THE NOT-SO-STRANGE MODAL LOGIC OF INDETERMINACY

by
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P.F. Gibbins closes his article ("The Strange Modal Logic of Indeterminacy" *Logique et Analyse* #100:443-446) with

> But indeterminacy generates a strange modal logic. The semantical business of there being classes of indeterminate worlds accessible to no worlds not even to themselves is strange and not intuitively attractive.

I wish to suggest that the logic of indeterminacy is not so strange as that! While I agree Gibbins' final conclusion

> ...that the modal logic of indeterminacy, construed as an extension of maximally determinate classical logic, affords a poor model for the deep idea of vagueness *de re*

my reasons have rather to do with the idea that vagueness *de re* – that is, vagueness inhering in an object – is not plausibly construed by *any* operator on sentences. To say that an *object* is vague is to say at least that some predicate neither applies nor doesn't apply to it; and this seems to call for some construal of sentences like *Fa* in a manner opposed to treating it first as meaningful and then prefixing an indeterminacy operator to it. But this is not the point of the present note. Rather, I content myself with showing that Gibbins' argument about the "strangeness" of the semantics of indeterminacy is ill-founded. (1)

(1) I also ignore his direct reconstruction of Evans' argument, preferring not to comment on his use of "the expected Indeterminacy thesis $A \rightarrow \triangle A$", which thesis seems to me to be completely implausible as a truth about indeterminacy. (Evans, G. "Can There Be Vague Objects?" *Analysis* 38:208).

Following Gibbins, (2) we use ∇ as a sentential operator meaning "it is indeterminate whether" and introduce Δ as its dual "it is determinate that". ∇ and Δ are genuine duals, that is

[Def ∇] $\nabla A \leftrightarrow \neg\Delta\neg A$

It is also plausible to suppose, along with Gibbins

[RE] if $\vdash(A \leftrightarrow B)$ then $\vdash(\Delta A \leftrightarrow \Delta B)$
[RN] if $\vdash A$ then $\vdash \Delta A$

That is, if the equivalence of A and B is provable, so is the equivalence of whether they are definite; and if a formula is provable, then it is definite. Some further theorems not mentioned by Gibbons, but seemingly plausible for "determinateness" are (3)

[C] $\vdash(\Delta A \& \Delta B) \rightarrow \Delta(A \& B)$
[I] $\vdash(\Delta A \leftrightarrow \neg\nabla A)$
[I′] $\vdash(\Delta A \leftrightarrow \Delta\neg A)$
[I″] $\vdash(\nabla A \leftrightarrow \nabla\neg A)$

(Theorems [I], [I′] and [I″] are equivalent in the presence of [Def ∇]). Since this logic has [Def ∇] and [RE] it is *classical* in the sense of Segerberg, (4) and therefore can be given an analysis by "possible worlds". Gibbins mentions some principles that fail in this logic, such as

[T] $\Delta A \rightarrow A$
[P] $A \rightarrow \nabla A$

Other principles, not mentioned by Gibbins, that fail in this logic are

[D] $\Delta A \rightarrow \nabla A$
[M] $\Delta(A \& B) \rightarrow (\Delta A \& \Delta B)$

(Principle [D] obviously fails in the intended understanding of Δ and ∇. [M] fails because, for example, $(p \& \neg p)$ is definite (definitely

(2) Who follows Evans in this. A similar use of Δ and ∇ as "determinate" and "contingent" operators can be found in a series of papers from the late 1960's in *Logique et Analyse* by R. Routley and G. Montgomery.

(3) I use the names of the rules of inference and axioms found in B. Chellas *Modal Logic*, Cambridge U.P. 1980. The "I" is new and stands for "indeterminacy".

(4) K. SEGERBERG *An Essay in Classical Modal Logic* (1971), Uppsala.

false, that is, but definite nonetheless) while neither p nor $\neg p$ are definite. Strangely, Gibbins thinks that

[K] $\Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$

should be a theorem of the logic, thereby making the logic be *normal*. But it obviously should not be a theorem: let $A = (p \& \neg p)$ and $B = q$; $\Delta((p \& \neg p) \rightarrow q)$ is true and $\Delta(p \& \neg p)$ is true, but Δq needn't be. (5)

Since [K] is not in the logic, it is not normal and hence there is no normal, relational possible world semantics for the logic. But there can nonetheless be a possible world semantics, done by the "Montague-Scott" (or "neighborhood" or "minimal model") method. We first consider the logic as axiomatized by the propositional logic, [C], [I] with the rules of inference [RE] and [RN]. (In Chellas' notation it would be the logic ECNI.) Arguably there are more principles that should be valid in a logic of indeterminacy. We shall shortly consider them. For now we concentrate on just these few.

A model is a triple $M = \langle W,N,P \rangle$ such that W is a set of indices ("worlds"), P is mapping from natural numbers to subsets of W (i.e., $P(n) \subseteq W$ for each natural number n - telling us for each atomic proposition P(n) which subset of W it is true in), and N is a mapping from W to sets of subsets of W (i.e., $N\alpha \subseteq \mathscr{P}(W)$ for every world $\alpha \in W$ – that is, what propositions (subsets of worlds) are necessary at α). Define ΔA to be true at an index α in M iff the set of indices at which A is true, $|A|$, is a member of $N\alpha$, and ∇A to be true at α iff $(W-|A|) \notin N\alpha$. It is well known that propositional logic, [RE] and [Def ∇] are valid in any class of such models. It remains only to find that subclass determined by [RN], [C] and [I]. It is again well known that [RN] holds when M *contains the unit*, i.e.,

(n) $W \in N\alpha$

for every $\alpha \in M$ (anything true at all worlds is an element of the necessitation of any world); and that [C] holds if M is *closed under intersections*, i.e.,

(c) if $X \in N\alpha$ and $Y \in N\alpha$ then $(X \cap Y) \in N\alpha$

(5) Besides, if [K] were in the system then [M] would be also, which we have already seen to be wrong.

for every $\alpha \in M$ and all sets of indices X and Y. I dub the property which validates [I] as *contrariety* (if something is necessary so is its opposite)

(i) $X \in N\alpha$ iff $[(W\text{-}X) \in N\alpha]$

Standard methods (cf. Chellas *op cit* ch. 7) would clearly suffice to show that ECNI is determined by the class of contrary models that are closed under intersections and contain the unit. It is also obvious that principles [K], [M], [T] and [P] are not universally valid in this class of models.

In the scheme of modal logics we find ECNI located

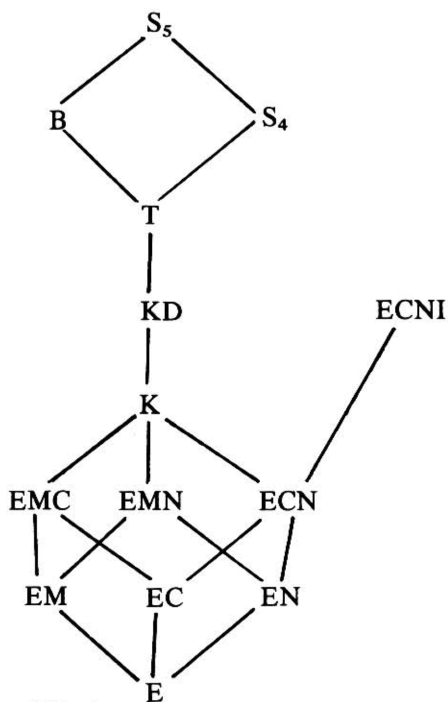

FIG. 1: A map of some modal logics

Far from being "at least as strong as S_5" (as Evans said), it is seen to be independent of it. And far from being "trivial" (as Gibbins says), there are no thesis of the form Δ A unless A is a propositional theorem or derived by repeated applications of rules [RE] and [RN] from propositional theorems. For example, if p is an atomic proposition, then Δp is not a theorem. ECNI is not so strange - it does just about what one would expect of a logic of indeterminacy.

What other theses might one suggest for a logic of indeterminacy than [RE], [RN], [Def ∇], [C] and [I]? We've seen that principle [M] does not hold, but a closely related one does seem to be valid, namely

[M*] $\vdash (\Delta(A \& B) \& (A \& B)) \rightarrow (\Delta A \& \Delta B)$

which says that if a conjunction is not only definite but also true then each conjunct has to be definite. I say "seems to be valid" when one understands 'Δ' as "definitely", but I would not insist on it. The (semantic) idea behind this feeling is that the antecedent of [M*] claims that $\Delta(A \& B)$ is true. Hence by our understanding of 'Δ', (A & B) must either be (universally) true or else (universally) false. But the antecedent also claims that (A & B) is (actually) true; so it must be universally true. The question then becomes: how can a conjunction be universally true? Certainly one way is if each conjunct is universally true, which would entail $(\Delta A \& \Delta B)$. There may be other ways, however, depending on other aspects one might think of for "definiteness"; but I can think of none. So it (tentatively) seems to me that the antecedent of [M*] should be allowed to imply its consequent when 'Δ' is understood as "definitely". The semantic condition corresponding to [M] is called *supplementation*

(m) if $x \in N\alpha$ and $X \subseteq Y$ then $Y \in N\alpha$

For our weaker [M*], I recommend the name *partial supplementation*

(m*) if $X \in N\alpha$ and $\alpha \in X$ and $X \subseteq Y$ then $Y \in N\alpha$

Since the logic ECNM* (without the I) is a sublogic of K and a superlogic of ECN, it falls on the line between K and ECN in Fig. 1. Our logic for vagueness is now ECNM*I, still independent of K (due to the presence of [I]), but a superlogic of ECNM*.

Other principles that might be thought of, seem to me to have considerably less plausibility as truths about vagueness.

[4] $\Delta A \rightarrow \Delta\Delta A$
[B] $A \rightarrow \Delta\nabla A$
[G] $\nabla\Delta A \rightarrow \Delta\nabla A$
[5] $\nabla A \rightarrow \Delta\nabla A$

[U] $\Delta(\Delta A \rightarrow A)$

(For some reason Gibbins thinks [4] and [5] are obviously valid principles of a logic of indeterminacy - but then he also thinks that $A \rightarrow \Delta A$ is too, so who knows why he thinks anything.) If one takes the view that all indices are "accessible" to any other, so that Δ means "is either true at all indices or false at all indices" and ∇A means "is true at some index and false at some index", then one will have the principles (all of which are equivalent, given [I] and [Def ∇])

[V_1] $\vdash \Delta \Delta A$
[V_2] $\vdash \neg \nabla \Delta A$
[V_3] $\vdash \Delta \nabla A$
[V_4] $\vdash \neg \nabla \nabla A$

i.e., whether A is definite or vague is itself always definite. For, if A is definite then A is either true at all indices or false at all, and hence ΔA is true at all indices - i.e., $\Delta \Delta A$. On the other hand if A is vague then it is true at each index that A is true at some index and false at some index, i.e., $\Delta \nabla A$. Given then one of the principles [V_1] - [V_4] we can see why the other principles ([4], [B], [G], [5], and [U]) hold - the [V_1] - [V_4] principles are the consequents of those conditionals. (6) This is the logic that Evans and Gibbins apparently wish to employ for "indeterminacy". The sense in which it is "at least as strong as S_5" is that, given S_5 we can define this system's Δ operator:

$$\Delta A = \text{df} (\Box A \vee \Box \neg A)$$

And given the present logic we can define the S_5 $\Box$:

$$\Box A = \text{df} (\Delta A \,\&\, A)$$

On the other hand, of course, the present logic of indeterminacy is independent of S_5, since for example it does not have [K]. (7)

(6) Actually, it takes a bit of an argument to show that [U] holds. Perhaps the following informal argument will suffice. Either (1) A is true at each index, or (2) A is false at each index, or (3) A is true at some and false at another index. In case (3), ΔA is false at each index, so $(\Delta A \rightarrow A)$ is true at each index and hence [U]. In case (1), since A is true at each index $(\Delta A \rightarrow A)$ is always true. Hence [U]. In case (2), since A is false in each index ΔA is true and A is false, so $(\Delta A \rightarrow A)$ is false at each index, hence [U].

(7) For further discussion on how these systems can both be and not be the same system, see my "Six Problems in Translational Equivalence" (*Logique et Analyse 108, pp. 423-434*).

This logic is axiomatized by ECNM*IV$_1$. The semantic condition on models for V$_1$ is *all-pervasiveness* (the necessity of each proposition is necessary)

(v1) $\{\beta : X \in N\beta\} \in N\alpha$

And this logic for indeterminacy is determined by the class of all-pervasive, contrary, partially supplemented models that are closed under intersections and contain the unit. Again, not the trivial logic. One can withhold [V$_1$], adding instead [4] to ECNM*I, and get a logic equivalent to S$_4$ – with the above definitions. Or, rather than [4], one could add

[B′] $A \rightarrow \Delta(\nabla A \vee A)$

and have a logic equivalent to B, under the above definitions. The logic ECNM* is in fact just logic T in disguise, under the definitions given above, as can be seen by the following argument. Substitute $(\Box A \vee \Box \neg A)$ for ΔA in the axioms of ECNM*I, and the result will be theorems of T; substitute $(\Delta A \mathbin{\&} A)$ for $\Box A$ in the axioms of T and the result will be theorems of ECNM*I. The same substitution in the inference rules of one system will yield derivable rules of the other. Here are a few examples to support these claims. Consider

$$\vdash \Box A \rightarrow A$$

(the t-axiom). After substitution we get

$$(\Delta A \mathbin{\&} A) \rightarrow A$$

which is a theorem of ECNM*I (a propositional theorem). Now consider

$$\vdash (\Delta A \leftrightarrow \Delta \neg A)$$

(the i-axiom). After substitution we get

$$((\Box A \vee \Box \neg A) \leftrightarrow (\Box \neg A \vee \Box \neg\neg A))$$

which is a theorem of T. The final requirements on equivalence are that "double substitutions" in any formula of one system are provably equivalent to the original formula of that system. For example, starting with $\Box A$, replacing this by $(\Delta A \mathbin{\&} A)$, and then substituting $(\Box A \vee \Box \neg A)$ for the ΔA therein, is equivalent to our original $\Box A$. I.e.,

$$\vdash \Box A \leftrightarrow ((\Box A \vee \Box \neg A) \& A)$$

– obviously a theorem of T. And similarly with a "double substitution" for ΔA

$$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta \neg A \& \neg A))$$

we get a theorem of ECNM*I (shown by the following argument)

$\vdash \Delta A \leftrightarrow \Delta A$	– prop. logic
$\vdash \Delta A \leftrightarrow (\Delta A \& (A \vee \neg A))$	– prop. equivalence
$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta A \& \neg A))$	– prop. logic distribution
$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta \neg A \& \neg A))$	– by [I]

Further details of these equivalence proofs can be found in the paper mentioned in the previous footnote.

I said before that none of [4], [B], [G], [5], [U], [V_1] - [V_4], etc., seem plausible candidates for a logic of indeterminacy. This is because of "higher order indeterminacy". It seems to me that a proposition might be definite, but not definitely so. Thus

$$\Delta A \& \neg \Delta \Delta A$$

seems possible, as does

$$\nabla A \& \neg \Delta \nabla A$$

and so on, for any number of iterations of the operators Δ and ∇. If we allow that all of these can happen, we shall want no reduction laws of the sort mentioned in [4], [5], etc.

I therefore recommend ECNM*I as a logic for indeterminacy. Is this a strange modal logic? No - at least no stranger than system T is. Is it appropriate for vagueness *de re*? Probably not, as indicated earlier - but that has nothing to do with whether it is strange. (8)

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