Separation properties and the group von Neumann algebra

by

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Abstract

A group endowed with a topology compatible with the group operations and for which every point has a neighborhood contained in a compact set is called a locally compact group. On such groups, there is a canonical translation invariant Borel measure that one may use to define spaces of p-integrable functions. With this structure at hand, abstract harmonic analysis further associates to a locally compact group various algebras of functions and operators. We study the capacity these algebras to distinguish closed subgroups of a locally compact group. We also characterize an operator algebraic property of von Neumann algebras associated to a locally compact group. Our investigations lead to a concise argument characterizing cyclicity of the left regular representation of a locally compact group.

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Chapter 1

Introduction and overview

A locally compact group is a group with a Hausdorff topology for which the group multiplication and inversion are continuous maps and such that every point has a neighborhood contained in a compact set. It follows from this that the neighborhood base at any point is a translate of the neighborhood base at the identity and that each point has a neighborhood base consisting of precompact sets. From this topological data, one may obtain a nonzero left translation invariant Borel measure with various regularity properties — the Haar measure — that is unique, and hence canonical. This measure is used to construct the spaces of p-integrable functions on a locally compact group G, of which we will be concerned with the space of integrable functions $L^1(G)$, the Hilbert space $L^2(G)$, and the von Neumann algebra $L^\infty(G)$. Considering the operators implementing left translation of $L^2(G)$ functions, we are naturally led to consider the weak operator topology closed algebra in $B(L^2(G))$ that they generate: the group von Neumann algebra $VN(G)$. The predual of this von Neumann algebra can be identified with an algebra of complex valued functions on G, the Fourier algebra $A(G)$, which is also identified with certain functions arising from the left regular representation of G. Considering functions arising in the analogous way from arbitrary representations produces the larger Fourier-Stieltjes algebra $B(G)$. Finally, we consider the Fourier multiplier algebra $M_{cb}A(G)$ that consists of functions on G that multiply the Fourier algebra into itself and for which the multiplication map satisfies a strong form of continuity.

Given a locally compact group G, we are interested in determining whether the operator algebra $VN(G)$ and the function algebras $A(G)$, $B(G)$, and $M_{cb}A(G)$ can distinguish a closed subgroup H. Thus we seek conditions on the group G, the group H, or the pair G and H that guarantee that certain separation properties are satisfied. This is the first theme of the thesis. The group von Neumann algebra $VN(G)$ contains a copy of $VN(H)$ for each closed subgroup H of G and it is natural to consider the existence of a projection of $VN(G)$ onto $VN(H)$. Since these algebras are canonically dual $A(G)$-modules, we may further require that there be a projection invariant under this module action. Such invariant projections are known to exist in a variety of settings, for example if H is an amenable group, then there exists an invariant projection $VN(G) \to VN(H)$ for any locally compact group G containing H as a closed subgroup [11]. If G is a SIN group, meaning the identity has a neighborhood base of conjugation invariant sets, then there is an invariant projection $VN(G) \to VN(H)$ for every closed subgroup H of G [20, Proposition 3.10]. The proof of this latter implication implicitly uses the notion of a bounded approximate indicator for H, which is a bounded net in the algebra $M_{cb}A(G)$ that converges in an appropriate sense to χ_H, the characteristic function of H in G, and the existence of which always guarantees the existence of an invariant projection $VN(G) \to VN(H)$. We study bounded approximate indicators for closed subgroups and establish new conditions for their existence.

We stated above that, if H is a closed subgroup of a locally compact group G, then an invariant projection for H exists whenever H is amenable. That amenability is inherited by closed subgroups entails that there is an invariant projection onto $VN(H)$ for every closed subgroup H of an amenable group G. These results display a connection between separation properties and amenability conditions, and along these lines we show that amenability of H is equivalent to the assertion that χ_H can be approximated in a certain topology by functions in $A(G)$. We investigate further when the characteristic function of a closed subgroup can be approximated by various algebras of functions and show that weaker forms of amenability suffice in several cases.

The second theme of this work, which is independent of the first, studies the notion of adapted normal states on the group von Neumann algebra of a locally compact group G as introduced by Neufang and Runde in [47]. We present the well known state of affairs for the von Neumann algebra $L^\infty(G)$, which asserts that the existence of adapted normal states is equivalent to the existence of faithful normal states, i.e. to σ-finiteness of the von Neumann algebra, and develop the analogous characterization for $VN(G)$. Our perspective is a novel one and produces a new (and we believe more natural) argument characterizing the existence of cyclic vectors for the left regular representation of G, originally shown by Greenleaf and Moskowitz in [25] and [26].

The document is organized as follows. In Chapter 2, we give an overview of the basic theory of C^*- and von Neumann algebras, providing references for the main foundational results of these subjects. We also review the relevant portions of the theory of projections in von Neumann algebras and of the supports of positive normal functionals, providing proofs of the key technical lemmas that will be needed. Chapter 3 provides the necessary background on abstract harmonic analysis, defining locally compact groups and the algebras associated to them. We also provide an introduction to the theory of operator spaces, a necessary technical tool in studying these algebras. The chapter ends with a short introduction to amenability theory for locally compact groups, defining and characterizing amenability and defining two weaker notions of amenability. Chapter 4 studies separation properties of closed subgroups. We develop various conditions for the approximability of characteristic functions of closed subgroups in Sections 4.1 and 4.2. A technical device for improving convergence properties of bounded nets of Fourier multipliers is presented in Section 4.3 and is employed in Section 4.4 to provide weaker conditions for the existence of bounded approximate indicators. The chapter ends with Section 4.5, where bounded approximate indicators are shown to produce invariant projections and a novel connection to operator amenability of a certain completely contractive Banach algebra is proven. Chapter 5 describes a gap in a proposition of [1] and gives counterexamples to a related claim. Chapter 6 characterizes the existences of adapted normal states on the von Neumann algebras $L^{\infty}(G)$ and $VN(G)$ associated to a locally compact group G, showing that in both cases this condition is equivalent to σ-finiteness of the algebra and to certain topological smallness properties of the group G. We characterize cyclicity of the left regular representation in Section 6.2. The final chapter lists some unresolved questions that arose during our investigations.

Chapter 2

Preliminaries on operator algebras

This chapter defines C^*- and von Neumann algebras and related concepts, states their basic properties, and provides examples that will be relevant to our investigations. We make no claim to providing a comprehensive treatment of these subjects and sometimes provide definitions in less than maximal generality that are sufficient and convenient for our purposes. Where proofs of significant results are omitted, specific references are provided, and more general references are provided at the end of each section.

All the topics discussed in this thesis can be broadly categorized as lying within the scope of functional analysis, with which we assume the reader is familiar. In particular, we will freely draw upon foundational results regarding Banach and Hilbert spaces as well as Banach algebras. A working knowledge of abstract measure theory is also prerequisite, as many of the spaces and algebras we consider arise from Borel measures on topological spaces. Note that our vector spaces will unanimously be over the complex field $\mathbb{C}$, that spaces of functions on a topological space will always consist of complex valued functions, and that every topology we consider will be Hausdorff, allowing us to couch our discussion of topological notions in the language of nets and convergence. For basic definitions and results from topology, functional analysis, measure theory, and Banach algebra theory that we take for granted, we refer the reader to [41], [52], [18], and [48], respectively.

We now fix some general notation that will be used throughout. A neighborhood of a point in a topological space always means a set containing an open set containing that point. Given a Banach space E, we denote its open and closed unit balls by $E_{<1}$ and $E_{\leq 1}$, respectively. If $X \subset E$ is any subset, then we write $\langle X \rangle$ the norm closed linear span of X and for a family of subsets $\mathcal{X}$ we let $\langle X : X \in \mathcal{X} \rangle = \langle \bigcup_{X \in \mathcal{X}} X \rangle$. For $\varphi \in E^*$ and $x \in E$ we write $\langle \varphi, x \rangle$ or, when we want to be explicit about the duality being applied, write $\langle \varphi, x \rangle_{E^*,E}$ for the value $\varphi(x)$. The ***annihilator*** of $X \subset E$ in E^* is the set $X^\perp = \{\varphi \in E^* : \langle \varphi, x \rangle = 0 \text{ for all } x \in X\}$ and the ***preannihilator*** of $Y \subset E^*$ is the set $Y_\perp = \{x \in E : \langle \varphi, x \rangle = 0 \text{ for all } \varphi \in Y\}$.

2.1 C^*-algebras

Let us recall the definition of an involution on a Banach algebra. We convene that a Banach algebra A always has a contractive multiplication, so that $\|ab\|_A \leq \|a\|_A \|b\|_A$ for all $a, b \in A$, and that the unit of a unital Banach algebra has norm one.

Definition 2.1.1. Let A be a Banach algebra. An ***involution*** on A is a map $A \to A : a \mapsto a^*$ satisfying:

1. $(\lambda a + b) = \overline{\lambda} a^* + b^*$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

2. $(ab)^* = b^* a^*$ for all $a, b \in A$.

3. $a^{**} = a$ for all $a \in A$.

4. $\|a^*\|_A = \|a\|_A$.

A Banach algebra A equipped with an involution is called a $*$***-algebra*** and is called a C^****-algebra*** if the norm on A satisfies $\|a^* a\|_A = \|a\|_A^2$ (the C^****-identity***) for all $a \in A$. We call a subalgebra B of A a C^****-subalgebra*** if it is closed under the involution and norm closed.

Example 2.1.2. (The C^*-algebra $\mathcal{C}_0(X)$) Let X be a locally compact Hausdorff topological space. A complex valued function f on X is said to ***vanish at infinity*** if $\sup_{x \in X \setminus K} |f(x)| \to 0$ as we let the compact subset K of X increase to X, i.e. where we have ordered the compact subsets of X by inclusion and require that the limit along this directed set be zero. The space $\mathcal{C}_0(X)$ denotes the ***continuous functions vanishing at infinity*** on X. This space is a commutative C^*-algebra when given the pointwise algebra operations, complex conjugation as involution, and the ***uniform norm*** defined by $\|f\|_{\mathcal{C}_0(X)} = \sup_{x \in X} |f(x)|$.

The ***support*** of a continuous function f on X, denoted $\operatorname{supp}(f)$, is defined to be the norm closure of $\{x \in X : f(x) \neq 0\}$. The ***continuous compactly supported functions*** $\mathcal{C}_c(X)$ are dense in $\mathcal{C}_0(X)$, but in general do not form a C^*-algebra, failing to be complete under the uniform norm unless X is compact, in which case $\mathcal{C}_c(X) = \mathcal{C}_0(X)$. When X is discrete, the compact subsets are the finite subsets, so that $\mathcal{C}_c(X)$ consists of the functions on X that are nonzero on only finitely many points of X, i.e. of finite support. To provide a concrete example, when $X = \mathbb{N}$ the space $\mathcal{C}_0(X)$ is the familiar space of complex valued sequences tending to zero, usually denoted $c_0(\mathbb{N})$ or c_0. When X is both discrete and compact, meaning that X is a finite set $\{x_1, \ldots, x_n\}$, we have $\mathcal{C}_c(X) = \mathcal{C}_0(X) = \mathbb{C}^n$.

Example 2.1.3. (The C^*-algebra $B(\mathcal{H})$) Let $\mathcal{H}$ be a Hilbert space. The space $B(\mathcal{H})$ of bounded linear operators on H given composition of operators for product, Hilbert space adjoint for involution, and the operator norm is a C^*-algebra. When $\mathcal{H}$ is finite dimensional, fixing a basis for $\mathcal{H}$ allows us to identify $B(\mathcal{H})$ with the n by n complex valued matrices $M_n(\mathbb{C})$, where $n = \dim \mathcal{H}$. The C^*-algebra $B(\mathcal{H})$ is only commutative when $\mathcal{H}$ has dimension 1. Any norm closed self-adjoint subalgebra of $B(\mathcal{H})$ is also a C^*-algebra.

The involution on a C^*-algebra produces the following classes of distinguished elements.

Definition 2.1.4. Let A be a C^*-algebra. We call $a \in A$:

1. ***Normal*** if $a^*a = aa^*$.

2. ***Self-adjoint*** if $a = a^*$.

3. ***Positive*** if $a = b^*b$ for some $b \in A$.

4. A ***projection*** if $a^2 = a^* = a$.

5. ***Unitary*** if A is unital and $a^*a = aa^* = 1$.

We write $a \geq 0$ to indicate that a is positive and define a partial order on the self-adjoint elements of A by saying that $a \geq b$ when $a - b \geq 0$. The set of positive elements of A is denoted A_+.

The positive elements of a C^*-algebra do indeed form a ***proper positive cone***, meaning that $\lambda a + b \in A_+$ when $a, b \in A_+$ and $\lambda \geq 0$ and that $a, -a \geq 0$ implies $a = 0$ for any self-adjoint $a \in A$.

The spectral radius of an element a of a Banach algebra A with unitization $A^\#$ is the quantity

$$\sup\left\{|\lambda| : \lambda \in \mathbb{C} \text{ and } a - \lambda 1 \text{ is not invertible in } A^\# \right\},$$

which is determined entirely by the algebraic structure of A. The spectral radius formula from Banach algebra theory and the C^*-identity together yield the following result.

Proposition 2.1.5. *Let A be a C^*-algebra. If $a \in A$ is normal, then $\|a\|_A$ is equal to its spectral radius.*

This has significant consequences for the theory of C^*-algebras: the norm of an arbitrary element a in a C^*-algebra A is given by $\|a\|_A = \|a^*a\|_A^{\frac{1}{2}}$ and therefore the norm on A is determined entirely by the algebraic structure of A. In particular, there is at most one norm on a $*$-algebra making it a C^*-algebra.

The terminology of Definition 2.1.4 is motivated by the following two examples.

Example 2.1.6. (Elements of $\mathcal{C}_0(X)$) Let X be a locally compact Hausdorff topological space. For the C^*-algebra $\mathcal{C}_0(X)$ of Example 2.1.2, the notions of Definition 2.1.4 correspond to familiar properties of functions as follows: every function is normal because $\mathcal{C}_0(X)$ is commutative; the self-adjoint, positive, and unitary functions are those that are real valued, nonnegative valued, and taking values in the complex unit circle $\mathbb{T}$, respectively; and a projection in $\mathcal{C}_0(X)$ is a function taking values in $\{0, 1\}$ and thus is the ***characteristic function*** χ_E of some subset E of X. When $\mathcal{C}_0(X)$ contains a nontrivial projection χ_E, i.e. one distinct from 0 or 1, so that E is not empty or X, the continuity of χ_E implies that the space X fails to be connected. Thus in general $\mathcal{C}_0(X)$ may not contain nontrivial projections.

Example 2.1.7. (Elements of $B(\mathcal{H})$) Let $\mathcal{H}$ be a Hilbert space. We discuss how definition 2.1.4 applies to the bounded linear operators $B(\mathcal{H})$ of Example 2.1.3 as follows:

1. We limit ourselves to observing that normality for operators in $B(\mathcal{H})$ is a generalization of its finite dimensional counterpart: normality of matrices over $\mathbb{C}$, and that a generalization of the fact that such matrices are diagonalizable carries over to (a certain class of operators on) infinite dimensional Hilbert spaces. See [38, Section 5.2] for details.

2. Self-adjoint operators have spectrum contained in $\mathbb{R}$, and normal operators with spectrum contained in $\mathbb{R}$ are self-adjoint.

3. Positive operators T in $B(\mathcal{H})$ are those satisfying $\langle T\xi|\xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$.

4. Projections are the orthogonal projections onto closed subspaces of $\mathcal{H}$. In fact, the association of its range to a projection yields a one to one correspondence between projections in $B(\mathcal{H})$ and the closed subspaces of $\mathcal{H}$.

5. Unitary operators in $B(\mathcal{H})$ are the isomorphisms of the Hilbert space $\mathcal{H}$, meaning the bijective inner product (equivalently, norm) preserving maps.

The maps between C^*-algebras that preserve the pertinent structures are the following.

Definition 2.1.8. Let A and B be C^*-algebras. A bounded linear map $\Psi : A \to B$ is:

1. A ***$*$-homomorphism*** if $T(ab) = T(a)T(b)$ and $T(a^*) = T(a)^*$ for all $a, b \in A$.

2. A ***$*$-isomorphism*** if T is a bijective $*$-homomorphism.

3. ***Positive*** if $T(A_+) \subset B_+$.

A positive linear functional of norm one on A is called a ***state***.

Example 2.1.9. (The duality between $\mathcal{C}_0(X)$ and $M(X)$) Let X be a locally compact topological space. By a ***Radon measure*** we mean a Borel measure on X that is outer regular on measurable sets, inner regular on open sets, and that takes finite values on compact sets. Given a complex Radon measure μ, integration $f \mapsto \int_X f d\mu$ defines a bounded linear functional on $\mathcal{C}_0(X)$ of norm $|\mu|(X)$. The Riesz representation theorem [18, Theorem 7.2] asserts that all elements of $\mathcal{C}_0(X)^*$ arise this way and moreover that the map $\mu \mapsto [f \mapsto \int_X f d\mu]$ is isometric. Let $M(X)$ denote the complex Radon measures on X, so that $M(X) = \mathcal{C}_0(X)^*$.

That a measure $\mu \in M(X)$ corresponds to a positive linear functional on $\mathcal{C}_0(X)$ is the assertion that $\int_X f d\mu = \langle \mu, f\rangle \geq 0$ for all positive $f \in \mathcal{C}_0(X)$, from which one may deduce that $\mu(E) \geq 0$ for all measurable subsets E of X. In particular, μ is a state exactly when μ is a positive measure of mass one, i.e. a probability measure on X.

See [18, Chapter 7] for a concise exposition of Radon measures on locally compact spaces and the duality between $\mathcal{C}_0(X)$ and $M(X)$. An interesting alternative treatment is given in [4].

Example 2.1.10. (Vector functionals on $B(\mathcal{H})$) Let $\mathcal{H}$ be a Hilbert space and let ξ and η be vectors in $\mathcal{H}$. The map $T \mapsto \langle T\xi|\eta\rangle$ defines a bounded linear functional on $B(\mathcal{H})$ of norm $\|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}}$, denoted $\omega_{\xi,\eta}$ or ω_ξ if $\xi = \eta$. We call such functionals on $B(\mathcal{H})$ ***vector functionals***, and if we may write a given functional $\varphi \in B(\mathcal{H})^*$ in the form $\omega_{\xi,\eta}$ for some $\xi, \eta \in \mathcal{H}$ then we say that φ is ***implemented by ξ and η*** and refer to these as the ***implementing vectors*** for φ. It is clear from the characterization of positivity in $B(\mathcal{H})$ given in Example 2.1.7 that a vector functional is positive when it is of the form ω_ξ for some $\xi \in \mathcal{H}$ and is a state when ξ is a unit vector.

The following analogue of the Cauchy-Schwarz inequality holds for positive linear functionals.

Proposition 2.1.11. ***(Cauchy-Schwarz inequality)*** *Let A be a C^*-algebra and let ω be a positive linear functional on A. Then $|\langle \omega, a^*b\rangle|^2 \leq \langle \omega, a^*a\rangle \langle \omega, b^*b\rangle$ for all $a, b \in A$.*

We will need the following useful results.

Proposition 2.1.12. *Let A be a unital C^*-algebra. The following hold:*

1. *A linear functional ω on A is positive if and only if $\langle \omega, 1\rangle = \|\omega\|_{A^*}$.*
2. *A linear functional ω on A is a state if and only if $\langle \omega, 1\rangle = \|\omega\|_{A^*} = 1$.*
3. *The set of states on A is a weak* compact, convex set.*

An important consequence of Proposition 2.1.12: if A is a unital C^*-algebra and B is a C^*-subalgebra of A that contains the unit of A, then the restriction to B of a state on A is a state on B.

The following result, a manifestation of the fact that the algebraic structure of a C^*-algebra determines its norm structure, implies that the range of a $*$-homomorphism is always a C^*-algebra.

Proposition 2.1.13. *Let A and B be C^*-algebras and $\Psi : A \to B$ a linear map.*

1. *If Ψ is a $*$-homomorphism, then Ψ is contractive.*
2. *If Ψ is an injective $*$-homomorphism, then Ψ is isometric.*

The following two results are fundamental to the theory of C^*-algebras. The latter motivates the interpretation of C^*-algebras as noncommutative topological spaces.

Theorem 2.1.14. *Let A be a C^*-algebra. The following hold:*

1. *[58, I.9.18] There exists a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism $A \to B(\mathcal{H})$.*
2. *[58, I.4.4] If A is commutative, then there exists a locally compact Hausdorff space X and an isometric $*$-isomorphism $A \to \mathcal{C}_0(X)$.*

We refer to a $*$-homomorphism of a C^*-algebra A into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ as a ***representation*** and say that A is ***represented*** in $B(\mathcal{H})$. A representation is called ***faithful*** when it is injective, in which case it is automatically isometric by virtue of Proposition 2.1.13.

An especially readable introduction to C^*-algebras is given in [38], while [58] provides a more comprehensive treatment.

2.2 Von Neumann algebras

Amongst the C^*-algebras, those that are dual spaces form an especially rich class.

Definition 2.2.1. A C^*-algebra M that is a dual space is called a ***von Neumann algebra***, in which case M has a unique predual, denoted M_*. A C^*-subalgebra N of M is called a ***von Neumann subalgebra*** if it is closed in the weak* topology of M.

Example 2.2.2. (The L^p-spaces) Let X be a locally compact space and μ a Radon measure on X. Recall that, for $1 \leq p < \infty$, the space $L^p(X)$ is the set of measurable complex valued functions f on X such that $\|f\|_{L^p(X)} = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$ is finite, where we identify functions that agree μ-almost everywhere. The analogous space for $p = \infty$ is that of the essentially bounded measurable functions: those f for which the essential supremum norm $\|f\|_{L^\infty(X)} = \inf\{\alpha \geq 0 : \{x \in X : |f(x)| > \alpha\} \text{ is } \mu\text{-null}\}$ is finite. We record some important facts:

1. The space $L^2(X)$ is a Hilbert space with inner product $\langle \xi | \eta \rangle = \int_X \xi \overline{\eta} d\mu$.

2. The continuous compactly supported functions $\mathcal{C}_c(X)$ are dense in $L^p(X)$ for $1 \leq p < \infty$.

3. When μ is counting measure, the spaces we have defined are the familiar spaces $\ell^p(X)$.

4. When μ is ***σ-finite***, meaning X is the union of countably many sets of finite measure, the space $L^\infty(X)$ is the dual of $L^1(X)$ via $\langle \phi, f \rangle_{L^\infty(X), L^1(X)} = \int_X \phi f d\mu$. This duality holds more generally under a slightly modified definition of $L^\infty(X)$ that coincides with ours for σ-finite measures, see [18, Chapter 6] for details. We use this duality in the greater generality, the distinction in the definition of $L^\infty(X)$ will not arise.

Example 2.2.3. (The von Neumann algebra $L^\infty(X)$) Let X be a locally compact space and μ a Radon measure on X. The space $L^\infty(X)$ under pointwise algebra operations, complex conjugation, and with essential supremum norm is a commutative von Neumann algebra with predual $L^1(X)$. This algebra acts on $L^2(X)$ via pointwise multiplication: given $\phi \in L^\infty(X)$, the map $M_\phi : \xi \mapsto \phi\xi$ is bounded and linear. The mapping

$$L^\infty(X) \to B\left(L^2(X)\right) : \phi \mapsto M_\phi \tag{2.2.1}$$

realizes $L^\infty(X)$ as a von Neumann subalgebra of $B\left(L^2(X)\right)$, i.e. is a faithful $*$-representation. If X is moreover a topological space and μ a Borel measure, then $\mathcal{C}_0(X)$ is a C^*-subalgebra of $L^\infty(X)$, since the supremum and essential supremum norms coincide for continuous functions.

The normal, self-adjoint, positive, and unitary functions in $L^\infty(X)$ are characterized, similar to Example 2.1.6, as those with range in $\mathbb{C}$, $\mathbb{R}$, $[0, \infty)$, and $\mathbb{T}$, respectively. The projections in $L^\infty(X)$ are the characteristic functions of measurable subsets of X. Thus $L^\infty(X)$ has an abundance of projections — their span is in fact norm dense, a property common to all von Neumann algebras. When X is a topological space, this is in sharp contrast to the generic situation for the C^*-algebra $\mathcal{C}_0(X)$, which may fail to have nontrivial projections.

The predual $L^1(X)$ consists of those complex Radon measures on X that are absolutely continuous with respect to μ, i.e. those for which every μ-null set is null. Positive normal functionals on $L^\infty(X)$ correspond to positive Radon measures on X that are absolutely continuous with respect to μ, and states to probability measures with this property. If we represent $L^\infty(X)$ in $B(L^2(X))$ as in (2.2.1), then for $\xi, \eta \in \mathcal{H}$ the vector functional $\omega_{\xi,\eta}$ is given by

$$\langle M_\phi, \omega_{\xi,\eta} \rangle = \langle \phi\xi | \eta \rangle = \int_X \phi\xi\overline{\eta} = \langle \phi, \xi\overline{\eta} \rangle_{L^\infty(X), L^1(X)},$$

so that $\omega_{\xi,\eta}|_{L^\infty(X)}$ corresponds to the $L^1(X)$ function $\xi\overline{\eta}$.

Example 2.2.4. (The von Neumann algebra $B(\mathcal{H})$) Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_\alpha)_\alpha$. The trace class operators $T(\mathcal{H})$, defined to be those $T \in B(\mathcal{H})$ such that $\sum_\alpha \langle |T| e_\alpha | e_\alpha \rangle$ is finite, where $|T| = (T^*T)^{\frac{1}{2}}$, form a predual for $B(\mathcal{H})$ under the duality $\langle T, S \rangle_{B(\mathcal{H}), T(\mathcal{H})} = \sum_\alpha \langle TSe_\alpha | e_\alpha \rangle$. Thus $B(\mathcal{H})$ is a von Neumann algebra. Below we provide an alternative description of the predual of $B(\mathcal{H})$ that is more suitable for our purposes.

Definition 2.2.5. Let $\mathcal{H}$ be a Hilbert space and let $(T_\alpha)_\alpha$ be a net in $B(\mathcal{H})$ and $T \in B(\mathcal{H})$. We say that T_α converges to T in the:

1. **weak operator topology** if $\langle T_\alpha\xi | \eta \rangle \to \langle T\xi | \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$.

2. **strong operator topology** if $\|T_\alpha\xi\|_{\mathcal{H}} \to \|T\xi\|_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$.

3. **σ-weak operator topology** if $\sum_{n=1}^\infty \langle T_\alpha\xi_n | \eta_n \rangle \to \sum_{n=1}^\infty \langle T\xi_n | \eta_n \rangle$ for all sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}$ such that $\sum_n \|\xi_n\|_{\mathcal{H}}^2$ and $\sum_n \|\eta_n\|_{\mathcal{H}}^2$ are finite.

Some comments regarding these topologies are in order. First, specifying the convergent nets as we have done in each of (1) to (3) does indeed determine a unique topology, see [41, p.73]. Second, the σ-weak topology is often called the ultraweak topology (a term we avoid), and we will see shortly that this is exactly the weak* topology of $B(\mathcal{H})$. Third, notice that convergence in the weak operator topology is exactly pointwise convergence on the vector functionals on $B(\mathcal{H})$, since $\langle T\xi | \eta \rangle = \langle T, \omega_{\xi,\eta} \rangle$. In (3), the conditions on the sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ assert exactly that $\sum_{n=1}^\infty \omega_{\xi_n,\eta_n}$ is a convergent series in $B(\mathcal{H})^*$, so convergence in the σ-weak operator topology is pointwise convergence against such series of vector functionals, because $\sum_{n=1}^\infty \langle T\xi_n | \eta_n \rangle = \langle T, \sum_{n=1}^\infty \omega_{\xi_n,\eta_n} \rangle$.

Proposition 2.2.6. *Let $\mathcal{H}$ be a Hilbert space.*

1. *The σ-weak operator topology and the weak* topology on $B(\mathcal{H})$ coincide.*

2. *Weak $\subset$ strong $\subset$ norm and weak $\subset$ σ-weak $\subset$ norm, where $\subset$ indicates that the left topology is weaker than the right.*

3. *The multiplication on $B(\mathcal{H})$ is separately continuous in the weak, strong, norm and σ-weak topologies.*

Definition 2.2.7. Let $\mathcal{H}$ be a Hilbert space. The ***commutant*** of a subset E of $B(\mathcal{H})$ is the set of operators that commute with each element of E.

The commutant of any nonempty subset of $B(\mathcal{H})$ is a subalgebra closed in the weak operator topology and, when E is self-adjoint, is self-adjoint and therefore a von Neumann algebra. In particular, this occurs when E is itself a von Neumann algebra.

Example 2.2.8. (The commutant of $B(\mathcal{H})$ in itself) For a Hilbert space $\mathcal{H}$, the commutant of $B(\mathcal{H})$ in itself is the centre of the algebra, which is $\mathbb{C}I$: if $T' \in B(\mathcal{H})'$ and $\xi, \eta \in \mathcal{H}$ are nonzero, then, letting $P_{\xi,\eta}\nu := \langle \nu|\xi\rangle \eta$ for $\nu \in \mathcal{H}$, we have $\langle \nu|\xi\rangle T'\eta = T'P_{\xi,\eta}\nu = P_{\xi,\eta}T'\nu = \langle T'\nu|\xi\rangle \eta$. Setting $\nu = \xi$ yields $T'\eta = \langle \xi|\xi\rangle^{-1} \langle T'\xi|\xi\rangle \eta$, implying that $\langle \xi|\xi\rangle^{-1} \langle T'\xi|\xi\rangle$ is independent of the choice of nonzero ξ, so is a scalar depending only on T' and $T' \in \mathbb{C}I$.

Example 2.2.9. (The commutant of $L^\infty(X)$) Let X be a locally compact space and μ a Radon measure on X. We show that the commutant of $L^\infty(X)$ in $B(L^2(X))$ is itself. Since $L^\infty(X)$ is commutative, we have $L^\infty(X) \subset L^\infty(X)'$. Let $T \in L^\infty(X)'$ and for each compact subset K of G let $\phi_K = T\chi_K$. If $\xi \in L^2(X) \cap L^\infty(X)$ with $\operatorname{supp}(\xi) \subset K$, so that $\xi = \xi\chi_K$, then

$$T\xi = T\xi\chi_K = TM_\xi\chi_K = M_\xi T\chi_K = M_\xi\phi_K = \xi\phi_K.$$

Set $E = \{s \in K : |\phi_K(s)| \geq \|T\| + 1\}$. Since $\chi_E \in L^2(X) \cap L^\infty(X)$ and $\operatorname{supp}(\chi_E) \subset K$, we have

$$\|T\chi_E\|^2_{L^2(X)} = \|\chi_E\phi_K\|^2_{L^2(X)} = \int_E |\phi_K|^2 \geq (\|T\| + 1)^2 \int_E 1 > \|T\|^2 |E| = \|T\|^2 \|\chi_E\|^2_{L^2(X)},$$

a contradiction unless $\|\chi_E\|_{L^2(X)} = 0$. Thus E must be a null set and consequently $\|\phi_K\|_{L^\infty(X)} \leq \|T\| + 1$. Directing the compact subsets of G by inclusion, the bounded net $(\phi_K)_K$ in $L^\infty(X)$ must have a weak* cluster point, and passing to a subnet we may assume a weak* limit, say $\phi \in L^\infty(X)$. If $\xi \in \mathcal{C}_c(G)$, say with support K, then for any compact $L \supset K$ we have $\xi \in L^2(X) \cap L^\infty(X)$ and $\operatorname{supp}(\xi) \subset L$, so that $T\xi = \xi\phi_L$. Therefore $T\xi = \lim_L^{w^*} \xi\phi_L = \xi\phi$ by weak* continuity of multiplication. Since $\mathcal{C}_c(X)$ is norm dense in $L^2(G)$, it follows that $T\xi = \xi\phi$ for all $\xi \in L^2(G)$, hence $T = M_\phi \in L^\infty(G)$ and $L^\infty(X)' = L^\infty(X)$.

Theorem 2.1.14 implies that C^*-algebras are exactly the norm closed subalgebras of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Amongst them, the von Neumann algebras are those that are closed in any of the weaker topologies of Definition 2.2.5.

Theorem 2.2.10. *Let A be a C^*-algebra in $B(\mathcal{H})$. The following are equivalent:*

1. *A is a von Neumann algebra.*

2. *A is closed in the weak, strong, or σ-weak operator topology of $B(\mathcal{H})$.*

3. *$A'' = A$.*

The equivalence of (1) and (3) in the preceding theorem is referred to as the ***double commutant theorem***.

In the context of von Neumann algebra theory, maps that are weak*-weak* continuous between von Neumann algebras are referred to as ***normal***. The normal linear functionals are therefore those in the predual.

The following is the von Neumann algebraic analogue of Theorem 2.1.14, and motivates the view of von Neumann algebras as noncommutative measure spaces.

Theorem 2.2.11. *Let M be a von Neumann algebra. The following hold [58, III, Section 1]:*

1. *There is a Hilbert space $\mathcal{H}$ and a weak* continuous isometric $*$-homomorphism $M \to B(\mathcal{H})$.*

2. *If M is commutative, then there exists a compact Hausdorff space X and a positive Radon measure μ such that there is a weak* continuous isometric $*$-isomorphism $M \to L^\infty(X, \mu)$.*

Theorem 2.2.11 faithfully represents any von Neumann algebra M concretely as bounded operators on some Hilbert space $\mathcal{H}$. Since M is then weak* closed in $B(\mathcal{H})$, this allows us to describe the predual of M in terms of that of $B(\mathcal{H})$: the normal functionals on M are exactly the restrictions of the normal functionals on $B(\mathcal{H})$ to M, so are of the form $\sum_{n=1}^{\infty} \omega_{\xi_n,\eta_n}$ for sequences $(\xi_n)_{n=1}^{\infty}$ and $(\eta_n)_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\sum_n \|\xi_n\|_{\mathcal{H}}^2$ and $\sum_n \|\eta_n\|_{\mathcal{H}}^2$ are finite. This description of the predual of a von Neumann algebra makes it clear that if N is a von Neumann subalgebra of M, then normal functionals on N extend to normal functionals on M, and by Proposition 2.1.12 that normal states extend to normal states. Every von Neumann algebra admits a faithful representation on some Hilbert space such that all of its positive normal functionals are vector functionals, which we call the ***standard representation*** [33].

A gentle introduction to the portions of von Neumann algebra theory presented here is available in [38]. An encyclopedic reference is [58].

2.3 Projections in von Neumann algebras

In this section we collect together the the facts about projections in von Neumann algebras that will be required in Chapter 6. Proofs are included for many basic results as the techniques they expose have much in common with those used in Chapter 6.

By Theorem 2.2.11, we may faithfully represent any von Neumann algebra M as bounded operators on a Hilbert space $\mathcal{H}$ and thereby view projections in M as orthogonal projections onto closed subspaces of $\mathcal{H}$, as per Example 2.1.7. Let us record some elementary facts about these projections.

Lemma 2.3.1. *Let $\mathcal{H}$ be a Hilbert space. For orthogonal projections P and Q in $B(\mathcal{H})$:*

1. *$\|P\|_{B(\mathcal{H})} = 1$ unless $P = 0$.*

2. *The range and kernel of P are closed orthogonal subspaces of $\mathcal{H}$ with direct sum $\mathcal{H}$.*

3. $rng(I-P) = ker(P)$ *and* $ker(I-P) = rng(P)$.

4. *If* $PQ = 0$, *then* $P+Q$ *is orthogonal projection onto* $rng(P) \oplus rng(Q)$.

5. $PQ = 0 \Longleftrightarrow rng(P) \perp rng(Q)$.

6. $PQ = Q \iff QP = Q \iff rng(Q) \subset rng(P)$.

These last two properties can be stated for projections in a von Neumann algebra without reference to a particular representation.

Definition 2.3.2. Let M be a von Neumann algebra and P and Q projections in M. We call P and Q ***orthogonal*** and write $P \perp Q$ if $PQ = 0$ and call Q a ***subprojection*** of P if $PQ = QP = Q$.

Example 2.3.3. (Projections in $L^\infty(X)$) Let X be a locally compact space and μ a Radon measure on X. We saw in Example 2.2.3 that the projections in $L^\infty(X)$ are characteristic functions of measurable subsets of X. Given measurable subsets E and F, we have $\chi_E \chi_F = \chi_{E\cap F}$, so that χ_E and χ_F are orthogonal exactly when $E \cap F$ is null. The projection χ_E is a subprojection of χ_F when $\chi_{E\cap F} = \chi_E$, equivalently when $\chi_{E\setminus F} = 0$, i.e. when $E \subset F$ up to a null set.

Projections in a von Neumann algebra are self-adjoint and the partial order of Definition 2.1.4 on self-adjoint elements of a C^*-algebra restricted to projections agrees with the notion of subprojection, i.e. $Q \le P$ if and only if Q is a subprojection of P.

Proposition 2.3.4. *Let $\mathcal{H}$ be a Hilbert space and M a von Neumann algebra in $B(\mathcal{H})$. For a family of projections $\mathcal{P}$ in M, define the supremum and infimum of $\mathcal{P}$ to be the orthogonal projections onto* $\langle rng(P) : P \in \mathcal{P}\rangle$ *and* $\bigcap_{P\in\mathcal{P}} rng(P)$, *respectively. With these operations, the projections in M form a complete lattice.*

In certain contexts, the following provides a means of reducing arguments involving positive operators to projections.

Definition 2.3.5. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. The projection onto $\overline{\operatorname{rng}(T)}$ is called the ***range projection*** of T and is denoted $R(T)$.

Proposition 2.3.6. *Let M a von Neumann algebra.*

1. *If* $T \in M$, *then* $R(T) \in M$.

2. *[38, 7.2.5] If* $T \in M$ *is positive and* $\omega \in M_*$ *is positive, then* $\langle \omega, T\rangle = 0$ *if and only if* $\langle \omega, R(T)\rangle = 0$.

Definition 2.3.7. Let M be a von Neumann algebra. A projection P in M is called ***σ-finite*** if any family of nonzero mutually orthogonal subprojections of P must be countable. We call M ***σ-finite*** if the identity is a σ-finite projection, equivalently if every family of nonzero mutually orthogonal projections in M is countable.

It is clear that subprojections of σ-finite projections are themselves σ-finite.

We provide proofs for the following routine lemmas to exhibit some of the common features of arguments appearing in the projection theory of von Neumann algebras.

Lemma 2.3.8. *Let $\mathcal{H}$ be a Hilbert space and M a von Neumann algebra in $B(\mathcal{H})$. The projection onto $\langle M'\xi\rangle$ is in M for any $\xi \in \mathcal{H}$.*

Proof. Let P be orthogonal projection onto $\langle M'\xi\rangle$, which is in $B(\mathcal{H})$. If $T', S' \in M'$, then $T'PS'\xi = T'S'\xi = PT'S'\xi$, implying that $T'P = PT'$ on $\langle M'\xi\rangle$, equivalently that $T'P = PT'P$, for every $T' \in M'$. Taking adjoints and applying this result to the adjoint, we obtain $P(T')^* = P(T')^* P = (T')^* P$. Therefore $P \in M'' = M$. $\square$

Proposition 2.3.9. *Let $\mathcal{H}$ be a Hilbert space and M a von Neumann algebra in $B(\mathcal{H})$. Any projection P in M can be written as $\sum_\alpha P_\alpha$ for mutually orthogonal projections P_α, each of which is orthogonal projection onto $\langle M'\xi_\alpha\rangle$ for some unit vector $\xi_\alpha \in \mathcal{H}$.*

Proof. For $\xi \in \mathcal{H}$, let P_ξ denote orthogonal projection onto $\langle M'\xi\rangle$. The result is trivial if $P = 0$, so assume P is nonzero and let ξ be a unit vector in the range of P. Then $PT'\xi = T'P\xi = T'\xi$ for every $T' \in M'$, hence $\langle M'\xi\rangle$ is contained in the range of P and Lemma 2.3.1 yields $P_\xi \leq P$. We have shown that every nonzero projection in M contains a nonzero subprojection of the form P_ξ for some $\xi \in \mathcal{H}$. The collection $\mathcal{P}$ of families of mutually orthogonal nonzero subprojections of P of the form P_ξ for $\xi \in \mathcal{H}$ is partially ordered by inclusion of families and is nonempty. If $\mathcal{C}$ is a chain in $\mathcal{P}$, then it is clear that $\bigcup_{F\in\mathcal{P}} F \in \mathcal{P}$ and that $\bigcup_{F\in\mathcal{P}} F$ dominates each family in $\mathcal{C}$ under the inclusion ordering, whence is an upper bound in $\mathcal{P}$ for $\mathcal{C}$. By Zorn's lemma, $\mathcal{P}$ contains a maximal family, say $(P_{\xi_\alpha})_\alpha$ for $\xi_\alpha \in \mathcal{H}$. If $\sum_\alpha P_{\xi_\alpha} < P$, then $P - \sum_\alpha P_{\xi_\alpha}$ is a nonzero subprojection of P, hence contains a nonzero subprojection of the form P_η for some $\eta \in \mathcal{H}$. Then $P_\eta \leq P - \sum_\alpha P_{\xi_\alpha}$, so $P_\eta + \sum_\alpha P_{\xi_\alpha} \leq P$, and we have $P_\eta = P_\eta\left(P - \sum_\alpha P_{\xi_\alpha}\right) = P_\eta - P_\eta \sum_\alpha P_{\xi_\alpha}$, implying that $\sum_\alpha P_\eta P_{\xi_\alpha} = 0$ and that P_η is orthogonal to each P_{ξ_α}. But then the family $\{P_\eta\} \bigcup (P_{\xi_\alpha})_\alpha$ strictly dominates $(P_{\xi_\alpha})_\alpha$ in the partial order on $\mathcal{P}$, a contradiction. Therefore $\sum_\alpha P_{\xi_\alpha} = P$. $\square$

Example 2.3.10. (σ-finiteness of $B(\mathcal{H})$) Let $\mathcal{H}$ be a separable Hilbert space. Let $(P_\alpha)_\alpha$ be a family of mutually orthogonal projections in $B(\mathcal{H})$. Choosing a unit vector e_α in the range of P_α, we have $e_\alpha \perp e_\beta$ for all $\alpha \neq \beta$, so $(e_\alpha)_\alpha$ is an orthonormal set in $\mathcal{H}$ and therefore must be countable, because separable Hilbert spaces are those with countable orthonormal bases. Thus $(P_\alpha)_\alpha$ is countable and $B(\mathcal{H})$ is σ-finite. (It follows that any von Neumann algebra that can be faithfully represented on $B(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$ is σ-finite.)

Conversely, let $\mathcal{H}$ be a Hilbert space for which $B(\mathcal{H})$ is σ-finite. By Proposition 2.3.9 we may write $I = \sum_{n=1}^\infty P_n$, where each P_n is orthogonal projection onto $\left\langle B(\mathcal{H})' \xi_n\right\rangle$ for some unit vector ξ_n. We have $\left\langle B(\mathcal{H})' \xi_n\right\rangle = \mathbb{C}\xi_n$ by Example 2.2.8, implying that $P_n\eta = \langle \eta|\xi_n\rangle \xi_n$. Then $(\xi_n)_{n=1}^\infty$ is an orthonormal set in $B(\mathcal{H})$ and $\eta = \sum_{n=1}^\infty P_n\eta = \sum_{n=1}^\infty \langle \eta|\xi_n\rangle \xi_n$ for all $\eta \in \mathcal{H}$, so this set is an orthonormal basis and $\mathcal{H}$ is separable.

We will characterize σ-finiteness for a certain class of von Neumann algebras in Chapter 6. For now, we develop the connection between σ-finiteness of von Neumann algebras and the existence of certain positive normal functionals.

Definition 2.3.11. Let ω be a positive normal functional on a von Neumann algebra M. The ***support*** of ω is the minimal projection S_ω in M satisfying $\langle \omega, S_\omega \rangle = \langle \omega, I \rangle$.

If ω is a positive normal functional, then it is clear that $\alpha\omega$ has the same support for any $\alpha > 0$.

Example 2.3.12. (Support of positive normal functionals on $L^\infty(X)$) Let X be a locally compact space, μ a Radon measure on X, and f a nonzero positive normal functional on $L^\infty(X)$, which is a positive $L^1(X)$ function by Example 2.2.3. We noted in Example 2.3.3 that projections in $L^\infty(X)$ are characteristic functions of measurable subsets of X. Recall that the ***support*** of the $L^1(X)$ function f is defined to be the closed set

$$\operatorname{supp}(f) = X \setminus \bigcup \{U \subset X : U \text{ is open and } f\mu(U) = 0\},$$

see [4, Chapter III, Section 2]. Since f is nonzero and positive, $\operatorname{supp}(f)$ is not null and $\left\langle f, \chi_{\operatorname{supp}(f)} \right\rangle = \int_{\operatorname{supp}(f)} f = \int_X f = \langle f, I \rangle$, implying that $S_f \leq \chi_{\operatorname{supp}(f)}$. Letting $S_f = \chi_E$, we established that $E \setminus \operatorname{supp}(f)$ is null. Since $\langle f, I \rangle = \langle f, \chi_E \rangle = \left\langle f, \chi_{\operatorname{supp}(f)\setminus E} \right\rangle + \left\langle f, \chi_{\operatorname{supp}(f)} \right\rangle = \left\langle f, \chi_{\operatorname{supp}(f)\setminus E} \right\rangle + \langle f, I \rangle$, we have $0 = \left\langle f, \chi_{\operatorname{supp}(f)\setminus E} \right\rangle = \int_{\operatorname{supp}(f)\setminus E} f$ and thus $\operatorname{supp}(f) \setminus E$ is null. Therefore $S_f = \chi_{\operatorname{supp}(f)}$.

We will frequently use the following elementary facts about supports of positive normal functionals.

Lemma 2.3.13. *Let $\mathcal{H}$ be a Hilbert space and M a von Neumann algebra in $B(\mathcal{H})$.*

1. *For a projection P in M and a unit vector $\xi \in \mathcal{H}$, we have that*

$$\xi \in rng P \iff P\xi = \xi \iff \langle \omega_\xi, P \rangle = 1 \iff S_{\omega_\xi} \leq P.$$

2. *The projection S_{ω_ξ} has range $\langle M'\xi \rangle$, for any $\xi \in \mathcal{H}$.*

3. *For $\xi, \eta \in \mathcal{H}$, we have that*

$$\langle \omega_\xi, S_{\omega_\eta} \rangle = 0 \iff S_{\omega_\eta}\xi = 0 \iff \xi \perp \langle M'\eta \rangle \iff \eta \perp \langle M'\xi \rangle \iff S_{\omega_\eta} S_{\omega_\xi} = 0.$$

   It follows that $\langle \omega_\xi, S_{\omega_\eta} \rangle = 0$ exactly when $\left\langle \omega_\eta, S_{\omega_\xi} \right\rangle = 0$.

4. *For any positive normal functional ω on M and any $T \in M$, we have that*

$$\langle \omega, T \rangle = \langle \omega, S_\omega T \rangle = \langle \omega, T S_\omega \rangle = \langle \omega, S_\omega T S_\omega \rangle.$$

Proof. (1) The first equivalence is clear and the third follows from the definition of S_{ω_η}. If $1 = \langle \omega_\xi, P\rangle = \langle P\xi|\xi\rangle$, then

$$0 \le \|P\xi - \xi\|_{\mathcal{H}}^2 = \langle P\xi - \xi | P\xi - \xi\rangle = \|P\xi\|_{\mathcal{H}} - \langle P\xi|\xi\rangle - \langle \xi|P\xi\rangle + \|\xi\|_{\mathcal{H}} = \|P\xi\|_{\mathcal{H}} - 1$$

implying that $1 \le \|P\xi\|_{\mathcal{H}} \le \|P\| \, \|\xi\|_{\mathcal{H}} = 1$, hence $P\xi = \xi$. Conversely, if $P\xi = \xi$, then $\langle \omega_\xi, P\rangle = \langle P\xi|\xi\rangle = \|\xi\|_2^2 = 1$, establishing the middle equivalence.

(2) The claim is trivial for zero vectors, we assume ξ is nonzero and further assume it has unit length. By definition of S_{ω_ξ} we have $\left\langle \omega_\xi, S_{\omega_\xi}\right\rangle = 1$ so that $S_{\omega_\xi}\xi = \xi$ by (1) and $S_{\omega_\xi}T'\xi = T'S_{\omega_\xi}\xi = T'\xi$ for any $T' \in M'$, so that $\langle M'\xi\rangle \subset \operatorname{rng}\left(S_{\omega_\xi}\right)$. Letting Q denote the projection onto $\langle M'\xi\rangle$ (which is in M by Lemma 2.3.8), we have shown that $Q \le S_{\omega_\xi}$. But $\langle \omega_\xi, Q\rangle = \langle Q\xi|\xi\rangle = \langle \xi|\xi\rangle = 1$ implies $S_{\omega_\xi} \le Q$ by minimality, so these projections coincide.

(3) Again, we may assume ξ and η are unit length. The equalities $\langle \omega_\xi, S_{\omega_\eta}\rangle = \langle S_{\omega_\eta}\xi|\xi\rangle = \langle S_{\omega_\eta}\xi|S_{\omega_\eta}\xi\rangle = \|S_{\omega_\eta}\xi\|_{\mathcal{H}}^2$ establish the first equivalence. If $S_{\omega_\eta}\xi = 0$, then ξ is in the kernel of S_{ω_η}, which is orthogonal to the range of S_{ω_η}, which is $\langle M'\eta\rangle$ by (2), and conversely if $\xi \perp \langle M'\eta\rangle$, then for any $\nu \in \mathcal{H}$ we have $0 = \langle S_{\omega_\eta}\nu|\xi\rangle = \langle \nu|S_{\omega_\eta}\xi\rangle$ and therefore $S_{\omega_\eta}\xi = 0$, so the second equivalence holds. The third equivalence is clear given that $(M')^* = M'$ and $\langle \xi|T'\eta\rangle = \langle (T')^*\xi|\eta\rangle$ for $T' \in M'$. If$\eta \perp \langle M'\xi\rangle$, then $\langle T'\eta|S'\xi\rangle = \langle \eta|\,(T')^* S'\xi\rangle = 0$ for all $T', S' \in M'$, so S_{ω_ξ} and S_{ω_η} have orthogonal ranges, and conversely $S_{\omega_\eta}S_{\omega_\xi} = 0$ implies $0 = S_{\omega_\eta}S_{\omega_\xi}\xi = S_{\omega_\eta}\xi$, using (1). Finally, that the third equivalence is symmetric in ξ and η shows last sentence of the claim holds.

(4) By the Cauchy-Schwarz inequality (Proposition 2.1.11),

$$|\langle \omega, T(I - S_\omega)\rangle|^2 \le \langle \omega, TT^*\rangle \langle \omega, (I - S_\omega)^* (I - S_\omega)\rangle = \langle \omega, TT^*\rangle \langle \omega, I - S_\omega\rangle = 0,$$

showing that $\langle \omega, T\rangle = \langle \omega, TS_\omega\rangle$. The remaining equalities are proven in a similar way. □

Definition 2.3.14. A positive normal functional ω on a von Neumann algebra M is called ***faithful*** if $S_\omega = I$, the identity in M.

Example 2.3.15. (Faithful positive normal functionals on $L^\infty(X)$) Let X be a locally compact space, μ a Radon measure on X, and f a faithful positive normal functional on $L^\infty(X)$. We saw in Example 2.3.12 that $S_f = \chi_{\operatorname{supp}(f)}$. Faithfulness of f is therefore the assertion that $\operatorname{supp}(f) = X$ up to a null set.

Example 2.3.16. (Positive normal functionals are faithful on the corner defined by their support) Let M be a von Neumann algebra and let ω be a nonzero positive normal functional on M. By a ***corner*** in M, we mean a von Neumann subalgebra of M of the form PMP for a projection P in M. Note that P is the unit in the corner PMP. The restriction of ω to $S_\omega M S_\omega$ is nonzero since $S_\omega \in S_\omega M S_\omega$, clearly remains positive and normal, and by definition is faithful.

We establish a lemma that will be needed in characterizing faithfulness of positive normal functionals.

Proposition 2.3.17. *Let ω be a nonzero positive normal functional on a von Neumann algebra M and let U be a unitary in M. Then $\langle \omega, U \rangle = \langle \omega, I \rangle$ if and only if $US_\omega = S_\omega U = S_\omega$.*

Proof. Recall from Proposition 2.1.12 that $\|\omega\|_{M_*} = \langle \omega, I \rangle$, so we may normalize ω and assume that $\langle \omega, I \rangle = 1$.

Suppose that $\langle \omega, U \rangle = 1$. Let $\mathcal{H}$ be a Hilbert space such that M is standardly represented in $B(\mathcal{H})$, so that there exists a unit vector $\xi \in \mathcal{H}$ for which $\omega = \omega_\xi$. Then $\langle U\xi | \xi \rangle = 1$ together with $\|U\xi\|_{\mathcal{H}} = 1$ entails $U\xi = \xi$. If $\eta \in \mathcal{H}$ is of the form $T'\xi$ for some $T' \in M'$, then $U\eta = UT'\xi = T'U\xi = T'\xi = \eta$, so that U fixes every vector in $\langle M'\xi \rangle$, the range of S_{ω_ξ}. If $\eta \in \mathcal{H}$ and we write $S_{\omega_\xi}\eta = \lim_\alpha T'_\alpha \xi$, then $US_{\omega_\xi}\eta = U(\lim_\alpha T'_\alpha \xi) = \lim_\alpha T'_\alpha \xi = U\eta$ and thus $US_{\omega_\xi} = S_{\omega_\xi}$. It follows that $S_{\omega_\xi} = S_{\omega_\xi} U^* U = \left(US_{\omega_\xi}\right)^* U = S_{\omega_\xi} U$.

Conversely, if $US_\omega = S_\omega$, then by Lemma 2.3.13(4) we have $1 = \langle \omega, S_\omega \rangle = \langle \omega, US_\omega \rangle = \langle \omega, U \rangle$. □

Corollary 2.3.18. *Let ω be a nonzero positive normal functional on a von Neumann algebra M. The set $\mathcal{U}_\omega$ of unitaries on which ω takes the value $\langle \omega, I \rangle$ is a weak* closed subgroup of the unitaries in M.*

Proof. The claim is clear given the characterization of elements of $\mathcal{U}_\omega$ as those unitaries U satisfying $US_\omega = S_\omega U = S_\omega$. □

Proposition 2.3.19. *Let ω be a nonzero positive normal functional on a von Neumann algebra M. The following are equivalent:*

1. *ω is faithful.*
2. *$\langle \omega, T \rangle > 0$ for every nonzero positive T in M.*
3. *$\langle \omega, P \rangle > 0$ for every nonzero projection P in M.*
4. *$\langle \omega, U \rangle = \langle \omega, I \rangle$ implies $U = I$, for any unitary U in M.*

Proof. We may normalize ω to assure that $\langle \omega, I \rangle = 1$.

If ω is faithful and $U \in M$ is unitary with $\langle \omega, U \rangle = 1$, then $U = US_\omega = S_\omega = I$ by Proposition 2.3.17, so (1) implies (4).

Suppose (4) holds. Given a projection $P \in M$, a routine calculation shows that $I - 2P$ is unitary and we have $\langle \omega, I - 2P \rangle = 1 - 2\langle \omega, P \rangle$, so that $\langle \omega, P \rangle = 0$ exactly when $\langle \omega, I - 2P \rangle = 1$, which implies that $I - 2P = I$, by (4), and therefore that $P = 0$. Thus (3) holds.

For $T \geq 0$ in M Proposition 2.3.6 asserts that $\langle \omega, T \rangle = 0$ if and only if $\langle \omega, R(T) \rangle = 0$, where $R(T)$ is the range projection of T, and $R(T) = 0$ exactly when $T = 0$, so (2) and (3) are equivalent.

Finally, if (2) holds, then $\langle \omega, I - S_\omega \rangle = \langle \omega, (I - S_\omega) S_\omega \rangle = 0$ implies $I - S_\omega = 0$, so (1) holds. □

We require one more lemma in order to characterize σ-finiteness of von Neumann algebras. Lemma 2.3.13 is used prodigiously and without comment in establishing the following.

Lemma 2.3.20. *Let $\mathcal{H}$ be a Hilbert space and M a von Neumann algebra in $B(\mathcal{H})$. If $\omega = \sum_{n=1}^{\infty} \gamma_n \omega_{\xi_n}$ is a convex combination of the normal states ω_{ξ_n} on M, where $\xi_n \in \mathcal{H}$ are unit vectors, then the range of S_ω is $\langle M'\xi_n : n \geq 1\rangle$.*

Proof. If $\eta \in \mathcal{H}$ is orthogonal to $\langle M'\xi_n : n \geq 1\rangle$, then $\left\langle \omega_\eta, S_{\omega_{\xi_n}} \right\rangle = 0$ and hence $\langle \omega_{\xi_n}, S_{\omega_\eta}\rangle = 0$ for each $n \geq 1$, implying that $\langle \omega, S_{\omega_\eta}\rangle = 0$ and so $S_\omega S_{\omega_\eta} = 0$, whence $\eta \in \operatorname{rng}(S_{\omega_\eta}) \subset \ker(S_\omega)$. Thus the range of S_ω is contained in $\langle M'\xi_n : n \geq 1\rangle$. Now if $\eta \in \operatorname{rng}(S_\omega)$ is a unit vector, then $S_\omega \eta = \eta$ and hence $\langle \omega_\eta, S_\omega\rangle = 1 \neq 0$ and thus $\langle \omega, S_{\omega_\eta}\rangle \neq 0$, implying that $\langle \omega_{\xi_n}, S_{\omega_\eta}\rangle \neq 0$ for some $n \geq 1$. Then ξ_n is not orthogonal to $\langle M'\eta\rangle$, so η is not orthogonal to $\langle M'\xi_n\rangle$, showing that no unit vector in the range of S_ω is orthogonal to $\langle M'\xi_n : n \geq 1\rangle$. Therefore the range of S_ω and $\langle M'\xi_n : n \geq 1\rangle$ coincide. □

Proposition 2.3.21. *Let M be a von Neumann algebra. Then M is σ-finite if and only if there exists a faithful positive normal functional on M.*

Proof. Let $\mathcal{H}$ be a Hilbert space such that $M \subset B(\mathcal{H})$. Suppose M is σ-finite. By Proposition 2.3.9 we have $I = \sum_{n=1}^{\infty} P_n$, where P_n is orthogonal projection onto $\langle M'\xi_n\rangle$ for some unit vector $\xi_n \in \mathcal{H}$ and $P_n P_m = 0$ for $n \neq m$. By Lemma 2.3.13(2), the projection P_n is exactly $S_{\omega_{\xi_n}}$. The positive normal functional $\omega = \sum_{n=1}^{\infty} 2^{-n} \omega_{\xi_n}$ has support projection with range $\langle M'\xi_n : n \geq 1\rangle$, by Lemma 2.3.20, and since $\eta = \sum_{n=1}^{\infty} P_n \eta \in \bigoplus_n \langle M'\xi_n\rangle$ for every $\eta \in \mathcal{H}$, we conclude that $\langle M'\xi_n : n \geq 1\rangle = \mathcal{H}$ and that ω is faithful.

Conversely, suppose that ω is a faithful positive normal functional on M. If $(P_\alpha)_\alpha$ is any family of nonzero mutually orthogonal projections in M, then $\langle \omega, P_\alpha\rangle > 0$ for each α and that $\sum_\alpha \langle \omega, P_\alpha\rangle = \langle \omega, \sum_\alpha P_\alpha\rangle \leq \langle \omega, I\rangle = 1$ is finite entails only countably many of the terms $\langle \omega, P_\alpha\rangle$ are nonzero, whence $(P_\alpha)_\alpha$ is countable. Therefore M is σ-finite. □

We may now provide the example that motivates the choice of term σ-finite for general von Neumann algebras.

Example 2.3.22. **(σ-finiteness of $L^\infty(X)$)** Let X be a locally compact space and μ a Radon measure on X. We saw in Example 2.3.15 that a positive normal functional f on $L^\infty(X)$ is faithful exactly when $\operatorname{supp}(f) = X$ up to a null set. The support of any function g in $L^1(X)$ is σ-finite: for $n \geq 1$ set $X_n = \left\{x \in X : |g(x)| \in \left[\frac{1}{n+1}, \frac{1}{n}\right)\right\}$ and set $X_0 = \{x \in X : |g(x)| \in [1, \infty)\}$, which have finite measure since $\frac{1}{n+1}\mu(X_n) \leq \int_{X_n} |g(x)| \leq \|g\|_{L^1(X)} < \infty$ for each $n \geq 0$. Then $\operatorname{supp}(f) = X_0 \cup \bigcup_n X_n$ is the union of countably many sets of finite measure. Therefore $\operatorname{supp}(f) = X$ implies (X, μ) is a σ-finite measure space. Conversely, if (X, μ) is σ-finite, say $X = \bigcup_{n=1}^{\infty} X_n$ for sets X_n of finite measure, then $\sum_{n=1}^{\infty} 2^{-n} \mu(X_n)^{-1} \chi_{X_n}$ is a positive function in $L^1(X)$ with support X, hence a faithful positive normal functional on $L^\infty(X)$.

References for the topics of this section are the same as those for Section 2.2.

Chapter 3

Preliminaries on abstract harmonic analysis

This chapter introduces the basic theory of locally compact groups and defines the associated algebras of functions and operators with which our later investigations will be concerned. We also outline the relevant theory of operator spaces.

Definition 3.0.1. Let G be a group and f a complex valued function on G. Define

$$\check{f}(t) = f\left(t^{-1}\right) \qquad (t \in G)$$

and for $s \in G$ define

$${}_s f(t) = f(st) \text{ and } f_s(t) = f(ts) \qquad (t \in G).$$

For an algebra A of functions on a set X and a subset S of X, we write $I_A(S)$ for the functions in A that are zero on S.

3.1 Locally compact groups

Abstract harmonic analysis is the study of groups endowed with a locally compact Hausdorff topology compatible with the group operations.

Definition 3.1.1. A group G is ***locally compact*** if there is a locally compact Hausdorff topology on G under which the maps $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous.

The topology of any locally compact group G is entirely determined by the neighborhood system at the identity element e: the neighborhood system at any x in G is the image of the neighborhood system at e under translation by x, a homeomorphism of G. Collecting together results of [29, Chapter 2], we have the following.

Proposition 3.1.2. *Let G be a locally compact group. For any neighborhood U of the identity e in G, there exists a compact neighborhood V of e such that $V = V^{-1}$ (the set V is* **symmetric***) and $VV^{-1} \subset U$. Consequently, there is a neighborhood base for e consisting of compact, symmetric sets.*

For these groups, there is a canonical translation invariant Radon measure that is unique up to scalar multiples. A proof of this fact can be found in [18, Chapter 11].

Definition 3.1.3. Let G be a locally compact group. A ***left (right) Haar measure*** on G is a nonzero positive Radon measure μ that is left (right) translation invariant, meaning that $\mu(xE) = \mu(E)$ $(\mu(Ex) = \mu(E))$ for all $x \in G$ and Borel subsets E of G. If μ' is any positive Radon measure μ that is left translation invariant, then $\mu' = \alpha\mu$ for some $\alpha \geq 0$.

Once a left Haar measure μ on a locally compact group G is chosen, it is referred to as *the* Haar measure on G and the L^p-spaces with respect to this measure may be defined as in Example 2.2.2. All integrals over a locally compact group are implicity taken with respect to the Haar measure on G. If G is a discrete group, then μ is taken to be the counting measure on G, which is trivially both left and right invariant, and if G is compact, then we assume that the Haar measure is normalized to have total mass one. It is straight forward to verify that a right Haar measure on G is defined by $E \mapsto \mu(E^{-1})$ for Borel subsets E of G. When the group and measure are clear from context, we adopt the notation $|E|$ for the Haar measure of the Borel subset E of G.

Example 3.1.4. (When Haar measure is finite) Let G be a locally compact group. We show that Haar measure is finite if and only if G is compact. One implication is clear: if G is compact, then, because a Radon measure takes finite values on compact sets, Haar measure is finite. Conversely, suppose G is not compact and let $K \subset G$ be compact with $|K| > 0$. There must exist $s_1 \in G \setminus K$ such that $s_1K \cap K = \emptyset$: otherwise, given $t \in G$, choose $k \in tK \cap K$ and write $k = tk'$ for some $k' \in K$, in which case and $t = k(k')^{-1} \in KK^{-1}$, implying that $G = KK^{-1}$, a compact set by [29, (4.4)]. Let $s_0 = e$, the identity of G. For $n \geq 1$, given $s_0, \ldots, s_n \in G$ such that the sets s_jK are pairwise disjoint, we may find $s_{n+1} \in G \setminus \bigcup_{j=0}^{n} s_jK$ such that $s_{n+1}K$ is disjoint from $s_1K, \ldots, s_nK$, otherwise the preceding argument would entail that G is compact. Thus we may find countably many pairwise disjoint translates of the set K in G, whence Haar measure is not finite.

Example 3.1.5. (When Haar measure has atoms) Let G be a locally compact group. We show that Haar measure assigns a singleton strictly positive measure if and only if G is discrete. One implication is trivial, since Haar measure is counting measure on discrete groups. Thus suppose Haar measure assigns some singleton strictly positive measure, in which case $|\{s\}| > 0$ for all $s \in G$ by left translation invariance, and suppose towards a contradiction that G is not discrete, in which case no singleton is open because translation is a homeomorphism. By outer regularity of Haar measure, there exists an open neighborhood U of e such that $|U| - |\{e\}| < \frac{1}{2}|\{e\}|$. Since U cannot be a singleton, choose $s \in U$ distinct from e, and choose disjoint neighborhoods U_e and U_s of e and

s, respectively, that are contained in the open set U. Then $2\left|\{e\}\right| \leq |U_e| + |U_s| \leq |U| < \frac{3}{2}\left|\{e\}\right|$, a contradiction, so that G must be discrete.

It follows from Example 3.1.5 that any countable locally compact group, being the disjoint union of countably many singletons, must be a discrete.

That Haar measure is translation invariant and inner regularity on open sets yields the following.

Lemma 3.1.6. *Let G be a locally compact group. If U is an open nonempty subset of G, then the Haar measure of U is strictly positive.*

Example 3.1.7. (Haar measure for closed subgroups) Let G be a locally compact group and H a closed subgroup. We show that Haar measure μ_H on H is restriction of Haar measure μ_G on G if and only if H is an open subgroup of G, otherwise H is a μ_G-null set. If H is open in G, then Lemma 3.1.6 asserts that $\mu_G(H) > 0$ and it is straight forward to see that μ_G defines a left translation invariant positive Radon measure on H, so we may take $\mu_H = \mu_G|_H$. Conversely, if H fails to be open in G, then H cannot contain a nonempty open set in G, otherwise H would be the union of translates of this open set and would itself be open in G. If it were the case that $\mu_G(H) > 0$, then by regularity of μ_G we may choose $U \subset G$ open and $K \subset H$ compact with $K \subset H \subset U$ such that $\mu_G(U \setminus K)$ is arbitrarily small. Thus we may find a compact subset K of H such that $\mu_G(H \setminus K)$ is arbitrarily small, so that we may assure $0 < \mu_G(K \cap H) < \infty$. Then [29, (20.17)] asserts that $(K \cap H)^2$ contains a nonempty open subset of G, a contradiction since $(K \cap H)^2 \subset H$. Therefore H is μ_G-null whenever H is not open in G, so certainly μ_H is not the restriction of μ_G.

Definition 3.1.8. Let G be a locally compact group with left Haar measure μ. The ***modular function*** of G is the continuous group homomorphism $\Delta : G \to (0, \infty)$ satisfying $\mu(Ex) = \Delta(x)\mu(E)$.

The left translation invariance of Haar measure manifests in the following useful formulas for integration.

Proposition 3.1.9. *Let G be a locally compact group. If $f \in L^1(G)$ and $s \in G$, then*

$$\int_G f(y)\,dy = \int_G f(sy)\,dy = \Delta(s)\int_G f(ys)\,dy = \int_G f\left(y^{-1}\right)\Delta\left(y^{-1}\right)dy.$$

Definition 3.1.10. Let G be a locally compact group. The ***group algebra*** of G is the Banach space $L^1(G)$ endowed with the convolution product and involution given by

$$(f * g)(s) = \int_G f(y)\,g\left(y^{-1}s\right)dy \text{ and } f^*(s) = \overline{f(s^{-1})}\Delta\left(s^{-1}\right) \qquad \left(f, g \in L^1(G), s \in G\right),$$

under which it is a Banach $*$-algebra.

Example 3.1.11. (Identities and approximate identities for the group algebra) Let G be a locally compact group. The group algebra of G is unital when 1_G is an integrable function, i.e.

when Haar measure is finite, which by Example 3.1.4 occurs exactly when G is compact. We show that $L^1(G)$ always has a bounded approximate identity. For each open neighborhood U of the identity e in G, let $f_U \in L^1(G)$ be positive, norm one, with support contained in U, and such that $f_U = \check{f_U}$. (For example, we may set $f_U = |V|^{-1} \chi_V$ for a symmetric neighborhood V of e contained in U.) If $g \in L^1(G)$, then

$$g * f_U(s) - g(s) = \int_G g(y) f_U\left(y^{-1}s\right) dy - \int_G f_U(y)\, dy = \int_G (g(sy) - g(s)) f_U(y)\, dy \qquad (s \in G),$$

so that

$$\|g * f_U - g\|_{L^1(G)} = \left\| \int_G (g_y - g) f_U(y)\, dy \right\|_{L^1(G)} \le \int_G \|g_y - g\|_{L^1(G)} f_U(y)\, dy \le \sup_{y \in U} \|g_y - g\|_{L^1(G)}.$$

It is shown in [17, Proposition 2.41] that translation is norm continuous on $L^1(G)$, so that $\sup_{y \in U} \|g_y - g\|_{L^1(G)} \to 0$ as $U \to \{e\}$ and $(f_U)_U$ is a right bounded approximate identity for $L^1(G)$. That this net is also a left approximate identity follows by a similar computation.

It is a classical result of Wendel that the group algebra, as a Banach algebra, is a complete invariant for G [61].

Definition 3.1.12. A ***unitary representation*** of a locally compact group G is a representation $\pi : G \to B(\mathcal{H})$ of the group G as unitary operators on the Hilbert space $\mathcal{H}$ that is continuous in the strong operator topology on $B(\mathcal{H})$, meaning that $G \to \mathcal{H} : s \mapsto \pi(s)\xi$ is continuous for each $\xi \in \mathcal{H}$.

A vector $\xi \in \mathcal{H}$ is ***cyclic*** for the unitary representation π if $\langle \pi(G)\xi \rangle = \mathcal{H}$, where $\pi(G)\xi$ denotes the set $\{\pi(s)\xi : s \in G\}$.

It is equivalent to ask that a representation of a locally compact group as unitary operators be continuous in the weak and the strong operator topologies [17, p.68].

Example 3.1.13. (Regular representations of G) The ***left*** *and* ***right regular representation*** of a locally compact group G are given, respectively, by

$$\lambda : G \to B\left(L^2(G)\right) : \lambda(s)\xi(t) = \xi\left(s^{-1}t\right) \quad \text{and} \quad \rho : G \to B\left(L^2(G)\right) : \rho(s)\xi(t) = \xi(ts)\,\Delta(s)^{\frac{1}{2}},$$

It is straight forward to verify that these are indeed unitary representations of G, see for example [17, Chapter 3], where the basic properties of unitary representations of locally compact groups are outlined. We show in Lemma 6.2.1 that λ and ρ are unitarily equivalent.

We now describe how a unitary representation $\pi : G \to B(\mathcal{H})$ as in Definition 3.1.12 induces a $*$-representation of the $*$-algebra $L^1(G)$ by integration. For a fixed vector $\xi \in L^2(G)$ the function $G \to L^2(G) : s \mapsto \pi(s)\xi$ is bounded and continuous, so that for $f \in L^1(G)$ the weak integral

$\int_G f(y)\,\pi(y)\,\xi dy$ exists: it is the unique element of $L^2(G)$ satisfying

$$\left\langle \int_G f(y)\,\pi(y)\,\xi dy|\eta \right\rangle = \int_G f(y)\,\langle \pi(y)\,\xi|\eta\rangle\, dy \qquad \left(\eta \in L^2(G)\right),$$

see [17, Appendix 3]. The map $L^2(G) \to L^2(G) : \xi \mapsto \int_G f(y)\,\pi(y)\,\xi dy$ is clearly linear and is bounded, and letting $\pi(f)$ denote this operator in $B\left(L^2(G)\right)$, it is routine to verify that

$$\pi : L^1(G) \to B\left(L^2(G)\right) : f \mapsto \pi(f)$$

is a $*$-representation, the ***induced $*$-representation of*** $L^1(G)$.

The text [17] contains a more detailed introduction to the theory of locally compact groups that we have expounded here. A comprehensive reference for classical noncommutative harmonic analysis is [29, 30].

3.2 Operator spaces and completely contractive Banach algebras

This section outlines the very basics of operator space theory and defines the related algebraic notions — algebras and modules — that will arise in the sequel. An operator space structure on a Banach space is a refinement on the norm structure achieved by simultaneously considering norms on all the matrix spaces over the Banach space. The significance of this refinement to harmonic analysis will become apparent in the subsequent sections.

Operator spaces were originally defined to be closed linear subspaces of the bounded operators on Hilbert spaces and were characterized axiomatically by Ruan in 1988, giving the following definition:

Definition 3.2.1. An ***operator space*** is a vector space E equipped with a complete norm $\|\cdot\|_n$ on the matrix space $M_n(E)$ for each $n \geq 1$, subject to the following compatibility criteria:

$$\left\| \begin{matrix} x & 0 \\ 0 & y \end{matrix} \right\|_{n+m} = \max\left\{\|x\|_n, \|x\|_m\right\} \text{ and } \|\alpha x \beta\|_n \leq \|\alpha\|\,\|x\|_n\,\|\beta\|,$$

for every $x \in M_n(E), y \in M_m(E)$, and $\alpha, \beta \in M_n(\mathbb{C})$. Given a vector space E, we refer to a choice of complete norms $\left(\|\cdot\|_n\right)_{n\geq 1}$ on the matrix algebras over E that satisfy the above conditions as an ***operator space structure*** on E.

It is plain to see that a closed subspace of an operator space is an operator space.

Example 3.2.2. (Banach spaces are operator spaces) Every Banach space E may be endowed with a canonical operator space structure — the maximal operator space structure on E — under which every bounded map from E into an operator space is completely bounded [14, Section 3.2]. Thus a theorem that holds for all operator spaces is a literal generalization of its analogue for Banach spaces.

Example 3.2.3. (Closed subspaces of $B(\mathcal{H})$ are operator spaces) Let $\mathcal{H}$ be a Hilbert space and E a closed subspace of $B(\mathcal{H})$. For each $n \geq 1$, the matrix space $M_n(E) \subset M_n(B(\mathcal{H}))$ is given a norm by identifying $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$ as vector spaces via

$$[T_{ij}] \mapsto \left[[\xi_j] \mapsto [T_{ij}][\xi_j] = \left[\sum_j T_{ij}\xi_j \right] \right].$$

An elementary computation shows that the resulting norms satisfy the conditions of Definition 3.2.1 and are therefore an operator space structure on E.

Example 3.2.4. (C^*-algebras are operator spaces) Every C^*-algebra A can be faithfully represented as bounded operator on a Hilbert space $\mathcal{H}$, by Theorem 2.1.14. The amplifications of such a faithful representation are representations of $M_n(A)$ as a closed $*$-subalgebra of $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, from which we may pull back a C^*-norm on $M_n(A)$. The matrix algebra $M_n(A)$ is a $*$-algebra under the natural algebra operations on matrices and involution given by $[a_{ij}]^* = \left[a_{ji}^*\right]$, which are entirely determined by the $*$-algebraic structure of A and, as seen in Section 2.1, uniquely determine the C^*-norm on $M_n(A)$. Since the algebraic structure of $M_n(A)$ is determined by that of A, it follows that the C^*-norm on $M_n(A)$ is independent of the faithful representation of A that we began with. Consequently, each matrix space over A carries a unique C^*-norm determined by A. It can be verified that these yield an operator space structure on A.

The maps of interest to operator space theory are as follows.

Definition 3.2.5. Let $T : E \to F$ be a linear map between vector spaces. The ***n*th amplification** of T is the linear map $T^{(n)} : M_n(E) \to M_n(F) : [x_{ij}] \mapsto [Tx_{ij}]$.

If E and F are operator space, then T is called:

1. ***Completely bounded*** if $\|T\|_{cb} := \sup_{n\geq 1} \left\|T^{(n)}\right\| < \infty$.

2. ***Completely contractive*** if $\|T\|_{cb} \leq 1$.

3. A ***complete isometry*** if $T^{(n)}$ is an isometry for every $n \geq 1$.

4. A ***complete quotient map*** if $T^{(n)}$ is an quotient map for every $n \geq 1$.

The norm $\|\cdot\|_{cb}$ is referred to as the ***cb-norm***. The space of completely bounded maps from E to F, denoted $CB(E, F)$, is a Banach algebra under composition and the cb-norm.

Example 3.2.6. ($*$-homomorphisms of C^*-algebras are completely bounded) The amplifications of a $*$-homomorphism between C^*-algebras are $*$-homomorphisms when the relevant matrix spaces are given $*$-algebra operations as in Example 3.2.4, so are automatically contractions by Proposition 2.1.13. Consequently, $*$-homomorphisms between C^*-algebras are complete contractions.

It can be shown that bounded maps from any operator space into a commutative C^*-algebra are completely bounded and that the operator norm and the cb-norm coincide for such maps, see for example [14, Proposition 2.2.6]. In particular, bounded linear functionals on an operator space E are completely bounded and $E^* = CB(E, \mathbb{C})$ isometrically.

Example 3.2.7. (Duals of operator spaces are operator spaces) Let E be an operator space. The Banach space dual of E is given an operator space structure by identifying $M_n(E^*) = M_n(CB(E, \mathbb{C}))$ with $CB(E, M_n(\mathbb{C}))$ as vector spaces and using the cb-norm on the latter space. See [14, Chapter 2] for details. It follows that the bidual E^{**} carries an operator space structure and it can be verified that, with this structure, the canonical inclusion $E \to E^{**}$ is a complete isometry. If E is a dual Banach space, say $E = F^*$, the predual F thereby inherits an operator space, being a closed subspace of $F^{**} = E^*$. In combination with Example 3.2.4, it follows that the predual of a von Neumann algebra always carries a canonical operator space structure.

The classical concrete definition of operator spaces is now a representation theorem, which, for comparison, we state alongside its Banach space analogue.

Theorem 3.2.8. *[14, Theorem 2.3.5] Every operator space E is completely isometrically isomorphic to a closed subspace of $B(\mathcal{H})$ for some Hilbert space.*

Theorem 3.2.9. *Every Banach space E is isometrically isomorphic to a closed subspace of $\mathcal{C}(X)$ for some compact Hausdorff space X.*

The expected notions of complete boundedness and contractivity for bilinear maps between operator spaces allow operator space overtones to be added to the notions of Banach algebras and modules over Banach algebras, as follows.

Definition 3.2.10. A Banach algebra A that is also an operator space is called a ***completely contractive Banach algebra*** if the multiplication map $A \times A \to A$ is completely contractive, in which case a left Banach A-module X is called a ***completely contractive left Banach A-module*** if X is also an operator space and the module action of A on X define a completely contractive map $A \times X \to X$. Right modules and bimoduals are defined analogously.

Example 3.2.11. (Duals of completely contractive modules) Let A be a completely contractive Banach algebra and X a completely contractive left A-module. The dual X^* of X is an operator space by Example 3.2.7 and is a completely contractive right A-module under the natural dual action

$$\langle \varphi \cdot a, x \rangle = \langle \varphi, a \cdot x \rangle \qquad (\varphi \in X^*, a \in A, x \in X).$$

Duals of right modules and bimodules are defined analogously.

Example 3.2.12. ($L^1(G)$ is a completely contractive Banach algebra) Let G be a locally compact group. The group algebra $L^1(G)$, being the predual of the von Neumann algebra $L^\infty(G)$, is an operator space by in Example 3.2.7. It can be shown that the operator space structure arising this way is the maximal operator space structure [14, Section 3.3], from which it follows that $L^1(G)$ is trivially a completely contractive Banach algebra.

There is a natural construction of linearizers for completely bounded bilinear maps, giving rise to the ***operator space projective tensor product***, which we denote by $\widehat{\otimes}$. See [14, Chapter 7] for details.

A much more detailed account of what is presented here may be found in [14].

3.3 The group von Neumann algebra and the Fourier algebra

Let G be a locally compact group and $\pi : G \to B(\mathcal{H})$ a unitary representation of G as operators on a Hilbert space $\mathcal{H}$. The image $\pi(G)$ of G in $B(\mathcal{H})$ generates a von Neumann algebra, which, by Proposition 2.2.10, equals $\pi(G)'' = \overline{\operatorname{span}}^{wot}\pi(G)$. For the left regular representation of G, we name the resulting algebra.

Definition 3.3.1. The ***group von Neumann algebra*** $VN(G)$ of a locally compact group G is the von Neumann algebra generated in $B\left(L^2(G)\right)$ by the left translation operators.

It is clear that the right translations on $L^2(G)$ — the operators $\rho(s)$ for $s \in G$ — commute with left translations, so that $\rho(G) \subset VN(G)'$ and thus $\rho(G)'' \subset VN(G)'$. It can be shown that equality holds, which we record for later reference. A proof can be found in Chapter 5 of [13] or Chapter 7, Section 3 of [59].

Proposition 3.3.2. *Let G be a locally compact group. The commutant of the group von Neumann algebra in $B\left(L^2(G)\right)$ is the von Neumann algebra generated by $\rho(G)$.*

At the end of Section 2.2 we mentioned that every von Neumann algebra can be represented on some Hilbert space for which all its normal functionals are vector functionals. For the group von Neumann algebra, the inclusion of $VN(G)$ into $B\left(L^2(G)\right)$ is this representation and consequently every element of $VN(G)_*$ may be written in the form $\omega_{\xi,\eta}$ for $\xi, \eta \in L^2(G)$. Normality of the functional $\omega_{\xi,\eta}$ implies that it is determined by its values on the weak* dense spanning set $\lambda(G)$ for $VN(G)$, and we may view $\langle \lambda(s), \omega_{\xi,\eta}\rangle$ as a function on G, using the notation $\omega_{\xi,\eta}(s)$ for this value. Since $\omega_{\xi,\eta}(s) = \langle \lambda(s)\xi | \eta\rangle$, the continuity of λ in the weak operator topology implies $\omega_{\xi,\eta}$ is continuous on G and moreover we have

$$\|\omega_{\xi,\eta}\|_{\mathcal{C}_0(G)} = \sup_{s\in G} |\langle \lambda(s), \omega_{\xi,\eta}\rangle| \leq \|\omega_{\xi,\eta}\|_{VN(G)_*}.$$

The space of functions on G obtained this way is in fact an algebra:

Theorem 3.3.3. *[16] Let G be a locally compact group. The predual of $VN(G)$ may be identified with a dense subalgebra of $\mathcal{C}_0(G)$. For $\omega \in VN(G)_*$, we have*

$$\|\omega\|_{VN(G)_*} = \inf\left\{ \|\xi\|_{L^2(G)} \|\eta\|_{L^2(G)} : \xi, \eta \in L^2(G) \text{ such that } \omega = \omega_{\xi,\eta} \right\}$$

and under this norm $VN(G)_*$ is a commutative semisimple completely contractive Banach algebra, the **Fourier algebra** *of G, denoted $A(G)$.*

The Fourier algebra has the following useful properties, all of which may be found in [16].

Proposition 3.3.4. *Let G be a locally compact group. The following hold:*

1. *The nonzero multiplicative linear functionals on $A(G)$ are exactly the elements of $\lambda(G)$ and the spectrum of $A(G)$ is homeomorphic to G.*

2. *The compactly supported functions in $A(G)$ are dense.*

3. *For any $K \subset G$ compact and $U \supset K$ open, there exists $u \in A(G)$ with compact support taking values in $[0,1]$ and satisfying $u|_K = 1$ and $u|_{G\setminus U} = 0$.*

The Fourier algebra respects closed subgroups in the following natural sense.

Theorem 3.3.5. *[27] **(Herz's restriction theorem)** Let G be a locally compact group and H a closed subgroup. The restriction map $r_H : A(G) \to A(H)$ is a complete quotient map.*

Example 3.3.6. ($VN(H)$ for a closed subgroup H of G) Let G be a locally compact group and H a closed subgroup. By Herz's restriction theorem the adjoint of the restriction map is a normal isometry $VN(H) \to VN(G)$. For $s \in H$,

$$\langle r_H^* (\lambda_H (s)), \omega \rangle = r_H(\omega)(s) = \omega(s) = \langle \lambda_G(s), \omega \rangle \qquad (\omega \in A(G)),$$

implying that $r_H^*(\lambda_H(s)) = \lambda_G(s)$. It follows that $r_H^* : VN(H) \to VN(G)$ is a unital $*$-homomorphism on $\mathrm{span}\lambda_H(H)$, so also on $VN(H)$ by normality. The group von Neumann algebra of H may thus be identified with the unital von Neumann subalgebra $r_H^*(VN(H))$ of $VN(G)$, for which we write $VN_H(G)$.

Example 3.3.7. (Ideals vanishing on closed subgroups) Let G be a locally compact group and H a closed subgroup. The $A(G)$ norm dominates the $L^\infty(G)$ norm, so that norm convergence in $A(G)$ implies pointwise convergence, from which it follows that the ideal $I_{A(G)}(H) = \{u \in A(G) : u|_H = 0\}$ is closed. For $s \in H$ the functional $\lambda(s)$ annihilates $I_{A(G)}(H)$ and conversely any element of $A(G)$ that is annihilated by $\lambda(H)$ is in $I_{A(G)}(H)$. Thus $I_{A(G)}(H) = \lambda(H)_\perp$ and it follows from the bipolar theorem that $I_{A(G)}(H)^\perp = (\lambda(H)_\perp)^\perp = \overline{\mathrm{span}}^{w^*}\lambda(H) = VN_H(G)$.

A subset E of a locally compact group G is said to be a set of ***spectral synthesis for*** $A(G)$ if the compactly supported functions in $I_{A(G)}(E)$ with support disjoint from E are dense in $I_{A(G)}(E)$.

Example 3.3.8. (Closed subgroups are sets of spectral synthesis for $A(G)$) Let G be a locally compact group and H a closed subgroup. Let $I^0_{A(G)}(H)$ denote the closure of the compactly supported functions in $I_{A(G)}(H)$ with support disjoint from H. If $s \in H$, then clearly $\lambda(s) \in I^0_{A(G)}(H)^\perp$, and if $s \in G \setminus H$, then Proposition 3.3.4(3) implies that there exists $u \in A(G)$ with

$u(s) = 1$ and compact support disjoint from H, so that $u \in I^0_{A(G)}(H)$ and $\langle u, \lambda(s) \rangle = 1$, implying $\lambda(s) \notin I^0_{A(G)}(H)^\perp$. Therefore $\left\{ s \in G : \lambda(s) \in I^0_{A(G)}(H)^\perp \right\} = H$ and it follows from Theorem 6 of [60] that $VN_H(G) = I^0_{A(G)}(H)^\perp$, whence

$$I_{A(G)}(H) = \left(I_{A(G)}(H)^\perp \right)_\perp = VN_H(G)_\perp = \left(I^0_{A(G)}(H)^\perp \right)_\perp = I^0_{A(G)}(H).$$

For a locally compact group G and closed ideal I of $A(G)$, the multiplication of $A(G)$ naturally makes I a completely contractive $A(G)$-module. It follows that $I^\perp$ is a completely contractive $A(G)$-module when given the natural dual module action of Example 3.2.11.

Definition 3.3.9. Let G be a locally compact group and let I be a closed ideal of $A(G)$. A bounded map $\Psi : VN(G) \to VN(G)$ is called a ***projection onto*** $I^\perp$ if it has range $I^\perp$ and satisfies $\Psi^2 = \Psi$ and is called an ***invariant projection*** if it is moreover an $A(G)$-module map.

Given a locally compact group G, there is a natural map $A(G) \otimes A(G) \to A(G \times G)$ given by $u \otimes v \mapsto [u \times v : (s,t) \mapsto u(s)\, v(t)]$. Both of these spaces are completely contractive $A(G)$-bimodules by the natural actions

$$u \cdot (v \otimes w) \cdot x = uv \otimes wx \text{ and } u \cdot (v \times w) \cdot x = uv \times wx \qquad (u, v, w, x \in A(G)).$$

It follows from a result of Effros and Ruan [15] that this map induces a completely isometric isomorphism of $A(G)$-bimodules on the operator space projective tensor product, which we record for later reference.

Proposition 3.3.10. *If G is a locally compact group, then $A(G) \widehat{\otimes} A(G) = A(G \times G)$ as completely contractive $A(G)$-bimodules.*

For a unitary representation $\pi : G \to B(\mathcal{H})$, functions on G of the form $s \mapsto \langle \pi(s)\, \xi | \eta \rangle$ for $\xi, \eta \in \mathcal{H}$, which are always continuous and bounded, are called ***coefficient functions*** of the representation π. Thus $A(G)$ is the space of all coefficient functions of the left regular representation. The space of coefficient functions of all unitary representations of G forms a unital commutative completely contractive Banach algebra of functions on G, the ***Fourier-Stieltjes algebra*** $B(G)$, under the norm

$$\|u\|_{B(G)} = \inf \left\{ \|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}} : \xi, \eta \in \mathcal{H} \text{ such that } u = \langle \pi(\cdot)\, \xi | \eta \rangle \right\}. \qquad (3.3.1)$$

Taking the direct sum of all unitary representations of G (that is, one for each equivalence class under unitary equivalence) yields a faithful representation $\pi_u : G \to B(\mathcal{H}_u)$, the ***universal representation*** of G, and it is not hard to see that $B(G)$ is the space of coefficient functions of π_u. The induced representation $\pi_u : L^1(G) \to B(\mathcal{H}_u)$ is a faithful $*$-representation of the $*$-algebra $L^1(G)$ and taking the completion of $L^1(G)$ under the C^*-norm obtained by identifying it with its image yields the ***group*** C^****-algebra*** $C^*(G)$. An element u of $B(G)$ may be written as $\langle \pi_u(\cdot)\, \xi | \eta \rangle$

for some $\xi, \eta \in \mathcal{H}_u$, so $u(s) = \langle \pi_u(s), \omega_{\xi,\eta} \rangle$ for $s \in G$, where $\omega_{\xi,\eta}$ is the vector functional on $B(\mathcal{H}_u)$. For $f \in L^1(G)$ we have

$$\int_G uf = \int_G \langle \pi_u(y), \omega_{\xi,\eta} \rangle f(y)\, dy = \left\langle \int_G \pi_u(y) f(y)\, dy, \omega_{\xi,\eta} \right\rangle_{B(\mathcal{H}),B(\mathcal{H})_*} = \langle \pi_u(f), \omega_{\xi,\eta} \rangle_{B(\mathcal{H}),B(\mathcal{H})_*}$$

and, since $\pi_u(L^1(G))$ is norm dense in $C^*(G)$, it follows that $\langle u, \pi_u(f) \rangle := \int_G uf$ extends to a bounded linear function on $C^*(G)$. It is shown in [16] that every element of $C^*(G)^*$ arises this way and that $\|u\|_{B(G)} = \|u\|_{C^*(G)^*}$, identifying $B(G)$ with the dual of $C^*(G)$. The Fourier-Stieltjes algebra therefore carries a natural weak* topology and it is straight forward to verify that the multiplication of $B(G)$ is separately weak* continuous.

The following relations between $A(G)$ and $B(G)$ hold.

Proposition 3.3.11. *Let G be a locally compact group. The Fourier algebra $A(G)$ is a closed ideal in $B(G)$ and coincides with the closure of the compactly supported functions in $B(G)$.*

In Chapter 4 we will be concerned with determining which subsets of a locally compact group G have characteristic functions lying in various function algebras. For the Fourier-Stieltjes algebra, these subsets are known.

Theorem 3.3.12. ***(Cohen-Host idempotent theorem)*** *Let G be a locally compact group and $E \subset G$. The function χ_E is in $B(G)$ if and only if E is in boolean ring of subsets generated by the left cosets of open subgroups of G.*

Eymard defined the Fourier and Fourier-Stieltjes algebra of a locally compact group G in [16]. This article remains the most complete reference on these algebras.

3.4 The Fourier multipliers

A ***Fourier multiplier*** is a complex valued function m on G such that $mA(G) \subset A(G)$ and the multiplication $u \mapsto mu$ is a completely bounded map on $A(G)$. The set of all such functions is a completely contractive Banach algebra of continuous, bounded functions under pointwise multiplication and the norm $\|u \mapsto mu\|_{cb}$, denoted $M_{cb}A(G)$. This algebra is often referred to as the completely bounded multiplier algebra of the Fourier algebra, a verbose name that we avoid. Since $A(G)$ is an ideal in $B(G)$, which itself has completely contractive multiplication, we have $B(G) \subset M_{cb}A(G)$. The following representation theorem, originally due to Gilbert [24], will be our main tool when working with Fourier multipliers. A short proof may be found in [35].

Theorem 3.4.1. ***(Gilbert's theorem)*** *Let G be a locally compact group. A complex valued function m on G is in $M_{cb}A(G)$ if and only if there exists a Hilbert space $\mathcal{H}$ and bounded continuous maps $P, Q : G \to \mathcal{H}$ such that $m(s^{-1}t) = \langle P(t) | Q(s) \rangle$ for all $s, t \in G$, in which case $\|m\|_{M_{cb}A(G)}$ is the infimum of the quantities $\|P\|_\infty \|Q\|_\infty$ taken over all such maps P and Q and Hilbert spaces $\mathcal{H}$.*

If u is in $B(G)$ and $u = \langle \pi(\cdot)\xi|\eta\rangle$ is any representation of u as a coefficient function of a unitary representation of G, then $u(s^{-1}t) = \langle \pi(t)\xi|\pi(s)\eta\rangle$. It follows from (3.3.1) and Theorem 3.4.1 that $\|\cdot\|_{M_{cb}A(G)} \leq \|\cdot\|_{B(G)}$ on $B(G)$. We write $A_{cb}(G)$ for the norm closure of $A(G)$ in $M_{cb}A(G)$, which is a commutative completely contractive Banach algebra with spectrum G [23, Proposition 2.2].

As Fourier multipliers lie in $L^\infty(G)$, we may consider $L^1(G)$ as a subspace of the dual of $M_{cb}A(G)$. Taking the completion of $L^1(G)$ with respect to the norm

$$\|f\|_{Q(G)} = \sup\left\{\left|\int_G fm\right| : m \in M_{cb}A(G) \text{ with } \|m\|_{M_{cb}A(G)} \leq 1\right\} \qquad \left(f \in L^1(G)\right)$$

yields a predual $Q(G)$ for $M_{cb}A(G)$ [12, Proposition 1.10]. Thus $M_{cb}A(G)$ is a dual space and has a weak* topology, and it can be checked that the multiplication of $M_{cb}A(G)$ is separately weak* continuous [23, p.969]. For $f \in L^1(G)$ we have

$$\|f\|_{C^*(G)} = \sup\left\{\left|\int_G fm\right| : m \in B(G) \text{ with } \|m\|_{B(G)} \leq 1\right\},$$

from which it follows that $\|f\|_{C^*(G)} \leq \|f\|_{Q(G)}$ and therefore that the identity map on $L^1(G)$ induces a contraction $Q(G) \to C^*(G)$. The adjoint is the inclusion map $B(G) \to M_{cb}A(G)$, which is therefore weak* continuous, meaning that any weak* convergent net in $B(G)$ is weak* convergent as a net in $M_{cb}A(G)$. Since $\|\cdot\|_{Q(G)} \leq \|\cdot\|_{L^1(G)}$ on $L^1(G)$, it follows that $\|\cdot\|_{L^\infty(G)} \leq \|\cdot\|_{M_{cb}A(G)}$. In summary,

$$A(G) \subset A_{cb}(G) \quad \text{and} \quad A(G) \subset B(G) \subset M_{cb}A(G) \subset L^\infty(G), \tag{3.4.1}$$

and each containment may be (often is) strict. For $u \in A(G)$ and $v \in B(G)$,

$$\begin{gathered}\|u\|_{L^\infty(G)} \leq \|u\|_{A_{cb}(G)} = \|u\|_{M_{cb}A(G)} \leq \|u\|_{A(G)} = \|u\|_{B(G)}, \\ \|v\|_{L^\infty(G)} \leq \|v\|_{M_{cb}A(G)} \leq \|v\|_{B(G)},\end{gathered}$$

so the first, third, and fourth inclusions in (3.4.1) are in general contractive while the second is isometric.

The following are shown in [12]:

Proposition 3.4.2. *Let G and G' be locally compact groups and let H a closed subgroup of G. The following hold:*

1. *The restriction map $r_H : M_{cb}A(G) \to M_{cb}A(H)$ is a well-defined contraction.*
2. *The natural map $M_{cb}A(G)\widehat{\otimes}M_{cb}A(G') \to M_{cb}A(G \times G') : m \otimes n \mapsto m \times n$ is a well-defined contraction.*

Using these results, Herz's restriction theorem and Proposition 3.3.10, it is straight forward to verify that both maps of Proposition 3.4.2 preserve the algebra $A_{cb}(G)$. Sets of ***spectral synthesis*** for $A_{cb}(G)$ are defined as for $A(G)$: those subsets E of G for which the functions in $I_{A_{cb}(G)}(E)$ with compact support disjoint from E are norm dense. ***Projections*** and ***invariant projections*** are defined analogously as well.

In Chapter 4 we will have need to consider a locally compact group G equipped with the discrete topology, which we indicate by G_d. The inclusions $B(G) \subset B(G_d)$ and $M_{cb}A(G) \subset M_{cb}A(G_d)$ hold and are in fact isometric, see [16] and [55, Corollary 6.3], respectively. For each $x \in G$ the evaluation functional δ_x lies in $\ell^1(G_d)$, which is contained in both $C^*(G_d)$ and in $Q(G_d)$, whence convergence in the weak* topology of $B(G_d)$ or $M_{cb}A(G_d)$ implies pointwise convergence. For bounded nets, the converse holds, see [16] for the assertion regarding $B(G_d)$ and [23, Lemma 2.6] regarding $M_{cb}A(G_d)$. We record this formally for later reference.

Proposition 3.4.3. *Let G be a discrete locally compact group. Convergence in the weak* topology of either $B(G)$ or $M_{cb}A(G)$ implies pointwise convergence, and on bounded subsets of either space these two topologies coincide.*

The Fourier multiplier algebra of a locally compact group has a long and rich history. Some of the basic properties of this algebra were established by Haagerup and others in [7, 12]. A modern treatment focusing on the connection to Herz-Schur multipliers may be found in [55]. The useful Appendix A of [42] records many folklore facts about the Fourier multipliers.

Aside from Theorem 3.4.4(1), the following result will not be needed in the sequel, but we believe its mention is warranted as it provides some indication of how one may interpret general Fourier multipliers. For the moment, let us call a (not necessarily unitary) representation $\pi : G \to B(\mathcal{H})$ of G as invertible operators that is continuous in the strong operator topology an ***invertible representation***, and call an invertible representation $\pi : G \to B(\mathcal{H})$ satisfying $\sup_{s\in G} \|\pi(s)\|_{B(\mathcal{H})} < \infty$ a ***uniformly bounded representation***. Define coefficient functions of these more general representations as we did before. Recall that a topological space is ***second countable*** if it has a countable base. Amenable locally compact groups are defined in Section 3.5.

Theorem 3.4.4. *Let G be a locally compact group. The following hold:*

1. *$M_{cb}A(G) = B(G)$ if and only if G is amenable.*

2. *[12, Theorem 2.2] The coefficient functions of uniformly bounded representations of G are Fourier multipliers.*

3. *[57] If G is second countable and m is a Fourier multiplier of G, then there exists an invertible representation $\pi : G \to B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$ such that $m(s^{-1}t) = \left\langle \pi(t)\xi | \pi(s^{-1})^* \eta \right\rangle$ for all $s, t \in G$.*

One direction of the first result is due to Losert [46], the other due to unpublished work of Ruan. Steenstrup [56] gives a proof that for many familiar nonamenable groups, e.g. the free groups on at least two generators, there exist Fourier multipliers that are not coefficients of uniformly bounded representations.

3.5 Amenability and related notions for groups and algebras

This section records the basic definitions regarding amenability for locally compact groups and for (completely contractive) Banach algebras and gives several characterizations of amenable locally compact groups.

Definition 3.5.1. A locally compact group G is called ***amenable*** if there exists a ***left invariant mean*** on $L^{\infty}(G)$, i.e. a state ω on $L^{\infty}(G)$ satisfying $\langle \omega, {}_s\phi \rangle = \langle \omega, \phi \rangle$ for all $\phi \in L^{\infty}(G)$ and $s \in G$.

Example 3.5.2. (Compact and abelian groups are amenable) Let G be a locally compact group. If G is compact, then $1_G \in L^1(G)$ is a normal left invariant mean on $L^{\infty}(G)$. If G is abelian, then the left translation action of G on the weak* compact, convex set $S(L^{\infty}(G))$ of states on $L^{\infty}(G)$ yields a family of commuting, weak* continuous, affine maps of $S(L^{\infty}(G))$ into itself and therefore has a fixed point, by the Markov-Kakutani fixed point theorem [10, p.109]. This fixed point is a left invariant mean.

Example 3.5.3. ($\mathbb{F}_2$ is not amenable) Let $\mathbb{F}_2$ denote the free group on two generators a and b. For $x \in \{a, b, a^{-1}, b^{-1}\}$ let W_x denote the words in $\mathbb{F}_2$ that begin with the symbol x. It is not too difficult to see that $\mathbb{F}_2 = W_a \cup aW_{a^{-1}} = W_b \cup bW_{b^{-1}}$. If ω were a left invariant mean on $\ell^{\infty}(\mathbb{F}_2)$, then $\left\langle \omega, \chi_{aW_{a^{-1}}} \right\rangle = \left\langle \omega, {}_{a^{-1}}\left(\chi_{W_{a^{-1}}}\right) \right\rangle = \left\langle \omega, \chi_{W_{a^{-1}}} \right\rangle$ implies

$$1 = \langle \omega, 1_{\mathbb{F}_2} \rangle \leq \langle \omega, \chi_{W_a} \rangle + \left\langle \omega, \chi_{W_{a^{-1}}} \right\rangle$$

and similarly for b. Since $\mathbb{F}_2$ equals the disjoint union $\{e\} \cup W_a \cup W_b \cup W_{a^{-1}} \cup W_{b^{-1}}$, it follows that

$$1 = \langle \omega, 1_{\mathbb{F}_2} \rangle = \left\langle \omega, \chi_{\{e\}} \right\rangle + \langle \omega, \chi_{W_a} \rangle + \langle \omega, \chi_{W_b} \rangle + \left\langle \omega, \chi_{W_{a^{-1}}} \right\rangle + \left\langle \omega, \chi_{W_{b^{-1}}} \right\rangle \geq 2.$$

Therefore $\mathbb{F}_2$ cannot be amenable.

It was shown by Johnson that a locally compact group G is amenable if and only if the Banach algebra $L^1(G)$ has a certain cohomological property [37], which became a definition for general Banach algebras. We give an equivalent formulation, also due to Johnson [36], that is more suitable for our purposes, and we give its operator space adaptation, due to Ruan [51].

Definition 3.5.4. A completely contractive Banach algebra A with product $\Delta : A \widehat{\otimes} A \to A$ is called ***operator amenable*** if there exists a bounded net $(d_\alpha)_\alpha$ in $A \widehat{\otimes} A$ satisfying:

1. $\|a\Delta(d_\alpha) - a\|_A \to 0$ for all $a \in A$, and
2. $\|a \cdot d_\alpha - d_\alpha \cdot a\|_{A\widehat{\otimes} A} \to 0$ for all $a \in A$,

where in (2) we have used the canonical completely contractive A-bimodule action on $A \widehat{\otimes} A$ given on elementary tensors by $a \cdot (b \otimes c) = ab \otimes c$ and $(b \otimes c) \cdot a = b \otimes ca$, for $a, b, c \in A$. Such a net is called a ***bounded approximate diagonal*** for A.

Of the many characterizations of amenability for locally compact groups, the following will be relevant to our investigations.

Theorem 3.5.5. *Let G be a locally compact group. The following are equivalent:*

1. *G is amenable.*

2. *[37] $L^1(G)$ is amenable.*

3. *[51] $A(G)$ is operator amenable.*

4. *[44] (**Leptin's theorem**) $A(G)$ has a bounded approximate identity.*

5. *There is a state ω on $VN(G)$ such that $\langle \omega, \lambda(s) \rangle = 1$ for all $s \in G$.*

If we ask that $A(G)$ have an approximate identity bounded in the potentially smaller Fourier multiplier norm, we obtain a strictly weaker condition.

Definition 3.5.6. A locally compact group G is called ***weakly amenable*** if $A(G)$ has an approximate identity bounded in the Fourier multiplier norm.

This notion was first studied by Haagerup in [31], who showed that $\mathbb{F}_2$ is weakly amenable. Since then, many nonamenable groups have been found to be weakly amenable, for example [32] showed that the nonamenable group $SL(2, \mathbb{R})$ is weakly amenable.

Proposition 3.5.7. *Let G be a locally compact group. The following are equivalent:*

1. *G is weakly amenable.*

2. *$A_{cb}(G)$ has a bounded approximate identity.*

3. *1_G is in the weak* closure of $A(G) \cap M_{cb}A(G)_{\leq C}$ in $M_{cb}A(G)$ for some $C > 0$.*

4. *$A(G) \cap M_{cb}A(G)_{\leq C}$ is weak* dense in $M_{cb}A(G)_{\leq 1}$ for some $C > 0$.*

Proof. The lengthy but routine approximation argument establishing the equivalence of (1) and (2) is Proposition 1 of [21].

If (2) holds, then by density of $A(G)$ we may assume there is a bounded approximate identity $(e_\alpha)_\alpha$ for $A_{cb}(G)$ consisting of functions in $A(G)$. Gien $f \in \mathcal{C}_c(G)$ with compact support K, Proposition 3.3.4(3) asserts the existence of $u \in A(G)$ with $u|_K = 1$ and we have

$$|\langle e_\alpha - 1_G, f \rangle| = \left| \int_K (e_\alpha - 1_G)\, uf \right| = |\langle e_\alpha u - u, f \rangle| \to 0.$$

Since $\mathcal{C}_c(G)$ is dense in $L^1(G)$, which is dense in $Q(G)$, and since the net $(e_\alpha)_\alpha$ is bounded in $A_{cb}(G)$, a straight forward approximation argument shows that e_α converges weak* to 1_G and (3) holds.

If $(u_\alpha)_\alpha$ is a net in $A(G)$ that converges weak* to 1_G in $M_{cb}A(G)$ with $\sup_\alpha \|u_\alpha\|_{M_{cb}A(G)} \leq C$, and if $m \in M_{cb}A(G)_{\leq 1}$, then $u_\alpha m \overset{w^*}{\to} m$ by the weak* continuity of multiplication in $M_{cb}A(G)$ and $u_\alpha m \in A(G)$ with $\|u_\alpha m\|_{M_{cb}A(G)} \leq C$, whence (3) implies (4).

If (4) holds, then let $(u_\alpha)_\alpha$ be a net in $A(G)$ that converges weak* to 1_G in $M_{cb}A(G)$ with $\sup_\alpha \|u_\alpha\|_{M_{cb}A(G)} \leq C$. Let $f \in \mathcal{C}_c(G)$ such that $f \geq 0$ and $\int_G f = 1$. Note that, for $u \in A(G)$ we have $f * u = \int_G f(y) \left({}_{y^{-1}}u\right) dy$, where the integral is the weak Banach space valued integral of [17, Appendix A], and consequently $f * u \in A(G)$. By Theorem 4.3.1, the net $(f * (f * u_\alpha))_\alpha$ has the same norm bound as $(u_\alpha)_\alpha$ and is an approximate identity of $A(G)$. Thus (1) holds. □

In [34], Haagerup and Kraus developed a further weakening of condition (3) of Proposition 3.5.7.

Definition 3.5.8. A locally compact group G is said to have the ***approximation property*** if 1_G is in the weak* closure of $A(G)$ in $M_{cb}A(G)$, equivalently if $A(G)$ is weak* dense in $M_{cb}A(G)$.

The group $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ fails to be weakly amenable, but does have the approximation property, and it has been recently shown that $SL(3,\mathbb{R})$ fails to have the approximation property. The introduction to [32] provides a brief history of these weaker forms of amenability for locally compact groups. References for the claims of this paragraph may be found in that article.

Comprehensive references for amenability of locally compact groups are [49] and [50]. The text [53] provides a thorough overview of amenability theory of (completely contractive) Banach algebras and its relation to amenability for groups.

Chapter 4

Separation properties for closed subgroups

In this chapter we investigate separation properties for closed subgroups of locally compact groups. Our main objectives are to determine conditions on a closed subgroup H of a locally compact group G that imply:

1. $VN_H(G)$ is complemented in $VN(G)$ as a completely contractive $A(G)$-bimodule, that is, there is a completely bounded $A(G)$-bimodule projection $VN(G) \to VN_H(G)$.

2. The characteristic function χ_H, which is always in $M_{cb}A(G_d)$, may be approximated in the weak* topology by elements of $M_{cb}A(G)$.

The norm of the Fourier-Stieltjes algebra dominates the Fourier multiplier norm, meaning it is easier for a net to be bounded in the Fourier multiplier norm. The following definition is therefore, strictly speaking, a generalization of the definition given in [1], where bounded nets in the Fourier-Stieltjes algebra were considered.

Definition 4.0.1. Let G be a locally compact group and H a closed subgroup. A ***bounded approximate indicator*** for H is a bounded net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$ satisfying

1. $\|ur_H(m_\alpha) - u\|_{A(H)} \to 0$ for all $u \in A(H)$, and

2. $\|um_\alpha\|_{A(G)} \to 0$ for all $u \in I_{A(G)}(H)$.

As we will see in Section 4.5, bounded approximate indicators consisting of Fourier multipliers provide a means of obtaining projections as in (1) above. In Section 4.1 we show, amongst other things, that (1) is a stronger condition than (2), at least when H satisfies certain weak forms of amenability. It is noted in Chapter 5 that a closed subgroup can satisfy (1) without satisfying (2).

Example 4.0.2. (Bounded approximate indicators for the diagonal subgroup of $G \times G$) Let G be a locally compact group and consider the closed subgroup $G_\Delta = \{(x, x) : x \in G\}$, which is isomorphic to G as a topological group. We show that a a bounded approximate diagonal for $A(G)$ is exactly a bounded approximate indicator for G_Δ that lies in $A(G)$ and is bounded there. Under the identification $A(G)\widehat{\otimes}A(G) = A(G\times G)$, the completely contractive Banach algebra product $\Delta : A(G)\widehat{\otimes}A(G) \to A(G)$ corresponds to restriction to the diagonal $r_{G_\Delta} : A(G\times G) \to A(G_\Delta)$. Moreover, the kernel of Δ is identified with the ideal $I_{A(G\times G)}(G_\Delta)$. The algebra $A(G)$ has the property that $\ker\Delta = \langle uv\otimes w - u\otimes vw : u,v,w \in A(G)\rangle$ [9, Theorem 6.5], which together with the fact that $A(G\times G) = \langle u\times v : u,v\in A(G)\rangle$ yields

$$\begin{aligned} I_{A(G\times G)}(G_\Delta) &= \langle uv\times w - u\times vw : u,v,w\in A(G)\rangle \\ &= \langle v\cdot(u\times w) - (u\times w)\cdot v : u,v,w\in A(G)\rangle \\ &= \langle u\cdot w - w\cdot u : u\in A(G) \text{ and } w\in A(G\times G)\rangle \\ &= \langle (u\times 1_G - 1_G\times u)\, w : u\in A(G) \text{ and } w\in A(G\times G)\rangle . \end{aligned}$$

It follows that a net $(d_\alpha)_\alpha$ in $A(G\times G)$ satisfies $\|u\cdot d_\alpha - d_\alpha\cdot u\|_{A(G\times G)} \to 0$ for all $u\in A(G)$ if and only if it satisfies $\|ud_\alpha\|_{A(G\times G)} \to 0$ for all $u\in I_{A(G\times G)}(G_\Delta)$. Collecting these facts together, we deduce that for $A(G)$, a bounded approximate diagonal is a net $(d_\alpha)_\alpha$ in $A(G\times G)$ such that

$$\|ur_{G_\Delta}(d_\alpha) - u\|_{A(G_\Delta)} \to 0 \quad \text{and} \quad \|wd_\alpha\|_{A(G\times G)} \to 0$$

for all $u\in A(G_\Delta)$ and $w\in I_{A(G\times G)}(G_\Delta)$, i.e. a bounded approximate indicator for G_Δ in $A(G\times G)$.

Example 4.0.3. (A weaker homological property of $A(G)$) Let G be a locally compact group. The preceding example showed that the existence of a bounded approximate indicator for the diagonal in $A(G\times G)$ characterizes operator amenability of $A(G)$. The notion of bounded approximate indicators for subgroups was introduced in [1] to provide a characterization of the operator biflatness of $A(G)$, a weaker condition than operator amenability which may be defined as asserting the existence of an invariant projection $VN(G\times G) \to VN_{G_\Delta}(G\times G)$. We will see in Section 4.5 that the existence of a bounded approximate indicator for a closed subgroup H of G yields an invariant projection $VN(G) \to VN_H(G)$, from which it follows that if a bounded approximate indicator exists for G_Δ, then $A(G)$ is operator biflat. It was recently shown in [8] that operator biflatness of $A(G)$ is in fact characterized by the existence of a bounded approximate indicator for G_Δ in $B(G\times G)$.

Example 4.0.4. (Bounded approximate indicators for subgroups of amenable groups) Let G be an amenable locally compact group. By Leptin's theorem, the algebra $A(G)$ has a bounded approximate identity and [19, Proposition 6.4] then asserts that an invariant projection $VN(G) \to VN_H(G) = I_{A(G)}(H)^\perp$ exists exactly when the ideal $I_{A(G)}(H)$ has a bounded approximate identity. By [22], this occurs for every closed subgroup of G and it follows that an approximate indicator exists for every closed subgroup, since $(1_G - e_\alpha)_\alpha$ is an approximate indicator for H when $(e_\alpha)_\alpha$ is a bounded approximate identity for $I_{A(G)}(H)$.

The following result suggests a strong connection between amenability properties of locally compact groups and approximability of characteristic functions.

Theorem 4.0.5. *A locally compact group H is amenable if and only if χ_H is in the weak* closure of $A(G)$ in $B(G_d)$ for some (equivalently, any) locally compact group G containing H as a closed subgroup.*

Proof. Fix a locally compact group G that contains H as a closed subgroup.

Suppose H is amenable. By Leptin's theorem there exists a bounded approximate identity $(e_\alpha)_\alpha$ for $A(H)$ and by [11] there exists an invariant projection $\Psi : VN(G) \to VN_H(G)$. We showed in Example 3.3.6 that the adjoint of the restriction map $r_H : A(G) \to A(H)$ is a $*$-isomorphism $\tau := r_H^* : VN(H) \to VN_H(G)$. It is shown in Example 4.1.6 that $\Psi(\lambda_G(s)) = \chi_H(s)\lambda_G(s)$ for all $s \in G$ and the argument of Example 4.1.2 shows that $e_\alpha \overset{ptw}{\to} 1_H$. It follows that the composition

$$VN(H)^* \overset{(\tau^*)^{-1}}{\to} VN_H(G)^* \overset{\Psi^*}{\to} VN(G)^*$$

satisfies, for $s \in G$,

$$\begin{aligned}
\left\langle \Psi^*(\tau^*)^{-1}(e_\alpha), \lambda_G(s)\right\rangle &= \left\langle (\tau^*)^{-1}(e_\alpha), \chi_H(s)\lambda_G(s)\right\rangle \\
&= \left\langle e_\alpha, \tau^{-1}(\chi_H(s)\lambda_G(s))\right\rangle \\
&= \begin{cases} \langle e_\alpha, \lambda_H(s)\rangle, & s \in H \\ 0, & s \notin H \end{cases} \\
&\to \chi_H(s),
\end{aligned}$$

Let E be a weak* cluster point of the bounded net $\left(\Psi^*(\tau^*)^{-1}(e_\alpha)\right)_\alpha$ in $VN(G)^*$, so that $\langle E, \lambda_G(s)\rangle = \chi_H(s)$ for all $s \in G$, by the above computation. Letting $(u_\alpha)_\alpha$ be a bounded net in $A(G)$ converging weak* to E, we have that $u_\alpha \overset{ptw}{\to} \chi_H$ and therefore $u_\alpha \overset{w^*}{\to} \chi_H$ in $B(G_d)$ by Proposition 3.4.3.

Conversely, if $(u_\alpha)_\alpha$ is a bounded net in $A(G)$ such that $u_\alpha \overset{w^*}{\to} \chi_H$ in $B(G_d)$, then $u_\alpha \overset{ptw}{\to} \chi_H$ and thus $r_H(u_\alpha) \overset{ptw}{\to} 1_H$. A routine argument shows that we may assume the $r_H(u_\alpha)$ are states on $VN(H)$, in which case any weak* cluster point $\omega \in VN(H)^*$ of the net $(r_H(u_\alpha))_\alpha$ is a state on $VN(H)$ satisfying $\langle \omega, \lambda_H(s)\rangle = \lim_\alpha \langle r_H(u_\alpha), \lambda_H(s)\rangle = 1$ for all $s \in H$. Therefore H is amenable by Theorem 3.5.5. □

4.1 The discretized Fourier multiplier algebra

Let G be a locally compact group. In [1], the ***discretized Fourier-Stieltjes algebra*** $B^d(G)$ is defined to be the weak* closure of $B(G)$ in $B(G_d)$ and it is noted that there is a weak* continuous quotient map $B(G)^{**} \to B^d(G)$ extending the inclusion $B(G) \subset B^d(G)$. We make the analogous definition for the Fourier multipliers of G.

Definition 4.1.1. The ***discretized Fourier multiplier algebra*** $M_{cb}^{d}A(G)$ of a locally compact group G is the weak* closure of $M_{cb}A(G)$ in $M_{cb}A(G_d)$. Let $Q^d(G)$ denote the predual $Q(G_d)/M_{cb}^{d}(G)_{\perp}$ of $M_{cb}^{d}A(G)$. Let $A^d(G)$ denote the weak* closure of $A(G)$ in $M_{cb}A(G_d)$.

Given a subset A of $M_{cb}A(G)$, a subset E of G is called **A-approximable** if χ_E is in the weak* closure of A in $M_{cb}A(G_d)$. We call E ***approximable*** if $\chi_E \in M_{cb}^{d}A(G)$.

In this section, we investigate when subsets of a locally compact group G are approximable. For a closed subgroup H of G, the assertion that H is approximable may be viewed as a very weak form of subgroup separation. It should note that, for a subset E of G to be approximable, it must already be that $\chi_E \in M_{cb}A(G_d)$, and the subsets of G for which this occurs are not well understood when G_d is not amenable. In the amenable case, the spaces $M_{cb}A(G_d)$ and $B(G_d)$ coincide by Theorem 3.4.4 and the Cohen-Host idempotent theorem (Theorem 3.3.12) provides a complete description of the subsets of G whose characteristic functions lie in $B(G_d)$. For discrete groups, determining the approximable sets is exactly the question of determining the sets with characteristic function in the Fourier multipliers.

Example 4.1.2. (Approximate indicators and approximable subgroups) Let G be a locally compact group and H a closed subgroup for which a bounded approximate indicator $(m_\alpha)_\alpha$ exists. If $s \in H$, then by Proposition 3.3.4(3) we may find $u \in A(H)$ such that $u(s) = 1$, in which case

$$|m_\alpha(s) - 1| \leq \|ur_H(m_\alpha) - u\|_{L^\infty(H)} \leq \|ur_H(m_\alpha) - u\|_{A(H)} \to 0.$$

If $s \in G \setminus H$, then, using Proposition 3.3.4(3), we may find $w \in I_{A(G)}(H)$ such that $w(s) = 1$, and

$$|m_\alpha(s)| \leq \|wm_\alpha\|_{L^\infty(G)} \leq \|wm_\alpha\|_{A(G)} \to 0.$$

Thus the bounded net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$ converges pointwise to χ_H. Since $M_{cb}A(G) \subset M_{cb}A(G_d)$ isometrically and since weak* and pointwise convergence coincide on bounded subsets of $M_{cb}A(G_d)$, it follows that m_α has weak* limit χ_H in $M_{cb}A(G_d)$, hence H is approximable.

The algebra $M_{cb}^{d}A(G)$ is a weak* closed subalgebra of $M_{cb}A(G_d)$ and has separately weak* continuous multiplication, since $M_{cb}A(G_d)$ does. If we impose a rather weak condition on the discrete group G_d, then every function in $M_{cb}A(G_d)$ may be approximated in the weak* topology by elements of $A(G)$.

Proposition 4.1.3. *Let G be a locally compact group. The inclusion $A_{cb}(G_d) \subset A^d(G)$ always holds. Consequently, if G_d has the approximation property, then $A^d(G) = M_{cb}A(G_d)$.*

Proof. By Proposition 1 of [3] we have $A(G_d) \subset \overline{A(G)}^{B(G_d),w^*}$ and, because weak* convergence in $B(G_d)$ implies weak* convergence in $M_{cb}A(G_d)$, it follows that $A(G_d) \subset \overline{A(G)}^{B(G_d),w^*} \subset A^d(G)$. Taking norm closures in $M_{cb}A(G_d)$ yields $A_{cb}(G_d) \subset A^d(G)$. If $(e_\alpha)_\alpha$ is a net in $A(G_d)$ converging weak* to 1_G in $M_{cb}A(G_d)$ and if $m \in M_{cb}A(G_d)$, then $me_\alpha \in A(G_d) \subset A^d(G)$ for each α and $me_\alpha \xrightarrow{w^*} m$ in $M_{cb}A(G_d)$, implying that $m \in A^d(G)$. □

The following lemma is well known.

Lemma 4.1.4. *Let E and F be Banach spaces and let $\Psi : E \to F$ be a bounded map. If F is a dual space, then Ψ extends uniquely to a bounded, weak* continuous map $\tilde{\Psi} : E^{**} \to F$.*

Proof. Let $\kappa : F_* \to F^*$ be the canonical isometric inclusion and let $\tilde{\Psi} = \kappa^* \Psi^{**}$, which, being the adjoint of $\kappa \Psi^*$, is weak* continuous. For $x \in E$,

$$\left\langle \tilde{\Psi}(x), \varphi \right\rangle_{F, F_*} = \langle \Psi^*(\varphi), x \rangle_{E^*, E} = \langle \Psi(x), \varphi \rangle_{F, F_*} \qquad (\varphi \in F_*),$$

showing that $\tilde{\Psi}$ extends Ψ. If Ψ' is any weak* continuous extension of Ψ to E^{**}, then Ψ' and $\tilde{\Psi}$ agree on the weak* dense subset E of E^{**}, hence are equal. □

We provide a more concrete construction of the canonical map $M_{cb}A(G)^{**} \to M_{cb}^d A(G)$ that exploits the relation between $M_{cb}A(G)$ and $M_{cb}^d A(G)$. Let $\iota_d : M_{cb}A(G) \to M_{cb}A(G_d)$ and $\kappa_Q : Q(G_d) \to Q(G_d)^{**}$ denote the inclusion maps. By the bipolar theorem $M_{cb}^d A(G)_\perp = M_{cb}A(G)_\perp$ and we have

$$\kappa_Q \left(M_{cb}A(G)_\perp \right) \subset M_{cb}A(G)^\perp = \operatorname{im}(\iota_d)^\perp = \ker(\iota_d^*),$$

which together imply that the composition

$$Q(G_d) \overset{\kappa_Q}{\to} Q(G_d)^{**} \overset{\iota_d^*}{\to} Q(G)^{**}$$

induces a map $Q^d(G) \to Q(G)^{**}$. Denote the adjoint of this induced map by $\tau : M_{cb}A(G)^{**} \to M_{cb}^d A(G)$.

Let $q : Q(G_d) \to Q^d(G)$ denote the quotient map, which has adjoint the inclusion map $q^* : M_{cb}^d A(G) \to M_{cb}A(G_d)$. If $m \in M_{cb}^d A(G)$, then

$$m(s) = \langle q^*(m), \delta_s \rangle = \langle m, q(\delta_s) \rangle,$$

where $s \in G$ and $\delta_s \in \ell^1(G_d) \subset Q(G_d)$ is the point mass at s.

For $m \in M_{cb}A(G)$ and $s \in G$ we have

$$\begin{aligned}
\tau(m)(s) &= \langle \tau(m), q(\delta_s) \rangle_{M_{cb}^d A(G), Q^d(G)} \\
&= \langle m, \iota_d^* \kappa_Q(\delta_s) \rangle_{M_{cb}A(G)^{**}, M_{cb}A(G)^*} \\
&= \langle \iota_d(m), \delta_s \rangle_{M_{cb}A(G_d), Q(G_d)} \\
&= m(s),
\end{aligned}$$

showing that τ is the canonical extension of the inclusion $M_{cb}A(G) \subset M_{cb}^d A(G)$. It follows that if $m_\alpha \in M_{cb}A(G)$ converges weak* to $\omega \in M_{cb}A(G)^{**}$, then

$$\tau(\omega)(s) = \langle \tau(\omega), q(\delta_s) \rangle = \lim_\alpha \langle \tau(m_\alpha), q(\delta_s) \rangle = \lim_\alpha \tau(m_\alpha)(s) = \lim_\alpha m_\alpha(s) \quad (s \in G), \quad (4.1.1)$$

so the map τ can be thought of as extracting pointwise limits from nets in $M_{cb}A(G)$ that are convergent in the bidual.

The range of τ consists of exactly those elements of $M_{cb}^{d}A(G)$ that are limits of bounded nets in $M_{cb}A(G)$: if $m \in M_{cb}^{d}A(G)$ is the limit of a bounded net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$, then by passing to a subnet we may assume that $(m_\alpha)_\alpha$ converges weak* to an element ω of $M_{cb}A(G)^{**}$, in which case $\tau(\omega) = m$ by (4.1.1), and conversely if $\omega \in M_{cb}A(G)^{**}$, then we may find a bounded net $(m_\alpha)_\alpha$ in $M_{cb}A(G)$ converging weak* to ω, in which case $m_\alpha \overset{ptw}{\to} \tau(\omega)$ by (4.1.1) and hence $m_\alpha \overset{w^*}{\to} \tau(\omega)$ in $M_{cb}A(G_d)$ by boundedness.

Proposition 4.1.5. *Let G be a locally compact group and $E \subset G$. If there is a bounded map $\Psi : VN(G) \to VN(G)$ satisfying $\Psi(\lambda(s)) = \chi_E(s)\lambda(s)$ for all $s \in G$, then $\chi_E A(G) \subset M_{cb}^{d}A(G)$. If, moreover, $\chi_E \in M_{cb}A(G_d)$ and G is $A(G)$-approximable, then E is approximable.*

Proof. Let $\kappa_A : A(G) \to A(G)^{**}$ and $\iota_A : A(G) \to M_{cb}A(G)$ be the inclusions and let σ denote the composition

$$A(G) \xrightarrow{\kappa_A} A(G)^{**} \xrightarrow{\Psi^*} A(G)^{**} \xrightarrow{\iota_A^{**}} M_{cb}A(G)^{**} \xrightarrow{\tau} M_{cb}^{d}A(G).$$

For $u \in A(G)$ and $s \in G$,

$$\begin{aligned} \sigma(u)(s) &= \langle \sigma(u), q(\delta_s)\rangle_{M_{cb}^{d}A(G),Q^d(G)} \\ &= \langle \iota_A^{**}\Psi^*\kappa_A(u), \iota_d^*\kappa_Q(\delta_s)\rangle_{M_{cb}A(G)^{**},M_{cb}A(G)^*} \\ &= \langle \Psi^*\kappa_A(u), \lambda(s)\rangle_{A(G)^{**},VN(G)} \\ &= \langle \Psi(\lambda(s)), u\rangle_{VN(G),A(G)} \\ &= \chi_E(s)\,u(s), \end{aligned}$$

where we have used that $\iota_A^*\iota_d^*\kappa_Q(\delta_s) = \lambda(s)$. Thus $\chi_E u = \sigma(u) \in M_{cb}^{d}A(G)$ for all $u \in A(G)$.

If G is $A(G)$-approximable, say $(e_\alpha)_\alpha$ is a net in $A(G)$ converging weak* to 1_G in $M_{cb}A(G_d)$, then that $\chi_E \in M_{cb}A(G_d)$ implies $\chi_E e_\alpha \overset{w^*}{\to} \chi_E$, by w^*-continuity of multiplication in $M_{cb}A(G_d)$. Since $\chi_E e_\alpha \in M_{cb}^{d}A(G)$, we conclude that E is approximable. □

By the Cohen-Host idempotent theorem, $\chi_H \in B(G_d) \subset M_{cb}A(G_d)$ for any subgroup H of a locally compact group G, so that the second part of Proposition 4.1.5 is applicable to characteristic functions of subgroups. It is shown in Corollary 4.1.12 below that, when $\chi_H A(G) \subset M_{cb}^{d}A(G)$, we need only require $1_H \in A^d(H)$ to deduce that H is approximable.

Example 4.1.6. (Invariant projections on $VN(G)$) Let G be a locally compact group and H a closed subgroup. We show that an invariant projection Ψ of $VN(G)$ onto $VN_H(G)$ satisfies the hypothesis of Proposition 4.1.5. It is clear that $\Psi(\lambda(s)) = \lambda(s)$ for $s \in H$. Let $s \in G \setminus H$ and suppose $T_\alpha \in \text{span}\lambda(H)$ such that $T_\alpha \overset{w^*}{\to} \Psi(\lambda(s))$. If $u \in A(G)$ with $u(s) = 1$ and $u|_H = 0$, then

$$\langle u \cdot \lambda(s), v\rangle = \langle \lambda(s), vu\rangle = v(s)\,u(s) = v(s) = \langle \lambda(s), v\rangle \qquad (v \in A(G)),$$

so $u \cdot \lambda(s) = \lambda(s)$. If $S = \sum_j \alpha_j \lambda(s_j) \in \operatorname{span}\lambda(H)$, then

$$\langle u \cdot S, v \rangle = \sum_j \alpha_j v(s_j) u(s_j) = 0 \qquad (v \in A(G)),$$

so that $u \cdot S = 0$. Thus $0 = u \cdot T_\alpha \overset{w^*}{\to} u \cdot \Psi(\lambda(s)) = \Psi(u \cdot \lambda(s)) = \Psi(\lambda(s))$ and $\Psi(\lambda(s)) = 0$. Therefore $\Psi(\lambda(s)) = \chi_H(s)\lambda(s)$ for all $s \in G$.

Example 4.1.7. (Natural projections on $VN(G)$) In [45], Lau and Ülger define a projection Ψ on $VN(G)$ to be ***natural*** if $\Psi(\lambda(s)) = \chi_E(s)\lambda(s)$ for some subset E of G. We may interpret Proposition 4.1.5 as stating restrictions on which subsets of G can arise from a natural projection.

Definition 4.1.8. Let G be a locally compact group. A bounded net $(e_\alpha)_\alpha$ in $A(G)$ or $A_{cb}(G)$ is called a ***Δ-weak bounded approximate identity*** if e_α converges pointwise to 1_G.

A bounded approximate identity for $A_{cb}(G)$ is a Δ-weak one, by the reasoning of Example 4.1.2, so that $A_{cb}(G)$ has Δ-weak bounded approximate identity whenever G is weakly amenable. Observe also that the set G is $A(G)$-approximable when $A_{cb}(G)$ has a Δ-weak bounded approximate identity because, by density, we may assume such a net lies in $A(G)$ and on bounded subsets of $M_{cb}A(G_d)$ pointwise and weak* convergence coincide (Proposition 3.4.3).

The construction of the map τ above and the proof of Proposition 4.1.5 can be carried out with $M_{cb}A(G)$ replaced by $B(G)$. To be able to conclude that χ_H is in the weak* closure of $B(G)$ in $B(G_d)$ using this result, we would require 1_G to be in the weak* closure of $A(G)$ in $B(G_d)$. It appears difficult to satisfy this condition without utilizing a Δ-weak bounded approximate identity: any *bounded* net in $A(G)$ with the desired weak* limit is already such an approximate identity, and any attempt to produce an unbounded net cannot use the only tool at hand, that on bounded sets pointwise convergence implies weak* convergence in $B(G_d)$. But when $A(G)$ has a Δ-weak bounded approximate identity the group G is already amenable [40, Theorem 5.1]. It is the availability of Δ-weak bounded approximate identities in $A_{cb}(G)$ for a larger class of groups — containing at least the weakly amenable group — that is responsible for the utility of Proposition 4.1.5. Whether the existence of a Δ-weak bounded approximate identity in $A_{cb}(G)$ implies weak amenability of G appears to be an open question.

Definition 4.1.9. Let G be a locally compact group and H a closed subgroup. Let

$$r_H : M_{cb}A(G_d) \to M_{cb}A(H_d) \quad \text{and} \quad e_H : M_{cb}A(H_d) \to M_{cb}A(G_d)$$

denote the restriction map and the map that extends by zero, respectively. For a function f on H let $\mathring{f}$ denote its extension by zero to G.

The restriction r_H is a complete quotient and extension e_H a complete isometry (see [55, Corollary 6.3] or [54, Proposition 4.1]), and it is clear that $r_H e_H = \mathrm{id}_{M_{cb}A(H_d)}$ and $e_H r_H = M_{\chi_H}$, the multiplication by χ_H.

Lemma 4.1.10. *Let G be a locally compact group and H a closed subgroup. The maps r_H and e_H are weak*-continuous.*

Proof. If $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \ell^1(H_d) \cap \mathcal{C}_c(H_d)$, then $\overset{\circ}{f} \in \ell^1(G_d)$ and

$$\langle r_H^*(f), m \rangle = \sum_{j=1}^{n} \alpha_j m(x_j) = \left\langle \overset{\circ}{f}, m \right\rangle \qquad (m \in M_{cb}A(G_d)),$$

showing that $r_H^*\left(\ell^1(H_d) \cap \mathcal{C}_c(H_d)\right) \subset Q(G_d)$. Since $\ell^1(H_d) \cap \mathcal{C}_c(H_d)$ is dense in $Q(H_d)$, it follows that r_H is weak*-continuous.

Now if $f = \sum_{j=1}^{n} \alpha_j \delta_{x_j} \in \ell^1(G_d) \cap \mathcal{C}_c(G_d)$, then

$$\langle e_H^*(f), m \rangle = \left\langle \overset{\circ}{m}, f \right\rangle = \sum_{j=1}^{n} \alpha_j \chi_H(x_j) m(x_j) = \left\langle \sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j}, m \right\rangle \qquad (m \in M_{cb}A(H_d)),$$

and so $\sum_{j=1}^{n} \alpha_j \chi_H(x_j) \delta_{x_j} \in \ell^1(H_d)$. Therefore $e_H^*\left(\ell^1(G_d) \cap \mathcal{C}_c(G_d)\right) \subset Q(H_d)$ and the claim follows by density, as above. $\square$

Proposition 4.1.11. *Let G be a locally compact group and H a closed subgroup. Then r_H maps $M_{cb}^d A(G)$ into $M_{cb}^d A(H)$ and the following are equivalent:*

1. *$\chi_H A(G) \subset A^d(G)$.*
2. *$e_H\left(A^d(H)\right) \subset A^d(G)$.*
3. *$A^d(G) = I_{A^d(G)}(G \setminus H) \oplus I_{A^d(G)}(H)$ (algebraic direct sum).*

Proof. Since the restriction of a Fourier multiplier of G to the closed subgroup H yields a Fourier multiplier of H by Proposition 3.4.2, the first claim follows from weak*-continuity of r_H.

If $\chi_H A(G) \subset A^d(G)$, then, because $A(H) = r_H(A(G))$ by Herz's restriction theorem,

$$e_H(A(H)) = e_H(r_H(A(G))) = \chi_H A(G) \subset A^d(G)$$

and (2) follows by weak*-continuity of e_H. Thus (1) implies (2).

If $e_H\left(A^d(H)\right) \subset A^d(G)$, then given $m \in A^d(G)$, the weak*-continuity of r_H implies $r_H(m) \in A^d(H)$ and it follows that $\chi_H m = e_H r_H(m) \in A^d(G)$, whence $\chi_{G \setminus H} m = m - \chi_H m \in A^d(G)$ and therefore $m = \chi_H m + \chi_{G \setminus H} m \in I_{A^d(G)}(G \setminus H) + I_{A^d(G)}(H)$. These ideals clearly have trivial intersection. Thus (2) implies (3).

If (3) holds, then we may write $m \in A(G)$ as $m = m_1 + m_2$ for $m_1 \in I_{A^d(G)}(G \setminus H)$ and $m_2 \in I_{A^d(G)}(H)$, in which case $\chi_H m = m_1 \in A^d(G)$. Thus (3) implies (1). $\square$

When the equivalent conditions of Proposition 4.1.11 hold, condition (2) implies that $e_H\left(A^d(H)\right) = I_{A^d(G)}(G \setminus H)$ and, because e_H is a complete isometry, it follows from condition (3) that $A^d(G) = A^d(H) \oplus I_{A^d(G)}(H)$.

Corollary 4.1.12. *Let G be a locally compact group and H a closed subgroup. If $\chi_H A(G) \subset A^d(G)$ and $1_H \in A^d(H)$, then H is $A(G)$-approximable. The latter condition holds, in particular, whenever H is weakly amenable or H_d has the approximation property.*

Proof. It follows from Proposition 4.1.11(2) that $\chi_H = e_H(1_H) \in A^d(G)$. We noted following Definition 4.1.8 that weak amenability of H implies $1_H \in A^d(H)$, while Proposition 4.1.3 implies $1_H \in A^d(H)$ when H_d has the approximation property. □

Combining the results of this section, we are able to partially address the second point of the introduction to this chapter.

Corollary 4.1.13. *A closed subgroup H of a locally compact group G is approximable when either of the following conditions is satisfied:*

1. *G_d has the approximation property.*

2. *There is a bounded map $\Psi : VN(G) \to VN_H(G)$ such that $\Psi(\lambda(s)) = \chi_H(s)\lambda(s)$ for $s \in G$, which occurs in particular if Ψ is a natural or invariant projection onto $VN_H(G)$, and $1_H \in A^d(H)$, which occurs in particular when H is weakly amenable or H_d has the approximation property.*

4.2 The discretized H-separation property

In this section, we provide a characterization of the condition that a closed subgroup H of a locally compact group G be approximable in the spirit of the H-separation property of Kaniuth and Lau [39]. We first review the H-separation property, which is a condition involving the positive definite functions on G.

Definition 4.2.1. Let G be a locally compact group. A bounded continuous function ϕ on G is called ***positive definite*** if $\int_G (f^* * f)\,\phi \geq 0$ for all $f \in L^1(G)$. The set of positive definite functions on G is denoted by $P(G)$. For a closed subgroup H of G, let $P_H(G) = \{u \in P(G) : u|_H = 1\}$.

The following basic properties of positive definite functions are established in Section 3.3 of [17].

Theorem 4.2.2. *Let G be a locally compact group. The following hold:*

1. *The positive definite functions on G coincide with the:*

   (a) *Coefficient functions of the form $\langle \pi(\cdot)\xi|\xi\rangle$ for $\xi \in \mathcal{H}$, where $\pi : G \to B(\mathcal{H})$ is a unitary representation of G. Consequently, $P(G) \subset B(G)$, the Fourier-Stieltjes algebra of G.*

   (b) *Positive linear functionals on the group C^*-algebra $C^*(G)$. In particular, $\lambda u + v \in P(G)$ for all $u, v \in P(G)$ and $\lambda \geq 0$.*

(c) *Continuous functions $u : G \to \mathbb{C}$ satisfying*

$$\sum_{j,k} \alpha_j \overline{\alpha_k} u\left(s_k^{-1} s_j\right) \geq 0 \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{C} \text{ and } s_1, \ldots, s_n \in G. \tag{4.2.1}$$

2. *$P(G)$ is closed under multiplication.*

3. *For $u \in P(G)$, we have :*

   (a) $\|u\|_{L^\infty(G)} = \|u\|_{B(G)} = u(e)$.
   (b) $u(s^{-1}) = \overline{u(s)}$ *for all* $s \in G$.

Kaniuth and Lau introduced the following separation property.

Definition 4.2.3. Let G be a locally compact group and H a closed subgroup. The group G is said to have the ***H-separation property*** or to ***separate*** H if for each $s \in G \setminus H$ there exists $u \in P_H(G)$ such that $u(s) \neq 1$.

It was shown by Forrest [20] that if a locally compact group G has a neighborhood base for the identity consisting of compact sets that are invariant under conjugation, i.e. if G is a SIN group, then G separates every closed subgroup.

Example 4.2.4. (Open, compact, and normal subgroups are separated) Let G be a locally compact group and H a closed subgroup.

1. Suppose H is open in G. We have $1_H \in P(H)$ since $1_H = \langle \pi(\cdot) 1 | 1 \rangle$, where $\pi : G \to \mathbb{C}$ is the trivial representation, and by [30, (32.43)] the extension by zero of a function on H satisfying (4.2.1) to a function on G also satisfies (4.2.1). Thus χ_H satisfies (4.2.1) and is continuous, since H is open, so is in $P_H(G)$.

2. Suppose H is compact. Fix $s_0 \in G \setminus H$ and a neighborhood U_0 of H with $s_0 \notin U_0$. Choose a neighborhood U of H with $UU^{-1} \subset U_0$. Choose an open precompact neighborhood V of the identity such that $HV \subset U$. Since the set $HV = \bigcup_{h \in H} hV$ is open and $HV \subset H\overline{V}$, the latter being a compact set, we have $0 < |HV| < \infty$ and thus χ_{HV} is in $L^2(G)$ and is nonzero. Let

   $$u(s) := \langle \lambda(s) \chi_{HV} | \chi_{HV} \rangle = \int_G \chi_{HV}\left(s^{-1} y\right) \chi_{HV}(y)\, dy = |sHV \cap HV| \qquad (s \in G).$$

   If $h \in H$, then $hHV \subset HV$ implies $u(h) = |hHV| = |HV|$. If $s_0 HV \cap HV \neq \emptyset$, then $s_0 hv = h'v'$ for some $h, h' \in H$ and $v, v' \in V$, whence $s_0 = h'v'(hv)^{-1} \in HV(HV)^{-1} \subset UU^{-1} \subset U_0$, a contradiction. Therefore $u(s_0) = 0$ and $|HV|^{-1} u \in P_H(G)$.

3. Suppose H is normal. Fix $s_0 \in G \setminus H$. The argument above applied to the trivial subgroup of G/H yields $u \in P(G/H)$ such that $u(eH) = 1$ and $u(s_0 H) \neq 1$. Letting $q : G \to G/H$ denote the quotient map, the composition $uq : G \to \mathbb{C}$ is continuous and is in $P(G)$ by [16, Théorème 2.20], because q is a group homomorphism. It is clear that $uq(h) = 1$ for $h \in H$ and $uq(s_0) = u(s_0 H) \neq 1$.

In [39], a fixed point argument is used to show that an invariant projection $VN(G) \to VN_H(G)$ exists when the locally compact group G separates a closed subgroup H. In fact, it follows from their argument that a bounded approximate indicator for H exists.

Proposition 4.2.5. *Let G be a locally compact group and H a closed subgroup. Then G has the H-separation property if and only if there exists a bounded approximate indicator for H in $P_H(G)$.*

Proof. Suppose G has the H-separation property. By the proof of [39, Proposition 3.1], there exists an invariant projection $P : VN(G) \to VN_H(G)$ such that $P = w^*ot - \lim_\alpha M_{u_\alpha}$, where $u_\alpha \in P_H(G)$ and $M_{u_\alpha} : VN(G) \to VN(G)$ is the adjoint of the multiplication map $u \mapsto u_\alpha u$ on $A(G)$. Given $u \in A(H)$, we may find $\tilde{u} \in A(G)$ with $r_H(\tilde{u}) = u$ by Herz's restriction theorem and by Example 3.3.6 we have $r_H^*(VN(H)) \subset VN_H(G)$, so that for $T \in VN(H)$,

$$\langle T, ur_H(u_\alpha)\rangle = \langle T, r_H(\tilde{u}u_\alpha)\rangle = \langle M_{u_\alpha}(r_H^*(T)), \tilde{u}\rangle \to \langle P(r_H^*(T)), \tilde{u}\rangle = \langle r_H^*(T), \tilde{u}\rangle = \langle T, u\rangle .$$

If $w \in I_{A(G)}(H)$, then

$$\langle T, wu_\alpha\rangle = \langle M_{u_\alpha}(T), w\rangle \to \langle P(T), w\rangle = 0 \qquad (T \in VN(G))$$

since $P(T) \in VN_H(G) = I_{A(G)}(H)^\perp$ by Example 3.3.7. Therefore $ur_H(u_\alpha) \to u$ weakly in $A(H)$ for all $u \in A(H)$ and $wu_\alpha \to 0$ weakly in $A(G)$ for all $w \in I_{A(G)}(H)$. Passing to convex combinations yields a bounded approximate indicator for H which is in $P_H(G)$, since this set is convex.

Conversely, if $(u_\alpha)_\alpha$ is a bounded approximate indicator for H in $P_H(G)$, then, given $s \in G \setminus H$ use Proposition 3.3.4 to find $w \in I_{A(G)}(H)$ such that $w(s) = 1$, in which case

$$|u_\alpha(s)| = |u_\alpha(s)w(s)| \le \|u_\alpha w\|_{L^\infty(G)} \le \|u_\alpha w\|_{A(G)} \to 0$$

implies $u_\alpha(s) \neq 1$ eventually. ☐

For a closed subgroup H of a locally compact group G, we consider a weaker form of the H-separation property that replaces the algebra $B(G)$ with $B^d(G)$. We show that this weakened condition characterizes when χ_H may be approximated by elements of $B(G)$ and provide a related characterization of amenability of H.

Definition 4.2.6. Let G be a locally compact group and H a closed subgroup. For a subalgebra A of $B(G)$, we say that G has the ***A-discretized H-separation property*** if for any $s \in G \setminus H$ there exists $u \in \overline{A}^{B(G_d),w^*} \cap P_H(G_d)$ such that $u(s) \neq 1$.

Proposition 4.2.7. *Let G be a locally compact group and H a closed subgroup. Then G has the $B(G)$-discretized H-separation property if and only if $\chi_H \in B^d(G)$.*

Proof. Suppose G has the $B(G)$-discretized H-separation property and for each $s \in G \setminus H$ let $u_s \in B^d(G) \cap P_H(G_d)$ with $u_s(s) \neq 1$. Replacing u_s by $\frac{1}{2}(1_G + u_s)$, which is in $B^d(G) \cap P_H(G_d)$

by Theorem 4.2.2(1b), we may assume $|u_s(s)| < 1$. By (2) and (3) of Theorem 4.2.2, the net $(u_s^n)_{n=1}^{\infty}$ is in $B^d(G) \cap P_H(G_d)$ with $\|u_s^n\|_{B(G_d)} = u_s^n(e) = 1$ and therefore has a weak* cluster point $u_s^0 \in B^d(G)_{\leq 1}$. Then $u_s^0|_H = 1$ and $|u_s^0(s)| \leq \limsup_n |u_s(s)|^n = 0$, so $u_s^0(s) = 0$. Let $\mathcal{F}$ be the collection of finite subsets of G and for each $F \in \mathcal{F}$ let $u_F = \prod_{s \in F} u_s^0$, which is in $B^d(G)_{\leq 1}$. Ordering $\mathcal{F}$ by inclusion, we have that $u_F|_H = 1$ and $u_F(s) = 0$ eventually for each $s \in G \setminus H$, so that $u_F \overset{ptw}{\to} \chi_H$ and by boundedness $u_F \overset{w^*}{\to} \chi_H$ in $B(G_d)$, whence $\chi_H \in B^d(G)$.

The converse is clear once we note that the characteristic function of a subgroup H is always in $P_H(G_d)$, as was shown in Example 4.2.4. $\square$

Proposition 4.2.8. *Let G be a locally compact group and H a closed subgroup. Then G has the $A(G)$-discretized H-separation property if and only if H is amenable.*

Proof. In the argument establishing Proposition 4.2.7, substituting $\frac{1}{2}(u_s^2 + u_s)$ for the element $\frac{1}{2}(1_G + u_s)$ yields a proof that G has the $A(G)$-discretized H-separation property if and only if χ_H is in the weak* closure of $A(G)$ in $B(G_d)$. This latter condition is equivalent to amenability of H by Theorem 4.0.5. $\square$

4.3 Convergence of Fourier multipliers and averaging over closed subgroups

It is folklore that the convergence properties of nets of Fourier multipliers can be improved by convolving them with probability measures in $\mathcal{C}_c(G)$. For example, Knudby recently recorded the following, the second part of which originated in an argument of Cowling and Haagerup [7].

Theorem 4.3.1. *[42, Lemma B.2] Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$, $m \in M_{cb}A(G)$, and let $f \in \mathcal{C}_c(G)$ such that $f \geq 0$ and $\int_G f = 1$. Convolution on the left with f is a contraction on $M_{cb}A(G)$ and the following hold:*

1. *If $m_\alpha \overset{w^*}{\to} m$ in $M_{cb}A(G)$, then $f * m_\alpha \overset{ucs}{\to} f * m$.*
2. *If $m_\alpha \overset{ucs}{\to} m$, then $\|(f * m_\alpha)u - (f * m)u\|_{A(G)} \to 0$ for all $u \in A(G)$.*

In Section 4.4, we will be interested in upgrading the convergence properties of the restrictions of such nets to closed subgroups. Towards that end, this section develops an analogue of the convolution technique relative to a closed subgroup.

Throughout this section, let G be a locally compact group, H a closed subgroup, and fix a function $f \in \mathcal{C}_c(H)$ such that $f \geq 0$ and $\int_H f = 1$. The continuous, bounded functions on G are denoted by $\mathcal{C}_b(G)$.

Definition 4.3.2. For $u \in \mathcal{C}_b(G)$, define a function $\Omega_f(u)$ on G by the formula

$$\Omega_f(u)(s) = \int_H f(h)\, u\left(h^{-1}s\right) dh \qquad (s \in G).$$

We will show that Ω_f defines a bounded map on $M_{cb}A(G)$. For a Hilbert space $\mathcal{H}$, let $\mathcal{C}_c(G,\mathcal{H})$ and $\mathcal{C}_b(G,\mathcal{H})$ denote the continuous functions $G \to \mathcal{H}$ that are compactly supported and bounded, respectively. The continuous, compactly supported functions from G into a Hilbert space are uniformly continuous.

Lemma 4.3.3. *Let $\mathcal{H}$ be a Hilbert space. If $u \in \mathcal{C}_c(G,\mathcal{H})$ then for any $\epsilon > 0$ there is an open neighborhood U of the identity e such that $\sup_{t\in G} \|u(st) - u(t)\| < \epsilon$ for all $s \in U$.*

Proof. The standard proof in the case that $\mathcal{H} = \mathbb{C}$, for example [17, Proposition 2.6], works for any Hilbert space. □

Lemma 4.3.4. *Let $\mathcal{H}$ be a Hilbert space. If $u \in \mathcal{C}_b(G,\mathcal{H})$, $s_0 \in G$, and $\epsilon > 0$, then there is an open neighborhood U of s_0 in G such that $\sup_{h\in H} \|f(h)u(sh) - f(h)u(s_0h)\| < \epsilon$ for all $s \in U$.*

Proof. Since H is closed in G, the function f extends to a continuous compactly supported function f' on G. Assume first that $s_0 = e$. Since $f'u$ is compactly supported, Lemma 4.3.3 yields an open neighborhood U of e such that

$$\sup_{t\in G} \|f'u(st) - f'u(t)\| < \frac{\epsilon}{2} \quad \text{and} \quad \sup_{t\in G} |f'(st) - f'(t)| < \frac{\epsilon}{2\|u\|_\infty}$$

for all $s \in U$. Then

$$\begin{aligned}
\sup_{h\in H} \|f(h)u(sh) - f(h)u(h)\| &\leq \sup_{t\in G} \|f'(t)u(st) - f'(t)u(t)\| \\
&\leq \sup_{t\in G} \left(\|f'(t)u(st) - f'u(st)\| + \|f'u(st) - f'(t)u(t)\|\right) \\
&< \|u\|_\infty \sup_{t\in G} |f'(st) - f'(t)| + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

for all $s \in U$. For $s_0 \neq e$, the above argument with u replaced by ${}_{s_0}u$ yields a neighborhood U of e and s_0U is then the desired neighborhood of s_0. □

In proving the following, we use standard facts regarding Banach space valued integration that may be found in [17, Appendix 3].

Proposition 4.3.5. *If $u \in M_{cb}A(G)$, then $\Omega_f(u) \in M_{cb}A(G)$. The map $\Omega_f : M_{cb}A(G) \to M_{cb}A(G)$ is a linear contraction.*

Proof. Let $u \in M_{cb}A(G)$ and apply Gilbert's theorem (Theorem 3.4.1) to obtain a Hilbert space $\mathcal{H}$ and functions $P, Q \in \mathcal{C}_b(G,\mathcal{H})$ such that $u(s^{-1}t) = \langle P(t) | Q(s)\rangle$ for all $s, t \in G$. Then

$$\Omega_f(u)\left(s^{-1}t\right) = \int_H f(h)\,u\left(h^{-1}s^{-1}t\right) dh = \left\langle P(t) \middle| \int_H f(h)\,Q(sh)\,dh \right\rangle \qquad (s,t \in G).$$

We show that $q(s) = \int_H f(h)\,Q(sh)\,dh$ defines a bounded continuous function on G, from which it will follow that $\Omega_f(u)$ is in $M_{cb}A(G)$, again by Gilbert's theorem. Define $Q' : G \to L^1(H,\mathcal{H})$ by

$Q'(s) = f({}_sQ)$ (product, not evaluation), which maps into $L^1(H,\mathcal{H})$ since f has compact support. Set $K = \mathrm{supp} f$. Given $s_0 \in G$ and $\epsilon > 0$, Lemma 4.3.4 yields an open neighborhood U of s_0 in G such that

$$\|Q'(s) - Q'(s_0)\|_{L^\infty(H,\mathcal{H})} = \sup_{h\in H} \|f(h) Q(sh) - f(h) Q(s_0 h)\| < \frac{\epsilon}{|K|}$$

for all $s \in U$. Since $Q'(s)$ is supported in K for every $s \in G$, it follows that

$$\begin{aligned} \|Q'(s) - Q'(s_0)\|_{L^1(H,\mathcal{H})} &= \|\chi_K (Q'(s) - Q'(s_0))\|_{L^1(H,\mathcal{H})} \\ &\leq \|\chi_K\|_{L^1(H,\mathcal{H})} \|Q'(s) - Q'(s_0)\|_{L^\infty(H,\mathcal{H})} \\ &< \epsilon \end{aligned}$$

for all $s \in U$. Thus Q' is continuous and so too is q, the latter being the composition of Q' with the bounded map $L^1(H,\mathcal{H}) \to \mathcal{H} : g \mapsto \int_H g$.

Using that f is nonnegative with mass one, if $s \in G$, then $\|q(s)\| \leq \int_H f(h) \|Q(sh)\| \, dh \leq \|Q\|_\infty$, so q is bounded with $\|q\|_\infty \leq \|Q\|_\infty$. By the norm characterization of Gilbert's theorem, $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|P\|_\infty \|q\|_\infty \leq \|P\|_\infty \|Q\|_\infty$ and since P, Q, and $\mathcal{H}$ are an arbitrary representation of u, we conclude that $\|\Omega_f(u)\|_{M_{cb}A(G)} \leq \|u\|_{M_{cb}A(G)}$. □

The construction just undertaken does indeed average over the subgroup H and achieve our purpose.

Proposition 4.3.6. *$r_H \Omega_f(u) = f * r_H(u)$ for all $u \in M_{cb}A(G)$.*

Proof. If $u \in M_{cb}A(G)$, then

$$r_H \Omega_f(u)(s) = \int_H f(h) u\left(h^{-1}s\right) dh = \int_H f(h) r_H(u)\left(h^{-1}s\right) dh = f * r_h(u)(s) \qquad (s \in G).$$

□

Theorem 4.3.7. *Let $(m_\alpha)_\alpha$ be a bounded net in $M_{cb}A(G)$, let $m \in M_{cb}A(H)$. The following hold:*

1. *If $r_H(m_\alpha) \overset{w^*}{\to} m$ in $M_{cb}A(H)$, then $r_H \Omega_f(m_\alpha) \overset{ucs}{\to} f * m$.*
2. *If $r_H(m_\alpha) \overset{ucs}{\to} m$, then $\|\Omega_f(m_\alpha) u - (f * m) u\|_{A(H)} \to 0$ for all $u \in A(H)$.*

Proof. These follow immediately from Theorem 4.3.1 and Proposition 4.3.6. □

In our applications, the preceding theorem will be applied with $m = 1_H$, which is fixed under convolution with f on the left. We conclude the section with some additional observations about the map Ω_f.

1. An argument very similar to that establishing Proposition 4.3.5 shows that $\Omega_f(u)$ is bounded and continuous for any bounded continuous function u on G. It is clear that Proposition 4.3.6 holds for functions in $\mathcal{C}_b(G)$ as well.

2. If $u = \langle \pi(\cdot)\xi | \eta \rangle \in B(G)$, where π is a continuous unitary representation of G on a Hilbert space $\mathcal{H}$ and $\xi, \eta \in \mathcal{H}$, then

$$\Omega_f(u)(s) = \int_H f(h) \langle \pi(s)\xi | \pi(h)\eta \rangle \, dh = \left\langle \pi(s)\xi \,\middle|\, \left(\int_H f(h)\pi(h)\, dh \right) \eta \right\rangle,$$

so $\Omega_f(u) \in B(G)$, and from $\left\| \int_H f(h)\pi(h)\, dh \right\| \leq \int_H f(h) \|\pi(h)\| \, dh = 1$ it follows that $\|\Omega_f(u)\|_{B(G)} \leq \|u\|_{B(G)}$. Thus Ω_f restricts to a contraction on $B(G)$ and moreover restricts to a contraction on $A(G)$, since $\Omega_f(u)$ is a coefficient of the same representation as u.

3. An argument similar to that establishing the weak* continuity of the map Φ_f in the proof of [34, Lemma 1.16] shows that Ω_f is weak* continuous on $M_{cb}A(G)$ with preadjoint mapping $g \in L^1(G)$ to the $L^1(G)$ function $s \mapsto \int_H f(h) g(hs)\, dh$.

4.4 Existence of approximate indicators for closed subgroups

This section provides weaker sufficient conditions for the existence of a bounded approximate indicator for a closed subgroup of a locally compact group.

Say that a net $(m_\alpha)_\alpha$ of functions on a topological space X ***converges locally eventually to zero on*** $A \subset X$ and write $m_\alpha \overset{le}{\to} 0$ if for any compact subset K of A there is an index α_0 such that $m_\alpha|_K = 0$ for all $\alpha \geq \alpha_0$.

Proposition 4.4.1. *Let G be a locally compact group and H a closed subgroup. Let $f \in \mathcal{C}_c(H)$ such that $f \geq 0$ and $\int_H f = 1$. If $(m_\alpha)_\alpha$ is a bounded net in $M_{cb}A(G)$ and $m'_\alpha = \Omega_f(m_\alpha)$, then the net $(m'_\alpha)_\alpha$ has the same norm bound as $(m_\alpha)_\alpha$ and*

1. *if $r_H(m_\alpha) \overset{ucs}{\to} 1_H$, then $\|u \cdot r_H(m'_\alpha) - u\|_{A(H)} \to 0$ for all $u \in A(H)$;*
2. *if $m_\alpha \overset{le}{\to} 0$ on $G \setminus H$, then $m'_\alpha \overset{le}{\to} 0$ on $G \setminus H$.*

If the bounded net $(m_\alpha)_\alpha$ satisfies the hypotheses of both (1) and (2), then $(m'_\alpha)_\alpha$ is a bounded approximate indicator for H.

Proof. The claim regarding norm bounds holds since the map Ω_f of Section 4.3 is a contraction on $M_{cb}A(G)$.

(1) If $r_H(m_\alpha) \overset{ucs}{\to} 1_H$, then, since restriction is a contraction from $M_{cb}A(G)$ into $M_{cb}A(H)$, the net $(r_H(m_\alpha))_\alpha$ is bounded and (1) follows from Theorem 4.3.7.

(2) Suppose that $m_\alpha \overset{le}{\to} 0$ on $G \setminus H$. Let $K \subset G \setminus H$ be compact and choose α_0 such that $\alpha \geq \alpha_0$ implies $m_\alpha = 0$ on the compact set $(\operatorname{supp}(f))^{-1} K$. For $\alpha \geq \alpha_0$, if $s \in K$ and $h \in H$, then $f(h) m_\alpha(h^{-1}s) = 0$ since either $h \notin \operatorname{supp}(f)$ or $h^{-1}s \in (\operatorname{supp}(f))^{-1} K$, implying that $m'_\alpha(s) = \int_H f(h) m_\alpha(h^{-1}s)\, dh = 0$. Therefore $m'_\alpha = 0$ on K, for all $\alpha \geq \alpha_0$. This shows that (2) holds.

If $(m_\alpha)_\alpha$ satisfies the hypotheses of both (1) and (2), then $(m'_\alpha)_\alpha$ satisfies condition (1) of Definition 4.0.1. The second condition of Definition 4.0.1 follows by a standard argument. If

$u \in I_{A(G)}(H)$ has compact support, then $m'_\alpha u = 0$ eventually by (2), so certainly $\|um'_\alpha\|_{A(G)} \to 0$. Using that H is of spectral synthesis in $A(G)$ (Example 3.3.8), we now approximate: if $u \in I_{A(G)}(H)$, then given $\epsilon > 0$ choose $u_0 \in I_{A(G)}(H)$ of compact support with $\|u - u_0\|_{A(G)} < \epsilon$. For sufficiently large α,

$$\|um'_\alpha\|_{A(G)} \leq \|u_0 m'_\alpha\|_{A(G)} + \|u_0 m'_\alpha - um'_\alpha\|_{A(G)} < \epsilon \|m'_\alpha\|_{M_{cb}A(G)},$$

and thus $\|um'_\alpha\|_{A(G)} \to 0$, because $(m'_\alpha)_\alpha$ is a bounded net in $M_{cb}A(G)$.

□

Proposition 4.4.1 allows us to obtain approximate indicators consisting of Fourier multipliers by verifying the same conditions that yielded approximate indicators in [1]. In particular, given a net of Fourier multipliers, we are able to pass to an averaged net in place of directly satisfying condition (1) of Definition 4.0.1, which is difficult in practice.

4.5 Invariant projections from bounded approximate indicators

The following result motivates our interest in bounded approximate indicators.

Proposition 4.5.1. *Let G be a locally compact group and H a closed subgroup. If there is a bounded approximate indicator for H, then there is a completely bounded invariant projection $VN(G) \to VN_H(G)$.*

Proof. Let $(m_\alpha)_\alpha$ an approximate indicator for H of norm bound C. For $m \in M_{cb}A(G)$, let M_m denote the adjoint of the map $u \mapsto mu$ on $A(G)$, so that M_m is a completely bounded $A(G)$-module map on $VN(G)$. The net $(M_{m_\alpha})_\alpha$ in $CB_{A(G)}(VN(G))$ is then bounded by C and thus has a weak* operator topology cluster point $\Psi \in CB_{A(G)}(VN(G))_{\leq C}$. Passing to a subnet if necessary, we may assume that $M_{m_\alpha} \overset{w^*ot}{\to} \Psi$. If $T \in VN_H(G)$, then $T = r_H^*(S)$ for some $S \in VN(H)$ and

$$\langle \Psi(T), u\rangle = \lim_\alpha \langle S, r_H(uu_\alpha)\rangle = \lim_\alpha \langle S, r_H(u)\, r_H(u_\alpha)\rangle = \langle S, r_H(u)\rangle = \langle T, u\rangle \qquad (u \in A(G)),$$

showing that Ψ is the identity on $VN_H(G)$. For any $T \in VN(G)$ and $u \in I_{A(G)}(H)$ we have $\langle \Psi(T), u\rangle = \lim_\alpha \langle S, uu_\alpha\rangle = 0$, so Ψ maps into $I_{A(G)}(H)^\perp = VN_H(G)$ and is therefore a projection onto $VN_H(G)$. □

An analogous result holds for $A_{cb}(G)$ when the locally compact group G is weakly amenable.

Proposition 4.5.2. *Let G be a weakly amenable locally compact group and H a closed subgroup. If there is a bounded approximate indicator for H, then there is a completely bounded invariant projection $A_{cb}(G)^* \to I_{A_{cb}(G)}(H)^\perp$.*

Proof. Let $(m_\alpha)_\alpha$ an approximate indicator for H of norm bound C. Since $A(G)$ is an ideal in $M_{cb}A(G)$, so too is its closure $A_{cb}(G)$, so that multiplication by m_α is a completely bounded

$A_{cb}(G)$-module map on $A_{cb}(G)$, say with adjoint denoted by M_{m_α}. As in the proof of Proposition 4.5.1, we may suppose that $M_{m_\alpha} \overset{w^*ot}{\to} \Psi$ for some $\Psi \in CB_{A_{cb}(G)}(A_{cb}(G)^*)$. Let $\Psi_A \in CB_{A(G)}(VN(G))$ be the projection constructed in Proposition 4.5.1 and let $\iota : A(G) \to A_{cb}(G)$ be the inclusion. If $T \in A_{cb}(G)^*$ and $u \in A(G)$, then

$$\langle \Psi_A \iota^*(T), u\rangle = \lim_\alpha \langle \iota^*(T), um_\alpha\rangle = \lim_\alpha \langle T, \iota(u) m_\alpha\rangle = \langle \Psi(T), \iota(u)\rangle = \langle \iota^*\Psi(T), u\rangle$$

and $\Psi_A\iota^* = \iota^*\Psi$ by density of $A(G)$ in $A_{cb}(G)$. Injectivity of ι^* and

$$\iota^*\Psi^2 = \Psi_A\iota^*\Psi = \Psi_A^2\iota^* = \Psi_A\iota^* = \iota^*\Psi$$

imply that $\Psi^2 = \Psi$. If $T \in I_{A_{cb}(G)}(H)^\perp$, then $\iota^*(T) \in I_{A(G)}(H)^\perp$ and

$$\langle \Psi(T), \iota(u)\rangle = \langle \Psi_A\iota^*(T), u\rangle = \langle \iota^*(T), u\rangle = \langle T, \iota(u)\rangle \qquad (u \in A(G)),$$

whence $\Psi(T) = T$, again by density of $A(G)$. Therefore $I_{A_{cb}(G)}(H)^\perp$ is contained in the range of Ψ. Finally, for any $T \in A_{cb}(G)^*$ if $u \in I_{A(G)}(H)$, then $\langle \Psi(T), \iota(u)\rangle = \langle \Psi_A\iota^*(T), u\rangle = 0$, so $\Psi(T) \in I_{A(G)}(H)^\perp = \left(\overline{I_{A(G)}(H)}^{A_{cb}(G)}\right)^\perp$. That $A_{cb}(G)$ has bounded approximate identity implies every set of synthesis for $A(G)$ is one for $A_{cb}(G)$ [23, Proposition 3.1] and, because compactly supported elements of $A_{cb}(G)$ are in $A(G)$, that H is of spectral synthesis for $A_{cb}(G)$ is exactly the assertion $\overline{I_{A(G)}(H)}^{A_{cb}(G)} = I_{A_{cb}(G)}(H)$. It follows that Ψ has range in $I_{A_{cb}(G)}(H)^\perp$ and is thus a projection onto $I_{A_{cb}(G)}(H)^\perp$. $\square$

Let Λ_G denotes the smallest norm bound of an approximate identity for $A_{cb}(G)$.

Corollary 4.5.3. *Let G be a weakly amenable locally compact group and H a closed subgroup. If an approximate indicator for H of norm bound C exists, then $I_{A_{cb}(G)}(H)$ has an approximate identity of norm bound $(1+C)\Lambda_G$.*

Proof. The argument of Proposition 4.5.2 yields an invariant projection $A_{cb}(G)^* \to I_{A_{cb}(G)}(H)^\perp$ of norm bound C and, because the Banach algebra $A_{cb}(G)$ has a bounded approximate identity, it follows from [19, Proposition 6.4] and its proof that the ideal $I_{A_{cb}(G)}(H)$ has an approximate identity of norm bound $(1+C)\Lambda_G$. $\square$

For weakly amenable groups, Corollary 4.5.3 allows us to strengthen a convergence property of bounded approximate indicators at the cost of increasing their norm bounds.

Corollary 4.5.4. *Let G be a weakly amenable locally compact group and H a closed subgroup. If an approximate indicator for H of norm bound C exists, then an approximate indicator for H of norm bound $1+(1+C)\Lambda_G$ exists that is identically one on H.*

Proof. Corollary 4.5.3 yields an approximate identity $(e_\alpha)_\alpha$ for $I_{A_{cb}(G)}(H)$ of norm bound $(1+C)\Lambda_G$ and it is routine to verify that $(1_G - e_\alpha)_\alpha$ is an approximate indicator for H with the desired norm bound. $\square$

We showed in Example 4.0.2 that bounded approximate diagonals for $A(G)$ and bounded approximate indicators for the diagonal subgroup of $G \times G$ are closely related. We close this section by recording a connection between operator amenability of $A_{cb}(G)$ and the existence of bounded approximate diagonals.

Proposition 4.5.5. *Let G be a locally compact group. If $A_{cb}(G)$ is operator amenable, then there is a bounded approximate indicator for G_Δ in $A_{cb}(G \times G)$.*

Proof. Write $\Delta : A_{cb}(G) \widehat{\otimes} A_{cb}(G) \to A_{cb}(G)$ for the product map, $r : A_{cb}(G \times G) \to A_{cb}(G)$ for restriction to the diagonal G_Δ in $G \times G$, and $\Lambda : A_{cb}(G) \widehat{\otimes} A_{cb}(G) \to A_{cb}(G \times G)$ for the complete contraction defined on elementary tensors by $\Lambda(u \otimes v) = u \times v$, so that $\Delta = r\Lambda$. Let $(d_\alpha)_\alpha$ be a bounded approximate diagonal for $A_{cb}(G)$ and set $m_\alpha = \Lambda(X_\alpha)$. We show that the net $(m_\alpha)_\alpha$ is a bounded approximate indicator for G_Δ. Let $u \in A(G)$ have compact support and choose $v \in A(G)$ with $v = 1$ on $\operatorname{supp}(u)$ (use Proposition 3.3.4(3)), so that $u = uv$ and

$$\|ur(m_\alpha) - u\|_{A(G)} = \|u\Delta(d_\alpha) - u\|_{A(G)} \leq \|u\|_{A(G)} \|v\Delta(d_\alpha) - v\|_{A_{cb}(G)} \to 0.$$

As the compactly supported functions in $A(G)$ are dense by Proposition 3.3.4 and the net $(r(m_\alpha))_\alpha$ is bounded in $\|\cdot\|_{A_{cb}(G)}$, a routine estimate shows that the above holds for all $u \in A(G)$. We saw in Example 4.0.2 that

$$I_{A(G \times G)}(G_\Delta) = \langle (u \times 1_G - 1_G \times u)\, w : u \in A(G) \text{ and } w \in A(G \times G) \rangle,$$

and for elements of this dense spanning set,

$$\begin{aligned} \|(u \times 1_G - 1_G \times u)\, w m_\alpha\|_{A(G \times G)} &\leq \|w\|_{A(G \times G)} \|u \cdot m_\alpha - m_\alpha \cdot u\|_{A_{cb}(G \times G)} \\ &\leq \|w\|_{A(G \times G)} \|u \cdot d_\alpha - d_\alpha \cdot u\|_{A_{cb}(G) \widehat{\otimes} A_{cb}(G)} \to 0, \end{aligned}$$

where the second inequality uses that Λ is a contractive $A(G)$-bimodule map. That $\|wm_\alpha\|_{A(G \times G)} \to 0$ for all $w \in I_{A(G \times G)}(G_\Delta)$ follows from the density claim above and the boundedness of $(m_\alpha)_\alpha$. $\square$

For a locally compact group G, it seems plausible that, as in the $A(G)$ case, the converse of Proposition 4.5.5 may hold, i.e. that the existence of a bounded approximate indicator for G_Δ in $A_{cb}(G \times G)$ characterizes operator amenability of $A_{cb}(G \times G)$. Indeed, an argument similar to Example 4.0.2 shows that the converse holds under the hypothesis that the identity $A_{cb}(G) \widehat{\otimes} A_{cb}(G) = A_{cb}(G \times G)$ is valid. However, beyond the amenable case — when $A_{cb}(G)$ and $A(G)$ coincide — it remains unclear when this identity holds.

Chapter 5

A proposition of Aristov, Runde, and Spronk

This short chapter presents a gap that was discovered in the proof of Proposition 3.6 of [1]. Recall from Section 4.1 that for a locally compact group G the discretized Fourier-Stieltjes algebra $B^d(G)$ is the weak* closure of $B(G)$ in $B(G_d)$.

Proposition 5.0.1. *[1, Proposition 3.6] Let G be a locally compact group and H a closed subgroup. If $\chi_H \in B^d(G)$, then for $K \subset H$ and $L \subset G \setminus H$ compact and $\epsilon > 0$ there exists $f \in B(G)_{\leq 1}$ such that $f|_L = 0$ and $|f(s) - 1| < \epsilon$ for all $s \in K$.*

The main application of this result was the following.

Theorem 5.0.2. *[1, Theorem 3.7] Let G be a locally compact group and H a closed subgroup. If $\chi_H \in B^d(G)$, then there is a bounded approximate indicator for H in $B(G)$.*

The proof given in [1] involves manipulations with polars of subsets of $B^d(G)$.

Definition 5.0.3. Let E be a Banach space and let $F \subset E$ and $S \subset E^*$ be any subsets. The ***polar*** of F is the set $F^\circ = \{\varphi \in E^* : |\langle \varphi, x\rangle| \leq 1 \text{ for all } x \in F\}$ and the ***prepolar*** of S is the set $S_\circ = \{x \in E : |\langle \varphi, x\rangle| \leq 1 \text{ for all } \varphi \in S\}$.

It is clear that $(E_{\leq 1})^\circ = E^*_{\leq 1}$ and $\left(E^*_{\leq 1}\right)_\circ = E_{\leq 1}$ for a Banach space E. The contentious portion of the proof Proposition 3.6 of [1] is stated in the language of polars, but the sets involved are linear subspaces, for which polars take the following form.

Example 5.0.4. (Polars and prepolars of subspaces are annihilators) Let E be a Banach space and F a linear subspace of E. If $\varphi \in F^\circ$, then for $x \in F$ nonzero we have $\alpha |\langle \varphi, x\rangle| = |\langle \varphi, \alpha x\rangle| \leq 1$ for all $\alpha \geq 0$, implying that $\langle \varphi, x\rangle = 0$. Therefore $F^\circ \subset F^\perp$. The reverse inclusion being trivial, we have $F^\circ = F^\perp$. A similar argument shows that $S_\circ = S_\perp$ for a linear subspace S of E^*. In particular, polars of subspaces are subspaces.

The proof of Proposition 3.6 in [1] makes use of the following standard results on annihilators.

Lemma 5.0.5. *Let E be a Banach space. The following hold:*

1. *If F a linear subspace of E, then $\left(F^{\perp}\right)_{\perp} = \overline{span}^{\|\cdot\|_E} F$.*
2. *If F is a linear subspace of E^*, then $(F_{\perp})^{\perp} = \overline{span}^{w^*} F$.*
3. *If F_1 and F_2 are linear subspaces of E, then $F_1^{\perp} \cap F_2^{\perp} = (F_1 \cup F_2)^{\perp}$.*
4. *If F_1 and F_2 are weak* closed linear subspaces of E^*, then $(F_1 \cap F_2)_{\perp} = \overline{(F_{1\perp} + F_{2\perp})}^{\|\cdot\|_E}$.*

Proof. The first two claims are the content of the bipolar theorem as applied to the locally convex topological spaces F with its weak topology and F^* with its weak* topology, see [6, p.127] for a proof.

The third is straight forward to verify: if $\varphi \in F_1^{\perp} \cap F_2^{\perp}$, then $\langle \varphi, x \rangle = 0$ for $x \in F_1$ or $x \in F_2$, whence $F_1^{\perp} \cap F_2^{\perp} \subset (F_1 \cup F_2)^{\perp}$, and if instead $\varphi \in (F_1 \cup F_2)^{\perp}$, then $\langle \varphi, x \rangle = 0$ for all $x \in F_1$ or $x \in F_2$, so $\varphi \in F_1^{\perp} \cap F_2^{\perp}$ and the claim holds.

For the fourth claim, since both F_1 and F_2 coincide with their respective weak* closed linear spans, $F_1 = (F_{1\perp})^{\perp}$ and $F_2 = (F_{2\perp})^{\perp}$, so that

$$
\begin{aligned}
(F_1 \cap F_2)_{\perp} &= \left((F_{1\perp})^{\perp} \cap (F_{2\perp})^{\perp}\right)_{\perp} \\
&= \left((F_{1\perp} \cup F_{2\perp})^{\perp}\right)_{\perp} \\
&= \overline{\text{span}}^{\|\cdot\|_E} (F_{1\perp} \cup F_{2\perp}) = \overline{(F_{1\perp} + F_{2\perp})}^{\|\cdot\|_E}.
\end{aligned}
$$

□

In [1], Lemma 5.0.5(4) is invoked when the subspaces F_1 and F_2 are not closed in the weak* topology. We, as well as the authors of [1], have been unable to fill the resulting gap and it remains unknown whether Proposition 3.6 or the results of [1] that rely on it are correct. Note, however, that no counterexample is known. A jointly authored corrigendum has now been published [2].

We close this chapter by showing that the Fourier multiplier analogue of Theorem 3.7 in [1] is false.

Example 5.0.6. The locally compact group $SL(2,\mathbb{R})$ contains $\mathbb{F}_2$ as a closed subgroup. It has recently been shown that $SL(2,\mathbb{R})$ is weakly amenable as a discrete group [43], so that $\mathbb{F}_2$ is $A(G)$-approximable by Proposition 4.1.3, in particular $\chi_{\mathbb{F}_2} \in M_{cb}^d(G)$. If there were a bounded approximate indicator for $\mathbb{F}_2$, then there would exist a completely bounded projection $VN(SL(2,\mathbb{R})) \to VN_{\mathbb{F}_2}(SL(2,\mathbb{R}))$ by Proposition 4.5.1. Since $SL(2,\mathbb{R})$ is connected, its group von Neumann algebra is injective [49, (2.35)], meaning that there exists a completely bounded projection $B(L^2(SL(2,\mathbb{R}))) \to VN(SL(2,\mathbb{R}))$. Composing these projections yields a completely bounded projection $B(L^2(SL(2,\mathbb{R}))) \to VN_{\mathbb{F}_2}(SL(2,\mathbb{R}))$, implying that $VN_{\mathbb{F}_2}(SL(2,\mathbb{R})) = VN(\mathbb{F}_2)$ (see Example 3.3.6) is an injective von Neumann algebra [5]. But, for discrete groups, injectivity of the group von Neumann algebra is equivalent to amenability of the group [49, (2.35)] and $\mathbb{F}_2$ is not amenable by Example 3.5.3. Therefore no bounded approximate indicator exists for $\mathbb{F}_2$.

Example 5.0.7. Let $G = SL(2, \mathbb{R})$ and consider the diagonal subgroup G_Δ of $G \times G$. It was recently shown by Crann and the author that $A(G)$ is not operator biflat [8], meaning that no invariant projection $VN(G \times G) \to VN_{G_\Delta}(G \times G)$ exists (see Example 4.0.3) and consequently that no bounded approximate indicator for G_Δ exists (Proposition 4.5.1). But the weak amenability of G_d, noted in Example 5.0.6, implies the weak amenability of $G_d \times G_d = (G \times G)_d$, so that $\chi_{G_\Delta} \in M_{cb}^d(G \times G)$ by Proposition 4.1.3.

Both of the preceding examples show that a closed subgroup H of a locally compact group G may be approximable when no invariant projection $VN(G) \to VN_H(G)$ exists.

Chapter 6

Adapted states on von Neumann algebras associated to a locally compact group

This chapter studies the relationship between adapted and faithful normal states on two classes of von Neumann algebras associated to a locally compact group. For such a group G, we define adapted normal states on $L^{\infty}(G)$ and on the group von Neumann algebra $VN(G)$ and for each show that the existence of an adapted normal state is equivalent to σ-finiteness of the algebra and to certain topological properties of the group G. For the group von Neumann algebra, our methods yield a new argument characterizing the existence of cyclic vectors for the left regular representation of G.

6.1 The von Neumann algebra $L^{\infty}(G)$

A probability measure μ on a locally compact group G is called adapted if the closed subgroup generated by $\operatorname{supp}(\mu)$ is as large as possible, that is, equal to G. Restricting attention the the probability measures in $L^{1}(G)$ produces a condition on the normal states of $L^{\infty}(G)$.

Definition 6.1.1. Let G be a locally compact group. A normal state ω on $L^{\infty}(G)$ is called ***adapted*** if, viewed as a function in $L^{1}(G)$, the support of ω generates G as a topological group, meaning that $\bigcup_{n=1}^{\infty}\left(\operatorname{supp}(\omega)\bigcup\operatorname{supp}(\omega)^{-1}\right)^{n}$ is dense in G.

Recall that a topological space is ***σ-compact*** if it is the union of countably many compact subsets and a measure space is ***σ-finite*** if it is the union of countably many subsets of finite measure. Most of the work needed to establish our main result was completed in Section 2.3.

Lemma 6.1.2. *Let X be a locally compact space and let μ be an outer regular Borel measure on X which is strictly positive on nonempty open sets. If X is σ-finite under μ, then X is σ-compact.*

Proof. We show first that X is Lindelöf, i.e. that open covers have countable subcovers. Let $(E_n)_{n\geq 1}$ be a cover of X by sets of finite measure and for each n choose an open set V_n of finite measure such that $E_n \subset V_n$. If $(U_\alpha)_\alpha$ is an open cover of X, then the union $V_n = \bigcup_\alpha U_\alpha \cap V_n$ has finite measure and, because each of the open sets $U_\alpha \cap V_n$ has positive measure whenever it is nonempty, only countably many of these are nonempty for each n, hence only countably many of the U_α are needed to cover each V_n. It follows that countably many of the U_α suffice to cover the union of the sets V_n, which is X. Thus X is Linelöf. Now, choosing for each $x \in X$ a compact neighborhood K_x, the open cover $(\text{int}K_x)_{x\in X}$ of X has a countable subcover and therefore countably many of the K_x cover X, so X is σ-compact. □

The following result is Theorem 1.2.1 of [28], to which we refer for a proof.

Lemma 6.1.3. *Let G be a locally compact group. If $f, g \in L^1(G)$ are positive, then $supp\,(f * g) = \overline{supp\,(f)\,supp\,(g)}$.*

Recall from Definition 3.1.10 that the involution on $L^1(G)$ is given by $f^*(s) = \overline{f(s^{-1})}\Delta(s^{-1})$ for $f \in L^1(G)$ and $s \in G$. If f is a probability measure on a locally compact group G, i.e. a positive function of $L^1(G)$ norm one, then so too is f^* since this function is also positive and $\int_G f^* = \int_G f(y^{-1})\,\Delta(y^{-1})\,dy = \int_G f = 1$, using Proposition 3.1.9. The convolution of probability measures $f, g \in L^1(G)$ is one as well, since

$$\int_G f * g = \int_G \int_G f(z)\,g\left(z^{-1}y\right) dzdy = \int_G f(z) \int_G g\left(z^{-1}y\right) dydz = \int_G f(z)\,dz \int_G g(y)\,dy = 1.$$

We now establish the main result of this section.

Theorem 6.1.4. *Let G be a locally compact group. The following are equivalent:*

1. *G is σ-compact.*
2. *There is an adapted normal state on $L^\infty(G)$.*
3. *There is a faithful normal state on $L^\infty(G)$.*
4. *$L^\infty(G)$ is a σ-finite von Neumann algebra.*
5. *G equipped with Haar measure is a σ-finite measure space.*

Proof. If G is σ-compact, say with countable compact cover $(K_n)_{n=1}^\infty$, then, because Haar measure is finite on compact sets, $\sum_{n=1}^\infty 2^{-n}\mu(X_n)^{-1}\chi_{X_n}$ is a well defined positive function with $L^1(X)$ norm one and support X, so is an adapted state (in fact a faithful state) on $L^\infty(X)$. Thus (1) implies (2).

Let f be an adapted normal state on $L^\infty(G)$ and set $E = \text{supp}(f)$. Since both f and f^* are positive and $\text{supp}(f^*) = \text{supp}(f)^{-1}$, we have $\text{supp}(f * f^*) = \overline{EE^{-1}}$, and since both f and f^* are probability measure in $L^1(G)$, so too is $f * f^*$. Thus $\frac{1}{2}(f + f * f^*)$ is a normal state on

$L^{\infty}(G)$ and has support containing both E and $\overline{EE^{-1}}$. The latter set contains the identity e of G, so, by replacing f with $\frac{1}{2}(f+f*f^*)$ if necessary, we may assume $e \in E$, in which case $E \cup E^{-1} \subset \overline{EE^{-1}} = \operatorname{supp}(f*f^*)$. Letting $(f*f^*)^{*n}$ denote the convolution of $f*f^*$ with itself $n \geq 1$ times, an induction argument yields $\left(E \cup E^{-1}\right)^n \subset \operatorname{supp}\left((f*f^*)^{*n}\right)$. The normal state $\sum_{n=1}^{\infty} 2^{-n}(f*f^*)^{*n}$ therefore contains $\bigcup_{n=1}^{\infty}\left(E \cup E^{-1}\right)^n$ in its support and so has support G, because supports are closed sets. Thus (2) implies (3).

Proposition 2.3.21 asserts that (3) and (4) are equivalent for any von Neumann algebra and Example 2.3.22 shows that (4) and (5) are equivalent. That (5) implies (1) follows from Lemma 3.1.6 and Lemma 6.1.2. □

6.2 The group von Neumann algebra

This section shows that the analogue of Theorem 6.1.4 holds for the group von Neumann algebra. In addition, we provide a concise argument characterizing the existence of cyclic vectors for the left regular representation of a locally compact group G. Part of the characterization we establish was first given by Greenleaf and Moskowitz in [25] and [26] using reduction techniques in combination with a complicated operator algebraic argument for a particular class of locally compact groups. We, too, establish the result in the framework of operator algebras, but our use of the notion of adapted states on the group von Neumann algebra furnishes a much shorter and, we believe, more natural argument. Some additional results of interest are established along the way and as a consequence of our techniques.

In the interest of completeness, we prove the following routine lemma.

Lemma 6.2.1. *The left and right regular representation of a locally compact group G are unitarily equivalent, i.e. are intertwined by a unitary operator in $B\left(L^2(G)\right)$. Consequently, λ has a cyclic vector if and only if ρ does.*

Proof. Define an operator $U \in B\left(L^2(G)\right)$ by $U\xi(s) = \xi\left(s^{-1}\right)\Delta(s)^{-\frac{1}{2}}$, which is clearly linear and is bounded since for $\xi \in L^2(G)$,

$$\|U\xi\|_{L^2(G)}^2 = \int_G \left|\xi\left(y^{-1}\right)\Delta(y)^{-\frac{1}{2}}\right|^2 dy = \int_G \left|\xi\left(y^{-1}\right)\right|^2 \Delta(y)^{-1}\, dy = \int_G |\xi(y)|^2\, dy = \|\xi\|_{L^2(G)}^2 .$$

It is easy to see that $U^2 = I$ and we have

$$\begin{aligned}
\langle U\xi|\eta\rangle &= \int_G \xi\left(y^{-1}\right)\Delta(y)^{-\frac{1}{2}}\,\overline{\eta(y)}\,dy \\
&= \int_G \xi(y)\,\Delta(y)^{\frac{1}{2}}\,\overline{\eta(y^{-1})}\,\Delta(y)^{-1}\,dy \\
&= \int_G \xi(y)\,\overline{\eta(y^{-1})\,\Delta(y)^{-\frac{1}{2}}}\,dy \\
&= \langle \xi|U\eta\rangle ,
\end{aligned}$$

so that $U^* = U = U^{-1}$ and U is unitary. The operator U indeed intertwines λ and ρ: if $\xi \in L^2(G)$ and $s, y \in G$,

$$\begin{aligned}
U\lambda(s)\,U\xi(y) &= \lambda(s)\,U\xi\left(y^{-1}\right)\Delta(y)^{-\frac{1}{2}} \\
&= U\xi\left(y^{-1}s\right)\Delta\left(s^{-1}y\right)^{-\frac{1}{2}} \\
&= \xi(ys)\,\Delta(y)^{-\frac{1}{2}}\,\Delta\left(s^{-1}y^{-1}\right)^{-\frac{1}{2}} \\
&= \xi(ys)\,\Delta(s)^{\frac{1}{2}} \\
&= \rho(s)\,\xi(y).
\end{aligned}$$

Finally, if $\xi \in L^2(G)$ is cyclic for λ and if $\eta \in L^2(G)$ and $T_\alpha \in \mathrm{span}\lambda(G)$ with $\|T_\alpha\xi - U\eta\|_{L^2(G)} \to 0$, then $UT_\alpha U \in \mathrm{span}\rho(G)$ and

$$\|(UT_\alpha U)\,U\xi - \eta\|_{L^2(G)} = \|T_\alpha\xi - U\eta\|_{L^2(G)} \to 0,$$

showing that $\eta \in \langle\rho(G)\,U\xi\rangle$ and hence that $U\xi$ is cyclic for ρ. A similar argument establishes the converse of the last sentence of the proposition. □

Making use of the fact that $VN(G)'$ is the strong operator topology closure of $\mathrm{span}\rho(G)$ (Proposition 3.3.2), we see that $\langle\rho(G)\,\xi\rangle = \left\langle VN(G)'\,\xi\right\rangle$ for any $\xi \in L^2(G)$. This simple observation yields a characterization of the existence of cyclic vectors for the right regular representation in terms of the existence of certain faithful states on $VN(G)$.

Proposition 6.2.2. *Let G be a locally compact group. The vector $\xi \in L^2(G)$ is cyclic for ρ if and only if the vector functional ω_ξ implemented by ξ is faithful on $VN(G)$.*

Proof. We have, using Lemma 2.3.13(2), that

$$\begin{aligned}
\xi \text{ is cyclic for } \rho &\iff \langle\rho(G)\,\xi\rangle = L^2(G) \\
&\iff \left\langle VN(G)'\,\xi\right\rangle = L^2(G) \\
&\iff S_{\omega_\xi} = I \\
&\iff \omega_\xi \text{ is faithful on } VN(G).
\end{aligned}$$

□

For a normal state ω on $VN(G)$, it was argued in [47] that the appropriate analogue of adaptedness is the condition that $\{s \in G : \omega(s) = 1\}$ be as small as possible, that is, for $\omega(s) = 1$ to imply $s = e$.

Definition 6.2.3. Let G be a locally compact group. A normal state ω on $VN(G)$ is called ***adapted*** if $\omega(s) = 1$ implies $s = e$, for all $s \in G$.

For a normal functional ω on $VN(G)$ and $s \in G$ we have $\omega(s) = \langle \omega, \lambda(s) \rangle$, so the assertion that a normal state is adapted says that, if it takes the value 1 on any unitary in $\lambda(G)$, then that unitary must be the identity. Consequently, Proposition 2.3.19 implies that any faithful normal state on $VN(G)$ is adapted.

Note that if $u \in A(G)$, then the function $\check{u}(s) = u(s^{-1})$ is in $A(G)$, since we may write $u = \omega_{\xi,\eta}$ for some $\xi, \eta \in L^2(G)$ and

$$\check{u}(s) = \left\langle \lambda\left(s^{-1}\right)\xi | \eta \right\rangle = \langle \xi | \lambda(s)\eta \rangle = \overline{\langle \lambda(s)\eta | \xi \rangle} = \left\langle \lambda(s)\overline{\eta} | \overline{\xi} \right\rangle = \omega_{\overline{\eta},\overline{\xi}}(s) \qquad (s \in G). \quad (6.2.1)$$

Using the characterization of the $A(G)$ norm given in Theorem 3.3.3, it follows that $\check{}: A(G) \to A(G)$ is a contraction. We will need the following action of $VN(G)$ on $A(G)$ defined in [16]: for $T \in VN(G)$ and $u \in A(G)$, let $\check{T}$ be the image of T under the adjoint of the map $\check{}: A(G) \to A(G)$ and define $T \check{\cdot} u \in A(G)$ by

$$\langle T \check{\cdot} u, S \rangle = \left\langle u, \check{T} S \right\rangle \qquad (S \in VN(G)).$$

It is shown in [16, Proposition 3.17] that when $u \in A(G) \cap L^2(G)$ we have $T \check{\cdot} u = Tu$, where the right hand side is evaluation of the operator $T \in B\left(L^2(G)\right)$ at the vector $u \in L^2(G)$. In particular, if $\xi \in L^2(G)$ has compact support and is positive, then

$$\omega_\xi(s) = \langle \lambda(s)\xi | \xi \rangle = \int_G \xi\left(s^{-1}y\right)\xi(y)\,dy = \xi * \check{\xi}(s) \qquad (s \in G), \qquad (6.2.2)$$

so ω_ξ has support the compact set $\operatorname{supp}(\xi)\operatorname{supp}(\xi)^{-1}$ by Lemma 6.1.3 and is therefore in $A(G) \cap L^2(G)$. Note also that $\check{\omega_\xi}(s) = \omega_{\overline{\xi}}(s) = \omega_\xi(s)$ by (6.2.1).

The following lemma is key to our main result.

Lemma 6.2.4. *Let G be a locally compact group. Every nonzero positive operator in $VN(G)$ has a nonzero continuous function in its range.*

Proof. Let $P \in VN(G)$ be a nonzero projection and choose a unit vector ξ in its range, so that $S_{\omega_\xi} \leq P$ by Lemma 2.3.13(1). Since positive functions span $\mathcal{C}_c(G)$, which is in turn dense in $L^2(G)$, we may find a positive $f \in \mathcal{C}_c(G)$ of norm one in $L^2(G)$ and not orthogonal to $\langle \rho(G)\xi \rangle$, so that $\left\langle \omega_f, S_{\omega_\xi} \right\rangle \neq 0$ by Lemma 2.3.13(3). The comments above show that $\check{\omega_f} = \omega_f$ and $\omega_f \in A(G) \cap L^2(G)$, so

$$S_{\omega_\xi}(\omega_f)(e) = \left(S_{\omega_\xi} \check{\cdot} \omega_f\right)(e) = \left\langle S_{\omega_\xi} \check{\cdot} \omega_f, \lambda(e) \right\rangle = \left\langle \omega_f, \check{S_{\omega_\xi}} \right\rangle = \left\langle \check{\omega_f}, S_{\omega_\xi} \right\rangle = \left\langle \omega_f, S_{\omega_\xi} \right\rangle \neq 0.$$

Thus $S_{\omega_\xi}(\omega_f) = S_{\omega_\eta} \check{\cdot} \omega_f$ is nonzero and in $A(G)$, hence continuous. For $T > 0$ in $VN(G)$ apply the preceding argument to the range projection of T (Definition 2.3.5). $\square$

Recall that a topological space is ***first countable*** if every point has a countable neighborhood base. For a locally compact group, in which the neighborhood base at any point is determined by that of the identity (see Section 3.1), it is equivalent to require that the identity element have a countable neighborhood base.

Theorem 6.2.5. *Let G be a locally compact group. The following are equivalent:*

1. *G is first countable.*
2. *There is an adapted normal state on $VN(G)$.*
3. *There is a faithful normal state on $VN(G)$.*
4. *VN(G) is a σ-finite von Neumann algebra.*
5. *The left (equivalently, right) regular representation has a cyclic vector.*

Proof. Suppose G is first countable and let $(U_n)_{n=1}^{\infty}$ be a neighborhood base at the identity in G. We claim the normal state $\omega = \sum_{n=1}^{\infty} 2^{-n}\omega_n$ is faithful, where $\omega_n = |U_n|^{-1}\,\omega_{\chi_{U_n}}$. Let T be a positive operator in $VN(G)$ with $\langle \omega, T\rangle = 0$ and let P be the range projection of T, so that $\langle \omega, P\rangle = 0$ by Proposition 2.3.6. Given any vector f in the range of T we have $S_{\omega_f} \leq P$ and thus $0 \leq \left\langle \omega_n, S_{\omega_f}\right\rangle \leq \langle \omega_n, P\rangle \leq \langle \omega, P\rangle = 0$, implying that f is orthogonal to $\langle \rho(G)\,\chi_{U_n}\rangle$ for each $n \geq 1$. If f is continuous it follows that

$$f(s) = \lim_n |U_n s|^{-1} \int_{U_n s} f = \lim_n |U_n s|^{-1} \left\langle f | \rho\left(s^{-1}\right)\chi_{U_n}\right\rangle \Delta(s)^{\frac{1}{2}} = 0$$

for every $s \in G$. Thus $T = 0$ by Lemma 6.2.4 and ω is faithful, establishing that (1) implies (3).

Conditions (3) and (4) are equivalent by Proposition 2.3.21. Any faithful normal state on $VN(G)$ is a vector state by the comments of Section 3.3, so that (3) is equivalent to (5) by Proposition 6.2.2. That faithful normal states on $VN(G)$ are adapted follows from Proposition 2.3.19, so (3) implies (2).

It remains to show that (2) implies (1), for which we provide the argument of [47]. Suppose ω is an adapted normal state on $VN(G)$. Fix a compact neighborhood K of the identity in G and for each $n \geq 1$ define $U_n = \left\{x \in K : |\omega(x) - 1| < \frac{1}{n}\right\}$, which is open by continuity of the $A(G)$ function ω. Let V be any open neighborhood of the identity contained in K. We show that $(U_n)_{n=1}^{\infty}$ forms a neighborhood base at the identity, for which it suffices to establish that some U_n is contained in V. Since ω is continuous, adapted, and $K \setminus V$ is compact, the value $\epsilon = \inf\{|\omega(x) - 1| : x \in K \setminus V\}$ is strictly positive and we may find $n \geq 1$ such that $\frac{1}{n} < \epsilon$. If $x \in U_n$ then $x \in K$ and that $|\omega(x) - 1| < \epsilon$ implies $x \notin K \setminus V$, so $x \in V$, as required. ☐

If we call an arbitrary (not necessarily normal) state on a von Neumann algebra faithful when it satisfies condition (2) of Proposition 2.3.19, i.e. it takes strictly positive values on strictly positive operators, then it can be shown that, for any von Neumann algebra, the existence of a faithful state implies the existence of a faithful normal state. The analogous result for adaptedness of states on $VN(G)$ is false.

Example 6.2.6. Let G be a locally compact group. Let $\mathcal{U}$ be a neighborhood base at the identity e consisting of compact sets and for each $U \in \mathcal{U}$ let V be a neighborhood of e such that $VV^{-1} \subset U$ (Proposition 3.1.2) and set $\xi_U = |V|^{-\frac{1}{2}} \chi_V$. Then ω_{ξ_U} is a normal state on $A(G)$ and the computation (6.2.2) shows that $\omega_{\xi_U} = \xi_U * \check{\xi_U}$, so that $\operatorname{supp}(\omega_{\xi_U}) \subset \overline{\operatorname{supp}(\xi_U)\operatorname{supp}(\xi_U)^{-1}} = VV^{-1} \subset U$ by Lemma 6.1.3. Directing $\mathcal{U}$ by reverse inclusion and passing to a subnet of $(\omega_U)_U$ if necessary, we may assume this net has a weak* limit $\omega \in VN(G)^*$, which is then a state on $VN(G)$ by Proposition 2.1.12. Let $s \in G$ be distinct from the identity, let K be a compact neighborhood of e and W an open neighborhood of e with $K \subset W$ and $s \notin W$, and use Proposition 3.3.4(3) to obtain $u \in A(G)$ such that $u(K) = 1$ and $u|_{G\setminus W} = 0$, so that $u(s) = 0$. If $U \in \mathcal{U}$ with $U \subset K$, then u is identically one on $\operatorname{supp}(\omega_{\xi_U})$ and hence $\omega_{\xi_U} u = \omega_{\xi_U}$, implying that

$$\langle \omega, \lambda(s) \rangle = \lim_U \langle \omega_{\xi_U}, \lambda(s) \rangle = \lim_U \langle \omega_{\xi_U} u, \lambda(s) \rangle = \lim_U \omega_{\xi_U}(s)\, u(s) = 0.$$

Therefore ω takes that value one at no unitary in $\lambda(G)$ except the identity, but no adapted normal state exists for any G that fails to be first countable.

Proposition 6.2.7. *Let G be a locally compact group. The following are equivalent:*

1. *G is first countable.*

2. *There exists a sequence $(\xi_n)_{n=1}^{\infty}$ in $L^2(G)$ such that $\langle \rho(G)\xi_n : n \geq 1 \rangle = L^2(G)$.*

When these hold, given any neighborhood base $(U_n)_{n=1}^{\infty}$ at the identity in G, we have $\langle \rho(G)\chi_{U_n} : n \geq 1 \rangle = L^2(G)$.

Proof. If G is first countable, then Lemma 2.3.20 applied to the faithful state constructed in the proof of Theorem 6.2.5 establishes (2). Given a sequence $(\xi_n)_{n=1}^{\infty}$ as in (2), by omitting zero vectors and normalizing we may assume the ξ_n are unit vectors, in which case the normal state $\sum_{n=1}^{\infty} \gamma_n \omega_{\xi_n}$ on $VN(G)$ has support projection I by Lemma 2.3.20, hence is faithful and G is first countable by Theorem 6.2.5.

The final claim follows from the proof of Theorem 6.2.5 and Lemma 2.3.20. □

Chapter 7

Open problems

In this chapter we collect unresolved problems that arose in the preceding chapters. To our knowledge, each is an open problem.

7.1 Preliminaries

Question 1: Are the positive linear functionals on a C^*-algebra taking the value one at no unitary except the identity exactly the faithful ones?

Here, a positive linear functional a C^*-algebra is called faithful if it satisfies condition (2) of Proposition 2.3.19, i.e. takes strictly positive values on strictly positive elements.

7.2 Chapter 4

Let G be a locally compact group.

Question 2: Does the existence of a Δ-weak bounded approximate identity for $A_{cb}(G)$ imply the existence of a bounded approximate identity, i.e. the weak amenability of G?

Kaniuth and Ülger have shown [40, Theorem 5.1] that the analogous result holds for $A(G)$, but their argument exploits operator algebraic techniques available because $A(G)$ is the predual of a von Neumann algebra. For $A_{cb}(G)$, one has only general Banach algebraic techniques to work with.

Question 3: Is there an example of a locally compact group G and closed subgroup H for which an invariant projection $VN(G) \to VN_H(G)$ exists, but not approximate indicator for H exists?

Question 4: When is the natural map $M_{cb}A(G)^{**} \to M_{cb}^d(G)$ a surjection?

The discussion of Section 4.1 shows that it is the same to ask: when is every element of $M_{cb}^d(G)$ the limit of a bounded net in $M_{cb}A(G)$? When G_d is amenable (which implies G is amenable), the Fourier multiplier algebras reduce to the Fourier-Stieltjes algebras, and the natural map $B(G)^{**} \to B^d(G)$ is always a quotient map, in particular surjective, without any amenability hypotheses. Let us briefly sketch this argument, which (unfortunately) exposes the assertion as a consequence of operator algebraic phenomena that are absent in the Fourier multiplier setting. Let $\tau : B(G)^{**} \to B^d(G)$ be the natural map, which is weak* continuous by definition, so has a preadjoint $\tau_* : C^*(G_d)/B(G)_\perp \to B(G)^* = C^*(G)^{**}$. The bidual of the group C^*-algebra — its enveloping von Neumann algebra — can be shown (see Section 3 of [1]) to be the von Neumann algebra generated by the universal representation $\pi_u : G \to B(\mathcal{H}_u)$ mentioned in Section 3.3. Since $C^*(G_d)$ is a completion of $\ell^1(G_d)$, it is generated by the set $\{\delta_s : s \in G_d\}$ and the map $\delta_s \mapsto \pi_u(s)$ extends to a $*$-homomorphism $C^*(G_d) \to C^*(G)^{**}$ with kernel contained in $B(G)_\perp$. The induced map on the quotient $C^*(G_d)/B(G)_\perp$ is exactly τ_*, which is then an injective and therefore isometric $*$-homomorphism. That τ_* is an isometry entails its adjoint τ is a quotient map.

Question 5: Is there a generalization of the Cohen-Host idempotent theorem that characterizes the subsets E of G for which χ_E is a Fourier multiplier?

Question 6: When does the identity $A_{cb}(G) \widehat{\otimes} A_{cb}(G) = A_{cb}(G \times G)$ hold?

The analogous result for $A(G)$ is, in keeping with the theme, a consequence of operator algebraic results: if M and N are von Neumann algebras, and if their preduals are given operator space structures as in Example 3.2.7, then, denoting the spatial tensor product of von Neumann algebras by $M \bar{\otimes} N$, we have $(M \bar{\otimes} N)_* = M_* \widehat{\otimes} N_*$ [15]. It is also noted in [15] that $VN(G) \bar{\otimes} VN(G) = VN(G \times G)$, whence

$$A(G) \widehat{\otimes} A(G) = VN(G)_* \widehat{\otimes} VN(G)_* = (VN(G) \bar{\otimes} VN(G))_* = VN(G \times G) = A(G \times G).$$

7.3 Chapter 5

Question 7: If $\chi_H \in B^d(G)$ for a closed subgroup H of a locally compact group G, does it follow that there is an approximate indicator for H in $B(G)$?

We showed that the corresponding question for the Fourier multipliers has a negative answer, however the Fourier-Stieltjes algebra is the dual of a C^*-algebra and therefore carries operator algebraic structure that may produce a positive answer.

7.4 Chapter 6

Question 8: Does the analogue of the last claim in Proposition 6.2.7 hold for an arbitrary locally compact group G?

Explicitly, if $(U_\alpha)_{\alpha \in I}$ is a neighborhood base at the identity for an arbitrary locally compact group G, is it true that $\langle \rho(G) \chi_{U_\alpha} : \alpha \in I \rangle = L^2(G)$?

Question 9: If G is a locally compact group and $\mathcal{U}$ is a neighborhood base at the identity, and if $E \subset G$ has finite measure and $\epsilon > 0$, can we always find $U_1, \ldots, U_n \in \mathcal{U}$ and $x_1, \ldots, x_n \in G$ such that the sets $U_j x_j$ are pairwise disjoint and $\left|\bigcup_j U_j x_j \triangle E\right| < \epsilon$, where $\triangle$ denotes symmetric difference of sets?

A positive answer to this problem would yield a positive answer to the preceding problem.

Question 10: How can the notions of adaptedness for normal states on $L^\infty(G)$ and $VN(G)$ be unified?

These von Neumann algebras are naturally dual objects when viewed in the context of locally compact quantum groups. Less extravagantly, they are also both left von Neumann algebras arising from a certain left Hilbert algebra. Can adaptedness of normal states be characterized in either of these more general contexts? A reasonable first step towards an answer would be to abstract the definition of adaptedness for $L^\infty(G)$ out of the language of subsets of the group G. The following is a small but more concrete manifestation of this problem.

Question 11: If ω is an adapted normal state on $VN(G)$, is the normal state $\sum_{n=1}^{\infty} 2^{-n} |\omega|^{2n}$ faithful?

Given an adapted normal state f on $L^\infty(G)$, the proof of Theorem 6.1.4 shows that the normal state $\sum_{n=1}^{\infty} 2^{-n} (f * f^*)^{*n}$ is faithful. For elements of $A(G)$, the analogue of the involution on $L^1(G)$ is the map $\omega \mapsto \overline{\check{\omega}}$, which is simply the complex conjugation for a state in $A(G)$ by the comments preceding Lemma 6.2.4. Thus for a normal state $\omega \in A(G)$ we have $\left(\omega\overline{\check{\omega}}\right)^n = |\omega|^{2n}$, so that our question does indeed ask for a $VN(G)$ analogue of the construction of Theorem 6.1.4.

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