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A STUDY OF TWO CHILDREN'S LEARNING OF BASE COMPLEMENT ADDITIONS

By

PETER MCCARTHY ©

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled *A Study of Two Children's Learning of Base Complement Additions* submitted by *Peter McCarthy* in partial fulfillment of the requirements for the degree of *Master of Education.*

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ABSTRACT

This is a qualitative study, which describes two students' learning of the base complement additions method for compound subtraction of whole numbers. In particular the research addresses the questions: 1. How does base complement additions (BCA) compare to decomposition (D), as an algorithm for teaching/learning compound subtraction, in terms of skill development? 2. Which method do students choose to use when subtracting? Why do they say they chose the method? 3. In what ways does the learning of both methods increase students' arithmetic competence?

The BCA is a derivative of equal additions algorithm, which was based on compensation. That is, if the same number is added to both minuend and subtrahend the difference remains the same. Equal additions was intensively used in the USA a generation ago but its distinct disadvantage was its difficulty to model subtraction meaningfully using manipulatives.

The data collection covered the period of November 2000 to January 2001. Pre-intervention, intervention and post-intervention design was used and computational tests and interviews were two main evaluation instruments used for the study.

Results of the study indicate that participants were more confident with base complement additions than with decomposition, that students made fewer errors, recognized errors in worked examples, and chose base complement additions over decomposition as their method for compound subtraction.

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DEDICATION

This work is dedicated to my dear wife, Mrs. Grace McCarthy and my children Ezekiel and Esther McCarthy and Kofi Ampiah. It is also dedicated to Esi Benyiwa, Adjoa Kakraba and Kwesi Ahen and my most affectionate brother, Mr. Paul McCarthy.

It is also dedicated to my friends who have sustained me throughout my program.

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CHAPTER 1 - STATEMENT OF THE PROBLEM

Background of the Problem

Chief examiners and assistant examiners of the West African Examinations Council, mathematics teachers and employers who are in a position to criticise the capability of young people who have passed through the public elementary schools, have all experienced some uneasiness about the condition of arithmetical knowledge at the present time. They have observed, for instance, that accuracy in manipulation of figures, by the pupils, is not as expected (Gyening, 1993).

According to Gyening (1993) employers sometimes express concern at the inability of young persons to perform simple numerical operations, without the use of calculators. From my experience, when a problem like, "9-7", is given to learners, they use a variety of strategies to solve it. Some of these strategies are: counting down to, counting on and take away. However, when it comes to compound subtraction like, "701-467", they don't usually find it easy and they seek the help of calculators. Why? I conjecture that the most common method students use for computing compound subtraction is the "decomposition method" and they find it difficult; they tend to make a lot of errors with this method. The report of the National Assessment of Education (Carpenter, 1975), found that only 55% of the nine year olds could correctly complete two-digit subtraction problems with regrouping. This is not encouraging because one of the aims of the school curriculum is to help children develop computational skills. This problem has actually been a concern for many governments, especially that of Ghana. In England and Wales, for instance, concern for the apparent lack of basic computational skills and other mathematical skills and abilities in many children led to the appointment of a committee chaired by W. H. Cockroft (1982). The committee was charged with considering the teaching of mathematics in primary and secondary schools in England and Wales, with particular regard to the mathematics required in further and higher education, employment and adult life, and to make recommendations (Cockroft, 1982).

In Ghana, today, the situation is not different. The apparent lack of basic mathematics computational skills has led to poor performance in mathematics examinations. In Ghana, the 1998 Senior Secondary School Certificate Examination results in mathematics is a typical example. In my school, Diabene Secondary-Technical School, only 55% of the total number of students who sat for the core mathematics examination passed.

According to the study, entitled "*Cognitive Arithmetic Across Cultures*," (Campbell & Xue, 2001), the performance of Asian Chinese was better than the non-Asian counterparts. This was associated with the frequency of the latter's use of calculators in elementary schools. In addition the authors reiterated further on simple mathematics, that Asian Chinese students and Chinese Canadian students were nearly equivalent and both performed better than the non-Asian students. The authors offered the following explanation: If students exclusively solve arithmetic problems using a calculator, then they will become proficient using a calculator, but this will not provide much exercise for the long-term and short-term memory processes that mental arithmetic depends on. On the other hand, use of calculators was associated with poor performance of the complex arithmetic task. Working complex arithmetic without the aid of a calculator promotes better working memory skills for complex arithmetic procedures such as carrying, borrowing, and place keeping. In conclusion, Campbell and Xue (2001) state

> A calculator is a very powerful tool for learning arithmetic and math, because it provides students with instantaneous means to explore, discover and confirm mathematical facts and principles. Undoubtedly, calculator use can contribute very positively to math education in many ways, but mental arithmetic expertise will develop only through extensive practice of the specific cognitive processes that underlie that skill (pp. 312-315).

According to Gyening (1993), poor arithmetic skills in many children are due to a number of causes. One of the possible causes is the teaching of certain topics; and one such topic is compound subtraction. The conventional method of teaching compound subtraction is the method of "decomposition". He argues that this method has many inherent drawbacks, which adversely affect pupils in their performance on arithmetical tasks involving compound subtraction. Some of these drawbacks are limited preparation in time, difficulties with presentation forms and frequency of errors.

a. *It takes too much time to prepare the pupil's mind for the decomposition method.*

Teaching of compound subtraction by decomposition requires knowledge of place value, expansion, and renaming or regrouping. These are pre-requisites in the teaching of the decomposition method, and each takes a lot of time. Thus it is too time-consuming when teaching children this method. The adverse effects, among others, are that materials cannot be covered conveniently in the time allocated for the topic and pupils become handicapped with respect to mastery, for there is not enough time for discussion and practice.

b. *It does not lend itself conveniently to the representation of the task in the Horizontal form.*

For the difference of 63 – 28, the written representation, $\overset{5}{\not{6}}$13-28, after "borrowing" looks like 513-28. On account of this shortcoming the standard textbooks do not usually present the treatment in the horizontal form. According to Gyening (1993), they do not only shy away from it but they also exhort others to follow suit. Hence pupils are seldom taught compound subtraction by horizontal presentation, and this has affected pupils computation regarding subtraction in horizontal forms.

Habor and Nworgu (1989) conducted an experiment using the same set of test items on three different groups of pupils of relatively the same academic abilities. To one group the items were put in vertical arrangement only; to another group the items given were all in horizontal form; and the last group was given a blend of both horizontal and vertical forms. It was discovered that the group that was given the set of questions in vertical form did best, followed by the group that were given the mix of both vertical and horizontal forms of subtraction.

c. Errors associated with the decomposition method

Despite the fact that treatment of compound subtraction by the method of decomposition takes a lot of time, pupils eventually make a lot of errors in their compound subtraction computations (as mentioned above). Common among the errors I have observed from my teaching experience are:

1. Reversal Errors;

```
e.g.   8344
     - 6877
       2533
```

This is subtracting smaller number from a bigger number irrespective of which is subtrahend. In this example the person mistakenly subtracts 4 from 7 and 3 from 8.

2. Borrowing without reducing the minuend;

```
e.g.   7586
     - 5647
       2949
```

In this example, the student treats the 6 as 16 in order to subtract but does not "borrow" any unit from the 8 tens. Similarly the student treats the 5 in the hundreds column as 15 in order to subtract but does not "borrow" any unit from the 7 hundreds.

3. Borrowing across zero;

```
e.g.   4 0 1
      -2 0 4
       1 0 7
```

In this example the student borrows but rather than taking ten tens from the hundreds and then ten ones from the tens to the ones, simply reduced the hundreds by 1 and send it straight across the zero to the ones.

Other Methods Used a Generation Ago

Another widely used method for dealing with this compound subtraction was equal additions (Muller, 1964). The equal additions method was a method that was extensively used in the U.S.A. a generation ago and it was generally used throughout Europe until 1964. The rationale for the equal additions technique is based on compensation. That is, if the same number is added to both minuend and subtrahend the difference remains the same.

Now the decomposition method is more widely used than the equal additions mainly because the decomposition method is easier to rationalise (Jerman & Beardslee 1978). Both decomposition and equal additions have been compared in many research studies with inconclusive results as to which one is superior to the other. Brownell and Moser (1949) showed, in a classic experiment comparing the effectiveness of the decomposition and equal additions methods, that the decomposition when taught meaningfully was more successful; the equal additions method was difficult to rationalise.

According to Jerman and Beardslee (1978), the equal additions method is difficult to model with manipulatives and does not really model the subtraction process. They went on to say that equal additions has been compared with the decomposition method in many research studies and in most cases equal additions has been shown to be a superior teaching tool when meaning is not a factor but computation is what is considered important. Hence the superiority of one method over the other depends on the desired outcome, meaning or computational effectiveness.

Research Questions

Consideration of the weakness of equal additions and drawbacks and errors associated with the decomposition methods led to the study of the problem of compound subtraction by Gyening of the Department of Science Education, University of Cape Coast. On April 23, 1993, at a departmental seminar, Gyening presented a paper titled "Facilitating Compound Subtraction by Equivalent Zero Additions (EZA)." At the end of the presentation, the consensus of the audience was that this method (EZA) has potential to compare favourably with the method of decomposition (D) which is now the standard method used in most parts of the world, including Ghana. Hence, it was recommended that a research study be conducted to determine how this EZA method compares with the D method. This sparked my interest and I decided to pursue this study as part of my Master's program at the University of Alberta. A close study of the EZA method (by myself) indicated that base complement additions (BCA) is a better title since the model is based on *base complements*. Hence I pose the research questions below:

1. How does base complement additions (BCA) compare to decomposition (D), as an algorithm for teaching/learning compound subtraction, in terms of skill development?

2. Which method do students choose to use when subtracting? Why do they say they chose the method?

3. In what ways does the learning of both methods increase students' arithmetic competence?

The Purpose of the Study

The purpose of the study is to learn more about children's ways of doing subtraction that involve renaming. The study is an attempt to throw some light on the teaching and learning of compound subtraction by BCA. It is also an attempt to determine whether the teaching of compound subtraction by the method of BCA compares favorably with the method of D in terms of skill development if manipulatives are used to teach both algorithms.

Definition of Terms

For the purpose of the study, the following terms used in the report are defined as shown below.

Simple subtraction

This is subtraction that does not need renaming.

E. g. 1. 8 – 4   2. 435
                     -214

Compound Subtraction

This is subtraction that requires renaming. It is when at least one of the digits of the minuend is less than the corresponding digits of the subtrahend.

E. g. 1. 536   2. 615
         249   -378

Subtrahend

The number that is being subtracted. For instance, if $a - b = c$, then b is said to the subtrahend.

Minuend

This is a number from which another number is taken away. For instance if $a - b = c$, then a is said to be the minuend.

Algorithm

An algorithm is a step-by-step procedure for accomplishing a task, which is done mentally and/or by paper-and-pencil. Its form distinguishes such procedure, in which the first action in the sequence is usually precisely specified and after that the next is also indicated. (Algorithms are discussed in further detail in chapter 2.)

Subtraction by Decomposition

Subtraction by "decomposition" is subtraction done by renaming the minuend. It is a rule of procedure or algorithm for solving compound subtraction problems by decomposing the minuend e.g. by increasing the ones place by 10 units and decreasing the tens place by 1 ten. Subtraction by decomposition is commonly called "borrowing".

Subtraction by Equal Additions

The equal additions algorithm is a rule of procedure for solving compound subtraction problems by changing both the minuend and the subtrahend. The rationale for the equal additions technique is based upon a general property of the operation of subtraction commonly referred to as compensation. That is if the same number is added to both the minuend and the subtrahend the difference remains the same. However, in practice, in equal additions strategy, one is only required to add ten to the minuend and a ten to the subtrahend. A typical example will be this "If the ones place of the minuend is increased by 10 ones to allow for subtraction within the set of whole numbers it is compensated by adding 1 ten to the tens place of the subtrahend".

E g. 7 3 ⇒ 7 (13)
- 5 8 - (6) 8

Subtraction by Base Complement Additions

This is a rule of procedure that transforms compound subtraction into simple subtraction by way of base complement (*s*). By base complement of a number *s* relative to the base *b*, we mean the positive number that should be added to *s* to obtain *b*. In other words *c* is the base complement of *s* if $s + c = b$, where *b* is the base.

Thus with respect to base 10, we have the following

Number	Base complement
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1

Table 1-1. *Digits and Their Corresponding Base Complements*

E. g. 2 4 (+3) ⟶ 2 7
-1 7 -(+3) - 2 0

The base complement of 7 (i. e., 3) is added to both the minuend (24) and the subtrahend (17). This gives a transformed minuend and subtrahend of 27 and 20 respectively.

Crutch

This is a symbolic learning aid when doing a paper and pencil computation. Crutches are characterized by the use of markings through numerals followed by a subscript and/or a superscript.

E.g. 1) $\begin{array}{r} \overset{7}{\cancel{8}}\,{}_{1}7 \\ \underline{-5\;\;9} \end{array}$ 2) $\begin{array}{r} \overset{6}{\cancel{7}}\,{}_{1}2 \\ \underline{-3\;\;7} \end{array}$

Deep-end Approach

Deep-end approach according to Gagné et al. (1963) is learning from complex to simple. With respect to mathematics instruction, this is when one or two prerequisites or phases or stages of instruction is/are skipped. A typical example is instruction using symbolic phase of instruction, where iconic or enactive phases could be used to buttress understanding only.

It must be recalled that certain individuals can skip prerequisites or phases or stages of instruction and learn complex algorithms without explicitly practicing the sub-skill(s) (Dienes & Golding, 1971). Factors that bring about this include: high motivation; presentation of tasks that might involve some of skills acquired somewhere; and that all the sub-skills might have been learned already at home or taught by a friend(s) or at school.

Instructional Sublimation

This is instructional process whereby sub-skills or phases are skipped during instruction. Instructional direction here is from simple to complex and *vice-versa*. For this study instruction from symbolic to enactive and *vice-versa* are typical examples. They are processes to help the learner to fold back and forth for meaningful construction of knowledge.

Transferability

This is the ability to demonstrate successful carry-over to untaught processes (Brownell & Moser, 1949). In this study transferability was measured by the ability to solve from 4 up to 11 digits following an intervention involving 2 and 3-digit examples of compound subtraction of whole numbers.

Instrumental Understanding

Instrumental understanding, according to Skemp (1978), is the rote performance of a procedural learning.

Procedural Knowledge

This is learning that focuses on skills and step-by-step procedures and sometimes without explicit reference to mathematical ideas.

Meaning

In this study, the instruction is designed to help students understand why renaming, by use of BCA, is valid. To do this BCA is modeled with a manipulative (see chapter 4). To the extent that this modeling offers an explanation of why renaming does not change the equality of the expression, it has the potential to be meaningful to the students. In this study meaning is used to speak about how the BCA algorithm works as a procedure. The students' understanding is assessed strictly on the basis of demonstrating conceptual understanding or procedural knowledge of BCA.

Delimitations

The study was delimited to two students in fourth and fifth grades and of the same parents. This was for convenience sake. The study was delimited to whole number paper-and-pencil subtraction. Most of the questions were selected from MathQuest (1986) because this was the textbook the students brought home from school.

Limitations

The study was limited to the effects of five days of instruction, fourteen days retention period during which no BCA was taught, twenty-eight day post- post intervention to confirm retention during which no BCA was taught or intentionally reinforced. However, students may have been practicing at school when responding to subtraction questions.

The sequence of instruction was developed by the researcher, and is limited to the various reactions of the participants. The researcher's teaching experience of 12 years included 3 years in a training college, 4 years in a secondary –technical school, 2 years in a middle school and 3 years in a primary school. All his teaching experience was limited to the use of D algorithm. His familiarity with the BCA algorithm was limited to insight given to him by Gyening of the Mathematics Education Department (the then Science Education Department), Faculty of Education, University of Cape Coast, Ghana.

This study is a descriptive, qualitative one, in which only two students' activities and responses are studied. Hence no claims are made to suggest that the results are generalizable.

Assumptions

The participants used as subjects come from the same family, attended the same school but they took different end of term examinations because one was in grade 4 and the other was in grade 5. It was assumed:

- That they would perform differently depending on their understanding of the algorithm.
- Those participants already know how to do simple addition of whole numbers.
- That participants could use their logico-mathematical thinking to reason at the formal stage of instruction; thus they would be able to transfer knowledge from two- or three digit number to multi-digit numbers.
- that the test items were valid and reliable since they were selected from a standard mathematics textbook, MathQuest 3, 4 & 5. [Note-This book is no longer the recommended for use in Alberta schools. However, as already noted, this is the text the students were using in school.]

Significance of the Study

If BCA is found to be meaningfully understood to participants, then this study offers pupils an additional tool for their mathematical toolkit. From this toolkit they can choose the most appropriate strategy based on their comfort level. Thus it will expand the knowledge base of learners to improve upon their learning. For teachers, the study has the potential to inform them about an alternative algorithm for multi-digit subtraction that involves renaming.

Organization of the Report

The report of this research study is organized in six chapters. Chapter 1 presents the problems and purpose of the study. Chapter 2 is a review of literature related to the study. Chapter 3 describes the method and the procedures used for the study. Chapter 4 is reports on the development of the intervention. Chapter 5 is about the results followed by a summary of the chapter. The last is Chapter 6, which presents a summary of the whole study, the conclusions and recommendations.

CHAPTER 2 - REVIEW OF RELATED RESEARCH AND LITERATURE

Introduction

From early to the middle 1900s, textbooks used any one of the three different algorithms, for teaching subtraction that involves renaming: the equal additions, the decomposition and the Austrian method. However, the algorithms used did not show any marking through or numbers being rewritten. The students were expected to remember when a number was "borrowed" and reduce the number "borrowed from" accordingly. After William Brownell modified the decomposition algorithm in 1949, to include the use of the crutch, the use of other algorithms was practically eliminated in mathematics textbooks (Ross & Pratt-Cotter, 1999).

This chapter is a review, which describes the related research that has been done regarding the two algorithms–decomposition and equal additions. This chapter is divided into four parts. The first part discusses pre-algorithmic subtraction (i.e., subtraction without renaming) and subtraction algorithms. It is followed by section two, which reports on some early research efforts. Reviews on some other research efforts and justification for the research study are the third part. The fourth section of the chapter deals with the learning theories that underpin the study.

Pre-algorithmic Subtraction/Subtraction Without Renaming

According to Carpenter and Moser (1982), children bring to school well developed counting procedures, some knowledge of numbers, and some understanding of physical operations. Carpenter and Moser, therefore, encourage researchers to examine variations in how children process information prior to, during, and after formal instruction.

According to Carpenter Hiebert, & Moser (1983) before formal instruction children can analyze and solve simple subtraction tasks by directly presenting the operation with physical objects or by using counting strategies. However, these strategies are too cumbersome to be effective with more complex problems and large numbers. According to their research findings, there is evidence that at the time children are introduced to writing mathematical sentences they see no connection between them and their informal strategies (Carpenter, et. al., 1983). From the study, it was learned that counting backwards (down) is more difficult than counting up for the first grade children, especially when the count goes beyond three numbers or when the count starts with a number greater than 10 (Baroody, 1984; Carpenter & Moser 1984). According to Baroody (1984), in order for children to use counting up, or counting back procedures successfully to solve subtraction problems, they must be able to count on and back from given numbers "with ease".

Boulton-Lewis (1993) noted that teaching children to count backwards as a procedure for finding the difference between two numbers as opposed to forward counting of subtrahend and minuend has been discussed in detail. Backward counting caused problems, with both small and large subtrahends but more particularly with subtrahends greater than 3 or 4 (e.g. 12–9, 16–8, 25–12, 31–6). Children who attempted and succeeded with more difficult subtraction tasks were generally not using backward counting. It was also noted that the practice of teaching subtraction algorithms to 2 or 3 digits without regrouping before those that require regrouping caused some children to use "buggy" algorithms and others not to use the written algorithm at all. Only a small number of responses in the year 2 group made correct use of the written algorithm and that was for items that required no regrouping.

In classroom based research three distinct classes of subtraction strategies: separating from, counting down from, and counting up from given or adding on have been identified (Carpenter & Moser 1982). Their results indicate that children tend to model the action or relationship described in the problem rather than attempt to relate the problem to a single operation of subtraction. These researchers hypothesized that children do not recognize the interchangeability of subtraction strategies. This accounts for the close match between problem structure and strategy. The authors believe that the transition from the informal modeling and counting strategies that children invent to solve basic addition and subtraction problems to the use of the number facts and formal algorithms they learn in school is a critical stage in children's learning of mathematics. Thus the investigators encourage efforts to ease the transition from informal problem solving strategies to formal algorithms.

There is the need to precede any basic fact instruction involving counting with conceptually-based learning activities that enable children to reflect on and refine their own counting understandings and skills (Fuson & Secada, 1986; Fuson & Willis, 1988; and Thornton, 1989). When children are having difficulty in computing basic subtraction problems, it is very important to begin diagnosis with an examination of their informal subtraction procedures. Understanding the complicated nature of children's entry behaviour will go a long way in the designing of specific remedial instruction.

Strategies Based Approaches to "Basic Facts" Learning

Rathmell (1978) found that teaching children thinking strategies facilitates their learning and retention of basic facts. According to Chambers (1996), more recent studies have confirmed this effect. Chambers explains that a strategies approach helps students organize the facts in a meaningful network so that they are easily remembered and accessed. Adults also use strategies and rules for certain facts (Thornton, 1990).

Algorithms

An algorithm is "a process, or set of rules, usually one expressed in algebraic notation, now used especially in computing, machine translation and linguistics" (O. E. D, 00005610).

Mathematics educators typically may offer something like "a step-by-step process that guarantees the correct solution to a given problem, provided the steps are executed correctly" as their definition of algorithm (Barnett, 1998 p. 69). An algorithm is also explained as a finite, step-by-step procedure for accomplishing a task that we wish to complete (Boyer, Brumfield & Higgins 1960; Usiskin, 1998; Ashlock, 2002). Maurer (1998) defines algorithm as precise and systematic method for solving a class of problems, which follows a determinate set of rules, and finite number of steps to give outputs that provide conclusive answers.

It appears there are many definitions or explanations for an algorithm though the meanings might be similar. For the purpose of this research an algorithm will be understood as step-by-step procedures for accomplishing tasks, which are done mentally and/or by paper-and-pencil. Such procedures are distinguished by their form, in which the first action in the sequence is usually precisely specified and after that the next is also indicated. These procedures are said to be conclusive, for their outputs are solutions to the specified tasks (Maurer 1998).

Historical Account of Algorithm

Gerbert d'Aurillac (ca. 940-1003) is known to be the "father" of the use of Hindu-Arabic numerals in Western Europe. D'Aurillac used the numerals 1, 2, 3, ...9 in columns on the counting board using counters. Zero was represented on the board with a blank column. He appeared not to have algorithms for calculating with these counters. It was rather Fibonacci, also known as Leonardo da Pisa (ca. 1180-1250), and other Western European medieval mathematicians who learned these algorithms from the works of al-Kwarizmi, on Hindu Arabic system. Algorithm is a word that means a rule or procedure for solving a problem. The word was chosen as an honor to al-Kwarizmi, who supposedly wrote the first elementary textbook on algebra (Hughes, 1998; Barnett, 1998).

In the Renaissance Italy students competed within the commercial setting to earn their living by computation. These students were the children of the Italian abacists who were mathematical practitioners. These practitioners earned their living by teaching methods of computation. But there was much disagreement between those who supported the use of the counting board for computational purposes and the proponents of the Hindu-Arabic system- after al-Kwarizmi's arithmetic had been translated to Latin in the twelfth century the first time (Barnett, 1998).

According to Boyer, Brufiel and Higgins (1960), algorithm was synonymous with positional numeration between the tenth and the fifteenth centuries. Emphasis, at that time was on symbols and arithmetic then called algorism. The people who used algorism to calculate were called algorists. Today, the Oxford English Dictionary indicates that the word algorithm and algorism are synonymous (NCTM, 1960, p. 252; OED, 00005610).

Eventual Adoption of al-Kwarizmi's Arithmetic

From records of Barnett (1998), the Hindu-Arabic numerals and their algorithms (of al-Kwarizmi's arithmetic) was preferred to the use of the counting board for the following reasons:

- The counting board system was logically awkward, since it required both a large board and a bag of counters
- Nontransportability of the medieval counting board
- Recording the computation with the computing board was difficult, since each step had to be eliminated before the next could be effected on the board.
- Hindu-Arabic algorithms allowed one to record each step of the process, making verification of the result much easier. (pp. 70-71).

Types of Algorithms

In the mathematics education literature we find two main types of algorithms: *student developed algorithms* (see page 16) and *standard algorithms* (Ashlock, 2002). According to Ashlock (2002) the paper-and-pencil procedures that individuals learn and use differ over time and among cultures. On this account, curriculum designers make choices in selecting algorithms as "standard". Algorithms are usually acceptable as standard when they always provide the correct answers. From the point of view of Usiskin (1998) these paper-and-pencil algorithms that are being used in school mathematics can be put into three categories. These are: *arithmetic, algebra, and calculus and drawing algorithms.* It is the arithmetic algorithms that are pertinent to this research study. *Arithmetic algorithms* including those for subtraction, finding square roots, doing long division, standard deviation, and so on (Usiskin, 1998). Specifically this study is on an algorithm for doing compound subtraction.

Reasons for Using Algorithms

For a procedure to be used as an algorithm Usiskin (1998) has this to say: "If we are going to answer one question of a particular type we don't tend to think of the procedure we used to answer that question as an algorithm. However, if there are many questions for which the same procedure works, then it becomes useful to learn a procedure. Once we are able to identify the sequence of steps in the procedure, it becomes an algorithm" (p.10). Usiskin then asserts, "Algorithms are generalizations that embody one of the main reasons for studying mathematics to find ways of solving classes of problems. When we know of an algorithm, we can complete not just one task but all tasks of a particular kind and we are guaranteed an answer or answers" (p. 10). He then analyzed reasons for having algorithms, some of which are as follows:

> *Power.* This applies to the breadth of the applicability of an algorithm.
>
> *Reliable and Accurate.* An algorithm always provides the correct answer, when done correctly.
>
> *Speed.* An algorithm proceeds directly to the answer.
>
> *A record.* A paper-and-pencil algorithm provides a record of "how" the answer was determined.
>
> *Instruction.* An algorithm is instructive. " They don't just announce the result; they construct it." (pp. 10-14).

Why Teach Algorithms as Another Tool in Toolkit

Recent educational reforms emphasize the importance of the development of number sense, conceptual understanding and problem solving (Hiebert et al. 1991). Now is the time for us to evaluate the importance of arithmetic computation and the standard algorithms. A student might perform a mental calculation, develop an invented procedure, choose an appropriate written algorithm, or use a calculator when doing arithmetic computation. When asked to do a simple calculation, like 34-16, students might use a mental procedure. But for more complex calculations the calculator and the written algorithm may be more appropriate. Estimation technique may also be appropriate if the exact answer is not needed. Students, however, are usually encouraged to reason about which approach might best fit the problem at hand and then consider the reasonableness of their answers.

To enhance number sense students must be provided the opportunities in the primary grade levels, to develop and discuss invented algorithms (Kami, Lewis, & Livingstone 1993; Sowder 1992). In these situations, such students were found to be aware of several approaches and were flexible in their thinking and in choosing a procedure that best fits a problem. Very importantly, they were found to develop the ability to share their strategies and thus improve their reasoning and communication skills (Carroll & Denise 1998). It was apparent from Carroll and Denise's studies that teaching of specific algorithms should be diminished or delayed in the early years of the child's primary education. But it is also obvious that as students get to later stages in the primary grades levels and beyond, written algorithms become indispensable for several reasons.

> First, some calculations are not handled as well by a mental method. As the number becomes greater, demand on memory grows and often requires written

> record keeping. Although calculators and estimation can often be used, written algorithms are often the most reasonable method to use for "middle size" numbers. Second, some students are not successful in "inventing" addition and subtraction algorithms, and some alternative, like the traditional written method, may be beneficial to them. Methods that build on their thinking and early experience can help them use written algorithm with meaning. Third, tests and other societal expectations still make it necessary for most students to know some written method for each operation. Not only do many standardized tests include a computation section, but also parents and other adult still see competence in written computation as a benchmark of success in mathematics. Students benefit from having a range of options to choose from, including paper-and-pencil methods. This is supported by our observations where students are introduced to several methods for the subtraction of whole numbers (p. 107).

The researchers confirmed that "although it is advantageous for all students to know at least one written procedure for each of the operations, the standard algorithms taught in school are often not the most appropriate or understandable. Although they are efficient, the meaning of these standard algorithms is often unclear to students who learn them without understanding (e.g. why the ones are being carried). Consequently, 'buggy' procedure may develop" (Carroll & Denise, 1998 p. 107). This study offers an alternative subtraction algorithm for students. Once that has potential for enabling students to be more proficient at multi-digit compound subtraction and it provides another tool in their math toolkit. It avoids such procedures as "borrowing" or "carrying" and appears to be powerful, reliable and accurate and provides good records for instructional purposes.

Subtraction Algorithms

In the history of mathematics education, teaching children to subtract has long been considered a problem area. Winch (1920), noted that "No methods give more trouble and are less successful than those of teaching subtraction" (p.20). Thorndike (1921) believed that the controversy regarding how children should be taught to subtract centered on the argument of whether to use the "subtractive" or "additive" method. Today, the subtraction algorithm continues to be a source of difficulty for many students and the question of how best to teach it to youngsters is gaining renewed interest (Martin, 1992).

Currently, the decomposition algorithm, is predominantly used in the United States, Canada and other parts of the world, as the standard compound subtraction algorithm. However, several others, mostly students developed algorithms, have been endorsed as efficient for learning compound subtraction (Ross & Prat-Cotter, 1999). The following discussion of algorithms will be limited to decomposition and equal additions. These two have been singled out based on their worldwide recognition as standard subtraction algorithms (Suydam & Dessart, 1976; Thiessen et al., 1989).

Student Developed Algorithms

Fuson et al. (1997) provided a good summary of informal algorithms students use for solving subtraction problems that involves renaming. The following provide protocol for finding the difference 62-28 identified in this analysis:

> *Sequential:* Sixty take away twenty is forty. Then put back the two; that's forty-two. Now take away the eight from the forty-two. First, take away the two, that's forty, and then six more makes thirty-four.
>
> *Combining units separately*: Sixty take away twenty is forty. You can't take eight from two. If you take two from the two, you still have six more to take away. Now take the six from the forty; that's thirty-four.
>
> *Compensation:* Sixty-two take away thirty, but we're to take away twenty-eight so it's two more. The answer is, therefore, thirty-four. (on compensation, refer to pages 18- 19)

We accept the above as a comprehensive analysis of categories for students' invented algorithms for computing compound subtraction (Sowder, 1998). However, from different methods students use for subtraction, cited in Gordon Calvert's hand-out 1.1.3, (2002) there are many others we may find difficult to put in the above categories. For example,

54	54	
-38	+61	Subtract each number in the bottom from 9 i.e. 3 becomes 6, 8 becomes 1. Now add. When done, drop the 1 in the largest place and add 1 to the ones place to get the answer of 16.
	115	
	+1	
	16	

(Gordon Calvert, 2002, p. 3)

Both standard and student invented algorithms reduce a complex procedure to simpler components that can be operated on using available knowledge or procedures.

The Decomposition Algorithm

The D algorithm is commonly called the "borrowing method". However, the term "borrow" may be a misnomer. By definition, whatever is borrowed must be paid back, and this is something that is not clearly seen when using the D algorithm. Interestingly, the word "borrow" was not used when this method was first introduced in the United States around 1821. The term seemed to have appeared some years later (Johnson, 1938, p. 14).

The instructional sequence for subtraction (See Milne, 1895; Ray, 1857), might be described as follows:

1. Write the subtrahend under the minuend, units under units, tens under tens, etc.

2. Begin at the right and subtract each figure of the subtrahend from the corresponding figure of the minuend, writing the result beneath.

3. If a figure in the minuend has a less value than the corresponding figure in the subtrahend, increase the former by ten, and subtract; then diminish by one the units of the next higher order in the minuend, and subtract as before.

4. Proof : Add together the difference and subtrahend. If the result is equal to the minuend, the work is correct (Milne, 1895, p. 47).

Subtraction with zeros was handled separately. The explanation given for the problem 9,000 - 7,685 describes the D method.

> Since five units cannot be subtracted from 0 units, and since there are no tens nor hundreds, 1 thousand must be changed into hundreds, leaving 8 thousand; 1 of the hundreds must be changed into tens, leaving 9 hundreds and 1 of the tens into units, leaving 9 tens. The expression 8 thousand, 9 hundreds, 9 tens, and 10 units is thus equivalent to the minuend, from which the units of the subtrahend can be readily subtracted (Milne, 1895 p. 49).

There is one very important advantage for this algorithm: the ability to demonstrate the regrouping procedure using bundles of sticks. This has made the algorithm very popular, especially following the "meaningful teaching" at the turn of the century (Brownell & Moser, 1949). Many mathematics educators claim that this algorithm is easy to rationalize (Martin, 1992).

The most obvious drawback of D is that it takes too much time for students to learn it. Some students stick to their informal strategies thus subtraction becomes extremely difficult when it involves large numbers and renaming is required more than once (Carpenter et al., 1983).

The Use of a Crutch in Decomposition Algorithm

Brownell (1947) conducted a study to determine if a "crutch" in the D algorithm would be helpful to students. The crutch consisted of the use of markings through numerals from which an amount would be borrowed in order to keep track of the different steps when working. There was no evidence that the crutch was being used when Brownell conducted his research on its effectiveness. The only example that could be found was from a text published in 1857, in which markings were used to keep track of the renaming process. This was done in only one problem in the text, with all other problems worked without any markings (Ross & Pratt-Cotter, 1999).

According to the study the crutch was very beneficial to the users. Users of the crutch were more accurate, at the beginning of instruction, on borrowing as well as at the end. The recorded speed for the users was relatively better, at the beginning of the study, as compared to the students not using the crutch. Toward the end of the study, however, there was not as much significant difference in speed between the users of the crutch and non-users (Brownell & Moser, 1949).

Use of the place value initials H, T, and O at the top of the algorithm were used to help students identify H for hundreds, T for tens and O for ones. Many textbooks used this followed by the D method which involves borrowing – for the crutch caught on very quickly (Buswell, Brownell, & John, 1947).

Today this method of subtraction is used in most textbooks that treat compound subtraction. While some textbooks may present alternative algorithms for subtraction, the D algorithm is considered the primary algorithm in the United States, Canada and other parts of the world (Ross & Pratt-Cotter, 1999).

The Equal Additions Algorithm

The equal additions method was used extensively in United States a generation ago and it is still used in some European countries (Gyening, 1993). The rationale for equal additions technique is based on compensation. That is if the same number is added to both minuend and the subtrahend, the difference, however remains the same. The compensation in subtraction expressed generally is that for all $a, b, k \in R$, if $a - b = k$, then $(a + c) - (b + c) = k$. This is because given $a - b = k$, we have

$$
\begin{aligned}
&a - b = k \\
&a - b + c - c = k \\
&a + c - b - c = k \\
&(a + c) - (b + c) = k
\end{aligned}
$$

For example, consider 21 – 17 = 4 (in table 2 below); note that the invariance of the difference as the minuend and the subtrahend are successively increased by 5, 7, 9, 10, 15, 1000.

	+5	+ 7	+ 9	+ 10	+ 15	+ 1000
2 1 - 1 7	2 6 - 2 2	2 8 - 2 4	3 0 -2 6	3 1 - 2 7	3 6 - 3 2	1 0 2 1 -1 0 1 7
4	4	4	4	4	4	4

Table 1-2. *A Set of Subtraction Expressions for 21 –17 (Based on Compensation)*

From the writings of Johnson (1938), the equal additions algorithm could be traced back to the 15th and 16th centuries. Johnson explained that equal additions could also be called the "borrow and repay method" (p. 16). With respect to the meaning of the term "borrow", it is clear that it fits better in equal additions algorithm than the D algorithm. This is because in this method, a power of 10 is borrowed to add to the necessary place in the minuend and is repaid by adding to the digit in the next place of the subtrahend. An example of equal additions is shown in the table above.

Although the D and Austrian algorithms did appear in some textbooks, EA was the predominant algorithm used before 1940s. A textbook, authored by Willetts (1819), explains subtraction using the following rule:

1. Place the less number under the greater, with units under units, tens under tens, etc.

2. Begin at the right hand and take the lower figure from the one above it and set the difference down.

3. If the figure in the lower line is greater than the one above it, take the lower figure from 10 and add the difference to the upper figure which sum set down.

4. When the lower figure is taken from 10 there must be one added to the next lower figure. (p. 11).

Willetts (1819) was using "upper figure" and "lower figure" to mean minuend and subtrahend respectively. Though this method is "equal additions", in principle, to my observation, the strategy being used is base complement additions. This gives me an indication that the base complement additions was once used without modeling the mathematics with manipulatives.

The Base Complement Additions Algorithm (BCA)

Like the equal additions algorithm, the BCA is a rule of procedure for solving compound subtraction problems by changing both the minuend and the subtrahend. The BCA is a rule of procedure that transforms compound subtraction into simple subtraction by way of base complement(s). By base complement of a number s relative to the base b, we mean the positive number that should be added to s to obtain b. In other words c is the base complement of s if $s + c = b$, where b is the base (see page 6).

The rationale for BCA heuristic is based on compensation. That is, if the same number is added to both the minuend and the subtrahend the difference is not changed though there is change in arithmetic expression. (For more information on BCA refer to page 6 and chapter 4.)

Categories for Comparison: Decomposition and Base Complement Additions as Algorithms for Computing Multi-digit Subtraction

DECOMPOSITION	BASE COMPLEMENT ADDITIONS
1. Use is made of place value	1. Use is made of place value
2. Could be modeled using Manipulatives	2. Could be modeled using Manipulatives
3. Symbolic representation has three phases: Renaming/expansion The crutch The formal	3. Symbolic representation has three phases: Renaming The crutch The formal

Table 1-3. *Commonalties Between Decomposition and Base Complement Methods*

DECOMPOSITION	BASE COMPLEMENT ADDITIONS
1. Computation is usually subtractive and compound	1. Computation is additive and usual simple
2. Renaming is based on reducing a place value in the minuend	2. Renaming is based on adding base complement, of the subtrahend, to both minuend and subtrahend
3. The deep-end approach may not be appropriate for instructional sequence	3. The deep-end approach may be appropriate for instructional sequence (Not yet researched)
4. Uses expansion to rename/ transform compound to simple subtraction	4. Uses the idea of balancing to rename/ transform compound to simple subtraction
5. With respect to use of manipulatives: Use is made of Dienes blocks, abacus, sticks, bottle tops, number tray, etc.	5. With respect to use of manipulatives: balance scale, numeration cards, Dienes blocks and algebra tiles are useful
6. The following errors are common: reversal, borrowing without reducing the minuend and borrowing across zero	6. The following errors are absent: reversal, borrowing without reducing the minuend and borrowing across zero
7. Renaming by way of expansion is unique with compound subtraction	7. Renaming by way of balancing could be transferred to the concept of "equivalent equations" which is useful in algebraic manipulations

8. Borrowing when subtraction is simple is not possible	8. Application of base complement addition when subtraction is simple could be possible. [This could be a problem if not well handled]
9. Subtraction is usually not easy with multi-digit subtraction (subtraction involving more than four digits)	9. Subtraction is always easy with multi-digit subtraction of whole numbers

Table 1-4. *Differences Between Decomposition and Base Complement Methods*

Some Early Research Efforts

The D algorithm dates back to 1140 and was first introduced to America in 1822 (Smith, 1925). Johnson (1938) indicated that by 1875 many textbooks in United States were using this approach. The equal additions algorithm (EA) dates back to the writings of Fibonacci in 1202. And there have been more than 16 studies and many articles investigating the relative merit of these approaches of teaching compound subtraction since the turn of the century. Below are the most significant ones.

Smith (1925) believed the equal additions algorithm was "justly looked upon as one of the best plans by most of the fifteenth and sixteenth century writers, and we have none that is superior to it at present time" (p.100). Smith also noticed that prior to 1820 this approach was the only method used in the United States. According to Johnson (1938) the equal additions method prevailed in Illinois until 1850. Proponents of this method claim that it results in faster and more accurate computation, solving the difficulty present with the D algorithm when successive zeroes occur in the minuend.

A distinct disadvantage of this method, however, is the difficulty of rationalizing it. Commenting on the difficulty of developing the equal additions algorithm meaningfully, Gagg (1956) wrote:

> The equal additions method is not so logical as the other. . . The children will see that adding ten to each does not alter the answer (p.30). The author continues to endorse the algorithm stating that "This is just one of those rare occasions when many children will just have to believe what you say about the method being accurate" (p.30).

Which Algorithm To Use

The controversy over the best algorithm to teach compound subtraction to students began in the United States early in the 1900s (Trafton, 1970). In 1914, P. B. Ballard conducted an experiment in England exploring the three algorithms. Ballard conducted his investigation with 18,600 pupils ranging, in age, from 8 to 14. The study was based on a timed arithmetic test. The result indicated that the EA subjects were more accurate and faster than the D pupils. The investigator indicated that an obvious disadvantage occurred when zeroes were in the minuend. According to Ballard, the EA method was superior to the D method (Osburn, 1927).

In 1918, W. W. McClelland compared the equal additions method and the D method and stated:

> The complete result of the comparison of the two methods may thus be summarised by saying that the method of equal additions appears superior in speed, accuracy, and adaptability to new conditions, while the method of D is superior in speed after long practice (Osburn, 1927, p. 239).

Johnson's preliminary efforts involving several methods of subtraction took place in 1924 and 1931. After investigating speed and accuracy with college students, adults, and fifth through eighth graders, he concluded that subtraction using EA method resulted in more accurate and faster computation than the D method. Johnson's most influential work was his doctoral study conducted in 1938. His subjects included 1054 third through eighth graders and 43 adults. He compared the performance efforts using the D, EA and a third method, the Austrian (A) approach to subtraction. Accuracy and time were the criteria used to measure the effectiveness of each algorithm.

Johnson (1938) summarized two of his significant findings as follows:

> Other things being equal, the D technique in subtraction of whole numbers is, by its own intrinsic nature, by far the poorest method to employ from the standpoint of both accuracy and time. When compared to the equal additions method, the D method produces 18% more errors and requires 15% more time (p. 66).

Murray studied the speed and accuracy of over 3000 children comparing the D, EA and A methods of subtraction. The subjects included 1622 youngsters just learning compound subtraction and 1675 subjects, ages 10-11, already familiar with subtraction. Murray found that the D group were significantly inferior to both EA and A groups in accuracy and rate for both age groups. Murray's findings influenced the Committee on Primary School Subjects to recommend the EA approach to be adopted as the sole method taught in the schools of Scotland (Murray, 1941).

In Ray's Arithmetic (1857), the EA method was described in approximately the same way as above and the remark made is that the EA method of proceeding is the one generally used in practice, it is more convenient and less liable to error, especially when the upper number contains one or more zero.

The Transition

Two researchers Brownell and Moser (1949), comparing the effectiveness of D and EA methods, showed that the D method when taught meaningfully was the more successful method. According to the report the EA method was difficult to rationalise, the use of crutch facilitated the teaching and learning of the D method, and children discarded the crutch when encouraged to do so by teacher. No comparative research of such import on the merit of the best algorithm for teaching subtraction has been conducted since the Brownell-Moser (1949) study (Suydam & Dessart, 1976; Trafton, 1970). Whereas the old method studies focused on speed and accuracy, the more recent research studies focused on understanding.

Neureiter (1965) advocated D for its effectiveness in reinforcing place value concepts and continued that:

> The long controversy, of the past, ended with the all-out American choice of the decomposition method of subtraction (borrowing), between composition and equal additions methods. Although research had shown certain advantages in terms of speed and accuracy for the additive method, American educators agreed that decomposition should be preferred because it fits best into the concept of meaningful instruction since its mathematical bases is easily understood by children (p. 277).

As indicated above the EA method is difficult to model with manipulatives and does not really model the subtraction process. Though it had been compared with the D method in many research studies and in most cases it had been shown to be a superior teaching tool when meaning and understanding were not factors (Jerman & Beardslee, 1978). Similarly studies by Grossnickle and Reckzeh (1973) showed that the D was preferred, for EA method was very difficult to objectify.

Rheins and Rheins (1955) conducted a comparative interview study of subtraction methods with eighth graders. They concluded that D, generally, was not inferior for the more intelligent subjects in their sample, and for less intelligent portion, D proved to be superior. However, Weaver (1956) criticised this research. He pointed out that Rheins data did not tell anything about the value of meaningful teaching. In their computation for accuracy, Weaver noted that 36% of the problems in Rheins study requires no regrouping.

It is clear from the above literature that extensive research work has been done in the area of subtraction algorithms. Though D started about a century before the introduction of EA, the early investigations preceding the Brownell-Moser (1949) study indicated a general preference for the EA algorithm, based primarily on speed and accuracy. Brownell's (1949) classic research endorsed the teaching of subtraction via the meaningful development of the D procedure. This research led to the adaptation of the D method for the entire United States and many parts of the world. But according to Johnson (1938) and Murray (1941), it was the poorest method if one is looking for speed and accuracy of computation. I wonder how one research study could convince teachers, curriculum developers, mathematicians and mathematics educators to use the decomposition method even though it had been found in earlier studies as the poorest method with respect to speed, accuracy and retention? It appears that it is because of its meaningfulness to learning through its modelling with manipulatives that it was so widely accepted. If this is the case, then why not give the EA method a trial by researching its strengths when taught with manipulatives?

Other Research Efforts

Martin (1992) conducted a comparative study of 178 students in second and third grade in a rural setting of South Illinois. There were D and EA groups for each grade level. The study covered three weeks of instruction followed by ten days of retention during which period there was no compound subtraction taught. The results were as follows:

> Immediately following instruction there were no significant difference between the computation post-test of the D and EA among the second and the third graders. However, the difference which was shown favoured the D group. With respect to post-test it was concluded that the D group produced better computation accuracy among the second and the third graders than the EA group. The difference was greater for the third graders than for the second graders. The transfer post test and post-post test favoured the D group. The mean for the D group at both grade levels were higher than those for the EA group (pp. 48-49).

A weakness in this research was that the teacher-implementers did not know anything about EA algorithm and Martin (1992) had to brief them before each instructional period.

In their article published in 1999, Ross and Prat-Cotter conducted a thorough research study and stated that:

> The algorithms used for subtraction vary. In the United States, three different algorithms were used almost equally until the 1940s: the equal additions, the decomposition, and the Austrian method. After William Brownell modified the decomposition algorithm, the use of other algorithms was practically eliminated in mathematics textbooks. Today, some texts introduce the equal equal additions, but the decomposition is still the predominant algorithm (p.389).

The above statement is an indication that the EA algorithm is now being introduced in some textbooks, even though its mathematics is not modeled with manipulatives.

Justification for the Research Study

The algorithms used and the language used to describe subtraction have changed since 1940s. One of the most dramatic changes in algorithms seen in the last century was the introduction of Brownell's crutch. This new version of the D algorithm completely changed the way subtraction was taught in the United States and Ghana. The rapidity with which the D algorithm became dominant could be due to many factors, one of which was the way research was distributed. Throughout the 1920s and 1930s, research indicated the D algorithm was the weakest of the three algorithms. The research at that time might not have had the widespread impact that it had in Brownell's day, possibly due to publication practices (Martin, 1992; Ross & Pratt-Cotter, 1999). But it was quite clear that most of the research studies were in favour of the EA. I, therefore, strongly agree with Gyening's recommendation (1993) that a research study should be conducted to investigate the relative efficiency of D and BCA methods in terms of speed, accuracy and retention. BCA method, a modification of EA method, is rather easy to objectify with a model in order to make subtraction more meaningful.

If BCA could compete favourably with D, then it could be recommended to teachers as an alternative approach to teaching compound subtraction, when necessary (as stated above). After all we are usually not restricted to use only one method in solving a problem in mathematics.

Learning Theories Underpinning the Instructional Methods for this Study

Procedural Learning

Procedural learning, unlike conceptual learning, which focuses on ideas and on generalizations that make connections among ideas, focuses on skills and step-by-step procedures without necessarily making explicit reference to mathematical ideas.

Procedural knowledge has two distinct parts: representation system and algorithms, for completing arithmetic tasks. The first part is sometimes called the "form" of mathematics (Byers & Erlwanger, 1984). This is the symbols for mathematical ideas and syntactic rules for writing the symbols. The second part of procedural learning consists of the step-by-step instructions for completing tasks (i.e. algorithms). Procedural systems are usually structured and hierarchically arranged. Thus some procedures are embedded in others as sub-procedures. The super-procedures are entire sequence of step-by-step prescriptions- i.e. the target tasks (Hiebert & Lefevre, 1986).

The relationship between conceptual and procedural knowledge is obviously crucial during the early years of the child's education (Baroody, 1982; Ginsburg, 1977; Hiebert & Lefevre, 1986; Sinclair & Sinclair, 1984). It may result in symbols with meaning and procedures that can be remembered better and used more effectively (Hiebert & Lefevre, 1986). Proponents of "meaningful" learning and teaching of arithmetic suggest that understanding of the paper-and-pencil computational procedure is achieved through a solid conceptual basis (Brownell, 1935; Byers & Herscovics, 1977; Skemp,1978). In other words the procedural knowledge must rest on a conceptual knowledge base. The implication here is that conceptual knowledge is to form a support system for procedural knowledge (among others). But it is also known that procedural knowledge can be achieved without conceptual knowledge. Such can be quite limited in terms of solving non-routine problems unless it is connected to a conceptual knowledge base. Hiebert (1986) offers an interesting example, if a person knows how to prepare a meal only by following explicit cookbook directions the person will be helpless. The person is unlikely to modify a recipe according to taste especially when the needed ingredient is unavailable or when the cookbook fails to talk about it. But when the person's procedural knowledge of cooking is enriched with conceptual information about the spices, and the role of various ingredients in the cooking process, then the person is likely to apply the knowledge to novel situations (pp. 184-185).

Algorithms and Procedural Knowledge

Ashlock (2002) also points out that procedural learning may be based on concepts already learned. He supports the claim that standard algorithms can be taught so that students understand the concept and reasoning associated with procedures. This means paper-and-pencil procedures could involve both conceptual and procedural knowledge. Therefore as indicated above students are not merely mechanical processors. Ashlock (2002) maintains that when students use specific paper-and-pencil algorithm, the procedure becomes more automatic. Repeated use of an algorithm over time brings about less conceptual knowledge and more procedural knowledge. He termed this as "proceduralization" (p. 9). As support for the teaching of computation skills in school, Ashlock quoted NCTM's *Principles and Standards for School Mathematics as follows*:

> ...students must become fluent in arithmetic computation- they must have efficient and accurate methods that are supported by an understanding of numbers and operations. "Standard" algorithms for arithmetic computation are one means of achieving this fluency (p. 9).

From the context of the relationship between conceptual and procedural knowledge (above), the use of paper-and-pencil algorithms, even in this contemporary world of technology, is appropriate when it makes sense to do so.

Instrumental and Relational Understanding

Skemp (1978) has clearly distinguished between "instrumental" and "relational" understanding. He refers to instrumental understanding as a kind of rote procedure and relational understanding as the procedure with "understanding". He further explained that relational understanding has a rich supply of conceptual knowledge base for the procedures. However, some students have procedural competence only (as indicated above) and thus cannot use their procedural knowledge to solve non-routine problems.

These students, according to Skemp, have only the instrumental understanding of their procedures. Thus with respect to transfer of knowledge, for solving non-routine problems, relational understanding is preferred (Hiebert, 1986; Skemp, 1978). But despite all odds, many teachers teach instrumentally. Are there some reasons why teachers might teach for instrumental understanding? Could there be some advantages for teaching instrumentally? Gordon Calvert (2002) summarized Skemp's work as follows.

Instrumental Understanding	**Relational Understanding**
Maths is a series of rules, laws or algorithms to remember	Maths is about relationships and asking "why" or "what if?"
If you do lots of examples of the same type you will learn the rule	Examples are not repeated often. They tend to be different to each other
Children taught in this way often want to be "shown one like it" at the start of the lesson	Children expect to understand rather than remember
Teachers doing this are often thought to be "good teachers" because they "explain"	Children sometimes complain, "she won't explain!", "She always asks another question!"

Gordon Calvert

Table 2-1. *Instrumental and Relational Understanding*

Why Teachers Might Teach for Instrumental Understanding

Though mathematics taught relationally is easier to remember, more adaptable to new tasks, is organic and a goal in itself, many teachers do not teach this way (Skemp, 1976). We might ask why so many children are taught for instrumental understanding? Skemp (1976) gave some reasons for which individual teachers might make the choice to teach for instrumental understanding.

- That relational understanding takes too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
- That relational understanding of a particular topic is too difficult.
- That a skill is needed for use in higher mathematics and other subjects like science before it can be understood relationally with the schemas presently available to the pupils.
- That all the other mathematics teaching is instrumental.
- The quantity of information that is required by syllabi. (p. 24)

The implication of the above is that the teacher must consider alternative goals, of instrumental and relational understanding on their merit in relation to a particular situation, to make a choice. To make a reasoned choice, according to Skemp (1976), one must be aware of the distinction between relational and instrumental understanding.

Advantages of Instrumental Mathematics

Skemp (1976) suggested reasons, by offering three advantages, for teaching instrumental mathematics.

- With instrumental thinking one can often get the right answer more quickly and reliably than with relational.

- Instrumental mathematics, within its own context, is *easier to understand.* Some topics are very difficult to understand relationally. A typical example, here, is "Minus times minus is plus".

- From the reasons above the *rewards are more immediate, and more apparent*. It is nice to get a page of right answers and have the feeling of success from it (p. 23).

It must be recalled that a school is well known to chalk successes if very high percentage of students from the school pass final exams. From the above if what a student wants is to simply get the right answers, one could understand how a teacher may believe that instrumental mathematics may be quicker and easier rate.

Transfer of Learning

One of the traditional conceptions of associationist psychologists has been the view that learning one task successfully helps one to learn another task in so far as the tasks have the same set of associations. This is the concept of transfer of learning known as "the identity element theory of learning". Gagné (1963) represents the most successful person to apply this idea of transfer rigorously to math's curriculum, which he refers to as "the cumulative learning theory (CLT) or the theory of learning hierarchies".

The CLT of Gagné starts with a task analysis of ordered sub-skills. He begins his hierarchy by asking, "What does the child need to know how to do in order to be able to perform the task if given only instruction?". This generates subordinate skills or sub-skills to the target task. The same question is asked again to generate further pre requisites or sub-skills to the preceding sub-skills. This procedure according to Gagné is repeated until a successively simpler set of skills or hierarchy is generated. The skills do not go beyond the level where the student is assumed to have mastered relevant skills. A few features and assumptions are built into the learning hierarchies. A task position higher in a hierarchy does not mean that it is harder to learn than skills below it. The skill above may in fact be more complex than a successively lower skill because it incorporates all the lower skills. A sub-skill identified in the hierarchy may also play a role in other hierarchies. The target task, also, is supposed to incorporate all the components of the subordinate skill (Gagné et al, 1963).

The Implications of Transfer of Learning for this Study

Gagné's theory of hierarchies has a number of implications for mathematics instructors. In the first place, it makes the researcher aware of the need to break down the entire instruction into smaller units of a teaching sequence and to give treatment from simple to complex so that "transfer of learning" can enhance learning. The pre-intervention test involved the use of learning hierarchies must be diagnostic in nature. This can help the researcher to capture the capabilities of the participants and assist the researcher in determining where to start the intervention. Testing also reveals individual capabilities and therefore calls on the researcher to adopt individualization of instruction whereby individual attention is given to participants according to their need. Research evidence from students in learning hierarchies indicates that some students benefit from the deep-end approach; this implies that treating all your participants in the same manner can constrain some able participants, or in fact cause harm to others. Gagné's theory of hierarchies, therefore calls the attention of the researcher to individual differences in participants and therefore ensures that the researcher adopts instructional approaches beneficial to all participants.

Use of Manipulatives

Zoltan P. Dienes (1973), is credited with an idea that involved preparation of *materials* for the achievement of instructional objectives. His theory of learning cycle therefore starts with giving children a large variety of *structural materials* concerning the concept to be taught. Initially, the child is expected to interact with these materials in a free *play way*. Dienes believes that children are naturally attracted to the properties of structural materials: shapes, size, color, pattern etc. This process he believes, must not be hurried since it is an essential process and concept is embedded in it.

The next stage according to Dienes (1973) is the process of *structural play*. In this process the learner starts with the abstractions of a concept. He, therefore, extracts properties common to a given principle while discarding those that are not common to the principle or concept. Dienes contends that the materials should be presented to the child in *multiple embodiments*. That is, different sets of materials each embodying the concept of interest should be presented to the child to ensure *perceptual variability*. Central to this, according to Dienes, is recognition and stressing of *mathematical variability*. In this case, a concept is developed using various relevant forms and structures. In teaching place value for instance variability may be place value involving other bases like base five, six etc. This helps to ensure that application is not restricted, in a sense, to the child. This leads to mathematical variability. It is after these processes that Dienes suggested the use of graphs, images, maps, etc to abstract knowledge that has been gained. This stage according to Dienes is *the pictorial representation*. Even in this case, he believes, *symbolism* could be initiated by the child, using or adopting his/her own symbols (Dienes, 1966; 1970; 1973).

Implications of the Use of Manipulatives for this Study

Dienes' approach to mathematics instruction has relevance for this study. Children are able to abstract knowledge better if they are given the opportunity to interact with teaching/learning materials. In giving the intervention, therefore, the researcher should provide teaching/learning materials to enhance abstraction. The child who goes through a lot of activities for preparation and then uses the manipulatives this way for solution to the problem practically is more likely to work better with meaning than the one who has just been taught to apply the algorithm. On the other hand, the use of manipulatives is essential to meaningful teaching and learning of BCA.

The Dienes concept also implies that the researcher does not hurry through the activities but progresses from one stage to the other such that participants can rely on experiences in previous activities to solve new problems. Instructions need not be tied to only one type of instructional material for teaching a concept. In order to assist children to learn with meaning without attaching properties or principles wrongly to certain properties of material, several materials embodying the concept of interest should be employed in the treatment. The researcher should not start treatment with abstract ideas. There should always be a basis for abstraction that makes it easier for children to store images of objects they interacted with as a transition to abstraction and symbolism.

Bruner's Phases of Instruction

In his theory of "the sequence of conceptual development" Bruner (1964) describes three modes of representation. These modes of capturing experience in the memory are enactive, iconic and symbolic. Enactive representation is the concrete level. Here children manipulate with materials directly. In the classroom, this is the level that involves concrete objects. "Iconic" is the second phase of representation. Iconic phase takes a step away from the concrete and physical to the mental imagery. According to Bruner, iconic representation is what happens when the child "pictures" an operation as a way of remembering the act and recreating it mentally when necessary. Such mental pictures do not include every detail of what happened but summarize events by representing only their important characteristics. In the classroom, this is when we might use pictorial representations including drawings, sketches, diagrams and pictures. The symbolic representation is the third way of capturing experience in the memory. This phase is made possible by the advent of language competence. A symbol is a word or a mark that stands for something but in no way resembles that thing. For example, the numeral 9 does not look at all like an actual array of object having that property and neither does the word "eight"- each of these symbols are completely abstract.

According to Bruner writing mathematical operations using numerals, simple equations, and operation signs (+, -, =, etc) is the beginning of symbolic representation. Children soon learn to think about their performance in terms of the same symbols, which opens up the new possibility of abstract thinking. The three modes of representation are related developmentally (Bruner, 1964). Each depending on the one preceding it and requiring a great deal of practice before the transition to the next mode occurs.

Bruner claims that any idea or problem or body of knowledge can be presented in a form simple enough so that any particular child can understand it in a recognizable form; that is, there are ways to present complicated concepts such that children of any age would understand them at any level. He continues that any teaching, of a new concept, that follows the order enactive, iconic and symbolic is the best. And even though some students might be quite "ready" for a pure symbolic presentation it seemed wise to present at least the iconic mode as well so that the learners would have mental images to fall back on in case their symbolic manipulations fail (Bruner, 1959; Bruner, 1966, p.44).

Implications of Bruner's Phases of Instruction for this Study

Bruner's "phases of instruction" creates awareness for the researcher regarding the planning of the instructional sequence. It indicates that when teaching a new concept, the three phases are very necessary to bring about meaningful learning, enhance retention and retrieving information from the memories. This approach will help the participants of the study. Furthermore, the three phases of instruction promote discovery learning. For participants can see, feel, manipulate and play with physical materials to abstract concepts. Since each phase depends on the one preceding it and it requires a great deal of practice before the transition to the next phase can occur, the researcher should allow participants to do a lot of exercises during treatment.

CHAPTER 3 - METHODS AND PROCEDURES

Introduction

The study investigated the effects of an intervention based on subtraction of multi-digit whole numbers, when the new algorithm, BCA, was taught using manipulatives. Two students participated in the study. One was in grade 4 and the other in grade 5.

In this chapter information on the research design and recruitment of participants are presented. Following this general outline of study, procedures used in analyzing the data and the instructional setting (the development of instruction sequences for the D and the BCA approaches to solving subtraction that involves renaming) are discussed. The instructional sessions, evaluation instruments and the summary of activities are also featured in this chapter.

Design of the Study

The study, which took approximately eight school weeks for each participant, was divided into three phases. The procedure was the same for both participants for they both sat together to receive instruction, at the same time, and at their own residence.

Phase I consisted of a pre-intervention test and an interview. The pre-intervention test assessed entry behavior of participants and the pre-intervention interview helped to know more about participants and how they did subtraction in general. Phase I also included a review of some basic subtraction facts, place value and a review of doing 2-digit subtraction requiring both grouping and non-grouping. The D method was reviewed.

Phase II was the intervention and it covered five contact periods. A sequence of instructions was given to participants. It was mainly based on the BCA. The BCA was meaningfully developed with participants, to cover the enactive, iconic and symbolic phases of representation. The intervention was followed by an intervention test; but there was no interview at this phase.

Phase III was conducted 14 days after phase II for both participants. The same post-intervention test was given to each participant in the study. A day after the test, post-intervention interviews were conducted with each participants. The purpose was to determine how meaningful the instruction was to participants. It was conducted to find out if students were still sticking to their old methods and thus, it was a measure to observe if there was any change in behavior of participants.

The fourth phase was the post-post intervention session. This was approximately four weeks after phase III. At this phase unexpected tests were administered to the participants to confirm the authenticity of the findings of the study at phase III - which were far beyond the expectations of the researcher. On account of that, immediately after the post-post tests, the third interviews followed and audiotaped. Table 2-2, below, gives a diagrammatic view of the design of the study.

Sessions	**Description of Activities**	**Date**
1st	Pre-intervention test	November 13, 2000
2nd	1st Interview	November 14, 2000
3rd	1st Instructional Period Using numeration cards to model compound subtraction	November 15, 2000
4th	2nd Instructional Period Diagrammatic representation of activities of the 3rd session	November 17, 2000
5th	3rd Instructional Period Renaming of subtraction expressions	November 18, 2000
6th	4th Instructional Period Revision of activities of the 5th session. The three stages of the symbolic phase of renaming- i. e. expansion, the crutch and the formal.	November 20, 2000
7th	5th Instructional Period The formal analyzes of the third phase was stressed. Discussions based on transfer of learning from two to three and four-digit numerals	November 21, 2000
8th	Intervention Test	November 22, 2000
9th	Post- Intervention Test	December 6, 2000
10th	Post- Intervention Interviews	December 7, 2000
11th	Post-Post- Intervention Tests Post-Post- Intervention Interviews	January 5, 2001

Table 2-2. *The Design of the Study*

Recruitment of Participants

The researcher, unfamiliar with local schools and school personnels, registered and participated in a study buddy program with the aim of getting participants for the study. But some questions came to his mind when he was in the program. These were:

- *How long will it take for Cooperative Activities Program (CAP) to go through before the research could start?*

- *Students have already been introduced to using the D algorithm in computing compound subtraction. The question here was "what if learning BCA brings interference and thus causes students to be worse off in arithmetic skill development, forgetting their old method D and not understanding the BCA?"*

- *Can I get volunteers for the study?*

- *When and where could be appropriate for both participants and the researcher to do the study?*

The above questions troubled the mind of the researcher. Eventually the thought of recruiting from the church members so that the study could be done outside the normal classroom came. Recruiting students in this way offered the possibility for the researcher, an international student, to carry out the study in Canada rather than returning to Ghana.

By arranging with the presiding elder of the church of Pentecost, Edmonton, an announcement was made to ask for interested students. Four students came forward to participate in the study. However, only two were chosen after a preliminary interview based on:

- Their involvement in the activities during this introductory interview;

- Their willingness to participate in further interviews;

- Their willingness to identify, reflect upon, and discuss issues on the subject matter;

- Students who already knew how to do subtraction that involves renaming;

- Their availability at any time.

The two students who were selected were of the same parents. Both children were enrolled in the same primary school during the fall of 2000. One child was in grade 4 and the other child in grade 5. Prior to the study the two students had been introduced to working compound subtraction by way of D. Thus it was assumed that students already knew what place value was, how to do simple subtraction and also how to do exercises based on compound subtraction.

The parents were asked for their children's participation in the study and fortunately they requested that all sessions be conducted in their residence. The parents were informed of their option to withdraw their wards from the study at any time, without affecting the children in any way. Examples of the letters sent to the parents are found in Appendix A. The parents, however, did not withdraw the children from the study.

Procedure

The study began on November 13, 2000 and ended on January 5, 2001. First, the participants were given the same set of test items to do. This was then followed by interviews. Each interview was conducted in such a way that there was one person at a time with the researcher. The researcher met participants everyday for 8 days, during which the participants were given instructions on a new algorithm for doing compound subtraction. After which there was a break for 14 days. After the fourteenth day the post-intervention tests and interviews followed consecutively. Four weeks after the post-intervention test and interviews came the post-post intervention tests and interviews in the same session. All these occurred at the end of the day after the participants' formal school hours.

The instructional periods started from the third session and ended on the eighth session with a test. The duration for this intervention, which covered 5 instructional sessions, was approximately 35 minutes for each contact period. All the tests, instructions and interviews were centered on subtraction that involves renaming. All the interviews were audio-taped; the transcribed audio-tapes, the test papers and the working papers form the bases for the data required for the study.

Instructional Intervention

After a pre-intervention test and pre-intervention interviews, the researcher and the participants had a short discussion of how to solve compound subtraction using the D method. This was followed by the intervention, which was BCA for compound subtraction. From my point of view, the intervention and the interviews were the hallmark of the study. Each session was characterized by the researcher's explicit goal to foster the generation and evolution of the participants' understanding of the BCA algorithm by judiciously selecting tasks and, second, by researcher-participants interdependent activity through verbal and nonverbal communication. Consequently, researcher-participants interaction was the result of collaborative communication in which language was used as a tool to achieve and express cognitive processes. Theoretical bases for the instructions, discussed in chapter two as the "learning theories underpinning the instructional methods for this study" were the key to developing each session. However, it should be noted that instruction in the sessions was modified continually to the students' ways of understanding of the topic at hand as revealed by his/her answers and explanations.

Application of the Learning Theories to the Instruction

The instructional activities for BCA involved concrete representation of the transformation as well as diagrammatic and symbolic representations (of BCA). To master the new algorithm, BCA, the Diene's principles for mathematics instruction (on structural materials for achievement of instructional objectives) and Bruner's phases of instruction were used. The two theories cover representations using concrete materials, followed by image formation of the first representation, which is done by using diagrams. These, mentally, bring the objects, which are not physically present to the classroom for further discussions. The third phase is the symbolic representation. This carries knowledge to abstract realms. For concepts to be understood meaningfully, depends on proper integration of these three representations or at least two of them (see next chapter).

In Dienes (1973), theory of preparation of materials for the achievement of instructional objectives, there were representations that were not directly mentioned in Bruner's approach; but all were considered in the study. One representation is "free play" in which children are naturally attracted to properties of the physical materials: shapes, size, color, pattern etc. Another representation is "multiple embodiment" in which different sets of materials each embodying the concept of interest is presented to the child to ensure "perceptual variability". There is also "mathematical variability" in which a concept is developed using numeration cards. The researcher presented the learning situation first in free play before it became structured. At the free play stage Rubin wrote numerals at the back of some of the small cards- the ones. He and his sister used these as a game on one of the flat cards, which represents a hundred. Their play was such that it worth studying; it has the potential of promoting interpersonal communicative skills. The researcher also used different colors and different sets of teaching/learning aids (e.g. straws; algebra tiles; ones, longs and flats of Dienes blocks; cents, ten cents and dollars) for multiple embodiment and perceptual variability.

At the symbolic phase of representation are three stages: renaming, the crutch and the formal. The target task of the BCA is to operate at the formal stage of the symbolic representation. The BCA algorithm was procedurally developed; but more importantly conceptually informed through the enactive and iconic phases of representations for all the eight pre-requisite skills (see chapter 4). Thus for meaningful transfer from two/three-digits to multi-digits compound subtraction, the researcher put the entire intervention into smaller units of teaching sequence and gave treatments from simple to complex. Application of BCA to many different compound subtraction examples made its use become automatic and appeared less conceptual and more procedural (Ashlock 2002) and therefore more appropriately "instrumental" in use.

Instructional Sequence for Discussion on Decomposition Algorithm

Step1. Development of the concept of renaming using manipulatives such as bundles of sticks and numeration flash cards;
Step 2. Revision of renaming process by way of the manipulatives mention above;
Step 3. Use of number-words to express the renaming of numbers, for example: 72 = 7 tens, 2 ones renamed as 6 tens, 12 ones.
Step 4. Application of D in horizontal notation; e. g. 456 – 238
Step 5. Application of D in vertical arrangement;

e. g. 8 6
-5 9

Step 6. Renaming at both the developmental and final stages, in the vertical form. For example:

(8) (12)	(8) (12)
9 tens, 2 ones	9 2
- 5 tens, 6 ones	- 5 6

Step 7. Extension of algorithm to 3-digit numbers; special attention paid to zeroes in the minuend.

		(69) (14)
For example,	7 0 4	7 0 4
	- 3 6	- 3 6

Step 8. Using the D algorithm with word problems.

It is important to note that participants had been introduced to the D algorithm for at least three years so the above revision took one contact period only. After revision of the D algorithm with participants, the BCA algorithm was introduced to participants. Below are the main features of the BCA.

The Main Features of the Instructional Sequence for BCA

Step 1. Use of numeration cards
- for expansion of numbers (the idea of place value)
- for the concept of base complement
- modeling subtraction

Step 2. Discussion of types of subtraction
- simple
- compound

Step 3. Identification of simple and compound subtraction from group of expressions.
Step 4. Renaming, using the idea of
- equivalent equations (from equivalent zero addition)

- base complement additions (Use of numeration cards for the enactive representation before proceeding to diagrammatic and symbolic representations.)

Step 5. Stages of base complement additions

a. 21 +3 → 24
\- 17 -3 → - 20 (renaming)
4

b. 24
21 (using crutches)
-17
20

c. 21
-17 (formal application)
4

Step 6. Exercises based on step 5, above
Step 7. Word problems that will involve compound subtraction (two digits only; use must be made of BCA only, in computation).
Step 8. Extending BCA to three digit-numbers.
- identification of which subtraction at a place value is simple or compound
- typical examples of how three-digit compound subtraction are worked using BCA

Step 9. Exercises based on step 8 above
Step 10. Word problems related to step 9 (above).

Instructional Sessions

1st Session: pre-intervention test. A set of subtraction questions was given to participants to try. The purpose was to observe individual participants' various approaches to subtraction. These observations were used to provide a clue to the interviewer about the line of questioning that would be used in subsequent sessions. The interviews were to find out participants' thinking regarding subtraction that involves renaming. They were also conducted to find out the various procedures students employ when doing subtraction that involves renaming. (Refer to appendix B.)

2nd Session: pre-intervention interviews. These interviews were conducted and audio-taped in the house of participants. A participant was called and interviewed whilst the other was asked to play in another room. A complete interview took 40 minutes for the first participant and 35 minutes for the second participant. Children bring to school some informal counting procedures, some knowledge of numbers, and some understanding of physical operations (Carpenter & Moser 1982). Thus there was the need to examine variations in how children process information prior to, during, and after formal instruction. The interviews, in this session, were conducted purposely to find out participant's entry behaviors. They were also conducted mainly to find out the informal strategies students already have and also their level of understanding of the D algorithm for solving compound subtraction. (This strategy was developed because of Gagné's (1963) suggestion that instruction should be informed by an awareness of the student's sub-skills for the new skill to be acquired.)

3rd Session: It was the first instructional period of the instructional sequence. This session covered a discussion of types of subtraction, identifying simple and compound subtraction from a group of expressions, an introduction to the numeration cards and using numeration cards to model subtraction. Dienes blocks, algebra tiles, straws, etc. were also displayed and discussed as alternatives to numeration cards. The duration for this period was 40 minutes and activities were based on two-digit whole numbers. Students participated together.

4th Session: This was the second instructional period of the instructional sequence. This session was mainly on diagrammatic representation of the modeling activities with numeration cards of the third session. Activities of the first instructional period were revisited with more intensive work concentrated on the iconic representations. Whereas the first instructional period was characterized by the use of manipulatives, this session was characterized by student drawings of the manipulatives that model specific compound subtraction questions. This was done to link the manipulatives to the iconic phase (Bruner, 1966). Activities were based on two-digit whole numbers only and the duration was 30 minutes.

5th Session: This was the third instructional period of the instructional sequence. This session covered "renaming of subtraction expressions" using the idea of

- equivalent equations and
- base compliment additions

Use was made of numeration cards, diagrammatic and symbolic representations. After using 10 minutes to discuss the enactive and the iconic representations, 20 to 25 minutes was used to discuss the symbolic representations. To help participants to effectively connect phases of instruction, activities were based on two-digit whole numbers only.

6th Session: It was recorded as the fourth instructional period of the intervention. This session was used to revise the activities of the fifth session and also to cover the three stages of the symbolic phase of renaming compound subtractive expressions- i. e. renaming, the crutch and the formal stages. Here, Dienes blocks and algebra tiles were displayed and discussed as alternatives for numeration cards. Activities were based on two-digit whole numbers. The duration was 37 minutes.

7th Session: This was the fifth instructional period. This session was mainly a revision of the sixth session activities. Use was made of word problems and application of base complements. The formal analysis of the third phase of procedural development was stressed. Discussions were based on transfer of learning from two to three and four-digit numerals. Application to numeration systems (bases 2, 3, 4, and 5) was discussed briefly. This was to avoid interference with participants thinking processes, with respect to base 10. (There was no revision for the numeration systems. It was discussed only once).

8th Session: It was recorded as session for the intervention test. This intervention test was the second set of compound subtraction questions given to the participants. The purpose was to find out if the students made sense of the intervention and were able to use the BCA algorithm. The duration was supposed to be 30 minutes but participants used less than 15 minutes to complete the test. (Please, refer to test B in Appendix B).

9th Session: This was the post-intervention test session. This was conducted fourteen days after intervention test. The purpose was to find out if the students were still using BCA algorithm and to determine what method they spontaneously selected after the intervention. According to Martin (1992), a fourteen-day period is enough for students to fall back to their old methods, if the new method had not been understood. (Please, refer to test C in Appendix B).

10th Session: This session was about the post-intervention interviews. The interviews, which were audio-taped, were conducted in the house of participants. A participant was called and interviewed individually, whilst the other was asked to be far away from the scene. The interview took approximately 30 minutes for the first interviewee who was in grade 4 and 40 minuets for the second who was in grade 5. These interviews were conducted purposely to find out whether participant's entry behaviors had undergone some changes. In addition, they were conducted to find out how BCA contributed to their skill development, in terms of accuracy, speed and especially retention. As well, these interviews were conducted to find out what kind of meaning the participants made out of the intervention. On the other hand, the interviews were conducted to find out whether participants were still clinging to the use of the D algorithm for solving compound subtraction problems.

11th Session: This was the final session with participants; and it was recorded as the post-post-intervention tests and interviews period. The researcher conducted the instructions to cover mainly two-digits numbers and tried to help participants to transfer to three and four-digit subtraction. However, the two children were able to use the BCA algorithm to solve subtraction questions with as many as eleven digits. This was first discovered when, in a discussion they were asked to set questions for each other to solve. This ability was clearly pronounced during interviews with the interviewees in the 10th session. This was far beyond the expectations of the researcher; thus these post-post-intervention tests and interviews were conducted, to find out how authentic the participants' claim was (Refer to Appendix B).

Evaluation Instruments and Procedures

Two evaluation instruments were used to assess computational skill in solving subtraction problems offered in a number of forms: horizontal presentation, vertical presentation and word problems. The evaluation instruments (i.e., the computational tests and interviews) were all designed and administered by the researcher. Copies of the tests can be found in Appendix B. The evaluation instruments used in this study included:

Phase I: <u>Pre-intervention Phase</u>
- Pre-intervention test
- Pre-intervention interview

Phase II: <u>Intervention Phase</u>
- Intervention test

Phase III: <u>Post-intervention Phase</u>
- Post-intervention test
- Post-intervention interviews

Phase IV: <u>Post-post-intervention Phase</u>
- Post-post-intervention tests and interviews

Administration of the Tests

The Tests

The objective of the tests was to ascertain the participants' way of operating with natural numbers and their current methods of subtraction that involves renaming.

The study started with a pre-intervention test. The same test items were given to each of the two participants. Though the time limit was 40 minutes the researcher was very flexible with the time set for the test. Students were then given enough time to complete the test. There were 20 test items in this pre-intervention test: 6 involved two-digit subtraction, 3 had the subtraction expressed in the horizontal form, 3 questions involved money and there was one word problem. All questions involved compound subtraction. (Refer to Appendix B.)

The next test to be administered in the study, after the pre-intervention test, was the intervention test. Each of the two participants was given the same test items to do. The time limit for the intervention test was also 40 minutes. Students were assured of enough time even if they were not able to complete the test within the stipulated time. Like the pre-intervention test, the intervention test consisted of 20 items: 6 involved two-digit subtraction, 3 had the subtraction question expressed in the horizontal form, 3 question involved money and there was one word problem. These all involved compound subtraction. (Refer to Appendix B.)

According to the researcher's schedule, the last test to be administered was the post-intervention test. Like the pre-intervention test and the intervention sets, the same test items were given to each of the two participants in the post-intervention test. The time limit was again 40 minutes, but was made very flexible for participants. There were 20 test items, in this test; 6 were on two-digit subtraction, 3 were on subtraction in the horizontal form, 3 on money and one word problem. They were all on compound subtraction. (Refer to Appendix B.)

Post-post-intervention tests were unscheduled according to researcher's initial program of activities. The post-post-intervention tests were administered as a follow-up to the findings of the post-test. Unlike the pre-intervention, intervention and post-intervention tests, the post-post-intervention test items were two different sets of test items. These test items consisted of four, five, six and seven-digit numbers on compound subtraction. They contain more word problems and questions that require investigations and reasoning; they did not, however, require students to explain their reasoning. There were no time limits for participants in these test items. (Refer to Appendix B.)

Analysis of the Tests

As seen above the learning of computational strategies was measured by subtraction computation tests, which required at least an understanding of subtraction. The test items were based primarily on the MATHQUEST (1984) text series. The researcher selected test items from the subtraction computation sections found in the third, fourth and fifth grade textbooks. The researcher assumed these questions were at an appropriate level given they were found in the text resource the students were using in their mathematics classes.

The researcher's test consisted of 20 items involving whole numbers, problems on money and some word problems. Items were presented in both vertical and horizontal forms. The money items were all in the vertical formation. Out of the 20 test items, 4 were in horizontal form, 15 in the vertical form and a word problem. There were 6 two-digit number items, 9 three-digit number items and 5 four-digit items. All the problems were written in large print to allow for the crutch of either algorithm. Though 40 minutes was indicated on the test sheet, participants were told not to mind it, but to finish their work at their own rate. Matchsticks, bottle tops numeration cards, straws etc. were put before participants to use when necessary (during the tests).

The pre-intervention, intervention and the post-intervention tests were identical for each has the same number of items for two, three and four digits, vertical and horizontal digits, money and word problems. The same set of questions was given to each participant for the pre-intervention, intervention and the post-intervention tests. The post-post intervention tests were different, but parallel forms. Each set had two sections; the first section comprises of one example each on three, four, five, six and ten-digit numbers. The second section had two aspects, the first aspect was a set of questions based on the ability of participants to determine which question is wrong/right and to show the part(s) which is/are wrong. The second aspect was based on a value on a paycheck. It required participants to find out if their expected value on the paycheck was right or wrong. It asked them to write down how they were able to determine their expectations and the method used. It also asked participants to write down reasons for choosing a method.

The Interviews

According to Piaget (discussed in Ginsburg, 1981), an interview may have three purposes: discovery of cognitive activities, the identification of cognitive activities and the evaluation of competence levels. The first purpose enables the researcher to elucidate students' error patterns and ways of thinking during instances of problem solving. The second purpose enables the researcher to investigate more completely patterns of behavior in a variety of problem solving situations. The final purpose enables the researcher to elucidate the highest mathematical level at which the subjects are able to perform. An interview may focus on any one or more of these purposes in a given interview or a series of related interviews. In this study, it was necessary to know the entry behavior of participants and then learn about changes that had occurred during and after the intervention. Thus a tool was necessary for the investigation of how students do subtraction that involves renaming. Hence interviews were conducted for this purpose. The interviews enabled the researcher to elucidate students' error patterns and ways of thinking during instances of problem solving in compound subtraction. In addition, they were used so that the researcher could investigate more completely patterns of behavior, since entry behavior was expected to change at the end of the intervention.

It must be recalled that one of the initial concerns a researcher faces is a concern for the reliability and validity of the emerging data. Swenson et al. (1981) argue that: (a) verbal data have a place in cognitive research, (b) there are no important limits and constraints on their use: the level of questioning imposed upon subjects must be restricted to that at which answers can reasonably be provided, for example, subjects cannot reasonably be asked to comment on their neural functioning, (c) effective use of verbal data requires paying careful attention to the limit of constraints, (d) provided this is done, any of the remaining problems using verbal reflections are the same as those which apply to traditional research methods. From this perspective the authors agree that interview is a viable research methodology (Swenson, 1981).

The interviews were conducted over a period of 8 weeks. Although it was important to keep the structure of the interview open-ended in order to accommodate participants' reflection and prioritization of issues, these questions were also developed in order to focus on the participants' thoughts. The following questions formed the backbone of the interviews:

1. What is your personal experience regarding the use of subtraction? Can you please describe your experience both at school and home?

2. What can you tell me about the use of technology (calculators and computers) in the learning of subtraction?

3. i. Here is a question 85 − 37 a. Will you do it for me? b. Can you tell me why you did…?

   ii. Here is another question 90 − 48 Can you use your strategy?

4. Students use different strategies in calculating subtraction of all kinds. Could you explain the strategies you are using?

5. Which strategy, in calculating subtraction, are you most comfortable with? Why?

6. Do you feel that one method can promote arithmetic better than another can?

7. Do you feel you have a choice in choosing your subtractive algorithm?

8. In your opinion does the learning of both your own methods and that of teacher increase your arithmetic competence? Why?

9. If some body does this work:

$$\begin{array}{r} 9024 \\ -\underline{4034} \\ \underline{4090} \end{array}$$

   Is the person right or wrong? What makes it right or wrong?

10. Is this correct or wrong?

$$\begin{array}{r} 4003 \\ -\underline{2007} \\ \underline{1006} \end{array}$$

   How did you check your answer?

The interviews were taped and transcribed for analysis. The final clinical interview probed the participants' current knowledge of compound subtraction, after receiving the intervention. A complete interview took approximately 40 minutes.

Analysis of the Interviews

The purpose of the interviews in this study was to determine the effects of the BCA on participants' computational skills. The interviews were conducted at each stage to assess strategies used for subtraction. Interviews were conducted at the first stage to find out the entry behavior of participants with respect to subtraction strategies they already used. But the interviews at the post-intervention phase were conducted to assess whether participants have understood and could use the novel algorithm. As well the interviews were conducted to find out if participants were sticking to their old algorithm, the D, and if so, why? On the interview days, each student was called and interviewed one at a time. The two were not paired since the researcher wanted to know the thinking of each child with respect to the new approach to solving compound subtraction. The interview times were pre-arranged with the parents and the interviews were not conducted during regular school hours, but in the afternoons after participants had finished school. The interviews ranged from a length of forty to forty-five minutes. The interviewer went to participants' residence and walked with participants to their big hall, the location of interviews. The location maximized privacy and minimized external distractions.

The interviews were made as open-ended as possible. Ginsburg (1981) says interviews must be unstructured and open-ended to give the child the opportunity to display his natural 'inclination' (p. 6). The interviews, of this study, were task-oriented and flexible. The interviewer sometimes followed and pursued the student's thinking, asking questions until the student's reasons for some responses were understandable to the researcher. The direction of the interviews was, therefore, largely based upon the responses given by the subject. In this way, the interviews were unstructured and open-ended. The interviews began by asking participants to provide their full names. The participants were informed that the researcher is not interested in their answers being correct or wrong but how they understand and do subtraction work. They were informed that the interviews will be audio-taped using a small recorder so that the researcher could remember exactly what the participants had said/done when solving the problems. Each question was presented to the participant one at a time and each question was read aloud by the researcher. Participants were given several pieces of manipulatives to help them. But the participants, generally, did not use the manipulatives during the interviews.

Summary

Pre-intervention, intervention and post intervention interviews were used to collect data. In addition interviews were conducted, which were audio-taped, in each phase to investigate learners' thinking about the subject matter. Students' working papers were also collected and analyzed. In the pre-intervention session a set of subtraction questions was given to participants to try. The purpose was to observe individual participants' various strategies and then follow it by interviews to find out how they approach subtraction that involves renaming. After this intervention, a second set of questions was given to participants to do, to find out if the participants made sense of the researcher's intervention. The post-intervention came a week after the intervention. In this session participants were given a third set of questions, on subtraction. After the test they were interviewed again. The post-post intervention test was unscheduled and comprised of two, different but parallel, sets of questions and interviews.

After completion of all the interviews the tapes were transcribed and analyzed as follows:

a. All the transcripts were read and re-read.

b. All the questions and their corresponding answers were numbered for each set of interviews.

c. Themes arising out of the data were written down.

d. Now the transcripts were slowly read and the numerals pertaining to any particular theme were written under each theme

e. The process explained in d. (above) was used for each of the three interviews.

f. After reading and distributing the numbers to each respective theme, the computer was used to select the items to their respective themes.

g. After the process of f. (above), the themes were carefully read and some were merged and the transcribed items that appeared more than once were put under appropriate themes.

CHAPTER 4 - REPORTS ON DEVELOPMENT OF THE INTERVENTION

Use of Numeration Cards

Numeration cards were made and named by the researcher for use in the intervention. They were mainly used for developing meaningful understanding of place values, base complements, compound subtraction expressions and transformation of compound subtraction expressions. The numeration cards were displayed and named as: small, long and flat cards.

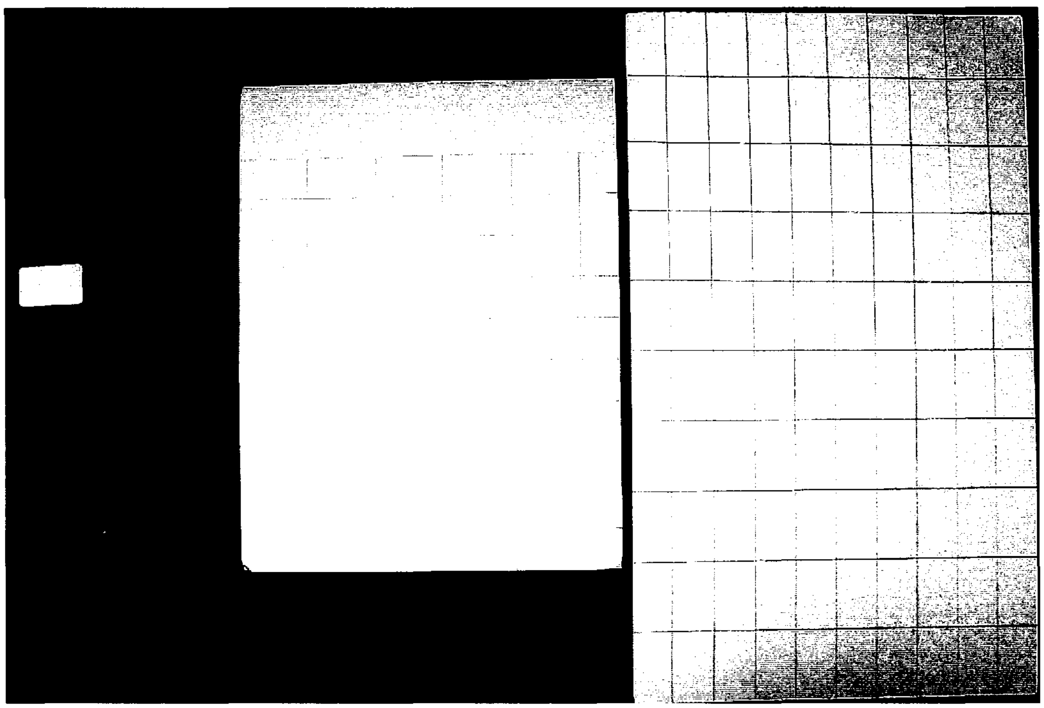


Figure 4 -1. Display of numeration cards as small, long and flats (from left to right)

The numeration cards work in the following way:

- Ten smalls (ones) are equivalent to one long. Thus 10 smalls (ones) could be exchanged for 1 long.
- Five longs are equivalent to one flat (first flat). The first flat has an area for 50 (fifty) to be written. But it was left empty (this was on top of the card) for students to find out for themselves.
- Ten longs are equivalent to one flat (the second flat). Ten longs could be exchanged for one flat. Thus it contains 100 smalls (ones).
- The cards are used to model numerals on the flats as well as modeling subtraction by placing and taking away smalls (ones) and/or longs, on the flats.

Place Value

Participants were asked to play with the cards, for a while to satisfy their curiosities. After playing with the cards participants were asked to find out the number of small ones that will make a long and the number of longs that will make a flat.

Participants did not waste time in finding out that 10 small equal 1 long and 10 longs equal 1 flat. To help participants review place value, the following arrangements were made and participants were asked to name them.

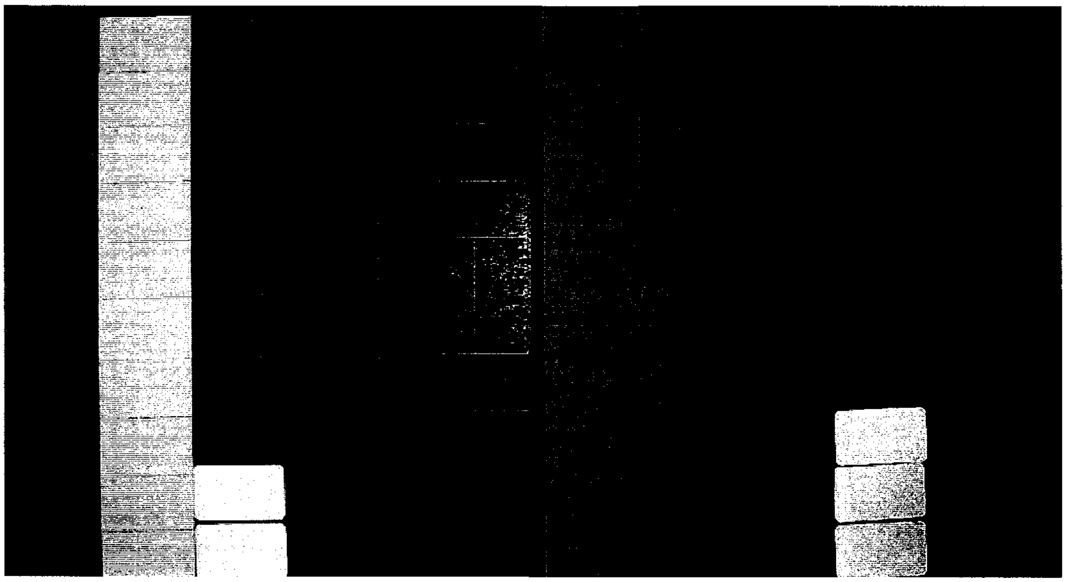


Figure 4-2. Numeration cards arrangements for 22 (left) and 33 (right)

On the left side of Figure 4-2 two tens, 20 smalls (ones), are placed from bottom-left to the top, to cover the first two columns, and two smalls (ones) are placed after them on a flat. The number of columns and small spaces covered on the flat indicate the idea of place value for tens and ones respectively. Subtraction, here, is modeled by taking smalls away, starting from the bottom-left. On the right side in Figure 4-2 three longs are placed from left to right to cover the first three columns and three smalls (ones) placed after them, on a flat. To take away more than 3 small ones, some of the longs must be exchanged for smalls (ones) to allow for subtraction of the ones.

Students were asked to exchange 10 ones for 1 long, which they easily did. Since participants had already studied the concept of place value they easily answered questions of the form: How many tens are in this mathematical expression? How many ones are in this mathematical expression? How many small ones are left to fill this long?

Base Complements

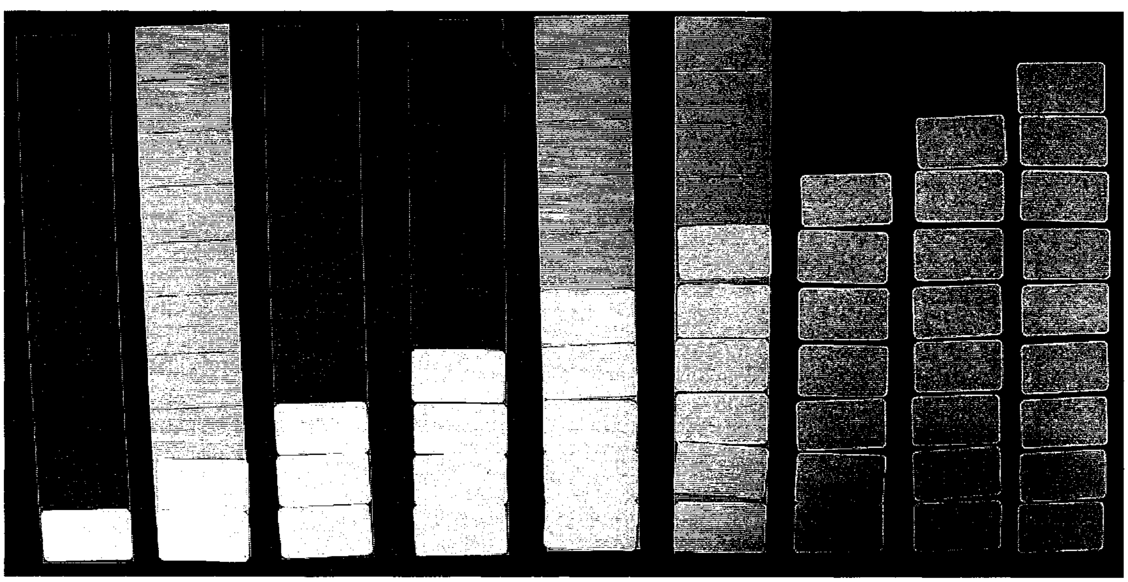
To help participants understand base complement and be able to apply it in the latter stage of the intervention, the researcher went through the following with the participants.

a. How many small ones can fill in figure 4-3?

Figure 4-3. Numeration cards arrangement to show base complement of one

b. What is the base complement of 1 (one)? (Referring to figure 4-3 above)

Participants were able to look at the empty spaces, count them and say the right answers. But for clarification they were asked to fill the long with small ones. The statements for a. and b. (above) were repeated for the following:



1 2 3 4 5 6 7 8 9

Figure 4-4. Numeration cards arrangements to show base complements of 1, 2, 3, , 9

The iconic representation of numbers 4, 5, … 9 above were not covered, for participants showed good understanding of the base complements. Thus, iconic transformation of 1, 2, 3 to their symbolic representation only, of the figure 4-4 (above, numbered 1, 2, 3, …9), were covered.

Participants were asked to draw a "long", show one "small" on it and write the base complement besides it.

Figure 4-5. Grace's work showing base complement of 1 is 9 (left) and Rubin's work showing base complement of 1 is 9 (right)

Participants were asked to draw two "longs"; put, on one long, two (2) small; and the other, three (3) small ones. After that they were told to write the base complements beside each.

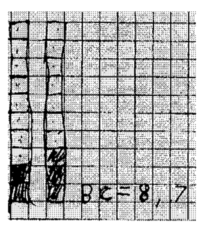


Figure 4-6. Grace's work showing base complement of 2 and 3 are 8 and 7 respectively

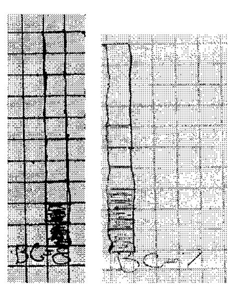


Figure 4-7. Rubin's work showing base complement of 2 and 3 are 8 and 7 respectively

After the above, participants were asked to fill in a table to show the base complements of the numbers from one to ten (Figure 4-8).

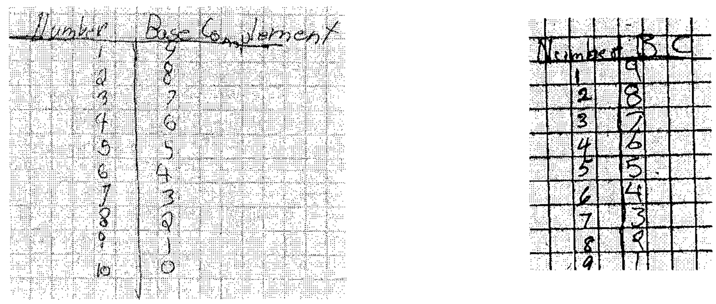


Figure 4-8. Grace's work: Digits and the corresponding base complements (left); and Rubin's work: Digits and the corresponding base complements (right)

Modeling Subtraction

To effectively model subtraction with numeration cards the researcher symbolically discussed simple and compound subtraction and the idea of equivalent equations (by way of equivalent zero addition- a form of balancing equations).

Simple and Compound Subtraction

The researcher wrote down the following and had a discussion about them with the participants. They were asked to look at the following and tell the difference in terms of difficulties when working them.

1. 9–2   2. 7–4   3. $\begin{array}{r} 24 \\ \underline{-\ 16} \end{array}$ 4. $\begin{array}{r} 412 \\ \underline{-\ 276} \end{array}$

From the discussion, the participants made it clear that items 1 and 2 are less difficult than 3 and 4. For the latter involves borrowing. At this point the researcher explained "borrowing" as a method for renaming such expressions as 3 and 4 (above). Thus 3 and 4 involve renaming and therefore are "compound subtraction expressions." 1 and 2 however, do not involve renaming and thus are called "simple subtraction expressions." Several subtraction expressions were written for participants to identify as simple or compound; and this posed no problem to participants.

Simple and Compound Subtraction Expressed in a Vertical Form

The researcher wrote down the following and asked participants to identify which subtraction expressed vertically (i. e., place-value subtraction) is simple and which is compound.

a. 583 b. 647
-327 - 456

Participants find no problem in saying that:

1. In a) subtraction at "ones column" is compound; at "tens" is simple and at "hundreds" is simple.

2. In b) subtraction at "ones" is simple; at "tens" is compound and "hundreds" is simple.

Several discussions, of this type, went on to help participants identify simple and compound subtraction in vertical arrangements of numbers.

Equivalent Subtraction Expressions

Equivalent zeros were introduced as "balancing subtractive expressions". Participants were told that if the same number is added to both the minuend and the subtrahend the answer remains unchanged. However, the expression is changed (transformed) to a more difficult or less difficult one depending upon the equivalent zero used. To help participants understand this a set of equivalent expressions was written down for discussion.

21 [22 23 24 25 26 27 28 121]
-17 [-18 -19 -20 -21 -22 -23 -24 … - 117]

Figure 4-9. A set of equivalent expressions

For discussion, participants were told to look at each expression and say the answer for each case, and then also say what has brought the differences in expressions and which expression is easiest to work. Participants were able to use borrowing to get the answer 4 for all. They could not, however, easily see that equal numbers were added to both minuend and subtrahend for each case. Thus the researcher took the expressions one at a time and discussed the difference (of each expression) with respect to the original expression. After participants understood it, they were asked to pick, from among the equivalent expressions, the one that looks easiest to calculate. Here they were asked to do independent work. They both selected the third equivalent expression: 24
-20.

This led to the introduction of BCA as a method for transforming a compound subtraction for easier computation of subtraction of multi-digit whole numbers.

Numeration: Using the Cards

The researcher asked the participants to play with the cards for a while (and they played their own game for some time). After that they were asked to show 21 and 32 with the cards. The participants were a bit confused; they did not know how to arrange the cards- whether from right to left and from top to bottom or otherwise. Grace started from left to right and bottom to mid-way to the top, whereas Rubin started from the middle top to right and filled to bottom. The researcher then explained that in our traditional writing we start from left to right and top to bottom. We are starting from left to right, as usual, but from bottom to the topmost part (as conventional method for our arrangement of numeration cards). After the explanation participants easily arranged the cards as done below:

Figure 4-10. Numeration cards arrangements for 21(left) and 32 (right)

After the above participants were asked to show 22, 34 and 45. This was no big deal for them.

Subtraction: Using Numeration Cards

The researcher referred participants to figure 4-10 above and asked them to take away 17 cards from each. Interestingly, they each took away 17 cards the same way; they both started from the right leaving in the first instance 4 single cards on the bottom left hand corner and in the second instance 15 cards, 10 in the first column on the left and 5 more in the next column. However, this is not useful for this model since the intent is to leave some evidence of what the original question was after taking away cards.

At this point the researcher reminded the students that since the arrangement of cards starts from the bottom left, taking away the cards must also start the same way.

After this rule was told to the participants they were able to model the subtraction as done below.

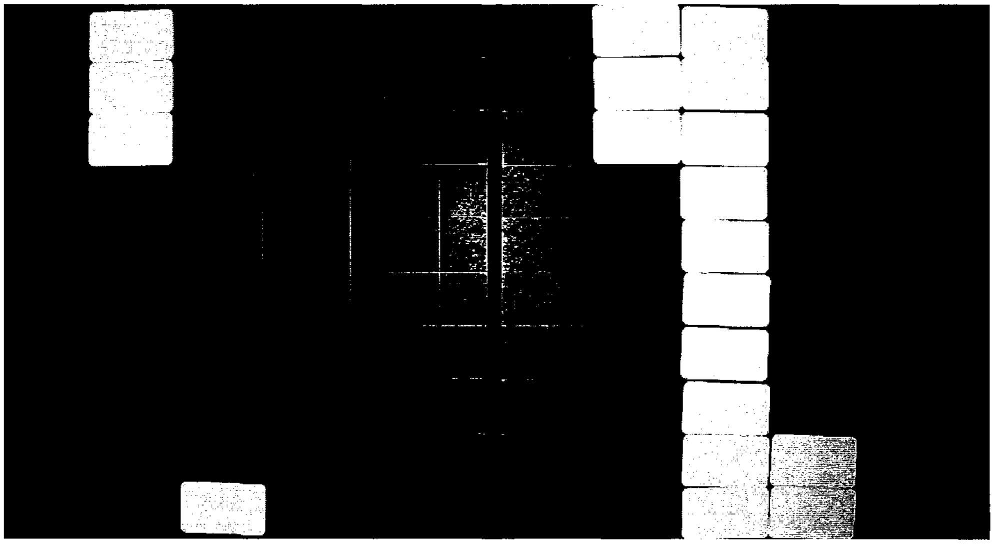


Figure 4 —11. Subtraction models for 21-17 (left) and 32-17 (right)

Many examples were given to participants to do. Some of the expressions included: Use the numeration cards to represent each of the following and What subtraction expression does each of the following represents?

After several examples participants were asked to write symbolic expressions beside each numeration arrangement as done below:

21-17 32-17

Figure 4-11 i. Subtraction models and their symbols written on pieces of paper. Twenty-one take away seventeen (left) and thirty-two take away seventeen (right).

Transformation of Compound Subtraction to Simple Subtraction: Application of the Base Complement, of the Subtrahend

For meaningful transformation of compound subtraction to simple subtraction, the researcher used numeration cards and involved participants in discussions. For continuity, the researcher used figure 4-11 i (above). The discussion went on as described below:

Students were given the two models, in Figure. 4-11 i and asked which aspects show 21 and 32, and which show 17. This appeared to be of no problem for the participants to show. But the more difficult aspect was adding the same value to transform subtractive expressions (Refer to discussion below).

Peter: Do you remember which expression from among the set of equivalent expressions, we said, is easier to work with?
Rubin: When there is zero at the end of number we are subtracting.
Grace: [joining in] when there is zero
Peter: What was added to both minuends and subtrahends? (referring to the set of equivalent subtractive expressions).

Grace forgot it but Rubin easily remembered that the base complement must be added to both sides. After accepting what Rubin said participants were asked to demonstrate it using the numeration cards. But this was not done properly as shown below:

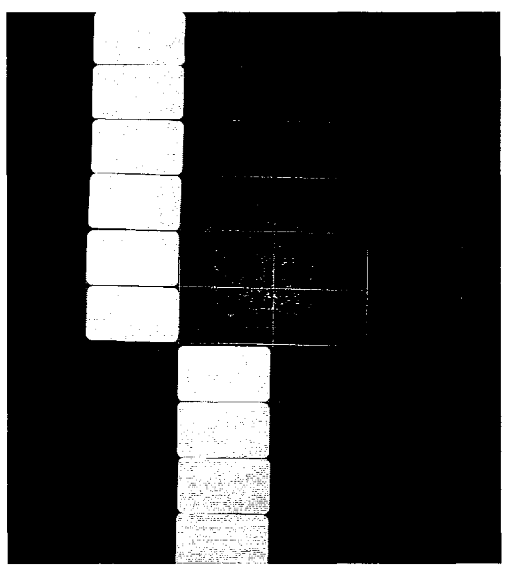


Figure 4-12. Participants model showing 24-14 (instead of 24-20)

As shown each participant added 3 cards to each side. But this does not really demonstrate equivalent zero + 3 - 3. They rather showed 24-14 which does not have the same answer as the original expression 21-17 . At this point the researcher explained what is meant by adding to a minuend and a subtrahend. Using the expression 21 — 17 again the explanation of the procedure went on, symbolically, as follows:

Consider: 21
-17

21 is the minuend; adding 3 to it increases it to 24. 17 is the subtrahend; adding 3 to it, to read 20 (minus 20) means three (3) more have been taken away. Though we see it to be added to the number more are rather taken away. The implications are that we added 3 and have taken 3 back. Thus the original expression is balanced. Hence the expressions i and ii (below) are also equivalent and thus have the same answer.

i. 32 ii. 35
-17 -20

With the explanation above participants were able to correctly transform the expressions, 21-17 and 32 — 17, using numeration cards as done below

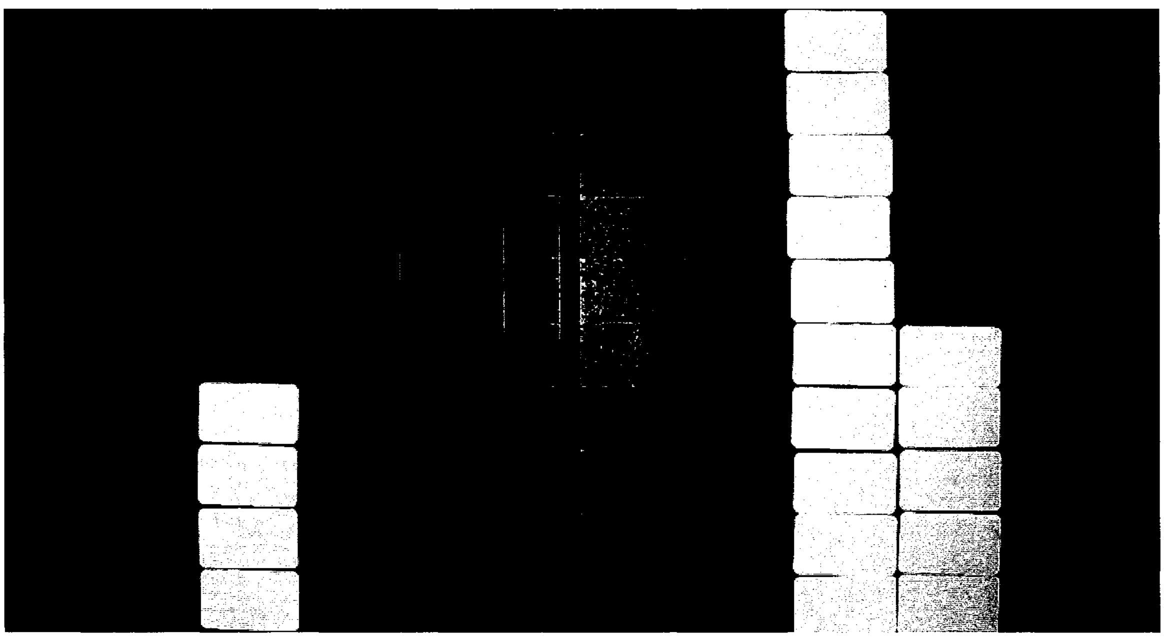


Figure 4-13. Transformation of 21-17 to 24-20 (left); and 32-17 to 35-20 (right)

Note, to identify the compound subtraction that was transformed, one of the following is required: the base complement used for the transformation, the minuend before transformation or the subtrahend before transformation.

The question of Which of the two arrangements, i and ii above, is easier to tell the answer was answered by Grace. Grace after arranging for the transformation said, I can look at these and tell the answer more quickly . The above statement of Grace brought relief to the researcher; and several examples were given to the participants, which covered all phases of instructional representations.

Iconic Representations

Each lesson was introduced with iconic representations of the previous enactive-based lesson. Thus iconic representations became recapitulations of the preceding lessons. Most of the homework was presented in iconic form but participants were asked to present their answers in symbols. For iconic representations refer to appendix C. It must be recalled that the illustrative works in the lessons were mostly done using two- and three-digit numeration. Participants were, however, able to transfer to multi-digit after they had meaningfully developed the BCA skill.

Renaming Subtraction Expressions

The target task of using BCA is to operate at the formal stage of the symbolic representation, during subtractive computation. Students will eventually have to think for example at the right:

"1-7 is compound"	21
"Base complement of 7 is 3, 3 plus 1 is 4"	-<u>17</u>
"1 increases by 1. Now 2; 2 take away 2 is 0"	
"Write 0 and 4 in their proper places as 04 or 4"	

Initially participants were not taught to use this verbal pattern. Rather, they were guided to use a series of steps in their thinking:

1. a. 21   +3   b. 24
   -<u>17</u>   —- 3—→   -<u>20</u>

This step can be thought of as *renaming the subtraction expression*

The rationale is as follows: "If the subtraction is compound, add the base complement of the subtrahend to both minuend and subtrahend in order to transform the expression from compound to simple". That is, we add 3 to 21 to give us 24; and add the 3 to 17 to give us 20- in order to "balance account". Hence expression 1a is transformed to 1b (above).

Subtraction procedures of this kind require that pupils learn a new and very important mathematical *concept* and some *pre-requisites.* The new concept is "balancing". That is, the difference between two numbers is not altered if the same amount is added to both minuend and subtrahend. Thus the answer for 1a and 1b (above) is 04 in each case. The pre-requisites, already discussed, include place value, simple addition and subtraction of whole numbers, equivalent zeros and base complements (see page 94).

Insert below (figure 4-14 and figure 4-15) are the works of Grace and Rubin respectively on compound subtraction transformed to simple subtraction.

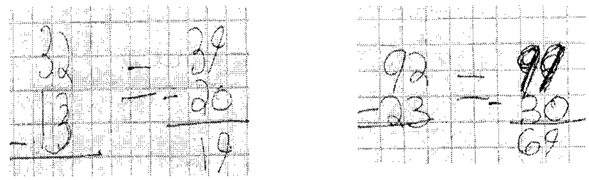


Figure 4-14. Grace's work illustrating transformation of compound to simple subtraction

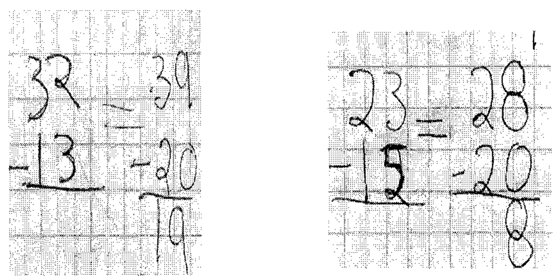


Figure 4-15. Rubin's work illustrating transformation of compound to simple subtraction

The renaming in BCA wastes time as compared to its formal application. However, for meaningful computation of multi-digit subtraction, this is a necessary pre-requisite. Participants had already had some understanding of "place value" and could solve "simple addition and simple subtraction" problems. On account of that they understood the renaming without any problem. They did all exercises correctly, including the seatwork and the homework.

The Base Complement Additions Crutch (The BCA crutch)

After much work on the renaming of subtraction expressions, participants were introduced to the use of "BCA crutch". The BCA crutch is a bit different from "D crutch" or "Crutch for borrowing". Borrowing renames the minuend only. Thus in D the crutch is only used in the minuend. BCA renames both minuend and subtrahend thus BCA crutch is applied to both minuend and subtrahend (see figure 4-16, below).

1 ~~2~~13 -1 6	*(The D crutch)*	7 2 ~~3~~ -~~1 6~~ 2 0 or 2 7 ~~23~~ -~~16~~ -2 0	*(The BCA crutch)*

Figure 4-16. Renaming compound subtraction by way of crutches

From renaming to crutch was illustrated as follows:

i. 21 +3 ii. 24
- 17 -3 → - 20 *(renaming)*

2 4
iii. 21
- 17 *(BCA crutch)*
2 0

Figure 4-17. Transformation process of compound subtraction expression: i. the original expression; ii. the renamed expression and iii. the BCA crutch

At first the experimenter thought this would be a problem to participants, since it was their first time of doing such work. But it was not the case; they later did the work as expected of them, by the researcher. Figure 4-18 and figure 4-19 are some exercises done by Grace and Rubin on BCA crutch.

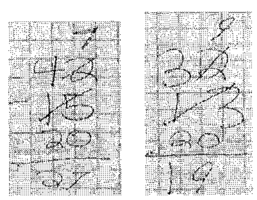


Figure 4-18. Grace's work on BCA crutch

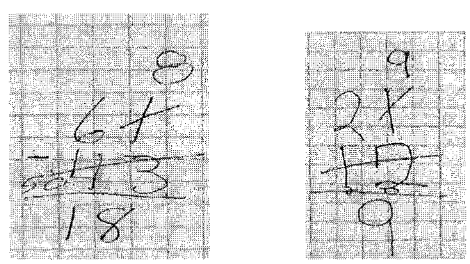


Figure 4-19. Rubin's work on BCA crutch

The crutch taught in connection with the BCA procedure apparently served useful ends. The participants during the course of the instruction used the crutch in almost all their written work. Effort to encourage them not to use the crutch was not very successful initially, for they continue to use the renaming and the crutch. This was particularly true for Grace.

The Formal Representation

The practice of using the renaming and the BCA crutch went on for some time. Immediately after they had been shown the shorter ways, the formal representation, there was a pronounced drop in the frequency of the use of the crutch, by Rubin. However, Grace continued to use the renaming (especially) and the crutch. At the formal stage participants have to note and make some observations from the renaming and/or the crutch expressions as follows:

At the formal stage of base complement additions (two-digit numbers)

i. We consider two numbers at a time.

ii. We rename the compound subtraction using base complement of the subtrahend.

Observations

Minuend and subtrahend increase or change by the value of the base complement i.e.
i. *minuend:* increases by the base complement of the subtrahend

ii. *subtrahend:* 1. increases by the value of base complement; 2. ends with zero;
3. next number (on the left, the higher place value) increases by one.

iii. When base complement is not applied the next number, on the left, in the subtrahend, does not increase.

For example:
1. Given two-digit numbers

a. 72 − 36 ⟶ b. 76 − 40 *(renaming)*

c. 76 over ~~72~~; − ~~36~~ with 40 below *(BCA crutch)*

2. a. 91 − 58 ⟶ b. 93 − 60 *(renaming)*

c. 93 over ~~91~~; − ~~58~~ with 60 below *(BCA crutch)*

3. 2 7
 - 1 6 *(7- 6 is simple; no transformation)*

Extending BCA to three digit-numbers

i. We consider two numbers at a time.
If three-digit numbers, then note that we have two types: the first is renaming once and the second is renaming twice.

ii. We rename using base complement of the subtrahend

Observations

Minuend and subtrahend increase or change by the value of the base complement i.e.

h. *minuend:* increases by the base complement of the subtrahend

ii. *subtrahend:* 1 increases by value of base complement; 2 ends with zero; 3. next number (on the left, the higher place value) increases by one.

iii. When base complement is not applied the next number, on the left, the higher place value in the subtrahend, does not increase.

Renaming once
Here either the ones column or the tens column is compound.
E.g.
1. 647 – 456 which gives us

647
456 ....*(Three digits, renaming once; for 4-5 is compound subtraction)*

Here we have

a. 647 → b. 697
 - 456 → - 506 *(renaming)*

  69
c. 6~~4~~7
 - ~~4~~5 6 *(BCA crutch)*
  5 0

2. 583- 327 which gives us

583
 - 327 *(Three digits, renaming once; for only 3-7 is compound subtraction)*

a. 583 ⟶ b. 586
- <u>327</u>   - <u>330</u>   *(renaming)*

6
c. 5 8 ~~3~~
- 3 ~~2~~ ~~7~~   *(BCA crutch)*
30

<u>Renaming twice</u>
Here both ones and tens columns are compound; or after renaming the ones column the tens column, which was simple subtraction, changes to become compound.
E.g.
1. 600 - 512 which is

600
<u>-512</u> .....*(Three digits, renaming twice; for 0-1 and 0-2 are all compound subtraction)*

a. 600 ⟶ b. 608 ⟶ c. 688
-<u>512</u>   - <u>520</u>   - <u>600</u>   *(renaming)*

8 8
d. 6 ~~0~~ ~~0~~
-~~5~~ 1 ~~2~~
~~2~~ 0   *(BCA crutch)*
6 0

2. 700 – 403 which is

700
- <u>403</u>   *(Three digits, renaming twice; for 0-0 was simple and 0-3 was compound but after renaming the ones column the second column changes to become compound subtraction)*

a. 700 ⟶ b. 707 ⟶ c. 797
- <u>403</u>   - <u>410</u>   - <u>500</u>   *(renaming)*

9 7
d. 7 ~~0~~ ~~0~~
- ~~4~~ ~~0~~ ~~3~~
~~1~~ ~~0~~   *(BCA crutch)*
5 0

It is important to consider the units and the last two numbers (first), if the units are of compound. Then the tens and the hundred columns follow in the computation of three-digit numbers (This, from the researcher's observation was a bit confusing to participants. However, they were able to transfer their knowledge of BCA, from two- and three-digit numbers, to all multi-digit subtractions; and it works.)

As already mentioned, Grace initially did not easily connect the observations to the final step though she tells everything as expected of her (Please refer to excerpt 1 below).

Excerpt 1

(P: Peter G: Grace)
P: What observation do you make?
G: We added base complement to 72 to make 76 and 36 to make 40.
P: What else?
G: 36 becomes 40.
P: Is that all?
G: (Kept Quiet)
P: Consider the 2 (in 72), the BC of 6 (in the 76) and the answer 6, with respect to addition.
G: BC added to 2 to give 6.
P: Good, note that BC added to the 2 to give the answer at that side. What else?
G: The 3 becomes 4.
P: Very good. Now do this, using renaming and crutch.

$$\begin{array}{r} 63 \\ -\underline{25} \end{array}$$

Grace does it as shown in figure 4-20.

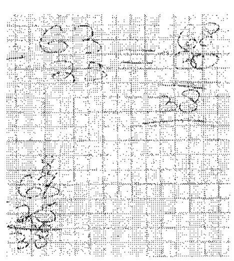


Figure 4-20. Grace's subtraction process using the renaming and the crutch

P: What observation do you make?
G: Like before?
P: Can you do the work without the renaming and the crutch?
G: No.

A lot of discussion went on but Grace could not complete the task without the renaming or the crutch. However, Rubin understood it quickly and set three examples on his own. He did the work in the formal stage of the symbolic representation (see insert below).

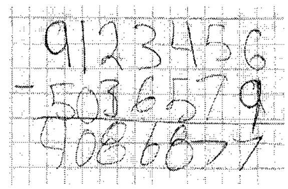


Figure 4-21. The work of Rubin: Question set by himself

Grace's difficulty in moving from the renaming/crutch to formal stage leads the researcher to admit that "crutches of any kind are usually opposed on the ground that: i. they are themselves difficult to learn, sometimes being more difficult than the processes they are expected to simplify; ii. once learned, they actually contribute little to efficiency; and iii. once learned they become permanent parts of the total process; that is they are seldom discarded" (Brownell & Moser, 1949, p. 65). Despite all odds, the crutches are an integral part of skill-development procedures. Hence it is difficult to isolate their influence from the procedures as a whole. The crutches give vivid pictures of the transformation and procedural steps between the renaming and the formal stage of the BCA. However, participants were advised to discard it and use the formal stage which was the target task. Eventually, this target task of the formal stage of the symbolic representation was hit successfully by way of subliming instructional procedure from the enactive to the symbolic

The Sublimation of Instructional Procedures (Enactive-symbolic Procedure)

Transformation of compound subtraction to simple subtraction was revisited using manipulatives (refer to page 55-57 above). Participants were asked to compare expressions i. and ii. (attention was more on Grace, this time). Please refer to the excerpt 2 below.

Excerpt 2

(Peter & Grace)

P: Let us consider i. (refer to page 56). How do you describe it?
G: We have 32 take away 17.
P: Right. Can you make any observation with respect to tens and ones of both minuend and subtrahend?
G: We have three tens and two ones; we have taken one ten and seven ones.
P: Okay, we want to read the answer faster so let us make adjustments. We will take the base complement cards and in order not to change the answer we will put them back, but on the minuend ones. Is ii. describing what I am saying?
G: Exactly.
P: Will you read ii.?
G: 35 take away 20
P: Now let's make comparisons of the two. First the minuends (what was already there) and then the subtrahends (what was taken away).

G: 32, now 35. Increased by 3.
P: Where from the 3?
G: The rest of 7.
P: That is the base complement of 7. Good. What about the side taken away?
G: It was 17 but now 20.
P: Now compare the two rods for subtraction (the part taken away). Our attention now is on the number of ten rods.
G: 1 ten now 2 tens.
P: Is there an increase?
G: Yeah. Increased by one.
P: Now go through all what we have done and see whether what we are saying works.
G: Okay.

At this point she set her own questions and answered them without using the renaming or the crutch. She then began to sing a song; the context of which was not known to the researcher. She continued to sing and dance till their parents came back home, from work.

CHAPTER 5 - RESULTS OF THE STUDY

The two students Grace and Rubin, came to the intervention with some knowledge related to simple addition and subtraction which was useful because the BCA algorithm requires such knowledge as prerequisite. However, neither of them said that math was their best subject. Three main categories of observations arising out of the development of the intervention, the tests, and the interviews regarding participants' knowledge of subtraction were made: (a) observations before the intervention (b) observations during the intervention and (c) observations after the intervention. The purpose of this chapter is to report on the results of the research by way of the key observations of participants' behavior before, during and after intervention.

Observations Before the Intervention

Students Had Some Understanding of Place Value

During the development of the intervention students were found to be using the D crutch. When questioned about the use of the crutch Rubin explained that "10 ones is a ten, 10 tens is a hundred, and 10 hundreds is a thousand; so if you cannot subtract you can go to the next place which is ten and borrow one ten which equals 10 ones." This explanation indicates Rubin has some understanding of the concept of place value. To build on the participants' knowledge of place value, in the intervention, numeration cards were used for instruction of place values, base complements, compound subtraction expression and transformation of compound subtraction to simple subtraction (Please refer to chapter 4). The development of the instructional sequence went on in the direction as suggested by Bruner (1964) i.e. enactive, iconic and symbolic phases. Enactive phase is the concrete representation level. At this stage, students manipulated materials directly, modeling various numbers with single cards, longs and flats. At the iconic phase the students took a step away from the concrete materials and physical actions to mental images. In this phase the child pictures an operation, such as subtraction, by remembering his actions in the enactive phase and recreating those actions with pictures or other images (Refer to Appendix C). Once the child learns to think about his performance in terms of the symbols then we note he is acting symbolically.

Since participants had already formed a concept of place value exchanging 1 ten for 10 ones, 1 hundred for 10 tens was not a problem. They were able to form representations of various amounts with the cards as directed by the researcher, without many problems. Their initial difficulties were not with place value per se but rather on the order of arrangement of the cards. The students were initially confused as to how to start to arrange the cards on the flats. But after telling them of each column as a ten and the fact that arrangements ought to start from the left to the right, the numeration continued smoothly. Participants were able to show 21, 32, 23, 72, etc. therefore some of these examples were used to advance the instruction.

For BCA algorithm, an understanding of place value is necessary, but no extensive use is made of it, during computation directly. When the next higher place value of the subtrahend is 9, application of base complement makes it 10– even here the rule of place value is adhered to (That is, ten numbers at a place is equal to one at the next higher place and vise versa). A typical example is "732-195". Here the transformed expression, after applying BCA, is "737-200". This was where I anticipated some problems. But as indicated above participants were able to transfer their learning situation successfully without my intervention in that situation.

Participants Had Some Informal Knowledge Related to Simple Subtraction

In the first interview session with Rubin, when he was asked to do 9-6 he started with "I will take 9 away from 6 and that makes 3 left". Though there was a mistake in his language, the first expression indicates a take away model (separation from whole), as did his second response. When asked whether that is all, he continued, "You could also put 9 and to this you can take away 6, and that will make 3" using his fingers (Refer to excerpt 3, lines 14 and 16). When the researcher mentioned yet another strategy, counting on from 6 to 9, Rubin indicated he knew that method, too (Excerpt 3, lines 23 - 24). Rubin also indicated he has seen a counting down strategy, as well (Excerpt 3, lines 27 - 28). When Grace was asked to do 8-5 she made a mental computation. She figured out what number had to be added to the 5 to give 8 (Excerpt 3, line 187). She might be using "counting on" or she might have recalled a number fact. Neither of the students demonstrated more than one strategy; they only acknowledged their knowing of other strategies when the researcher raised them.

BCA is based principally on compensation and use is made of simple addition and simple subtraction. With simple addition students could even use any of their traditional additive strategies, since operation is based on the digits 1, 2, 3, …9, and sometimes on 0 and 10. The base complement plus the minuend is so easy that a pupil could use counting on, counting all, counting up to and many others which are all simple for pupils to handle with or without manipulatives. The excerpt below informs us that students may use one particular strategy but have others that they may not often use unless prompted (Refer to Excerpt 3).

Excerpt 3

(Peter & Rubin)	(Peter & Grace)
Interview # 1	Interview # 1
11. P: Oh that is good- you have good memories. E e mm how many ways can you work this? 9-6	186. P: How will you do this work? 8-5
12. R: E-e…	187. G: I can now think in my head. How umm you try to figure out a number that you have plus the 5 that equals to 8.
13. P: Can I see how you do it?	
14. R: I will take 9 away from 6 and that makes 3 left.	188. P: Okay.

15. P: Okay. Is that the only way? We have got other ways. 16. R: You could also put 9 and to this you can take away 6, and that will make 3. 17. P: Okay. And so you mean you can use your fingers? 18. R: Yeah. 19. P: Then you count 9 using your fingers you take away 3. Is that what you mean? 20. R: You take away 6? Good! 21. P: Is that all what you know? 22. R: Yeah. 23. P: Now a certain boy used this method; he started from 6 and then he counted on 7, 8, and 9. 24. R: Oh yeah. I have done it before. 25. P: You used that before? 26. R: Yeah. 27. P: It seems this is counting on. We also count down to i.e. from 9 you count how many steps to arrive at 6. So we have 8, 7 and 6. That gives 3. Have you used that before? 28. R: Yeah	189. G: And that will be the number 190. P: Is that all? 191. G: Yeah. 192. P: Think of other ways that you can use to get this. 193. G: That is what I am using. 194. P: Are you sure? 195. G: Yeah. ... 198. P: Okay. What about when you start from 5 and count on "6, 7, and 8"? Can you give me the answer? 199. G: Yeah. 200. P: Do that and let me see. 201. G: 6, 7 and 8 (keeping track using the fingers). 202. P: Okay. Can you use it for this? 9-4 203. G: 5, 6, 7, 8, and 9. 204. P: So, the answer is? 205. G: 5 206. P: Good. You see that there are so many ways of doing it? Some people also can count down. From 9 you can count 8, 7, 6, 5 (G comes in here to count down together). 207. G: I think I know that.

Participants Used the Decomposition Crutch

The formal stage for each of the algorithms is purely abstract and is based on abstraction from the renaming and/or the crutch. At the formal stage, of the symbolic phase, the target task for each algorithm is such that it is extremely difficult for one to determine whether the author is using the D or the BCA- unless the one interviews the author. The crutch, however, could be determined. For D renames the minuend in compound subtraction computation, the D crutches are marks in the minuend. The BCA renames both minuend and subtrahend and thus has "crutch-marks" in both minuend and subtrahend.

Before intervention participants were tested to find out their entry behavior with respect to how they understand and do subtraction. Participants were observed to be using the D crutch throughout their work. The following are extracts from the pre-intervention test and discussion on D algorithm.

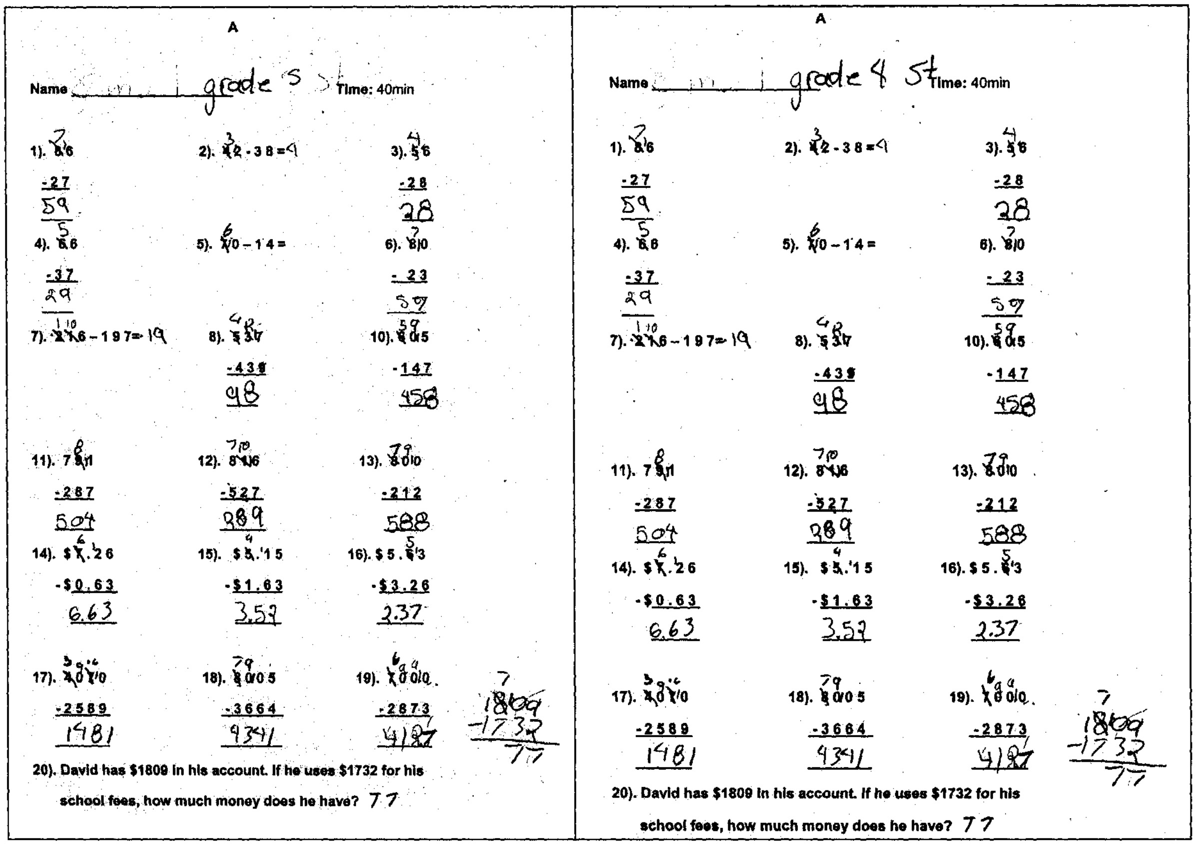


Figure 5-1. Pre-intervention-test work sheet for Rubin (left) and pre-intervention-test work sheet for Grace (right)

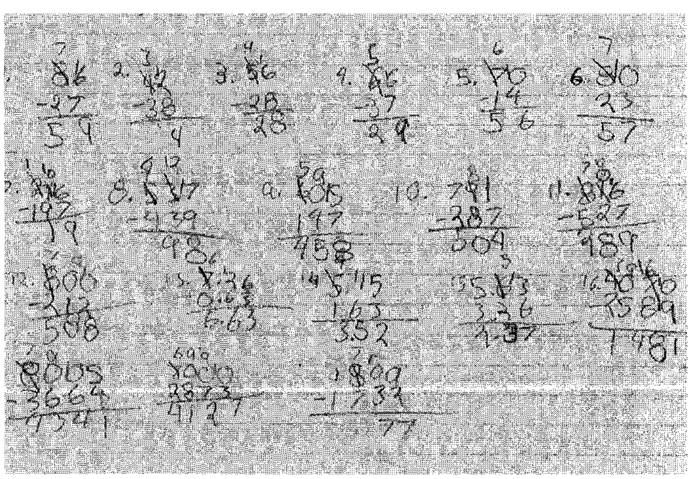


Figure 5-2. Revision exercise on the D algorithm (Rubin's work)

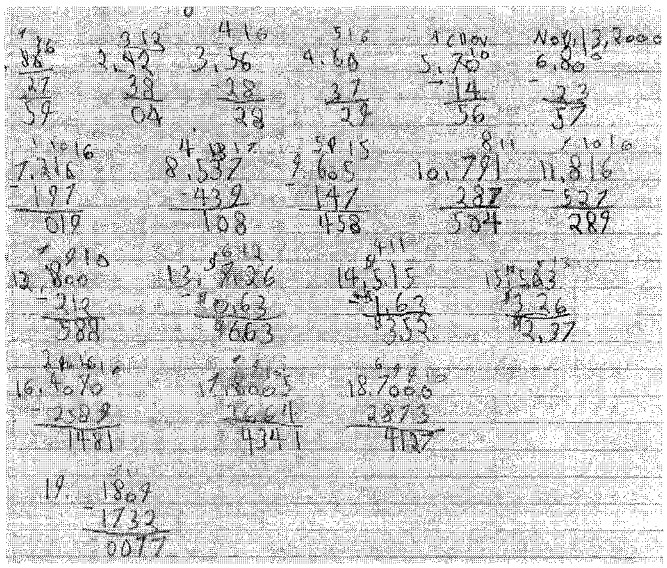


Figure 5-3. Revision exercise on the D algorithm (Grace's work)

The crutch is a symbolic aid in subtraction computation as shown above. In D, it is characterized by marks in the minuend and superscript(s) and/or subscript(s). The crutch can also be seen as the compressed form of the renaming. For example 24 – 18 is (2 tens + 4 ones) – (1 ten + 8 ones). Since subtraction at ones place cannot be done (for whole numbers), the minuend (2 tens + 4 ones) is renamed as 1 ten +14 ones; and from here the subtraction continues. Thus we have:

(1 ten +14 ones) – (1 ten + 8 ones) or

$$\begin{array}{rl} & 1 \text{ ten} + 14 \text{ ones} \\ - & (\underline{1 \text{ ten} + 8 \text{ ones}}) \quad \text{(D renaming)} \end{array}$$

The above expressions are compressed as

$$\begin{array}{rl} & \overset{1}{\not{2}}14 \\ - & \underline{1\ 8} \end{array} \quad \text{(D crutch)}$$

At this stage, in grade 5, the researcher was expecting Rubin to be using the formal stage of the D, which is a mental reflection on the renaming or the crutch procedure. But contrarily he was also using the D crutch, just like his younger sister. It was not clear to the researcher whether Rubin was using the D crutch because the younger sister was using it, or he was not discouraged to depend on it at school.

The Place Value Strategy Had Been Taught But Was Not Used

In the discussion of subtraction, participants were given a set of subtraction questions to do and then give explanation for their work. In her work, Grace explained 86 – 27: subtracting 7 from 6 is not possible (She was doing whole number subtraction), so I go to 8 and borrow a one to 6. Six is now sixteen and I take 7. Eight reduces to 7 and 7 take away 2 is 5. Rubin also explained 86 – 27 the same way as Grace did above. He also explained 1809 – 1732, as he has done it: 9 take away 2 is 7, write 7 down. 0 take away 3 we can not do, take one from the eight and 8 becomes 7 and 1 turns to a 10. 10 +0 is 10 and 10 take away 3 is 7. 7 – 7 is 0 and 1-1 is 0. All the explanations were made using simple subtraction; and the use of simple subtraction is very important sub-skill for BCA algorithm.

Rubin was asked whether he had employed any other method apart from the above. His answer was "No". The researcher then introduced the place-value method where subtraction is done first by considering the highest place-value item, then the next, and so on (Refer to example below).

1809 – 1732	86 - 47
1809 – 1000 809 – 700 109 – 30 79-2 77	86 – 40 46 – 7 46 –6 40 – 1 39

Figure 5-4. The place value strategy

When asked whether they had been using it before Rubin said "Not really" and Grace replied "No". While Rubin said he had been taught to subtract that way before, Grace said she had not been introduced to that method of subtraction. Rubin was then asked why he was not using such a method which appear to be simple- he gave no reply.

Subtraction like 86-40 in figure 5- 4 (above) is a bit confusing to some students. This type is one that may end up with a kind of compound subtraction. Here subtraction pattern changes, getting to the end of the subtraction procedure. Also, it may not be economical, to some students, for it is a waste of time and paper, for a trivial question of that sort (according to researcher's experience). Moreover, it was not of his interest to use it for the intervention. But they need to know of it. After all, in mathematics we may use more than one method for computing any particular problem.

Students Used Only the Decomposition Algorithm for Compound Subtraction Computations

During the pre-intervention discussion the researcher reminded the participants of the possibility of using the "place value" strategy, when doing compound subtraction (figure 5-4). However, they did not use it, not even once. Rubin claimed to have been taught it but he did not use it. Rather he continued with the use of the D crutch, which was an indication that he was using the D algorithm. For Grace, she made it clear that she had not been taught the place value strategy and that was her first encounter with it.

Please refer to figures. 5-2 and 5-3 above; they show Rubin and Grace's work on compound subtraction. Notice how they used D even after being reminded of the possibility of using the "place-value" strategy.

It appears that the participants of this study must have been exposed to the D algorithm because that was precisely what they used in their subtraction computations before the intervention. Other approaches, if there were any taught in their math classes, might not have been emphasized for their use at school.

Participants Made Many Subtraction Errors

There are lots of errors associated with the use of D algorithm. Some of these, according to my own experience, are reversal errors, borrowing without reducing the minuend and borrowing across zero. Referring to question 8 of Grace's work in the pre-intervention test she did her work as shown below:

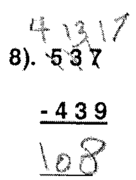


Figure 5-5. Grace's work on borrowing

Her work shows clearly that she borrowed one from the minuend 3 (3 in the minuend) without reducing it. Hence her answer zero (0) in the tens column. It was clear that she was using the D crutch but she forgot to reduce the 5 in the minuend- though she had in mind that she reduced it to 4 looking at the superscript 4. The 13 superscript again shows that she was borrowing without reducing.

During the pre-intervention interview Grace was given 537 – 439 to work using any strategy she could think of. Working it she got "-108". In her reply to a question from the researcher she said "The answer is a hundred and ninety eight". This answer shows that she had no idea of integers. But since the research is delimited to whole numbers only, the researcher did not say anything about integers. He rather used questions as prompts, to help her realize her mistakes and correct her work without tackling integers.

Excerpt 4

(Peter & Grace)
Interview # 1
241. P: That is

537
-439

242. P: Okay. Let's see your strategy

243. G: (Working on it)
244. P: You got what… -1…
245. G: Hundred and ninety eight
246. P: Go ahead; explain it
247. G: When we get minus that means below zero. So that is how
248. P: You see which one is below zero 500 or 400?
249. G: Umm a the whole number
250. P: Okay now. How did you get your 8?
251. G: From 17-9
252. P: Okay How did you get the 9?
253. G: 12-3
254. P: And now how did you get the minus one?
255. G: Ma
256. P: You see you have got –1 because you took 5 from 4
257. G: (She laughs)

The above question (line 241) was lifted from the pre-intervention test for discussion with Grace, during the pre-intervention interview. But here too she displayed the same type of errors. Doing away with her negative sign, we clearly observed that she subtracted from 5 without reducing it to 4. She had also made a reversal error- that is subtracting the numbers irrespective of which is the subtrahend or minuend. When she was further asked whether 623 – 435 = 212 is right her answer was "I think that is right" (Refer to excerpt 5 below)

Excerpt 5

(Peter & Grace)
271. P: Okay. Now if somebody does it this way
If we have 623
- 435 and the person got 212
272. G: I think that is right
273. P: Hmm?
274. G: I think that is right
275. P: Why?
276. G: Here he went borrowing and also you have 3 and you can't take 5 from 3 because is a higher number.

Did Grace really understand how to subtract using the D algorithm, at all? Obviously she did. Considering the fact that she got almost all of the pre-intervention tests and the seatwork correct, using the D algorithm. Rubin got all the pre-intervention test questions right. However, when he was asked about whether 537 – 439 = 108 (which was Grace's work; only that he did not know) is right or wrong he said the 0 (zero) is wrong, the rest are right. At the researcher's prompting he saw that the "1" is also wrong. As Rubin was not able to see at a glance that 537 – 439 = 108 was wrong, he was given 721 – 364 = 443 to endorse it as right or wrong. But interestingly he said it is right (excerpt 6, below).

Excerpt 6

(Peter & Rubin)

Interview # 1

51. P: Now what about if the person did it this way?
    The question is now 537-439 and the person wrote 108 as the answer
52. R: …Still the 0 is wrong.
53. P: The 0 is wrong. Is that all?
54. R: Yeah.
55. P: Are you sure this side only… is wrong?
56. R: Yeah.
57. P: Are you sure? Look at it carefully
58. R: Yeah.
59. P: Are you sure? Look at it carefully
60. P: You see 5…
61. R: Ooh.
62. P: E h e e you see–
63. R: Okay, the 1 there is wrong

…

104. P: O. K. Now what is wrong with this. A student does this

$$\begin{array}{r} 721 \\ -\underline{364} \end{array}$$

And the student's answer is 443. You look at it.

105. R: It's right.
106. P: What is wrong?
107. R: Nothing is wrong.
108. P: Nothing is wrong? Rubin you see …
109. R: Oh, I see. Yes.
110. P: Yeah I am glad that you've seen something wrong. And now what is wrong?
111. R: Okay. So 4-1 can't do so we have to borrow from the 2, and then 2 turns into 1 and the 1 turns into 11. So 11-4 is umm 7 so the 3 is wrong and 6-1 we can't do; take away from the 7 so this is already wrong.
112. P: That is wrong. Oh, yeah.
113. R: Okay, then.
114. P: So, what should be the answer?
115. R: The answer should be– let's see. The answer should be– Oh yeah, the other 4 is wrong.
116. P: You look at this side "What is it?"
117. R: 3.
118. P: You have done the same mistake.

Rubin's performance during the pre-intervention interview was quite surprising. He did not get any question wrong during the test and the general discussion. However, he was turning things upside down and back to front. Even in his speech instead of "one take away six" he said "six take away one." (Refer to line 111.) Could there be any reasons for these mistakes?

Observations During the Intervention

Fifth Grade Participant was Faster in Being Able to Perform the Base Complement Additions Algorithm

In terms of participants' rate of learning, Rubin was observed to be faster in getting the new algorithm. During the fourth session, for instance, he got all the exercises correct using the BCA algorithm at the formal stage. From the renaming and crutch to the formal stage took Grace a bit longer time. Whereas Rubin understood them and was jubilating over his achievement (understanding of BCA at the formal stage), Grace kept on using the renaming and the crutch. She wept as her brother teased her- using a "tongue language". She continued to weep until their parents came and her father consoled her explaining that she should not expect to understand everything all the time. It was difficult for her to connect the prerequisite knowledge she had acquired to the target task- the formal stage. Must I end the lesson? I asked myself. On account of Grace's emotional response, I ended the session immediately and advised that her older brother himself teach her before the next session. I had in mind that as a brother he could use more appropriate language for her so that she could understand the formal stage – which is one of the essences of peer discussions.

Rate of learning, to some extent, depends on the learner's understanding of the subject matter in relation to the manipulatives. Because physical interaction is essential to effective learning, the researcher made extensive use of physical materials at the enactive phase of representation to help Grace catch up with her brother, Rubin.

BCA algorithm has eight sub-skills: place value, simple addition of whole numbers, simple subtraction of whole numbers, equivalent zeros, base complements, identification of simple and compound subtraction, transformation of compound subtraction to simple subtraction and the use of base complement crutch. In each of these sub-skills Rubin was observed to be faster. He was faster at arranging the manipulatives and in answering questions based on on-going activities. However some questions remain for another study.

1. Are these rates of learning differences dependant on age?
2. Is it a factor of experience, thus the grade level?
3. Is it a matter of gender?

Grace Kept on Using the Base Complement Additions Crutch

While the researcher expected the participants to use the formal stage of the BCA algorithm, Grace continued to use the renaming and the BCA crutch. The participants entered the research using D crutch (Refer to pre-intervention test work sheets: Figure 5-1). After going through the eight pre-requisites for base complement algorithm the participants were expected to use its formal stage of the symbolic phase of the instructional sequence for BCA. But they continued to use the renaming or the crutch for some time, especially Grace. She understood all the pre-requisites but could not connect to the target task, which was formal use of the algorithm. One may be tempted to conclude that the students used the crutch because they were already familiar with the use of the D crutch. But this is not an acceptable explanation since each of the two algorithms is based on different concepts. The borrowing algorithm is based on renaming values in the minuend (or place values in the minuend). But BCA algorithm is based on compensation (or balancing or equivalent zeros) to rename both minuend and subtrahend.

In order to avert the problem of holding on to a crutch long after an algorithm is first learnt (Brownel & Moser, 1949), the researcher de-emphasized the use of BCA crutch.

"Instructional Sublimation" Contributed to Grace's Understanding of the Target Task, in the Use of Base Complement Additions Algorithm

Instructional sublimation (Refer to chapter 2: definition of terms) is skipping parts of the instructional sequence backward and forward (or forward and backward) to help the learner to move back and forth in developing procedural fluency. The researcher sublimed instruction forward and backward to help Grace, especially, to move back and forth in her understanding of the formal stage of the use of BCA algorithm in subtraction that involves renaming. As has been indicated earlier, her worry about not getting the last stage of the instruction while her brother showed incredible mastery of it made her weep bitterly and this led to an abrupt end of the day's lesson. The researcher went home thinking of the best way of connecting Grace's knowledge to the formal stage, for she had been successful in all the eight sub-skills. At the next contact period the researcher decided to start all over from the enactive phase, then skipping the iconic come straight to symbolic, in all aspects of instructions. (When attention shifted to Grace, Rubin was just drawing unnecessary things on his worksheet.) At last this instructional strategy of sublimation worked for Grace.

Participants Used Only the Formal Stage in Computing Compound Subtraction in Their Intervention Test

Before intervention students were using the D crutch, as heuristics, to help them to complete the test that was based on compound subtraction. With the introduction of BCA algorithm, initially, participants were using the renaming and the crutch (the crutch especially). This was really supporting the claim by Brownell and Moser that "crutches of any kind are usually opposed on the ground that once learned they become permanent parts of the total process; that is they are seldom discarded" (Brownell & Moser, 1949, p. 65). The good news is that after the researcher stressed on discarding the BCA crutch, the students were able to do so when the formal stage was introduced– though Grace found it a bit problematic (see figures 5-6 below).

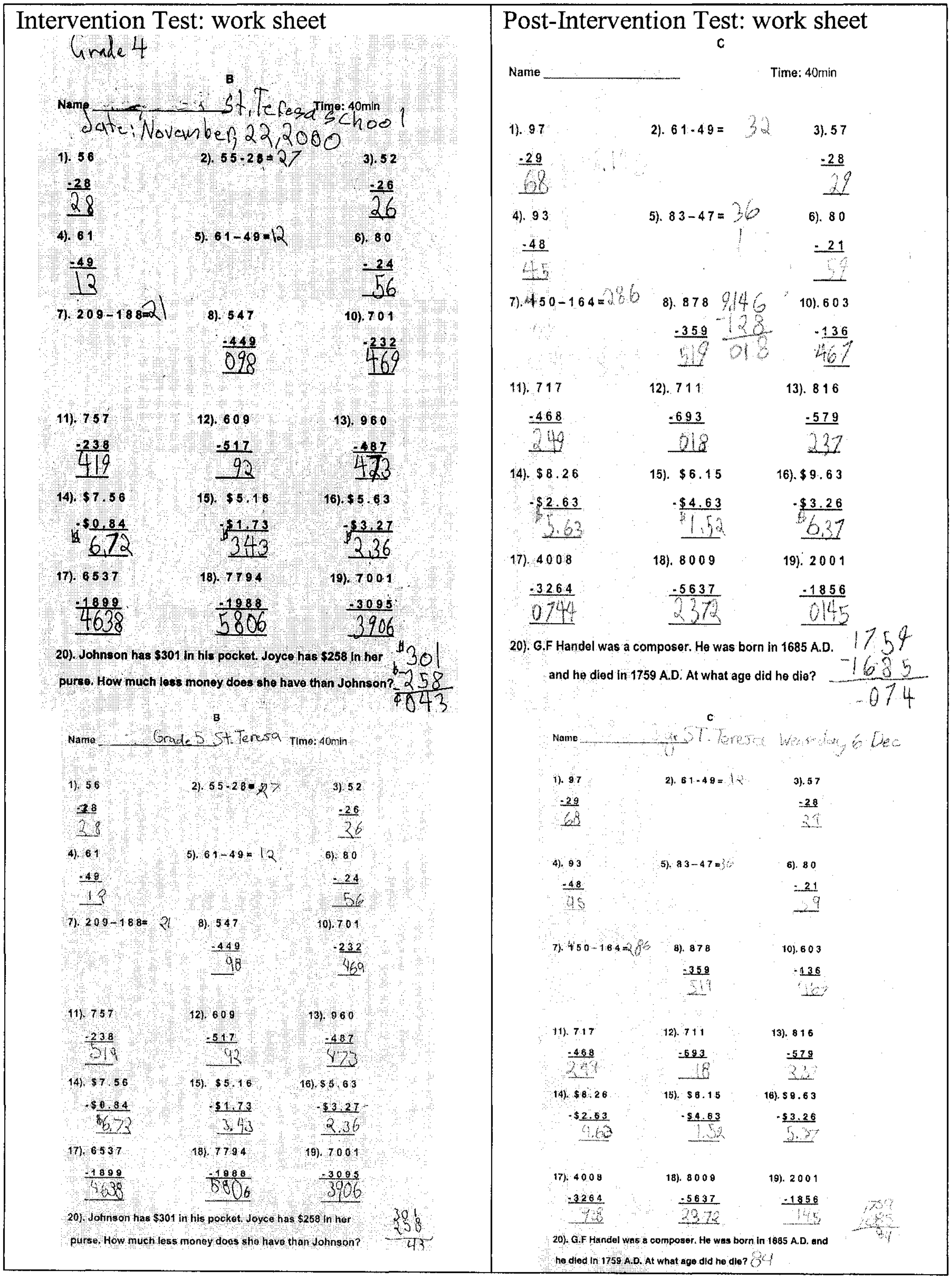


Figure 5-6. Participants' worksheets: For the intervention (B) and post-intervention (C) tests.

The formal stages of D and BCA algorithms are such that they can only be determined by interviewing the person who did the computation. From the work above one may ask, "Did the participants use the D or the BCA?" "How can one know whether the students used D, since they have been using that for at least four years?" It must be recalled that Rubin and Grace could not use the formal stage of the D algorithm, during the pre-intervention test and class discussions before the intervention (refer to "Pre-intervention-test work sheets" above, page 69). During these times Rubin was expected to be using the formal stage of the D algorithm, at his grade level but he was not. However, the researcher did not encourage them to discard the D crutch during these periods. Therefore, it was quite clear that at the time of the post intervention test if they exhibited "A formal stage calculation" of multi-digit subtraction then participants used the BCA algorithm. Furthermore, the participants saw borrowing to be time consuming and thus it was quite obvious that they would not use it, even at the formal stage, during a test, once they knew of an easier approach.

Observations After the Intervention

Participants Could Use Both Decomposition and Base Complement Additions Algorithms

One remarkable thing about problem solving in mathematics is that a person can use more than one method in solving a problem. In compound subtraction computations, the students were observed to be using the D algorithm, prior to the intervention. At that point, Rubin was observed to be sticking to his traditional algorithm, D, even when he was reminded of an alternative strategy (refer to pp. 71-72 above). But after the intervention he claimed to be able to use both D and the BCA algorithms (Refer to line 543-544 of the transcribed interview under excerpt 12 below.) Looking at the comments of the participants one may conclude that these students might be exaggerating. It is possible that they wanted to satisfy the researcher as well as their parents. Consider the comment made by Rubin as a response to a question by the researcher:

1. P: Yeah. Okay. Umm tell me why you used this method
   Can you tell me something about it?
2. R: Because you taught us isn't it? And it is much more easier than borrowing;
   because borrowing you have to do all the stuff and you get mixed up.
3. P: Have you been using "Borrowing" since we started this study?
4. R: Yes, I have tried it sometimes

The above is an extract from a discussion with Rubin during the intervention. After the question, 7102 – 4235, was given to Rubin he was asked to work and then explain it. (He was not told which algorithm to use.) After working it he used BCA algorithm to explain, hence line 1 above. The statement "Because you taught us" (line 2) may suggest that participants were using the BCA algorithm simply because the person who taught them was there. The other part of the same line 2 indicates that he gets mixed up. This part could be explained by the fact that there might be interference in knowledge of the D, by introduction of something meaningful to their understanding (otherwise they would not even use the new one). However, from item 4 Rubin has been trying to do subtraction work using the D algorithm. In addition, he demonstrated his ability to do subtraction that involves renaming using D algorithm– though there were errors here and there. (Refer to excerpt 7 below.)

Excerpt 7

(Peter & Rubin)
669. P: Can you use borrowing to check the answer?
670. R: Yes.
671. P: Go ahead.
672. R: 5-7 we can't do so we go to the 4 and take 1; 1 is 10 so 5 turns to 15. 15-7 is 8 the 4 turns to 3, 3-6 we can't do so we borrow 1 from the 2; 1 is 10 so 4 turns to a 14; 14 – 6 is 8. Now 2 reduces to 1; 1-8 we can't do so we borrow 1 form 0, we can't do so we go to 7 and borrow 1; 1 is 10 and 0 turns 10. 10 take away 1 we can do; 1 is 10 so 1 turns to a 11, 11-8 is 3. 0 is now 9, 9-1 is 8; 7 take away 1 is 6, 6-3 is 3.
673. R: Borrowing takes longer time. It is too long.

It must be recalled that errors in computation using borrowing were exhibited even before the intervention. Hence the errors in item 672 above are not a surprise. It appears that he knew exactly what he was doing but he made some mistakes. Rubin's errors might be due to his nervousness or over ambition. Rubin may have been nervous in this context; he was able to do the subtraction sums correctly when no one was watching him. He may also feel that subtraction is an easy task and was therefore careless.

There was something remarkable about Rubin's way of doing subtraction, after the intervention. Whenever he was asked to do subtraction work using D method, he would first use the BCA algorithm, get the answer down, and then use that as a check for the work done using the D algorithm (Excerpt 8).

Excerpt 8

(Peter & Rubin)
572. P: Now I have seen that you've used BCA.
573. R: I fit [sic] do it in borrowing method.
574. P: Okay. (The question was 9124102-6146747)
575. R: 2 take away 7 you can't do, so you have to borrow from the 0. You can't borrow from the 0 so have to borrow from the next digit, which is 1, and bring it to 0. The 0 behind it turns into 9; then the digit behind it turns into 12. 7 take away 12 is 5. Now it's 4 take away 9 (P: comes in "9 take away 4"). 9 take away 4 is 5. Now it's 4 take away 9 (P: comes in again "9 take away 4"). 9 take away 4 is 5. 0 take away 7 you can't do so you have to borrow from the 4. 4 turns to 3 so 10 take away 7 is 3. Now 3 take 6 you can't do, so have to borrow from the 2. Nnn has to borrow from the 2. 2 turns to 1 and the 3 turns into a 13. 6 take away 13 is 7. Now 1 take away 4 you can't do, so you have to borrow from 1 ahead of it. 1

turns to 0, 2 turns into 12. 12 take away 4 is 8. 11 take away 4is 8. ...1 (not clear) so 11. So 4 take away 11 is 7. O.K. now 0 take away 1 you can't do. Have to borrow from the 9. Turns into a 8. 0 turns into a 10. 10 take away 1 is 9. Then 6 take away 8 is 2.

576. P: 6 take away 8? Is it 8 take away 6?

577. R: Oh, 8 take away 6

...

580. P: Okay. Now let's continue. So now what I have observed is that I told you to use "Borrowing" but you used BCA [Rubin comes in quickly and said "Yes"]; you made use of "Base complement" as a check to your answer for the "Borrowing". That's very good, because it is one of what I expect. Sit well and let's continue.

581. R: Okay.

The question is "Why did Rubin not use decomposition to get the answer and then check with base complement additions?" (line 580, excerpt 8). Note the student's response to being asked to subtract with the D method (line 571, excerpt 9). The D method appears to require too much effort. As well, it takes a longer time to do the computation (refer to line 673 exert 1). Two things must be noted: *A method that appears to be more complicated when using it* and *a method that takes longer time to do.* Complicated work does not necessarily imply it takes a long time to work out; if the student knows exactly what he/she is doing will be able to do it in a reasonable amount of time. The former will, of course, imply the later when there is lack of understanding. Using the place value method in doing multi-digit subtraction could provide a useful example here. A student who does not have mastery over its use has a problem. Such a method appears longer and a bit problematic to some students. However, it is a nice method, and though long, may not take much time.

Students Could Do Subtraction of Whole Numbers with Less Difficulty than Before Intervention

During the post-intervention interview with Rubin, he was asked to use both D and BCA algorithms. He agreed, and then asked if he could be given more digits to work on. He requested a question with ten or more digits. This really shows his confidence in doing multi-digit subtraction. He has now got an algorithm (BCA) which once applied can be checked by performing the second algorithm (D).

There was one very important observation made in the interviews. Rubin breathed deeply whenever he was asked to use both algorithms. Did this indicate that making the digits longer was a problem? Since the researcher did not know which of the possibilities was related to Rubin's deep breath, the best thing, the researcher could have done was to ask Rubin why he breathed like that. The researcher did not. Rather, he gave a seven-digit question to Rubin rather than a 10 or 11 digit one. The researcher was concerned that if Rubin failed to do the questions properly he might become discouraged. Thus to avoid any embarrassment a seven-digit subtraction question was set for Rubin (excerpt 9, lines 563-566, below).

8123102 – 6146747

Figure 5-7. A seven-digit question set for Rubin

After the question was set, Rubin insisted that he was able to do as many as 11 digits in a subtraction question. This was an indication that, his deep breath was not really a matter of difficulty in doing the work but rather satisfaction. He intended to use the BCA as a check and vise versa, so he has no problem at all; and this was what he did when the question was given to him. The moment he started to explain his work it became clear that he was using BCA algorithm. Meanwhile he was asked to use the D algorithm. A close study of participants' behaviors during and after the intervention clearly indicated that they find it easier to do multi-digit subtraction computation using the BCA algorithm. Below are exercises that were set by each participant for the other. Grace's work was not well arranged so the researcher re-arranged them for Rubin to do (figure 5-8, left). The one set by Rubin (on the right, figure5-8) was okay.

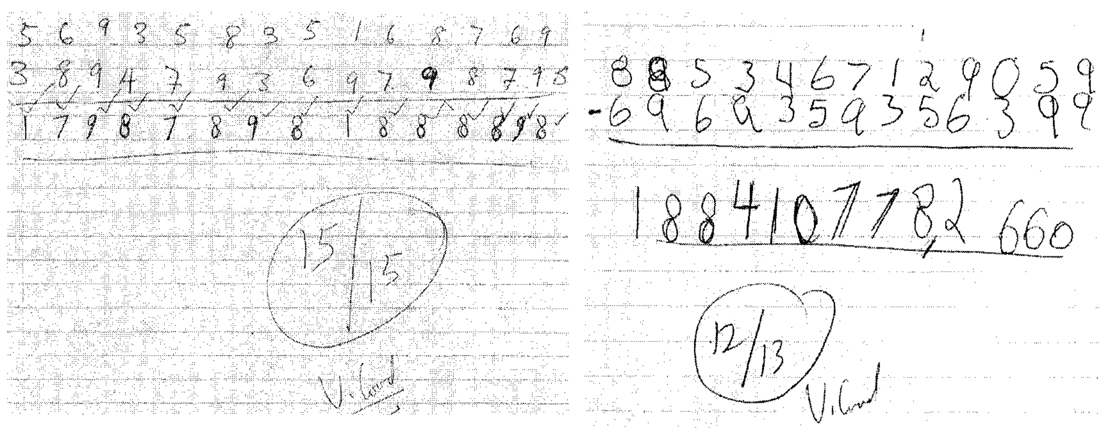


Figure 5-8. Participants' further work on multi-digit compound subtraction

The theory of hierarchies has contributed a lot to this study. The awareness, that there is the need to break down the entire instruction into smaller units of a teaching sequence and to give treatment from simple to complex, so that transfer of knowledge could enhance learning, has made the whole learning situation conducive to understanding of the BCA algorithm; and thus a very successful intervention. The BCA algorithm, was taught using only two and three-digit numbers; but participants were able to transfer this knowledge to 4, 5, ...11 digits (excerpt 9, lines 553-559 below).

Excerpt 9

(Peter & Rubin)

Interview # 2

549. P: I know that you already know "Borrowing".
550. R: Yes.
551. P: And this has come to add to it.
552. R: Okay.
553. P: So I will like you to use both borrowing and base complement additions.
554. R: Okay. Can you make it a little bit more longer?
555. P: (He laughs). Longer! You mean more digits?
556. R: Yeah.
557. P: Okay. (Then Rubin sings.)
558. P: No problem. I will make it for you.
     About how many digits do you like it?
559. R: Ten. Actually eleven this time.
560. P: And you are going to use both "Borrowing" and "BCA" (Rubin comes in immediately saying, "Yes")
561. P: Okay.
562. R: (He breathed deeply.)
563. P: This one is 7 digits. I want you to do only 7 digits (The question was initially 812-614, but because of his challenge it was changed to 8123102-6146747)
564. R: I want to do 11. A special challenge.
565. P: You do this first; use the Borrowing (S: Okay) and then BCA.
566. R: Okay (He silently works the 7-digit- number subtraction work). At the end he said, "Okay."
567. P: Explain it the borrowing way.
568. R: N hu hu. 2 take away 7 you can't do so the Base Complement of 7–
569. P: –no, no, no. I say use borrowing (They all laughed.)
570. P: I say use borrowing I didn't say use BCA.
571. R: O o o h! Too long. Oh, okay (he shouted).

Participants Were Able to Determine Errors in Multi-digit Subtraction with Less Difficulty than Before the Intervention

Before the intervention students could not easily determine errors in compound subtraction computation (excerpt 10, lines 66-69 and lines 316-332). With the introduction of BCA, after the intervention, they could easily look at an expression and determine whether it was right or wrong and why it was wrong (excerpt 10, lines 649-660 and 687-694 below). This was achieved by way of discussion during the intervention.

Interpretive discussion could be organized in several ways. The most common is student-to-teacher discussion in the whole class situation and student-to-student discussion in large or small group or student pairs. Student-to-teacher discourse was mainly used in this study. Student-to-teacher discussion is generally diagnostic and enhances student self-communication. Questions like, "What is wrong with the second one?" (line 66), "How did you check your answer?" (line 612), "If somebody does this ...

is the person right or wrong?" (line 649), "How did you do it? How did you get your answer?" (484), "Is this correct or wrong?" (line 610), "What makes it wrong?" (lines 653, 691), help the students to reflect on their thinking processes; and the end result is reasoning out.

Excerpt 10

(Peter & Rubin) <u>Interview # 1</u> (The question is now 537-339 and the person wrote 108 as the answer) 66. P: Now let's continue. What is wrong with the second one? 67. R: Actually this is right 68. P: Just like the first one? 69. R: I see no, no, no, it's not right, it's not right. 70. P: E h e e 71. R: This is not right, 1 is right but the 0 is not right. 72. P: The 1 is right? Is the 1 right? 73. R: And the 8 is right but the 0 is not right. 74. P: This time 1 is right, 0 is not right but 8 is also right So the person did not reduce the 3. That is why we got the 8. Thank you. Have you ever done such mistakes before? 75. R: Yes. <u>Interview # 2</u> 610. P: Is this correct or wrong? 4003 -<u>2007</u> <u>1006</u> 611. R: No the answer is 1996 612. P: How did you check your answer? 613. R: I used base complement	(Peter & Grace) <u>Interview #1</u> 316. P: If we have 7001 -<u>5004</u> and the person writes 1007. Look at it. In the first place is it correct? 317. G: Hmmm no 318. P: Yeah, what is wrong? Which parts are wrong and which parts are correct? 319. G: These two parts. (She shows.) 320. P: Which parts? 321. G: The…[the tape is not clear] 322. P: Are they wrong? 323. G: Yeah. 324. P: Are you referring to the answer or the figure 7001? 325. G: The one that I am looking is this…then…[not clear] 326. P: Ah? Again– 327. G: … 328. P: Aha. Look at it well. We have 7001 and then 5004. 329. G: This is right (referring to the work). 330. P: Is it right? 331. G: Yeah (They all laughed) 332. P: Are you sure it is right? (She is not certain. At first she said no it is wrong. Now she says yeah, it is right.) 333. G: Yeah.

Interview # 3

649. P: Okay, if somebody does this work:

9024
- 4034
4090

Is the person right or wrong?

650. R: (Kept quiet but looking at the work.)
651. P: You look at it and say what happened.
652. R: Wrong.
653. P: What makes it wrong?
654. R: The answer is 4990.
655. P: How did you get the answer?
656. R: 4-4 is simple, 4-4 is 0; 2-3 we can't do the base complement of 3 is 7, 7+2 is 9; 0 goes up 1, the base complement of 1 is 9, 9+0 is 9; 4 goes up by 1, 9-5 is 4.
657. P: So what makes it wrong?
658. R: Umm, the zero there is 9.
659. P: Which zero?
660. R: The one near 4.

Interview # 2

482. P: If somebody does this work is the person right?

723
-456
233

483. G: No it must be 267.
484. P: How did you get your answer?
485. G: I used base complement.

Interview #3

687. P: Okay, if some body does this work

9036
-4058
5088

Is the person right or wrong?

688. G: (Kept quiet but looking at the work and smiled.)
689. P: You look at it well.
690. G: It is wrong!
691. P: What makes it wrong?
692. G: 6-8 is compound, so the base complement of 8 is 2, 2+6 is 8; 5 goes up 1, 3-6 we can't do, the base complement of 6 is 4, 4+3 is 7; 0 turns 1, 0-1 we can't do, the base complement of 1 is 9, 9+0 is 9. 4 goes up 1, 9-5 is 4.
693. P: So what makes it wrong?
694. G: The answer is 4978.

Learning is a product of our thinking and talking through social interaction and physical contact with the environment. These experiences stimulate our thinking. However, the learning that occurs is dependent on how the physical activity is related to the subject matter. The manipulatives, the thinking, as well as the talking are all interrelated. The ability to determine errors is dependent on the understanding of the subject matter, which is the product of the interaction with manipulatives, the thinking and talking about the manipulatives.

Excerpt 11

(Peter & Grace)

695. P: Well done. Okay, another person does this work

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 84205
-33827
 51622
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If you are asked to check the answer which method will you use?

696. G: I will use base complement.

697. P: Why?

698. G: Because it takes less time to do the question with base complement.

699. P: What makes it wrong?

700. G: 5-7 is compound so base complement of 7 is 3, 3+5 is 8; 2 turns to a 3, 0-3 we can't do, base complement of 3 is 7, 7+0 is 7; 8 turns to a 9, 2-9 we can't do, so BC of 9 is 1, 1+2 is 3; 3 turns to a 4, 4-4 we can do, 4-4 is 0; 8-3 we can do, 8-3 is 5.

701. P: Compare your answer to what is there. Is there any difference?

702. G: Yes. My answer is 50378. Only the 5 up there is correct.

703. P: Can you use borrowing to check the answer?

704. G: Yes.

705. P: Do it by borrowing.

706 G: 5-7 we can't do so we borrow 1 from the next number but 0 has nothing. Go to the 2 and take 1; 1 means 10 so 0 becomes 10 and we take away 1; 1 is 10 so 5 turns to 15. 15-7 is 8. 10 take away 1 is 9, 9-2 is 7. 2 take away 1 is 1, 1-8 we can't do, so we borrow 1 from the 4, 1 is 10 so 1 turns 11, 11-8 we can do, 11-8 is 3. 4 turns to 3, 3-3 we ca do so 3-3 is 0; 8-3 we can do, 8-3 is 5.

706. P: Thank you Gladys.

707. P: Now, let us go back a bit. What is base complement additions?

708. G: You taught us.

709. P: Do you understand it?

710. G: Yeah!

711. P: Can you explain it?

712. G: Give me an example.

713. P: Take any example and explain it.

714. G: Hmm ..can I use 84205-33827?

715. P: Yes.

716. G: 5-7 we cannot do so base complement of 7 is 3, 3+5 is 8 (P comes in here)

717. P: Do you remember, from the beginning, we discussed that

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 21     is equal to each of the following? [22-18; 23-19; 24-20; 25-21; 26-22 …]
-17
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718. G: Yeah!

719. P: How do you understand it?

720. G: We add the same number up and down and the answer is not changed. Okay, in base complement additions you add base complements up and down to subtraction we cannot do; and the same work turns to be easier to work.

721. P: Good! But how do you do this: 21-17=?
722. G: 1-7 we cannot do so base complement of 7 is 3, 3+1 is 4; 1 becomes 2, 2-2 is 0.
723. P: Why 3+1 is 4 and 1 becomes 2 but not working with 24-20 since you say the work is compound so we add base complement up and down…
724. G: Yes that is it. Look here (she was using numeration cards to explain) the complement added to the 1 at ones side is 4 making 24-20. Simply add the complement to the ones there and the number of tens taken away increases by one… you see?
725. P: Good. Thank you

The use of numeration cards, in the intervention, gave the participants the opportunity to interact meaningfully on both social and physical levels. Questions were posed to participants for discussion– especially to determine whether each subtraction entry was right or wrong and then to allow students to provide reasons for their answers (refer to excerpts 10 and 11 above). Discussion, during the intervention, helped students clarify their thinking. In many cases, it encouraged students to articulate their thoughts and also helped in formulating them. Many a time when we talk (mathematics talk) we are put in a position in which we have to explain concepts or principles to others. Determining errors with explanation helped students to develop meaning for their knowledge. Talking exposes our thinking to others who can examine it and comment on it. In this way, the interpretive discussion was diagnostic. In addition, participants' analysis of subtraction expression showed that the speaker had knowledge of the discussion at stake.

Participants Appeared to be More Confident and Efficient with Base Complement Additions than with Decomposition.

During the post-intervention interviews, participants made it clear that they prefer BCA to D, in computation of multi-digits compound subtraction. The students used BCA with confidence and air of satisfaction and efficiency. This gave the researcher the impression that BCA is a superior teaching and learning tool for compound subtraction. Hence post-post-tests and interviews, which happened almost a month after the post-tests, were used to assess retention of the newly learned algorithm (BCA). Excerpt 9 shows participants' confidence with the BCA. It appears that once they understood and used BCA they had difficulty in using the borrowing method even when directed to do so (see excerpt 12, lines 439-463). Rubin's expression, "Yeah borrowing involves some kind of many take away, add, take away like that before the subtraction itself" suggests how relatively uneasy it was for him to use the D at this point in time. He expressed that BCA is much easier (excerpt 12, lines 646-648). As discussed, when restricted to use the D algorithm, the participants usually used BCA and recorded their answers down before using the D algorithm. It must be recalled that when the place value method was introduced Rubin claimed to know it but he never used it either. It could be, assumed, that the strategy was not efficiently learned or alternatively that they use the method with which they feel most comfortable.

Efficient learners, usually, are learners who assume responsibility for their own learning and set their own goals. Apparently all learners do this but the teacher's role is to enhance this process. Learners who know the purpose of their learning have control of it. Many students do not know why they use certain strategies. A typical example can be seen in the use of equal additions. Before 1949 students used equal additions without knowing why. Hence Gagg (1954) wrote about it as not so logical and that many children only have to believe that it is accurate.

Purpose can exist at many levels. An overall purpose might be "Why am I using this strategy? Does it go with the question or the context? Why must I use a simpler but equivalent question? The focus must be on the task. (Refer to Figure 4-9: A set of equivalent expressions, under renaming subtraction expressions, in chapter 4).

Teaching students about equivalent expressions (see page 52) helped them learn to distinguish simple forms of expressions and to realize that any expression could be transformed into a simpler form for subtraction. The listing of the 8 pre-requisites, before the students, for the target task became very useful and important for them. Usefulness and importance of any computation process can come from the teacher's explanation. Thus purpose is largely a motivational device but it also serves in an operational sense. When learners know what they are studying and why they are engaged in a particular activity, they are more likely to participate fully and whole-heartedly in the activity.

Excerpt 12

(Peter & Grace)	(Peter & Rubin)
<u>Interview # 2</u> 439. P: Now I want you to use the borrowing for this 9124 - <u>4156</u> 440. G: The base complement of 6 441. P: No, no I say use borrowing. 442. G: The 4 is not bigger than the 6 so we have to borrow from the 2. With that one is a 14 (then she got stuck) 443. P: Okay. You work it using the BCA after that explain. ... 461. P: Okay. Now umm I will like you to use the strategy to work this one. This time it is 73451 - <u>35672</u> Use your strategy. The method that you used to work the other one; the one that you like	<u>Interview # 2</u> 536. P: The question is 7102 - <u>4235</u> Will you do it for me? 537. R: Okay. (He took in deep breath.) Am I supposed to do it? 538. P: You will do it and then explain it as well. 539. R: Okay. (He then did the work quietly.) 540. R: Okay. 2 take away 5 can't do it; BC of 5 is 5; add to 2 is 7. 3 increased by 1, so to take away 4 can't do it. So BC of 4 is 6 add to 0 i.e. 6. Then 2 increases by 1 take away 3 we can't do so BC of 3 is 7; 7+1 is 8 and the 4 goes up by 1. 7 take away 5 we can do, so 7 take away 5 is 2. 541. P: Yeah. Okay. Umm tell me why you used this method. Can you tell me

462. G: I am going to use the one that you don't have to borrow.
463. P: Okay.
464. G: The BC of 2 is 8. 8+1 is 9. Change 7 to an 8. BC of 8 is 2. 5+2 is 7 You change the 6 to a 7. BC of 7 is 3. 3+ 4 is 7. You change the 5 to a 6. BC of 6 is 4. 3+4 is 7. You change that to a 4. 7 take away 4 is 3.
465. P: Now why didn't you use, borrowing?
466. G: I didn't use borrowing because I think it's much harder to do and sometimes I make simple mistakes.

<u>Interview # 3</u>

679. P: Which questions did you use borrowing and which did you use base complement additions?
680. G: I did not use borrowing.
681. P: Which method did you use?
682. G: Base complement for all for all my work.
683. P: Why?
684. G: Base complement is much easier and faster.
685. P: Are you sure?
686. G: Yes.

something about it?
542. R: Because you taught us isn't it? And it is much more easier than borrowing; because borrowing you have to do all. The stuff and you get mixed up.
543. P: Have you been using "Borrowing" since we started this study?
544. R: Yes, I have tried it sometimes.
545. P: Are you getting mixed up?
546. R: Oh yeah.

<u>Interview # 3</u>

641. P: Which questions did you use borrowing and which did you use base complement additions?
642. R: None of them for borrowing.
643. P: Which method did you use?
644. R: I used base complement for all.
645. P: Why?
646. R: Base complement is much easier
647. P: Are you sure?
648. R: Yeah. Borrowing involves some kind of many take away, add, take away like that before the subtraction itself.

...

661. P: Good. Okay. Another person does this work:

$$\begin{array}{r} 70245 \\ -\ \underline{31867} \\ \underline{38482} \end{array}$$

If you are asked to check the answer which method will you use?
662. R: Base complement.
663. P: Why?
664. R: That is easier to do.

In the pre-intervention, reversal errors as well as borrowing across zero and borrowing without reducing minuend were observed. After learning BCA, all these became minimal in participants' subtraction calculations. After learning the BCA algorithm, participants were able to do multi-digit compound subtraction more easily. Furthermore, participants were able to determine errors in multi-digit subtraction with less difficulty than they did before the intervention. This was really a mark of improvement in participants' subtraction-skill development.

CHAPTER 6 - SUMMARY, CONCLUSION AND RECOMMENDATION

Summary of the Study

According to research studies on the learning of subtraction algorithms, students have shown vast differences in success using the decomposition (D) and equal additions (EA) algorithms. Most of the research dates back to the late 1930s and 1940s. An updated investigation of the subtraction algorithm seemed appropriate. In this study an algorithm, in compound subtraction, was introduced to participants. Then the researcher investigated, by way of testing and interviewing, whether the intervention made sense to participants. D was the traditional algorithm commonly used in United States, Canada and most part of the world. Base Complements Additions (BCA), though not new, was a novel algorithm for the students.

In the pre-intervention session two equal sets of subtraction questions were given to participants to try. The purpose was to observe the participants' various strategies for subtraction and then follow it up with interviews to find out strategies they use for subtraction that involves renaming. It was assumed that students enter the classroom with some informal knowledge about what is to be taught. The main objective of this session was to determine their entry behavior.

Intervention sessions followed the initial interviews. These were a sequence of instructions that were given to the participants. It took five contact-hours in the residence of the participants. After this intervention a second set of questions was given to participants to do. The purpose was to find out if the participants made sense of the intervention. It was assumed that students would use their traditional D algorithm if the intervention was not so meaningful to them. The post-intervention came two weeks after the intervention. In this session participants were given a third but parallel set of subtraction questions on compound subtraction. After the test they were interviewed the next day, for investigations pertaining to retention and subtraction skill-development.

The post-post intervention was an unscheduled session to test for the reliability of findings in the post intervention interviews. It was comprised of two, different but parallel, sets of questions and interviews. The computational tests and interviews were two main evaluation instruments used to assess participants' computational skills in solving subtraction problems, their understanding of the subject matter and ability to solve word problems.

All the interviews were audiotaped, in each phase, to investigate the subject matter. Students' working papers were also collected and analyzed. The evaluation instruments were all designed and administered by the researcher. Pre-intervention, intervention and post-intervention was the design used for the study.

Findings From the Study

Analyses from participants' working papers, the tests and the transcribed interviews brought out the following findings from the research study:

1. *Students have more than one strategy for doing simple subtraction; but with compound subtraction there was only one method students used, before the intervention.*

The implication of this finding was that with simple subtraction there was flexibility in the paper-and-pencil approach. However, with compound subtraction, failure to understand the D algorithm might lead to inability to correctly complete computation, in terms of paper-and-pencil *work*. It could even discourage the students from doing mental arithmetic. But in problem solving students are usually encouraged to find alternative approaches to solving any given problems. For students to use only one method for solving compound subtraction obviously does not encourage problem solving in students with respect to paper-and-pencil multi-digit compound subtraction. During the revision lesson Rubin disclosed, after researcher's prompt, that he was once taught how to subtract by way of place-value strategy. But even after discussing it he never used it. As the adage goes "You can lead a horse to water but you cannot make it drink". It is clear that students must be exposed to a variety of methods and they have to decide which one to use to solve a given problem.

2. *The notion that there was only one method used for solving compound subtraction, has changed after the intervention. The interviews indicated the possibility of an additional strategy which students were more confident with.*

The main purpose of this research was to introduce BCA to participants and find out whether they would make sense of it. Participants' confidence in BCA is a clear indication that they really mastered the intervention procedure. Otherwise they would have reverted back to using D (borrowing) a week or two after the intervention.

3. *Students made many errors when working with "borrowing" (the previously learnt algorithm). With the introduction of the new algorithm students were seen to make few mistakes with respect to multi-digit subtraction computations. Reversal errors, for instance, in participants' expressions and answers, did not happen with the use of BCA. Participants, in the study, were able to do eleven-digit subtraction exercises without making other errors associated with D (borrowing).*

Compound subtraction that involves, at least, three renaming appears complex when D is applied. Sometimes the subtraction could involve a series of subtracting and adding before the actual subtraction is made. Thus when the student forgets to keep track of the procedure, this brings about errors such as borrowing without reducing the minuend, borrowing across the zero and reversal errors. BCA, however, uses only simple addition and simple subtraction. Further, the numbers involved in BCA calculations are the digits 1, 2, 3, ... 9. With BCA, compound subtraction is transformed to simple subtraction. Thus I speculate at this point, that many people (students and non-students) would feel comfortable with BCA, in terms of multi-digit subtraction that involves renaming.

4. *Before the intervention students could not easily determine errors in compound subtraction computations. With the introduction of BCA, after the intervention, they could easily look at subtraction expressions and determine whether they are right or wrong and why. Thus it was easier for students to determine right or wrong subtraction expressions with BCA than with borrowing.*

Students' ability to easily determine right or wrong subtraction expressions with BCA, is an indication that there is improvement in their arithmetic competence. Their knowledge base in compound subtraction may have been expanded. Another implication is that participants understand the computational procedures of BCA from the beginning to the end. And they used their answers they got from working with BCA to check those they got by way of D.

The general implication, here, is that BCA is a powerful tool for computing subtraction that involves renaming, for these participants. Thus BCA algorithm compares very favorably with the conventional D algorithm.

Conclusion

The purpose of this study was to explore students' understanding of multi-digit subtraction given an intervention. However, the study began by determining students' existing strategies for subtraction. From the study it was found that:

- *With respect to simple subtraction participants used "separating from whole or take away", "counting on" and "counting down to" models in their computation of simple subtraction;*

- *For compound subtraction participants used only the D algorithm.*

In this study the researcher was able to shed some light on the teaching and learning of compound subtraction by BCA. Particular effort was made to determine whether the teaching and learning of compound subtraction by the method of BCA would be useful to participants for skill development, if objectified. From the findings, the following general conclusion could be made.

- *The base complement additions algorithm was an effective teaching and learning tool for subtraction that involves renaming for the participants in this study.*

The post-intervention test and interviews and the post-post intervention test and interviews indicated a change in participants' entry behavior; hence one can argue that participants have a preferred algorithm to use in the computation of multi-digit compound subtraction. In all cases, post-intervention, they self selected BCA when asked to subtract.

Participants did subtraction computation, even, up to eleven digits. It might be inferred that the instructional sequence, the mental math and the numerous practice problems offered thorough and comprehensive intervention coverage for participants. The end result was that with BCA students could easily determine wrong or right subtraction work, at a glance, which they had found difficult when D strategy was used. It must be recalled that prior to the study the participant in grade 4 had already been taught the method of D when she was in grade 3. Similarly, the participant in grade 5 had been introduced to the traditional D algorithm in grades 3 and 4.

Interestingly, the BCA intervention covered only five contact periods but the students did not go back to using the D even after six weeks. There is little doubt that these students found BCA to be a powerful tool for compound subtraction. This is consistent with the observation made by early researchers, including Jerman and Beardslee (1978); when comparing the relative efficiency of D and equal additions, equal additions had been shown to be a superior teaching tool. In their observation, however, meaning and understanding were not factors. But in this study the researcher attempted, by way of physical representations, to build in opportunities for BCA to be meaningful to the students. This research also supports the consensus of the audience of Gyening's presentation (1993) that Equivalent Zero Addition (from which BCA was derived) has the potential to compare favorably with the D algorithm.

The equal additions and the BCA are similar for they are both based on compensation. In equal additions one is required to add 10 to both minuend and subtrahend, whereas in BCA one is only required to add base complement(s) of the subtrahend to both minuend and subtrahend. The instructional activities for BCA cover the enactive (through the use of numeration cards, Dienes blocks, straws etc.), iconic (pictorial and diagrammatic representations) and symbolic phases of conceptual representation as suggested by Bruner (1966). The instructional sequence developed for this research, therefore offers possibilities for modeling the BCA subtraction algorithm; hence attending to the deficiency noted in earlier research (Brownwell & Moser, 1949; Gagg, 1956). Given a way of modeling BCA, it has potential to be a very useful tool for teaching and learning subtraction that involves renaming.

Recommendations

For Instruction

Teachers should break down the target task into sub-tasks and make sure each is adequately treated and meaningfully understood before tackling the target task. A successful intervention depends on the adequacy of pre-requisites. For a successful intervention of BCA, at the formal stage, the following pre-requisites are recommended:

a. Place value

b. Simple addition of whole numbers

c. Simple subtraction of whole numbers

d. Equivalent zeros

e. Base complements

f. Identification of simple and compound subtraction

g. Renaming compound subtraction expression by way of base complement additions (transformation of compound subtraction to simple subtraction).

h. The use of BCA crutch.

Since the adequacy of each of the sub-procedures depends on the adequacy of the frames for concepts involved it (Halabisky, 1981, p.108), the subject matter of each subordinate task must be developed in such a way that they will be most beneficial to learning of the target task. The use of physical materials is, therefore, recommended for each subordinate task.

For Further Research

It is recommended that, a thorough and comprehensive study be made in grades 1, 2 and 3 to find out what stage of the instructional sequence (above) is viable at these grade levels. This recommendation is made in light of the fact that the participant in this study, who was in grade 4, was not, initially, finding it easy to operate at the formal level of the symbolic phase of the instructional sequence.

The results also suggest several related areas for possible research studies; the possible target-task of which might be a focus on the formal stage of the symbolic phase of BCA. Why? The reason is that it appears operating at the formal stage might not be possible at these grade levels. It seems that introducing BCA to first, second and third graders must cover certain depths of the instructional sequence only, in order not to over burden learners with instructions. Teaching and learning of BCA at these grade levels must be "conceptual" to provide a rich knowledge base for future "procedural" learning of BCA.

The researcher furthermore recommends that, a thorough and comprehensive study be made in grades 6, 7 and 8, using the "deep-end approach to mathematical instruction". This must be conducted to find out whether the symbolic phase of the instructional sequence (above) could be applied, at these grade levels, without the enactive and the iconic phases. This recommendation is in light of the fact that the participant in this study, who was in grade 5, found the whole instructional processes very comprehensive; and was able to easily transfer the knowledge to eleven and more digits and also to numeration systems (i. e. bases 5, 4, 3). It seems, to the researcher, that it is easier to operate at the symbolic phase of the instructional sequence if students understand "balancing". The principle of balancing, also known as *compensation*, is applied in many subject areas and disciplines. These include algebra, calculus, mechanics and physics. Thus the application of only symbols at the above grade levels must be considered meaningful, if students understand and thus can explain what they are doing. At these grade levels remedial instructions in the form of iconic representation is recommended, should learners find problems in getting the formal stage of the symbolic phase.

A study of two children is not sufficient to generalize findings. Thus for validity of the BCA intervention to uncover exaggerations and oversimplifications in this study, a large scale comparative study of D and BCA in grades 4 and 5, in terms of subtraction-skill development is recommended.

For Dissemination of Knowledge on the Base Complement Additions Algorithm

Based on this study, the researcher believes that BCA is a useful strategy for elementary aged school children. However, it does not appear to be a common strategy developed in textbooks or known to math teachers. Hence educational administrators and teacher educators should consider refresher courses, workshops and/or seminars for classroom teachers to acquaint them with this method of working compound subtraction sums.

With respect to problem solving, it is recommended that BCA be promoted as a strategy for computing multi-digit compound subtraction. It both provides students with another algorithm for subtraction and it helps pupils appreciate the notion of balance (equivalence) at their initial school ages.

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APPENDIX A

Ethics Review Approval Letters
Parents' Letters

FACULTIES OF EDUCATION AND EXTENSION
RESEARCH ETHICS BOARD

Graduate Student Application for Ethics Review

Name: **Peter McCarthy** **Student ID: 397588**

E-mail: **pm@ualberta.ca**

Project Title: ***Exploring students' understanding of subtraction given an intervention.***

Project Deadlines:

Starting date: **January, 2000** Ending date **August, 2001**

If your project goes beyond the ending date, you must contact the REB in writing for an extension.

Status:

___ Master's Project __√__ Master's Thesis ___ Doctoral Thesis ___ Other____________ (Specify)

The applicant agrees to notify the Research Ethics Board in writing of any changes in research design after the application has been approved.

[signature]
Signature of Applicant

November 6, 2000
Date

The supervisor of the study or course instructor approves submission of this application to the Research Ethics Board.

[signature]
Signature of Supervisor/Instructor

Nov 6 20 00
Date

ETHICS REVIEW STATUS

___ Application not approved

__✓__ Expedited review approved by Unit Statutory member/Alternate

___ Review approved by Research Ethics Board

[signature]
Signature of REB Member

Nov 7 20 00
Date

PARENTS LETTER

Date:

Dear Parents,
Your child will be involved with a study about how he/she reacts to a new method of subtraction for a period of two months or twelve contact hours. The instructor Peter McCarthy, is a fully trained teacher with 12 years of classroom teaching experience. He developed the sequence of instruction in the lessons in partial fulfillment of his master's research project.

The purpose of the study is:
That I am trying to learn about how students subtract
That I am trying to explore students' understanding of subtraction that involves renaming, given an instruction.

Your child will be given some renaming-subtraction questions to do, interviewed as to how he/she did the work and then a new approach to solving renaming-subtraction will be introduced to him/her. A day after the intervention your child will be given another renaming-subtraction questions to do but will not be interviewed. Two weeks after the intervention a third set of renaming-subtraction questions will be given to your child to do after which he/she will be interviewed. Time for each set of questions will be 40 minutes and each interview will take 40-60 minutes. The researcher is not interested in the scores of your child but how he/she understands and does the work. All interviews, will be audio taped and transcribed; and will be outside the normal class periods. Transcripts will be given to participants for confirmation. Participation in this project is entirely voluntary. It is your decision whether or not to allow the results to be used in the research study.

The data collected will only be used for the purpose of my research. Only the results will be reported, with no reference to individual children. The experimenter will assign pseudonyms to ensure confidentiality and anonymity.

If you have any questions regarding the instructional sequence, or you want to withdraw your child from the project, contact Mr. McCarthy (at U of A, 492-3760 or at home, 988–2418) or his advisor, Prof. Elaine Simmt (Department of Secondary Education, Education South, University of Alberta, Edmonton, at 492-0881).
This project has been reviewed and approved by the Faculties of Education and Extension Research Ethics Board for research involving Human subjects.
Thank you for your cooperation.

Sincerely

PETER MCCARTHY

University of Alberta

Research Consent Form

I, ____, hereby consent
(print name of parent/legal guardian or independent student)

for ____ to be
(print name of student)

interviewed by **Peter McCarthy.**

I understand that:

- my child may withdraw from the research at any time without penalty
- all information gathered will be treated confidentially and discussed only with your supervisor
- any information that identifies my child will be destroyed upon completion of this research
- my child will not be identifiable in any documents resulting from this research thesis.

I also understand that the result of this research will be used only in the research thesis, presentations and written articles for other educators.

For further information concerning the completion of the form, please contact **Peter McCarthy** (Department of Secondary Education, University of Alberta, at 492-3760) Or his advisor, **Prof. Elaine Simmt** (University of Alberta at 492-0881).

____
Signature of parent/legal guardian or independent student

Date signed: ________________________________

APPENDIX B

Pre-Intervention Test Items -A

Intervention Test Items -B

Post-Intervention Test Items -C

Post-Post-Intervention Test Items

A

Name ______________________ **Time:** 40min

1). 8 6
- 2 7

2). 4 2 - 3 8 =

3). 5 6
- 2 8

4). 6 6
- 3 7

5). 7 0 – 1 4 =

6). 8 0
- 2 3

7). 2 1 6 – 1 9 7=

8). 5 3 7
- 4 3 9

10). 6 0 5
- 1 4 7

11). 7 9 1
- 2 8 7

12). 8 1 6
- 5 2 7

13). 8 0 0
- 2 1 2

14). $ 7 . 2 6
- $ 0 . 6 3

15). $ 5 . 1 5
- $ 1 . 6 3

16). $ 5 . 6 3
- $ 3 . 2 6

17). 4 0 7 0
- 2 5 8 9

18). 8 0 0 5
- 3 6 6 4

19). 7 0 0 0
- 2 8 7 3

20). David has $1809 in his account. If he uses $1732 for his school fees, how much money does he have?

B

Name ______________________ **Time:** 40min

1). 56
 - 28
 ____

2). 55 - 23 =

3). 52
 - 26
 ____

4). 61
 - 49
 ____

5). 61 – 49 =

6). 80
 - 24
 ____

7). 209 – 188=

8). 547
 - 449
 ____

10). 701
 - 232
 ____

11). 757
 - 238
 ____

12). 609
 - 517
 ____

13). 960
 - 487
 ____

14). $7.56
 - $0.84
 ____

15). $5.16
 - $1.73
 ____

16). $5.63
 - $3.27
 ____

17). 6537
 - 1899
 ____

18). 7794
 - 1988
 ____

19). 7001
 - 3095
 ____

20). Johnson has $301 in his pocket. Joyce has $258 in her purse. How much less money does she have than Johnson?

C

Name ____________________ **Time:** 40min

1). 9 7
 -2 9
 ____

2). 6 1 - 4 9 =

3). 5 7
 -2 8
 ____

4). 9 3
 -4 8
 ____

5). 8 3 – 4 7 =

6). 8 0
 - 2 1
 ____

7). 1 5 0 – 1 6 4 =

8). 8 7 8
 -3 5 9
 ____

10). 6 0 3
 -1 3 6
 ____

11). 7 1 7
 -4 6 8
 ____

12). 7 1 1
 -6 9 3
 ____

13). 8 1 6
 -5 7 9
 ____

14). $ 8 . 2 6
 -$ 2 . 6 3
 ____

15). $ 6 . 1 5
 -$ 4 . 6 3
 ____

16). $ 9 . 6 3
 -$ 3 . 2 6
 ____

17). 4 0 0 8
 -3 2 6 4
 ____

18). 8 0 0 9
 -5 6 3 7
 ____

19). 2 0 0 1
 -1 8 5 6
 ____

20). G.F Handel was a composer. He was born in 1685 A.D. and he died in 1759 A.D. At what age did he die?

Unexpected Test.

Name:

School: St. Teresa

class: 4

Age: 9

Date: January, 5 2001

Do the following ex.

1.
```
  4 1 4
- 2 3 6
  1 7 8
```

(2)
```
  6 2 3 1
- 3 6 7 2
  2 5 5 9
```

(3)
```
  7 0 1 0 5
- 3 1 2 5 6
  3 8 8 4 9
```

(4)
```
  6 7 1 4 8 0
- 4 7 2 4 9 1
  1 9 8 9 8 9
```

(5)
```
  8 1 2 1 3 2 5 0 4 3
- 4 1 2 4 3 4 6 4 5 8
  3 9 9 6 9 7 8 5 8 5
```

Grace

Name :
School : ST. TERESA
Class : 5
Age : 11
Date : January 5, 2001

Do the following ex.

1).
```
  3 2 4
- 1 3 5
  -----
  1 8 9
```

2).
```
  5 1 3 2
- 3 5 6 4
  -------
  1 5 6 8
```

3).
```
  8 0 2 1 4
- 4 1 3 6 7
  ---------
  3 8 8 4 7
```

4).
```
  7 1 4 7 2 5
- 5 1 5 7 3 6
  -----------
  1 9 8 9 8 9
```

5)
```
  9 2 3 4 3 6 0 4 2
- 4 3 3 6 3 8 8 6 7
  -----------------
  4 8 9 7 9 7 1 7 5
```

Name:

A (a) Write besides each question whether right or wrong

(b) Circle the wrong part(s).

E.g.

```
  400
- 201
  109  -- wrong
```

```
  7001
- 5234
  1767  -- right
```

1)

```
  832
- 475
  443  — wrong
```

Which did you use to check your answer: "D" or "BCA"?

2).

```
  5904
- 4007
  1097  — wrong
```

Which did you use to check your answer: "D" or "BCA"?

3).

```
  670
- 472
 1198  — wrong
```

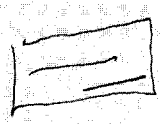
Which did you use to check your answer: "D" or "BCA"?

4).

```
  9004
- 4246
  5768  — wrong
```

Which did you use to check the answer: "D" or "BCA"?

NB: D is Borrowing ; BCA is Base complement addition

D - borrowing 

B.1) If you expect a pay check of \$624 but you are given \$454, what difference do you expect?

$$\begin{array}{r} 624 \\ -\,454 \\ \hline 170 \end{array}$$

2). If the pay difference brought to you reads \$128. (a). Are you O.K with it? (b) why?

a.) I'm not okay. Boys are mean!! I expect more money.

3). How did you check your answer? Did you use "Base Complement Addition (BCA)" or "Decomposition (D)".

I used (BCA).

4). Write your reason(s) why you like "D" or "(BCA)"?

I like it (BCA) because it takes less time to do the question!

5). Can you use both methods?

Yes you can, you could use one for the question and a

Name:

A. (a) Write besides each question whether right or wrong
(b) Circle the wrong part(s).

Eg
```
  450
- 201
  109   -- wrong
```

```
  7001
- 5234
  1767   -- right
```

1)
```
  834
- 475
  441
```
Wrong

Which did you use to check your answer: "D" or "BCA"

2)
```
  6003
- 4006
  1007
```
wrong

Which did you use to check the answer: "D" or "BCA"?

(3)
```
  671
- 472
 1198
```
Wrong

Which did you use to check the answer: "D" or "BCA"?

(4)
```
  8002
- 4235
  4767
```

Which did you use to check the answer: "D" or "BCA".

D = Borrowing   BCA = Base Complement Addition

B. (1) If you expect a pay check of $513 and you are given $356, what difference do you expect?

356   513
513 −356
      157

(2) If the pay difference brought to you reads $143. Are you O.K with it? Why? No because the answer doesn't match the answer that I got.

(3) How did you check your answer? Did you use "Base Complement Addition" or "Decomposition"?

I used BCA

(4) Write your reason(s) why you like "Decomposition" or "Base Complement Addition".

Because its faster

(5) Can you use both methods?

yes

APPENDIX C

Iconic Representations of Activities-BCA

Numeration Flash cards

(Iconic Representations of Activities-BCA)

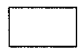


Figure i
Small = one

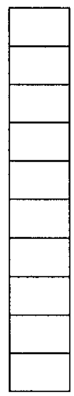


Figure i
Long = ten

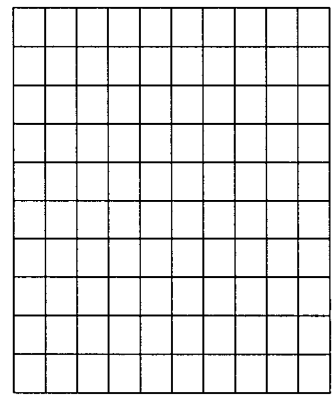
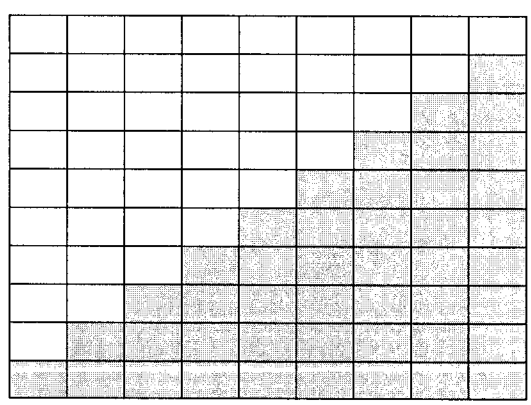


Figure iii
Flat = hundred

The small card is the unit of the numeration flash cards. The small represents one (Figure i).

The long could be a unit to the flat numeration card. It contains 10 small numeration cards. The long represents a ten (Figure ii).

One flat contains 100 ones or 10 longs. 10 flats are equivalent to a thousand (Figure iii).



1 2 3 4 5 6 7 8 9

Figure iv
Base complements

For any digit, what adds up to ten is the base complement of the digit, in base ten. To each of these digits the number of empty spaces on the column is the base complement.

Numeration

Numeration flash cards are displayed according to the number indicated

Mode of representation:
The conventional method of representation is from the bottom left to the top. After the first column on the leftmost side is filled then comes the next. As we fill from left to right, bottom left to the top, we keep track with the number till the counting is complete. Figures v, vi, and vii are array of 21, 23, and 32 respectively.

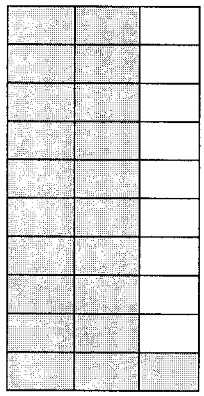


Figure v
21

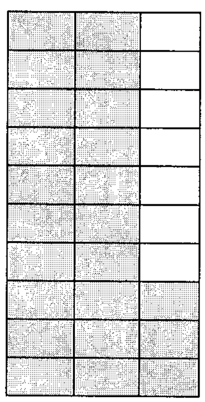


Figure vi
23

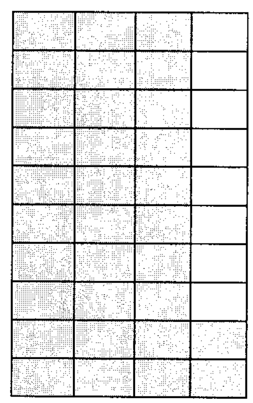


Figure vii
32

Modeling Subtraction

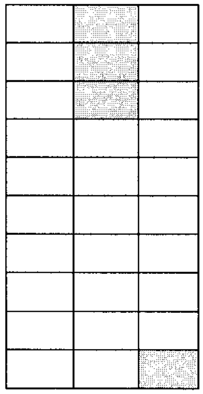


Figure viii
21-17

Subtracting a number is generated the same way as displaying the numeral- from left to right and bottom to the top. Description on the left shows that originally 21 cards were displayed and latter 17 of them were taken away- thus a model of 21-17.

Figures ix, x, xi and xii (below) represent the models 22-17, 23-15, 21-12 and 32-13 respectively.

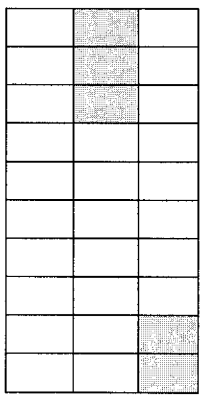


Figure ix
22-17

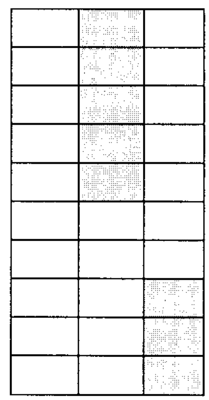


Figure x
23-15

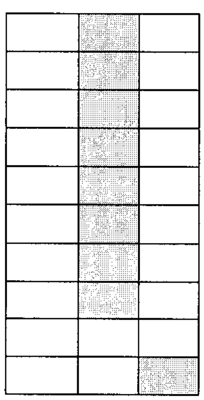


Figure xi
21-12

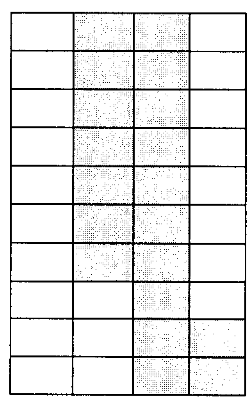


Figure xii
32-13

THE TRANSFORMATION (RENAMING)

This is regrouping without altering the number of cards present.

Figure xi a, shows 21 cards of which 17 are taken away. This leaves 4 cards; three, as the complement of the seven cards taken away on the middle column and one, on the last column. In the transformation process the complement cards are removed to the last column. Figure xi b (below) is a transformation of figure xi a (below).

Figures xii a and xii b, xiii a and xiii b, and xiv a and xiv b (below) are subtraction models and their corresponding transformations.

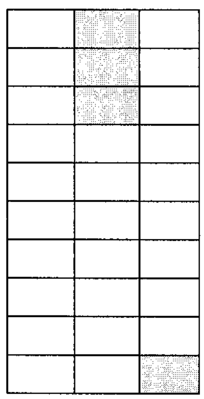


Figure xi a
21-17

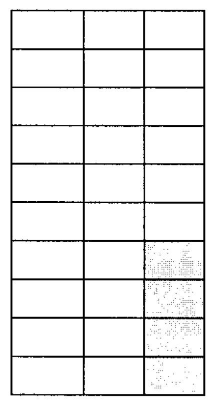


Figure xi b
24-20

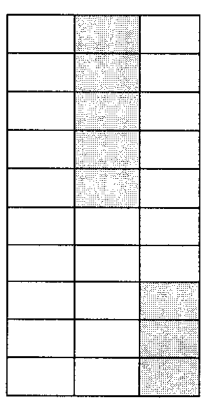


Figure xii a
23-15

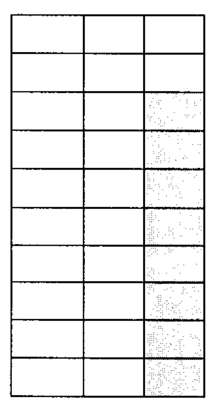


Figure xii b
28-20

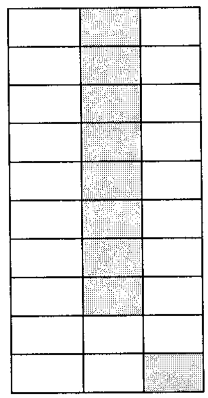


Figure xiii a
21-12

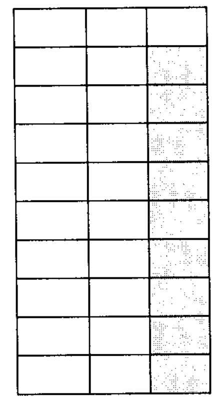


Figure xiii b
29-20

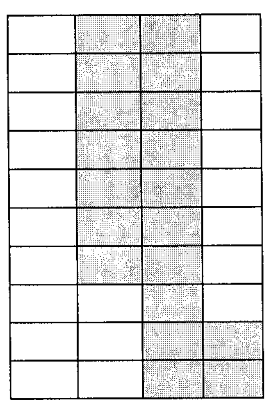


Figure xiv a
32-13

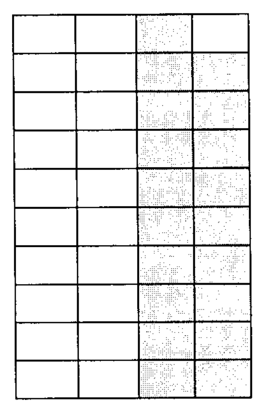


Figure xiv b
39-20