University of Alberta

THE NEED TO PROVE

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Thus, far from being an exercise in reason, a convincing certification of truth, or a device for enhancing the understanding, a proof in a textbook on advanced topics is often a stylized minuet which the author dances with his readers to achieve certain social ends. What begins as reason soon becomes aesthetics and winds up as anaesthetics.

—Philip J. Davis

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled The Need To Prove submitted by David Alexander Reid in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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DEDICATION

This thesis is dedicated to the memory of

Nicolas Herscovics (1935-1994)

who pointed me to the path I now lay down.

ABSTRACT

Mathematics education is essential, both in helping students to function in a Rationalist world, and in assisting them in making that world a better place. The deductive reasoning which typifies mathematical proving is the basis for Rationalism, and so is important in the achievement of both of these goals. At present, however, the teaching of proving is largely unsuccessful. This lack of success seems to be related to an incompatibility between the picture of proving portrayed in schools, and the role of deductive reasoning in professional mathematics and in students' lives. The research reported here is concerned with developing a better understanding of students' *need to prove*, with the aim of identifying aspects of teaching which might be improved.

The research studies involved the observation and interviewing of high school and undergraduate university students as they investigated problem solving situations. Their mathematical activity is described using a vocabulary developed during the research that identifies (1) needs which motivate reasoning, (2) types of reasoning, and (3) degrees of formulation of proving and of proofs. Categories of needs include explanation, exploration, and verification. Reasoning can be inductive, deductive, or analogical. Proving can be unformulated, formulated, mechanical, or formulaic. Proofs can be preformal, or semi-formal.

Three main observations are derived from the research studies: (1) The participants were able to reason deductively, and, with help, to formulate their proving. (2) Proving was applied primarily to exploration and explanation. Verification seemed to be a very poor motivation to prove. (3) The reasoning used by the participants was influenced by the activities of those around them, both observers and other participants.

These observations lead to two suggestions for teaching: (1) The current presentation of proving as deductive reasoning employed to verify statements should be expanded to include the use of proving to explain and explore. (2) The organization of class activities should accommodate the development of a "culture of proving," in which students feel that deduction is an appropriate way to reason about mathematics.

ACKNOWLEDGMENTS

Writing is a journey. Like any journey the trip is more fun, and one is less likely to get hopelessly lost, if there are others along to help. I have had the good fortune to have a great deal of company on my journey, which has made the resulting dissertation, the metaphorical slide show, better in uncountable ways.

Tom Kieren, as supervisor, co-researcher, reader, and friend, created a climate of fellowship and intellectual activity, an occasion of which I have been fortunate to be a part. Lynn Gordon-Calvert, and Elaine Simmt, the other half of the Enactivism Research Group, offered ideas, comments, and support which was integral to my research and writing. Other members of the wonderful place in which I found myself include: Al Olson and Ted Lewis, whose presence on my supervisory committee somehow both broadened and focused my thinking; Heidi Kass, Daiyo Sawada, Anna Sierpinska, and Sol Sigurdson, who were always willing to lend an ear, and a critical eye, to my ideas; and the people who made the community, Brent, Dennis, Hridaya, Ingrid, John, Judy, Kgomotso, Laura, Leo, Paul, Ralph, Ray, Rebecca, Roshan, Sandra, Tim, Tim, and Vi, the friends and inhabitants of 948. I must also acknowledge the students and teachers who took part in my studies, whose names are hidden, but whose importance is manifest.

My journey has been an emotional one as well as an intellectual one, and the existence of my dissertation owes as much to the support of my friends as it does to the ideas of my colleagues. Of course, some people contributed in both of these roles. Without Constance I would not have begun. Without Patrick, Tim, Kelly, Johwanna, Elaine, Samantha, Gwen, Peter, Gina, Ben, Allison, Drew, Lynn, Chris, Ralph, Sean, Bonnie, Elyse, Penny, and my parents, I would not have been able to go on. Jennifer, Sarah, and Drue were and are always with me, even from far away.

Finally, I must acknowledge the importance of the financial support given to me by the University of Alberta and by the Social Sciences and Humanities Research Council of Canada (Grant #752 93 3268). I have been fortunate to be among the Canadians who have benefited from community support for research and education, support which I hope we will have the wisdom to maintain.

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INTRODUCTION

*Caminante, son tus huellas el camino, nada más**
— Antonio Machado

"The need to prove" has a double meaning for me. Both of its meanings come into play in this dissertation. In the first half I concentrate on the need to prove felt by students engaged in mathematical activity. In the second half I consider the need to prove on a societal level. These two perspectives are linked by a consideration of the teaching of proving, which must blend students' needs and society's needs in order to be successful.

The relevance of my research to the field of mathematics education will be addressed in the main body of the text. Here I would like to mention the relevance of my research to me, and some of my personal assumptions that have motivated me to conduct this research in the way I have.

As a teacher and researcher in the psychology on mathematics education I have become convinced that learning very much depends on students' prior knowledge, abilities, and beliefs; what is, in the language of constructivism** and Enactivism, called their "structures." Given this, the practice of teaching mathematical reasoning as if it were unconnected to students' prior experiences of reasoning in other domains, and without considering students' prior ability to reason deductively, seems nonsensical to me. I feel that the way to teach students to prove must lie between assuming they know nothing and teaching logic as rules of procedure, and assuming they know everything and penalizing them when they fail to apply their abilities to reason to the peculiar contexts of mathematics.

With this assumption in mind, the first step in improving the teaching of proving must be the development of an understanding of how students reason in mathematical situations so we know where to begin. Such an understanding cannot be a general understanding of how *all* students reason. Students reason in quite individual ways. What this understanding can and must be is a sense of the range of possibilities in individuals' reasoning, combined with some way of noticing and talking about this reasoning. I believe my research is an important step toward this understanding, combining what is already available in the mathematics education literature with the results of my own empirical studies of proving.

A large part of my motivation to be a teacher is the conviction that schooling can play a role in preparing students to survive and improve the world in which they live. This conviction is also a motivation for my research. The adoption of proving in mathematics as the model for correct thinking in all domains has been seen as marking the beginning of the modern era, and is central to the Rationalist attitude that continues to affect the way decisions are made in our society. Meanwhile, the limitations of proving have come to be understood through work in mathematics, analytic philosophy, and linguistics. It is in these fields, that depend heavily on proving as their method of discovery, that the determination that proving

* Wanderer, the road is your footsteps, nothing else.

** Constructivism the theory of learning, not the philosophy of mathematics or the art movement.

has limits first became possible. It is also in these fields that these limits have been analyzed and understood. Teaching students to survive in a Rationalist world must involve teaching proving, and helping them improve this world must involve teaching proving well enough for the limits of proving to be seen and understood.

It is not only students who need to take responsibility for improving the world. Educators and researchers must also engage in the continuous process of considering how our methods arose, and what limits their origins have passed on to them. Research in education has been strongly influenced by the Rationalist attitude, and this influence, although weakening, continues. The particular limits of proving as a model for research in education have been revealed in two ways. Many researchers have noted that the predictive power of proving seems not to apply in educational research, and that there are many aspects of education that seem to be inexpressible in Rationalist terms. This is a partial understanding of the limits of proving as a model for research, as it detects limits, but does not provide any analysis of the origins of those limits. In some cases this has led researchers to advocate a complete rejection of Rationalist methods in educational research. Other researchers have come to an understanding of the limits of Rationalist methods by a careful use of the methods themselves. This process is analogous to the processes applied in mathematics to reveal the limits of proving. It has the advantage of revealing not only the limits of Rationalist methods, but also the origins of these limits. Limits which have not yet been detected can be predicted, and so areas in which Rationalist methods can be reasonably applied can be identified. In my final chapter I will describe a methodology for research in mathematics education that comes out of such a self-reflective analysis of method.

In an earlier draft of this dissertation I structured my chapters and sections as if they were part of a proof. My arguments were broken down into definitions and lemmas, some of them quite involved, that led up to short sections with ambitious titles in which I asserted my conclusions. These sections referred back to the preceding chapters for the lemmas required to support their conclusions. This format was singularly inappropriate to the message I am attempting to communicate. In my final chapter you will find me asserting that research into the thinking of human beings cannot be like proving, and so casting the results of such research into the shape of a proof was not only a confusing act on my part, it was a serious contradiction.

One of my indulgent readers pointed out this problem, and unlike many people who point out problems, she also provided me with a solution. Of course I must take responsibility for the success or failure of this attempt to implement her idea. The restrictions of text and my own lack of creativity led to the linear form of my writing. I admit that I have written the sections with the lower page numbers as the beginning, and have proceeded in the usual way through a middle, to an end. I think it reads pretty well this way, but I will leave it to you to make the final judgment. The structure ends up resembling my own progress in my research. First I considered what I knew and had read about proving by students and by professional mathematicians. Then I observed students proving, and noted what I felt was important in what they did. Next I considered how the proving of the students I observed might relate to the teaching of proving, and to the role of proving in society. Finally, I considered how the ideas of Enactivism informed and were clarified by what I had learned in my research. When I came to write about what I had learned there were inevitably things which needed to be written, but which played supporting rather than central roles in my thinking. Such things have been included in appendices.

I would like to emphasize that there is no reason to read from beginning through middle to end. There is a degree of connection between the end and the beginning, that lends a circular aspect to the whole work. While writing in my mundane linear manner I have tried to make connections forward and backward, so that an adventurous reader might start anywhere and read in either direction.

Before you decide how adventurous you would like to be, let me describe the territory you will be exploring. In the following you will find me setting forth some fundamental questions related to the need to prove, reporting the results of my attempts to answer these questions, relating my results to the teaching of mathematics, discussing the role of proving in society, and offering some ideas on research and the need to prove.

In Chapter I the basic questions underlying my research are introduced. They are "What is proving?" and "Why do people prove?" I take some preliminary steps to answer the first question, suggesting that proving should include deductive reasoning used for any purpose, in order to fit with the role of proving in mathematics. I also report further on proving in mathematics to address the question "Why do mathematicians prove?" The chapter ends with a sketch of the methods I used in my attempt to answer the question, "Why do students prove?"

In Chapter II, I turn to the question "Why do students prove?" and report some results of the research studies I undertook in order to investigate students' need to prove. This chapter is organized into several sections addressing specific needs; explaining, exploring, verification, and teacher-games. A network of terms is used to clarify the relationships between needs and proving.

In Chapter III, I use the language developed in Chapter II to describe the proving of two students who participated in one of my studies. This example shows both the application of the language and also expands on the relationships between terms.

In Chapter IV I report on circumstances that constrained the use of proving in my research studies. These include individuals' structures, social constraints, and the problem situations in which the participants found themselves.

In Chapter V, I discuss the teaching of proving. This includes a consideration of the importance of proving seen by curriculum designers, a critique of current teaching practices, a description and critique of several innovative experiments in teaching proving, my own speculations as to ways in which the teaching of proving might be improved, and finally a reinterpretation of the need to teach proving.

In Chapter VI, I turn to the role of proving in society, including the rise and influence of Rationalism, some problems with Rationalism, and alternative modes of thinking.

In my final chapter, Chapter VII, I describe Enactivism, as an extension of Rationalism that acknowledges its limits. I show how Enactivism can be used to provide a theoretic basis, and a methodology, for educational research into proving. I conclude with a summary of my thoughts on proving, in education and in research.

There are several appendices that provide details of my research that did not fit into the structure of the main body of the text, but that some people might find interesting. They include an annotated bibliography of research on teaching proof (as opposed to teaching proving), details of the design of my research studies, and several different summaries of my data.

As noted above, I have written the text in what I feel is the best way for it to be read. If you prefer to read a more traditional dissertation, or a more deductive argument of my points, Table 1 gives a concordance of the Chapter and sections included here, in an order suitable for those two alternate readings.

Chr.	Traditional Dissertation	Deductive argument
Intro.	Introduction, I-1, V-1	Introduction, V-1
I	I-2, A	VI
II	B	V-5, V-2, I-2, VII-2
III	II, III, IV	B, VII-3, II
IV	V-4, VII-4	V-4, VII-4

Table 1: Alternate readings.

To clarify, a reader who wished to read a traditional dissertation should begin by reading this introduction, the first section of Chapter I, and the first section of Chapter V. Taken together these sections cover much of what is usually presented in the introduction to a dissertation. On the other hand, the first section of Chapter I is not really needed for the deductive argument reading, and can be omitted.

Note that two sections that are usually found at the beginning of a dissertation, a review of related literature and a description of the design of the studies, have been relegated to appendices. The reasons for this move are given at length at the end of Chapter I and at the beginning of Appendix A. Briefly, the traditional exhaustive review of the literature has been rendered superfluous by the introduction of electronic indexes to the literature, so I restrict the references I make in the main text to those that are directly related to the topics under discussion. For example, the extensive work of Balacheff on teaching students to create proofs is not mentioned until Chapter V, when teaching proving is considered. The custom of describing in detail the design of research studies is taken from the style of reporting research used in the sciences, where reproducibility is an important issue. The complexity of human reasoning makes reproducibility in detail impossible, so in the main body of the text I limit my descriptions of my studies to what is needed for understanding the results I report.

A note on transcripts and diagrams

In presenting excerpts from the words spoken and the writing of the participants in my research studies I have attempted to balance clarity of presentation with completeness. While I recognize that transcripts and writing pulled out of context are already a long way from the situations in which they occurred, I realize that some readers will wish to consider how the examples I give might be interpreted differently, and I do not wish to discourage them. At the same time, transcripts and reduced images of written work are more difficult to understand than the original voices and full sized writings. I do not wish to make my examples any more difficult to decipher than they need be. With this in mind I have made some editorial changes to the transcripts and diagrams included as examples.

In the case of transcripts, I have omitted many of the inevitable "hmms," "uhs," and other sounds that punctuate normal speech. Such omissions are marked with ellipses (...). I have used two conventions in an effort to capture some of the rhythm of spoken language in text. Utterances that were interrupted or left unfinished are marked with a short dash (-) at the point of interruption. Long pauses are marked with long dashes (—). Longer pauses are marked with several long dashes. In a very few cases I have omitted several lines from transcripts where they do not contribute directly to the point I am attempting to illustrate. Such omissions are noted in the analyses of the transcripts, and glosses of the omitted matter are provided there.

The participants in the studies were quite careful not to use more paper than absolutely necessary, which resulted in pages covered with writing, often overlapping or oriented in strange directions. As it is impossible, and unhelpful, to reproduce such pages at their actual size in the space defined by my margins, I have either reduced them in size, or selected smaller areas of pages that are of particular interest. I have also erased stray lines, figures, etc. that do not relate to my purpose in providing the illustration. There are cases where my main interest is in the content of the participants' writings, and they do not include drawings. In such cases I have typed the participants' writings. Such passages are italicized.

CHAPTER I

PROOF AND PROVING

Prove all things; hold fast that which is good.
— I Thessalonians 5, 21

This chapter explores two questions, by way of introducing the ideas I will be considering as part of my larger exploration of the need to prove. The first of the questions is, "What is proving?" The second is, "Why do people prove?" In the two sections of this chapter I will only be able to begin trying to answer these questions, but I do hope to clarify exactly what is being asked.

1. What is proving?

The simplest answer to this question might be "Proving is making a proof." This answer leaves us free to talk about proofs. Talking about proofs is easier than talking about proving, just as talking about books is easier than talking about writing. If my original question had been "What is a proof?" we could have begun with a few examples. This is precisely the approach taken by the professor in a vignette by Davis and Hersh (1981. p. 39), when asked "What is a mathematical proof?". There has been a great deal of research done in mathematics education on proof, especially on teaching students to read and write proofs, and on their difficulties in doing so. There has been very little research on proving, the reasoning processes that the proof embodies. It is not that proving is uninteresting. It is just that proofs are a lot easier to observe, to talk about, and to write about. If I were researching proofs I could show you the proofs that were produced in my studies. Strictly speaking, I cannot show you proving (although I will be trying to do so with transcripts of students' proving).

Nevertheless, it is important that I am researching proving. Rather than saying "Proving is making a proof," I would rather say "A proof is what results from proving." This emphasis on proving is a consequence of my belief that the teaching of proving must begin with students' existing reasoning processes, and with an awareness of the circumstances in which they reason deductively. In the next chapter it will become apparent how this emphasis affects my research.

So, what is proving? In Thessalonians, Paul the Apostle advised "Prove all things." Paul was not, so far as I know, an obsessed mathematician. His suggestion simply means that we should investigate. "Prove" is derived from *probare*, which means to test, to try. The verb "probe" still carries this meaning, although it also conjures images of poking with sticks. The sense of *probare* is part of what proving is: investigating. But proving is investigating in a certain way, and to get at that aspect of what proving is, I will, after all, have to talk about proof.

In some common phrases, "proofread," "proof of the pudding," "100 proof," the word "proof" still holds onto the meaning of investigation, but there is another common usage of proof. When we doubt a statement, we may ask, "Do you have any proof of that?" In this question "proof" means evidence. Often the evidence is expected to take the form of a deductive argument from some agreed upon premises to the desired conclusion. This especially true in the sciences and in mathematics. This deductive aspect of proof indicates the way of investigating that I call proving.

The contrast between everyday uses of the word "prove" and the more precise meaning I give it is illustrated by the expression "the exception which proves the rule." This expression is usually taken in the paradoxical sense of asserting that the presence of a single counterexample to a generalization establishes the universal truth of that generalization. "Prove" is taken to mean providing evidence, without any reference to reasoning about the situation. This expression was not always so paradoxical. In fact, if we remember that proving originally referred to investigating a situation, saying "the exception proves the rules" amounts to suggesting that examining exceptions closely, reasoning out the way they occur, can lead to a clarification and improvement of the rule. Lakatos (1976) elaborates this process in some detail, in his analysis of the ways in which counterexamples and proving interact to improve theorems in mathematics.

In summary, proving, for me, is investigating using deductive reasoning. Deductive reasoning refers to reasoning that proceeds from agreed upon premises to conclusions, using logical arguments. I will not be using proving to mean investigating in non-deductive ways, nor will I be restricting proving to reasoning deductively to provide evidence. I hope the reader will agree that this interpretation of proving is useful, in light of the ideas it permits me to present in the following chapters.

Although proving is a part of reasoning in many fields, I am particularly concerned with proving in mathematics, and so it seems advisable to look at what proving is in mathematics. As mathematics in schools is necessarily different from what professional mathematicians do, I will briefly mention what proving is in schools, both in the curriculum and in students' own understandings, before describing what proving is to mathematicians.

What is proving in school mathematics?

In schools "prove" is often used loosely, as it is in everyday life. It can also have a much more restricted meaning. The authors of Alberta's curriculum documents define "prove" in this way: "Prove: to substantiate the validity of an operation, solution, formula or theorem in general and to provide logical arguments for each step in the process" (Alberta Education, 1991, p. 5). This meaning of proving is concerned with providing evidence, with substantiating validity. No longer are the experimental techniques once used to "proof" rum appropriate. Proving here is logical, deductive, certain, and general. In addition the stress has shifted from the action of proving to the result, certain knowledge of the validity of a statement.

Students often use "proving" to mean providing evidence, without distinguishing how that evidence is obtained. This is startlingly illustrated by the comments of students interviewed by Finlow-Bates (1994). Consider the following exchange (K is Finlow-Bates, T is a university student):

K: And the examples, what are they there for?

T: Just to *prove*, *prove* the statement.

K: What does that mean, "they prove the statement"?

T: They *prove*, that means they make it true.

K: So the ...

T: (interrupting) under all conditions. (p. 348, emphasis in original)

For this student "prove" means to make a statement true, and the inductive evidence provided by examples is sufficient to do so. We will see other examples of this association of "proof" with verifying in the next chapter.

What is proving for professional mathematicians?

> Proving is a means of coming to understand, and of coming to know what understanding is. In trying to prove something new, one is asking what makes it tick; in trying alternative proofs, rejecting them, modifying them, one is discovering things about its structure—and solidifying one's knowledge in the process. This is the deep reason for much of the emphasis on proof in mathematics. The mathematician comes to accept proving as a way (if not the way) of thinking, a way of demanding and insuring that he does indeed understand. (Schoenfeld, 1982, p. 168, emphasis in original)

Professional mathematicians prove as an integral part of their occupation. Lakatos (1976) describes the process of mathematical discovery as a cycle of conjecturing, making a proof, and testing with counterexamples. This process can begin at any stage in the cycle. For example, a conjecture can be made for which a proof is offered, and then a counterexample is found that forces a revision of the conjecture or the proof, so a new conjecture or proof is made, and the cycle continues. Alternately, proving could lead to a new result, so that the proof and the conjecture arise together. The discovery of a counterexample then returns the cycle to proving anew. While Lakatos does not consider beginning the cycle with a counterexample, a cycle could begin there, as in the case of a counterexample to an implicit generalization. An example is De Morgan's discovery that the digit 7 occurs less often than one would expect in the decimal expansion of π, which marks the origin of a cycle of proofs and refutations involving the degree of randomness to be expected in π.

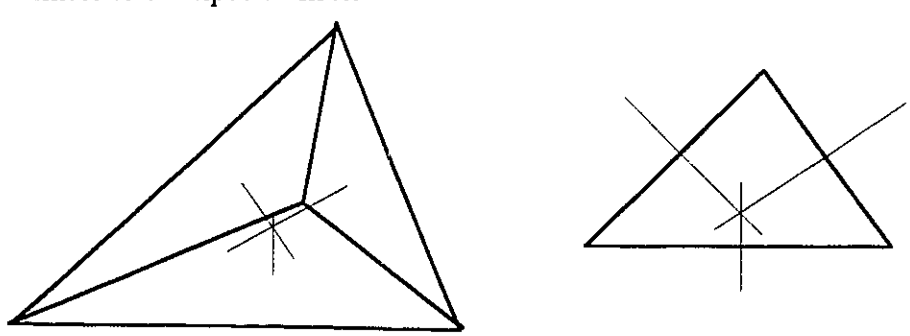


Figure 1: Intersecting perpendiculars.

While conjecture, proofs, and counterexamples can all arise from proving, proving is not the only way of investigating used by mathematicians. The importance of analogical and inductive reasoning in mathematics has been described at length by Polya (1968). Reasoning by analogy involves making a conjecture based on similarities between two situations. For example, one might conjecture that the perpendiculars through the centroids of the faces of an irregular tetrahedron meet in a point, by analogy to the perpendicular bisectors of a triangle (see Figure 1).

Inductive reasoning is characterized by the making of a generalization from a pattern noticed in several specific cases. The classic example is making the generalization "The sun will rise every day" from several million specific cases. A more mathematical example would be generalizing "The product of two consecutive numbers is always even" from the cases $3\times4=12$, $4\times5=20$, $12\times13=156$, and $37\times38=1406$.

Lakatos developed his cycle of proofs and refutation to accurately represent what proving is to mathematicians. He did so in opposition to what he called the Euclidean model, which portrays mathematical research as a process of beginning with a set of assumptions, and then proving theorems from them with absolute certainty. This image might be derived from the assertions of Formalist philosophers of mathematics (see Chapter VI, section 2). The Euclidean model bears a superficial resemblance to a cycle that begins with making a proof. It lacks, however, any role for counterexamples. In the Euclidean model mathematical discovery is seen as a steady forward progress from truth to truth, not a recurring cycle of proof and refutation.

Another mistaken image of what proving is to mathematicians portrays proving as the verification of mathematical discoveries that are made through other ways of reasoning. This image is often associated with the teaching of mathematics. For example the NCTM *Standards* (NCTM, 1989) state:

> A mathematician or a student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in particular cases (inductive reasoning) and then tests the conjecture by constructing either a logical verification or a counterexample (deductive reasoning). (p. 143)

If a cycle of proofs and refutations begins with the making of a conjecture then that cycle looks something like this "discover, then prove" image. It differs from this image both in its cyclical character, and in that conjectures can arise through proving, as well as through inductive or analogical investigations.

This may be an appropriate place to mention that the question "What is a proof?" is an important one in mathematics. This question is related to two issues, the degree of formality of a proof, and the sort of proving that produced the proof (see Chapter VI, section 2). Formality became important in the early twentieth century, when formal proofs were seen as more reliable than informal proofs. The sort of proving involved in producing a proof has become important with the increase in the use of computers in mathematics. The character of proving by working through an argument oneself differs from proving done by setting up a computer to check all possible cases. Concern over the nature of proof also arises when a proof is the product of many individuals working on related topics, so that a conclusion might be reached without any one person ever having proved it entirely.

Lakatos (1978, p. 61) introduced the terms *pre-formal, formal*, and *post-formal* to describe proofs of different degrees of formality. I would further divide formal proofs into *semi-formal* and *completely formal* proofs, a distinction pointed out to me by Uri Leron. A pre-formal proof might appear in the working notes of a mathematician. It may involve hidden assumptions, and use informal language and notation. It might also include references to analogical or inductive evidence for the conjecture. A semi-formal proof is presented in a form suitable for publication in a professional journal or a textbook. The arguments are purely deductive, and unusual assumptions are made explicit. Some steps might be omitted however, with a note to the reader suggesting how they may be worked out. The proof is written in a mixture of formal symbols and natural language. A completely formal proof might also appear in a journal, or as a computer program. In a completely formal proof all steps are included, and all assumptions are made explicit. The language of the proof is entirely symbolic. A post-formal proof talks about the nature of formal proofs, from a meta-mathematical perspective. They may resemble pre-formal or semi-formal proofs, but they will also include elements from the formal system that is the object of the proof. The proof of Gödel's Theorem is a well known example (see Chapter VI, section 2).

2. Why do people prove?

It is impossible to give a single reason why people prove. Proving occurs in widely different circumstances, with different goals. Within groups, however, one can begin to see some common purposes for proving. Because proving is so important in mathematics, I would like to begin by focusing on the question "Why do mathematicians prove?" The more complicated question, "Why do students prove?" is central to my research. I will consider it at the end of this section and in the next three chapters. In Chapter VI I will return again to the general question of why people prove, and by that point it may be possible to hint at some answers.

Why do mathematicians prove?

Ulam (1976) hints at the answer to this question when he states that "Georg Cantor proved (i.e., discovered) that the continuum is not countable" (p. 282). Cantor's discovery came through proving. Lakatos' (1976) historical analysis of the use of proving in mathematics reveals that mathematicians in general employ proving to both discover and improve propositions. The use of proving to discover, which Lakatos calls "deductive guessing," involves a cycle of proofs and refutations in which the proving is both the source of the conjecture and part of the process of testing it. As Lakatos wrote:

> There is a simple pattern of mathematical discovery — or of the growth of informal mathematical theories. It consists of the following stages:
> (1) Primitive conjecture.
> (2) Proof (a rough thought-experiment or argument, decomposing the primitive conjecture into subconjectures or lemmas).
> (3) 'Global" counterexamples (counterexamples to the primitive conjecture) emerge.
> (4) Proof re-examined: the 'guilty lemma' to which the global counterexample is a 'local' counterexample is spotted. This guilty lemma may have previously remained 'hidden' or may have been misidentified. Now it is made explicit, and built into the primitive conjecture as a condition. The theorem — the improved conjecture — supersedes the primitive conjecture with the new proof-generated concepts as its paramount new feature.

> These four stages constitute the essential kernel of proof analysis. But there are some further standard stages which frequently occur:
>
> (5) Proofs of some other theorems are examined to see if the newly found lemma or new proof-generated concept occurs in them: this concept may be found lying at cross-roads of different proofs, and thus emerge as of basic importance.
>
> (6) The hitherto accepted consequences of the original and now refuted conjecture are checked.
>
> (7) Counterexamples are turned into new examples — new fields of inquiry open up. (1976, p. 127)

This use of proving to discover is doubtless the central reason why mathematicians prove. Thurston (1995), in relating something he learned as a graduate student about mathematicians, mentions other reasons. "I thought what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than knowledge, people want *personal understanding*. And in our credit driven system, they also want and need *theorem credits* (pp. 35-36, emphasis in original).

The importance of proving as a mark of mathematical activity should not be underestimated. The criticism of Mandelbrot by Krantz (1989), in which Krantz charged that Mandelbrot's investigations of fractal geometry were not a part of mathematics because Mandelbrot proves no theorems, illustrates this use of proving as a marker. The exclusion of non-Europeans from the history of mathematics, on the same basis that they did not prove their work (Gheverghese Joseph, 1991), is another example. Proving in this context can be seen as conferring status on a mathematician, as a stethoscope does on a doctor. Mathematicians form a society with customs and rituals, just as other groups of people do, and the rite of initiation is the creation of an original proof.

I have not mentioned a reason to prove that many people would see a central to mathematics: verifying that theorems are true. The influence of this idea can be measured by reference to Crowe's (1988) list of "ten misconceptions about mathematics." It is included in Crowe's list twice, once as "mathematics provides certain knowledge," and the second time as "mathematical statements are invariably correct." It is a misconception for two reasons: 1) Proving does not always verify, and 2) Methods other than proving are often used to verify theorems.

<u>Proving does not always verify</u>

The status of proving as the path to absolute certainty has suffered some serious setbacks in the last two centuries, and this process continues. The discovery of non-Euclidean geometries, which contradicted the claim that Euclidean geometry describes with certainty, was the first setback. The paradoxes of set theory offered the second setback, and Gödel's Theorem set bounds on the certainty proving could provide (Kline, 1980; see also Chapter VI, section 2). More recently, the proliferation of proofs in mathematics journals, the increasing length of proofs, the specialization of the field, and the increased use of computers, have highlighted the human and social elements of the uncertainty of proving.

Ulam (1976, p. 288) estimated that in the early 1970s, almost 200 000 theorems were published each year. Davis (1972/1986) pointed out the stress this puts on the process of refereeing proofs, leading to the suggestion that half of the proofs published might be flawed. Although this suggestion was originally made to Davis in jest by an editor of Mathematical Review, it has been widely quoted as correct, presumably because it seems quite plausible to members of the mathematical community. Recently, the case of Wiles' proof of Fermat's Last Theorem (see below) has added further evidence as to the unreliability of published proofs. The careful checking of Wiles' proof resulted in the discovery of errors in several of the proofs to which he made reference. Given this, it is hard to justify a claim that we are certain of the 200 000 theorems published in 1970, even though they have been proved. At best we can ascribe a probability of certainty to them.

Part of the difficulty in deriving certainty from contemporary proofs in mathematics is their length. The elusive property of elegance in proofs includes an inclination towards short proofs. A joke claims that a Ph.D. thesis in mathematics should be rejected if it runs over ten pages. This preference for brevity is not merely aesthetic, however. There are sound practical reasons for mathematical proofs to be short. Chief among these is the requirement of surveyability. For a proof to be surveyable it ought to be possible for a suitably trained mathematician to consider the whole proof at one time. Many contemporary proofs stretch this requirement. The most extreme example thus far is the cataloguing of the simple finite groups. The proofs cover over 5000 journal pages, and none of the mathematicians involved can be said to have surveyed the complete proof (Davis & Hersh, 1981, p. 388).

A further difficulty in achieving certainty through proving is the increased specialization of mathematics. Many proofs concern topics or employ techniques so abstruse as to be incomprehensible to the vast majority of mathematicians. This problem is a steadily worsening one. It is said that Poincaré, who died in 1912, was the last mathematician to have a sound grasp of the entire field (Boyer, 1968/1985, p. 650).

The problems of length and specialization can be illustrated by a consideration of the recent proof of Fermat's Last Theorem by Wiles (originally described by him at Cambridge in 1993, and outlined by Ribet & Hayes in American Scientist in 1994. The original, flawed, proof has been repaired, but is as yet unpublished). The proof is very long, running to about 200 pages (Ribet & Hayes, 1994, p. 156), and makes use of several mathematical specializations: elliptic curves, Galois groups, deformation theory, modular forms, etc. There are few mathematicians with the background to referee Wiles' proof, and its complexity and length make their task difficult. If the proposition were a less celebrated one, it is doubtful the resources being devoted to checking the proof would have been available, and the errors detected so far might have gone uncorrected.

The proof of Fermat's Last Theorem also demonstrates another aspect of the role of proving in providing certainty. Prior to the announcement of Wiles' proof few mathematicians would have doubted that Fermat's Last Theorem is true, based on the considerable empirical evidence amassed in the three centuries since Fermat proposed it. It is known, for example, that Fermat's Last Theorem is true for all numbers less than 4 million. Fermat's Last Theorem also has a quality, which could be described as *plausibility*. In mathematics, exceptions to simple generalizations usually are discovered quickly, if they exist. The truth of Fermat's Last Theorem for $n = 3$ or 4 made it plausible that no other exceptions would be found. Given that everyone expected Fermat's Last Theorem to be true, in what sense can Wiles' proof be said to have increased the certainty of its truth?

The use of computers in math has radically changed the way we see proof in mathematics. This has occurred in two ways. Computers have introduced a powerful new tool for proving mechanically, analogous to algebra and calculus, but not as yet enjoying the same degree of acceptance. In addition, computers have allowed mathematicians to visualize mathematical situations before proving in those situations.

I have been fortunate to been working in mathematics in the years when two fascinating theorems first were proved: Fermat's Last Theorem, and the Four Colour Theorem. Both of these propositions are easily stated, but difficult to prove. The proof of Fermat's Last Theorem is long and complicated, but traditional in form. The proof of the Four Colour Theorem, on the other hand, provoked controversy because of the extensive use of computer algorithms in it. Because of the use of this novel technique of mechanical deduction many mathematicians rejected the proof as invalid. This added another element of uncertainty to use of proving to verify.

The proof of the Four Colour Theorem could be criticized on the basis of its unsurveyability, due to its length, or the specialist backgrounds required to understand it; however, the chief critique focused on the possibility of programming or computer error. This is a general weakness in any form of mechanical deduction. A misprint in an algebraic derivation or a computer program can be made easily, have radical effects, and be almost undetectable. As a result, in practice the validity of a proof must often be determined in other ways.

Other ways mathematicians verify

Hanna (1983) describes five ways in which she believes mathematicians verify propositions:

> Most mathematicians accept a new theorem when some combination of the following factors is present:
> 1. They understand the theorem, the concepts embodied in it, its logical antecedents, and its implications. There is nothing to suggest it is not true;
> 2. The theorem is significant enough to have implications in one or more branches of mathematics (and thus important and useful enough to warrant detailed study and analysis);
> 3. The theorem is consistent with the body of accepted mathematical results;
> 4. The author has an unimpeachable reputation as an expert in the subject matter of the theorem;
> 5. There is a convincing argument for it (rigorous or otherwise), of a type they have encountered before.
>
> If there is a rank order of criteria for admissibility, then these five criteria all rank higher than rigorous proof. (p. 70)

Note that most of these are based in the human and social nature of mathematics, not on the use of proving to produce certainty.

Why do students prove?

Some readers may question the wisdom of asking "Why do students' prove?" when it seems quite possible that students do not prove. To begin, then, I will give some examples of students' proving from my own work and from the mathematics education literature. I will then review the answers given in that literature to the question "Why do students prove?" with some comments on the plausibility of those answers. The next chapter explores my own attempts to investigate students' need to prove in detail.

Do students prove?

When "proving" is taken in the restricted sense of "producing semi-formal proofs" very little proving is witnessed. But if "proving" is taken to refer to deductive reasoning, evidence abounds that students can and do prove in and out of mathematical contexts. In this section I will be mainly concerned with what I call "unformulated proving," proving that is informal and only partially articulated. A more detailed description of unformulated proving occurs in the next chapter.

In the mathematics education literature the main focus is on proofs rather than proving. This has limited the amount of published discussion of unformulated proving. There are, however, several indications that unformulated proving is a known phenomenon. Balacheff (1991, p. 179) mentions that students show "some awareness of the necessity to prove and some logic" in their behavior outside of school. Edwards (1992) comments:

> Some students at the beginning of high school, even without instruction in formal proof, will go beyond empirical reasoning and offer informal proofs, or explanations, of their findings. (p. 215)

Blum and Kirsch (1991) describe "preformal" proofs, which they claim students are generally able to construct, based on "intuitions" that are common to all students. Moore (1990, 1994) also observed students who could prove informally, and examined some of the elements of their difficulties in making their proving more formal. "Examples, concept images, and informal approaches were helpful, and often necessary, for *discovering* a proof, they did not guarantee that a student could *write* a correct proof" (1994, p. 257, emphasis in original).

In my own research (Reid 1992, 1993) I have seen unformulated proving by students from a wide range of school levels and mathematical abilities. A particularly clear example appears in Reid (1992). Beth, a university humanities undergraduate, who last took mathematics in grade 11, gave this argument that every third Fibonacci number* is even:

(1) Beth: This one, at least I think I know why, the multiples of three work out to be even because the, the other two, when you add the Fibonacci numbers the other two are odd and then so it would come out to be even.

(2) DR: How do you know the other two are going to be odd?

---

* The Fibonacci numbers are the elements of the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... in which each term is the sum of the previous two terms. The first two terms are both 1, by definition.

(3) Beth: I don't — that again is looking at the little charts and they seem to work out that way —

(4) DR: So you've made a conjecture that, the two Fibonacci numbers before one that is a multiple of three will both be odd.

(5) Beth: Because, no, because you- If each Fibonacci number is the first one plus the second one equals the third one, — the first, it starts out, well, then you would be adding two odd numbers together and get an even number, and then you add, oh, that's the same thing, I see, you'd say, then the next one then is odd, so you'd add that to the even and then you'd come out to another odd, but then I don't necessarily know that the, that the next number after an even number would be odd so —

(6) DR: Can you think of any reason why the next one after an even number should be odd?

(7) Beth: — because the one before the even number was odd (p. 319)

In my current research (described in the next chapter) unformulated proving was used by all the participants, including Sandy, a mathematically talented student in grade six, and Bill and John, two mathematically weak students in grade 10.

Uses of proving

Many uses of proving have been mentioned in the mathematics education literature. They include:

Verification — Fischbein and Kedem (1982), Bell (1976), and many others
Explanation — Hanna (1989), de Villiers (1991), Bell (1976), Moore (1990)
Exploration — de Villiers (1990)
Systematization — Bell (1976)
Communication — de Villiers (1990), Arsac, Balacheff and Mante (1992)
Aesthetics — de Villiers (1990)
Personal self-realization — de Villiers (1990)
Developing logical thinking — de Villiers (1991)
A "teacher-game" — Alibert (1988), Schoenfeld (1987)

In a survey of prospective teachers de Villiers (1991, p. 23) found that most (61%) felt that the main function of proof is verification. Other popular categories were explanation (7%), systematization (11%), and developing logical thinking (4%). Moore (1990) found that college students only listed verification and explanation as functions of proof. In contrast to de Villiers' teachers, Moore's students proposed explanation and verification in approximately equal numbers (5 explanation, 6 verification, p. 113). Each of the uses listed above will be described in more detail in the following paragraphs.

Teachers often tell students that proving *verifies* that a mathematical proposition is true. In doing so we echo a traditional definition of proof as something that establishes truth. The need to verify as a motivation for proving appears in almost all research about proof and proving (see Appendix A). This motivation for proving has also provided mathematics education researchers with a methodology to determine understanding of proving. For example, Fischbein and Kedem (1982) tested understanding by asking students who had seen a proof whether they would care to examine other confirming examples of the proposition. Students who requested additional empirical data were deemed not to understand proof, as they had not understood that proving establishes certainty. According to this criteria, mathematicians such as Crowe (1988), who does not believe that proving establishes certainty, would also be deemed not to understand proof.

Finlow-Bates (1994) has done some research that suggests that the students Fischbein and Kedem studied might have learned to request additional examples in school. In his study five students were asked to select the "best" proof from a set of four. The four proofs included a set of examples, a proof, a proof preceded by examples, and a proof followed by examples. Although three students chose the proof alone when first asked, the suggestion that they provide a reason for their choice caused them to switch to the proof followed by examples. In the final rankings all five students rated best the proof followed by examples, with the examples followed by the proof ranked second best. Although the reason for the students' choice is not obvious, it seems plausible that they were reflecting the normal presentation they had seen in school where teachers often state a general principle, explain it, and then give examples.

Research has suggested that only a few students see verifying as a use of proving; most students do not (Bell, 1976; Braconne & Dionne, 1987; Fischbein, 1982; de Villiers, 1992; Senk, 1985). Using verification to motivate proving in schools may play a significant role in students' difficulties in learning to prove. The fiction of proving as the path to complete certainty *is* a fiction, and students may be quicker than their teachers to recognize this.

Hanna (1989) and de Villiers (1991) both stress the importance of proving as a way of *explaining* in educational contexts. Hanna asserts that proofs used by teachers in lessons should be picked on the basis of both their explanatory and verificatory qualities. De Villiers claimed that students have a need for explanations and will accept proofs as explanations. Unfortunately, de Villiers' research is not sufficient to indicate that students would feel a need for explanation in all mathematical contexts, nor did he consider whether students themselves would employ proving to explain. In my research I have attempted to investigate these questions further (see Chapter II).

De Villiers (1990) asserts that proving is an important means of *exploring* in mathematics.

> Even within the context of such formal deductive processes as *a priori* axiomatization and defining, proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of *a posteriori* verification, but often also a means of exploration, analysis, discovery and invention. (p. 21)

De Villiers goes on to give examples of theorems in geometry that students could discover through deductive exploration. The use of proving to explore is also implicit in the teaching methods proposed by Fawcett (1938) and Lampert (1990).

Systematization consists of "the organisation of results into a deductive system of axioms, major concepts and theorems, and minor results derived from these" (Bell, 1976, p. 24). De Villiers (1990) mentions the importance of systematization for mathematicians, but he presents no evidence that students would prove to satisfy a need to systematize.

Communication is also suggested by de Villiers (1990) as a possible reason to prove. Arsac, Balacheff & Mante (1992) provide examples of students proving as communication, in the context of an activity which asked them to focus specifically on communicating mathematical ideas.

De Villiers (1990) briefly mentions *aesthetics* and *personal self-realization* as reasons to prove, but he does not elaborate. Presumably he meant these to correspond to needs felt by professional mathematicians.

Developing logical thinking was once an often stated goal of teaching proof (see Fawcett, 1938, for a summary of such assertions). Although research (e.g., Sekiguchi, 1991, p. 26) indicates that there is little transference of proof skills learned in mathematics to other contexts, some teachers (4% according to de Villiers, 1991, p. 23) still believe that this is the primary function of proof.

A *teacher-game* is an activity that earns marks and acceptance, but is seen as being otherwise useless. Alibert (1988) and Schoenfeld (1987) describe proof having this function for students. Teacher-games are described in more detail in Chapter II, section 4.

Researching the need to prove

The needs listed above are all possible answers to the question "Why do students prove?" but a more precise answer is needed. If teaching is to be based on an understanding of students' needs, then that understanding must include an idea of which needs are most important, and what circumstances occasion those needs. Developing this understanding is the goal of the studies I report in the next three chapters. Before considering those studies, however, let me recall two basic assumptions on which my research is based.

First, I assume that what people learn is based on what they already know. In fact, to make this statement a bit stronger, I assume that what we *can* learn is based on what we already know. This assumption is at the base of constructivist learning theory, and Enactivism, a theory of learning I will be describing at length in Chapter VII. A consequence of this assumption for education is that teaching ought to be based on what students know. In the context of teaching mathematical proof, this means that the proving we would like students to do should be based on the proving they already do, not developed as a disconnected skill unrelated to any other way of thinking. This assumption also has a consequence for my research, in that I concentrate on the proving in which students engage without any instruction from me. The studies I conducted occasionally touched on the possible effects of teacher interventions on students' proving, but the main focus was the reasoning students were inclined to do based on whatever previous experiences they had.

Second, I assume that people reason in different ways in different contexts and that seemingly similar contexts can, and often do, turn out to be different enough to give rise to different reasoning. For this reason I am not interested in searching for *the* context in which students' prove. I am interested in exploring possibilities, not generalities. This interest had consequences for the way I conducted my studies. It occasioned the use of open problem situations, in which deductive reasoning could be used, but was not required. It also occasioned the involvement of a wide range of students, both in age and ability.

Before describing the studies, and my results, a brief note on the contents of the next few chapters is in order. It is usual in dissertations to describe in detail the design of the studies undertaken before presenting the results of the studies. This is sound practice if an important feature of the research being conducted is reproducibility. For a legitimate attempt at reproducing a study to be made, it is necessary that the design of the study be clearly understood.

In the case of my studies the situation is a bit different. My conclusions are of two kinds: observations and speculations. My observations consist of claims that one or more of the participants in my studies reasoned in a particular way. Such observations are clearly not reproducible, as the context can never be replicated. To do so would require a replica of the participant involved, and not even the original participants continue to encompass all the aspects of who they were at the time. My observations show what is possible, but do not allow predictions of what will happen. All the same, suggesting ways to improve teaching must be based on some sort of reasons, and in many cases these reasons take the form of predictions of the wonderful things that will result if such-and-such a reform is introduced.

This is where my speculations come in. Based on both the observations I have made, and philosophical considerations, I indulge in some speculations as to ways the teaching of proving could be reformed. I do not, however, claim that these speculations are based on reproducible evidence. In fact, I would suggest that the best way to test my conclusions as they apply to teaching is not to attempt to replicate my results (I would be far more interested in studies that expanded the bounds of the possible by observing proving in situations I failed to investigate), but rather to attempt to implement my speculations in practice. It could be suggested that experimenting with new methods of teaching, without having "scientific" evidence of their effectiveness beforehand, is irresponsibility on the part on an educator. In response I would note that the results of research on students' understanding of proving in mathematics (e.g., Bell, 1976; Braconne & Dionne, 1987; Fischbein, 1982; de Villiers, 1992; Senk, 1985) indicates that current methods are so unsuccessful that it is difficult to imagine students suffering much under a change. I would also note that it may be impossible to acquire "scientific" evidence in some contexts (I expand on this idea in Chapter VI) and so limiting educational reform to those based on such evidence might permanently cripple our educational systems.

With the above comments in mind, I have left the details of the design of my studies to an appendix (Appendix B). Below I will give an introduction to the studies, and in the following chapters I will describe aspects of the studies as they become needed. In doing so I will concentrate on the particular, in keeping with overall focus of my research. Those who would prefer an overview should consult Appendix B now, and Appendix D when a summary of results is desired.

In Chapter VII I make some important connections between how I did my research and the theoretical basis for my thinking. It could be suggested that an understanding of my methodology and the theoretical basis of it is important to the reading of the results I have included in the next three chapters. This may be, but it is also the case that the contents of the next three chapters are important to the reading of my final chapter. Enactivism, the topic of Chapter VII, is not a theory or a methodology in the abstract. It must be *about* something. In Chapter VII it is about my research into proving. The reading of the next three chapters may raise important questions about theory and methodology, the existence of which will give the answers given to them in Chapter VII a context and motivation.

Although there is a great deal of research in mathematics education on the types of proofs students make and accept, on the teaching of proving, and on students' acceptance of proofs as absolute verification, the only studies that have concentrated on students' need to prove have been those of Bell (1976), Hanna (1989), and de Villiers (1990, 1991). None of these observed students' proving in contexts in which proving was possible but not required. As it seems to me that it is only in such situations that students' needs to prove will become apparent, I have attempted in a series of research studies to observe students in such contexts.

The students involved in my studies were volunteers from high school and university classes. They represent a wide range of mathematical backgrounds, from students with undergraduate degrees in mathematics, studying to become teachers, to students in the non-academic stream of grade 10. In addition to taking part in problem solving sessions, high school students were also observed in their regular classrooms, to get a sense of what their typical experience of proving in mathematics is like.

The studies included observations of three high school classes, observations of students in problem solving situations, and interviews with those students. Three problem situations were investigated: the Arithmagon, the Fibonacci sequence, and GEOworld. Prompts used for the Arithmagon problem and the Fibonacci situation are given in Figures 2 and 3. GEOworld is a computer microworld, that will be described when students' activities in that situation are discussed. These problems were selected based on their having occasioned proving in previous studies (e.g., Reid, 1992) and pilot studies. It might be helpful for the reader to investigate these problems before continuing, in order to have a better feel for the students' reasoning*.

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.

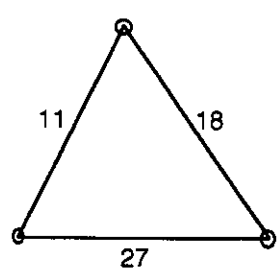


Figure 2: The Arithmagon prompt.

* "Answers" appear at the back of the book, in Appendix E.

The Fibonacci sequence begins:

$$1, 1, 2, \ldots$$

and continues according to the rule that each term is the sum of the previous two (e.g., 1+1=2).

The Fibonacci sequence has many interesting properties.

Can you find an interesting property of every third Fibonacci number?

Can you find other interesting properties?

Figure 3: The Fibonacci prompt.

After investigating the problems, in separate sessions, the students were interviewed.

As I mentioned above, the next three chapters elaborate on the studies I conducted into students' proving and need to prove, and the results of these studies. Most of the examples in these chapters are transcribed from video tapes of the problem sessions and interviews. A few examples are taken from the observations of high school mathematics classes. Introducing all the students I observed at once could be confusing, and is unnecessary, as my analysis concentrated on the particular rather than the general. I have taken most of my examples from the observations of three pairs of university students, two pairs of high school students at North School, and Sandy, a male grade six student. The university students included one male-male pair (Ben and Wayne), one female-female pair (Eleanor and Rachel), and one female-male pair (Stacey and Kerry). Both pairs of high school students are male. Bill and John were in grade 10. Colin and Anton were in grade 12. Descriptions of the students and their mathematical backgrounds is given in Appendix B.

CHAPTER II

NEEDS AND PROVING

Come, my Celia, let us prove
— Ben Jonson, *Volpone*, III.v

One object and result of my research into the need to prove is a vocabulary for describing needs, and various ways of proving. The needs suggested in the literature in mathematics education have been described in the previous chapter. They included: verification, explanation, exploration, systematization, communication, aesthetics, personal self-realization, developing logical thinking, and "teacher-games." Not all of these needs seemed to be felt by the students involved in my studies. In this chapter I introduce the needs I did identify in my studies: explaining, exploring, verifying, and teacher-games. After a short introduction each of these needs will be illustrated with examples drawn from the studies, which show the various ways in which proving and other kinds of reasoning were used to satisfy these needs.

Exploration extends the bounds of what is known. Questions such as "How can I find the measure of this angle? and "I wonder what happens if I add the sides" indicate a need to explore. In the case of the first question it is a need to explore with a goal in mind. In the case of the second question the exploration does not have a goal. De Villiers (1990) suggests that the need to explore motivates proving.

The following example of exploring is taken from the work done by two university students, Stacey and Kerry, while investigating the Arithmagon situation. It is episode 2 in the mathematical activity trace in Appendix C* and also appears as part of the case study in Chapter III. Stacey and Kerry solved the original puzzle by reducing a system of equations. In this episode Stacey explores the situation further.

(1) Stacey: What happens if you add the middle numbers together? —

(2) Kerry: Well I guess we could, hmm.

(3) Stacey: I just want to try something. If you take 27, 18, and 11. 2, 4, 5, 56. Right?

(4) Kerry: Sure.

(5) Stacey: And you have — So you add each of those twice, right? — Yeah you do. That's not going to help you either. That's what you end up doing right?

---

* Mathematical activity traces are summaries of the activities of the participants in a session. They were created for some of the problem sessions and interviews as part of the analysis of the studies. They also provide a context for the episodes I will be referring to, and I will make reference to them when describing episodes for which a MAT exists. Appendix C is divided into two sections. The first included MATs for the students from North school. The second includes MATs for the university students. Within these sections the MATs are grouped according to which pair of students was involved. The Table of Contents lists the exact page number for each MAT in Appendix C

(6) Kerry: What'd you do?

(7) Stacey: You add A, B, C. Then you multiply them by 2. You get this answer. —

As I reconstruct them, Stacey's thoughts ran as follows: "What happens if you add the middle numbers together? 27+18+11 is 56. And how is that related to the numbers on the corners? Each corner number is added into two of the middle numbers, so in the total of the middle numbers you add each corner number twice. Therefore the sum of the middle numbers is two times the sum of the corner numbers. But just knowing the sum of the corner numbers doesn't help us figure them out."

Stacey's initial comment "What happens if you add the middle numbers together?" indicates an exploratory frame of mind, as does her making observations without a particular goal. That she was not expecting the conclusion she reached is indicated by her dissatisfaction with it: "That's not going to help you either."

Verifying involves the determination of the truth or falsity of a statement whose truth value is in doubt. A question like "Is the sum of the sides always even?" indicates a need to verify. Many researchers, including Bell (1976) have identified verifying as a source of a need to prove.

The following example continues from the transcript of Stacey and Kerry investigating the Arithmagon situation, given above.

(8) Kerry: Do you add?

(9) Stacey: 22, and 34. Yup. Do you know what I mean?

(10) Kerry: Sorry. So you add this and multiply by 2 so, like, the sum of this is 28 times 2. And it's 56. Good one. What's that mean?

(11) Stacey: Nothing. [laughing]

(12) Kerry: Is that-

(13) Stacey: That was just-

(14) Kerry: Is that true for all of them?

(15) Stacey: Yeah.

(16) Kerry: I guess so. It must be. It can't just be fluke.

When Kerry asked, "Is that true for all of them?" Stacey could respond that it was, because the exploring she had done also verified that the relationship is generally true for Arithmagons. As Kerry did not see how Stacey had arrived at her conclusion, his verification of it was based on the low probability of such a relationship occurring by chance in this case. Both of them verified, but in different ways.

Explaining provides connections between what is known in a way that clarifies *why* a statement is true. A question like "Why is the sum of the sides always even?" expresses a need to explain. Bell (1976) and de Villiers (1990) suggest explaining as a need to prove.

The following example, taken from a lesson observed at Central High School, illustrates explaining in a mathematical context.

The teacher, Mr. C, presented his grade 12 class with the problem of showing that $\binom{8}{3}=\binom{7}{3}+\binom{7}{2}$. He asked, "Can someone explain why the number of ways of selecting 3 from 8 is the same as $\binom{7}{3}+\binom{7}{2}$?" The students gave no response, and Mr. C explained. He asked them to consider the particular problem of choosing three people from a committee of eight. Assuming the role of one member of the committee, he reasoned that the number of ways of choosing the three people would be the sum of the number of ways three people could be chosen without including him, and the number of ways the three could be chosen if he were among those chosen. The number of ways the three could be chosen to exclude Mr. C is $\binom{7}{3}$ and the number of ways they could be chosen to include Mr. C is $\binom{7}{2}$. As the total number of ways of choosing three people from a group of eight is known to be $\binom{8}{3}$ the equality $\binom{8}{3}=\binom{7}{3}+\binom{7}{2}$ is explained by this argument.

The *need to explain* provided the motivation for the proving Mr. C did for his class. This is indicated by his question "Can someone explain why...", and by his role as a teacher with a responsibility to explain. The possibility that verifying might have been a motivation is unlikely, considering that verification of this particular sum could have been achieved much more quickly using arithmetic techniques.

A *teacher-game* is a situation in which students act in a particular way in order to satisfy the implicit or explicit demands of a teacher. Playing a teacher game can be based on attempting to achieve a high grade, or simply facilitating the smooth running of the class, to avoid social discomfort. Alibert (1988) and Schoenfeld (1983) point to the importance in schools of conforming to the expectations of the teacher, playing a 'teacher-game', as a motivation to engage in proving.

As an example consider the continuation of the lesson taught by Mr. C. After explaining that $\binom{8}{3}=\binom{7}{3}+\binom{7}{2}$ Mr. C assigned the students the task of proving algebraically $\binom{n}{r}=\binom{n-1}{r}+\binom{n-1}{r-1}$. As they were proving I asked several students why they were doing so. Six of the seven students I asked indicated they were proving only because it was an assigned task (the seventh said it was for fun). In this example it was a *teacher-game* that created the need to prove. Other possible needs had been satisfied by Mr. C's explanation and authority as a teacher.

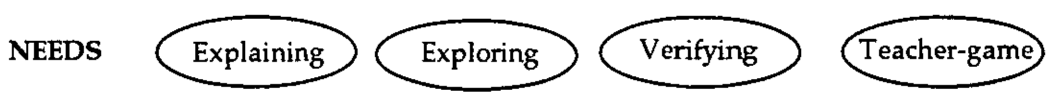


Figure 4: First rank of the proving network.

These four needs form the first rank of a network of paths relating aspects of proving (see Figure 4). As each of these needs is discussed, the relevant part of the network will be expanded. Section 1 elaborates and gives examples of proving to explain, and of the use of reasoning by analogy to explain. A distinction is made between proving that is unformulated, of which the prover is unaware, and formulated proving. Section 2 concerns exploring, and the use of proving, inductive reasoning, and reasoning by analogy to explore. Proving by mechanical deduction, using a deductive tool like algebra, is described in this section. Section 3 includes examples and explanations of the use of proving and inductive reasoning to verify. Section 4 describes proving that takes place as part of a teacher-game. The chapter concludes with a summary of needs and the proving associated with them.

1. Explaining

Explaining provided a definite need to prove for the participants in my studies, but proving was not the only sort of reasoning motivated by a need to explain. Reasoning by analogy was also used to explain and was preferred to proving in some cases. In addition, I observed significant differences in the formulation of their proving. In Figure 5 the paths involving explaining in the network are shown. Those involving proving are shown by thick lines. The three remaining stages of the network appear for the first time in this figure. The second stage includes different types of reasoning. The third distinguishes proving by the degree of formulation involved in it. The fourth stage differentiates between proofs, on the basis of the formality of their presentation. In the following examples and discussion of these paths the distinction between "formulated" and "unformulated" proving will be clarified. Pre-formal proofs, described above in Chapter I, section 1, will be elaborated upon, and illustrated with examples. At the end of this section examples of explaining by analogy will be discussed and

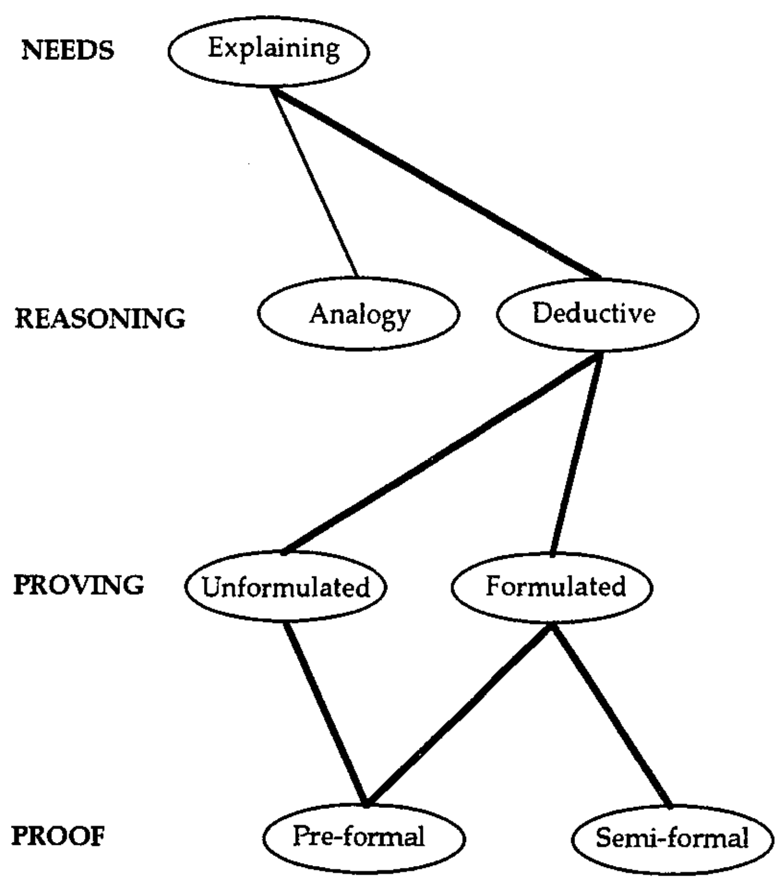


Figure 5: Paths related to explaining.

contrasted with explaining by proving. In the following section the network will be extended, as new nodes in each stage are discussed.

Explaining — Formulated proving — Preformal proof

Perhaps the most important distinction suggested by my research is that between formulated and unformulated proving. *Formulation* refers to the provers' knowledge or awareness that they are proving. It could also be described as the degree to which the proving is thought-of and thought-out. Formulation is related to two other characteristics of proving: its articulation and the hidden assumptions made while proving. Articulation and hidden assumptions provide valuable clues to formulation, in addition to being important characteristics of proving in and of themselves.

The extent and clarity of the spoken or written articulation of proving has implications for the possibility of the proving being interpreted by others and for the formulation of the proving. Being aware of one's own proving and being able to articulate that proving are interrelated. Articulating proving assists in formulating since articulation makes aspects of proving tangible. At the same time, formulated proving is more easily articulated.

All proving involves some hidden assumptions. These assumptions can range from wrong assumptions through implausible and plausible assumptions to assumptions that are known within a community. The formulation of proving reveals hidden assumptions, making the presence of wrong or implausible assumptions less likely.

Explaining using formulated proving can be quite successful; however, it seems to require a suitable social context. Rachel and Eleanor, two undergraduate students, provide an example of the use of formulated proving to explain. Their case, and those of Ben (an undergraduate student), Colin and Anton (two grade 12 students), and Bill (a grade 10 student), illustrate the different social contexts in which formulated proving to explain seems to occur.

Rachel and Eleanor explain the formulae found in the Arithmagon situation

Rachel and Eleanor developed two different methods for solving the Arithmagon. Rachel derived the formula $x = \frac{b+c-a}{2}$ algebraically from the relations between the three known values, *a*, *b*, and *c*. Eleanor's method was based on two observations: that the sum of each side and the corner opposite it is a constant for a given triangle, and that the sum of the corners is half the sum of the sides, and is the same constant. Her method was to find the sum of the sides, divide by 2, and subtract the side opposite the corner she wished to discover.

At the end of the session (MAT episodes E22, R14) Eleanor and Rachel began to wonder why the relations Eleanor's method is based on work, and how their two methods are connected. They derived the relation $a + b + c = 2(x + y + z)$ independently (see Figure 6 for the diagrams corresponding to Eleanor and Rachel's equations). Eleanor derived it from the given relations between the sides *a*, *b*, and *c* and the unknown corners. Rachel derived it from her formula $x = \frac{b+c-a}{2}$ (see Figures 7 and 8).

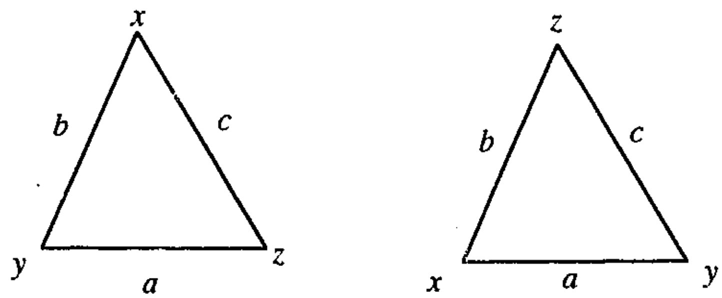


Figure 6: Diagrams corresponding to Rachel's (left) and Eleanor's (right) equations.

$$x + y = a$$

$$y + z = c$$

$$\underline{x + z = b}$$

$$2x + 2y + 2z = a + b + c$$

$$2(x + y + z) = a + b + c$$

$$y + b = y + (x + z) = \frac{1}{2}(a + b + c)$$

Figure 7: Eleanor's proving to explain her method.

Eleanor then asked: "How come this plus this — adds up to the sum of all these?" In other words, why is it that the relation ($y + b = y + x + z$), which relates the sum of a side with the corner opposite to it to the sum of the sides, holds? She quickly derived it from the relation $b = x + z$. Meanwhile, Rachel derived her formula from a formulation of Eleanor's method.

$$\frac{(a + \cancel{c} - \cancel{b}) + (\cancel{a} + \cancel{b} - \cancel{c}) + (c + b - \cancel{a})}{2} = x + y + z$$

$$\frac{a + b + c}{2} - c = \frac{a + b + c - 2c}{2} = \frac{a + b - c}{2} = y$$

Figure 8: Rachel's proving to explain her method.

Eleanor and Rachel did not begin suddenly to engage in formulated proving. Rachel had begun early in the session, as a way of exploring (see section 2). Eleanor had been working with either Ben and Wayne, with whom she worked inductively, or Rachel, with whom she engaged in formulated proving. Eleanor's sensitivity to the reasoning of those with whom she was communicating illustrates one characteristic of a social context for formulated proving. If others are communicating by way of formulated proving, then this activity might be picked up. This point is elaborated further in section 2.

Ben's formulated proving to explain to others

When Rachel, Eleanor, Ben, and Wayne investigated the Arithmagon situation, the first person to solve the original puzzle was Ben. When Eleanor asked how he found the solution so quickly his first response was "Don't ask me that! I don't know. I just saw it right away." A short while later, he attempted to explain his method, which he seems to have reconstructed deductively (MAT episode 7). Unlike his attempt to explain using unformulated proving (described below), this attempt was formulated, and much more successful as an explanation.

(1) Eleanor: But you saw that right away.

(2) Ben: Yeah.

(3) Eleanor: Why don't you try another one and see if -

(4) Ben: It'd have to be do-able though — I don't know. I kind of looked at 27. I don't know what I did actually. No idea how I got that. — Well I knew, well OK, I kind of knew how I did it. The number, the number between. You know how I did that? The number here [*C*] had to be less than 27, and less, it had to be less than 18, the number here, right, — had to be less than 18. And the number here [*B*] had to be less than 11,— right?

(5) Wayne: Yeah, otherwise they'd add to more than 27

(6) Ben: So then the number here [*A*] had to be less than 18 and less than 11. So, I mean, I just said — This 27 right and this 18 since this number is being added, right, that's one of the adding factors.

(7) Eleanor: You mean that this number here?

(8) Ben: Well no, I don't know what I looked at first, but I looked at, I noticed that, it [*C*] has to be lower than 27 and 18, to be added to each other, right?

(9) Eleanor: Yeah

(10) Ben: And I noticed that 11 and 18 had to be a number less than 11 and 18. And I noticed that it had, the third number, 10, had to be less than 11 and 27.

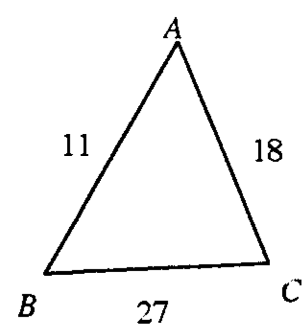


Figure 9: Labeling of Arithmagon for description of Ben's explanation.

In line 4 Ben observes that *C* must be less than both 27 and 18 (see Figure 9 for labeling of corners). This is true only if the secret numbers are assumed to be whole numbers. Both Ben and the others seem to have made this assumption at first. He goes on in lines 4 and 6 to specify the constraints on *A* and *B*. In line 5

Wayne explicitly gives the reason for the constraints. Ben repeats this reason in line 8.

All of the things Ben noticed "had to be" involve making deductions from an assumption that the solution is limited to whole numbers. This allows limits to be placed on the potential values of the unknowns. Once he had completed his explanation, Eleanor showed she had understood it by attempting to use his method to solve another puzzle.

In the previous two cases it was the questions of other participants that provided the social context for the use of formulated proving to explain. In the next three cases it is the interviewer who sets up the occasion for proving to explain.

<u>Colin and Anton explain the origin of their Arithmagon formula</u>

Colin and Anton were two grade 12 students in an academic stream mathematics course. After two problem sessions, in which they worked on the Arithmagon and the Fibonacci situations I interviewed them twice. In the first interview session, after Colin and Anton had derived and verified their formula for the Arithmagon, I asked them why their formula works (MAT episode 3). While Anton referred to the examples they had tested it with, Colin proposed "proving" it:

(1) Anton:	This is the relationship between the corners and their sides.
(2) Colin:	Right, but how come that works? — They have to be related because-
(3) Anton:	Because 11
(4) Colin:	Yeah. The one number, defines them both. In this corner.
(5) DR:	How do you mean it defines them?
(6) Colin:	Well, See, if this was 3 and 13, whatever this number [*Y*] is, is going to affect both of these numbers. [See Figure 10]
(7) DR:	Okay
(8) Colin:	So.
(9) Anton:	'Cause if this is 10 more, then this has to be 10 more, in order to get the same numbers, you know. So we have this main point right here. So, if this is more, more, this has to be 10 more, in order to get,
(10) Colin:	'Cause this number will be the same. So it could be 12, and still this would be 10 apart. So that's why those two are 10 apart, and that's where we got this first formula. And then this one-

After Anton identified one of their equations as the difference relation they had found ($Z - X = x - z$, line 1), Colin asked why it worked (line 2). They then made the connection with the common corner Y, which in this case is 11 (lines 3-4), which requires that any difference in the sides ($(13+Y) - (3+Y)$) be due to the difference between the other two corners (13–3) (lines 9-10).

After explaining the relation generally, "The one number, defines them both," Colin repeated the explanation using specific values. This use of general numbers in proving is less formulated than his general explanation but has greater explanatory power. The use of numbers makes his explanation clearer. He might instead have rephrased his general explanation into something like "Each side is the sum of this corner number and another number, any difference must come from the other number, as this number is shared by both sums." In conversation, because of its rapid pace, it is more important that an explanation come quickly and clearly than that it be perfectly formulated. This seems to have been behind Colin's switch from a general statement to the use of general numbers. Anton's less formulated way of expressing himself may also have had some influence.

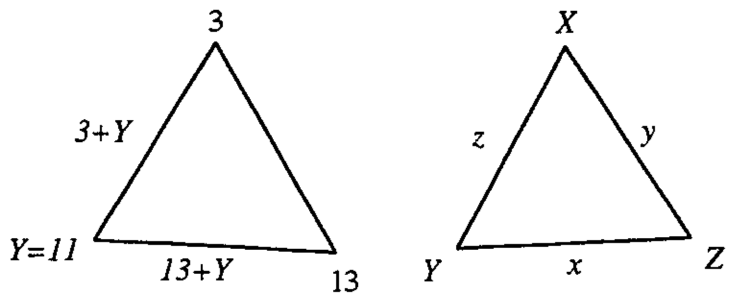


Figure 10: Labeling of triangles for description of Colin and Anton's explanation.

Colin and Anton's formulated explanation that $n^3 - n$ is a multiple of 6

In the second interview, Colin and Anton were guided through a formulated proving that $n^3 - n$ is a multiple of 6. They made this conjecture after examining a number of specific values for the expression. I began by asking them if they would prefer to explore, explain, or verify this conjecture. They indicated that they were more concerned with explaining it. By asking guiding questions, like "What do you know about $(n-1)(n)(n+1)$?" I guided them through a deductive argument. After we had completed the derivation verbally, I asked if it explained their conjecture. They said it did. I then asked them to write out the argument, and Colin did so (see Figure 11, MAT episode 6). Colin omits the last step of stating that the product of an even number and a multiple of 3 must be a multiple of 6.

Colin's argument, and the verbal argument that preceded it, occurred in a particular social context, that of a teacher guiding students through an argument. This context encouraged the reasoning to remain formulated and deductive. An interesting aspect of this particular case is that when I asked Colin and Anton if they had discovered anything through the reasoning used to explain their conjecture they said no even though they had remarked on the discovery that $n^3 - n$ is the product of three consecutive numbers, with n as the middle number. In this case, the need to explain seems to have kept Colin and Anton from seeing that a need to explore could also be addressed by the same reasoning. One need to prove might interfere with the satisfaction of another.

n^3 - n

n (n+1) (n-1) ——— means multiplying 3 consecutive #'s

One of the 3 #'s must be even, another must be a multiple of 3. Even because when you choose 3 consecutive #'s 1 must be even.

odd, even, odd (7, 8, 9)

even, odd, even (4, 5, 6)

Multiple of 3 because when choosing 3 consecutive #'s one is divisible by 3.

if n = 3x then

if n = 3x-2 then n-1 = 3x-3 which is a multiple

if n = 3x-1 then n+1 = 3x

Figure 11: Colin's written proof from the second interview.

Bill and John's explanations by formulated proving

Bill and John were two grade 10 students at North School. In their first interview session they were guided through the derivation of a formula to solve Arithmagons. Bill followed this derivation, but John had trouble with it. In the derivation of the formula *A*, *B*, and *C* were used to represent the unknown corner numbers. The formula at which we arrived was $\frac{(A+C)-(A+B)+(B+C)}{2}$. It is fairly easy to show that this formula simplifies to *C*, by rearranging the terms: $\frac{(A+B)-(A+B)+(C+C)}{2}$. Bill used this simplification to show the equivalence twice. John suggested that the formula should be rewritten using variables to stand for the known numbers on the sides: *D*, *E*, and *F*. His formula was: $\frac{E-D+F}{2}$. Figure 12 shows the labeling they used. In the following transcript they compare the two formulae (MAT episode 14).

(1) John: Plus B plus C, which would be F. So in other words, E minus D plus F.

(2) Bill: Yeah. That's an easy way to think of it.

(3) John: So, 63. So we just look. We know E is 63. D is 18. And F is 3. So it'll make it much easier to work with. I guess.

(4) Bill: Yeah.

(5) John: Then we can just go from there. We know that's divided by 2. So.

(6) Bill: But, You, you are aware of why it is divided by 2, right? The ... reason this, this would make it kind of easier is 'cause you would know how much is left behind. You would see that the A's cancel each other out. The B cancels each other out. You would know that

you would have 2C. With this you wouldn't really know that you had to divide it by 2. With this you would.

(7) John: That's true. — OK. So then would it be

(8) Bill: But, uh. Once ... you already knew that you had to divide by 2, some brilliant genius ... could go like E take way D plus F and then have it divided by 2 and that would be the whole formula. That's how they would word it. From the start. But they won't know why it works. But they would know it does. This shows why it works. That's, that's all I can say. But yeah, yours is pretty good.

(9) John: OK. So it just works. This is why it works.

(10) Bill: Yeah.

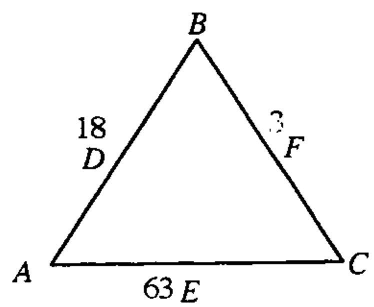


Figure 12: Labeling used by Bill and John for their formulae.

The contrast they make is between John's "much easier to work with" formula, and Bill's, which "shows why it works." Bill's formula makes it simple to prove that the calculation must end with the division by 2. This association with the formulated proving Bill had done makes it more explanatory. Bill acknowledges that John's formula is easier to use, but he feels that given that formula, "they won't know why it works." This case is similar to the previous case involving Colin and Anton, in that the guided, formulated proving the students engaged in satisfied a need to explain. The need to explain by itself was not enough to motivate the proving, but the addition of a social context that encouraged it occasioned activity that satisfied that need to explain.

In my second interview with them I guided Bill and John through two attempts to prove that the sum of two odd numbers is even. The second attempt was fairly successful, and after they had proven even+even=even and even+odd=odd with my guidance, they were able to prove odd+odd=even independently. After this proving concluded, I asked them if the proof explained why the sum of two odd numbers is even (MAT episode 18).

(1) Bill: Um. Yeah. It does explain it, 'cause, ... this would be the same as, ... an even plus an even. ... In the end result, which were given, even, 'cause we found that out here, already. ... This is an even plus an odd, which gives you an odd 'cause, well we found that out. ... 'Cause it's plus 1.

(2) DR: Umhmm.

(3) Bill: It's an even number plus 1 that's why it's an odd. Um, yeah. That explains it.

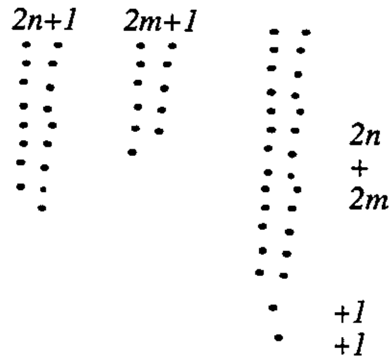


Figure 13: Representation of addition of odd numbers used in Bill's proving.

Here Bill is summarizing the formulated proving they had done. They represented an odd number both as a column of pairs of dots, with a single dot added, and with the expressions $2n + 1$ and $2m + 1$ (see Figure 13). In line 1 he is saying that $2n + 2m$ is even because it is "an even plus an even" which they know is even " 'cause we found that out here, already." That even number $(2n+2m)$ plus 1 "is an even plus an odd, which gives you an odd " 'cause, well we found that out." Bill's summary is a bit different from what they actually did since in their original proving they considered $(2n+2m) + 2$, which is even, according to they rule they proved earlier. His addition of 1 may have been a step in a derivation involving the principle that one more than a odd number is even, or he may have been making connections with both of the rules they knew as part of an effort to systematize their knowledge. In any case, I interrupted his summary at this point. What is clear is that he considered their formulated proving to be an explanation of the odd+odd=even rule.

Explaining—Unformulated Proving

As an explanation for others, unformulated proving is inadequate. The lack of articulation and hidden assumptions that come with unformulated proving prevents other people from being able to understand it. For an individual, however, unformulated proving can be used to provide a personal explanation. This is successful when the deductive chain is not too long. Of the cases that follow, Kerry's concerns a short, unformulated, deduction that explained. Other such episodes probably occurred in the studies, but the very fact that they are short and involve unformulated proving makes it difficult to be sure of their nature, and especially to be sure that they involve a need to explain. The other two cases of explaining using unformulated proving described here illustrate how such attempts can fail when the deductive chain is too long. The first involves Ben, an undergraduate student. The second involves Bill, a grade 10 student.

Kerry's short explanation

When working in the Fibonacci situation Kerry noticed a pattern in the sums of sequences of three consecutive Fibonacci numbers (MAT episode 6, see Appendix C). He noted that the sum was always twice the last number. For example, he saw that 55+89+144=288. He wondered to himself why this should be so and quickly observed that as the rule defining the Fibonacci sequence told him that 55+89 would be the next Fibonacci number, 144, the sum of three consecutive Fibonacci numbers would always be the same as adding the last of the three to itself.

Kerry's explaining here is barely articulated, but his assumptions and reasoning are fairly clear. The brevity of his proving contributes to its lack of formulation. A quick deduction like this one does not require any formulation to be continued. Longer chains of deduction become increasingly difficult without formulation.

Ben's unformulated explaining to others

A group of four undergraduate students, Wayne, Ben, Eleanor, and Rachel, worked together in the Arithmagon situation. Rachel and Eleanor solved the initial puzzle using systems of equations. Ben gave the correct answer almost immediately, and later reconstructed a solution method based on selective guessing guided by the given numbers. At the time this transcript begins (MAT episode 8), I offered him a challenge. My description of the 1-4-12 triangle as "simple" is ironic, as I chose it precisely because the solution includes negative numbers and fractions.

(1) DR: Here's a simple one: Can you do 1, 4, 12?

(2) Ben: 1, 4, 12, like that?

(3) DR: Well, they're on the sides, because you're supposed to be figuring out -

(4) Ben: On the sides, 1, 4, 12. Well that's 0 or 1. One of them has to be 0 — No, That's impossible — Because, I mean if this one is 0, that one has to be 1, that one has to be 3, this is adds up to 3. If this one is 0, this one has to be 4 and that one has to be 1.

(5) Wayne: Who said it's got to be 0 though?

(6) Ben: Well, Yeah — It still shouldn't matter — if you go down on the number line you still have to go up on the number line

***

(7) Eleanor: 4 and a negative 3

(8) Wayne: Umhmm

(9) Ben: The difference is still a 1 —

(10) Eleanor: but this doesn't have to be 0

(11) Ben: But even if it is, like let's say negative 4 and negative 3, right? You still have to get this to be 4 it has to be 7, all right? It's still minus. So it will still be like, 3. — You know where I'm coming from?

(12) Eleanor: Say it again

(13) Ben: The difference, the difference between these two is still always going to be 1, right? No matter if you represent it with negative or adding.

Ben's first explanation (line 4), is a fairly well formulated deduction from the principles he had been using to solve other puzzles. He explains why the triangle is impossible by reasoning deductively from the implicit assumption that the secret numbers are natural numbers. When Wayne questions his hidden assumption, Ben immediately offers further explanation (line 6). Note that this explanation has a different character from the ones he has offered before (in line 4). It is much less articulated, making it difficult to judge how aware Ben was of his reasoning. His language suggests that his proving is based on an image of the relationship between the values. These features lead me to characterize this explanation as unformulated proving.

Eleanor's request to "say it again" (in line 12) marks the failure of Ben's unformulated proving to explain to her. Ben has based his argument (in line 11) on a hidden assumption, which in this case is wrong. He seems to believe that the difference between the two secret numbers is 1. This is true in the case where one of them is zero, which he had just been considering. The two numbers he names, ⁻4 and ⁻3, have a difference of 1, and these numbers do not work. In fact, if the difference must be 1, there is no way that a difference as large as (12–4) could occur. This provides the basis for Ben's belief that the puzzle can not be solved.

Bill's unformulated explanation that F_{3n} is even

When investigating the Fibonacci situation, Bill and John identified the recursive rule defining the sequence, and found patterns in F_{3n} and F_{4n} inductively. The pattern they saw for F_{3n} was that all such Fibonacci numbers are even. John accepted that this was generally true, based on inductive evidence, although Bill did not. Bill was unusually resistant to accepting inductive verifications. At the end of the session I asked Bill if he could see any reason why every third Fibonacci number is even (MAT episode 16.2).

(1) Bill: Um. Why? Um. When you add two odd numbers it goes into an even. That's one theory. I don't know. If you. If you add two odds you'd get an even, wouldn't you?

(2) DR: Yeah.

(3) Bill: So, let me see. It starts with, uh, with uh one. One, one plus one gives you two.

(4) DR: Umhmm.

(5) Bill: And then this it would give you an odd. And then since, uh, then that's. Oh, wait wait wait wait wait wait. — What was I seeing? [laughs] — —Every third, third number. Oh yeah, Oh no. I'm thinking, right. To get the third number you would add the two odds. Since this is the third number, this would be the next third, third number. There's two odds, you get an even. This is the third number. There's two odds, you get an even.

(6) DR: Would it work- It wouldn't work if there wasn't two odds?

(7) Bill: It wouldn't work if there wasn't two odds. But since there's two odds—you can make an even. —

Bill identified part of the deductive argument that shows that every third Fibonacci number is even. The argument involves three steps. The first is to observe that every third Fibonacci number is even because it is preceded by two odd numbers. The second step is to see that this odd-odd-even pattern must recur since it forces the two numbers after an even Fibonacci number to be odd. The final step is to generalize this recurring pattern to the entire sequence. Bill saw the first step, and in line 5 he may have briefly seen the second step. Then he became confused, and soon after he reverted to inductive reasoning to establish that there is always a pair of odd numbers before F_{3n}.

Explaining—Analogy

Reasoning by analogy to explain occurred only a few times in the studies, but those occasions indicated that analogy can be a powerful method of explaining in mathematical situations. The analogies the participants offered can be described as 'weak' or 'strong' analogies. Their acceptance as explanations was related to this strength. The two cases I will describe in detail involve Bill, and Rachel, Ben, and Wayne. In Bill's case his analogy is very strong. Rachel, Ben and Wayne each offered explanations of which Rachel's was deductive, Ben's was a strong analogy, and Wayne's was a weak analogy.

Bill's explanation by analogy

In my second interview with Bill and John, the grade 10 students at North High School, we examined a question that had come up in the first interview session: "Why is the sum of two odd numbers even?"

When the question was first asked (MAT episode 9) Bill gave an explanation by analogy, that he had hinted at in the first interview session. His analogy relates even and odd numbers to positive and negative integers. The rule 'An odd number plus an odd number is an even number' corresponds to the rule 'A negative number times a negative number is a positive number.' The sums of even numbers and odd numbers are related in a similar way to products of integers.

This analogy is actually quite strong. The integers can be divided in half in various ways, two of which are the division into even and odd, and the division into positive and negative. In both cases there are ethical connotations attached to the words used to describe the two halves, which makes one half "better" than the other. For example, both 'even' (as in 'even-handed') and 'positive' have good connotations. As well, in both cases there is a binary operation that combines two like numbers into a number in the "better" half, and that combines two unlike numbers into a number in the other half. All of these features mark links between the two domains of the analogy. The number of links, and the degree of match between the features they link, is a measure of the strength of the analogy. I am not asserting that Bill would have been able to explicate these links himself, only that they contribute to the strength of his analogy.

After Bill offered his analogy, I led Bill and John through a deductive exploration of the question, using a syncopated algebra to deduce the rules. This was accepted, but not with enthusiasm. I then proposed a pictorial model of the sum of two even numbers and a formal notation related to it. In this context Bill and John were able to prove the odd+even=odd principle with some assistance, and the odd+odd=even principle independently. I then asked Bill and John if what they had done explained the principle odd+odd=even. Bill said it did, and repeated the gist of the argument in order to illustrate how it explained the principle.

About ten minutes later Bill made an interesting comment (MAT episode 20.3). He stated that he did not like proofs, and that he had no interest in explanations of mathematical principles. This statement contradicts comments he made in the first interview session, in which he expressed a preference for his Arithmagon formula and derivation on the basis that they allowed him to see why the division by 2 was needed. This change of heart may reflect the differing circumstances in the two situations. In the first interview session, the proving through which I guided Bill explained his new formula, which was otherwise unexplained. In the second interview, the statements they proved were already known and accepted, and Bill had already offered his analogy as an explanation for them. If Bill judged his analogy to be a better explanation than the proof, then the proving in this case could be seen as superfluous. As his analogy is a strong one, and he showed enthusiasm for it, I suggest that he did choose it over the deductive explanation offered by the proof.

A strong analogy can indicate the possibility of a deductive link although in the case of Bill this is not so. In the following case of Rachel, Ben and Wayne an analogy that does indicate a deductive link will be described and strong and weak analogies will be contrasted.

Rachel, Ben and Wayne attempt to explain "Why 2?"

The following episode (MAT episodes B19.3, W17.3, E18, R12.1) took place toward the end of a problem session in the first clinical study, in which four undergraduates (Ben, Wayne, Eleanor and Rachel) were working in the Arithmagon situation. It shows both strong and weak analogies, and explaining by proving. Rachel had discovered a formula for determining the value at a vertex x. It is:

$$x = \frac{b + c - a}{2}$$

where a, b, and c are the values on the sides of the triangle, with a opposite the vertex x. Wayne gave a verbal rendition of this formula:

(1) Wayne: You pick any vertex and it's going to be the two sides that make the angle, subtract the side opposite the angle, and divide by two. I understand everything except why you divide by 2.

***

(2) Ben: You know why you divided by 2, is because-

(3) Rachel: Because there's two sides.

(4) Ben: No. No, it's because-

(5) Wayne: There's two other points, to be solved for, no?

(6) Ben: No. No. No. We found out that Y, X + Y + Z is half of the outside points.

(7) Wayne: That's right!

***

(8) Ben: So if you're trying to find one point you add the two-

(9) Wayne: The two sides-

(10) Ben: -adjacent sides and then-

(11) Wayne: -that angle. The two sides that come in-

(12) Ben: -Yeah, the angle.-

(13) Wayne: -to that point to make that angle, OK Adjacent.

(14) Ben: -minus the opposite.

(15) Wayne: And divide by 2 because we found out that the ratio for sides added together, plus points added together was 2.

(16) Rachel: Was 2.

(17) Ben: The ratio is one half.

(18) Wayne: And since we're trying to find a point,

(19) Eleanor: This is half of this.

(20) Wayne: -that's why it's a half, over a half.

In line 1 Wayne wants an explanation: "Why you divide by 2?" After a short discussion of the differences between the way Wayne describes the process and Rachel's equation (omitted from the transcript), Ben attempts an explanation (see lines 2-7). He is interrupted by explanation from both Rachel and Wayne.

Rachel's explanation (line 3) might be analogical or deductive. She may be referring to an analogy between the two sides adjacent to the vertex to be solved and the divisor 2. But it is more likely that her explanation is deductive, given that she derived the original equation algebraically. She began her derivation by adding the two equations referring to the vertex x:

$$x + y = b$$

$$x + z = c$$

$$2x + y + z = b + c$$

$$2x + a = b + c$$

$$2x = b + c - a$$

$$x = \frac{b + c - a}{2}$$

When the 2 appears in the third line, it is because x is involved in the totals of two sides, b and c. When the derivation is completed the 2, which came from the combination of two sides at the beginning, becomes the 2 that is divided by at the end. Rachel's explanation is deductive for no one but her since her short comment is not sufficient to really communicate it to them or to Eleanor. In fact, it is likely that if they considered her explanation at all, they took it to be a weak analogy.

Wayne's explanation (line 5) works only by analogy. There are two more vertices to be solved, once the first is known, but there is no connection between the division by 2 and the number of vertices remaining to be solved, other than the number 2. This makes this a weak analogy. It is interesting that even though Wayne had been the first to voice a need to understand the division by 2, a minute and a half later he seems more anxious to suggest his own explanation than to hear Ben's.

Ben's explanation (line 6) might be analogical or deductive, but it seems more likely that it is analogical. There is no evidence that Ben spent time making a deductive connection between Rachel's equation and the relation which he had discovered empirically with Eleanor and Wayne: $a + b + c = 2(x + y + z)$. Here the analogy is stronger than in Wayne's explanation since the analogy is between two equations with variables instead of between an equation and a state of affairs. This strength is likely to have led to Wayne's acceptance of Ben's explanation over his own.

The explanations that were rejected were a weak analogy (Wayne's) and a deductive explanation that was taken by the rest of the students to be a weak analogy (Rachel's). The students preferred the strong analogy, which was based on several points of connection. This is sensible since a strong analogy could have (and in this case does have) the potential to be developed into a deductive proof. In the end (at line 15) it is this strong analogy that is accepted as explaining why the division by 2 occurs.

It is worth noting that even though Rachel's explanation was the most thought out and based on deduction rather than analogy, which might suggest it was a more certain explanation, it was apparently not even considered by the others. This illustrates a weakness of proving versus analogy for explaining. Proving is a process that must be formulated to be communicated and must be followed with some care to be understood. In this situation the social dynamic did not afford Rachel the opportunity to make her case clearly. Ben's analogy, on the other hand, could be understood immediately by Wayne and Eleanor, who were familiar with the context to which he was making links. Rachel could also see these links after Eleanor showed her what formula was being referred to (line 19).

Summary

Explaining can be done by proving and by analogy. Explaining by proving can be more or less formulated. Whether explaining is successful depends not only of the method of explaining but also on the social context.

Unformulated explanations are limited precisely because they are unformulated. As explanations for other people they are useless, as Ben's attempt reveals. As explanations for an individual, they may work, but only if the argument required is very short. These same weakness show up in unformulated proving used to explore (see section 2).

Formulated proving allows extended explanations beyond what analogy can provide. At the same time, formulated proving is not necessarily preferred over explaining by analogy (as in the case of Bill). In some contexts, where no strong analogies occur, formulated proving may be the only method of explaining possible. At the same time, formulated proving to explain seems to require an appropriate social context, either one in which it is already occurring to address another need, or one in which there is a strong need to explain to others, or one in which a teacher (present or in the past) indicates that formulated proving should be used.

Analogies can be described as strong or weak. A strong analogy can satisfy a need for explanation. In fact, a strong analogy can be preferable to a deductive explanation, either because it is more easily communicated, or because it occurs first and removes the need to prove to explain.

2. Exploring

Given the exploratory nature of open ended problem solving it is not surprising that the need to explore arose often in my studies. This need occasioned both unformulated and formulated proving as well as an important kind of proving that I call mechanical deduction. Mechanical deduction involves the use of a technique or technology that is based on deductive principles, but that conceals the operation of these principles in a set of mechanistic rules. Algebraic manipulations are included in mechanical deduction, as is computer programming. Both reasoning by analogy and inductive reasoning were also used to explore, and examples of such reasoning are discussed at the end of this section. Figure 14 shows the paths in the network related to exploring, with paths involving proving marked by thicker lines.

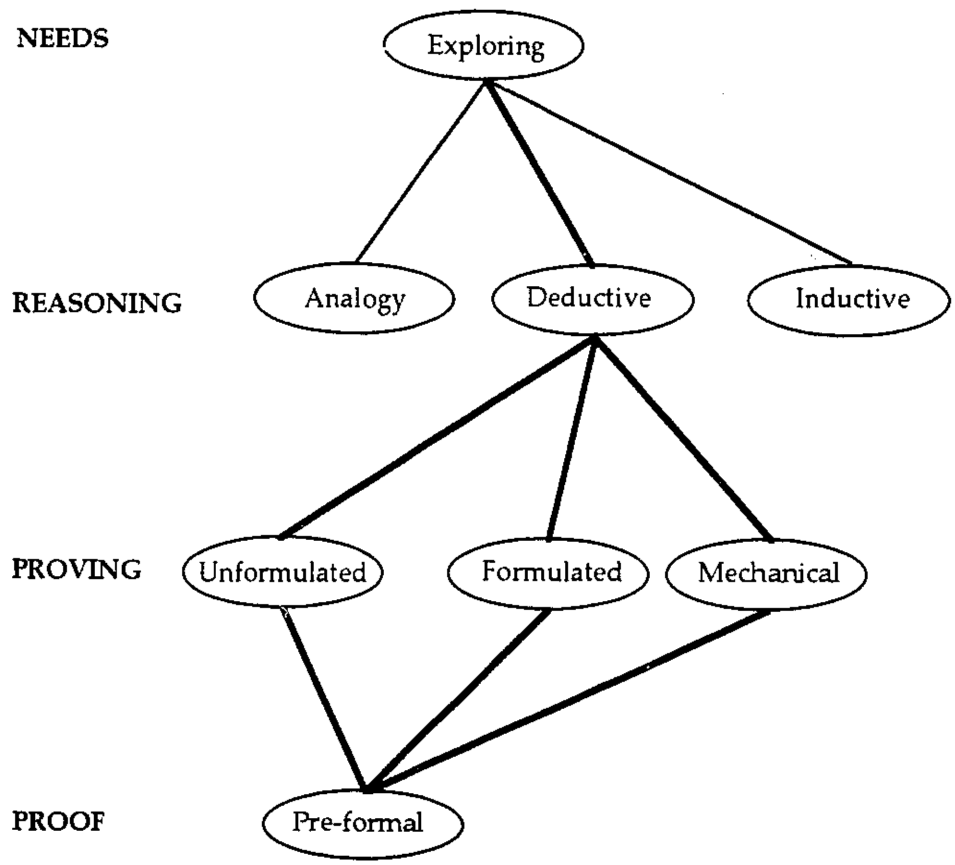


Figure 14: Paths related to exploring.

<u>Exploring — Unformulated proving</u>

Several episodes of unformulated proving used to explore were observed. In each of these it seems that such explorations must be successful quickly or not be successful at all. It seems also that the results of such explorations are not remembered unless they are later formulated. The cases here show a successful short term exploration (the case involving Bill), and a case of unformulated explorations that were remembered because they were later formulated (the case of Sandy, a grade 6 student).

Bill's unformulated exploration

This case is taken from Bill and John's investigation of the Fibonacci situation. After they had been working for about 30 minutes, I asked them to summarize their results (MAT episode 11). After listing three of the relations they had discovered, Bill made this claim: "The first number here would have to be 0, because 0 + 1 = 1." This claim is an example of a simple deduction of a value in the sequence from the recursive rule defining the sequence. Any extension of the sequence, such as determining the value of F_9 from the values of F_8 and F_7, would be logically the same. Bill's claim is different in that it uses the recursive rule in an unusual way, to extend the sequence backwards beyond its "beginning". He found that the recursive rule and deductive reasoning allowed him to circumvent the given situation: "the sequence of Fibonacci numbers begins: 1, 1 ...;" and to invent new mathematical objects, in this case F_0. It is the novelty of the result obtained that makes this case an example of exploration.

Sandy's formulation of his unformulated proving to explore

I made several informal visits to meet with Sandy at his school when he was in grade 6. His ability in mathematics was impressive and inspired my visits. On my first visit I asked him to investigate the Arithmagon situation. He solved the original puzzle quickly through unformulated proving. As he solved a second puzzle his reasoning became more formulated and when asked he could produce a formula and provide a proof of it.

When Sandy was first given the 11-18-27 Arithmagon, he guessed 5 as the value at the top corner, but then rejected it without trying. He then adopted a more systematic approach. He tried 0 and mentally determined that it didn't work because 11 + 18 ≠ 27. He then observed that the choice of the value at the top corner determines the values of the other two which must add up to 27: "That would make these two numbers and then you just have to find one that makes 27." He also commented: "I knew it was a fairly small number because this [27] is almost these two [11+18] together." Although he did not say anything at the time, based on his later comments, I believe that when he tried 1 and found that it worked, he also observed that the sum 11+18 contained the top corner number twice and each of the others once. This allowed him to deduce that the value of the top number would be half of the difference between 11+18 and 27, that is, 1.

Sandy was then asked to make up an Arithmagon of his own, and he chose 13-21-42. He added 13+21 mentally and then subtracted 34 from 42. This gave him 8, which he took half of to arrive at a value of 4. These calculations came quickly and confidently, indicating that he had worked out a method similar to that described above while solving the original puzzle. When Sandy checked his answer, however, he discovered a problem. 4 was not the correct value.

He then tried 2 and when it didn't work he asked: "Are there only certain combinations that work out here?" He was reassured that the puzzle should work, and then realized the value would have to be negative: "The number's too high — unless you put ⁻4." He immediately tried ⁻4 and found that it worked.

(1) DR: Why did you go straight to ⁻4?

(2) Sandy: Because these two [13+21] were less than that [42], you need negative numbers.

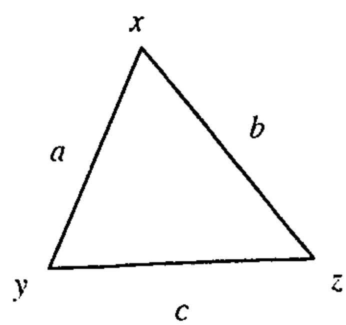
(3) DR: But how come 4?

(4) Sandy: I subtracted 34, the total of this, from 42, which gave 8 divided in half. ... I tried 4 first, remember?

(5) DR: Yeah.

(6) Sandy: I figured 1's not enough, 0's not even enough, so I tried $^{-}4$.

Here Sandy's reasoning is much clearer. He took the sum of the two adjacent sides, found the difference from the opposite side, and divided by 2. At first he took the positive difference which gave him 4. When this failed, he briefly turned to a systematic inductive approach, dividing by 2 again and then reasoning about the size of the number he was seeking. When he realized the answer must be negative, he returned to his original method, seeing that it was sound. He then repeated his calculations in an organized way to show that they gave $^{-}4$.



$$\frac{a + b - c}{2} = x$$

Figure 15: Sandy's formula for the Arithmagon.

After repeating his calculations for the original puzzle, to show how they worked in that case, he was asked to solve two more Arithmagons and to write his method in formal terms. This he did. He solved a triangle with fractional values as secret numbers without comment and finally produced the diagram and formula in Figure 15.

Sandy's original proving, which led him to the discovery of his method, was unformulated, but the problems he had with the second puzzle, and the questions he was asked led him to formulate it somewhat. This enabled him to used his method intelligently in cases with fractional solutions and later when exploring Arithmagon squares.

<u>Exploring — Formulated proving — Preformal proof</u>

In all the studies I did as part of this research, exploring using formulated proving occurred in only one case: Rachel and Eleanor's investigation of the Arithmagon situation with Ben and Wayne. The association of this proving with a need to explore should be qualified by the observation that Rachel's initial choice to explore by proving was motivated by the expectations she ascribed to one of the observers. Eleanor's choice to prove as a way of exploring may be linked to her observation of Rachel's activities.

Rachel's explorations by formulated proving

For the first half hour of the session, Rachel engaged in activities similar to those of the others. She solved the original puzzle by using a system of equations, she tried solving other triangles, she experimented with Ben's constraints method, and she searched for patterns inductively. In the second half hour of the session, however, she worked by herself, or with the help of an observer, Tom Kieren, using formulated proving to explore. Her first exploration was of the situation in which two of the known sides are equal. By working with the known relations she was able to deduce that two of the corners would be equal in that case (see Figure 16). After she came to this conclusion, she described her work to the others MAT episode 8). This description may have inspired Eleanor's explorations by formulated proving (see below).

Rachel then explored the case of all the known sides being equal. When she completed her deductions, the observer closest to her, Tom, suggested that she explore the situation without putting any constraints on the situation. With a few hints from Tom she then derived a general formula, $x = \frac{b+c-a}{2}$, for an unknown corner.

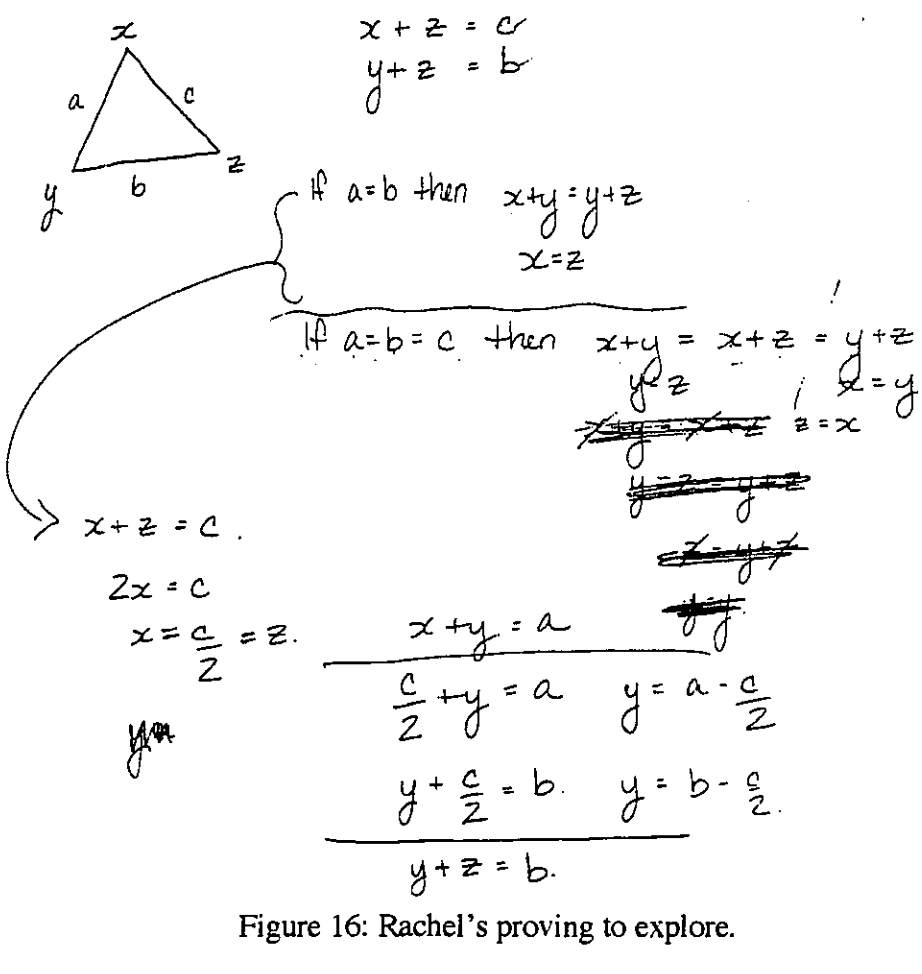


Figure 16: Rachel's proving to explore.

While Rachel's proving led to her discovery of new aspects of the Arithmagon situation, it would be misleading to claim that exploring was her sole motivation to prove. Her comments in the interview session make it clear that she had other considerations in mind.

(1) DR: Why were you doing that?

(2) Rachel: Oh, you want me to answer right now? [laughter] Because I was stuck at them. I didn't know where to go. And Tom was sitting beside me saying, "Well, what can you do now?"

(3) Eleanor: Nothing, nothing. [laughter]

(4) Rachel: So I was thinking I'd better think of something, or else that question's going to keep coming. So I just thought, well hey, in math you always get that right? You always get those conditions. Every teacher's listing these conditions. Now, if we have this condition where this equals that. You know what I mean. So that's. It just. You know. It comes from my head. Something I knew of already that I thought I could apply to that problem.

Here it seems that the form of Rachel's proving was taken from her past school experience, in the hope of satisfying her need to answer Tom's question "What can you do now?"

Eleanor's explorations by formulated proving

While Rachel was exploring the Arithmagon situation modified by various conditions, Eleanor was working with Ben and Wayne. Together they found inductively the relations $a+z = b+y = c+x$ and $a+b+c = 2(x+y+z)$ (see Figure 17). Eleanor then worked independently, trying to invent a general method of solution based on these relations (MAT episodes 11-13). Her work was interrupted on several occasions but she did manage to develop an alternative solution method to Rachel's. Her method involves finding the sum of the three sides (34, in the example shown in Figure 17), and dividing by 2 (17 in her example, but not shown). This calculation gives her "middle number," (note the expression in the middle of the upper triangle). Because the sum of a side and the corner opposite is this middle number, the corner numbers are obtained by subtracting across the triangle (e.g., 17–11=6).

Eleanor's explorations differed from Rachel's in an important respect. Rachel began from the conditions known to her and those she added, and explored to see what she could find out. She had no particular goal in mind. In Eleanor's case, she had the specific goal of finding a general solution method. One difference between exploring with a goal and exploring without one is the criteria for satisfaction. In Eleanor's case, her proving could only satisfy her need to explore if she found a general method. In Rachel's case, whatever particular results she found satisfied her need to explore. It was only the teacher-game she felt she was playing with Tom that kept her proving (Other teacher-games are described in section 4).

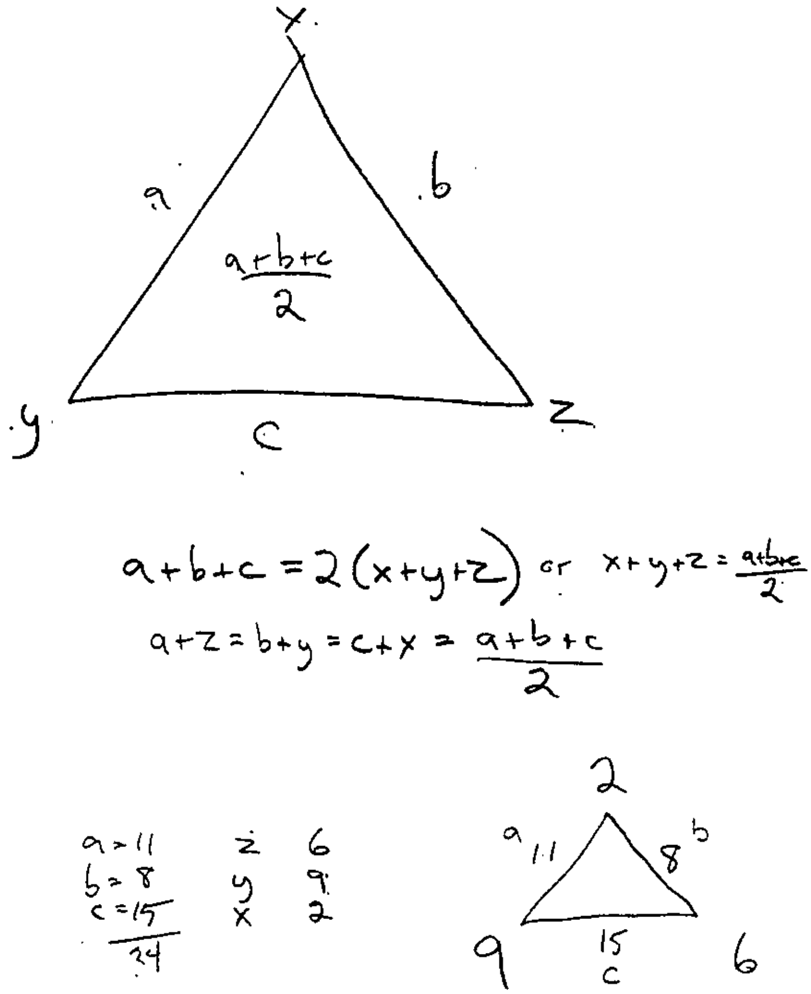


Figure 17: Work related to Eleanor's formulated proving to explore.

<u>Exploring — Mechanical Deduction — Preformal proof</u>

Particular circumstances are necessary for mechanical deduction to be used to explore. The situation has to be one in which what is known about the situation can easily be cast into a form to which a particular technique is suited. The Arithmagon situation is one such situation. Kerry, Jane, Chris, Rachel, and Eleanor in the first clinical study, and Colin in the North school study, used mechanical deduction to solve the original puzzle (see Chapter III for Stacey and Kerry's uses of mechanical deduction, and Appendix E for descriptions of ways of solving the Arithmagon puzzle).

When the information given in the problem is cast into an algebraic form a system of three equation in three unknowns is generated:

$$a + b = 11$$
$$b + c = 18$$
$$a + c = 27$$

This system of equations can be solved by elimination, using procedures that were known to all the participants in the studies except Sandy, Bill and John. The system can also be solved as a matrix, and this was done by Kerry and Chris.

Chris's explorations (his partner, Jane, was not very actively involved) were unusual for the strong preference he had for mechanical deduction. In the Arithmagon situation (see MAT) he solved the original puzzle using a system of equations and felt he had established a general method. I then suggested he try an Arithmagon square. Some Arithmagon squares have an infinite number of solutions, and some have no solution. He began working with a matrix, which he row reduced correctly, as far as was possible. He was not able, however, to give meaning to his result. He continued wondering what was wrong with his matrix for an extended period. At no point did he attempt an example, which might have indicated to him the nature of his difficulties with his matrix. In the Fibonacci situation he formalized the rule as $F_1=a$, $F_2=b$, $F_n=F_{n-1} + F_{n-2}$. He then derived a wide variety of statements from this rule, without interpreting any of them in terms of the actual sequence. At one point he obtained a sequence of expressions, that included the Fibonacci numbers as coefficients, but he did not notice. The mechanical deduction he was engaged in kept him from seeing any meanings in what he was doing.

That mechanical deduction was not employed for exploration more extensively is probably an indication of the unusual conditions present at the beginning of the investigation of the Arithmagon situation, compared to the conditions later in that situation and generally in the Fibonacci situation. In the Arithmagon situation an algebraic expression of the situation is easily obtained, and is of a form familiar to most of the participants. These problem situations are unusual when compared with the exercises the students at Central, North and South Schools saw in their mathematics classes. In the schools the majority of exercises the students are assigned are set up to provide a fairly easy entry for the algebraic techniques used for mechanical deductions.

For example, at South High School the students were assigned problems asking them to determine, given the coordinates of four points, if those points were the vertices of a parallelogram. In class they were specifically instructed that a careful graph of the points would not suffice to answer the question, even if the graph clearly showed two sides were not parallel. A correct solution to the exercise required that the students employ the slope formula to determine the slopes of the two segments in question. The exercise was set up to make mechanical deduction using the slope formula easy, and the teacher's instructions required it.

Exploring — Analogy

Many participants in the studies considered whether the triangle in the original Arithmagon puzzle might offer a clue to its solution. Wayne was unusual in that he took this clue seriously, even when told that the situation had nothing to do with triangles. Wayne explored the Arithmagon situation by drawing analogies between the situation, and his knowledge of triangle geometry (MAT episodes 1, 3, 5, 6, & 16). This made triangle properties like area, Pythagorean triples, and angle measure relevant to his exploring (see Figure 18).

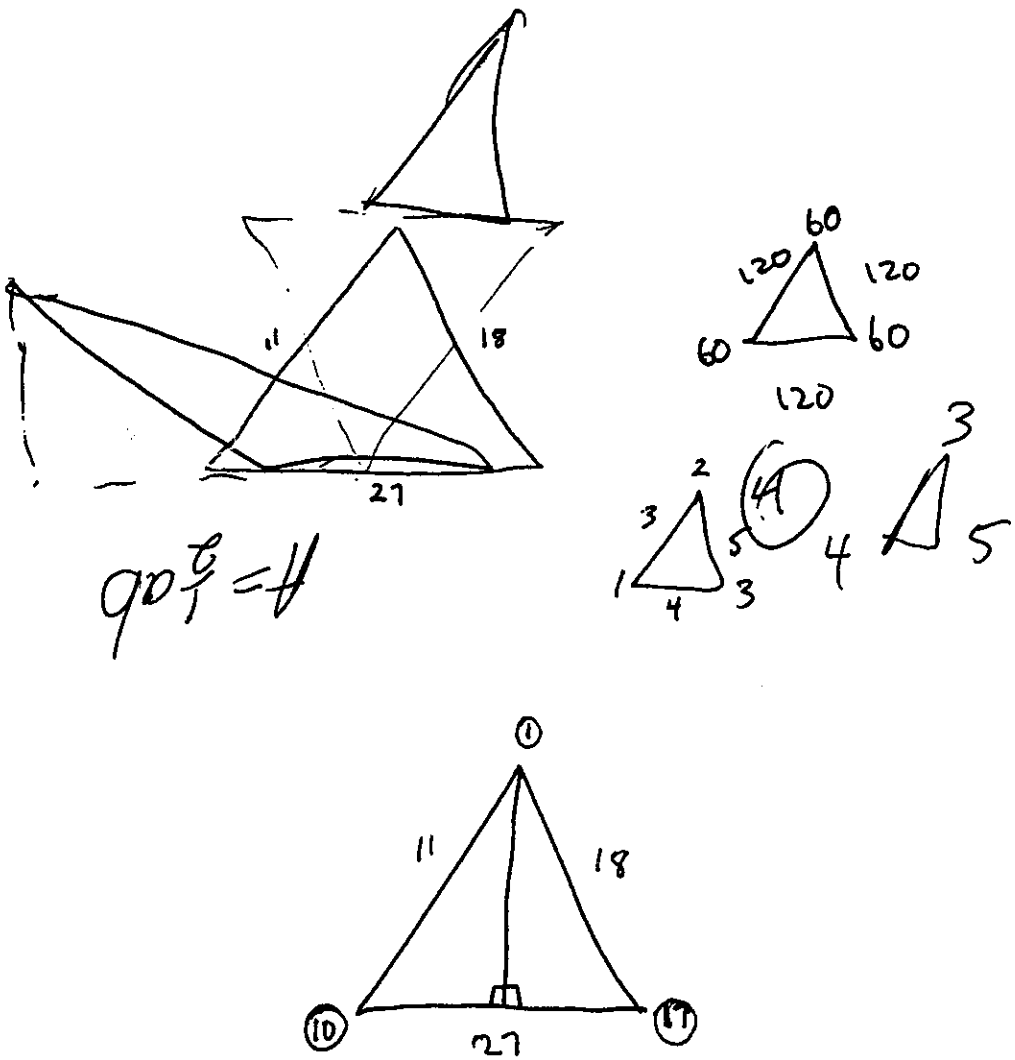


Figure 18: Triangles drawn by Wayne while exploring, relating to geometric properties.

This unusual use of analogy to explore interfered with Wayne's ability to identify patterns inductively and so interfered with the satisfaction of his need to explore. His lack of success need not have been the case, however. The main limitation of his use of analogy was the weakness of the analogy. Nothing but the arrangement of the puzzle suggested that triangle geometry might be involved. Perhaps if the analogy had been stronger, Wayne might have enjoyed successes like those of Euler, who used analogy extensively in his mathematical explorations (Polya, 1968).

Exploring — Inductive reasoning

Exploring inductively was used extensively by all the participants in the studies. This is not surprising. In many circumstances the information given provided a weak base for both analogy and proving. In addition, inductive exploring was the way of exploring most often modeled in the schools.

Inductive exploring was used more often in the Fibonacci situation than in the Arithmagon situation. This may be due to the ease of generating new data in the Fibonacci situation. In the Arithmagon situation new triangles could be generated by starting with known corner numbers and adding to find the side numbers, but this technique produced Arithmagons with special properties. To generate general examples, new puzzles had to be made, and then solved. Even with the use of mechanical deduction this process was time consuming. Most participants examined only a few Arithmagon triangles. On the other hand, most participants examined from 20 to 40 terms of the Fibonacci sequence. Another feature of the Fibonacci situation, that may have discouraged deductive reasoning, is the recursive formulation of the rule defining the sequence. While unformulated reasoning was not hampered by this feature of the situation, the participants are unlikely to have had much experience reasoning formally on recursive sequences.

In the GEOworld situation the students were asked to try to identify features of a geometric world defined by a computer procedure that drew geometric figures based on three numeric inputs. For example, Figure 19 shows the results of the inputs 100, 100, 3. In this situation the students had no general knowledge of the situation at all. Any general principles had to be developed inductively by them. I was interested to see whether the establishment of such principles would give rise to a need to prove from them, or whether some need to prove would motivate the participants to establish general principles.

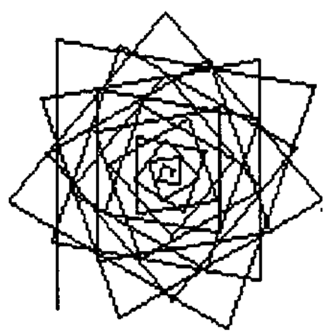


Figure 19: Output of GEO 100 100 3.

In the end, the GEO situation provided little information about proving, because almost all the reasoning employed in that situation was inductive. This reflects the complete lack of known relations available at the beginning of the participants' investigations. In a few cases reasoning by analogy and unformulated proving were used, once some patterns had been established, but these were fleeting.

Summary

Exploring is the need that was satisfied in the widest variety of ways by the participants in my studies. Both inductive reasoning and reasoning by analogy were used to explore, but exploring inductively was far more common and far more successful. The proving used include unformulated and formulated proving and mechanical deduction. The problem situations themselves seemed to be the strongest constraints on the method of exploration. The exception to this is the case of formulated proving, which seems to require a supportive social context to occur.

Unformulated proving can satisfy a need to explore, but only in cases where the situation is not too complex. Simple discoveries like Bill's occur in isolation and are unlikely to be connected with other aspects of the situation. Sandy provides an example of unformulated proving which became sufficiently formulated to be used and extended as exploration of the situation continued. As with explaining by unformulated proving, the proving that occurred in these examples is difficult to detect and lacks the permanence that formulated proving seems to possess.

Exploring by formulated proving was an uncommon method of exploring for the participants in the studies. This seems odd considering its usefulness as a method of exploration. It may be that other methods of exploration are just as useful, or that students have more experience with other methods of exploring. Exploring can be directed to a particular goal or undirected. Formulated proving is useful in both cases although satisfaction is likely to come more quickly if no particular goal is sought. Finally, as in the case of explaining by formulated proving, social factors seem to be important in the use of formulated proving to explore.

3. Verifying

Verifying is traditionally and most commonly thought to be the need which proving satisfies. As I noted in the previous chapter, this is not the case in mathematics nor for many students. The participants in my studies did prove to verify, but only in particular circumstances. For example, unformulated proving was an integral part of "guess and check" inductive explorations, forming the

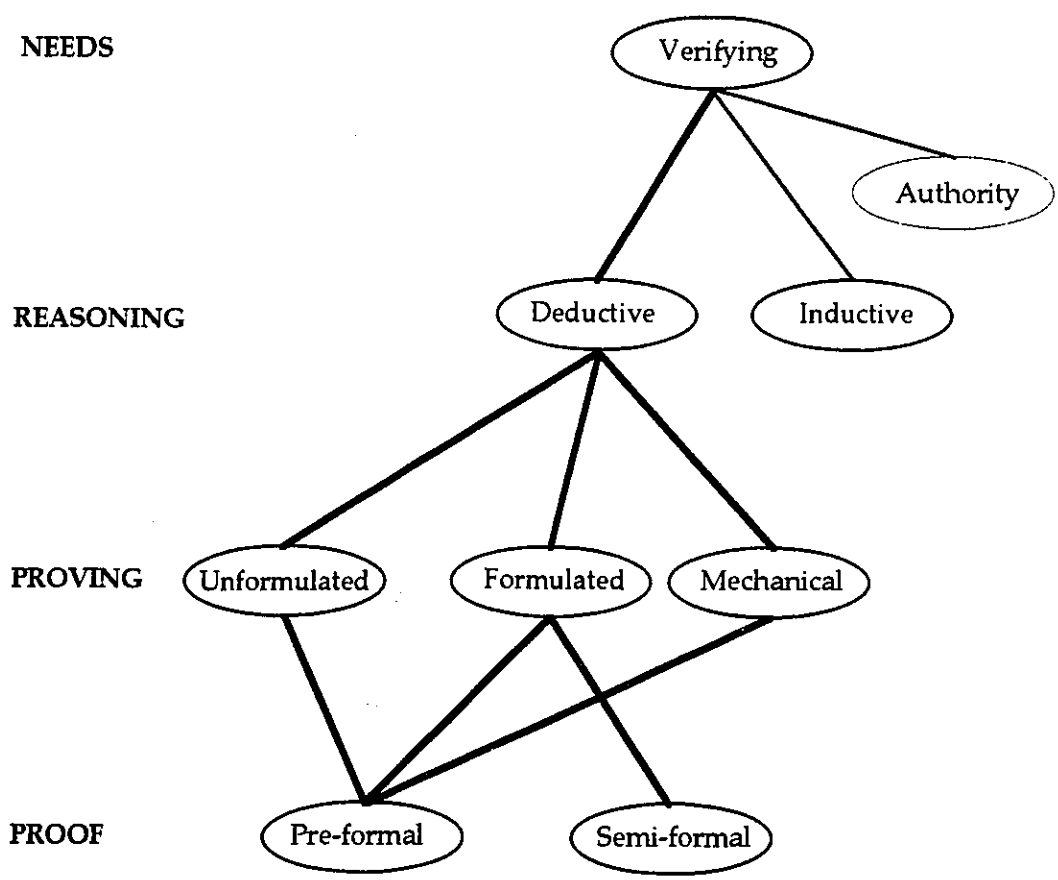


Figure 20: Paths related to verification.

"check." Proving which was begun to satisfy some other need can also verify. All three kinds of proving I identified are include in the network shown in Figure 20 because of this incidental verifying which occurs. Formulated proving to verify also occurred as part of a teacher-game. These occurrences are described and discussed in the next section, which is focused directly on teacher-games as a need to prove.

Inductive reasoning can also be used to verify and an example and discussion of this will also be given. While not a kind of reasoning, appeal to authority can also be an important way of verifying in mathematics, and so I include it as part of the network. Appeal to authority was only observed with the high school students. Figure 20 shows the paths associated with the need to verify. Those involving proving are marked by thick lines.

Verifying — Unformulated proving

Unformulated proving was used in conjunction with inductive exploration in a 'guess and check' cycle. While many 'checks' were simple calculations, which I do not count as deductions (although technically they are), others were more involved. A particularly clear example is Bill's verification of a method he found for solving the Arithmagon. Bill had been experimenting with various operations on the three known sides to try to derive a secret number. When he calculated $27 + 18 \times 11 \approx 16.5$ he felt he was near a solution (MAT episode 6). Once he had rounded 16.5 to 17 he could determine the other secret numbers by subtracting from the sums on the sides. Bill had at this point found a solution method that worked for a special case. Now he needed to test it in general. He asked "But would it work for any one?" and tried the case of a 3-6-17 triangle. His method gave him a value of 9, which he felt was wrong.

To verify that 9 did not work Bill used unformulated proving by *reducio ad absurdum**. "No, how could it be 9? Because you, you get 6. [9+?=6] Unless this was, um, negative three. [9+(⁻3)=6. Another secret number must then be ⁻3.] But then you get 3 from here. [(⁻3)+?=3] You would have to have, um. To get a negative, you have to have 6. [(⁻3)+6=3. The third secret number would have to be 6.] But then 9 and 6 is 15. [Not 17, as it should be. A contradiction has been reached.] It would not work."

While most participants were not so clearly using unformulated proving to verify results, this does seem to have been fairly common. This seems reasonable since the main weakness of unformulated proving, the impermanence of the process and result, does not matter in cases where the result will be discarded if found wrong. It should also be noted that unformulated proving to explain or explore can also verify the results involved, and some examples included under those headings could also appear here.

Verifying — Mechanical deduction — Preformal proof

Colin was observed using mechanical deduction to verify statements during the first interview session at North School. There is an interesting contrast between Colin's use of mechanical deduction and Anton's use of inductive reasoning to verify.

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* Deriving a contradiction.

In the first interview session I asked Colin and Anton to reexamine a calculation they had made in the Arithmagon situation. They did so, and Colin noticed an error in it. This allowed him to perform a correct calculation which produced the formula $Z = \frac{x-z+y}{2}$. When I asked Colin what this formula meant, he said with confidence that it gave one of the secret corner numbers. Anton then suggested trying it to see if it worked. They did so. Anton then suggested trying it on a different corner. They did so, and I asked if they had expected the formula to work on one corner but not on all. Colin asserted that the formula should work the same way for all corners (MAT episode 2.5). Anton then asked me if the formula was correct, and I said it was one of several correct formulae. These episodes were followed by Colin's use of formulated proving to explain, described in section 1.

What distinguishes this case from those of formulated proving described above is the use of mechanical deduction to discover and simultaneously verify the formula. The mechanical aspect of Colin's derivation was revealed more clearly later in the session when I asked them to interpret the derivation in terms of what it meant to subtract the two equations relating sides to corners. They were unable to do so, indicating that the original subtraction was not a formalized act of deductive reasoning but a mechanical act of deductive reasoning.

<u>Verifying — Inductive reasoning</u>

Verifying using inductive reasoning was observed far more often than verifying by proving. This seems to be the method of choice for verifying in mathematical situation for the participants in the studies. Colin and Anton provide a sharp contrast between these two methods of verifying (described under Verifying—Formulated proving). Such a sharp contrast is not visible in many other cases, since few people verified by proving while everyone verified inductively.

Inductive verification was used often even when a deductive verification could easily have been provided. For example, when Rachel announced her formula for solving Arithmagon puzzles, she made it clear that she had derived it. Ben and Wayne, however, did not ask her how the derivation went, but instead tried her formula on a example and concluded that it worked.

This overwhelming preference for inductive reasoning as a way of verifying has been demonstrated by many studies (e.g., Fischbein & Kedem, 1982) that claim that students do not understand proofs because they prefer to verify statements inductively. The possible implications this could have for teaching proving are discussed in Chapter V.

<u>Verifying — Authority</u>

The way of verifying mathematical statements that was second in popularity to inductive reasoning was a form of non-reasoning: making reference to an authority. Two of the participants at North School, Bill and Anton, provide good examples of verifying by making reference to authority. The participants in the clinical studies seem to have assumed that simply asking for answers was not permitted. They may have been playing a research-game more than a teacher-game. These two high school students, however, were willing to ask, and did so. The case of Anton gives a simple example of asking for verification. The case of Bill is more complex.

<u>Anton's reference to authority</u>

Immediately after Colin and Anton solved the Arithmagon puzzle, in the Arithmagon session, Anton asked if their solution was correct (MAT episode 1.4). As they found their solution by means of a mechanical deduction, it is not surprising that Anton might have doubts about their solution. Mistakes happen in mechanical deduction, and are hard to notice. What is surprising is that Anton chose to verify their solution by asking an authority, rather than checking it himself. He seems at this point to have a preference for authority over induction. This was not the general case, however, as Anton verified statements inductively in the other sessions.

<u>Bill's rejection of analogy, induction, and deduction in the face of authority</u>

In the second interview with Bill and John, we proved that the sum of two odd numbers is even, twice. They both seemed quite confident that this was a general principle. Bill, in particular, had a number of reasons to believe in the generality of this principle. He had seen inductive evidence for it in the Fibonacci session. In the first interview session I had assured him that it is generally true. In both interview sessions he had offered explanations by analogy of the principle. And finally he had just seen two proofs of the principle, one of which he developed himself. When I made the claim, at the end of the second interview session, that the assertion fails for large numbers, their response surprised me. In the face of all the evidence they had seen, Bill and John rejected the generality of the odd+odd=even principle (MAT episode 19).

Both Bill and John's first response to my assertion that the principle failed for large numbers was to ask "How large are we talking?" I replied that the principle failed for 117 digit numbers. Bill then said: "I don't see how come that is, but of course I'd have to see a number that long. [laughter] And it would take like a year to really find out why. But, um, That's really kind of neat." His comments do not indicate that he couldn't see how the principle could fail, that he doubted it could, but rather that he could not understand why it would fail. He acted as if the failure of mathematics to make sense in this case was a failure on his part to make sense of it. Against all the inductive, deductive and analogical arguments he had seen, my authority, acting in the role of a teacher, was overwhelming. Neither the possibility that I was lying or mistaken was voiced, and neither possibility seemed even to be considered. (Lest there be concerns about the ethics of my research, I did admit my deception before the session ended.)

Bill's beliefs about the origins of mathematics may be related to his attitude towards authorities. In line 8 of the transcript in the sub-section Explaining — Formulated proving, in section 1, he refers to "they", the "brilliant genius" who could develop the formula and then rephrase it in a useful form. Of course, this is exactly what Bill and John did, but Bill quickly removed himself from this creative activity, substituting the anonymous geniuses who create mathematics.

Verifying in school

It seems plausible that Bill and Anton might have developed their attitudes towards verifying by making reference to authority in school. This example from South School illustrates how this might have come about.

Ms. E was teaching a lesson on slopes of perpendicular lines. She had the class suggest relationships between the slopes of perpendicular lines. The first suggestion was that the slopes would be opposites. The example Ms. E had on the board involved slopes of 1 and $^-1$, so in this case the suggested relationship worked. Ms. E then looked an example of a line of slope 3. She drew in a line of slope $^-3$ and noted that the line did not look perpendicular. A student suggested the negative reciprocal. Ms. E. drew in a line of slope $\frac{-1}{3}$, and noted that it did look perpendicular. She then said, "That doesn't look bad. Again, it's not a proof, but it's good inductive reasoning, and in this case it is true." In this case the students have been given some visual, inductive evidence that the relationship is what Ms. E says it is, but she has then explicitly rejected that evidence as a verification. The students are left to rely on the teacher's authority as verification.

Summary

Verification, the traditional reason to prove, seems not to be a major motivation for the students I observed. Some unformulated proving and mechanical deduction was used to verify, but either as a side effect of proving for some other reason, or as part of an inductive, guess and check, process. Verifying by induction was quite popular, as one would expect from past research. The popularity of verifying by making reference to authority, especially among the high school students surprised me. I suspect I was being naive. On reflection verifying by making reference to authorities makes a lot of sense.

Verifying by reference to authority is not necessarily a poor method of verification. Matters of historical fact, for example, cannot be verified without consulting and analyzing various texts, which act as authorities. In mathematics verifying by making reference to published proofs saves considerable time and effort, although at some risk, as Wiles found when some of the proofs on which he had based his first proof of Fermat's Last Theorem turned out to have flaws. The elevation of verification by reference to authority over reasoning as a method of verifying has problems, however. Authorities make errors (or lie as I did with Bill) and reasoning in various ways can discover these errors (as occurred in the case of Wiles' proof of Fermat's Last Theorem; see Chapter I, section 2).

4. Teacher-games

Formulated proving, specifically intended to verify, occurred only as a result of an interaction with someone in the role of a teacher. In these cases the need to verify becomes entangled with a teacher-game as a motivation to prove. In addition, proving to verify as part of a teacher game can lead to what I call "formulaic" proof-making. Formulaic proof-making results in a proof, but it is not proving. Instead the proof was constructed according to principles which are associated with the creation of the sorts of proof teachers like. Two of the examples given below illustrate formulated proving as part of a teacher-game. The third illustrates formulaic proof-making. Figure 21 shows the paths associated with these examples.

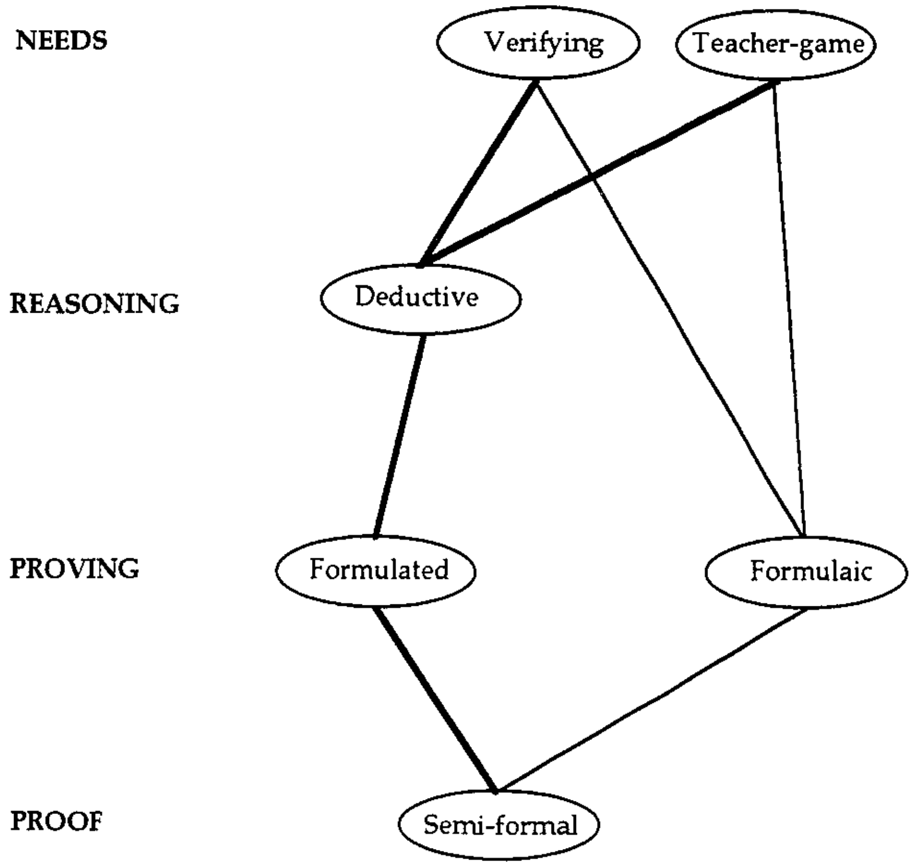


Figure 21: Paths related to teacher-games.

<u>Teacher-game/Verifying — Formulated proving</u>

The cases described here come closest to the 'official' model of proving of any of the paths I have described. Here the purpose of the proving is to verify, and the proving is formulated. Another characteristic of these cases is that in both of them the proving is occasioned by an observer. By questioning the participants' confidence a teacher-game is initiated. The first case is taken from the second interview with Colin and Anton at North School. The second case is taken from Kerry's investigation of the Fibonacci situation.

<u>Colin verifies a particular case by reference to formulated proving</u>

In the second interview session, Colin and Anton determined inductively that n^3-n is always a multiple of six. I then guided them through a formulated proving of this statement (described in section 1). After Colin had written out the argument I asked them if they now knew that 417^3-417 is a multiple of 6 (MAT episode 7). While Anton checked on a calculator, Colin applied the reasoning he had just written out in general to verify this particular case. He argued that 417^3-417 is $416\times417\times418$, which includes an even number and a multiple of 3. Colin's use of formulated proving to verify stands in contrast to Anton's use of inductive reasoning.

Kerry proves to verify that F_{3n} is even

After spending a few minutes examining cases, Kerry and Stacey determined that F_{3n} is even, inductively. Having solved this puzzle to their satisfaction, they began exploring F_p, the Fibonacci numbers with prime indexes. After about 7 minutes of this they had concluded that F_p is always prime. The observer, Tom Kieren, then asked them about F_{3n} again (MAT episode 4). They looked once more at their table of Fibonacci numbers. Tom then asked, "How sure are you about that?" In response to this Kerry proved the statement in a formulated way, making explicit use of the odd+odd=even and odd+even=odd relations for addition of even and odd numbers:

Kerry: Oh, I'm positive. Because, well, you can see that the- that you'll add this odd number to an even number to get an odd number. Then you'll add a- Then you'll add that odd number to the even number to get an odd number. Then you'll have two odd numbers to add together to get an even number.

Kerry had already verified his statement inductively, and it seems unlikely he would have produced his clear argument to satisfy some residual doubt he might have had.

Teacher-game/Verifying — Formulaic proof making

Laura discovered the formula $\frac{a + b - c}{2}$ for solving the Arithmagon inductively. When I asked her if she was sure her formula worked for all Arithmagons, she replied "Oh, you want me to prove it" and she wrote out the "proof" in Figure 22.

It should be noted that this proof is not correct. Laura shows that if her formula works, then the known relations between the sides and corners will hold. She has proved the converse of her statement. Her proof-making is formulaic, not a result of formulated proving, since she has lost track of the sense of her steps and is 'going through the motions.' In a different context she proved differently. When she was shown this proof in the interview session she reported that she had seen a better proof when she got home, and she described a correct proof based on the given relations.

PROOF

$$\frac{a+b-c}{2}=D$$

$$\frac{b+c-a}{2}=E$$

$$\frac{a+c-b}{2}=F$$

$$D+F=a$$

$$\frac{a+b-c}{2}+\frac{a+c-b}{2}=$$

$$\frac{a+\cancel{b}-\cancel{c}+a+\cancel{c}-\cancel{b}}{2}=\frac{2a}{2}=a$$

$$D+E=\frac{b+c-a}{2}+\frac{a+b-c}{2}=\frac{2b}{2}=b$$

$$E+F=\frac{b+c-a}{2}+\frac{a+c-b}{2}=\frac{2c}{\cancel{2}}=c$$

Figure 22: Laura's "proof."

Summary

In all of these cases, the use of formulated proving to verify was triggered by an observer's question. The second interview involved my explicitly guiding Colin and Anton through formulated proving, and I raised the question of the truth of their conclusion in a particular case. Kerry had already verified his statement inductively and it seems unlikely he would have produced his clear argument to satisfy some residual doubt he might have had. Similarly Laura's "proof" could not have verified her formula for her, as she did not unpack the meaning of her manipulations. I have listed these cases under the heading of "Teacher-game/Verifying" to signal that an important aspect of these episodes is their role as part of a teacher-game, of producing "proofs" to verify a statement for a teacher.

5. Synthesis

This section attempts to summarize what I can now suggest about the need, or needs, to prove displayed by the participants in my studies. The different activities of the participants provide a rich source of possible connections and distinctions between needs, forms, and contexts of proving. I have chosen to summarize the connections that seem most significant to me under the headings of needs. I begin with verification, since it is the traditional purpose ascribed to proving. I then move to explanation, which has been a popular alternate purpose for proving in mathematics education research. Third, I look at exploration, which is important in Lakatos' (1976) account of professional mathematicians' reasoning. Finally I consider teacher-games and other social situations.

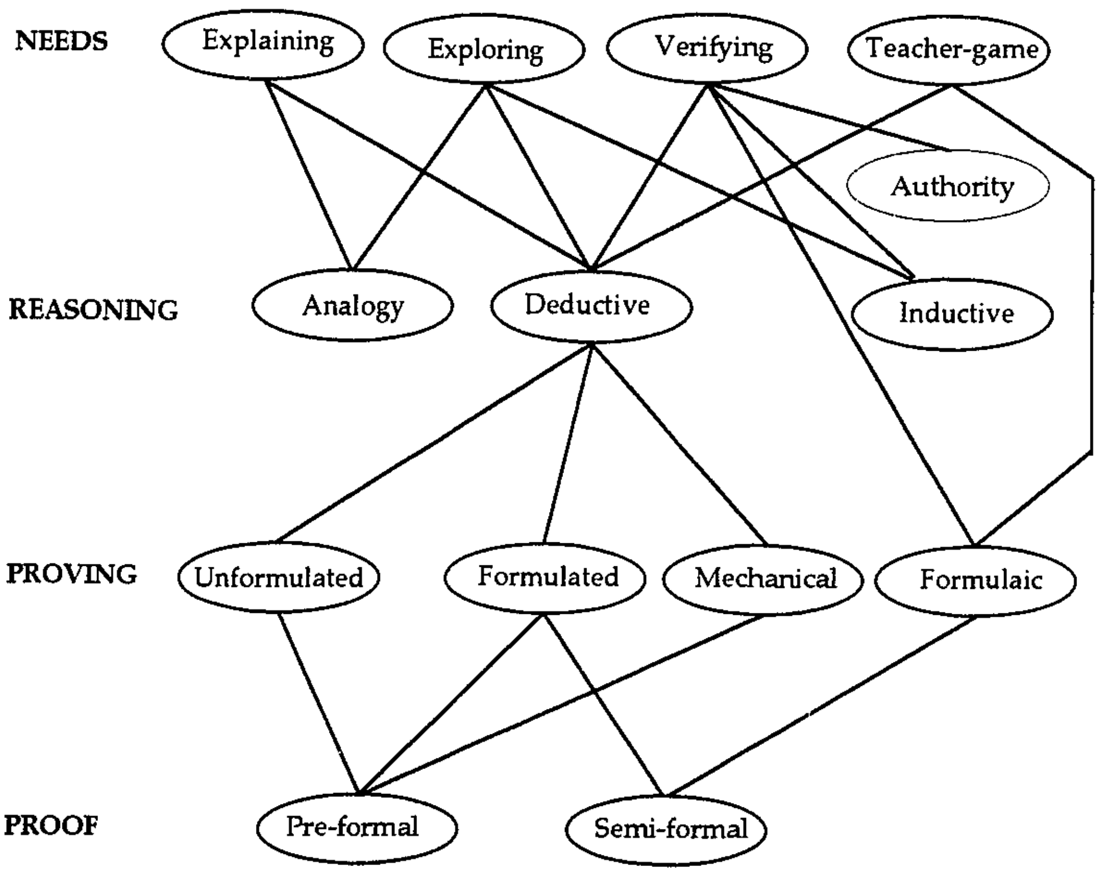


Figure 23: The complete network.

The relationships between needs, reasoning, proving, and proofs can now be gathered together in a single network. An episode of reasoning connects a need with a way of reasoning and if the reasoning is deductive, the connection continues down through a kind of proving and a type of proof (see Figure 23).

Verification

According to a common understanding of it, proving is about knowing with absolute certainty, verifying without doubt. The participants in my studies, however, did not use proving to verify except in certain special circumstances. Instead they relied on inductive evidence and authorities to verify their conjectures. This is not unreasonable, considering that in many other fields the accepted form of verification is induction or appeal to an authority. These are also the methods of verification modeled by mathematics teachers, at least at North and South schools. It could be argued that it is precisely the fact that proving is used to verify in mathematics that distinguishes mathematics from other fields and so proving should be taught as the only acceptable method of verifying in mathematics. There is, however, significant evidence raised by quasi-empiricist philosophers of mathematics that it is not the case. They point to contemporary and historical cases that illustrate that professional mathematicians do not verify by proving (see Chapter I, section 2).

It is important to note, however, that the participants in the study did prove to verify in some circumstances. As part of inductive cycles of guessing and checking, unformulated proving was used to test conjectures in specific cases (e.g., Bill in section 3). Unformulated proving is well adapted to this usage. Formulated proving was used to verify by some participants who have learned that proving is the only acceptable form of verification in mathematics (e.g., Kerry and Laura in section 4). While they did not prove to verify for themselves or their colleagues, they did so when an observer asked for verification. It would seem that there are two mathematics going on in these situations: the mathematics of the participant and an 'official' mathematics represented by the observer.

Explanation

Proving to explain has been suggested as a good way to introduce proving to students (Hanna, 1989). The participants in my studies did use proving to explain with varying success and accepted deductive explanations. At the same time, explaining by analogy was an unexpectedly successful method of explaining.

The deductive explanations observed in the studies involve unformulated proving, formulated proving producing preformal proofs, and formulated proving to interpret semi-formal proofs. Of these, unformulated proving was not usually successful in explaining to others (e.g., Ben in section 1) although short unformulated proving was used in explanations to the prover (e.g., Kerry in section 1). Formulated proving was more successful as a way of explaining (e.g., Eleanor in section 1). Its main weakness was the time and attention it required of the listener. Semi-formal proofs were also accepted as explanations by some participants but not by all (e.g., Kerry in Chapter III, section 12).

The main rival of proving for explaining was the use of reasoning by analogy. Explaining by analogy was more or less successful, depending on the strength of the analogy. A strong analogy was accepted over a deductive explanation in some cases (e.g., Ben, Wayne, and Rachel in section 1). Some explanations by analogy made connections which could have been established deductively although no participants attempted to transform an analogy in this way.

Exploration

Some exploring by proving was observed but for the most part inductive reasoning was used for exploration. The deductive explorations that were successful were formulated or, in certain conditions, mechanical deductions. Both the problem situation and the social context played a significant role in the choice of deduction for exploration.

Unformulated proving was occasionally used for exploring, but these explorations cannot be counted as successful, since they were forgotten soon after (e.g., Stacey in Chapter III, section 2). Formulated proving was more successful (e.g., Rachel in section 2). The use of proving for exploration required that initial conditions suitable for deduction be clearly accessible to the participants. If these conditions happened to match the requirements of a particular deductive technique, then mechanical deduction was usually used (e.g., solving the Arithmagon using a system of equations). In other cases the proving was formulated and either focused on a particular goal or on reaching novel conclusions. The role of social conditions on proving to explore is discussed in Chapter IV, section 2.

In situations where the participants did not perceive sufficient initial conditions for proving they chose to explore inductively. This occurred in most situations. Exploring by induction was generally successful, occasionally leading to discoveries the observers were not expecting.

<u>Teacher games and other social contexts.</u>

The occurrence of formulated proving whether to verify, explain or explore was usually related to a social context. In the case of verification, the social context was what I call a teacher-game. That is, it was a situation where the perceived expectations of someone in the role of a teacher guide the actions of the prover. There were differences between the social contexts occasioning proving to explore and those which occasioned proving to explain.

The need to explain to others requires that the explanation be in a form that can be understood by others. This encourages formulation of the proving process. The participants' skills in formulating explanations varied, as did their success in explaining to others. The semi-formal proofs I offered as explanations were also accepted as such by some participants. The interpretation of proofs seems to require similar skills to the formulating of proving, and so formal proofs could not be accepted as explanations by all participants.

Proving to explore occurred both as part of a teacher game and because of the generation of a social context that supported proving to explore. It is this second context that is the most interesting. In Eleanor's case, at least, the sort of exploring she did depended on the exploring being done by those around her. When working with people exploring inductively, she explored inductively. When working with Rachel, who was proving to explore, Eleanor proved.

CHAPTER III

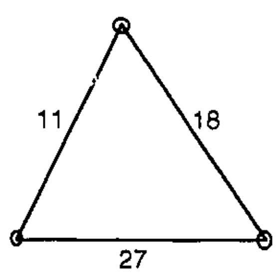
KERRY, STACEY AND THE ARITHMAGON: REASONING IN ACTION

I gave her one, they gave him two,
You gave us three or more;
They all returned from me to you,
Though they were mine before.
— Lewis Carroll,
Alice's Adventures in Wonderland.

In this chapter I will describe the reasoning of a pair of undergraduate students, Stacey and Kerry, as they investigate the Arithmagon situation. The terms I introduced in the previous chapter will be used in this description, clarifying the relationships between needs and proving, while presenting a more situated view of students' proving.

Stacey was a student in mathematics education when she volunteered to participate in the first clinical study. She asked if she could work with her friend, Kerry, who was an undergraduate student in Finance. It was plain in the problem solving sessions, and I hope it will be clear in the transcripts given below, that Stacey and Kerry enjoyed investigating the situations, and were comfortable working together. They investigated the Arithmagon in their first problem session. They were interviewed three weeks later, after investigating the other two problem situations.

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.



Generalize the problem and its solution

Figure 24: The Arithmagon prompt.

1. Mechanical deduction to explore*

After Stacey and Kerry were given the Arithmagon prompt (Figure 24) they negotiated how begin. The operant need at this point is a need to explore towards

* The numbers of the sections in this chapter correspond to MAT episode numbers in Stacey and Kerry's MAT for the Arithmagon in Appendix C.

the specific goal of finding an answer to the puzzle. Stacey proposed exploring inductively, by "trial and error" (line 2). Kerry rejected this suggestion, preferring to "deduce" the answer by mechanical deduction, using a system of equations (line 8).

(1) Kerry:	I guess this is uh, — uh
(2) Stacey:	Trial and error?
(3) Kerry:	No. We'll go. Let's assign variables to these then. Do you want? We can write on this paper right?
(4) DR	Sure
(5) Stacey:	Go for it
(6) Kerry:	We'll start with A B and C.
(7) TK:	We've got plenty of paper
(8) Kerry:	OK. Yeah. And I guess we'll probably get a system of equations out of that. And we'll try and- And we'll subtract one equation from the other. And try and deduce one variable, plug it back in — OK So that's the sum of A and B
(9) Stacey:	Yup.
(10) Kerry:	OK [Kerry starts writing equations]
(11) Stacey:	A and B is 11. B and C is 18. And A and C is 27 —
(12) Kerry:	OK — These are our three formulas. Our three equations.

After solving the equations they checked their answer against the original puzzle, verifying the result of a mechanical deduction by simple, unformulated proving. Kerry then noted the general rule that if the system of equations gave a solution, then the answer must work in the original puzzle. This is a second case of proving to verify, specifying from a general rule in a formulated way. In both cases the proving involves only a single deductive step, and is applied to a very simple situation.

2. Unformulated proving to explore

Stacey then began exploring again, this time without a clear goal (this episode is also described at the beginning of Chapter II).

(1) Stacey:	What happens if you add the middle numbers together? —
(2) Kerry:	Well I guess we could, hmm.
(3) Stacey:	I just want to try something. If you take 27, 18, and 11. 2, 4, 5, 56. Right?
(4) Kerry:	Sure.
(5) Stacey:	And you have — So you add each of those twice, right? — Yeah you do. That's not going to help you either. That's what you end up doing right?
(6) Kerry:	What'd you do?

(7) Stacey: You add A, B, C. Then you multiply them by 2. You get this answer. —

(8) Kerry: Do you add?

(9) Stacey: 22, and 34. Yup. Do you know what I mean?

(10) Kerry: Sorry. So you add this and multiply by 2 so, like, the sum of this is 28 times 2. And it's 56. Good one. What's that mean?

(11) Stacey: Nothing. [laughing]

(12) Kerry: Is that-

(13) Stacey: That was just-

(14) Kerry: Is that true for all of them?

(15) Stacey: Yeah.

(16) Kerry: I guess so. It must be. It can't just be fluke.

Stacey, in lines 1 through 5, is exploring, as indicated by her initial comment "What happens if you add the middle numbers together?" She continues, proving in an unformulated way. After determining the sum of the three numbers (56, line 3) she reasons that each of the secret numbers must be included in the sum twice. This occurs because "you add each of those twice," "those" being the secret numbers. Although she does not see any connection with her conclusion and the problem of generalizing the Arithmagon, she checks with Kerry to see if he has followed her reasoning (line 5). Because it is unformulated, he has not understood either her conclusion, or how she reached it (line 6).

She manages to communicate her conclusion in the specific case that they are considering (lines 7-10), at which point Kerry asks "Is that true for all of them?" provoking a need to verify. Stacey's unformulated proving, although it was originally begun to satisfy a need to explore, also satisfies this new need to verify but only for her. Kerry, who has no access to her unformulated reasoning, instead verifies by a probabilistic argument, based on the inductive evidence that most patterns observed in mathematical situations turn out to be general patterns, rather than "flukes."

An important aspect of this case is the ease with which Stacey's conclusion was forgotten. When Stacey and Kerry were searching for an explanation for the division by 4 in the method the discovered later in the session (see section 9 or Appendix E), they made no link between the relation Stacey discovered and the division by 4. Given their failure to come up with any explanation other than a weak analogy, it would be surprising if they remembered Stacey's relation but failed to make use of it. Kerry, in fact, claimed in the interview session that he had never even seen Stacey's relation.

3. Mechanical deduction to explore

After dismissing Stacey's relation between the sum of the sides and the sum of the secret numbers, Kerry proposed solving the puzzle again, using a matrix instead of a system of equations. This is another example of mechanical deduction used to explore, but as the solution to the puzzle was already known to them, this exploration must not have had that as its goal. Instead Kerry was exploring to find something new, without any idea of what it might be.

While Kerry seemed comfortable working with his matrices, and was proceeding in a mechanical way, Stacey seemed uncomfortable, and as Kerry worked she interpreted the matrices as equations. This became evident when Kerry made an arithmetic error resulting in the matrix:

1	1	0	11
0	1	1	18
0	0	1	12

Kerry saw nothing significant in this matrix, but Stacey, who was interpreting Kerry's mechanical deduction saw 0 0 1 12 as C=12, which she knew contradicted the value for C they had found earlier. This reveals an important aspect of mechanical deduction, that the meaning of the symbols being manipulated is suspended which the deduction proceeds.

After Stacey pointed out the contradiction, they carefully checked Kerry's calculations but did not find the error. They were uncertain what to do next, but eventually they continued reducing this matrix, arriving at a second solution to the Arithmagon. The existence of this second solution made them aware of an implicit assumption they had made, that only one solution existed. They then checked their new solution against the original puzzle (verifying by a short, unformulated act of proving) and rejected it, concluding that they had made an error somewhere in the reduction of the matrix. Kerry ended this episode by specifying from the (false) generalization that reducing a matrix gives a unique solution if the number of variable equals the number of equations.

4. "Generalizing"

Turning to the cryptic instruction "Generalize the problem and its solution" they were uncertain what to do. Kerry decided that generalizing meant describing what they had done in solving the system in general terms (e.g., "We added the first two equations.") This rehashing of their actions is not described as reasoning in my terminology, although it could be considered a kind of formulating of the rules of their mechanical deduction.

5. Inductive and deductive exploring

This is another episode of Stacey exploring, this time involving a mixture of unformulated proving, mechanical deduction, and inductive reasoning. As Kerry was winding up his "generalization" Stacey had been looking thoughtfully at the original problem. As soon as he finished, she began exploring:

(1) Stacey: OK. This is what we have. — — 18 11 and 27 and we're given three other numbers. — Right? What can we do with three other numbers? We can, extend lines. — — —

(2) Kerry: What'ya doing? Making another big triangle?

(3) Stacey: Yeah. — I don't know what I'm doing yet. ... And we'll call— What's 11, 27? Right. This one. What was this? 10?

(4) Kerry: Yeah. Let's get rid of this. The other answers
(5) Stacey [laughs] Well what-
(6) Kerry: This, this was 17. And, what, what was this?
(7) Stacey: It was-
(8) Kerry: This side here was
(9) Stacey: 17
(10) Kerry: C, was 17. Yeah.
(11) Stacey: And this is 17. What was the top one? 1.
(12) Kerry: 1. —

Here she begins exploring, without a goal: "I don't know what I'm doing yet." She adds a second triangle around the original one, and fills in the values which they now know on her new diagram (see Figure 25). The "other answers" Kerry refers to in line 4 are the values obtained from their matrices.

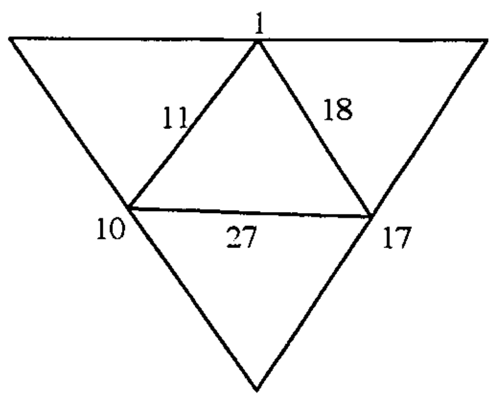


Figure 25: Stacey's triangle with "extended lines."

(13) Stacey: You keep going.
(14) Kerry: Where you going?
(15) Stacey: You keep going. We could find numbers for this. —
(16) Kerry: For that?
(17) Stacey: Yeah. —
(18) Kerry: OK. So you want to solve that then?
(19) Stacey: Umhmm
(20) Kerry: OK.
(21) Stacey: It will go right to zero. —
(22) Kerry: Are you saying-
(23) Stacey: This I don't, I don't know.
(24) Kerry: Are you saying the numbers would keep getting smaller?
(25) Stacey: Yeah, these ... would have to be-
(26) Kerry: Yeah I guess the will be getting smaller

(27) Stacey: -like decimals. 'Cause you got 1 on one side. — ...

Stacey proposes here that they solve the new 1-10-17 triangle revealed by her diagram. This generation of new data about triangles is part of exploring by inductive reasoning. At the same time her prediction "It will go right to zero" is arrived at by unformulated proving, possibly based on her earlier conclusion (in episode 2) which implies that the sum of the sides will be halved for each new triangle she adds by "extending lines." As the sums must have a limit of zero, she reasons that the individual numbers on the sides must also have the same limit, and that eventually they "would have to be ... decimals." This is likely to happen soon, as "you got 1 on one side."

(28) Kerry: You, want, you want to try to solve it then?

(29) Stacey: Yeah sure, Away you go. ...

(30) Kerry: OK, then let's label it.

(31) Stacey: I like drawing it

(32) Kerry: OK, well label these anyways. So we've gone. So go. D E

(33) Stacey: and F

(34) Kerry: F. OK. D plus F equals 10. D plus E equals 1. E-

(35) Stacey: E plus F

(36) Kerry: plus F

(37) Stacey: 17

(38) Kerry: Equals 17 — OK So uh,

(39) Stacey: ... How are we doing here?

(40) Kerry: OK, we'll subtract, we'll take uh,

(41) Stacey: 1 minus 2

(42) Kerry: 1 minus 2. So we've got

(43) Both: F minus E is 9

(44) Kerry: and then we'll go, uh.

(45) Stacey: That plus 3

(46) Kerry: Hmm?

(47) Stacey: Plus 3

(48) Kerry: Plus 3?

(49) Stacey: Yeah.

(50) Kerry: OK. E plus F equals 17 and then cancel, cancel. 2 F equals 26

(51) Stacey: F equals 13.

(52) Kerry: 13

(53) Stacey: So it's 13 here.

(54) Kerry: 13 there. And, want another one?

(55) Stacey: Yeah.

(56) Kerry: OK. Then . Oh I think we're going to get a nice negative number.

(57) Stacey: Oooh

(58) Kerry: D equals minus 3 —

(59) Stacey: Negative 3 and E is then

(60) Kerry: E is

(61) Stacey: [whispers "4"]

(62) Kerry: [whispers "4"]

(63) Stacey: Sorry. 4. [laughs]

(64) Kerry: OK. So we got. We did get a negative number.

This long recital is the spoken manifestation of mechanical deduction. Throughout it is Kerry who writes the equations, although Stacey is involved in what he does. Note that the equations are referred to by numbers (lines 41, 42, 45, 47, and 48). They are being added as formal strings, although Stacey is using some aspects of their structure to decide which ones to add first. At line 54 Kerry continues to work with the equations, in spite of the fact that knowing one secret number means that they could return to the meaningful situation of the triangle to find the rest of the values. Because the meanings of the equations have been suspended while doing the mechanical deduction, there is no prompt for him to return to the original situation. Stacey is connecting what Kerry is writing to the original triangle, and when she whispers the final value, 4, she has seen the answer in the triangles rather than in the equations. Her apology in line 63 is directed towards the observers, who have asked them to speak clearly to make deciphering the videotape easier.

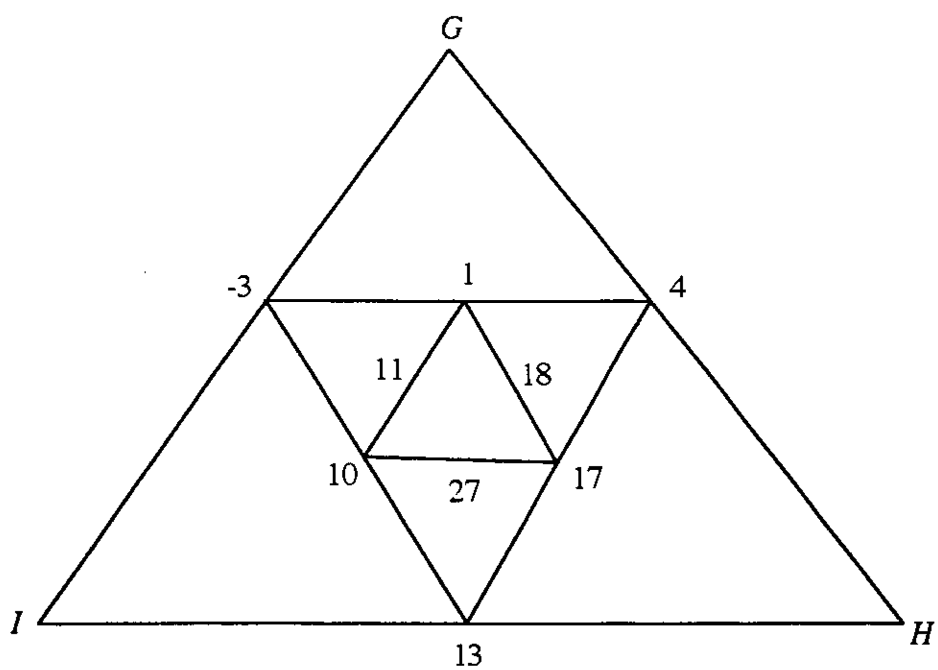


Figure 26: Stacey's third triangle.

(65) Stacey: Oooh. Let's keep going.

(66) Kerry:	OK. —
(67) Stacey:	This is negative 3 — This is 4. And they join each other.
(68) Kerry:	You didn't drop this low enough. Soon you'll need a huge sheet of paper.
(69) Stacey:	OK
(70) Kerry:	Pretend the corner's there.
(71) Stacey:	That was 4. And this is —13.
(72) Kerry:	OK. G H and I. — So I plus G equals negative 3. G plus H equals 4. H plus I equals.

Here Stacey extends lines again, producing the diagram shown in Figure 26. Kerry immediately begins setting up the equations to solve this new triangle by mechanical deduction (line 72).

6. Inductive reasoning to explore

In this episode Stacey and Kerry explore inductively, and reject conjectures based on short unformulated proving, or verify them inductively. The transcript continues directly from the transcript in episode 5.

(73) Stacey:	Do you know what?
(74) Kerry:	What?
(75) Stacey:	— OK. You want a prediction?
(76) Kerry:	OK. Sure.
(77) Stacey:	Well I don't know about this. This is just the one little step. This is decreased by 14. This is decreased by 14. And this is decreased by 14. —
(78) Kerry:	What's that mean?
(79) Stacey:	From here to here.
(80) Kerry:	I see that. That's-
(81) Stacey:	Yeah
(82) Kerry:	-pretty neat.
(83) Stacey:	Yeah.
(84) Kerry:	Yes.
(85) Stacey:	Keep going. Does that mean this will decrease by 14? For this line here?
(86) Kerry:	OK. For H. Well.
(87) Stacey:	Is that a prediction that H is 3? Go for it.

Stacey's prediction comes as the result of exploring by inductive reasoning. She has seen that 27–14=13, 18–14=4, and 11–14=⁻3 in this case. She is aware of the weakness of generalizing from one case "This is just the one little step" and so she urges Kerry to find the solution for the ⁻3-4-13 triangle in order to provide another case on which to base an inductive argument. Kerry solved his system of equations (again using mechanical deduction to explore towards a specific goal) and soon found that H=10, contradicting Stacey's conjecture. The rejection of her conjecture at this point qualifies as verifying the falsehood of her conjecture by unformulated proving involving a single deductive step.

After Kerry had found the values of *G, H,* and *I*, Stacey and Kerry advanced new predictions. Kerry predicted that the next difference would be 3.5, reasoning inductively that the differences, 14 and 7, were being halved at each stage. Stacey suggested that the differences might alternate: 14,7,14,7,... To test Stacey's prediction they switched from their previous pattern of solving each new triangle using a system of equations, and instead tried the values predicted by Stacey directly on the new ⁻6-3-10 triangle. When the predicted values failed to work this verified, by way of simple unformulated proving, that Stacey's prediction was wrong. Stacey observed that her prediction was based on inductive reasoning from very little data, so it was not surprising when it failed. Kerry then returned to his systems of equations to find the correct values. They conformed to his prediction of a decrease of 3.5, verifying his conjecture inductively. To provide further inductive verification, Kerry predicted the values for the next triangle, and tested them. At this stage they both accepted as a general principle that the difference for each triangle was half of the difference for the triangle contained in it.

7. Unformulated proving to explore

Having verified a generalization inductively, Stacey and Kerry were now in a position to explore by proving from their generalization. In episode 7 they did so. In fairly quick succession they made two predictions based on unformulated proving. They predicted that if they drew a triangle inside the original 11-18-27 triangle, the values on its sides could be found by adding 28 (14 times 2) to the values on the 11-18-27 triangle's vertices. They did so, which provides an example of exploring deductively, followed by verifying inductively, precisely the opposite of the pattern suggested in curriculum documents (e.g., NCTM, 1989) and the research literature (e.g., Fischbein and Kedem, 1982). They then reasoned that there would be a limiting value for the numbers on the triangles if the process of adding triangles outside was continued indefinitely. Aside from a quickly rejected suggestion that Stacey calculate this limit, they made no additional attempt to verify this conjecture, presumable because an inductive verification would have been laborious, and the unformulated proving which suggested it also verified it. This episode ended with Stacey suggesting that they find a formula for their conclusion, but the confusion of variables in their diagram convinced them that this would be difficult.

8. Reasoning by analogy to explain

This episode marks the first time Stacey and Kerry evidenced a serious need to explain. Kerry wondered why the initial difference in their triangles had been 14, and also why his matrices had not produced the correct answer. Stacey experimented with the numbers involved in the original puzzle, and discovered that 56, that is (11+18+27), is evenly divisible by 14. This lead them to conclude (by the same inductive "It can't just be fluke" reasoning Kerry employed earlier) that the number 4 must be significant in some way. Their interest then turned to explaining the association of 4 with the original situation. They each noted an analogy which would explain 3 being important, as if they were attempting to find analogies for 4 and failing. Stacey noted that triangles have three sides, which Kerry noted that finding the sum of three numbers and dividing by 3 would be taking the average of them. Kerry even went so far as multiplying 14 by 3 to see if that product would be related to the situation in some way. These associations with 3 are imperfect examples of explaining by analogy, as they fail to explain the occurrence of 4. In episode 10 Stacey offers a more reasonable analogy to explain the occurrence of 4.

9. Inductive reasoning to explore

One of the observers, Tom Kieren, interrupted them at this point, and asked "Is 14 special, or is one fourth of the sum special?" This provoked further inductive explorations directed towards the goal of seeing whether 14, or 4, was a general feature of the Arithmagon situation. The checked one of the triangles they had already produced by extending lines, and a new triangle with sides 8, 19, and 21. These explorations led them to make a new conjecture, of a general method for solving Arithmagons:

(1) Kerry: Hmm? Yeah, OK. — Well, we figured out that's how it goes, eh?

(2) Stacey: Yeah, it's quite constant.

(3) Kerry: You take the. So if we. If we were first given this, we could've — found the sums, right off the bat. We found the sums. Found the sum, sorry, the sum of these three. — Of these: 11 plus 18 plus 27 equals 56. We divided that by 4, right off the bat. We got our 14 to start with. —

Stacey interrupts Kerry's description at this point, but to continue and clarify it, their method is as follows. First they extend the diagram by adding a second triangle around the first one. Next they add together the three known sides ("11 plus 18 plus 27 equals 56"), and divide by 4. In this case this gave them 14. Subtracting this result from each of the known sides gives values which they assign to the corners of their outer triangle (see Figure 27). Adding these gives numbers for the sides of the outer triangle, which are also the values for the secret numbers on the corners of the inner triangle.

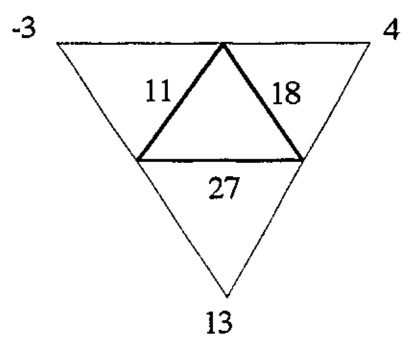


Figure 27: Values on the outer triangles.

10. Reasoning by analogy to explain

Having developed a general method for solving Arithmagons, they then returned to the problem of explaining the importance of the number 4. This transcript continues the transcript in episode 9.

(4) Stacey: But why did we divide by 4?

(5) Kerry: Because 4 is the. We figured out 4 is the magic thing here.

(6) Stacey: Yeah, but why?

(7) Kerry: Oh yeah, OK, so we figured it out anyways. – But we now have to figure out, why 4? So we figured out the system, now we've just got to understand: why the system?

(8) Stacey: Yeah.

(9) Kerry: Why is 4-

(10) Stacey: Yeah.

(11) Kerry: So important?

(12) Stacey: A triangle has 3 sides, and 3 points. When you cut that in 1, 2, 3, 4,

(13) Kerry: What? How are you cutting that in 4?

(14) Stacey: Why didn't I just see that? But does that have anything to do with it? Here you have a triangle. There you have a triangle. Here you have a triangle. There you have a triangle.

(15) Kerry: Well that's 4 triangles. –

(16) Stacey: We'll just use that, OK? [laughs]

In lines 10-14 Stacey has finally found something in the problem situation which is associated with the number 4. She observed that the act of nesting the original triangle in a larger triangle created four triangles approximately the same size as the original (see Figure 28).

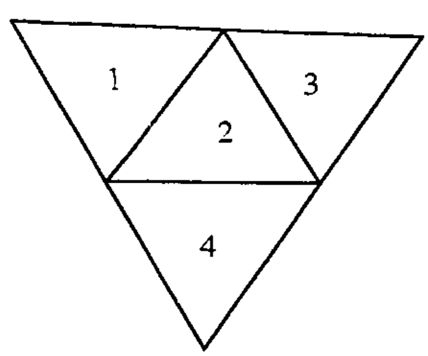


Figure 28: Stacey's four triangles.

Kerry was unhappy with this explanation however, "You can't just say that, you have to explain that. Why are those 4 triangles important?" Stacey's analogy is a weak one, between the geometry of the situation and the method of its solution, so it is not surprising that neither of them had much faith in it.

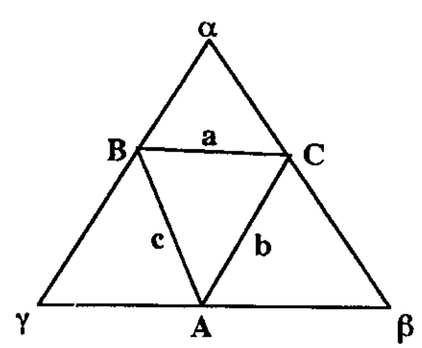
11. Further exploring

At this point the problem session ended. Before they left Stacey and Kerry asked if they had done what we had expected; if they had found the answer, as it were. After congratulating them on following a path which we had not anticipated in the least, we commented that investigating Arithmagon squares was one generalization we had thought that some participants might make. Stacey and Kerry reported when they came in for the GEOworld session the next week that they had gone to the library and spent many hours exploring how their method could be extended to Arithmagons of four or more sides, evidencing a strong need to explore at that time.

12. Formulated proving to explain, interpreting a semi-formal proof

When Stacey and Kerry left the Arithmagon problem session, they were still wondering why the number 4 had come up in their general method for solving the Arithmagon . In the interview session, they were shown a formal proof (see Figure 29), which derives a formula equivalent to their solution method from the relations given in the problem. For Kerry this proof explained the need to divide by 4 in their method. Several times he made comments like "That's where we get the $\frac{1}{4}$ from, Neato" and "That's the $\frac{1}{4}$ rule." For him the formulated proving involved in interpreting the proof satisfied his need to explain.

My claim that formulated proving is involved in the interpretation of proofs is based on my own experiences, and the following analysis by Freudenthal (1973).



$$
\begin{aligned}
a+b+c &= B+C+C+A+A+B \\
&= 2(A+B+C) \\
&= 4(\alpha+\beta+\gamma) \\
a &= B+C \\
&= \alpha+\gamma+\alpha+\beta \\
&= \alpha+\alpha+\beta+\gamma \\
&= \alpha+\frac{a+b+c}{4} \\
\alpha &= a-\frac{a+b+c}{4}
\end{aligned}
$$

Figure 29: The proof shown to Stacey and Kerry in the interview session.

> The only mathematician I can tell precisely of how he reads mathematical papers, am I myself. I never read mathematical papers from the first to the last word. I start with the results. I appreciate them being neatly exhibited. Then I think about them. If I cannot confirm them, I look through the paper for some indication how they can be proved. Maybe I then succeed in confirming the results. Otherwise I look for lemmas I understand and try to derive the main theorems from them. Maybe I have to take a closer look at some proof; if an earlier result is referred to, I go back to it. If finally by my own means and a bit of cribbing I have confirmed the results, that is, if I master all the connections, I am likely to read the papers through once again systematically. Others have told me that they also read papers written by others in approximately the same way. There are people who can read papers systematically, page by page, line by line, letter by letter. To do so testifies to a strong discipline of mind which is not everybody's attribute. I think it is the rule that in trying to understand papers written by others, people behave as if they are making original investigations. They try to reinvent the contents of the paper; this is a bit easier than brand-new inventions because you can crib as much as you want. (1973, p. 115)

Interpreting proofs involves a process of proving, guided by the proof. This process need not begin with the definitions and premises of the proof. In fact, in the case of many proofs, this approach would be quite difficult. The process of interpretation involves a loose reading of the proof, identifying the basic structure, and looking for aspects which might cause difficulties. Once this is done, tricky parts of the proof might be examined in more detail. It is at these times that the definitions and premises chosen often turn out to be relevant, because they were chosen by the originator of the proof not before proving, but during the process, as the need for them arose.

CHAPTER IV

CONSTRAINTS ON PROVING

> *"I know what your thinking about," said Tweedledum; "but it isn't so, nohow." "Contrariwise," continued Tweedledee, "if it was so, it might be, and if it were so, it would be; but as it isn't, it ain't. That's logic."*
>
> — Lewis Carroll,
> *Through the Looking Glass, and What Alice Found There.*

My research would be much more simple if the same needs always led to the same reasoning. If exploring meant proving, and verifying meant reasoning inductively, and explaining meant reasoning by analogy, then the need to prove would be all that needed to be said about it. Unfortunately, the examples in the previous chapters show that this is not so. In addition to the needs that motivate proving there are also other factors that constrain proving. In Chapter II these constraints are mentioned as they occurred, but the focus of that chapter is needs, and so constraints are not treated systematically there. This chapter attempts to organize what I learned about three important constraints on proving into a useful summary.

1. Individual differences

Not everyone proves in the same way. It would be foolish to try to generalize from a few people's proving behaviors to a larger group. In the case of the participants in my studies there is ample evidence of considerable variation in proving style and in the needs felt (see Appendix D for a summary of this variation). In the previous chapter I described Stacey and Kerry's approaches to the Arithmagon in which there are clear differences between them as individuals. In this section I will describe two other participants, Bill and John, in order to further illustrate the range of activity I observed.

Bill and John were two students in a Math 13 class at North School. Math 13 is the grade 10 mathematics class in which students who have done poorly in grade 9 find themselves placed. Bill was among the better students in his class and seemed to be actively engaged in following the teacher's lessons and in doing the assigned work. He responded readily to the teacher's questions and asked questions if he did not understand something. John was more quiet but also seemed to be actively following the lessons. He did have difficulty with the assigned exercises on some occasions and was more likely to turn to Bill for help than to ask the teacher.

Bill and John were not only different in many ways from the other participants in the studies (as one would expect given the differences between the Math 13 program and undergraduate mathematics) but also different from each other. At the same time the proving they did is similar in many ways to that done by other participants.

Bill explored generally toward some goal and explained both to himself and to John but verified only by making reference to my authority. In his explorations he made use of both unformulated proving and inductive reasoning. He had a preference for an explanation by analogy for the sum of two odd numbers being even but on other occasions accepted unformulated and formulated proving as explanation. He was the only participant in the study to explicitly reject inductive verification, accepting verification by authority only.

Bill's spontaneous efforts at proving to explain or explore were generally short and unformulated. When I guided him through more formulated proving however, he displayed a clear understanding of the arguments and, in the case of the formula we deduced for the Arithmagon he accepted the proving as explaining. While proving is something Bill is capable of, at least unformulated proving, it is not his first choice for satisfying a need to explain, explore or verify. In explaining the fact that two odd numbers always add up to an even number, he preferred his analogy to the product of two negative numbers to either of the pre-formal proofs we produced. In that case he explicitly said he did not like explanations like the pre-formal proofs. In the case of the formula for solving Arithmagons, he had said he preferred the form of the formula which made it easy to explain how it worked. This seeming contradiction illustrates the importance of the need to prove. In the case of the Arithmagon formula we were dealing with a new piece of mathematics, which lacked an explanation. In the case of the sum of two odd numbers, Bill had provided an explanation by analogy, and had it on my authority that the rule was a general one. There was no need for proving the rule.

In exploring Bill was like the other participants in preferring to explore inductively. In verifying he was quite different. Most of the participants would use either inductive reasoning or proving to verify. In Bill's case reasoning did not verify. He relied completely on my authority to establish truth. This was strongly suggested by Bill's reaction when I claimed that 117 digit odd numbers do not generally add up to even numbers. Bill responded, "I don't see how come that is, ... but that's really kind of neat." Against my authority, two pre-formal proofs, an analogy, and a wealth of inductive evidence were insufficient to convince Bill.

John was less involved and less vocal than Bill and so there is less that can be said about his reasoning. He explored inductively but whether he had a goal in mind was not clear. He explained when asked to, and asked for explanations from Bill. Explaining by proving was acceptable to him, and he used proving to explain in simple situations. His verifying was tentative, but he was more willing than Bill to accept inductive verifications.

The clearest indication of John's thinking came after Bill and I had deduced a formula for the Arithmagon: $\frac{(A+C)-(A+B)-(B+C)}{2}$ (MAT episodes 10-14). Our formula involves variables representing the three unknown corners, grouped in added pairs according to sides. For Bill this formula was good because it had explanatory power. While he recognized this power, John preferred a different formulation: $\frac{E-D+F}{2}$ This formula is easy to use. John choice of formula illustrates his preference for mathematics that is easy to use, as opposed to mathematics that is easy to understand.

Summary

While the cases of Stacey and Kerry (described in the previous chapter) and Bill and John do not begin to cover the range of individual differences seen in the participants in my studies, they do point out certain kinds of differences that influence the need to prove and the proving that is done. An important difference between Stacey and Kerry was in what they knew, or at least the technical skills with which they felt comfortable. Kerry's use of mechanical deduction could not have occurred if he had been unfamiliar with solving systems of equations and might not have occurred if he had been just as proficient but less comfortable. Table 2 shows the initial method used to solve the original puzzle for all the participants in the studies, and whether they had been taught to solve systems of equations. The entries "10+" and "10–" indicate the accelerated Math 10 program at South school, and the Math 13 class at North School, respectively. The cases marked with a question mark (?) were in the process of learning to solve systems of equation at the time of the Arithmagon session, and it is not clear exactly what they had been taught at the time of the session.

Group	Grade	Solution method	Taught?
Ben & Wayne	U	Trial and Error	Yes
Jane & Chris	U	System of Equations	Yes
Kerry & Stacey	U	System of Equations	Yes
Eleanor & Rachel	U	System of Equations	Yes
Roger & Marie	U	Trial and Error	Yes
Trisha & James	U	Trial and Error	Yes
Laura & Donald	U	Trial and Error	Yes
Colin & Anton	12	System of Equations	Yes
Joseph, Stephen, & Scott	10+	System of Equations	Yes
Alec & Darrell	10+	System of Equations	Yes
Tara & Topaz	10+	Trial and Error	?
Ann, Lynda, & Joanna	10+	Trial and Error	?
Bill & John	10–	Trial and Error	No
Sandy	6	Unformulated proving	No

Table 2: Use of systems of equations in solving the Arithmagon.

Bill and John indicate the importance of individual's beliefs about mathematics and learning to the need to prove. Bill's reliance on authorities as the ultimate source of verification limited the importance of that need in motivating both proving and inductive reasoning. While proving to verify is not common in general, most of the participants in the studies did verify inductively, and Bill's reluctance to do so may be limiting his possibilities for learning from his own experiences. John's preference for useful mathematics over explanations could make learning mathematics more difficult for him. It provides him with a disincentive to develop relational understandings as opposed to instrumental understandings (to use terminology from Skemp, 1987). Instrumental understandings are not as useful a basis for learning new concepts as relational understandings and are harder to maintain.

2. Social constraints

In all the sessions the participants worked in a social context. The observers and the other participants defined an environment in which each participant reasoned. In my final chapter I describe a theory in which the development of deductive reasoning is tied to social relations between people, mediated by language. If deductive reasoning develops out of social relations then it is not surprising that reasoning in problem solving is constrained by social factors. In this section I will describe how social constraints were linked to proving activities in the cases of Rachel and Eleanor, two university students, and Bill and John, two Math 13 students.

Eleanor and Rachel

Eleanor and Rachel provide an excellent example of social constraints, in part because of their manner of working together and in part because of the different social contexts in which they worked. The Fibonacci session was their first session. In it they established a pattern of working somewhat independently but consulting each other regularly. This permitted them to act independently when they wished but also to work with each other's ideas. In the Arithmagon session they worked with Ben and Wayne, but the dynamic between them was similar to that they established in the Fibonacci session. Other social constraints, however, meant that they acted quite differently in the two sessions.

Rachel

Rachel spent the Fibonacci session exploring and verifying the conjectures she made inductively (see MAT). She noticed a pattern in the sequence of every third Fibonacci number, which she then described to Eleanor. Eleanor formalized the pattern as $F_{3n} = 4F_{3n-3} + F_{3n-6}$. For example, $F_9 = 34$, $F_6 = 8$, and $F_3 = 2$, and $34 = 4(8) + 2$. Rachel continued to explore inductively, looking for similar patterns for F_{4n}, F_{5n}, F_{6n}, etc. Whenever she found a pattern she described it in terms of actions, e.g., "Multiply by 4, add the previous one, and you get the next one." She then helped Eleanor formalize her pattern, or in the case of the later patterns, formalized it herself. This pattern of discovery, verification and formalization was her way of working throughout the session.

In the Arithmagon situation, Rachel was sitting between Eleanor and Tom Kieren, who was observing. She began by solving the puzzle using a system of equations, and then began looking for patterns, exploring inductively as she had in the Fibonacci situation. She interrupted her explorations to listen to Ben describe his method of solving by systematic trial and error. She then attempted to use his method. When I suggested to Ben that he try to solve a 1-4-12 triangle (because his method would need to be modified to solve it) Rachel also attempted to solve that triangle.

About halfway through the session, Rachel's activity changed (MAT episode 8). She stopped exploring inductively and began using formulated proving to explore special cases of the problem (e.g., when two sides are equal). Her explorations are described in Chapter II, section 2. In the interview session I asked her why she had begun to explore in this way.

(1) DR: Why were you doing that?

(2) Rachel: Oh, you want me to answer right now? [laughter] Because I was stuck at them. I didn't know where to go. And Tom was sitting beside me saying, "Well, what can you do now?"

(3) Eleanor: Nothing, nothing. [laughter]

(4) Rachel: So I was thinking I'd better think of something, or else that question's going to keep coming. So I just thought, well hey, in math you always get that right?. You always get those conditions. Every teacher's listing these conditions. Now, if we have this condition where this equals that. You know what I mean. So that's. It just. You know. It comes from my head. Something I knew of already that I thought I could apply to that problem.

At first, Rachel had addressed her need to explore by way of inductive reasoning, but the presence of Tom was a social constraint that led her to use formulated proving. A bit later, Tom became a more explicit social constraint, by suggesting that she see what she could derive for the general case. This led to her derivation of a general formula for solving the Arithmagon.

Eleanor responded to Rachel's announcement of her formula by asking for an explanation. This social context led to a need to explain, and Rachel's immediate history of formulated proving led her to use formulated proving to explain in this context.

Eleanor

In the Fibonacci session Eleanor played off Rachel's ability to see patterns by formalizing them, and then searching for patterns in the formalization. Without Rachel's participation her activities would have been different, since her focus would have had to include the original sequence as well as the relations Rachel found in it.

In the Arithmagon situation Eleanor was sitting between Rachel and Ben. She began solving the original puzzle using a system of equations and then stopped when she saw that Rachel had found the answer. After comparing methods with Ben and Wayne she then began trying to use Ben's method to solve the triangle. When I proposed the 1-4-12 triangle she discussed with Ben whether it was possible or not, and then, at Ben's request, solved it using a system of equations. She then joined Ben and Wayne in exploring inductively. They found the relations $A+a = B+b = C+c$ and $a+b+c = 2(A+B+C)$. Eleanor then worked independently, eventually discovering a general method of solution based on these relations. Her method is described in Chapter II, section 1 and in Appendix E. After Rachel announced her formula and explained its derivation to Eleanor, Eleanor described her method, and began explaining, to herself for the most part, how it related to Rachel's formula.

Eleanor changed the way she was working depending on the way the people around her were working. She used mechanical deduction while Rachel was doing so. She tried Ben's method, and solved a triangle with a system of equations when Ben asked her to. She explored inductively when working with Ben and Wayne, who had been working inductively (and, in Wayne's case, by analogy) all along. She explored using unformulated proving when working alone and then explained using formulated proving when working with Rachel.

Bill and John

Bill and John are included here because of the contrast between their inductive and analogical reasoning in the two problem sessions, compared to their use of formulated proving in the interview sessions when I encouraged them to reason deductively. In the problem sessions they occasionally used unformulated proving, but it required the social context of my encouragement for them to formulate their proving, and to prove for extended periods. This is heartening in terms of the potential influence a teacher has on students' reasoning. The proving they did in the interview sessions is described in Chapter II, section 1.

3. The prompts

The factor that might have been expected to have an influence of the participants' proving, the problems they were investigating, did turn out to constrain the activities in which they engaged. Investigations of the Arithmagon problem involved proving, and especially mechanical deduction, more often than either of the other two situations. In the GEOworld situation almost all the activity was inductive exploration. In the Fibonacci situation, some participants did nothing but inductive exploration while others engaged in some, usually unformulated, proving.

Arithmagon

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.

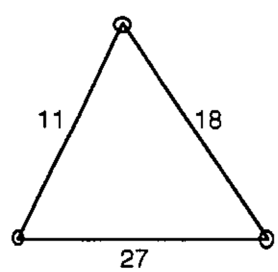


Figure 30: The Arithmagon prompt.

The Arithmagon problem was chosen for its potential to be generalized in many ways, and the variety of solution methods and interest it created in informal pilot testing of it. In the studies it lived up to my expectations, and in some cases it occasioned truly unexpected mathematical activity (see Appendix E). The discovery that other researchers (Simpson, 1994; Duffin & Simpson, 1993) were [illegible]sing the problem for research into proof was fortuitous and added another point of view to my research.

All the participants in the studies investigated the Arithmagon. The activities in which they engaged ranged from inductive exploration to formulated proving. Some participants discovered a general method for solving Arithmagons, some discovered a formula, some did both, and some did neither. The activities of all the participants in my studies is summarized in Table 3.

The variety of solution methods listed in Table 3 only begins to suggest the openness of the Arithmagon situation. For example, Kerry and Stacey's general method for solving Arithmagons, based on adding a new triangle around the original, is very different from Eleanor's method of adding the sides together, dividing by 2, and using this number, which is also the sum of a side and the corner opposite it, to find the corners. Appendix E lists a number of other methods of solving the original puzzle and deriving a general method.

Group	Solution method	Means of deriving a general method or formula
Ben & Wayne	Inductive	Told by Rachel
Jane & Chris	Mechanical deduction	None
Kerry & Stacey	Mechanical deduction	Inductive
Eleanor & Rachel	Mechanical deduction	Formulated proving
Roger & Marie	Inductive	Guided proving
Trisha & James	Inductive	Inductive
Laura & Donald	Inductive	Inductive
Colin & Anton	Mechanical deduction	Mixed inductive and deductive reasoning
Joseph, Stephen, & Scott	Mechanical deduction	None
Alec & Darrell	Mechanical deduction	None
Tara & Topaz	Inductive	Inductive
Ann, Lynda, & Joanna	Inductive	None
Bill & John	Inductive	Guided proving
Sandy	Formulated proving	Formulated proving

Table 3: Summary of participants' activities in the Arithmagon situation.

Fibonacci

> The Fibonacci sequence begins:
>
> 1, 1, 2, ...
>
> and continues according to the rule that each term is the sum of the previous two (e.g., 1+1=2).
>
> The Fibonacci sequence has many interesting properties.
>
> Can you find an interesting property of every third Fibonacci number?
>
> Can you find other interesting properties?

Figure 31: The Fibonacci prompt.

The Fibonacci numbers are famous as a rich source of patterns, all derived from a simple rule. In my research for my master's thesis (Reid, 1992), I found that the pattern of every third Fibonacci number being even was easily discovered and proved. For this reason I included the Fibonacci situation in my research for this dissertation. Surprisingly, the pattern of every third Fibonacci number being even was either too simple (some participants noticed it but continued looking for something more significant) or missed entirely. One unexpected outcome was the discovery, by several participants, of a relation between every third Fibonacci number which was new to me. Rachel's discovery of it is described above (section 2). The wording of the prompt caused some trouble, and it was changed for each of the three studies for which it was used, but in all cases the presence of several suggestions for patterns to notice led to participants moving quickly on from one inductive exploration to the next. This fragmented pattern of exploration is described in more detail in Kieren, Pirie, and Reid (1994). The Fibonacci situation was not used at South School because of this problem.

My chief expectation for the Fibonacci situation was that the participants would notice that every third Fibonacci number is even, and prove this, noting that the rule defining the sequence forces a pattern of Odd, Odd, Even, Odd, Odd, Even, onto the sequence. While this pattern was noticed in several cases, it was only Kerry and Bill who proved the pattern, and in both cases it was as a result of an observer's intervention. What I had expected to be an occasion for proving to explain became an occasion for proving in a teacher-game.

GEOworld

The GEOworld offered the possibility of seeing the participants prove from postulates of their own creation, as a way of exploring and explaining. The situation is similar to a scientific investigation, as initial theories must be established inductively, but then proving can be used both to test the theories, and to explore in a more directed way. None of the participants did anything in the GEOworld situation other than exploring inductively and occasionally making a prediction and testing it. To allow additional time for interviews, the GEOworld situation was not used at North School.

Summary

The sampling of examples in this dissertation reveals that the Arithmagon situation was much more conducive to proving than either the Fibonacci situation or GEOworld. That some situations are better for proving than others is not a surprise, but the exact features that made the Arithmagon different are not entirely clear. It was important that the situation gave some initial relations on which to build deductive arguments, which GEOworld did not do. The imprecision of the prompt, compared to the Fibonacci prompt, may also have been an advantage.

It should be noted, however, that having a problem situation conducive to proving is not enough. As has been pointed out above, social and personal factors are also important, and the variety of activities in which participants engaged in the Arithmagon situation is an indication of this.

CHAPTER V

TEACHING PROVING

Histories make men wise; poets, witty; the mathematics, subtle; natural philosophy, deep; moral, grave; logic and rhetoric, able to contend.

— Francis Bacon, *Essays*,
50. Of Studies

The importance of teaching proving has long been recognized by mathematics educators and curriculum designers. The difficulty of teaching proving also has been recognized, and research focused on improving the teaching of proving dates back at least sixty years (e.g., Fawcett, 1938). The insights into students' need to prove provided by my research suggest ways of modifying and extending teaching methods to help students develop mathematical thinking from their own ways of reasoning.

1. Proving in the curriculum

In North America three curricular positions regarding proving can be identified. In some places proving is taught as part of geometry, the teaching of which occupies the second year of high school. In other places the curriculum has been reorganized, downplaying geometry and proving. This move may have been inspired in part by the poor results of teaching proving in geometry (Senk, 1985). Other places have adopted the NCTM *Standards* (1989) as the basis for their curricula. The *Standards* place considerable emphasis on mathematical reasoning in general, and proving in particular, and encourage the teaching of proving in all mathematical contexts, not just geometry.

Alberta

The current curriculum of Alberta illustrates the second curricular position, the downplaying of geometry and proving. In the introductory material to both the Courses of Studies, and the Teacher Resource Manuals for grades 10, 11, and 12, the following definitions occur:

> Throughout the learner expectations, the words verify and prove appear. For the purposes of the Senior High Mathematics Program, they are interpreted as:
>
> - Verify: to substantiate the validity of an operation, solution, formula or theorem through the use of examples that may or may not be generalized;

- Prove: to substantiate the validity of an operation, solution, formula or theorem in general and to provide logical arguments for each step in the process. (Alberta Education, 1991, p. 5; 1990, p. 6; 1989 p. 5; emphasis in original.)

Note that 'proving' refers only to deductive reasoning used to verify. The use of 'proving' in this restricted sense is common in mathematics education. Even though attention is paid to the definitions of these words, which are supposed to occur "throughout the learner expectations," proving is rarely mentioned in these documents.

In the Program of Studies (Alberta Education, 1989) for Math 10*, the word "verify" occurs six times in 102 pages. "Prove" does not occur, but expectation 1.1.1 in the topic area Coordinate Geometry and Graphing reads: "Students will be expected to be able to deduce the distance formula from the Pythagorean theorem." This is the only explicit reference to proving in the Math 10 curriculum documents.

In the Course of Studies (Alberta Education, 1990) for Math 20, "prove" does not occur; however, students are expected to provide two "logical arguments" in the context of geometry (pp. 12, 13). The word "verify" occurs four times in that document.

In Math 30, things improve somewhat, as proving is mentioned in non-geometric contexts. Students are expected to prove the Remainder Theorem and the Factor Theorem in the unit on Polynomial functions (Alberta Education 1991 p. 37) and to prove trigonometric identities (p. 44).

The discussion of the expectations related to proving in the Math 30 Teacher Resource Manual (Alberta Education, 1991) are interesting as an illustration of the ways proving is seen by curriculum planners.

> Manipulating trigonometric identities provides an excellent opportunity for students to learn to "prove" that a given relationship is true. The nature of proof should be discussed, particularly in terms of the difference between a verification using particular values of the variable and a complete argument that demonstrates truth in general. A discussion of the nature of deductive and inductive proofs would fit well here....
>
> Students should be encouraged to present logical arguments to show that the quotient and Pythagorean identities are true. Note that this does not necessitate the use of the T proof processes that were so common in the teaching of deductive geometry. This is an excellent place to discuss deduction and show students that a proof is a

* The high school mathematics courses in Alberta are numbered as follows: The 10/20/30 stream is grade 10/11/12 mathematics (respectively) for college bound students. The 13/23/33 stream was intended to be the regular stream for most students, but the desire to have the option of attending a post-secondary institution means that many students opt for the 10/20/30 route instead. The 13/23/33 route has become the route for students who have difficulties in mathematics. Specialized courses (business math, calculus) are given the tens digit appropriate to their grade level, and a units digit other than 0 and 3.

> logical, cogent sequence of statements beginning with what is given or is known to be true, followed by statements based on previously established knowledge and concluding by what is to be proved. (p. 45, quote marks in original)

The curriculum planners clearly see proving as a process whose function is verification. Although the definition of "prove" they gave in the introductory material implied that proving is a deductive process, they imply here that "proofs" can be either deductive or inductive. Proofs must also be semi-formal, hence the requirement that variables be used instead of specific values. The second paragraph concentrates on the form of a proof. The warning against "T" proofs reflects the practice of requiring such proofs in the past* .

It should be noted that even in a curriculum which pays scant attention to proving, mathematics is described as useful in developing logical reasoning. Alberta Education's "Program Rationale and Philosophy", which appears in all the senior high school curriculum documents, states that: "an understanding of mathematical techniques or processes ... will enable [students] to ... acquire higher order skills in logical analysis and methods for making valid inferences." (Alberta Education, 1991, p. 1; 1990, p. 1; 1989, p. 1)

During my research studies I had the opportunity to see how three well respected teachers interpret the Alberta curriculum in their classroom. Two of the three made no mention of proving to verify in the time I observed them. The third, Ms E, described algebraic determinations of slopes of lines as "proofs". Examples of her teaching appear Chapter II, section 3. In most cases mathematical statements made by the teacher were verified inductively by examples or by reference to the teacher's knowledge and authority.

The *Standards*

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) reflect a different approach to proving. Of the 14 standards for senior high school mathematics, standard 3 is "mathematics as reasoning". This is also one of the 13 standards listed for the lower grades. The *Standards* document is a publication of North America's largest mathematics education organization, which is intended to "guide reform in school mathematics in the next decade" (NCTM, 1989, p. v). The inclusion of reasoning as one of the key standards in this document indicates the importance attached to proving in mathematics education.

Standard 3 states:

> In grades 9-12, the mathematics curriculum should include numerous and varied experiences that reinforce and extend logical reasoning skills so that all students can—
> - make and test conjectures;
> - formulate counterexamples;
> - follow logical arguments;
> - judge the validity of arguments;
> - construct simple valid arguments;

* A "T" proof was once the required form of proofs in high school geometry. A large "T" was drawn, and the steps of the proof were written on the left side of the horizontal, with references to the theorems or axioms which justified each step written on the right.

and so that, in addition, college-intending students can—

- construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction. (p. 143)

The function of these reasoning skills that the NCTM envisages can be seen in their description of the practice of mathematics:

> A mathematician or a student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in particular cases (inductive reasoning) and then tests the conjecture by constructing either a logical verification or a counterexample (deductive reasoning)....Furthermore, all students, especially the college-intending, should learn that deductive reasoning is the method by which the validity of a mathematical assertion is finally established. (p. 143)

This vision of proving is well within what Dawson (1969, p. 142) called the "naive heuristic of mathematical inquiry." Based on the work of Lakatos (1963/1976) Dawson identified a second heuristic, the deductive heuristic, in which the function of proving is to explore rather than to verify. Proving to explore is described in detail in Chapter II.

2. Current practices in teaching proving

Proving has been a part of teaching mathematics since at least the time of Plato. But millennia of experience does not mean teaching is as good as it can be. In fact, the current methods fail to teach many students to prove (Senk, 1985; Schoenfeld, 1985; Fischbein, 1982). What, then, is wrong with the way we teach?

People learn when they have a need which learning might fulfill. As Vygotsky observed, learning is adaptation and it is a truism that "all adaptations are regulated by needs" (1986, p. 37) There have been two needs that teachers by and large have invoked in order to motivate students to prove: the need to succeed in school and the need to know with certainty.

Of the three curricular approaches to proving mentioned in the previous section, the two most common are the teaching of proving in a year long course in Euclidean Geometry, and the downplaying of proving, as in the Alberta program of studies. Schoenfeld (1985) describes teaching in the first of these contexts. I will illustrate teaching in the Alberta context with observations from my studies in high schools (see Appendix B for descriptions of these studies).

Schoenfeld on teaching for examinations

At the school Schoenfeld studied, the teaching of proving involved a requirement that students prove propositions on a timed examination. Schoenfeld traced the effects this requirement had on what teachers taught and what students believed. Students were expected to prove one of a set of 30 propositions in Euclidean geometry on New York's Regent's examination. For Schoenfeld this expectation explained the adoption by their teacher of drill and memorization as his main teaching methods. In this context the students became adept at the speedy production of precise constructions and at memorizing proofs.

In addition to encouraging memorization, Schoenfeld found that the examination system motivated the development of beliefs about the role of proving in mathematics. The two beliefs Schoenfeld mentions which are most closely related to proving are these:

> The processes of formal mathematics (e.g., "proof") have little or nothing to do with discovery or invention.
>
> Only geniuses are capable of discovering creating or really understanding mathematics. (Schoenfeld, 1988, p. 151)

The antecedents of these beliefs in teaching probably include the practice of ignoring proving in the context of constructions and definitions, the other main topics of students' work in Euclidean geometry. The verification of constructions, which could be made an important context for proving, is often done visually — hence the stress observed by Schoenfeld on precision in constructions. Definitions are usually presented as *fait accompli* rather than evolving from the needs of proving, as advocated by Borasi (1991).

The second belief noted by Schoenfeld, that mathematics is created only by geniuses, might also be related to the requirements of examinations. Examinations require that a certain collection of facts, procedures, and skills be mastered. This content is not organized as if it arose from a probable sequence of mathematical explorations, but rather as the endpoints of many explorations which have found application in some context deemed important at some time or another. As a result a teacher allowing students to engage in any creation of mathematics would be doing them a disservice since it would detract from mastering the content required by the examinations. Even if the content were such that a teacher could expect students to create it in the course of exploration, such a course would still be irresponsible since the course of exploration can never be entirely controlled, and the presence of digression can only serve to distract students from the required content. Given the beliefs about math and the prevalence of memorized proofs encouraged by examinations it is not surprising that teaching based on using scholastic success to motivate students has failed to result in many students learning to prove.

Teaching at North and South Schools

The teaching of mathematics I observed at North School made no reference to proof or proving. Verification of answers was done inductively or by reference to the authority of the teacher or textbook. The focus was on learning procedures for obtaining answers quickly and accurately. In the Math 13 class, this was a conscious decision of Mr. A, who felt the students would be best served by extensive practice of procedures without being confused by proofs. In the Math 30 class Mr. B had made explanatory proving a part of his teaching, but the students, concerned with performance on their final examinations, had asked him to limit himself to what was going to be on the test.

At South School, proving was identified with algebraic methods and was associated with verification. On several occasions Ms E pointed out that a graphical rendering of a situation is not a proof and that proving required algebraic manipulations based on formulae. On other occasions, while graphs were rejected as proofs, the only alternative offered was the authority of the teacher.

3. Experiments in teaching proving

Fawcett (1938) suggests these assumptions as the basis of the teaching of proving:

> 1. That a senior high school pupil has reasoned and reasoned accurately before he begins the study of demonstrative geometry.
> 2. That he should have the opportunity to reason about the subject matter in his own way.
> 3. That the logical processes which should guide the development of the work should be those of the pupil and not those of the teacher. (p. 21)

In Chapter I, section 2, I have indicated my reasons for believing that students can prove, in agreement with Fawcett's first assumption. His second and third assumptions could well have been listed by any present day proponent of the constructivist theory of learning. In the following discussion of the teaching of proving I will be accepting Fawcett's assumptions, as well as two more:

4. The proving which is taught in mathematics should reflect the nature of proving in professional mathematics.
5. The teaching of proving should take into consideration not only the form of proving used, but also the need which proving is satisfying in that context.

In my examination of the research literature on proof and proving, I have encountered only three studies of teaching in which proving was taught it a manner consistent with Fawcett's assumptions. These studies are those by Fawcett himself, Balacheff and his coworkers (Balacheff, 1991; Arsac, Balacheff, & Mante, 1992) and Lampert (1990). I would like now to describe the work of these researchers and to comment on them in light of assumptions #4 and #5.

Fawcett's research

Fawcett's research is the subject of an NCTM Yearbook (Fawcett, 1938). In it he describes in detail his methods of teaching and the results he obtained, according to both interviews with students and standardized tests. In general he was quite successful and it is not clear why his straightforward suggestions for improving teaching were not implemented more widely. His description of the teaching methods he hoped to replace are quite similar to those described by Schoenfeld (1985) which I related in the previous section.

In some respects Fawcett's teaching seems quite traditional. He makes no reference to students working together, except in the context of whole class discussions led by the teacher. The context for teaching proving is geometry and although Fawcett does note the importance of students being able to transfer their ability to prove to non-mathematical domains, he does not discuss proving in other areas of mathematics. Other aspects of his teaching are fairly radical, at least compared to current practice. He summarizes his methods as follows:

> 1. No formal text is used. Each pupil writes his own text as the work develops and is able to express his own individuality in

organization, in arrangement, in clarity of presentation and in the kind and number of implications established.

2. The statement of what is to be proved is not given the pupil. Certain properties of a figure are assumed and the pupil is given an opportunity to discover the implications of these assumed properties.
3. No generalized statement is made before the pupil has had an opportunity to think about the particular properties assumed. This generalization is made by the pupil after he has discovered it.
4. Through the assumptions made the attention of all pupils is directed toward the discovery of a few theorems which seem important to the teacher.
5. Assumptions leading to theorems that are relatively unimportant are suggested in mimeographed material which is available to all pupils but not required of any.
6. The major emphasis is not on the statement proved, but rather on the *method of proof.*
7. The extent to which pupils profit from the guidance of the teacher varies with the pupil and the supervised study periods are particularly helpful in making it possible to care for these variations. In addition individual conferences are planned when advisable. (p. 62, emphasis in original)

Fawcett was quite successful in achieving the objectives he set for himself. His teaching cannot be criticized on the basis that it does not work. In fact, in many ways his methods seem deserving of application in the teaching of mathematics in general, not just proving. At the same time I have some concerns related specifically to the aims of his teaching. Fawcett assumes that the purpose of proving is the determination of truth. He makes no reference to proving as explaining or exploring although the students in his course did a fair bit of both. He is also concerned that his students be able to transfer their ability to prove to "non-mathematical material" (p. 21). In his classes the non-mathematical material examined consisted of advertisements, political arguments, and legislation. He achieved some success in persuading his students to reason deductively outside of mathematical contexts, as indicated by these comments he received from parents by way of another teacher who conducted interviews with them:

> The parents fear that the course may tend to inhibit in the boy the power of imagination for creative writing in English. For example, when he was writing of a personal experience for an English assignment he resented some suggestions his mother made in order to add interest to the composition on the basis that the suggestions were not facts. He wished to write only in a scientific manner.
>
> The mother fears that the girl may carry her criticism to the point of quibbling, however. In some cases she has gone to the point of criticising authorities on subjects about which she knew nothing. (p. 109)

My point in quoting these comments is to suggest that presenting proving as verifying, and then encouraging students to employ proving in a wide range of contexts, could lead them to apply proving in cases where it is inappropriate and also to miss occasions when proving might be used to explain or explore but not to verify. In the next chapter I have some further comments on the misapplication of proving to verify outside of mathematics.

Research by Balacheff et al.

An example of the studies being done in France by Balacheff and others is the study done by Arsac, Balacheff, & Mante (1992). Students were presented with this task:

> Write for other students a message allowing them to come to know the perimeter of any triangle a piece of which is missing. To do so, your colleagues will have at disposal only the paper on which is drawn a triangle and the same instruments as you have (rulers, etc.) (pp. 10-11)

The lesson was divided into two phases. In the first phase the students solved the problem, working in groups.

> During the second phase, called the *debate period*, aiming at a collective discussion about the proposed solutions, the organization is the following: Students' solutions are written on a large sheet of paper and are then displayed as posters on the wall of the classroom. Each team has to analyze the posters and their spokes-person tells the class their criticism and suggestions. The criticism must be accepted by the team whose poster is discussed. Since the students involved are 13 to 14 years old, it is not possible to leave them free of any regulation. The management of the activity is then left to the teacher.... The social situation, as a whole, constitutes here the didactical milieu of the students' mathematical activity. But such a milieu is not sufficient by itself to guarantee the quality of the debate. We can then foresee that the teacher will have to play a role especially when the student group might come to an incorrect agreement or to an impasse. (p. 9, emphasis in original)

I liken this process of presentation of arguments followed by consensus decision making to a judicial trial, with the students acting as both lawyers presenting arguments and as jurors evaluating the argument. The teacher plays the role of the judge, advising the jury on the admissibility of evidence. Teaching based on this "courtroom" metaphor has much to recommend it . The body of mathematical knowledge provides a codified basis for argument, much as the body of laws provides a basis for legal arguments, especially in legal systems based on the Napoleonic code, such as France and the United States. In addition, the process of evaluating major mathematical propositions within the mathematical community has characteristics of a legal proceeding. The evidence for a proposition, a proof, is offered to the community, experts comment on it and point out flaws. These are then corrected, and perhaps new arguments are brought forth, until finally the proposition is accepted into the body of mathematical knowledge.

The research done on the basis of the courtroom model also places a strong emphasis on communication. This is important as an encouragement to formulate unformulated proving. This practice could help students to overcome the difficulty in formulating I observed in my studies.

Research by Lampert in the United States

Lampert (1990) has taken the important steps of attempting to teach proving to students in the early grades. She taught a grade 5 class using a method similar to that employed in France, with modifications appropriate to younger students.

> As students volunteered their solutions to a given problem, I write them on the board for consideration, and I put a question mark next to all of them.... Once the list of students' solutions was up on the board, they were open for discussion and revision.... If they wanted to disagree with an answer that was up on the board, the language that I have taught them to use is, "I want to question so-and-so's hypothesis." ... I always ask them to give reasons why they questioned the hypothesis, so that their challenge took the form of a logical refutation rather than a judgment. (p. 40)

Lampert also placed some emphasis on portraying mathematics as exploratory, in keeping with Lakatos' (1976) historical analysis (see Chapter I, section 2). She is especially aware of the "cultural" side of teaching proving:

> I assumed that changing students' ideas about what it means to know and do mathematics was in part a matter of creating a social situation that worked according to rules different from those that ordinarily pertain in classrooms, and in part respectfully challenging their assumptions about what knowing mathematics entails. (p. 58)

Teaching based on the "courtroom" metaphor provides a context for the development of a culture of proving. The importance of such an atmosphere is indicated in the encouragement to prove it provided the participants in my studies.

Weaknesses of teaching based on the courtroom metaphor

Teaching based on the courtroom metaphor does have some shortcomings. One of these is pointed out by Arsac, Balacheff, & Mante (1992), who report that in classrooms the arguments offered are often not entirely founded on mathematical bases, but include appeals to social and personal factors. Students rely on their personal authority as members of the social structure of the class to verify their statements by reference to their own authority. This is entirely in keeping with the metaphor since the decisions of juries are as much determined by the persuasive abilities of lawyers as they are based on the code of law.

A second flaw in the courtroom metaphor is found in the need a trial serves: the establishing the truth or falsity of charges. A legal proceeding verifies. In the same way teaching based on the courtroom metaphor verifies the conjectures that students make. As I noted above, professional methematicians use proving to satisfy needs other than verifying, and the results of my studies indicate that proving to verify is contrary to students' inclinations. It is the stress on proving to verify which is, in my mind, the major problem with the courtroom approach to teaching proving.

4. Speculations on improving teaching

The research on teaching proving I have described above contains many excellent suggestions for improving teaching, which I can only reiterate. There are shortcomings in the basic assumptions of these efforts, however. Chief among these is that the need to prove is a need to verify. I believe that the teaching of proving ought to center on the importance of proving as a way of explaining and exploring in mathematics and also in the sciences and in dealing with a technological world.

Teaching with this aim must still include several of the important features of Fawcett's teaching, and of teaching in the courtroom metaphor. These features are the following:

1. Providing situations for proving in which results are discovered by the students, and in which the need to prove arises out of a need to explore, explain, or verify in the situation.
2. Statements which are proved are proposed by students, at the level of precision the students find necessary.
3. The need to define precisely arises out of the requirements of proving, not as an arbitrary imposition by authority.
4. "The major emphasis is not on the statement proved, but rather on the *method of proof*" (Fawcett, 1938, p. 62, emphasis in original)
5. Providing occasions for students to express their reasoning in their own way, and in ways which permit communication with their peers and with the larger mathematical community.
6. Proving occurs in a social context in which there is an expectation that explanations will be deductive and in which accommodation is made for the time and attention explaining by proving requires.

In addition to these principles, I would propose an alternative to the courtroom metaphor implicitly used by Lampert, and Balacheff, et al. Inspired by the deductive methods of Sherlock Holmes and by the mathematical acumen of his nemesis, Prof. Moriarty, I would propose a metaphor of mathematics as detective work. This metaphor shares important features with the courtroom metaphor like communication and the fostering of a "culture of proving." At the same time it emphasizes the importance of proving as a way of explaining and exploring in mathematics and provides a basis for exploration by formulated proving by inviting the question, "What do you know? What clues do you have?"

A small shift in emphasis can bring teaching based on the courtroom metaphor closer to the detective metaphor. In the courtroom the objective of the lawyers' arguments is to convince the jury. A detective is more concerned with proving to explain than to convince, and this suggests some changes to the teaching methods of the courtroom metaphor. The same process of generating conjectures occurs, but instead of the students attempting to convince each other, they attempt to explain to each other. They try to help the other students understand why their conjecture is true, not just that it is true. This shift in focus builds on the use of proving to explain which was indicated in my studies, and in previous research (Hanna, 1989; de Villiers, 1991). It also makes the decision process less one of conflict and more one of consensus building.

A more complete development of the detective metaphor involves the students in significant investigations in mathematics. Such investigations require complex and open situations, teacher support and guidance, and a culture of proving in the classroom, such as some of those suggested by Fawcett (1938).

The problem prompts employed in my research begin to suggest the sorts of situations which would permit mathematical investigations involving proving. They could be criticized on the basis of being unconnected to the topics in many high school curricula. I consider this less of a difficulty than the limited scope of the problems. The Fibonacci situation can be expanded to include other topics (e.g., the golden mean) but it is practically limited to the properties of the sequence itself. The Arithmagon situation can include a wide range of uses of systems of equations and linear algebra, but at the same time it imposes constraints on the interpretations of these topics.

I would propose that the best problem prompts might already appear in our textbooks, either as mathematical problems which involve most of the content of a unit, or as "add-on" or "enrichment" activities linking the content with genuine applications of mathematics in business, science, or the arts. For example, the problems Ms E presented at the end of her unit on equations of lines involved finding equations of altitudes, medians, and perpendicular bisectors of the sides of triangles. These problems could have been presented at the beginning of the unit as problems for investigation. Determining the point of intersection for these lines could have extended the same problems into the next unit on linear systems.

Fawcett (1938) provides other examples of good problems for investigation:

> Referring to the diagram [Figure 32] let us assume that *AP* and *AQ* are tangents drawn to circle *O* from an external point, *A*. What are the implications of this assumption?

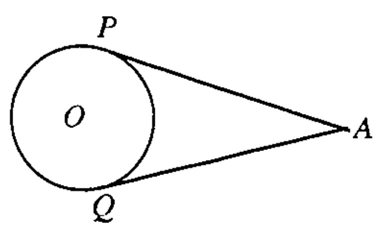


Figure 32: Fawcett's diagram.

> Draw a right triangle and from the vertex of the right angle draw a perpendicular to the hypotenuse. What properties of this figure can you discover and establish by deductive proof? (p. 91)

I would replace the phrase "establish by deductive proof," in Fawcett's second problem with the word "prove" but otherwise these problems seem to be excellent starting points for mathematical investigations. Fawcett's students' investigations led them to a wide range of discoveries, including the Pythagorean Theorem, which is suggested by the second problem.

To contrast the results of current teaching methods with what could occur if students spent more time proving to investigate problems like these, consider the behavior of students interviewed by Schoenfeld (1985). When given the diagram in Figure 33 and asked to prove that PA ≅ QA and OA bisects ∠PAQ the students could do so easily. When given the same diagram with the circle omitted a few minutes later, and asked to construct a circle tangent to PA at P and also tangent to PQ, they attempted to do so by constructing various arcs and lines with the inaccurate compass Schoenfeld provided, judging whether they had succeeded by the appearance of the result. No attempt to use the proving they had done previously was made. Would students who had learned to prove to explore, not just to verify, have behaved the same way?

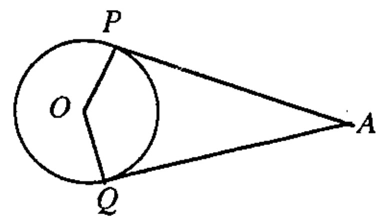


Figure 33: Perpendiculars to tangents meet at the center.

Throughout such investigations the teacher plays vital roles, as observer, guide, resource, and co-investigator. These roles focus the teacher on the process of doing mathematics, on reasoning and understanding, and on the mathematical worlds of the students. They stand in contrast with more traditional teacher roles as an authority on mathematical knowledge and the active agent in the classroom.

As an observer the teacher assesses the reasoning and understanding of students so as to be better able to guide them in their investigations. This role also plays an important part in the investigation itself since an observer can record and recollect parts of the investigation which might be useful later. This recollection is also an aspect of the teacher's role as a resource. In acting as an observer, the teacher is modeling an activity which the students themselves should learn as part of mathematical and scientific investigation. The role of observer might be usefully assigned to students as one way of contributing to a group investigation.

Guiding students should not be confused with "funneling" them towards a known goal along a familiar path. Guiding students' investigations is more concerned with pointing out important signs, reflecting on reasoning, raising unthought of possibilities and, on occasion, warning students away from unprofitable paths. It must be emphasized that investigation is not about efficiency and optimization. In every path there is some value. The teacher does have a responsibility, however, to proscribe the range of investigation to exclude paths which are known to be generally unproductive, or even misleading.

In addition to recalling students' recent actions and discoveries, the teacher is also a source of information and ideas. Of course, this is a large part of teachers' traditional role but with an important difference. While the usual pattern is for teachers to anticipate what skills and knowledge students' will need (as Ms E did in her unit on graphing lines), a teacher acting as a resource for an investigation must learn to wait and listen and to provide students with information when it is needed, not before. This was the role my co-researchers and I adopted in problem sessions. While we asked questions to point out aspects of the situation for the participants to continue their investigations, we gave answers only to the participants' own direct questions. Interestingly, those participants who were most willing to ask questions asked only to allow themselves to continue investigating without the impediment of stopping to make calculations. They asked no questions that would dispel the mystery they were investigating.

The openness of the problem prompts used in my research created situations in which the participants knew as much about what was going on as the observers did. This was especially true in the Arithmagon situation, in which almost all the participants, including both the first and the last groups observed, surprised me with aspects of the situation I had not anticipated. Classroom investigations should also afford the opportunity for teachers to join their students in genuine mathematical activity, complete with the uncertainty of exploration. This places a demand on teachers unlike those often encountered. What is most important for teaching in investigations is a willingness to learn and to live with the uncertainty which come with learning.

Permitting the development of a culture of proving in a classroom is a difficult task. Some studies (e.g., Perry, 1981; Belenky, Clinchy, Goldberger & Tarule, 1986) suggest that students are not prepared to participate in such a culture prior to their university education, or that current schooling practices delay the development of the necessary attitudes towards knowledge until students enter universities. Lampert (1990) offers some hope that the real situation is the latter. She reports success in developing in grade 4 and grade 5 students attitudes appropriate to mathematical investigations. These attitudes are described by Polya (1968) as "intellectual courage," "intellectual honesty," and "wise restraint."

> First, we should be ready to revise any one of our beliefs.
> Second, we should change a belief when there is a compelling reason to change it ...
> Third, we should not change a belief wantonly, without some good reason. (p. 8)

In addition to these attitudes, classrooms also need to present opportunities for mathematical discourse, such as are described in the NCTM *Professional Standards for Teaching Mathematics* (1991, see also Reid, 1994).

> In order for students to develop the ability to formulate problems, to explore conjecture and reason logically, to evaluate whether something makes sense, classroom discourse must be founded on mathematical evidence.
>
> Students must talk, with one another as well as in response to the teacher. When the teacher talks most, the flow of ideas and knowledge is primarily from teacher to students. When students make public conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively, revealing mathematics as constructed by human beings within an intellectual community. (p. 34)

5. Why teach proving?

You will have noticed that the critiques of the teaching methods proposed by Fawcett, Balacheff, and Lampert (above), could as easily be applied to the curricular objectives I described at the beginning of this chapter. The justification given for teaching proving is that proving is the way to verify in mathematics. If I argue that this justification misrepresents mathematics and proving, and leads to teaching in ways which neglect students' needs to prove, what then is the reason to teach proving at all?

Perhaps the best way to begin to answer this question is to look at the reason we teach anything. Bruner sums it up nicely:

> A culture is as much a *forum* for negotiating and renegotiating meaning and for explicating action as it is a set of rules or specifications for actions. Indeed, every culture maintains specialized institutions or occasions for intensifying this 'forum-like' feature. Storytelling, theatre, science, even jurisprudence are all techniques for intensifying this function—ways of exploring possible worlds out of the context of immediate need. Education is (or should be) one of the principle forums for performing this function—though it is often timid in doing so. It is the forum aspect of a culture that gives its participants a role in constantly making and remaking the culture. (Bruner, 1986, p. 123, emphasis in original)

According to Bruner, and I would have to agree with him, education has two aims. The first is simple. We want to pass on aspects of our culture which we consider to be important to our children. In doing so we are presenting them with "a set of rules or specifications for actions" which will allow them to continue to be a part of a world defined by our culture. For this reason we teach children to speak the language we speak, to listen to music like the music we listen to, to read books we have read, and to appreciate the dramatic arts as much as we do.

Passing on our culture to students takes many forms, from providing basic skills and knowledge we know all of them will need, to exposing them to the more esoteric aspects of human culture, giving them opportunities to go where they might otherwise never have ventured. Among the basic skills of living in our society is an ability to reason deductively as a basis for problem solving in encounters with technology and as a tool for evaluating scientific, legal, and statistical arguments used to justify public policy. Proving is also an introduction to the esoteria of mathematics, physics, and analytic philosophy, which are as much a part of our culture as the poems of Milton and the music of Bach, and so as important to students' education.

It is vital not to be confused about the importance of basic skills versus exposure to esoteria. Teaching students the relevance of proving to mathematics or philosophy is not the most central of our aims and teaching needs to reflect this. Reasoning deductively to determine the functions of, or flaws in, a product of rational design science is a more central aim. This should be apparent in the contexts for reasoning we choose for our students. Students may learn to prove as well in thinking about set theory as in thinking about microwave oven programming, but the reasons to prove communicated will be quite different.

The second aim of education, according to Bruner, is to give students "a role in constantly making and remaking the culture." Deductive reasoning is an important part of our culture and a part of our culture in need of being remade (as I discuss further in the next chapter). An important part of remaking this part of our culture is detecting misuses of proving. In some cases these become apparent from the unsatisfactory results obtained, but it is better to notice misuses of proving before they cause harm, and in a way which includes an understanding of the weaknesses which make proving unsuitable for a particular use. The ability to do this depends on an ability to prove.

The misuse of proving might be a result of flawed proving, or it might result from the use of proving in an inappropriate context. When the misuse of proving is a result of errors in the proving process it is only through proving that the errors can be discovered. This use of proving might be compared with Lakatos' "proving to improve" (1976, p. 37) in mathematics, broadened to include proving in other contexts. When a correct proving process is used in an unsuitable context, proving provides the basis for a precise understanding of its own limits.

The use of proving in our culture, its misuses in inappropriate contexts, and the ways in which proving can define its own limits, are the topic of the next chapter.

CHAPTER VI

PROVING IN SOCIETY

Logical consequences are the scarecrows of fools and the beacons of wise men.
— T. H. Huxley, *Science and Culture,* ix, On the Hypothesis that Animals are Automata.

The usual reason given for teaching proving, the importance of proving as the method of verification is mathematics, does not agree with either the role of proving in mathematics or with students' needs to prove. This observation led me to raise the question "Why teach proving?" at the end of the previous chapter. It could have as easily led to the question "Why are the curriculum designers asserting that proving has a role that it does not have either in mathematics or for students?" The answer to this question lies in the role of Rationalism in our society. The problems that have resulted from the application of proving in inappropriate contexts and weaknesses in the basic assumptions of Rationalism suggest that the role of Rationalism in society needs to be reconsidered. In fact, I would assert that if education involves preparing students to play "a role in constantly making and remaking the culture" (Bruner, 1986, p. 123), then Rationalism is a part of our culture that needs remaking.

In this chapter I describe Rationalism and some problems it has given rise to. I then analyze some of its weaknesses and limits. These limits suggest both a need for other modes of thinking and a need for a remaking of Rationalism. In the next chapter I describe one possible remaking of Rationalism, and its implications for teaching and research. My critique cannot pretend to be exhaustive. A thorough description and critique of Rationalism would fill many volumes. I will be ignoring the critical perspectives of feminism, post-modernism, and phenomenology, among others. These perspectives are certainly valuable, but my purpose here is simply to suggest that a strong critique of Rationalism exists from a Rationalist perspective, and so I will be limiting myself to that perspective. I will also be providing a simplified description of Rationalism, which contains what I believe are its central points, but which necessarily neglects subtleties which would be included in a more thorough history.

1. What is Rationalism?

Rationalism is based on two beliefs: that deductive reasoning can determine absolute truths, and that deduction is applicable to all situations. The implication of these two beliefs is that in any situation in which we want to know something, the best way to reason is deductively. The basic ideas of rationalism can be traced to Descartes. Descartes published his *Discourse on the Method for Rightly Conducting One's Reason and for Seeking Truth in the Sciences* in 1637. The "Method for Rightly Conducting One's Reason" he wrote of is deductive, rational thought.

As an aside, I should note that my tracing of Rationalism to a few words of Descartes could be seen as a misrepresentation of his work. Descartes' ideas occurred in a context and have been reinterpreted many times in many other contexts. My use of his name and words here is largely a rhetorical and heuristic device. It allows me to describe in a simple way a belief or attitude held by many people at many times. Modern Rationalism can be, as I have done here, traced to Descartes, but just as easily traced to Russell, or Leibniz, or Plato.

Descartes' method was inspired by the proofs of Euclid. His innovation was to imagine that such deductions might illuminate other areas:

> Those long chains of reasoning, each of them simple and easy, that geometricians commonly use to attain their most difficult demonstrations, have given me an occasion for imagining that all the things that can fall within human knowledge follow one another in the same way and that, provided only that one abstain from accepting anything as true that is not true, and that one always maintains the order to be followed in deducing the one from the other, there is nothing so far distant that one cannot finally reach nor so hidden that one cannot discover. (Descartes, 1637/1993, p. 11)

In fact, Descartes believed that the simple and easy reasoning that geometricians use was the only way of reasoning which could succeed in revealing truth:

> Of all those who have already searched for truth in the sciences, only the mathematicians were able to find demonstrations, that is, certain and evident reasons. (p. 11)

The influence of Rationalism has extended beyond its origins in mathematics, science, and philosophy. In the eighteenth century, the Age of Reason, the successes of Rationalist science became known to all educated Europeans and had effects on their vision of the world:

> Science was for them ... living growing evidence that human beings, using their "natural" reasoning powers in a fairly obvious and teachable way, could not only understand the way things really are in the universe; they could understand what human beings are really like, and by combining this knowledge of nature and human nature, learn how to live better and happier lives. (Brinton, 1967, p. 519)

This Enlightenment vision has continued into present day rhetoric, curriculum documents, textbooks, and teaching. Rationalism also continues to play an important role in research, both in the definition of reasoning and as the basis of methodology. Lakoff (1987) describes the dominant understanding of what "reasoning" means in this way:

> In this century reason has been understood by many philosophers, psychologists, and others as roughly fitting the model of formal deductive logic:
>
> > Reason is the mechanical manipulation of abstract symbols which are meaningless in themselves, but can give meaning by virtue of their capacity to refer to things either in the actual world or in possible states of the world. (p. 7)

2. Rationalism's weaknesses

The two beliefs that form the basis of Rationalism, that deductive reasoning can determine absolute truths and that deduction is applicable to all situations, turn out to be problematic. The weaknesses in these fundamental beliefs permeate all of Rationalism (because Rationalism has a deductive structure). In this section I describe the weaknesses of Rationalism, and also suggest that Rationalism is not only flawed, but dangerous when applied to many situations.

Deduction and absolute truth

Descartes modeled his method on the proofs of Euclid, and saw them as the ultimate example of thinking which produced certainty. For this reason it seems to me that the relationship of absolute truth to proving in mathematics is a sensible place to explore this aspect of Rationalism in general. In the past two centuries the relationship of absolute truth in mathematics has shifted radically. In the late eighteenth century mathematics, and especially geometry, was seen as the most absolute of truths. By the late nineteenth century, it was acknowledged that what was true depended on the assumptions, the axioms and postulates, which form the basis of a mathematical system. There could be two equally valid systems, based on different assumptions, but within each system proving could reveal all truth and engender no contradiction. By the mid-twentieth century, even this hope was lost since it was shown that all mathematical systems are necessarily incomplete; there are truths that can be known but not proven. What Kline (1980) called the "loss of certainty" in mathematics has implications for Rationalism as a whole although they are barely beginning to be felt.

In the Age of Enlightenment Euclid's geometry was often held to be the epitome of certainty. Descartes based Rationalism of Euclid's proofs, and Kant (1781/1927) used the certainty of geometry to support the necessity of space being *a priori*.

> On this necessity of an *a priori* representation of space rests the apodictic certainty of all geometric principles, and the possibility of their construction *a priori*. For if the intuition of space were a concept gained *a posteriori*, borrowed from general external experience, the first principles of mathematical definition would be nothing but perceptions. They would be exposed to all the accidents of perception, and there being one straight line between two points would not be a necessity, but only something taught in each case by experience. Whatever is derived from experience possesses a relative generality only, based on induction. We should therefore not be able to say more than that, so far as hitherto observed, no space has been found to have more than three dimensions. (p. 19)

The "apodictic certainty of all geometric principles"* was undermined by the discovery, in early nineteenth century, of non-Euclidean geometries. These geometries begin with different assumptions than Euclid's, but rather than collapsing in a mess of contradictions, as Kant might have predicted, they turn out to be just as consistent as Euclidean geometry.

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* "Apodictic" means "established on incontrovertible evidence. (By Kant applied to a proposition enouncing a necessary and hence *absolute* truth.)" (Oxford English Dictionary, 1971)

The first of these non-Euclidean geometries, hyperbolic geometry, was independently discovered by Lobachevsky (published in 1829), Bolyai (published in 1832), and the renowned mathematician Gauss (never published, but he claimed he thought of it first). Hyperbolic geometry replaces Euclid's "parallel postulate" (There is exactly one line through a given point parallel to a given line) with the postulate "There is *more than* one line through a given point parallel to a given line." This postulate leads to many surprising results, like the sum of the angles in a triangle is *less than* 180°, but it does not lead to contradictions.

Strictly speaking, what was shown was not that the non-Euclidean geometries are consistent, but rather that they are consistent *if* Euclidean geometry is consistent. This raised for the first time the question "How do we know Euclidean geometry is consistent?" The old answer, that it is the absolutely true geometry of space and so must be consistent, was no longer acceptable. Instead, a mathematician named Hilbert answered this question by proving that Euclidean geometry is consistent *if* basic arithmetic is consistent. Now the problem was to show that arithmetic is consistent.

Hilbert presented this problem, along with about twenty others, at the Second International Congress of Mathematicians held in Paris in 1900. At that time great progress had been made in making mathematical reasoning more formal, which made gaps in logic easier to spot and fix. The mathematical community had great confidence that the formal structures they were developing would, for mathematics at least, achieve what Leibniz had dreamed of in the eighteenth century, "an exhaustive collection of logical forms of reasoning—a *calculus ratiocinator*—which would permit any possible deductions from initial principles" (Kline, 1980, p. 183).

It quickly became apparent that the problem of verifying the consistency of arithmetic was not going to be a simple one. And the problem was not just showing consistency. By selecting a very small number of initial assumptions or axioms, it was easy to produce a system that was consistent. But a small number of axioms was not enough to allow the derivation of all the statements one might make about arithmetic. In this case the system would be a consistent but incomplete arithmetic.

Most attempts to develop a formal structure for arithmetic used axioms about sets as their basis. But the theory of sets, which was chosen for its simplicity and obviousness, turned out to produce paradoxes. The central problem involves sets that contain themselves. The set of all apples does not contain itself because a set is not an apple. The set of all mathematical objects does contain itself because a set is a mathematical object. We can distinguish between sets that do contain themselves and sets that do not. But what of the set of all sets which do not contain themselves? Does it contain itself?

This paradox is called the Barber Paradox, which was first noticed by Bertrand Russell in 1902. The name comes from the following story, which expresses the same paradox in different terms.

> In a village there is a barber, who claims that he shaves every man who does not shave himself. Of course, he does not shave those who do shave themselves. Who shaves the barber?

Implicit in the paradox is a fashion statement: all men are shaved. Either Russell's barber shaves himself, or he doesn't. If he does, then he belongs to the class of men who shave themselves, and he does not shave those men so he cannot shave himself. If he doesn't shave himself, then he belongs to the class of men who do not shave themselves, and he shaves all such men. Either way, a contradiction arises.

This amusing story caused great consternation in the small, but famous, circle of mathematicians working on the problem of showing that arithmetic is consistent. Their proofs became more and more formal. Solutions to the paradoxes were proposed, but many of these solutions had undesirable side effects since they barred methods of proving that had been used with much success in the past. It began to look like showing arithmetic to be consistent might require redoing most of mathematics or even rejecting parts of it.

Hilbert, Russell, Whitehead, Peano, Frege, Zermelo, Brouwer, Weyl, and many others worked on the problem of consistency. They divided into various schools, employing different bases and limitations on logic in developing arithmetic. The conflict between these schools raised another issue. Critics from other schools raised the point that it might be possible that such and such a school's position might guarantee consistency but only at the cost of completeness. There might be important parts of mathematics that would left out. The baby might go with the bath water.

Russell and Whitehead decided to approach the problem by basing arithmetic on logic itself. As logic would be used in any proof of consistency, using logic as a basis added no new potential source of contradiction. Russell and Whitehead's efforts resulted in their *Principia Mathematica*, first published in 1913. Although no one felt that they had completely solved the problem of consistency, Russell and Whitehead had, through careful use of formal proving, clarified the problem further. Progress continued, and things looked hopeful.

In 1931 the situation changed radically. Kurt Gödel published a paper entitled "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems." In this paper Gödel delivered a powerful double whammy. The first blow related specifically to "*Principia Mathematica* and Related Systems." Gödel proved that the task which had occupied the greatest mathematical minds of the first three decades of the century could not be done. The consistency of arithmetic cannot be established using the logical principles of Russell and Whitehead. The second blow was even worse. Gödel proved that *any* system which does manage to show that arithmetic is consistent, must be incomplete. That is, we can use formal deductive logic to know that *part* of arithmetic is free of contradictions, but we can never know that *all* of arithmetic is free of contradictions.

Gödel's proof depends on producing a true statement, which he then shows cannot be proved without resulting in a contradiction. He does this by encoding the familiar Epimenides paradox into formal mathematics. The simplest formulation of this paradox is the sentence "This sentence is false." If this sentence is true, then it is false. If it is false then it is true. Gödel produced a formal sentence (encoded as a number), which asserted that it could not be proved. If the statement is assumed to be true, then there is a true statement that cannot be proved and mathematics is incomplete. If it is false, then there is a false statement that can be proved and mathematics is inconsistent. (A readable description of Gödel's Theorem is

Hofstadter, 1980. Readers with backgrounds in computer science may prefer the description in Penrose, 1994, based on computability. The empty set {readers who are mathematicians and read German, but are unacquainted with Gödel's Theorem} will find a reference to the original paper in the list of References.)

Gödel's Theorem established that deductive reasoning has limits even in mathematics, the original model of Rationalism. Its significance lies in establishing that there are some questions that deductive reasoning is powerless to answer.

Significance of Gödel's Theorem outside of mathematics

Gödel's Theorem undermines the foundations of Rationalism by invalidating Descartes' original assumption that mathematics is complete. This is enough to cast doubts on any sort of Rationalist description of the world. Gödel's Theorem and analogous arguments can also be applied directly to Rationalist world views. As examples, consider Penrose's critique of Artificial Intelligence research, and Putnam's critique of Objectivist semantics.

Penrose (1989, 1994) argues that Gödel's Theorem implies that Artificial Intelligence (AI), as it is usually understood, is impossible, and that a scientific understanding of the mind will require major revisions to current theories in physics.

His argument against AI is essentially that a computer based AI is a formal system, and so by Gödel's Theorem there are statements that are true, but which the AI cannot know because they cannot be proven within the formal system defined by the AI. Penrose asserts that an intelligent being could understand Gödel's Theorem, but that an AI could not, at least as far as Gödel's Theorem applies to the AI itself. If it could, then it would know some statement was true but be unable to prove it. Knowing the statement is true, however, implies that it *is* proven within the formal system of the AI.

Penrose's argument for the need to revise the theories of physics is based on his AI argument. According to current physical theories, the human brain operates according to physical laws which, in theory, could be represented by an incredibly complicated formal system. If this were the case, then the same argument he used to show an AI cannot exist would show that human intelligence cannot exist. Given that human intelligence does seem to exist, there must be some physical property of brains that makes them essentially unlike a formal system. If such a property exists, however, current theories of physics must undergo a radical modification.

Lest you be tempted to assume that Penrose is a crackpot, which would be reassuring given the sweeping nature of his conclusions, I should assure you that he is a well respected mathematical physicist. He has had many critics, which is a measure of the significance of what he has to say. People may not agree on what significance Gödel's theorem has, but at the very least there is no doubt that its significance is not restricted to mathematics.

Semantics is the part of linguistics that shows how abstract symbols are related to the world and that characterizes 'meaning'. In what Lakoff (1987) calls Objectivist semantics, these two processes are one and the same. Recall the description of reason according to Objectivism:

> Reason is the mechanical manipulation of abstract symbols which are meaningless in themselves, but can give meaning by virtue of their capacity to refer to things either in the actual world or in possible states of the world. (Lakoff, 1987, p. 7)

A symbol has meaning only because of the way it is related to the world. This relation is based on the idea of truth value. A sentence has meaning if it has a well defined truth value, which in turn is supposed to be based on the way the terms in it are related to the world. Putnam (1981) argues against this position by showing that a sentence might have two interpretations; that is, its terms might relate to the world in two different ways while the truth value, and hence meaning, of the sentence remain the same. If this is the case, then meaning cannot be based on the relation between symbols and the world.

Putnam's argument is analogous to Gödel's Theorem in at least two ways. It plays a similar role in limiting Rationalism, and the arguments are similar in structure. Gödel's Theorem does not show that all of mathematics is inconsistent, or even that deduction cannot be used to safely determine truth in mathematics. What Gödel showed is that there are limits to the power of deduction to reveal truth. Similarly Putnam showed not that it is impossible for symbols to be related to the world in a way that gives them meaning, but instead that there are limits to this process of making relations to the world to give meaning. According to Gödel there is a class of true statements that cannot be accounted for by deduction. According to Putnam there is a class of meaningful statements that cannot be accounted for by reference to the world.

Recall that Gödel's proof involved producing a sentence that asserted that it could not be proved. Putnam produced a sentence that could be given two interpretations. In both interpretations the sentence is true, so it has a truth value and is meaningful. But the two interpretations use very different references for the terms involved, making its meaning under the two interpretations different. In other words, Putnam produced a well-defined, meaningful sentence (according to the Objectivist idea of 'meaning'), with a completely ambiguous meaning (according to common sense). Thus there must be something more to ascribing meaning to a sentence than the criteria employed by Objectivism.

Putnam illustrates his proof with an example, and I am not able to provide a clearer synopsis of his proof, so I will quote his example in full. Further details can be found either in Putnam's work, or in Lakoff (1987).

> Consider the sentence
>
> > (1) A cat is on a mat. (Here and in the sequel 'is on' is *tenseless*, i.e. it means 'is, was, or will be on'.)
>
> Under the standard interpretation this is true in those possible worlds in which there is at least one cat on at least one mat at some time, past, present, or future. Moreover, 'cat' refers to cats and 'mat' refers to mats. I shall show that sentence (1) can be reinterpreted so that in the *actual* world 'cat' refers to *cherries* and 'mat' refers to *trees* without effecting the truth-value of sentence (1) in any possible world. ('Is on' will keep its original interpretation.)

The idea is that sentence (1) will receive a new interpretation in which what it will come to mean is:

(a) A cat* is on a mat*.

The definition of the property of being a cat* (respectively, a mat*) is given by cases, the three cases being:

(a) Some cat is on some mat, and some cherry is on some tree.
(b) Some cat is on some mat, and no cherry is on any tree.
(c) Neither of the foregoing.

Here is the definition of the two properties:

DEFINITION OF 'CAT*'
x is a cat* if and only if case (a) holds and x is a cherry; or case (b) holds and *x* is a cat; or case (c) holds and *x* is a cherry.
DEFINITION OF 'MAT*'
x is a mat* if and only if case (a) holds and *x* is a tree; or case (b) holds and *x* is a mat; or case (c) holds and *x* is a quark.

Now, in possible worlds falling under case (a), 'A cat is on a mat' is true, and 'A cat* is on a mat*' is also true (because a cherry is on a tree, and all cherries are cats* and all trees are mats* in worlds of this kind). Since in the actual world some cherry is on some tree, the actual world is a world of this kind, and in the actual world 'cat*' refers to cherries and 'mat*' refers to trees.

In possible worlds falling under case (b), 'A cat is on a mat' is true, and 'A cat* is on a mat*' is also true (because in worlds falling under case (b) 'cat' and 'cat*' are coextensive terms and so are 'mat' and 'mat*'). (Note that although cats are cats* in some worlds — the ones falling under case (b) — they are *not* cats* in the actual world.)

In possible worlds falling under case (c), 'A cat is on a mat' is false and 'A cat* is on an mat*' is also false (because a cherry can't be on a *quark*).

Summarizing, we see that *in every possible world* a cat is on a mat if and only if a cat* is on a mat*. Thus, reinterpreting the word 'cat' by assigning to it the intension we just assigned to 'cat*' and simultaneously reinterpreting the word 'mat' by assigning to it the intension we just assigned to 'mat*' would only have the effect of making 'A cat is on a mat' mean what 'A cat* is on a mat*' was defined to mean; and this would be perfectly compatible with the way truth-values are assigned in every possible world. (Putnam, 1981, pp. 33-35, emphasis in original)*

* The asterisk (*) in Putnam's example is used to distinguish the word 'cat' from the word 'cat*', and does not indicate a reference to a footnote.

Rationalism is not universally applicable

Examples of cases in which Rationalism seemed not to be applicable have occurred many times in the past three hundred years. Mere examples, however, are not enough to shake the Rationalist belief that proving is universally applicable. In some cases it was suggested that human weakness had introduced some bias into the course of the deduction, which made the problem people rather than Rationalism. In other cases, it was acknowledged that Rationalism could say nothing about a phenomenon, but that was taken to indicate that the phenomenon did not really exist. This approach has been taken by some Artificial Intelligence researchers to deal with the phenomenon of consciousness (Searle, 1992, pp. 6-7).

In order to show that Rationalism is not universally applicable, it is necessary to prove it, just as Gödel proved that Rationalism cannot establish all truths. In order to do so, we need to consider the relationship between logical implication and causation. The application of Rationalism to events in the world requires that physical causes must be almost as certain as logical implications. Otherwise Rationality has no predictive power, which is its whole point. Not even the most die hard Rationalist would assert that causes can be used to predict effects exactly, but there is an underlying assumption that causes predict effects approximately. The great successes of Rationality in the physical sciences bear this out. The orbits of the planets are almost exactly what Newton's laws predicts, and Einstein's theories improve the accuracy even further. In the Rationalist world all phenomena can be predicted, and things like the weather, for which prediction is currently very approximate, will be predicted with more and more accuracy as science progresses.

I bring up the planets and the weather as examples because they are two examples of dynamical systems, which are the topic of chaos and complexity theory. The simplest description of chaos theory is that it is the study of how chaos can emerge from order. Complexity theory considers how order can arise out of chaos. To take the planets as an example, physics provides precise laws which govern the motion of the planets. However, the interacting gravities of several bodies in motion result in a system for which questions like "Will the moon fall out of the sky one day?" cannot be answered. Not only is it impossible to predict the path of the moon exactly, it is not even possible to do so approximately over long periods of time. In weather systems this phenomenon is more obvious. In fact it was in attempts to simulate weather systems that the emergence of chaos from order was first observed (Gleick, 1987, p. 16). Approximate prediction is not possible because very tiny changes to initial conditions result in radical changes in final conditions. One might expect that knowing approximately what the current state of things is would be sufficient to predict approximately future events. This would allow refinements in our knowledge of the present to improve predictions of the future. In dynamical systems, however, the sensitivity to initial conditions is such that prediction is simply impossible.

Complexity theory examines dynamical systems in order to describe how order emerges from the chaos produced by the interactions within them. To return to the example of weather, one might expect that a system which changes radically in response to slight variations in initial conditions would be essentially random. However, when we examine satellite photographs, for instance, we see patterns in this chaos. Not all chaotic systems give rise to patterns, but some of the ones that do are very significant in our lives. They are those systems whose internal interactions are such that they are self-sustaining. This characteristic means that within such systems the second law of thermodynamics, entropy, does not apply. Some examples are living things, species, herds, and societies.

The emergence of such orderly systems out of the chaos of dynamical systems implies that they are, for all practical purposes, unpredictable. This sharply limits the use of Rationalism in understanding them. At the same time the basis of dynamical systems is interactions, which are causal when taken individually. This means that Rationalism has a role to play in the study of such systems. But it is a role that involves some fundamental changes in the use of proving. It is a role that replaces proving to verify with proving to explain and explore. It is proving that the students who participated in my research might relate to. An example of such a transformation of Rationalism is Enactivism, which is the topic of the next chapter.

Dangers of misapplied Rationalism

I would like to briefly mention some of the effects attempts to treat Rationalism as if it can be applied in all domains have had. I do this in order to make it plain that applying Rationalism to "all the things that can fall within human knowledge," as Descartes suggested, is not only a logical error, but dangerous to individuals and societies.

Descartes spent a number of years as a gentleman soldier of fortune, and so it is perhaps appropriate to begin by describing the Rationalism of war. War has become increasingly "scientific," especially in the past hundred years. When Gilbert and Sullivan wrote *The Pirates of Penzance* in 1880 mathematics was as important to the training of a "modern Major General" as statecraft or strategy. By the First World War, the planning of the generals was so rationally perfect that the Austrian declaration of war on Serbia led to the British declaration of war on Germany eight days later with all the inevitability of one of Euclid's proofs (Taylor, 1974, pp. 25-28).

Modern technology provides further scope for Rationalist military planning. Consider the age old problem of ground troops becoming frightened or disturbed at the carnage of war. The U.S. military is developing remote control, "telerobotic," tanks and planes, which can be operated from sufficient distances to eliminate the risk to the driver or pilot operating them (Rheingold, 1991, pp. 357-358), and simulators, like SIMNET which is capable of connecting 200 four person tank crews, each in their own simulated M-1 tank, into a virtual tank battle. The combination of these two technologies could solve the problem of troop morale. If the simulators used to train soldiers are equipped to operate the robot tanks and planes, then the combatants need never *know* that the images on their simulator screens are real people, and they will feel no remorse at their deaths. One more human element will have been eliminated from the planning of war on Rationalist principles.

Many people have realized that, "if there are 'objective' criteria on which to base a decision, then one cannot be blamed for being biased, and consequentially one cannot be criticized, demoted, fired, or sued" (Lakoff, 1987, p. 184). The association of mathematics with objectivity, which is an integral part of Rationalism, has lead to the use of mathematics as an "objective" way of determining which people are accepted to some desirable position. For example, in universities many academic programs include a mathematics course as part of their requirements. These "service" courses have high failure rates. In the case of the introductory calculus course at one major Canadian university about a third of the students who register either fail or drop the course. This limits the number of students enrolled in programs with mathematics requirements. Mathematics serves as a social filter, imbued with the appearance of rationality and objectivity.

Davis and Hersh (1986) describe this situation in the case of the calculus requirement of the typical business school:

> There seems to be no necessity to make math a requirement. There *is* a practical necessity to make a selection among the students who want to go to the business school. The business professors decide to use math for that purpose. Is that OK? How should we (math teachers) feel about it? First of all, there is nothing inevitable about the choice of math as a filter. Some other filters that could be used, or that have been used are: family connections, political connections, income, ability in sports, personal charm, brutality and aggressiveness, trickiness and sneakiness, devotion to public welfare, etc. The first five have been relevant criteria in admission of students to U.S. institutions of higher learning; the last three are suggested, somewhat in jest, for particular relevance to a school of business. (pp. 101-102)

Davis and Hersh go on to consider the effects on mathematicians that result from spending a great deal of effort teaching students who have no desire to learn mathematics, and who see it mainly as an impediment to their success in some unrelated academic program. In this case, it is not only the students who are denied access to their chosen career on the basis of a Rationalist criteria who suffer. The mathematicians who participate in this process also find it demoralizing and try to avoid teaching such courses.

Gould (1981) described the unfortunate effects of Rationalist attempts to quantify mental attributes have had on individuals and groups. The connection between mathematics and Rationalism leads to situations such as Gould described, in which the use of statistical methods in psychometrics gave the field an air of objectivity. This Rationalist claim has been the basis for the acceptance of psychometrics as the fair way to determine employment, immigration, and scholastic opportunity since the development of statistical techniques in the late nineteenth century.

Gould's most disturbing example describes the effects of the IQ testing of 1 750 000 U.S. Army recruits during the First World War on social policy. One important "discovery" that came out of the interpretation of this data was the mental inferiority of immigrants, especially immigrants from Mediterranean and Slavic backgrounds, including Jews. This led to the U.S. Immigration Restriction Act of 1924, which sharply limited immigration in general, especially from southern and eastern Europe. This Act prevented the immigration to the U.S. of millions of people, including Jews attempting to leave Hitler's Germany. As Gould (1981) says:

> We know what happened to many who wished to leave but had nowhere to go. The paths to destruction are often indirect, but ideas can be agents as sure as guns and bombs. (p. 233)

That particular misapplication of Rationality is still a serious problem is indicated by the periodic appearance of books that make use of data from psychometric testing to argue for the inferiority of various groups within our society, or globally (e.g., Jensen, 1979; Herrnstein & Murray, 1994).

Our model of thinking defines what we can think. Not surprisingly, Rationalism most strongly determines how scientists and mathematicians can think. This can limit the possible explanations scientist can provide for phenomena, perhaps resulting in false conclusions. This is illustrated by the existence of something known in psychology as the 'base-rate fallacy' which is usually illustrated by studies done by Kahneman & Tversky (see, for example, Bruner 1986, p. 89 or Holland, Holyoak, Nisbett & Thagard, 1986, pp. 217-222). In a typical study subjects are shown psychological profiles drawn from a sample of 70 engineers and 30 lawyers. They are then asked to guess whether the profile is that of an engineer or a lawyer. When the subjects ignore the information that 70% of the profiles are of engineers, even when the individuating information is completely useless for making a decision, the researchers label this irrational behavior the "base-rate fallacy." The subjects are not thought to be thinking differently, but incorrectly. The interpretation given by the scientists involved is an illustration of the powerful hold Rationalism has on the paths that their thoughts can take and cannot take. As Wittgenstein points out: "So much is clear: when someone says: 'If you follow the *rule*, it *must* be like this', he has not any *clear* concept of what experience would correspond to the opposite" (1956, §III-29, p. 121, emphasis in original).

3. Rationalism and other modes of thought

The previous sections could be very discouraging if we believe that thinking means thinking deductively. And we would not be alone in believing that:

> In this century reason has been understood by many philosophers, psychologists, and others as roughly fitting the model of formal deductive logic:
>
> > Reason is the mechanical manipulation of abstract symbols which are meaningless in themselves, but can give meaning by virtue of their capacity to refer to things either in the actual world or in possible states of the world. (Lakoff, 1987, p. 7)

Any mode of thought defines how one sees the world and acts in the world, which in turn defines what one is in the world. Rationalism is no different. It acts as a filter and a lens for perception, eliminating some objects and relationships from view, distorting others, and bringing some into clear focus. It "enable[s] us to keep an enormous amount in mind while paying attention to a minimum of detail" (Bruner, 1986, p. 48) in much the same way that a wide angle lens provides an enormous view, but distorts details.

As Rationalism has developed, it has become more and more difficult to see the world in other ways. This is a general feature of "ideas of mind."

> Perhaps once a culture has become gripped by an idea of mind, its uses, and their consequences, it is impossible to shed the idea, even when one has lost faith in it.

> For the impact of ideas about mind does not stem from their truth, but seemingly from the power they exert as possibilities embodied in the practices of culture. (Bruner, 1986, p. 138)

The adoption of Rationalism gives it power to affect the way that society develops. As Rationalism is more generally considered to be *the* model of thinking, we construct our society in such a way that it *must* be our model of thinking.

> Isn't it like this: so long as one thinks it can't be otherwise one draws logical conclusions. This presumably means: so long as *such-and-such is not brought into question at all.*
>
> The steps which are not brought into question are logical inferences. But the reason why they are not brought into question is not that they 'certainly correspond with the truth'—or something of the sort,—no, it is just this that is called 'thinking', 'speaking', 'inferring', 'arguing'. There is not any question at all here of some correspondence between what is said and reality; rather is logic *antecedent* to any such correspondence; in the same sense, that is, as that in which the establishment of a method of measurement is *antecedent* to the correctness or incorrectness of a statement of length. (Wittgenstein, 1956, §I-155, p. 45, emphasis in original)

If we accept that all thinking is deductive, and combine that idea with the knowledge that systems of deductive logic are essentially incomplete, we might be tempted to believe that there are things we cannot think about at all. Rather than do that I would take up Bruner's (1986) suggestion that thinking can occur in a number of modes.

Bruner identified two main modes of thinking in our society, paradigmatic (which is Rationalism), and narrative.

> The 'reality' of most of us is constituted roughly into two spheres: that of nature and that of human affairs, the former more likely to be structured in the paradigmatic mode of logic and science, the latter in the mode of story and narrative. The latter is centered around the drama of human intentions and their vicissitudes; the first around the equally compelling, equally natural idea of causation. (p. 88)

It should be noted that these modes of thought are complementary. While it seems that some individuals have developed one of these modes of thought to a higher degree than the other, all humans possess the ability to think in these ways. In this modes of thought seem to correspond to what Lakoff (1987) calls "conceptual schemes".

The paradigmatic mode of thought is closely allied to Rationalism.

> [It] attempts to fulfill the ideal of a formal, mathematical system of description and explanation. It employs categorization or conceptualization and the operations by which categories are established, instantiated, idealized, and related one to the other to form a system. Its armamentarium of connectives includes on the formal side such ideas as conjunction and disjunction, hyperonymy and hyponymy, strict implication, and the devices by which general

> propositions are extracted from statements in their particular contexts. At a gross level, the logico-scientific mode ... deals in general causes, and in their establishment, and makes use of procedures to assure verifiable reference and to test for empirical truth. Its language is regulated by requirements of consistency and noncontradiction. Its domain is defined not only by observables to which its basic statements relate, but also by the set of possible worlds that can be logically generated and tested against observables—that is, it is driven by principled hypotheses. (Bruner, 1986, pp. 12-13)

The narrative mode, on the other hand, "deals in human or human-like intention and action and the vicissitudes and consequences that mark their course. It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place" (Bruner, 1986, p. 13).

Bruner briefly touches on one other mode of thinking, faith, and notes its power in the Middle Ages. This was: "an unmediated knowing of eternal truths revealed by God (or ... by virtue of man's endowment with an intuition of pure knowledge). It was revelation" (p. 108). Barrow (1992) goes into more detail on the subject of faith, or theological thinking:

> Abstract ideas and concrete realities were once interwoven and interdependent to such an extent that no significant wedge could be driven between them. For the ancients and the medievals symbolic meanings of things assumed a natural significance that rests upon associations of ideas that we no longer possess. ... In this way numbers came to possess one aspect that was within the reach of human computation, whilst always possessing others which could be fathomed only by divine revelation. ... Every user of numbers adds their own subjective ingredient to the question of their true *meaning* and its link to the meanings of other aspects of reality. (p. 106-107, emphasis in original)

Barrow's description of faith in the Middle Ages resembles what Gödel called "the theological worldview" which is:

> the idea, that the world and everything in it has meaning and reason, and in particular a good and indubitable meaning. It follows immediately that our worldly existence, since it is in itself at most a very dubious meaning, can only be the means to the end of another existence. The idea that everything in the world has a meaning is an exact analogue of the principle that everything has a cause, on which rests all of science. (quoted in Barrow, 1992, p. 124)

Note that just as paradigmatic thought is based on causality, and narrative thought is based on intention, theological thought has its basis, which Gödel called meaning.

Narrative, paradigmatic and theological thinking do not necessarily exhaust the possible modes of thinking. Davis' (1993) statement "mathematics also displaced religion, history, and narrative to become the primary model of reason for the modern era" (p. 3) suggests that history could also be seen as a mode of thinking. In an historical mode of thought, truth would be derived from the truths of the past, from traditions and customs. The basis of the historical mode thought could be called 'conservation of truth over time'. If scientists best typify paradigmatic thinking (as Bruner asserts, 1986, p. 15), then perhaps members of conservative political parties and movements best typify the historical mode of thinking.

The existence of different modes of thought, each suited to thinking about the world in different ways, provides one important answer to the question "How can we know what Rationalism cannot tell us?" We can know in many ways. At the same time it is important to ask, "If Rationalism cannot tell us everything, what *can* it tell us?" The answer to this question is the topic of the next chapter.

CHAPTER VII

REASONING AND RESEARCH FROM AN ENACTIVIST PERSPECTIVE

> *"Well, it's no use* your *talking about waking him," said Tweedledum, "when you're only one of the things in his dream."*
> — Lewis Carroll, *Through the Looking Glass, and What Alice Found There.*

In this final chapter I attempt to describe Enactivism, and to relate it to my research. I begin by describing some of the key ideas of Enactivism and giving examples from my studies of these ideas in action. I then develop a theory of the development of deductive reasoning, based on Enactivist principles. This is followed by a description of an Enactivist methodology for research in education, which reflects the current state of development of the underlying methodology of my studies. Finally I make a few comments on the relationship between my research and this dissertation and on the teaching of proving in schools.

1. Enactivism

"Enactivism" is used by Bateson, Maturana, Rosch, Thomson, and Varela to label their theories. The "Experientialism" of Lakoff and Johnson is closely related to Enactivism, and I will not distinguish them here. Enactivism is a theory of mind, but, as Bateson (1987) notes, from an Enactivist perspective "epistemology and theories of mind and theories of evolution are very close to being the same thing" (p. 38) so discussions of Enactivism range through the traditional disciplines of philosophy, psychology, and biology. Elements of the psychology of Piaget and Vygotsky are compatible with Enactivism, and I will draw on their writings occasionally, especially in considering Enactivism in relation to learning. The philosophical basis of Enactivism can be found, with some effort, in the writings of Wittgenstein on the philosophy of psychology, and I will make connection with his work wherever possible.

A good starting point to understanding Enactivism is the problem of the relationship between an entity and its surroundings. The first part of this problem is specifying what it is that makes us see the entity as separate from its surroundings. The term *organization* is used to describe those features of an entity which allow an observer to distinguish it from everything else. Note that this implies that an entity's organization varies from observer to observer.

The participants in the studies, the pens they wrote with, the tables they write on, all have particular organizations that make them people, or pens, or tables. These entities have fairly stable organizations, but other entities do not. During the problem solving sessions many groups worked together as a group. The groups themselves can be distinguished as entities, which had a certain organization in the problem solving situations. In most cases however, those entities no longer exist. The organization that defined them no longer relates the people involved. Stacey and Kerry were an interesting case, partly because the organization that defined them as a pair existed before they became participants in my studies. This implies a different organization for the Stacey-Kerry pair than for the others, one that continued to exist.

Some entities have an organization that is *complex*. Complexity is a term borrowed from complexity theorists (e.g., Kauffman, 1993, see also the previous chapter). A system is complex if "a great many independent agents are interacting with each other in a great many ways" (Waldrop, 1992, p. 11). Complex systems create themselves, in the sense that they come into being and remain in existence through their own internal interactions.

Systems that continually create themselves are referred to in Enactivism as *autopoetic*. The components of autopoetic systems "must be dynamically related in a network of ongoing interactions" (Maturana & Varela, 1992, pp. 43-44). That is, the components interact in ways which are continually changing, but which at the same time allow for the continuation of interactions so that the system continues to exist. In addition, the interactions of the components of an autopoetic system are responsible for the production of the components themselves. In summary, an autopoetic system is an emergent phenomenon arising from the interaction of components which, by way of these interactions, give rise to new interactions and new components, while preserving the system's autopoetic character.

Human beings, and living beings in general, are autopoetic. I change continuously, but at the same time all these changes permit me to continue existing as me. In the past three years many of the changes in me have involved the evolution of my ideas on proof, proving, and thinking. These changes have been a part of my continuing existence as a Ph.D. candidate. The interactions which make me a Ph.D. candidate are now changing as well as the complex system that is me prepares to orient itself to a different environment in a way that will permit my continued existence.

Adapting involves changes to a system's *structure*. It is important to distinguish between the structure of a system and its *organization*. A system's organization includes the invariant features without which it would cease to be what it is. An autopoetic system must maintain its organization. The structure of a system includes all its features at a given moment. Interactions with its environment and within the system itself result in a continuous modification of a system's structure.

> The problem is how to handle the problem of structural change and to show how an organism, which exists in a medium and which operates adequately to its need, can undergo a continuous structural change such that it goes on acting adequately in its medium, even though the medium is changing. Many names could be given to this; it could be called learning. (Maturana, 1987, pp. 74-75)

In the problem sessions the participants' structures were changing continuously as their understandings of the situations changed and as they reasoned in different ways. At the same time they remained themselves. When Eleanor switched from reasoning inductively with Ben and Wayne to proving with Rachel she did not become a new person. Nor was there any danger in confusing who was Eleanor and who was Rachel when they were thinking in the same way. The way they were thinking was a part of their structure. That they *could* think is a part of their organization. I have been careful not to say: "Rachel is an deductive thinker," which implies thinking deductively is a part of her organization. If this were so then she would cease to be the entity she is whenever she thought in another way. Thinking deductively was a part of her structure for much of the time she was investigating the Arithmagon situation, but that is as much about the situation and the social context as it is about her.

Living systems achieve autopoesis by *acting* in some way to adjust to local conditions. It is this acting that indicates cognition, so in Enactivism, cognition is a feature of all living systems. This idea is encapsulated in the phrase "Knowing is being is doing." The word "enactivism" is derived from this idea of knowing in action. The way a living system comes to know about the medium it is in is through interaction with that medium. This implies that the system's knowledge of its world depends not only on the medium, but also on the actions the system is capable of.

To take an example from the Arithmagon situation, some of the participants who knew how to solve systems of equations acted in the Arithmagon situation by solving it using such a system. In doing so they learned that the Arithmagon is, in a sense, about a system of equations. That is, for those participants who solved the puzzle using a system of equations, the Arithmagon is a system of equations. For the participants who could not act in that way, either because they did not know how, or for some other reason, the Arithmagon is not about a system of equations. What they could know about the medium they were in, the Arithmagon situation, depended on what they could do.

An autopoetic system is "an active self-updating collection of structures capable of informing (or shaping) its surrounding medium into a world through a history of structural coupling with it" (Varela, 1987, p. 52). As noted above, a system only knows about those aspects of its medium with which it can interact in some way. This means that in being, doing, and knowing, a system defines the world in which it lives.

As I noted above, for those participants who solved the Arithmagon using a system of equations, the Arithmagon situation included systems of equations. For others it included other features. When Eleanor showed her solution method to the rest of us, the Arithmagon included a "middle number" that was simultaneously the sum of the corners, half the sum of the sides, and the sum of a corner and the side opposite. For the participants in other sessions, the Arithmagon did not have a "middle number." For us it did. Similarly other groups were in an Arithmagon situation which included only the sum of the side and the corner opposite or which included the relationship between the sums of the sides and corners. For Wayne, the Arithmagon situation included properties of triangles, and he explored the Arithmagon using those properties.

The activity of coming to know, of learning, is a modification of structure. At the same time it is the system's structure that limits what actions it can take in the environment, and therefore what it can come to know. This limitation of a system's possible actions is called *structure determinism*. What a system does in response to a trigger from its medium is determined entirely by its structure.

When I asked Laura if she was sure her formula worked for all Arithmagons, I asked because I wanted to know if she was sure. Her reply "Oh, you want me to prove it" indicates that her structure was such that the trigger provided by my question did not produce the effect I expected. My question was just a trigger. Laura's response was a product of her structure. If she has spent some period of time in a medium where the question "Are you sure?" is used as a trigger for proving in a teacher-game, for example in a mathematics class where proving is thought to be the only way to be sure, then Laura would have had to modify her structure to remain viable in that medium. This modified structure then determined her response to my question. Of course the action of producing her "proof" changed her structure, and the realization she related later, that she could prove her formula in a better way which explained it, changed her structure again, and so I could not predict how she would respond in a similar situation now, just as I could not predict how she would respond when I asked the question in the first place. Because she has a complex structure her actions may be determined from moment to moment, but they are never predictable.

> If I have a living system ... then this living system is in a medium with which it interacts. Its dynamics of state result in interactions with the medium, and the dynamics of state within the medium result in interactions with the living system. What happens in interaction? Since this is a structure determined system ... the medium triggers a change of state in the system, and the system triggers a change of state in the medium. What change of state? One of those which is permitted by the structure of the system. (Maturana, 1978, p. 75)

In this passage Maturana introduces a central idea of Enactivism: co-emergence. The interaction between a system and a medium (which may include other autopoetic systems) is the mechanism by which both the system and the medium change. As long as a system and a medium continue to be able to interact they are said to be *structurally coupled* and they co-emerge. It should be emphasized that co-emergence does not imply that the system and the medium are becoming more fully adapted to each other. All that is certain is that their structures allow them to interact. It is possible that a history of structural coupling may lead to a situation in which the system and the medium are no longer able to interact. In this case they cease to be structurally coupled. This may be because the system migrates to another medium or because the interaction between the medium and the system disrupts the organization of one or the other, and it dies.

In any of the sessions the participants and the observers co-emerged with each other and the situation. Tom's presence changed Rachel's actions, and hence her being, her structure. At the same time her actions changed his structure, as he learned about mathematical understanding, among other things, in the Arithmagon situation. What Eleanor knew at the end of the session developed through her interactions with everyone else, and what the rest of us knew at the end of the session developed in part through her interactions with us. As what we know is embodied in our structures, our structures co-emerged throughout the time we were interacting; that is to say, throughout the time we were structurally coupled.

In describing the relationship between an entity and its environment, the mistake is sometimes made of seeing the environment as *prescribing* the structure of the entity. For example, in the popular understanding of Darwin's theory of evolution animals are seen as having certain features *because* their environment requires that feature. So polar bears are white, unlike most other bears, because they live in snowy surroundings. The enactivist view of evolution is one of natural drift, based on an animal's environment *proscribing* certain features. This proscription is simply another way of looking at the breakdown of the structural coupling between the animal and its environment. If the animal's structure does not allow for interaction with its environment, then it dies. In effect it is not allowed to have that structure. This is not the same as the environment requiring that it have a certain structure, and in fact many different structures are possible within the constraints imposed by the need to remain structurally coupled. The full range of possible structures defines a *sphere of behavioral possibilities* within which animals can act.

The problem situations the participants investigated defined a sphere of behavioral possibilities for them. The Arithmagon situation can be investigated in many ways, but not, as far as I know, by singing arias. If the participants had chosen to sing arias during a problem situation, then I would judge that they were no longer *in* the situation. Their behavior would break the structural coupling between them and the situation. When Kerry began to investigate the negative Fibonacci sequence, Tom intervened and said the sequence did not work like that. The Fibonacci situation, which for Kerry included Tom, was proscribing the investigation Kerry had attempted to initiate.

2. Enactivism and reasoning

Enactivism is, among other things, a theory of learning, and the following is my attempt to use the ideas of Enactivism to elaborate a theory of the development of deductive reasoning in children growing up in a Rationalist society. This is a theory in progress, as all good theories should be, but this one is so early in its progress that I expect its structure to undergo some serious modifications in the future. I include it here both as an application of Enactivism, and as an indication of the theoretical perspective which co-emerged with the methodology and results of my research. It is also a hint of future research I plan, involving studies of younger students reasoning deductively, with the aim of tracing the development of deductive reasoning more closely.

In developing a theory of the development of deductive reasoning I have drawn on the ideas of Piaget, Vygotsky, Wittgenstein, and the Enactivists. Piaget's ideas form the basis of constructivism, a theory of learning that informs my work and the work of many others in educational research. The central idea of constructivism is that the individual constructs the world in which he or she lives.

> ...when language and thought begin, [the child] is for all practical purposes but one element or entity among others in a universe that he has gradually constructed himself and which hereafter he will experience as external to himself. (Piaget, 1967, p. 9)

For Piaget the world is constructed by the individual, and the influence that the world has on the individual's development is not a major focus.

For Vygotsky, the external world, especially the social world, plays a central role in the development of the individual. This is especially evident in the relationship between thought and language.

> Thought development is determined by language, i.e., by the linguistic tools of thought and by the sociocultural experience of the child. (Vygotsky, 1986, p. 94)

Vygotsky holds that language development and thought development occur separately in very young children. At a certain point however, these two paths of development meet, and "thought becomes verbal and speech rational" (1986, p. 83); that is, thought and language develop together from that point on. The development of language cannot depend solely on the individual, as language takes place in a social context. The linking of thought and language means that thought is similarly constrained by social context.

Wittgenstein also considered the relationship between thought and language, and contributed the important idea of a "language game". A language game is the context, the "form of life" in which words are spoken, and actions are made.

> ...we make a radical break with the idea that language always functions in one way, always serves the same purpose: to convey thoughts—which may be about houses, pains, good or evil, or anything else you please. (Wittgenstein, 1958, §304)

The way we use language in a particular context determines the meaning of the words we use in that context. Meaning cannot be established, once and for all, as dictionaries attempt. The best we can hope for is to be aware of language games, and perhaps make use of the "family resemblances" between language games to know what meanings are in play.

Varela and Maturana introduced the Enactive approach to cognition to escape from the "chicken and egg" situation of trying to decide whether the individual constructs a world or if the world constructs the individual. In a sense, they agree with everyone. The material world and the social world do affect the cognitive structures of the individual, as they must if the individual is going to survive embedded in a material and social context. At the same time, the cognitive structures of the individual guide the individual's actions and interactions in and with the material and social worlds. "World and perceiver, specify each other" (Varela, Thompson, and Rosch, 1991, p. 172). We as beings co-emerge with the worlds we inhabit.

The mechanism by which this co-emergence takes place has two parts:

> ... the structured nature of bodily and social experience and ... our innate capacity to imaginatively project from certain well-structured aspects of bodily and interactional experience to abstract conceptual structures. (Lakoff, 1988, quoted in Varela, Thompson, and Rosch, 1991, p. 178)

Lakoff and Johnson give the name "experiential gestalt" to our perception of structures in our material and social worlds. They call our ability to "imaginatively project" from our experiential gestalts to abstract concepts "metaphoric projection." I believe these two mechanisms can be used to provide a theoretical path from human experience, at its most basic level, to the production of formal proofs.

The perception that lies at the base of deductive reasoning is *coincidence*. When two unusual events happen at roughly the same time, or in the same place, we perceive them as linked. For example, Freudenthal (1973) tells a story of walking with a child past a railway crossing. The previous day, when they had passed the crossing, there had been a friendly dog there. On this day they came to the crossing, and the child asked "But where is the dog?" The child had perceived a link in the coincidence of the dog and the crossing.

The capability to notice and remember coincidences is vital for perception. Without this capability, babies, for example, could not learn that the particular sensory stimulations associated with faces often coincide with those associated with voices, a necessary step to the perception of voices emanating from faces.

The studies of the behaviorist psychologists seem to have established that mammals and birds are capable of noticing and remembering coincidences. It is the coincidence of two stimuli that allows the transference of a response from one to the other. Note that noticing and remembering coincidences is an act of perception, not one of conscious, thoughtful, action. Given the empirical evidence, it seems reasonable to accept that observing coincidences is a biological feature of human beings.

One coincidence that babies might observe and remember is that shaking a rattle, and the sound of a rattle coincide. In fact, babies not only observe and remember such coincidences, they practice them (Lakoff & Johnson, 1980, p. 70). These coincidences are important for babies, and so they shake their rattles, not continuously, but in short bursts, pausing to delight in another occurrence of the coincidence.

Bruner notes research showing that babies have a sense of causation (1986, p. 17). Given that such a sense exists, how might it have come to exist? Lakoff and Johnson (1980) consider causation to be an experiential gestalt. Just as coincidence involves observing and remembering *when* events occur, an experiential gestalt involves observing and remembering *how* events occur. The event of shaking a rattle shares an experiential gestalt with many other coincidences in babies' lives. Pulling blankets, dropping things, throwing things, all share features with shaking a rattle.

Lakoff & Johnson (1980) give a list of the features of the prototypical experiential gestalt for causation:

> The agent has as a goal some change of state in the patient.
> The change of state is physical.
> The agent has a "plan" for carrying out this goal.
> The plan requires the agent's use of a motor program.
> The agent is in control of that motor program.
> The agent is primarily responsible for carrying out the plan.
> The agent is the energy source (i.e., the agent is directing his energies toward the patient), and the patient is the energy goal (i.e., the change in the patient is due to an external source of energy).
> The agent successfully carries out the plan.
> The change in the patient is perceptible.
> The agent monitors the change in the patient through sensory perception.
> There is a single specific agent and a single specific patient. (pp. 70-71)

In the case of the rattle the agent is the baby; the patient is the rattle; the energy transfer is the shaking; and the goal is the sound.

While the experiential gestalt of causation has been presented here in a propositional form, babies, of course, do not think of causation in this way. I would characterize babies' pre-verbal thinking as *sensing*, and speak of a *sense* of causation. A parallel might be made between this sense of causation and the sense of direction we derive from hearing sounds:

> I may be able to tell the direction from which a sound comes only because it affects one ear more than the other, but I don't feel this in my ears; yet is has its effect: I *know* the direction from which the sound comes; (Wittgenstein, 1958, IIviii, p. 185)

Similarly, when we know that an event caused some other event, it may be because we recognize that the condition for the experiential gestalt of causation are present, but we do not know it in that way. We just know it, as a sense. In the case of the pre-verbal baby this sense can exist, even though the concept of causation cannot.

> The fact of becoming conscious of a category will alter its actual nature.... When the child "is cause," or acts as though he knew one thing was the cause of another, this, even though he has not consciously realized causality, is an early type of causality, and, if one wishes, the functional equivalent of causality. (Piaget, 1959, pp. 229-230)

When a baby begins to learn language, the sense of causation the baby has developed from coincidences is changed by the way language talks about causation. "The rattle made noise because I shook it" seems to be nothing more than an expression of the sense the baby already possessed, but the very act of expressing that sense changes it. As Vygotsky (1986, p. 219) puts it: "It does not merely find expression in speech; it finds its reality and form."

Talking about causation is an act with causation itself as its object. The very act of talking about causation makes causation an object in our world. "As language arises, objects also arise as linguistic distinctions of linguistic distinctions that obscure the actions they coordinate" (Maturana & Varela, 1992, p. 210). In talking about causation we are constrained by our language. A child's idea of causation, once articulated, becomes subject to the rules of already existing language-games. These language-games are the context in which verbal thought develops. It is important to consider, as well, that this development has effects on both the individual's thinking, and the language-game. Verbal thought and language-games coemerge. So, even as we become able to think about causation by becoming able to talk about causation, what "causation" could mean to us is changed by our new ability to think about it.

What is the "new form" of causation? We can now say "We will eat now because it is six o'clock." Such a sentence casts time as an agent, and ourselves as patients, changing who we are. Such a sentence is an example of the metaphoric projection of causation. The casting of an experiential gestalt into verbal form permits such projections of meaning to new domains. This process both changes the concept we extend (causation) and the domains into which we extend it (time, and ourselves). Such metaphoric projections can be made by individuals, but more often they are suggested to us by others. The language used around us leads us to make certain metaphoric projections and not others.

As we learn language we also learn to use language to refer to abstract entities like time. These abstract entities, and the ways in which we can talk about them, are constrained by the language games in which we find ourselves embedded. When the language game involves making links between abstract entities, analogous to causal links between material entities, we have the occasion to make inferences. *Inference* is the metaphoric projection of causation to abstract entities and energies.

To emphasize the importance of language games in the development of the ability to make inferences, consider Belenky et al.'s (1986) Silent Knowers. They experience situations of extreme social instability. In such situations language games do not include inferences, and so making inferences is both useless and inconceivable. It is only in retrospect that a silent way of knowing can be described, if such a description assumes a way of knowing that includes sufficient stability of abstract entities to allow inferences to be made.

The ability to make inferences probably precedes the concept of inference, just as the ability to sense causation precedes the concept of causation. Unlike causation, however, the concept need not come after the sense. Just as the sense of inference develops from the sense of causation, the concept of inference can develop from the concept of causation by metaphoric projection.

The experiential gestalt of causation deals for the most part with physical agents, physical energy transfer, and physical patients. In the example of "We will eat now because it is six o'clock" we encountered an abstract agent, time. Such a metaphoric projection is comprehensible to us because of our ability to perceive coincidences, in this case between aspects of abstract entities and physical entities. We cannot perceive coincidence between just any aspects; however, the aspects available to our perception are those that are reified by language. By such a metaphoric projection the concept of inference can develop. In the case of inference, the sense need not precede the concept, nor must the concept precede the sense. It seems likely that when language is full of abstract entities linked by inference there would also be occasions to refer to these inferences, and in such a context the sense and concept of inference could coemerge.

The concept of inference is the basis for deduction. Deduction involves a perception of inferences as meaningful.

> The school child passes from unformulated to verbalized introspection, he perceives his own psychic processes as meaningful. (Vygotsky, 1986, p. 170)

Being able to make inferences, being able to refer to inferences, and the use of inferences in paradigmatic language games (Bruner, 1986), gives the inferences themselves meanings, beyond those of the concepts involved. Part of the meaning of an inference is the idea of logical necessity. An inference with such a meaning is a deduction. When these meaningful inferences or deductions occur in sequences they constitute what I call proving.

An inference need not be a deduction. In the case of narrative language games inferences are related to intention, rather than logical necessity (Bruner, 1986). Because of this distinction, inferences about the actions of human beings and other intentional beings refer to choices made by them, rather than actions forced on them by material or logical constraints.

Deduction permits the conscious choice to deduce, with particular ends in mind. Deductive "chains" can be purposefully constructed. The concept of a formal proof can be developed by metaphoric projection of deduction and chains, into an abstract entity which can be both analyzed and self-directed.

> Learning to direct one's own mental processes with the aid of words or signs is an integral part of the process of concept formation. (Vygotsky, 1986, p. 108)

> Analysis of reality with the help of concepts precedes analysis of the concepts themselves. (Vygotsky, 1986, p. 141)

A chain consists of links, each of which is a separate entity. The links are not materially connected to each other. They could continue to exist outside of the context of the chain. Their structure, however, is such that they are constrained by their neighbors. The combined effect of these constraints can be seen by an observer to constitute a single object, a chain, where many objects exist. Similarly the deductions in a proof can stand alone, but their structure allows them to be joined together into what seems to an observer to be a single object, a proof.

Because proofs are metaphoric projections of chains, they can be analyzed as if they were chains, one link at a time. The strength of a proof, like the strength of a chain, is that of its weakest link. A missing link in a proof or a chain makes it completely functionless as a way of connecting its two ends. The making of a proof, like the making of a chain, is a self-directed activity. At each link there is a decision made as to what link to attach next, how it should be attached, and whether there might be some shorter chains (lemmas) lying about that might be incorporated into the proof chain under construction.

Because proofs are self-directed and analyzable, they must be generated by a formulated act of proving. Formulated proving is a sub-category of the proving which I described above as sequences of deductions. In formulated proving the next link is chosen. In unformulated proving the next link is whatever is at hand. This means that an observer could judge the strength of a proof produced by unformulated proving (by analyzing it, proving in a formulated way with the proof as a guide), the person who proved could not.

3. Enactivism and research

Enactivism is not only a logical extension of Rationalism made aware of its weaknesses, nor only a theory of learning with which to interpret the proving of students, nor only the basis for theories such as the one I just outlined. It is also, and must also be, if it is accepted as a theory of learning, the basis of a methodology for research. Research is learning, and educational research which employs one theory of learning to interpret a student's actions, and a different theory to motivate a researcher's actions, undermines its own basis. In this section I describe my methodology, which arose out of this research and the research and teaching I have done in the past, and which co-emerged with the research methodologies of my colleagues who form a loose Enactivist research group. Kieren, Gordon-Calvert, Reid & Simmt (1995), or Gordon-Calvert, Kieren, Reid & Simmt (1995) are examples of research done by the group as an emergent entity of structurally coupled researchers.

Basic principles

The overall methodology, which connects the various methods and analytic procedures outlined below, I call 'bricological.' Bricological research, in short, combines the flexibility and creativity of *bricolage*, with an underlying logic of inquiry. Given the critique of Rationalism in Chapter VI, I hope you will not be surprised that some Rationalist assumptions are missing from this methodology. The idea of objective truth, and the application of deductive reasoning to nondeterministic complex systems, like people and societies, are the two most important omissions.

For me, the key point of Enactivism is the co-emergence of individuals with their environments. The distinction between individual and environment must be blurred, as each is an active entity whose actions occasion modifications of the other's structure. A related idea is that of proscriptive constraints. Any individual acts within a sphere of possibilities, which proscribe some actions, but dictate no action in particular. This play in the interaction between individual and environment makes the usual assignations of cause and effect impossible. What the individual does is 'caused' both by its own dynamic structure and by the constraints of the environment. At the same time there are 'effects' on both the individual and the environment as their structures are changed by the (inter)action.

Bricolage, as it is used in conceptualizing bricological research, favors the production of complex structures, theories, models, etc. because there is no need to reject possibilities that are 'too expensive', or 'too long'. It can be contrasted with a technological attitude that favors production of lots of results through straightforward, 'clean' techniques. Complex theories are appropriate because the topics of my research are complex systems in and of themselves. It is important to note that just as complex systems are self-organizing, so are complex theories. They organize themselves in a medium that is defined by my thinking, and the thinking that takes place in the groups with whom I do research. They also adapt as part of the process of reporting research, as writing involves a structural coupling with an imaginary reader, whose thinking joins into the theories' medium.

The logic of the bricological methodology comes from the questions chosen for research, and the theories and models with which the research begins. These questions, models, and theories reflect expectations of what might be seen. In the adaptation of Enactivism to research, these expectations correspond to the plastic structure that determines the actions of an individual in a context. Just as an individual's structure changes in changing the context, so our expectations change even as we observe, interview, and analyze according to our expectations.

My favorite metaphor for bricological research is the medieval method of cathedral building. Unlike modern office towers, the design of a cathedral was not the work of an individual whose plans determined the actions of a crew of workers. In the construction of a cathedral every worker had a general idea of the final appearance of the building, but no single individual knew exactly what it would look like. In the time scale of cathedral building, the master builder who sketched out the initial design might well be dead by the time the cathedral was completed, and changes in finance, style, technology, and workers might have resulted in considerable alterations. Each worker's contributions were to a small portion of the building and were governed by the possibilities created by the actions of previous workers. The combined efforts of the workers constituted a *bricolage* of what was possible with the materials. skills, and prior work present. At the same time their work was drawn together by the idea, the logic, of the cathedral as a final form, dedicated to the glory of God.

<u>Some specifics</u>

Bricological research is research in the Rationalist tradition, though much of what was wrong with Rationalism is missing, and much of what is left is changed. All the same, I can write of my "data" and the "analysis" of it.

The data generated in my research include field notes, video tapes, audio tapes, participants' writings, transcripts, notes based on viewing video tapes, mathematical activity traces (MATs, see Appendix C for examples) which summarize the actions in a video taped session, research reports, conference presentations, and notes from discussions with other researchers. These artifacts can be lumped together, as they are here, as 'data', but at the same time all of them record acts of interpretation. In a sense it can be said that there is no data, only interpretations and interpretations of interpretations. That said, I will refer to any artifact of the research process as 'data'.

The analysis of the data is tied up with the idea of multiple interpretations. This means several things. It means that the same event was interpreted in several kinds of data. It means that the data was interpreted by several researchers. It means that the data was interpreted many times by one researcher. It means that the problem prompts were interpreted by a number of participants. It means that the participants were interpreted by several problem prompts. A rough chronology of this process follows.

The first stage of analysis in all the studies was the recording of field notes and video tapes (or in the case of the classroom observations, audio tapes) and the collection of participants' writings. The field notes record the initial impressions of the researcher as to what was important in the session. The video tape records what was visible and audible from a particular point of view. The participants' writings record what they felt it was necessary to record during the course of their investigation of the situations.

The second stage of analysis was the viewing of video tapes and participants' writings by the researchers, either alone or in groups. The notes produced through this process reflect again what was important to the researcher at the time, and the significant points introduced by other researchers. This stage was repeated a number of times by various researchers, according to the perceived significance of the data produced.

The third stage of analysis included the production of transcripts and mathematical activity traces from the video tapes and notes taken during viewing. Not every word spoken, nor every action taken by the participants was transcribed or entered into a MAT. The selection of significant episodes was yet another interpretive action coming out of the viewing of the tapes.

The fourth stage of analysis was the preparation and presentation of the research. The organization and expression of data in publications, research reports, and presentations, occasioned the modification of theories and the reevaluation of data. The comments of respondents at presentations marked their involvement as co-participants in the research whose contributions form a part of the interpretation and data of the research.

As is discussed in the methodology section above, the analysis of data is seen here not as the re-presentation of objective facts, but as a process of co-evolution of theory and data. This point of view comes out of a recognition that such a process is implied in all research, and a belief that adopting a methodology that makes use of this inevitability is the best way to accommodate it. I take seriously the claims of philosophers of science (e.g., Kuhn, 1970, and Chalmers, 1982) that no observation is free of a bias introduced by the theoretical position of the observer. At the same time the necessity of theory to account for data results in a dialog between theory and data, with each one affecting the other. A methodology that attempts to make use of this interaction transforms the analysis of data into a continual process of change and encourages this process as the mechanism of theory improvement.

4. A few parting words on Enactivism, proving, and teaching

I hope this chapter points out the Enactivist notion of "coemergence" in the relation between my empirical and theoretical work on reasoning, and the underlying methodology of my work. In researching reasoning I was reasoning about research, and what I found out about reasoning and what I reasoned about what I found interacted throughout my work. The circularity in my writing, which I noted in my introduction, is not a simple ploy to get you to read more or differently, but instead an accurate reflection of the relationship between methodology, theory, and data in my work. Perhaps research can be done in the traditional models of picking a methodology, gathering data, and developing a theory (although Enactivism suggests that this is not so); in my research the methodology, data and theory emerged together in the interactions which define my research.

Those who have thought about the place of proving in mathematics education, and the place of mathematics in education, have typically arrived at one of two contradictory positions. Some argue that mathematics is the best context for the teaching of rational thought. Others have pointed to the damage done by scientific and mathematical thinking, and have wondered whether we might not all be better off not knowing how to think rationally. I believe that both positions ignore the feature which makes rational thought so far unique. Of the various modes of thinking we have, rational thought is the only one which has demonstrated, within its own criteria, that it has limits. Its strength is in identifying its weakness.

Rationalism has been a horrible choice for the status it has been given as sole mode of correct thinking. But forgetting how to think rationally is not the answer even if it were possible. Rationalism was embraced with enthusiasm because of its power to make predictions about the natural world. That power is worth something in itself, and even if we never helped our children to think rationally, some would discover how to and lead humanity down the same path again. Instead we must try to make sure that we teach deductive reasoning well enough that its limits are understood.

In the past we have been unsuccessful in teaching deductive reasoning in mathematics classes. At the root of this failure are two misconceptions about proof and proving. When students ask why they must prove, the common answer is that proving verifies statements. This answer neglects both the importance of other factors in convincing us of truth, and the importance of other uses of proving. We also expect proofs to be individual works, expressed in formal language, but this expectation ignores the vital role that social interaction plays in supporting proving and its formulation. If we can teach in a way that acknowledges that importance of explaining and exploring as motivations to prove, and that creates social contexts that allow the development of a culture of proving, then we may find that our students prove and understand proving well enough to understand that other ways of thinking are sometimes better.

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Reid, D. (1993). *Pre-formal, formal, and formulaic proving.* Paper presented at the meeting of the Canadian Mathematics Education Study Group, Toronto.

Reid, D. (1994). Classroom discourse. In A. Olson & D Reid (Eds.), *Applying the NCTM Standards in Alberta* (pp. 9-25). Edmonton: Centre for Mathematics, Science, and Technology Education.

Rheingold, H. (1991). *Virtual Reality.* New York: Summit Books

Ribet, K. & Hayes, B. (1994, March-April). Fermat's last theorem and modern arithmetic, *American Scientist, 82*, 144-156.

Schoenfeld, A. (1982). Psychological factors affecting students' performance on geometry problems. In S. Wagner (Ed.), *Proceedings of the Fourth PME-NA Conference*, (pp. 168-174). Athens, GA.

Schoenfeld, A. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive Science, 7*, 329-363. (revised version of a paper presented at the 1982 meeting of AERA).

Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic

Schoenfeld, A. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215) Hillsdale NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. (1988). When good teaching leads to bad results: The disasters of "well taught" mathematics courses. *Education Psychologist, 23*(2), 145-166.

Searle, J. (1992). *The rediscovery of the mind.* MIT Press.

Sekiguchi, Y. (1991) *An investigation on proofs and refutations in the mathematics classroom.* Unpublished doctoral dissertation, University of Georgia, Athens.

Senk, S. (1985, September). How well do students write geometry proofs? *Mathematics Teacher,* pp. 448-456.

Simpson, A. (1994). Student attitudes to proof. In G. Ervynck & D. Tall (Eds.) *Abstracts of the sessions of the working group on advanced mathematical thinking (A.M.T.).* Unpublished monograph prepared for the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education. Lisbon.

Skemp, R. (1987) *The psychology of learning mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Taylor, A. (1974). War by time-table. In A. Taylor, (Ed.).*History of World War I.* (pp. 25-29). London: Octopus Books.

Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29-37. Originally published in *Bulletin of the American Mathematical Society, n.s., 30*(2), 161-177.

Tymoczko, T. (Ed.), (1986). *New directions in the philosophy of mathematics: An anthology.* Boston: Birkhäuser.

Ulam, S. (1976). *Adventures of a mathematician.* New York: Scribner.

Varela, F. (1987). Laying down a path in walking. In W. Thompson (Ed.), *Gaia: A way of knowing.* (pp. 48-64). Hudson, NY: Lindisfarne Press.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience.* Cambridge MA: MIT Press.

Vygotsky, L. (1986). *Thought and language.* Cambridge MA: MIT Press

Waldrop, M. (1992). *Complexity: The emerging science at the edge of order and chaos. New York: Touchstone.*

Wittgenstein, L. (1956). *Remarks on the foundations of mathematics*, (G. E. M. Anscombe, Trans.). Oxford: Basil Blackwell.

Wittgenstein, L. (1958). *Philosophical investigations.* (G. E. M. Anscombe, Trans.). Oxford: Basil Blackwell.

APPENDICES

APPENDIX A

RELATED LITERATURE

It was once customary for a dissertation to include a section entitled "Literature Review" in which the author attempted to describe all the research which had been done on the topic of the dissertation. The bibliographic search required to assemble this section was often a large part of the effort involved in preparing a dissertation. This effort was worthwhile, both as basic preparation for beginning research in an area, and as a service to the research community. Such reviews provided a starting point for other researchers interested in quickly becoming acquainted with the literature of a particular area.

The introduction of services like ERIC and Dissertation Abstracts International, especially in their electronic formats, have made the preparation of a literature review much easier, at the same time they have made it largely superfluous. Assembling a list of most of the works in educational research related to proof and proving is now a matter of a few keystrokes, and the information available includes extensive abstracts and detailed information on availability of unpublished manuscripts, research reports, and government documents, dissertations, as well as journal publications. Given this level of service it would be surprising if a researcher interested in proof and proving went to the trouble of requesting a copy of my dissertation, either as an interlibrary loan or from University Microfilms International, just to read my literature review, when the same information, in more detail and more up to date, is available in any university library and over the Internet.

For this reason this appendix is an appendix, and slightly different in form than the traditional literature review. The research which is directly related to mine, and which played an important role in the development of my ideas, is described in the appropriate sections of the main text. Other work on proof and proving which is interesting, but not directly related to my interest in the need to prove, is gathered together here. The one exception to this is the first section, in which I have listed works by researchers who have been very productive, and published their research in a wide range of publications. Rather than list a reference for every occurrence of a researcher's ideas when I mention them in the main text, I have chosen to list only the most accessible or complete presentations of the ideas. For completeness, and in the event that some sources are not as accessible as I thought, I have listed other publications of these researchers here.

1. Other publications by researchers referenced in the main text

Arsac, G. (1990). Les recherches actuelles sur l'apprentissage de la demonstration et les phenomes de validation en France. *Recherches en Didactique des Mathématiques, 9*(2), 247-280.

Arsac, G., Chapiron, G., Colonna, A., Germain, G. Guichard, Y., & Mante, M. (1992). *Initiation au raisonnement déductif au collège*. Presses Universitaires de Lyon.

Balacheff, N. (1986). Cognitive versus situational analysis of problem-solving behaviors. *For the Learning of Mathematics, 6*(3), 10-12.

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics, 18,* 147-176.

Balacheff, N. (1990a). A study of students' proving processes at the junior high school level. In I. Wirszup & R. Streit (Eds.), *Developments in School Mathematics Education Around the World, Vol. 2.* Reston VA: NCTM. (Originally presented in 1988 at the joint conference 66th NCTM and UCSMP project, Chicago.)

Balacheff, N. (1990b). Beyond a psychological approach: The psychology of mathematics education. *For the Learning of Mathematics,10*(3), 2-8.

Balacheff, N. (1990c). Towards a problematique for research on mathematics teaching. *Journal for Research in Mathematics Education, 21*(4), 258-272.

Borasi, R. (1990). The invisible hand operating in mathematics instruction: Student's conceptions and expectations. In T. Cooney (Ed.), *Teaching and learning mathematics in the 1990s: NCTM yearbook 1990.* Reston VA: National Council of Teachers of Mathematics.

Bruner, J. (1990). *Acts of meaning.* Cambridge MA: Harvard University Press.

Davis, P. (1993). Visual theorems. *Educational Studies in Mathematics, 24,* 333-344.

Hanna, G. (1989a). More than formal proofs. *For the Learning of Mathematics. 9*(1), 20-23

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13

Hanna, G. (1991). Mathematical proof. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic.

Hanna, G. & Jahnke, N. (1993). Proof and application. *Educational Studies in Mathematics, 24,* 421-438.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24,* 389-399.

Schoenfeld, A. (1982). *Psychological factors affecting students' performance on geometry problems.* In S. Wagner (Ed.), Proceedings of the Fourth PME-NA Conference, (pp. 168-174). Athens, GA.

Schoenfeld, A. (1986). On having and using geometric knowledge. In: *Conceptual and Procedural Knowledge: the Case of Mathematics* pp. 225-264

Schoenfeld, A. (1987a). Confessions of an accidental theorist. *For the Learning of Mathematics, 7*(1), 30-38

Schoenfeld, A. (1987b). Understanding and teaching the nature of mathematical thinking. In I. Wirszup & R. Streit (Eds.), *Developments in School Mathematics Education Around the World.* Reston VA: NCTM.

Schoenfeld, A. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education, 20*(4), 338-355

Tymoczko, T. (1986) Making room for mathematicians in the philosophy of mathematics. *Mathematical Intelligencer 8*(3), 44-50.

2. Discussions of the nature of proof in mathematics

Barbeau, E. (1990). Three faces of proof. *Interchange, 21*(1), 24-27.

Barbeau mentions the two everyday uses of "proof": verification, and testing or trying. He also points out that proofs can satisfy a need for explanation in mathematics. He makes a distinction between verification and convincing, which he uses in the sense of explanation. He defines verification as making a mechanically checkable argument, and convincing "a revelation of underlying structure, appropriate level of generality, comprehensiveness, and a degree of satisfaction and appreciation aroused in the listener" (p.24).

Neubrand, M. (1989). Remarks on the acceptance of proofs: The case of some recently tackled major theorems. *For the Learning of Mathematics, 9*(3), 2-6.

Neubrand quotes Hanna's (1983) assertion that verification is supposed to be the business of mathematics, but this is practically impossible, and convincing is what mathematicians actually engage in. He states that "a 'convincing argument' is not simply a sequence of correct inferences. One always expects some 'qualitative' reason, or an intuitive capable basic idea, behind the—nevertheless necessary—single steps of the proof" (p.4). He adds, however, that in mathematics "to be convinced depends on the high standards of argumentation which mathematicians have reached during a long historical development" (p.3).

Wheeler, D. (1990). Aspects of mathematical proof. *Interchange, 21*(1) 1-5.

Wheeler seems to assert that proving does not create new mathematical knowledge, which would imply that it is not useful for exploration. "We can no longer assume that proofs establish knowledge, because in fact most proofs come after the knowledge of the things they prove. They could be said perhaps to substantiate knowledge or to validate it, or confirm it, but proofs, on the whole, do not establish knowledge" (p.2). He does not believe there is much of a role for proving in mathematics classrooms.

3. Research on students' understanding of the concept of proof

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

Chazan reports interviews with high school students describing their views of proofs as evidence, versus their acceptance of examples as verification.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the international group for the psychology of mathematics education* (pp. 70-95). Cambridge: Cambridge University Press.

Hershkowitz discusses proving in the context of van Heile's level of understanding of geometry. She suggests that students' difficulties in proving may stem from a lack of understanding of the necessity to prove in mathematics. She advocates a strategy of presenting proofs to trigger the students' "intellectual curiosity", in which empirical discovery acts as a source of a need to verify by proving. "It is a common belief now that inductive, empirical discoveries in geometry are necessary because ... by regarding the generalization as a conjecture in itself, the learner feels the necessity to prove what he or she has conjectured to be true; and ... inductive experiences are the intuitive base upon which the understanding and the generation of a deductive proof can be built" (p.89).

Martin, G. & G. Harel (1989) Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51

Martin & Harel (1989) conducted a quantitative study in which they claim: "Many students who correctly accepted a general-proof verification did not reject a false proof verification; they were influenced by the appearance of the argument—the ritualistic aspects of the proof—rather than the correctness of the argument" (p.49). Unfortunately their statistics seem not to back this up. They presented preservice teachers with two statements, one of which the teachers had seen in class three weeks previously. The teachers were given empirical evidence for each statement, as well as a correct and an incorrect deductive proof. They were asked to rate each on a scale of 1 to 4, with 4 indicating that they felt the evidence constituted a proof. For the statement they had seen in class 52% rated the incorrect proof either 3 or 4, indicating they accepted it as a proof. 75% gave the correct proof either 3 or 4. This would argue that the proof-like form of the incorrect proof was influencing them. If this were the case one would expect a similar result in the case of the unfamiliar statement. In that case however, only 38% gave the incorrect proof either 3 or 4 while 63% gave the correct proof 3 or 4. It may have been that the teachers remembered the familiar statement as correct, and so were predisposed to assume that a proof of it was correct. In that case of the unfamiliar statement the proofs would have been checked more closely, resulting in a drop in the acceptance of the false proof.

Movshovits-Hadar, N. (1988). Stimulating presentation of theorems followed by responsive proofs. *For the Learning of Mathematics, 8*(2), 12-19, 30.

Movshovits-Hadar proposes presenting conjectures in surprising ways in order to inspire a need to verify and explain in students, in a manner similar to that proposed by Hershkowitz.

O'Daffer, P. G. & Thornquist, B. A. (1993). Critical thinking, mathematical reasoning, and proof. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 39-56). New York: Macmillan.

A general discussion of research and the meanings of the various terms used in the NCTM *Standards* (1989) to describe reasoning.

Sierpinska, A. (1995). Mathematics: "In context", "pure", or "with applications"? *For the Learning of Mathematics, 15*(1), 2-15.

A discussion of the role of real life contexts in mathematics teaching. Sierpinska critiques the use of social contexts in the teaching of proof, as proposed by Arsac, Balacheff, and Lampert, and suggests instead an apprenticeship model. Her focus seems to be more on the education of future mathematicians than the general population.

4. Useful literature reviews

Rather than duplicate the efforts of other researchers who have assembled reviews of the literature related to the aspects of proof and proving they find most interesting, I have gathered together here a few works which contain excellent reviews. The two dissertations include the traditional exhaustive literature review, and in addition to the interesting research they report, they also fulfill the traditional purpose of making other researchers' lives easier. These two are particularly strong in listing older research, which is not always covered in the electronic databases.

Dreyfus, T. (1990) Advanced mathematical thinking. In P. Nesher & J. Kilpatrick (Eds.) *Mathematics and Cognition.* (pp. 113-134).

An overview of current research.

Smith, E. P. (1959). *A developmental approach to teaching the concept of proof in elementary and secondary school mathematics.* Unpublished doctoral dissertation, Ohio State University.

Smith's dissertation was written in the early days of the development of the "New Math" and he sets out clearly the agenda of that movement.

Williams, E. R. (1979). *An investigation of senior high school students' understanding of the nature of mathematical proof.* Unpublished doctoral dissertation, University of Alberta.

Williams did his research at the close of the "New Math" era, and he covers most of the research done within that approach to mathematics and mathematics education. His research is a series of statistical studies of students' performance.

APPENDIX B

DESIGN AND DÉROULEMENT OF THE RESEARCH STUDIES

The empirical basis of my research includes four research studies, and several isolated observations and interviews. The basic structure of each study was the same. Typically, a pair of students was observed in two problem solving situations, and then interviewed. Each problem solving session and the interview took about one hour and the sessions were spaced one week apart. Differences from this typical outline will be indicated below for each of the studies.

1. The North School study

Context

The study took place over a seven week period in the spring of 1994. The site was a large urban high school, which will be referred to as North School. The classes involved were both normally taught by the same teacher (called Mr. B here), but the period of the study overlapped the last four weeks of a student teacher's practicum in Mr. B's classes. This student teacher (called Mr. A here) had sole responsibility for teaching one of the classes chosen for observation. Mr. B taught the other class. Both teachers are highly competent. Mr. A's lack of experience was compensated for by his enthusiasm and knowledge of mathematics and teaching methods. Mr. B. is a highly respected teacher both in his school district and in the community at large.

The classes chosen were selected to be as different as possible in the high school context. One class was a non-academic 10th grade class (Math 13), which was taught by Mr. A. It had a total enrollment of nearly 30, but normal attendance ranged between 20 and 25. The other was a 12th grade class for university bound students (Math 30). It had an enrollment of just over 20, almost all of which were present every day. Each class met three times a week, for 65 minutes each session.

Outline

The study was conducted in two phases, a classroom observation phase, and a small group phase.

In the first phase each class was observed engaged in their normal mathematical activities, over a period of three weeks. In this period field notes were kept of the general character of the classes' activities, to provide context, and of particular observations of deductive reasoning employed by the students in the course of their normal activities. On some occasions audio tapes were made of class sessions which were then used to expand the records in the field notes.

In the second phase a pair of students who volunteered from each class engaged in problem solving activities. There were four sessions in this phase: two problem sessions, and two interviews. Each session was video taped for later analysis. During the problem sessions the students worked together on a single problem for about one hour. The researcher observed, interacting with the students only when they asked questions. During the interview sessions the researcher asked the students about particular aspects of their problem solving, making use of video tapes of the problem sessions where appropriate. The students were also asked to continue work on the problems they had been given in the problem sessions, and to solve a new problem, with some help from the researcher. They were then assisted in making formal the reasoning they had employed in solving the problems. The problem situations used in the problem sessions were the Arithmagon and Fibonacci. The prompts for these problems are shown below (in section 7).

Participants

The students who participated in the second phase were selected from their classes on the basis of their willingness to participate, their involvement in class, and the reasoning they displayed in class.

The pair chosen from the Math 13 class I have called Bill and John. Bill sat behind John in class. At the time of the study Bill was doing very well in Math 13. He was normally attentive in class, and appeared to catch on quickly to the concepts presented to him. On two occasions, when Mr. A was absent and Mr. B was teaching the class, Bill responded at length to requests for explanations and alternative methods. Bill normally worked on seat work by himself. John had more difficulty than Bill, but was still able to succeed on most of the work required in the class. He rarely spoke in class, and worked by himself, except for rare occasions when he would consult with Bill on seat work with which he was having difficulty. On one of the uncommon occasions when John asked a question, he had noticed an unusual pattern in a linear equation, and wondered if it were general. (Briefly, John noticed that the solution, $\frac{11}{7}$, of the equation $2 = \frac{3}{7} + b$, could be obtained by calculating $2 \times 7 = 14$, and then subtracting $14 - 3 = 11$, and placing this result over the denominator 7.)

The pair from the Math 30 class I have called Colin and Anton. They often worked together, or with others in the group of students sitting near them. Colin was the top student in his class. He was attentive, did all assigned work, and understood new concepts quickly. Anton was in the top half of the class, was less enthusiastic about doing seat work, and often needed to ask Colin or someone else to explain the procedures required by the assigned tasks. This was more due to his lack of attention to Mr. B's explanations and instructions than to difficulties in comprehension. Anton was more talkative in class than Colin, although much of his talking was not related to mathematics.

2. The South School study

Context

The study at South School took place in the fall of 1994. South School is a large urban high school. One class was observed, taught by Ms E. Ms E is a highly competent and well respected teacher, who has made considerable efforts to promote mathematics at South School and through professional organizations. The class chosen was an "Academic Challenge" grade 10 (Math 10 AC) class. The students had been selected from among the most successful students in their grade 9 programs. The class met every day, for 80 minutes each session.

Outline

The general outline of the study was the same as for the North School study. The class was observed for two weeks, and then students were observed and interviewed. In the small group phase four groups of students participated. There were three sessions with each group, two problem sessions and an interview. Each session was video taped for later analysis. In keeping with the enactive methodology (described in Chapter VII) several researchers were involved in each session. Researcher interventions in the problem sessions were limited to answering the participants' questions, and asking questions designed to encourage further investigations. During the interview session the researchers asked the participants about particular aspects of their problem solving, and proposed additional problems. The Arithmagon and GEOworld situations were used in the problem sessions (see below for prompts).

Participants

The students who participated in the problem sessions and interviews were selected from their classes on the basis of their willingness to participate, their involvement in class, and the reasoning they displayed in class. In general all the students did well in class.

Group I included three female students, Ann, Lynda, and Joanna. Ann was active in the class, asking and answering questions, and working with the students around her, including Lynda, who sat in front of her. Lynda was quiet, and participated in class only in her interactions with Ann. Joanna was active primarily in responding to Ms E's questions, and working with the students who sat around her.

Group II included two female students, Tara and Topaz. Tara was active in working with the students around her, and in answering questions. Topaz was occasionally absent, and her main involvement in the class was working with the students around her and asking Ms E questions.

Group III include three male students, Joseph, Stephen and Scott. The three of them sat together, and often worked together. Scott was interesting in that he often talked to himself while working, and seemed quite involved in his work.

Group IV included two male students, Alec and Darrell. They normally worked by themselves, occasionally interacting with those around them.

3. The first clinical study

Context and outline

Two of the main studies were conducted with undergraduate students. These studies will be referred to as the 'clinical studies'. The first clinical study occurred in the fall of 1993.

The clinical studies involved three problem solving sessions, followed by an interview. The problem situations used were the Arithmagon, Fibonacci, and GEOworld. In the first session each pair of participants worked separately on either the Arithmagon or the Fibonacci situation. In the second week each pair worked in GEOworld. In the third week each pair worked in the situation they had not yet seen. The prompts used are given below, in section 7. For the sessions in the second and third weeks the pairs were grouped in two sets. Each set worked in the same situation at the same time, and could communicate with the other pair in the set. The situations and sets are summarized in Table 4. Each pair was interviewed separately in the fourth week.

Pair	Week 1	Week 2 (set)	Week 3 (set)
I (B&W)	Fibonacci	GEOworld (A)	Arithmagon (A)
II (S&K)	Arithmagon	GEOworld (B)	Fibonacci (B)
III (E&R)	Fibonacci	GEOworld (B)	Arithmagon (A)
IV (J&C)	Arithmagon	GEOworld (A)	Fibonacci (B)

Table 4: Schedule of the sessions for the first clinical study.

Participants

The students who participated (with one exception, Kerry) were volunteers from a pre-service mathematics teacher education course. A total of eight students participated. The participants worked in the following pairs:

Pair I: Ben and Wayne. Ben and Wayne volunteered as a pair. Each of them had completed the mathematics requirements of the B. Ed. degree.

Pair II: Stacey and Kerry. When she volunteered Stacey asked to work with her friend Kerry. Kerry was student in Finance, and had completed courses in linear algebra and calculus as part of his degree. Stacey had completed the mathematics requirements of the B. Ed. degree.

Pair III: Eleanor and Rachel. Eleanor and Rachel volunteered separately and were paired by default. Eleanor had completed a bachelor's degree in mathematics 15 years previously, which provided her with the mathematics entrance requirements for the B. Ed. degree. Rachel had completed the mathematics courses required as part of the B. Ed. degree.

Pair IV: Jane and Chris. Jane and Chris volunteered as a pair. They had both completed bachelor's degrees in the past which provided them with the mathematics entrance requirements for the B. Ed. degree. Chris had recently completed a physics degree. Jane had completed a mathematics degree.

4. The second clinical study

The second clinical study took place in the fall of 1994. The context and general outline was identical to that of the first clinical study, except that the pairs never worked as a set of four. The schedule of sessions is summarized in Table 5.

Pair	Week 1	Week 2	Week 3
I (J&T)	Fibonacci	GEOworld	Arithmagon
II (M&R)	Arithmagon	GEOworld	Fibonacci
III (L&D)	Fibonacci	GEOworld	Arithmagon

Table 5: Schedule of sessions for the second clinical study.

Participants

As with the first clinical study, the students who participated were volunteers from a pre-service mathematics teacher education course. Six students participated. The participants worked in the following pairs:

Pair I: James and Trisha. James and Trisha were paired by chance. Each of them had completed the mathematics requirements of the B. Ed. degree.

Pair II: Roger and Marie. Roger and Marie were paired by chance. Each of them had completed the mathematics requirements of the B. Ed. degree.

Pair III: Laura and Donald. Laura and Donald were paired by chance. Each of them had completed the mathematics requirements of the B. Ed. degree.

5. Other Studies

Sandy

In the winter of 1993-1994 several interviews were done with Sandy, a mathematically gifted student in grade 6. In one of these sessions Sandy was given the Arithmagon problem to explore. This session differed from the typical problem sessions as Sandy was questioned about his reasoning as he worked on the problem.

Central High School

In May 1993 I spent two weeks observing two mathematics classes at an academically oriented high school, as a preliminary study to the school studies. One of the classes observed was a grade 10 class studying linear equations. The other was a grade 11 International Baccalaureate class studying combinatorics and probability. The organization of the classes was flexible, but a typical period would begin with a lecture by the teacher, followed by work on assigned exercises, either individually or in small groups. Occasionally a period would begin with a problem solving exercise, to be done individually and as quickly as possible.

6. Methods

Three contexts for research were used in the studies. Different methods were appropriate for each of these. The classroom observations, and observations of the problem sessions involved passive observation techniques. The interviews employed techniques from traditional clinical interview methods (see below). All of these methods were used in the context of the Enactivist methodology described in Chapter VII.

The observers' role in the observations of the problem sessions can be called 'passive', as the observers interrupted the participants' work only when asked to do so by the participants. This usually took the form of the participants asking a factual question of the observers. This passivity permitted the students participating to 'own' the situation to a greater extent than in the interview sessions. Their explorations and interactions were governed more by their own interests and needs, than by those of the observers. In some of the problem sessions it is clear that the number of observer interruptions is related to the degree to which the participant reasoned and understood in a connected manner (see Kieren, Pirie, and Reid, 1994, for details).

The clinical interview is a research method derived from the work of Piaget. Piaget developed the clinical interview in order to investigate reasoning in young children. The method has since been modified in many ways, involving more or less standardization. Opper (1977) gives this description:

> The essential character of the method is that it constitutes a hypothesis-testing situation, permitting the interviewer to infer rapidly a child's competence in a particular aspect of reasoning by means of observation of [the child's] performance at certain tasks.... The interviewer presents to the child an "experiment" that has been selected as suitable for the study of the specific aspect of cognition of interest.... [The interviewer] asks a series of related questions which are aimed at leading the child to predict, observe, and explain ... It is these predictions, observations and explanations that provide useful information on the child's views of reality and his thought processes. The verbal explanations are particularly valuable for inferring the underlying mental processes ... If further clarifications are required [the interviewer] asks additional questions or introduces extra items.... The information at any point may substantiate or invalidate the original hypothesis. In the former case, the interviewer may ask additional questions so as to satisfy himself of the stability and consistency of the child's responses ... If the original hypothesis is not confirmed, the interviewer reformulates it to take into account the child's responses and asks further questions or introduces additional items to clarify these responses. (pp. 92-93)

In the case of the interviews in my studies the "experiment" presented to the students were questions, notes and transcripts based on my previous observations.

7. Prompts

The situations the participants explored resemble those employed in problem solving research (e.g., Schoenfeld, 1985). While this research owes much to traditional problem solving research, the underlying assumptions concerning the nature of the participants' activities differ. What would otherwise be seen as "problem solving" is seen here as a process of transforming a text (the prompt) into a situation and investigating and extending that situation. This process depends as much on the participants' histories as it does on the originating text. Within this process we see both the participants' cognitive structures being modified through learning about the situation, and also the situation itself being modified through interaction with the participants' cognitive structures. Thus, these situations can be seen as co-emerging* with the participants during the course of a session.

The mathematical situations explored were based on three "prompts"; the texts on which the participants founded their explorations. Two of these,

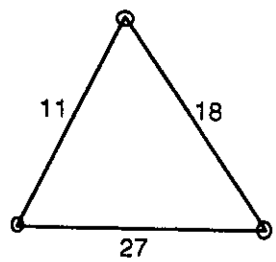
* See Chapter VII for a description of co-emergence, and structure.

Arithmagon and Fibonacci, are traditional paper and pencil problems. The third is a LOGO microworld.

Arithmagon

The Arithmagon situation is derived from a problem in Mason, Burton, & Stacey (1985). The text used in the North School study was slightly different from the text used at South School, and in the clinical studies. The prompt used at North School was:

> The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.



> Make up a triangle of your own, and solve it.
> Can you describe a general way to solve all triangles?
> Make up a square and solve it.
> Can you find a general way to solve all squares?

The last four lines were replaced by the instruction "Generalize the problem and its solution" in the other studies. The questions were added to the North School prompt to try to focus the participants' investigations along paths which I thought at the time to be the most fruitful. I do not believe this was the case, and so the prompt reverted to its original form for the second clinical and the South School studies.

Fibonacci

The prompt for the Fibonacci problem was modified for each study. The prompt used in the first clinical study was:

> The Fibonacci sequence begins:
> 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
> If you are familiar with the recursive rule defining the sequence write it down. If you are not, try to discover the rule.
> Use the notation F_n to stand for the n^{th} Fibonacci number. For example, F_7 is 13 and F_{10} is 55.
> Look for patterns which relate the index n to the Fibonacci number F_n. For example, is there anything special about F_n when n is a multiple of 3, or a multiple of 4, or prime?

The prompt used at North School was:

> The sequence of Fibonacci numbers begins:
> 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

> Try to discover a rule which will tell you the next number in the sequence.
> Is there anything special about every third Fibonacci number (2, 8, 34, ...)?
> Is there anything special about every fourth Fibonacci number (3, 21, ...)?

The prompt used in the second clinical study was:

> The Fibonacci sequence begins:
>
> 1, 1, 2, ...
>
> and continues according to the rule that each term is the sum of the previous two (e.g., 1+1=2).
>
> The Fibonacci sequence has many interesting properties.
> Can you find an interesting property of every third Fibonacci number?
> Can you find other interesting properties?

The Fibonacci prompt in all its forms had the problem of being too directive. The intention of the questions was to inspire investigations in particular areas known to be suitable for deductive reasoning. They usually resulted in superficial investigations, since the participants were quick to go on to the next question.

In both the Arithmagon and Fibonacci situations the participants are given an initial puzzle to solve. In the Arithmagon the puzzle is the determination of the secret numbers. In the Fibonacci situation the puzzle is determining the recursive rule defining the sequence or finding a pattern in every third Fibonacci number. These tasks are puzzles in that they have definite answers, and are well within the capabilities of the participants. The giving of initial puzzles was intended as a means of giving the participants' investigations an initial basis, motivation and direction. This was more successful in the case of the Arithmagon than in the case of the Fibonacci numbers.

GEOworld

The LOGO microworld, called GEOworld, is created by a simple recursive program (reproduced below) which accepts three numerical parameters and produces a geometric figure. In this case no initial puzzle was proposed. The situation is not really suited to the construction of simple puzzles. Instead an initial prompt was given. In the clinical studies the example of GEO 100 100 2 was included in the instruction sheet (Figure 34 shows the output of this input). In the second school study the functions of the three parameters were described vaguely, and three examples were proposed: GEO 100 100 2, GEO 135, 100 -3, and GEO 15 25 1. These prompts do focus the participants on certain aspects of GEOworld, to the neglect of others, but it was possible to ignore them.

```
to geo :a :b :c
if :b <1 [stop]
fd :b
rt :a
geo :a :b-:c :c
lt :a
bk :b
end
```

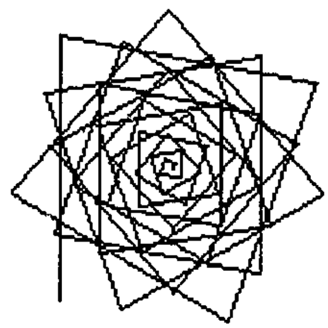


Figure 34: Output of GEO 100 100 3.

APPENDIX C

MATHEMATICAL ACTIVITY TRACES

As is mentioned in Chapter VII, part of the analysis of the data involved making Mathematical Activity Traces (MATs) which chart the episodes which occurred in a session. This made it easier to see shifts in reasoning, and to associate reasoning with needs. The analysis of data progressed to different stages, depending on its usefulness to my research at that time. For this reason not every group had a MAT made for them. These are included both to give examples of this way of recording the events in a problem solving session, and to organize the episodes I have referred to in the text.

1. MATs from the study at North School

Mathematical Activity Traces for the Math 13 Pair, Bill & John

Fibonacci

(minutes elapsed)

1.1 Given sheet. (01)
1.2 Looked for difference pattern
1.3 B saw recursive rule pattern.
1.4 Calculated 55+89=144. (02)

2 Both immediately observed that F_{3n} is even.

3.1 J conjectured F_{4n} is odd
3.2 B pointed out 144 is not odd (03)
3.3 J investigated relationships between F_{4n}; nearly found $F_{4n} + 4F_{4n+4} = F_{4n+8}$ (04-05)
3.4 J extended sequence
3.5 B observed sum of digits in F_{4n} is divisible by 3; assembled empirical evidence. (07)
3.6 B suggested finding the next F_{4n}; J did so. (08)
3.7 B saw pattern in 144 233 377, contradicted by 610. (09)
3.8 B again expressed his sum of digits rule, with a reservation about its generality based on the inadequacy of empirical evidence. (10)
3.9 J suggested looking at F_n for $n \neq 4$. Looked instead at 3, 21, 144, 233, 377. B expressed confusion about this sequence. (11-12)

4 B checked digit sum conjecture for larger values of F_{4n} but continued to reject generality because of limitations of empirical evidence. (13)

5.1 J calculated differences in 2, 8, 34, 144, 610. B examined differences of differences. (14)
5.2 B again mentioned digit sum conjecture, and limited its generality: "It could be a coincidence" (15)

5.3 J suggested dividing sums of digits by 3. B responded that they will divide evenly by 3, but no one did the division. (17-18)

6 B checked F_{3n} is even conjecture for larger values.

7.1 J observed that $F_{4n} \div F_{4n-4} \approx 6$. He used this to calculate:
$987 \div 144 \approx 6; \quad 6 \times 21 \approx 144$ (20)
7.2 B found next F_{4n} to test J's conjecture. (20)
7.3 J checked conjecture with 4181. It failed as 4181 is not the right number. (21)
7.4 B checked sequences to see if they had the right number.
7.5 They checked J's conjecture. (23)
7.6 J formulated his conjecture. (24)
7.7 B commented on its lack of practical utility. (24)

8.1 B commented that his digit sum conjecture had not been disproved. J suggested reexamining it. B suggested a counter example would soon occur. (25)
8.2 J noticed a pattern in the sums of digits: 3, 3, 9, 24, 24,
8.3 B calculated the next numbers. (26)
8.4 They checked the next number. J took 10946 to be the next and found it didn't work. B took 46368 to be the next number and found it worked. They clarified which number is F_{4n}. (27-28)

9.1 They discussed J's ratio conjecture again. J commented on its inexact nature, B commented on its uselessness. (28)

10.1 B checked F_{3n} for even numbers again. J watched. The conjecture was confirmed for several more values, but B was still not certain.
10.2 B considered his use of digit sums. He was uncertain as to its applicability to mathematics as he had only seen the technique used in astrology/numerology. (30)
10.3 B on F_{3n}: "We could go on forever but we can't know that it's always even" (31)

11.1 J asked DR if they are on the "right track". DR said yes, and asked for a recap.
11.2 They listed 3 conjectures they found:
1. the recursive rule for the sequence
2. B's digit sum conjecture
3. J's ratio conjecture.
11.3 B formulated J's conjecture (35)
11.4 B added 0 to beginning of sequence and gave a deductive justification for its being correct. (35)

12.1 B noticed pattern in sums: 3,3,9,6,6,9. (36)
12.2 J became confused between F_n and F_{4n}. B corrected J's confusion (36-37)
12.3 B began to extend sequence systematically, recognizing need for some organization in their work. J was working on a dividing pattern using digit sums which he never described. (38-40)

12.4 B found a contradiction to his digit sum conjecture; 260497, which was due to an addition error in extending the sequence. (41)
12.5 B rejected his conjecture on the basis of his counter example.
12.6 Both began looking for patterns in differences of the digit sums. They noted they are all multiples of 3. (44)
12.7 B suggested a pattern of alternating divisibility by 3 and 2, in groups of six.

13.1 DR pointed out addition error in sequence. (47)
13.2 B recalculated sequence. J watched.
13.3 B checked next value of F_{4n}: 317811. "It still works" (50)
13.4 DR talked about potential for surprise in the sequence (51)
13.5 B proposed trying next value of F_{4n} to test digit sum conjecture, and did so. (52)

14.1 DR asked about surprise.
14.2 B commented that there is no reason to be surprised at a counter example and expressed doubt of F_{3n} even conjecture. (56)
14.3 J investigated sums of digits for other numbers.

15.1 B suggested that differences between F_{4n}s are divisible by 3
15.2 Checked several cases, but doubted generality.

16.1 DR pointed out that 3, 21, 144, ... are all multiples of 3. B didn't think 144 was. (62)
16.2 DR asked B if he can give a reason for every third number to be even. B suggested it is because the sum of two odd numbers is even. (64)
16.3 B checked sequences to see if pairs of odds continue, and to see if O+O=E rule holds. (66)

<u>Arithmagon</u>

(time of day, tape started at 10:52)

0.1 Discussed previous session (Fibonacci). J commented that it helped him learn that many pattern might exist, rather than just 1 simple answer. (10:53)
0.2 B asked if the ratio F_n:F_{n+1} was ever the same for two values of n, in response to DR's commenting that the ratio was approximately the same and more so as n gets bigger.

1.0 Given sheet (10:56)
1.1 B formulated how puzzle works, while J read entire sheet. (10:57)
1.2 B announced solution, used inference. (10:58)
1.3 B created 3-3-4 triangle, by choosing corners and adding. (10:59)
1.4 B looked for patterns in solved puzzles (11:01)
1.5 B asked whether negative numbers are allowed. (11:03)
(from 1.2-1.5 J is working silently)

2.1 J asked for a protractor, B found a tentative rule. (11:04)

3.1 B returned to examining the original puzzle. J asked him for the numbers. (11:05)

3.2 Looked for patterns (11:06-11:08)
3.3 J wondered if 11, 18, 27 are all multiples of something. B formulated the task as finding a simpler way (better than trial and error) to find the solution. (11:08)

4.1 B returned to his "old theory" (2.1) which worked in two cases. (11:09)
4.2 J reported that the sum of the sides is 56. He divided by 3, because the triangle has three sides, to get 18. (11:10)
4.3 B looked for factors in common.

5.1 J suggested looking at square (they didn't) (11:13)
5.2 Looked for patterns (11:14-11:16)

6.1 B found a possible method: $27 \div 18 \times 11 \approx 16.5 \approx 17$, which is one of the numbers needed. He tried this method with other sides, and it failed, but he reasoned that he only needed one corner to find the others. He tried his method on a 3-6-17 triangle . J worked in silence. (11:16)
6.2 B tried his method on a 41-86-92 triangle (11:19)
6.3 J asked B what he was doing. B described his conjecture that $A \times B \div C = x$, and said he was trying other examples. (11:22)
6.4 B tried a 14-29-56 triangle. (11:22)

7.1 J said that the "minus 1 thing" doesn't work. They had conjectured earlier that one vertex was a side minus 1. B gave a reason "It only worked here [11-18-27] because this is 1". (11:23)
7.2 B: "Maybe this would work with the 56 theory" (11:25)

8.1 B wondered if J tried the square. J said he did but it "didn't go"
8.2 B considered solving squares, but decided to do triangles first. Conjectured that the solution was related to the number of sides.

9.2 B conjectured the pattern of odd-odd-even is important. (11:28)
9.3 B conjectured $(A - C) + (B \div C)$, rounded off, which in 11-18-27 triangle gives him B, 18, again. He recognized that this is not his goal. (11:30-11:31)
9.4 J conjectured $2 + 27$ (in 14-29-56 triangle) gives 29, but realized he was reasoning from corners to sides, and he didn't know corners to start with. (11:32)

10.1 J showed B the 11-17-21-15 square he was working on at some previous time. B worked on it, conjecturing that the solution of the square might help solve the triangle. (11:33)
10.2 J suggested adding all the numbers.
10.3 B commented squares are harder than triangles. Wondered if there is really a link between squares and triangles. Decided there is, based on both occurring in the same situation. (11:34-11:35)
10.4 B returned to triangles, suggesting that the only reason he has for thinking there is an easier way, the need to solve large number triangles, could be eliminated by employing SI units, or scientific notation. (11:37-11:38)

11.1	DR suggested looking at triangles with smaller numbers.	(11:38)
11.2	B worked on 2-3-3 triangle.	(11:39)
11.3	B conjectured, based on 3-2=1, $3 \times 1 = 3$. He generalized and checked with another triangle	(11:40)

Session ended due to time constraint.

First Interview

In this trace each group of episodes is given a heading indicating the general nature of the episodes, or the stage in deductive exploration.

		(Time of day)
1	Stating unknowns and givens:	(1:59-2:05)
1.1	DR pointed to $B+C=18$; B added $A+C=27$	(2:00-2:01)
1.2	DR pointed to working with a particular corner, as it relates to the others.	(2:03-2:05)
2	Building on givens:	(2:05-2:08)
2.1	DR: combined 27+11=38, related to $A+B+A+C$	(2:05-2:07)
2.2	B: made a false start solving from new relation.	(2:07-2:08)
3	Stating more givens:	(2:09-2:12)
3.1	DR pointed to B & C	(2:09-2:10)
3.2	DR wrote known relations	(2:10)
3.3	B asserted that finding one corner A will be enough to solve all.	(2:10)
3.4	J summed 11+17+27+38 and was corrected by B	(2:11-2:12)
4.	Building from givens with a new triangle	(2:13-2:16)
4.1	B stated that knowing the solution interferes with solving in a new way. DR offered 1-4-12 triangle as an alternative puzzle, and established known relations.	(2:12-2:13)
4.2	DR suggested subtracting $(A+B)-(A+C)=3$ and simplified to $B-C=3$. B claimed to follow, J was uncertain.	(2:14)
4.3	B pointed out that now that the sum and difference of B & C were both known, a solution could be found.	(2:16)
5	Review: Digression as DR reviewed simplification of difference for J.	(2:17-2:18)
6	Building from givens continued.	(2:19-2:23)
6.1	B continued: Now that sum and difference of B & C were both known, solution could be found, but he was stumped when no pairs of whole numbers summing to 12 had a difference of 3.	(2:19-2:21)
6.2	DR suggested looking for numbers in between, and B arrived at 7.5 and 4.5	(2:21)
6.3	B verified solution	(2:22-2:23)
7	B solved a new triangle at DR's suggestion.	(2:24-2:30)
7.1	DR offered 3-18-63 triangle for solution. Suggested B "explain" to J	(2:24)
7.2	B consulted previous work, and recreated the derivations.	(2:24-2:26)
7.3	B asked for confirmation that he was proceeding correctly	(2:27)

7.4 B was worried about *C-B*=45, as difference should be less than sum. (2:28)

7.5 DR suggested adding *C-B* and *C+B*. B did so and simplified to *C*=24 (2:28-2:30)

7.6 B verified *C*=24 in puzzle. (2:31)

8 Review (2:31-2:34)

8.1 J asked how B found the other sides once *C* was known. B explained (2:31-2:32)

8.2 J suggested replacing *A+B* with a single variable. (2:33)

8.3 B recapitulated the derivation. (2:33-2:34)

9 Generalization (2:35-2:37)

9.1 B asked if the process will always simplify to 2C (2:35)

9.2 DR pushed him to try to explain why it would be. (2:35)

9.3 B explained his goal: to produce a simple equation. (2:36)

9.4 DR asked where the 2 came from. (3:36)

9.5 B recapitulated the simplification, and still wondered if the 2C is general. (2:36-2:37)

10 Formulating (2:38-2:39)

10.1 B wrote $\frac{(A+C)-(A+B)-(B+C)}{2}$ (2:38)

10.2 B explained that this formula simplifies to *C*. (2:38)

10.3 B used formula with numbers. (2:39)

11 Checking formula (2:40-2:41)

11.1 J suggested solving 11-18-27 triangle with formula (2:40)

11.2 B did so. (2:40-2:41)

12 Testing B's strength of conviction (2:41-2:48)

12.1 B stated that formula works for the second triangle, but that says nothing about general case. (2:41)

12.2 DR pointed to the canceling in B's derivation of the formula and asked if it indicated anything about when the formula would work. B said "I couldn't say". (2:42-2:43)

12.3 B commented that the canceling eliminated the negatives; the "angles" are positive. (2:43)

12.4 DR asked if substituting -21 for *B* would effects anything. B asserted that the formula would still work with numbers.

12.5 DR asked if there are circumstances in which the formula wouldn't work.

12.6 B digressed into a comparison with the area formula for the triangle. (2:46)

12.7 B commented: he couldn't see why it wouldn't work for all triangles, but he might be wrong. (2:47)

12.8 J commented that they don't need to figure out the other sides once they've figured out *C* with the formula. (2:47)

12.9 DR asserted that he can see no reason the formula would not work in general. B concluded that they had found it out. (2:47)

13 Review of derivation

13.1	J went through derivation trying to understand where the 48 came from.	(2:49)
13.2	B explained the origin of the 48 to J.	(2:49)
14	Formulating	(2:52-2:54)
14.1	J replaced B's formula with $\frac{E\text{-}D+F}{2}$	(2:52)
14.2	B asked J if he understood where the 2 came from.	(2:52)
14.4	B claimed his notation shows why the formula works, while J's is tidier, easier to learn.	(2:53)
14.5	DR asked if this was related to their school experience. B commented he liked to know why.	(2:54)
15	Videotape follow-up	(2:55-2:58)
15.1	DR showed video tape of 13-F; 16.2-16.3	(2:55-2:57)
15.2	DR asked why two odds make an even. B gave examples of *n+n=even* and asserted all numbers would end in digits which determine parity.	
15.3	J commented that figuring out why is hard, so it's easier "just to believe"	(2:58)
16	Starting from a proof	(2:59-3:00)
16.1	D asked why there's always two odds in a row in F_n. B questioned *odd+even*; claimed it wouldn't always give an odd, then checked a single case and claimed it would.	
17	Proof analysis, Lemma	(3:00-3:04)
17.1	DR asked why *odd+even=odd*? No reply.	(3:00)
17.2	DR asked what "even" means. B replied "divide by 2"	(3:00)
17.3	B related *odd+even* to *positive times negative*.	(3:01-3:02)
17.4	J made a comment on evenness.	(3:03)
17.5	DR gave a weird explanation about *odd+even*.	(3:04)
18	Return to an example	(3:04-3:06)
18.1	DR listed the Fibonacci numbers and labeled each *O* or *E* as appropriate.	(3:04)
18.2	B asserted that F_{3n} is even and F_{4n} is odd.	(3:04)
18.3	B commented that the pattern is "working out so far ... if I'm correct"	(3:06)
19	Testing conviction	(3:06-3:08)
19.1	DR asked if the *OOE* pattern would continue forever.	(3:06)
19.2	B said "Yes", and explained that the sequence started out that way and the pattern repeats, but qualified with "I think"	(3:06-3:07)
19.3	DR asked how the pattern could change. B said that from what they had seen that would be inconceivable.	(3:08)
19.4	DR asserted that the pattern did continue.	
20	Reflections on school math.	
20.1	B & J asked what DR could say about them	(3:09)
20.2	DR commented they didn't work well together, and asked if they had done group work in class.	(3:10)
20.3	B commented that math is an individual activity.	(3:10)

Second Interview

		(time of day)
1.	Conjecture	(12:59)
1.1	Given sheet for n^3-n	(12:59)
1.2	B conjectured that they are all factors of 6.	(12:59)
2.	Exploring factors	(1:00?-1:05)
2.1	DR gave choice: to verify, explain or explore.	(1:00?)
2.2	B and J chose "why"	(1:02)
2.3	B observed the values of n^3-n are multiples of 3 as well.	(1:02)
2.4	DR asked if all multiples of 6 are even.	(1:03?)
2.5	J gave explanation: 6 is even and adding evens makes evens. B extended this with the example 6×4=6+6+6+6.	(1:03?)
2.6	DR asked if all multiples of 6 are multiples of 3.	(1:04)
2.7	J replied that 3 goes into 6 so 3 goes into "these". B observed that 3 can be a factor of both even and odd numbers.	(1:04-1:05)
3	Failed formalization	
3.1	DR asked what they knew about n^3-n.	(1:06)
3.2	B responded that it was a general expression for expressions like 3^3-3	(1:06)
3.3	DR asked if they knew about factoring. They said maybe; probably not.	(1:06-1:07)
4.	Exploring n^3-n	(1:07-1:11)
4.1	DR asked if 5^3-5 is a multiple of 5	(1:07)
4.2	B responded "yes" and noted that 60 (4^3-4) is a multiple of 4.	(1:08)
4.3	J suggested finding 6^3-6. DR said it is 210.	(1:08)
4.4	B observed that 210 is a multiple of 3, and asked DR what he was trying to get them to see.	(1:08-1:11)
5.	Exploring factors of 120	(1:11-1:15)
5.1	DR asked what goes into 120	(1:11)
5.2	B and J listed all the factors and thought in silence.	(1:12-1:14)
5.3	DR observed 120=5×24, and asked if there is anything special about the factors of 24	(1:14)
5.4	B listed the factors. DR asked if 5 was among them, and B replied "No".	(1:15?)
6	Exploring factors of other numbers	(1:16-1:20)
6.1	DR suggested looking at the factors of other numbers. B and J wrote out factors.	(1:16-1:17)
6.2	B suggested a false pattern.	(1:18)
6.3	DR pointed out the sequence 123, 1234, 12345, 123456	(1:18)
6.4	B predicted, 1234567, tried it and rejected the conjecture.	(1:19-1:20)
7.	Exploring groups of factors	(1:20-1:23)
7.1	DR asked what the factors of 210 are.	(1:20)
7.2	B wrote them out.	(1:21)

7.3	DR pointed out the groups 123 and 567.	(1:22)
7.4	B Saw no pattern, and suggested investigating differences.	(1:22-1:23)

8.	Formalizing with a generic example	(1:24-1:34?)
8.1	DR factors 6 out of 6^3-6, and then redistributes 6^2-1	(1:24-1:28)
8.2	DR asked if breakdown of 6^3-6 into (6-1)(6)(6+1) would work for 5	(1:28)
8.3	B said "yes" and checked.	(1:28-1:29)
8.4	DR asked if it would work in general. B said yes	(1:30)
8.5	DR attempted to explain that n^3-n is always a multiple of 6, based on the factoring.	(1:30-1:32)
8.6	B wondered if all the work was really needed. He pointed out that it only gave them a few factors, which DR claimed was enough to show the conjecture.	(1:32-1:33)

9	Explaining O+O=E	(1:35-1:38?)
9.1	DR asked why adding rules for odd and evens work.	(1:35)
9.2	B gave his analogy with integer multiplication, including implicit use of the parity adding rules to justify the integer multiplication rules.	(1:35-1:36)
9.3	DR rejected his argument because it was based on the notation which is of recent introduction.	(1:36)
9.4	DR asked J for his ideas. He had nothing to add.	(1:38?)

10	Exploring evenness	(1:39)
10.1	DR asked what they knew about even numbers.	(1:39)
10.2	B said they were multiples of 2, except for 0, and they can be divided by 2.	(1:39)
10.3	J argued that E+E=E because apple+apple=apple. DR gave O+O, orange+orange, as a counter example.	(1:39)

11	Explaining E+E formally	(1:40-1:43)
11.1	DR went through argument that E+E=E based on denoting E as "2×something" and redistributing.	(1:40-1:42)
11.2	J asked for clarification: if you multiply 2 by an odd you get an even.	(1:43)

12	Explaining E+O formally	(1:44-1:48?)
12.1	DR asked B to do E+O	(1:44)
12.2	B wondered if the factor of 2 for an even would be 1 or 3 for an odd.	(1:44)
12.3	DR asked what they knew about odds. J offered that they had two factors, and then realized he was thinking of primes and composites.	(1:44)
12.4	DR went through argument with generic example of 2+3=5	(1:45)
12.5	DR generalized to 2n+2m+1	(1:45)

13	Explaining formally O+O	(1:48?-1:51)
13.1	DR asked B to do O+O	(1:48?)
13.2	B began with 2(3)+2(3)	(1:48?)
13.3	DR prompted "odds are...?" B Replied "1 more than even"	(1:49)
13.4	DR went through argument	(1:51)

14	Explaining graphically E+E	(1:53-1:54?)
14.1	DR drew arrays of pairs of dots, and asked whether the total number was even or odd.	(1:53)
14.2	J explained that there are 2 dots in every group, so "it'll always be even"	(1:53)
14.3	DR formalized the number of pairs to be n and m and asked how many dots in each column ($2n$: J) and how many in the total ($2n+2m$: B)	(1:53-1:54?)
15	Explaining graphically E+O	(1:55-1:57)
15.1	DR drew arrays for E+O	(1:55)
15.2	J said the total would be odd. B began assigning symbols to the number of dots.	(1:55)
15.3	B worked through the formal argument for odd	(1:56)
15.4	DR asked for clarification of why the answer was odd, and B gave it, based on the total being one more than a multiple of 2.	(1:57)
16	Explaining graphically and formally O+O=E	(1:58-1:59)
16.1	DR asked what the pictures for O+O would look like.	(1:58)
16.2	B drew picture, adding formal labels as he went, and argued formally that the sum was even.	(1:58)
16.3	DR asked why 2 more than an even number would be even.	(1:59)
16.4	J explained it based on the alternation of evens and odds.	(1:59)
16.5	B explained it based on it being the sum of two even numbers.	(1:59)
17	Debriefing	(2:00-2:01)
17.1	DR explained that his plan had been to take them from unformulated proving to formulated proving.	(2:00)
17.2	DR commented on the value of pictures to make things less formal.	(2:01)
18	Does proof explain?	(2:02)
18.1	DR asked if proof explained that O+O=E B said yes, with a good summary of argument.	(2:02)
19	Testing conviction	(2:03)
19.1	DR proposed that for very large odds their sums are odd.	(2:03)
19.2	B accepted this idea.	(2:03)
19.3	DR said it's not actually true.	(2:03)
20	Winding down	(2:04-2:16)
20.1	DR discussed transfinite numbers	(2:04)
20.2	B wondered why he and J were picked. DR explained.	(2:05-2:07)
20.3	B said he didn't like proofs, he doesn't care why.	(2:13)
20.4	B wondered if new math is always being discovered.	(2:16)

Mathematical Activity Traces for the Math 30 Pair, Colin & Anton

Fibonacci

		(Time of Day)
1	Finding Answers	(10:57-11:01)
1.1	Given problem sheet	(10:57)
1.2	Found recursive rule for sequence and formalized it.	(10:58)

1.3 Claimed F_{3n} is always even. (10:59)
1.4 Claimed F_{4n} is always a multiple of 3. (11:00-11:01)
1.5 C wrote out "answers" neatly on problem sheet. A asked if there was more to it. DR said yes. (11:02)

2 Data gathering. (11:02-11:06)

2.1 They extended the sequence to 24 terms, making an error at F_{10}(11:02-11:06)

3 Finding a rule for F_{3n} (11:07-11:12)

3.1 C noticed 8x4+2=32. Conjectured $F_{3n} = 4F_{3n-3} + F_{3n-6}$. (11:07)
3.2 Encountered counter examples due to error of F_{10}=600. (11:08-11:12)
3.3 Found error, formalized rule for F_{3n} (11:12)

4 Finding a rule for F_{4n} (11:13-11:20)

4.1 Conjectured $F_{4n} = 7F_{4n-4} - F_{4n-8}$. (11:14)
4.2 Encountered problems verifying due to errors in sequence. Re-calculated the sequence correctly. (11:15-11:17)
4.3 Encountered counter example due to miscalculation. (11:18)
4.4 Tried more cases. (11:19)
4.5 C formalized rule for F_{4n} (11:20)

5 Looking for another F_{3n} rule (11:21-11:24)

5.1 A considered looking at F_{5n} but didn't as it was not on the sheet. (11:21)
5.2 They both looked for other patterns in F_{3n}, without finding any.(11:21-11:24)

6 Finding a rule for F_{5n} (11:25-11:31)

6.1 C made a variety of calculations attempting to find a new rule for F_{5n}. (11:25-11:27)
6.2 A joined him in considering F_{5n} (11:28-11:31)
6.3 C described F_{5n} rule. A commented it was the same rule as they had found for F_{3n} and F_{4n}. (11:31)

7 Finding a rule for F_{7n} (11:32-11:33)

7.1 C made a variety of calculations attempting to find a rule for F_{7n}. (11:32-11:33)

8 Formalizing (11:33-11:35)

8.1 C wrote out the rules for F_{3n}, F_{4n} and F_{5n} (11:33-11:34)
8.2 C saw a pattern in 4, 7, 11 as 4+3=7, 7+4=11, 11+5=16. A saw the alternating addition and subtraction in the rules. (11:35)

9 Predicting from a formalism. (11:36-11:38)

9.1 C predicted $F_{6n} = 16F_{6n-6} - F_{6n-12}$. (11:36)
9.2 When they're calculations failed they rechecked them several times for errors. (11:36-11:38)

10 Summary (11:39-11:41)

10.1 DR asked them to summarize their results and how they had arrived at them (11:39)
10.2 C Attempted to locate the formalisms he had written down, and read them. A indicated that they had tried their conjectures out and they had worked. (11:40-11:41)

Arithmagon

		(Time of day)
1	Solving by mechanical deduction	(12:50-12:53)
1.1	Given sheet	(12:50)
1.2	A labeled triangle and set up system of equations	(12:51)
1.3	A found an incorrect solution.	(12:52)
1.4	A asked DR of their solution was correct. DR asked them to check it, and they did.	(12:53)
2	A second approach	(12:54-12:55)
2.1	C attempted to solve by isolating x and equating the expressions 18-y and 27-z.	(12:54-12:55)
3	Redoing the system of equations	(12:55-12:57)
3.1	C asked A to explain how he had obtained x-z=7 in his derivations.	(12:55)
3.2	A solved the system of equations again, obtaining the correct solution and checked it.	(12:56-12:57)
4	Considering the general case	(12:57-12:58)
4.1	C set up a general triangle with equations	(12:57)
4.2	A suggested solving C's equations, but decided they couldn't.	(12:58)
5	Searching for patterns	(12:59-1:04)
5.1	C set up a 23-30-33 triangle, by beginning with known corners and adding to find the sides. Both looked for patterns in this triangle.	(12:59-1:00)
5.2	A noticed that the sum of the sides is equal to twice the sum of the corners. C wrote out this relation formally.	(1:00-1:01)
5.3	They attempted to use the sum relation to solve a 23-60-100 triangle. They determined the sum of the corners, but concluded that the triangle had no solution.	(1:01-1:03)
5.4	C solved a 4-4-4 triangle, then proposed a 4-4-6 triangle. A systematically checked all positive pairs of numbers adding to 6, and then rejected the triangle as unsolvable.	(1:03-1:04)
6	Searching for a pattern in the original triangle	(1:05-1:09)
6.1	C Suggested that solvable triangles might require the 'prime plus two multiples of three' pattern. He attempted to solve a 13-9-12 triangle. C found solution by guessing, but A wanted a formula.	(1:05)
6.2	They re-examined the solution to the original triangle, and observed that 17=18-11+10. They conjectured a general formula of x=a-b+10.	(1:06-1:08)
6.3	A suggested trying another triangle. C drew 3-15-24 triangle and they tried it with their formula.	(1:08-1:09)
7	Searching for more patterns	(1:09-1:16)
7.1	C suggested investigating 13-16-17 triangle, which he had set up from known corners.	(1:09)
7.2	They calculated the sum of the sides, and A divided the triangle into sum triangles and calculated sums of corners for them.	(1:10-1:11)
7.3	Solved 10-16-26 triangle	(1:13-1:14)
7.4	A solved 20-30-22 triangle using a system of equations.	(1:14-1:15)

7.5	C noticed relationship between differences of sides and differences of corners. He related this to the sharing of corners.	(1:15-1:16)
8	**Investigating squares**	**(1:17-1:18)**
8.1	C assumed opposite sides must be equal.	(1:17)
8.2	A divided a 3-4-3-4 square to make a triangle, and found a solution.	(1:18)
9	**Discovering a method**	**(1:19-1:22)**
9.1	Found solution to 5-12-13 triangle based on differences.	(1:19)
9.2	Found solutions to 20-30-36 triangle, and 24-14-16 triangle.	(1:20-1:22)
10	**Formalizing**	**(1:23-1:27)**
10.1	C labeled 11-18-27 triangle.	(1:23)
10.2	A suggested using x, X to represent opposite sides and corners.	(1:24)
10.3	C wrote out $Z-X = x-z$	(1:25)
10.4	C commented that their method was not a general solution, and began deriving formally.	(1:26)
10.5	C derived $Z-X+Y=2x$, but thought that they had already found this.	(1:27)
11	**Back to squares**	**(1:28-1:31)**
11.1	Investigated squares with all sides equal. Found that there were many solutions. Formalized this conclusion.	(1:28-1:29)
11.2	Introduced diagonals. They found a conflict in a 3-3-3-3 square with 1-2-1-2 as the solution. C concluded that squares could be solved in various ways.	(1:30-1:31)
12	**Formalizing their method for triangles**	**(1:32-1:38)**
12.1	C drew a general triangle	(1:32-1:33)
12.2	A attempted to investigate triangles by adding sides to make a square.	(1:34)
12.3	C wrote difference relationship as a ratio relationship, and began trying to solve 11-18-27	(1:35-1:36)
12.4	A asked if they were close to a formula, and then set up another general triangle.	(1:37)
12.5	C concluded that his ratios wouldn't work.	(1:38)
13	**Refining the method**	**(1:39-1:41)**
13.1	A declared that they were "stuck". They looked for more patterns.	(1:39)
13.2	They solved 10-16-44 triangle. C solved by taking half the difference ($\frac{16-10}{2}=3$) and adding or subtracting it from half the other side (22), to find the two adjacent corners (he found 20 and 24, by mistake, and did not check).	(1:40)
13.3	C concluded some triangles were impossible because the differences would have to be divisible by 2.	(1:41)
14	**Formalizing the method**	**(1:42-1:43)**
14.1	A wrote out difference relation as $B-C = b<c$	(1:42)
14.2	C changed it to $\lvert B-C \rvert = \lvert b-c \rvert$.	(1:43)

15	Refining the method again	(1:43-1:46)
15.1	C gave A a 21-24-33 triangle to solve. A did a systematic search based on 33-24=9 and found solution.	(1:43-1:45)
15.2	A described the method as taking the difference of the largest side and another side, and finding two numbers with that difference, whose sum is the third side.	

Session ended due to time constraint.

First Interview

		(Time of day)
1	Tape Viewing	(12:04-12:16)
1.1	Watched tape of Arithmagon session from 1:20-1:22.	(12:04-12:06)
1.2	C described how they solved triangles using their method.	(12:06)
1.3	Watched tape of Arithmagon session from 1:22-1:24.	(12:07-12:09)
1.4	Discussed which triangle they had been solving on tape. It was a 11-18-27 triangle.	(12:09)
1.5	DR asked how their differences method worked. C explained.	(12:10)
1.6	DR asked how they knew the differences were equal. A replied that they had tested many cases.	(12:10)
1.7	DR asked how knowing the difference helped in solving the triangles. C replied that they plugged in numbers with that difference until a pair worked.	(12:11)
1.8	DR asked what they would do if the difference were a million. A replied that they would know where to start checking based on the sum they were trying to get.	(12:12)
1.9	Watched tape of Arithmagon session from 1:24-1:28.	(12:12-12:16)
1.10	A commented "We were stuck so we were just trying to do anything", referring to their derivation of Z-X+Y=2x.	(12:15)
1.11	DR asked for an interpretation of the derivation. C attempted to explain it, but discovered an error.	(12:16)
2	Mechanical deduction of the formula	(12:17-12:22)
2.1	C corrected second line to $y = X+Z$.	(12:17)
2.2	C derived new formula. DR asked what it meant. C noted that the formula gave one of the corners.	(12:18)
2.3	A suggested trying the formula on an example to see if it worked. C tried it with the original 11-18-27 triangle and concluded the formula worked.	(12:19-12:20)
2.4	DR asked which side would be subtracted in the formula. C answered that the side opposite the corner the formula gave would be subtracted.	(12:20)
2.5	A suggested using the formula to find a different corner. After they had done so DR asked if they had expected the formula to fail. C claimed that the formula should work in the same way for all the corners, and wrote a verbal formulation of it in terms of adjacent and opposite sides.	(12:21)
2.6	A asked if the formula was the correct one. DR said it was one correct formula.	(12:22)
3	"Proving" the formula	(12:23-12:26)
3.1	DR asked why the formula worked. A said they had tested it with examples. C suggested "proving" it.	(12:23)

3.2 C identified the difference relation as important to the formula, and asked why it worked. He then made the connection with the common corner, which requires that any difference in the sides be due to the difference between the other two corners. (12:24-12:25)

3.3 DR commented on the informality and meaningfulness of C's argument, compared with the mechanics of the algebraic derivation. A connected this with the contrast between explaining and finding a formula. (12:26)

<u>4 Extending new ideas to squares (12:27-12:29)</u>

4.1 C gave an example of a square with negative corners to show negative numbers could occur. (12:27)

4.2 DR suggested they drop their requirement that all sides of a square should be equal. (12:28)

4.3 A asserted that they could draw in a diagonal and use their triangle formula to solve squares. He tried to do this with a 3-4-3-4 square, arriving at a false solution. C concluded that the formula didn't apply to squares. (12:28-12:29)

<u>5 Searching for a pattern for squares. (12:30-12:33)</u>

5.1 They tried to solve a 3-4-5-6 square by trial and error. (12:30-12:31)

5.2 C set up a 7-13-16-10 square by starting with known corners. They looked for patterns in this square, and labeled the corners and sides. (12:31-12:33)

5.3 DR asked what relations they had found, and asked them to write equations for them. (12:33-12:34)

<u>6 Mechanical deductions on squares (12:34-12:40)</u>

6.1 A derived expressions from the relations he wrote. (12:34-12:36)

6.2 DR pointed out that they also knew that corners added up to sides, and A added equations for these relations to his. (12:36)

6.3 A produced the equation $27 = 2C-2A+w+z-y$. DR asked if that equation meant anything. A continued with his derivations. (12:36-12:37)

6.4 DR asked how A had arrived at 27, and pointed out that it was based on the sides of the particular square they were investigating. A replaced the numerical values of the sides with the variables which stood for them. (12:37-12:38)

6.5 A continued his derivations, mentioning that he was trying to isolate the variables for the corners. (12:39-12:40)

<u>7 Formalizing the proof for the triangle formula. (12:40-12:45)</u>

7.1 DR asked them to return to the derivation of the triangle formula, and to try to formalized the informal explanation of the difference relation, or to try to explain informally the formal derivation of the formula. (12:40)

7.2 A commented that some things were more easily done one way than the other. (12:41)

7.3 C explained the difference relation informally. (12:42)

7.4 A set up equations for the relations along the sides in question. (12:43)

7.5 C commented that one could see from the equations that the differences must be the same. (12:44)

7.6 DR asked him to show it algebraically, and C subtracted the two equations to arrive at the difference relation. (12:45)

8 Giving meaning to the algebra (12:46-12:49)

8.1 DR asked them to say what the algebra of their formula derivation meant. (12:46)

8.2 C attempted to do so, but ended up reciting the algebra, rather than giving meaning to it. (12:47)

8.3 C focused on the meaning of the subtraction of the two equations. A commented that they had learned to isolate variables. C pointed out that they still didn't know what it meant. They continued to consider it. (12:48-12:49)

Session ended due to time constraint.

Second Interview

(Time of day)

1 Gathering data (11:03-11:07)

1.1 A and C both thought they had seen something similar before. Tried to remember. (11:03-11:04)

1.2 A conjectured they are all even, and calculated the next term to confirm. (11:04)

1.3 C calculated several terms and differences between them. (11:05)

1.4 A calculated all the terms up to n=10. (11:05-11:06)

1.5 DR asked why the needed more terms. C answered that it kept them from identifying "fluke patterns". (11:06)

1.6 A calculated terms for $-4 < n < 0$ (11:07)

2 Noticing patterns (11:08-11:10)

2.1 C conjectured they are all even. (11:08)

2.2 A noticed 0 0 0 6 4 pattern in final digits and pointed it out to C. (11:08-11:09)

2.3 A calculated terms for $10 < n < 15$ (11:10)

2.4 C conjectured that they are all multiples of 6 and DR said that was the pattern he was thinking of. (11:10)

3 What do we know about n^3-n? (11:11-11:13)

3.1 DR asked if they would like to explain or verify that n^3-n is always a multiple of 6, or explore for more. They chose to explain. (11:11)

3.2 DR asked what they knew about expressions like n^3-n. They replied they could graph it or factor it. (11:12)

3.3 C factored n^3-n. (11:13)

4 What do we know about those three numbers? (11:13-11:15)

4.1 DR asked what they knew about $n(n-1)(n+1)$. C replied that they were three consecutive numbers. (11:13)

4.2 DR asked why the product of three consecutive numbers would be a multiple of 6. (11:13)

4.3 A observed that either they would have two odd numbers or two even numbers in the three. (11:13)

4.4 C pointed out that they needed to show that the product was divisible by both 3 and 2. (11:14)

4.5 DR asked if they knew one of the numbers is even. C answered that the numbers were either even-odd-even, or odd-even-odd. A concluded this meant the product must be even. (11:14)

4.6 C commented that the middle number of the three is the original n from n^3-n. (11:15)

5 Is there a factor of 3? (11:15-11:17)

5.1 DR asked if they could find a factor of 3 in the three numbers. C examined several examples. He then claimed that they would always have a number divisible by 3 because every third number is divisible by 3. (11:15-11:16)

5.2 DR asked if that explained why n^3-n is always a multiple of 6. A said yes, and C explained that n^3-n is always the product of three consecutive numbers, at least one of which is even and one of which is a multiple of 3. (11:17)

6 Formalizing (11:17-11:22)

6.1 DR asked if they could write out the argument. (11:17)

6.2 C wrote out the argument. (11:17-11:21)

6.3 C gave examples of the two cases of n not a multiple of 3. (11:22)

7 Testing confidence (11:23)

7.1 DR asked them if they now knew that 417^3-417 is a multiple of 6. (11:23)

7.2 A checked on a calculator. (11:23)

7.3 C argued that it would be 416×417×418, which includes an even number and a multiple of 3. (11:23)

8 Did we explain/explore? (11:24-11:25)

8.1 DR asked if they would use their argument to explain why n^3-n is always a multiple of 6. C said yes. (11:24)

8.2 DR asked if they had discovered anything new about n^3-n by working out the argument. They didn't think so. DR commented on the discovery that n^3-n is always the product of three consecutive numbers. (11:25)

9 What about the converse? (11:25-11:16)

9.1 DR asked if the product of three consecutive numbers would always be n^3-n for some n. C answered yes, and that n would be the middle number of the three. He also worked an example. (11:25-11:26)

10 Another statement to explain (11:27-11:30)

10.1 DR asked why the sum of two odd numbers is even. (11:27)

10.2 C and A independently determined the sum of ($2n$-1) and ($2n$-1), concluding that $4n$-2 must be even. (11:27-11:28)

10.3 DR pointed out that they had only shown that the sum of two identical odd numbers is even. C calculated ($2n$-1)+($2n$-7). A calculated ($2n$-1)+($2n$+1). (11:28)

10.4 DR pointed out that these were still special cases, and suggested using a different variable for the second number. C and A independently added ($2n$-1)+($2x$-1) arriving at 2(n+x-1). (11:29-11:30)

11 Winding down (11:30+)

Various discussions occurred after the second interview. Only key episodes are listed here.

11.1 DR showed how Arithmagon squares work. (11:34)
11.2 A commented on doing math in school. "I can't remember a formula unless I understand it." (11:37)
11.3 A commented that Mr. B was happy to explain anything they asked about, at great length, and they were disappointed when they found an elaborate explanation would not be on the examination. A described this as a "waste of time." (11:38)

2. MATs from the first clinical study

Group I: Ben and Wayne

Arithmagon

Ben

(time elapsed from start of tape)

0. Given Problem Sheet (4:08)

1. Solved Problem "intuitively" (4:38)

2.1 Explored relations between the numbers. (~6:00)
2.2 Found pattern in differences: A-B=a-b, etc. (8:11)

3 W: "Can you use negative numbers?" (9:40)
B: "Sure."
W: "You can't have negative length of a side though."
B: The triangle is irrelevant.

4 Compared Solutions with R & E

5.1 B asked E & R how they used algebra to solve problem;
5.2 B Explained his constraints method to E & W

6 W said he was "playing" with properties of triangles.
E & K told W the triangle was irrelevant.
E: "I guess you could [treat sides as lengths]"
B: "I don't think you could."
B rejected taking triangle as importani as angles didn't work.

7 Discussion of B's method. B reconstructed his thinking and solved another triangle.

8.1 DR gave 1-4-12 triangle. (~22:30)
8.2 B declared it impossible. He explained that only 0+1 gives 1, and neither order works. E & W suggested negative numbers (~23:00)

9 W: "Do the three numbers represent angles or something?"
E, B, & DR: "No." (~23:30)

10.1 B Decided 1-4-12 triangle could be solved with negative numbers. (~24:30)

10.2 B Proposed that E or R solve 1-4-12 algebraically, E started

10.3 B: *"I've determined that it is impossible."* (25:30)
E: "You think it's impossible?"
B: (to DR) Is it impossible?

10.4 B Asked for E's algebraic solution. (25:40)
E got it wrong.

10.5 B Suggested fractions involved. (26:11)

10.6 All worked on 1-4-12 independently ~ (26:30-29:00)

10.7 E gave B solution to 1-4-12 triangle. (~29:00)

10.8 B checked her solution (~30:00)

11 Everyone listed to R describe progress.

12.1 W described what he was doing to E and B. (~32:00)

12.2 E noticed 6-6-6 [A+a=B+b=C+c] in W's work. (~32:30)

12.3 W checked [A+a=B+b=C+c] on other triangles. They continued to explore. (~33:00)

12.4 E noticed that a+b+c=12=A+a in a particular triangle (Announced that a+b+c=12) (37:14)

12.5 W enunciated rule: a+b+c=2(A+B+C) (~37:30)

12.6 Several examples were checked. (38:50)

13 K & W worked on solving 11-8-15 using new found relations. Found that sum/2 =12, but then became stuck. (~40:00)

14.1 E explained her method. W interrupted with a new problem.

14.2 E gave solution. W: "No."

15 B, W & E worked on relation of division by 2 to area formula for a triangle.

16 R announced her formula $\frac{a+b+c}{2}$ to the group.

17 Tried with W to Confirm R's formula for 0-1-2 triangle

18 Everyone discusses connection of R's formula with cosine law

19.1 Discussed R's formula. W: "I understand everything except why you divide by 2"

19.2 W repeated operational version of R's formula.

19.3 Exchange of explanations for division by 2. B's link to a+b+c=2(A+B+C) accepted as explanation.

20. Watched W work examples

21 All explained E's method to TK

22 Discussed relation with angles, with E & W

23 Worked out 3-7-9 triangle by E's method.

Wayne

		(time elapsed from start of tape)
0	Given Problem Sheet	(4:08)
1.1	Explored properties of triangles.	(~5:30-9:30)
1.2	*W: "Can you use negative numbers?"*	(9:40)
	B: "Sure."	
	W: "You can't have negative length of a side though."	
	B: The triangle is irrelevant.	
2	Compared Solutions with R & E	(~10:30)
3.1	Explored properties of triangles.	(~11:00-1200)
3.2	W said he was "playing" with properties of triangles.	(~13:00)
	E & K told W the triangle was irrelevant.	
	E: "I guess you could [treat sides as lengths]"	
	B: "I don't think you could."	
	B rejected taking triangle as important as angles didn't work.	
3.3	Working on diagrams of triangles.	(14:30)
3.4	Announced that he was: "*Frustrated...no idea what to do.*"	(~15:00)
4.1	DR gave 1-4-12 triangle.	(~22:30)
4.2	B declared it impossible. He explained that only 0+1 gives 1, and neither order works. E & W suggested negative numbers	(~23:00)
5	*W: "Do the three numbers represent angles or something?"*	
	E, B, & DR: "No."	(~23:30)
6.	Explained his "page 1" to DR. Began with 3-4-5 because it is a right triangle. Expanded out. Shrunk in. Proposed 0-1-2 triangle to B.	(~26:30-29:00)
7	Everyone listed to R describe progress.	
8.1	W described what he was doing to E and B.	(~32:00)
8.2	E noticed 6-6-6 [A+a=B+b=C+c] in W's work.	(~32:30)
8.3	W checked [A+a=B+b=C+c] on other triangles. They continued to explore.	(~33:00)
8.4	E noticed that a+b+c=12=A+a in a particular triangle (Announced that a+b+c=12)	(37:14)
8.5	W enunciated rule: a+b+c=2(A+B+C)	(~37:30)
8.6	Several examples were checked.	(38:50)
9	K & W worked on solving 11-8-15 using new found relations. Found that sum/2 =12, but then became stuck.	(~40:00)
10.1	E explained her method. W interrupted with a new problem.	(~40:30)
10.2	E gave solution. W: "No."	
11	Confirmed that A+a rule holds for original Arithmagon. Recorded rule.	
12	B, W & E worked on relation of division by 2 to area formula for a triangle.	

13	Suggested link to Golden Ratio. DR discouraged this idea.	
14	R announced her formula $\frac{a+b+c}{2}$ to the group.	
15	Tried with B to confirm R's formula for 0-1-2 triangle	(49:30)
16	Everyone discussed connection of R's formula with cosine law	(~50:00-51:00)
17.1	Discussed R's formula. W: "I *understand everything except why you divide by 2"*	(~52:30)
17.2	W repeated operational version of R's formula.	(~53:30)
17.3	Exchange of explanations for division by 2. B's link to a+b+c=2(A+B+C) accepted as explanation.	(~54:00)
18	Worked examples, others watched.	(~55:00)
19	All explained E's method to TK	(~57:00)
20	Discussed relation with angles, with E & B	(~58:30-59:30)
21	Wrote out rule. Gave verbal version of rule to DR.	(~61:00)

<u>Group II: Stacey and Kerry</u>

<u>Arithmagon</u>

(time elapsed from start of tape)

0.	Given problem sheet.	(2:10)
1.1.	Kerry chose to "deduce"; to use algebra rather than trial and error.	(2:45)
1.2	Solved by using simultaneous equations.	(3:15-4:30)
1.3	Checked their solution: Stacey—"Is that right?"	(4:30)
1.4	Kerry observed that if method gives solution, it must work in original puzzle.	
2.1	Stacey—"What happens if you add the middle numbers together?"	(6:50)
2.2	She added up the "middle" numbers (those on the sides), and then considered how that is related to the secret numbers on the corners.	(7:10)
2.3	As each secret number is added into two of the numbers on the sides ("So you add each of those twice") she deduced that the sum of the numbers on the sides is twice the sum of the numbers on the corners.	(7:30)
2.4	Kerry checked her assertion, but did not see her argument.	(8:10)
2.5	For Stacey the relationship must hold "for all of them" because she has deduced it. For Kerry it is only the unlikeliness of such a relationship occurring by chance that convinces him that the relationship is a general one.	(8:20)

3.1 Kerry solved again, by using matrices. (8:30-15:00)
3.2 Stacey observed that 0 0 1 12 is wrong, as she attached the meaning C=12 to it. Kerry was proceeding formally. (c. 11:00)
3.3 Due to an arithmetic error, they obtained a different "solution". After checking over their work, they briefly considered the possibility of two solutions, but rejected the idea when they checked their answer in the original problem. (14:15)
3.4 Kerry claimed matrix should give solution as the number of variables equals the number of equations. (16:15)

4 "Generalized" their solution by describing their actions in general terms. (18:00-21:00)

5.1 Stacey extended sides: "just trying something". (21:45)
Around the original triangle she drew a sequence of triangles, using the corner numbers from each triangle as the side numbers for the next larger triangle.
5.2 Explored relationships between nested triangles. (c. 23:00)

6.1 Stacey observed that the differences 27-13, 17-3, and 11-(-3), are all 14. She predicted that the numbers in the next triangle would also be 14 less than those in the 1-10-17 triangle. For example, she predicted that the number at the lower right hand corner of the largest triangle would be 3. (25:15)
6.2 When this prediction was disproved (26:40)
both Stacey and Kerry advanced new predictions. Kerry predicted that the next difference would be 3.5, an induction based on 7 being half of 14. Stacey suggested that the differences might alternate: 14,7,14,7,... (30:00)
6.3 Tested Stacey's prediction (33:25)
6.4 Kerry suggested trial and error to determine next triangle's solution (33:40)
6.5 Stacey observed that her prediction was only based on one trial, so it was not surprising it failed. (33:50)
She suggested they work out the next triangle's solution to give them another trial to base predictions on.
6.6 Kerry's prediction of 3.5 was confirmed. (36:00)
6.7 Kerry predicted 1.75, tested and confirmed his prediction. (36:30-37:45)

7.1 They extended halving principle to a doubling principle (going in). (41:00)
7.2 They discussed the limit of the values for the triangles. (43:00)
7.3 Stacey suggested deriving a formula. This idea was rejected due to the large number of variables. (45:15)

8.1 Kerry expressed interest in finding a reason for the initial difference being 14 and for failure of the matrix. (47:00)
8.2 Stacey observed that (11+18+27)÷14=4. (48:00)
8.3 Stacey compared 4 with the number of sides of the triangle. (48:45)
8.4 Kerry compared dividing by 4 with averaging. (50:00)

9.1 TK intervened—"Is 14 special, or is one fourth of the sum special?" (50:30)

9.2 Investigated an exterior triangle, and another triangle based on new numbers. (c. 51:00)
9.3 Described a general method for solving triangles. (59:00)

10.1 Both expressed continuing concern over the number 4. (60:30)
10.2 Stacey observed that the act of nesting the original triangle in a larger triangle created four triangles approximately the same size as the original. (61:00)
10.3 Kerry was unhappy with this as an explanation—"You can't just say that, you have to explain that. Why are those 4 triangles important?" (61:30)

11 Continued exploration (of Arithmagons of more than 3 sides) after research session ended.

12 In a follow up interview three weeks later, Kerry and Stacey were shown a formal proof of the correctness of their method. For Kerry this proof explained the occurrence of the 4. He commented "That's where we get the 1/4 from". It is not clear whether Stacey understood the proof as an explanation.

Fibonacci

(time elapsed)

1.1 They tried to remember the rule, arriving at $F_n = F_{n-1} + F_{n-2}$. (began 1:35)
1.2 Kerry rephrased as $F_{n+2} = F_{n+1} + F_n$
1.3 Kerry added $F_n = F_{n+2} - F_{n+1}$ to allow determination of F_1 and F_2
1.4 Verified by cases. (until 6:00)
1.5 Looked for other rules. (7:00-7:45)

2.1 Examined F_{3n} for pattern. Tried to used differences and ratios. (began 8:00)
2.2 Determined they are even, by induction. (finished at 12:40)

3.1 Examined F_p for pattern (began 12:45). Stacey determined they are all odd, by induction. (12:55)
3.2 Revised conjecture to: F_p is always prime. (13:45) (14:15)
3.3 Made a list of Fibonacci numbers to examine. (14:15-15:15)
3.2 Observed that converse does not hold. (15:50) (16:15)
3.4 Tested more cases. (16:15-20:30)

4.1 TK asked which are even (20:30). Examined F_{3n} in table.
4.2 Kerry provided O+O=E proof when prompted. (23:25-25:30)

5.1 Examined groups of four consecutive Fibonacci numbers at TK's prompting. (27:00-28:00)
5.2 Investigated sequence starting with 7,7 at Stacey's suggestion. (28:00)
5.3 Found sums of groups of four make a Lucas sequence. (28:40-32:00)
5.4 Investigated negative sequence (32:00-35:00)

6. Explored groups of three consecutive Fibonacci numbers at Kerry's suggestion. Found sums make a Lucas sequence. (36:00-43:00)

7. DR suggested investigating products of three consecutive Fibonacci numbers. (43:00)

7.2 Found a false pattern. (47:00)
7.3 Found product of end numbers equal to middle number squared ± 1 (47:30)
7.4 Checked for 7,7 sequence at Stacey's suggestion. (48:15-50:00)
7.5 Find product of end numbers equal to middle number squared $\pm F_1$. (50:00)
7.6 Investigated other sequences. (53:00)
7.7 TK and Kerry debate whether sequences start *n,n* or 0,*n* . (56:00)
7.8 TK suggests making a list of sequences considered so far.

8. Gave F_{3n} even and F_p prime as their discoveries when asked to summarize. (100:00+)

<u>Group III: Eleanor and Rachel</u>

<u>Arithmagon</u>

Eleanor

(time elapsed from start of tape)

0. Given Problem Sheet (3:55)

1.1 Worked with R on solution by system of equations. (-6:00)
1.2 Worked independently on solution by system of equations. (6:00-7:55)
1.3 Wondered if the solution is unique. (8:00)
1.4 Decided that the algebraic solution showed only one solution is possible. (9:38)

2.1 Compared Solutions with R, B & W (10:20)
2.2 B asked E & R how they used algebra to solve problem; (~11:00)

3 W said he was "playing" with properties of triangles. E & K told W the triangle was irrelevant.
E: "I guess you could [treat sides as lengths]"
B: "I don't think you could."
B rejected taking triangle as important as angles didn't work.

4.1 Discussion of B's method.
4.2 E & R watched B (1400-1500)
B reconstructed his thinking and solved another triangle.

5. Worked independently (1500-1600)

6.1 R & E analyzed B's method. (16:30-~17:00)
6.2 Tried a triangle by B's method, with R (~17:00-21:00)
(~20:00 B gave solution)

7.1 DR gave 1-4-12 triangle. (22:15)
7.2 B declared it impossible. He explained that only 0+1 gives 1, and neither order works. E & W suggested negative numbers
7.3 W: "Do the three numbers represent angles or something?" E, B, & DR: "No." (~23:30)
7.4 B Proposed that E or R solve 1-4-12 algebraically, E began to do so. (~25:00)
7.5 B: "I've determined that it is impossible."

E: "You think it's impossible?"
B: (to DR) Is it impossible?
B Asked for E's algebraic solution. (25:40)

7.6 worked on 1-4-12 independently (~26:30-29:00)

7.7 E gave B solution to 1-4-12 triangle.

8 Everyone listed to R describe progress.

9 W described what he was doing to E and B.

10.1 E noticed 6-6-6 [A+a=B+b=C+c] in W's work.

10.2 W checked [A+a=B+b=C+c] on other triangles. They continued to explore.

10.3 E noticed that a+b+c=12=A+a in a particular triangle (Announced that a+b+c=12) (37:14)
W enunciated rule: a+b+c=2(A+B+C)

10.4 Several examples were checked. (38:50)

11.1 Worked on inventing a method of solution.

11.2 E explained her method. W interrupted with a new problem. (40:00)

12.1 E gave solution. W: "No."

13.1 Tried to clarify the relations she was working with.

14 B, E, & W worked on relation of division by 2 to area formula for a triangle. (~44:30-48:00)

15 R announced her formula $\frac{a+b+c}{2}$ to the group.

16 R explained the derivation of her formula to E by recapitulating her calculations. (~49:30-51:00)

17 Tried to relate R's formula to her own relations.

18 General discussion of R's formula. W: "I understand everything except why you divide by 2"

19.1 Announced they have found two different methods.

19.2 Worked on clarifying her method.

20 All explained E's method to TK (~56:00-58:00)

21 Discussed relation to angles with B&W (~58:30-59:30)

22.1 Derived R's formula from her equations. (~59:30-60:30)

22.2 Compared results w/ R (~60:30)

22.3 Tried to explain equations by algebraic derivation. (~62:00)

Rachel

(time elapsed from start of tape)

0. Given Problem Sheet (3:55)

1.1	Worked with E on solution by system of equations.	(-6:00)
1.2	Worked independently on solution by system of equations.	(6:00-7:55)
1.3	Solved puzzle	(7:55)
2.1	Compared Solutions with E, B & W	(10:20)
2.2	B asked E & R how they used algebra to solve problem;	(~11:00)
3.1	Made a new puzzle	(~11:00)
3.2	Looked for patterns.	(~13:00-14:30)
4.1	Discussion of B's method.	
4.2	E & R watched B B reconstructed his thinking and solved another triangle.	(1400-1500)
5.	Worked independently	(1500-1600)
6.1	R & E analyzed B's method.	(16:30-~17:00)
6.2	Tried a triangle by B's method, with E	(~17:00-21:00) (~20:00 B gave solution)
6.3	Tried a triangle by B's method, alone	(~21:00-22:15))
7.1	DR gave 1-4-12 triangle.	(22:15)
7.2	Worked on solving 1-4-12 triangle	(~23:00-25:30)
7.3	Watched E & B ~	(25:30-26:00)
7.4	worked on 1-4-12 independently	(~26:30-29:00)
8.1	Began working on algebraic derivations.	(~29:00)
8.2	Determined that if two sides are equal then two corners are equal.	(29:37)
8.3	Everyone listed to R describe progress.	(~31:00)
9	Continued to explore deductively.	(~32:00-48:00)
9.2	Worked with TK's help	(~34:00-35:00)
9.3	deduced that if all sides are equal all corners are too.	(~37:30)
9.4	TK suggested deducing with no constraints.	(42:20)
9.5	TK Suggested focus on $\frac{1}{2}$.	(4:30)
9.6	TK helped	(46:00)
9.7	Found formula, and tested it.	(~46:30)
9.8	R announced her formula $\frac{a+b+c}{2}$ to the group.	(~48:00-49:00)
10	R explained the derivation of her formula to E by recapitulating her calculations.	(~49:30-51:00)
11	Watched as:	
11.1	All discussed R's formula. W: "I understand everything except why you divide by 2"	
11.2	W repeated operational version of R's formula.	(~53:30)
12.1	Exchange of explanations for division by 2. B's link to a+b+c=2(A+B+C) accepted by W, and R, as explanation.	(~54:00)
12.2	Watched W work an example.	(~55:00)
13	All explained E's method to TK	(~56:00-59:00)

14.1	Derived a+b+c = 2(A+B+C) algebraically	(59:00)
14.2	Compared results w/ E	(~60:30)
14.3	Continued derivations	(~62:00)

Fibonacci

		(time of day, or tape counter)
1	Making conjectures	(Clk 3:10-3:15)
1.1	**Conjecture (R): F_{3n} is even**	**(Clk 3:12)**
1.2	**Conjecture (E): F_{4n} is odd, also a multiple of 3**	**(Clk 3:12)**
1.3	Noted pattern OOEOOE	(Clk 3:15)
2.	*Discovery of 3s rule*	
3.	**Making conjecture (E?): F_p is prime, and F_c is not prime**	**(Clk: 3:20)**
4.1	Exploring: R looking at 4s, E looking at 3s	
4. 2	*4s rules discovered*	*(100)*
	verified,	*(130)*
	formulated	*(160-730 by R)*
5.	**Making conjecture: $F_{-n} = -F_n$**	**(Ctr: 660-700)**
5.2	Summarized results	(780-880)
5.3	TK talked about Fibonacci Quarterly	(880-1000)
6	*3s rule formulated*	*(1050 by E)*
	(Why 4?, need to explain)	*(Ctr: 970-1130)*
	Worked independently:	
	E tried to relate the 4s rule and the 3s rule	(1150-1350)
	TK asked for clarification, R answered	(1250-1320)
	E gave report to TK	(1350-1400)
	?	(1400-1625)
7	E wondering about 4 in 3s rule. R looking at factors	(1625)
	Searching for explanations	(1625, E)
	R looked at 2s rule briefly	(1660)
8	*5s rule discovered,*	*(1760)*
	verified,	*(1760)*
	formulated	*(1860-1940 by E; 2070 by R)*
9	**Making conjecture (R): An n-rule exist for all n**	**(Ctr: 1920)**
10	*Cycles of discovery, verification, and formulation.*	
10.1	*6s rule discovered, verified,(*	*2040)*
	formulated	*((2120 by E; 2190 by R)*
10.2	*2s rule discovered and verified*	*(2150)*
	R formulated her method for discovering rules	(2200)

11 **Making conjecture (R): 7s rule will not work as 7 is prime.** E: 2&3 are prime. (2300-2310)

12 Cycles of discovery, verification, and formulation.
12.1 7s rule verified (2360)
12.2 8s rule predicted & verified, (2420)
alternative verification suggested. (2390-2480)

They were about to check for an 11 rule when they ran out of time.

Group IV: Jane & Chris

Arithmagon

(tape counter)

Solving

1.1 Using system of equations (140-450)
1.2 Check solution, it doesn't work in original problem (it does, C mis-added in his head.) (450)

2.1 Second attempt, with matrix (580-940)
C knows matrix is the same as equations, but doesn't have any better ideas (580)
2.2 C Predicts matrix will not reduce, as otherwise it should produce a solution which should work. (640)

3 Temporary halt in matrix work, based on knowledge that it's the same as the equations. Search for error in the equations. (770-880)

4 Discovery of solution with matrix (880-940)

Generalizing

5 Conjecture that all numbers which work are of the form 2n, 3n, n+2. Rejected. (1200-1550)

6 trying another example: 3-5-2. C predicts any triangle solvable based on 3 equations with 3 unknowns. (1580-1590)

7 J Considers 1-1-1 triangle, concludes need for fractions. (1720-1760)

Is the Arithmagon always solvable?

8 C on general solvability: 3x3=>solution (1840)
Squares work too. (1870)
Equations must be linearly independent (1890)

9 Is the triangle always solvable?
9.1 Worked to solve general square. J using equations, C using matrix. (1920-2200)
Matrix does not reduce. This casts doubt on general solvability of triangles. (2200)
9.2 J Solving triangle in general. C continuing to investigate square's matrix (2270)
J concludes square never works (2375)
9.3 C solves general triangle with matrix (2600-2680)

10 <u>Is the square never solvable?</u>

10.1 Solution of triangle now casts doubt on calculations related to square (2680)

10.2 C reviewed calculations for square (2680-2750)

10.3 Conjecture made: Square is different as corners are unconnected. Conjecture Squares never work. (2750-2800)

10.4 Counter example 1-1-1-1 (2825)
Revised conjecture: It works for some values.

11 <u>What is going on with our matrix?</u>

11.1 Search for error in square's matrix: Confusion in writing v and y (2930)
Relation: v=y+z-x discovered, confirmed, and considered. (3000, 3060 &3170)

11.2 Conclusion that C and D are arbitrary (3110, 3160&3300)
rejected on grounds it "doesn't make sense" (3130&3300)

12 Interacting with TK (3350-3400)

13 Continuing to explore square's matrix (3415-3460)

APPENDIX D

TABLES OF RESULTS

The following tables summarize some of the information presented in Chapters II, III and IV. Table 6 shows the distribution of needs and reasoning according to problem situations. Table 7 shows the distribution of needs and reasoning according to the participants involved. The remaining tables show distributions for individual participants, indicating problem situations.

1. Needs and proving in different problem situations

	Proving					
	Unformulated	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self	F	A				
Explaining to others	A, F	A, N, O		A, O		
Exploring to a goal	A	A	A	A	A, F, G, N, O	
Exploring without goal	A, F	A	A, F		A, F, G	
Verifying	A		A		A, F, G, N, O	A, O
Teacher-game		A, N				

Table 6: Distribution of needs and reasoning according to problem situations.

Key to problems:

A: Arithmagon
F: Fibonacci
G: GEOworld
N: Patterns in n^3-n
O: Sum rule, Odd+Odd=Even

2. Needs and proving by different participants

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self	K	E, K, R				
Explaining to others	Be, Bi	A, Be, Bi, Co, E, Jo, R		Be, Bi, St, W		
Exploring to a goal	Sa	E, Sa	Co, E, K, St, R	W	A, Be, Bi, Co, E, Jo, K, R, St, W	
Exploring without goal	St	R	Ja, Ch		A, Be, Bi, Co, E, Jo, K, R, St, W	
Verifying	Bi		Co		A, Be, Bi, Co, E, K, R, St, W	A, Bi
Teacher-game		Co, K, R				

Table 7: Distribution of needs and proving according to participants involved.

<u>Key to participants:</u>

- A: Anton (Math 30 student, worked with Colin)
- Be: Ben (undergraduate student, worked with Wayne)
- Bi: Bill (Math 13 student, worked with John)
- Ch: Chris (undergraduate student, worked with Jane)
- Co: Colin (Math 30 student, worked with Anton)
- E: Eleanor (Undergraduate student, worked with Rachel)
- Ja: Jane (undergraduate student, worked with Chris)
- Jo: John (Math 13 student, worked with Bill)
- R: Rachel (undergraduate student, worked with Eleanor)
- Sa: Sandy (grade 6 student, informal interview)
- St: Stacey (undergraduate student, worked with Kerry)

3. Needs and proving for individual participants

In the following tables letter codes for the problem situations in which the reasoning took place are used. The key for Table 6 should be consulted. An "X" indicates that such reasoning occurred several times, but that no specific example is present in Chapter II. In all other cases a description of the episode in question can be found in the appropriate section of Chapter II or, in the case of Stacey and Kerry, in Chapter III.

First clinical study

Ben

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others	A	A		A		
Exploring to a goal					X	
Exploring without goal					X	
Verifying					X	
Teacher-game						

Table 8: Needs and reasoning — Ben.

Wayne

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others				A		
Exploring to a goal				A	X	
Exploring without goal					X	
Verifying					X	
Teacher-game						

Table 9: Needs and reasoning — Wayne.

Stacey

	Proving					
	Unformulated	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others				A		
Exploring to a goal			A		X	
Exploring without goal	A				X	
Verifying					X	
Teacher-game						

Table 10: Needs and reasoning — Stacey.

Kerry

	Proving					
	Unformulated	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self	F	A				
Explaining to others						
Exploring to a goal			A		X	
Exploring without goal					X	
Verifying					X	
Teacher-game		A				

Table 11: Needs and reasoning — Kerry.

Eleanor

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self		A				
Explaining to others		A				
Exploring to a goal		A	A		X	
Exploring without goal					X	
Verifying					X	
Teacher-game						

Table 12: Needs and reasoning — Eleanor.

Rachel

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self		A				
Explaining to others		A				
Exploring to a goal			A		X	
Exploring without goal		A			X	
Verifying					X	
Teacher-game		A				

Table 13: Needs and reasoning — Rachel.

Jane

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others						
Exploring to a goal			A		X	
Exploring without goal			X		X	
Verifying					X	
Teacher-game						

Table 14: Needs and reasoning — Jane.

Chris

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others						
Exploring to a goal			A			
Exploring without goal			X			
Verifying						
Teacher-game						

Table 15: Needs and reasoning — Chris.

<u>North School study</u>

<u>Bill</u>

	Proving					
	Unformulated	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others	F	O		O		
Exploring to a goal					X	
Exploring without goal	F				X	
Verifying	A					O
Teacher-game						

Table 16: Needs and reasoning — Bill.

<u>John</u>

	Proving					
	Unformulated	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others		O				
Exploring to a goal					X	
Exploring without goal					X	
Verifying					X	
Teacher-game						

Table 17: Needs and reasoning — John.

Colin

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others		A, N				
Exploring to a goal			A	X		
Exploring without goal				X		
Verifying			A	X		
Teacher-game		N				

Table 18: Needs and reasoning — Colin.

Anton

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others		N				
Exploring to a goal					X	
Exploring without goal					X	
Verifying					A	A
Teacher-game						

Table 19: Needs and reasoning — Anton.

Sandy

	Proving *Unformulated*	*Formulated*	*Mech. Deduction*	Reasoning by analogy	Inductive reasoning	Referring to an authority
Explaining to self						
Explaining to others						
Exploring to a goal	A	A				
Exploring without goal						
Verifying						
Teacher-game						

Table 20: Needs and reasoning — Sandy.

APPENDIX E

METHODS OF SOLUTION

While the methods the participants used to solve the initial puzzles and the generalizations they made in the problem situations were not objects of my studies, they are of sufficient interest to warrant their inclusion here. They may serve to provide those readers who have not had time to investigate the problem situations themselves an opportunity to see the openness and possibility of the problems.

1. Arithmagon

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.

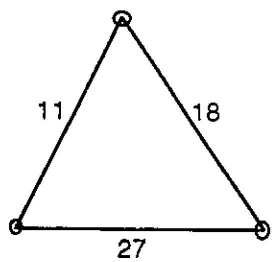


Figure 35: The Arithmagon prompt.

Most of the participants were given a problem prompt containing nothing but the initial puzzle shown in Figure 35 and the cryptic instruction to "Generalize the problem and its solution." In the following sections I will describe, in turn, specific solutions to the puzzle, general solutions, and general problems.

Specific solutions to the puzzle

I have seen three solutions to the puzzle which I do not consider to be general solutions, for various reasons. The constraints method involves an assumption which is not generally true, so that while it does solve the given puzzle, it cannot solve all puzzles. Using a system of equations is quite general, in fact too general, and it tells us nothing about the Arithmagon itself. If it is solved for the general case with three variables on the sides, I consider the resulting formula to be a general solution to the Arithmagon. If every Arithmagon requires solving a new system, however, that is not general enough for me. The method of false position is quite general, but the one person I have seen come up with it did not see it that way, so I include it here as a specific method.

The constraints method

This method was used by about half the participants. It is based on an assumption that the secret numbers are whole numbers, which they are in the case of the initial puzzle, so they can be found this way. The method is an intelligent guess and check. Ben described this method to Rachel, Eleanor, and Wayne as the way he had solved the puzzle (see Chapter II, section 1).

The assumption that the secret numbers are whole numbers means they cannot be less than 1. How large they can be depends on the numbers on the sides. The secret number at the top can be at most 10, because it must add to another number, which we know is at least 1, to give 11. Similar reasoning gives maximum values for all the secret numbers. It is now a simple matter of trying all the values in the allowed range of one secret number until we find the right one. If we start with the value 1 for the top number (which most participants did) we quickly find that the secret numbers are 1, 17, 10.

Systems of equations

If we label the diagram as shown in Figure 36 we can express the relationships between the secret numbers *A*, *B*, *C*, and the number on the sides in these three equations:

(1) $A + B = 18$
(2) $B + C = 27$
(3) $C + A = 11$

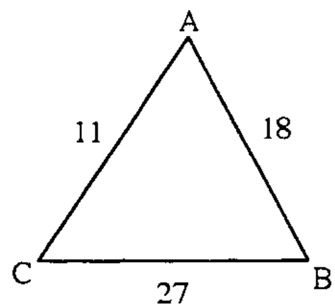


Figure 36: Labeling the Arithmagon for a system of equations.

Subtracting equation (1) from equation (2) yields: (4) $C - A = 9$

Adding equation (4) to equation (3) yields: (5) $2C = 20$

Dividing by 2 yields: $C = 10$. The remaining values can be found by substituting 10 into the equations wherever *C* occurs, or by writing in 10 on the diagram and working around the triangle, subtracting to find the unknown numbers.

The method of false position

One solution which none of the participants used, but which I have seen in another context, is similar to a historical technique for solving equations known as the method of false position. Begin with a corner, say the top, and pick a number, say 5. If the top were 5 then the lower left would be 6, and the lower right would be 13. The sum of these is only 19, so they are too small. That means our original guess was too big. We might simply try reducing it by some amount at random, and we would quickly zero in on the right value. However, we can be a little bit more clever. Our sum for the base was wrong by 8. Changing the top number makes changes in each of the bottom corners equally, so it makes sense that an answer which is too low by 8 means that each of the bottom corners is too low by 4. So the lower left is 6+4 = 10 and the lower right is 13+4 =17.

General solutions

The usual formula

As I noted above, solving the Arithmagon using a system of equations with variables in place of the known side numbers yields a general formula, $A = \frac{y+z-x}{2}$, where x is the side opposite the secret number A. Expressed verbally, it is "Add the two adjacent sides and subtract the opposite one." This formula was also discovered inductively by some participants, and by proving in the case of Sandy (see Chapter II, section 2).

Stacey and Kerry's method

This method is quite unusual, and based on the extended explorations undertaken by Kerry and Stacey. It is described in Chapter III. I will repeat the gist of it here.

Begin by drawing another triangle around the original triangle (see Figure 37). Add the known numbers and divide by 4 (in this case we get 56÷4=14). Subtract this number from each of the known sides, and write the values you get on the corners of the new, larger, triangle (see Figure 37).

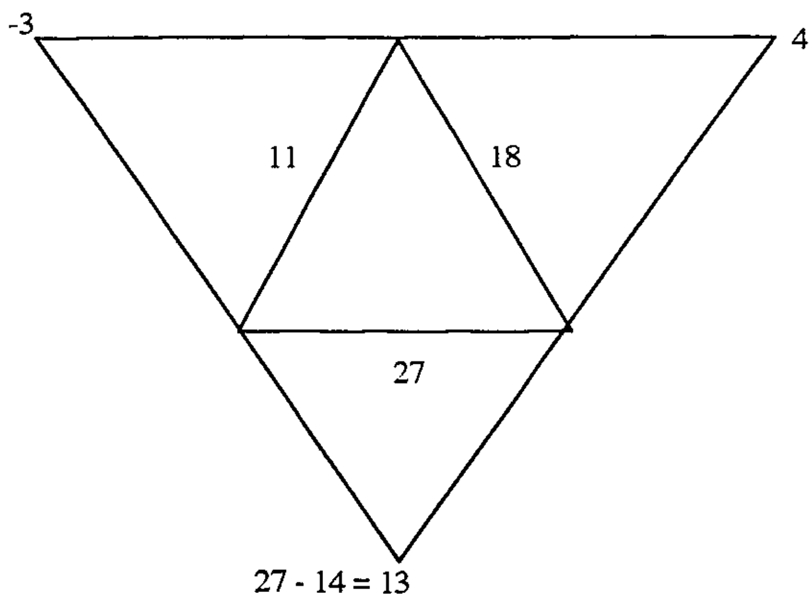


Figure 37: Stacey and Kerry' general solution.

Adding these values in pairs, (e.g., 13+4 =17) gives the side numbers for the larger triangle, which are the secret corner numbers for the original triangle. It really works! If you will not take my word for it (verifying by authority) try it a few times (verifying inductively) or see the proof in Chapter III (verifying by proving).

Colin and Anton's method

Colin and Anton noticed that the difference of two corner numbers is the same as the difference of the numbers on the two sides they do not have in common. (see Figure 38). They discovered this relationship inductively and later explained it deductively (See Chapter II, section 1). Once this relationship is known it establishes enough information to quickly find an answer. For example, in the original triangle, it tells us the difference of the secret numbers on the base is 18-11 = 7 (Notice that the equation $C - A = 9$ which came up in the solution using system of equations expresses the same relationship for two other secret numbers). We know their sum is 27 and their difference is 7. Anton would usually find the numbers quickly by guess and check at this point, but on one occasion Colin suggested subtracting the difference from the sum to find twice one of the numbers, and then dividing by 2. They had a great deal of difficulty formulating this method, but had they succeeded it would have produced a formula like $A = \frac{y - (z - x)}{2}$, which is similar to the "usual" formula, though it reveals its much different origins.

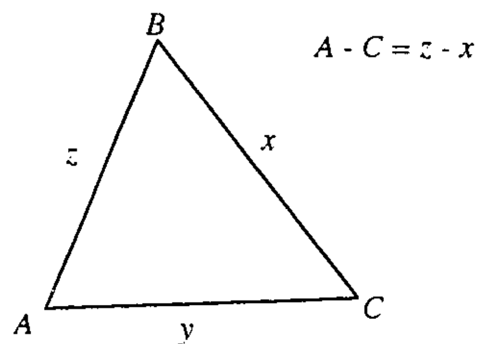


Figure 38: Colin and Anton's difference relation.

Eleanor's method

Working with Ben and Wayne, Eleanor notice two important relationships between the numbers in the Arithmagon. They are: $a+z = b+y = c+x$ and $a+b+c = 2(x+y+z)$ (see Figure 39).

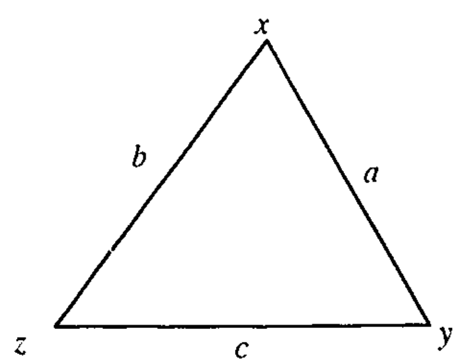


Figure 39: Labeling of triangle for the basic relations in Eleanor's method.

Using this relation she developed a general method for solving the Arithmagon. Her method begins by adding up the known numbers, and dividing by 2 (yielding 27 in the original puzzle). She then writes this in the center of the triangle (see Figure 40). Subtracting each of the known sides from this number yields the secret number on the corner across from it.

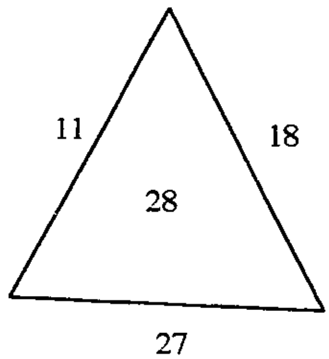


Figure 40: Eleanor's "middle" number.

General problems

The Arithmagon problem can be generalized in several ways. Generalizations of the problem did not occur spontaneously to any of the participants. The idea of Arithmagon squares was suggested to several groups (and included in the prompt used at North School), and some investigations of Arithmagon squares did occur. Stacey and Kerry were the only participants to look at Arithmagons of more than four sides.

Other than generalizing the number of sides of the Arithmagon, one can also generalize to higher dimensions, and investigate Arithmagon polyhedra (Arithmahedra?), or, in the abstract, figures of 4 or more dimensions. Or one could look at the effect of using the product or difference of the two secret numbers to produce the numbers on the sides.

2. Fibonacci

The Fibonacci sequence begins:

1, 1, 2, ...

and continues according to the rule that each term is the sum of the previous two (e.g., 1+1=2).

The Fibonacci sequence has many interesting properties.

Can you find an interesting property of every third Fibonacci number?

Can you find other interesting properties?

Figure 41: The Fibonacci prompt.

The investigation of the Fibonacci sequence usually turned into a property hunt. All this hunting did result in the discovery of some interesting properties. In addition some participants generalized the sequence by loosening some of the requirements of its definition.

Properties of every n^{th} Fibonacci number

Most groups noticed that every third Fibonacci number is even, but in many cases this was not considered interesting enough to be a property. In general every n^{th} Fibonacci number is a multiple of F_n, so F_{3m} is even because F_3 is 2. Some participants, after noting the evenness property went on to notice the relation $F_{3m} = 4F_{3m-3} + F_{3m-6}$. For example $F_9=34$, $F_6=8$, and $F_3=2$, and $34 = 4 \times 8 + 2$. Similar relations hold for F_{4m}, F_{5m}, etc. with the multiplier '4' being replaced by increasing terms of the Lucas sequence, 4, 7, 11, 18, ... Several groups noticed that Fibonacci numbers with prime indexes (F_p where p is prime) are also prime. No group noticed that this property fails for F_{19}.

Generalizations of the Fibonacci sequence

Stacey and Kerry were the most enthusiastic generalizers of the Fibonacci sequence, but others also explored some of the possibilities. The sequence can be generalized by changing the initial values, changing the range the sequence is defined on, or changing the rule. No one tried changing the rule, perhaps because it is the most unusual aspect of the sequence. Stacey and Kerry examined the sequence without constraining the values to positive integers. They developed the integer sequence ... -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, ... They, along with other groups, also explored sequences beginning with values other than 1, 1.