University of Alberta

The static spherically symmetric Einstein-Yang-Mills equations

by

Todd A. Oliynyk ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics.

Department of Mathematical and Statistical Sciences

Edmonton, Alberta
Spring 2002

0-612-68610-8

University of Alberta

Release Form

Name of Author: Todd A. Oliynyk

Title of Thesis: The static spherically symmetric Einstein-Yang-Mills equations

Degree: Doctor of Philosophy

Year This Degree Granted: 2002

Permission is hereby granted to the University of Alberta Library to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as hereinbefore provided neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

(Signed) Todd Oliynyk

Todd A. Oliynyk
10851-33 Avenue
Edmonton, Alberta
T6J 2Z3
Canada

Date: March 12, 2002

UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **The static spherically symmetric Einstein-Yang-Mills equations** submitted by **Todd A. Oliynyk** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Dr. Martin Legaré (Chair)

Dr. Hans Kunzle (Supervisor)

Dr. Robert Moody

Dr. Terry Gannon

Dr. Valeri Frolov (Physics)

Dr. Joel Smoller (University of Michigan)

January 4, 2002

To my parents, Roland and Marilyn Oliynyk

Abstract

In general, there does not exist a unique action of the rotation group (or $SU(2)$) on a principal bundle over spacetime whose structure group is a compact semisimple Lie group G. Each possible action is uniquely determined by a vector Λ_0 in the Cartan subalgebra of $\mathfrak{g}$ where $\mathfrak{g}$ is the complexification of the Lie algebra of G. When one of the vectors Λ_0 induces a Π-system, the Lie algebra $\mathfrak{g}$ can be replaced by a subalgebra in which Λ_0 lies in the interior of a Weyl chamber. In this situation, considerable simplifications occur in the static spherically symmetric Einstein-Yang-Mills (EYM) equations. However, we cannot generally expect such simplifications as we show that defining vectors Λ_0 which induce Π-systems are rare. We prove the existence and uniqueness of bounded local solutions to the static spherically symmetric EYM equations near the singularities located at the center $r = 0$, the black hole horizon $r = r_H > 0$, and at spatial infinity $r = \infty$ and we establish the free parameters that characterize these local solutions. Under the assumption that a global solution exist, we establish bounds on the solution and determine the asymptotic behavior as $r \to \infty$. That some special global solutions exist is easily derived from the fact that $\mathfrak{su}(2)$ is a subalgebra of any compact semisimple Lie algebra.

Acknowledgements

First and foremost, I would like to thank Hans Künzle for providing support and encouragement during my doctoral studies. His patience and generosity with his time will always be appreciated. I would also like to thank Michael Li for helpful discussions on the asymptotic behavior of differential equations. I wish to thank Monica Nevins for introducing me to nilpotent orbits and their relation to $\mathfrak{sl}_2\mathbb{C}$ subalgebras. Finally, I would like to thank my wife Romalynn for her patience and support.

Contents

List of Tables

Chapter 1

Introduction

It is a well known result of Deser [13] and Coleman [11] that the four dimensional flat space Yang-Mills (YM) equations have no static solution of finite energy. Deser also showed that static solutions cannot exist within three dimensional Einstein-Yang-Mills (EYM) theory [14]. It came, therefore, as a surprise when Bartnik and Mckinnon numerically constructed globally regular, asymptotically flat, static, spherically symmetric solutions to the four dimensional EYM equations with gauge group $SU(2)$ [3]. As is standard, we will refer to these static, globally regular solutions as *solitons*. Shortly after the solitons were discovered, static spherically symmetric $SU(2)$-EYM black holes were found numerically [5,32,64].

To understand the importance of these solutions we first recall the *no hair conjecture*. The no hair conjecture loosely stated says the only allowed characteristics of a black hole equilibrium configuration are the mass, angular momentum, and the electric and magnetic charges. The $SU(2)$-EYM black hole solutions provided counter examples to this conjecture and stimulated investigations into other matter fields coupled to gravity for the purpose of finding other solutions that violated the no hair conjecture. Consequently, it has been realized that violation of the no hair conjecture is typical for gravity coupled to non-Abelian gauge theories. More recently [27,28], static, axisymmetric black holes and globally regular solutions to the $SU(2)$-EYM equations have been constructed numerically, providing dramatic examples of violations to the no hair conjecture. All of these solutions have shown that equilibrium configurations of black holes can be much more complicated than had been previously thought.

Although the static spherically symmetric solutions to the $SU(2)$-EYM equations were shown to be unstable [10,61,62], they may still be physically relevant due to their similarities with sphalerons [16]. Indeed, sphalerons, which are unstable static solutions of the classical equations for the bosonic sector of the electroweak theory, are believed to be responsible for violations of the conservation of baryon numbers at high temperatures [1,35,50]. Therefore, it is possible that the EYM solitons could play a role in the violation of the conservation of baryon and lepton numbers at high temperatures.

Existence of the soliton and black hole solutions to the $SU(2)$-EYM equations was first established analytically by Smoller, Wasserman, Yau, and McLeod [52-54]. Global existence was also established by Breitenlohner, Forgács, and Maison in [6] using different methods. Smoller and Wasserman have extensively studied the $SU(2)$-EYM equations [55-60] and have completely classified [66] the solutions which are defined in the far field, i.e. for large radius r. One surprising result that they have discovered is that any solution that is defined in the far field is actually defined on the whole interval $(0, \infty)$. This is not the usual situation for solutions to non-linear systems of differential equations where one normally expects global existence for only a small subset of the initial conditions.

For gauge groups G other than $SU(2)$, much less is known and the investigations have almost exclusively focused on $SU(n)$ and only for the most obvious ansatz for the spherically symmetric gauge field. For $SU(3)$ and $SU(4)$ solitons and black hole solutions which are not embedded $SU(2)$ solutions, have been found numerically [23, 26, 34], but little work has been done analytically with the exception of the papers [4, 42]. For arbitrary compact gauge groups even less in known. No numerical solutions have been constructed for $G \neq SU(n)$ and the only analytical work is contained in the papers [2, 9, 10]. However, in these papers, they restrict themselves to the so called *regular actions* which as we shall see later is a strong condition. For a review of these developments in EYM theory, see [18]. The EYM equations continue to attract attention. Rotating $SU(2)$-EYM black holes have been constructed numerically [29] and the $SU(2)$-EYM equations with a cosmological constant have been studied [36–38, 49, 51, 67].

Recall that the YM field is determined by a connection on a principal bundle over spacetime. So a YM field is spherically symmetric if and only if the connection is invariant under an action by principal bundle automorphisms of the rotation group. Because there is no unique way to lift an isometry from the base manifold to a principal bundle, the notion of spherical symmetry is more complicated for YM fields than for tensor fields. It turns out a conjugacy class of the principal bundle automorphisms is characterized by a generator Λ_0 which is an element of a Cartan subalgebra $\mathfrak{h}$ of the complexified Lie algebra $\mathfrak{g}$ of G [2, 8]. Under certain assumptions such as regularity of the center and vanishing of the total magnetic charge, the generator Λ_0 is forced to be a semisimple element of a $\mathfrak{sl}_2\mathbb{C}$ subalgebra of $\mathfrak{g}$. If Λ_0 lies in the interior of a Weyl chamber, then the action determined by Λ_0 will be called *regular*. The assumption that Λ_0 is regular has been used in all previous work on the EYM equations. The reason for this is that the EYM equations take on a relatively simple form with this assumption. Without this assumption, the equations are extremely complicated and much more difficult to analyze. Unfortunately, as will be seen in chapter 4, the set of regular actions is very small in the set of all possible actions implying that the assumption that the action is regular is a strong one. The purpose of this thesis is to investigate the EYM equations under the assumptions of spherical symmetric and staticity for arbitrary compact gauge groups but without assuming that the action is regular. Our long term goal is to characterize the solution space of these equations as has been done for $G = SU(2)$.

This thesis is organized as follows. Chapter 2 contains a review of the results from the theory of three dimensional semisimple Lie algebras that will be needed. In chapter 3 we review the YM formalism and describe how to classify spherically symmetric YM fields. The static spherically symmetric EYM equations are presented in chapter 4 along with a classification of the regular actions. Before global existence of solutions can be proved via a shooting technique, local existence must first be established. In our case we are interested in local existence near the origin $r = 0$, the black hole horizon $r = r_H$, and spatial infinity $r = \infty$. The proof of local existence is the content of chapter 5. All the results contained in this chapter are taken from the two papers [45, 46]. Finally, in chapter 6 we establish some a priori estimates on global solutions. The estimates serve two purposes. First, they aid in constructing numerical solutions by providing insight into what sort of behavior to expect from the solution and this allows one to greatly increase the efficiency of the search for numerical solutions. Secondly, these estimates will be necessary in proving global existence.

Chapter 2

Lie algebra theory

In this chapter we review the necessary definitions and results that we require from Lie algebra theory. Of particular importance to us are the three dimensional semisimple subalgebras. These algebras will play a distinguished role in our later work as they arise from the study of spherically symmetric Yang-Mills fields. Fortunately, a complete classification of these subalgebras is available and this information will be used to investigate the class of spherically symmetric Yang-Mills fields.

2.1 Notation and conventions

None of the assertions made in this section will be proved. Most are well known and can be found, for example, in [21] and [30]. Throughout this thesis G will always denote a real compact semisimple Lie group with Lie algebra $\mathfrak{g}_0$. The adjoint action of G on $\mathfrak{g}_0$ will be denoted by Ad, while ad will denote the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_0$, i.e. $\mathrm{ad}(X)(Y) = [X,Y]$ for all X,Y in $\mathfrak{g}_0$. The complexification of $\mathfrak{g}_0$ will be denoted by $\mathfrak{g}$. The ad action can then be extended by complex linearity to an action of $\mathfrak{g}$ on $\mathfrak{g}$ so that $\mathrm{ad}(X)(Y) := [X,Y]$ for all X,Y in $\mathfrak{g}$. We will use the notation $\mathfrak{g}^X$ for the centralizer of a single element $X \in \mathfrak{g}$. In other words,

$$\mathfrak{g}^X := \{ Y \in \mathfrak{g} \mid [X,Y] = 0 \} .$$

Similarly, $\mathfrak{g}_0^X$ is the centralizer of an element $X \in \mathfrak{g}_0$.

We will let $(\cdot|\cdot)$ be any non-degenerate ad-invariant bilinear form on $\mathfrak{g}$ that restricts to a negative definite inner product on $\mathfrak{g}_0$. By ad-invariance we mean that

$$([X,Y]|Z) = (X|[Y,Z]) \quad \forall\, X,Y,Z \in \mathfrak{g} .$$

For example, we could take $(\cdot|\cdot)$ to be the Killing form on $\mathfrak{g}$. For later use, we introduce a non-degenerate Hermitian inner product $\langle\cdot|\cdot\rangle$ on $\mathfrak{g}$ defined by

$$\langle X|Y \rangle := -(c(X)|Y) \quad \forall\, X,Y \in \mathfrak{g} ,$$

where $c : \mathfrak{g} \to \mathfrak{g}$ is the conjugation operator determined by the compact real form $\mathfrak{g}_0$. From the ad-invariance of $(\cdot|\cdot)$ and the fact that conjugation is an automorphism of

$\mathfrak{g}$ it follows that $\langle \cdot | \cdot \rangle$ satisfies

$$\begin{aligned} \langle X|Y \rangle &= \overline{\langle Y|X \rangle} , \\ \langle c(X)|c(Y) \rangle &= \overline{\langle X|Y \rangle} , \\ \langle [X, c(Y)]|Z \rangle &= \langle X|[Y, Z] \rangle \end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$. Treating $\mathfrak{g}$ as a $\mathbf{R}$-linear space by restricting scalar multiplication to multiplication by reals, we can introduce a positive definite inner product $\langle\langle \cdot | \cdot \rangle\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ on $\mathfrak{g}$ defined by

$$\langle\langle X|Y \rangle\rangle := \mathrm{Re}\langle X|Y \rangle \quad \forall\, X, Y \in \mathfrak{g} .$$

Let $\|\cdot\|$ denote the norm induced on $\mathfrak{g}$ by $\langle\langle \cdot | \cdot \rangle\rangle$, i.e.

$$\|X\| := \sqrt{\langle\langle X|X \rangle\rangle} \quad \forall\, X \in \mathfrak{g} . \tag{2.1.1}$$

From the invariance properties satisfied by $\langle \cdot | \cdot \rangle$, it is straightforward to verify that $\langle\langle \cdot | \cdot \rangle\rangle$ satisfies

$$\begin{aligned} \langle\langle X|Y \rangle\rangle &= \langle\langle Y|X \rangle\rangle , \\ \langle\langle c(X)|c(Y) \rangle\rangle &= \langle\langle X|Y \rangle\rangle , \\ \langle\langle [X, c(Y)]|Z \rangle\rangle &= \langle\langle X|[Y, Z] \rangle\rangle \end{aligned} \tag{2.1.2}$$

for all $X, Y, Z \in \mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $R \subset \mathfrak{h}^*$ the roots determined by $\mathfrak{h}$. Then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathbf{C}\mathbf{e}_\alpha$$

where the nonzero vectors $\mathbf{e}_\alpha$ satisfy

$$[H, \mathbf{e}_\alpha] = \alpha(H)\mathbf{e}_\alpha \quad \forall\, H \in \mathfrak{h} . \tag{2.1.3}$$

Note that

$$\mathbf{C}\mathbf{e}_\alpha = \{\, X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \quad \forall\, H \in \mathfrak{h} \,\} . \tag{2.1.4}$$

A straightforward consequence of (2.1.4), the Jacobi identity, and the ad-invariance of $(\cdot|\cdot)$ is that

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] \in \mathbf{C}\mathbf{e}_{\alpha+\beta} \tag{2.1.5}$$

and

$$(\mathbf{e}_\alpha|\mathbf{e}_\beta) = 0 \quad \text{if } \alpha + \beta \neq 0. \tag{2.1.6}$$

Following [21], we define $\mathbf{t}_\alpha \in \mathfrak{h}$ as the unique vector in $\mathfrak{h}$ that satisfies

$$(\mathbf{t}_\alpha|H) = \alpha(H) \quad \forall\, H \in \mathfrak{h} .$$

Then

$$(\alpha|\beta) := (\mathbf{t}_\alpha|\mathbf{t}_\beta) \quad \forall\, \alpha, \beta \in R \tag{2.1.7}$$

defines a positive definite inner product on the space $\mathrm{span}_{\mathbf{R}}\{\, \alpha \mid \alpha \in R \,\}$. We will use $|\cdot|$ to denote the norm of this inner product. It will be useful to introduce the "dual roots" $\alpha^\vee$ defined by

$$\alpha^\vee := \frac{2\alpha}{|\alpha|^2} \,.$$

We can use the dual roots to define the angle bracket

$$\langle \alpha, \beta \rangle := (\alpha|\beta^\vee) \ ,$$

and the vectors

$$\mathbf{h}_\alpha := \mathbf{t}_{\alpha^\vee} \,.$$

Choosing a base Δ for R, we then have

$$\mathfrak{h} = \bigoplus_{\alpha\in\Delta} \mathbb{C}\mathbf{h}_\alpha \,.$$

Also the Cartan matrix C is defined via

$$C_{\alpha\beta} := \langle \alpha, \beta \rangle \quad \forall\, \alpha, \beta \in \Delta \,.$$

A useful relation that is an easy consequence of the above definitions is

$$[\mathbf{h}_\alpha, \mathbf{e}_\beta] = C_{\beta\alpha}\mathbf{e}_\beta \quad \forall\, \alpha, \beta \in \Delta \,. \tag{2.1.8}$$

Since $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}$, the vectors $\{\, \mathbf{h}_\alpha,\ \mathbf{e}_\alpha \mid \alpha \in R \,\}$ can always be chosen to satisfy the following relations

$$c(\mathbf{h}_\alpha) = -\mathbf{h}_\alpha, \quad c(\mathbf{e}_\alpha) = -\mathbf{e}_{-\alpha} \quad \forall\, \alpha \in R \,, \tag{2.1.9}$$

and

$$[\mathbf{e}_\alpha, \mathbf{e}_{-\alpha}] = \mathbf{h}_\alpha \tag{2.1.10}$$

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = N_{\alpha,\beta}\mathbf{e}_{\alpha+\beta} \qquad \text{if } \alpha+\beta \in R \tag{2.1.11}$$

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0 \qquad \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin R \tag{2.1.12}$$

where the constants $N_{\alpha,\beta}$ are real and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. We are also free to normalize the vectors $\mathbf{e}_\alpha$ as follows

$$(\mathbf{e}_\alpha|\mathbf{e}_\alpha) = \frac{2}{|\alpha|} \quad \forall\, \alpha \in R \,.$$

A basis satisfying these conditions will be called a *Chevalley-Weyl basis.* The compact real form $\mathfrak{g}_0$ can then be written as

$$\mathfrak{g}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R}i\mathbf{h}_\alpha \oplus \bigoplus_{\alpha\in R^+} \mathbb{R}(\mathbf{e}_\alpha - \mathbf{e}_{-\alpha}) \oplus \bigoplus_{\alpha\in R^+} \mathbb{R}i\,(\mathbf{e}_\alpha + \mathbf{e}_{-\alpha}) \ ,$$

where R^+ is the set of positive roots. The subspace

$$\mathfrak{h}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R}i\mathbf{h}_\alpha \tag{2.1.13}$$

is called the real Cartan subalgebra of $\mathfrak{g}_0$. Notice that $\mathfrak{h}$ is the complexification of $\mathfrak{h}_0$. As in the complex case, a real Cartan subalgebra can be defined independently as a maximal Abelian subalgebra of $\mathfrak{g}_0$.

From (2.1.13) and the fact that $(\alpha|\beta) \in \mathbb{R}$ for all $\alpha, \beta \in R$, it is clear that $\alpha(H) \in$

$i\mathbb{R}$ for every $H \in \mathfrak{h}_0$ and $\alpha \in R$. This allows us to define a subset $\mathcal{W}_{\mathbb{R}}$ of $\mathfrak{h}_0$ called the *real fundamental open Weyl chamber* by

$$\mathcal{W}_{\mathbb{R}} := \{\, H \in \mathfrak{h}_0 \mid -i\alpha(H) > 0 \quad \forall \alpha \in \Delta \,\} .$$

We will also need a related subset $\mathcal{W}$ of $\mathfrak{h}$ called the *(complex) fundamental open Weyl chamber* which is defined by

$$\mathcal{W} := \{\, H \in \mathfrak{h} \mid \alpha(H) > 0 \quad \forall \alpha \in \Delta \,\} .$$

Observe that we have the inclusion $\mathcal{W} \subset i\mathfrak{h}_0$.

If we let $\exp : \mathfrak{g}_0 \to G$ denote the exponential map, then the kernel of exp is by definition

$$\ker(\exp) = \{\, X \in g_0 \mid \exp(X) = \mathbb{1} \,\} .$$

The subset of ker(exp) given by

$$\mathcal{I} := \ker(\exp) \cap \mathfrak{h}_0$$

is known as the *integral lattice.*

2.2 Three dimensional semisimple Lie subalgebras

Later on we will see that classifying spherically symmetric Yang-Mills potentials is related to the problem of classifying three dimensional semisimple Lie subalgebras of $\mathfrak{g}$ up to conjugation by inner automorphisms. This problem of classifying three dimensional semisimple Lie subalgebras has been studied extensively by many authors beginning with Mal'cev [39] and Dynkin [15]. For a modern presentation and relations to nilpotent orbits see [12].

It is well known that any three dimensional semisimple Lie algebra is isomorphic to $\mathfrak{sl}_2\mathbb{C}$ and is spanned by three vectors $\{\,\Omega_0, \Omega_+, \Omega_-\,\}$ that satisfy the commutation relationships

$$[\Omega_0, \Omega_\pm] = \pm 2\,\Omega_\pm \quad \text{and} \quad [\Omega_+, \Omega_-] = \Omega_0 . \tag{2.2.1}$$

The vectors $\{\Omega_0, \Omega_+, \Omega_-\}$ are known collectively as a *complex standard triple.* Instead of working directly with $\mathfrak{sl}_2\mathbb{C}$-subalgebras, we will often find it more convenient to work with A_1-vectors. An A_1*-vector* is a vector $\Omega_0 \in \mathfrak{g}$ for which there exists two vectors Ω_+, Ω_- such that the commutation relationships (2.2.1) are satisfied. The set of all A_1-vectors will be denoted by $\mathcal{A}_1^\vee$. A distinguished subset of $\mathcal{A}_1^\vee$ is the set $\mathcal{A}_1^{\vee,\mathbb{R}}$ of *real* A_1*-vectors.* These are the A_1-vectors Ω_0 for which Ω_+, Ω_- can be chosen so that

$$c(\Omega_0) = -\Omega_0 \quad \text{and} \quad c(\Omega_+) = -\Omega_- \tag{2.2.2}$$

are also satisfied. Notice that in the real case if we define vectors Ω_1, Ω_2, and Ω_3 in $\mathfrak{g}_0$ via

$$\Omega_+ = 2i\Omega_3 \quad \text{and} \quad \Omega_\pm = \mp\Omega_1 - i\Omega_2 , \tag{2.2.3}$$

then Ω_1, Ω_2, and Ω_3 satisfy

$$[\Omega_i, \Omega_j] = \epsilon_{ij}{}^k \Omega_k . \tag{2.2.4}$$

This shows that $\mathrm{span}_{\mathbb{R}}\{\Omega_1, \Omega_2, \Omega_3\}$ is isomorphic to $\mathfrak{so}_3\mathbb{R}$.

Let

$$\mathrm{Aut}(\mathfrak{g}) := \{\, \phi \in \mathrm{GL}(\mathfrak{g}) \mid [\phi(X), \phi(Y)] = \phi([X, Y]) \text{ for all } X, Y \in \mathfrak{g} \,\}$$

denote the automorphism group of $\mathfrak{g}$. The *group of inner automorphisms* $\mathrm{Int}(\mathfrak{g})$ is defined to be the subgoup of $\mathrm{Aut}(\mathfrak{g})$ generated by automorphisms of the form $\exp(\mathrm{ad}(X))$ where X is any element of $\mathfrak{g}$ for which $\mathrm{ad}(X)$ is nilpotent. It is a standard result in Lie algebra theory that $\mathrm{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$. With these conventions we define

$$[\mathcal{A}_1^\vee] := \{ \text{ Int}(\mathfrak{g}) \text{ conjugacy classes of } A_1\text{-vectors of } \mathfrak{g} \ \}, \tag{2.2.5}$$

and

$$[\mathcal{A}_1] := \{ \text{ Int}(\mathfrak{g}) \text{ conjugacy classes of } \mathfrak{sl}_2\mathbb{C}\text{-subalgebras of } \mathfrak{g} \ \}. \tag{2.2.6}$$

Conjugacy classes of an element x will be denoted by $[x]$.

It is well known [12] that the map

$$[\mathcal{A}_1] \longrightarrow [\mathcal{A}_1^\vee] \ : \ [\mathrm{span}_{\mathbb{C}}\{\Omega_0, \Omega_+, \Omega_-\}] \longmapsto [\Omega_0] \tag{2.2.7}$$

is a bijection. In [15] Dynkin proved that for fixed Cartan subalgebra $\mathfrak{h}$ and base $\Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_\ell \}$ there exists a unique A_1-vector Ω_0 in a conjugacy class $[\Omega_0']$ such that

$$\alpha(\Omega_0) = 0, \ 1, \text{ or } 2 \text{ for all } \alpha \in \Delta.$$

He then defined the *characteristic* $\chi([\Omega_0'])$ of the conjugacy class $[\Omega_0']$ by

$$\chi = \chi([\Omega_0']) := (\alpha_1(\Omega_0), \ldots, \alpha_\ell(\Omega_0)) .$$

The importance of the characteristic is that it is a complete invariant, i.e. $[\Omega_0''] = [\Omega_0']$ if and only if $\chi([\Omega_0'']) = \chi([\Omega_0'])$. Consequently,

$$\mathcal{A}_1^\vee \cap \overline{\mathcal{W}} \text{ is in one-to-one correspondence with } [\mathcal{A}_1^\vee]. \tag{2.2.8}$$

It is worthwhile to note that not every combination of 0, 1, and 2 defines the characteristic of some conjugacy class $[\Omega_0]$. In fact, the total number of conjugacy classes is far less than the potential 3^ℓ. For example, the number of characteristics for $A_{\ell-1}$ is equal to to the number of partitions of ℓ and this is asymptotically equivalent to

$$\frac{1}{4\sqrt{3}\ell} e^{\pi\sqrt{\frac{2\ell}{3}}} ,$$

which is much smaller than $3^{\ell-1}$.

It is not difficult to show that for every Lie algebra $\mathfrak{g}$ there is always a characteristic of the form

$$\chi = (2, 2, \ldots, 2) .$$

In other words there always exists an A_1-vector Ω_0 such that $\alpha(\Omega_0) = 2$ for all $\alpha \in \Delta$. These distinguished elements will be called *principal A_1-vectors*.

The Dynkin diagram of a Lie algebra $\mathfrak{g}$ labeled with the characteristic numbers $\alpha_k(\Omega_0)$ above the nodes is called a *weighted Dynkin diagram*. All the possible weighted Dynkin diagrams of the exceptional Lie algebras G_2, F_4, E_6 and E_8 were determined by Dynkin in [15]. A listing of these diagrams can be found in [12] section 8.4 .

For the classical Lie algebras the weighted Dynkin diagrams are not optimal for classifying the conjugacy classes of $[\mathcal{A}_1^\vee]$. Instead, a different method based on the "partitions of n" is used. To describe this method, we first consider $\mathfrak{sl}_n\mathbb{C} = A_{n-1}$ for which the classification problem can be solved by elementary methods. A *partition* of n is an k-tuple $\mathbf{d} := (d_1, d_2, \ldots, d_k)$ such that

$$d_1 \geq d_2 \geq \ldots \geq d_k > 0 \quad \text{and} \quad n = \sum_{j=1}^{k} d_j \,. \tag{2.2.9}$$

If a number s is repeated q times in a partition we will denote this by s^q and q will be called the *multiplicity* of s. For example, the partition $(9, 9, 9, 6, 4, 4, 2, 1, 1, 1)$ will also be written as $(9^3, 6, 4^2, 2, 1^3)$. The set of all partitions of n will be denoted by $\mathcal{P}(n)$. Using $\mathfrak{sl}_2\mathbb{C}$ representation theory, it not hard to show that the there exists a bijection from $[\mathcal{A}_1^\vee]$ to $\mathcal{P}(n)$. Moreover, for each partition $(d_1, \ldots, d_k)$ a canonical representative $\Omega_0^{(d_1,\ldots,d_k)}$ of the conjugacy class can be constructed as follows. For each $s \in \mathbb{N}$ let

$$\Omega_0^s = \begin{pmatrix} s & 0 & 0 & \cdots & 0 & 0 \\ 0 & s-2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s-4 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & -s+2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -s \end{pmatrix} . \tag{2.2.10}$$

Then

$$\Omega_0^{(d_1,\ldots,d_k)} = \bigoplus_{j=1}^{k} \Omega_0^{d_j - 1} \tag{2.2.11}$$

is the canonical representative. There also exists simple formulas for $\Omega_\pm$. For each $s \in \mathbb{N}$ let

$$\Omega_+^s = \begin{pmatrix} 0 & \sqrt{1(s)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2(s-1)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sqrt{3(s-2)} & & \\ 0 & 0 & 0 & 0 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & \sqrt{(s)1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.2.12}$$

and

$$\Omega_-^s = (\Omega_+^s)^t \tag{2.2.13}$$

where ^{t} denotes the transpose of a matrix. Then

$$\Omega_\pm^{(d_1,\ldots,d_k)} = \bigoplus_{j=1}^{k} \Omega_\pm^{d_j - 1} \,. \tag{2.2.14}$$

From these formulas it is easy to verify that

$$\operatorname{span}_\mathbb{C}\{\Omega_0^{(d_1,\ldots,d_k)}, \Omega_+^{(d_1,\ldots,d_k)}, \Omega_-^{(d_1,\ldots,d_k)}\} \cong \mathfrak{sl}_2\mathbb{C} \,.$$

Similar results can be obtained for the other classical Lie algebras, with the conjugacy classes of $[\mathcal{A}_1^\vee]$ being parametrized by a subset of $\mathcal{P}(n)$ for appropriate n. A canonical representative of the conjugacy class can also be constructed although the formulas are more complicated. All of this can be found in chapter 5 of [12], we only state the results.

Theorem 2.2.1. *If $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ then there exists a bijection between $[\mathcal{A}_1^\vee]$ and $\mathcal{P}(n)$.*

Theorem 2.2.2. *If $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$ then there exists a bijection between $[\mathcal{A}_1^\vee]$ and the set of partitions of $2n+1$ in which the even parts occur with even multiplicity.*

Example: $\mathfrak{so}_7\mathbb{C}$ contains six conjugacy classes parametrized by the partitions (7), $(5,1^2)$, $(3,1^4)$, $(3,2^2)$, $(3^2,1)$, and $(2^2,1^3)$.

Theorem 2.2.3. *If $\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{C}$ then there exists a bijection between $[\mathcal{A}_1^\vee]$ and the set of partitions of $2n$ in which the odd parts occur with even multiplicity.*

Example: $\mathfrak{sp}_6\mathbb{C}$ contains seven conjugacy classes parametrized by the partitions (6), $(4,2)$, $(4,1^2)$, (3^2), (2^3), $(2^2,1^2)$, and $(2,1^4)$.

Theorem 2.2.4. *If $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ then there exists a bijection between $[\mathcal{A}_1^\vee]$ and the set of partitions of $2n$ in which the even parts occurs with even multiplicity except that the "very even" partitions $(d_1,\ldots,d_k)$(those with only even parts, each having even multiplicity) correspond to conjugacy classes labeled $(d_1,\ldots,d_k)_I$ and $(d_1,\ldots,d_k)_{II}$.*

Example: $\mathfrak{so}_8\mathbb{C}$ contains eleven conjugacy classes parametrized by the partitions $(7,1)$, $(5,3)$, $(4^2)_I$, $(4^2)_{II}$, $(5,1^3)$, $(3^2,1^2)$, $(3,2^2,1)$, $(2^4)_I$, $(2^4)_{II}$, $(3,1^5)$, and $(2^2,1^4)$.

Chapter 3

Spherically Symmetric Yang-Mills fields

In this chapter we review the Yang-Mills formalism and introduce the notion of spherically symmetric Yang-Mills fields and Yang-Mills potentials. We also establish the validity of the temporal gauge. Throughout this thesis a *spacetime* will refer to a connected four dimensional manifold M equipped with a Lorentzian metric g. By Lorentzian we mean that there exists a frame $\mathbf{f}_a$ at every point in M for which $g(\mathbf{f}_a, \mathbf{f}_b) = \eta_{ab}$ and

$$[\eta_{ab}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Any non-zero tangent vector $v \in \mathrm{T}M$ is called *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$, and *null* if $g(v,v) = 0$. A hypersurface $\Sigma \subset M$ will be called *spacelike* if every vector $v \in \mathrm{T}\Sigma$ is spacelike. From this definition, it is clear that g restricts to a Riemannian metric on a spacelike hypersurface. A *Killing vector field* ξ is a vector field for which the Lie derivative $L_\xi g$ vanishes, so that the flow generated by ξ defines isometries of the spacetime. A spacetime is *static* if there exists a timelike Killing vector field ξ that is hypersurface orthogonal. That is $L_\xi g = 0$ and for every $x \in M$ there exists a spacelike hypersurface Σ_x containing x that is everywhere orthogonal to ξ. In this situation, it is always possible to introduce coordinates (x^a) so that

$$\xi = \partial_0 , \quad g_{0b} = 0 , \quad \text{and} \quad \partial_0 g_{ab} = 0$$

where $g = g_{ab} dx^a \otimes dx^b$. Here we are using the standard notation $\partial_a = \frac{\partial}{\partial x^a}$.

Suppose M is a static spacetime. Let Σ be any hypersurface that is orthogonal to the timelike Killing vector ξ and let $\mathrm{i} : \Sigma \to M$ denote inclusion. Then M will be said to be *asymptotically flat* if there exist a compact set $K \subset \Sigma$ and a coordinate system (x^i) $i = 1, 2, 3$ on $\Sigma \backslash K$ in which $\lim_{r\to\infty} (\mathrm{i}^* g)_{ij} = \delta_{ij}$ and $\lim_{r\to\infty} g(\xi, \xi) = -a$ $(a > 0)$ where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We note that this is a weak form of asymptotic flatness. Often, the rate at which $(i^* g)_{ij}$ approaches δ_{ij} and $g(\xi, \xi)$ approaches $-a$ as $r \to \infty$ is specified. For example, one often assumes that $(i^* g)_{ij} = \delta_{ij} + \mathrm{O}(1/r)$ and $\partial_k (i^* g)_{ij} = \mathrm{O}(1/r^2)$ as $r \to \infty$. However, such consideration will not concern us here

3.1 Review of the Yang-Mills formalism

In this section, we will quickly review the Yang-Mills formalism. Let $P = (P, \pi, M, G)$ be a principal G-bundle over M and ω a connection 1-form on P. We will use R_k , $k \in G$ to denote the right action of G on fibres of P. Let $\sigma : U \subset M \to P$ be a local section of P. The *Yang-Mills (or gauge) potential* A^σ is defined by

$$A^\sigma = \sigma^* \omega \, . \tag{3.1.1}$$

The gauge potential is a local object that depends on the local section σ. Different choices of local sections yield different gauge potentials. Choosing a specific local section is also known as ***fixing the gauge.*** Once a gauge is chosen, to reduce notation we will write A instead of A^σ. If $\sigma_1 : U \subset M \to P$ and $\sigma_2 : U \subset M \to P$ are two local sections, then there exists a map $g : U \to G$ such that $\sigma_2(x) = R_{g(x)}\sigma_1(x)$ for all $x \in U$. Realizing G as a matrix group, which we can do since G is compact, the relation between the two gauge potential A^{σ_1} and A^{σ_2} is

$$A^{\sigma_2} = g^{-1} A^{\sigma_1} g + g^{-1} dg \, . \tag{3.1.2}$$

The curvature Ω of ω is defined by

$$\Omega = d\omega + \tfrac{1}{2}[\omega \wedge \omega] \, .$$

The *Yang-Mills (or gauge) field strength* F^σ is then defined by

$$F^\sigma := \sigma^* \Omega \, . \tag{3.1.3}$$

It is not hard to verify that F^σ is also given by

$$F^\sigma = dA^\sigma + [A^\sigma, A^\sigma] \, . \tag{3.1.4}$$

and the relation between F^{σ_1} and F^{σ_2} is

$$F^{\sigma_2} = g^{-1} F^{\sigma_1} g \, .$$

As with the gauge potential, once a gauge is fixed we will write F instead of F^σ. In local coordinates (x^a) we can write

$$A = A_a dx^a$$

where the A_a are $\mathfrak{g}_0$-valued maps. The the field strength is then given by

$$F = \tfrac{1}{2} F_{ab} dx^a \wedge dx^b$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \, .$$

For any spacelike submanifold S^2 in M diffeomorphic to a two sphere, we can define two gauge invariant quantities

$$Q_E^{S^2} := \tfrac{1}{4\pi} \int_{S^2} \| * F_{ab} \epsilon^{ab} \| \, \epsilon \quad \text{and} \quad Q_M^{S^2} := \tfrac{1}{4\pi} \int_{S^2} \| F_{ab} \epsilon^{ab} \| \, \epsilon \, ,$$

where ϵ is the associated area element of the sphere S^2. The quantities $Q_M^{S^2}$ and $Q_E^{S^2}$ can, in analogy with electromagnetic theory, be thought of as the magnetic and electric charge, respectively, of the gauge field contained inside of S^2.

If M is static and and asymptotically flat, let Σ be a spacelike hypersurface that is orthogonal to the timelike Killing vector ξ. Then Σ can be foliated by a family of two spheres S^2_r where r is the asymptotic radial coordinate. The total magnetic and electric charges Q_M and Q_E of Σ are then defined by

$$Q_E := \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2_r} \left\| *F_{ab}\epsilon^{ab}_r \right\| \epsilon_r \quad \text{and} \quad Q_M := \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2_r} \left\| F_{ab}\epsilon^{ab}_r \right\| \epsilon_r \,. \tag{3.1.5}$$

3.2 Spherically symmetric Yang-Mills fields

Let M be a spherically symmetric spacetime, i.e. $SU(2)$ acts on M by isometries with orbits diffeomorphic to two spheres S^2 so that locally $M \cong \tilde{M} \times S^2$. We will denote this action by $\tilde{\psi} : SU(2) \times M \to M$. The Lorentz metric g on M can be decomposed as $g = \tilde{g} + r^2\hat{g}$ where $\tilde{g}$ is a Lorentz metric on $\tilde{M}$, r is a scalar on $\tilde{M}$, and $\hat{g}$ is the standard metric on S^2. Locally we can always introduce a Schwarzschild type coordinate system (t, r, θ, ϕ) in which (r, t) are coordinates $\tilde{M}$, (θ, ϕ) are the usual angular coordinates on S^2, and

$$g = -N(r,t)S(r,t)^2dt^2 + N(r,t)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.2.1}$$

Let $P = (P, \pi, M, G)$ be a principal G-bundle. Let $\psi : SU(2) \times P \to P$ be a left action of $SU(2)$ on P by principal bundle automorphism such that the induced action of $SU(2)$ on M is equal to $\tilde{\psi}$. In other words, the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\psi_k} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\tilde{\psi}_k} & M \end{array} \qquad\qquad \begin{array}{ccc} P & \xrightarrow{R_g} & P \\ {\scriptstyle\psi_k}\downarrow & & \downarrow{\scriptstyle\psi_k} \\ P & \xrightarrow{R_g} & P \end{array}$$

commute for all $k \in SU(2)$ and $g \in G$. A *spherically symmetric connection* ω is defined to be any connection that satisfies

$$\psi_k^*\omega = \omega \quad \forall\, k \in SU(2)\,.$$

Spherically symmetric gauge potentials and gauge field strengths are those that are derivable from a spherically symmetric connection, i.e. see (3.1.1) and (3.1.3).

To determine all spherically symmetric connections and hence all spherically symmetric gauge fields two problems must be solved. The first is to classify all principal G-bundles over M that admit an action of $SU(2)$ by principal bundle automorphisms. The second is, given such a bundle P, determine all the $SU(2)$ invariant connections on P. These two problems have been solved independently by Bartnik [2] and Brodbeck and Straumann [8].

To explain Brodbeck and Straumann's results, we first explain how classifying principal G-bundles over M that admit an action of $SU(2)$ by principal bundle automorphisms is equivalent to classifying the conjugacy classes of homomorphisms λ from $U(1)$ into G. To see this, fix a point $x_0 \in M$ and consider the orbit $\tilde{\mathcal{O}} := \tilde{\psi}_{SU(2)}(x_0)$. We will denote the isotropy group of x_0 by $SU(2)_{x_0} \cong U(1)$. Now consider the sub-bundle $P_{\tilde{\mathcal{O}}} = \pi^{-1}(\tilde{\mathcal{O}})$ over $\tilde{\mathcal{O}}$. Observe that the $SU(2)$ action is now fibre transitive on this bundle. The isotropy group $SU(2)_{x_0}$ maps the fibre $\pi^{-1}(x_0)$ to itself. Fix $u_0 \in \pi^{-1}(x_0)$. Then for any $k \in SU(2)_{x_0}$ there exists a $\lambda(k) \in G$ such that $\psi_k(u_0) = R_{\lambda(k)}(u_0)$. It is easy to verify that the map $\lambda : SU(2)_{x_0} \to G$ is a homomorphism. If we change the point $u_0 \in \pi^{-1}(x_0)$ then the resulting homomorphism will be conjugate to λ and this is also true for any $SU(2)$-isomorphic bundle. This shows that the equivalence classes of principal G-bundles admitting a fiber transitive $SU(2)$ action are in one to one correspondence with the conjugacy classes of homomorphisms $\lambda : U(1) \to G$. Brodbeck and Straumann then proved

Theorem 3.2.1. *The set of conjugacy classes of homomorphisms $\lambda : U(1) \to G$ is in one-to-one correspondence with the set $\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbb{R}}$. The conjugacy class of λ is characterized by $\lambda'(\mathbf{e}) \in \mathcal{I} \cap \overline{\mathcal{W}}_{\mathbb{R}}$ where $\lambda' : \mathfrak{u}(1) \to \mathfrak{g}_0$ is the induced Lie algebra homomorphism and $\mathbf{e} = 2\pi i$ is the standard basis vector in the integral lattice of $\mathfrak{u}(1)$.*

Since the lattice point cannot jump when we change orbits, the above theorem shows that the lattice points within the closed fundamental Weyl chamber classify the principal G-bundles with $SU(2)$-actions. So, once a principal G-bundle with an $SU(2)$-action is fixed, it determines a element $\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbb{R}}$ which we will denote as $-4\pi\Lambda_3$. Brodbeck and Straumann then proved that in a Schwarzschild type coordinate system (t, r, θ, ϕ) a gauge can always be chosen such that a spherically symmetric Yang-Mills-connection on P is locally given by

$$A = \tilde{A} + \hat{A} \tag{3.2.2}$$

where

$$\tilde{A} = N(t,r)S(t,r)\mathcal{A}(t,r)dt + \mathcal{B}(t,r)dr \tag{3.2.3}$$

is a $\mathfrak{g}_0^{\Lambda_3}$-valued 1-form, and

$$\hat{A} = \Lambda_1(t,r)d\theta + (\Lambda_2(t,r)\sin\theta + \Lambda_3\cos\theta)d\phi \tag{3.2.4}$$

where Λ_1 and Λ_2 are $\mathfrak{g}_0$-valued maps that satisfy

$$[\Lambda_2, \Lambda_3] = \Lambda_1 \quad \text{and} \quad [\Lambda_3, \Lambda_1] = \Lambda_2. \tag{3.2.5}$$

The field strength is given by

$$\begin{aligned} F = {} & (\partial_t\mathcal{B} - \partial_r(NS\mathcal{A}) + NS[\mathcal{A}, \mathcal{B}])\, dt \wedge dr + (\partial_t\Lambda_1 + NS[\mathcal{A}, \Lambda_1])\, dt \wedge d\theta \\ & + (\partial_t\Lambda_2 + NS[\mathcal{A}, \Lambda_2]) \sin\theta\, dt \wedge d\phi + (\partial_r\Lambda_1 + [\mathcal{B}, \Lambda_1])\, dr \wedge d\theta \\ & + (\partial_r\Lambda_2 + [\mathcal{B}, \Lambda_2]) \sin\theta\, dr \wedge d\phi + ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin\theta\, d\theta \wedge d\phi\,. \end{aligned} \tag{3.2.6}$$

3.3 Temporal gauge

In this section we show that the $\mathcal{B}$ part of the gauge potential (3.2.3) can always be gauged away, at least locally. We assume that $\mathcal{B}(t,r)$ is defined and smooth on a neighborhood $\mathcal{N}_{(t_0,r_0)}$ of (t_0, r_0). Let $\mathcal{U}$ be a neighborhood of $0 \in \mathfrak{g}_0$ such that

$$\exp|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{V} := \exp(\mathcal{U}) \tag{3.3.1}$$

is a diffeomorphism. Then

$$\Psi : \mathcal{U} \longrightarrow \mathrm{GL}(\mathfrak{g}_0) \ : \ Y \longmapsto (\mathrm{T}_Y \exp)^{-1} \circ \mathrm{T}_e \mathrm{L}_{\exp(Y)} \tag{3.3.2}$$

defines a smooth map. Let

$$\mathcal{U}^{\Lambda_3} := \mathcal{U} \cap \mathfrak{g}_0^{\Lambda_3} \tag{3.3.3}$$

and define a smooth map

$$f : \mathcal{U}^{\Lambda_3} \times \mathcal{N}_{(t_0,r_0)} \longrightarrow \mathfrak{g}_0 \ : \ (Y, t, r) \longmapsto -\Psi(Y) \circ \mathrm{Ad}_{\exp(-Y)}(\mathcal{B}(t,r))\,. \tag{3.3.4}$$

Lemma 3.3.1.

$$f(\mathcal{U}^{\Lambda_3}, \mathcal{N}_{(t_0,r_0)}) \subset \mathfrak{g}_0^{\Lambda_3} \tag{3.3.5}$$

Proof. Suppose $Y \in \mathcal{U}^{\Lambda_3}$ and $(t,r) \in \mathcal{N}_{(t_0,r_0)}$. Then

$$\begin{aligned}
[\mathrm{Ad}_{\exp(-Y)}(\mathcal{B}(t,r)), \Lambda_3] &= \mathrm{Ad}_{\exp(-Y)}[\mathcal{B}(t,r), \mathrm{Ad}_{\exp(Y)}(\Lambda_3)] \\
&= \mathrm{Ad}_{\exp(-Y)}[\mathcal{B}(t,r), \Lambda_3] && \text{since } Y \in \mathfrak{g}_0^{\Lambda_3} \\
&= 0 && \text{since } \mathcal{B}(t,r) \in \mathfrak{g}_0^{\Lambda_3}
\end{aligned}$$

Therefore

$$\mathrm{Ad}_{\exp(-Y)}(\mathcal{B}(t,r)) \in \mathfrak{g}_0^{\Lambda_3} \,. \tag{3.3.6}$$

Since G is compact, we can assume that $G \subset \mathrm{GL}(\mathbf{R}^n)$ for some $n > 0$. Suppose $Z \in \mathcal{U}^{\Lambda_3}$. Then

$$\begin{aligned}
\Psi(Y)^{-1}(Z) &= \mathrm{T}_{\exp(Y)}\mathrm{L}_{\exp(-Y)} \circ \mathrm{T}_Y \exp(Z) \\
&= \frac{d}{dt}\bigg|_{t=0} \exp(-Y)\exp(Y+tZ) \,,
\end{aligned}$$

and hence

$$[\Psi(Y)^{-1}(Z), \Lambda_3] = \frac{d}{dt}\bigg|_{t=0} (\exp(-Y)\exp(Y+tZ)\Lambda_3 - \Lambda_3\exp(-Y)\exp(Y+tZ)) \,.$$

But $\exp(-Y)\exp(Y+tZ)\Lambda_3 = \Lambda_3\exp(-Y)\exp(Y+tZ)$ since $Y, Z \in \mathfrak{g}_0^{\Lambda_3}$, and so we find that $[\Psi(Y)^{-1}(Z), \Lambda_3] = 0$. Thus $\Psi(Y)^{-1}(\mathfrak{g}_0^{\Lambda_3}) \subset \mathfrak{g}_0^{\Lambda_3}$. Since $\Psi(Y)^{-1}$ is invertible it follows that $\Psi(Y)^{-1}(\mathfrak{g}_0^{\Lambda_3}) = (\mathfrak{g}_0^{\Lambda_3})$ or equivalently

$$\Psi(Y)(\mathfrak{g}_0^{\Lambda_3}) = (\mathfrak{g}_0^{\Lambda_3}) \,. \tag{3.3.7}$$

The lemma now follow immediately from (3.3.6), (3.3.7) and the definition of f. □

By lemma 3.3.1, the initial value problem

$$\partial_r X(t,r) = f(X(t,r),t,r) \quad : \quad X(t_0,r_0) = 0 \tag{3.3.8}$$

has a unique smooth solution $X(t,r)$ defined on a neighborhood $\tilde{\mathcal{N}}_{(t_0,r_0)} \subset \mathcal{N}_{(t_0,r_0)}$ of (t_0,r_0) that satisfies

$$X(\tilde{\mathcal{N}}_{(t_0,r_0)}) \subset \mathfrak{g}_0^{\Lambda_3} \,. \tag{3.3.9}$$

The solution $X(t,r)$ to (3.3.8) can be used to generate a gauge transformation via

$$g(t,r) = \exp(X(t,r)) \,. \tag{3.3.10}$$

It is worthwhile to notice that if $\mathcal{B}$ is independent of t, then the differential equation (3.3.8) will also be t-independent. So then the solution X and hence the gauge transformation (3.3.10) will be t-independent.

Proposition 3.3.2. *The gauge potential* (3.2.2) *transforms under the gauge transformation* (3.3.10) *into*

$$NS\tilde{\mathcal{A}}dt + \tilde{\Lambda}_1 d\theta + (\tilde{\Lambda}_2 \sin\theta + \Lambda_3\cos\theta)d\phi \tag{3.3.11}$$

where $\tilde{\mathcal{A}} = Ad_{g^{-1}}(\mathcal{A}) + T_{\exp(X)}L_{\exp(-X))} \circ T_X \exp\partial_t X$, $\tilde{\Lambda}_1 = Ad_{g^{-1}}(\Lambda_1)$ *and* $\tilde{\Lambda}_2 =$

$Ad_{g^{-1}}(\Lambda_2)$. *Moreover,* $\widetilde{\Lambda}_1$, $\widetilde{\Lambda}_2$ *and* $\widetilde{\mathcal{A}}$ *satisfy*

$$[\widetilde{\Lambda}_2, \Lambda_3] = \widetilde{\Lambda}_1\,, \quad [\Lambda_3, \widetilde{\Lambda}_1] = \widetilde{\Lambda}_2\,, \quad \textit{and} \quad [\widetilde{\mathcal{A}}, \Lambda_3] = 0\,. \tag{3.3.12}$$

Proof. Under the gauge transformation (3.3.10) the gauge potential (3.2.2) transforms as

$$\begin{aligned} A \longmapsto NS\,\mathrm{Ad}_{\exp(-X(t,r))}\mathcal{A}dt + \widetilde{\Lambda}_1 d\theta + \widetilde{\Lambda}_2 \sin\theta + \mathrm{Ad}_{\exp(-X(t,r))}(\Lambda_3)\cos\theta)d\phi \\ + \mathrm{Ad}_{\exp(-X(t,r))}(\mathcal{B}(t,r))dr + \Theta(t,r) \end{aligned} \tag{3.3.13}$$

where

$$\begin{aligned} \Theta(t,r) &= \mathrm{T}_{\exp(X(t,r))}\mathrm{L}_{\exp(-X(t,r))} \circ \mathrm{T}_{X(t,r)} \exp\left(\partial_t X(t,r)dt + \partial_r X(t,r)dr\right) \\ &= \Psi(X(t,r))^{-1}\left(\partial_t X(t,r)dt + \partial_r X(t,r)dr\right) \qquad \text{by (3.3.2)} \end{aligned}$$

Now,

$$\mathrm{Ad}_{\exp(-X)}\mathcal{B}dr + \Psi(X)^{-1}\partial_r X dr = \Psi(X)^{-1}\left(\partial_r X - f(X,r)\right)dr = 0$$

by (3.3.4) and by (3.3.8). Also

$$\mathrm{Ad}_{\exp(-X(r))}(\Lambda_3) = \Lambda_3 \qquad \text{by (3.3.9)}\,.$$

The above two result show that (3.3.13) reduces to (3.3.11) as required.

Now,

$$\begin{aligned} [\mathrm{Ad}_{\exp(-X(r))}\mathcal{A}, \Lambda_3] &= \mathrm{Ad}_{\exp(-X(r))}[\mathcal{A}, \mathrm{Ad}_{\exp(X(r))}\Lambda_3] \\ &= \mathrm{Ad}_{\exp(-X(r))}[\mathcal{A}, \Lambda_3] && \text{by (3.3.9)} \\ &= 0 && \text{since } [\mathcal{A}, \Lambda_3] = 0 \end{aligned}$$

and

$$[\mathrm{T}_{\exp(X)}\mathrm{L}_{\exp(-X)} \circ \mathrm{T}_X \exp \partial_t X, \Lambda_3] = \left.\frac{d}{ds}\right|_{s=0} [\exp(-X(t,r))\exp(X(t+s,r)), \Lambda_3]\,.$$

But G is compact, so we can again assume that $G \subset \mathrm{GL}(\mathbf{R}^n)$ for some $n > 0$. Then $\exp(-X(t,r))\exp(X(t+s,r))\Lambda_3 = \Lambda_3 \exp(-X(t,r))\exp(X(t+s,r))$ by (3.3.9). Therefore

$$[\mathrm{T}_{\exp(X)}\mathrm{L}_{\exp(-X)} \circ \mathrm{T}_X \exp \partial_t X, \Lambda_3] = 0$$

and so it follows that $[\widetilde{\mathcal{A}}, \Lambda_3] = 0$. Similar arguments can be used to show that $[\widetilde{\Lambda}_2, \Lambda_3] = \widetilde{\Lambda}_1$ and $[\Lambda_3, \widetilde{\Lambda}_1] = \widetilde{\Lambda}_2$. □

Chapter 4

The Einstein-Yang-Mills equations

The Einstein-Yang-Mills equations can be derived from the action principle

$$S(g, A) = \int \sqrt{|g|} d^4x \left(\tfrac{1}{16\pi} R - \tfrac{1}{2} \langle F_{ab} | F^{ab} \rangle \right) . \tag{4.0.1}$$

We are using relativistic units where the speed of light and the gravitational constant have been set to one and we have absorbed the gauge coupling constants into the definition of the inner product $\langle \cdot | \cdot \rangle$. Varying the action (4.0.1) with respect to the metric g yields the Einstein equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{4.0.2}$$

where the stress-energy tensor T_{ab} is given by

$$T_{ab} = \langle F_{ac} | F_b{}^c \rangle - \tfrac{1}{4} g_{ab} \langle F_{cd} | F^{cd} \rangle . \tag{4.0.3}$$

Varying the action (4.0.1) with respect to the gauge potential A yields the Yang-Mills equations

$$D_a F^a{}_b = 0 \tag{4.0.4}$$

where

$$D_a F_{bc} := \nabla_a F_{bc} + [A_c, F_{bc}] \tag{4.0.5}$$

and ∇ is the metric connection.

The stress-energy tensor T^a_b defines a linear operator on TM. Although T_{ab} is symmetric, the linear operator T^a_b is not necessarily diagonalizable due to the fact that the metric g is not Riemannian. If T^a_b happens to be diagonalizable, then the invariant physical quantities of the Yang-Mills field can be extracted from the stress energy tensor as follows. Since T^a_b is a diagonalizable by assumption, we can find an orthonormal basis $\{U^a, X^a, Y^a, Z^a\}$ of eigenvectors with U^a timelike and eigenvalues e, ρ_X, ρ_Y, and ρ_Z, respectively. The eigenvalue e can be interpreted as the rest energy density of matter. The other eigenvalues ρ_X, ρ_Y, and ρ_Z are called the *principal pressures*.

Now that spherically symmetric Yang-Mills fields and spherically symmetric space-times have been defined, we are ready to introduce the static spherically symmetric Einstein-Yang-Mills (EYM) equations. The first section of this chapter will be devoted to defining these equations and some related quantities. As we remarked earlier in the introduction, previous studies of the static spherically symmetric Einstein-Yang-Mills equations have been carried out under the assumption that the action is *regular*. The rest of this chapter will be spent determining exactly which actions are *regular*. As we will see, the *regular* actions form a very small subset of the total actions and hence are quite special.

4.1 Static spherically symmetric field equations

We now assume that all the fields are static and spherically symmetric. From the discussion in section 3.2, we can introduce a Schwarzschild type coordinate system (t, r, θ, ϕ) for which the metric takes the form

$$g = -N(r)S(r)^2dt^2 + N(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \quad (4.1.1)$$

and the gauge potential can be written as

$$A = \tilde{A} + \hat{A} \quad (4.1.2)$$

where

$$\tilde{A} = N(r)S(r)\mathcal{A}(r)dt + \mathcal{B}(r)dr \quad (4.1.3)$$

is a $\mathfrak{g}_0^{\Lambda_3}$-valued 1-form, and

$$\hat{A} = \Lambda_1(r)d\theta + (\Lambda_2(r)\sin\theta + \Lambda_3\cos\theta)d\phi \quad (4.1.4)$$

where Λ_1 and Λ_2 are $\mathfrak{g}_0$-valued maps that satisfy (3.2.5). As was shown in section 3.3 we are free to use the temporal gauge and therefore set

$$\mathcal{B} = 0 . \quad (4.1.5)$$

We will make one more assumption on the form of the gauge potential, namely that $\mathcal{A} = 0$. In analogy with the electromagnetic theory, we call $\tilde{A}$ and $\hat{A}$ the electric and magnetic parts of the gauge potential, respectively. Thus (4.1.5) and the assumption $\mathcal{A} = 0$ means that the gauge potential is purely magnetic. For solutions that are bounded at the origin, it can be shown by analyzing the initial value problem at $r = 0$ using the techniques of chapter 5 that $\mathcal{A} = 0$ is a consequence of the EYM equations. Therefore no generality is lost by setting $\mathcal{A} = 0$ when looking for solutions that are bounded at $r = 0$. However, for black hole solutions it is known that there exists solutions with $\mathcal{A}$ not identically zero [17, 25]. Therefore $\mathcal{A} = 0$ is a restriction in this case.

With the above assumption, the gauge potential takes the form

$$A = \Lambda_1(r)d\theta + (\Lambda_2(r)\sin\theta + \Lambda_3\cos\theta)d\phi . \quad (4.1.6)$$

Using (4.1.1) and (4.1.6), the EYM equations (4.0.2)-(4.0.5) reduce to

$$m' = (NG + r^{-2}P), \quad (4.1.7)$$

$$S^{-1}S' = 2r^{-1}G, \quad (4.1.8)$$

$$r^2N\Lambda_+'' + 2(m - r^{-1}P)\Lambda_+' + \mathcal{F} = 0, \quad (4.1.9)$$

$$[\Lambda_+', \Lambda_-] + [\Lambda_-', \Lambda_+] = 0 \quad (4.1.10)$$

where $' := d/dr$ and

$$\Lambda_\pm := \mp\Lambda_1 - i\Lambda_2, \quad \Lambda_0 := 2i\Lambda_3, \tag{4.1.11}$$

$$N =: 1 - \frac{2m}{r}, \quad G := \tfrac{1}{2}\langle \Lambda'_+ | \Lambda'_+ \rangle \quad P := \tfrac{1}{2}\langle \hat{F} | \hat{F} \rangle, \tag{4.1.12}$$

$$\hat{F} := \tfrac{1}{2}(\Lambda_0 - [\Lambda_+, \Lambda_-]), \tag{4.1.13}$$

$$\mathcal{F} := -i[\hat{F}, \Lambda_+]. \tag{4.1.14}$$

Using the norm (2.1.1), G and P can can be written as

$$G = \tfrac{1}{2}\|\Lambda'_+\|^2 \quad \text{and} \quad P = \tfrac{1}{2}\|\hat{F}\|^2 \tag{4.1.15}$$

We obviously have $G \geq 0$ and $P \geq 0$. A useful variant of (4.1.7) is

$$N' = \frac{1}{r}\left(1 - N - 2NG - \frac{2}{r^2}P\right). \tag{4.1.16}$$

Observe that equation (3.2.5) becomes

$$[\Lambda_0, \Lambda_\pm] = \pm 2\Lambda_\pm , \tag{4.1.17}$$

and Λ_0 satisfies

$$c(\Lambda_0) = -\Lambda_0 . \tag{4.1.18}$$

Defining

$$S_\lambda = \{\, \alpha \in R \mid \alpha(\Lambda_0) = 2 \,\} \tag{4.1.19}$$

it follows easily from (4.1.17) that

$$\Lambda_+(r) \in \bigoplus_{\alpha \in S_\lambda} \mathbb{C}e_\alpha \quad \forall\, r . \tag{4.1.20}$$

It is straightforward to verify that the stress-energy tensor T^a_b in the Schwarzschild coordinates is given by

$$T^a_b = \operatorname{diag}\left(-\frac{NG}{r^2} - \frac{P}{r^4}, \frac{NG}{r^2} - \frac{P}{r^4}, \frac{P}{r^4}, \frac{P}{r^4}\right)$$

So the energy density, radial and tangential pressure are given by

$$4\pi e = r^{-2}(NG + r^{-2}P), \quad 4\pi p_r = r^{-2}(NG - r^{-2}P), \quad \text{and} \quad 4\pi p_\theta = r^{-4}P, \tag{4.1.21}$$

respectively. We see from (3.2.6) that the field strength can be written as

$$F = \frac{1}{2}(\Lambda'_+ - \Lambda'_-)dr \wedge d\theta + \frac{i}{2}(\Lambda'_+ + \Lambda'_-)\sin\theta\, dr \wedge d\phi + \hat{F}\sin\theta\, d\theta \wedge d\phi . \tag{4.1.22}$$

A short calculation shows that

$$*F = \frac{1}{2}(\Lambda'_+ - \Lambda'_-)SN\sin\theta\, dt \wedge d\phi - \frac{i}{2}(\Lambda'_+ + \Lambda'_-)SN dt \wedge d\theta + \frac{S}{r^2}\hat{F} dt \wedge dr . \tag{4.1.23}$$

Assuming that spacetime M is asymptotically flat so that $N, S \to 1$ as $r \to \infty$, we see from (3.1.5) that total magnetic and electric charges are given by

$$Q_M = \lim_{r\to\infty} \|\hat{F}(r)\| \quad \text{and} \quad Q_E = 0\,. \tag{4.1.24}$$

Following [6], we find it useful to introduce a new independent variable τ via

$$\frac{dr}{d\tau} = r\sqrt{N} \tag{4.1.25}$$

and dependent variables

$$\mu := \sqrt{N}, \quad U_+ := \sqrt{N}\Lambda'_+, \quad \text{and} \quad \kappa := \frac{1}{2\mu}(1 + \mu^2 + 2\mu^2 G - 2r^{-2}P)\,. \tag{4.1.26}$$

In these variables, equations (4.1.7)-(4.1.10) become

$$\dot{r} = r\mu\,, \tag{4.1.27}$$
$$\dot{\Lambda}_+ = rU_+\,, \tag{4.1.28}$$
$$\dot{\mu} = (\kappa - \mu)\mu - 2\mu^2 G\,, \tag{4.1.29}$$
$$(S\mu)^{\cdot} = S(\kappa - \mu)\mu\,, \tag{4.1.30}$$
$$\dot{\kappa} = -\kappa^2 + 1 + 2\mu^2 G\,, \tag{4.1.31}$$
$$\dot{U}_+ = -(\kappa - \mu)U_+ - \frac{1}{r}\mathcal{F}\,, \tag{4.1.32}$$

where $\dot{(\cdot)} = \frac{d(\cdot)}{d\tau}$. One advantage of this system of equations over (4.1.7)-(4.1.10) is that it is no longer singular at $\mu = N^2 = 0$. However, this will not be important to us here. Instead, we shall exploit in section 6.2 the fact that the system (4.1.27)-(4.1.32) is asymptotically autonomous to determine the behaviour of bounded solutions as $r \to \infty$.

4.2 Restrictions on Λ_0

In section 3.2 is was shown that Λ_0 must lie in the set $\frac{1}{2\pi i}\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbb{R}}$. We shall now see how boundary conditions restrict Λ_0 to lie in an even smaller set. For the geometry to be regular at the origin, it is necessary that

$$\lim_{r\to 0} N(r) = 1\,. \tag{4.2.1}$$

For a global solution that is defined for all $r \in [0,\infty)$, the physical boundary conditions at the origin $r = 0$ are that the energy density and the radial and tangential pressures are finite there. From (4.1.21) and (4.2.1), is clear that these boundary conditions imply

$$G(r) = \mathrm{O}(r^2) \quad \text{and} \quad P(r) = \mathrm{O}(r^4) \quad \text{as } r \to 0\,. \tag{4.2.2}$$

An immediate consequence of this is that

$$[\Lambda_+(0), \Lambda_-(0)] = \Lambda_0\,.$$

This result combined with (4.1.17) and (4.1.18) shows that $\{\Lambda_0, \Lambda_+(0), \Lambda_-(0)\}$ form a standard triple satisfying (2.2.2). Thus $\Lambda_0 \in \mathcal{A}_1^{\mathrm{v},\mathbb{R}} \cap (\frac{1}{2\pi i}\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbb{R}})$.

Lemma 4.2.1.
$$\mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}} = \mathcal{A}_1^{v,\mathbf{R}} \cap \left(\frac{1}{2\pi i}\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbf{R}}\right)$$

Proof. Suppose $\Omega_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \mathcal{W}$. Then $2\pi i\Omega_0 \in \mathcal{W}_{\mathbf{R}}$. Moreover, $\exp(2\pi i\Omega_0) = 1 \in G$ as Ω_0 is the neutral element of a $\mathfrak{sl}_2\mathbf{C}$ subalgebra and $2\pi i\Lambda_0 \in \mathfrak{h}_0$. Therefore $\Omega_0 \in \frac{1}{2\pi i}\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbf{R}}$ and hence

$$\mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}} \subset \mathcal{A}_1^{v,\mathbf{R}} \cap \left(\frac{1}{2\pi i}\mathcal{I} \cap \overline{\mathcal{W}}_{\mathbf{R}}\right).$$

The reverse inclusion is simple to establish and will be left to the reader. □

This lemma shows that with the above boundary conditions at the origin $r = 0$, we can assume that
$$\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}. \tag{4.2.3}$$

As we remarked before, the assumption that the spacetime is asymptotically flat implies that
$$\lim_{r\to\infty} N(r) = \lim_{r\to\infty} S(r) = 1. \tag{4.2.4}$$
A common boundary condition that is adopted at $r = \infty$ is that the total magnetic charge vanishes. The vanishing of the total magnetic charge is equivalent to

$$\lim_{r\to\infty} \hat{F} = 0 \tag{4.2.5}$$

by (4.1.24). Assuming the limit $\lim_{r\to\infty}\Lambda_+(r)$ exists, 4.2.5 implies

$$[\Lambda_+(\infty), \Lambda_-(\infty)] = \Lambda_0$$

where $\Lambda_+(\infty) = \lim_{r\to\infty}\Lambda_+(r)$. The same argument as above shows that $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$. Therefore, if the magnetic charge vanishes, then we can assume that $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$.

The condition (4.2.5) does not seem to be a necessary one as purely magnetic black hole solutions have been constructed numerically with nonzero magnetic charge [25]. However, for globally regular solutions defined on $[0,\infty)$ it is unknown if this condition is necessary. Indeed, as we shall see in chapter 6 for certain choices of $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$, it is not clear that the limit $\lim_{r\to\infty}\Lambda_+(r)$ actually exists, and even if it does exists we have not been able to prove that (4.2.5) is automatically satisfied. With this said, we will for the remainder of the thesis assume that 4.2.3 holds.

4.3 Regular $\mathcal{A}_1$-vectors and Π-systems

If $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \mathcal{W}$ then we will call Λ_0 a *regular* A_1*-vector* and the action of $SU(2)$ determined by Λ_0 according to theorem 3.2.1 will be called a *regular action*. Previously, all the results in the literature concerning the EYM equations have be derived under the assumption that Λ_0 is regular. There are two main reasons for this assumption. First, equation (4.1.10) can be solved exactly and secondly the remaining equations (4.1.7)-(4.1.9) can be expanded out in a Chevalley-Weyl basis $\{\mathbf{h}_\alpha \mid \alpha \in \Delta\} \cup \{\mathbf{e}_\alpha \mid \alpha \in R\}$ without having to explicitly compute any of the brackets $[\mathbf{e}_\alpha, \mathbf{e}_\beta]$. As we shall see below, this simplification can be traced back to the fact that S_λ is a Π*-system* whenever Λ_0 is regular. So in fact the simplification is not dependent on Λ_0 being regular but only on S_λ being a Π-system. We recall [15] that a subset $\Sigma \subset R$ is called a Π*-system* if and only if

(i) if $\alpha, \beta \in \Sigma$ then $\alpha - \beta \notin R$

(ii) Σ is linearly independent

For a proof of the fact that Λ_0 regular implies that S_λ is a Π-system see [9].

If we assume that S_λ is a Π-system, then $\{ \mathbf{h}_\alpha, \mathbf{e}_\alpha, \mathbf{e}_{-\alpha} \mid \alpha \in S_\lambda \}$ generates a semisimple Lie subalgebra of $\mathfrak{g}$ denoted $\mathfrak{g}_\lambda$ for which S_λ is a base [15]. By the definition of S_λ, it is clear that Λ_0 is a principal A_1-vector in $\mathfrak{g}_\lambda$. Also from (4.1.20) and the definition of $\mathfrak{g}_\lambda$, we see that $\Lambda_+(r) \in \mathfrak{g}_\lambda$ for all r. The above discussion shows that if $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ is chosen so that S_λ is a Π-system, then the field equations (4.1.7)-(4.1.10) can be reduced to a subalgebra of $\mathfrak{g}$ for which Λ_0 is a principal. Therefore, when S_λ is a Π-system we can, without loss of generality, assume that Λ_0 is a principal A_1-vector in $\mathfrak{g}$.

So assume now that $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ is principal. We have the expansion

$$\Lambda_+(r) = \sum_{\alpha \in S_\lambda} w_\alpha(r) \mathbf{e}_\alpha , \tag{4.3.1}$$

by (4.1.20) where the $w_\alpha(r)$ are complex valued functions and $\Delta = S_\lambda$. From (2.1.9) and (4.1.11) it follows that

$$\Lambda_-(r) = \sum_{\alpha \in S_\lambda} \bar{w}_\alpha(r) \mathbf{e}_{-\alpha} . \tag{4.3.2}$$

Substituting (4.3.1) and (4.3.2) into (4.1.10) and using (2.1.10)-(2.1.12) yields

$$w_\alpha \bar{w}'_\alpha = w'_\alpha \bar{w}_\alpha \quad \forall\, \alpha \in S_\lambda \tag{4.3.3}$$

since α, $\beta \in S_\lambda$ with $\alpha \neq \beta$ implies that $\alpha - \beta \notin R$. Solving equation (4.3.3) shows that w_α must have constant phase. We are free to choose these phases, which amounts to a choice of gauge, and so we will demand that the phases are all zero. Hence the $w_\alpha(r)$ are all real valued functions. We can substitute (4.3.1) and (4.3.2) into (4.1.7) and (4.1.9) to get

$$m' = (NG + r^{-2}P), \tag{4.3.4}$$

$$r^2 N w''_\alpha + 2(m - r^{-1}P) w'_\alpha + \tfrac{1}{2} \sum_{\beta \in S_\lambda} w_\alpha C_{\alpha\beta}(\lambda_\beta - w_\beta^2) = 0 \tag{4.3.5}$$

where $(C_{\alpha\beta}) := (\langle \alpha, \beta \rangle)$ is the Cartan matrix of the reduced structure group, and

$$P = \tfrac{1}{8} \sum_{\alpha,\beta \in S_\lambda} (\lambda_\alpha - w_\alpha^2) h_{\alpha\beta} (\lambda_\beta - w_\beta^2) , \tag{4.3.6}$$

$$G = \sum_{\alpha \in S_\lambda} \frac{{w'_\alpha}^2}{|\alpha|^2} , \tag{4.3.7}$$

$$h_{\alpha\beta} = \frac{2C_{\alpha\beta}}{|\alpha|^2} , \tag{4.3.8}$$

and

$$\lambda_\alpha = 2 \sum_{\beta \in S_\lambda} (C^{-1})_{\alpha\beta} . \tag{4.3.9}$$

As before, in deriving the above expression we have used α, $\beta \in S_\lambda$ with $\alpha \neq \beta$, implies $\alpha - \beta \notin R$.

On the other hand, if S_λ is not a Π-system then equation (4.1.10) can no longer be solved exactly. This is due to the fact that for $\alpha,\beta \in R$ with $\alpha \neq \beta$ it is no longer necessary that $\alpha - \beta \notin R$. This implies in particular that the bracket $[\mathbf{e}_\alpha, \mathbf{e}_\beta]$ may no longer be zero. The inability to solve (4.1.10) implies that the system of equations (4.1.7)-(4.1.10) can no longer be written in the standard form $y'(r) = f(y(r), r)$ which provides a serious complication. Also the non-vanishing of the brackets $[\mathbf{e}_\alpha, \mathbf{e}_\beta]$ also greatly increases the complexity of the equations.

In view of the above discussion, it would be desirable to classify all those $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ for which S_λ is a Π-system.

Lemma 4.3.1.

$$\{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid \ S_\lambda \textit{ is a } \Pi\textit{-system} \,\} = \{\, \Lambda_0 \in \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}} \mid \ S_\lambda \textit{ is a } \Pi\textit{-system} \,\}$$

Proof. Since $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \subset \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ the inclusion $\{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\} \subset \{\, \Lambda_0 \in \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\}$ is clear. To show the reverse inclusion we note that by the above discussion there exists a subalgebra $\mathfrak{g}_\lambda \subset \mathfrak{g}$ such that Λ_0 is principal in $\mathfrak{g}_\lambda$ and S_λ is a base for $\mathfrak{g}_\lambda$. We can expand Λ_0 in a Chevalley-Weyl basis $\{\, \mathbf{h}_\alpha \mid \alpha \in S_\lambda \,\} \cup \{\, \mathbf{e}_\alpha \mid \alpha \in R_\lambda \,\}$ as follows

$$\Lambda_0 = \sum_{\alpha \in S_\lambda} \lambda_\alpha \mathbf{h}_\alpha \tag{4.3.10}$$

where λ_α is defined above by (4.3.9). From the Cartan matrix it follows that $\lambda_\alpha > 0$ for each $\alpha \in S_\lambda$. Define

$$\Omega_+ := \sum_{\alpha \in S_\lambda} \sqrt{\lambda_\alpha} \mathbf{e}_\alpha \quad \text{and} \quad \Omega_- := -c(\Omega_+) = \sum_{\alpha \in S_\lambda} \sqrt{\lambda_\alpha} \mathbf{e}_{-\alpha} \,. \tag{4.3.11}$$

Then using (2.1.10)-(2.1.12) it is easy to verify that $\{\Lambda_0, \Omega_+, \Omega_-\}$ is a complex standard triple that satisfies (2.2.2). So $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}}$ and therefore $\{\, \Lambda_0 \in \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\} \subset \{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\}$. □

From section 2.2, we know that the sets $\mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ can be completely parametrized. We can use this parametrization to determine all the $\Lambda_0 \in \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ such that S_λ is a Π-system. Then by the above lemma this will completely determine the set $\{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\}$.

Suppose $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ is such that S_λ is a Π-system. Let $\mathfrak{g} = \bigoplus_j \mathfrak{g}^j$ denote the decomposition of $\mathfrak{g}$ into simple ideals and $R^j \subset R$ denote the roots of $\mathfrak{g}^j$. This determines a decomposition of $S_\lambda = \dot{\cup} S_\lambda^j$ into a disjoint union of sets $S_\lambda^j \subset R^j$ such that S_λ^j is a Π-system in $\mathfrak{g}^j$. Moreover, if we let $\Lambda_0 = \sum_i \Lambda_0^j$ denote the corresponding decomposition of Λ_0 then it is not difficult to show that $S_\lambda^j = \{\, \alpha \in R^j \mid \alpha(\Lambda_0^j) = 2 \,\}$ and $\Lambda_0^j \in \mathcal{A}_1^{\vee,\mathbf{R}}(\mathfrak{g}^j) \cap \overline{\mathcal{W}}(\mathfrak{g}^j)$. This proves that if we can parametrize the set $\{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid S_\lambda \text{ is a } \Pi\text{-system} \,\}$ for simple Lie algebras $\mathfrak{g}$ then we can parametrize it for all semisimple Lie algebras. The next theorem provides such a parametrization for the classical simple Lie algebras.

Theorem 4.3.2.
$\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$

S_λ is a Π-system if and only if the partition $\mathbf{d}$ *that determines* Λ_0 *satisfies one of the following*

(i) $\mathbf{d} = (2v, 2u+1)$ *with* $2v > 2u+1 \geq 3$,

(ii) $\mathbf{d} = (2u+1, 2v)$ *with* $2u+1 > 2v \geq 2$,

(iii) $\mathbf{d} = (2v, 1, 1, \ldots, 1)$ *with* $v \geq 1$.

$\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$

S_λ is a Π-system if and only if the partition $\mathbf{d}$ *that determines* Λ_0 *satisfies one of the following*

(i) $\mathbf{d} = (2u+1)$ *with* $u \geq 1$,

(ii) $\mathbf{d} = (2u+1, 2v, 2v)$ *with* $u \geq 1$, *and* $2u+1 > 2v$,

(iii) $\mathbf{d} = (2v, 2v, 2u+1)$ *with* $u \geq 1$ *and* $2v > 2u+1$,

(iv) $\mathbf{d} = (2v, 2v, 1, 1, \ldots, 1)$ *with* $v \geq 1$.

$\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{C}$

S_λ is a Π-system if and only if the partition $\mathbf{d}$ *that determines* Λ_0 *is of the form* $\mathbf{d} = (2v, 1, 1, \ldots, 1)$ *where* $v \geq 1$.

$\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$

S_λ is a Π-system if and only if the partition $\mathbf{d}$ *that determines* Λ_0 *satisfies one of the following*

(i) $\mathbf{d} = (u, u)$ *with* $u \geq 1$,

(ii) $\mathbf{d} = (2u+1, 1)$ *with* $u \geq 1$,

(iii) $\mathbf{d} = (2u+1, 2v, 2v, 1)$ *with* $2u+1 > 2v \geq 2$,

(iv) $\mathbf{d} = (2v, 2v, 2u+1, 1)$ *with* $2v > 2u+1 \geq 1$,

(v) $\mathbf{d} = (2v, 2v, 2u+1, 2u+1)$ *with* $2v > 2u+1 \geq 3$,

(vi) $\mathbf{d} = (2u+1, 2u+1, 2v, 2v)$ *with* $2u+1 > 2v \geq 2$.

Proof. We will only prove the theorem for simplest case $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$. The other classical algebras $\mathfrak{so}_{2n+1}\mathbb{C}$, $\mathfrak{so}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$ can be analyzed in a similar fashion using the formulas from chapter 5 of [12]. However, due to the increase in complexity of the formulas over those for $\mathfrak{sl}_n\mathbb{C}$, the proofs become much more difficult and tedious.

To proceed, let $\mathfrak{D}$ denote the set of diagonal $n \times n$ complex matricies. Then

$$\mathfrak{h} = \{ H \in \mathfrak{D} \mid \mathrm{trace}(H) = 0 \}$$

is a Cartan subalgebra for $\mathfrak{sl}_n\mathbb{C}$. Define $\epsilon_j \in \mathfrak{D}^*$ by

$$\epsilon_j(\mathrm{diag}(H_1, H_2, \ldots, H_n)) = H_j\,.$$

The set of roots determined by $\mathfrak{h}$ is $R = \{ \epsilon_i - \epsilon_j \mid 1 \leq i,j \leq n \quad i \neq j \}$ and $\Delta = \{ \epsilon_i - \epsilon_j \mid j = 1, 2, \ldots, n-1 \}$ is a base for R. Suppose Λ_0 is the A_1-vector determined by the partition $\mathbf{d} = (d_1, d_2, \ldots, d_k)$ according to the formulas (2.2.10) and (2.2.11).

Lemma 4.3.3. *If there exists $r, s \in \{1, 2, \ldots, k\}$ with $r < s$ such that d_r and d_s are both even, then S_λ is not a Π-system.*

Proof. Since d_r and d_s are both even, and $r < s$, it follows that $d_r \geq d_s \geq 2$. Let $\hat{d}_j := \sum_{i=1}^{j-1} d_i$, $I = \hat{d}_r + d_r/2$, and $J = \hat{d}_s + d_s/2$. Then it is not difficult to verify that $\epsilon_I - \epsilon_{I+1}$ and $\epsilon_I - \epsilon_{J+1}$ are in S_λ. But $(\epsilon_I - \epsilon_{I+1}) - (\epsilon_I - \epsilon_{J+1}) = \epsilon_{J+1} - \epsilon_{I+1} \in R$ and hence S_λ is not a Π-system by definition. □

Lemma 4.3.4. *If there exists $r, s \in \{1, 2, \ldots, k\}$ with $r < s$ such that d_r and d_s are both odd and $d_r > 1$, then S_λ is not a Π-system.*

Proof. Since $r < s$, d_r and d_s are odd, and $d_r > 1$, we must have $d_r \geq 3$ and $d_r \geq d_s$. Let $I = \hat{d}_r + (d_r - 1)/2$ and $J = \hat{d}_s + (d_s - 1)/2$ with $\hat{d}_j$ defined as in lemma 4.3.3. Then it is easy to show that $\epsilon_I - \epsilon_{I+1}$ and $\epsilon_I - \epsilon_{J+1}$ are in S_λ. But then $(\epsilon_I - \epsilon_{I+1}) - (\epsilon_I - \epsilon_{J+1}) = \epsilon_{J+1} - \epsilon_{I+1} \in R$ and hence S_λ is not a Π-system by definition. □

From these two lemmas it is clear that the only possibilities left for a partition $\mathbf{d}$ to give rise to a S_λ that is a Π-system is if $\mathbf{d}$ satisfies (i), (ii), or (iii) from the statement of the theorem. It can be verified that each of these leads to a S_λ that is a Π-system. We will only verify the case (i). Now, straightforward computation shows that $S_\lambda = \{\, \epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, d_1 - 1 \,\} \cup \{\, \epsilon_{d_1+i} - \epsilon_{d_1+i+1} \mid i = 1, \ldots, d_2 - 1 \,\}$. Thus $S_\lambda \subset \Delta$ which implies that S_λ is a Π-system. □

We have not yet proved a corresponding theorem to theorem 4.3.2 for the exceptional Lie algebras. From the tables in chapter 8 of [12], it is clear that a proof of such a theorem would involve only straightforward computation.

Theorem 4.3.5. *For the simple algebras B_ℓ, C_ℓ, D_ℓ, G_2, F_4, E_6, E_7, E_8, and $A_{\ell'}$ with ℓ' odd, the only regular A_1-vector is the the principal one. For ℓ' even, the regular A_1-vectors are classified by partitions of the form $\mathbf{d} = (\ell' + 1 - k, k)$ for $k = 1, 2, \ldots, \ell'/2$.*

Proof. The statements concerning the classical simple algebras A_ℓ, B_ℓ, C_ℓ, and D_ℓ can be verified using theorem 4.3.5 and the formulas from chapter 5 of [12]. For the the exceptional algebras G_2, F_4, E_6, E_7, and E_8, the conclusion follows immediately from the tables in chapter 9 of [12]. □

The above considerations show how to parametrize the set $\{\, \Lambda_0 \in \mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \mid S_\lambda$ is a Π-system$\}$. An obvious question is how can one parametrize the set $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ which we know is in one-to-one correspondence with the set of all non-conjugate spherically symmetric Yang-Mills potentials that satisfy the boundary conditions 4.2.2 and 4.2.5. Since $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} \subset \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$, we could use the parametrization of $\mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ discussed in section 2.2 to parametrzie $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$. The difficulty with this approach is that we have no useful characterization with respect to this parametrization of when an element Λ_0 of $\mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ is in $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$. However, in the case $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, the formulas (2.2.10)-(2.2.14) can be used to show that $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} = \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ and so the parametrization problem is solved. For the other classical Lie algebras there exist explicit formulas for Λ_0 similar to (2.2.10) and (2.2.11) although they are more complicated. There does not, however, exist explicit formulas for $\Omega_\pm$ analogous to (2.2.14). Therefore we cannot use the same method to prove that $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} = \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$ for the other classical Lie algebras. Calculations for low dimensional simple Lie algebras shows that $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} = \mathcal{A}_1^{\vee} \cap \overline{\mathcal{W}}$. This seems to indicate that $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}} =$

$\mathcal{A}_1^\vee \cap \overline{W}$ for all Lie algebras. If this is the case, then the parametrization problem is solved. Note also that this would imply that Λ_0 for which S_λ is a Π-system are rare.

Chapter 5

Initial value problem

The main result of this chapter is that the EYM equations (4.1.7)-(4.1.10) admit bounded local solutions in the neighborhood of the origin $r = 0$, a black hole horizon $r = r_H > 0$, and spatial infinity $r = \infty$. The boundary conditions that we will adopt are:

- at the origin $r = 0$, all the physical quantities (4.1.21) are finite,
- at the black hole horizon $r = r_H > 0$, $N(r_H) = 0$, $N'(r_H) > 0$, and $S(r_H)$ finite,
- the spacetime is asymptotically flat and the total magnetic charge vanishes which implies that $N \to 1$, $S \to 1$, and $\hat{F} \to 0$ as $r \to \infty$.

Our interest in local solutions is because they provide a starting point for construction of global solutions both numerically and analytically via the shooting method. Indeed, the local existence proofs isolate the free parameters in the solutions and provide a well defined local solution which one can try to extend. Numerically, this means that there exists a convergent Taylor series expansion from which numerical integration can start.

To prove local existence to the EYM equations (4.1.7)-(4.1.10), we proceed in three steps.

1. First we prove the existence of local solutions $\{\Lambda_+(r), m(r)\}$ to the equations (4.1.7) and (4.1.9).
2. Then we determine which solutions from step 1 satisfy equation (4.1.10).
3. Finally, equation (4.1.8) can be integrated for all solutions from step 2 to obtain the metric function $S(r)$.

Our strategy for carrying out the first step will be to prove that there exists a change of variables so that the field equations (4.1.7) and (4.1.9) can be put into a form to which the following theorem applies. An analytic version of this theorem was first proved by Breitenlohner, Forgács and Maison in [6].

Theorem 5.0.6. *The system of differential equations*

$$t\frac{du_i}{dt} = t^{\mu_i} f_i(t,u,v) \qquad i = 1,\ldots,m$$
$$t\frac{dv_j}{dt} = -\lambda_j v_j + t^{\nu_j} g_j(t,u,v) \qquad j = 1,\ldots,n$$

where μ_i, ν_j are integers greater than 1, $\lambda_j > 0$, and f_i and g_j are analytic functions in a neighborhood of $(0, c_0, 0) \in \mathbf{R}^{1+m+n}$, has a unique C^1 solution $t \mapsto (u_i(t), v_j(t))$ defined on $0 \le t \le T$ for some $T > 0$ that satisfies $u(0) = c$ provided $|c - c_0|$ is small enough. Moreover this solution is analytic in (c, t) for $|t| < T$ and $|c - c_0|$ small enough and it satisfies

$$u_i(t) = c + \mathcal{O}(t^{\mu_i}) \quad \textit{and} \quad v_i(t) = \mathcal{O}(t^{\nu_j}) \quad \textit{as } t \to 0 \, .$$

Proof. We will only prove the case $m = n = 1$. By replacing u with $u - c$ we can assume that $u(0) = 0$. To start, we will also assume that $\mu = \nu = 1$. We note that the proof of this theorem is similar to the local existence proof in [54].

Consider the integral equation

$$u(t) = \int_0^t f(s, u(s), v(s))dt \quad \text{and} \quad v(t) = \frac{1}{t^\lambda}\int_0^t g(s, u(s), v(s))ds \, . \tag{5.0.1}$$

For $t > 0$ it is easy to see that this is equivalent to the differential equation. Here we are assuming that $u(t)$ and $v(t)$ are continuous for $t \ge 0$. At $t = 0$ the only problem that can arise is that $v(t)$ may not be differentiable there due to the t^λ in the denominator. Consider the limit

$$\lim_{t\searrow 0} \frac{v(t)}{t} = \lim_{t\searrow 0} \frac{\int_0^t g(s, u(s), v(s))ds}{t^{\lambda+1}} = \lim_{t\searrow 0} \frac{g(t, u(t), v(t))}{\lambda + 1} = \frac{1}{\lambda+1} g(0, 0, v(0)) \, .$$

where in getting the second equality we have used l'Hospital's rule. This shows that $v(t)$ is differentiable at $t = 0$. Note also that $v(t)$ is C^1 for $t > 0$. Similar calculations show that $\lim_{t\searrow 0} v'(t) = (\lambda + 1)^{-1} g(0, 0, v(0))$ and so we see that $v(t)$ is actually C^1 for $t \ge 0$.

Define a map

$$\Psi(u, v)(t) = \left(\int_0^t f(s, u(s), v(s))dt, \frac{1}{t^\lambda}\int_0^t g(s, u(s), v(s))ds\right) \tag{5.0.2}$$

and let

$$\mathcal{D}_{R,T} = \{ \, (u, v) \in C^0([0, T]) \times C^0([0, T]) \mid \|(u, v)\| \le R \, \}$$

where

$$\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\} \, .$$

Then using standard arguments, it can be shown that there exists $T_*, R_* > 0$ such that $\Psi(\mathcal{D}_{R,T}) \subset \mathcal{D}_{R,T}$ and Ψ is a contraction for all $0 < T \le T_*$ and $0 < R \le R_*$.

Let $D_T = \{ \, z \in \mathbf{C} \mid \|z\| \le T \, \}$ and define

$$\mathcal{D}^a_{R,T} = \{ \, (u, v) \in H_T \times H_T \mid \|(u, v)\| \le R \, \}$$

where

$$H_T = \{ \, u : D_T(\mathbf{C}) \to \mathbf{C} \mid \text{ u is continuous on } D_T \text{ and analytic in the interior} \, \} \, .$$

Let Ψ_a denote the same map as (5.0.2), where now the functions u and v lie in H_T and the integration can be taken over the straightline path between two points in D_T. Again using standard arguments, it can be shown that there exists $T^a_*, R^a_* > 0$ such that $\Psi_a(\mathcal{D}^a_{R,T}) \subset \mathcal{D}^a_{R,T}$ and Ψ_a is a contraction for all $0 < T \le T^a_*$ and $0 < R \le R^a_*$.

Let $T_0 = \min\{T^a_*, T_*\}$, $R_0 = \min\{R^a_*, T_*\}$, and choose $(u^0, v^0) \in \mathcal{D}^a_{R_0,T_0}$ such that

$(u^0|_{[0,T_0]}, v^0|_{[0,T_0]}) \in \mathcal{D}_{R_0,T_0}$. Letting $\Psi_a^n = \Psi_a \circ \Psi_a \circ \cdots \circ \Psi_a$ (n times) and using similar notation for Ψ^n, we see that if we define

$$(u^n, v^n) := \Phi_a^n(u^0, v^0)$$

then

$$(u^n|_{[0,T_0]}, v^n|_{[0,T_0]}) = \Phi^n(u^0|_{[0,T_0]}, v^0|_{[0,T_0]}) \,. \tag{5.0.3}$$

Now,

$$(u(z), v(z)) := \lim_{n\to\infty} (u^n(z), v^n(z)) \quad z \in D_{T_0}(\mathbb{C})$$

is the unique fixed point in $\mathcal{D}^a_{R_0,T_0}$ for Φ_a. But from (5.0.3) it is clear that

$$(u(t), v(t)) = \lim_{n\to\infty} (u^n(t), v^n(t)) \quad t \in [0, T_0]$$

and hence $(u(t), v(t))$ is the unique fixed point in $\mathcal{D}_{R_0,T_0}$ for Φ. This implies that differential equation

$$t\frac{du}{dt} = tf(t,u,v) \quad \text{and} \quad t\frac{dv}{dt} = -\lambda v + tg(t,u,v)$$

has a unique solution (u, v) in $\mathcal{D}_{R_0,T_0}$ that satisfies $u(0) = 0$. Moreover, we get from the above arguments that $u(t)$ and $v(t)$ are actually analytic for $|t| < T_0$.

To prove the general case, let $\hat{f}(t,u,v) = t^{\mu-1}f(t,u,v)$ and $\hat{g}(t,u,v) = t^{\nu-1}g(t,u,v)$. Then

$$t\frac{du}{dt} = t\hat{f}(t,u,v) \quad \text{and} \quad t\frac{dv}{dt} = -\lambda v + t\hat{g}(t,u,v)$$

and we are back to the situation with $\mu = \nu = 1$.

The easiest way to verify the fall off conditions,

$$u(t) = c + \mathcal{O}(t^\mu) \quad \text{and} \quad v(t) = \mathcal{O}(t^\nu) \quad \text{as } t \to 0$$

is to substitute a powerseries representation about $t = 0$ for $u(t)$ and $v(t)$ into the differential equation and then solve for the powerseries coefficients. □

The next lemma shows that if $\{\Lambda_+(r), m(r)\}$ is a solution to the field equations (4.1.7) and (4.1.9) then the quantity $[\Lambda'_+, \Lambda_-] + [\Lambda'_-, \Lambda_+]$ satisfies a first order linear differential equation. This unexpected result is what allows us to carry out step 2 and thereby construct local solutions.

Lemma 5.0.7. *If $\{\Lambda_+(r), m(r)\}$ is a solution to the field equations* (4.1.7) *and* (4.1.9) *then*

$$\gamma(r) := [\Lambda_+(r), \Lambda'_-(r)] + [\Lambda_-(r), \Lambda'_+(r)]$$

satisfies the differential equation

$$\gamma' = -\frac{2}{r^2N}\left(m - \frac{1}{r}P\right)\gamma\,.$$

Proof. Differentiating γ yields

$$\begin{aligned}\gamma' &= [\Lambda_+, \Lambda''_-] + [\Lambda_-, \Lambda''_+] \\ &= -\frac{2}{r^2N}\left(m - \frac{1}{r}P\right)\gamma + \frac{i}{r^2N}([\Lambda_-, [\hat{F}, \Lambda_+]] + [\Lambda_+, [\Lambda_-, \hat{F}]])\,,\end{aligned}$$

by (4.1.7) and (4.1.9) while

$$[\Lambda_-, [\hat{F}, \Lambda_+]] + [\Lambda_+, [\Lambda_-, \hat{F}]] = 0$$

by (4.1.13), (4.1.17), and the Jacobi identity. Combining the above two results proves the lemma. □

5.1 Algebraic results

In this section we collect all of the algebraic results needed to prove the local existence theorems. We will employ the same notation as in [45] section 6. For this section, we will assume that $\Lambda_0 \in \mathcal{A}_1^{\vee,\mathbb{R}} \cap \overline{\mathcal{W}}$ is fixed. Let $\Omega_+, \Omega_- \in \mathfrak{g}$ be two vectors such that

$$[\Lambda_0, \Omega_\pm] = \pm 2\Omega_\pm\,, \quad [\Omega_+, \Omega_-] = \Lambda_0 \quad \text{and} \quad c(\Omega_+) = -\Omega_-\,. \tag{5.1.1}$$

Then

$$\mathfrak{q} := \text{span}_{\mathbb{C}}\{\Lambda_0, \Omega_+, \Omega_-\} \cong \mathfrak{sl}_2\mathbb{C} \tag{5.1.2}$$

and we will often use the dot notation to denote the adjoint action of $\mathfrak{q}$ on $\mathfrak{g}$, i.e.

$$X.Y := \text{ad}(X)(Y) \qquad \forall\, X \in \text{span}_{\mathbb{C}}\{\Lambda_0, \Omega_+, \Omega_-\},\ Y \in \mathfrak{g}.$$

Because Λ_0 is a semisimple element, $\text{ad}(\Lambda_0)$ is diagonalizable and it follows from $\mathfrak{sl}(2)$-representation theory [21] that the eigenvalues are integers. Let V_n denote the eigenspaces of $\text{ad}(\Lambda_0)$, i.e.

$$V_n := \{X \in \mathfrak{g} \mid \Lambda_0.X = nX\} \quad n \in \mathbb{Z}\,. \tag{5.1.3}$$

Observe that

$$V_2 = \bigoplus_{\alpha \in S_\lambda} \mathbb{C}e_\alpha\,. \tag{5.1.4}$$

It also follows from $\mathfrak{sl}_2\mathbb{C}$-representation theory that if $X \in \mathfrak{g}$ is a highest weight vector of the adjoint representation of $\text{span}_{\mathbb{C}}\{\Lambda_0, \Omega_+, \Omega_-\}$ with weight n, and we define $X_{-1} = 0$, $X_0 = X$ and $X_j = (1/j!)\Omega_-^j.X_0 \quad (j \geq 0)$, then

$$\begin{aligned} \Lambda_0.X_j &= (n - 2j)X_j\,, \\ \Omega_-.X_j &= (j+1)X_{j+1}\,, \\ \Omega_+.X_j &= (n - j + 1)X_{j-1} \quad (j \geq 0)\,. \end{aligned} \tag{5.1.5}$$

Proposition 5.1.1. *There exists M highest weight vectors $\xi^1, \xi^2, \ldots, \xi^M$ for the adjoint representation of $\mathfrak{q}$ on $\mathfrak{g}$ that satisfy*

(i) the ξ^j have weights $2k_j$ where $j = 1, 2, \ldots, M$ and $1 = k_1 \leq k_2 \leq \cdots \leq k_M$,

(ii) if $V(\xi^j)$ denotes the irreducible submodule of $\mathfrak{g}$ generated by ξ^j, then the sum $\sum_{j=1}^M V(\xi^j)$ is direct,

(iii) if $\xi_l^j = (1/l!)\Omega_-^l.\xi^j$ then

$$c(\xi_l^j) = (-1)^l \xi_{2k_j - l}^j\,, \tag{5.1.6}$$

(iv) $M = |S_\lambda|$ and the set $\{\xi_{k_j-1}^j \mid j = 1, 2, \ldots M\}$ forms a basis for V_2 over $\mathbb{C}$.

Proof. *(i)* and *(ii)*: The conjugation operator c satisfies

$$c([X,Y]) = [c(X),c(Y)] \quad \forall\, X,Y \in \mathfrak{g}\,. \tag{5.1.7}$$

Using (5.1.1), (5.1.7), and (4.1.18), it is easy to see that

$$c\circ \mathrm{ad}(\Omega_\pm)^n = (-1)^n \mathrm{ad}(\Omega_\pm)^n \circ c \;\forall n \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad c\circ \mathrm{ad}(\Lambda_0) = -\mathrm{ad}(\Lambda_0)\circ c\,. \tag{5.1.8}$$

As usual, define the Casimir operator $\mathcal{C}$ by

$$\mathcal{C} = \tfrac{1}{2}\mathrm{ad}(\Lambda_0)^2 + \mathrm{ad}(\Omega_+)\mathrm{ad}(\Omega_-) + \mathrm{ad}(\Omega_-)\mathrm{ad}(\Omega_+)\,.$$

Then $\mathfrak{g}$ can be decomposed as follows [44]

$$\mathfrak{g} = \bigoplus_p V(s_p, v^p)\,, \tag{5.1.9}$$

where $V(s_p, v^p)$ is a highest weight module generated by the highest weight vector v^p of weight s_p, and it has the property

$$\mathcal{C}|_{V(s_p,v^p)} = \left(\tfrac{1}{2}s_p^2 + s_p\right)\mathrm{id}_{V(s_p,v^p)} \quad \forall\, p\,. \tag{5.1.10}$$

From (5.1.8) it follows that $\mathcal{C}\circ c = c\circ\mathcal{C}$. Using this result and (5.1.10), we see that

$$c(V(s_p, v^p)) \subset V(s_p, v^p) \quad \forall\, p\,. \tag{5.1.11}$$

Let $\{s_{p_1}, s_{p_2}, \ldots, s_{p_M}\}$ be the set of weights from the decomposition (5.1.9) that are even and greater than zero. We will assume that they are ordered so that $s_{p_1} \leq s_{p_2} \leq \ldots \leq s_{p_M}$. Define $k_j = s_{p_j}/2$. Then the k_j are positive integers that satisfy $k_1 \leq k_2 \leq \ldots \leq k_M$. Note that $k_1 = 1$ because Ω_+ is a highest weight vector with weight 2. To simplify notation, set $v^j := v^{p_j}$. As before with highest weight vectors (see (5.1.5)), we let $v^j_l = (1/l!)\Omega_-^l.v^j$. Define

$$\xi^j := \begin{cases} iv^j + c(iv^j_{2k_j}) & \text{if } c(v^j_{2k_j}) = -c(v^j) \\ v^j + c(v^j_{2k_j}) & \text{otherwise} \end{cases} \tag{5.1.12}$$

for $j = 1, 2, \ldots M$. Then $\Lambda_0.\xi^j = 2k_j\xi^j$ and $\Omega_+.\xi^j = 0$ for $j = 1, 2, \ldots M$ by (5.1.8) and (5.1.5). Therefore all the ξ^j are all highest weight vectors of weight $2k_j$. Let $V(\xi^j)$ denote the irreducible submodule generated by ξ^j. From (5.1.11) and (5.1.12) it is clear that $\xi^j \subset V(2k_j, v^j)$ and hence $V(\xi^j) = V(2k_j, v^j)$. Thus the decomposition (5.1.9) shows that the sum $\sum_{j=1}^M V(\xi^j)$ is direct.

(iii): The relationship (5.1.6) follows from (5.1.8) , (5.1.12), and (5.1.5).

(iv): Because the numbers $2k_1, 2k_2, \ldots, 2k_M$ exhaust all the positive even weights and the sum $\sum_{j=1}^M V(\xi^j)$ is direct, it follows from $\mathfrak{sl}_2\mathbb{C}$-representation theory that $\{\xi^j \mid j = 1, 2, \ldots M\}$ is a basis over $\mathbb{C}$ for V_2. But $\{\mathbf{e}_\alpha \mid \alpha \in S_\lambda\}$ is also a basis over $\mathbb{C}$ for V_2. Therefore $M = |S_\lambda|$. □

The next proposition shows that when S_λ is a Π-system then the span the of the highest weight vectors form an Abelian subalgebra of $\mathfrak{g}$.

Proposition 5.1.2. *Suppose S_λ is a Π-system. Then $\mathrm{span}_\mathbb{C}\{\xi^1, \xi^2, \ldots, \xi^M\}$ is an Abelian subalgebra of $\mathfrak{g}_\lambda$ and hence also an Abelian subalgebra of $\mathfrak{g}$.*

Proof. From the definition of $\mathfrak{g}_\lambda$ given in section 4.3, it follows that $\mathrm{span}_\mathbb{C}\{\Lambda_0, \Omega_+, \Omega_-\}$

$\subset \mathfrak{g}_\lambda$ and $V_2 \subset \mathfrak{g}_\lambda$. But by proposition 5.1.1, $V_2 = \text{span}_{\mathbf{C}}\{\xi^1_{k_1-1}, \xi^2_{k_2-1}, \ldots, \xi^M_{k_M-1}\}$, and hence

$$\frac{(k_l+1)!}{(2k_l)!}\Omega_+^{k_l-1}.\xi^l_{k_l-1} = \xi^l \in \mathfrak{g}_\lambda$$

for $i = 1, 2, \ldots, M$. Therefore $\text{span}_{\mathbf{C}}\{\xi^1, \xi^2, \ldots, \xi^M\} \subset \mathfrak{g}_\lambda$. The ξ^j are highest weight vectors, consequently

$$\text{span}_{\mathbf{C}}\{\xi^1, \xi^2, \ldots, \xi^M\} \subset \mathfrak{g}_\lambda^{\Omega_+} \tag{5.1.13}$$

where $\mathfrak{g}_\lambda^{\Omega_+} = \{ X \in \mathfrak{g}_\lambda \mid [\Omega_+, X] = 0 \}$. Define $V_{\lambda,n} := \{ X \in \mathfrak{g}_\lambda \mid \Lambda_0.X = nX \}$. Clearly, $V_{\lambda,2} = V_2$. Using $\mathfrak{sl}_2\mathbf{C}$-representation theory, it is not hard to show that $\dim_{\mathbf{C}} \mathfrak{g}_\lambda^{\Omega_+} = \dim_{\mathbf{C}} V_{\lambda,2}$. But $\dim_{\mathbf{C}} V_2 = |S_\lambda|$, and therefore $\dim_{\mathbf{C}} \mathfrak{g}_\lambda^{\Omega_+} = |S_\lambda|$. By proposition 5.1.1, $|S_\lambda| = M$ and hence we get from (5.1.13) that

$$\text{span}_{\mathbf{C}}\{\xi^1, \xi^2, \ldots, \xi^M\} = \mathfrak{g}_\lambda^{\Omega_+} . \tag{5.1.14}$$

Since S_λ is a base, $|S_\lambda| = \dim_{\mathbf{C}} \mathfrak{h}_\lambda$ which in turn gives, via the above result, $\dim_{\mathbf{C}} \mathfrak{g}_\lambda^{\Omega_+} = \dim_{\mathbf{C}} \mathfrak{h}_\lambda$. Applying lemma 2.1.15 of [12] then shows that

$$\dim_{\mathbf{C}} \mathfrak{g}_\lambda^{\Omega_+} = \min\{ \dim_{\mathbf{C}} \mathfrak{g}_\lambda^X \mid X \in \mathfrak{g}_\lambda \} . \tag{5.1.15}$$

We can identify $\mathfrak{g}_\lambda$ with the dual $\mathfrak{g}_\lambda^*$ using the form $(\cdot|\cdot)$, i.e.

$$\iota : \mathfrak{g}_{\lambda,0} \rightarrow \mathfrak{g}_{\lambda,0}^* \ ; \ \iota(X)(\cdot) = (X|\cdot) .$$

So if $f \in \mathfrak{g}_\lambda^*$ and we define $\mathfrak{g}_\lambda^f = \{ X \in \mathfrak{g}_\lambda \mid \text{ad}_X^*(f) = 0\}$, then

$$\mathfrak{g}_\lambda^{\iota(X)} = \mathfrak{g}_\lambda^X \quad \forall\, X \in \mathfrak{g} . \tag{5.1.16}$$

Let G_λ be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}_\lambda$. Then for $f \in \mathfrak{g}_\lambda^*$, $\mathfrak{g}_\lambda^f$ is the Lie algebra of coadjoint isotropy group $G_{\lambda,f} = \{a \in G_\lambda \mid \text{Ad}_a^*(f) = f \}$. But then (5.1.14), (5.1.15), (5.1.16) and a straightforward generalization of theorem 9.3.10 in [41] to complex Lie groups imply that $\text{span}_{\mathbf{C}}\{\xi^1, \xi^2, \ldots, \xi^M\}$ is an Abelian subalgebra. □

Define an $\mathbf{R}$-linear operator $A : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$A = \tfrac{1}{2}\text{ad}(\Omega_+) \circ \big(\text{ad}(\Omega_-) + \text{ad}(\Omega_+) \circ c \big) . \tag{5.1.17}$$

Proposition 5.1.3. *The* $\mathbf{R}$*-linear operator* A *is symmetric with respect to the inner product* $\langle\langle\ |\ \rangle\rangle$, *i.e.* $\langle\langle A(X)|Y \rangle\rangle = \langle\langle X|A(Y) \rangle\rangle \quad \forall\, X, Y \in \mathfrak{g}$.

Proof. From (5.1.1) and the invariance properties (2.1.2) of the inner product $\langle\langle\ |\ \rangle\rangle$, it follows that $\langle\langle [\Omega_+, [\Omega_-, X]]|Y \rangle\rangle = \langle\langle X|[\Omega_+, [\Omega_-, Y]] \rangle\rangle$ and $\langle\langle [\Omega_+, [\Omega_+, c(X)]]|Y \rangle\rangle = \langle\langle X|[\Omega_+, [\Omega_+, c(Y)]] \rangle\rangle$ for every $X, Y \in \mathfrak{g}$. From the definition of A, it is then obvious that $\langle\langle A(X)|Y \rangle\rangle = \langle\langle X|A(Y) \rangle\rangle$ for every $X, Y \in \mathfrak{g}$. □

An immediate consequence of this proposition is that A is diagonalizable. The next lemma shows that V_2 is an invariant subspace for A and hence A restricts to a diagonalizable operator on V_2. We denote this operator by

$$A_2 := A|_{V_2} . \tag{5.1.18}$$

As we shall see, the diagonalizability of A_2 is essential in proving local existence.

Lemma 5.1.4.

$$A(V_2) \subset V_2 \tag{5.1.19}$$

Proof. It follows from $\mathfrak{sl}_2\mathbb{C}$-representation theory that $\Omega_\pm.V_n \subset V_{n\pm 2}$. From (5.1.8) it is clear that $c(V_n) \subset V_n$. Thus $\Omega_+.\Omega_-.V_2 \subset V_2$ and $\Omega_+.\Omega_+.c(V_2) \subset V_2$ which implies that $A(V_2) \subset V_2$. □

We label the integers k_j from proposition 5.1 1 as follows

$$1 = k_{J_1} = k_{J_1+1} = \cdots = k_{J_1+m_1-1} < k_{J_2} = k_{J_2+1} = \cdots = k_{J_2+m_2-1}$$
$$< \cdots < k_{J_I} = k_{J_I+1} = \cdots = k_{J_I+m_I-1}\,,$$

where $J_1 = 1$, $J_l + m_l = J_{l+1}$ for $l = 1, 2, \ldots, I$ and $J_{I+1} = M - 1$. Define

$$\mathsf{k}_l := k_{J_l} \quad l = 1, 2, \ldots, I\,. \tag{5.1.20}$$

The set $\{\xi^j_{k_j-1} \mid j = 1, 2, \ldots M\}$ forms a basis over $\mathbb{C}$ of V_2 by proposition 5.1.1 (iv). Therefore the set of vectors $\{X^l_s, Y^l_s \mid l = 1, 2, \ldots, I\,;\ s = 0, 1, \ldots, m_l - 1\}$ where

$$X^l_s := \begin{cases} \xi^{J_l+s}_{\mathsf{k}_l-1} & \text{if } \mathsf{k}_l \text{ is odd} \\ i\xi^{J_l+s}_{\mathsf{k}_l-1} & \text{if } \mathsf{k}_l \text{ is even} \end{cases} \quad \text{and} \quad Y^l_s := iX^l_s\,, \tag{5.1.21}$$

forms a basis of V_2 over $\mathbb{R}$. The next lemma shows that

$$\{X^l_s, Y^l_s \mid l = 1, 2, \ldots, I\ s = 0, 1, \ldots, m_l - 1\}$$

is an eigenbasis of A_2.

Lemma 5.1.5.

$$A_2(X^l_s) = \mathsf{k}_l(\mathsf{k}_l + 1)X^l_s \quad \textit{and} \quad A_2(Y^l_s) = 0 \tag{5.1.22}$$

for $l = 1, 2, \ldots, I$ *and* $s = 0, 1, \ldots, m_l - 1$.

Proof. Calculations using formulas (5.1.5) and proposition 5.1.1 (iii) show that $A(\xi^j_{k_j-1}) = \frac{1}{2}k_j(k_j+1)\left(1 + (-1)^{k_j-1}\right)\xi^j_{k_j-1}$ and $A(i\xi^j_{k_j-1}) = \frac{1}{2}k_j(k_j+1)\left(1 + (-1)^{k_j}\right)i\xi^j_{k_j-1}$ for $j = 1, 2, \ldots M$. The proposition then follows from (5.1.20) and 5.1.21. □

An immediate consequence of this lemma is that $\operatorname{spec}(A_2) = \{0\} \cup \{\mathsf{k}_j(\mathsf{k}_j+1) \mid j = 1, 2, \ldots I\}$ and m_j is the dimension of the eigenspace corresponding to the eigenvalue $\mathsf{k}_j(\mathsf{k}_j + 1)$. Note that I is the number of distinct positive eigenvalues of A_2 while M is the total number of positive eigenvalues including multiplicities. Define

$$\mathcal{E} := \{\mathsf{k}_1, \mathsf{k}_2, \ldots \mathsf{k}_I\} \tag{5.1.23}$$

so that we can write

$$\operatorname{spec}(A_2) = \{0\} \cup \{s(s+1) \mid s \in \mathcal{E}\}. \tag{5.1.24}$$

When Λ_0 is principal, the set $\mathcal{E}$ can be computed for the simple Lie algebras. Refer to table 5.1 for a list.

Define

$$E^l_0 = \operatorname{span}_{\mathbb{R}}\{Y^l_s\}_{s=0}^{m_l-1}\,,\ E^l_+ = \operatorname{span}_{\mathbb{R}}\{X^l_s\}_{s=0}^{m_l-1}\,, \tag{5.1.25}$$

Lie algebra	$\mathcal{E}$
A_ℓ	j
B_ℓ	$2j-1$
C_ℓ	$2j-1$
D_ℓ	$\begin{cases} 2j-1 & \text{if } j \leq (\ell+2)/2 \\ \ell-1 & \text{if } j = (\ell+2)/2 \\ 2j-3 & \text{if } j > (\ell+2)/2 \end{cases}$
E_6	$1,4,5,7,8,11$
E_7	$1,5,7,9,11,13,17$
E_8	$1,7,11,13,17,19,23,29$
F_4	$1,5,7,11$
G_2	$1,5$

Table 5.1: The set $\mathcal{E}$ for the simple Lie algebras with Λ_0 principal.

and

$$E_0 = \bigoplus_{l=1}^{l} E_0^l \,, \qquad E_+ = \bigoplus_{l=1}^{l} E_+^l \,. \tag{5.1.26}$$

Then

$$E_0 = \ker(A_2) \tag{5.1.27}$$

and E_+^l is the eigenspace of A_2 corresponding to the eigenvalue $k_l(k_l+1)$. Moreover, using proposition 5.1.1 (iv), it is clear that

$$V_2 = E_0 \oplus E_+ \,. \tag{5.1.28}$$

To simplify notation in what follows, we introduce

$$E^l := \bigoplus_{q=0}^{l} E_0^q \oplus E_+^q \,. \tag{5.1.29}$$

When S_λ is a Π-system, subspace E_+ can be described much more simply. Observe that if S_λ is a Π-system then Ω_+ can always be chosen so that $\Omega_+ \in \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$ by (4.3.11). Once this is done we then have:

Proposition 5.1.6. *If $\Omega_+ \in \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$ and S_λ is a Π-system, then $E_+ = \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$.*

Proof. As discussed in section 4.3, we can restrict to the subalgebra $\mathfrak{g}_\lambda$ in which Λ_0 is principal. So without loss of generality we can assume that Λ_0 is principal.

Introduce a basis $\{ Z_j \mid 1 \leq j \leq M \}$ over $\mathbf{R}$ for E_+ by defining

$$Z_j = \begin{cases} \xi^j_{k_j-1} & \text{if } k_j \text{ is odd} \\ i\,\xi^j_{k_j-1} & \text{if } k_j \text{ is even} \end{cases} \qquad 1 \leq j \leq M \,.$$

Equations (5.1.5) and proposition 5.1.1 (iii) can be used to show that

$$\Omega_+ . c(Z_j) = \Omega_- . Z_j \quad 1 \leq j \leq M \,. \tag{5.1.30}$$

By assumption $\Omega_+ = \sum_{\alpha\in S_\lambda} w_\alpha \mathbf{e}_\alpha$ for some set of constants $w_\alpha \in \mathbf{R}$. Note that 4.3.10 implies that $w_\alpha \neq 0$ for all $\alpha \in S_\lambda$ because otherwise $\lambda_\alpha = 0$ for some $\alpha \in S_\lambda$ in the expansion 4.3.10. But this is impossible since we know that $\lambda_\alpha > 0$. Because $c(\Omega_+) = -\Omega_-$ and $c(\mathbf{e}_\alpha) = -\mathbf{e}_{-\alpha}$, $\Omega_- = \sum_{\alpha\in S_\lambda} w_\alpha \mathbf{e}_{-\alpha}$. Since $Z_j \in V_2$, $Z_j = \sum_{\alpha\in S_\lambda} a_{j\alpha}\mathbf{e}_\alpha$ for some set of constants $a_{j\alpha} \in \mathbf{C}$. So then $c(Z_j) = -\sum_{\alpha\in S_\lambda} \bar{a}_{j\alpha}\mathbf{e}_{-\alpha}$. Now, since Λ_0 is principal, $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0$ for all $\alpha,\beta \in S_\lambda$, $\alpha \neq \beta$. Therefore

$$\Omega_-.\, Z_j = \sum_{\alpha\in S_\lambda} w_\alpha a_{j\alpha}[\mathbf{e}_{-\alpha}, \mathbf{e}_\alpha] = \sum_{\alpha\in S_\lambda} -w_\alpha a_{j\alpha}\mathbf{h}_\alpha \,, \tag{5.1.31}$$

while

$$\Omega_+.\, c(Z_j) = \sum_{\alpha\in S_\lambda} -w_\alpha \bar{a}_{j\alpha}[\mathbf{e}_\alpha, \mathbf{e}_{-\alpha}] = \sum_{\alpha\in S_\lambda} -w_\alpha \bar{a}_{j\alpha}\mathbf{h}_\alpha \,. \tag{5.1.32}$$

The three results (5.1.30), (5.1.31), and (5.1.32) then yield

$$\sum_{\alpha\in S_\lambda} w_\alpha (a_{j\alpha} - \bar{a}_{j\alpha})\mathbf{h}_\alpha = 0 \,.$$

But $w_\alpha \neq 0$ for all $\alpha \in S_\lambda$ and the set $\{\mathbf{h}_\alpha \,|\, \alpha \in S_\lambda\}$ is linearly independent. Thus $a_{j\alpha} - \bar{a}_{j\alpha} = 0$ for all $\alpha \in S_\lambda$ and $j = 1, 2, \ldots, M$. So $Z_j \in \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$ for $j = 1, 2, \ldots, M$ which implies that $E_+ \subset \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$. However, $\dim_{\mathbf{R}} E_+ = \dim_{\mathbf{R}}(\sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha) = |S_\lambda|$ and therefore $E_+ = \sum_{\alpha\in S_\lambda} \mathbf{R}\mathbf{e}_\alpha$. □

The operator A_2 also has a simple description when S_λ is a Π-system. Indeed, writing $\Omega_+ = \sum_{\alpha\in S_\lambda} w_\alpha \mathbf{e}_\alpha$ where $w_\alpha \in \mathbf{R}$ and using (2.1.9), (2.1.10)-(2.1.12), and $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0$ for all $\alpha,\beta \in S_\lambda$, $\alpha \neq \beta$, we get

$$A_2(\mathbf{e}_\alpha) = \sum_{\beta\in S_\lambda} w_\beta \langle \beta, \alpha \rangle w_\alpha \mathbf{e}_\beta \,.$$

This result along with (5.1.28) and proposition 5.1.6 shows that $\{\mathbf{e}_\alpha \,|\, \alpha \in S_\lambda\}$ can be completed to a basis over $\mathbf{R}$ of V_2 so that the matrix of A_2 with respect to this basis takes the form

$$[A_2] = \begin{pmatrix} 0 & 0 \\ 0 & [A_{\alpha\beta}] \end{pmatrix} , \tag{5.1.33}$$

with

$$A_{\alpha\beta} = w_\alpha \langle \alpha, \beta \rangle w_\beta \,. \tag{5.1.34}$$

Lemma 5.1.7. *Suppose $X \in V_2$. Then $X \in E^l$ if and only if $\Omega_+^{k_l}.\, X = 0$.*

Proof. From the formulas (5.1.5), we get

$$\Omega_+^{q-1}.\, \xi^l_{k_l-1} = \begin{cases} 0 & \text{if } q > k_l \\ d(q, k_l)\xi^l_{k_l-q} & \text{if } q \leq k_l \end{cases} \,.$$

where $d(q,r) = (q+r)!/(r+1)!$. This implies that

$$\Omega_+^{l-1}.\, X^q_p = \begin{cases} 0 & \text{if } l > \mathrm{k}_q \\ \beta_q\, d(l, \mathrm{k}_q)\, \xi^{J_q+p}_{\mathrm{k}_q-l} & \text{if } l \leq \mathrm{k}_q \end{cases} \,, \tag{5.1.35}$$

and

$$\Omega_+^{l-1}.\, Y^q_p = \begin{cases} 0 & \text{if } l > \mathrm{k}_q \\ \bar{\beta}_q\, d(l, \mathrm{k}_q)\, i\, \xi^{J_q+p}_{\mathrm{k}_q-l} & \text{if } l \leq \mathrm{k}_q \end{cases} \,, \tag{5.1.36}$$

where

$$\beta_q = \begin{cases} 1 & \text{if } \mathsf{k}_q \text{ is odd} \\ i & \text{if } \mathsf{k}_q \text{ is even} \end{cases} .$$

Suppose $X \in V_2 = \bigoplus_{q=1}^{I} E_0^q \oplus E_+^q$. Then there exists real constants a_{qp} and b_{qp} such that

$$X = \sum_{q=1}^{I} \sum_{p=0}^{m_q-1} \left(a_{qp} Y_p^q + b_{qp} X_p^q \right) . \tag{5.1.37}$$

Suppose $\Omega_+^{\mathsf{k}_l}.X = 0$. Then (5.1.35), (5.1.36), and (5.1.37) imply that

$$\Omega_+^{\mathsf{k}_l}.X = \sum_{q=l+1}^{I} \sum_{p=0}^{m_q-1} \left(a_{qp} \overline{\beta}_q d(\mathsf{k}_l+1,\mathsf{k}_q)\, i\, \xi^{J_q+p}_{\mathsf{k}_q-\mathsf{k}_l-1} + b_{qp} \beta_q d(\mathsf{k}_l+1,\mathsf{k}_q)\, \xi^{J_q+p}_{\mathsf{k}_q-\mathsf{k}_l-1} \right) = 0 .$$

But the set of vectors

$$\left\{ \overline{\beta}_q\, i\, \xi^{J_q+p}_{\mathsf{k}_q-\mathsf{k}_l-1}, \beta_q\, \xi^{J_q+p}_{\mathsf{k}_q-\mathsf{k}_l-1} \mid q = l+1, l+2, \ldots I\ ,\ p = 0, 1, \ldots, m_q - 1 \right\}$$

is linearly independent over $\mathbf{R}$. Therefore $X = \sum_{q=1}^{l} \sum_{p=0}^{m_q+1} (a_{qp} Y_p^q + b_{qp} X_p^q)$ which implies that $X \in \bigoplus_{q=1}^{l} E_0^q \oplus E_+^q$.

Conversely, suppose $X \in \bigoplus_{q=1}^{l} E_0^q \oplus E_+^q$. Then X can be written in the form (5.1.37) and it is easy using (5.1.35) and (5.1.36) to verify that $\Omega_+^{\mathsf{k}_l}.X = 0$. □

Lemma 5.1.8. *Suppose $X \in V_2$. Then $X \in E^l$ if and only if $\Omega_+^{\mathsf{k}_l+2}.c(X) = 0$.*

Proof. Proved in a similar fashion as lemma 5.1.7. □

Lemma 5.1.9. *Let $\tilde{\ }: \mathbb{Z}_{\geq -1} \to \{1, 2, \ldots, I\}$ be the map defined by*

$$\widetilde{-1} = \tilde{0} = 1 \textit{ and } \tilde{s} = \max\{\, l \mid \mathsf{k}_l \leq s \,\} \textit{ if } s > 0.$$

Then

(i) $\mathsf{k}_{\tilde{s}} \leq s$ *for every* $s \in \mathbb{Z}_{\geq 0}$.

(ii) $\mathsf{k}_{\tilde{s}} \leq s < \mathsf{k}_{\tilde{s}+1}$ *for every* $s \in \{0, 1, \ldots \mathsf{k}_I - 1\}$.

Proof. (i) This is obvious from the definition of $\tilde{\ }$.

(ii) From part (i), $\mathsf{k}_{\tilde{s}} \leq s$. So suppose $\mathsf{k}_{\tilde{s}+1} \leq s$. Then from the definition of $\tilde{\ }$ it is clear $\mathsf{k}_{\tilde{s}+1} \leq \mathsf{k}_{\tilde{s}}$. But because $\mathsf{k}_1 < \mathsf{k}_2 < \cdots < \mathsf{k}_I$, it follows that $\tilde{s} + 1 \leq s$ which is a contradiction. Thus $\mathsf{k}_{\tilde{s}+1} > s$ and we are done. □

Lemma 5.1.10. *If $X \in V_2$, $\mathsf{k}_{\tilde{p}} + s < \mathsf{k}_{\tilde{p}+1}$ $(s \geq 0)$, and $\Omega_+^{\mathsf{k}_{\tilde{p}}+s}.X = 0$, then $\Omega_+^{\mathsf{k}_{\tilde{p}}}.X = 0$.*

Proof. Assume $s > 0$, otherwise we are done. Because $X \in V_2$, we have $\Omega_+^{\mathsf{k}_{\tilde{p}}+s-1}.X \in V_{2(\mathsf{k}_{\tilde{p}}+s)}$. By assumption $\Omega_+^{\mathsf{k}_{\tilde{p}}+s}.X = 0$, so

$$\Omega_+^{\mathsf{k}_{\tilde{p}}+s-1}.X \in V_{2(\mathsf{k}_{\tilde{p}}+s)} \cap \ker(\mathrm{ad}(\Omega_+)) .$$

But, if $n \in \mathbb{Z}_{>0}$, then

$$V_{2n} \cap \ker(\mathrm{ad}(\Omega_+)) \neq \{0\} \iff n \in \{\mathsf{k}_1, \mathsf{k}_2, \ldots, \mathsf{k}_I\} ,$$

because otherwise $\mathfrak{g}$ would contain an irreducible $\mathfrak{q}$-submodule with weight $2n \in \mathbb{Z}_{>0}\backslash\{2k_1, 2k_2, \ldots, 2k_l\}$. This is impossible as the set $\{2k_1, 2k_2, \ldots, 2k_l\}$ exhausts all the positive even weights of the irreducible $\mathfrak{q}$-submodules in $\mathfrak{g}$. Therefore $\Omega_+^{k_{\tilde{p}}+s-1}.X = 0$ as $k_{\tilde{p}} < k_{\tilde{p}}+s < k_{\tilde{p}+1}$ implies that $(k_{\tilde{p}}+s)$ is not in $\{k_1, k_2, \ldots, k_l\}$. Repeat the above argument with $s' = s - 1$ to arrive at $\Omega_+^{k_{\tilde{p}}+s'-1}.X = \Omega_+^{k_{\tilde{p}}+s-2}.X = 0$. Continuing in this manner, we find $\Omega_+^{k_{\tilde{p}}}.X = 0$. □

We will frequently use the following fact

$$(l \pm 1)^{\sim} = \tilde{l} \pm 1 \quad \forall l \in \mathcal{E}\,. \tag{5.1.38}$$

Proposition 5.1.11. *If $X_a \in E^{\tilde{a}}$, $Y_b \in E^{\tilde{b}}$, and $Z_c \in E^{\tilde{c}}$ then $[[c(X_a), Y_b], Z_c] \in E^{(a+b+c)^{\sim}}$*

Proof. Suppose $X_a \in E^{\tilde{a}}$, $Y_b \in E^{\tilde{b}}$, and $Z_c \in E^{\tilde{c}}$. Then

$$\Omega_+^{k_{\tilde{a}}+2}.c(X_a) = \Omega_+^{k_{\tilde{b}}}.Y_a = \Omega_+^{k_{\tilde{c}}}.Z_c = 0 \tag{5.1.39}$$

by lemmas 5.1.7 and 5.1.8. Now,

$$\Omega_+^p.[[c(X_a), Y_b], Z_c] = \sum_{l=0}^{p}\sum_{m=0}^{l}\binom{p}{l}\binom{l}{m}W_{pabclm}$$

where

$$W_{pabclm} = [[\Omega_+^m.c(X_a), \Omega_+^{l-m}.Y_b], \Omega_+^{p-l}.Z_c]\,.$$

It then follows from (5.1.39) that $W_{pabclm} = 0$ if $m \geq k_{\tilde{a}} + 2$ or $l - m \geq k_{\tilde{b}}$ or $p - l \geq k_{\tilde{c}}$. Thus $W_{pabclm} = 0$ unless $p < l + k_{\tilde{c}} < m + k_{\tilde{b}} + k_{\tilde{c}} < k_{\tilde{a}} + k_{\tilde{b}} + k_{\tilde{c}} + 2$. But this can never be satisfied if $p = k_{\tilde{a}} + k_{\tilde{b}} + k_{\tilde{c}}$ and so we arrive at $\Omega_+^{k_{\tilde{a}}+k_{\tilde{b}}+k_{\tilde{c}}}.[[c(X_a), Y_b], Z_c] = 0$. But $k_{\tilde{a}} + k_{\tilde{b}} + k_{\tilde{c}} \leq a + b + c$ by lemma 5.1.9 and hence it follows that $\Omega_+^{a+b+c}.[[c(X_a), Y_b], Z_c] = 0$. But then lemma 5.1.10 implies that $\Omega_+^{k_{(a+b+c)^{\sim}}}.[[c(X_a), Y_b], Z_c] = 0$ and hence $[[c(X_a), Y_b], Z_c] \in E^{(a+b+c)^{\sim}}$. □

Proposition 5.1.12. *If $X_a \in E^{\tilde{a}}$ and $Y_b \in E^{\tilde{b}}$ then $[[X_a, c(Y_b)], \Omega_+]$, $[[\Omega_+, X_a], Y_b]$, $[[\Omega_-, X_a], Y_b] \in E^{(a+b)^{\sim}}$.*

Proof. This proposition can be proved using the same techniques as proposition 5.1.11. □

Lemma 5.1.13. *If $l \in \mathcal{E}$ and $Z \in E_+^{\tilde{l}} \oplus E^{\tilde{l}-1}$ then $\Omega_+^{l+1}.c(Z) = l(l+1)\Omega_+^{l-1}.Z$.*

Proof. Since $Z \in E_+^{\tilde{l}} \oplus E^{\tilde{l}-1}$ there exist real constants a_q^s, b_q^s such that

$$Z = \sum_{q=1}^{\tilde{l}-1}\sum_{s=0}^{m_q-1}(a_q^s + ib_q^s)X_s^q + \sum_{s=0}^{m_{\tilde{l}}-1}a_{\tilde{l}}^s X_s^{\tilde{l}} \tag{5.1.40}$$

where

$$X_s^q = \begin{cases} \xi_{k_q-1}^{J_q+s} & \text{if } k_q \text{ is odd} \\ i\xi_{k_q-1}^{J_q+s} & \text{if } k_q \text{ is even} \end{cases}.$$

But

$$\Omega_+^{l-1}.X_s^q = 0 \quad \text{for } q \leq \tilde{l} - 1 \tag{5.1.41}$$

by lemma 5.1.7, and so we get

$$\Omega_+^{l-1}.Z = \sum_{s=0}^{m_{\bar{l}}-1} a_{\bar{l}}^s \Omega_+^{l-1}.X_s^{\bar{l}} . \tag{5.1.42}$$

Now,

$$c(X_s^q) = \begin{cases} \xi_{k_q+1}^{J_q+s} & \text{if } k_q \text{ is odd} \\ i\xi_{k_q+1}^{J_q+s} & \text{if } k_q \text{ is even} \end{cases},$$

by proposition 5.1.1 and so

$$\Omega_+^2.c(X_s^q) = k_q(k_q+1)X_s^q . \tag{5.1.43}$$

Since $l \in \mathcal{E}$ implies that $k_{\bar{l}} = l$, it follows easily form (5.1.40),(5.1.41), and (5.1.43) that

$$\Omega_+^{l+1}.c(Z) = l(l+1) \sum_{s=0}^{m_{\bar{l}}-1} a_{\bar{l}}^s \Omega_+^{l-1}.X_s^{\bar{l}} . \tag{5.1.44}$$

Comparing (5.1.42) and (5.1.44), we see that $\Omega_+^{l+1}.c(Z) = l(l+1)\Omega_+^{l-1}.Z$. □

Proposition 5.1.14. *If Z_l with $l = 0, 1, \dots k$ is a sequence of vectors such that*

$$Z_0 = \Omega_+ , \quad Z_l \in E^{\bar{l}} \quad l = 1, 2, \dots k \quad \text{and} \quad Z_l \in E_+^l \oplus E^{\bar{l}-1} \quad \text{if } l \in \mathcal{E}$$

then for any $j = 1, \dots, l$

$$\sum_{s=1}^{k-1} [[c(Z_{j-s}), Z_s], \Omega_+] \in \begin{cases} E^{\bar{k}} & \text{if } k \notin \mathcal{E} \\ E^{\bar{k}-1} & \text{if } k \in \mathcal{E} \end{cases}.$$

Proof. Suppose Z_l is as in the hypotheses of the proposition, then

$$\Omega_+^{k_{\bar{l}}+2}.c(Z_l) = \Omega_+^{k_{\bar{l}}}.Z_l = 0 . \tag{5.1.45}$$

Now,

$$-\Omega_-^p . \sum_{s=1}^{k-1} [[c(Z_{k-s}, Z_s], \Omega_+] = \sum_{s=1}^{k-1} \sum_{l=0}^{p+1} \binom{p+1}{l} W_{pkls} \tag{5.1.46}$$

where

$$W_{pkls} = [\Omega_+^l.c(Z_{k-s}), \Omega_+^{p+1-l}.Z_s] .$$

From (5.1.46) we see that $W_{pkls} = 0$ if $l \geq k_{\overline{(k-s)}}$ or $p + 1 - l \geq k_{\bar{s}}$. Thus

$$W_{pkls} = 0 \text{ unless } p + 1 < l + k_{\bar{s}} < k_{\bar{s}} + k_{\overline{(k-s)}} + 2 . \tag{5.1.47}$$

Now, suppose $p = k$. Then using the fact that $k_{\bar{s}} \leq s$ and $k_{\overline{(k-s)}} \leq k - s$, we see from (5.1.47) that $W_{pkls} = 0$ unless $k + 1 < l + k_{\bar{s}} < k + 2$. Since this is impossible to satisfy $W_{pkls} = 0$ for all l, s. Thus the sum (5.1.46) vanishes. i.e.

$$-\Omega_-^k . \sum_{s=1}^{k-1} [[c(Z_{k-s}, Z_s], \Omega_+] = 0 .$$

and we get

$$\sum_{s=1}^{k-1}[[c(Z_{k-s}, Z_s], \Omega_+] \in \bigoplus_{q=1}^{\bar{k}} E_0^q \oplus E_+^q \tag{5.1.48}$$

by lemmas 5.1.7 and 5.1.10.

Now, suppose further that $k \in \mathcal{E}$ and let $p = k - 1$. Then by (5.1.47) we have $W_{pkls} = 0$ unless $k < l + \mathsf{k}_{\bar{s}} < \mathsf{k}_{\bar{s}} + \mathsf{k}_{(k-s)^-} + 2$. Now $\mathsf{k}_{\bar{s}} \leq s$ and $\mathsf{k}_{(k-s)^-} \leq k - s$, so suppose $\mathsf{k}_{\bar{s}} < s$ or $\mathsf{k}_{(k-s)^-} < k - s$. Then $\mathsf{k}_{(k-s)^-} + \mathsf{k}_{\bar{s}} < k - s + s = k$ which will make the inequality $k < l + \mathsf{k}_{\bar{s}} < \mathsf{k}_{\bar{s}} + \mathsf{k}_{(k-s)^-} + 2$ impossible to satisfy. Therefore we see that $W_{pkls} = 0$ unless $\mathsf{k}_{(k-s)^-} = k - s$ and $\mathsf{k}_{\bar{s}} = s$ (i.e. $k - s, s \in \mathcal{E}$). However, if $\mathsf{k}_{(k-s)^-} = k - s$ and $\mathsf{k}_{\bar{s}} = s$, then $k < l + \mathsf{k}_{\bar{s}} < \mathsf{k}_{\bar{s}} + \mathsf{k}_{(k-s)^-} + 2$ will satisfied only if $l + s = 1$. So $W_{pkls} = 0$ unless $k - s, s \in \mathcal{E}$ and $l + s = 1$. This allows us to write the sum (5.1.46) as

$$\begin{aligned}
-\Omega_-^{k-1}.\sum_{s=1}^{k-1}&[[c(Z_{k-s}), Z_s], \Omega_+] \\
&= \sum_{s=1}^{k-1}\binom{k}{k-s+1}(k-s)(k-s+1)[\Omega_+^{k-s-1}.Z_{k-s}, \Omega_+^{s-1}.Z_s] \quad \text{by lemma 5.1.12} \\
&= \sum_{s=1}^{k-1}\frac{k!}{(k-s-1)!(s-1)!}[\Omega_+^{k-s-1}.Z_{k-s}, \Omega_+^{s-1}.Z_s]\,.
\end{aligned}$$

Assume now that k is odd. Then we can write the above sum as

$$\begin{aligned}
-\Omega_-^{k-1}.\sum_{s=1}^{k-1}[[c(Z_{k-s}), Z_s], \Omega_+] &= \sum_{s=\frac{k-1}{2}-1}^{k-1}\frac{k!}{(k-s-1)!(s-1)!}[\Omega_+^{k-s-1}.Z_{k-s}, \Omega_+^{s-1}.Z_s] \\
&\quad + \sum_{s=1}^{\frac{k-1}{2}}\frac{k!}{(k-s-1)!(s-1)!}[\Omega_+^{k-s-1}.Z_{k-s}, \Omega_+^{s-1}.Z_s] \\
= \sum_{s=1}^{\frac{k-1}{2}}\frac{k!}{(k-s-1)!(s-1)!}&\left\{[\Omega_+^{k-s-1}.Z_{k-s}, \Omega_+^{s-1}.Z_s] + [\Omega_+^{s-1}.Z_s, \Omega_+^{k-s-1}.Z_{k-s}]\right\} \\
&= 0\,.
\end{aligned}$$

Similar arguments show that $-\Omega_-^{k-1}.\sum_{s=1}^{k-1}[[c(Z_{k-s}, Z_s], \Omega_+] = 0$ if k is even. We then get

$$\sum_{s=1}^{k-1}[[c(Z_{j-s}), Z_s], \Omega_+] \in E^{\bar{k}-1}$$

by lemmas 5.1.7 and 5.1.10 and (5.1.38). □

Proposition 5.1.15. *If Z_l with $l = 0, 1, \ldots k$ is a sequence of vectors such that*

$$Z_0 = \Omega_+\,, \quad Z_l \in E^{\bar{l}} \quad l = 1, 2, \ldots k \quad \text{and} \quad Z_l \in E_+^l \oplus E^{\bar{l}-1} \quad \text{if } l \in \mathcal{E}$$

then for any $j = 1, \ldots, l$

$$\sum_{j=1}^{k-1}\sum_{s=0}^{j}[[c(Z_{j-s}), Z_s], Z_{k-j}] \in \begin{cases} E^{\bar{k}} & \text{if } k \notin \mathcal{E} \\ E^{\bar{k}-1} & \text{if } k \in \mathcal{E} \end{cases}.$$

Proof. Proved using similar arguments as for proposition 5.1.14. □

5.2 Local existence proofs

For $q = 1, 2, \ldots I$ let

$$\mathrm{pr}^q_+ : V_2 \to E^q_+ , \quad \mathrm{pr}^q_0 : V_2 \to E^q_0 , \quad \text{and} \quad \mathrm{pr}^q : V_2 \to E^q_0 \oplus E^q_+$$

denote the projections determined by the decomposition (5.1.28), (5.1.26) of V_2.

5.2.1 Solutions bounded at the origin

Theorem 5.2.1. *Fix $X \in E_+$ and $\Omega_+ \in E_+$ that satisfies $[\Omega_+, \Omega_-] = \Lambda_0$ where $\Omega_- := -c(\Omega_+)$. Then for some $\epsilon > 0$ there exist a unique $C^2 \times C^1$-solution $\{\Lambda_+(r, X), m(r, X)\}$ to the system of differential equations* (4.1.7) *and* (4.1.9) *defined on* $[0, \epsilon)$ *that satisfies $\Lambda_+(0) = \Omega_+$ and*

$$pr^s_+(\Lambda_+ - \Omega_+) = X_s r^{s+1} + O(r^{s+2}) , \quad pr^s_0(\Lambda_+) = O(r^{s+2}) \quad \forall s \in \mathcal{E},$$
$$pr^s_+(\Lambda_+ - \Omega_+) = X_s r^{s+1} + O(r^{s+2}) , \quad pr^s_0(\Lambda_+) = O(r^{s+2}) \quad \forall s \in \mathcal{E},$$

a

^{a}*where $X_s := pr^s_+(X)$. Moreover, these solutions are defined and analytic on $(-\epsilon, \epsilon)$, $G = O(r^\ell)$ as $r \to 0$.*

Before we proceed with the proof of this theorem we will first present an example of its consequences. Suppose $\mathfrak{g} = E_8$ and Λ_0 is principal. From table 5.1 we see that $\mathcal{E} = \{1, 7, 11, 13, 17, 19, 23, 29\}$. Since $\dim_{\mathbb{R}} E_+ = |\mathcal{E}|$, it follows from theorem 5.2.1 that 8 parameters will have to be chosen to single out a unique solutions to (4.1.7) and (4.1.9) in a neighborhood of $r = 0$. Moreover, the last parameter does not show up in the Taylor expansion of a solution until the 30^{th} power in r. This illustrates the highly singular nature of the equations (4.1.7) and (4.1.9) at $r = 0$

Proof of theorem 5.2.1. Introduce new variables $\{ u^+_s, u^0_s \mid s \in \mathcal{E} \}$ via

$$u^+_s := \mathrm{pr}^s_+(\Lambda_+ - \Omega_+) r^{-s-1} \quad \text{and} \quad u^0_s := \mathrm{pr}^s_0(\Lambda_+) r^{-s-2} \tag{5.2.1}$$

where $\Omega_+ = \Lambda_+(0)$. This allows us to write Λ_+ as

$$\Lambda_+(r) = \Omega_+ + \sum_{s \in \mathcal{E}} (u^+_s(r) + r u^0_s(r)) r^{s+1} . \tag{5.2.2}$$

Lemma 5.2.2. *For every $s \in \mathcal{E}$ there exists analytic maps*

$$\mathcal{F}^1_s : E_+ \longrightarrow E^s_0 \oplus E^s_+ \quad \text{and} \quad \mathcal{F}^2_s : E_0 \times E_+ \times \mathbb{R} \longrightarrow E^s_0 \oplus E^s_+$$

such that, for $\mathcal{F}$ given in (4.1.14),

$$\mathrm{pr}^s \mathcal{F} = -s(s+1) u^+_s r^{s+1} + r^{s+2} \mathcal{F}^1_s(u^+) + r^{s+3} \mathcal{F}^2_s(u^0, u^+, r)$$

where

$$u^0 := \sum_{a \in \mathcal{E}} u^0_a \quad \text{and} \quad u^+ := \sum_{a \in \mathcal{E}} u^+_a . \tag{5.2.3}$$

Proof. Let $u_s = ru_s^0 + u_s^+$. Then from (4.1.14) we find

$$\mathcal{F} = \sum_{a\in\mathcal{E}} A_2(u_a) r^{a+1} + \tfrac{1}{2} \sum_{a,b\in\mathcal{E}} \left([[u_a, c(u_b)], \Omega_+] + [[\Omega_+, c(u_a)], u_b] + [[\Omega_-, u_a], u_b]\right) r^{a+b+2}$$
$$+\tfrac{1}{2} \sum_{a,b,c\in\mathcal{E}} [[u_a, c(u_b)], u_c] r^{a+b+c+3} .$$

But

$$A_2(u_a) = a(a+1)u_a^+ . \tag{5.2.4}$$

by lemma 5.1.5. Also, $k_{\bar a} = a$ for $a \in \mathcal{E}$ by lemma 5.1.9. So

$$\bar a \le \bar b \Longleftrightarrow a \le b . \tag{5.2.5}$$

Using (5.2.4) and (5.2.5), we get

$$\mathrm{pr}^{\bar s}\mathcal{F} = -s(s+1)u_0^+ r^{s+1} + \frac{1}{2} \sum_{\substack{a,b\in\mathcal{E}\\ a+b\ge s}} \mathrm{pr}^{\bar s}\Big([[u_a, c(u_b)], \Omega_+] + [[\Omega_+, c(u_a)], u_b]$$
$$+[[\Omega_-, u_a], u_b]\Big) r^{a+b+2} + \frac{1}{2} \sum_{\substack{a,b,c\in\mathcal{E}\\ a+b+c\ge s}} \mathrm{pr}^{\bar s}\Big([[u_a, c(u_b)], u_c]\Big) r^{a+b+c+3}$$

by propositions 5.1.11 and 5.1.12. Substituting $u_a = ru_a^0 + u_a^+$ into the above expression completes the proof. □

For every $s \in \mathcal{E}$ define

$$v_s^+ := u_s^{+\prime} \quad \text{and} \quad v_s^0 := (ru_s^0)' . \tag{5.2.6}$$

Lemma 5.2.3. *There exists analytic functions*

$$\hat P : E_0 \times E_+ \times \mathbb{R} \longrightarrow \mathbb{R} \quad \text{and} \quad \hat G : E_0 \times E_0 \times E_+ \times E_+ \times \mathbb{R} \longrightarrow \mathbb{R}$$

such that

$$P = r^4 \left\|u_1^+\right\|^2 + r^5 \hat P(u^0, u^+, r) \quad \text{and} \quad G = r^2 \left\|u_1^+\right\|^2 + r^3 \hat G(u^0, v^0, u^+, v^+, r)$$

where

$$v^0 := \sum_{a\in\mathcal{E}} v_a^0 \quad \text{and} \quad v^+ := \sum_{a\in\mathcal{E}} v_a^+ . \tag{5.2.7}$$

and u^0, u^+ *are defined by* (5.2.3).

Proof. The existence of the analytic function $\hat G$ follows easily from the definition (4.1.12) of G and equations (5.2.3) and (5.2.7). From the definition (4.1.12) of P it follows that

$$P = \frac{r^4}{8} \left\|[\Omega_+, c(u_1^+)] + [\Omega_-, u_1^+]\right\| + r^5 Q(u^0, u^+, r)$$

where Q is a polynomial in r, u^0 and u^+. Using (2.1.2), (5.1.17), (5.1.18), and $A_2(u_1^+) = 2u_1^+$, we get

$$\frac{1}{8} \left\|[\Omega_+, c(u_1^+)] + [\Omega_-, u_1^+]\right\| = \left\|u_1^+\right\|$$

and this completes the proof. □

From (5.1.25)-(5.1.27) we have

$$A_2(u_s^+) = s(s+1)u_s^+ \quad \text{and} \quad A_2(u_s^0) = 0$$

since $u_s^+ \in E_+^s$ and $u_s^0 \in E_0^s$. Using this result, lemma 5.2.2, and equations (5.2.3) and (5.2.7), the field equations (4.1.7) and (4.1.9) can be written as

$$ru_s^{+\prime} = rv_s^+ \,, \tag{5.2.8}$$

$$\begin{aligned} rv_s^{+\prime} &= -2(s+1)v_s^+ - \frac{2}{rN}\left(m - \frac{1}{r}P\right)v_s^+ - \frac{s(s+1)}{r}\left(\frac{1}{N} - 1\right)u_s^+ \\ &\quad - \frac{2(s+1)}{r^2N}\left(m - \frac{1}{r}P\right)u_s^+ - \frac{r}{N}\mathrm{pr}_+^s\mathcal{F}_s^2(u^0, u^+, r) \\ &\quad - \left(\frac{1}{N} - 1\right)\mathrm{pr}_+^s\mathcal{F}_s^1(u^+) - \mathrm{pr}_+^s\mathcal{F}_s^1(u^+)\,, \end{aligned} \tag{5.2.9}$$

$$ru_s^{0\prime} = -u_s^0 + v_s^0\,, \tag{5.2.10}$$

$$\begin{aligned} rv_s^{0\prime} &= -2(s+1)v_s^0 - s(s+1)u_s^0 - \frac{2}{N}\left(m - \frac{1}{r}P\right)v_s^0 \\ &\quad - \frac{2(s+1)}{rN}\left(m - \frac{1}{r}P\right)u_s^0 - \frac{r}{N}\mathrm{pr}_0^s\mathcal{F}_s^2(u^0, u^+, r) \\ &\quad - \left(\frac{1}{N} - 1\right)\mathrm{pr}_0^s\mathcal{F}_s^1(u^+) - \mathrm{pr}_0^s\mathcal{F}_s^1(u^+)\,, \end{aligned} \tag{5.2.11}$$

(5.2.12)

where $s \in \mathcal{E}$. For every $s \in \mathcal{E}$, introduce two new variables

$$x_s := -(s+1)u_s^0 - \frac{s+1}{s}v_s^0\,, \quad y_s := (s+1)u_s^0 + v_s^0\,,$$

and define

$$\begin{aligned} f_s := &-\frac{2}{N}\left(m - \frac{1}{r}P\right)v_s^0 - \frac{2(s+1)}{rN}\left(m - \frac{1}{r}P\right)u_s^0 \\ &- \frac{r}{N}\mathrm{pr}_0^s\mathcal{F}_s^2(u^0, u^+, r) - \left(\frac{1}{N} - 1\right)\mathrm{pr}_0^s\mathcal{F}_s^1(u^+)\,. \end{aligned}$$

Then equations (5.2.10) and (5.2.11) can be written as

$$rx_s' = -(s+2)x_s + \frac{s+1}{s}\mathrm{pr}_0^s\mathcal{F}_s^1(u^+) + \frac{s+1}{s}f_s, \tag{5.2.13}$$

$$ry_s' = -(s+2)y_s - \mathrm{pr}_0^s\mathcal{F}_s^1(u^+)f_s\,, \tag{5.2.14}$$

for every $s \in \mathcal{E}$. Define

$$\mu = \frac{1}{r^3}\left(m - r^3\left\|u_1^+\right\|^2\right)\,. \tag{5.2.15}$$

Then the mass equation (4.1.7) can be written as

$$r\mu' = -3\mu + r\Big\{\hat{P}(u^0,u^+,r) + \hat{G}(u^0,v^0,u^+,v^+,r) - 2\langle\langle u_1^+|v_1^+\rangle\rangle$$
$$-2r\left(\mu + \|u_1^+\|^2\right)\left(2\|u_1^+\|^2 + r\hat{G}(u^0,v^0,u^+,v^+,r)\right)\Big\}\,. \quad (5.2.16)$$

For every $s \in \mathcal{E}$, introduce one last change of variables

$$\hat{v}_s^+ := v_s^+ + \frac{1}{2(s+1)}\mathrm{pr}_0^s\mathcal{F}_s^1(u^+)\,, \quad \hat{x}_s := x_s - \frac{s+1}{s(s+2)}\mathrm{pr}_0^s\mathcal{F}_s^1(u^+)\,,$$

and

$$\hat{y}_s := y_s + \frac{1}{s+1}\mathrm{pr}_0^s\mathcal{F}_s^1(u^+)\,.$$

Define

$$\hat{v}^+ := \sum_{s\in\mathcal{E}} v_s^+\,, \quad \hat{x} := \sum_{s\in\mathcal{E}} x_s\,, \quad \hat{y} := \sum_{s\in\mathcal{E}} y_s\,,$$

and

$$\eta(r) = (\hat{x}(r), \hat{y}(r), u^+(r), \hat{v}^+(r), \mu(r), r)\,.$$

Fix $X \in E_+$ and let $\mathcal{N}_X$ be a neighborhood of X in E_+. Define a set $D(\mathcal{N}_X,\epsilon)$ by

$$D(\mathcal{N}_X,\epsilon) := E_0 \times E_0 \times \mathcal{N}_X \times E^+ \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon)\,.$$

Then using lemmas 5.2.2 and 5.2.3 and equations (5.2.8), (5.2.9), (5.2.13), (5.2.14), and (5.2.16) it is not hard to show that there exists an $\epsilon > 0$ and analytic maps

$$\mathcal{U}_s\,,\ \mathcal{V}_s : D(\mathcal{N}_X,\epsilon) \longrightarrow E_+^s\,, \quad \mathcal{X}_s\,,\ \mathcal{Y}_s : D(\mathcal{N}_X,\epsilon) \longrightarrow E_0^s \quad \forall\, s \in \mathcal{E}\,,$$

and

$$\mathcal{M} : D(\mathcal{N}_X,\epsilon) \longrightarrow \mathbf{R}$$

such that

$$ru_s^{+\prime}(r) = r\mathcal{U}_s(\eta(r)), \quad r\hat{v}_s^{+\prime}(r) = -2(s+1)\hat{v}_s^+(r) + r\mathcal{V}_s(\eta(r)),$$
$$r\hat{x}_s{}'(r) = -(s+2)\hat{x}_s(r) + r\mathcal{X}_s(\eta(r)), \quad r\hat{y}_s{}'(r) = -(s+1)\hat{x}_s(r) + r\mathcal{Y}_s(\eta(r)),$$

for every $s \in \mathcal{E}$ and

$$r\mu'(r) = -3\mu(r) + r\mathcal{M}(\eta(r))\,.$$

This system of differential equations is in the form to which theorem 5.0.6 applies. Therefore for fixed $X \in E_+$ there exist a unique C^1-solution $\{u_s^+(r,Y), \hat{v}_s^+(r,Y), \hat{x}_s(r,Y), \hat{v}_s(r,Y), \mu(r,Y)\}$ that is analytic in a neighborhood of $(r,Y) = (0,X)$ and that satisfies

$$u_s^+(r,Y) = Y_s + \mathrm{O}(r), \quad \hat{v}_s^+(r,Y) = \mathrm{O}(r), \quad \hat{x}_s(r,Y) = \mathrm{O}(r), \quad \hat{y}_s(r,Y) = \mathrm{O}(r).$$

for all $s \in \mathcal{E}$, and

$$\mu(s,Y) = \mathrm{O}(r)\,,$$

where $Y_s = \mathrm{pr}_+^s(Y)$. From these results it is easy to verify that

$$m(r) = \mathrm{O}(r^3) \quad \text{and} \quad u^0(r) = \mathrm{O}(r^0)$$

Also

$$P = O(r^4) \quad \text{and} \quad G = O(r^2)$$

by lemma 5.2.3. □

It is important to realize that the fall off conditions $m = O(r^3)$, $P = O(r^4)$ and $G = O(r^2)$ as $r \to 0$ imply that the physical quantities (4.1.21) are finite at $r = 0$. Therefore all of these local solutions satisfy our boundary conditions at $r = 0$.

Theorem 5.2.4. *Every solution from theorem 5.2.1 satisfies equation* (4.1.10) *on a neighborhood of* $r = 0$.

Proof. Let $\{\Lambda_+(r), m(r)\}$ be a solution of the equations (4.1.7) and (4.1.9) on a neighborhood $\mathcal{N}$ of $r = 0$, which we know exists by theorem 5.2.1. Let γ be defined as in lemma 5.0.7. Observe that $\Lambda'_+(0) = 0$ by theorem 5.2.1 and so $\gamma(0) = 0$. Also, because $m = O(r^3)$ and $P = O(r^4)$ for these solutions we see, by shrinking $\mathcal{N}$ if necessary, that

$$f(r) = -\frac{2}{r^2 N}\left(m - \frac{1}{r}P\right)$$

is analytic on $\mathcal{N}$. But γ satisfies the differential equation

$$\gamma' = f(r)\gamma \, .$$

Solving, we find

$$\gamma(r) = \gamma(0)e^{\int_0^r f(\tau)d\tau} \quad \forall r \in \mathcal{N} \, .$$

Therefore $\gamma(r) = 0$ for all $r \in \mathcal{N}$, since $\gamma(0) = 0$. □

5.2.2 Asymptotically flat solutions

In proving that local solutions exist near $r = 0$, we were able to "guess" the appropriate transformations needed to bring the field equations (4.1.7) and (4.1.9) in to a form for which theorem 5.0.6 applies. Near, $r = \infty$ the equations become more difficult to analyze and guessing the appropriate transformation is no longer possible. Instead, we will show that the field equations (4.1.7) and (4.1.9) admit a formal power series solutions about the point $r = \infty$. This formal power series solution will then be used to construct a transformation to bring the equations (4.1.7) and (4.1.9) in to a form for which theorem 5.0.6 applies.

Let $z = \frac{1}{r}$, and define

$$\mathring{f} = \frac{df}{dz}$$

for any function f. Then (4.1.7) and (4.1.9) can be written as

$$z^2\mathring{m} + NG + z^2 P = 0 \, , \tag{5.2.17}$$

$$z^2 N\mathring{\mathring{\Lambda}}_+ + 2z(1 - 3zm + z^2P)\mathring{\Lambda}_+ + \mathcal{F} = 0 \, . \tag{5.2.18}$$

Assume a powerseries expansion of the form

$$\Lambda_+ = \sum_{k=0}^{\infty} \Lambda_{+,k} z^k \quad \text{and} \quad m = \sum_{k=0}^{\infty} m_k z^k \, . \tag{5.2.19}$$

We will define

$$\Lambda_{-,k} := -c(\Lambda_{+,k}) \quad \text{and} \quad \Omega_\pm := \Lambda_{\pm,0} \, . \tag{5.2.20}$$

We assume that $[\Omega_+, \Omega_-] = \Lambda_0$. As we discussed in section 4.2, $[\Omega_+, \Omega_-] = \Lambda_0$ is a consequence of the vanishing of the total magnetic charge. Substituting the powerseries (5.2.19) in the equations (5.2.17) and (5.2.18) yields the recurrence equations

$$m_1 = m_2 = 0\,, \quad m_k = \frac{1}{k}\left(-G_{k-3} + 2\sum_{j=0}^{k-4} m_j G_{k-j-4} - P_{k-1}\right) \quad k = 3, 4, 5, \ldots \tag{5.2.21}$$

$$A_2(\Lambda_{+,k}) - k(k+1)\Lambda_{+,k} = h_k + f_k \quad k = 1, 2, 3, \ldots \tag{5.2.22}$$

where

$$G_k := \frac{1}{2}\sum_{j=0}^{k}(j+1)(k+1-j)\langle\langle \Lambda_{+,k+1-j}|\Lambda_{+,j+1}\rangle\rangle \quad k \geq 0\,, \tag{5.2.23}$$

$$\hat{F}_0 := 0\,, \quad \hat{F}_k := \frac{1}{2}\sum_{j=1}^{k}\sum_{s=0}^{j}[[\Lambda_{-,j-s}, \Lambda_{+,s}], \Lambda_{+,k-j}] \quad k \geq 1\,, \tag{5.2.24}$$

$$P_0 = P_1 = 0, \quad P_k := \frac{1}{2}\sum_{j=1}^{k-1}\langle\langle \hat{F}_j|\hat{F}_{k-j}\rangle\rangle \quad k \geq 2\,, \tag{5.2.25}$$

$$h_1 = 0, \quad h_k := 2\sum_{j=0}^{k-1}(k-j-1)\left(P_{j-2} - (k-j+1)m_j\right)\Lambda_{+,k-j-1} \quad k \geq 2\,, \tag{5.2.26}$$

and

$$f_1 = 0, \quad f_k := \frac{1}{2}\left\{\sum_{j=1}^{k-1}\sum_{s=0}^{j}[[\Lambda_{-,j-s}, \Lambda_{+,s}], \Lambda_{+,k-j}] + \sum_{s=1}^{k-1}[[\Lambda_{-,k-s}, \Lambda_{+}s], \Omega_+]\right\} \quad k \geq 2\,. \tag{5.2.27}$$

Note that with these definitions $\hat{F} = \sum_{k=0}^{\infty}\hat{F}_k z^k$ and $P = \sum_{k=0}^{\infty} P_k z^k$ while $G = \sum_{k=0}^{\infty} G_k z^{k+4}$.

Theorem 5.2.5. *Fix $X \in E_+$ and $m_\infty \in \mathbb{R}$. Then there exists a unique solution $\{\Lambda_{+,k}, m_k\}_{k=0}^{\infty}$ to the recurrence equations (5.2.21) and (5.2.22) that satisfies*

$$m_0 = m_\infty\,, \quad m_1 = m_2 = 0$$
$$pr_+^{\bar{k}}\Lambda_{+,k} = X_k \quad \forall\, k \in \mathcal{E}$$

and

$$\Lambda_{+,k} \in \begin{cases} E^{\bar{k}} & \text{if } k \notin \mathcal{E} \\ E_+^{\bar{k}} \oplus E^{\bar{k}-1} & \text{if } k \in \mathcal{E} \end{cases},$$

where $X_k = pr_+^{\bar{k}} X$.

Proof. Fix $X \in E_+$, $m_\infty \in \mathbf{R}$, and let

$$X_k = \mathrm{pr}_+^{\bar{k}} X \quad \forall\, k \in \mathcal{E}\,.$$

We will use induction to prove that the recurrence equations (5.2.21) and (5.2.22) can be solved. When $k = 1$, the equations (5.2.21) and (5.2.22) reduce to

$$m_1 = 0 \quad \text{and} \quad A_2(\Lambda_{+,1}) - 2\Lambda_{+,1} = 0\,.$$

This can be solved in $E_0^{\bar{1}} \oplus E_+^{\bar{1}}$ by letting $\Lambda_{+,1} = X_1$. Note that since $\min \mathcal{E} = 1$, we have $\bar{1} = 1$.

We now assume that for $k \le l$, $\{\Lambda_{+,k}, m_k\}$ is a solution to the recurrence equations (5.2.21) and (5.2.22) that satisfies

$$\Lambda_{+,k} \in \begin{cases} E^{\bar{k}} & \text{if } k \notin \mathcal{E} \\ E_+^{\bar{k}} \oplus E^{\bar{k}-1} & \text{if } k \in \mathcal{E} \end{cases}.$$

It is clear from (5.2.22) that m_{l+1} is then determined. From (5.2.23-5.2.27) and propositions 5.1.14 and 5.1.15 it follows that

$$h_{l+1} + f_{l+1} \in \begin{cases} E^{\bar{l}+1} & \text{if } l+1 \notin \mathcal{E} \\ E^{\bar{l}} & \text{if } l+1 \in \mathcal{E} \end{cases}. \tag{5.2.28}$$

Equation (5.2.22) implies that

$$(A_2 - (l+1)(l+2)\mathbb{1}_{V_2})\,\Lambda_{+,l+1} = h_{l+1} + f_{l+1}\,. \tag{5.2.29}$$

Suppose $l + 1 \notin \mathcal{E}$. Then $A_2 - (l+1)(l+2)\mathbb{1}$ is invertible and

$$\Lambda_{+,l+1} = (A_2 - (l+1)(l+2)\mathbb{1}\,)^{-1}\,(h_{l+1} + f_{l+1})\,.$$

But then (5.2.28) implies that $\Lambda_{+,l+1} \in E^{\bar{l}+1}$.

Alternatively, suppose $l + 1 \in \mathcal{E}$. Then $\ker(A_2 - (l+1)(l+2)\mathbb{1}\,) = E_+^{\bar{l}+1}$ by (5.1.25)-(5.1.27) and (5.1.38). Therefore, (5.2.28) shows that

$$\Lambda_{+,l+1} = ((A_2 - (l+1)(l+2)\mathbb{1}\,)|_{E^{\bar{l}}})^{-1}\,(h_{l+1} + f_{l+1}) + X_{l+1}$$

solves (5.2.29) since $X_{l+1} \in \ker(A_2 - (l+1)(l+2)\mathbb{1}$. It also clear from (5.2.28) and $X_{l+1} \in E_+^{\bar{l}+1}$ that $\Lambda_{+,l+1} \in E^{\bar{l}+1} \bigoplus_{q=1}^{\bar{l}} E_0^q \oplus E_+^q$. This prove that $\Lambda_{+,l+1}$ satisfies the induction hypothesis and so the proof is complete. □

Theorem 5.2.6. *Fix $X \in E_+$, $m_\infty > 0$, and $\Omega_+ \in E_+$ that satisfies $[\Omega_+, \Omega_-] = \Lambda_0$ where $\Omega_- := -c(\Omega_+)$. Then for some $\epsilon > 0$ there exist a unique $C^2 \times C^1$-solution $\{\Lambda_+(r, m_\infty, X), m(r, m_\infty, X)\}$ to the system of differential equations (4.1.7) and (4.1.9) defined on $[0, \epsilon)$ that satisfies $\Lambda_+(0) = \Omega_+$, $m = m_\infty + O(r^{-3})$ and*

$$pr_+^{\bar{s}}(\Lambda_+ - \Omega_+) = \frac{X_s}{r^s} + O(r^{-s})\,, \quad pr_0^{\bar{s}}(\Lambda_+) = O(r^{-s}) \quad \forall s \in \mathcal{E}$$

where $X_s := pr_+^{\bar{s}}(X)$. Moreover, these solutions are defined and analytic on $(-\epsilon, \epsilon)$ and depend analytically on the initial data (m_∞, X).

Comparing this theorem to theorem 5.2.1, we see that the singular behavior at

$r = \infty$ is similar to the behavior at $r = 0$ except at $r = \infty$ an extra parameter m_∞ is present. The parameter m_∞ measures the "mass" of the spacetime. Recall our example where $\mathfrak{g} = E_8$ and Λ_0 is principal so that $\mathcal{E} = \{1, 7, 11, 13, 17, 19, 23, 29\}$. Since $\dim_{\mathbb{R}} E_+ = |\mathcal{E}|$, it follows from theorem 5.2.6 that 9 parameters will have to be chosen to single out a unique solutions to (4.1.7) and (4.1.9) in a neighborhood of $r = \infty$. Furthermore, the last parameter will not show up in the Taylor expansion of a solution until the 29$^{\text{th}}$ power in $\frac{1}{r}$.

Proof of theorem 5.2.6. Fix $X \in E_+$ and let

$$\Lambda_{+,k} = \Lambda_{+,k}(X, m_\infty) \quad \text{and} \quad m_k = m_k(X, m_\infty)$$

be solutions to the recurrence equations (5.2.21) and (5.2.22) which satisfy

$$m_0 = m_\infty\,, \quad m_1 = m_2 = 0 \tag{5.2.30}$$

$$\mathrm{pr}_+^{\bar{k}} \Lambda_{+,k} = X_k \quad \forall\, k \in \mathcal{E} \tag{5.2.31}$$

and

$$\Lambda_{+,k} \in \begin{cases} \bigoplus_{q=1}^{\bar{k}} E_0^q \oplus E_+^q & \text{if } k \notin \mathcal{E} \\ E_+^{\bar{k}} \oplus \bigoplus_{q=1}^{\bar{k}-1} E_0^q \oplus E_+^q & \text{if } k \in \mathcal{E} \end{cases}, \tag{5.2.32}$$

where $X_k = \mathrm{pr}_+^{\bar{k}} X$. Define

$$U := \sum_{k=0}^{n} \Lambda_{+,k} z^k \quad \text{and} \quad M := \sum_{k=0}^{n} m_k z^k \tag{5.2.33}$$

and introduce new variables $\phi(z)$ and $\sigma(z)$ via

$$\Lambda_+ = U + z^{n-3}\phi \quad \text{and} \quad m = M + z^{n-3}\sigma\,, \tag{5.2.34}$$

where the integer n is to be chosen later. Define

$$N_p := 1 - 2Mz\,, \quad \hat{F}_p := \frac{i}{2}(\Lambda_0 + [U, c(U)])\,, \tag{5.2.35}$$

$$\mathcal{F}_p := -i[\hat{F}_p, U]\,, \quad P_p := \frac{1}{2}\|\hat{F}_p\|^2\,, \quad \text{and} \quad G_p := \frac{1}{2} z^4 \|\mathring{U}\|^2\,. \tag{5.2.36}$$

From these definitions it is clear that the quantities N_p, $\hat{F}_p$, $\mathcal{F}_p$, P_p and G_p are all polynomial in the variables X and m_∞. Now, because U and M are the first n terms in the powerseries solution to the field equations (5.2.17) and (5.2.18) about the point $z = 0$ they satisfy

$$z^2 N_p \mathring{\mathring{U}} + 2z(1 - 3zM + z^2 P_p)\mathring{U} + \mathcal{F}_p = z^{n-1}(a_1(X, m_\infty) + a_2(X, m_\infty)y)\,,$$

$$z^2 \mathring{M} + N_p G_p + z^2 P_p = z^n b(X, m_\infty)\,,$$

where $a_1, a_2 : V_2 \times \mathbf{R} \to \mathbf{R}$ and $b : V_2 \times \mathbf{R} \to \mathbf{R}$ are polynomial in their variables.

From (5.2.36) and (4.1.14) we get

$$\mathcal{F} = \mathcal{F}_p - z^{n-3}A_2(\phi) + z^{n-2}\sum_{j=1}^{3}\mathcal{F}_{R,j}(\phi, X, m_\infty, z)$$

where

$$\mathcal{F}_{R,j} : V_2 \times E_+ \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \quad j = 1,2,3$$

are analytic maps that satisfy

$$\mathcal{F}_{R,j}(\epsilon Y_1, Y_2, x_1, x_2) = \epsilon^j \mathcal{F}_{R,j}(Y_1, Y_2, x_1, x_2)$$

for all $\epsilon \in \mathbf{R}$. It is also not difficult to see from (5.2.36) and (4.1.12) that

$$G = \frac{z^4}{2}\|\mathring{U}\|^2 + z^4\langle\langle \mathring{U}|(n-3)z^{n-4}\phi + z^{n-3}\mathring{\phi}\rangle\rangle + \frac{1}{2}\|(n-3)z^{n-4}\phi + z^{n-3}\mathring{\phi}\|$$

and

$$P = P_p + z^{n-2}\sum_{j=1}^{4}P_{R,j}(\phi, X, m_\infty, z)$$

where

$$P_{R,j} : V_2 \times E_+ \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \quad j = 1,2,3,4$$

are analytic maps that satisfy

$$P_{R,j}(\epsilon Y_1, Y_2, x_1, x_2) = \epsilon^j P_{R,j}(Y_1, Y_2, x_1, x_2)$$

for all $\epsilon \in \mathbf{R}$. Note also that

$$N = N_p - 2z^{n-2}\sigma .$$

Let

$$\omega := \mathring{\phi} \quad \text{and} \quad \theta := z^{-1}\phi . \tag{5.2.37}$$

Using the above results, straightforward calculation shows that there exists analytic maps

$$\mathcal{G} : V_2 \times V_2 \times E_+ \times \mathbf{R}^3 \longrightarrow V_2 \quad \text{and} \quad \mathcal{M} : V_2 \times V_2 \times E_+ \times \mathbf{R}^3 \longrightarrow \mathbf{R}$$

such that equation (5.2.17) and (5.2.18) can be written as

$$z\mathring{\sigma} = -(n-3)\sigma + z^2\mathcal{M}(\theta, \omega, X, m_\infty, \sigma, z) , \tag{5.2.38}$$

$$z\mathring{\theta} = -\theta + \omega , \tag{5.2.39}$$

and

$$\begin{aligned} N(z\mathring{\omega} + 2(n-3)\omega + (n-3)(n-4)\theta) + 2(n-3)\theta \\ + 2\omega - A_2(\theta) - z\mathcal{G}(\theta, \omega, X, m_\infty, \sigma, z) = 0 . \end{aligned} \tag{5.2.40}$$

We can rewrite (5.2.40) as

$$z\mathring{\omega} = -2(n-2)\omega + (A_2 - (n-3)(n-2)\mathbb{1})\theta + z\hat{\mathcal{G}}(\theta, \omega, X, m_\infty, \sigma, z) \tag{5.2.41}$$

where

$$\hat{\mathcal{G}}(Y_1, Y_2, Y_3, x_1, x_2, x_3) = \frac{1}{x_3}\left(\frac{1}{1-2(M(Y_3,x_1)+x_3^{n-3}x_2)x_3} - 1\right)(2(n-3)Y_1 + 2Y_2 - A_2(Y_1)) + \mathcal{G}((Y_1, Y_2, Y_3, x_1, x_2, x_3) .$$

Because $\mathcal{G}$ is analytic, $\hat{\mathcal{G}}$ is analytic in a neighborhood of $(0,0,X,m_\infty,0,0) \in V_2 \times V_2 \times E_+ \times \mathbf{R}^3$. For $Y \in V_2$ and $s \in \mathcal{E}$, define

$$Y_s^+ := \mathrm{pr}_+^s Y \quad \text{and} \quad Y_s^0 := \mathrm{pr}_0^s Y .$$

Recalling that $\mathrm{pr}_0^s \circ A_2 = A_2 \circ \mathrm{pr}_0^s = 0$ and $\mathrm{pr}_+^s \circ A_2 = A_2 \circ \mathrm{pr}_+^s = s(s+1)\mathrm{pr}_+^s$ for every $s \in \mathcal{E}$, we can write (5.2.39) and (5.2.41) as

$$z\mathring{\theta}_s^+ = -\theta_s^+ + \omega_s^+ , \tag{5.2.42}$$

$$z\mathring{\omega}_s^+ = -2(n-2)\omega_s^+ + (s(s+1)-(n-3)(n-2))\theta_s^+ + z\hat{\mathcal{G}}_s^+(\theta,\omega,X,m_\infty,\sigma,z) , \tag{5.2.43}$$

$$z\mathring{\theta}_s^0 = -\theta_s^0 + \omega_s^0 , \tag{5.2.44}$$

and

$$z\mathring{\omega}_s^0 = -2(n-2)\omega_s^+ + -(n-3)(n-2)\theta_s^0 + z\hat{\mathcal{G}}_s^0(\theta,\omega,X,m_\infty,\sigma,z) , \tag{5.2.45}$$

for all $s \in \mathcal{E}$.

For every $s \in \mathcal{E}$, introduce one last change of variables

$$\zeta_s^1 := (-n+3)\theta_s^0 - \omega_s^0 , \quad \zeta_s^2 := (n-2)\theta_s^0 + \omega_s^0 ,$$

$$\eta_s^1 := \frac{1}{2s+1}((s-n+3)\theta_s^+ - \omega_s^+) , \quad \eta_s^2 := \frac{1}{2s+1}((s+n-2)\theta_s^+ + \omega_s^+) .$$

and let

$$\zeta^j := \sum_{s\in\mathcal{E}} \zeta_s^j \quad \text{and} \quad \eta^j := \sum_{s\in\mathcal{E}} \eta_s^j \quad j = 1,2.$$

Using this transformation we can write (5.2.42)-(5.2.45) for all $s \in \mathcal{E}$ and $j = 1,2$ as

$$z\mathring{\zeta}_s^j = -(n-j)\zeta_s^j + z\mathcal{K}_s^j(\zeta^1,\zeta^2,\eta^1,\eta^2,X,m_\infty,\sigma,z) , \tag{5.2.46}$$

$$z\mathring{\eta}_s^j = -(n-(-1)^j s - j)\eta_s^j + z\mathcal{H}_s^j(\zeta^1,\zeta^2,\eta^1,\eta^2,X,m_\infty,\sigma,z) , \tag{5.2.47}$$

where $\mathcal{K}_s^j$ and $\mathcal{H}_s^j$ ($j = 1,2$) are E_0 and E_+ valued maps, respectively, that are analytic in a neighborhood of $(0,0,X,m_\infty,0,0) \in V_2 \times V_2 \times E_+ \times \mathbf{R}^3$.

The system of differential equations given by (5.2.38), (5.2.46), and (5.2.47) is equivalent to the original system (5.2.17), (5.2.18). Moreover, if we choose $n = \max\{3, 3+\max\mathcal{E}\}$, then (5.2.38), (5.2.46), and (5.2.47) are in a form to which theorem 5.0.6 applies. Applying this theorem shows that there exist a unique C^1-solution $\{\sigma(z,a,Y), \zeta_s^j(z,a,Y), \eta_s^j(z,a,Y)\}$ that is analytic in a neighborhood of $(z,a,Y) = (0,m_\infty,X)$ and that satisfies

$$\zeta_s^j(z) = O(z) , \quad \zeta_s^j(z) = O(z) , \quad \text{and} \quad \sigma(z) = O(z) . \tag{5.2.48}$$

From (5.2.30-5.2.32), (5.2.33), (5.2.34) and (5.2.48) it follows that

$$\mathrm{pr}_+^{\bar{s}}(\Lambda_+ - \Omega_+) = Y_s z^s + O(r^s)\,, \quad \mathrm{pr}_0^{\bar{s}}(\Lambda_+) = O(z^s) \quad \forall s \in \mathcal{E}$$

and

$$m = a + 0(z^3)\,.$$

□

Theorem 5.2.7. *Every solution from theorem 5.2.6 satisfies equation* (4.1.10) *on a neighborhood of* $r^{-1} = 0$.

Proof. Let $z = 1/r$ and $\{\Lambda_+(z), m(z)\}$ be a solution of the equations (4.1.7) and (4.1.9) on a neighborhood $\mathcal{N}$ of $z = 0$, which we know exists by theorem 5.2.6. Lemma 5.0.7 shows that

$$\gamma(z) = -z^2([\Lambda_+(z), \mathring{\Lambda}_-(z)] + [\Lambda_-(z), \mathring{\Lambda}_+(z)]) \tag{5.2.49}$$

and $\gamma(z)$ satisfies

$$\mathring{\gamma}(z) = f(z)\gamma(z) \tag{5.2.50}$$

where

$$f(z) = \frac{2}{N}(m - 2zP)\,. \tag{5.2.51}$$

By theorem 5.2.6 and shrinking $\mathcal{N}$ if necessary, we see that $f(z)$ is analytic on $\mathcal{N}$. Solving (5.2.50) we find

$$\gamma(z) = \gamma(0)e^{\int_0^z f(\tau)d\tau} \quad \forall z \in \mathcal{N}\,.$$

But $\gamma(0) = 0$ by (5.2.50). Therefore $\gamma(z) = 0$ for all $z \in \mathcal{N}$. □

5.2.3 Bounded black hole solutions

We now prove a local uniqueness result for the black hole boundary conditions. The proof is less difficult than at $r = 0$ or $r = \infty$ because the singularity that occurs at $r = r_H$ is relatively mild.

Theorem 5.2.8. *Let* $t = r - r_H$ *and suppose* $X \in V_2$ *satisfies*

$$\nu(X) := \frac{1}{r_H} - \frac{1}{r_H}\|\Lambda_0 + [X, c(X)]\|^2 > 0\,.$$

Then for some $\epsilon > 0$ *there exist a unique* $C^2 \times C^1$*-solution* $\{\Lambda_+(t,X), N(t,X)\}$ *to the system of differential equations* (4.1.7) *and* (4.1.9) *defined on* $[0,\epsilon)$ *that satisfies*

$$N(t) = \nu(X)t + \mathcal{O}(t^2) \quad \text{and} \quad \Lambda_+(t) = X + \mathcal{O}(t)\,.$$

Moreover, these solutions are defined and analytic on $(-\epsilon, \epsilon)$, *and depend analytically on the initial data* X.

Proof. Introduce new variables t, μ, and v via

$$t = r - r_H\,, \quad N = t(\mu + \nu)\,, \quad \text{and} \quad v = (\mu + \nu)\Lambda_+'\,, \tag{5.2.52}$$

where ν is a constant. Then

$$t\frac{d\Lambda_+}{dt} = t\left(\frac{v}{\mu+\nu}\right) , \tag{5.2.53}$$

and it is clear that there exists analytic maps

$$\hat{\mathcal{F}} : V_2 \longrightarrow E_+ \quad \text{and} \quad \hat{P} : V_2 \longrightarrow \mathbf{R}$$

such that

$$\hat{\mathcal{F}}(\Lambda_+) = \mathcal{F} \quad \text{and} \quad \hat{P}(\Lambda_+) = P$$

where $\mathcal{F}$ and P are defined by (4.1.14) and (4.1.12), respectively. Assume $|\nu| > 0$. Define an analytic map

$$\hat{G} : V_2 \times I_{|\nu|}(0) \longrightarrow \mathbf{R} \, ; \; (X, a) \longmapsto \frac{1}{2(a+\nu)^2} \|X\|^2 \, .$$

Then $G = \hat{G}(v, \mu)$. Using these new variables, we can write the field equations (4.1.7) and (4.1.9) as

$$\begin{aligned} t\frac{d\mu}{dt} =& -(\mu+\nu) + \frac{1}{r_H} - \frac{2}{r_H^3}\hat{P}(\Lambda_+) + t\left[\frac{1}{t}\left(\frac{1}{t+r_H} - \frac{1}{r_H}\right)\right. \\ &\left. -\frac{2}{t}\left(\frac{1}{(t+r_H)^3} - \frac{1}{r_H^3}\right)\hat{P}(\Lambda_+) + \left(\frac{\mu+\nu}{t+r_H}\right)(1+2\hat{G}(v,\mu))\right] , \end{aligned} \tag{5.2.54}$$

and

$$t\frac{dv}{dt} = -v - \frac{1}{(t+r_H)^2}\hat{\mathcal{F}}(\Lambda_+) - t\left(\frac{2\hat{G}(v,\mu)}{t+r_H}\right) v \, , \tag{5.2.55}$$

respectively. Introduce two new variables μ and $\hat{v}$ via

$$\hat{\mu} = \mu + \nu - \frac{1}{r_H} + \frac{2}{r_H^3}\hat{P}(\Lambda_+) \, . \tag{5.2.56}$$

$$\hat{v} = v + \frac{1}{r_H^2}\hat{\mathcal{F}}(\Lambda_+) \, . \tag{5.2.57}$$

Define an analytic map

$$\Gamma : V_2 \times \mathbf{R} \longrightarrow \mathbf{R} \, ; \; (X, a) \longmapsto a - \nu + \frac{1}{r_H} - \frac{2}{r_H^3}\hat{P}(X) \, .$$

Fix a vector $X \in V_2$ that satisfies $\|\frac{1}{r_H} - \frac{2}{r_H^3}\hat{P}(X)\| > 0$. Then if we set

$$\nu = \frac{1}{r_H} - \frac{2}{r_H^3}\hat{P}(X) \, ,$$

we get $\Gamma(X, 0) = 0$. So we can define an open neighborhood D of $(X, 0) \in V_2 \times \mathbf{R}$ by

$$D = \{ \, (Y, a) \mid \|\Gamma(Y, a)\| < \|\nu\| \, \}.$$

Then from (5.2.53), (5.2.54), (5.2.55), (5.2.56), and (5.2.57), it is not hard to show that there exists an $\epsilon > 0$ and analytic maps

$$\mathcal{G} : V_2 \times D \longrightarrow \mathbf{R} \quad \text{and} \quad \mathcal{H}, \mathcal{K} : V_2 \times D \times I_\epsilon(0) \longrightarrow \mathbf{R}$$

such that

$$t\frac{d\Lambda_+}{dt} = t\mathcal{G}(\hat{v}, \Lambda_+, \hat{\mu}) \,, \tag{5.2.58}$$

$$t\frac{d\hat{v}}{dt} = -\hat{v} + t\mathcal{H}(\hat{v}, \Lambda_+, \hat{\mu}, t) \,, \tag{5.2.59}$$

$$t\frac{d\hat{\mu}}{dt} = -\hat{\mu} + t\mathcal{K}(\hat{v}, \Lambda_+, \hat{\mu}, t) \,. \tag{5.2.60}$$

The system of differential equations (5.2.58), (5.2.59) and (5.2.60) is in the form for which theorem 5.0.6 applies. Applying this theorem shows that exists a unique C^1-solution $\{\Lambda_+(t,Y),\ \hat{v}(t,Y),\ \hat{\mu}(t,Y)\}$ to this system of differential equations that is analytic in a neighborhood of $(0, X)$ and that satisfies

$$\Lambda_+(t,Y) = Y + O(t) \,, \quad \hat{v}(t,Y) = O(t) \,, \quad \text{and} \quad \hat{\mu}(t,Y) = O(t) \,.$$

It follows that $N(t)$ and $\Lambda_+(t)$ are analytic in a neigborhood of $t = 0$. Expanding in a convergent Taylor series we have

$$N(t) = \sum_{k=0}^{\infty} N_k t^k \quad \text{and} \quad \Lambda_+(t) = \sum_{k=0}^{\infty} Y_k t^k \quad (Y_0 = Y) \,.$$

Substituting these powerseries into (4.1.7) and (4.1.9) shows

$$N_0 = 0 \quad \text{and} \quad N_1 = \frac{1}{r_H} - \frac{1}{r_H} \|\Lambda_0 + [Y, c(Y)]\|^2 \,.$$

□

Theorem 5.2.9. *Every solution from theorem 5.2.8 satisfies equation* (4.1.9) *on a neighborhood of* $r = r_H$.

Proof. Let $t = r - r_H$ and suppose $X \in V_2$ satisfies

$$\nu := \frac{1}{r_H} - \frac{1}{r_H} \|\Lambda_0 + [X, c(X)]\|^2 > 0 \,. \tag{5.2.61}$$

Then we know by the previous theorem that there exist a solution $\{\Lambda_+(t), N(t)\}$ to the system of differential equations (4.1.7) and (4.1.9) that is analytic in a neighborhood of $t = 0$ and satisfies

$$N(t) = \nu t + 0(t^2) \quad \text{and} \quad \Lambda_+(t) = X + 0(t) \,. \tag{5.2.62}$$

Let

$$f(t) := -\frac{2}{(t + r_H)^2 N(t) t^{-1}} \left(m(t) - \frac{1}{t + r_H} P(t) \right) \,.$$

Then (5.2.62) shows that $f(t)$ is analytic in a neighborhood of $t = 0$. A short calculation shows that $f(0) = -1$, and therefore we can write

$$f(t) = -1 + tg(t)$$

where $g(t)$ is analytic near $t = 0$. Consider the differential equation on V_2,

$$t\frac{d\eta(t)}{dt} = -\eta(t) + tg(t)\eta(t) , \qquad (5.2.63)$$

It has $\eta(t) = 0$ as a solution, and therefore this is the unique analytic solution near $t = 0$ by theorem 5.0.6. But lemma 5.0.7 shows that $\gamma(t) = [\Lambda_+(t), \frac{d\Lambda_-}{dt}(t)] + [\Lambda_-(t), \frac{d\Lambda_+}{dt}(t)]$ also solves (5.2.63) in an neighborhood of $t = 0$. Because $\gamma(t)$ is analytic near $t = 0$, it follows by uniqueness of analytic solutions to (5.2.63) that $\gamma(t) = 0$ near $t = 0$. □

Chapter 6

Global behavior

In the previous chapter we have established that the EYM equations are locally solvable near $r = 0$ and $r = r_H$. If one of these solutions can be continued out to $r = \infty$, we would like to know its behaviour. Knowing the global behaviour is important for two reasons. The first is that numerical solutions can be constructed much more efficiently when one knows what to expect. The second is that we believe that these global estimates will be necessary in proving the existence of global solutions as was the case when $G = SU(2)$.

Suppose that $\{\Lambda_+(r), m(r)\}$ is a local solution to (4.1.7), (4.1.9), and (4.1.10) in a neighborhood of $r = r_*$ where $r_* = 0$ or $r_* = r_H > 0$. We are interested in the local solutions that can be continued out to $r = \infty$ with $N(r) > 0$ for $r > r_*$. For the moment we will assume that there exists a $r_0 > r_*$ so that the conditions

$$N(r_0) < 1, \quad \|\Lambda_+(r)\| \le \frac{1}{\sqrt{2}}\|\Lambda_0\| \,. \tag{6.0.1}$$

and

$$[\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0, \tag{6.0.2}$$

are satisfied. At the end of section 6.3 we will show that all local solutions that can be continued out to $r = \infty$ with $N(r) > 0$ for $r > r_*$ will necessarily have to satisfy these conditions. The goal of this chapter will be to determine the global behavior of these type of solutions. Before we state the main theorem that characterizes the global behavior, we first need to introduce a technical condition. The space V_2 (see (5.1.4)) is uniquely determined by the choice of Λ_0 in $\mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$. Therefore the bilinear form

$$B : V_2 \times V_2 \longrightarrow V_2 \,:\, (X, Y) \longmapsto [X, c(Y)] \,. \tag{6.0.3}$$

depends implicitly on Λ_0. Our results require that Λ_0 is chosen so that the following coercive condition is satisfied

$$\frac{4}{\|\Lambda_0\|^2} \le \inf_{X \in V_2 \setminus \{0\}} \frac{\|B(X, X)\|^2}{\|X\|^4} \,. \tag{6.0.4}$$

We show in the next section that there exists Λ_0 in $\mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$ for which the inequality (6.0.4) is satisfied. In fact we have some evidence that (6.0.4) is satisfied for all Λ_0 in $\mathcal{A}_1^{v,\mathbf{R}} \cap \overline{\mathcal{W}}$ although we have no proof of this fact.

We now state our main result:

Theorem 6.0.10. *Suppose* $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{W}$ *is such that the inequality* (6.0.4) *is satisfied. If* $\{\Lambda_+(r), m(r)\}$ *is a solution to equations* (4.1.7) *and* (4.1.9) *defined on* $[r_0, \infty)$ $(r_0 > 0)$ *that satisfies*

$$N(r_0) < 1\,, \quad \|\Lambda_+(r_0)\| \leq \frac{1}{\sqrt{2}}\|\Lambda_0\|\,, \quad [\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0\,,$$

at the point r_0 *and*

$$N(r) > 0 \textit{ for all } r \geq r_0,$$

then

(i) *there exist a* $m_\infty > 0$ *such that* $m(r) \to m_\infty$ *as* $r \to \infty$,

(ii) $0 < N(r) < 1$ *for all* $r \geq r_0$,

(iii) *equation* (4.1.10) *is automatically satisfied for all* $r \geq r_0$,

(iv) *equation* (4.1.8) *can be integrated to obtain* $S(r)$ *and* $S(r_0)$ *can be chosen so that* $S(r) \to 1$ *as* $r \to \infty$,

(v) $\|\Lambda_+(r)\| \leq \|\Lambda_0\| / \sqrt{2}$ *for all* $r \geq r_0$,

(vi) $r\Lambda'_+(r) \to 0$ *and* $\|\Lambda_+(r) - \mathfrak{F}^\times\| \to 0$ *as* $r \to \infty$,

where

$$\mathfrak{F}^\times := \{\, X \in V_2 \setminus \{0\} \mid [[c(X), X], X] = 2X \,\}.$$

Moreover if S_λ *is a* Π*-system then* $\Lambda_+(r) \in E_+$ *for all* $r \geq r_0$ *and* $\lim_{r\to\infty} \Lambda_+(r) = \Omega_+^\infty$ *for some* $\Omega_+^\infty \in \mathfrak{F}^\times \cap E_+$.

When Λ_0 is such that S_λ is a Π-system, this theorem is a natural generalization of the $SU(2)$ results. However, if S_λ is not a Π-system, then there is a possibility for a new type of behavior as (vi) leaves open the possibility that $\Lambda_+(r)$ does not actually approach a limit as $r \to \infty$. The reason that this possibility exists is that when S_λ is not a Π-system $\mathfrak{F}^\times$ forms a $|S_\lambda|$-dimensional real variety. On the other hand, the existence of the limit $\Lambda_+(r)$ as $r \to \infty$ when S_λ is a Π-system is due to the fact that the set $\mathfrak{F}^\times \cap E_+$ is discrete (see lemma 6.1.2).

From the definition of $\mathfrak{F}^\times$, it is clear that every $X_+ \in \mathfrak{F}^\times$ determines a real standard triple $\{X_0, X_-, X_+\}$ where $X_- := -c(X_+)$ and $X_0 := [X_+, X_-]$. As $\Lambda_0 \in \mathcal{A}_1^{v,\mathbf{R}} \cap \overline{W}$, we know that there exist an $\Omega_+ \in V_2$ such that $\{\Lambda_0, \Omega_+, \Omega_-\}$ $(\Omega_- := -c(\Omega_+))$ is a real standard triple. Let

$$\mathfrak{C} := \{\, \Omega_+ \in V_2 \setminus \{0\} \mid [c(\Omega_+), \Omega_+] = \Lambda_0 \,\}.$$

Then $\mathfrak{C} \subset \mathfrak{F}^\times$, and it follows from (3.1.5) that the magnetic charge $Q_M \to 0$ as $r \to \infty$ if and only if $\|\Lambda_+(r) - \mathfrak{C}\| \to 0$ as $r \to \infty$. Therefore $\mathfrak{F}^\times \setminus \mathfrak{C}$ characterizes the asymptotic values of $\Lambda_+(r)$ for which the magnetic charge does not vanish. If $G = SU(2)$ then $\mathfrak{C} = \mathfrak{F}^\times$ and so we recover the known fact that the global solutions cannot have any magnetic charge. For $G \neq SU(2)$, in general $\mathfrak{C}$ is a proper subset of $\mathfrak{F}$ and hence there exist a possibility for solutions with magnetic charge. As we mentioned earlier in section 4.2, purely magnetic black hole solutions with nonzero magnetic charge have been found numerically [25]. However, it is not clear if solitons with nonzero magnetic charge exist. No numerical solutions of this type have been found. Assuming that $\lim_{r\to\infty}\Lambda_+(r)$ exists, the initial value problem at $r=\infty$ may provide some insight. For $\mathfrak{g}=\mathfrak{sl}_n\mathbb{C}$ and Λ_0 principal, the possiblility of magnetic charge has been studied by Künzle in [34]. To describe his results, we first expand $\Lambda_+(r)$ as (see (4.3.1))

$$\Lambda_+(r)=\sum_{\alpha\in S_\lambda} w_\alpha(r)e_\alpha\ ,$$

where the w_α are real valued functions. We note this expansion is possible since Λ_0 is regular. If the magnetic charge does not vanish, then it can be shown that $\lim_{r\to\infty} w_\alpha(r)=0$ for some of the $\alpha\in S_\lambda$. Assuming analyticity of the solution about $r=\infty$, the power series expansion then shows that $w_\alpha=0$ for r near $r=\infty$. We expect , although we have no proof, that $w_\alpha=0$ near $r=\infty$ actually implies that $w_\alpha=0$ for all r. For black hole solutions this is not a problem. In fact the magnetically charged black hole solutions of [25] were found by setting $w_\alpha=0$ for certain $\alpha\in S_\lambda$. But for solitons, $w_\alpha=0$ for any $\alpha\in S_\lambda$ is not compatible with the boundary conditions at $r=0$. This may explain why no magnetically charged solitons have been found. Our analysis of the initial value problem at $r=\infty$ has been done under the assumption that $\lim_{r\to\infty}\Lambda_+(r)\in\mathfrak{C}$. In view of the above discussion, it would be desirable to generalize the existence and uniqueness proof at $r=\infty$ to allow for $\lim_{r\to\infty}\Lambda_+(r)$ in $\mathfrak{F}^\times$.

It is worthwhile at this point to mention how the existence proof for the gauge group $SU(2)$ can be used to imply the existence of global solutions for any compact gauge group and any generator Λ_0 in $\mathcal{A}_1^{\vee,\mathbf{R}}\cap\overline{\mathcal{W}}$. To see this, choose Ω_+ so that $\{\Lambda_0,\Omega_+,\Omega_-\}$ is a real standard triple. Let

$$\Lambda_+(r)=\mathfrak{w}(r)\Omega_+\quad \text{with } \mathfrak{w}(0)=1. \tag{6.0.5}$$

Then it follows from (4.1.10) that $\mathfrak{w}(r)=e^{i\gamma_0}u(r)$ for a constant γ_0 and a real function $u(r)$. The remaining equations (4.1.7)-(4.1.9) become

$$m'=g_0^2(Nu'^2+\tfrac{1}{2}r^{-2}(1-u^2)^2), \tag{6.0.6}$$

$$S^{-1}S'=2g_0^2r^{-1}u'^2. \tag{6.0.7}$$

$$r^2Nu''+(2m-g_0^2r^{-1}(1-u^2)^2)u'+(1-u^2)u=0, \tag{6.0.8}$$

where $g_0:=\frac{1}{2}\|\Lambda_0\|$. They reduce with

$$\rho:=r/g_0,\quad \mu:=m/g_0 \tag{6.0.9}$$

to the equations for the $SU(2)$ theory,

$$d\mu/d\rho=N(du/d\rho)^2+\tfrac{1}{2}\rho^{-2}(1-u^2)^2, \tag{6.0.10}$$

$$S^{-1}dS/d\rho=2r^{-1}(du/d\rho)^2, \tag{6.0.11}$$

$$\rho^2Nd^2u/d\rho^2+(2\mu-\rho^{-1}(1-u^2)^2)du/d\rho+(1-u^2)u=0, \tag{6.0.12}$$

where now $N=1-2\mu/\rho$. In view of the existence theorems for the $G=SU(2)$ case [6,52–54] it now follows that the system (4.3.4) and (4.3.5) always admits some global solutions

Theorem 6.0.11. *There exists a countably infinite family of globally regular solutions of the Einstein-Yang-Mills-equations for any simply connected compact semisimple gauge group G on a static spherically symmetric asymptotically flat space-time diffeomorphic to $\mathbf{R}^4$. Similarly, for any $r_H>0$ there exists an infinite family of* *asymptotically flat black hole solutions with black hole radius r_H.*

This of course leaves open the question of what are all the possible global solutions.

6.1 A coercive condition

In this section we show that there exist Λ_0 in $\mathcal{A}_1^{\vee,\mathbb{R}} \cap \overline{W}$ so that (6.0.4) is satisfied. To start, we first derive an inequality that is equivalent to (6.0.4) but easier to work with. Let

$$S(V_2) := \{\, Y \in V_2 \mid \|Y\| = 1 \,\}$$

and define

$$\mathcal{J} := \{\, X \in S(V_2) \mid [[c(X), X], X] = \|[X, c(X)]\|^2\, X \,\}$$

Lemma 6.1.1.

$$\inf_{X \in \mathcal{J}} \|B(X,X)\|^2 = \inf_{X \in V_2\setminus\{0\}} \frac{\|B(X,X)\|^2}{\|X\|^4}$$

Proof. Define

$$Q(Y) := B(Y,Y)^2\,,$$

and let $\mathcal{C}$ denote the set of critical points of $Q|_{S(V_2)}$. Then it is clear that

$$\inf_{X \in \mathcal{C}} \|B(X,X)\|^2 = \inf_{X \in V_2\setminus\{0\}} \frac{\|B(X,X)\|^2}{\|X\|^4}\,.$$

Therefore to prove the theorem we need to show that $\mathcal{J} = \mathcal{C}$. So suppose X is a critical point of $Q|_{S(V_2)}$ and let $f(Y) := \|Y\|^4$. By the method of Lagrange multipliers there exists a $\beta \in \mathbb{R}$ such that

$$DQ(X) = \beta Df(X)\,.$$

Straightforward calculation shows that

$$Df(Z)\cdot Y = 4\,\|Z\|^2\,\langle\langle Z|Y\rangle\rangle \quad \text{and} \quad DQ(Z)\cdot Y = -4\langle\langle\, [[Z, c(Z)], Z] | Y\rangle\rangle\,.$$

Therefore X must satisfy

$$\|X\| = 1 \quad \text{and} \quad [[c(X), X], X] = \beta X\,. \tag{6.1.1}$$

Taking the norm on both sides of $[[c(X), X], X] = \beta X$ and using $\|X\| = 1$ yields

$$\beta = \langle\langle\, [[c(X), X], X] | X\rangle\rangle = \|[c(X), X]\|^2\,. \tag{6.1.2}$$

Let $X_+ := X$, $X_- := -c(X)$, and $X_0 := [X_+, X_-]$. Then $[X_0, X_\pm] = \pm\,\|X_0\|^2\,X_\pm$ by (6.1.1) and (6.1.2). This proves that $\mathcal{C} \subset \mathcal{J}$. The reverse inclusion is straightforward to verify. □

Define

$$\mathfrak{J} := \{\, X \in V_2 \mid [[c(X), X], X] = 2X \,\} \tag{6.1.3}$$

At this point we will prove a result about the structure of $\mathfrak{J}$ that will be required later on. This result will not be used in this section.

Lemma 6.1.2. *If S_λ is a Π-system, then $\mathfrak{J} \cap E_+$ is a discrete set.*

Proof. Proposition 5.1.6 shows that $E_+ = \sum_{j=1}^{\ell} \mathbf{R}\mathbf{e}_{\alpha_j}$, so we can expand $X_+ \in E_+$ as

$$X_+ = \sum_{j=1}^{\ell} x_j \mathbf{e}_{\alpha_j}$$

where $x_j \in \mathbf{R}$. So

$$X_- := -c(X_+) = \sum_{j=1}^{\ell} x_j \mathbf{e}_{-\alpha_j}$$

and hence

$$X_0 := [X_+, X_-] = \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} x_j x_k [\mathbf{e}_{\alpha_j}, \mathbf{e}_{-\alpha_k}] = \sum_{j=1}^{\ell} x_j^2 \mathbf{h}_{\alpha_j}$$

as $[\mathbf{e}_{\alpha_j}, \mathbf{e}_{-\alpha_k}] = \delta_{jk}\mathbf{h}_{\alpha_j}$. Using $[\mathbf{h}_{\alpha_j}, \mathbf{e}_{\pm\alpha_k}] = \pm C_{kj}\mathbf{e}_{\pm\alpha_k}$ where C_{kj} is the Cartan matrix of $\mathfrak{g}_\lambda$, we get

$$[X_0, X_\pm] \mp 2X_\pm = \pm \sum_{k=1}^{\ell} \left(\sum_{j=1}^{\ell} C_{kj} x_j^2 - 2 \right) x_k \mathbf{e}_{\pm\alpha_k} .$$

Because the vectors $\mathbf{e}_{\pm\alpha_j}$ are linearly independent, it is clear that $X_+ \in \mathfrak{J} \cap E_+$ if and only if

$$\left(\sum_{j=1}^{\ell} C_{kj} x_j^2 - 2 \right) x_k = 0 \quad \text{for } k = 1, 2, \ldots \ell.$$

Using the invertibility of the Cartan matrix C, the above equation can be solved to give

$$x_k = 0 \quad \text{or} \quad x_k = \pm \left(2 \sum_{j=1}^{\ell} (C^{-1})_{kj} \right)^{\frac{1}{2}} \quad k = 1, 2, \ldots, \ell .$$

This solution set is obviously finite and therefore the proof is complete. □

Define

$$\mathfrak{J}^\times := \mathfrak{J} \backslash \{0\}. \tag{6.1.4}$$

Lemma 6.1.3.

$$\inf_{X \in \mathcal{J}} \|B(X,X)\|^2 = \inf_{X \in \mathfrak{J}^\times} \frac{4}{\|B(X,X)\|^2}$$

Proof. Suppose $X_+ \in \mathcal{J}$. Let $X_- := -c(X_+)$ and $X_0 := [X_+, X_-]$. Then

$$2\|X_+\|^2 = \langle\langle 2X_+ | X_+ \rangle\rangle = \langle\langle [X_0, X_+] | X_+ \rangle\rangle = \langle\langle X_0 | [c(X_+), X_+] \rangle\rangle = \|X_0\|^2 . \tag{6.1.5}$$

Also note that if $X \in V_2$ and $[c(X), X] = 0$ then

$$0 = \langle\langle [c(X), X] | \Lambda_0 \rangle\rangle = \langle\langle X | [X, \Lambda_0] \rangle\rangle = -2\|X\|^2 .$$

Therefore

$$\text{if } X \in V_2 \text{ then } [c(X), X] = 0 \text{ if and only if } X = 0. \tag{6.1.6}$$

Define a map

$$\hat{\ }: \mathcal{J} \longrightarrow \mathfrak{J}^\times$$

by

$$X_+ \longmapsto \hat{X}_+ := \frac{\sqrt{2}}{\|[c(X_+), X_+]\|}\, X_+ .$$

Then using (6.1.5) and (6.1.6), it is straightforward to verify that the above map is well defined and bijective. The proof now follows since

$$\|B(X_+, X_+)\|^2 = \frac{\|B(\hat{X}_+, \hat{X}_+)\|^2}{\|\hat{X}_+\|^4},$$

and

$$\frac{\|B(\hat{X}_+, \hat{X}_+)\|^2}{\|\hat{X}_+\|^4} = \frac{4}{\|B(\hat{X}_+, \hat{X}_+)\|^2}$$

by (6.1.5) and the fact that $\|B(\hat{X}_+, \hat{X}_+)\| = \|X_0\|$. □

The above two lemmas show that the coercive condition (6.0.4) is equivalent to

$$\frac{4}{\|\Lambda_0\|^2} \le \inf_{X \in \mathfrak{Z}^\times} \frac{4}{\|B(X, X)\|^2}. \tag{6.1.7}$$

We will now show that there exist generators Λ_0 in $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ that satisfy the inequality (6.0.4).

Theorem 6.1.4. *If S_λ is a Π-system then the inequality* (6.0.4) *is satisfied.*

Proof. Since S_λ is a Π-system, the discussion in section 4.3 shows that we can without loss of generality assume that Λ_0 is a principal A_1-vector. Note that if $X_+ \in \mathfrak{Z}$ then $X_0 := [X_+, X_-] \in \mathcal{A}_1^\vee$ where $X_- := -c(X_+)$. Also note that since X_0 satisfies $c(X_0) = -c(X_0)$ it follows from the definition of $\|\cdot\|$ and B that $(X_0|X_0) = \|B(X_+, X_+)\|$. Therefore

$$\inf_{X \in \mathcal{A}_1^\vee} \frac{4}{(X|X)} \le \inf_{X \in \mathfrak{Z}^\times} \frac{4}{\|B(X, X)\|^2}, \tag{6.1.8}$$

as it can be easily shown that $(X|X) \in \mathbf{R}$ for all $X \in \mathcal{A}_1^\vee$.

Lemma 6.1.5. *If $Y_0 \in \mathcal{A}_1^\vee$ then $(Y_0|Y_0) \le \|\Lambda_0\|^2$.*

Proof. Since Λ_0 is principal, there exists a base Δ such that

$$\alpha(\Lambda_0) = 2 \text{ for all } \alpha \in \Delta. \tag{6.1.9}$$

Also there exists an automorphism ϕ of $\mathfrak{g}$ such that

$$\alpha(\phi(Y_0)) = 0,\ 1, \text{ or } 2 \text{ for every } \alpha \in \Delta. \tag{6.1.10}$$

Now

$$\Lambda_0 = \sum_{\alpha \in \Delta} \lambda_\alpha \mathbf{h}_\alpha \quad \text{and} \quad \phi(Y_0) = \sum_{\alpha \in \Delta} y_\alpha \mathbf{h}_\alpha$$

where

$$\lambda_\alpha = 2 \sum_{\beta \in \Delta} (C^{-1})_{\alpha\beta}, \quad y_\alpha = \sum_{\beta \in \Delta} (C^{-1})_{\alpha\beta}\, \beta(\phi(Y_0))$$

and C^{-1} is the inverse of the Cartan matrix $C = (\langle \alpha, \beta \rangle)$. Using the above expansions it is easy to show that

$$(\Lambda_0|\Lambda_0) = 2\sum_{\alpha,\beta\in\Delta} \frac{4}{|\alpha|^2}(C^{-1})_{\alpha\beta}\,. \tag{6.1.11}$$

and

$$(Y_0|Y_0) = 2\sum_{\alpha,\beta\in\Delta} \frac{1}{|\alpha|^2}\alpha(\phi(Y_0))\,(C^{-1})_{\alpha\beta}\,\beta(\phi(Y_0))\,. \tag{6.1.12}$$

But $(C^{-1})_{\alpha\beta} \geq 0$ for all α, $\beta \in \Delta$. Therefore $(\phi(Y_0)|\phi(Y_0)) \leq (\Lambda_0|\Lambda_0)$ by (6.1.10), (6.1.11), and (6.1.12). Finally, observe that $(\phi(Y_0)|\phi(Y_0)) = (Y_0|Y_0)$ and $(\Lambda_0|\Lambda_0) = \|\Lambda_0\|^2$ since ϕ is an automorphism and $c(\Lambda_0) = -\Lambda_0$. Therefore $(Y_0|Y_0) \leq \|\Lambda_0\|^2$ and the proof is complete. □

From this lemma and (6.1.8), we see that the inequality (6.1.7) is satisfied. By the above results this implies that (6.0.4) is also satisfied. □

Since we now know that there exists Λ_0 in $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ such that the inequality (6.0.4) is satisfied, it would be desirable to determine exactly which Λ_0 satisfy (6.0.4). In general, this appears to be a difficult question. However, for low-dimensional Lie algebras, computations show that every Λ_0 in $\mathcal{A}_1^{\vee,\mathbf{R}} \cap \overline{\mathcal{W}}$ satisfies (6.0.4). This gives some evidence to our belief that (6.0.4) is always satisfied. If this were the case, then our later proofs that rely on (6.0.4) would be general.

6.2 Asymptotic Yang-Mills equations

The flat space spherically symmetric Yang-Mills equations can be written as

$$\ddot{\Lambda}_+ - \dot{\Lambda}_+ + \mathcal{F} = 0 \tag{6.2.1}$$

where $\dot{(\cdot)} = \frac{d(\cdot)}{d\tau}$ and $\tau = \ln(r)$. However, for the purpose of this section we will consider equation (6.2.1) in its own right, and let τ denote a parameter that is not necessarily related to the radial coordinate r. We will be interested in $\tau \to \infty$ behavior of bounded solutions to equations of the form

$$\ddot{\Lambda}_+ - \dot{\Lambda}_+ + \mathcal{F} = \delta(\tau)\dot{\Lambda}_+\,, \tag{6.2.2}$$

where δ is any C^1 function that satisfies

$$\lim_{\tau\to\infty} \delta(\tau) = 0\,. \tag{6.2.3}$$

To determine this behavior, we use the results of Markus [40] concerning the long time behavior of solutions to asymptotically autonomous differential equations. See also [43,63]. To describe these results we first recall that a nonautonomous system of differential equations in $\mathbf{R}^N$

$$\dot{x}(\tau) = h(\tau, x(\tau)) \tag{6.2.4}$$

is said to be *asymptotically autonomous* with *limit equation*

$$\dot{y}(\tau) = k(y(\tau)) \tag{6.2.5}$$

if

$$h(\tau, x) \to k(x) \text{ as } \tau \to \infty \text{ uniformly on compact subsets of } \mathbf{R}^N.$$

We note that the maps h and g are assumed to be continuous and locally Lipschitz on $\mathbf{R}^N$. The ω-limit set $\omega(\tau_0, x_0)$ of a bounded solutions $x(\tau)$ to (6.2.4) on $[\tau_0, \infty)$ satisfying $x(\tau_0) = x_0$ is defined by

$$\omega(\tau_0, x_0) = \{\, y \mid y = \lim_{j \to \infty} x(\tau_j) \text{ for some sequence } \tau_j \to \infty \,\}.$$

The fundamental result of Markus is:

Theorem 6.2.1. *The ω-limit set $\omega(\tau_0, x_0)$ of a bounded solutions $x(\tau)$ to (6.2.4) on $[\tau_0, \infty)$ satisfying $x(\tau_0) = x_0$ is nonempty, compact, and connected. Moreover,*

$$\operatorname{dist}(x(\tau), \omega(\tau_0, x_0)) \to 0 \text{ as } \tau \to \infty,$$

and $\omega(\tau_0, x_0)$ is invariant under (6.2.5).

Define maps

$$\hat{\mathcal{F}} : V_2 \longrightarrow V_2 \;:\; X \longmapsto \frac{1}{2}[\Lambda_0 + [X, c(X)], X], \tag{6.2.6}$$

$$f : \mathbf{R} \times V_2 \times V_2 \longrightarrow V_2 \times V_2 \;:\; (\tau, X_1, X_2) \longmapsto (X_2, X_2 - \hat{\mathcal{F}}(X_1) + \bar{\delta}(\tau) X_2), \tag{6.2.7}$$

and

$$g : V_2 \times V_2 \longrightarrow V_2 \times V_2 \;:\; (\tau, X_1, X_2) \longmapsto (X_2, X_2 - \hat{\mathcal{F}}(X_1)). \tag{6.2.8}$$

Using these maps we can write (6.2.1) and (6.2.2) in first order form as

$$(\dot{\Lambda}_+, \dot{\Gamma}_+) = g(\Lambda_+, \Gamma_+) \tag{6.2.9}$$

and

$$(\dot{\Lambda}_+, \dot{\Gamma}_+) = f(\tau, \Lambda_+, \Gamma_+), \tag{6.2.10}$$

respectively.

Proposition 6.2.2. *$f(\tau, X, Y) \longrightarrow g(X, Y)$ as $\tau \to \infty$ uniformly on compact subsets of $V_2 \times V_2$.*

Proof. Let $B_\rho(V_2) = \{\, X \in V_2 \mid \|X\| \leq \rho \,\}$ and suppose $K \subset V_2 \times V_2$ is compact. Then there exist a $\rho > 0$,such that $K \subset B_\rho(V_2) \times B_\rho(V_2)$. Fix $\epsilon > 0$. Since $\delta(\tau) \to 0$ as $\tau \to \infty$, there exists a τ_ϵ such that $|\delta(\tau)| \leq \rho^{-1}\epsilon$ for all $\tau \geq \tau_\epsilon$. Then for anv $(X, Y) \in K$, we have

$$\|f(\tau, X, Y) - g(X, Y)\| = |\delta(\tau)|\,\|Y\| \leq \frac{\epsilon}{\rho}\rho = \epsilon.$$

Thus $\|f(\tau, X, Y) - g(X, Y)\| \leq \epsilon$ for all $\tau \geq \tau_\epsilon$ and $(X, Y) \in K$. □

This proposition shows that the nonautonomous system (6.2.10) is asymptotically autonomous with limit equation (6.2.9).

Proposition 6.2.3. *Suppose $\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))$ is a bounded solution to (6.2.10) that is defined for all $\tau \geq \tau_0$ and satisfies $\mathbf{X}(\tau_0) = \mathbf{X}_0$. Then*

(i) $\omega(\tau_0, \mathbf{X}_0)$ *is non-empty, compact and connected,*

(ii) $\|\mathbf{X}(\tau) - \omega(\tau_0, \mathbf{X}_0)\| \to 0$ *as $\tau \to \infty$,*

(iii) $\omega(\tau_0, \mathbf{X}_0)$ *is invariant under* (6.2.9).

Proof. Follows directly from theorem 6.2.1 by proposition 6.2.2. □

Define

$$H : V_2 \times V_2 \longrightarrow \mathbf{R} : (X_1, X_2) \longmapsto \frac{1}{2}\|X_2\|^2 - \frac{1}{2}\|\hat{F}(X_1)\|^2, \tag{6.2.11}$$

where

$$\hat{F}(X) := \frac{i}{2}(\Lambda_0 + [X, c(X)]). \tag{6.2.12}$$

Proposition 6.2.4. *If* $\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))$ *is a bounded solution to* (6.2.10), *then there exists a* $\beta \in \mathbf{R}$ *such that* $H(\omega(\tau_0, \mathbf{X}_0)) = \beta$.

Proof. Straightforward calculation using (4.1.18), (5.1.3), the properties (2.1.2) of the inner product $\langle\langle \cdot | \cdot \rangle\rangle$, and (6.2.10) shows that

$$(H(\mathbf{X}(\tau))^{\cdot} = \|X_2(\tau)\|^2 (1 + \delta(\tau)).$$

But $\delta(\tau) \to 0$ as $\tau \to \infty$, which shows that $(H(\mathbf{X}(\tau))^{\cdot} \geq 0$ for τ large enough. Thus $\lim_{\tau\to\infty} H(\mathbf{X}(\tau))$ exists and we will denote it by β. Therefore for any sequence $\tau_k \to \infty$, we also have $\lim_{k\to\infty} H(\mathbf{X}(\tau_k)) = \beta$. By continuity of H, we have $H(\lim_{k\to\infty} \mathbf{X}(\tau_k)) = \beta$. From the definition of $\omega(\tau_0, \mathbf{X}_0)$ it is clear that $H(\omega(\tau_0, \mathbf{X}_0)) = \beta$. □

The fixed points of (6.2.9) are

$$\mathfrak{F} \times \{0\} \tag{6.2.13}$$

where $\mathfrak{F}$ was previously defined in (6.1.3).

Theorem 6.2.5. *If* $\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))$ *is a bounded solution to* (6.2.10), *then*

(i) $\|X_1(\tau) - \mathfrak{F}\| \to 0$ *as* $\tau \to \infty$

(ii) $X_2(\tau) \to 0$ *as* $\tau \to \infty$

Proof. Suppose $\mathbf{Y}_0 = (Y_{1,0}, Y_{2,0}) \in \omega(\tau_0, \mathbf{X}_0)$. Let $\mathbf{Y}(\tau) = (Y_1(\tau), Y_2(\tau))$ be a solution to (6.2.9) with $\mathbf{Y}(0) = \mathbf{Y}_0$. Then $\mathbf{Y}(\tau) \in \omega(\tau_0, \mathbf{X}_0)$ and $H(\mathbf{Y}(\tau)) = \beta$ for all $\tau \geq 0$ by propositions 6.2.3 and 6.2.4. Using (4.1.18), (5.1.3), the properties (2.1.2) of the inner product $\langle\langle \cdot | \cdot \rangle\rangle$, and (6.2.9), it is not difficult to show that $(H(\mathbf{Y}(\tau))^{\cdot} = \|Y_2(\tau)\|^2$. Therefore we must have $\|Y_2(\tau)\| = 0$ and hence $Y_2(\tau) = \dot{Y}_2(\tau) = 0$. It then follows form the differential equation (6.2.9) that $\dot{Y}_1(\tau) = 0$ and $[c(Y_1(\tau)), Y_1(\tau)] = 2Y_1(\tau)$. Therefore, $\mathbf{Y}(\tau) = (Y_{0,1}, 0)$ and $[c(Y_{1,0}), Y_{1,0}] = 2Y_{1,0}$. This proves that $\omega(\tau_0, \mathbf{X}_0) \subset \mathfrak{F} \times \{0\}$. The proof now follows easily since $\|\mathbf{X}(\tau) - \omega(\tau_0, \mathbf{X}_0)\| \to 0$ as $\tau \to \infty$ by proposition (6.2.3). □

Theorem 6.2.6. *If* $\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))$ *is a non trivial bounded solution to* (6.2.10), *then*

(i) $\|X_1(\tau) - \mathfrak{F}^{\times}\| \to 0$ *as* $\tau \to \infty$

(ii) $X_2(\tau) \to 0$ *as* $\tau \to \infty$

Proof. If $\lim_{\tau\to\infty} \mathbf{X}(\tau) \neq 0$ then we are done by the above theorem. So assume that $\lim_{\tau\to\infty} \mathbf{X}(\tau) = 0$. Let $W = V_2 \times V_2$ and define a linear operator $\mathbf{T}$ on W by

$$\mathbf{T}(\mathbf{Z}) := \mathbf{D}g(0) \cdot \mathbf{Z} \,.$$

A short calculation shows that

$$\mathbf{T} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{bmatrix} ,$$

and that $\mathbf{T}$ has two distinct eigenvalues $(1 \pm i\sqrt{3})/2$ each with multiplicity $\dim_{\mathbf{R}} V_2$. Therefore there exists constants K, $\alpha > 0$ such that

$$\left|e^{-\tau\mathbf{T}}\right| \leq K e^{-\alpha\tau} \quad \forall\, \tau \geq 0 \,. \tag{6.2.14}$$

Choose $l > 0$ so that

$$l < \frac{\alpha}{K} \,. \tag{6.2.15}$$

Because $g(0) = 0$ and $\mathbf{D}g(0) = 0$ it can be shown using appropriate smooth bump functions that for any $\mu > 0$ there exists an $\epsilon > 0$ and a C^∞ map $\hat{g} : W \to W$ such that

$$\|\hat{g}(\mathbf{Z}_1) - \hat{g}(\mathbf{Z}_2)\| \leq l\,\|\mathbf{Z}_1 - \mathbf{Z}_2\| \quad \forall\, \mathbf{Z}_1, \mathbf{Z}_2 \in W \,, \tag{6.2.16}$$

$$\|\hat{g}(\mathbf{Z})\| \leq \mu \quad \forall \mathbf{Z} \in W \tag{6.2.17}$$

and

$$\hat{g}(\mathbf{Z}) = g(\mathbf{Z}) - \mathbf{T}(\mathbf{Z}) \quad \forall\, \mathbf{Z} \in B_\epsilon(W) \,. \tag{6.2.18}$$

Also because $\lim_{\tau\to\infty} \delta(\tau) = 0$, there exists a τ_0 and a C^∞ function $\hat{\delta}(\tau)$ such that

$$|\hat{\delta}(\tau)| \leq l \quad \forall \tau \in \mathbf{R} \,, \tag{6.2.19}$$

and

$$\hat{\delta}(\tau) = \delta(\tau) \quad \forall \tau \geq \tau_0 \,. \tag{6.2.20}$$

Letting $\mathrm{pr}_2 : W \to V_2$ denote projection onto the second factor, it is clear from (6.2.18) and (6.2.20) that

$$f(\tau, \mathbf{Z}) = \mathbf{T}(\mathbf{Z}) + \hat{g}(\mathbf{Z}) + \hat{\delta}(\tau)\mathrm{pr}_2(\mathbf{Z}) \quad \forall\, \mathbf{Z} \in B_\epsilon(W),\ \tau \geq \tau_0 \,. \tag{6.2.21}$$

Because $\lim_{\tau\to\infty} \mathbf{X}(\tau) = 0$, there exists a $\tau_1 > \tau_0$ such that $\mathbf{X}(\tau) \in B_\epsilon(W)$ for all $\tau \geq \tau_1$. So $\mathbf{X}(\tau)$ must be a solution to the differential equation

$$\dot{\mathbf{Y}} = \mathbf{T}(\mathbf{Y}) + \hat{\delta}(\tau)\mathrm{pr}_2(\mathbf{Y}) + \hat{g}(\mathbf{Y})$$

for $\tau \geq \tau_1$ by (6.2.21). Define

$$\mathbf{T}(t) := \mathbf{T} + \hat{\delta}(\tau)\mathrm{pr}_2 \tag{6.2.22}$$

so that Y satisfies

$$\dot{\mathbf{Y}} = \mathbf{T}(t)(\mathbf{Y}) + \hat{g}(\mathbf{Y}) \,. \tag{6.2.23}$$

Now $|\mathrm{pr}_2| \leq 1$ so $|\hat{\delta}(\tau)\mathrm{pr}_2| \leq l$ for all $\tau \in \mathbf{R}$ by (6.2.19). Consequently,

$$\int_{\tau}^{\tau_0} |\hat{\delta}(s)\mathrm{pr}_2| ds \leq l(\tau_0 - \tau) \quad \forall \tau_0 \geq \tau . \tag{6.2.24}$$

Let $\Psi(\tau)$ be a fundamental matrix associated to $\mathbf{T}(\tau)$. In other words $\Psi(\tau)$ is an invertible matrix solution to

$$\dot{\Psi}(\tau) = \mathbf{T}(\tau)\Psi(\tau) .$$

It then follows by (6.2.14), (6.2.15), (6.2.24), and theorem 2.3 page 86 of [19] that $\Psi(\tau)$ satsifies

$$|\Psi(\tau)\Psi(\tau_0)^{-1}| \leq e^{-\alpha(\tau_0-\tau)} \quad \forall\, \tau_0 \geq \tau . \tag{6.2.25}$$

The inequalities (6.2.16) and (6.2.17) guarantee that any solution of (6.2.23) is defined for all τ. Let $\tilde{\mathbf{X}}(\tau)$ denote the unique global solution to (6.2.23) that satisfies $\tilde{\mathbf{X}}(\tau) = \mathbf{X}(\tau)$ for all $\tau \geq \tau_1$. Since $\lim_{\tau\to\infty} \mathbf{X}(\tau) = 0$, $\tilde{\mathbf{X}}(\tau)$ is bounded on $[0,\infty)$. Notice, that because $\hat{g}(0) = g(0) = 0$, $Z(\tau) = 0$ is also a solution to (6.2.23). However, a slight generalization of lemma 1.5, page 54 of [22] shows that any solution to (6.2.23) bounded on $[0,\infty)$ is unique by (6.2.15), (6.2.16), (6.2.17), and (6.2.25). Therefore, $\tilde{\mathbf{X}}(\tau) = 0$ for all $\tau \in \mathbf{R}$ and this implies that $\mathbf{X}(\tau) = 0$ for $\tau \geq \tau_1$. But $\mathbf{Z}(\tau) = 0$ is a solution to (6.2.10) and so $\mathbf{X}(\tau) = 0$ for all $\tau \in \mathbf{R}$. This contradicts the assumption that $\mathbf{X}(\tau)$ is a non-trivial solution to (6.2.10). Therefore $\lim_{\tau\to\infty} \mathbf{X}(\tau) \neq 0$. □

6.3 Global estimates

At the end of this section we prove theorem 6.0.10. However, we first need to prove a number of preliminary results.

Proposition 6.3.1. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to equations* (4.1.7) *and* (4.1.9) *defined on an interval* $[r_0, r_1)$ $(r_0 > 0)$ *and that satisfies* $N(r) > 0$ *for all* $r \geq r_0$ *and* $[\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0$ *then* Λ_+ *also satisfies* (4.1.10).

Proof. Lemma 5.0.7 shows that $\gamma(r) = [\Lambda_+(r), \Lambda'_-(r)] + [\Lambda_-(r), \Lambda'_+(r)]$ satisfies the differential equation $\gamma' = -\frac{2}{r^2 N}\left(m - \frac{1}{r}P\right)\gamma$. Integrating this equation yields

$$\gamma(r) = \gamma(r_0)\exp\left(\int_{r_0}^{r} -\frac{2}{s^2 N}\left(m - \frac{1}{s}P\right) ds\right) .$$

But $\gamma(r_0) = 0$ by assumptions, hence $\gamma(r) = 0$ for all $r \geq r_0$. □

Proposition 6.3.2. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to equations* (4.1.7) *and* (4.1.9), *that satisfies* $N(r_0) > 0$, $[\Lambda_+(r_0), \Lambda_-(r_0)] = \Lambda_0$, *and* $\Lambda'_+(r_0) = 0$ *for some* $r_0 > 0$, *then* $\Lambda_+(r) = \Lambda_+(r_0)$, $m(r) = \frac{r_0}{2}(1 - N(r_0))$, *and* $S(r) = S(r_0)$ *for all* $r > \max\{2m(r_0), 0\}$.

Proof. It is straightforward to check that if $N(r_0) > 0$, $[\Lambda_+(r_0), \Lambda_-(r_0)] = \Lambda_0$, and $\Lambda'_+(r_0) = 0$, then $\Lambda_+(r) := \Lambda_+(r_0)$ and $m(r) := \frac{r_0}{2}(1 - N(r_0))$ solve (4.1.7), (4.1.9), and (4.1.10). By standard uniqueness results for systems of differential equations, this is the only solution satisfying $N(r_0) > 0$, $[\Lambda_+(r_0), \Lambda_-(r_0)] = \Lambda_0$, and $\Lambda'_+(r_0) = 0$. □

The next two propositions generalize propositions 8 and 9 in [6] which are valid for $G = SU(2)$. .

Proposition 6.3.3. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to equations* (4.1.7) *and* (4.1.9), *on* $[r_0, r_1)$ $(r_0 > 0)$ *and* $0 < N(r_0) < 1$ *then* $N(r) < 1$ *for all* $r \in [r_0, r_1)$.

Proof. This can be proved in the exact same manner as when $G = SU(2)$. See [6] proposition 8 for details. ☐

Proposition 6.3.4. *If $\{\Lambda_+(r), m(r)\}$ is a solution to equations (4.1.7) and (4.1.9), on $[r_0, r_1)$ $(r_0 > 0)$ with $0 < \epsilon \leq N(r) < 1$ then there exists a $\delta > 0$ such that the solutions exists and is analytic on $[r_0, r_1 + \delta)$.*

Proof. First note that if the solution $\{\Lambda_+(r), m(r)\}$ exists on some open interval $I \subset (0, \infty)$ on which $N > 0$ then the Cauchy-Kowalevski theorem will guarantee the solution will be analytic. From standard theorems on differential equations, it follows that the solution will continue to exist at $r = r_1$ unless $N \to 0$ or one of the variables $\{m, \Lambda_+, \Lambda'_+\}$ becomes unbounded as $r \to r_1$. By assumption N does not approach zero and $0 < N(r) < 1$ implies that

$$0 < 2m(r) < r_1 \quad \forall\, r \in [r_0, r_1)\,. \tag{6.3.1}$$

Therefore we only need to show that Λ_+ and Λ'_+ are bounded as $r \to r_1$.
Integrating (4.1.7) yields

$$m(r) - m(r_0) = \int_{r_0}^{r} (NG + \rho^{-2}P)d\rho \geq \int_{r_0}^{r} NGd\rho \tag{6.3.2}$$

since $P \geq 0$ and $r_0 > 0$. From (6.3.1), (6.3.2), and $N(r) \geq \epsilon$ it follows that

$$2\epsilon \int_{r_0}^{r} Gd\rho \leq 2\int_{r_0}^{r} NGd\rho \leq r_1 \qquad \forall r \in [r_0, r_1)\,,$$

which implies that

$$2\int_{r_0}^{r} \rho^{-1}Gd\rho \leq \frac{2}{r_0}\int_{r_0}^{r} Gd\rho \leq \frac{r_1}{\epsilon r_0} \qquad \forall r \in [r_0, r_1)\,. \tag{6.3.3}$$

Integrating (4.1.8) yields

$$S(r) = S(r) = S_0 \exp(2\int_{r_0}^{r} \rho^{-1}Gd\rho) S_0 > 0)\,,$$

and hence

$$0 < S_0 \leq S(r) \leq S_0 \exp\left(\frac{r_1}{\epsilon r_0}\right) \qquad \forall r \in [r_0, r_1)\,. \tag{6.3.4}$$

by (6.3.3).
Now,

$$\|\Lambda_+(r) - \Lambda_+(r_0\| = \left\|\int_{r_0}^{r} \Lambda'_+(\rho)d\rho\right\| \leq \int_{r_0}^{r} \|\Lambda'_+(\rho)\|\, d\rho\,.$$

But,

$$\begin{aligned}\int_{r_0}^{r} \|\Lambda_+'(\rho)\|\, d\rho &\le \left(\int_{r_0}^{r} \|\Lambda_+'(\rho)\|^2 d\rho\right)^{\frac{1}{2}} \sqrt{r-r_0} && \text{by Hölders inequality}\\ &= \left(2\int_{r_0}^{r} G d\rho\right)^{\frac{1}{2}} \sqrt{r-r_0} && \text{by def. of } G\\ &\le \left(\frac{r_1(r_1-r_0)}{\epsilon}\right)^{\frac{1}{2}} .\end{aligned}$$

The above two results show that

$$\sup_{r\in[r_0,r_1)} \|\Lambda_+(r)\| < \infty . \tag{6.3.5}$$

We can rewrite (4.1.9) as

$$(NS\Lambda_+')' = -\frac{S\mathcal{F}}{r^2} . \tag{6.3.6}$$

So then

$$\begin{aligned}\|N(r)S(r)\Lambda_+'(r) - N(r_0)S(r_0)\Lambda_+'(r_0)\| &= \left\|\int_{r_0}^{r} (NS\Lambda_+')' d\rho\right\|\\ = \left\|\int_{r_0}^{r} \frac{S\mathcal{F}}{\rho^2} d\rho\right\| &\le \int_{r_0}^{r} \frac{S\,\|\mathcal{F}\|}{\rho^2} d\rho .\end{aligned} \tag{6.3.7}$$

But

$$\|\mathcal{F}\| = \left\|\frac{i}{2}(\Lambda_0 - [\Lambda_+,\Lambda_-])\right\| \le \frac{1}{2}(\|\Lambda_0\| + \|[\Lambda_+,\Lambda_-]\|) \le \frac{1}{2}\,\|\Lambda_0\| + k\,\|\Lambda_+\|^2 \tag{6.3.8}$$

for some constant $k > 0$ since $(X,Y) \to [X, c(Y)]$ is a continuous bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$. It follows from (6.3.4), (6.3.5), (6.3.7), and (6.3.8) that

$$\sup_{r\in[r_0,r_1)} \|\Lambda_+'(r)\| < \infty .$$

□

From this point onward, we will assume that Λ_0 satisfies the coercive condition (6.0.4). This next theorem can be use to generalize theorem 7 of [45] to any Λ_0 that satisfies (6.0.4).

Proposition 6.3.5. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *on* $[r_0, r_1)$ ($r_0 > 0$) *with* $N(r) > 0$, *then* $\|\Lambda_+(r)\|^2$ *can not achieve a local maximum in the region where* $\|\Lambda_+(r)\|^2 > \frac{1}{2}\|\Lambda_0\|^2$.

Proof. Let $v(r) := \|\Lambda_+(r)\|^2$ and suppose $v(r)$ achieves a local maximum at $r_\bullet$. Then

$$v'(r_\bullet) = 0 \quad\text{and}\quad v''(r_\bullet) \le 0 .$$

From (4.1.9), it is not hard to show that $v(r)$ satisfies

$$r^2 N(r)v''(r) + \Phi(r)v'(r) + 2v(r) - \|[\Lambda_+(r),\Lambda_-(r)]\|^2 = 2r^2 N(r)\,\|\Lambda_+'(r)\|^2 , \tag{6.3.9}$$

where

$$\Phi(r) := 2(m(r) - r^{-1}P(r)) = r(1 - N(r)) - 2r^{-1}P(r)\,. \tag{6.3.10}$$

It follows from the above equations that $v(r_*) \geq \frac{1}{2}\,\|[\Lambda_+(r_*), \Lambda_-(r_*)]\|^2$ while (6.0.4) implies that $\frac{4}{\|\Lambda_0\|^2}v(r_*)^2 \leq \|[\Lambda_+(r_*), \Lambda_-(r_*)]\|^2$. Therefore $v(r_*) \leq \frac{1}{2}\,\|\Lambda_0\|^2$ and the proof is complete. □

The next proposition is very similar to the previous one, however its slightly different conclusion will be useful in proving the next result.

Proposition 6.3.6. *Suppose* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *on* $[r_0, r_1)$ $(r_0 > 0)$ *with* $N(r) > 0$ *and let* $v(r) = \|\Lambda_+(r)\|^2$. *If* $v(r_0) > \frac{1}{2}\,\|\Lambda_+(r)\|^2$ *and* $v'(r_0) > 0$ *then* $v(r) > \frac{1}{2}\,\|\Lambda_+(r)\|^2$ *and* $v'(r) > 0$ *for all* $r \geq r_0$.

Proof. Let r_1 be the first $r > r_0$ such that $v'(r) = 0$. Then (6.3.9) shows that $r_1^2 N(r_1)v''(r_1) \geq \|[\Lambda_+(r_1), \Lambda_-(r_1)]\|^2 - 2v(r_1)$ while it follows from (6.0.4) that $v(r_1)^2 \leq \frac{\|\Lambda_0\|^2}{4}\,\|[\Lambda_+(r_1), \Lambda_-(r_1)]\|^2$. Therefore

$$r_1^2 N(r_1)v''(r_1) \geq \frac{4}{\|\Lambda_0\|^2}v(r_1)^2 - 2v(r_1)\,.$$

But $v(r_1) > \frac{1}{2}\,\|\Lambda_0\|^2$ implies that $\frac{4}{\|\Lambda_0\|^2}v(r_1)^2 - 2v(r_1) > 0$ and hence $v''(r_1) > 0$ since $N(r_1) > 0$ by assumption. This implies that $v'(r_1) = 0$ is impossible. □

The next proposition is a generalization of proposition 2.2 of [56]. The key to the proof is the observation that the equation (6.3.9) governing $\|\Lambda_+(r)\|^2$ can be analyzed in the region where $\|\Lambda_+(r)\|^2 > \|\Lambda_0\|^2/2$ using the techniques developed in [56] for $G = SU(2)$. It is remarkable that the $SU(2)$ proof can be adapted to the general case with such ease.

Proposition 6.3.7. *Suppose* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *defined in a neighborhood of* r_0 *and let* $v(r) = \|\Lambda_+(r)\|^2$. *If* $0 < N(r_0) < 1$, $v(r_0) > \frac{1}{2}\,\|\Lambda_0\|^2$ *and* $v'(r_0) > 0$ *then there exists a* $r_1 > r_0$ *with* $N(r_1) = 0$, $0 < N < 1$ *on* $[r_0, r_1)$, *and* $[r_0, r_1)$ *is the maximal interval of existence.*

Proof. Assume that the solution is defined on $[r_0, \infty)$ and $N(r) > 0$. Then (6.3.9) and (6.0.4) imply that

$$r^2Nv'' + 2\Phi v' + 2v - \frac{4}{\|\Lambda_0\|^2}v^2 \geq 0\,. \tag{6.3.11}$$

Consider the differential equation

$$r^2N\tilde{v}'' + r\tilde{v}' + 2\tilde{v} - \frac{4}{\|\Lambda_0\|^2}\tilde{v}^2 = 0\,, \tag{6.3.12}$$

$$\tilde{v}(r_0) = v(r_0) \quad \text{and} \quad \tilde{v}'(r_0) = v'(r_0)\,. \tag{6.3.13}$$

Lemma 6.3.8. $v'(r) > \tilde{v}'(r)$ *for all* $r > r_0$ *and hence* $v(r) > \tilde{v}(r)$ *for all* $r > r_0$.

Proof. First note that it follows from proposition 6.3.6 that

$$v'(r) > 0 \quad \text{and} \quad v(r) > \frac{1}{2}\,\|\Lambda_0\|^2 \quad \forall\, r \geq r_0\,. \tag{6.3.14}$$

Because $N(r_0) > 0$, $P(r_0) \geq 0$, and $v'(r_0) > 0$, we get from (6.3.11), (6.3.12), and (6.3.13) that

$$r_0^2 N(r_0)(v''(r_0) - \bar{v}''(r_0)) \geq -(\Phi(r_0) - r_0)v'(r_0) > 0\,. \tag{6.3.15}$$

Thus $v''(r_0) > \bar{v}''(r_0)$ and hence $v'(r) > \bar{v}'(r)$ for $r > r_0$ with r near r_0. Suppose r_1 is the first $r > 0$ for which $v'(r_1) = \bar{v}'(r_1)$. Then it follows from (6.3.13), (6.3.14), and the fact that $v'(r) > \bar{v}'(r)$ for all $r \in [r_0, r_1)$ that

$$\frac{4}{\|\Lambda_0\|^2}v(r_1)^2 - 2v(r_1) > \frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1)\,. \tag{6.3.16}$$

since the function

$$k(x) = \frac{4}{\|\Lambda_0\|^2}x^2 - 2x > 0 \tag{6.3.17}$$

is monotonically increasing in the region $x > \frac{1}{2}\|\Lambda_0\|^2$. Then

$$r_1^2 N(r_1)(v''(r_1) - \bar{v}''(r_1)) \geq \frac{4}{\|\Lambda_0\|^2}v(r_1)^2 - 2v(r_1) - \left(\frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1)\right) > 0$$

by (6.3.11), (6.3.12), (6.3.16), $N(r_1) > 0$, and $P(r_1) \geq 0$. Therefore $v''(r_1) > \bar{v}''(r_1)$ and this implies that $v'(r_1) = \bar{v}'(r_1)$ is impossible. □

Lemma 6.3.9. *$\bar{v}'(r) > 0$ and $\bar{v}(r) > \frac{1}{2}\|\Lambda_0^2\|$ for $r > r_0$.*

Proof. Proved in similar fashion as proposition 6.3.6. □

Lemma 6.3.10. *If $f(r) = r\bar{v}' + 2\bar{v} - \frac{4}{\|\Lambda_0\|^2}\bar{v}^2$, then there exists an $R > r_0$ such that $f(r) < 0$ for all $r \geq R$.*

Proof. Suppose $f(r_1) = 0$ for some $r_1 > r_0$. Differentiating f yields $f' = r\bar{v}'' + \left(3 - \frac{8}{\|\Lambda_0\|^2}\bar{v}\right)\bar{v}'$. Since $f(r_1) = 0$, the differential equation (6.3.12) shows that

$$r_1^2 N(r_1)\bar{v}''(r_1) = 0$$

and hence $\bar{v}''(r_1) = 0$ as $N(r_1) > 0$. Thus

$$f'(r_1) = \left(3 - \frac{8}{\|\Lambda_0\|^2}\bar{v}(r_1)\right)\bar{v}'(r_1) < 0$$

by lemma 6.3.9. This shows that f can cross zero at most once. Thus f is either always positive for $r > r_0$ or there exist an $R > r_0$ such that $f(r) < 0$ for all $r \geq R$. Suppose $f(r) > 0$ for all $r > r_0$. Then

$$r\bar{v}' + 2\bar{v} - \frac{4}{\|\Lambda_0\|^2}\bar{v}^2 > 0$$

or equivalently

$$\frac{d\bar{v}}{-2\bar{v} + \frac{4}{\|\Lambda_0\|^2}\bar{v}^2} > \frac{dr}{r}$$

by lemma 6.3.9. But

$$\int_{\bar{v}(r_0)}^{\infty} \frac{d\bar{v}}{-2\bar{v} + \frac{4}{\|\Lambda_0\|^2}\bar{v}^2} < \infty$$

while

$$\int_{r_0}^{\infty} \frac{dr}{r} = \infty$$

which is a contradiction. □

Consider the differential equation

$$r^2\bar{v}'' + r\bar{v}' + 2\bar{v} - \frac{4}{\|\Lambda_0\|^2}\bar{v}^2 = 0 \tag{6.3.18}$$

$$\bar{v}(R) = \bar{v}(R) \quad \text{and} \quad \bar{v}(R) = \bar{v}'(R) \tag{6.3.19}$$

where R is defined in lemma 6.3.10.

Lemma 6.3.11. *$\bar{v}(r) > \bar{v}(r)$ and $\bar{v}'(r) > \bar{v}'(r)$ for all $r > R$.*

Proof. From (6.3.12), (6.3.18) and (6.3.19) we have

$$\bar{v}''(R) - \bar{v}''(R) = \frac{f(R)}{R^2}\left(1 - \frac{1}{N(R)}\right). \tag{6.3.20}$$

Since $0 < N(r_0) < 1$ and we are assuming that $N > 0$, it follows from proposition 6.3.3 that $0 < N < 1$. Therefore (6.3.20) shows that $\bar{v}''(R) > \bar{v}''(R)$ and hence $\bar{v}'(r) > \bar{v}'(r)$ for $r > R$ with r near R. Suppose there exists a smallest $r_1 > R$ for which $\bar{v}(r) = \bar{v}(r)$. Using similar arguments as in proving lemma 6.3.8, it can be shown that

$$\bar{v}'(r) > 0 \text{ and } \bar{v}(r) > \frac{1}{2}\|\Lambda_0\|^2 \text{ for all } r > R. \tag{6.3.21}$$

Thus

$$\bar{v}(r_1) > \bar{v}(r_1) > 0 \quad \text{and} \quad \frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1) > \frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1)$$

and hence using (6.3.12) and (6.3.18) we see that

$$r_1^2 N(r_1)\bar{v}''(r_1) - r_1^2 N(r_1)\bar{v}''(r_1) = \frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1) - \left(\frac{4}{\|\Lambda_0\|^2}\bar{v}(r_1)^2 - 2\bar{v}(r_1)\right) > 0$$

which implies that

$$\bar{v}''(r_1) > N(r_1)\bar{v}''(r_1) > \bar{v}''(r_1).$$

Therefore $\bar{v}'(r_1) = \bar{v}'(r_1)$ is impossible. □

Lemma 6.3.12. *There exists a $\bar{r} > R$ for which $\lim_{r\to\bar{r}} \bar{v}(r) = \infty$ and $\lim_{r\to\bar{r}} \bar{v}'(r) = \infty$.*

Proof. Let $t = \ln(r)$. Then we can write (6.3.18) as

$$\ddot{\bar{v}} + 2\bar{v} - \frac{4}{\|\Lambda_0\|}^2 \bar{v}^2 = 0 \qquad (\dot{\cdot}) := \frac{d(\cdot)}{dt}. \tag{6.3.22}$$

So $\ddot{\bar{v}} > 0$ by (6.3.21) and (6.3.22). This implies that $\dot{\bar{v}}$ is increasing. If $T = \ln(R)$ then $\dot{\bar{v}}(T) > 0$ by (6.3.21), and thus $\dot{\bar{v}}(t) > \dot{\bar{v}}(T) > 0$ for $t > T$ as $\dot{\bar{v}}$ is increasing. It follows that $\lim_{t\to\infty} \bar{v}(t) = \infty$.

The differential equation (6.3.22) admits a first integral

$$H(t) = \frac{1}{2}\dot{\bar{v}}^2 + \bar{v}^2 - \frac{4}{3\,\|\Lambda_0\|^2}\bar{v}^3 . \tag{6.3.23}$$

Therefore if we let $H_0 = H(T)$, then

$$\frac{1}{2}\dot{\bar{v}}(t)^2 = H_0 + \bar{v}(t)^2\left(\frac{4}{3\,\|\Lambda_0\|^2}\bar{v}(t) - 1\right)$$

as H is a constant of the motion. Since $\bar{v}(t)$ is increasing and $\lim_{t\to\infty} \bar{v}(t) = \infty$, there exists a $t_1 > T$ such that

$$\frac{1}{2}\dot{\bar{v}}^2 > \frac{2}{3\,\|\Lambda_0\|^2}\bar{v}^3 \quad \forall\, t \geq t_1$$

As $\bar{v} > 0$ and $\dot{\bar{v}} > 0$, the above expression is equivalent to

$$\frac{\dot{\bar{v}}}{\bar{v}^{\frac{3}{2}}} > \frac{4}{3\,\|\Lambda_0\|^2} \quad \forall\, t \geq t_1 .$$

Integrating both sides yields

$$-\frac{2}{\sqrt{\bar{v}(t_2)}} + \frac{2}{\sqrt{\bar{v}(t_1)}} \geq \frac{4}{3\,\|\Lambda_0\|^2}(t_2 - t_1) ,$$

or equivalently

$$\sqrt{\bar{v}(t_2)} \geq \frac{2}{\frac{2}{\sqrt{\bar{v}(t_1)}} + \frac{4}{3\|\Lambda_0\|^2}(t_1 - t_2)} .$$

This shows that there exists a $\bar{t}$ such that $\lim_{t\to\bar{t}} \bar{v}(t) = \infty$. But $r = e^t$, so if we let $\bar{r} = e^{\bar{t}}$ then it follows that $\lim_{r\to\bar{r}} \bar{v}(r) = \infty$ and $\lim_{r\to\bar{r}} \bar{v}'(r) = \infty$. □

The above lemmas show that there exist a $\tilde{r} \leq \bar{r}$ such that $\lim_{r\nearrow\tilde{r}} v'(r) = \infty$ which proves that $\Lambda_+(r)$ or $\Lambda'_+(r)$ becomes unbounded as $r \to \tilde{r}$. This contradicts the solution existing on $[r_0, \infty)$. In view of proposition 6.3.4, we must have $N(\tilde{r}) = 0$. Let r_1 be the smallest r such that $N = 0$. Then (4.1.16) implies that

$$r_1 N'(r_1) = 1 - \frac{2}{r_1^2}P(r_1) \tag{6.3.24}$$

while while it follows from (6.3.9) and (6.0.4)

$$\begin{aligned}\left(r_1 - \frac{2}{r_1}P(r_1)\right) v'(r_1) + 2v(r_1) - \frac{4}{\|\Lambda_0\|^2}v(r_1)^2 \\ \geq \left(\frac{\|[\Lambda_+(r_1), c(\Lambda_+(r_1))]\|}{\|\Lambda_+(r_1)\|^4} - \frac{4}{\|\Lambda_0\|^2}\right) v(r_1)^2 \geq 0 .\end{aligned}$$

But $v(r_1) > \|\Lambda_0\|^2/2$ implies that $4v(r_1)^2/\|\Lambda_0\|^2 - 2v(r_1) > 0$ and therefore

$$\left(r_1 - \frac{2}{r_1}P(r_1)\right)v'(r_1) > 0\,.$$

Since $v'(r_1) > 0$ the above inequality implies that $r_1 > 2P(r_1)/r_1$ and hence $1 - 2P(r_1)/(r_1^2) > 0$. Combining this with (6.3.24) we see that $N'(r_1) > 0$ which contradicts $N(r_1) = 0$. Therefore $[r_0, r_1)$ must be the maximal interval of existence. □

Theorem 6.3.13. *If $\{\Lambda_+(r), m(r)\}$ is a solution to* (4.1.7) *and* (4.1.9) *defined on $[r_0, \infty)$ ($r_0 > 0$) and it satisfies $N > 0$, $\|\Lambda_+(r_0)\| < \|\Lambda_0\|/\sqrt{2}$ and $N(r_0) < 1$ then $\|\Lambda_+(r)\| < \|\Lambda_0\|/\sqrt{2}$ for all $r \geq r_0$.*

Proof. Since N cannot cross 1 from below by proposition 6.3.3, we have $0 < N(r) < 1$ for all $r \geq r_0$. Let $v(r) = \|\Lambda_+(r)\|^2$ and suppose there exists a $r_1 > r_0$ such that $v(r_1) > \|\Lambda_0\|^2/2$. Then by the mean value theorem there exists a $r_* \in (r_1, r_2)$ such that $v'(r_*) > 0$ and $v(r_*) > \|\Lambda_0\|^2/2$. The proof then follows from proposition 6.3.7. □

The next theorem, which guarantees that the mass is bounded, is a generalization of theorem 2 from [32] and the proof uses similar methods.

Theorem 6.3.14. *If $\{\Lambda_+(r), m(r)\}$ is a solution to* (4.1.7) *and* (4.1.9) *defined on $[r_0, \infty)$ with $N > 0$, $N(r_0) < 1$ and $\|\Lambda_+(r_0)\| \leq \|\Lambda_0\|/\sqrt{2}$ then*

$$\int_{r_0}^{\infty} NG\,dr < \infty\,.$$

Proof. It follows from theorem 6.3.13 that

$$\|\Lambda_+(r)\|^2 \leq \frac{1}{2}\|\Lambda_0\|^2 \quad \forall\, r \geq r_0\,. \tag{6.3.25}$$

Let $\{X_k\}_{k=1}^{n}$ be a orthogonal basis for V_2 with normalization $\|X_k\| = 1/\sqrt{2}$. Define
Let $\{X_k\}_{k=1}^{n}$ be a orthogonal basis for V_2 with normalization $\|X_k\| = 1/\sqrt{2}$. Define

$$w_k(r) := \langle\langle X_k | \Lambda_+(r) \rangle\rangle\,.$$

Then it follows from (4.1.9) that for any q,

$$qr^{q-1}Nw_k' = (r^q N w_k')' + 2r^{q-1}NGw_k' + r^{q-2}\langle\langle X_k|\mathcal{F}\rangle\rangle\,. \tag{6.3.26}$$

Lemma 6.3.15. *If w_k has a critical point $c \in [r_0, \infty)$ then*

$$\int_{r_0} N|w_k'|^2 dr < \infty\,.$$

Proof. Let C denote the set of critical points of $w_k(r)$. Since $w_k(r)$ is analytic by proposition 6.3.4 the set C can have no limit points. There are two cases to consider. either C is bounded or C is unbounded. Note that C is not empty by assumption.

If C is bounded, let $\bar{c} = \sup C$. Then w_k' must be either greater than zero or less than zero for $r > \bar{c}$. We first assume that $w_k'(r) > 0$ for $r > \bar{c}$. Then integrating (6.3.26) with $q = 0$ yields

$$Nw_k'(r) = -\int_{\bar{c}}^{r} \frac{2}{\rho} NGw_k' d\rho - \int_{\bar{c}}^{r} \rho^{-2}\langle\langle X_k|\mathcal{F}\rangle\rangle d\rho \leq \frac{1}{\sqrt{2}}\int_{\bar{c}}^{r} \rho^{-2}\|\mathcal{F}\|\,d\rho\,.$$

But it is clear from (6.3.8) and (6.3.25) that there exists a $\beta > 0$ such that $\|\mathcal{F}(r)\| \leq \sqrt{2}\beta$ for all $r \geq r_0$. Consequently, $Nw_k'(r) \leq \beta(\bar{c}^{-1} - r^{-1})$ and using

$$\sup_{r_0 \leq \rho \leq r} \beta\left(\frac{1}{\bar{c}} - \frac{1}{\rho}\right) = \beta\left(\frac{1}{\bar{c}} - \frac{1}{r}\right)$$

we get

$$\begin{aligned}\int_{\bar{c}}^{r} N|w_k'|^2 d\rho &\leq \beta\left(\frac{1}{\bar{c}} - \frac{1}{r}\right)\int_{\bar{c}}^{r} w_k'(\rho)d\rho \leq \beta\left(\frac{1}{\bar{c}} - \frac{1}{r}\right)|w_k(r) - w_k(\bar{c})| \\ &\leq \beta\left(\frac{1}{\bar{c}} - \frac{1}{r}\right)|\langle\langle X_k|\Lambda_+(r) - \Lambda_+(\bar{c})\rangle\rangle| \leq \frac{\beta}{\sqrt{2}}\left(\frac{1}{\bar{c}} - \frac{1}{r}\right)\|\Lambda_+(r) - \Lambda_+(\bar{c})\| \\ &\leq \frac{\beta}{\sqrt{2}}\left(\frac{1}{\bar{c}}\right)\alpha \end{aligned} \tag{6.3.27}$$

for some $\alpha > 0$ by (6.3.25). Letting $r \to \infty$ in the above expression shows that

$$\int_{\bar{c}}^{\infty} N|w_k'|^2 dr \leq \frac{\alpha\beta}{\sqrt{2}\bar{c}} < \infty .$$

Similar arguments show that the above inequality continues to hold if $w_k < 0$.

If C is unbounded, there there exists a sequence of critical points $\{c_j\}$ such that $c_j < c_{j+1}$, $[c_1, \infty) = \bigcup_{j=1}^{\infty}[c_j, c_{j+1}]$, and w_k does not change sign on (c_j, c_{j+1}). From (6.3.27) we see that

$$\int_{c_j}^{c_{j+1}} N|w_k'|^2 dr \leq \frac{\alpha\beta}{\sqrt{2}}\left(\frac{1}{c_j} - \frac{1}{c_{j+1}}\right)$$

and so

$$\int_{c_1}^{c_j} N|w_k'|^2 dr \leq \frac{\alpha\beta}{\sqrt{2}}\left(\frac{1}{c_1} - \frac{1}{c_j}\right) .$$

Letting $j \to \infty$ gives

$$\int_{c_1}^{\infty} N|w_k'|^2 dr \leq \frac{\alpha\beta}{\sqrt{2}c_1} < \infty .$$

□

Lemma 6.3.16. *If $w_k' > 0$ or $w_k' < 0$ for $r > r_0$ then for any $q > 1$ and $r > r_0$*

$$r^q N(r)|w_k'(r)| \leq r_0^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}(r^q - r_0^q) + \int_{r_0}^{r} \|\mathcal{F}\| \rho^{q-2} d\rho .$$

Proof. Using Young's inequality it is not difficult to verify that

$$q\int_{r_0}^{r} |w_k'|N\rho^{q-1}d\rho \leq 2\int_{r_0}^{r} |w_k'|^3 N\rho^{q-1}d\rho + \frac{2}{3}\sqrt{q}(r^q - r_0^q) . \tag{6.3.28}$$

Assume $w_k' > 0$. Then integrating (6.3.26) yields

$$\begin{aligned} r^q N(r)w_k'(r) &= r_0^q N(r_0)w_k'(r_0) + \\ &\quad q\int_{r_0}^{r} \left(\rho^{q-1}Nw_k' - 2\rho^{q-1}NGw_k'\right) d\rho - \int_{r_0}^{r} \rho^{q-2}\langle\langle X_k|\mathcal{F}\rangle\rangle \, d\rho . \end{aligned}$$

But

$$|w_k'|^2 = |\langle\langle \Lambda_+' | X \rangle\rangle|^2 \leq \frac{1}{2} \left\|\Lambda_+'\right\|^2 = G \quad \text{and} \quad |\langle\langle X_k | \mathcal{F} \rangle\rangle| \leq \|X_k\| \, \|\mathcal{F}\| \leq \|\mathcal{F}\|$$

and therefore

$$\begin{aligned} r^q N(r) w_k'(r) &\leq r_0^q N(r_0) w_k'(r_0) + \\ &\quad q \int_{r_0}^{r} \left(\rho^{q-1} N w_k' - 2\rho^{q-1} N |w_k'|^2 w_k'\right) d\rho + \int_{r_0}^{r} \|\mathcal{F}\| \, \rho^{q-2} d\rho \\ &\leq r_0^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}(r^q - r_0^q) + \int_{r_0}^{r} \|\mathcal{F}\| \, \rho^{q-2} d\rho \quad \text{by (6.3.28)} \end{aligned}$$

Similar arguments show that if $w_k' < 0$ then

$$-r^q N(r) w_k'(r) \leq -r_0^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}(r^q - r_0^q) + \int_{r_0}^{r} \|\mathcal{F}\| \, \rho^{q-2} d\rho \,.$$

□

Lemma 6.3.17. *If $w_k' > 0$ or $w_k' < 0$ for $r > r_0$ then there exists a constant $h > 0$ such that $N|w_k'| \leq h$ for $r > r_0$.*

Proof. Now, $\|\mathcal{F}\| = \|[\hat{F}, \Lambda_+]\| \leq h_1 \|\hat{F}\|$ for some constant $h_1 > 0$ since $[\cdot,\cdot]$ is a continuous bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$. So then

$$N(r)|w_k'(r)| \leq \left(\frac{r_0}{r}\right)^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}\left(1 - \left(\frac{r_0}{r}\right)^q\right) + \frac{h_1}{r^q}\int_{r_0}^{r} \|\mathcal{F}\| \, \rho^{q-2} d\rho \,.$$

by lemma 6.3.16. But

$$\begin{aligned} \int_{r_0}^{r} \|\mathcal{F}\| \, \rho^{q-2} d\rho &\leq \left(\int_{r_0}^{r} \|\mathcal{F}\|^2 \rho^{-2} d\rho\right)^{\frac{1}{2}} \left(\int_{r_0}^{r} \rho^{2(q-1)} d\rho\right)^{\frac{1}{2}} \\ &= \left(\int_{r_0}^{r} \|\mathcal{F}\|^2 \rho^{-2} d\rho\right)^{\frac{1}{2}} \sqrt{\frac{r^{2q-1} - r_0^{2q-1}}{2q-1}} \,. \end{aligned}$$

Combining the above two inequalities yields

$$\begin{aligned} N(r)|w_k'(r)| &\leq \left(\frac{r_0}{r}\right)^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}\left(1 - \left(\frac{r_0}{r}\right)^q\right) + \\ &\quad \frac{h_1}{\sqrt{2q-1}} \left(\int_{r_0}^{r} \|\mathcal{F}\|^2 \rho^{-2} d\rho\right)^{\frac{1}{2}} \sqrt{\frac{r^{2q-1} - r_0^{2q-1}}{r^{2q}}} \,. \end{aligned} \tag{6.3.29}$$

From (4.1.7) we have $r^2 P = r^{-2}\|\hat{F}\|^2/2 \leq m'$, while $N > 0$ implies $2m(r) < r$ and hence

$$\int_{r_0}^{r} \|\mathcal{F}\|^2 \rho^{-2} d\rho \leq 2m(r) - 2m(r_0) \leq r \,. \tag{6.3.30}$$

So

$$N(r)|w_k'(r)| \leq \left(\frac{r_0}{r}\right)^q N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{q}\left(1 - \left(\frac{r_0}{r}\right)^q\right) + \frac{h_1}{\sqrt{2q-1}}\sqrt{1 - \left(\frac{r_0}{r}\right)^{2q-}}$$

by (6.3.29) and (6.3.30). Setting $q = 2$ in the above expression yields

$$N(r)|w_k'(r)| \le N(r_0)|w_k'(r_0)| + \frac{2}{3}\sqrt{2} + \frac{h_1}{\sqrt{3}}\,.$$

□

Lemma 6.3.18. *If* $w_k' > 0$ *or* $w_k' < 0$ *for* $r > r_0$ *then*

$$\int_{r_0}^{\infty} N|w_k'|^2 dr < \infty\,.$$

Proof. Suppose $w_k' > 0$. Then

$$\begin{aligned}\int_{r_0}^{r} N|w_k'|^2 d\rho &\le h\int_{r_0}^{r} w_k'\, d\rho &&\text{by lemma 6.3.17}\\ &\le h|w_k(r) - w_k(r_0)| = h|\langle\langle X_k|\Lambda_+(r) - \Lambda_+(r_0)\rangle\rangle| \\ &\le \frac{h}{\sqrt{2}}(\|\Lambda_+(r)\| + \|\Lambda_+(r_0)\|) \le K\end{aligned}$$

for some constant $K > 0$ by (6.3.25). Letting $r \to \infty$ in the above expression completes the proof. □

Now,

$$G = \frac{1}{2}\left\|\Lambda_+'\right\|^2 = \frac{1}{2}\sum_{k=1}^{n}|\langle\langle \Lambda_+' \,|\, \|X_k\|^{-1} X_k\rangle\rangle| = \sum_{k=1}^{n}|w_k'|^2\,.$$

Therefore

$$\int_{r_0}^{\infty} NG\,dr = \sum_{k=1}^{n}\int_{r_0}^{\infty} N|w_k'|^2 dr < \infty$$

by lemmas 6.3.15 and 6.3.18. □

Corollary 6.3.19. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *defined on* $[r_0, \infty)$ *with* $N > 0$, $N(r_0) < 1$ *and* $\|\Lambda_+(r_0)\| \le \|\Lambda_0\|/2$ *then* $\lim_{r\to\infty} m(r)$ *exists and* $\lim_{r\to\infty} N(r) = 1$

Proof. Since $N > 0$, $P \ge 0$,$G \ge 0$, and $\|\Lambda_+(r)\| \le \|\Lambda_0\|/2$ it follows from (4.1.7) that

$$0 \le m' \le NG + \frac{K}{r^2}$$

for some constant $K > 0$. Integrating yields

$$m(r) \le m(r_0) + \int_{r_0}^{\infty} NG\,dr + \frac{K}{r_0} < \infty$$

by theorem 6.3.14. Thus $\lim_{r\to\infty} m(r)$ exists as m is increasing and bounded above. From the definition of N it is then clear that $\lim_{r\to\infty} N(r) = 1$. □

Proposition 6.3.20. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *defined on* $[r_0, \infty)$ *with* $N > 0$, $N(r_0) < 1$ *and* $\|\Lambda_+(r_0)\| \le \|\Lambda_0\|/2$ *then* (4.1.8) *can be solved for* S *and* $S(r_0)$ *can be chosen so that* $\lim_{r\to\infty} S(r) = 1$.

Proof. We can solve equation (4.1.8) to get $S(r) = S_0 \exp(\int_{r_0}^{r} 2\rho^{-1} G\,d\rho)$, where $S_0 > 0$ is an arbitrary constant. Because $N > 0$ and $\lim_{r\to\infty} N(r) = 1$ by corollary 6.3.19, N is bounded below on $[r_0, \infty)$ by a positive constant $\hat{N}$. Then

$$\int_{r_0}^{\infty} \frac{2}{r} G\,dr = \int_{r_0}^{\infty} \frac{2}{Nr} NG\,dr \leq \frac{2}{r_0 \hat{N}} \int_{r_0}^{\infty} NG\,dr < \infty$$

by theorem 6.3.14. So we can let $S_0 = \exp(-\int_{r_0}^{\infty} 2r^{-1} G\,dr)$ which then implies that $\lim_{r\to\infty} S(r) = 1$. □

Proposition 6.3.21. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *defined on* $[r_0, \infty)$ *with* $N > 0$, $N(r_0) < 1$ *and* $\|\Lambda_+(r_0)\| \leq \|\Lambda_0\|/2$ *then there exists a constant* $h > 0$ *such that* $rN\|\Lambda'_+\|^2 < h$ *for all* $r \geq r_0$.

Proof. From corollary 6.3.19 and theorem 6.3.13, we get that $P(r)$ is bounded and $\lim_{r\to\infty} m(r) = m_\infty$ for some constant $m_\infty > 0$. Then from the definition of $\Phi(r)$ (see (6.3.10)) it is clear that there exists a r_* and an $\epsilon > 0$ such that $\Phi(r) \geq \epsilon > 0$ for all $r \geq r_*$. Thus

$$\frac{\Phi(r)}{r^2 N} + \frac{2G}{r} > 0 \quad \forall\, r > r_* . \tag{6.3.31}$$

Because $N > 0$ and $\lim_{r\to\infty} N(r) = 1$ by corollary 6.3.19, N is bounded below on $[r_0, \infty)$ by a positive constant $\hat{N}$. Also note that theorem 6.3.13 and (6.3.8) imply that $\|\mathcal{F}\|$ is bounded. So then

$$\begin{aligned}\int_{r_*}^{\infty} \frac{1}{r} |\langle\langle \Lambda'_+ | \mathcal{F} \rangle\rangle|\,dr &\leq \int_{r_*}^{\infty} \sqrt{N}\,\|\Lambda'_+\|\,\frac{1}{\sqrt{N}r}\,\|\mathcal{F}\|\,dr \\ &\leq \frac{1}{2}\int_{r_*}^{\infty} \left(NG + \frac{1}{Nr^2}\|\mathcal{F}\|^2\right) dr < \infty\end{aligned} \tag{6.3.32}$$

by theorem 6.3.14.

From (4.1.9) and (4.1.16) it follows that

$$(rNG)' = -\left(\frac{\Phi(r)}{r^2 N} + \frac{2G}{r}\right) rNG + NG - \frac{1}{r}\langle\langle \Lambda'_+ | \mathcal{F} \rangle\rangle .$$

Therefore

$$rN(r)G(r) = e^{-\Psi(r)}\left(rN(r_*)G(r_*) + \int_{r_*}^{r}\left(NG - \frac{1}{\rho}\langle\langle \Lambda'_+ | \mathcal{F} \rangle\rangle\right) e^{\Psi(\rho)}\,d\rho\right) ,$$

where $\Psi(r) = \int_{r_*}^{r} (\rho^{-2} N^{-1} \Phi + 2\rho^{-1} G)\,d\rho$. It then follows from theorem 6.3.14, (6.3.31), (6.3.32), and

$$\sup_{r_* \leq \rho \leq r} \exp(\Psi(\rho)) = \Psi(r)$$

that there exists a constant $\tilde{h}$ such that $rN(r)G(r) < \tilde{h}$ for all $r \geq r_*$. Letting

$$h = \max\{\, \tilde{h}\, , \max\{\, \rho N(\rho) G(\rho) \,|\, r_0 \leq \rho \leq r_* \,\}\,\}$$

completes the proof. □

Proposition 6.3.22. *If* $\{\Lambda_+(r), m(r)\}$ *is a solution to* (4.1.7) *and* (4.1.9) *defined on* $[r_0, \infty)$ *with* $N > 0$, $N(r_0) < 1$ *and* $\|\Lambda_+(r_0)\| \leq \|\Lambda_0\|/2$ *then there exist a constant* $h > 0$ *such that* $r\|\Lambda'_+\| \leq h$ *for all* $r \geq r_0$.

Proof. From theorem 6.3.13 and (6.3.8) we see that

$$\sup_{r_0 \le r < \infty} S(r)\,\|\mathcal{F}\| < K$$

for some constant $K > 0$. So then

$$\|N(r)S(r)\Lambda'_+(r) - N(r_1)S(r_1)\Lambda'_+(r_1)\| \le \frac{K}{r_0} \quad \forall r, r_1 \in [r_0, \infty) \tag{6.3.33}$$

by (6.3.6). An immediate consequence of corollary 6.3.19 and propositions 6.3.20 and 6.3.21 is that

$$\lim_{r\to\infty} N(r)S(r)\Lambda'_+(r) = 0\,.$$

So then letting $r_1 \to \infty$ in (6.3.33) yields $\|N(r)S(r)\Lambda'_+(r)\| \le K/r_0$ for all $r \ge r_0$. However, we know from corollary 6.3.19 and proposition 6.3.20 that both $S(r)$ and $N(r)$ are bounded below by a positive number $\epsilon > 0$. Therefore if we let $h = K/(r_0\epsilon^2)$ then $\|\Lambda'_+(r)\| \le h$ for all $r \ge r_0$. □

We are now ready to prove theorem 6.0.10.

Proof of theorem 6.0.10. (i)-(v) : These are just a restatement of corollary 6.3.19, theorem 6.3.13, and propositions 6.3.1, and 6.3.3.
(vi) : Since N is bounded below away from zero and $N \to 1$ as $r \to \infty$, the change of variable from r to τ given by (4.1.25) is well defined and $\tau \to \infty$ as $r \to \infty$. Therefore to prove *(vi)* we can show instead that $\|\Lambda_+(\tau) - \mathfrak{F}^\times\| \to 0$ and $\dot{\Lambda}_+(\tau) \to 0$ as $\tau \to \infty$. Using (4.1.27)-(4.1.32) it is easy to show that $\Lambda_+(\tau)$ satisfies

$$\ddot{\Lambda}_+ - \dot{\Lambda}_+ + \mathcal{F} = \delta(\tau)\dot{\Lambda}_+\,, \tag{6.3.34}$$

where

$$\delta(\tau) = 2\mu(\tau) - \kappa(\tau) - 1\,. \tag{6.3.35}$$

Since $\lim_{r\to\infty} N(r) = 1$ we have $\lim_{\tau\to\infty} \mu(\tau) = 1$ and hence it follows from proposition 6.3.22 that $\lim_{\tau\to\infty} \Lambda'_+(\tau) = 0$. Therefore $\lim_{\tau\to\infty} \mu(\tau)^2 G(\tau) = 0$. Also, because $\Lambda_+(r)$ is bounded and hence $\Lambda_+(\tau)$ is also bounded, we get $\lim_{\tau\to\infty} r(\tau)^{-2} P(\tau) = 0$. From the definition (4.1.26) of κ it is then clear that $\lim_{\tau\to\infty} \kappa(\tau) = 1$ and hence

$$\lim_{\tau\to\infty} \delta(\tau) = 0\,.$$

Another consequence of proposition 6.3.22 is that $\dot{\Lambda}_+(\tau)$ is bounded. Therefore we see that $\mathbf{X}(\tau) = (\Lambda_+(\tau), \dot{\Lambda}_+(\tau))$ is a bounded, non-trivial solution to the differential equation (6.2.10). So $\|\Lambda_+(\tau) - \mathfrak{F}^\times\| \to 0$ and $\dot{\Lambda}_+(\tau) \to 0$ as $\tau \to \infty$ by theorem 6.2.6.

If S_λ is a Π-system then it follows from the discussion in section 4.3 that

$$\Lambda_+(r) = \sum_{\alpha\in S_\lambda} w_\alpha(r)\mathbf{e}_\alpha$$

where the $w_\alpha(r)$ are real valued functions. Therefore $\Lambda_+(\tau) \in E_+$ for all τ. Since $\mathfrak{F}^\times \cap E_+$ is a discrete set by lemma 6.1.2 and $\|\Lambda_+(\tau) - \mathfrak{F}^\times\| \to 0$ as $\tau \to \infty$, there exists a $\Omega_+^\infty \in \mathfrak{F}^\times \cap E_+$ such that $\lim_{r\to\infty} \Lambda_+(r) = \Omega_+^\infty$. □

We now show that any local solution that can be continued out to a global solution necessarily satisfies (6.0.1) and (6.0.2).

Proposition 6.3.23. *Suppose $\{\Lambda_+(r), m(r)\}$ is a local solution to* (4.1.7) *and* (4.1.9) *defined in a neighborhood of $r = r_*$ where $r_* = 0$ or $r_* = r_H > 0$. If the local solution can be continued out to $r = \infty$ with $N(r) > 0$ for $r > r_*$, then*

$$N(r_0) < 1\,, \quad \|\Lambda_+(r_0)\| \le \frac{1}{\sqrt{2}}\|\Lambda_0\|\,, \quad [\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0\,,$$

for some $r_0 > r_*$.

Proof. $\mathbf{r_* = 0}$: We know by theorem 5.2.1 that $m(r)$ and $\Lambda_+(r)$ are analytic in a neighborhood of $r = 0$ and

$$m(r) = O(r^3) \quad \text{and} \quad \Lambda_+(r) = \Omega_+ + Xr^2 + O(r^3) \quad \text{as } r \to 0$$

for some $X \in E_1$. Substituting powerseries representation for $m(r)$ and $\Lambda_+(r)$ into the field equations (4.1.7)and (4.1.9) shows that $m(r) = \|X\|^2 r^3 + O(r^4)$ near $r = 0$ and hence

$$N(r) = 1 - 2\|X\|^2 r^2 + 0(r^3) \text{ as } r \to 0. \tag{6.3.36}$$

Let $v(r) = \|\Lambda_+(r)\|^2$. Then

$$\begin{aligned} v(0) &= \tfrac{1}{2}\langle\langle [\Lambda_0, \Omega_+] | \Omega_+ \rangle\rangle = \tfrac{1}{2}\langle\langle \Lambda_0 | [\Omega_+, \Omega_-] \rangle\rangle && \text{by (2.1.2) and } \Omega_+ = -c(\Omega_-) \\ &= \tfrac{1}{2}\|\Lambda_0\|^2 && \text{since } [\Omega_+, \Omega_-] = \Lambda_0 \end{aligned}$$

The solution is trivial for $X = 0$ by proposition 6.3.2. If we assume the solution is non-trivial, then we must have $X \neq 0$. So (6.3.36) shows there exists an $\epsilon > 0$ such that $0 < N(r) < 1$ for $0 < r < \epsilon$. Suppose there exists a $r_0 \in (0, \epsilon)$ for which $v(r_0) > \|\Lambda_0\|^2/2$. By the mean value theorem there exist a $r_1 \in (0, r_0)$ such that $v(r_1) > \|\Lambda_0\|^2/2$ and $v'(r_1) > 0$. It then follows from proposition 6.3.7 that the maximal interval of existence for the solutions is finite which contradicts the assumption that it is defined on $(0, \infty)$. Therefore we conclude that $v_r \le \|\Lambda_0\|^2/2$ for all $0 < r < \epsilon$. To complete the proof for $r_* = 0$. we observe that $[\Lambda'_+(r), \Lambda_-(r)] + [\Lambda'_-(r), \Lambda_+(r)] = 0$ near $r = 0$ by theorem 5.2.4.

$\mathbf{r_* = r_H}$: From the boundary conditions at $r = r_H > 0$ we have $N(r_H) = 0$ and $N'(r_H) > 0$ and hence $0 < N(r) < 1$ for $r > r_H$ with r near r_H. We know by theorem 5.2.8 that $\Lambda_+(r)$ and $N(r)$ are analytic in the variable $t = r - r_H$ near $t = 0$ and there exists a $X \in V_2$ so that

$$N(t) = \nu t + 0(t^2) \quad \text{and} \quad \Lambda_+(t) = X + 0(t)\,,$$

where

$$\nu = \frac{1}{r_H} - \frac{1}{r_H^3}\|\Lambda_0 + [X, c(X)]\|^2 > 0\,.$$

Expanding $N(t)$ and $\Lambda_+(t)$ in powerseries about $t = 0$, it follows from the field equations (4.1.7) and (4.1.9) that

$$\Lambda'_+(r_H) = \frac{1}{2\nu r_H^2}[X, \Lambda_0 + [X, c(X)]]\,. \tag{6.3.37}$$

Note also that $[\Lambda'_+(r), \Lambda_-(r)] + [\Lambda'_-(r), \Lambda_+(r)] = 0$ for r near r_H by theorem (5.2.9).
Let $v(r) = \|\Lambda_+(r)\|^2$. If $v(r_H) < \|\Lambda_0\|^2/2$ then $v(r) < \|\Lambda_0\|^2/2$ for r near r_H and

we are done. So assume that $v(r_H) \geq \|\Lambda_0\|^2/2$. Now,

$$v'(r_H) = 2\langle\langle \Lambda'_+(r_H) | \Lambda_+(r_H) \rangle\rangle = \frac{1}{\nu r_H^2} \langle\langle [X, [\Lambda_0 + [X, c(X)]]] | X \rangle\rangle$$

by (6.3.37). Using (2.1.2)and $X \in V_2$, we can write the above expression as

$$v'(r_H) = \frac{1}{\nu r_H^2} (\|[X, c(X)]\|^2 - 2\|X\|^2) . \tag{6.3.38}$$

But

$$0 \leq \|\Lambda_0 + [X, c(X)]\|^2 = (\|\Lambda_0\|^2 - 2\|X\|^2) + (\|[X, c(X)]\|^2 - 2\|X\|^2) \tag{6.3.39}$$

by (2.1.2) and the fact that $X \in V_2$. Since $X = \Lambda_+(r_H)$,

$$v'(r_H) \leq 0 \iff \|\Lambda_+(r_H)\|^2 \leq \frac{1}{2}\|\Lambda_0\|^2 \tag{6.3.40}$$

by (6.3.38) and (6.3.39). Suppose $v(r_H) > \|\Lambda_0\|^2/2$. Then $v'(r_H) > 0$ by (6.3.40). But this implies that the maximum interval of existence for the solutions is finite by proposition 6.3.7. So $v(r_H) = \|\Lambda_0\|^2/2$ as the solution is assumed to exist on (r_H, ∞). Suppose now there exists a $r_0 > r_H$ with r_0 near r_H for which $v(r_0) > \|\Lambda_0\|^2/2$. Then by the mean value theorem there exist a $r_1 \in (r_H, r_0)$ so that $v(r_1) > \|\Lambda_0\|^2/2$ and $v'(r_1) > 0$. But this is impossible by proposition 6.3.7. Therefore we must have $v(r) \leq \|\Lambda_0\|^2/2$ for $r > r_H$ and r near r_H. □

Chapter 7

Conclusion

In this thesis, it has been established that the static spherically symmetric EYM equations admit bounded local solutions in the neighborhood of the origin $r = 0$, a black hole horizon $r = r_H > 0$, and spacial infinity $r = \infty$. As we mentioned earlier, this local existence result provides a necessary starting point for the shooting method both analytically and numerically. We have also determined the behavior near $r = \infty$ and established global bounds for solutions satisfying $N(r) > 0$ for all r and $N(r_0) < 1$, $\|\Lambda_+(r_0)\| \leq \|\Lambda_0\| / \sqrt{2}$ and $[\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0$ at some point $r_0 > 0$. As we discussed in chapter 6, if any of the local solutions near $r = 0$ or $r = r_H$ could be extended to $r = \infty$ with $N(r) > 0$ then there certainly would be a point $r_0 > 0$ at which $N(r_0) < 1$, $\|\Lambda_+(r_0)\| \leq \|\Lambda_0\| / \sqrt{2}$ and $[\Lambda'_+(r_0), \Lambda_-(r_0)] + [\Lambda'_-(r_0), \Lambda_+(r_0)] = 0$. For the behavior near $r = \infty$ there still remains the question of whether or not $\Lambda_+(r)$ always has a limit as $r \to \infty$. This is an interesting one and should be resolved. A numerical solutions for a Λ_0 for which S_λ is not a Π-system would aid in settling this problem. The model with the smallest number of free parameters and for which S_λ is not a Π-system is $\mathfrak{g} = \mathfrak{so}_5\mathbb{C}$ where Λ_0 has the characteristic $\chi = (2, 0)$. However, even in this simple case if a numerical solution were to be constructed by shooting from both ends, say $r = 0$ and $r = \infty$, and then matching somewhere in the middle, the number of free parameters would be 10. Of these, 3 would come from the local solution near $r = 0$, 4 from the local solution near $r = \infty$, and the remaining 3 would be due to the fact that we can only fix $\Lambda_+(r)$ at only one end. If we fixed $\Lambda_+(r)$ at $r = 0$, then at $r = \infty$ $\Lambda_+(r)$ could take values in $\mathfrak{F}^\times$ which turns out to be a 3-dimensional real variety. This implies that a search of a 10-dimensional parameter space would be required to construct a global solution. This is a difficult problem. On the theoretical side, a good place to start would be with the limit equation (6.2.9). The set $\mathfrak{F}^\times$ consists entirely of fixed points of (6.2.9). Therefore by linearizing (6.2.9) about the points of $\mathfrak{F}^\times$ it may be possible to determine the behavior of solutions to (6.2.9) in a neighborhood of $\mathfrak{F}^\times$. Hopefully, this information could be used to infer the behavior of the EYM equations near $\mathfrak{F}^\times$.

Although we do not yet have a clear indication on how classify all the possible global soliton and black hole solutions for arbitrary compact gauge groups, we have shown that it is possible to generalize many of the results from the $SU(2)$ analysis. We have already started to consider how to generalize the $SU(2)$ results contained in the papers [56, 59, 60, 66] by Smoller and Wasserman, with the aim of classifying all solutions that are defined in the far field, i.e for $r >> 1$, as was done for $G = SU(2)$.

Bibliography

[1] P. Arnold and L. McLerran, *Sphalerons, small fluctuations, and baryon-number violation in electroweak theory*, Phys. Rev. D **36** (1987), 581–595.

[2] R. Bartnik, *The spherically symmetric Einstein Yang-Mills equations*, Relativity Today (Z. Perjés, ed.), 1989, Tihany, Nova Science Pub., Commack NY, 1992, pp. 221–240.

[3] R. Bartnik and J. Mckinnon, *Particlelike solutions of the Einstein-Yang-Mills equations*, Phys. Rev. Lett. **61** (1988), 141–144.

[4] R. Bartnik, *The structure of spherically symmetric su(n) Yang-Mills fields*, J. Math. Phys. **38** (1997), 3623–3638.

[5] P. Bizon, *Colored black hole*, Phys. Rev. Lett. **64** (1990), 2844-2847.

[6] P. Breitenlohner, P. Forgács, and D. Maison, *Static spherically symmetric solutions of the Einstein-Yang-Mills equations*, Comm. Math. Phys. **163** (1994), 141–172.

[7] O. Brodbeck, *Gravitierende Eichsolitonen und schwarze Löcher mit Yang-Mills-Haar für beliebige Eichgruppen*, Ph.D. thesis, Universität Zürich, 1995.

[8] O. Brodbeck and N. Straumann, *A generalized Birkhoff theorem for the Einstein-Yang-Mills system*, J. Math. Phys. **34** (1993), 2412–2423.

[9] O. Brodbeck and N. Straumann, *Selfgravitating Yang-Mills solitons and their Chern-Simons numbers*, J. Math. Phys. **35** (1994), 899–919.

[10] O. Brodbeck and N. Straumann, *Instability proof for Einstein-Yang-Mills solitons and black holes with arbitrary gauge groups*, J. Math. Phys. **37** (1996), 1414–1433. (gr-qc/9411058)

[11] S. Coleman, *Classical lumps and their quantum descendants*, New Phenomena in Subnuclear Physics, ed A Zichichi, Plenum, New York, 1977.

[12] D.H. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York, 1993.

[13] S. Deser, *Absence of static solutions in source-free Yang-Mills theory*, Phys. Lett. **64B** (1976), 463-4.

[14] S. Deser, *Absence of static Einstein-Yang-Mills excitations in three dimensions*, Class. Quatum Grav. **1** (1984), L1-2.

[15] E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. **(2)6** (1957), 111–244.

[16] D.V Gal'tsov and M.S. Volkov, *Sphalerons in Einstein-Yang-Mills theory*, Phys. Lett. B **273** (1991), 255-259.

[17] D.V. Gal'tsov and M.S. Volkov, *Charged non-Abelian* $su(3)$ *Einstein-Yang-Mills black holes*, Phys. Lett. B **274** (1992), 173–178.

[18] D.V Gal'tsov and M.S. Volkov, *Gravitating non-Abelian solitons and black holes with Yang-Mills fields*, Phys. Rep. **319** (1999), 1-83. (hep-th/9810070)

[19] J.K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969.

[20] S.P. Hastings, J.B. McLeod, and W.C. Troy, *Static spherically symmetric solutions of a Yang-Mills field coupled to a dilaton*, Proc. Roy. Soc. London Ser. A **449** (1995), 479–491.

[21] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, New York, 1972.

[22] U. Kirchgraber and K. J. Palmer, *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, Pitman Research Notes in Mathematics Series 233, Longman, 1990.

[23] B. Kleihaus, J. Kunz, and A. Sood, $SU(3)$ *Einstein-Yang-Mills sphalerons and black holes*, Phys. Lett. B **354** (1995), 240–246. (hep-th/9504053)

[24] B. Kleihaus, J. Kunz, and A. Sood, *Sequences of Einstein-Yang-Mills-dilaton black holes*, Phys. Rev. D (3) **54** (1996), 5070–5092. (hep-th/9605109)

[25] B. Kleihaus, J. Kunz, and A. Sood, *Charged* $SU(N)$ *Einstein-Yang-Mills black holes*, Phys. Lett. B **418** (1998), 284–293. (hep-th/9705179)

[26] B. Kleihaus, J. Kunz, A. Sood, and M. Wirschins, *Sequences of globally regular and black hole solutions in* $SU(4)$ *Einstein-Yang-Mills theory*, Phys. Rev. D (3) **58** (1998), 4006–4021. (hep-th/9802143)

[27] B. Kleihaus, and J. Kunz, *Static axially symmetric Einstein-Yang-Mills-Dilaton solutions: I regular solutions*, Phys.Rev. **D57** (1998), 834-856.

[28] B. Kleihaus, and J. Kunz, *Static axially symmetric Einstein-Yang-Mills-Dilaton olutions: II black hole solutions* , Phys.Rev. D57 (1998), 6138-6157.

[29] B. Kleihaus, and J. Kunz, *Rotating Hariy Black Holes*, Phys. Rev. Lett. **86** (2001), 3704-3707 .

[30] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics 140, Birkhuser, Boston, 1996.

[31] S. Kobayashi and K. Nomizu, *Foundations of differential geometry i*, Interscience, Wiley, New York, 1963.

[32] H.P. Künzle and A.K.M. Masood-ul-Alam, *Spherically symmetric static* $SU(2)$ *Einstein-Yang-Mills fields*, J. Math. Phys. **31** (1990), 928–935.

[33] H.P. Künzle, $SU(n)$*-Einstein-Yang-Mills fields with spherical symmetry*, Classical Quantum Gravity **8** (1991), 2283–22297.

[34] H.P. Künzle, *Analysis of the static spherically symmetric* $SU(n)$*-Einstein-Yang-Mills equations*, Comm. Math. Phys. **162** (1994), 371–397.

[35] V.A. Kuzmin, V.A. Rubakov and M. E. Shaposhnikov, *On anomalous electroweak baryon-number non-conservation in the early universe* , Phys. Lett B **155** (1985), 36–42.

[36] A.N. Linden, *A Classification of Spherically Symmetric Static Solutions of SU(2) Einstein Yang Mills Equations with Non-negative Cosmological Constant* , preprint gr-qc/0005006

[37] A.N. Linden, *Existence of noncompact static spherically symmetric solutions of the Einstein* $SU(2)$*-Yang-Mills equations* , Commun. Math. Phys. **221** (2001), 525–547.

[38] A.N. Linden, *Far field behavior of noncompact static spherically symmetric solutions of Einstein* $SU(2)$ *Yang Mills equations*, J. Math. Phys. **42** (2001), 1196–1201.

[39] A.I. Mal'cev, *Commutative subalgebras of semisimple Lie algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 291-300.

[40] L. Markus, *Asymptotically autonomous differential systems.* In: Lefschetz, S. (ed.) Contributions to the Theory of Nonlinear Oscillations Γ11, Ann. Math. Stud., Vol 36, 17-29, Priceton University Press, Princeton, 1956.

[41] J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Springer-Verlag, 1994.

[42] N.E. Mavromatos and E. Winstanley, *Existence theorems for hairy black holes in* $su(N)$ *Einstein-Yang-Mills theories*, J. Math. Phys. **39** (1998), 4849–4873. (gr-qc/9712049)

[43] K. Mischaikow, H. Smith, and H.R. Thieme, *Asymptotically autonomous semi-flows: Chain recurrence and Lyapunov functions*, Trans. Amer. Math. Soc. **347** (1995), 1669-1685.

[44] R.V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Wiley, New York, 1995.

[45] T.A. Oliynyk, and H.P. Künzle, *Local existence proofs for the boundary value problem for static spherically symmetric Einstein-Yang-Mills fields with compact gauge groups*, preprint gr-qc/0008048.

[46] T.A. Oliynyk and H.P. Künzle, *On all possible static spherically symmetric EYM solitons and black holes*, preprint gr-qc/0109075.

[47] K.J. Palmer, *A Generalization of Hartman's Linearization Theorem*, J. Math. Anal. Appl. **41** (1973), 753–758.

[48] K.J. Palmer, *Linearization near an Integral Manifold*, J. Math. Anal. Appl. **51** (1975), 243–255.

[49] E. Radu, *Static axially symmetric solutions of Einstein-Yang-Mills equations with a negative cosmological constant: the regular case*, preprint (2001), gr-qc/0109015.

[50] A. Ringwald, *Rate of anomalous electroweak baryon and lepton number violation at finite temperature* , Phys. Lett. B **201** (1988), 510–516.

[51] O. Sarbach and E. Winstanley, *On the linear stability of solitons and hairy black holes with a negative cosmological constant: the odd-parity sector* , Class. Quantum. Grav. **18** (2001), 2125-2146.

[52] J.A. Smoller, A.G. Wasserman, S.-T. Yau and J.B. McLeod, *Smooth static solutions of the Einstein/Yang-Mills equations*, Comm. Math. Phys. **143** (1991), 115–147.

[53] J.A. Smoller and A.G. Wasserman, *Existence of infinitely many smooth, static, global solutions of the Einstein-Yang-Mills equations*, Comm. Math. Phys. **151** (1993), 303–325.

[54] J.A. Smoller, A.G. Wasserman, and S.-T. Yau, *Existence of black hole solutions for the Einstein-Yang/Mills equations*, Comm. Math. Phys. **154** (1993), 377–401.

[55] J.A. Smoller and A.G. Wasserman, *An investigation of the limiting behavior of particle-like solutions to the Einstein-Yang-Mills equations and a new black hole solution*, Comm. Math. Phys. **161** (1994), 365–389.

[56] J.A. Smoller and A.G. Wasserman, *Regular solutions of the Einstein-Yang-Mills equations*, J. Math. Phys. **36** (1995), 4301–4323.

[57] J.A. Smoller and A.G. Wasserman, *Limiting masses of solutions of the Einstein-Yang-Mills equations*, Phys. D **93** (1996), 123–136.

[58] J.A. Smoller and A.G. Wasserman, *Uniqueness of zero surface gravity* $SU(2)$ *Einstein-Yang-Mills black holes*, J. Math. Phys. **37** (1996), 1461–1484.

[59] J.A. Smoller and A.G. Wasserman, *Reissner-Nordström-like solutions of the* $SU(2)$ *Einstein-Yang-Mills equations*, J. Math. Phys. **38** (1997), 6522–6559.

[60] J.A. Smoller and A.G. Wasserman, *Extendability of solutions of the Einstein-Yang-Mills equations*, Comm. Math. Phys. **194** (1998), 707–732.

[61] N. Straumann and Z.H. Zhou, *Instability of a colored black hole solution*, Phys. Lett. B **243** (1990), 33–35.

[62] N. Straumann and Z.H. Zhou, *Instability of the Bartnik-McKinnon solution of the Einstein-Yang-Mills equations*, Phys. Lett. B **237** (1990), 353-356.

[63] H.R. Thieme, *Convergence results and a Poicaré-Bendixson trichotomy for asymptotically autonomous differential equations*, J. Math. Biol. **30** (1992), 755–763.

[64] M.S. Volkov and D.V Gal'tsov, *Non-Abelian Einstein-Yang-Mills black hole*, JETP Lett. **50** (1989), 346–350.

[65] H.C. Wang, *On invariant connections over a principal bundle*, Nagoya Math. J. **13** (1958), 1–19.

[66] A.G. Wasserman, *Solutions of the spherically symmetric* $SU(2)$ *Einstein-Yang-Mills equations defined in the far field*, J. Math. Phys. **41** (2000), 6930–6936.

[67] E. Winstanley, *Existence of stable hairy black holes in* $SU(2)$ *Einstein-Yang-Mills theory with a negative cosmological constan*, Class. Quantum. Grav. **16** (1999), 1963-1978.