

RUSSELL'S NOTES ON FREGE FOR APPENDIX A OF *THE PRINCIPLES OF MATHEMATICS*

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This article presents notes that Russell made while reading the works of Gottlob Frege in 1902. These works include Frege's books as well as the packet of off-prints Frege sent at Russell's request in June of that year. Russell relied on these notes while composing "Appendix A: The Logical and Arithmetical Doctrines of Frege" to add to *The Principles of Mathematics*, which was then in press. A transcription of the marginal comments in those works of Frege appeared in the previous issue of this journal.

ORIGIN AND DESCRIPTION OF THE NOTES

The Bertrand Russell Archives possess extensive notes on the works of Gottlob Frege that Russell made in 1902. Twenty-one leaves of notes have been published as "Frege on the Contradiction" in the *Collected Papers* (4: 607–19).¹ The entire holding has been described by Gregory Moore (in *Papers* 3: 692–3).² Two leaves of these notes are printed in that volume as Plate VIII at (3: 39). Russell's marginal comments on Frege's works have been published.³ What follows is

¹ The notes are from RAI 230.030420–F2.

² Moore identifies two leaves each on *Begriffsschrift* and *Grundgesetze* as from RAI 220.010630. See the section on "Early Notes" below.

³ B. Linsky, "Russell's Marginalia in His Copies of Frege's Works", *Russell*, n.s. 24 (2004): 5–36.

a transcription of a further 36 pages of notes written on 34 leaves, the notes that Russell used in preparation for his appendix on Frege in *The Principles of Mathematics*. Thus remain 26 leaves of notes translating proofs from Frege's *Grundgesetze der Arithmetik* into Russell's notation, which are still unpublished.⁴

The notes published here were clearly made by Russell in preparation for "Appendix A: The Logical and Arithmetical Doctrines of Frege", which he added to the *Principles* late in 1902, after the body of the work had been sent to Cambridge University Press.⁵ That the notes were made in rapid succession, and in preparation for writing Appendix A, can be seen internally from the similarities of style and content between them, the systematic way in which they were prepared, and the many items in the notes that show up in the appendix.

The notes are presented below in groups. The first item is leaf (i), an outline which Russell titles "Appendix on Frege". The notes that follow are organized by the topics in this outline. Leaves (ii) to (xii), numbered 1 to 11 by Russell, systematically collect notes on the topics in the outline from various works of Frege, covering "Sinn und Bedeutung", "Wahrheitswerthe and Assertion", "Begriff und Gegenstand, Functions", and "Werthverläufe", from the first four items in the outline for the appendix. Then follow leaves (xiii) through (xxx), which are organized by source and appear to have been used in constructing the previous notes, as many entries appear in both lists. These sources include five papers by Frege: "Über Sinn und Bedeutung", "Über formale Theorien der Arithmetik", "Function und Begriff", "Über Begriff und Gegenstand" and "KB", and three books: *Begriffsschrift*, *Die Grundlagen der Arithmetik* and Volume 1 of *Die Grundgesetze der Arithmetik*. Finally there are the contents of five sheets found in Russell's copy of *Grundgesetze* when his library was acquired as part of the "Second Archives".⁶ From their contents, in particular the fact that in these notes Russell has questions about Frege's work that are answered in the other notes, it appears that these were the first notes that Russell made. Similarity in content suggests that they were made at the same time as, or soon after

⁴ From RAI 230.030420-F1.

⁵ See Moore, *Papers* 3: 693 and lvii, for the chronology.

⁶ As items RA2 220.148001b (removed from Gg. Vol. 1) and .148001c (removed from Gg. Vol. 2).

the marginalia which Russell presumably made on his first reading of the works. Copies of all of the works cited, except for the article by Benno Kerry, are to be found in Russell's library in the Archives. The Frege works, with the exception of *Grundgesetze*, Volume I, bound with the later Volume II, are together in a volume labelled "Pamphlets" on the spine. The marginalia strongly suggest these were the copies of Frege's publications from which Russell worked when composing the notes.

SIGNIFICANCE OF THE NOTES

The notes are extensive and primarily expository, copying Frege's own language, indeed sometimes preserving German terms such as "*Bedeutung*" for which Russell has no ready English equivalent. Russell saved his criticisms of Frege for the appendix. The simple extent of the notes, and their accuracy, by themselves should remove the suspicion that Russell had not studied Frege's work very carefully.

Every one of the over 70 references to Frege in appendix A can be found in the notes, but the cited passages are just a small part of the total notes and do not cover the whole range of Frege's works that Russell lists as sources. (The passages cited in the appendix are indicated in the transcription.) Discussions that compare the views of Frege and Russell on logic can now be read with a better appreciation of the nature of Russell's grasp of Frege's views.⁷

Russell's interests are primarily in logical issues such as Frege's notions of extension, (or *Werthverlauf*), concept, and assertion. There is no mention of the contradiction in the notes, although Russell does discuss it in the appendix. Indeed in the marginalia, and then in the first notes (at xxxiii below, *Gg*, p. 14⁸) Russell seems concerned about the truth of Principle v when the functions involved are not concepts. Principle v identifies the "course of values" (*Werthverläufe*) of functions which always have the same value for each argument. Russell seems to read it as applying only to concepts, which for Frege are functions having truth-

⁷ E.g., Michael Beaney, "Russell and Frege", in *The Cambridge Companion to Bertrand Russell*, ed. Nicholas Griffin (Cambridge: Cambridge U. P., 2003), pp. 128–70.

⁸ References to the notes will be by leaf (xxxiii in this case), then work, using Russell's abbreviation (*Gg* for *Grundgesetze der Arithmetik*) and finally the page number cited in the notes (p. 14 in the German original).

values as their values. In the final notes these worries are gone. The notes on *Grundgesetze* (at ix–xi) follow Frege’s discussion of these issues without remarking on the contradiction. Russell is more interested in Frege’s distinction between a singleton class and its sole member (x and ιx in Peano’s notation). Indeed, it is arguments from Frege (at ix, *KB*, p. 444, for example) which seem to have persuaded Russell to change his mind on this issue. In the body of the *Principles* Russell considers the “extensional” view of classes to be committed to identifying a class with its members, and so not distinguishing a singleton from its sole member. In Appendix A (*PoM*, p. 513), Russell acknowledges Frege’s arguments that this distinction must be made, even when one views classes extensionally.

Russell notes many passages in Frege’s works that have since figured importantly in Frege scholarship. Thus he notices Frege’s argument in *Grundlagen* that the implicit definition of “ u has the number n ” in terms of one-to-one correspondences does not yet settle what numbers are, in particular whether Julius Caesar is a number (at xvi, *Gl*, p. 68). This leads Frege to the explicit definition of numbers as extensions. Russell does not notice what has since come to be called the “Julius Caesar problem”, namely that identification of the extensions of co-extensive concepts in Principle v still does not settle whether Caesar is an extension or not. Solutions to contemporary problems of Frege scholarship will not be found in these notes. Rather, they will be best used as evidence of Russell’s grasp of Frege’s views in 1902.

Indeed, the notes only provide evidence of Russell’s interest in Frege in the period preceding the completion of the appendix in November 1902. Thus while Russell makes extensive notes on Frege’s notions of *Sinn* and *Bedeutung* (at ii–iv and xiii–xiv), he does not notice the issues about the shift of *bedeutung* of expressions in “indirect” contexts and whether objects are constituents of thoughts, which became of interest in his correspondence with Frege in 1903, and to philosophers of language since.⁹

Other issues looming large in contemporary discussions of Frege do not appear at all in the notes. In the “Introduction” to *Grundlagen*

⁹ Gottlob Frege, *Philosophical and Mathematical Correspondence*, ed. G. Gabriel *et al.* and Brian McGuinness (Chicago: U. of Chicago P., 1980), pp. 147–70. Frege raises these issues in letters beginning 20 October 1902.

(p. x), Frege cites three “fundamental principles” which he will follow: “always to separate sharply the psychological from the logical, the subjective from the objective; never to ask for the meaning of a word in isolation, but only in the context of a proposition; never to lose sight of the distinction between concept and object.” Russell only mentions the last of these in his notes on that page (at *xv*, *Gl*, p. x) but pays a good deal of attention to the distinction of concept and object in Frege’s thought. This includes his reading of Kerry’s criticisms of Frege. Kerry argued that Frege must hold that concepts cannot be named by singular terms, hence that Frege was committed to the notorious paradoxical view that “The concept horse” is not a concept (at *viii*, *BuG*, p. 196). Yet Russell appreciates Frege’s view, though he thinks that everything, including concepts, must be possible subjects of propositions, or *terms* in his own system. Frege’s principle “never to ask for the meaning of a word in isolation” is not mentioned in the notes.

Russell’s primary interest as exhibited in the notes is in those aspects of Frege’s logical views that have to do with the foundations of mathematics. The earlier notes (at *xxxii*, *Bs*, p. 76) cite Frege’s definition of the ancestral of a relation, and Russell remarks, “Frege says this defines ‘ x precedes y in series generated by R ’. [The whole proposition amounts merely to aRb].”¹⁰ He seems to suppose that R is a transitive relation, in which case it is identical with its ancestral. However, by the time of writing the final notes (at *xxii*, *Bs*, p. 61), his view has changed. Here we find: “This relation may be expressed ‘ x precedes y in the R -series.’ [It seems to be a non-numerical definition of R^N , and very ingenious: it is better than Peano’s mathematical induction.]” The passage from the earlier to the later notes thus records the moment when Russell came to appreciate Frege’s definition of the ancestral of a relation. This remark suggests, however, that at this point Russell simply sees Frege’s approach as a (superior) alternative to adopting an axiom of induction as

¹⁰ Frege defines the ancestral of a relation R as holding between an individual x and any individual y which has all the properties possessed by x which are hereditary on the R relation. A property F is inherited by the R relation if when some individual x has F then anything bearing R to x also possesses F . The natural numbers are defined as those objects bearing the ancestral of the successor relation to 0, and so it is easy to prove that the principle of induction holds for the natural numbers: any property possessed by 0 and hereditary with respect to the successor relation will be possessed by all numbers.

Peano proposed. Russell does not seem to see deriving the induction principle from logic alone as a primary goal for the logicist programme. It is simply “better”, albeit “very ingenious”.

One might think that Russell could have gotten the idea of the ancestral of a relation and its connection with mathematical induction from reading Dedekind’s *Was sind und was sollen die Zahlen?*¹¹ Dedekind’s work followed Frege’s *Grundlagen* and *Begriffsschrift*, but Russell does seem to have been studying them all together to supplement and annotate his work in the *Principles*, so the remarks on Frege do not prove that Frege was his only source. However, the Russell Archives possess Russell’s copy of Dedekind, dated March 1898, but probably read carefully also only in 1902. Russell made numerous marginal notes in his symbolism, and seven leaves of notes on the proofs.¹² In section 59 Dedekind proves a “Theorem of complete induction”, to the effect that: “In order to show that the chain A_0 is part of any system Σ ... it is sufficient to show that $A \in \Sigma$, and ... that the transform of every common element of A_0 and Σ is likewise an element of Σ .” In section 60 Dedekind says that “The preceding theorem, as will be shown later, forms the basis for the form of demonstration known as complete induction (the inference from n to $n + 1$)...” To this Russell remarks in the margin: “? Does not the definition of a chain involve mathematical induction in many cases?” Russell does not see in Dedekind the reduction of induction to logical principles that he identifies in Frege’s work.¹³

Although almost all of the appendix is devoted to foundational and philosophical questions, almost half of the notes cover the technical details of *Grundgesetze* and *Begriffsschrift*, and so by their sheer number show that Russell also studied Frege’s logical works with care. Indeed the amount of this material alone shows that Russell was a careful and systematic student of all of Frege’s works. Some of the objections to Frege’s

¹¹ Richard Dedekind, *Was sind und was sollen die Zahlen?* (Braunschweig: Biewig und Sohn, 1887; 2nd ed., 1893). Translated by W. W. Beman as “The Nature and Meaning of Numbers” in *Essays on the Theory of Numbers* (LaSalle: Open Court, 1901; reprinted New York: Dover Publications, 1963).

¹² In file RAI 230.030870. The pamphlet is found in a volume, “Pamphlets on Quantity” (Russell’s Library, no. 70).

¹³ I am grateful to Leonard Linsky for suggesting to me the importance of the issue of where Russell learned about induction, and for passing along the view that he should have learned about induction from Dedekind.

views in the appendix may seem confused, but they are based on a thorough grasp of much of his thought.

TEXT OF THE NOTES

The 32 leaves of notes have been numbered (*i*) to (*xxxiv*) in the transcription.

The first leaf in the transcription appears fourth in the file RAI 230.030420–F1. Leaves (*ii*) to (*xvi*) are in file RAI 230.030420–F2, followed there by the notes published in *Collected Papers* 4 as “Frege on the Contradiction”. Except for switching (*xiii*) and (*xix*), the transcription follows the order of the file. Leaves (*xvii*) to (*xxx*) are in RAI 230.030420–F1, where they are found in the following order: *xx–xxii*, *i*, *xix*, *xvii–xviii*, *xxiii–xxix*.

The two sheets containing leaves (*xxxi*) to (*xxxiv*) are in RAI 220.010630, and were found among somewhat earlier notes in the Russell Archives. Leaves (*i*) to (*xxx*) are written on a single side. Leaves (*xxxi*) to (*xxxiv*) are written on both sides (thus *xxxi* and *xxxii*, *xxxi* and *xxxiv* are on two sheets), each side divided in half, right and left, each half written on from top to bottom. These sides are indicated by “*rhs*” and “*lhs*”; e.g., “*xxxii, rhs*” is the right-hand side of the verso of its sheet.

The transcription of the notes below is preceded by a list of “Russell’s Sources”, with abbreviations, based on Russell’s own “List of Abbreviations” at page 500 of the *Principles*. The original German source that Russell cites is followed by a standard English translation. (The article by Kerry has been added. It is the only work not by Frege that is considered in the notes.) Russell begins each group of notes with the abbreviation for the work cited and then gives the page number for each specific note. The list of Sources gives a standard English translation for each and they provide the original German pagination, so that the reader does not need the German text to follow the notes. The exception is the Kerry article, which is, however, cited by Frege in *BuG*.

The transcription follows generally the indentation and other arrangements of notes on the page. Russell’s own foliation is indicated in the upper right-hand corner, where it is indicated in the original. Sometimes leaves were renumbered by him, and this is recorded as it appears, with the new number to the left of the old, e.g. “4 | 2”. Leaf (*iii*) on “Ueber

Sinn und Bedeutung” is clearly taken from before leaf (*xiii*), which continues with two final notes. Otherwise Russell seems to have moved entire leaves without breaking up notes from one source, to add to the final round of notes. Some notes on Kerry, perhaps a sheet, seem to be missing, as the notes begin far into the article (*xix* begins with p. 277 and the article is on pp. 249–307). As well one citation in Appendix A refers to page 272ff. of Kerry. This is the only citation in the appendix not taken from the notes. The passages cited in the appendix are indicated at their first occurrence with a superscripted “A”.

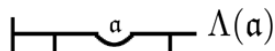
Sometimes Russell records a new page of a source in the middle of a note, suggesting that he is recording the turning of a page while taking notes. This is indicated in the transcription. Deletions are reproduced with a line through them, and in each case they appear to be initial, inaccurate notes that were quickly corrected. German terms are italicized. Russell’s underlining is replaced by italics, except for German terms where the underlining is retained in case emphasis was intended. Editorial comments are indicated with angle brackets, < ... >, as Russell uses square brackets for his own comments. Rare editorial interpolations are recorded in footnotes.

Rather than recreating Frege’s *Begriffsschrift* notation, formulas from Frege are written in a contemporary notation meant to suggest the original. Thus rather than the concavity to express universal quantifiers, $\forall$ is used, and a gothic font indicates what Russell calls the “German” (Fraktur) letters that Frege used for bound variables. Negation is indicated by $\neg$ (with parentheses to indicate its scope) rather than Frege’s negation stroke, and the complex arrangement of strokes indicating a conditional is replaced by $\rightarrow$ with complex consequents placed in brackets. Russell’s own $\hookrightarrow$ is also replaced by $\rightarrow$ in keeping with his own imminent change to the arrow. Special signs in Frege’s theory of one-one relations and number are replaced by new symbols, meant to suggest the original. Thus a raised “-1” indicates the converse of a relation in place of $\mathfrak{L}$. Russell’s own notes directly copy Frege’s notation. These two examples from the notes give the translation used below, followed by a reconstruction of the *Begriffsschrift* original, which Russell faithfully reproduces, only using German script letters where Frege has a Fraktur font.¹⁴

¹⁴ Thanks to Edward Zalta for assistance with the *Begriffsschrift* notation and to the

From leaf (xxii):

$\vdash \neg \forall \alpha \neg \Lambda(\alpha)$



From leaf (xxxi):

$\vdash \neg (B \rightarrow [\neg A])$



An image of Russell's reproduction of Frege's notation will be found at leaf (xxviii). If Russell's German script were retained, the transcription would look thus:

$\vdash \alpha \in [\neg (Ip \rightarrow [\neg \forall \mathfrak{A} (\forall \mathfrak{A} \mathfrak{A} \cap (\mathfrak{A} \cap p) \rightarrow [\neg (\mathfrak{A} \cap \alpha)]) \rightarrow [\neg (\mathfrak{A} \cap \epsilon)])]] = > p$

RUSSELL'S SOURCES

- Bs* *Begriffsschrift, Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle: Louis Nebert, 1879). Translated by S. Bauer-Mengelberg in Jean van Heijenoort, ed., *From Frege to Gödel* (Cambridge, Mass.: Harvard U. P., 1967), pp. 1–82.
- Gl* *Die Grundlagen der Arithmetik, Eine logisch-mathematische Untersuchung über den Begriff der Zahl* (Breslau: Verlag von Wilhelm Koebner, 1884). Translated by J. L. Austin as *The Foundations of Arithmetic: a Logico-mathematical Enquiry into the Concept of Number* (Oxford: Basil Blackwell, 1974).
- FT* “Über formale Theorien der Arithmetik”, *Sitzungsberichte der Jenaischen Gesellschaft für Medizin und Naturwissenschaft*, 19 (1885): suppl. 2, pp. 94–104. Translated by E.-H. W. Kluge as “On Formal Theories of Arithmetic” in Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, ed. Brian McGuinness (Oxford: Basil Blackwell, 1984).
- FuB* *Function und Begriff* (Jena: Hermann Pohle, 1891). Translated by P. Geach as “Function and Concept” in McGuinness, pp. 137–56.
- BuG* “Über Begriff und Gegenstand”, *Vierteljahrsschrift für wissenschaftliche Philosophie*, 16 (1892): 192–205. Translated by P. Geach as “On Concept and Object” in McGuinness, pp. 182–94.
- SuB* “Über Sinn und Bedeutung”, *Zeitschrift für Philosophie und philosophische Kritik*, 100 (1892): 25–50. Translated by Max Black as “On Sense and

Editor for the “German script” and Russell's special symbol $\curvearrowright$, which was originally printed as a dagger and eventually as an arrow. Plate VIII of *Papers* 3: 39 shows the “fancy U” letter, $\mathfrak{U}$, on folio 4, Russell's note on p. 57.

Meaning” in McGuinness, pp. 157–77.

- KB* “Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik”, *Archiv für systematische Philosophie*, 1 (1895): 433–56. Translated by Max Black as “A Critical Elucidation of Some Points in E. Schröder, *Vorlesungen über die Algebra der Logik* [Lectures on the Algebra of Logic]” in McGuinness, pp. 210–28.
- Bp* “Ueber die Begriffsschrift des Herrn Peano und meine eigene”, *Berichte der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig (Mathematische-Physischen Klasse)*, 48 (1896): 362–8. Translated by V. H. Dudman as “On Mr. Peano’s Conceptual Notation and My Own” in McGuinness, pp. 234–8.
- Gg* *Grundgesetze der Arithmetik: Begriffsschriftlich abgeleitet*, Vol. 1 (Jena: Hermann Pohle, 1893; reprinted with the same pagination, Hildesheim: Georg Olms, 1962). The first part, to §52, is translated and edited by Montgomery Furth as *Basic Laws of Arithmetic: Exposition of the System* (Berkeley and Los Angeles: U. of California P., 1967).
- Benno Kerry, “Ueber Anschauungen und ihre psychische Verarbeitung, IV”, *Vierteljahresschrift für wissenschaftliche Philosophie*, 11 (1887): 249–307.
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I. OUTLINE FOR THE APPENDIX¹⁵

(i)

<230.030420-F2>

Appendix on Frege.

- A. Logic. B. Arithmetic
- A. I. Theory of knowledge: separation of logic and Psychology.
- II. *Sinn und Bedeutung*, *Wahrheitswerthe* <truth-values>, etc. Assertion.
- III. Theory of functions, *Begriff* <concept>.
- IV. *Werthverläufe* <courses-of-values>. Are x and ux identical?
- V. The variable: Definitions always for all values of variable.
- VI. Implication, Symbolic Logic.
- B. I. Definition of Nc: Nc according to Frege belongs to *Begriff*, for me to *Werthverlauf*.
- II. Mathematical induction, theory of progressions, etc.

II. NOTES ORGANIZED BY TOPIC

(ii)

<230.030420-F2 fol.> 1

Frege.

Sinn und Bedeutung. <Sense and Reference>

Bs. p. 13^A *Inhaltsgleichheit* <sameness of content> differs from other cases by the fact that it concerns the signs and not what they signify.

p. 15^A $\vdash (A \equiv B)$ means: the sign A and the sign B have the same conceptual content, so that B can be substituted for A everywhere and vice versa. (?)

Gl. p. 69 “The number of Jupiter’s moons³⁴ is four” expresses the identity of objects denoted by two names.

FT. p. 3 2 + 5 and 3 + 4 are not merely equal, but same: opposite view rests on confusing form and content, sign and signified.

BuG. p. 198^A When I wrote my *Gl.*, I had not distinguished *Sinn u. Bedeutung*
SuB. [passim]

¹⁵ In the Appendix (p. 501) Russell describes the “principal heads under which Frege’s doctrines may be discussed” as “(1) meaning and indication <his translation of ‘*Bedeutung*’ for the Appendix>; truth-values and judgment; (3) *Begriff* and *Gegenstand*; (4) classes; (5) implication and symbolic logic; (6) the definition of integers and the principle of abstraction; (7) mathematical induction and the theory of progressions.”

- Gg. p. 11 “ $\Gamma = \Delta$ ” is to mean the true, when Γ is the same as Δ ; otherwise, the false.
- p. 32 I call a *name* whatever is to *bedeuten* something. Latin letters not names.
- p. 43 German and Greek letters also not names.

(iii)

<230.030420–F2 fol.> 2

Frege, *Ueber Sinn u. Bedeutung*; *Zeitschrift f. Phil. u. ph. Krit.* vol. 100

- p. 25^A Difficult questions about Identity: is it a relation? is it between objects, or between names or signs? I assumed the latter in *Begriffsschrift*.
- p. 26 $a = b$ seems to mean “ a and b denote the same object.” — must distinguish from *Bedeutung* the *Sinn*, in which is contained the way of being given.
- p. 27^A Thus evening star and morning star have same *Bedeutung*, different *Sinn*. I understand by sign or name what denotes an object, not a concept.
The *Sinn* is given by the words: the *Bedeutung* would only be fully known if we could say of every *Sinn* whether it belongs to it.
- p. 28^A Perhaps every grammatically correct expression standing for an object has a *Sinn*, but it may have no *Bedeutung*. A word ordinarily stands for its *Bedeutung*; if we wish to speak of its *Sinn*, we must use inverted commas.
- p. 30^A The *Bedeutung* of a proper name is the object which it denotes: the Presentation which goes with it is quite subjective; between lies the *Sinn*, which is not subjective yet not the object.
- p. 31^A A proper name *expresses* its *Sinn*, *denotes* its *Bedeutung* [*bedeutet oder bezeich<n>et*: thus these words are used as synonyms.]
- <The following five notes are marked with a line on the left margin with the remark: “*Wahrheitswerthe* and Assertion”>
- p. 32^A In a proposition, the *Sinn* is a *Gedanke* [the objective content of a thought]. In a dependent clause, or where a proper name such as Ulysses, which has no *Bedeutung*, occurs, a proposition may have no *Bedeutung*. But when a proposition has a truth-value, this is its *Bedeutung*. Thus every assertive proposition is a proper name, whose *Bedeutung* is the true or the false. A judgment is not the mere comprehension of a *Gedanke*, but the recognition of its truth.
- p. 35 A truth-value can never be part of a ~~*Sinn*~~ *Gedanke*, because it is
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- an object, not a *Sinn*. Knowledge requires both a *Gedanke* and its truth-value.
- p. 37 In a dependent clause, the *Bedeutung* is what would usually be *Sinn*.
- p. 40 In "Kepler died in poverty" it is presupposed that the name "Kepler" has a *Bedeutung*, but this is not part of *Sinn* of proposition, as may be seen from its negation.
- (iv) <230.030420-F2 fol.> 3
- Frege.
Wahrheitswerthe and Assertion.
- Bs. pp. 1-2 —*A* denotes the unasserted ~~prop~~ notion "the truth of *A*"; $\vdash A$ denotes "*A* is true".
- p. 4 $\vdash$ is predicate of all my propositions. Negation belongs to content of propositions.
- FuB. p. 13^A A proposition bedeutet the true or the false, which I call truth-values. " $2^2 = 4$ " ~~is a name for~~ *bedeutet* the true, just as 2^2 ~~is a name for~~ *bedeutet* 4.
- p. 14 Sameness of *Bedeutung* does not involve sameness of *Gedanke*.
 p. 21^A —*x* means the True if *x* is true, otherwise the false. —*x* expresses an Annahme: a special extra sign is required for actual judgment. The sign of judgment can't form part of a function, since it doesn't combine with other signs to denote an object. A judgment *bedeutet* nothing, but asserts something.
- BuG p. 198 When I wrote my *Gl.*, I had not distinguished *Sinn* and *Bedeutung*, and therefore called beurtheilbarer Inhalt the two things which I now distinguish as *Gedanke* and *Wahrheitswerthe*.
- Sub. p. 32 In a proposition, the *Sinn* is a *Gedanke* (the objective content of a thought). In a dependent clause, or where a proper name without *Bedeutung* occurs, such as Odysseus, a proposition may have no *Bedeutung*. But when a proposition has a truth-value, this is its *Bedeutung*. Thus every assertive proposition is a proper name, whose *Bedeutung* is the true or the false. A judgment is not the mere understanding of a *Gedanke*, but the recognition of its truth.
- p. 35 A truth-value can never be part of a *Gedanke*, because it is an object, not a *Sinn*. Knowledge requires both a *Gedanke* and its truth-value.
- p. 40 In "Kepler died in poverty", it is presupposed that "Kepler" has *Bedeutung*, but this is not part of *Sinn* of proposition, as may be seen from its negation.
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- (v) $\langle 230.030420\text{-F2 fol.} \rangle_4 \lfloor 2$

(vi) <230.030420-F2 fol.> 5 | I

By suppressing in like manner a proper name in the name of a function of first order with one argument, we get name of function of first order with two arguments. By suppressing a function in like manner, we get name of function of second order.

- Bs. p. 17^A "Every positive integer is sum of 4 squares." But "every positive integer is not a value of x in " x is sum of 4 squares." Meaning of "every positive integer" depends on context.—As long as proposition contains only constituents, distinction of argument and function depends on is only in point of view; but when one is variable, real distinction. Either separately may be varied. By varying second constituent get function of two variables, i.e. relation.
- p. 19 $\phi(A)$ may be regarded as function of argument ϕ . But the values taken by ϕ must all be functions.

(vii) <230.030420—F2 fol.> 6 | 2

Frege, *Begriff* u. *Gegenstand*, Function.

- Gl. p. 63 The word for a ~~concept~~ *Begriff* is not the name of a thing; it needs *the* or a demonstrative pronoun before it to become so.
- p. 64 A word with the indefinite article ~~before it~~ or in the plural without article is a word for a concept.
- p. 65 To assert existence is to deny 0, thus existence is a property of a concept. Hence failure of ontological argument. [Mistake.]
- ~~BuG. p. 200~~ Two meanings of existence (1) *Bedeutung* of proper name (2) $\exists$
- KB. p. 453. [there is a third]
- Gl. p. 65 Uniqueness [*Einzigkeit*] is a mark of concepts, namely of those under which only one object falls, e.g. of *earth's moon*, but not of the moon itself.
- p. 82^A Given a propositional content dealing with a and b , abstracting from these we have left a concept of relation. Referent and relatum both subjects in xRy .
- p. 83 Concept of relation, like simple concept, belongs to pure logic.
- p. 87^A Self-contradictories are also concepts. F is concept if " a falls under concept F " is a proposition whatever *Gegenstand* a may be.
- FuB. p. 5^A An arithmetical function, e.g. $2x^3 + x$, does not denote the result of an arithmetical operation. If it did, it would be merely a number.
- p. 6^A The essence of a function is what is left when x is taken away, i.e. $2()^3 + ()$. The argument does not belong to the function, and the two together make a whole. [How distinguish above from $2x^3 + y$?]
- p. 13^A A function may be a proposition: its value is then a truth-value.
- p. 17^A *Behauptungssätze* <statements> can be divided into two parts, of

which one is complete in itself, while the other is incomplete, as “Caesar ___ conquered Gaul”. *Bedeutung* of latter part I call a function. Must allow any object whatever as argument of a function.

(viii) <230.030420-F2 fol.> 7 | 3

Frege, *Begriff u. Gegenstand*. Function.

FuB. p. 18^A An object is anything not a function, i.e. whose expression leaves no empty place.

BuG. p. 193 My explanation of a concept is not intended as a proper definition.—A concept is predicative, an object never.

p. 195^A That some things can only occur as *Gegenstände*, others not, is an important distinction, even if, as Kerry thinks, concepts can occur also as objects.—We can have a concept falling under a higher one [$u \in k$]: in such cases, not concept itself, but its name, is in question. [Here the concept occurs as term]. “The concept horse” is not a concept. This use has to be indicated by inverted commas or italics.

p. 198^A A concept is the *Bedeutung* of a predicate; an object is what can never be the whole *Bedeutung* of a predicate, but can be that of a subject.

p. 201^A A concept is essentially predicative even when something is asserted of it. An assertion which can be made of a concept does not fit an object. In $x \in u$ and $u \in k$ the two $\in$ ’s have not the same meaning.

p. 205^A Of the parts of a *Gedanke* not all can be complete: one at least must be *ungesättigt* or predicative, otherwise they wouldn’t cohere.

Gg. p. 36ⁿ. A function which has always the same value is distinct from that value.

p. 42 Functions of second order can be replaced by functions of first order by substituting *Werthverläufe* for the functions which were arguments.

(ix) <230.030420-F2 fol.> 8

Frege. *Werthverläufe*.

FuB. The extension of a concept is the *Werthverlauf* of a function whose value for

p. 16^A every argument is a truth-value.

- p. 19^A *Werthverläufe* are objects, whereas functions are not.
 KB pp. 436–7^A No null class if classes taken in extension. This requires concepts, not regions <gebiete>.
- p. 439 Schröder, by not distinguishing ϵ and $\supset$, thinks ιx contains two terms, x and Λ . Hence $\Lambda = x$. Schröder infers Universe must contain no terms u such that $\exists i \wedge u$.
- p. 444^A Against $x\iota' \iota x : x \in \text{Cls} - 1 - 0 . y, z \in x . y \circ' z . \supset . y, z \in \iota x . \supset . \iota x \sim \epsilon 1$.
- p. 445 But $x\iota' \iota x$ follows if we take classes in extension.
- p. 451^A The extension of a concept has its being in concept itself, not in individuals.
- p. 454^A When I say something of all men, I say nothing about some person in the centre of Africa, who is in no way *bedeutet* and does not belong to *Bedeutung* of *man*.
- p. 455^A I hold concepts prior to their extensions, and I regard it as a mistake to try to base extension on individuals. This leads to Calculus of regions, not to logic.

(x) <230.030420–F2 fol.> 9 | 1

Frege on *Werthverläufe*.

Grundgesetze der Arithmetik.

- p. 7^A $\phi(\xi), \psi(\xi)$ have same *Werthverlauf* if they have same value for all values of ξ .
- p. 8 If $\phi(\xi)$ is a propositional function, we call it a *Begriff*, and speak of its *Werthverlauf* as *Umfang des Begriffes*.
- p. 14^A $\phi x . \equiv_x . \psi x : \equiv . x \ni \phi x = x \ni \psi x$ Pp. [I put $x \ni \phi x$ for *Werthverlauf* of ϕx .] An *Anzahl* is the *Umfang* of a *Begriff*. [My theory]
- p. 16^A §10 gives an account of *Werthverläufe*. We have as yet only determined *equality* of *Werthverläufe*: not what they are in themselves. If $X(\xi)$ be a function which never has the same value for different values of the variable, we shall have

$$X(x \ni \phi x) . \equiv . X(x \ni \psi x) : \equiv . x \ni \phi x = x \ni \psi x . \equiv : \phi x . \equiv_x . \psi x$$
- Thus $X(x \ni \phi x)$ fulfils the conditions hitherto laid down just as well as $x \ni \phi x$ does. This ambiguity is removed by deciding, with every function ϕx , what values it is to have when x is a *Werthverlauf*. Observe — $\xi . \equiv : \xi = (\xi = \xi)$: both hold when and only when ξ is a true proposition.
- p. 17^A Thus $\xi = \zeta$ is the fundamental form of function. We have to

decide whether a truth-value can be a *Werthverlauf*. We have still enough liberty to decided arbitrarily that a certain *Werthverlauf* is to be the true, and a certain other the false. We decide $\dot{\epsilon}(\neg\epsilon)$ [i.e. $p \supset (p)$] is to be the true, and $\dot{\epsilon}\{\epsilon = (\neg\forall\alpha \alpha = \alpha)\}$ [i.e. $p \supset \{p \cdot \equiv \cdot \exists x \supset (x \sim = x)\}$] is the false.

p. 18^A

Hence $\neg\dot{\epsilon}\phi(\epsilon)$ is the true, when and only when the true falls under the concept $\phi(\epsilon)$; in all other cases it is false. Also $\dot{\epsilon}\{\epsilon = (\neg\forall\alpha \alpha = \alpha)\}$ is the *Werthverlauf* of $\xi = (\neg\forall\alpha \alpha = \alpha)$, which is the true when the argument is false, and is the false in all other cases.

(xi)

<230.030420-F2 fol.> 10 | 2

Frege on *Werthverläufe*.

p. 18^A

[note.] It is tempting to regard every object as a *Werthverlauf*, i.e. as extension of a concept under which it alone falls. Such a concept, for Δ , is $\Delta = \xi$. Let us attempt $\dot{\epsilon}(\Delta = \epsilon) = \Delta$. This will do as long as Δ is not given as a *Werthverlauf*; but the way of being given is irrelevant logically. If $\Delta = \dot{\alpha}\phi(\alpha)$, we obtain $\dot{\epsilon}\{(\dot{\alpha}\phi(\alpha) = \epsilon)\} = \dot{\alpha}\phi(\alpha)$, which is equivalent to $\dot{\alpha}\phi(\alpha) = x \cdot \equiv_x \cdot \phi(x)$. This will hold if $\phi(\xi)$ is satisfied by only a single term, namely $\dot{\alpha}\phi(\alpha)$. Hence $\dot{\epsilon}(\Delta = \epsilon) = \Delta$ will not do. [The proposition in question is $\phi(x) \cdot \equiv_x \cdot \phi\{x \supset \phi(x)\}$ $x = y \supset \phi y$: i.e. ϕ is satisfied by no value except the class of its own values: this requires the class of its own values to be a class of only one term. Thus the proposition in question might be used to define units, if we identify these with their only terms.] [If we distinguish x and ιx the above proposition can only be true if, for some value of x , $x = \iota x$: i.e.

$\phi \supset \{\phi x \cdot \equiv_x \cdot x = y \supset \phi y\} = \phi \supset \{x \supset \phi x \in \mathbf{I} \cdot x \supset \phi x = \iota x \supset \phi x\}$
I do not know whether any ϕ satisfies such a proposition or not.]
The function we want is $\Omega(\xi, \zeta)$ where $\Delta \equiv_{\Delta} \xi \supset \{\Omega(\xi, \Delta)\}$.
Such a function is $\xi \cap \zeta$ (defined later); but this is defined by means of *Werthverläufe*.

p. 19

The function $\backslash \xi$ is defined as follows: (1) If $\exists \Delta \supset \{\xi = \dot{\epsilon}(\Delta = \epsilon)\}$, $\backslash \xi$ is to be Δ . (2) If $\sim \exists \Delta \supset \{\xi = \dot{\epsilon}(\Delta = \epsilon)\}$, $\backslash \xi$ is to be ξ . Thus if $\phi x \in \mathbf{I}$, we have $\backslash \dot{\epsilon}\phi(\epsilon) = \iota x \supset (\phi x)$. But if $\phi x \sim \in \mathbf{I}$, $\backslash \dot{\epsilon}\phi(\epsilon) = x \supset (\phi x)$. [This leaves it quite undecided whether $x = \iota x$ or not.]

p. 48

There are 8 ~~kinds of~~ functions involved: (A) Functions of first order with one argument: (1) $\neg \xi$ (2) $\neg \xi$ (3) $\backslash \xi$ (B) Ditto with two arguments: (4) $\zeta \supset \zeta \cdot \supset \cdot \zeta \supset \xi$ (5) $\xi = \zeta$ (C) Functions of

second order with arguments of second kind: (6) $\forall \alpha \phi(\alpha)$ (7) $x \ni \phi x$ (D) Functions of third order (8) $\forall f \mu_\beta f(\beta)$. Also (unused) (9) $\forall f \mu_{\beta\gamma} f(\beta, \gamma)$

(xii) <230.030420—F2 fol.> II 3

Frege on *Werthverläufe*.

p. 49^A Has $\dot{\epsilon} \phi(\epsilon)$ always a *Bedeutung*? We will confine ourselves to the case where $\phi \epsilon$ has a *Bedeutung*, is of the first order, and has one variable. We will call $\dot{\epsilon} \phi(\epsilon)$ in this case a *proper* range. [Range = *Werthverlauf* Df] Let us examine whether $\neg x \ni \phi x$ and $\neg x \ni \phi x$ has a meaning in such cases; also $\zeta \supset x \ni \phi x$, $x \ni \phi x \supset \zeta$, $\xi = x \ni \phi x$, $x \ni \phi x = \zeta$ ought to be names of proper functions when ϕ is given. In virtue of previous definitions, $\Gamma = \Delta$ always has a meaning when Γ and Δ are proper ranges or truth-values. Hence $\xi = (\xi = \xi)$ has a meaning when ξ is replaced by a proper range; so therefore has $\neg \xi$. [Put $\xi = x \ni \phi x$. I don't see how Frege helps us to decide whether this is true or not when $\exists \phi x \cdot \exists \sim \phi x$. Thus $\xi = (\xi = \xi)$ seems still indeterminate.]

p. 53^A $a \cap u = \text{< "}\exists \text{" deleted } \setminus x \ni [\exists \phi \ni \{u = y \ni (\phi y)\}. \phi a = x\}]$ Df [I have practically same definition in my notes]
[If the class defined does not consist of a simple term, $a \cap u$ is the class of propositions any one of which is obtained by putting a as argument in a propositional function whose range is u . I imagine Frege regards all these propositions as identical: in this case, $a \cap u$ is the proposition expressing the fact proposition that " a satisfies any propositional function whose range is u ". This is exactly equivalent to $a \in u$. It will be false when u is not a range, or when u is a range to which a does not belong. But it will always be a proposition, whatever a and u may be. But if u is not a range, the class before which $\setminus \text{"}\xi \text{" deleted}$ is placed is null, and therefore $a \cap u$ is the null-class. Thus $a \cap u$ in this case is not a proposition at all.] When u is not a range, $a \cap u$ is the range $\dot{\epsilon} (\neg \epsilon = \epsilon)$ [Bg p. 17, this is the false.] [Thus $a \cap u$ is always a proposition or a truth-value.]

p. 55^A $\Gamma \cap \Delta \cap \Theta$ corresponds to my xRy . If Θ is a single range, $\Gamma \cap \Delta \cap \Theta$ is the same as $\dot{\epsilon} (\neg \epsilon = \epsilon)$; if Θ is not a range at all, $\Gamma \cap \Delta \cap \Theta$ is the false. It can only be true when Θ is a double range [*Doppelwerthverlauf*]

III. NOTES ORGANIZED BY SOURCE

(xiii)

<230.030420—F2 fol.> 2

Frege, *Sinn und Bedeutung*.

p. 41 In ordinary language, can't always be sure whether an expression has *Bedeutung*.

p. 43^A In hypothetical judgment, don't have relation of two judgments, but of two *Gedanken*.

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Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik. Archiv f. syst. Phil. I, 4.

pp. 436–7 No null-class if classes taken in extension. This requires concepts, not regions.

p. 438 In "All *A*'s are *B*", *A* and *B* are not extensions of concepts.

p. 439 Contradiction quoted from Schröder: ιx contains x and nothing besides, i.e. x and Λ . Hence $\Lambda = x$. Schröder infers Universe must contain no terms which are classes having terms of universe as terms, i.e. $\sim \exists i \wedge u \ni (\exists i \wedge u)$.

p. 440 This requires distinction of ϵ and $\mathbf{C}$.

p. 444 Following argument against regarding every individual as a class: Consider ιu where $u \in \text{Cls}$. Then $\iota u' \iota u \cdot \supset :: x, y \in u \cdot \supset$.

$$x, y \in \iota u \cdot \supset \cdot x \iota' y$$

p. 445 But $\iota u' \iota u$ is a necessary consequence of the notion that classes are composed of individuals, which belongs to the Calculus of regions.

p. 446 If we identify ϵ and $\mathbf{C}$, a class of one term must be $\iota \Lambda$.

p. 449 If Λ an empty sign, it signifies nothing, and fails to be a sign. We can't by definition assign properties to an egg-shaped figure.

p. 451 The extension (*Umfang*) of a concept does not have its being in the individuals, but in the concept itself.

p. 453 Two meanings of existence: (1) *Bedeutung* of a proper name (2) $\exists$. [There is a third also]

p. 454 When I say something about all men, I say nothing about some creature in the centre of Africa, who is in no way *bedeutet*, and does not belong in any way to *Bedeutung* of *man*.

p. 455 I hold concepts prior to their extension and I regard it as a mistake to try to base extensions on individuals. This leads to Calculus of regions, not to logic. [*ib.* repeated]

p. 456 Can't create an object with arbitrary properties by definition.—Distinguish questions whether a proper name has *Bedeutung* and

whether $\exists u$. Proper names without *Bedeutung* are to be excluded, not so *A*.

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Ueber die Begriffsschrift des Herrn Peano und meine eigene: Berichte der mathematischen Classe der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, 6 July 1896.¹⁶

(xiv)

<230.030420—F2>

Frege, *Ueber Begriff und Gegenstand*; *Vierteljs. f. wiss. Phil.* XVI, 2.

p. 193 My explanation of concept is not intended as a proper definition.
— A concept is predicative, an object never.

p. 194 *Note*. Schröder is to be blamed for confounding ϵ and $\supset$.

p. 195 That some things can only occur as objects, others not, is an important distinction, even if, as Kerry thinks, concepts can occur also as objects.—We can have a concept falling under a higher one ($u \in k$): in such cases not the concept itself, but its name, is in question. “The concept *horse*” is not a concept.

p. 197 This use has to be indicated by inverted commas or italics.

p. 198 When I wrote my *Gl.*, I had not distinguished *Sinn* and *Bedeutung*, and therefore called *beurtheilbarer Inhalt* the two things which I distinguish now as *Gedanke* and *Wahrheitswerth*.—A concept is the ~~meaning~~ *Bedeutung* of a predicate; an object is what can never be the whole *Bedeutung* of a predicate, but can be that of a subject. Such words as *all every no some* stand before concept-words. “The concept *F*” *bezeichnet* <signifies> not a concept, but an object.

p. 199 In $\exists \sqrt{4}$, nothing is said about 2 or -2 , but only about concept $\sqrt{4}$. But if I say, “the concept $\sqrt{4}$ exists”, I make an assertion about an object. This is not same proposition as $\exists \sqrt{4}$.

p. 200 $\exists$ Julius Caesar is not false, but meaningless. But “Julius Caesar exists [*ist erfüllt*]” is false, for this can be said of objects but is

p. 201 only true of objects of a peculiar kind, i.e. such as “the concept *F*”.—A concept is essentially predicative even when something is asserted of it. An assertion which can be made of a concept does not fit an object. In $x \in u$ and $u \in k$, the two ϵ ’s have different

¹⁶ This citation is at the bottom of a page of notes. No other page in the notes makes reference to it, nor does Russell cite it in the Appendix beyond listing it in the “List of Abbreviations”. A sheet of notes may yet have been made.

- p. 202 though similar meanings.—I call *Eigenschaft* of an object any concept under which it falls.
But if $\phi x \supset \psi x$, I call ψ *Merkmal* of ϕ .
- p. 205 Of the parts of a *Gedanke* not all can be complete: one at least must be *ungesättigt* or predicative, otherwise they wouldn't cohere.

(xv)

<230.030420—F2 fol.> 1

Frege, *Grundlagen der Arithmetik*. Breslau, 1884 [Gl.]

- p. v Divorce of mathematics and philosophy due to psychologism.
- p. ix Sharp separation of logic and mathematics impossible.
- p. x Distinction of concept and object is to be remembered.
- p. 4 A proposition is *analytic* if its proof requires only general logical laws, and definitions.
- p. 29^A An object can have different numbers truly assigned to it as a rule, and is therefore not that of which number is asserted.
[Wants discussion]
- p. 33 *Two* is not the same as *one pair*.
- p. 34 Number is as little ~~the~~ an object of Psychology or a result of psychological processes as the North Sea. Number is something objective.
- p. 37 If 2 were a Presentation, it would be primarily mine: another's Presentation as such is other.
- p. 38 Number is neither spatial nor subjective, but unsensuous and objective.
- p. 40^A *One* is not a property of things, like *wise*: to begin with, it would have to be a property of every thing.
- p. 49 A number is not a collection of things; for if it were, it would depend upon which things there were.
- p. 50 The + of Arithmetic is not the *and* of *A* and *B*, or we should then have 1 and 1 and 1 is 1.
- p. 57 Numbers not got by abstraction: this wouldn't apply, especially, to 0 and 1.
- p. 59^A Numbers are asserted of *concepts*. This especially obvious in the case of 0.
- p. 60 $a \subset b$ deals with *concepts*, not things; for no particular thing is mentioned in it.
- p. 63 The word for a concept is not the name of a thing: it needs *the* or a demonstrative pronoun before it to become so.
- p. 64 A word with the indefinite article or in the plural without article

- is a word for a concept.
- p. 65 To assert existence is to deny 0; thus existence is a property of a concept. Hence failure of ontological argument. [Mistake.]—Uniqueness is a mark of concepts, namely of those under which only one object falls, e.g. of *earth's moon* but not of the moon itself.
- (xvi) <230.030420—F2 fol.> 2
- Frege, *Gl.*
- p. 67^A Every single number is an independent object. It would seem natural to say: A concept u has the number 0 when $x \sim \epsilon u$ is true for all values of x ; and similarly for 1 and for $n + 1$. But
- p. 68 This gives us no means of deciding whether Julius Caesar is a number, or if, of what. We have explained " u has the number 0" but not 0 itself. Numbers objects, not properties.
- p. 69 "The number of Jupiter's moons is 4" expresses identity of objects denoted by two names.
- p. 72 4 is an object, though it has no position in space: it is the same for all who think of it.
- p. 73 Must explain $Nc'u = Nc'v$ without employing $Nc'u$.
- p. 79^A Where symmetrical transitive relations are concerned, required object is class of relata for given referent. [My principle of abstraction.] Hence $Nc'u = v \ni (Nc'v = Nc'u)$ [i.e. $v \ni (v \text{ sim } u)$]
- p. 80^A [note] I believe we could substitute concept for its extension. Might be objected (1) that a number would then not be an object (2) that different concepts may have same extension. I think both could be answered.
- p. 82 Given a propositional content dealing with a and b , abstracting from these we have left a concept of relation. The referent and relatum are both subjects in xRy .
- p. 83^A Concept of relation, like simple concept, belongs to pure logic.
- p. 85^A $Nc'u = Nc'v . = . \exists 1 \rightarrow 1 \wedge R \ni (u \subset \rho . \check{\rho}u = v)$ Df. Thus concept of number is explained.
- p. 87 Self-contradictories are also concepts. F is a concept if " a falls under the concept F " is a proposition whatever object a may be. The existence of 0 follows from the fact that "not identical with Self" is a null-concept.
- p. 90 Existence of 1 follows from "identical with 0".
- p. 91 These propositions don't depend upon the existence of thinking beings, for the truth of a proposition is not its being thought.

- p. 92 Definition of R^N .
p. 93 This makes order logical, and allows argument from u to $u + 1$ to be reduced to logic.
p. 99 Arithmetical propositions analytical and à priori: Arithmetic only developed logic.
p. 107 A mathematician is no more creative than a geographer: he can only discover and name what is there.

(xvii)

<230.030420-FI>

Frege, *Ueber formale Theorien der Arithmetik: Sitzungsberichte der Jenaischen Gesellschaft für Medicin u. Naturwissenschaft*. Jahrg. 1885 [July 17.]

- p. 1 Two sorts of formal theories, one I agree with, the other not. One says Arithmetic can and therefore ought to be deduced from definitions by Logic.
p. 2 Hence no sharp separation of logic and Arithmetic; and hence logic not as unfruitful as was thought. No methods of proof peculiar to Arithmetic.
p. 3 Everything arithmetical must be reduced by definitions to logical terms.
p. 4 The other sort of formal theory says that the symbols for numbers are empty signs. [Good critique follows.]
p. 10 The only way I know of being sure that a set of attributes are eventually compatible is to find an object which has them.
1/2 is not a concept, but an object.
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Frege, *Function u. Begriff. ib.* 1891 [Separately published Jena, 1891]

- p. 3 $2 + 5$ and $3 + 4$ are not merely equal, but the same: opposite view rests on confusing form and content, sign and signified.
p. 5 A function, e.g. $2x^3 + x$, does not denote the result of an arithmetical operation. If it did, it would be merely a number.
p. 6 And $2x^3 + x$, *per se*, is merely a number, and is nothing different from x . The essence of a function is what is left when x is taken away, i.e. $2(\quad)^3 + (\quad)$. The argument does not belong to the function, and the two together make a whole. [How distinguish above from $2x^3 + y^2$?]
p. 9 $x^2 - 4x = x(x - 4)$ does not express equality of two functions, but of their values. We can express such an equation as equality of *Werthverläufe*.
p. 10 This is not demonstrable, but is a fundamental law of logic.
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Frege, *Function u. Begriff*.

- p. 11 $x^2 - 4x = x(x - 4)$ expresses a *general* truth (*Allgemeinheit*): this is why we can't put $y(y - 4)$ on the right, though this would have same meaning.
- p. 13 A function may be a proposition: its value is then the true or the false. Such I call truth-values. $2^2 = 4$ denotes (*bedeutet*) the true, just as <indecipherable deleted word> 2^2 denotes 4.
- p. 14 Sameness of *Bedeutung* does not involve sameness of *Gedanke*.
- p. 16 The extension of a concept is the *Werthverlauf* of a function whose value for every argument is a truth-value.
- p. 17 Propositions which make an assertion [*Behauptungssätze*] can be divided into two parts, of which one is complete in itself, while the other is incomplete. So "Caesar ... conquered Gaul" *Bedeutung* of latter part I call *function*.—Must thus also allow any object whatever as argument of a function.
- p. 18 An object is anything not a function, i.e. whose expression leaves no empty place.
- p. 19 *Werthverläufe* of functions are objects whereas functions are not.—To avoid senseless expressions, must define our functions for *all* values of arguments, so that e.g. "Sun + 1" shall not be meaningless. Otherwise $x \ni (x + 1 = 10)$
- p. 20 e.g. is not a precise entity.
- p. 21 — x is to mean truth if x is true, otherwise falsehood. This expresses an
- p. 22 *Annahme*: a special extra sign is required for actual judgment. The sign of judgment [*Urtheilsstrich*] can't form part of a function, since it doesn't combine with other signs to denote an object. A judgment denotes [*bezeichnet*] nothing, but asserts something.
- p. 26 $\neg \forall \alpha \neg f(\alpha)$ expresses existence: it is a function of f
- p. 27 This I call a function of second order.
- p. 27 $3 > 2$ may be divided into 3 and $x > 2$, latter into 2 and $x > y$. Hence function of two arguments.
- p. 28^A Such functions are called relations. Functions of two arguments may be of different orders with respect to the two, e.g. $f'(x)$,
- p. 29 where f and x are arguments.

Kerry, *Ueber Anschauungen u. ihre psychische Verarbeitung*, IV. Viertelj. f. wiss. Phil. XI (1887), pp. 249–307.

p. 277^A Definition of $Nc'u$ as $v \ni (v \text{ sim } u)$ is a *ὑστερον πρότερον* <*hysteron proteron*>. Must know every concept has only *one* extension: must know what *one* object is. F 's extension is a mere symbol for what is commonly meant by *Number*.

p. 280 Notion of extension of concept only to be made complete through number.

p. 281 Frege errs in identifying concept with his extension. [He doesn't: *Gl.* p. 80, note, has been misunderstood by Kerry.]

p. 287^A Similarity of classes seems to presuppose that they have terms.

p. 288^A "Every object which falls under F stands in relation ϕ to an object which falls under G " means for Frege " a falls under F and stands in relation ϕ to no term of G " are not both true, whatever a may be. This strikes Kerry as absurd. He doesn't understand non-existential import of universal propositions. He says: this contradicts "If a falls under G , while nothing falls under F , then ' a falls under G and no object falling under F stands in relation ϕ to a ' are both true for all values of ϕ ". (He doesn't understand variable.) In this way, Kerry thinks, any two objects could be shown to have the same number.

<The following note was inserted in the margin:>

p. 290n^A Any proposition in which a and b occur is for F a relation between a and b : hence can't deny that a and b are related without affirming it. Such a notion of relation is so general as to have no sense or purpose.

p. 291^A Frege's introduction of 1 from 0 also objectionable. Only the *concepts* of 0 and 1, not the objects, have been defined. There might be several 1's.

p. 292^A Frege's definition of $+1$ is also wrong. Depends on his theory of series.

p. 293 He defines " F is inherited in f -series" but not " F is inherited" nor " f -series".

p. 295 He defines " y follows x in f -series" as " y has all the properties inherited in f -series" [Kerry omits: "and belonging to x ".] This criterion is of doubtful value, since no catalogue of such properties exists: but further, following x is itself one of these properties: hence a circle. [Misunderstands deduction completely.]

(xx)

<230.030420—FI fol.> 1

Frege, *Begriffsschrift*, Jena Halle, 1879.

pp. 1–2 — A denotes the unasserted notion “the truth of A ”; $\vdash A$ denotes “ A is true”.

p. 2 I don’t distinguish subject and predicate in a proposition.

p. 4^A $\vdash$ is the predicate of all my propositions.

Negation for me belongs to content of proposition.

p. 5^A $B \rightarrow A$ means A is true or B false. [In the *Begriffsschrift*, A and B have to be “*beurteilbare Inhalte*”, i.e. propositional concepts: see p. 2, where $\vdash$ is said to be only possible before a propositional concept.]

p. 9^A I employ no mode of conclusion [*Schlussweise*] except to conclude q from p and $p \supset q$. [He has other principles of deduction, but this is his only informal principle.]

p. 10 Negation. [Introduced as primitive idea.]

p. 13 *Inhaltsgleichheit* differs from other cases by the fact that it concerns the signs and not what they signify.

p. 15 $\vdash (A \equiv B)$ means: the sign A and the sign B have the same conceptual content, so that B can be substituted everywhere for A and vice versa. [?]

§9. *Functions*.

p. 16 “If in an expression, whose content need not be propositional [*beurteilbar*], a simple or composite sign occurs in one or more places, and we regard it as replaceable, in one or more of these places, by ~~other than~~ something else, but by the same everywhere, then we call the part of the expression which remains invariable in this process ~~the~~ a *function*, and the replaceable part we call its *argument*.”

(xxi)

<230.030420—FI fol.> 2

Frege, *Begriffsschrift*.

p. 17^A “20 is sum of 4 squares” and “every positive integer is ditto”, but “every positive integer” is not a value of x in “ x is sum of 4 squares”. “Every positive integer” is a phrase whose meaning depends on context.

As long as proposition contains only constituents, distinction of argument and function in point of view; but when one is variable, real distinction. Either separately may be varied.

p. 18 By varying second constituent of a ~~prop~~ an expression we get a

- function of two variables. So $\psi(A, B)$ expresses a relation of A and B .
- p. 19 In the expression of any proposition, everything right of $\vdash$ is function of any of the signs appearing in it. $\forall\alpha \phi(\alpha)$ expresses ϕx always true: only restrictions that must remain propositional concept, and that if variable sign of function, must remember this.
- p. 21 $\phi(A)$ may be regarded as function of the argument ϕ .
A Latin letter has whole proposition for its field. May replace a Latin letter always by a German one not already occurring in proposition: thus $\vdash X(a)$ can be replaced by $\vdash \forall\alpha X(\alpha)$ and $A \rightarrow \phi(a)$ implies $\vdash A \rightarrow \forall\alpha \phi(\alpha)$ provided a does not occur in A and only occurs as argument in $\phi(a)$. [Note. Whenever $\vdash$ comes in part of a sentence, what is meant to be said is not said: for the part cannot be asserted. Or can, perhaps, an asserted proposition be part of another proposition?] [Latin letters express *any*, German letters *every*.]
- p. 23 $\vdash \neg\forall\alpha \neg A(\alpha)$ expresses existence-theorem.
- p. 25 Laws underlying our symbolism can't be expressed symbolically.

(xxii)

<230.030420—FI fol.> 3

Frege, *Begriffsschrift*

- p. 26 Kernel of following are 9 propositions, Numbers 1, 2, 8, 28, 31, 41, 52, 54, 58. [These are: 1. $pq \supset p$ 2. $cb \supset a . c \supset b . \supset . c \supset a$ 8. $db \supset a . \supset . bd \supset a$ 28. $b \supset a . \supset . \sim a \supset \sim b$ 31, 41 (see preface) $\vdash (\neg\neg a \equiv a)$ i.e. $\sim(\sim a) \equiv a$ 52. $c \equiv d . \supset . f(c) \supset f(d)$ 54. $c \equiv c$ 58. $\vdash \forall\alpha f(a) \rightarrow f(c)$.
These are Frege's Pp's <Primitive Propositions>. The last is substitution, but Peano's symbols won't express it. It says: What holds of every holds of any.] [*Observe*. Such expressions as "whatever holds of" demand variable functions.]
[Note. By always using implications, Frege wholly avoids the logical product. Thus he has nothing of the nature of Import and Export]
- p. 55^A *Theory of Series*. Frege begins with considering

$$\left. \begin{array}{l} F(x) . f(x, y) . \supset_{x, y} . F(y) \\ \text{or } \check{p}u \subset u \text{ in my notation.} \end{array} \right\} \text{ represented by } \left\{ \begin{array}{l} \delta \mid F(\alpha) \\ \alpha \mid f(\delta, \alpha) \end{array} \right.$$
- p. 58^A When $\check{p}u \subset u$, I say the property u is inherited in the R -series.
- p. 59 $\check{p}u \subset u . x \in u . xRy . \supset . y \in u$ [Is first proposition primitive?]
- p. 60^A $x \in u . \check{p}u \subset u . xRy . \supset . y \in u$ second

- $\overset{\gamma}{\beta}f(x_\gamma, y_\beta)$ is defined as $\check{\rho}u \subset u \cdot \check{\rho}x \cdot \subset u \cdot \supset_u \cdot y \in u$ [I shall write instead xR^Ny]
- p. 61 This relation may be expressed “ x precedes y in the R -series.” [It seems to be a non-numerical definition of R^N , and very ingenious: it is better than Peano’s mathematical induction.]
- p. 63 $x \in u \cdot \check{\rho}u \subset u \cdot xR^Ny \cdot \supset \cdot y \in u$
- p. 66 $xR^Ny \cdot \check{\rho}x \subset u \cdot \check{\rho}u \subset u \cdot \supset \cdot \check{\rho}y \subset u$
- p. 68 $R \supset R^N$
- p. 71 $R^NR \supset R^N \check{\rho} (\check{\rho}^N x) \subset \check{\rho}^N \rho x (R^N)^2 \supset R^N R' = R^N \cup I'$ Df.
- p. 72 $R'R \supset R^N$
- p. 73 $R' \cap \sim R^N \supset I' \cdot R^N \supset R'$
- p. 74 $R'R \supset R'$
- p. 74 $\check{\rho}(\check{\rho}'x) \subset \check{\rho}'x$
- p. 75 $R'R \supset R' \cup \check{R}^N$
- p. 76 $R' \supset \check{R}' \cup R^N$
- p. 79 $R \in Nc \rightarrow I \cdot \supset \cdot \check{R} R \supset R'$
- p. 80 $R \in Nc \rightarrow I \cdot \supset \cdot \check{R} R^N \supset R'$
- p. 81 $R \in Nc \rightarrow I \cdot \supset \cdot \check{R} R^N \supset R^N \cup \check{R}'$
- p. 83 $R \in Nc \rightarrow I \cdot y(R^N \cup \check{R}') m \cdot y R_x \cdot \supset \cdot x(R^N \cup \check{R}') m$
- p. 86 $R \in Nc \rightarrow I \cdot \supset \cdot \check{R}^N R^N \supset R' \cup \check{R}^N$

(xxiii)

<230.030420—FI fol.> I

Frege, *Grundgesetze der Arithmetik*, Vol. 1, Jena, 1893. [Gg]

- p. ix The assertion of a number is assertion concerning a *Begriff*.
- p. x There are three elements in a judgment (1) recognition of truth (2) *Gedanke* (3) *Wahrheitswerth*.
- p. xv Logical laws are not laws of thought, but of how people ought to think, since they are true.
- p. xvii Truth is something objective.
- p. xviii There is also an objective domain of the not-actual, e.g. the number 1. Presentations differ from man to man.
- p. xxv Existence is stated of concepts, and thus a concept of second order.
- p. 1 To show that Arithmetic is branch of Logic, must settle on certain modes of conclusion beforehand, and take no steps except in accordance with them.
- p. 2 Classes not composed of individuals: if they were, no null-class.
- p. 3 x and ux distinct.—Concept and relation are my foundation-stones.

- p. 5 Essence of function in connection of x and $\phi(x)$.
 p. 6 Argument doesn't belong to function.
 p. 7 A true proposition is a name of the true, a false of the false: these are bedeutet.
 p. 8 Functions of two arguments are to be called relations.
 p. 9 Assertion requires special symbol.
 p. 10 Assertion distinct and additional to truth-value.
 p. 11 " $\Gamma = \Delta$ " is to mean the true, when Γ is same as Δ , otherwise the false.
 p. 20 $\zeta \rightarrow \xi$ is a function of two arguments whose value is the false when ζ is the true and ξ isn't; in all other cases, the value is to be the true. [Then follow logical Pp's <primitive propositions>]
 p. 31 A Latin letter has a field embracing everything in the proposition except *Urtheilstrich*: hence it can't be used to deny generality, but can for generality of denial. A Latin letter doesn't bedeuten an object, but andeuten. Say same of German, when not over a hollow.
 p. 32 I call a *name* whatever is to *bedeuten* something. Latin letters not names.
 p. 33 [It seems to me $\vdash \phi(x)$ asserts *any* proposition $\phi(x)$, $\forall \alpha \phi(\alpha)$ asserts *all* such propositions.]
 p. 35 $\vdash \forall \alpha \phi(\alpha) \rightarrow \phi(a)$ means "what holds of all holds of any".
 p. 36n. Functions which always have same value are distinct from that value. (*F und B*. p. 8)
 p. 37 Functions of two arguments are as distinct from those of one as these from objects. Again $\exists \phi$ can only have functions as arguments: such I call functions of second order [*Stufe*]

(xxiv)

<230.030420—FI fol.> 2

Frege, *Gg*.

- p. 39 Differential coefficient is function of two variables, one a function, one an object. *Note*. We must assume addition multiplication etc. defined even when arguments are not numbers.
 p. 42 Functions of second order can be replaced by those of first order, by substituting *Werthverläufe* for the functions which were arguments [e.g. $\exists u$ in place of $\exists \phi$]
 p. 43 German, Latin and Greek letters are not names, because they *bedeuten* nothing. But " $\forall \alpha \alpha = \alpha$ " is a proper name for the true.
 "If from a proper name we exclude a proper name, which is part or the whole of the first, in some or all of the places where it
-

occurs, but in such a way that these places remain recognizable as to be filled by one and the same arbitrary proper name (as argument-positions of the first kind), I call what we thereby obtain the name of a function of the first order with one argument. Such a name, together with a proper name which fills the argument-places, forms a proper name."

By suppressing a proper name in like manner in the name of a function of first order with one argument, we get name of a function of first order with two arguments. By suppressing a function in like manner, we get name of function of second order.

p. 44 The *Urtheilstrich* belongs neither to names nor to marks: it is *sui generis*. A definition.

p. 45 asserts that the new sign is to have same *Sinn* (?) and *Bedeutung* as old ones. For definitions, principle is: Rightly formed names must always *bedeuten* something.

(xxv)

<230.030420—FI fol.> 3

Frege, *Gg*.

Pp's I. $ab \supset a, a \supset a$

IIa. $\vdash \forall \alpha f(\alpha) \rightarrow f(a)$

[What holds of all, holds of any]

IIb. $\vdash \forall f M_\beta \{f(\beta)\} \rightarrow M_\beta \{f(\beta)\}$

[This seems to mean: If, whatever function f may be, the assertion M_β holds of $f(\beta)$, then the same assertion holds of $f(\beta)$, where f is a particular function.]

III. $\vdash g(a = b) \rightarrow [g(\forall f f(b) \rightarrow f(a))]$

I.e. Any function of $a = b$ implies the corresponding function of $\phi(b) \supset_\phi \phi(a)$. [I should use this as definition of identity.]

IV. $\vdash \neg (\neg a) = (\neg b) \rightarrow (\neg a) = (\neg b)$

I.e. the truth-value of $\neg a$ is the same as that of $\neg b$ or else its opposite.

V. $(\dot{\epsilon} f(\epsilon) = \dot{\alpha} g(\alpha)) = (\forall \alpha f(\alpha) = g(\alpha))$

I.e. $x \ni \phi x = y \ni \psi y \equiv :$

$\phi x \equiv_x \psi x$

VI. $\vdash a = \backslash \dot{\epsilon} (a = \epsilon)$

I.e. $a = \backslash x \ni (x = a)$

The above are called *Grundgesetze*.

Next section is *Zusammenstellung der Regeln*. Then follows Arithmetic proper.

(xxvi) <230.030420—FI fol.> 3a

Frege, *Gg*. 1.

p. 19 $\backslash \xi$ [i.e. $\gamma \xi$]: If $\xi = \dot{\epsilon}(\Delta = \epsilon)$, $\gamma \xi = \Delta$; if not, $\gamma \xi = \xi$.

p. 53 $a \cap u$ [i.e. $a \in u$] = $\gamma \dot{\alpha} [\exists g \ni \{\alpha = g(a). u = \dot{\epsilon} g(\epsilon)\}]$ Df.

[With regard to Greek, German and Latin letters, the principle seems to be this: *Greek* letters are used when they are no real part of what is said, but are inserted only to fill the argument-place in a function: i.e. when something is to be said about the function itself, as in $\dot{\epsilon} \phi(\epsilon)$; *German* letters are used where something is asserted for all values of the variable; and *Latin* letters where a function of these letters is asserted.] The above definition, in the case where u is not a class, makes $a \cap u$ the null-class.

p. 55 $\vdash \Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha) = \dot{\epsilon}(\epsilon + \Delta)$ and $\vdash \Gamma \cap \{\Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha)\} = \Gamma \cap \dot{\epsilon}(\epsilon + \Delta) = \Gamma + \Delta$ [These follow from previous definitions] [See p. 3b]

$\forall e \forall d \ e \cap (d \cap \Delta) \rightarrow [\forall a \ (e \cap (a \cap \Delta) \rightarrow d = a)]$ is the truth value of $\neg \xi \cap (\zeta \cap \Delta)$ being $\text{Nc} \rightarrow \text{I}$, i.e. such that given ξ there is at most one ζ .

[Thus it expresses $eRd, eRa \cdot \supset_{e, d, a} \cdot d \text{ I } a$: in $\xi \cap (\zeta \cap \Delta)$, the ξ is the referent and the ζ relatum.]

$\text{Ip} \cdot = \cdot P \in \text{Nc} \rightarrow \text{I}$ Df

(xxvii) <230.030420—FI fol.> 3b

$\Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha) = \gamma \dot{\beta} [\exists g \ni \{\beta = g(\Delta) \cdot \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha) = \dot{\alpha} g(\alpha)\}]$

Thus $g(\alpha) = \dot{\epsilon}(\epsilon + \alpha)$, $g(\Delta) = \dot{\epsilon}(\epsilon + \Delta)$. Thus

$\Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha) = \dot{\epsilon}(\epsilon + \Delta)$

For $\Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha)$ means $g(\Delta)$, provided the range of g is $\dot{\alpha} \dot{\epsilon}(\epsilon + \alpha)$, i.e. provided $g(\Delta)$ is $\dot{\epsilon}(\epsilon + \Delta)$. We have

$\dot{\epsilon}(\epsilon + \Delta)$ is the range of values of $x + \Delta$ for varying x :

$\dot{\alpha} \dot{\epsilon}(\epsilon + \alpha) \dots \dot{\epsilon}(\epsilon + \Delta) \dots \Delta$.

$\Gamma \cap \dot{\epsilon}(\epsilon + \Delta)$ means $g(\Gamma)$, provided the range of g is $\dot{\epsilon}(\epsilon + \Delta)$, i.e. provided $g(\Gamma)$ is $\Gamma + \Delta$: thus $\Gamma \cap \dot{\epsilon}(\epsilon + \Delta)$ means $\Gamma + \Delta$, and so does $\Gamma \cap \{\Delta \cap \dot{\alpha} \dot{\epsilon}(\epsilon + \alpha)\}$.

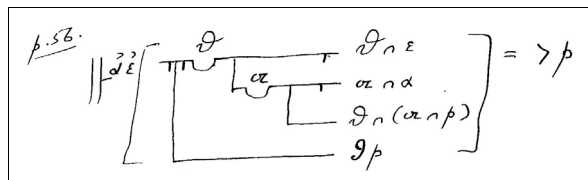
Thus $\Gamma \cap \{\Delta \cap \dot{\alpha} \dot{\epsilon} f(\epsilon, \alpha)\} = f(\Gamma, \Delta)$.

If $f(x, y)$ is a propositional function, $\dot{\epsilon} f(x, \epsilon)$ is the class of values of $f(x, y)$ for different values of y ; i.e. $y \ni \{f(x, y)\}$

I don't understand the meaning of $\dot{\alpha} \dot{\epsilon} f(\epsilon, \alpha)$ where f is a propositional function. It is not $(x \rightarrow y) \ni f(x, y)$, as one might suppose.

(xxviii) <230.030420—FI fol.> 4

Frege.



p. 56 $\vdash \dot{\alpha} \dot{\epsilon} [\neg (Ip \rightarrow [\neg \forall d (\forall \alpha d \cap (\alpha \cap p) \rightarrow [\neg (\alpha \cap \alpha)]) \rightarrow [\neg (d \cap \epsilon)])]] = > p$

i.e. $(u, v) \ni u > Pv . =$

$:: \{P \in Nc \rightarrow I : xPy . \supset_y . y \sim \epsilon v : \supset_x . x \sim \epsilon u\}$ Df

i.e. $u > Pv . = :: P \in Nc \rightarrow I : x \in u . \supset_x . \exists v \cap \check{\pi} x ::$ Df

i.e. $u > Pv . = :: P \in Nc \rightarrow I : u < \supset \text{deleted} \supset \pi . \check{\pi} u \subset v$ Df

[Thus if $P \sim \epsilon Nc \rightarrow I$, $>P = \Lambda$. If $\text{not } P \in Nc \rightarrow I$, $>P$ holds between any class u in π and any class v containing u .]

[Note. $\dot{\alpha} \dot{\epsilon} \phi(\alpha, \epsilon)$ is to be interpreted: $(x; y) \ni \phi(y, x)$, or better, $\gamma Rel \wedge R \ni \{xRy . \equiv . \phi(y, x)\}$. Thus]

p. 57^A $\vdash \dot{\alpha} \dot{\epsilon} (\alpha \cap (\epsilon \cap p)) = \neg p$

§39 i.e. $\gamma Rel \wedge R \ni \{xRy . \equiv . yPx\} = \check{P}$ Df

§40 $Nc' u = v \ni \{\exists P \ni (P \in Nc \rightarrow I . v \subset \pi . \check{\pi} v \subset u . P \in I \rightarrow Nc . u \subset \check{\pi} . \pi u \subset v)\}$ Df

i.e. $= v \ni \{\exists I \rightarrow I \wedge P \ni (v \subset \pi . \check{\pi} = u)\}$ Df

i.e. $= v \ni (v \text{ sim } u)$ which is my definition.

The first proposition proved by Frege (Sec A, §53 ff) is $u \text{ sim } v \supset Nc' u = Nc' v$.

This seems only to require $u \text{ sim } v . \supset . v \text{ sim } u$, i.e. $P \in I \rightarrow I . \supset . \epsilon P I \rightarrow I$. But Frege takes 16 pages to prove it.

(xxix) <230.030420—FI fol.> 4a

p. 56 $\vdash \dot{\alpha} \dot{\epsilon} [\neg (Ip \rightarrow [\neg \forall d (\forall \alpha d \cap (\alpha \cap p) \rightarrow [\neg (\alpha \cap \alpha)]) \rightarrow [\neg (d \cap \epsilon)])]] = > p$

[i.e. $>P = (x, y) \ni \{\sim (P \in \text{Nc} \rightarrow \text{I} : d Pa . \supset_a . a \sim \epsilon y : \sim \supset_d . d \sim \epsilon x)\}$
 $= (x, y) \ni \{\sim \exists d \ni (P \in \text{Nc} \rightarrow \text{I} : d Pa . \supset_a . a \sim \epsilon y : d \epsilon x)\}$ ¹⁷
 $= (x, y) \ni \{P \in \text{Nc} \rightarrow \text{I} : d Pa . d \epsilon x . \supset_{d,a} . a \epsilon y\}$
 $= (x, y) \ni \{P \in \text{Nc} \rightarrow \text{I} . d \epsilon x . \supset_d . \tilde{\pi} d \subset y\}$
 $= (x, y) \ni \{P \in \text{Nc} \rightarrow \text{I} . \tilde{\pi} x \subset y\}$
i.e. $>P$ is the relation of a class u contained in π to any class which contains $\tilde{\pi} u$.]

(xxx)

<230.030420-FI>

p. 56
$$\left. \begin{array}{l} u > Rv . = . R \in \text{Nc} \rightarrow \text{I} . u \subset \rho . \check{\rho} u \subset v \\ v > \check{R} u . \equiv . R \in \text{I} \rightarrow \text{Nc} . v \subset \check{\rho} . \rho v \subset u \quad \text{Df} \\ u > Rv . v > \check{R} u . \equiv . R \in \text{I} \rightarrow \text{I} . v = \check{\rho} u . \rho u = \rho v \end{array} \right\}$$

Note: $m \cap^{-1} \dot{\cup} q \cap (u \cap > p)$ appears to mean:
 $R \in \text{Nc} \rightarrow \text{I} . \check{\sigma}' m \subset \rho . \check{\rho} (\check{\sigma}' m) \subset u$
replacing p, q by R, S respectively: i.e. it means:
 $\check{\sigma}' m > Ru$

IV. EARLY NOTES ON “BEGRIFFSSCHRIFT” AND “GRUNDGESETZE”

<The remaining leaves are in RA1 220.010630, notes found with a half sheet on Boole, De Morgan and Venn, within a folded cover sheet marked “Frege etc.”>

(xxxi) (recto, rhs)

<220.010630 p.> I

Frege, *Begriffsschrift*, Halle, 1879.

- §1 Letters stand for variables.
§2 $\vdash A$ stands for A asserted; $\text{—}A$ for A unasserted.
§3 I make no distinction of subject and predicate. What is common to two equivalent propositions I call their *conceptual content* (*Begrifflichen Inhalt*); this alone relevant to *Begriffsschrift*. I make the whole content subject and $\vdash$ the predicate.
§4 Usual classifications of judgments belong to content, not to judgment itself.

¹⁷ The final “ x ” has been editorially supplied.

- (*xxxi*) (recto, *lhs*) <220.010630 p.> 2

§I0 $\vdash \psi(A, B)$ means “ B has ψ -relation to A ”.

§I1 $\vdash \forall \alpha \phi(\alpha)$, where α is a German letter, is to mean: “ $\phi(\alpha)$ holds for all values of α ”. If α is thus variable throughout the whole asserted proposition, replace it by a Latin letter, thus:
 $\vdash X(a)$ means $\vdash \forall \alpha X(\alpha)$ [Use α for German α]

§I2 $\vdash \neg \forall \alpha X(\alpha)$ means $\exists x \ni \{\sim X(x)\}$
 $\vdash \forall \alpha \neg X(\alpha)$ means $\sim \exists x \ni \{X(x)\}$
 $\vdash \neg \forall \alpha \neg X(\alpha)$ means $\exists x \ni \{X(x)\}$
 $\vdash \forall \alpha (X(\alpha) \rightarrow P(\alpha))$ means $X(\alpha) \supset_{\alpha} P(\alpha)$

§I3 Nine Pp’s <Primitive Propositions> in what follows.

§I4 (1) $a \supset b \supset a$ Pp.
(2) $c \supset b \supset a : c \supset b \supset c \supset a$ Pp
(3) $b \supset a \supset bc \supset a \supset b \supset c \supset a$ [Proved]

¹⁸ The second set of parentheses has been editorially supplied.

- (4) $b \supset a . \supset : c \supset . b \supset a :: \supset : ab \supset : c \supset b . \supset . c \supset a$
 [Proved]
 (5) $b \supset a . c \supset b . \supset . c \supset a$ [Professes to be proved]
 (6) $c \supset . b \supset a : \supset : c \supset : d \supset b . \supset . d \supset a$ [proved]
 (7) $b \supset a . \supset : d . \supset . c \supset b : \supset : d \supset . c \supset a$ [proved]
 §16 (8) $d \supset . b \supset a : \supset : b \supset . d \supset a$ Pp.
 (9) $c \supset b . \supset : b \supset a . \supset . c \supset a$ [5.8. $\supset$. Prop]
 (10) $ed \supset b . \supset a : \supset : de \supset b . \supset . a$
 (11) $c . \supset . b : \supset a :: \supset . b \supset a$
 (12) $dcb \supset a . dbc \supset a$
 (13) $dcb \supset a . \supset bdc \supset a$ (14) $edcb \supset a . \supset . ebdc \supset a$

(xxxii) (verso, lhs)

<220.010630 p.> 3

Frege, *Begriffsschrift*.

- (15) $edcb \supset a . \supset . bedc \supset a$ (16) $edcb \supset a . \supset . edbc \supset a$
 (17) $dcb \supset a . \supset . cbd \supset a$
 (18) $cb \supset a . \supset : d \supset c . \supset . bd \supset a$
 (19) $dc \supset b . \supset : b \supset a . \supset . dc \supset a$
 (20) $edc \supset b . b \supset a . \supset . edc \supset a$
 (21) $d \supset b . \supset . a : \supset : d \supset c . \supset : c \supset b . \supset a$
 (22) $fedcb \supset a . \supset . fedbc \supset a$
 [Note. All these proofs are vitiated by not proving $a, b, c \in \text{prop}$.
 $\supset . a, c, b \in \text{prop}$. They are not independent of this, for it is
 required in $a . \supset . b \supset c : \supset : b . \supset . a \supset c$. If a be a
 proposition, but not b , the first implication holds if a is false,
 while the second holds under no circumstances.]
 (23) $dcb \supset a . e \supset d . \supset . cbe \supset a$
 (24) $c \supset a . \supset . cb \supset a$ (25) $dc \supset a . \supset . dcb \supset a$ (26) $ba \supset a$
 (27) $a \supset a$
 §17 (28) $b \supset a . \supset . \sim a \supset \sim b$ Pp
 (29) $cb \supset a . \supset . c \sim a \supset \sim b$ (30) $bc \supset a . \supset . c \sim a \supset \sim b$
 §18 (31) $\sim(\sim a) \supset a$ Pp
 (32) $\sim b \supset a . \supset . \sim a \supset \sim(\sim b) : \supset : \sim b \supset a . \supset .$
 $\sim a \supset b$
 (33) $\sim b \supset a : \supset . \sim a \supset b$ [28,32] i.e. $a \cup b . \supset . b \cup a$ [This
 is not yet justified: we have $\sim(\sim a) \supset a$, not $\equiv a$]
 (34) $c \sim b \supset a . \supset . c \sim a \supset b$ (35) $c \sim b \supset a . \supset . \sim a \cap c$
 $\supset b$
 (36) $a \sim a \supset b$ (37) $c \sim \supset b . \supset . a : \supset . c \supset a$

- (38) $\sim a \cap a \supset b$ (39) $\sim a \supset a . \supset . \sim a \supset b$ (40) $\sim b . \sim a$
 $\supset a . \supset . a$
- §19 (41) $a \supset \sim(\sim a)$ Pp
 (42) $\sim\{\sim(a \supset a)\}$ (43) $\sim a \supset a . \supset a$ (44) $\sim a \supset c . c \supset a .$
 $\supset . a$
 (45) $\sim c \supset a . \supset . \sim a \supset c : \sim c \supset a . c \supset a : \supset . a$ (46) $\sim c$
 $\supset a . c \supset a . \supset . a$
 (47) $\sim c \supset b . b \supset a . c \supset a . \supset . a$ (48) $d \sim c \supset b . b \supset a .$
 $c \supset a . \supset . d \supset a$
 (49) $\sim c \supset b . c \supset a . b \supset a . \supset . a$ (50) $c \supset a . b \supset a . \sim c$
 $\supset b . \supset . a$
 (51) $dc \supset a . \supset . b \supset a . \supset . d . \supset \sim c \supset b . \supset . a$

(xxxii) (verso, rhs) <220.010630 p.> 4

Frege, *Begriffsschrift*.

§20 (52) $c \equiv d . \supset . f(c) \supset f(d)$ Pp. Note. c may be contained not only as variable in $f(c)$; hence c may still occur in $f(d)$.

§21 (54) $c \equiv c$ Pp. (55) $c \equiv d . \supset . d \equiv c$ [52.54]

§22 (58) $\forall \alpha f(\alpha) . \supset . f(c)$ Pp [Substitution]
 [This is a Pp, not deducible from dropping of a true hypothesis.]

(59) $g(b) . \sim f(b) . \supset . \sim(gx . \supset_x . fx)$ [58.30]

(61) $f(c) . \supset . a : \supset : \forall \alpha f(\alpha) . \supset . a$

(62) $g(x) : g(y) . \supset_y . f(y) : \supset . f(x)$ [Barbara for individual]

(64) $hy . \supset . gx : gz . \supset_z . fz : \supset : hy . \supset . fx$

(65) $hz . \supset_z . gz : gz . \supset_z . fz : \supset : hx . \supset . fx$ [Barbara for class]

§23 Theory of series: purely logical, not intuitional.

§24 (69) $\vdash \{ \{ \forall \delta F(\delta) \rightarrow [\forall \alpha f(\delta, \alpha) \rightarrow F(\alpha)] \} \equiv \delta \left(\begin{array}{l} F(\alpha) \\ \alpha \left(\begin{array}{l} f(\delta, \alpha) \end{array} \right) \end{array} \right) \}$

This is not a proposition, but a definition, indicated by $\vdash$. [In my notation, it gives an abbreviation for $\phi x . xRy . \supset_{x,y} . \phi(y)$] [Explanation of Greek letters very obscure: they seem to be any, not some.]

§26 (76) $\vdash \{ \{ \forall F \delta \left(\begin{array}{l} F(\alpha) \rightarrow [(\forall \alpha f(x, \alpha) \rightarrow F(\alpha)) \rightarrow [F(y)]] \end{array} \right) \} \equiv \alpha \left(\begin{array}{l} f(\delta, \alpha) \end{array} \right) \} \equiv \tilde{\beta}^{\gamma} f(x_{\gamma}, y_{\beta})$

[This is an abbreviation for $\phi x . xRy . \supset_{x,y} . \phi y : aRy . \supset_y . \phi y : \supset_y . \phi b$.

Frege says this defines “ x precedes y in series generated by R ”.]

[The whole proposition amounts merely to aRb]

- (91) $f(x, y) \cdot \supset \cdot \frac{\gamma}{\beta} f(x_\gamma, y_\beta)$ [Why not say so to begin with?]
- (95) $f(x, z) \cdot \supset \cdot \frac{\gamma}{\beta} f(x_\gamma, y_\beta) \cdot \supset \cdot \frac{\gamma}{\beta} f(x_\gamma, z_\beta)$
- (97) $\frac{\delta}{\alpha} \left(\frac{\gamma}{\beta} f(x_\gamma, \alpha_\beta) \right)$
 $\vdash \mid$
 $f(\delta, \alpha)$
- (98) $\frac{\gamma}{\beta} f(x_\gamma, y_\beta) \cdot \frac{\gamma}{\beta} f(y_\gamma, z_\beta) \cdot \supset \cdot \frac{\gamma}{\beta} f(x_\gamma, z_\beta)$
- §31 $\frac{\delta}{\epsilon} f(\delta, \epsilon)$ means f is $Nc \rightarrow I$ (115).
- (xxxiii) (recto, *rhs*) <220.010630 p.> 1

Frege, *Grundgesetze d. Arithmetik*, Jena, 1893.

- p. ix Numbers are asserted of concepts, not of classes.
 $2 + 2 = 4$ expresses identity of *Bedeutung* with difference of *Sinn*:
 cf. *Zeitschrift f. Phil. u. Phil. Krit.* Vol. 100 p. 25.
- p. 6 Propositional functions. $\xi^2 = 4$ is not, for me, an assertion.
- p. 7 [I shall translate *Sinn* and *Bedeutung* by *meaning* and *denoted object*, *bedeuten* by *denote*.] I shall say $2^2 = 4$, $2 > 1$ are names denoting *the true*, i.e. both denoting the same truth-value. So $3^2 = 4$, $1 > 2$ denote *the false*. But these have not same *meaning*. The *sense* meaning of the name of a truth-value I call a *Gedanken* [propositional concept]. The function $\xi^2 = 4$ can have only 2 values: true and false. [What is *meaning* of this function?] Any *object* can be argument of a function: an object is anything except a function. " ϕx and ψx have same range of values" $\cdot = : \phi x \cdot \equiv_x \cdot \psi x$ Df
- p. 8 Where the value of a function is always a truth-value, we can speak of "extension of concept" in place of "range of values of function"; in fact, this may be used as definition of concept. So Δ comes under concept $\phi(\xi)$ if $\phi(\Delta)$ is true. If $\psi(\xi, \zeta)$ always has a truth-value, it expresses a relation.
- p. 9 A judgment is recognition of truth of a *Gedanke*. If Δ is any
- p. 10 object whatever, $\neg \Delta$ denotes the truth when Δ is true, the false in all other cases. $\Delta = \neg \Delta$ is the truth-value of " Δ is a truth-value". $\neg \phi x$ and $\neg \psi(x, y)$ are respectively concept and relation whether ϕx and $\psi(x, y)$ are so or not. $\neg \xi$ is to be false when and only when $\neg \xi$ is true. Thus $\vdash \neg 2$.
- p. 12 $\forall x \phi x$ means ϕx always true, $\forall x \neg \phi x$ means ϕx always false, $\neg \forall x \phi x$ means ϕx not always true. $\vdash \neg \forall x \neg \phi x$ expresses $\exists \phi x$.

(xxxiii) (recto, lhs)

<220.010630 p.> 2

Frege, *Grundgesetze d. Arith.*

- p. 14 If $\phi x \cdot \equiv_x \cdot \psi x$, the two functions have same range of values.
This is a logical law [Pp] [My $\phi x \cdot \equiv_x \cdot \psi x : \supset \cdot x \ni \phi x \text{ I } x \ni \psi x$] ~~ϵ denotes $\epsilon \phi(\epsilon)$~~ denotes $x \ni \phi x$. [No! Frege uses $\epsilon \phi(\epsilon)$ in cases where $\phi(\epsilon)$ is not a propositional function: it means then merely the values of $\phi(\epsilon)$ for different values of ϵ . This being so, $\epsilon \phi(\epsilon) = \alpha \psi(\alpha)$ does not imply, though it is implied by, $\phi x \cdot \equiv_x \cdot \psi x$. This seems a mistake in Frege.]
- p. 19 Definition of *the*: takes us from concepts to proper names.
- p. 20 $\zeta \rightarrow \xi$ is to be false if ζ is the true and ξ is not; in *all* other cases, it is to be true. [$p \supset \neg q \zeta \supset \xi$, with the understanding that this holds when neither are propositions, when ζ is not a proposition, when ζ is false, or when ξ is true. This is more general than my idea of implication. It corresponds to $\zeta \in \text{Prop} \cdot \supset \cdot \zeta \supset \xi$. This makes it seem more complex than my relation. It is $\zeta \supset \zeta \cdot \supset \cdot \zeta \supset \xi$.]
- p. 21 $\vdash \neg \zeta \rightarrow \xi \cdot \equiv \cdot \zeta \sim \xi \quad \vdash \neg(\zeta \rightarrow \neg \xi) \cdot \equiv \cdot \zeta \xi \quad \vdash \neg \neg(\neg \zeta \rightarrow \xi) \cdot \equiv \cdot \xi \cup \xi$
- p. 25 Association [Pp]
- p. 26 *Association* is alone sufficient for conclusions, but here for convenience I use others. [Observe. Frege doesn't mean it is the only principle of deduction: he means my non-formal Association, which for me too is the only principle of *therefore*.]
- p. 26 $\Delta \supset \Gamma \cdot \Theta \supset \Delta \cdot \supset \cdot \Theta \supset \Gamma$ [presumably Pp]
- p. 27 $p \supset q \cdot \supset \cdot \sim q \supset \sim p$
- p. 28 $p \sim q \cap r \supset s \cdot \supset \cdot \sim(p \sim q \supset \sim r) \cdot \supset \cdot s$
- p. 29 $p \cdot \supset \cdot p \supset q : \supset \cdot p \supset q$
- p. 34 I. $pq \supset p$; hence $p \supset p \cdot p, q \in \text{Prop} \cdot \supset : \sim(p \equiv \sim q) \cdot \supset \cdot p \equiv q$ IV. $a = \text{I} x \ni (a = x) \text{ VI}$.
- p. 35 $\vdash (\forall \alpha f(\alpha)) \rightarrow f(a)$ IIa: What holds of all, holds of each.

(xxxiv) (verso, lhs)

<220.010630 p.> 3

Frege, *Grundgesetze d. Arithmetik*

- p. 36 $\vdash g(a = b) \rightarrow g(\forall f(f(b) \rightarrow f(a)))$ (III)
($\epsilon f(\epsilon) = \alpha g(\alpha) = (\forall \alpha f(\alpha) = g(\alpha))$ (V))
- p. 37 A function wholly different from an object: can never be argument to a function. Functions of two variables are as fundamentally distinct from those of one as these from objects.—

- $\neg\forall\alpha\neg\phi(\alpha)$ may be taken as a function of ϕ : for every value of ϕ , its value is a truth-value. When the argument is a function, I call the function itself of the second order (*Stufe*).
- p. 38 Another function of second order is $\phi a . \supset_a : \phi e . \supset_e . a = e$. This function of ϕ has truth for its value whenever ϕ is of the first order and defines an Elen.<Element?> Again $\phi(2)$ is of the second order (ϕ being variable). The value of this function is not always truth or falsehood. $\neg\phi(2)$ is distinguished from $\phi(2)$ by being always truth or falsehood. We may call it the property of the number 2; for the concepts under which 2 falls are those which fall under this concept.
- $\epsilon\phi(\epsilon)$ is a function of second order, but not a *Begriff*.
- p. 39 $f'(x)$ is function of f and x : of second order respect to f , first respect to x . $\neg\forall\alpha\forall e\neg\phi(\alpha, e)$ is concept of second order: equivalent to $\exists R$.
- p. 42 $\vdash (\forall\phi M_\beta(\phi\beta))\rightarrow M_\beta(f\beta)$ Iib i.e. what holds of all functions of the first order with one variable holds of each.
- p. 43 Ordinary letters don't denote (*bedeuten*) objects, and are therefore not names; but $\forall\alpha\alpha = \alpha$ is a name, for the true.
- p. 61 Collection of Pp's.
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