}\wedge\mathcal{B}$ does not occur in β or γ due to the assumption about the rank, any contraction must be part of the cut. All such contractions can be performed as part of the cut in the new proof. (We omit the rest of the details.) ◁

We state the obvious claim that is a consequence of the fact that any *implicit* contraction in [LLR$^{\boldsymbol{c}}$] is replicable by *explicit* contractions in LLR$^{\boldsymbol{c}}$.

Lemma 14. *If $\mathcal{A}$ is a theorem of* [LLR$^{\boldsymbol{c}}$]*, then $\mathcal{A}$ is a* theorem *of* LLR$^{\boldsymbol{c}}$.

Next, we want to make sure that hiding the contractions in the connective rules does not diminish the capacity of the calculus with respect to proving theorems.

Lemma 15. (Curry's lemma for [LLR$^{\boldsymbol{c}}$]) *If $\alpha \vdash \beta$ has a proof in* [LLR$^{\boldsymbol{c}}$] *with the height of the proof tree being n, and $\gamma \vdash \delta$ results from $\alpha \vdash \beta$ by one or more applications of the rules $(W\vdash)$ and $(\vdash W)$, then $\gamma \vdash \delta$ has* a proof *in* [LLR$^{\boldsymbol{c}}$]*, where the height of the proof tree is* not greater than n (*i.e., it is* $\leq n$).

Proof. This is a core lemma for decidability, hence, we give more details here. The base case concerns proofs of height 1.

1. The axioms (id), $(\vdash \boldsymbol{t})$ and $(\boldsymbol{f}\vdash)$ do not have instances to which a contraction rule could be applied; hence, the claim is true.

If the proof is an instance of $(\boldsymbol{F}\vdash)$, then it can be the case that $\boldsymbol{F}$ is the principal formula of $(W\vdash)$ or some other wff may have multiple copies in α or in β. However, a contraction on the left cannot lead to $\boldsymbol{F}$ being dropped altogether on the left-hand side of the $\vdash$. For instance, $\alpha';\boldsymbol{F};\boldsymbol{F};\mathcal{A};\mathcal{A} \vdash \mathcal{B};\mathcal{B};\beta'$ is an instance of the axiom, but so is $\alpha';\boldsymbol{F};\mathcal{A} \vdash \mathcal{B};\beta'$. The case of $(\vdash \boldsymbol{T})$ is similar, modulo $\boldsymbol{T}$ occurring on the right-hand side of the $\vdash$.

2. The rest of the cases make up the inductive step. There are three kinds of rules in [LLR$^{\boldsymbol{c}}$]. First, some rules have no contraction built in. The second group of rules has a type 1 situation on one side of the $\vdash$ and no contraction on the other side. Most rules are like this. Lastly, in three rules, there can be contraction hidden on both sides, one like type 2, the other like type 3. The concrete shape of the principal formula is really indifferent in this proof (though it is specific in each rule). Hence, we exemplify each case by detailing the step for one rule.

2.1. We will scrutinize the rules for the zero-ary constants, since, those rules (or the constants themselves) are not included in [36]. (We of course know that $\boldsymbol{t}$ does not cause a problem in [LR$^{\boldsymbol{t}}_{\rightarrow}$], as we had shown in Bimbó and Dunn [12].)

If the constant is a wff that could be contracted, then the application of the rule may be omitted. Any other contraction must involve parametric formulas, hence, the new proof is guaranteed to exist by the inductive hypothesis. Here is what happens in the case of the $(\vdash\boldsymbol{f})$ rule; the $(\boldsymbol{t}\vdash)$ rule behaves dually. (We only make explicit two pairs of parametric formulas and we assume that $\mathcal{B}$ is not $\boldsymbol{f}$. However, it should be clear that having more parametric formulas that could be contracted, or having them only on one side or having more copies of one particular formula does not change the general structure of this step in the proof.)

$$\dfrac{\vdots \atop \alpha';\mathcal{A};\mathcal{A} \vdash \mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A} \vdash \boldsymbol{f};\mathcal{B};\mathcal{B};\beta'} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\vdots \atop \alpha';\mathcal{A} \vdash \mathcal{B};\beta'}{\alpha';\mathcal{A} \vdash \boldsymbol{f};\mathcal{B};\beta'}$$

$$\dfrac{\vdots \atop \alpha';\mathcal{A};\mathcal{A} \vdash \boldsymbol{f};\mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A} \vdash \boldsymbol{f};\boldsymbol{f};\mathcal{B};\mathcal{B};\beta'} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad {\vdots \atop \alpha';\mathcal{A} \vdash \boldsymbol{f};\mathcal{B};\beta'}$$

2.2. In this situation, some duplicates of parametric formulas could be contracted, and additionally, the rule allows the contraction of the principal formula too, provided that it already occurred among the parametric formulas. The former sort of contraction can be dealt with by appeal to the inductive hypothesis, whereas the latter sort of contraction can result from the application of the same rule (to the new premise). As an illustration, we consider one of the $[\vdash\vee]$ rules; the other rules are similar.

$$\frac{\alpha';\mathcal{A};\mathcal{A}\vdash\mathcal{B};\mathcal{B};\mathcal{D};\mathcal{C}\vee\mathcal{D};\beta'}{\alpha';\mathcal{A};\mathcal{A}\vdash[\mathcal{B};\mathcal{B};\mathcal{C}\vee\mathcal{D};\mathcal{C}\vee\mathcal{D};\beta']} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \frac{\alpha';\mathcal{A}\vdash\mathcal{B};\mathcal{D};\mathcal{C}\vee\mathcal{D};\beta'}{\alpha';\mathcal{A}\vdash[\mathcal{B};\mathcal{C}\vee\mathcal{D};\mathcal{C}\vee\mathcal{D};\beta']}$$

2.3. The last situation is just a notch more complicated, primarily, due to the need to keep track of where the wff's that could be contracted come from. As an illustration, we choose the $[+\vdash]$ rule and we will assume that all the contractable formulas have been made explicit — with distinct letters standing for distinct formulas. (Adding more wff's only expands the size of the sequents, but it does not alter the proof step in a crucial way.) Thus, instead of the bracket notation, we write multisets in the lower sequents.

$$\frac{\gamma;\mathcal{A};\mathcal{A}+\mathcal{B};\mathcal{C};\mathcal{E};\mathcal{E}\vdash\mathcal{D};\delta \qquad \varepsilon;\mathcal{B};\mathcal{A}+\mathcal{B};\mathcal{C}\vdash\mathcal{G};\mathcal{G};\mathcal{D};\eta}{\gamma;\varepsilon;\mathcal{A}+\mathcal{B};\mathcal{A}+\mathcal{B};\mathcal{A}+\mathcal{B};\mathcal{C};\mathcal{C};\mathcal{E};\mathcal{E}\vdash\mathcal{G};\mathcal{G};\mathcal{D};\mathcal{D};\delta;\eta} \quad \overset{\text{i.h.}}{\rightsquigarrow}$$

$$\frac{\gamma;\mathcal{A};\mathcal{A}+\mathcal{B};\mathcal{C};\mathcal{E}\vdash\mathcal{D};\delta \qquad \varepsilon;\mathcal{B};\mathcal{A}+\mathcal{B};\mathcal{C}\vdash\mathcal{G};\mathcal{D};\eta}{\gamma;\varepsilon;\mathcal{A}+\mathcal{B};\mathcal{C};\mathcal{E}\vdash\mathcal{G};\mathcal{D};\delta;\eta}$$

It is easy to verify that the height of the new proof tree in **2.1.–2.3.** is not greater (in some cases, strictly less) than the height of the original proof tree. ◁

Cognate sequents are often defined for sequents that comprise a pair of sequences of formulas. However, the definition straightforwardly transfers to sequents based on pairs of multisets.

Definition 16. (Cognate sequents) The sequents $\alpha\vdash\beta$ and $\gamma\vdash\delta$ are *cognate* iff (1) and (2) hold for any formula $\mathcal{A}$.

(1) $\mathcal{A}$ occurs in α iff it occurs in γ. (2) $\mathcal{A}$ occurs in β iff it occurs in δ.

The *number of occurrences* is not mentioned in the definition at all, which reflects the idea that if we would view sequents as pairs of sets, then cognation means that the set-view turns the antecedents and succedents, respectively, into the same set.

Lemma 17. (Kripke's lemma for cognate sequents) *A sequence of distinct cognate sequents, in which, if $\alpha_n\vdash\beta_n$ precedes $\alpha_m\vdash\beta_m$, then the former does not result by one or more contractions of wff's in $\alpha_m\vdash\beta_m$, is* finite.

A possibly easier-to-understand phrasing of the lemma is in terms of natural numbers. Let a finite fixed set of prime factors be given, let us say, $\{2,5,13\}$. Contraction is the reduction of an exponent by 1, for instance, $2^6\cdot5^6\cdot13^4$ is a contraction of $2^7\cdot5^6\cdot13^4$. Then, a sequence of distinct natural numbers over the set of fixed prime factors (all with positive integer exponents), in which earlier numbers are not (single or multiple) contractions of later numbers, is finite.[10]

[10] Kripke's lemma is equivalent to lemmas from other parts of mathematics, e.g., to Dickson's lemma in number theory. The truth of none of these equivalent lemmas has been questioned. The connection to Dickson's lemma was discovered by Meyer, as noted in [18] and also in its expanded version [20]. (Both

Proof. We note that Kripke's lemma is not specific to the language of a logic, that is, it does not matter what connectives occur in the formulas. The numerical illustration clearly hints toward this. A proof of Kripke's lemma may be found in Anderson and Belnap [2, §13], (and we do not repeat that proof here). ◁

Another lemma that is general, in the sense that the shape of the components in the structure is unimportant, is Kőnig's lemma about trees. *Finite branching* or *finite forking* means that no node has infinitely many children, whereas having *finite branches* means that every maximal path is finite.

Lemma 18. (Kőnig's lemma) *A tree with finite branching and with finite branches is* finite.

Proof. A detailed proof of this lemma may be found in Smullyan [41], for example, and we do not repeat that proof here. ◁

Now we can put together the latter two lemmas with some facts about [LLR^c] to obtain the decidability of [LLR^c], thereby, of LLR^c.

Theorem 19. (Decidability for *L*LR^c) *The logic* LLR^c *is* decidable.

Proof. To start with, we note that each formula in the language of [LLR^c] has finitely many subformulas (under the usual understanding of subformulas), hence, finitely many proper subformulas. For example, if a sequent is by the ($\wedge\vdash$) rule, then there are only two possible choices as to what the subaltern in the premise could be.

Finiteness obtains in other respects too. Each sequent contains finitely many formulas, each occurring finitely many times. Given a sequent and fixating on a rule that could have resulted in that sequent, there are finitely many contractions that could have been part of the application of that rule.

The cut theorem provides the assurance that every theorem has a cut-free proof.

Let us assume that a wff $\mathcal{A}$ is given. We construct a *proof-search tree* to determine if $\mathcal{A}$ is or is not a theorem of [LLR^c]. The proof-search tree has two important properties, namely, it is *a finite tree*, and if the given formula is a theorem, then the proof-search tree contains a subtree that is *a proof of the formula* in the root sequent.

The proof-search tree is built from the bottom to the top by "backward applications of the rules." The root of the tree is the sequent $\vdash \mathcal{A}$. By "backward applications of rules" we mean the consideration of potential rules (and their premises), the applications of which could result in the sequent in a given node. We may assume that the potential premises are arranged into an ordered set of leaves, and on each level we proceed from left to right — taking a node after another one, and trying to expand the tree with new nodes (forming a new level in the tree). For a node in the tree, we consider which rules could have been applied and what the premises would be. We add all those premises as children of the given node (i.e., as new leaves) to the tree as long as they do not violate the condition in Kripke's lemma. Then we move on to consider the next node.

---

contain a persuasive visualization of a concrete instance of the lemma; so does [9, §9.1].) See also Riche and Meyer [38] and Kopylov [29].

A theorem $\mathcal{A}$ has a cut-free proof in [LLR$^{\boldsymbol{c}}$], hence, the exhaustive search through all the possible rules and potential premises guarantees that a proof is constructed within the proof-search tree (if the formula is a theorem). On the other hand, the tree is finite, because of the already mentioned finiteness properties together with Kripke's and Kőnig's lemmas. Finally, the equivalence of LLR$^{\boldsymbol{c}}$ and [LLR$^{\boldsymbol{c}}$] with respect to provable sequents guarantees that no theorems are misclassified as unprovable when we use [LLR$^{\boldsymbol{c}}$] in the proof search. ◁

4. Modalities Added to LLR$^{\boldsymbol{c}}$

Modalities could be added *explicitly* to LLR$^{\boldsymbol{c}}$, indeed, $\Box$'s addition to LR was considered in [36]. A way to proceed is to consider some usual rules for $\Box$, and their duals for $\Diamond$ together with the connecting rules from Kripke [31], which allow us to prove versions of the so-called modal De Morgan laws for the two modalities.

Definition 20. The sequent calculus LLR$^{\Diamond\Box}$ is defined by the axioms and rules of LLR$^{\boldsymbol{c}}$ and the following rules.

$$\frac{\alpha;\mathcal{A}\vdash\beta}{\alpha;\Box\mathcal{A}\vdash\beta}\ {\scriptstyle\Box\vdash}\qquad\frac{\Box\alpha\vdash\mathcal{A};\Diamond\beta}{\Box\alpha\vdash\Box\mathcal{A};\Diamond\beta}\ {\scriptstyle\vdash\Box}\qquad\frac{\Box\alpha;\mathcal{A}\vdash\Diamond\beta}{\Box\alpha;\Diamond\mathcal{A}\vdash\Diamond\beta}\ {\scriptstyle\Diamond\vdash}\qquad\frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash\Diamond\mathcal{A};\beta}\ {\scriptstyle\vdash\Diamond}$$

$\Box\alpha$ ($\Diamond\alpha$) is a multiset in which the main connective of each formula is $\Box$ ($\Diamond$). The notions of a *proof* and of a *theorem* are as for LLR$^{\boldsymbol{c}}$.

There is an obvious similarity between these rules and the $(\,!\vdash)$, $(\vdash !)$, $(\,?\vdash)$ and $(\vdash ?)$ rules in LCLL (cf. Definition 6). The analogy suggests taking $!$ to be $\Box$, and $?$ to be $\Diamond$, and this translation is very tempting. However, $\Box$ and $\Diamond$ have deeply engraved connotations in the presence of $\wedge$ and $\vee$. Especially, under the alethic reading of the connectives, it seems plausible that $\mathcal{A}$ is necessary and $\mathcal{B}$ is necessary exactly when $\mathcal{A}\wedge\mathcal{B}$ is necessary. In LLR$^{\Diamond\Box}$, it is not too difficult to prove half of this, namely, the sequent $\Box(\mathcal{A}\wedge\mathcal{B})\vdash\Box\mathcal{A}\wedge\Box\mathcal{B}$, and dually, the sequent $\Diamond\mathcal{A}\vee\Diamond\mathcal{B}\vdash\Diamond(\mathcal{A}\vee\mathcal{B})$. Moreover, neither proof requires an application of any structural rule. In other words, if we were to omit $(W\vdash)$ and $(\vdash W)$, the sequents would remain provable. We denote by LLRW the contraction-less sequent calculus derived from LLR$^{\boldsymbol{c}}$; its modalized version will be denoted by LLRW$^{\Diamond\Box}$.

Of course we know, though we have not yet stated it, that the cut theorem holds for LLR$^{\Diamond\Box}$; moreover, that this logic is decidable too. Nonetheless, after some proof attempts, one might convince oneself that $\Box\mathcal{A}\wedge\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})$ is not provable not only in LLRW$^{\Diamond\Box}$ but in LLR$^{\Diamond\Box}$ either. The analog sequent $!\mathcal{A}\wedge\,!\mathcal{B}\vdash\,!(\mathcal{A}\wedge\mathcal{B})$ is not provable in LCLL. This formula provides an example of how the proof of Theorem 10 proceeds. If $(\boldsymbol{t}\wedge\mathcal{A})\wedge(\boldsymbol{t}\wedge\mathcal{B})\vdash\boldsymbol{t}\wedge(\mathcal{A}\wedge\mathcal{B})$ would not be provable in LLR$^{\boldsymbol{c}}$, then we could immediately conclude that $!\mathcal{A}\wedge\,!\mathcal{B}\vdash\,!(\mathcal{A}\wedge\mathcal{B})$ is not provable in CLL. However, the translation is *provable* in LLR$^{\boldsymbol{c}}$, and so a proof search has to be carried out in [LLR$^{\boldsymbol{c}}$] taking into account all the constraints from Theorem 10. The proof search does not produce a proof, therefore, we may conclude that the sequent is not provable in CLL. We return to the provability of $\Box\mathcal{A}\wedge\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})$ in the next section, but now we turn to what is provable in LLR$^{\Diamond\Box}$.

The following four formulas are theorems of LLR$^{\Diamond\Box}$, and (R4) is the rule of "necessitation." (We omit the details of the proofs, which are straightforward.) (A18)–(A21) look like the earlier wff's (1)–(4), in which $\Box$ and $\Diamond$ were defined connectives. By numbering these formulas and the rule consecutively, we indicate that $H\mathrm{LR}^{\Diamond\Box}$ may be defined from $H\mathrm{LR}^{\boldsymbol{c}}$ by these additions.

(A18) $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$ (A20) $\Box\mathcal{A} \to \Box\Box\mathcal{A}$
(A19) $\Box\mathcal{A} \to \mathcal{A}$ (A21) $(\Diamond\mathcal{A} \to {\sim}\Box{\sim}\mathcal{A}) \wedge ({\sim}\Box{\sim}\mathcal{A} \to \Diamond\mathcal{A})$
(R4) $\vdash \mathcal{A}$ implies $\vdash \Box\mathcal{A}$

The cut theorem is true of $L\mathrm{LR}^{\Diamond\Box}$, which facilitates the proof of the equivalence of the sequent and axiomatic formulations as well as the proof of decidability.

Theorem 21. (Cut theorem for $L\mathrm{LR}^{\Diamond\Box}$) *The cut rule is* admissible *in* $L\mathrm{LR}^{\Diamond\Box}$.

Proof. The proof proceeds as usual. An important observation is that in the transformations of proofs no other rules are used than those already used. ◁

The modalities do not appear to be too intricate — even if they do not have all the usual properties that $\Box$ and $\Diamond$ have in S4. The latter logic (more precisely, the propositional part of S4) is known to be decidable. Therefore, we may wonder whether we can adapt and extend the proof of the decidability of $L\mathrm{LR}^{\boldsymbol{c}}$ to $L\mathrm{LR}^{\Diamond\Box}$.

Definition 22. We define the *sequent calculus* denoted as $[L\mathrm{LR}^{\Diamond\Box}]$ by taking $[L\mathrm{LR}^{\boldsymbol{c}}]$, and by adding the following connective rules for the modalities.

$$\frac{\alpha;\mathcal{A} \vdash \beta}{[\alpha;\Box\mathcal{A}] \vdash \beta}\ [\Box\vdash] \qquad \frac{\Box\alpha \vdash \mathcal{A};\Diamond\beta}{\Box\alpha \vdash \Box\mathcal{A};\Diamond\beta}\ \vdash\Box \qquad \frac{\Box\alpha;\mathcal{A} \vdash \Diamond\beta}{\Box\alpha;\Diamond\mathcal{A} \vdash \Diamond\beta}\ \Diamond\vdash \qquad \frac{\alpha \vdash \mathcal{A};\beta}{\alpha \vdash [\Diamond\mathcal{A};\beta]}\ [\vdash\Diamond]$$

We assume some earlier notions and notational conventions in an obvious way. Two of the rules have no bracketing at all, whereas the two others are of type 1.

Theorem 23. (Cut theorem for $[L\mathrm{LR}^{\Diamond\Box}]$) *The cut theorem is* admissible *in* $[L\mathrm{LR}^{\Diamond\Box}]$.

Proof. The proof proceeds as usual.[11] Here is a sample case from the transformation, where $\Box\mathcal{C}$ is the cut formula. If $\rho = 2$, then $\Box\mathcal{C}$ could not have been contracted in $[\Box\vdash]$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Box\alpha \vdash \mathcal{C};\Diamond\beta\end{array}}{\Box\alpha \vdash \Box\mathcal{C};\Diamond\beta} \qquad \dfrac{\begin{array}{c}\vdots\\ \gamma;\mathcal{C} \vdash \delta\end{array}}{\gamma;\Box\mathcal{C} \vdash \delta}}{[\Box\alpha;\gamma] \vdash [\Diamond\beta;\delta]} \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Box\alpha \vdash \mathcal{C};\Diamond\beta\end{array} \qquad \begin{array}{c}\vdots\\ \gamma;\mathcal{C} \vdash \delta\end{array}}{[\Box\alpha;\gamma] \vdash [\Diamond\beta;\delta]}$$

If the right rank $\rho_r > 1$ and all the contractions in the original proof resulted from the application of the cut, then the above transformation suffices. Otherwise, that is, if $\Box\mathcal{C}$, the principal formula of the $[\Box\vdash]$ rule, was contracted as part of the application of the rule, then we consider the number of occurrences of $\Box\mathcal{C}$ in γ. If there are several occurrences in γ, then we permute the applications of the cut rule and of the $[\Box\vdash]$ rule. If there is only one occurrence of $\Box\mathcal{C}$, then beyond the permutation, we also include a cut on the subaltern (which is of lower degree than $\Box\mathcal{C}$). Here are the resulting chunks of the proof. ($n \in \mathbb{N}$ and $n > 1$.)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Box\alpha \vdash \mathcal{C};\Diamond\beta\end{array}}{\Box\alpha \vdash \Box\mathcal{C};\Diamond\beta} \qquad \begin{array}{c}\vdots\\ \gamma';(\Box\mathcal{C})^n;\mathcal{C} \vdash \delta\end{array}}{[\Box\alpha;\gamma';(\Box\mathcal{C})^{n-1};\mathcal{C}] \vdash [\Diamond\beta;\delta]}}{[\Box\alpha;\gamma';(\Box\mathcal{C})^{n-1}] \vdash [\Diamond\beta;\delta]}$$

[11] Some details of a related proof are given in [8, §2].

$$\dfrac{\begin{matrix}\vdots\\ \Box\alpha\vdash\mathcal{C};\Diamond\beta\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \Box\alpha\vdash\mathcal{C};\Diamond\beta\end{matrix}}{\Box\alpha\vdash\Box\mathcal{C};\Diamond\beta} \qquad \begin{matrix}\vdots\\ \gamma';\Box\mathcal{C};\mathcal{C}\vdash\delta\end{matrix}}{[\Box\alpha;\gamma';\mathcal{C}]\vdash[\Diamond\beta;\delta]}}{[\Box\alpha;\gamma']\vdash[\Diamond\beta;\delta]}$$

◁

Next, we prove Curry's lemma, which is sometimes called the height-preserving admissibility of contraction.

Lemma 24. (Curry's lemma for [*L*LR$^{\Diamond\Box}$]) *If* $\alpha\vdash\beta$ *has a proof in* [*L*LR$^{\Diamond\Box}$] *with the height of the proof tree being n, and* $\gamma\vdash\delta$ *results from* $\alpha\vdash\beta$ *by one or more applications of the rules* $(W\vdash)$ *and* $(\vdash W)$*, then* $\gamma\vdash\delta$ *has* a proof *in* [*L*LR$^{\Diamond\Box}$]*, where the height of the proof tree is* not greater than *n* (*i.e., it is* $\leq n$).

Proof. The proof of this lemma seamlessly incorporates the proof of Lemma 15. We have four new rules — compared to [*L*LR^c]. Two of those do not permit contractions, hence, any contractions that could be applied to the lower sequent of those rules are guaranteed to exist by the hypothesis of the induction. We consider the remaining two rules, which expand case **2.2**.

2.2. If the last rule applied in the given proof is $[\Box\vdash]$, then $\Box\mathcal{A}$ may be contracted, provided that it already occurs in the antecedent, that is, in α. We consider $\mathcal{B}$ and $\mathcal{C}$ as other wff's that potentially could be contracted. The following is an illustration of a representative case, though concretely, there might be fewer or more formulas that could be contracted.

$$\dfrac{\begin{matrix}\vdots\\ \alpha';\mathcal{B};\mathcal{B};\Box\mathcal{A};\mathcal{A}\vdash\mathcal{C};\mathcal{C};\beta'\end{matrix}}{[\alpha';\mathcal{B};\mathcal{B};\Box\mathcal{A};\Box\mathcal{A}]\vdash\mathcal{C};\mathcal{C};\beta'} \qquad \overset{\text{i.h.}}{\rightsquigarrow} \qquad \dfrac{\begin{matrix}\vdots\\ \alpha';\mathcal{B};\Box\mathcal{A};\mathcal{A}\vdash\mathcal{C};\beta'\end{matrix}}{[\alpha';\mathcal{B};\Box\mathcal{A};\Box\mathcal{A}]\vdash\mathcal{C};\beta'}$$

$\mathcal{B}$ and $\mathcal{C}$ can be contracted above the application of the $[\Box\vdash]$ rule, by the inductive hypothesis, and $\alpha';\mathcal{B};\Box\mathcal{A}$ can be obtained using $[\Box\vdash]$.

If the last rule applied in the proof is $[\vdash\Diamond]$, then we have a dual situation. Here is the given segment, and the new chunk.

$$\dfrac{\begin{matrix}\vdots\\ \alpha';\mathcal{A};\mathcal{A}\vdash\mathcal{B};\Diamond\mathcal{B};\mathcal{C};\mathcal{C};\beta'\end{matrix}}{\alpha';\mathcal{A};\mathcal{A}\vdash[\Diamond\mathcal{B};\Diamond\mathcal{B};\mathcal{C};\mathcal{C};\beta']} \qquad \overset{\text{i.h.}}{\rightsquigarrow} \qquad \dfrac{\begin{matrix}\vdots\\ \alpha';\mathcal{A}\vdash\mathcal{B};\Diamond\mathcal{B};\mathcal{C};\beta'\end{matrix}}{\alpha';\mathcal{A}\vdash[\Diamond\mathcal{B};\Diamond\mathcal{B};\mathcal{C};\beta']}$$

Clearly, the height of the proof tree does not increase in either case. ◁

Theorem 25. (Decidability for *L*LR$^{\Diamond\Box}$) *The logic L*LR$^{\Diamond\Box}$ *is* decidable*.*

Proof. The proof of this theorem proceeds like the proof of Theorem 19. We have two sequent calculuses, [*L*LR$^{\Diamond\Box}$] and *L*LR$^{\Diamond\Box}$, in which the same theorems are provable. Additionally, we have proved Curry's lemma for [*L*LR$^{\Diamond\Box}$]. The whole structure of the proof is the same as before, that is, it is through performing an exhaustive search in a finite search space. ◁

5. Logics in the Neighborhood of $\mathrm{LR}^{\Diamond\Box}$

If we keep the four connective rules for $\Diamond$ and $\Box$ fixed, then we may wonder about the effects of the inclusion of the contraction or the thinning rules, or of their modalized versions (like those in CLL). In particular, the next proof suggests the usefulness of the modalized thinning rules with $L\mathrm{LR}^{\Diamond\Box}$. We labeled the steps where

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box K\vdash\;\dfrac{\dfrac{\mathcal{A}\vdash\mathcal{A}}{\mathcal{A};\Box\mathcal{B}\vdash\mathcal{A}}}{\Box\mathcal{A};\Box\mathcal{B}\vdash\mathcal{A}} \qquad \dfrac{\dfrac{\mathcal{B}\vdash\mathcal{B}}{\Box\mathcal{A};\mathcal{B}\vdash\mathcal{B}}}{\Box\mathcal{A};\Box\mathcal{B}\vdash\mathcal{B}}\;\Box K\vdash}{\Box\mathcal{A};\Box\mathcal{B}\vdash\mathcal{A}\wedge\mathcal{B}}}{\Box\mathcal{A};\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})}}{\Box\mathcal{A};\Box\mathcal{A}\wedge\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})}}{\Box\mathcal{A}\wedge\Box\mathcal{B};\Box\mathcal{A}\wedge\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})}}{\Box\mathcal{A}\wedge\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})}\;W\vdash$$

Figure 1. A proof of the distributivity of $\Box$ over $\wedge$

structural rules (modalized or plain ones) are applied. The proof sort of "explains" why the bottom sequent is not provable in $L\mathrm{LR}^{\Diamond\Box}$ or in LCLL (with $!$ instead of $\Box$ in other notation). The logic $L\mathrm{LR}^{\Diamond\Box}$ has no thinning (except for $\boldsymbol{t}$ and $\boldsymbol{f}$), whereas LCLL does not have plain contraction. More contemplation of the proof allows us to conclude that $!\mathcal{A}\otimes\,!\mathcal{B}\vdash\,!(\mathcal{A}\,\&\,\mathcal{B})$ is provable in LCLL because of $(\,!K\vdash)$, hence, $\Box\mathcal{A}\circ\Box\mathcal{B}\vdash\Box(\mathcal{A}\wedge\mathcal{B})$ is provable in $L\mathrm{BCK}^{\Diamond\Box}$, for example.

$L\mathrm{BCK}^{\Diamond\Box}$ is $L\mathrm{LRW}^{\Diamond\Box}$ with left and right thinning rules. The letters B, C and K are motivated by the provability of the principal (simple) type schemas of the combinators B, C and K in the implicational fragment of $L\mathrm{BCK}^{\Diamond\Box}$. If we add modalized contraction rules, then we get $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$, which is also known as *affine linear logic*, and it had been proved decidable by Alexei P. Kopylov, in 1995 (see [29]), using normal sequents and vector games. We give a new proof of the decidability of $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ (which is conceptually different), in the second half of this section. Our proof shows that the modalization of the contraction rules does not destroy decidability. (The modalization of the thinning rules is absolutely unproblematic.)

The converse of the previously considered sequent, $!(\mathcal{A}\,\&\,\mathcal{B})\vdash\,!\mathcal{A}\otimes\,!\mathcal{B}$ is also provable in LCLL, because of $(\,!W\vdash)$, hence, $\Box(\mathcal{A}\wedge\mathcal{B})\vdash\Box\mathcal{A}\circ\Box\mathcal{B}$ is provable in $L\mathrm{LR}^{\Diamond\Box}$. Thus, some of the prototypical sequents provable in LCLL are the next four ones (where $\alpha\dashv\vdash\beta$ indicates that both $\alpha\vdash\beta$ and $\beta\vdash\alpha$ are provable).

$$!\mathcal{A}\otimes\,!\mathcal{B}\dashv\vdash\,!(\mathcal{A}\,\&\,\mathcal{B}) \qquad\qquad !(\,!\mathcal{A}\,\&\,!\mathcal{B})\dashv\vdash\,!(\mathcal{A}\,\&\,\mathcal{B})$$

The proof in Figure 1 also shows that the distribution of $\Box$ over $\wedge$ is provable in $L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}$, that is, $L\mathrm{LR}^{\Diamond\Box}$ extended with a pair of modalized thinning rules. (We pointed out on page 104 that the sequent $\Box(\mathcal{A}\wedge\mathcal{B})\vdash\Box\mathcal{A}\wedge\Box\mathcal{B}$ is always provable.)

Figure 2 (below) shows seven logics that we consider in varying details. The arrows indicate that the set of axioms and rules of a logic is a proper subset of the set of axioms and rules of another one (assuming the identity translation throughout). As a result, inclusions between sets of theorems of those logics also obtain, and they can be shown to be proper.

$$\begin{tikzcd}[column sep=small]
 & & L\mathrm{K}^{\Diamond\Box} & & \\
 & L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K} \arrow[ur] & & L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W} \arrow[ul] & \\
L\mathrm{LR}^{\Diamond\Box} \arrow[ur] & & L\mathrm{CLL} \arrow[ul] \arrow[ur] & & L\mathrm{BCK}^{\Diamond\Box} \arrow[ul] \\
 & & L\mathrm{LRW}^{\Diamond\Box} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

FIGURE 2. Seven logics with modalities

For the sake of clarity, the non-modalized thinning rules are the following rules:

$$\frac{\alpha \vdash \beta}{\alpha; \mathcal{A} \vdash \beta}\ {\scriptstyle K\vdash} \qquad\qquad \frac{\alpha \vdash \beta}{\alpha \vdash \mathcal{A}; \beta}\ {\scriptstyle \vdash K}$$

The subscripts ${\scriptstyle \Box\Diamond K}$ and ${\scriptstyle \Box\Diamond W}$ in the labels of some logics indicate the addition of a pair of the modalized structural rules $(\Box K\vdash)$ and $(\vdash \Diamond K)$, or $(\Box W\vdash)$ and $(\vdash \Diamond W)$.

Definition 26. The modalized thinning and contraction rules are as follows.

$$\frac{\alpha \vdash \beta}{\alpha; \Box\mathcal{A} \vdash \beta}\ {\scriptstyle \Box K\vdash} \quad \frac{\alpha \vdash \beta}{\alpha \vdash \Diamond\mathcal{A}; \beta}\ {\scriptstyle \vdash\Diamond K} \quad \frac{\alpha; \Box\mathcal{A}; \Box\mathcal{A} \vdash \beta}{\alpha; \Box\mathcal{A} \vdash \beta}\ {\scriptstyle \Box W\vdash} \quad \frac{\alpha \vdash \Diamond\mathcal{A}; \Diamond\mathcal{A}; \beta}{\alpha \vdash \Diamond\mathcal{A}; \beta}\ {\scriptstyle \vdash\Diamond W}$$

First of all, we should note that $L\mathrm{K}^{\Diamond\Box}$ is a *baroque logic*, because it has duplicate symbols for two of its connectives, namely, for $\wedge$ and $\vee$ (or for $\circ$ and $+$). The two pairs of the zero-ary constants also match as $\boldsymbol{T}$ and $\boldsymbol{t}$, and $\boldsymbol{F}$ and $\boldsymbol{f}$. Furthermore, they are definable by the lattice connectives and $\sim$. Unlike in the six other logics, $\wedge$ and $\vee$ distribute over each other. In sum, $L\mathrm{K}^{\Diamond\Box}$ is classical logic with $\Box$ and $\Diamond$, which are S4-type modalities; that is, $L\mathrm{K}^{\Diamond\Box}$ is a notational variant of S4.

Normality means that $\Diamond$ preserves $\boldsymbol{F}$, or dually, $\boldsymbol{T} \to \Box\boldsymbol{T}$ is a theorem. It is not difficult to check that if $(\vdash K)$ and $(K\vdash)$ are in one of those logics, then both obtain, and vice versa. Thus, normality is a feature of modalities already in $L\mathrm{BCK}^{\Diamond\Box}$. The modal operators are *monotone*, and this is their feature in all seven logics. The proof of *additivity* of $\Diamond$ requires contraction and (modalized) thinning, whereas the *normality* of $\Diamond$, as we already mentioned, requires thinning. Thus, the modalities have some S4ish properties — reflected by (A18)–(A21) and (R4) — in all seven logics, but $\Diamond$ is normal and additive, and $\Box$ has the dual of both properties only in $L\mathrm{K}^{\Diamond\Box}$.

Propositional S4 is known to be *decidable*, and this remains true the duplicate symbols notwithstanding. The decidability of $L\mathrm{LRW}^{\Diamond\Box}$ and of $L\mathrm{BCK}^{\Diamond\Box}$ is immediate (because neither calculus has any contraction rules), and we have shown that $L\mathrm{LR}^{\Diamond\Box}$ is decidable. Three other logics are left to consider. First, we focus on $L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}$ and $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$, then we turn to $L\mathrm{CLL}$.

The sequent calculus $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$ is an extension of the sequent calculus $[L\mathrm{LR}^{\Diamond\Box}]$ by the two modalized thinning rules. There is no contraction included in those rules. The rationale is the same as with the $(\boldsymbol{t}\vdash)$ and $(\vdash\boldsymbol{f})$ rules, which may be viewed as special thinning rules. Namely, if the principal formula would be contracted, then an application of the rule may be simply omitted.

Theorem 27. (Cut theorem for $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$) *The cut rule is* admissible *in* $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$.

Proof. The proof extends the proof of the cut theorem for $[L\mathrm{LR}^{\Diamond\Box}]$. We consider one new case in detail, when $\rho = 2$ and the right premise is by $(\Box K\vdash)$ and $\Box\mathcal{C}$ is the cut formula. The transformation ensures that $\Box\mathcal{C}$ (occurring in the succedent of the left premise) disappears without an application of the cut rule.

$$\dfrac{\dfrac{\Box\alpha\vdash\mathcal{C};\Diamond\beta}{\Box\alpha\vdash\Box\mathcal{C};\Diamond\beta}\qquad\dfrac{\gamma\vdash\delta}{\gamma;\Box\mathcal{C}\vdash\delta}}{[\Box\alpha;\gamma]\vdash[\Diamond\beta;\delta]}\quad\rightsquigarrow\quad\dfrac{\dfrac{\gamma\vdash\delta}{\Box\alpha;\gamma\vdash\delta}}{[\Box\alpha;\gamma]\vdash[\Diamond\beta;\delta]}$$

The case, in which the pair of rules is $\langle(\vdash\Diamond K),(\Diamond\vdash)\rangle$, is the dual of this.

If the principal formula of the modalized thinning rules does not coincide with the cut formula, then the cut may be permuted upward without difficulty, because there are no side conditions on the applicability of the modalized thinning rules. (We omit the remaining details.) ◁

The cut theorem ensures the subformula property in cut-free proofs. The following lemma is preeminent for decidability.

Lemma 28. (Curry's lemma for $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$) *If $\alpha\vdash\beta$ has a proof in $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$ with the height of the proof tree being n, and $\gamma\vdash\delta$ results from $\alpha\vdash\beta$ by one or more applications of the rules $(W\vdash)$ and $(\vdash W)$, then $\gamma\vdash\delta$ has* a proof *in $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$, where the height of the proof tree is* not greater than n (*i.e., it is* $\leq n$).

Proof. Once again, we suppose the proof for the logic $[L\mathrm{LR}^{\Diamond\Box}]$. We have to extend the inductive step, namely, case **2.2.** There are two new rules, and we consider each.

2.2. Let us assume that there are some parametric wff's, $\mathcal{A}$ and $\mathcal{B}$, which have multiple occurrences that could be contracted, but $\Box\mathcal{C}$, the principal formula of the $(\Box K\vdash)$ rule is not among the contractable formulas. Then we have the following.

$$\dfrac{\alpha';\mathcal{A};\mathcal{A}\vdash\mathcal{B};\mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A};\Box\mathcal{C}\vdash\mathcal{B};\mathcal{B};\mathcal{B};\beta'}\quad\overset{\text{i.h.}}{\rightsquigarrow}\quad\dfrac{\alpha';\mathcal{A}\vdash\mathcal{B};\beta'}{\alpha';\mathcal{A};\Box\mathcal{C}\vdash\mathcal{B};\beta'}$$

It could happen that $\Box\mathcal{C}$ already has some occurrences in the premise. Although the rule does not have any built-in contraction, the resulting sequent could be contracted. Here is an example.

$$\dfrac{\alpha';\mathcal{A};\mathcal{A};\mathcal{A};\Box\mathcal{C};\Box\mathcal{C}\vdash\mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A};\mathcal{A};\Box\mathcal{C};\Box\mathcal{C};\Box\mathcal{C}\vdash\mathcal{B};\mathcal{B};\beta'}\quad\overset{\text{i.h.}}{\rightsquigarrow}\quad\alpha';\mathcal{A};\Box\mathcal{C}\vdash\mathcal{B};\beta'$$

Dually, we have two possibilities with the $(\vdash\Diamond K)$ rule. (We use two copies of $\mathcal{A}$ and $\mathcal{B}$ in these proof segments.)

$$\dfrac{\alpha';\mathcal{A};\mathcal{A}\vdash\mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A}\vdash\Diamond\mathcal{C};\mathcal{B};\mathcal{B};\beta'}\quad\overset{\text{i.h.}}{\rightsquigarrow}\quad\dfrac{\alpha';\mathcal{A}\vdash\mathcal{B};\beta'}{\alpha';\mathcal{A}\vdash\Diamond\mathcal{C};\mathcal{B};\beta'}$$

$$\dfrac{\alpha';\mathcal{A};\mathcal{A}\vdash\Diamond\mathcal{C};\mathcal{B};\mathcal{B};\beta'}{\alpha';\mathcal{A};\mathcal{A}\vdash\Diamond\mathcal{C};\Diamond\mathcal{C};\mathcal{B};\mathcal{B};\beta'}\quad\overset{\text{i.h.}}{\rightsquigarrow}\quad\alpha';\mathcal{A}\vdash\Diamond\mathcal{C};\mathcal{B};\beta'$$

The height of the proof does not increase in any of the cases. ◁

Theorem 29. *The logic* $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$ *is* decidable.

Proof. The proof proceeds as before, hence, we skip the details here. ◁

The sequent calculus $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ is defined by adding the modalized contraction rules to $L\mathrm{BCK}^{\Diamond\Box}$, and it differs from $L\mathrm{K}^{\Diamond\Box}$, which has non-modalized contractions. As we noted, $L\mathrm{K}^{\Diamond\Box}$'s language could be simplified, however, for our purposes now it is useful to retain both the extensional (i.e., lattice) connectives and the intensional (including the modal) connectives.

We have noted also that $L\mathrm{K}^{\Diamond\Box}$ is decidable. In particular, the decidability of $L\mathrm{K}^{\Diamond\Box}$ can be proved along the lines of the decidability proofs of $L\mathrm{LR}^{\Diamond\Box}$ and $L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}$. The presence of thinning (modalized or plain) does not constitute a problem at all, because it does not even require contraction to be built into the thinning rules in the contraction-free version of the sequent calculus. $[L\mathrm{K}^{\Diamond\Box}]$ is defined as $[L\mathrm{LR}^{\Diamond\Box}]$ with the full left and right thinning rules added. Definition 8 does not mention (explicitly) a calculus, hence, we may use the same notion here with the assumption that the heap numbers for $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ are calculated from the Curry–Kripke decision procedure for $L\mathrm{K}^{\Diamond\Box}$.

Theorem 30. *The logic* $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ *is* decidable.

Proof. Given a wff $\mathcal{A}$, we can determine if the wff is a theorem of $L\mathrm{K}^{\Diamond\Box}$; if it is not, then $\mathcal{A}$ is not a theorem of $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ either. On the other hand, we can also determine if $\mathcal{A}$ is a theorem of $L\mathrm{BCK}^{\Diamond\Box}$; if it is, then it is a theorem of $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ too. We apply a proof search procedure to the remaining wff's.

We construct a proof-search tree in $L\mathrm{BCK}^{\Diamond\Box}_{\Box\Diamond W}$ taking into account the heap numbers for the subformulas of $\mathcal{A}$ as upper bounds on the number of applications of the $(\Box W\vdash)$ and $(\vdash\Diamond W)$ rules. The resulting tree will be finite, because there are no other contractions than those that are instances of the modalized contraction rules, and the number of their applications is bounded by the heap numbers, which are finite numbers. ◁

6. The Decidability of Linear Logic

We have already proved that classical linear logic (CLL) is decidable — as Theorem 10. CLL has a certain familiarity to many people, and it had been claimed to be undecidable in [35] (see Theorem 3.7) and in [28] (see Corollaries 5.5 and 5.7) We think though that those proofs fall short of establishing the undecidability of CLL. Since the undecidability of CLL is widely believed in the computer science community, we give a more direct proof (than the previous proof) for the decidability of CLL.

To make the reading of this proof easier for those in the linear logic community, we define a sequent calculus, which we call $[\![L\mathrm{CLL}]\!]$, and we use Girard's notation. $[\![L\mathrm{CLL}]\!]$ *is not* classical linear logic though. (A careful reader will recognize this logic as $[L\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$ in non-standard notation.)

The notion of a sequent is as before; a *sequent* is a pair of multisets of wff's separated by $\vdash$. We use both single and double bracketing in this calculus for permissible contractions that are built into the operational rules. For some purposes the single and the double bracketing might be treated as the same (just blur your vision). But as we shall explain after we state the rules, the double brackets sometimes mark a crucial distinction.

Definition 31. $[\![\mathcal{L}\mathrm{CLL}]\!]$ comprises the following axioms and rules.

$$\alpha;\mathbf{0}\vdash\beta \;\; \mathbf{0}\vdash \qquad \mathcal{A}\vdash\mathcal{A} \;\; \mathrm{id} \qquad \alpha\vdash\top;\beta \;\; \vdash\top$$

$$\bot\vdash \;\; \bot\vdash \qquad \frac{\alpha\vdash\beta}{\alpha\vdash\bot;\beta}\;\vdash\bot \qquad \frac{\alpha\vdash\beta}{\alpha;\mathbf{1}\vdash\beta}\;\mathbf{1}\vdash \qquad \vdash\mathbf{1}\;\;\vdash\mathbf{1}$$

$$\frac{\alpha;\mathcal{A}\vdash\beta}{[\alpha;\mathcal{A}\,\&\,\mathcal{B}]\vdash\beta}\;[\&\vdash_1] \qquad \frac{\alpha;\mathcal{A}\vdash\beta}{[\alpha;\mathcal{B}\,\&\,\mathcal{A}]\vdash\beta}\;[\&\vdash_2] \qquad \frac{\alpha\vdash\mathcal{A};\beta \quad \alpha\vdash\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\,\&\,\mathcal{B};\beta]}\;[\vdash\&]$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \quad \alpha;\mathcal{B}\vdash\beta}{[\alpha;\mathcal{A}\oplus\mathcal{B}]\vdash\beta}\;[\oplus\vdash] \qquad \frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash[\mathcal{A}\oplus\mathcal{B};\beta]}\;[\vdash\oplus_1] \qquad \frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash[\mathcal{B}\oplus\mathcal{A};\beta]}\;[\vdash\oplus_2]$$

$$\frac{\alpha\vdash\mathcal{A};\beta}{[\alpha;\mathcal{A}^{\bot}]\vdash\beta}\;[{}^{\bot}\vdash] \qquad \frac{\alpha;\mathcal{A}\vdash\beta}{\alpha\vdash[\mathcal{A}^{\bot};\beta]}\;[\vdash{}^{\bot}]$$

$$\frac{\alpha;\mathcal{A};\mathcal{B}\vdash\beta}{[\alpha;\mathcal{A}\otimes\mathcal{B}]\vdash\beta}\;[\otimes\vdash] \qquad \frac{\alpha\vdash\mathcal{A};\beta \quad \gamma\vdash\mathcal{B};\delta}{[\![\alpha;\gamma]\!]\vdash[\![\mathcal{A}\otimes\mathcal{B};\beta;\delta]\!]}\;[\![\vdash\otimes]\!]$$

$$\frac{\alpha;\mathcal{A}\vdash\beta \quad \gamma;\mathcal{B}\vdash\delta}{[\![\alpha;\gamma;\mathcal{A}\parr\mathcal{B}]\!]\vdash[\![\beta;\delta]\!]}\;[\![\parr\vdash]\!] \qquad \frac{\alpha\vdash\mathcal{A};\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\parr\mathcal{B};\beta]}\;[\vdash\parr]$$

$$\frac{\alpha\vdash\mathcal{A};\beta \quad \gamma;\mathcal{B}\vdash\delta}{[\![\alpha;\gamma;\mathcal{A}\multimap\mathcal{B}]\!]\vdash[\![\beta;\delta]\!]}\;[\![\multimap\vdash]\!] \qquad \frac{\alpha;\mathcal{A}\vdash\mathcal{B};\beta}{\alpha\vdash[\mathcal{A}\multimap\mathcal{B};\beta]}\;[\vdash\multimap]$$

$$\frac{\alpha;\mathcal{A}\vdash\beta}{[\![\alpha;\,!\mathcal{A}]\!]\vdash\beta}\;[\![\,!\vdash]\!] \qquad \frac{!\alpha\vdash\mathcal{A};\,?\beta}{!\alpha\vdash\,!\mathcal{A};\,?\beta}\;\vdash!$$

$$\frac{!\alpha;\mathcal{A}\vdash\,?\beta}{!\alpha;\,?\mathcal{A}\vdash\,?\beta}\;?\vdash \qquad \frac{\alpha\vdash\mathcal{A};\beta}{\alpha\vdash[\![\,?\mathcal{A};\beta]\!]}\;[\![\vdash?]\!]$$

$$\frac{\alpha\vdash\beta}{\alpha;\,!\mathcal{A}\vdash\beta}\;!K\vdash \qquad \frac{\alpha\vdash\beta}{\alpha\vdash\,?\mathcal{A};\beta}\;\vdash?K$$

To start with, the brackets (whether single or double) indicate optional contractions as in Definition 12. Then $[\![\mathcal{L}\mathrm{CLL}]\!]$ is equivalent to $[\mathcal{L}\mathrm{LR}^{\Diamond\Box}_{\Box\Diamond K}]$. We may weaken the logic in two different ways, each time getting CLL. First, we may forget about all the brackets and add the rules $(\Box W\vdash)$ and $(\vdash\Diamond W)$ (with $!$ for $\Box$ and $?$ for $\Diamond$). This is the calculus that we denote by $\mathcal{L}$CLL. Second, we can omit the single brackets and change the meaning of the double brackets as follows.

2. If $!\mathcal{A}$ occurs both in α and γ, then it may be contracted in $[\![\alpha;\gamma]\!]$. Dually, if $?\mathcal{A}$ occurs both in β and δ, then it may be contracted in $[\![\beta;\delta]\!]$. The principal formula cannot be involved in the contraction.
3. In $[\![\,!\vdash]\!]$ and $[\![\vdash ?]\!]$, $!\mathcal{A}$ and $?\mathcal{A}$ may be contracted, respectively, if it occurs in α and β.

Obviously, the scope of $[\![\quad]\!]$ could be made narrower in the rules where the main connective of the principal formula of the rule is binary. We denote the logic obtained by omitting the single brackets as $[\![\mathcal{L}\mathrm{CLL}]\!]$.

Now we prove some useful theorems about the calculus $[\![L\mathrm{CLL}]\!]$. Namely, every theorem of LCLL is a theorem of $[\![L\mathrm{CLL}]\!]$, and every theorem of $[\![L\mathrm{CLL}]\!]$ has a cut-free proof (by Lemma 32). A suitable version of Curry's lemma (Lemma 33) holds too.

Lemma 32. (Cut theorem for $[\![L\mathrm{CLL}]\!]$) *The cut rule is* admissible *in* $[\![L\mathrm{CLL}]\!]$.

Proof. The proof is by *double induction* on the rank of the cut and the degree of the cut formula. The rank of the cut (ρ) is defined as in Gentzen [22], and the degree of the cut formula (δ) is the number of unary and binary logical connectives in the cut formula. We divide the cases within the induction into four groups, and provide some representative details.

I. Let $\delta = 0$ and $\rho = 2$. The cut formula is (1) a propositional variable (e.g., p), (2) $\mathbf{1}$, (3) $\bot$, (4) $\top$ or (5) $\mathbf{0}$. None of these formulas can be thinned into a sequent by the rules $(\,!K\vdash)$ or $(\vdash ?K)$, hence, both premises are by an axiom or by a rule for $\mathbf{1}$ or $\bot$. There are various ways to count the subcases; either way there are several cases, and it is straightforward to verify that the cut is directly eliminable. We give two sample cases here.

$$\dfrac{\vdash \mathbf{1} \qquad \dfrac{\begin{matrix}\vdots\\ \alpha \vdash \beta\end{matrix}}{\mathbf{1};\alpha \vdash \beta}}{\alpha \vdash \beta}$$

The proof of the premise of the application of the $(\mathbf{1}\vdash)$ rule is identical to the end sequent, hence, the cut may be omitted altogether.

$$\dfrac{\alpha \vdash \beta;\top;p \qquad p;\mathbf{0};\gamma \vdash \delta}{\alpha;\mathbf{0};\gamma \vdash \beta;\top;\delta}$$

The end sequent is an instance of $(\mathbf{0}\vdash)$ and also of $(\vdash\top)$, hence, both premises of the cut (and the cut itself) may be omitted.

II. Let $\delta = 0$ and $\rho > 2$, in particular, let $\rho_l > 1$. We note that the left premise cannot be the result of an application of the $(\vdash\,!)$ or $(\,?\vdash)$ rules. Furthermore, if it is by a rule for $^\bot$, $\otimes$, $⅋$, $\multimap$, $\&$, $\oplus$, $\mathbf{1}$ or $\bot$, then the principal formula cannot be contracted as part of the application of the cut. It is routine to check that the rule yielding the left premise and the cut may be permuted, and the contractions included in the given proof may be carried out after the rules have been swapped.

Let the left premise be by $(\,!\vdash)$. The given and the transformed proof segments are as follows.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{A};\alpha \vdash \beta;p\end{matrix}}{[\![\,!\mathcal{A};\alpha]\!] \vdash \beta;p} \qquad \begin{matrix}\vdots\\ p;\gamma \vdash \delta\end{matrix}}{[\![\,!\mathcal{A};\alpha;\gamma]\!] \vdash [\![\beta;\delta]\!]} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{A};\alpha \vdash \beta;p\end{matrix} \qquad \begin{matrix}\vdots\\ p;\gamma \vdash \delta\end{matrix}}{[\![\mathcal{A};\alpha;\gamma]\!] \vdash [\![\beta;\delta]\!]}}{[\![\,!\mathcal{A};\alpha;\gamma]\!] \vdash [\![\beta;\delta]\!]}$$

If the application of the $[\![\,!\vdash]\!]$ rule involved a contraction of $\,!\mathcal{A}$, then the same contraction may be performed in the transformed proof too. (The case of $[\![\vdash ?\,]\!]$ is dually similar.)

Let the left premise be by $[\![\vdash ?K]\!]$. The given and the transformed proof segments are as follows.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \alpha\vdash\beta;p\end{array}}{\alpha\vdash\beta;?\mathcal{A};p}\qquad\begin{array}{c}\vdots\\ p;\gamma\vdash\delta\end{array}}{[\![\alpha;\gamma]\!]\vdash[\![\beta;\delta;?\mathcal{A}]\!]}\quad\rightsquigarrow\quad\dfrac{\dfrac{\begin{array}{c}\vdots\\ \alpha\vdash\beta;p\end{array}\qquad\begin{array}{c}\vdots\\ p;\gamma\vdash\delta\end{array}}{[\![\alpha;\gamma]\!]\vdash[\![\beta;\delta]\!]}}{[\![\alpha;\gamma]\!]\vdash[\![\beta;\delta;?\mathcal{A}]\!]}$$

If $?\mathcal{A}$ was contracted in the given proof as part of the application of the cut rule, then the last step is omitted from the transformed proof. All other contractions can be carried out as in the given proof.

$[\![L\mathrm{CLL}]\!]$ is fully symmetric — save the $\multimap$ rules, which however, are unproblematic — when the connectives are dualized. Thus, we leave the details of the $\rho_r > 1$ case to the reader.

III. Let $\delta > 0$ and $\rho = 2$. We distinguish two groups of subcases, namely, when one of the premises is by (id) or by an axiom for $\top$ or $\mathbf{0}$, and when the two premises are by matching rules. The case when a premise is $\mathcal{A}\vdash\mathcal{A}$ is immediate. As an example, we consider $\langle(\vdash\top),[\![\,!\vdash]\!]\rangle$.

$$\dfrac{\alpha\vdash\beta;\top;!\mathcal{A}\qquad\dfrac{\begin{array}{c}\vdots\\ \mathcal{A};\gamma\vdash\delta\end{array}}{!\mathcal{A};\gamma\vdash\delta}}{[\![\alpha;\gamma]\!]\vdash[\![\beta;\top;\delta]\!]}$$

The bottom sequent is an instance of $(\vdash\top)$, hence, the proof simplifies to that sequent.

If the principal formulas in the rules in the left and right premises have as their main connective ${}^{\perp}$, $\otimes$, $⅋$ or $\multimap$, then the transformed proof contains cuts on proper subformulas of the principal formula. The principal formula may not be contracted as part of the application of the cut rule in the given proof. Further, the parametric formulas are combined in the transformed proof in the same way as in the given proof; therefore, all the earlier contractions can be carried out. (We omit the details.)

There are four subcases with modalized cut formulas, because such formulas may be introduced by thinning too. We consider two of these cases, and leave the two others (which are duals) to the reader.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ !\alpha\vdash ?\beta;\mathcal{A}\end{array}}{!\alpha\vdash ?\beta;!\mathcal{A}}\qquad\dfrac{\begin{array}{c}\vdots\\ \mathcal{A};\gamma\vdash\delta\end{array}}{!\mathcal{A};\gamma\vdash\delta}}{[\![\,!\alpha;\gamma]\!]\vdash[\![\,?\beta;\delta]\!]}\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\vdots\\ !\alpha\vdash ?\beta;\mathcal{A}\end{array}\qquad\begin{array}{c}\vdots\\ \mathcal{A};\gamma\vdash\delta\end{array}}{[\![\,!\alpha;\gamma]\!]\vdash[\![\,?\beta;\delta]\!]}$$

The transformation decreases the degree of the cut formula, and provides a possibility for the same contractions as before.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \alpha\vdash\beta\end{array}}{\alpha\vdash\beta;?\mathcal{A}}\qquad\dfrac{\begin{array}{c}\vdots\\ \mathcal{A};!\gamma\vdash ?\delta\end{array}}{?\mathcal{A};!\gamma\vdash ?\delta}}{[\![\alpha;!\gamma]\!]\vdash[\![\beta;?\delta]\!]}\quad\rightsquigarrow\quad\dfrac{\begin{array}{c}\vdots\\ \alpha\vdash\beta\end{array}}{[\![\alpha;!\gamma]\!]\vdash[\![\beta;?\delta]\!]}$$

In the transformed proof, the double brackets simply indicate that the $(\,!K\vdash)$ and $(\vdash ?K)$ steps are applied only to build up the same sequent as the bottom sequent in the given proof. (The thinning rules do not contain any contraction.) It may be useful to point out that $[\![\alpha;!\gamma]\!]\subsetneq\alpha$ and $[\![\beta;?\delta]\!]\subsetneq\beta$ are not possible, hence, we are justified to start the transformed proof with the premise $\alpha\vdash\beta$.

IV. Let $\delta > 0$ and $\rho > 2$, in particular, let $\rho_l > 1$.

Most of the subcases in this case are similar to those in **II**. (We omit the details of those cases, where the change amounts to replacing p with $\mathcal{A}$.) Now an additional possibility is that the left premise is by $(\vdash\,!)$ or $(\,?\vdash)$. The side conditions of the rules together with $\rho_l > 1$ imply that the principal formula of either rule is not the cut formula. We consider in detail the case when the left premise is by $(\vdash\,!)$; the other rule may be dealt with similarly.

If $\rho_r = 1$, then the only possibility (beyond an axiom) is that the right premise is by $(\,?\vdash)$.

$$\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \dfrac{\overset{\vdots}{\mathcal{C};\, !\gamma \vdash\, ?\delta}}{?\mathcal{C};\, !\gamma \vdash\, ?\delta}}{\llbracket !\alpha;\, !\gamma \rrbracket \vdash \llbracket ?\beta;\, ?\delta;\, !\mathcal{A} \rrbracket} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}} \qquad \dfrac{\overset{\vdots}{\mathcal{C};\, !\gamma \vdash\, ?\delta}}{?\mathcal{C};\, !\gamma \vdash\, ?\delta}}{\llbracket !\alpha;\, !\gamma \rrbracket \vdash \llbracket ?\beta;\, ?\delta; \mathcal{A} \rrbracket}}{\llbracket !\alpha;\, !\gamma \rrbracket \vdash \llbracket ?\beta;\, ?\delta;\, !\mathcal{A} \rrbracket}$$

The transformation is justified by a decrease in ρ_l.

If $\rho_r > 1$, then the right premise cannot be by $(\vdash\,!)$ or $(\,?\vdash)$ due to the shape of the cut formula and the side conditions in those rules. If the right premise is by a rule for $^{\perp}$, $\otimes$, $\multimap$, ⅋, $\mathbf{1}$ or $\perp$, then the cut is moved upward and the transformation is justified by a decrease in ρ_r.

The remaining possibilities are that the right premise is by $\llbracket\, !\vdash \rrbracket$, $\llbracket \vdash ?\, \rrbracket$, $(\,!K\vdash)$ or $(\vdash ?K)$.

$$\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \dfrac{\overset{\vdots}{?\mathcal{C}; \mathcal{B}; \gamma \vdash \delta}}{\llbracket ?\mathcal{C};\, !\mathcal{B}; \gamma \rrbracket \vdash \delta}}{\llbracket !\alpha;\, !\mathcal{B}; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A} \rrbracket} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \overset{\vdots}{?\mathcal{C}; \mathcal{B}; \gamma \vdash \delta}}{\llbracket !\alpha; \mathcal{B}; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A} \rrbracket}}{\llbracket !\alpha;\, !\mathcal{B}; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A} \rrbracket}$$

$$\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \dfrac{\overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta; \mathcal{B}}}{?\mathcal{C}; \gamma \vdash \llbracket \delta;\, ?\mathcal{B} \rrbracket}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A};\, ?\mathcal{B} \rrbracket} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta; \mathcal{B}}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A}; \mathcal{B} \rrbracket}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A};\, ?\mathcal{B} \rrbracket}$$

The transformations are justified by a reduction in ρ_r. All the earlier contractions may be carried out in the new proof segments too. The next two cases are justified similarly.

$$\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \dfrac{\overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta}}{?\mathcal{C};\, !\mathcal{B}; \gamma \vdash \delta}}{\llbracket !\alpha;\, !\mathcal{B}; \gamma \rrbracket \vdash \llbracket ?\beta;\, !\mathcal{A}; \delta \rrbracket} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta;\, !\mathcal{A}; \delta \rrbracket}}{\llbracket !\alpha;\, !\mathcal{B}; \gamma \rrbracket \vdash \llbracket ?\beta;\, !\mathcal{A}; \delta \rrbracket}$$

$$\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \dfrac{\overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta}}{?\mathcal{C}; \gamma \vdash \delta;\, ?\mathcal{B}}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A};\, ?\mathcal{B} \rrbracket} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\overset{\vdots}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C}; \mathcal{A}}}{!\alpha \vdash\, ?\beta;\, ?\mathcal{C};\, !\mathcal{A}} \qquad \overset{\vdots}{?\mathcal{C}; \gamma \vdash \delta}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A} \rrbracket}}{\llbracket !\alpha; \gamma \rrbracket \vdash \llbracket ?\beta; \delta;\, !\mathcal{A};\, ?\mathcal{B} \rrbracket}$$

This completes the proof of the admissibility of the cut rule in $\llbracket L\mathrm{CLL} \rrbracket$. $\triangleleft$

Lemma 33. (Curry's lemma for $[\![L\text{CLL}]\!]$) *If $\alpha \vdash \beta$ has a proof in $[\![L\text{CLL}]\!]$ with the height of the proof tree being n, and $\gamma \vdash \delta$ results from $\alpha \vdash \beta$ by one or more applications of the rules $(\,!W\vdash)$ and $(\vdash\,?W)$, then $\gamma \vdash \delta$ has* a proof *in $[\![L\text{CLL}]\!]$, where the height of the proof tree is* not greater than *n (i.e., it is $\leq n$).*

Proof. The proof of this theorem is a straightforward extension of Curry's lemma for the multiplicative–exponential fragment of CLL with six cases added. (See Theorem 14 in [8].) Namely, the basis of the induction is expanded to deal with $(\mathbf{0}\vdash)$ and $(\vdash\top)$, plus $(\&\vdash)$, $(\vdash\&)$, $(\oplus\vdash)$ and $(\vdash\oplus)$ are added to the inductive step. Each of these is quite routine (and we omit the details).

From another point of view, we can start with the proof of Lemma 28. We considered $(\Box K\vdash)$ or $(\vdash\Diamond K)$ there. Here we have to consider what happens if $(\,!K\vdash)$ or $(\vdash\,?K)$ are the last rules applied in a proof. We assume that $!\mathcal{A}$, $!\mathcal{C}$, $?\mathcal{B}$ and $?\mathcal{D}$ are (pairwise) distinct, and that the former two differ from elements of α', and the latter two are not among the elements of β'. We also assume that *three* is a representative number for the general situation (and it also allows us to fit everything on a page).

Let us assume that the last rule is $(\,!K\vdash)$. We have the following.

$$\dfrac{\overset{\vdots}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}}}{!\mathcal{A};\ !\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\overset{\vdots}{!\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D}}}{!\mathcal{A};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D}}$$

If the thinned in formula is the same as $!\mathcal{C}$, then the application of $(\,!K\vdash)$ may be simply omitted like in

$$\dfrac{\overset{\vdots}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}}}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \overset{\vdots}{!\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D}}.$$

The case of $(\vdash ?K)$ is dual to this. Here is what it looks like.

$$\dfrac{\overset{\vdots}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}}}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{B}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\overset{\vdots}{!\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D}}}{!\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{B}}$$

$$\dfrac{\overset{\vdots}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}}}{!\mathcal{C};\ !\mathcal{C};\ !\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D};\ ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \overset{\vdots}{!\mathcal{C};\alpha'\vdash\beta';\ ?\mathcal{D}}$$

Next, we note that in the proof of Lemma 24, we can restrict contractions to exponential formulas. Then some of the cases disappear, whereas the others go through as before. This completes the proof. ◃

Now we turn to the decidability proof for LCLL.

Theorem 34. *Classical linear logic* (CLL) *is* decidable.

Proof. Given a wff $\mathcal{A}$, we narrow down the question whether the wff is a theorem of LCLL by ensuring that $\mathcal{A}$ is not a theorem of $L\text{LRW}^{\Diamond\Box}$, and it is a theorem of $[\![L\text{CLL}]\!]$. (If $\mathcal{A}$ is not within that range, then we already know whether it is a theorem of LCLL. Namely, if $\mathcal{A}$ is a theorem of $L\text{LRW}^{\Diamond\Box}$, then it is a theorem of LCLL, and if $\mathcal{A}$ is not a theorem of $[\![L\text{CLL}]\!]$, then it is not a theorem of LCLL.) The proof search in $[\![L\text{CLL}]\!]$ generates all the irredundant proofs of $\mathcal{A}$. By Definition 8, we calculate the heap numbers for the subformulas of $\mathcal{A}$. Then we start to build a proof-search tree in LCLL. The root is the sequent $\vdash \mathcal{A}$, and we expand the tree by scrutinizing each rule that could result in the sequent in a particular node in the tree. If there is a possibility for contractions then we add each possibility separately to the tree. However, we limit the number of contractions on each formula by its heap number. The whole tree is finite and if $\mathcal{A}$ is provable in LCLL, then the search tree will contain a proof. If $\mathcal{A}$ is not a theorem, then we will find this out in finitely many steps, namely, when the (finite) proof-search tree is completed without containing a proof. $\triangleleft$

7. Remarks on "Decision problems for linear logic"

Lincoln et al. [35] present what they take to be a proof of the undecidability of what we call "classical linear logic" (CLL) and they call "full propositional linear logic" (or sometimes just "linear logic"). This paper is highly original and well-motivated, exploiting the notion of linear logic as a "resource conscious logic." The proof was seemingly well-presented, and seemed to have convinced many people that CLL is undecidable. But only the most naive logician thinks that something is a proof because it is called a proof. Maybe, someday the dream will be fulfilled that all proofs will be computer checkable, but for now, and even as proofs get more and more complicated, we are largely dependent on human intelligence and a mixture of formal language, natural language, and a sometimes conventional, sometimes creative, hybrid mixture of the two. Unfortunately, and we are apologetic about this to Lincoln, Mitchell, Scedrov and Shankar (all fine logicians), but we think that there are some mistakes in their proof. We shall outline their proof both to help the reader (and ourselves) understand the virtues of their attempt, and a flaw in the proof.

The rough idea of their proof is to reduce the question of the decidability of CLL to the problem of the solvability of a question about certain finite automata, which they introduce and call *And-Branching Two-Counter Machines* ***Without*** *Zero-Test* (*ACM* for short). These are a variant of the more standard *And-Branching Two-Counter Machines* ***With*** *Zero-Test*. They ingeniously replace the Zero-Test with something they call "Forking." The corresponding question for the former is known to be unsolvable, and they show that the halting problem for these two is the same. They then go on to translate the question of the decidability of linear logic into the solvability of ACMs, and use the fact that ACMs are unsolvable to show that LCLL is undecidable. The "trick" is the translation between computations in ACMs and proofs in LCLL.

They start by defining (p. 261) a *theory* to be a finite set of *axioms*, and they define an *axiom* to be "a linear logic sequent of the form $\vdash C, p_{i_1}^{\perp}, \ldots, p_{i_n}^{\perp}$, where C is a MALL formula (a linear logic formula without ! or ?) and the remainder of the sequent is made up of negative literals."[12] They make it clear that the negative literals are allowed to be absent and that the restrictive form of axioms is due to their wanting to "achieve strict control over the shape of a proof."

They define that "a sequent $\vdash \Gamma$ is provable in T exactly when we are able to derive $\vdash \Gamma$ using the standard set of linear logic proof rules, in combination with axioms from

[12]We use " ; " in the sequent calculus LCLL, but in this section we resort to " , " for easy comparison with Lincoln et al. [35]. Incidentally, they use a one-sided sequent calculus, however, in the case of CLL, this affects only the presentation.

T." It is evident from context and from their Appendix B that they assume the cut rule to be in the "standard set of linear logic proof rules," just as [23] does. They go on to Lemma 3.1 that states that cut can be replaced by what they call "directed cut." They make it clear that such a derivation would be just like a proof tree in linear logic except that the leaves can be axioms from T, and not just the usual logical axioms $\vdash p_i, p_i^\perp$. Let us write $T \vdash_{L\mathrm{CLL}} \Gamma$ for $\vdash \Gamma$ is provable from the theory T in LCLL. Note that they explicitly define this notion only for the case of LCLL, not for its multiplicative-additive fragment MALL. This is important, because later (p. 265) they say: "We have just shown how a decision problem for MALL with the addition of nonlogical axioms may be encoded in full propositional linear logic without nonlogical axioms. Thus the upcoming proof of undecidability of MALL with nonlogical axioms will yield undecidability for full propositional logic."

Notice that here they talk about "MALL with the addition of nonlogical axioms," but this has not been really defined. They actually defined provability from T in LCLL. We know this sounds like a picky point, and readily agree that we can make sense of MALL theories as just the obvious variant of LCLL theories that does not allow applying the rules for the exponentials. But they misdescribe what they showed. What they in fact showed was how a decision problem for full propositional linear logic (not just for the MALL fragment) with the addition of nonlogical axioms may be encoded in full propositional linear logic without nonlogical axioms. However, they say (p. 260) that "We now show that if nonlogical (MALL) axioms are added to MALL, the decision problem becomes recursively unsolvable. We also show that nonlogical MALL axioms may be encoded in full propositional linear logic without nonlogical axioms, and thus we hve the result that full propositional linear logic is undecidable."

Lemmas 3.2 and 3.3 each prove different directions of the following biconditional.

$$\text{For any finite set of axioms } T, \quad T \vdash_{L\mathrm{CLL}} \Gamma \quad \text{iff} \quad \vdash_{L\mathrm{CLL}} [T], \Gamma.$$

But they also seem to be saying (or tacitly implying) that

$$T \vdash_{\mathrm{MALL}} \Gamma \quad \text{iff} \quad \vdash_{L\mathrm{CLL}} [T], \Gamma.$$

To understand these claims we need to understand $[T]$, which translates a theory $T = \{t_1, t_2, \ldots, t_k\}$ into a multiset of linear logic formulas $?[t_1], ?[t_2], \ldots, ?[t_k]$, where $[t_i]$ is the translation of the axiom t_i into a single linear logic formula as follows. If t_i is $\vdash C, p_{i_1}^\perp, \ldots, p_{i_n}^\perp$, then $[t_i]$ is $\vdash C^\perp \otimes p_{i_1} \otimes \cdots \otimes p_{i_n}$.

Also, (p. 269) they say: "We give a translation from ACMS to linear logic with theories and show that our sequent translation of a machine in a particular state is provable in linear logic if and only if the ACM halts from that state. In fact our translation uses only MALL formulas and theories, thus with the use of our earlier encoding Lemma 3.2 and 3.3, we will have our result for propositional linear logic without nonlogical axioms. Since an instantaneous description of an ACM is given by a list of triples, it is somewhat delicate to state the induction we will use to prove soundness."

Lincoln et al. use *non-deterministic And-Branching Two-Counter Machines Without Zero-Test* (ACMs). An ACM has a set of *states* $\mathbf{Q}$, a finite set δ of *transitions*, and *initial* and *final* states Q_I and Q_F.

Depending on the state Q_i, the ACM can do various things. Thus, where A and B are natural numbers in the first and second registers, the rules can add 1 to them, subtract 1 from them (unless they are 0 in which case the rule is not applicable), and move to the state Q_j. Or the machine can continue computation from two states Q_j and Q_k, using as inputs the values A, B in the current state Q_i.

Rule	Transition	Translation
Q_i **Increment** A Q_j	$\langle Q_i, A, B\rangle \mapsto \langle Q_j, A+1, B\rangle$	$\vdash q_i^\perp, (q_j \otimes a)$
Q_i **Increment** B Q_j	$\langle Q_i, A, B\rangle \mapsto \langle Q_j, A, B+1\rangle$	$\vdash q_i^\perp, (q_j \otimes b)$
Q_i **Decrement** A Q_j	$\langle Q_i, A+1, B\rangle \mapsto \langle Q_j, A, B\rangle$	$\vdash q_i^\perp, a^\perp, q_j$
Q_i **Decrement** B Q_j	$\langle Q_i, A, B+1\rangle \mapsto \langle Q_j, A, B\rangle$	$\vdash q_i^\perp, b^\perp, q_j$
Q_i **Fork to** Q_j **and** Q_k	$\langle Q_i, A, B\rangle \mapsto \langle Q_j, A, B\rangle$ and $\langle Q_k, A, B\rangle$	$\vdash q_i^\perp, (q_j \oplus q_k)$

An *instantaneous description* (*ID*) of an ACM M is a finite tree of ordered triples $\langle Q_i, A, B\rangle$, where $Q_i \in \mathbf{Q}$ (Q_i is a state), and A and B are natural numbers. The *accepting triple* is $\langle Q_F, 0, 0\rangle$. An *accepting ID* is any ID where every leaf of the ID is the accepting triple. This means that no matter how the computation evolves it ends with an accepting triple, that is, in an accepting state (which is unique) and the counters containing 0.[13]

Given a triple $\langle Q_i, A, B\rangle$, its *translation* $\theta(\langle Q_i, A, B\rangle)$ is $\vdash q_i^\perp, (a^\perp)^A, (b^\perp)^B, q_F$, where the superscript ^{A} and ^{B} indicate the number of $a^\perp$'s and $b^\perp$'s in the sequent. The translation of an ID comprises the translations of the elements of the ID, that is, $\theta(E_1, E_2, \ldots, E_m) = \theta(E_1), \theta(E_2), \ldots, \theta(E_m)$.

Lincoln et al.'s main result is: "Theorem 3.7. The provability problem for propositional linear logic is recursively unsolvable." This is just a different way of saying that CLL is undecidable. Their proof consists literally of the single statement (p. 275) "From Lemmas 3.2–3.6 we obtain our main result." We shall try to construct a proof using these lemmas, and in the process end up deconstructing their proof.

As already mentioned, the first two of these lemmas can be put together as the two halves of the next biconditional.

Lemmas 3.2–3.3. *For any finite set of axioms T, $T \vdash_{LCLL} \Gamma$ iff $\vdash_{LCLL} [T], \Gamma$.*

And the last two are the two halves of the following biconditional.

Lemmas 3.5–3.6. *An* ACM *M accepts from the triple s iff the sequent $\theta(s)$ is provable, given the theory derived from M.*

And the middle lemma is the keystone.

Lemma 3.4. *It is undecidable whether an* ACM *accepts from the triple $\langle Q_I, A, B\rangle$.*

The rough idea would then be to combine these lemmas so that the undecidability of the ACM accepting from $\langle Q_I, A, B\rangle$ translates into the undecidability of provability in *L*CLL (without axioms).

So let us suppose that we have a method for deciding the provability of theorems in *L*CLL. Consider then an arbitrary ACM M, and its theory T_M that translates the instructions of the machine using the table above. Then, as a special case of Lemmas 3.2–3.3 we have:

$$T_M \vdash_{L\mathrm{CLL}} \Gamma \quad \text{iff} \quad \vdash_{L\mathrm{CLL}} [T_M], \Gamma.$$

Further, as a special case of Lemmas 3.5–3.6, we have:

[13] Lincoln et al. defined an accepting ID to be an ID each element of which is an accepting triple. They should have meant what is in this paragraph unless only one-step trivial computations are permitted.

An ACM M accepts from $\langle Q_I, A, B\rangle$ iff the sequent $\theta(\langle Q_I, A, B\rangle)$ is provable, given the theory T_M.[14]

What is the sequent $\theta(\langle Q_I, A, B\rangle)$? It is $\vdash q_I, (a^\perp)^A, (b^\perp)^B, q_F$. So the problem is to figure out whether this sequent is provable using the theory T_M, i.e., using the sequents in T_M together with applications of the cut rule.

While it is true that using the exponentials LCLL can emulate that a sequent from T_M is *not used*, *used once* or *used several times* in a MALL proof, the exponentials interact with the MALL vocabulary. In effect, the interaction implies reliance on the following claim.

$$T_M \vdash_{\text{MALL}} \Gamma \quad \text{iff} \quad \vdash_{L\text{CLL}} [T_M], \Gamma.$$

The following is an equivalent claim.

$$T_M \vdash_{\text{MALL}} \Gamma \quad \text{iff} \quad T_M \vdash_{L\text{CLL}} \Gamma.$$

From left to right, the claim is obvious and true, but the converse is less than obvious. The cut rule is *not eliminable* in the presence of proper axioms (the elements of T_M) — as Lincoln et al. [35] themselves point out on p. 262. Of course, using the cut rule in a proof is unproblematic in the sense that it is a rule and so the sequent proved is a theorem, but the cut rule causes problems for the analysis of the proof. Thus, when we try to prove that $T_M \vdash_{L\text{CLL}} \Gamma$ implies $T_M \vdash_{\text{MALL}} \Gamma$, we run into a problem, because a proof of Γ in LCLL may contain applications of the cut rule too. In other words, if the cut rule is not eliminable, then it is difficult to contemplate how the right-to-left conditional could be proved at all.

So far, we assumed that Lemmas 3.5–3.6 concerned provability in MALL. An alternative reading of the those lemmas is that they permit the use of all the rules of LCLL, but the occurrences of applications of the cut rule are limited because of Lemma 3.1, which reads as follows.

Lemma 3.1. (Cut standardization). *If there is a proof of $\vdash \Gamma$ in theory T, then there is a directed proof of $\vdash \Gamma$ in theory T.*

A *directed cut* is simply an application of the cut rule, in which at least one of the two premises is an axiom, and the cut formula is C (using the earlier notation). A *directed proof* is a derivation with only directed cuts (or no cuts at all). The cut standardization lemma holds in MALL proofs, and ensures that MALL theories in proofs in MALL can mimic the consumption of instructions in an ACM.

Before we turn to the discussion of the modeling of ACMs (and Minsky machines), we illustrate a problem with the proof of the admissibility of the cut rule in Appendix A. In the relevance logic literature, the use of a multi-cut rule is quite common, because many relevance logics contain a contraction rule (but not a thinning rule). The multi-cut rule is similar to Gentzen's mix rule in that it allows cutting out more than two formulas. On the other hand, these rules are different, because the multi-cut rule does not require the elimination of all occurrences of the cut formula. Lincoln et al. [35] opt to use both single cut and multi-cut in their elimination proof, however, the latter is only applicable to formulas that start with exponentials. In CLL, only certain

[14] Lemmas 3.5–3.6 do not make explicit the logic in which provability is meant. However, the first paragraph in §3.5 (p. 269) seems to suggest that it is MALL.

exponential formulas can be contracted, which explains why multi-cut is introduced for such formulas.

[35] defines the degree of the cut formula in a fairly standard manner. However, they also define the degree of a proof as the maximal degree of any cut in the proof or zero (if there is no cut). Unfortunately, the degree of the proof does not decrease in every step in the cut elimination proof.[15] Their crucial lemma reads as

Lemma A.1 (Reduce one cut). *Given a proof of the sequent $\vdash \Gamma$ in linear logic which ends in an application of* **Cut*** *of degree $d > 0$, and where the degree of the proofs of both hypotheses is less than d, we may construct a proof of $\vdash \Gamma$ in linear logic of degree less than d.*

The proof is divided into cases, and in each case a local modification of the proof is given. The transformations are similar to what is to be expected. However, it is completely obvious that several of the one-step transformations do not establish the claim of the lemma.

The following example shows an application of the single cut rule with $d = 7$. (We make explicit only the segment of the proof that is problematic.)

$$\dfrac{\dfrac{\vdash p \,\&\, (\,!q \oplus r)}{\vdash\, !(p \,\&\, (\,!q \oplus r))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, ?q^{\perp} \,\&\, r^{\perp},\, (p \,\&\, q) \oplus r}}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp}),\, (p \,\&\, q) \oplus r}\;{\scriptstyle ?D}}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, (p \,\&\, q) \oplus r}\;{\scriptstyle ?C}}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, (p \,\&\, q) \oplus r}}{\vdash (p \,\&\, q) \oplus r}\;{\scriptstyle \text{cut}}$$

The last step in the proof is an application of the cut rule. The cut formula is principal in both premises of the cut, hence by A.2.5, the cut is moved up by a sequent in the right premise. This requires that **Cut!** be applied. However, the cut formula is the same as before, hence, the degree of the proof is also the same.

Lincoln et al. [35] must have realized that they do not have an inductive proof of Lemma A.1, because they say on p. 299 that "by induction on the size of proofs, we can construct the desired proof of degree less than *d*." It is possible, perhaps, even plausible that one can do this. However, they do not give such a proof, indeed, they do not even define what the size of a proof is. It could be the number of propositional variables in the proof, the sum of the degrees of all formulas, the height of the proof tree, to name a few alternatives.

Another problem with the argument for Lemma A.1 is that once the above proof is transformed (as shown below), it is no longer clear which transformation is to be applied next. The cut formula is still principal in the left premise, however, it is both principal and non-principal in the right premise. There is no definition in [35] that would allow us to determine whether the cut formula in the right premise is principal or not, which is needed in order to apply (1) or (2) on p. 298. In fact, the situation is very typical, because principal formulas are (usually) unique in a rule. Then, a cut on

[15] Lambek [32] was able to prove a cut theorem for his calculuses by induction on one parameter that he called degree, which is however, not identical to either of the degrees just mentioned. Also, Lambek's calculuses do not contain any kind of contraction, which means that the admissibility of the single cut rule can be proved directly (without mix or multi-cut).

$$\dfrac{\dfrac{\vdash p \,\&\, (\,!q \oplus r)}{\vdash\, !(p \,\&\, (\,!q \oplus r))} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, ?q^{\perp} \,\&\, r^{\perp}, (p \,\&\, q) \oplus r}}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})), p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp}), (p \,\&\, q) \oplus r}}{\vdash\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})),\, ?(p^{\perp} \oplus (\,?q^{\perp} \,\&\, r^{\perp})), (p \,\&\, q) \oplus r}\ {?D}}{\vdash (p \,\&\, q) \oplus r}\ \text{cut!}$$

several formulas moved upward in a proof tree will likely come to a sequent in which the cut formula has both principal and non-principal occurrences. The usual notion of a principal formula is extended on p. 297. However, that expansion leaves one occurrence of the cut formula in the right premise of the application of the cut! rule above as a non-principal occurrence.

Presumably, we should apply the second transformation in A.2.6 now. The transformation yields two cuts in a new proof that have the same degree, and which repeat a whole branch of the proof tree. This is a point where the informal allusion to the size of the proofs would need to be made precise, because neither the height of the proof tree is decreasing nor the number of sequents or cuts does.[16]

The cut elimination proof would be the basis for the proof that directed cuts suffice. However, we believe that there is *no proof of the admissibility of the cut rule* in [35], hence, there is *no proof of Lemma 3.1* and further, of Theorem 3.7 in that paper.

The ***main problem*** with the alleged proofs in [35] and [27] (as well as in [21]) goes beyond what we have outlined so far. The two models, ACMs and Minsky machines, are very similar; they are both variations on what more simply are called counter machines. A particularly elegant formulation is termed *abacus machines* in Boolos and Jeffrey [15] with reference to Lambek [34].

Counter machines are "full-fledged" models of computation as proved in [15] and in [34]. However, the abacus machines *compute functions*, that is, starting with natural numbers in the counters the machine halts with some content (which may or may not be all 0's) in the counters. ACMs and Kanovich's Minsky machines do not compute any functions, rather, they *accept* a certain input. Furthermore, both models are modified to accept by a final state with *all the counters empty*.

Neither [35] nor [28] ([27]) prove that the machines that they intend to model have an undecidable halting problem. The undecidability of the halting problem for Minsky machines with a restricted halting problem was recently proved in [21, §7] via several reduction steps from the Post correspondence problem.[17]

One might wonder how the computation of one or another machine is *modeled* in propositional logic. There are well-known ways to model primitive recursive functions and computations of a Turing machine in the language of first-order arithmetic. We have explained at the beginning of this section how [35] intend to model the computations of ACMs; the rest of the authors follow a similar idea. We present in Figure 3

[16] Sequent calculuses are, perhaps, more difficult to understand than axiomatic systems. This may be one of the reasons behind [42], which shows that the author does not understand the proof of the cut theorem in [8]. He also seems to assume that the decision procedure for MELL should generate *all* the infinitely many proofs for a provable sequent. Of course, decision procedures, normally, do not yield all possible proofs.

[17] In all the papers that we mentioned in this paragraph, a proof that matches a computation starts with the final state. We will refer to the authors of all these papers as the authors, when we talk about this feature of the machines.

(below) a small ACM, which differs from the example in [35] in that it accepts an *infinite language* and it contains *three zero-tests*. (Their sample machine accepts the finite language $\{ a^0 b^0 \}$ and contains no zero-tests at all.)

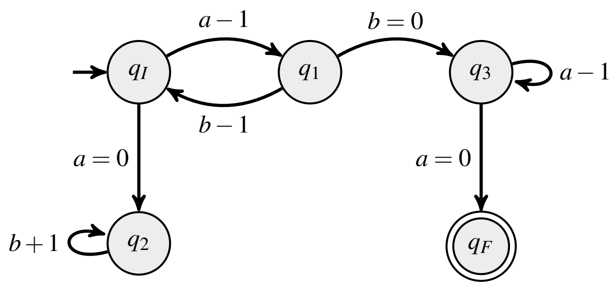

FIGURE 3. The ACM $\mathfrak{M}_1$

The picture of the ACM employs some notational conventions that are often used in visualizing finite state automata such as circles for states with the name of the state inside, and arrows with labeling for the actions of the machine. However, these similarities are somewhat superficial. The ACM receives input at the arrow pointing to q_I in the form of finitely many counters filled as desired. Then, the machine reads and occasionally *modifies* the content of the counters. The arrows labeled with $a = 0$ and $b = 0$ represent successful zero-tests. (The diagram hides the implementation of a zero-test via "and-branching.") The state q_3 is a seemingly spurious state; its function is simply to ensure that counter a is empty. The machine is so designed that if it reaches q_F, then it is guaranteed that the counters are empty, hence, the role of q_F is to indicate acceptance and halting. It is not difficult to see that $\mathfrak{M}_1$ accepts the language $\{ a^m b^n \colon m > n\}$ (where a and b are placeholders for "first" and "second" counter). This language is not very complicated, it's easily seen to be a CFL (context-free language). Alternatively, the machine can be thought to accept when the characteristic function of the $>$ relation (on $\mathbb{N}$) evaluates to true.

For example, the full description of the computation of the machine starting with a^3b^1 (i.e., 3 in the first counter, and 1 in the second counter) is the following sequence of triplets.

$$\langle q_I, a^3, b^1\rangle, \langle q_1, a^2, b^1\rangle, \langle q_I, a^2, b^0\rangle, \langle q_1, a^1, b^0\rangle, \langle q_3, a^1, b^0\rangle, \langle q_3, a^0, b^0\rangle, \langle q_F, a^0, b^0\rangle$$

The set of instructions for $\mathfrak{M}_1$ encoded as axioms for a CLL theory is as follows. Here we make explicit the hidden and-branching, which we use only with the zero states z_a and z_b.

Axioms for $\mathfrak{M}_1$:

1. $\vdash q_I^\perp, a^\perp, q_1$	2. $\vdash q_1^\perp, b^\perp, q_I$	3. $\vdash q_3^\perp, a^\perp, q_3$	4. $\vdash q_2^\perp, b \otimes q_2$
5. $\vdash q_I^\perp, z_a \oplus q_2$	6. $\vdash q_1^\perp, z_b \oplus q_3$	7. $\vdash q_3^\perp, z_a \oplus q_F$	
8. $\vdash z_a^\perp, b^\perp, z_a$	9. $\vdash z_b^\perp, a^\perp, z_b$	10. $\vdash z_a^\perp, q_F \oplus q_F$	11. $\vdash z_b^\perp, q_F \oplus q_F$

If we construct proofs with cuts, then these axioms cut out their negations from a sequent. If the proof is cut-free, then the same formulas have to be built-up.

Negations of axioms:

1. $q_I \otimes (a \otimes q_1^\perp)$	2. $q_1 \otimes (b \otimes q_I^\perp)$	3. $q_3 \otimes (a \otimes q_3^\perp)$	4. $q_2 \otimes (b^\perp \parr q_2^\perp)$
5. $q_I \otimes (z_a^\perp \,\&\, q_2^\perp)$	6. $q_1 \otimes (z_b^\perp \,\&\, q_3^\perp)$	7. $q_3 \otimes (z_a^\perp \,\&\, q_F^\perp)$	
8. $z_a \otimes (b \otimes z_a^\perp)$	9. $z_b \otimes (a \otimes z_b^\perp)$	10. $z_a \otimes (q_F^\perp \,\&\, q_F^\perp)$	11. $z_b \otimes (q_F^\perp \,\&\, q_F^\perp)$

It is easy to see that $\vdash z_a^\perp, (b^\perp)^n, q_F$ and $\vdash z_b^\perp, (a^\perp)^n, q_F$ are provable for any $n \in \mathbb{N}$. We will omit these parts of the proof to limit the size of the tree shown. The axioms that are used in applications of cuts are listed on the left.

$\vdash q_3^\perp, z_a \oplus q_F$

$\vdash q_3^\perp, a^\perp, q_3$

$\vdash q_1^\perp, z_b \oplus q_3$

$\vdash q_I^\perp, a^\perp, q_1$

$\vdash q_1^\perp, b^\perp, q_I$

$\vdash q_I^\perp, a^\perp, q_1$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\vdash z_a^\perp, q_F$}
\AxiomC{$\vdash q_F^\perp, q_F$}
\BinaryInfC{$\vdash z_a^\perp \,\&\, q_F^\perp, q_F$}
\UnaryInfC{$\vdash q_3^\perp, q_F$}
\UnaryInfC{$\vdash q_3^\perp, a^\perp, q_F$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\vdash z_b^\perp, a^\perp, q_F$}
\BinaryInfC{$\vdash z_b^\perp \,\&\, q_3^\perp, a^\perp, q_F$}
\UnaryInfC{$\vdash q_1^\perp, a^\perp, q_F$}
\UnaryInfC{$q_I^\perp, a^\perp, a^\perp, q_F$}
\UnaryInfC{$\vdash q_1^\perp, a^\perp, a^\perp, b^\perp, q_F$}
\UnaryInfC{$\vdash q_I^\perp, a^\perp, a^\perp, a^\perp, b^\perp, q_F$}
\end{prooftree}
$$

The proof *starts* in the *final* state. It is not accidental, because sequent calculus proofs are trees in which the root of the tree is the sequent that is proved. Hence, no tree branch in a proof can split downward.

A cut-free proof for the same sequent is the following. We indicate the negations of the axioms by their number in the listing above, and we omit the proofs leading to a z_x state from q_F together with the horizontal lines. (The two sequents that are not axioms, but easily provable, are *'d.)

$^*\vdash z_a^\perp, q_F$	$\vdash q_F^\perp, q_F$
$\vdash q_3^\perp, q_3$	$\vdash z_a^\perp \,\&\, q_F^\perp, q_F, 10$
$\vdash a^\perp, a$	$\vdash q_3^\perp, q_F, 10, 7$
$\vdash q_3^\perp, q_3$	$\vdash a \otimes q_3^\perp, q_F, 10, 7$
$^*\vdash z_b^\perp, a^\perp, q_F$	$\vdash q_3^\perp, a^\perp, q_F, 10, 7, 3$
$\vdash q_1^\perp, q_1$	$\vdash z_b^\perp \,\&\, q_3^\perp, a^\perp, q_F, 10, 7, 3, 11, 9$
$\vdash a^\perp, a$	$\vdash q_1^\perp, a^\perp, q_F, 10, 7, 3, 11, 9, 6$
$\vdash q_I^\perp q_I$	$\vdash a \otimes q_1^\perp, a^\perp, a^\perp, q_F, 10, 7, 3, 11, 9, 6$
$\vdash b^\perp, b$	$\vdash q_I^\perp, a^\perp, a^\perp, q_F, 10, 7, 3, 11, 9, 6, 1$
$\vdash q_1^\perp, q_1$	$\vdash b \otimes q_I^\perp, a^\perp, a^\perp, b^\perp, q_F, 10, 7, 3, 11, 9, 6, 1$
$\vdash a^\perp, a$	$\vdash q_1^\perp, a^\perp, a^\perp, a^\perp, b^\perp, q_F, 10, 7, 3, 11, 9, 6, 1, 2$
$\vdash q_I^\perp, q_I$	$\vdash a \otimes q_1^\perp, a^\perp, a^\perp, a^\perp, b^\perp, q_F, 10, 7, 3, 11, 9, 6, 1, 2$
	$\vdash q_I^\perp, a^\perp, a^\perp, a^\perp, b^\perp, q_F, 10, 7, 3, 11, 9, 6, 1, 2, 1$

The traditional claim is that if the proof is turned upside down, then it can be seen as a modeling of the computation from q_I (with 3 in a and 1 in b) to q_F. Of course, the upside down tree is *not a proof* in CLL at all. If we try to create an interpretation from the top of the proof, then it seems that the subproofs in the whole proof tree do not have an interpretation that is independent from the whole proof tree. Another way to look at this problem is that unless the proof has a sequent of the form $\vdash q_I^\perp, \ldots, q_F$ as its root, it is not a model of (any stage of) a computation of the machine.

To further illustrate the problem, let us assume that we add a new state q_4 to $\mathfrak{M}_1$. The new state has two outgoing arrows, one pointing to q_4 itself with a label $b-1$, the other pointing to q_3 with a label $b=0$. Our new machine $\mathfrak{M}_1'$ is equivalent in terms of acceptance to $\mathfrak{M}_1$. However, there is a proof of the sequent $\vdash q_4^\perp, b^\perp, b^\perp, b^\perp, q_F$, given its theory. The state q_4 — by design — is not accessible from q_I, which means that in $\mathfrak{M}_1'$ there is no computation that involves q_4. But it is true that if we picture the machine as a special graph (like $\mathfrak{M}_1$ in Figure 3), then there is *a path* between q_4 and q_F. And starting in state q_F, and by performing the inverses of the machine's instructions, it is possible to reach state q_4 with 3 in the second counter.

[26] and [21] number the final state with 0, which gives the appearance (at first) that a proof starts at an initial state (q_0). The latter paper models computation by getting from the 0th state (called PC value) to the 1st state.

Kopylov [29] noted that provable sequents (in the normal fragment) of CLL can be given *two computational readings*. Similarly, the provable sequents of CLL may be given two computational interpretations. The emptiness of the counters at halting, and halting in a unique final state are essential for the construction of sequent calculus proofs. In other words, proofs starting with $\vdash q_F^\perp, q_F$'s that contain forking cannot be replaced by proofs that start from $\vdash q_I^\perp, q_I$ (while proving the same sequent). However, the "non-traditional" interpretation means that there are *no zero-tests* in the machine that is modeled, and the decrement and increment instructions are swapped. According to this interpretation, that we think is *the correct one*, every subtree in a proof tree is a model of a step in *reverse computation*; that is, it is a model of "running" the machine backward. (This also means that the machine may get "stuck" in a state when the subtraction cannot be performed and there is no branch that takes care of the counter's emptiness.) In view of our decidability result, we think that the machines that emerge from this interpretation — *reverse* ACM*s* and *reverse Minsky machines*, etc. — do not have an undecidable halting problem. In other words, our decidability result supports the conjecture that the halting problem for reverse computation in ACMs and various counter machines, in general, is decidable.

To summarize, we think that each published "proof," most prominently, that by Lincoln et al. [35] and that by Kanovich [27], has *gaps* in it. Moreover, we think that there is a real reason to believe that some of those gaps cannot be filled to complete the proofs of the undecidability of *L*CLL, because there is a conceptual mismatch between (forward/normal) computational steps and steps in a sequent calculus proof in *L*CLL. Furthermore, our proofs demonstrate that *L*CLL is *decidable*.

8. Conclusion

This paper scrutinized the issue of modalities in lattice-R. To start with, the Ackermann and Church constants (hence, modalities defined from those constants) do not interfere with the decidability of lattice-R. The addition of *primitive modal operators* with some usual rules does not lead to undecidability either. If (modalized) versions of structural rules are added (or omitted) from LLR$^{\Diamond\Box}$, then the properties of modalities vary. Nonetheless, the *decidability* of the resulting logics — no matter with however unusual modalities — stays provable. We have also proved that classical propositional linear logic is decidable, and we have explained where the proofs of earlier undecidability claims in [35], in [28] (also, [27]) and in [21] are lacking.

Acknowledgments. We are grateful to Patrick Lincoln for providing us with a summary of his and his coauthors' views on the decidability of linear logic and its computational interpretation as well as their reactions to the first draft of our paper in 2014. We thank all of them.

We would like to thank Andre Scedrov for bringing to our attention A. Kopylov's and M. Kanovich's papers. We thank Max Kanovich for providing us with a "tutorial" on his interpretation of linear logic using Minsky machines.

We also thank Alasdair Urquhart for reading a draft of our paper, and especially, for his questions about the "heap numbers" and pointing out Roorda's work.

Of course, we do not mean to imply that these researchers endorse our paper.

We would like to thank audiences at the *2015 North American Annual Meeting of the Association for Symbolic Logic* in Urbana, IL (March 2015), at the *4th CSLI Workshop on Logic, Rationality and Intelligent Interaction* in Stanford, CA (May 2015), at the *POMSIGMA* session at the *Joint Mathematics Meeting* in Atlanta, GA (January, 2017), as well as, in the *Logic Seminar* of the *Indiana University Logic Group* in Bloomington, IN (February 2015 and February 2016), where we presented talks based on parts of this paper.

AFTERWORD

The first six and a half sections of this paper were written in 2015, and they remained basically the same since then. The last several "fault-finding" pages were rewritten and expanded several times to appease referees who repeated again and again that propositional linear logic is well known to be undecidable, and first of all, we should demonstrate mistakes in published proofs. It should be noted that no referee — in all those 6–7 years of refereeing — pointed out a mistake in our paper.

REFERENCES

[1] Allwein, G. and Dunn, J. M. (1993). Kripke models for linear logic, *Journal of Symbolic Logic* **58**(2): 514–545.

[2] Anderson, A. R. and Belnap, N. D. (1975). *Entailment: The Logic of Relevance and Necessity*, Vol. I, Princeton University Press, Princeton, NJ.

[3] Anderson, A. R., Belnap, N. D. and Dunn, J. M. (1992). *Entailment: The Logic of Relevance and Necessity*, Vol. II, Princeton University Press, Princeton, NJ.

[4] Avron, A. (1988). The semantics and proof theory of linear logic, *Theoretical Computer Science* **57**: 161–184.

[5] Belnap, N. D. (1993). Life in the undistributed middle, *in* K. Došen and P. Schroeder-Heister (eds.), *Substructural Logics*, Clarendon, Oxford, UK, pp. 31–41.

[6] Belnap, N. D. and Wallace, J. R. (1961). A decision procedure for the system $E_{\bar{I}}$ of entailment with negation, *Technical Report 11, Contract No. SAR/609 (16)*, Office of Naval Research, New Haven, CT, (published in *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **11** (1965): 277–289).

[7] Bimbó, K. (2007). Relevance logics, *in* D. Jacquette (ed.), *Philosophy of Logic*, Vol. 5 of *Handbook of the Philosophy of Science* of *Handbook of the Philosophy of Science* (D. Gabbay, P. Thagard and J. Woods, eds.), Elsevier, Amsterdam, pp. 723–789.

[8] Bimbó, K. (2015a). The decidability of the intensional fragment of classical linear logic, *Theoretical Computer Science* **597**: 1–17.

[9] Bimbó, K. (2015b). *Proof Theory: Sequent Calculi and Related Formalisms*, CRC Press, Boca Raton, FL.

[10] Bimbó, K. (2016). Review of M. Kanovich, "The undecidability theorem for the Horn-like fragment of linear logic (Revisited)", *Mathematical Reviews* . MR3492992.

[11] Bimbó, K. (2017). On the decidability of certain semi-lattice based modal logics, *in* R. A. Schmidt and C. Nalon (eds.), *Automated Reasoning with Analytic Tableaux and Related Methods. Proceedings of the 26th International Conference, TABLEAUX 2017, Brasília, Brazil, September 25–28, 2017*, number 10501 in *Lecture Notes in Artificial Intelligence*, Springer Nature, Switzerland, pp. 44–61.

[12] Bimbó, K. and Dunn, J. M. (2012). New consecution calculi for $R^{\boldsymbol{t}}_{\rightarrow}$, *Notre Dame Journal of Formal Logic* **53**(4): 491–509.

[13] Bimbó, K. and Dunn, J. M. (2013). On the decidability of implicational ticket entailment, *Journal of Symbolic Logic* **78**(1): 214–236.

[14] Bimbó, K. and Dunn, J. M. (2015). On the decidability of classical linear logic, (abstract), *Bulletin of Symbolic Logic* **21**(3): 358.

[15] Boolos, G. S. and Jeffrey, R. C. (1992). *Computability and Logic*, 3rd edn, Cambridge University Press, Cambridge, UK.

[16] Curry, H. B. (1963). *Foundations of Mathematical Logic*, McGraw-Hill Book Company, New York, NY. (Dover, New York, NY, 1977).

[17] Dunn, J. M. (1973). A 'Gentzen system' for positive relevant implication, (abstract), *Journal of Symbolic Logic* **38**(2): 356–357.

[18] Dunn, J. M. (1986). Relevance logic and entailment, *in* D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 1st edn, Vol. 3, D. Reidel, Dordrecht, pp. 117–224.

[19] Dunn, J. M. (1995). Positive modal logic, *Studia Logica* **55**: 301–317.

[20] Dunn, J. M. and Restall, G. (2002). Relevance logic, *in* D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd edn, Vol. 6, Kluwer, Amsterdam, pp. 1–128.

[21] Forster, Y. and Larchey-Wendling, D. (2019). Certified undecidability of intuitionistic linear logic via binary stack machines and Minsky machines, *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP '19), January 14–15, 2019, Cascais, Portugal*, ACM, New York, NY, pp. 1–14.

[22] Gentzen, G. (1935). Untersuchungen über das logische Schließen, *Mathematische Zeitschrift* **39**: 176–210.

[23] Girard, J.-Y. (1987). Linear logic, *Theoretical Computer Science* **50**: 1–102.

[24] Girard, J.-Y. (1995). Linear logic: Its syntax and semantics, *in* J.-Y. Girard, Y. Lafont and L. Regnier (eds.), *Advances in Linear Logic*, number 222 in *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, pp. 1–42.

[25] Hopcroft, J. E., Motwani, R. and Ullman, J. D. (2007). *Automata Theory, Languages, and Computation*, 3rd edn, Pearson Education, Boston, MA.

[26] Kanovich, M. (1995). The direct simulation of Minsky machines in linear logic, *in* J.-Y. Girard, Y. Lafont and L. Regnier (eds.), *Advances in Linear Logic*, number 222 in *London*

Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK, pp. 123–145.

[27] Kanovich, M. (2016). The undecidability theorem for the Horn-like fragment of linear logic (Revisited), *Mathematical Structures in Computer Science* **26**(5): 719–744.

[28] Kanovich, M. I. (1994). Linear logic as a logic of computations, *Annals of Pure and Applied Logic* **67**: 183–212.

[29] Kopylov, A. P. (2001). Decidability of linear affine logic, *in* A. R. Meyer (ed.), *Special issue: LICS 1995*, Vol. 164 of *Information and Computation*, IEEE, pp. 173–198.

[30] Kripke, S. A. (1959). The problem of entailment, (abstract), *Journal of Symbolic Logic* **24**: 324.

[31] Kripke, S. A. (1963). Semantical analysis of modal logic I. Normal modal propositional calculi, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **9**: 67–96.

[32] Lambek, J. (1958). The mathematics of sentence structure, *American Mathematical Monthly* **65**(3): 154–170.

[33] Lambek, J. (1961a). How to program an infinite abacus, *Canadian Mathematical Bulletin* **4**(3): 295–302.

[34] Lambek, J. (1961b). On the calculus of syntactic types, *in* R. Jacobson (ed.), *Structure of Language and its Mathematical Aspects*, American Mathematical Society, Providence, RI, pp. 166–178.

[35] Lincoln, P., Mitchell, J., Scedrov, A. and Shankar, N. (1992). Decision problems for linear logic, *Annals of Pure and Applied Logic* **56**: 239–311.

[36] Meyer, R. K. (1966). *Topics in Modal and Many-valued Logic*, PhD thesis, University of Pittsburgh, Pittsburgh, PA.

[37] Minsky, M. L. (1967). *Computation: Finite and Infinite Machines*, Automatic Computation, Prentice-Hall, Englewood Cliffs, NJ.

[38] Riche, J. and Meyer, R. K. (1999). Kripke, Belnap, Urquhart and relevant decidability & complexity, *in* G. Gottlob, E. Grandjean and K. Seyr (eds.), *Computer Science Logic (Brno, 1998)*, number 1584 in *Lecture Notes in Computer Science*, Springer, Berlin, pp. 224–240.

[39] Roorda, D. (1989). Investigations into classical linear logic, *ITLI Prepublication Series for Mathematical Logic and Foundations ML–89–08*, Institute for Language, Logic and Information, University of Amsterdam.

[40] Roorda, D. (1991). *Resource Logics: Proof-theoretical Investigations*, PhD thesis, University of Amsterdam.

[41] Smullyan, R. M. (1968). *First-order Logic*, Springer-Verlag, New York, NY.

[42] Straßburger, L. (2019). On the decision problem for MELL, *Theoretical Computer Science* **768**: 91–98.

[43] Thistlewaite, P. B., McRobbie, M. A. and Meyer, R. K. (1988). *Automated Theorem Proving in Non-classical Logics*, Pitman, London, UK.

[44] Troelstra, A. S. (1992). *Lectures on Linear Logic*, Vol. 29 of *CSLI Lecture Notes*, CSLI Publications, Stanford, CA.

DEPARTMENT OF PHILOSOPHY, UNIVERSITY OF ALBERTA, CANADA, *Email:* `bimbo@ualberta.ca`
LUDDY SCHOOL OF INFORMATICS, COMPUTING AND ENGINEERING, AND DEPARTMENT OF PHILOSOPHY, INDIANA UNIVERSITY, BLOOMINGTON, IN, U.S.A.