Event-Triggered Control Systems

by

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Abstract

Event-triggered control have increasingly become an active area of research in the last decade, thanks to their potential capability in reducing data communication between subsystems during control action. In this research we tackle some practical problems encountered in this field and endeavor to improve the state of the art.

We first start by designing periodic event-triggered control (PETC) for both state feedback and output feedback systems. PETC is a subclass of even-based systems, where the triggering conditions (TCs) are verified on a clock-driven basis. Despite their efficiency and advantages from practical of view, they have received less attention, compared to the other classes of event-driven systems. we first consider designing state feedback PETC for a class of nonlinear systems. We then, extend our result to the more challenging case of output feedback systems where, two independent TCs (with different sampling rates) are considered for plant and controller.

In the next part of this research we study the effect of noise in the implementation of event-based systems. Basically, the destructive impact of noise has been largely ignored in these class of systems. In this regard, first it is shown that how the output measurement noise can easily trigger unnecessary samples and deteriorate the performance of the systems. Then a novel triggering mechanism is proposed which reduces the effect of noise in the event-triggering scheme. Next, the same idea is extended to the observation problem, and a noise effective event-based observer is designed for LTI systems which assures H_∞ performance for the estimation error.

We then turn our attention to the fundamental problem of designing stable non-conservative event-triggered systems. We claim that most of the work reported in the literature is based on Lyapunov conditions that carry some intrinsic conservatism. We propose a novel integral-based condition that relaxes the strong conditions previously

reported and can reduce communication between plant and controller.

Finally, we consider decentralized event-triggered control for a class of nonlinear systems. We propose new decentralized TCs and show their efficiency compared to existent results.

Preface

Chapter 2 has been published in *Proceeding of 53rd IEEE Conference on Decision and Control, 2014*, as S. H. Mousavi and H. J. Marquez, "Event-Based Controller Design for a Class of Nonlinear Systems via Convex Optimization". I was responsible for the analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez had the supervision role throughout the work and also involved with the paper composition and drafting.

Chapter 4 has been published in the *International Journal of Control*, as S. H. Mousavi and H. J. Marquez, "Integral-Based Event Triggering Controller Design for Stochastic LTI Systems via Convex Optimization". I was responsible for the analysis, design, mathematical derivations, simulation part and also the work drafting. Dr. Marquez contributed in main the idea and also had the supervision role throughout the work. He was also involved with the paper composition and drafting.

An early result of chapter 6 for LTI systems has been published in *Proceeding of 53rd IEEE Conference on Decision and Control, 2014*, as S. H. Mousavi, M. Ghodrat and H. J. Marquez, "A Novel Integral-Based Event Triggering Control for Linear Time-Invariant Systems". I was responsible for the analysis, design, mathematical derivations, simulation part and also the work drafting. Mohsen Ghodrat contributed in main the idea and also assisted in the mathematical derivation. Dr. Marquez had the supervision role throughout the work and was also involved with the paper composition and drafting.

Chapter 6 has been published in the *IET Control Theory & Applications*, as S. H. Mousavi, M. Ghodrat and H. J. Marquez, "Integral-based event-triggered control scheme for a general class of non-linear systems". I was responsible for the analysis, design, mathematical derivations, simulation part and also the work drafting. Mohsen Ghodrat contributed in main the idea and also assisted in the mathematical derivation. Dr. Marquez had the supervision role throughout the work and was also involved with the paper composition and drafting.

To my beloved parents, brother, sister and my wife

for their endless love and support

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Notation

$\mathbb{R}$, $\mathbb{Z}$	The sets of real and integer numbers
$\mathbb{R}^+$, $\mathbb{Z}^+$	The sets of nonnegative real and integer numbers
$\mathbb{R}^n$	The set of real n-dimensional vector
$\mathbb{R}^{n \times m}$	The set of real $n \times m$ matrices
$\mathcal{L}_p$	Function space with well-defined p-norm
$\mathcal{L}_{p,T}$	Extended $\mathcal{L}_p$ space of truncated signals
$\exists$	Existential quantifier
$\forall$	Universal quantifier
$x \in X$	x is an element of set X
$X \subset Y$	X is a subset of Y
$A^\intercal$	Transpose of matrix or vector A
A^{-1}	Inverse of matrix A
$tr(A)$	Trace of matrix $A \in \mathbb{R}^{n \times n}$, defined as $tr(A) = \sum_{i=1}^{n} a_{ii}$
I	Identity matrix of appropriate dimension
$\lVert \cdot \rVert$ or $\lvert \cdot \rvert$	Euclidean norm of a vector or matrix
$\lVert z \rVert_2$	$\mathcal{L}_2$ norm of signal $z : \mathbb{R}^+ \to \mathbb{R}^n$, defined as $\lVert z \rVert_2 = (\int_0^\infty \lvert z(t) \rvert^2 dt)^{\frac{1}{2}}$
$\lVert z \rVert_{2,T}$	$\mathcal{L}_{2,T}$ norm of signal $z : \mathbb{R}^+ \to \mathbb{R}^n$, defined as $\lVert z \rVert_{2,T} = (\int_0^T \lvert z(t) \rvert^2 dt)^{\frac{1}{2}}$
$E\{\cdot\}$	Expected value operator
$\mathcal{N}(\mu, \sigma^2)$	Normal distribution of a random variable with expected value μ and variance σ^2

Abbreviations

LTI	Linear Time-Invariant
NCS	Networked Control Systems
TC	Triggering Condition
ISS	Input-to-State Stability
ZOH	Zero-Order Hold
PETC	Periodic Event-Triggered Control
IBTC	Integral-Based Triggering Condition
PIBTC	Periodic Integral-Based Triggering Condition
SBTC	Summation-Based Triggering Condition

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Chapter 1

Introduction

1.1 Background

It is nowadays well accepted that most control systems are implemented digitally using a computer. In a typical configuration, a sample device is placed at the plant output and the resulting discrete-time signal is transmitted to the controller. The controller operates in discrete-time and its output is fed to the plant through a hold device. In the classical approach, the sampling process is *periodic* or *time-triggered*, and signals in the loop are updated at each sampling instant. This natural approach makes control analysis and design rather simple, at least for linear time-invariant (LTI) systems. Discrete-time systems can be described using difference equations and control design can be carried out in discrete-time in a manner that is analogous to the continuous time.

The primary inconvenience of time-triggered systems is that they are inherent in the approach that information is transmitted from the sensors to the processor, and the control task is computed and executed (by the actuator) at regular time intervals. This happens regardless of whether measurement output changes require new data transmission and re-execution of control action to maintain the stability and performance of the system. For example, in a regulation-type problem under no disturbance action, a time-triggered system would continue to update the control signal even after the plant has reached steady-state. This issue may become critical in the systems where the sensors and processors have constrained energy, provided by batteries, and consequently, an optimal usage of resources is of a great importance.

On the other hand, regular data transmission during control implementation can be problematic where the subsystems are geographically distributed and exchange information through a shared communication channel; e.g. wireless networks. Although using such a network has several advantages such as low wiring cost, high reliabilities and reduced power requirement, it potentially introduces several challenges such as transmission delay and packet dropouts [2].

A key idea to cope with these troubles and prevent waste of energy and communication resources is to lower data transmission through the network channel. To achieve this goal, rather than sending periodically, the data exchange between the subsystems can be carried out based on an aperiodic and event-based scenario [3].

Fundamental idea in the *event-triggered* systems is that the data are sent through the channel just when it's inevitably needed. It can be argued that an event-triggered mechanism constitutes the most natural approach to control of certain systems. Consider for example biological systems, where the neurons interact by pulse transmission. In biological systems, "Electrical stimuli changes ion concentration in the neuron and a pulse is emitted when the potential reaches a certain level" [3]. Some efforts have been made to mimic and use such models to implement simple control systems [4]. A plant controlled by on-off relays is another example of event-based systems. In this system, the control action value is not changed unless the control error passes a certain threshold (*i.e.* just as an event occurs in the system) [5]. Event-triggered control is also applied where the control actions are costly. For instance, consider control of production volume in a chemical process. Since frequent changes in production rate is expensive and should be avoided, an applicable control idea in such system is to apply an event-based mechanism: the production rate would not be varied unless the production volume approaches the lower or upper limit of the storage tank. As another example for application of event-triggering mechanism, one can have a look at operation principle in typical accelerometers [6].

Mathematically speaking, in an event-based system the components do not exchange information unless a triggering condition (TC) is violated. The TC can be defined in different forms and varies depending on the nature of the system. For instance, in a power grid, a fault can be considered as an event, triggering a control action in the system [7]. Generally speaking, in most of systems, the event happens just as an error (representing

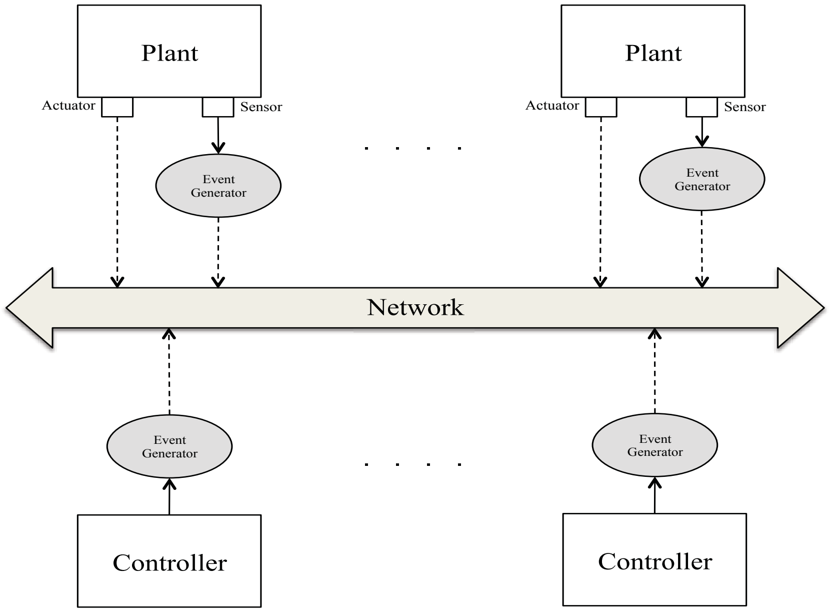


Figure 1.1: A general structure of event-based systems.

a specific performance of the system) exceeds a certain threshold [1]. In this regard, some kind of event detector hardware is required to generate an interruption and release the information through the network. Such a hardware should be devised locally for each component in proper places before the transmission channel (Fig.1.1)

As mentioned earlier, in addition to efficient use of communication, computation and energy resources, event-based systems results in less data transmission traffic, compared to the time-triggered systems. To have a better comparison between time-triggered and event-triggered control systems, one can refer to [8] which is one of the primary ideas proposed for the event-driven systems.

1.2 Literature Survey

Early works on event-based systems were proposed in late 1990's [9,10]. In [10], Arzen defines a simple triggering condition based on the tracking error and converts the standard PID control algorithm to an event-based form. The time-triggered and event-triggered controller are evaluated and compared both in simulations and in laboratory experiments for a double-tank process. The results confirm that using an event-based mechanism it is possible to obtain large reductions in the control execution with only minor control performance degradation. In [9], Åström introduces one of the primary theoretical development and evaluation for the event-triggered control systems. In the paper, he considers a first order stochastic system and designs both event-based and periodic controller so that the state stays close to the origin. He compares the results and analytically shows that for the same average sampling rates, the event-based controller gives remarkably smaller output variance than the periodic controller.

Over the following years, other researchers have extended the same principles for more control systems. Because of the use of different triggering strategies, similar idea appear in the literature under different names such as: level crossing sampling [11], event-driven systems [12] and state-triggered feedback systems [13].

Reference [9] sparked much interest for the research on event-based systems and marked the beginning of what has been an active area of research.

In [1], Tabuada studies asymptotic stability in the event-triggered control systems, using a Lyapunov-based approach. In particular, he applies the input-to-state stability (ISS) Lyapunov function and proposes a general framework for event triggering mechanism design for the following nonlinear systems:

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{1.1}$$

In this work, following the idea of event-based control mentioned earlier, instead of continuous information flow of states, it is assumed that the control law only receives intermittent information of states and so is only updated at time instants t_i ($i \in \mathbb{N}$) (where the TC is violated). The control input is held constant in between update times ($t_i \leq t \leq t_{i+1}$) using a zero-order-hold (ZOH) module. The basic assumption made by Tabuada is as follows: a feedback control

$$u = k(x) \tag{1.2}$$

has been already designed such that the system

$$\dot{x} = f(x, k(x + e)) \tag{1.3}$$

is Input-to-State stable with respect to measurement error e and there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, class $\mathcal{K}_\infty$ functions $\bar{\alpha}$, $\underline{\alpha}$, α and γ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \tag{1.4}$$

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq -\alpha(|x|) + \gamma(|e|). \tag{1.5}$$

The measurement error is defined as the difference between the last state information sent to the controller and current state value:

$$e = x(t_i) - x(t) \qquad t_i \leq t \leq t_{i+1}. \tag{1.6}$$

Having assumed the stability inequalities (1.4) and (1.5), the key idea to design event generator condition is to maintain the Lyapunov function V decreasing. To this aim, the following TC is proposed:

$$\gamma(|e|) \leq \sigma\alpha(|x|) \qquad \sigma > 0. \tag{1.7}$$

Substituting (1.7) in (1.5), it is readily obtained:

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq (\sigma - 1)\alpha(|x|). \tag{1.8}$$

Now, setting $\sigma < 1$ the time derivative of V is negative and so the system is stable. Simulation results, provided in the paper, clearly show the effectiveness of event-triggered control in data transmission reduction, while maintaining stability.

One of the important issues, related to hybrid dynamical systems in general, and event-based systems in particular, is the so called *Zeno* behavior [14]. Zeno is a response of system when infinite number of triggering happen in a finite amount of time. This issue is clearly undesirable from an implementation point of view. So, it is crucial in an event-based system that the designed TC guarantees a lower bound for the time intervals between triggering instants, and consequently ensures that there is no Zeno behavior in the system. This lower bound is also referred as *minimum inter-event time.*

Note that in [1], as another contribution of the work, the author analytically proves that such a minimum exists for the system.

Reference [1] is limited to the study of stability for centralized nonlinear systems, controlled by state feedback. References [15–23] extended the event-based idea for decentralized and distributed control systems. The main challenge in these class of systems is that the TC for each subsystem should only rely on its local information to trigger data through the channel, as it does not have access to the other subsystem's information.

In [15], using assumptions of $\mathcal{L}_p$ stability with respect to measurement error and also weak interconnection between subsystems, Wang et. al. examine event-triggered data transmission in distributed networked control systems (NCS). They propose an event-triggering scheme, where a subsystem broadcasts its state information to its neighbors only when a local triggering condition (defined similar to (1.7)) is violated. It is shown how the decentralized event-triggering scheme ensures asymptotic stability of the entire networked control system. However, no minimum bound on the inter-event times is guaranteed for the triggering modules. In [17], with a similar assumption of weakly coupled subsystems and also ISS stability, the authors take a Zeno-free approach and prove asymptotic stability of the distributed event-based system. In [18] and [19], the authors consider decentralized event-triggering problem for systems with a specific structure. In their models, it is assumed that the sensors are geographically distributed. However, the overall system is controlled by a centralized control unit. Although the basic structures of these two papers are almost the same, they exploit different forms of TCs and also design tools to prove stability of the system. In [21], Postoyan *et al.* present a general framework for the event-triggered control of distributed nonlinear systems. They take a quite different approach and model the event-triggered systems as a hybrid model. Then, they provide Lyapunov-based conditions to guarantee the stability of the resulting closed-loop system and explain how these conditions can be utilized to synthesize event-triggering rules. In [23], the authors investigate event-triggering design for output feedback control of LTI systems. Because of utilizing a dynamic controller, they introduce decentralized triggering modules for plant and controller sides, which guarantee global asymptotic stability of the closed-loop system. In [22], a more general case is studied. In the paper, Donkers and Heemels consider event-based scheme for disturbed LTI systems with dynamic output feedback controller, where the distributed sensors and actuators are assumed to be grouped into nodes. They model the proposed system in an impulsive form and then study both $\mathcal{L}_\infty$ performance and stability problem. In [24], Abdelrahim *et al.* design output feedback event-triggered controllers for stabilization of nonlinear systems. Despite using a centralized TC for the overall system, they develop a method to ensure minimum inter-event time for the scheme. Basically, they combine time-triggered and event-triggered control techniques. “The idea is to turn on the event triggering mechanism only after a fixed amount of time elapsed since the last transmission”.

Event-based scenario has also become much favorable in practical networked-based applications. One of the most popular applications is multi-agent systems [25]. Generally, in this class of systems each agent utilizes some local information and does a certain action such that the overall structure achieves a specific goal (such as consensus). Multi-agent systems can be categorized as a subset of distributed systems, where event-triggering scheme can be a beneficial tool to avoid probable data transmission problems [26–32].

In addition to the regulation problem, event-based control have also been proposed for the tracking control. In [33] and [34], Tallapragada and Chopra consider tracking problem for affine, and then general nonlinear systems, respectively. In the mentioned papers, they propose an event-based control algorithm which guarantees uniform ultimate boundedness of the tracking error. In [35], The authors consider the effect of disturbance and also network induced delay in the system dynamics, and examine the H_∞ tracking control problem for LTI systems. Applications of event-based tracking control can be found in [36] and [37]

Most references on the subject follow an approach similar in spirit to Tabuada's formulation, [1], and design the TC by relying in keeping the Lyapunov function decreasing along the system trajectories. In this regard, some attempts have been made to remove some of the intrinsic conservatism existent in this condition. In [38], the author introduces a dynamic mechanism and using an augmented Lyapunov function proves system stability. In [39], Wang and Lemmon introduce a new TC, based on a logic function of the Lyapunov function, which weakens the assumptions in [1] by considering an ISS approach and assuming a lower bound on the derivative of the Lyapunov function. In [40], the authors use a hybrid systems approach to model the event-based system in which the TC is a function of the derivative of the Lyapunov function. The approach requires computing the derivative of the Lyapunov function along system trajectories at all times.

In most of the works cited before, it is assumed that the TC is verified continuously which may demand high level of resource utilization in practical application. Alternatively, the TC can be checked periodically in a clock driven fashion [35, 41–44]. Periodic event-triggered control (PETC) systems are preferable, at least from an implementation point of view, since they have simpler hardware requirements. In [35], the authors propose a periodic triggering sampling scheme and use a delay system approach to study H_∞ state feedback control of linear systems. Using a similar idea, PETC have been designed for different problems [45–48]. In [42], Heemels *et al.* use an impulsive system approach to study the $\mathcal{L}_2$ gain properties of the proposed PETC system for LTI systems. In the mentioned paper, because of dealing with dynamic controller, two decentralized TCs are devised for the plant and controller. In [43], Meng and Chen study decentralized PETC for output feedback sampled-data systems. In the reference, the authors first develop a discretized model of the closed-loop system and then, propose a methodology for co-design of TCs and the controller parameters. In [44], Postoyan et. al. study the PETC idea for nonlinear systems. Reference [44] follows an emulation-based approach starting with a known (continuously evaluated) event-triggered controller and provide a technique to select the sampling period to approximately maintain the stability properties guaranteed by the original controller.

Before going forward to our research contributions, it must be noted that event-based control is an active area of research and in addition to the works reviewed in this section, several other results have been published in different fields such as: optimal control [49–56], and state estimation [57].

1.3 Research Motivation and Contributions

As discussed in the last section, several works have been conducted on different classes of even-based systems. However, there still exist several problems which require more study and research. In this thesis we have looked into some of these problems and made contributions on different aspects of the event-based systems. A brief description on the contributions of this research are given as follows. More detailed explanation on the results can be found in the related chapters.

PETC is an interesting and practical subclass of event-based systems, in which the triggering condition is verified in a clock-driven fashion. To the best of our knowledge, most of the references dealing with the periodic event-based problem, have concentrated on linear systems. Reference [44] proposes PETC for a class of nonlinear systems. In this reference, the authors use an emulation approach to turn a preexisting continuously verified event-based controller into periodic-based one, such that asymptotic stability is maintained. In the first part of our research, we focus on the PETC for nonlinear systems. The nonlinear system considered in this part, is modeled as a linear system plus a nonlinear term which satisfies a Lipschitz continuity condition, and is affected by some exogenous disturbance. Lipschitz systems are important in the sense that most nonlinear systems satisfy a Lipschitz condition, at least locally, and therefore can account for the effect of, at least, mild nonlinearities. In this work, the introduced scheme provides a way to directly design the state feedback control law and also the TC such that the stability and H_∞ performance are guaranteed for the closed-loop system.

Decentralized PETC has been developed for LTI output feedback systems ([42] and [43]). In these works it is assumed that the triggering modules of plant and controller side are synchronized in a way that their sampling rates and instants are exactly the same; something which may cause difficulties from a practical point of view. In the second part of this thesis our main interest is to address this issue. In particular, in this part, we consider PETC for LTI systems with dynamic controllers, where two separate and decentralized event modules are devised for the plant and controller sides, respectively. However, contrary to the references cited before, in this thesis the TCs act with non-equal sampling periods and in an asynchronous fashion. Therefore, here, it is not required to synchronize the triggering modules. This fact not only ease the implementation task, but also provides the flexibility to set each event-generator module sampling rate individually, possibly based on the local requirements and also hardware limitations. Using Lyapunov stability theorems the TCs are designed such that the stability and H_∞ performance (in the presence of disturbance) are ensured for the closed-loop system.

One issue of critical importance in event-triggered systems is the potential degradation of performance due to measurement noise. Indeed, in an event-triggered system the triggering mechanism directly depends on the instant values of the measured states, something which is also clear in the TC (1.7), proposed in Tabuada's general framework. Therefore, noise in event-driven systems plays a more fundamental detrimental role than in classical stochastic systems. Apart from output performance degradation, the existence of noise can trigger unnecessary samples making the triggering mechanism potentially less effective, an issue that has received very little attention in the literature. In [58], the authors consider norm-bounded noise on the output measurement. However, they use of classical forms of triggering schemes where the event generator condition depends on the instant values of the output and so is influenced by noise. In [59], the event-triggering condition for a single integrator multi-agent systems with noise is studied, but no modification is considered to decrease the effect of the noise on the TC.

The main purpose of the third part of this research is to directly address the effect of noise in the LTI event-triggered control systems. Our main contribution is a novel event-based scheme with a periodic integral-based triggering condition (PIBTC) that reduces the effect of noise in both the feedback loop and the triggering condition. We model the event-triggered system using a delay system approach [35], and then show that it is asymptotically stable in mean square sense. Finally, we propose a methodology to obtain the parameters of the event triggering system, minimizing data transmission in a networked-based control.

In the next part, this idea is extended to observer design. Generally, measurement noise has always been an indispensable part of the observation problem. So, noise-related issues can be more challenging in an event-based observer scheme, where an event generator module is located at the output of plant to trigger only necessary output samples towards the observer. In this part, event-triggered observer design for LTI system in the presence of disturbance and measurement noise is considered. We start by a traditional Luenberger observer and then develop a noise effective event-based scheme for it, such that the data transmission is efficiently reduced and also H_∞ performance is guaranteed for the estimation error.

As mentioned earlier in the previous section, most references in the literature follow the same idea given in [1] to design TC, which (as explained) is somewhat conservative. So, some works have been done to improve available event-based scenarios [38–40]. In the fifth part of this research, we propose a different method and re-examine Tabuadas main principle. Assuming that an analog controller has been designed and satisfies an ISS condition (the same as assumptions in [1]), we propose an alternative, less conservative, approach to the construction of the event-TC. The main idea consists of using an integral-based triggering condition (IBTC) that allows the Lyapunov function to be non-decrescent between triggering instants (thus allowing the time derivative of the Lyapunov function to have instantaneous positive values between triggering instants). Asymptotic stability of the proposed closed-loop system is ensured by Lyapunov stability theorems. The existence of a lower bound for the inter-event times is also proved and an explicit value for this bound is provided for a specific class of non-linear systems. An additional contribution of this work, is a rigorous proof showing that our proposed TC is more efficient than the existing results in terms of communication exchanged between plant and controller.

Finally, the proposed IBTC in the previous part is extended for a class of decentralized nonlinear PETC systems. Similar to the first part, the nonlinear term in the system dynamics is assumed to satisfy Lipschitz condition. The general control scheme of the system is similar to the one given in [18] and [19]. It is such that the sensors are geographically distributed; but their measurements are transmitted to a centralized control unit. So, event triggering modules are designed for sensors in a decentralized way. Moreover, the proposed form of TC is shown to be less conservative than the previous methods and avoids excessive data transmission.

1.4 Thesis Outline

The rest of this thesis is organized as follows.

Chapter 2: In this chapter, a method for designing PETC for Lipschitz nonlinear systems is proposed such that the stability and H_∞ performance are guaranteed. In addition to solving the stability and performance problem, our proposed design tackles the following objectives: 1- Enlarging the region of attraction which is directly related to enlargement of Lipschitz constant; 2- Minimizing the amount of data transmitted over the communication channel. In this regard using delay system approach [35], we present a multi-objective optimization problem in which the stability and performance conditions appear in the form of LMI constraints

Chapter 3: This chapter focuses on the PETC of output feedback systems with external disturbance. Because of dealing with a dynamic controller, two independent, decentralized triggering mechanisms are considered at the plant and controller output, respectively, which act with non-equal sampling periods and in an asynchronous fashion. The stability and performance conditions are provided in form of LMIs. Then, using a convex optimization approach, the event generator conditions are designed to reduce the $\mathcal{L}_2$ performance gain, while lowering data communication through the channel.

Chapter 4: This chapter studies event-triggered control for LTI systems in the presence of measurement noise. As the main contribution, a periodic integral-based triggering condition is proposed which is shown to be robust against measurement noise and reduces the data transmission effectively. Moreover, an optimization problem with LMI constraints is provided, which not only ensures the stability in the mean square for the closed-loop system, but also designs optimal values for the TC parameter.

Chapter 5: In this chapter, event-based observer for disturbed LTI systems is designed. To suppress the impact of measurement noise in the system, a TC, similar to the one introduced in Chapter 3, is proposed. Then, an optimization-based methodology is given to design the scheme parameters such that less data transmission, desired estimation error convergence rate and disturbance attenuation level are achieved.

Chapter 6: In this chapter, a novel event triggering scheme is given for a general class of nonlinear systems, which mainly improves the idea introduced by Tabuada [1]. The proposed integral-based triggering condition is less conservative in the sense that it does not require the derivative of Lyapunov function to be negative at all time instants. It is shown that the IBTC ensures a minimum inter-event time for the system. In addition, a well-proved comparison with traditional TC is given to represent the effectiveness of the proposed system in data transmission reduction.

Chapter 7: In this chapter, we study periodic decentralized event-triggered systems for nonlinear systems. It is assumed that the measurement sensors are geographically distributed. So, a centralized controller is employed while decentralized event triggering modules are used for each group of states. To avoid excessive data transmission, the IBTC proposed in the previous chapter is developed for this class of systems. Similar to the chapter 1, the class of nonlinear systems considered here is those whose nonlinear terms satisfy a Lipschitz continuity condition, or simply, Lipschitz systems. The main limitation encountered in this model is the size of the region of the space in which the Lipschitz condition is satisfied, which in turn limits the region of attraction of the trajectories. We propose a design procedure in the form of convex optimization problem to design the TC parameters, such that the closed-loop system is asymptotically stable and the region of stability is enlarged.

Chapter 2

Periodic Event-Triggered Control of Nonlinear Systems

In this section, a periodic event-based controller design scheme is proposed for a class of Lipschitz nonlinear systems. Using a delay system approach [35], the event-based system is modeled in a continuous time form. Then, the controller and event generator design methodology is formulated as a convex multi-objective optimization problem with LMI constraints, where the admissible Lipschitz constant of the system is enlarged and data communication between plant and controller is minimized simultaneously. Finally, the proposed system is evaluated via simulations.

The rest of this chapter is organized as follows. Section 2.1 presents the problem statement. In section 2.2, the proposed event-based system is modeled in a time delay form which is followed by the stability and performance analysis in section 2.3. Section 2.4 contains the main result, where the controller and event generator design is given in the form of an optimization problem. Simulation results are presented in section 2.5 to show the efficiency of the proposed mechanism and finally section 2.6 summarizes the results of this chapter.

2.1 Problem Statement

Consider the following class of nonlinear systems

$$\begin{cases} \dot{x} = Ax + Bu + \varphi(x,u) + B_w w \\ z = Cx \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^p$ is the controlled output and $w(t) \in \mathbb{R}^l$ is an exogenous disturbance input which is assumed to belong to $\mathcal{L}_2[0, +\infty)$. Moreover, $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, satisfying $\varphi(0,u) = 0$ is a locally Lipschitz function in its first argument on the region $\Omega \subset \mathbb{R}^n$ (which contains the origin in its interior), uniformly in u:

$$\|\varphi(x_1,u) - \varphi(x_2,u)\| \leq \ell \|x_1 - x_2\| \qquad \forall x_1, x_2 \in \Omega, u \in \mathbb{R}^m, \tag{2.2}$$

where ℓ is called Lipschitz constant. Consider now the following aperiodic discrete time control scheme: the controller receives fresh data of state values and updates the control law only if an event generator condition is triggered. Otherwise, the controller uses the previous received state information.

Based on this scenario, we consider a feedback controller of the form:

$$u = K\hat{x}, \tag{2.3}$$

where $\hat{x}$ is a piecewise constant signal, containing intermittent state information and as defined in Subsection 2.1.1. A general block diagram of the proposed event-based controller is shown in Fig.2.1

In this work, we assume that the event generator block is clock-driven, *i. e.* the event generator condition is verified periodically with the period of 'h'. In this regard, as shown in Fig.2.1 a sampler is used to feed the event generator block. A ZOH module located at the plant input generates a piecewise constant signal.

2.1.1 Event Generator Structure and Triggering Condition

Similar to the work [1], in this paper the event condition is triggered and fresh data is sent to the controller when the norm of the difference between the current state and the latest transmitted value exceeds a certain threshold. In this regard, define $\{k_j\}_{j=1}^{\infty}$ as the sequence of sample numbers at which the data is sent to the controller. Moreover, suppose

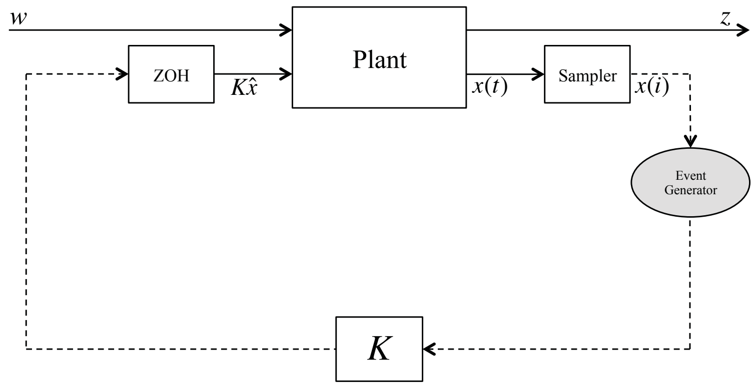


Figure 2.1: General block diagram of the proposed event-based system

$k_1 = 1$ and $x(k_1 h) = x_0$ where x_0 is the initial condition of system. For $i \geq k_j$ define the event generator function as:

$$f(x(ih), x(k_j h)) := (x(ih) - x(k_j h))^T (x(ih) - x(k_j h)) - x^T(ih)\beta x(ih), \tag{2.4}$$

where, β is a $n \times n$ positive definite matrix to be designed properly. Then, a fresh sample of x (*i.e.* $x(k_{j+1}h)$) is broadcast if the following periodic TC is violated :

$$f(x(i), x(k_j)) \leq 0, \tag{2.5}$$

where, for simplicity we have denoted $x(ih)$ and $x(k_j h)$ by $x(i)$ and $x(k_j)$ respectively.

In the other words, one can say that the following inequality holds for $k_j \leq i < k_{j+1}$

$$(x(i) - x(k_j))^T (x(i) - x(k_j)) \leq x(i)^T \beta x(i). \tag{2.6}$$

Having introduced $x(k_j)$ ($j \in \mathbb{N}$) as the output of event generator block, the piecewise constant signal $\hat{x}$ is defined as:

$$\hat{x}(t) := x(k_j h) \qquad \text{for} \quad t \in [k_j h, k_{j+1} h) \quad , \quad j = 1, 2, \ldots \tag{2.7}$$

2.1.2 Closed-Loop System Dynamics

Using the event-based controller (2.3), the closed-loop dynamics is given by:

$$\begin{cases} \dot{x} = Ax + Bu + \varphi(x,u) + B_w w \\ u = K\hat{x} \\ z = Cx \end{cases} \tag{2.8}$$

Our goal is to design the state feedback controller K and the event generator function (2.4) such that the system is asymptotically stable and for a given γ the H_∞ criterion (defined in Definition 2.1) is guaranteed while the following optimality criteria are achieved:

- Enlarging the region of attraction of the closed-loop system by enlargement of admissible Lipschitz constant ℓ. (Please refer to the last part of proof of Theorem 7.1 to see how increasing admissible ℓ leads to region of attraction expansion.)

- Reducing the number of transmitted data over the communication channel by maximizing β.

The design procedure is formulated as a convex optimization problem with LMI constraints.

Definition 2.1. *We say that H_∞ performance is guaranteed for the system (2.8) with gain γ if for any $w(t) \in \mathcal{L}_2[0, +\infty)$ the following inequality is satisfied:*

$$\|z\|_2 \leq \gamma \|w\|_2 + \delta, \tag{2.9}$$

where δ is a constant value that depends on the states initial condition.

In the next section the delay system approach is used to model the event-based system.

2.2 Modeling of the Event-Based System

In order to write the event-based system dynamics in a delay system form, an approach similar to [35] is exploited. we break down the time interval $[0, +\infty)$ as

$$[0, +\infty) = \bigcup_{j=1}^{+\infty} [k_j h, k_{j+1} h). \tag{2.10}$$

Based on the equation (2.7), the dynamics of event-based system (2.8) for $t \in [k_j h, k_{j+1} h)$ can be written as:

$$\begin{cases} \dot{x} = Ax + BKx(k_j h) + \varphi(x,u) + B_w w \\ z = Cx \end{cases} \tag{2.11}$$

Now, let l_t be the latest sample number before time instant 't' :

$$l_t = max\left\{i \in \mathbb{N} : ih \leq t\right\}. \tag{2.12}$$

Define

$$\tau(t) := t - l_t h \qquad \text{with} \quad 0 \leq \tau(t) \leq h, \tag{2.13}$$

$$e_t := x(k_j h) - x(l_t h). \tag{2.14}$$

Adding and subtracting $BKx(l_t h)$ to the dynamics (2.11) and using above definitions, the following time delay dynamics can be obtained:

$$\begin{cases} \dot{x} = Ax + Be_t + BKx(t - \tau(t)) + \varphi(x,u) + B_w w \\ z = Cx, \end{cases} \tag{2.15}$$

Similarly, the event-TC is written in time delay form.

Regarding the definition of l_t, for $t \in [k_j h, k_{j+1} h)$ one can readily find that

$$k_j \leq l_t < k_{j+1}. \tag{2.16}$$

So, based on equation (2.6) we have:

$$(x(l_t h) - x(k_j h))^T (x(l_t h) - x(k_j h)) \leq x(l_t h)^T \beta x(l_t h). \tag{2.17}$$

Then, using definitions (2.13) and (2.14), the following triggering inequality holds for $t \in [k_j h, k_{j+1} h)$:

$$e_t^T e_t \leq x(t - \tau(t))^T \beta x(t - \tau(t)). \tag{2.18}$$

Having formulated the closed-loop dynamics and the periodic TC in a continuous-time form, we are ready to give stability and performance conditions.

2.3 Stability and Performance Analysis of The Proposed Systems

Theorem 2.1 provides sufficient conditions in the form of LMIs to guarantee the asymptotic stability and H_∞ performance of the event-based system. First, we introduce three lemmas which are used in the proof of our result.

Lemma 2.1. *For any $x, y \in \mathbb{R}^n$ and $D = D^T \in \mathbb{R}^{n\times n} > 0$, the following inequality is satisfied:*

$$2x^T y \leq x^T D x + y^T D^{-1} y. \tag{2.19}$$

Proof: Introduce the matrix

$$G = (Dx - y)^T D^{-1}(Dx - y). \tag{2.20}$$

Since $G \geq 0$, inequality (2.19) immediately follows.

□

Lemma 2.2. *([35]) For any $\epsilon \in \mathbb{R} > 0$, $D = D^T \in \mathbb{R}^{n\times n} > 0$ and $X= X^T \in \mathbb{R}^{n\times n} > 0$:*

$$XD^{-1}X \geq -\epsilon^2 D + 2\epsilon X. \tag{2.21}$$

Proof: Introduce the matrix

$$G = (X - \epsilon D)D^{-1}(X - \epsilon D). \tag{2.22}$$

Since $G \geq 0$, inequality (2.21) is trivially satisfied.

□

Lemma 2.3. *Let $A_i \in \mathbb{R}^{n\times n}$, $\theta_i \in \mathbb{R}$ ($\forall i \in \{1, \ldots, k\}$) and define*

$$Conv(A_1, \ldots, A_k) = \{\theta_1 A_1 + \cdots + \theta_k A_k : \theta_i \geq 0 \ \ \forall i \in \{1, \ldots, k\}, \ \ \theta_1 + \cdots + \theta_k = 1\}. \tag{2.23}$$

Then,

$$A > 0 \quad \forall A \in Conv(A_1, \ldots, A_k),$$

if and only if $A_i > 0, \forall i \in \{1, \ldots, k\}$.

Proof: Let $A \in Conv(A_1, \dots, A_k)$. Then:

$$x^T Ax = \theta_1 x^T A_1 x + \cdots + \theta_k x^T A_k x \tag{2.24}$$

for some arbitrary $x \in \mathbb{R}^n$. Assume first that $A_i > 0 \;\; \forall i \in \{1, \dots, k\}$. Since $\theta_i > 0 \;\; \forall i \in \{1, \dots, k\}$ and x is arbitrary, we have that $x^T Ax > 0 \; \forall x \in \mathbb{R}^n$.

For the converse assume that $A > 0$ and set $\theta_1 = 1$ and $\theta_2 = \cdots = \theta_k = 0$ in (2.23) to obtain $A = A_1$. Since A is positive definite, so is A_1. In a similar way, positive definiteness of $A_i, \forall i \in \{2, \dots, k\}$ is also proved.

□

Theorem 2.1. *The event-based system (2.15) with Lipschitz constant ℓ and given periodic TC (2.18) is asymptotically stable and H_∞ performance (2.9) with gain γ is guaranteed if there exist matrices $P > 0, Q > 0, R > 0,$ M and N with appropriate dimensions and scalars $\eta_1 > 0$ and $\eta_2 > 0$ such that following LMIs are satisfied:*

$$\begin{bmatrix} \Gamma_0 + \Gamma_1 + \Gamma_1^T & \star & \star & \star \\ \Gamma_{2i}^T & -R & \star & \star \\ \Gamma_3 & 0 & -\Gamma_5 & \star \\ PJ_1 & 0 & 0 & -\eta_1 I \end{bmatrix} < 0 \qquad \textit{for} \;\; i = 1, 2, \tag{2.25}$$

where:

$$\Gamma_0 = \begin{bmatrix} \Gamma_{01} & PBK & 0 & PB_w & PBK \\ \star & \beta & 0 & 0 & 0 \\ \star & \star & -Q & 0 & 0 \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix}, \tag{2.26}$$

$$\Gamma_{01} = A^T P + PA + Q + C^T C + \ell^2(\eta_1 + h\eta_2), \tag{2.27}$$

$$\Gamma_1 = \begin{bmatrix} N & M - N & -M & 0 & 0 \end{bmatrix}, \tag{2.28}$$

$$\Gamma_{21} = \sqrt{h} M, \quad \Gamma_{22} = \sqrt{h} N, \tag{2.29}$$

$$\Gamma_3 = \sqrt{h} R \Lambda, \quad \Gamma_5 = R - \eta_2^{-1} R^2, \tag{2.30}$$

$$\Lambda = \begin{bmatrix} A & BK & 0 & B_w & BK \end{bmatrix}. \tag{2.31}$$

Proof: Define the following Lyapunov-Krasovskii functional [35]

$$V = x(t)^T P x(t) + \int_{t-h}^{t} x^T(s) Q x(s) ds + \int_{t-h}^{t} \int_{\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta. \tag{2.32}$$

For simplicity denote $\varphi(x, \hat{u})$ by φ and define

$$\nu(t) := \begin{bmatrix} x^T & x^T(t-\tau(t)) & x^T(t-h) & w^T & e_t^T \end{bmatrix}^T. \tag{2.33}$$

Then, the time derivative of V along the trajectories of x for $t \in [k_j h, k_{j+1} h)$ is derived as:

$$\begin{aligned}\dot{V} = & (\Lambda\nu + \varphi)^T P x + x^T P (\Lambda\nu + \varphi) + x^T Q x - x^T(t-h) Q x(t-h) + h\varphi^T R \varphi + 2h\varphi^T R \Lambda \nu \\ & - \int_{t-h}^{t} \dot{x}(s)^T R \dot{x}(s) ds + h\nu^T \Lambda^T R \Lambda \nu.\end{aligned} \tag{2.34}$$

Using Lemma 2.1, we can establish the following inequalities:

$$2h\varphi^T R \Lambda \nu \leq h\nu^T \Lambda^T R {W_2}^{-1} R \Lambda \nu + h\varphi^T W_2 \varphi, \tag{2.35}$$

$$2x^T P \varphi \leq \varphi^T W_1 \varphi + x^T P {W_1}^{-1} P x, \tag{2.36}$$

where $W_1 = \eta_1 I > 0$ and $W_2 = \eta_2 I - R > 0$ are matrices of appropriate dimensions. Furthermore, according to Leibniz-Newton formula, for any M and N of proper dimensions we have:

$$\nu^T M (x(t-\tau(t)) - x(t-h) - \int_{t-h}^{t-\tau(t)} \dot{x}(s) ds) = 0, \tag{2.37}$$

$$\nu^T N (x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s) ds) = 0. \tag{2.38}$$

Using the above equations and inequalities (2.35) and (2.36) and the Lipschitz property of φ we derive:

$$\begin{aligned}\dot{V} \leq & \nu^T \Lambda^T P x + x^T P \Lambda \nu + \ell^2 \eta_1 x^T x + \ell^2 h \eta_2 x^T x + x^T Q x - x^T(t-h) Q x(t-h) \\ & + \nu^T (\Gamma_1^T + \Gamma_1) \nu + h\nu^T \Lambda^T R (R^{-1} + W_2^{-1}) R \Lambda \nu + \eta_1^{-1} x^T P^2 x \\ & + (h - \tau(t)) \nu^T M R^{-1} M^T \nu + \tau(t) \nu^T N R^{-1} N^T \nu,\end{aligned} \tag{2.39}$$

where Γ_1 is defined as (2.28).

To simplify the term $R^{-1} + W_2^{-1}$ we proceed as follows [60]:

$$\begin{aligned}R^{-1} + W_2^{-1} &= R^{-1} + (\eta_2 I - R)^{-1} = (\eta_2 I - R)^{-1}((\eta_2 I - R) R^{-1} + I) \\ &= (R - \eta_2^{-1} R^2)^{-1} = \Gamma_5^{-1}.\end{aligned} \tag{2.40}$$

Now, using (2.40) and (2.18), by some calculation it can be shown that:

$$
\begin{aligned}
\dot{V} + z^T z - \gamma^2 w^T w \leq\ & \nu^T \Lambda^T P x + x^T P \Lambda \nu + +l^2 \eta_1 x^T x + l^2 h \eta_2 x^T x + x^T Q x \\
& -x^T(t-h) Q x(t-h) + x^T C^T C x - \gamma^2 w^T w - e_t^T e_t \\
& +x(t-\tau(t))^T \beta x(t-\tau(t)) + \eta_1^{-1} x^T P^2 x \\
& +h \nu^T \Lambda^T R \Gamma_5^{-1} R \Lambda \nu + (h-\tau(t)) \nu^T M R^{-1} M^T \nu \\
& +\tau(t) \nu^T N R^{-1} N^T \nu + \nu^T (\Gamma_1^T + \Gamma_1) \nu.
\end{aligned} \tag{2.41}
$$

A sufficient condition to have

$$\dot{V} + z^T z - \gamma^2 w^T w \leq 0 \tag{2.42}$$

is that the right hand side of equation (2.41) is negative, which is equivalent to:

$$\Gamma_0 + \Gamma_1 + \Gamma_1^T + \eta_1^{-1} J_1^T P^2 J_1 + \Gamma_3^T \Gamma_5^{-1} \Gamma_3 + \bar{\Gamma} \leq 0, \tag{2.43}$$

where

$$\bar{\Gamma} = \frac{(h-\tau(t))}{h} \Gamma_{21} R^{-1} {\Gamma_{21}}^T + \frac{\tau(t)}{h} \Gamma_{22} R^{-1} {\Gamma_{22}}^T, \tag{2.44}$$

and J_1 is a transformation matrix of proper dimension, defined such that:

$$x(t) = J_1 \nu(t). \tag{2.45}$$

Noting the fact that $\bar{\Gamma}$ is a convex combination of $\Gamma_{21} R^{-1} {\Gamma_{21}}^T$ and $\Gamma_{22} R^{-1} {\Gamma_{22}}^T$ and using Lemma 2.3, and using Schur complement [61], inequality (2.43) is true if and only if LMIs (2.25) hold.

Integrating (2.42) from k_j to $T \in [k_j h, k_{j+1} h)$ we have:

$$V(T) \leq V(k_j h) - \int_{k_j h}^{T} z^T z + \gamma^2 \int_{k_j h}^{T} w^T w. \tag{2.46}$$

In addition, integrating over the intervals $[k_i h, k_{i+1} h)$(for $i = 1, \ldots, j$) and noting $x(t)$ and so $V(t)$ are continuous, we get:

$$V(T) \leq V(0) - \int_0^T z^T z + \gamma^2 \int_0^T w^T w. \tag{2.47}$$

Regarding the fact that $V(T) > 0$, one can achieve the following H_∞ performance criteria:

$$\int_0^T z^T z \leq V(0) + \gamma^2 \int_0^T w^T w \qquad \forall\ T > 0. \tag{2.48}$$

To prove asymptotic stability, set $w = 0$ and let

$$\theta = \lambda_{min}(\Gamma_0 + \Gamma_1 + \Gamma_1^T + \Gamma_3^T \Gamma_5^{-1} \Gamma_3 + \bar{\Gamma} + \eta_1^{-1} J_1^T P^2 J_1). \tag{2.49}$$

By (2.41),

$$\dot{V} \leq -\theta \nu^T \nu. \tag{2.50}$$

So the system is asymptotically stable.

Note that to complete the proof, it should be ensured that $x(t)$ does not leave the region Ω where the Lipschitz condition applies. Using a similar approach to the last part of proof of Theorem 7.1, conditions on initial condition and disturbance can be provided to guarantee this issue.

□

2.4 Controller and Event Generator Design via Multi-Objective Optimization

Suppose that the Lipschitz constant ℓ, TC parameter β and the controller gain K are some unknown variables. In this section our goal is to give a procedure to design K and β in order to meet the following specifications while asymptotic stability and H_∞ performance are guaranteed.

- Enlarging ℓ to enlarge the operation region, following the idea proposed in [60]
- Maximization of β which is related to reduction of data transmission

This leads to a multi-objective optimization problem in which the optimal point is a trade-off between two optimality criteria.

Theorem 2.2. *Consider the Lipschitz nonlinear event-based system (2.15) with the TC (2.18). The closed-loop system is asymptotically stable with guaranteed H_∞ gain γ and the operating region and the data transmission through the feedback loop are simultaneously enlarged and reduced respectively, if there exist matrices $X > 0$, $Y > 0$, $\bar{Q} > 0$, $\bar{R} > 0$, $\bar{\beta}$, $\bar{M}$, $\bar{N}$ with proper dimensions and scalars $\alpha > 0$, $\eta_1 > 0$, $\eta_2 > 0$, $z_\beta > 0$, $z_l > 0$ and*

$0 < \lambda < 1$ *such that the following optimization problem has a solution for some arbitrary* ϵ_1 *and* ϵ_2*:*

$$\min \ \lambda z_\beta + (1-\lambda) z_l \tag{2.51}$$

s.t.

$$\begin{bmatrix} \bar{\Gamma}_0 + \bar{\Gamma}_1 + \bar{\Gamma}_1^T & \star & \star & \star & \star & \star \\ \bar{\Gamma}_{2i} & -\bar{R} & \star & \star & \star & \star \\ \bar{\Gamma}_3 & 0 & -\bar{\Gamma}_5 & \star & \star & \star \\ I & 0 & 0 & -\eta_1 I & \star & \star \\ XJ_1 & 0 & 0 & 0 & -\alpha I & \star \\ CXJ_1 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{2.52}$$

$$for \quad i = 1, 2,$$

$$\begin{bmatrix} z_\beta I & X \\ X & \bar{\beta} \end{bmatrix} \geq 0, \tag{2.53}$$

$$z_l \geq \alpha + h\eta_2 + \eta_1, \tag{2.54}$$

where

$$\bar{\Gamma}_0 = \begin{bmatrix} \Gamma_{01} & BY & 0 & B_w & BY \\ \star & \bar{\beta} & 0 & 0 & 0 \\ \star & \star & -\bar{Q} & 0 & 0 \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & -\epsilon_1^2 I - 2\epsilon_1 X \end{bmatrix}, \tag{2.55}$$

$$\Gamma_{01} = AX + XA^T + \bar{Q}, \tag{2.56}$$

$$\bar{\Gamma}_1 = \begin{bmatrix} \bar{N} & \bar{M} - \bar{N} & -\bar{M} & 0 & 0 \end{bmatrix}, \tag{2.57}$$

$$\bar{\Gamma}_{21} = \sqrt{h}\bar{M}, \quad \bar{\Gamma}_{22} = \sqrt{h}\bar{N}, \tag{2.58}$$

$$\bar{\Gamma}_3 = \sqrt{h} \begin{bmatrix} AX & BY & 0 & B_w & BY \end{bmatrix}, \tag{2.59}$$

$$\bar{\Gamma}_5 = -\epsilon_2^2 \bar{R} + 2\epsilon_2 X - {\eta_2}^{-1} I. \tag{2.60}$$

Having solved this optimization problem, controller gain is obtained by

$$K = YX^{-1}, \tag{2.61}$$

and the optimal values for β and ℓ are calculated as

$$\beta^* := X^{-1}\bar{\beta}X^{-1}, \quad \ell^* := \frac{1}{\sqrt{\alpha(h\eta_2+\eta_1)}}. \tag{2.62}$$

Proof: define new variable α as:

$$\alpha^{-1} := \ell^2(h\eta_2 + \eta_1). \tag{2.63}$$

Using this new variable, inequality (2.43) can be rewritten as:

$$\begin{aligned}&\Gamma_0' + \Gamma_1 + \Gamma_1^T + \eta_1^{-1}J_1^T P^2 J_1 + \Gamma_3^T\Gamma_5^{-1}\Gamma_3 + \alpha^{-1}J_1^T J_1 + C^T J_1^T J_1 C\\ &+\tfrac{(h-\tau(t))}{h}\Gamma_{21}R^{-1}\Gamma_{21}{}^T + \tfrac{\tau(t)}{h}\Gamma_{22}R^{-1}\Gamma_{22}{}^T \leq 0,\end{aligned} \tag{2.64}$$

where Γ_0' is defined as

$$\Gamma_0' = \begin{bmatrix} \Gamma_{01}' & PBK & 0 & PB_w & PBK \\ \star & \beta & 0 & 0 & 0 \\ \star & \star & -Q & 0 & 0 \\ \star & \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix}, \tag{2.65}$$

with

$$\Gamma_{01}' = A^T P + PA + Q. \tag{2.66}$$

To transform inequality (2.64) to LMI, let $X := P^{-1}$ and pre and post-multiply this inequality by $\bar{X} = \mathrm{diag}(X, X, X, I, X)$. Defining new variables

$$\begin{aligned}Y &:= KX, \qquad \bar{\beta} := X\beta X, \quad \bar{Q} := XQX,\\ \bar{M} &:= \bar{X}MX, \quad \bar{N} := \bar{X}NX, \quad \bar{R} := XRX,\end{aligned} \tag{2.67}$$

and using the following inequalities from Lemma 2.2 (similar to [35]):

$$\begin{aligned}-X^2 &\leq \epsilon_1^2 I - 2\epsilon_1 X \qquad \forall\epsilon_1 > 0,\\ X\bar{R}^{-1}X &\geq -\epsilon_2^2\bar{R} + 2\epsilon_2 X \qquad \forall\epsilon_2 > 0,\end{aligned} \tag{2.68}$$

by some calculation it can be proved that LMIs (2.52) are sufficient conditions to guarantee inequality (2.64).The rest of stability and the H_∞ performance proof is the same as the proof of Theorem 2.1.

Now based on definitions (2.67), to maximize β, one can equivalently minimize the term $X\bar{\beta}^{-1}X$. In order to express this objective in a convex form, define an auxiliary variable z_β such that

$$z_\beta I - X\bar{\beta}^{-1}X \geq 0, \tag{2.69}$$

and minimize z_β instead. Note that by Schur complement, inequality (2.69) is equivalent to LMI (2.53).

On the other hand, to enlarge ℓ, based on definition (2.63), we can shrink the summation term $\alpha + h\eta_2 + \eta_1$. So, defining the auxiliary variable z_l we can consider this variable as the objective while satisfying inequality (2.54).

Finally, having two optimality criteria, and using scalarization approach [61], we end up with optimization objective (2.51) where λ varies between 0 and 1.

Remark 2.1. *ϵ_1 and ϵ_2 are some design parameters which can be tuned properly using an iterative algorithm. For more details one can refer to [35].*

2.5 Simulation Results

In this section the effectiveness of the proposed method is illustrated by simulation results. Consider the following unstable plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.1(1 - cos(x_2^3)) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{2.70}$$

where, it is assumed that no disturbance is applied to the system and so we set $B_w = C = 0$.

Now, consider the event-based controller scheme, depicted in Fig.2.1, for this system. Suppose that the TC (2.5) is verified every $h = 0.01$ seconds. To design an optimal feedback gain K and event generator parameter β to increase the closed-loop system region of attraction and reduce data exchange between plant and controller, we set $\epsilon_1 = \epsilon_2 = 10$ and solve the convex optimization problem (2.51) for different values of $0 \leq \lambda \leq 1$.

The trade-off curve between β^* and ℓ^* over the range of λ is shown in Fig. 2.2. As seen, the maximum value for $\lambda_{min}(\beta^*)$ is 0.15, obtained for $\lambda = 1$, while the maximum value of ℓ^* is 0.52 for $\lambda = 0$.

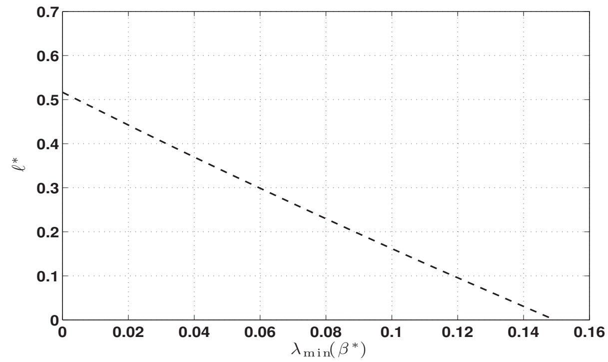


Figure 2.2: Trade-off curve between the optimal points of objective variables $\lambda_{min}(\beta)$ and ℓ.

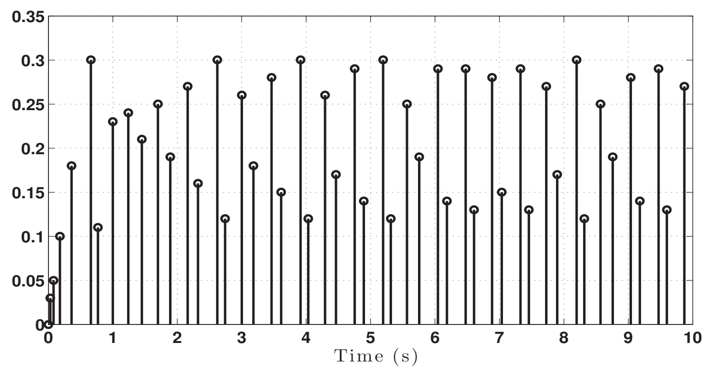


Figure 2.3: Triggering time-intervals versus event sampling instants.

Assuming $\lambda = 0.5$, the maximum Lipschitz constant, the optimal event-TC parameter and the state feedback gain are obtained as follows:

$$\ell^* = 0.28, \quad \beta^* = \text{diag}(0.13, 0.13), \quad K = [-6.93, -12.33]. \tag{2.71}$$

The value of ℓ^* determines a neighborhood of x_2, where the event-based system stability and performance are guaranteed. In the other words, one can say that the system is locally stable as long as the trajectory remains in the ball $|x_2| \leq 3.1$. simulation results for the values (2.71) and initial condition $x_0 = [0.5, 0]^T$ are shown in Figs.2.3 and 2.4.

In Fig. 2.3, time interval lengths $[k_{j-1}h, k_jh)$ are plotted versus k_jh (for $j = 2, 3, \dots$); where the number of data transmitted from plant to the controller in 10 seconds is counted to be only 49. Moreover, according to Fig. 2.4, system state trajectories tend to the origin over the time which confirms the asymptotic stability of the system.

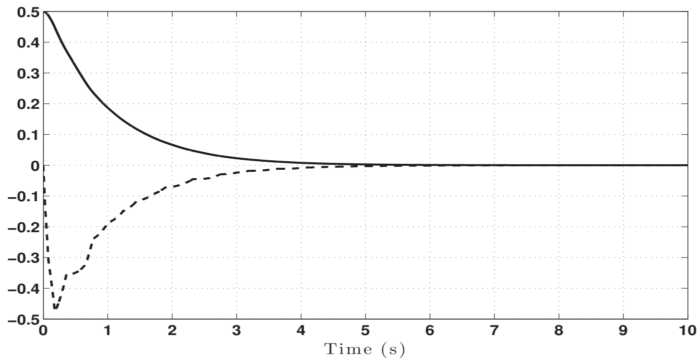


Figure 2.4: x_1 (solid) and x_2 (dashed) go to zero over the time, showing that the system is asymptotically stable using the proposed event-based mechanism.

2.6 Summary

Periodic event-based controller was investigated for a class of Lipschitz nonlinear systems. The design process was formulated into a multi-objective optimization problem, which enlarges the operating region of the system and decrease the information exchange between the plant and controller simultaneously . At the end, the simulation was implemented for a 2^{nd} order unstable plant to show the efficiency of the event-based controller.

Chapter 3

Asynchronous Event-Based H_∞ Control for Output Feedback Systems

In this chapter we study event-triggered control for output feedback linear systems. Assuming a pre-existing control design for the continuous-time plant, we design a periodic event generator that preserves stability and also the H_∞ performance of the system. Due to the dynamics of the output controller, two decentralized event generator modules are employed acting at the both controller and plant sides. The proposed TCs have the advantage to operate with non-equal sampling rates. Consequently, the triggering instants of the plant and controller event generators may be quite asynchronous. We first model the proposed event-based system in a continuous-time form and then stability and performance conditions are formulated using Linear Matrix Inequalities (LMIs). In addition, event generator conditions are designed to decrease data transmission through the network channel and also reduce the H_∞ gain of the system. Simulation results are given to illustrate the effectiveness of the proposed system.

The rest of the chapter is organized as follows: Section 3.1 introduces problem formulation and also the system structure. Section 3.2 contains modeling and analysis, and section 3.3 represents design issues for the proposed event-based system. Section 3.4 provides an illustrative example showing the efficiency of the proposed mechanism and finally summary is given as the last part.

3.1 Problem Formulation

Consider the following linear system

$$P : \begin{cases} \dot{x}_p = A_p x_p + B_p u + B_w w \\ y = C_p x_p \\ z = H_p x_p \end{cases} \tag{3.1}$$

where $x_p(t) \in \mathbb{R}^n$ is the plant state vector, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ are plant control input and measured output, respectively. $z(t) \in \mathbb{R}^p$ is the controlled output and $w(t)$ is an exogenous disturbance input which is assumed to belong to $\mathcal{L}_2[0, +\infty)$.

The system P is controlled using an output feedback controller C of the form

$$C : \begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c \end{cases} \tag{3.2}$$

where $x_c \in \mathbb{R}^n$ is the controller state vector.

Consider now the following scenario: assume that instead of continuous communication between plant and controller, data is transmitted asynchronously only upon the violation of certain conditions. More explicitly; two decentralized TCs will be considered: a TC is setup at the plant output and fresh data y is transferred to the controller only when the plant TC is violated. A second, independent, TC is setup at the controller output, and an updated controller signal is sent to the plant when the controller TC is violated. Based on this scenario, the plant and control inputs are piecewise constant signals that will be denoted $\hat{u}$ and $\hat{y}$, respectively:

$$\begin{cases} \dot{x}_p = A_p x_p + B_p \hat{u} + B_w w \\ \dot{x}_c = A_c x_c + B_c \hat{y} \\ y = C_p x_p \\ u = C_c x_c \\ z = H_p x_p \end{cases} \tag{3.3}$$

A general block diagram of the proposed event-based output feedback system is shown in Fig.3.1.

In this work we consider two separate and asynchronous periodic event-TCs. More precisely, we use two different samplers at the outputs of plant and controller , with sampling times h_p and h_c, respectively (Fig. 3.1).

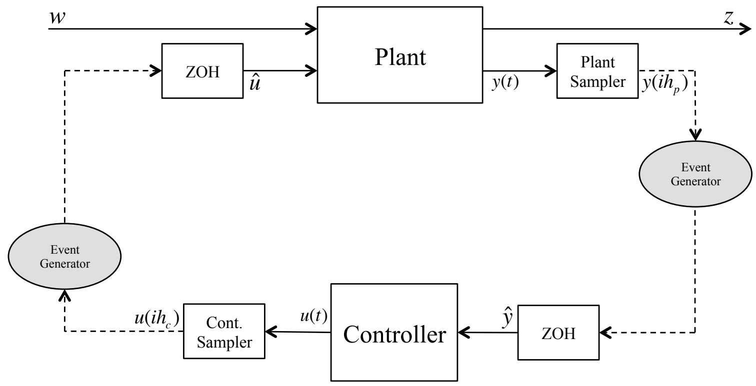


Figure 3.1: LTI systems controlled using event-based output feedback controller

Controller Implementation: To implement the continuous-time controller for the event-based system (3.3) on a digital platform we proceed by *emulation*; *i.e.* we find a discrete-time approximation of the controller implemented via integration methods, using an integration period typically much smaller than the smallest sampling period in the feedback loop.

In the following section we introduce the event generator mechanism applied in the event-based system. Our goal is to design event generator blocks such that the proposed event-based system remains stable and satisfies H_∞ performance, defined as Definition 2.1.

3.2 Periodic Event-Triggered Control for Output Feedback Systems

The first part of this section introduces the event triggering mechanisms used in the rest of the chapter. We then exploit a delay system approach to model the event-based system, and finally the stability analysis is given in section 3.2.3.

3.2.1 Event Generator Modules Structure

Plant Triggering Mechanism

As mentioned earlier, the plant event generator module samples the plant output periodically with period h_p. An updated output value is sent from the plant to the controller only whenever the normalized value of the difference between the latest transmitted output and the new sample surpasses a certain threshold. In other words, let $\hat{y}$ be the latest plant output value sent to the controller at time $t = \hat{t}_p$; Then, the plant event generator sends a fresh sample of y at the sample time $ih_p > \hat{t}_p$, if the following event triggering condition is violated:

$$(y(ih_p) - \hat{y})^T (y(ih_p) - \hat{y}) - \sigma_p y^T (ih_p) y(ih_p) \leq 0. \tag{3.4}$$

In the above inequality, the scalar $\sigma_p > 0$ is the plant side TC parameter to be designed to effectively reduce data communication from plant to controller.

Controller Triggering Mechanism

The TC on the controller side is also verified periodically; but with a different sampling time, denoted h_c. Let $\hat{u}$ be the latest controller output value sent to the plant at time $t = \hat{t}_c$; Then the controller event generator sends a new sample of u at the sample time $jh_c > \hat{t}_c$, if the following event triggering condition is violated:

$$(u(jh_c) - \hat{u})^T (u(jh_c) - \hat{u}) - \sigma_c u^T (jh_c) u(jh_c) \leq 0. \tag{3.5}$$

As in the plant side, the scalar $\sigma_c > 0$ is the controller side TC parameters which should be designed to effectively reduce data communication from controller to plant.

3.2.2 Formulating the event-based system using delay system approach

In order to write the event-based system dynamics (3.3) in a time-delay system form, let the subsequence $\{a_k\}_{k=0}^{\infty}$ represent sample numbers at which data is broadcast from the plant to the controller. Similarly, denote $\{b_k\}_{k=0}^{\infty}$ as the corresponding subsequence at which data is sent from the controller to the plant. Then the inputs of the controller and plant (shown in in Fig. 3.1) are expressed as follows:

$$\begin{aligned} \hat{y}(t) &= y(a_k h_p) \qquad \text{for} \quad t \in [a_k h_p, a_{k+1} h_p) \quad , \quad k = 1, 2, \ldots \\ \hat{u}(t) &= u(b_k h_c) \qquad \text{for} \quad t \in [b_k h_c, b_{k+1} h_c) \quad , \quad k = 1, 2, \ldots \end{aligned} \tag{3.6}$$

Now, let l_t^p be the latest sample number of the plant side before the time instant 't' :

$$l_t^p := max\left\{ i \in \mathbb{N} : ih_p \leq t \right\}, \tag{3.7}$$

and define

$$\tau_1(t) := t - l_t^p h_p \qquad \text{with} \quad 0 \le \tau_1(t) < h_p. \tag{3.8}$$

In a similar way, define l_t^c as the latest sample number of the controller side before the time instant 't':

$$l_t^c := max\left\{i \in \mathbb{N} : ih_c \le t\right\}, \tag{3.9}$$

and

$$\tau_2(t) := t - l_t^c h_c \qquad \text{with} \quad 0 \le \tau_2(t) < h_c. \tag{3.10}$$

Adding and subtracting $B_p u(l_t^c h_c)$ and $B_c y(l_t^p h_p)$ to the plant and controller dynamics, respectively, in equation (3.3) and using above definitions, the following time delay dynamics can be obtained:

$$\begin{cases} \dot{x}_p = A_p x_p + B_p e_t^c + B_p u(t - \tau_2(t)) + B_w w \\ \dot{x}_c = A_c x_c + B_c e_t^p + B_c y(t - \tau_1(t)) \\ y(t) = C_p x_p(t) \\ u(t) = C_c x_c(t) \\ z(t) = H_p x_p(t) \end{cases} \tag{3.11}$$

where:

$$\begin{aligned} e_t^p &:= y(l_t^p h_p) - \hat{y}(t), \\ e_t^c &:= u(l_t^c h_c) - \hat{u}(t). \end{aligned} \tag{3.12}$$

Using the same approach, the event-TCs are also expressed in time delay form: Based on the triggering mechanism introduced, the triggering inequalities (3.4) and (3.5) hold for every sampling time of plant and controller side, respectively. So, in particular, for the sample numbers l_t^p and l_t^c we have:

$$\begin{aligned} (y(l_t^p h_p) - \hat{y}(t))^T (y(l_t^p h_p) - \hat{y}(t)) &\le \sigma_p y^T(l_t^p h_p) y(l_t^p h_p), \\ (u(l_t^c h_c) - \hat{u}(t))^T (u(l_t^c h_c) - \hat{u}(l_t^c h_c)) &\le \sigma_c u^T(l_t^c h_c) u(l_t^c h_c), \end{aligned} \tag{3.13}$$

which can be readily expressed as:

$$\begin{aligned} e_t^{pT} e_t^p &\le \sigma_p y^T(t - \tau_1(t)) y(t - \tau_1(t)), \\ e_t^{cT} e_t^c &\le \sigma_c u^T(t - \tau_2(t)) u(t - \tau_2(t)). \end{aligned} \tag{3.14}$$

To put the event-based system equations in a compact form, define new variables

$$x := \begin{bmatrix} x_p^T, & x_c^T \end{bmatrix}^T \quad , \quad e := \begin{bmatrix} e_t^{pT}, & e_t^{cT} \end{bmatrix}^T, \tag{3.15}$$

and new matrices

$$\bar{A}_{d1} := \begin{bmatrix} 0 & B_pC_c \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{d2} := \begin{bmatrix} 0 & 0 \\ B_cC_p & 0 \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix},$$
$$\bar{B}_w := \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \qquad H = \begin{bmatrix} H_p, 0 \end{bmatrix}, \tag{3.16}$$

$$\beta_1 := \mathrm{diag}(\sigma_p C_p^T C_p, 0) \quad , \quad \beta_2 := \mathrm{diag}(0, \sigma_c C_c^T C_c) \quad , \quad \bar{A} := \mathrm{diag}(A_p, A_c) \ . \tag{3.17}$$

Then, for the interval $[a_k h, a_{k+1} h)$ the system equations (3.11) are expressed as:

$$\begin{aligned} &\dot{x} = \bar{A}x(t) + \bar{A}_{d1}x(t-\tau_1(t)) + \bar{A}_{d2}x(t-\tau_2(t)) + \bar{B}e + \bar{B}_w w, \\ &z(t) = Hx_p(t), \end{aligned} \tag{3.18}$$

where the following TC holds:

$$e^T e \leq x(t-\tau_1)^T \beta_1 x(t-\tau_1) + x(t-\tau_2)^T \beta_2 x(t-\tau_2). \tag{3.19}$$

3.2.3 Stability analysis of the proposed event-based system

In this section we present Theorem 3.1, our main result. This theorem provides sufficient conditions that guarantee asymptotic stability and H_∞ performance of the event-based system.

Theorem 3.1. *Consider the system (3.3) with the asynchronous periodic TCs (3.4) and (3.5). The event-based system is asymptotically stable with H_∞ performance (2.9), if there exist matrices $P > 0, Q_1 > 0, Q_2 > 0, R_1 > 0, R_2 > 0, U_1$ and U_2 with appropriate dimensions such that following LMIs are satisfied:*

$$\mathcal{L}_i^U = \begin{bmatrix} \Pi + S^T + S & \star & \star & \star \\ \Upsilon_{21}^i & -R_1 & \star & \star \\ \Upsilon_{31} & 0 & -R_2 & \star \\ \Upsilon_i^U & 0 & 0 & -R_i \end{bmatrix} < 0 \quad \mathit{for} \quad i = 1, 2,$$
$$\tag{3.20}$$
$$\mathcal{L}_i^V = \begin{bmatrix} \Pi + S^T + S & \star & \star & \star \\ \Upsilon_{21}^i & -R_1 & \star & \star \\ \Upsilon_{31} & 0 & -R_2 & \star \\ \Upsilon_i^V & 0 & 0 & -R_i \end{bmatrix} < 0 \quad \mathit{for} \quad i = 1, 2,$$

where:

$$\Upsilon_i^U = \sqrt{h_p + h_c}U_i,\ \Upsilon_i^V = \sqrt{h_p + h_c}V_i, \\ \Upsilon_{21} = \sqrt{h_p}R_1\Sigma,\ \Upsilon_{31} = \sqrt{h_c}R_2\Sigma, \tag{3.21}$$

$$S = \begin{bmatrix} V_1 + V_2, & U_1 - V_1, & U_2 - V_2, & -U_1, & -U_2, & 0, & 0 \end{bmatrix}, \\ \Sigma = \begin{bmatrix} \bar{A}, & \bar{A}_{d1}, & \bar{A}_{d2}, & 0, & 0, & \bar{B}_w, \bar{B} \end{bmatrix}, \tag{3.22}$$

and

$$\Pi = \begin{bmatrix} \Pi_{11} & P\bar{A}_{d1} & P\bar{A}_{d2} & 0 & 0 & P\bar{B}_w & P\bar{B} \\ \star & \beta_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \beta_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_2 & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 & 0 \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix}, \tag{3.23}$$

with

$$\Pi_{11} = \bar{A}^T P + P\bar{A} + Q_1 + Q_2 + HH^T.$$

Proof: Consider the following Lyapunov-Krasovskii functional

$$V = x^T(t)Px(t) + \int_{t-h_p}^t x^T(s)Q_1x(s)ds + \int_{t-h_p}^t \int_\theta^t \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \\ + \int_{t-h_c}^t x^T(s)Q_2x(s)ds + \int_{t-h_c}^t \int_\theta^t \dot{x}^T(s)R_2\dot{x}(s)dsd\theta. \tag{3.24}$$

Define

$$\zeta := \begin{bmatrix} x^T, & x^T(t-\tau_1(t)), & x^T(t-\tau_2(t)), & x^T(t-h_p), & x^T(t-h_c), & w^T, & e^T \end{bmatrix}^T. \tag{3.25}$$

Computing the time derivative of V along the trajectories of x, we have:

$$\dot{V} = \zeta^T\Sigma^T Px + x^T P\Sigma\zeta + x^T Q_1 x + x^T Q_2 x - x^T(t-h_p)Q_1x(t-h_p) \\ -x^T(t-h_c)Q_2x(t-h_c) + h_p\zeta^T\Sigma^T R_1\Sigma\zeta - \int_{t-h_p}^t \dot{x}(s)^T R_1\dot{x}(s)ds \\ +h_c\zeta^T\Sigma^T R_2\Sigma\zeta - \int_{t-h_c}^t \dot{x}(s)^T R_2\dot{x}(s)ds. \tag{3.26}$$

Based on Leibniz-Newton formula the following equations hold for any U_1, U_2, V_1 and V_2 of proper dimensions.

$$\Delta_1 := 2\zeta^T V_1(x(t) - x(t-\tau_1(t)) - \int_{t-\tau_1(t)}^t \dot{x}(s)ds) = 0, \\ \Delta_2 := 2\zeta^T U_1(x(t-\tau_1(t)) - x(t-h_p) - \int_{t-h_p}^{t-\tau_1(t)} \dot{x}(s)ds) = 0, \\ \Delta_3 := 2\zeta^T V_2(x(t) - x(t-\tau_2(t)) - \int_{t-\tau_2(t)}^t \dot{x}(s)ds) = 0, \\ \Delta_4 := 2\zeta^T U_2(x(t-\tau_2(t)) - x(t-h_c) - \int_{t-h_c}^{t-\tau_2(t)} \dot{x}(s)ds) = 0. \tag{3.27}$$

Adding above terms to the right hand side of (3.26), $\dot{V}$ can be rewritten as

$$\begin{aligned}\dot{V} = {} & \zeta^T \Sigma^T P x + x^T P \Sigma \zeta + x^T Q_1 x + x^T Q_2 x - x^T(t-h_p) Q_1 x(t-h_p) \\ & -x^T(t-h_c) Q_2 x(t-h_c) + h_p \zeta^T \Sigma^T R_1 \Sigma + h_c \zeta^T \Sigma^T R_2 \Sigma \zeta + \zeta^T (S^T + S)\zeta \\ & - \textstyle\int_{t-h_p}^{t-\tau_1(t)} (R_1 \dot{x}(s) + U_1^T \zeta(t))^T R_1^{-1} (R_1 \dot{x}(s) + U_1^T \zeta(t)) ds + \tau_1(t) \zeta^T V_1 R_1^{-1} V_1^T \zeta \\ & - \textstyle\int_{t-\tau_1(t)}^{t} (R_1 \dot{x}(s) + V_1^T \zeta(t))^T R_1^{-1} (R_1 \dot{x}(s) + V_1^T \zeta(t)) ds + \tau_2(t) \zeta^T V_2 R_2^{-1} V_2^T \zeta \\ & - \textstyle\int_{t-h_c}^{t-\tau_2(t)} (R_2 \dot{x}(s) + U_2^T \zeta(t))^T R_2^{-1} (R_2 \dot{x}(s) + U_2^T \zeta(t)) ds \\ & - \textstyle\int_{t-\tau_2(t)}^{t} (R_2 \dot{x}(s) + V_2^T \zeta(t))^T R_2^{-1} (R_2 \dot{x}(s) + V_2^T \zeta(t)) ds \\ & +(h_p - \tau_1(t)) \zeta^T U_1 R_1^{-1} U_1^T \zeta + (h_c - \tau_2(t)) \zeta^T U_2 R_2^{-1} U_2^T \zeta. \end{aligned} \tag{3.28}$$

Using the above inequality and the TC (3.19), it is easy to see that:

$$\dot{V} + z^T z - \gamma^2 w^T w \leq \zeta^T (S^T + S + \Pi + \Upsilon_{21}^T R_1^{-1} \Upsilon_{21} + \Upsilon_{31}^T R_2^{-1} \Upsilon_{31} + \Upsilon)\zeta, \tag{3.29}$$

where :

$$\begin{aligned}\Upsilon = {} & \tfrac{h_p - \tau_1(t)}{h_p + h_c} \Upsilon_1^U R_1^{-1} \Upsilon_1^{U^T} + \tfrac{\tau_1(t)}{h_p+h_c} \Upsilon_1^V R_1^{-1} \Upsilon_1^{V^T} + \tfrac{h_c - \tau_2(t)}{h_p+h_c} \Upsilon_2^U R_2^{-1} \Upsilon_2^{U^T} \\ & + \tfrac{\tau_2(t)}{h_p+h_c} \Upsilon_2^V R_1^{-1} \Upsilon_2^{V^T}. \end{aligned} \tag{3.30}$$

Equation (3.29) can be expressed as follows:

$$\dot{V} + z^T z - \gamma^2 w^T w \leq \zeta^T (\tfrac{h_p - \tau_1(t)}{h_p+h_c} \bar{\Upsilon}_1^U + \tfrac{\tau_1(t)}{h_p+h_c} \bar{\Upsilon}_1^V + \tfrac{h_c - \tau_2(t)}{h_p+h_c} \bar{\Upsilon}_2^U + \tfrac{\tau_2(t)}{h_p+h_c} \bar{\Upsilon}_2^V)\zeta, \tag{3.31}$$

where:

$$\begin{aligned} \bar{\Upsilon}_i^U &= S^T + S + \Pi + \Upsilon_{21}^T R_1^{-1} \Upsilon_{21} + \Upsilon_{31}^T R_2^{-1} + \Upsilon_i^U R_1^{-1} \Upsilon_i^{U^T} \quad \text{for } i = 1, 2, \\ \bar{\Upsilon}_i^V &= S^T + S + \Pi + \Upsilon_{21}^T R_1^{-1} \Upsilon_{21} + \Upsilon_{31}^T R_2^{-1} + \Upsilon_i^V R_1^{-1} \Upsilon_i^{V^T} \quad \text{for } i = 1, 2. \end{aligned} \tag{3.32}$$

Since :

$$\sum_{i=1}^{2} [\frac{h_i - \tau_i(t)}{h_p + h_c} + \frac{\tau_i(t)}{h_p + h_c}] = 1, \tag{3.33}$$

using Lemma 2.3 and applying Schur complement [61], the right hand side of equation (3.31) is negative definite if and only if LMIs (3.20) hold.

Now, define

$$-\mu = \max_{0 \leq \tau_1(t) \leq h_p, 0 \leq \tau_2(t) \leq h_c} [\tfrac{h_p - \tau_1(t)}{h_p+h_c} \theta_1^U + \tfrac{\tau_1(t)}{h_p+h_c} \theta_1^V + \tfrac{h_c - \tau_2(t)}{h_p+h_c} \theta_2^U + \tfrac{\tau_2(t)}{h_p+h_c} \theta_2^V] < 0, \tag{3.34}$$

where:

$$\theta_i^U = \lambda_{max}(\bar{\Upsilon}_i^U), \quad \theta_i^V = \lambda_{max}(\bar{\Upsilon}_i^V) \quad \text{for } i = 1, 2. \tag{3.35}$$

Taking account of (3.31) and (3.34), we have that

$$\dot{V} + z^T z - \gamma^2 w^T w \leq -\mu\zeta^T\zeta \leq -\mu x^T x. \tag{3.36}$$

Integrating above inequality from 0 to T, we get:

$$V(T) - V(0) \leq -\int_0^T z^T(s)z(s) + \gamma^2 \int_0^T w^T(s)w(s). \tag{3.37}$$

Since $V(T) > 0$, if $T \to \infty$, the H_∞ performance criteria (2.9) is satisfied:

$$\int_0^\infty z^T(s)z(s)ds \leq V(0) + \gamma^2 \int_0^\infty w^T(s)w(s)ds. \tag{3.38}$$

Assuming now that $w = 0$, the following inequality is achieved from (3.31):

$$\dot{V} \leq -\mu\zeta^T\zeta \leq -\mu x^T x. \tag{3.39}$$

Then, it follows from Barballat's lemma [62] that the origin is asymptotically stable. This concludes the proof.

□

3.3 Design Issues

The primary design element in our formulation are the parameters σ_p and σ_c that controls the trigger levels and indirectly controls the amount of data transferred over the network, and also the H_∞ gain γ. Considering σ_p, σ_c and γ^2 as extra LMI variables in (3.20), we can easily minimize the data transmission while lowering the impact of disturbance on the output performance.

Let $\sigma = \text{diag}(\sigma_p, \sigma_c)$. Based on (3.4) and (3.5), in order to achieve data transmission reduction, we should maximize $\lambda_{max}(\sigma)$ or equivalently minimize $\lambda_{min}(\sigma^{-1})$. To express this objective in a convex form, define an auxiliary variable z_σ such that

$$z_\sigma I - \sigma^{-1} \geq 0, \tag{3.40}$$

and minimize z_σ instead. Note that by Schur complement, inequality (3.40) is equivalent to:

$$\mathcal{L}_\sigma = \begin{bmatrix} z_\sigma I & I \\ I & \sigma \end{bmatrix} \geq 0. \tag{3.41}$$

Having two optimality criteria, and using scalarization approach [61], we end up with the following multi-objective optimization problem, where ϵ varies between 0 and 1.

$$\begin{aligned} &\min \epsilon z_\sigma + (1-\epsilon)\gamma^2 \\ &\mathcal{L}_1^U < 0, \quad \mathcal{L}_1^V < 0, \\ &\mathcal{L}_2^U < 0, \quad \mathcal{L}_2^V < 0, \\ &\mathcal{L}_\sigma < 0. \end{aligned} \tag{3.42}$$

Remark 3.1. *If we set $h_p = h_c = 0$ and also $\sigma_p = \sigma_c = 0$, the event-based system would turn into the traditional output feedback control scheme. Therefore, since this traditional system is already controlled and stabilized, it is expected that the optimization problem (3.42) has solution, at least for small values of h_p and h_c.*

To find proper values for the TC sampling periods, one can iteratively solve the optimization problem (3.42) for different values of h_p and h_c (up to the point where the constraints become infeasible), to tune them appropriately, according to the optimization results as well as hardware execution frequency limitations.

3.4 Simulation Results

To evaluate the operation of the proposed scheme, consider a satellite model, described in [63]. This system contains two rigid bodies which are connected through a flexible link. Modeling the link as a spring with torque constant k_s and viscous damping f, the equations of motion are given as follows:

$$\begin{aligned} &J_1\ddot{\theta}_1 + f(\dot{\theta}_1 - \dot{\theta}_2) + k_s(\theta_1 - \theta_2) = u(t) + 0.1w(t), \\ &J_2\ddot{\theta}_2 + f(\dot{\theta}_2 - \dot{\theta}_1) + k_s(\theta_2 - \theta_1) = 0, \end{aligned} \tag{3.43}$$

where, J_1 and J_2 denote the moment of inertia of two bodies, θ_1 and θ_2 represent the yaw angle of two bodies, and $u(t)$ and $w(t)$ are control torque and external disturbance, respectively. Now, define the state variables as $x = \begin{bmatrix}\theta_1, & \theta_2, & \dot{\theta}_1, & \dot{\theta}_2\end{bmatrix}^T$ and consider the output and controlled signals as $y = x_1$ and $z = 0.1x_1 + 0.1x_2$, respectively. Choosing $J_1 = J_2 = 1$, $f = 0.09$ and $k_s = 0.04$, the state space dynamics are expressed in the form of equations (3.1), with the following parameters:

$$A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3 & 0.3 & -0.004 & 0.004 \\ 0.3 & -0.3 & 0.004 & -0.004 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}, \tag{3.44}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad H_p = \begin{bmatrix} 0.1 & 0.1 & 0 & 0 \end{bmatrix}.$$

This system is already controlled by the output feedback controller (3.2), with the following matrices:

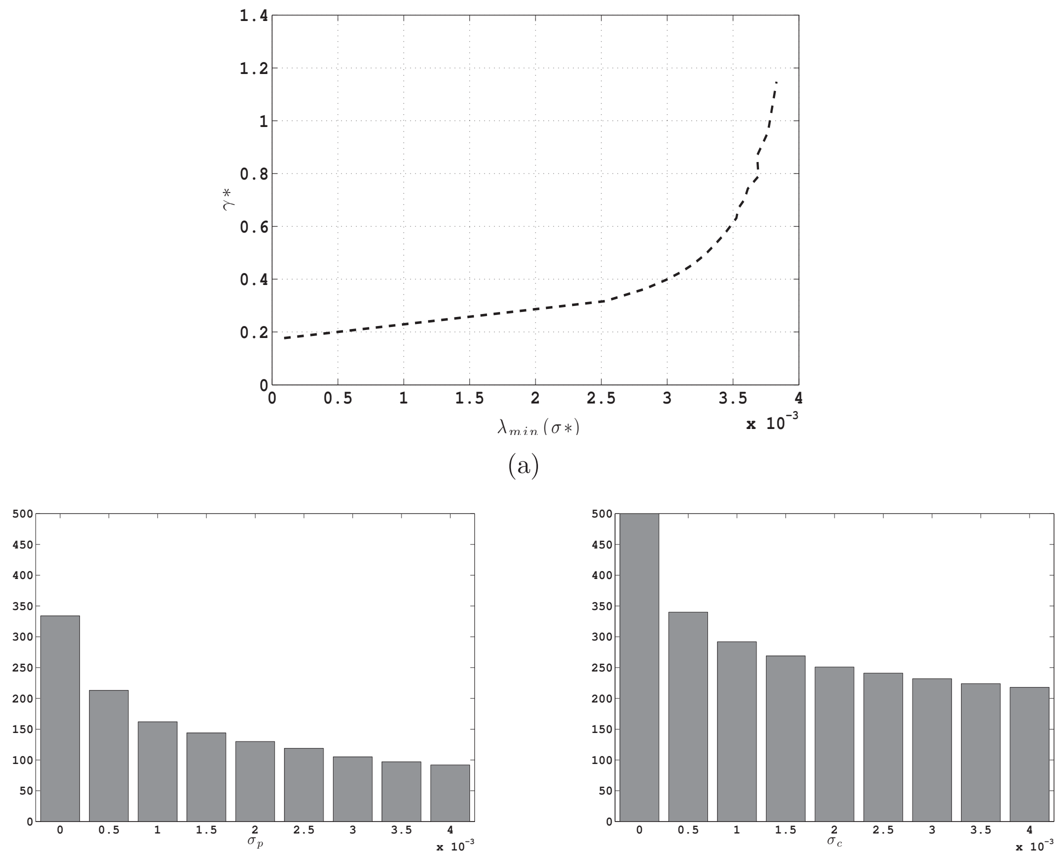
$$A_c = \begin{bmatrix} -2.9020 & 0.1727 & 1 & 0 \\ 0.0524 & -2.8900 & 0 & 1 \\ -6.7021 & 2.7198 & -3.6691 & -7.8290 \\ 0.2846 & -2.0820 & 0.0040 & -0.0040 \end{bmatrix}, \quad B_c = \begin{bmatrix} 2.9020 & -0.1727 \\ -0.0524 & 2.8900 \\ 1.8440 & -0.6144 \\ 0.0154 & 1.7820 \end{bmatrix}, \tag{3.45}$$

$$C_c = \begin{bmatrix} -4.5581 & 1.8054 & -3.6651 & -7.8330 \end{bmatrix}.$$

Assume now that instead of continuous data communication between plant and controller, an event-based mechanism is inserted. Consider the periodic TCs (3.4) and (3.5) along with the sampling periods of $h_p = 0.03$ and $h_c = 0.02$, respectively.

To design an optimal event-based mechanism for the above controller, The optimization problem (3.42) is solved, using YALMIP Toolbox in MATLAB, for different values of $0 \leq \epsilon \leq 1$. Fig 3.2.(a) represents the trade-off curve between $\gamma^\star$ and $\lambda_{min}(\sigma^\star)$. Based on the figure, it can be interpreted that the more data is exchanged between plant and controller, the better disturbance attenuation level is achieved by the control scheme.

Now, the initial conditions are set to $x_p(0) = [2, 1, 0, 0]^T, x_c(0) = [0.1, 0.4, 0, 0]^T$ and the disturbance is considered as $w(t) = e^{-0.3t}sin(3t)$. Simulations are carried out for different values of σ for 10 *sec.* Fig 3.2.(b) and (c) exhibit the number of data exchange between plant and controller in each simulation. According to the figures, as expected (since plant side sampling period is larger than the controller side's one), number of data transmitted from plant triggering module are quite smaller than the corresponding values for the controller triggering module. In addition, these figures illustrate that how increase



(a)

(b) (c)

Figure 3.2: (a) Trade-off curve between optmial values of $\lambda_{min}(\sigma)$ and γ. (b) Number of sampled data sent from plant to the controller. (c) Number of sampled data sent from controller to the plant.

of σ_p and σ_c from 0 to 4×10^{-3} effectively results in more than 50% data transmission reduction.

To evaluate the system state response performance, consider the optimization results for the case $\epsilon = 0.4$:

$$\sigma_p = \sigma_c = 3.4 \times 10^{-3}, \quad \gamma = 0.528.$$

Simulation results for this case are shown in Fig. 3.3. As seen in Fig. 3.3.(a), the impact of the disturbance on the output signal is appropriately attenuated by the proposed controller. Figs. 3.3.(b) and (c) represent inter-event sampling times for the plant and controller sides. It is observed that the minimum inter-event time for the plant and controller sides are equal to the sampling times $h_p = 0.03$ and $h_c = 0.02$, respectively, and the corresponding maximum values are 0.33 and 0.64. In addition, based on the

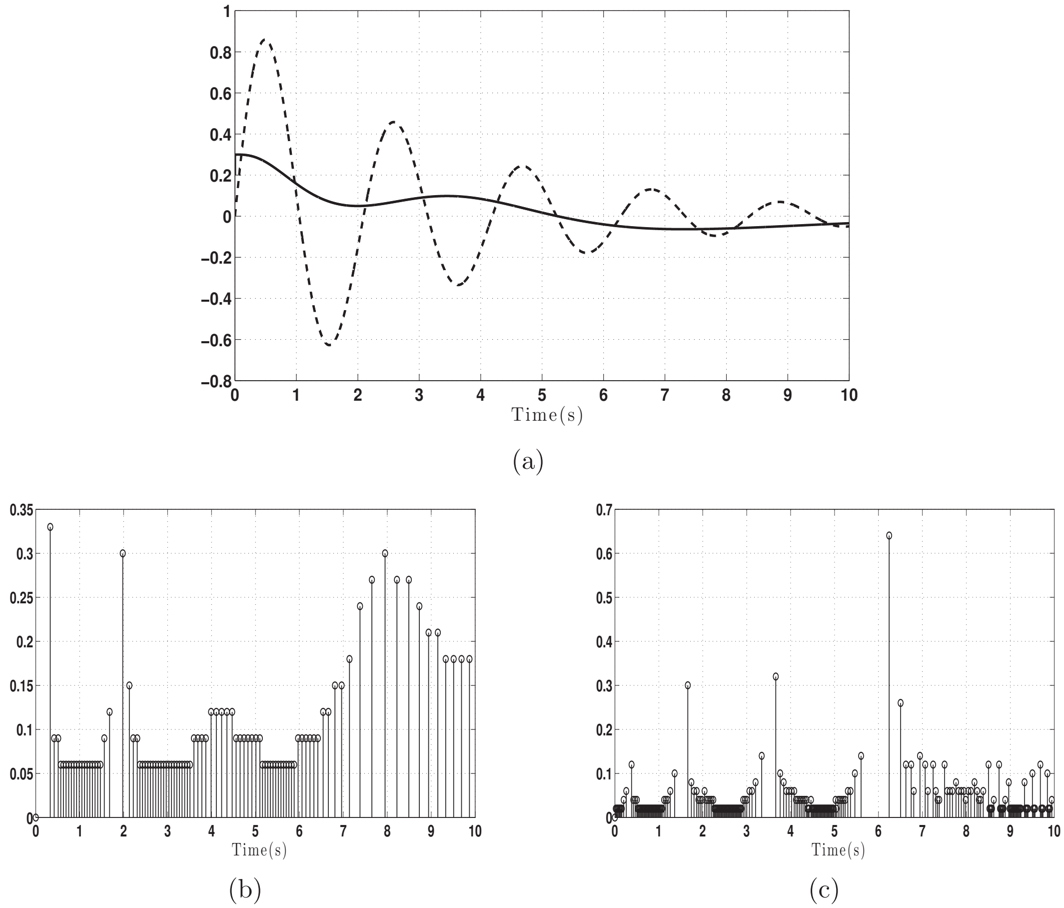


Figure 3.3: Simulation results for the case $\sigma_p = \sigma_c = 3.4 \times 10^{-3}$ (a) Trajectory of output signal z (solid) in the presence of disturbance w (dashed) (b) Inter-event sampling times for the plant side (c) Inter-event sampling times for the controller side

obtained results, the average of only 29% and 46% of sampled data are sent from plant and controller side respectively; something which validates the effectiveness of event-based mechanism for efficient energy resources usage and data transmission reduction.

3.5 Summary

Periodic event generator design was investigated for output feedback linear systems with disturbance. Two decentralized event generator schemes were considered at the controller and plant sides, respectively, operating asynchronously with non-equal sampling rates. stability and H_∞ performance analysis were carried out using LMIs. Both data transmission and performance gain reduction were formulated as a multi-objective convex optimization problem. At the end, the simulation was implemented for a satellite model to show the efficiency of designed event-based system.

Chapter 4

Event-Based Controller Design for LTI Systems in the Presence of Measurement Noise

The presence of measurement noise in the event-based systems can lower system efficiency both in terms of data exchange rate and performance. In this chapter a periodic integral-based event-triggered control system is proposed for LTI systems with stochastic measurement noise. We show that the new mechanism is robust against noise and effectively reduces the flow of communication between plant and controller, and also improves output performance. Using a Lyapunov approach, stability in the mean square sense is proved. A simulated example illustrates the properties of our approach.

The rest of this chapter is organized as follows. In section 4.1, problem statement and also the structure of the proposed system is provided. In section 4.2 the event-triggered system is modeled using a delay system approach, [45], and then in section 4.3 the conditions for asymptotic stability in mean square sense are given in form of LMIs. In section 4.4 we propose a convex optimization approach to obtain the parameters of the event triggering system minimizing data transmission in a networked-based control. Simulation results are represented in section 4.5 and finally, a summary of this chapter is given in section 4.6.

4.1 Problem Statement

Consider the following class of linear stochastic systems

$$\begin{cases} \dot{x} = Ax + Bu \\ x_\nu = x(t) + \nu(t) \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input. x_ν represent the measured states, contaminated by noise $\nu(t)$, assumed to be zero mean white Gaussian noise vector with the covariance matrix $\sigma^2 I$. Our main goal is to introduce an event-based control scheme that reduces data communication between plant and controller and is robust against measurement noise.

We begin by examining the traditional event-based paradigm and highlighting its possible shortcomings in the presence of noise. Following [1], the control law for the system (4.1) is defined as:

$$u = K\hat{x}_\nu(t), \tag{4.2}$$

where K is the state feedback gain and $\hat{x}_\nu(t)$ is the piecewise continuous signal containing intermittent measured state information:

$$\hat{x}_\nu(t) = x_\nu(t_k) \qquad \forall t \in [t_k, t_{k+1}). \tag{4.3}$$

Here, $t_k \ \ (k \in \mathbb{N})$ represents the time instants when an event condition is violated and a new state value is sent to the controller. Defining ([1]),

$$e_\nu := x_\nu(t) - x_\nu(t_k), \tag{4.4}$$

$$f(x_\nu, e_\nu) := e_\nu^T e_\nu - x_\nu^T Q x_\nu \tag{4.5}$$

for some positive definite matrix Q; the TC can be defined as:

$$f(x_\nu, e_\nu) < 0. \tag{4.6}$$

Equations (4.2) and (4.6) summarize the classical approach to event-triggered control. One problem with this methodology is that in a noisy environment not only the feedback value, but also the triggering instants directly depends on the instant values of the noise with the consequent potential to affect system performance and, more importantly, trigger unnecessary samples, making the event based control scheme less effective.

We now propose a novel event-triggered controller.

4.1.1 Noise Effective Event Generator Structure

Similar to the previous chapters, throughout this chapter we assume that the event generator block is clock driven with the period of h. To cope with the effect of noise, instead of instant measured values, our event-based mechanism consists of an average of the measured data over the past time interval $T_{int} < h$. Defining $\bar{x}_\nu(ih)$ as the input of event generator, we have:

$$\bar{x}_\nu(ih) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} x_\nu(\alpha) d\alpha \quad \text{for} \quad i = 1, 2, \dots \tag{4.7}$$

The event generator block transmits an updated value of $\bar{x}_\nu$ only when the normalized value of the difference between the latest transmitted value and the new sample surpasses a certain threshold. In the other words, let $\hat{\bar{x}}_\nu$ be the last value sent to the controller at the time instant $\hat{t}$. Then, the event generator transmits a new sample of $\bar{x}_\nu$ at the sample time $ih > \hat{t}$, if the following periodic integral based triggering condition is violated:

$$(\bar{x}_\nu(i) - \hat{\bar{x}}_\nu)^T(\bar{x}_\nu(i) - \hat{\bar{x}}_\nu) - \bar{x}_\nu^T(i)\beta\bar{x}_\nu(i) \leq 0, \tag{4.8}$$

where, for simplicity we have denoted $\bar{x}_\nu(ih)$ by $\bar{x}_\nu(i)$. Note that $\beta \in \mathbb{R}^{n\times n} > 0$ is the TC parameter to be designed.

Let the sequence $\{a_j\}_{j=1}^{\infty}$ represents the transmission instants when data is sent to the controller. Then, the proposed event based control law is

$$u(t) = K\hat{\bar{x}}_\nu(t), \tag{4.9}$$

where

$$\hat{\bar{x}}_\nu(t) := \bar{x}_\nu(a_j h) \qquad \text{for} \quad t \in [a_j h, a_{j+1}h) \;\;,\;\; j = 1, 2, \dots \tag{4.10}$$

A general block diagram of the proposed event based controller is shown in Fig. 4.1. To represent the robustness of our proposed control paradigm against noise, define

$$\bar{x}(i) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} x(\alpha) d\alpha, \tag{4.11}$$

$$\bar{\nu}(i) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} \nu(\alpha) d\alpha, \tag{4.12}$$

and from (4.1) and (4.7), we have:

$$\bar{x}_\nu(i) = \bar{x}(i) + \bar{\nu}(i), \quad \text{thus} \tag{4.13}$$

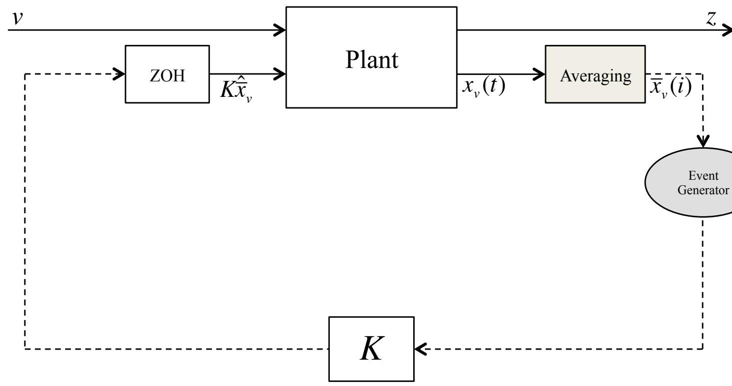


Figure 4.1: General block diagram of the event-based system

$$\hat{\bar{x}}_\nu(t) = \bar{x}(a_j) + \bar{\nu}(a_j) \qquad \text{for} \quad t \in [a_j h, a_{j+1}h). \tag{4.14}$$

According to equation (4.12), $\bar{\nu}(i)$ can be approximated as

$$\bar{\nu}(i) \approx S_n(i) = \frac{1}{n_s}\sum_{j=1}^{n_s} \nu_{ij}, \tag{4.15}$$

where the noise samples ν_{ij} are independent random variables with distribution $\mathcal{N}(0, \sigma^2)$ over the interval $[ih - T_{int}, T_{int}]$ and their number is n_s. Based on *Central Limit Theorem* [64], for large enough n_s, the distribution of S_n approximates $\mathcal{N}(0, \sigma^2/n_s)$; *i.e.*, $\bar{\nu}(i)$ approximates zero as the number of samples n_s increases. Thus, using the average measured states values rather than instant values effectively reduces the impact of noise on the both event triggering condition and feedback control law.

Remark 4.1. *In the proposed triggering structure, we have assumed zero mean white Gaussian noise as an important case encountered in many applications. Extensions are straightforward for as long as the time average coincides with the expected value, and the noise samples are independent and have the same distribution $\mathcal{N}(\mu, \sigma^2)$ (so that the conditions of the central limit theorem are satisfied). In this case, we can reformulate the PIBTC (4.8) and control law (4.9) as follows:*

$$(\bar{x}_\nu(i) - \hat{\bar{x}}_\nu)^T(\bar{x}_\nu(i) - \hat{\bar{x}}_\nu) - (\bar{x}_\nu(i) - \mu)^T\beta(\bar{x}_\nu(i) - \mu) \leq 0, \tag{4.16}$$

$$u(t) = K(\hat{\bar{x}}_\nu(t) - \mu). \tag{4.17}$$

A discussion similar to the one above shows that exploiting (4.16) and (4.17) can effectively reduce the effect of non-zero mean noise in the event-based system.

Having introduced the proposed event generator condition as (4.8) and the control feedback law as (4.9), the objective of this paper can be stated as follows: Designing both β and K such that the data transmission between plant and control units is effectively reduced. In this regard, as the first step, in the next section, the closed-loop system is modeled in continuous-time form.

4.2 Modeling The Event-Based System

Consider the system (4.1) with the proposed event-based controller (4.9). The closed-loop dynamics is given by:

$$\begin{cases} \dot{x} = Ax + Bu \\ u = K\hat{\bar{x}}_{\nu}(t) \end{cases} \tag{4.18}$$

To analyze the system dynamics using a delay system approach, the time interval $[0, +\infty)$ is broken down as

$$[0, +\infty) = \bigcup_{k=1}^{+\infty} [a_j h, a_{j+1} h). \tag{4.19}$$

Based on (4.10) and (4.14), the dynamics of event-based system (4.18) for $t \in [a_j h, a_{j+1} h)$ can be written as:

$$\dot{x} = Ax + BK\bar{x}(a_j) + BK\bar{\nu}(a_j). \tag{4.20}$$

Consider now the time instant $t \in [a_j h, a_{j+1} h)$ and let l_t be the latest sample number before 't' :

$$l_t = max \left\{ i \in \mathbb{N} : ih \leq t \right\}. \tag{4.21}$$

Define

$$\tau(t) := t - l_t h, \quad 0 \leq \tau(t) \leq h, \quad \text{and} \tag{4.22}$$

$$e_t := \bar{x}(l_t) - \bar{x}(a_j) \ \ . \tag{4.23}$$

Adding and subtracting $BK\bar{x}(l_t)$ to (4.20) and using the above definitions, we be obtain the time-delay system:

$$\dot{x} = Ax - BKe_t + BK\bar{x}(t - \tau(t)) + BK\bar{\nu}(a_j). \tag{4.24}$$

Similarly, the proposed PIBTC can be written in time delay form as follows: Based on (4.8) and (4.13) the following inequality is valid for $a_j \leq i < a_{j+1}$:

$$\begin{aligned} & e_t^T e_t + (\bar{\nu}(i) - \bar{\nu}(a_j))^T (\bar{\nu}(i) - \bar{\nu}(a_j)) + 2e_t^T (\bar{\nu}(i) - \bar{\nu}(a_j)) \\ & \leq \bar{x}^T(i)\beta\bar{x}^T(i) + \bar{\nu}^T(i)\beta\bar{\nu}(i) + 2\bar{x}^T(i)\beta\bar{\nu}(i). \end{aligned} \tag{4.25}$$

From the definition of l_t, for $t \in [a_j h, a_{j+1} h)$ one can readily find that:

$$a_j \leq l_t < a_{j+1}. \tag{4.26}$$

Therefore, based on (4.25) and definition (4.22) we have:

$$\begin{aligned} e_t^T e_t \leq\ & \bar{x}(t-\tau(t))^T \beta \bar{x}(t-\tau(t)) + \bar{\nu}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) + 2\bar{x}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) \\ & -2e_t^T(\bar{\nu}(t-\tau(t)) - \bar{\nu}(a_j)) - (\bar{\nu}(t-\tau(t)) - \bar{\nu}(a_j))^T(\bar{\nu}(t-\tau(t)) - \bar{\nu}(a_j)) \\ \leq\ & \bar{x}(t-\tau(t))^T \beta \bar{x}(t-\tau(t)) + \bar{\nu}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) + 2\bar{x}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) \\ & -2e_t^T(\bar{\nu}(t-\tau(t)) - \bar{\nu}(a_j)). \end{aligned} \tag{4.27}$$

Based on Lemma 2.1 and inequality (4.27), one can achieve the following inequality:

$$e_t^T D_1 e_t \leq \bar{x}(t-\tau(t))^T \beta \bar{x}(t-\tau(t)) + \Delta_\nu(t), \tag{4.28}$$

where

$$\begin{aligned} & \Delta_\nu(t) = \bar{\nu}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) - 2e_t^T\bar{\nu}(t-\tau(t)) + 2\bar{x}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) \\ & +\bar{\nu}(a_j)^T(I-D_1)^{-1}\bar{\nu}(a_j), \end{aligned} \tag{4.29}$$

and D_1 is a positive definite matrix satisfying $I - D_1 > 0$.

Before proceeding with the stability analysis and to simplify the equations, we need an approximation of the integral term $\bar{x}(t-\tau(t))$ appeared in the system dynamics (4.24) and the triggering condition (4.28). Using the Simpson's rule, [65], we have:

$$\int_{t-\tau(t)-T_{int}}^{t-\tau(t)} x(\alpha)d\alpha \approx \frac{T_{int}}{6}[x(t-\tau(t)-T_{int}) + 4x(t-\tau(t)-T_{int}/2) + x(t-\tau(t))]. \tag{4.30}$$

The approximation error is given by $\frac{1}{90}(\frac{T_{int}}{2})^5|x^{(4)}(\xi)|$, where ξ is some point in the interval $[t-\tau(t)-T_{int}, t-\tau(t)]$. A detailed discussion about the approximation error and precision of this integration method is given in Remark 4.3. Using (4.11) and (4.30), the integral term $\bar{x}(t-\tau(t))$ is approximated as follows:

$$\bar{x}(t-\tau(t)) \approx \alpha_1 x(t-\tau(t)-T_{int}) + \alpha_2 x(t-\tau(t)-T_{int}/2) + \alpha_3 x(t-\tau(t)), \tag{4.31}$$

where $\alpha_1 = 1/6,\ \alpha_2 = 2/3$, and $\alpha_3 = 1/6$.

Note that there are several alternatives to the Simpson rule. In particular, using the Trapezoidal rule results in the following approximation:

$$\int_{t-\tau(t)-T_{int}}^{t-\tau(t)} x(\alpha)d\alpha \approx \frac{T_{int}}{2}[x(t-\tau(t)-T_{int}) + x(t-\tau(t))], \tag{4.32}$$

with an approximation error given by $\frac{T_{int}^3}{12}|x^{(2)}(t-\tau(t)-T_{int})|$. Therefore, if the integration period is small enough, the trapezoidal rule can also provide a good approximation and has the advantage of making the analysis rather simple (Please refer to Remark 4.3). Using Trapezoidal rule, the integral term is written as equation (4.31) with the parameters $\alpha_1 = 1/2$, $\alpha_2 = 0$, and $\alpha_3 = 1/2$.

Having formulated the event-based system in a time delay form, stability and performance analysis are provided in the next section.

4.3 Stability and Performance Analysis of the Proposed Event-Based System

Theorem 4.1 provides sufficient conditions in the form of LMIs to guarantee the asymptotic stability (in the mean square sense) of the event-based system. Before proceeding to theorem 4.1, the following lemma is expressed which is exploited in the proof.

Theorem 4.1. *Consider the stochastic event-based system (4.18) with the proposed IBTC (4.8). As n_s (i.e. the number of samples over the integration period T_{int}) tends to infinity, the system is asymptotically stable in the mean square sense, if for given $D_1 > 0$ (satisfying $I - D_1 > 0$) and $D_2 > 0$ there exist matrices $P > 0$, $Q_i > 0$, $R_i > 0$, U_i and V_i $(i = 0, 1, 2)$*

with appropriate dimensions such that the following LMIs hold:

$$\mathcal{L}_j^U = \begin{bmatrix} \Gamma_0+\Gamma_1+\Gamma_1^T & \star & \star & \star & \star \\ \Gamma_0^R & -R_0 & \star & \star & \star \\ \Gamma_1^R & 0 & -R_1 & \star & \star \\ \Gamma_2^R & 0 & 0 & -R_2 & \star \\ \Gamma_j^U & 0 & 0 & 0 & -R_j \end{bmatrix} < 0$$

$$\mathcal{L}_j^V = \begin{bmatrix} \Gamma_0+\Gamma_1+\Gamma_1^T & \star & \star & \star & \star \\ \Gamma_0^R & -R_0 & \star & \star & \star \\ \Gamma_1^R & 0 & -R_1 & \star & \star \\ \Gamma_2^R & 0 & 0 & -R_2 & \star \\ \Gamma_j^V & 0 & 0 & 0 & -R_j \end{bmatrix} < 0 \tag{4.33}$$

$$\text{for} \quad j=0,1,2,$$

where:

$$\Gamma_0 = \begin{bmatrix} \Gamma_{01} & \alpha_1 F & \alpha_2 F & \alpha_3 F & 0 & 0 & 0 & -F & F \\ \star & \alpha_1^2\beta & \alpha_1\alpha_2\beta & \alpha_1\alpha_3\beta & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \alpha_2^2\beta & \alpha_2\alpha_3\beta & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \alpha_3^2\beta & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -Q_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -Q_2 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Gamma_{08} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \Gamma_{09} \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} \sum_{j=0}^{2} V_j & U_0-V_0 & U_1-V_1 & U_2-V_2 & -U_0 & -U_1 & -U_2 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{01} = A^TP + PA + \sum_{j=0}^{2} Q_j, \quad \Gamma_{08} = -D_1, \quad \Gamma_{09} = -D_2,$$

$$\Lambda = \begin{bmatrix} A & \alpha_1 BK & \alpha_2 BK & \alpha_3 BK & 0 & 0 & 0 & -BK & BK \end{bmatrix}, F = PBK,$$

$$\Gamma_j^U = \sqrt{3h+3T_{int}/2}U_j, \Gamma_j^V = \sqrt{3h+3T_{int}/2}V_j, \Gamma_j^R = \sqrt{h+jT_{int}/2}R_j\Lambda, \text{ for } \ j=0,1,2.$$

Proof. Consider the following Lyapunov Krasovskii functional:

$$V = \sum_{i=1}^{3} V_i, \tag{4.34}$$

$$
\begin{aligned}
V_1 &= x(t)^T P x(t),\\
V_2 &= \sum_{j=0}^{2} \int_{t-h-jT_{int}/2}^{t} x^T(s) Q_j x(s) ds, \\
V_3 &= \sum_{j=0}^{2} \int_{0}^{h+jT_{int}/2} \int_{t-\theta}^{t} \dot{x}^T(s) R_j \dot{x}(s) ds d\theta.
\end{aligned}
\tag{4.35}
$$

Let

$$
\zeta(t) \triangleq \begin{bmatrix} x(t)^T & x_\tau^T & x_M^T & e_t^T & \bar{\nu}^T \end{bmatrix}^T \tag{4.36}
$$

with

$$
\begin{aligned}
x_\tau^T &= \begin{bmatrix} x^T(t-\tau(t)) & x^T(t-\tau(t)-T_{int}/2) & x^T(t-\tau(t)-T_{int}) \end{bmatrix},\\
x_M^T &= \begin{bmatrix} x^T(t-h) & x^T(t-h-T_{int}/2) & x^T(t-h-T_{int}) \end{bmatrix}.
\end{aligned}
$$

Computing the time derivative of V_i along the trajectories of x for $t \in [a_j h, a_{j+1}h)$ we have:

$$
\begin{aligned}
\dot{V}_1 &= \zeta^T \Lambda^T P x + x^T P \Lambda \zeta,\\
\dot{V}_2 &= \sum_{j=0}^{2} x(t)^T Q_j x(t) - \sum_{j=0}^{2} x^T(t-h-jT_{int}/2) Q_j x(t-h-jT_{int}/2),\\
\dot{V}_3 &= \sum_{j=0}^{2} (h+jT_{int}/2)(\zeta^T \Lambda^T R_j \Lambda \zeta) - \sum_{j=0}^{2} \int_{0}^{h+jT_{int}/2} \dot{x}(t-\theta)^T R_j \dot{x}(t-\theta) d\theta.
\end{aligned}
\tag{4.37}
$$

Using Leibniz-Newton formula, for any free weighting matrices U_j and V_j [66] of proper dimensions we have:

$$
\zeta^T U_j (x(t-\tau(t)-jT_{int}/2) - x(t-jT_{int}/2-h) - \int_{t-jT_{int}/2-h}^{t-jT_{int}/2-\tau(t)} \dot{x}(s) ds) = 0, \tag{4.38}
$$

$$
\zeta^T V_j (x(t) - x(t-jT_{int}/2-\tau(t)) - \int_{t-\tau(t)-jT_{int}/2}^{t} \dot{x}(s) ds) = 0. \tag{4.39}
$$

Adding the above terms to the right hand side of the $\dot{V}_3$ in (4.37), we get:

$$
\begin{aligned}
\dot{V}_3 = &\sum_{j=0}^{2} [(h+jT_{int}/2)\zeta^T \Lambda^T R_j \Lambda \zeta] + \zeta^T(\Gamma_1 + \Gamma_1^T)\zeta\\
&+ \sum_{j=0}^{2} [(h-\tau(t))\zeta^T U_j R_j^{-1} U_j^T \zeta + (\tau(t)+jT_{int}/2)\zeta^T V_j R_j^{-1} V_j^T \zeta]\\
&- \sum_{j=0}^{2} [\int_{t-jT_{int}/2-h}^{t-jT_{int}/2-\tau(t)} (U_j^T \zeta(t) + R_j \dot{x}(s))^T R_j^{-1} (U_j^T \zeta(t) + R_j \dot{x}(s)) ds]\\
&- \sum_{j=0}^{2} [\int_{t-\tau(t)-jT_{int}/2}^{t} (V_j^T \zeta(t) + R_j \dot{x}(s))^T R_j^{-1} (V_j^T \zeta(t) + R_j \dot{x}(s)) ds],
\end{aligned}
\tag{4.40}
$$

which results in :

$$
\begin{aligned}
\dot{V}_3 \leq &\sum_{j=0}^{2} [(h+jT_{int}/2)\zeta^T \Lambda^T R_j \Lambda \zeta] + \zeta^T(\Gamma_1 + \Gamma_1^T)\zeta\\
&+ \sum_{j=0}^{2} [(h-\tau(t))\zeta^T U_j R_j^{-1} U_j^T \zeta + (\tau(t)+jT_{int}/2)\zeta^T V_j R_j^{-1} V_j^T \zeta].
\end{aligned}
\tag{4.41}
$$

Now, since the triggering inequality (4.28) holds for all $t \in [a_j h, a_{j+1} h)$, it can be derived that:

$$\dot{V} \leq \zeta^T(\Gamma_0 + \Gamma_1 + \Gamma_1^T)\zeta + \sum_{j=0}^{2}(h + jT_{int}/2)\zeta^T \Lambda^T R_j \Lambda \zeta$$
$$+ \sum_{j=0}^{2}[(h - \tau(t))\zeta^T U_j R_j^{-1} U_j^T \zeta + (\tau(t) + jT_{int}/2)\zeta^T V_j R_j^{-1} V_j^T \zeta] + \bar{\nu}(a_j)^T D_2 \bar{\nu}(a_j) + \Delta_\nu(t), - \tag{4.42}$$

where D_2 is some arbitrary positive definite matrix. Using definitions of Γ_j^R, Γ_j^U and Γ_j^V, the above inequality can be rewritten as follows:

$$\dot{V} \leq \zeta^T \bar{\Gamma} \zeta + \bar{\nu}(a_j)^T D_2 \bar{\nu}(a_j) + \Delta_\nu(t) \ , \tag{4.43}$$

where

$$\bar{\Gamma} = \Gamma_0 + \Gamma_1 + \Gamma_1^T + \sum_{j=0}^{2} \Gamma_j^{R^T} R_j^{-1} \Gamma_j^R + \sum_{j=0}^{2}[\tfrac{h-\tau(t)}{3h+3T_{int}/2}\Gamma_j^U R_j^{-1} \Gamma_j^{U^T} + \tfrac{\tau(t)+jT_{int}/2}{3h+3T_{int}/2}\Gamma_j^V R_j^{-1} \Gamma_j^{V^T}]. \tag{4.44}$$

It follows from equation (4.43) that:

$$E\{\dot{V}\} \leq E\{\bar{\nu}(a_j)^T D_2 \bar{\nu}(a_j)\} + E\{\Delta_\nu(t)\} + E\{\zeta^T(\Gamma_0 + \Gamma_1 + \Gamma_1^T + \sum_{j=0}^{2} \Gamma_j^{R^T} R_j^{-1} \Gamma_j^R + \bar{\Gamma})\zeta\}. \tag{4.45}$$

Since the elements of the random vector $\bar{\nu}$ are statistically independent, we have:

$$E\{\bar{\nu}(a_j)^T D_2 \bar{\nu}(a_j)\} = tr(D_2)\sigma^2/n_s, \tag{4.46}$$

$$\begin{aligned} &E\{\Delta_\nu(t)\} = (tr(\beta) + tr((I - D_1)^{-1}))\sigma^2/n_s \\ &+E\{2\bar{x}^T(t - \tau(t))\beta\bar{\nu}(t - \tau(t)) - 2e_t^T \bar{\nu}(t - \tau(t))\}, \end{aligned} \tag{4.47}$$

From the system dynamic (4.24), we see that the vectors $\bar{x}(t-\tau(t))$ and e_t, possibly depend on the noise samples $\nu(t')$, where '$t' < t - \tau(t) - h$'. Consequently, they are independent from $\bar{\nu}(t - \tau(t))$ and so the last term in the above equation is equal to zero. Therefore:

$$E\{\Delta_\nu(t)\} + E\{\bar{\nu}(a_j)^T D_2 \bar{\nu}(a_j)\} = c_\nu/n_s, \tag{4.48}$$

$$c_\nu = (tr(\beta) + tr((I - D_1)^{-1}) + tr(D_2))\sigma^2. \tag{4.49}$$

Since

$$\sum_{j=0}^{2}[\frac{h - \tau(t)}{3h + 3T_{int}/2} + \frac{\tau(t) + jT_{int}/2}{3h + 3T_{int}/2}] = 1, \tag{4.50}$$

then $\bar{\Gamma}$ can be rewritten as:

$$\bar{\Gamma} = \textstyle\sum_{j=0}^{2}[\tfrac{h-\tau(t)}{3h+3T_{int}/2}\bar{\Gamma}_j^U + \tfrac{\tau(t)+jT_{int}/2}{3h+3T_{int}/2}\bar{\Gamma}_j^V], \tag{4.51}$$

with

$$\bar{\Gamma}_j^U = \Gamma_0 + \Gamma_1 + \Gamma_1^T + \sum_{i=0}^{2} \Gamma_i^{R^T} R_i^{-1} \Gamma_i^R + \Gamma_j^U R_j^{-1} \Gamma_j^{U^T}, \tag{4.52}$$

$$\bar{\Gamma}_j^V = \Gamma_0 + \Gamma_1 + \Gamma_1^T + \sum_{i=0}^{2} \Gamma_i^{R^T} R_i^{-1} \Gamma_i^R + \Gamma_j^V R_j^{-1} \Gamma_j^{V^T}. \tag{4.53}$$

In other words, $\bar{\Gamma}$ is a convex combination of $\bar{\Gamma}_j^U$ and $\bar{\Gamma}_j^V$ (for $j = 0, 1, 2$). So, based on Lemma 2.3

$$\bar{\Gamma} < 0 \tag{4.54}$$

is equivalent to

$$\bar{\Gamma}_j^U < 0, \qquad \bar{\Gamma}_j^V < 0, \qquad \text{for } \ j = 0, 1, 2, \tag{4.55}$$

which in turn, using Schur complement, are equivalent to the LMIs (4.33).

Now, define:

$$-\theta_j^U := \lambda_{max}(\bar{\Gamma}_j^U), \qquad -\theta_j^V := \lambda_{max}(\bar{\Gamma}_j^V), \qquad \text{for } \ j = 0, 1, 2. \tag{4.56}$$

Using (4.51) and (4.55), we have:

$$\zeta^T \bar{\Gamma} \zeta \leq -\sum_{j=0}^{2} [\frac{h - \tau(t)}{3h + 3T_{int}/2} \theta_j^U + \frac{\tau(t) + jT_{int}/2}{3h + 3T_{int}/2} \theta_j^V] \|\zeta\|^2. \tag{4.57}$$

Defining [1]

$$\theta := \min_{0 \leq \tau(t) \leq h} \sum_{j=0}^{2} [\frac{h - \tau(t)}{3h + 3T_{int}/2} \theta_j^U + \frac{\tau(t) + jT_{int}/2}{3h + 3T_{int}/2} \theta_j^V], \tag{4.58}$$

then, we obtain:

$$E\{\dot{V}\} \leq -\theta E\{\|x(t)\|^2\} + c_\nu / n_s. \tag{4.59}$$

Taking n_s to infinity in equation (4.59) results in:

$$E\{\dot{V}\} \leq -\theta E\{\|x(t)\|^2\}. \tag{4.60}$$

Consequently, mean square asymptotic stability is ensured. This concludes the proof.

Remark 4.2. Theorem 4.1 assumes that the number of samples over the integration period, $n_s \to \infty$. The assumption ensures the cancellation of the positive term in equation

[1] Based on (4.50) and (4.56) such minimum exists and is greater than zero.

(4.59) used to prove asymptotic stability. In practice, n_s is finite in any practical implementation, violating the conditions of the theorem. Nevertheless, by increasing n_s, one can easily take this term down to a small value. In this situation, the state trajectories of the system are ultimately bounded in a (sufficiently small) region.

Remark 4.3. *Theorem 4.1 requires the use of a numerical integration technique. Of the many alternatives available, we chose the Simpson's rule as a good compromise between precision and simplicity. It is therefore an implicit assumption of Theorem 4.1 that the approximation error* $\frac{1}{32\times 90}T_{int}^5|x^{(4)}|$ *is negligible. This is reasonable assumption in any application in which the state trajectories are bounded, provided that the sampling time* T *is small enough. Theorem 4.1 can also be modified to accommodate a more elaborate integration technique such as, for example, the extended Simpson's rule [67]. The use of this approximation method, or any other more elaborate rule, will result in additional delayed state terms in the system equations and consequently the LMIs introduced in Theorem 1 will have larger dimensions.*

On the other hand, the Trapezoidal rule (4.32) provides a simpler solution that can also be used to approximate the integral term in Theorem 4.1. In this case, the solution contains only two delay terms and the LMIs can be easily simplified and reduced in size by removing the extra terms and matrices generated for the term $x(t-\tau(t)-T_{int}/2)$*. The prize paid for the simplification is a larger approximation error.*

4.4 Controller and TC Parameters Design

In this section, similar to Chapter 2, we assume that the state feedback gain K and the PIBTC gain β are unknown parameters to be designed. To this end, we reformulate Theorem 4.1 as an optimization problem, where effective data transmission reduction is achieved by maximization of β. Theorem 4.2 provides a convex optimization problem with LMI constraints to find optimal values for the control system parameters.

Theorem 4.2. *Consider the stochastic system (4.1). As* $n_s \to \infty$*, there exist an event-based control law (4.9) and a PIBTC (4.8) such that the closed-loop system is asymptotically stable in the mean square if there exist matrices* $X > 0$*,* $Y > 0$*,* $\hat{Q}_j > 0$*,* $\hat{R}_j > 0$*,* $(j = 0, 1, 2)$*,* $\hat{U}_j$ *and* $\hat{V}_j$ $(j = 0, 1, 2)$ *of appropriate dimensions, and scalar* $z_\beta > 0$ *such* *that the following optimization problem has solution for some $\varepsilon_i > 0$, $D_1 > 0$ (satisfying $I - D_1 > 0$), $D_2 > 0$:*

$$min \quad z_\beta \tag{4.61}$$

$$s.t.$$

$$\mathcal{L}_j^{\hat{U}} = \begin{bmatrix} \hat{\Gamma}_0 + \hat{\Gamma}_1 + \hat{\Gamma}_1^T & \star & \star & \star & \star \\ \hat{\Gamma}_0^R & -\hat{\Gamma}_{XR_0} & \star & \star & \star \\ \hat{\Gamma}_1^R & 0 & -\hat{\Gamma}_{XR_1} & \star & \star \\ \hat{\Gamma}_2^R & 0 & 0 & -\hat{\Gamma}_{XR_2} & \star \\ \hat{\Gamma}_j^U & 0 & 0 & 0 & -\hat{R}_j \end{bmatrix} < 0$$

$$\mathcal{L}_j^{\hat{V}} = \begin{bmatrix} \hat{\Gamma}_0 + \hat{\Gamma}_1 + \hat{\Gamma}_1^T & \star & \star & \star & \star \\ \hat{\Gamma}_0^R & -\hat{\Gamma}_{XR_0} & \star & \star & \star \\ \hat{\Gamma}_1^R & 0 & -\hat{\Gamma}_{XR_1} & \star & \star \\ \hat{\Gamma}_2^R & 0 & 0 & -\hat{\Gamma}_{XR_2} & \star \\ \hat{\Gamma}_j^V & 0 & 0 & 0 & -\hat{R}_j \end{bmatrix} < 0 \tag{4.62}$$

$$\text{for} \quad j = 0, 1, 2,$$

$$\begin{bmatrix} z_\beta I & X \\ X & \bar{\beta} \end{bmatrix} \geq 0, \tag{4.63}$$

where:

$$\hat{\Gamma}_1 = \left[\sum_{j=0}^{2} \hat{V}_j \quad \hat{U}_0 - \hat{V}_0 \quad \hat{U}_1 - \hat{V}_1 \quad \hat{U}_2 - \hat{V}_2 \quad -\hat{U}_0 \quad -\hat{U}_1 \quad -\hat{U}_2 \quad 0 \quad 0 \quad 0 \right],$$

$$\hat{\Gamma}_j^R = \sqrt{h + jT_{int}/2}\hat{\Lambda}, \quad \hat{\Gamma}_{XR_j} = -\varepsilon_j^2 \hat{R}_j + 2\varepsilon_j X,$$

$$\hat{\Gamma}_j^U = \sqrt{3h + 3T_{int}/2}\hat{U}_j, \ \hat{\Gamma}_j^V = \sqrt{3h + 3T_{int}/2}\hat{V}_j, \ \text{for } j = 0, 1, 2,$$

$$\hat{\Lambda} = \left[AX \quad \alpha_1 BY \quad \alpha_2 BY \quad \alpha_3 BY \quad 0 \quad 0 \quad 0 \quad -BY \quad BY \right].$$

In addition, $\hat{\Gamma}_0$ is defined in the same way as Γ_0, given in theorem 4.1, where $F, \beta, Q_j, \Gamma_{01}, \Gamma_{08}$ and Γ_{09} are replaced by $\hat{F}, \hat{\beta}, \hat{Q}_j, \hat{\Gamma}_{01}, \hat{\Gamma}_{08}$ and $\hat{\Gamma}_{09}$, respectively, where:

$$\hat{\Gamma}_{01} = XA^T + AX + \sum_{j=0}^{2} \hat{Q}_j,$$

$$\hat{\Gamma}_{08} = \varepsilon_3^2 D_1^{-1} - 2\varepsilon_3 X, \quad \hat{\Gamma}_{09} = \varepsilon_4^2 D_2^{-1} - 2\varepsilon_4 X, \quad \hat{F} = BY.$$

Having solved the optimization problem, the optimal values for controller gain and event generator parameter are calculated as:

$$K = YX^{-1}, \quad \beta = X^{-1}\hat{\beta}X^{-1}. \tag{4.64}$$

Proof. Considering β and K as optimization variables, the inequalities provided in Theorem 4.1 are no longer in LMI form. To express the conditions of the theorem in LMI form we use an approach similar to [35]. Let $X := P^{-1}$, $\hat{X} := \text{diag}(X, X, X, X, X, X, X, X, X)$ and define new variables

$$\begin{aligned} & Y := KX, \\ & \hat{\beta} := X\beta X, \quad \hat{Q}_j := XQX, \quad \hat{U}_j := \hat{X}U_jX, \\ & \hat{V}_j := \hat{X}V_jX, \quad \hat{R}_j := XR_jX, \qquad\qquad \text{for } j = 0, 1, 2. \end{aligned} \tag{4.65}$$

Then, pre and post-multiply (4.54) by $\hat{X}$. With this construction (4.54) is equivalent to:

$$\hat{\hat{\Gamma}} = \sum_{j=0}^{2} [\frac{h - \tau(t)}{3h + 3T_{int}/2} \hat{\hat{\Gamma}}_j^U + \frac{\tau(t) + jT_{int}/2}{3h + 3T_{int}/2} \hat{\hat{\Gamma}}_j^V] < 0, \tag{4.66}$$

where

$$\hat{\hat{\Gamma}}_j^U = \hat{\Gamma}_0 + \hat{\Gamma}_1 + \hat{\Gamma}_1^T + \sum_{i=0}^{2} \hat{\Gamma}_i^R R_i^{-1} \hat{\Gamma}_i^R + \hat{\Gamma}_j^U \hat{R}_j^{-1} \hat{\Gamma}_j^U, \tag{4.67}$$

$$\hat{\hat{\Gamma}}_j^V = \hat{\Gamma}_0 + \hat{\Gamma}_1 + \hat{\Gamma}_1^T + \sum_{i=0}^{2} \hat{\Gamma}_i^R R_i^{-1} \hat{\Gamma}_i^R + \hat{\Gamma}_j^V \hat{R}_j^{-1} \hat{\Gamma}_j^V. \tag{4.68}$$

Using the Schur complement, Lemmas 2.2 and 2.3, necessary conditions to guarantee inequality (4.66) are that LMIs (4.62) hold.

The rest of stability proof is the same as of the Theorem 1 and is thus omitted.

Based on definitions (4.65), maximizing β is equivalent to minimizing the term $X\hat{\beta}^{-1}X$. In order to express this objective in convex form, define an auxiliary variable z_β such that

$$z_\beta I - X\hat{\beta}^{-1}X \geq 0, \tag{4.69}$$

and minimize z_β instead. Note that by the Schur complement, inequality (4.69) is equivalent to LMI (4.63).

4.5 Simulation Results

In this section we present an illustrative example to substantiate the proposed approach.

Consider the unstable batch reactor system [68], with the following dynamics:

$$\dot{x} = Ax + Bu,$$

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}. \tag{4.70}$$

We assume that there are additive noise signals on the measured state values. To clarify our main points and to provide a proper comparison, different noise distributions are considered. To design the proposed event triggering control system for this plant, the period of TC verification is set to $h = 0.05$, and the integration period is chosen as $T_{int} = 0.04$. Since the integration period is small enough, the trapezoidal rule is exploited to replace the integral term, and values $\alpha_1 = 1/2$, $\alpha_2 = 0$, and $\alpha_3 = 1/2$ are used in the formulation (4.31). In order to solve the optimization problem (4.61) with the constraints (4.62)-(4.63), the weighting matrices D_1 and D_2 are selected as $D_1 = D_2 = 0.8I_{4\times4}$. Solving the optimization problem of Theorem 2, and using (4.64), the state feedback and the TC parameter are as follows:

$$\beta = \begin{bmatrix} 0.0251 & -0.0014 & -0.0001 & -0.0004 \\ -0.0014 & 0.0501 & 0.0026 & 0.0065 \\ -0.0001 & 0.0026 & 0.0252 & 0.0007 \\ -0.0004 & 0.0065 & 0.0007 & 0.0267 \end{bmatrix}, \quad K = \begin{bmatrix} 0.2965 & -0.3730 & 0.1512 & -0.4645 \\ 1.6031 & -0.0269 & 0.9792 & -0.7223 \end{bmatrix}.$$

Note that to find proper optimal solutions, using the iterative bisection approach, the coefficients ϵ_1-ϵ_4 are easily tuned, resulting in the values $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = \epsilon_4 = 19$. To simulate the results, the initial condition vector is set to $x(0) = [-12, 6, -3, -12]^T$. First, to represent the effect of the noise on the event generator module and also the output signal performance in the traditional event-based scheme, the system is implemented using the (classical) TC and feedback law:

$$\begin{gathered} (x_\nu(i) - \hat{x}_\nu)^T (x_\nu(i) - \hat{x}_\nu) - {x_\nu}^T(i)\beta x_\nu(i) \leq 0, \\ u = K\hat{x}_\nu(t). \end{gathered} \tag{4.71}$$

In the other words, instead of average values, instant values of the output are applied in both TC and the state feedback law. Having set the parameters, the simulation is implemented for 4 seconds, where the measurement noise distribution for all state channels are assumed to be $\mathcal{N}(0, 0.05^2)$. To have a fair evaluation of the system, simulation is run 10

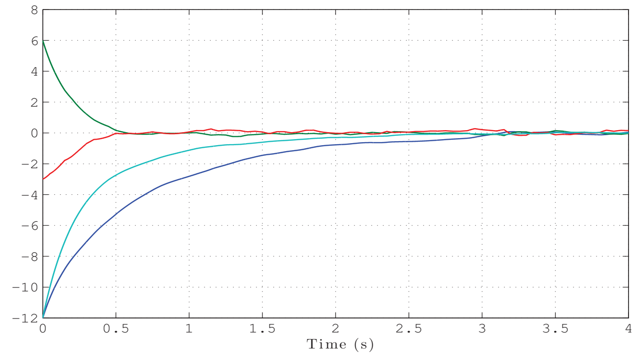


Figure 4.2: State responses of the reactor, controlled by the traditional event-based system

times (as a typical value) and finally the mean number of triggered samples between plant and controller is calculated. Based on the results obtained, the traditional event based system sends an average of 69 data points during each implementation. In the other words, almost 86% of the times that the TC was verified, it had been violated and data was sent to the controller, something which can mainly be attributed to presence of the noise. These conditions further deteriorate if the noise amplitude increases. Indeed, if standard deviation of the noise is increased and the noise distribution is changed to $\mathcal{N}(0, 0.3^2)$, the average data points transmitted increases to 77; *i.e.* the TC is violated at 96% of verification instants. The state response of this system is shown in Fig. 4.2, where, as expected, the existing noise has deteriorated the performance of output signals. Consider now the proposed PIBTC (4.8) and the control law (4.9), instead of the traditional scheme. Since calculation of the integral term $\bar{x}_\nu$ in our scheme directly depends on the sampling rate of the measurement sensors, the simulation is carried out for different sensor sampling frequencies. The results for two different measurement noise distributions are represented and compared with the traditional case in Table 4.1.

Table 4.1: Number of data exchanged between the reactor and the controller, while using event-based mechanism.

	Traditional mechanism	Proposed mechanism with different sensor sampling rate (Hz)					
		200	500	1000	2000	5000	10000
Noise distribution $\mathcal{N}(0, 0.3^2)$	77	71	63	59	56	51	46
Noise distribution $\mathcal{N}(0, 0.05^2)$	69	58	51	46	42	43	41

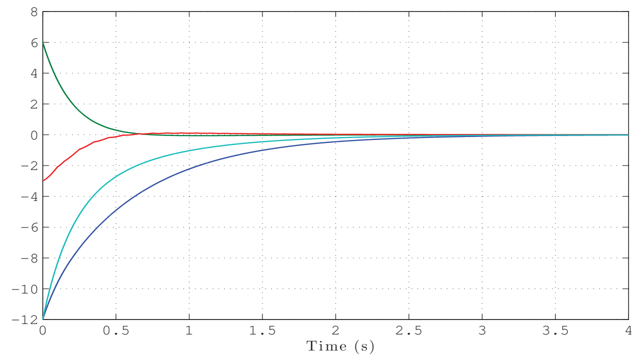


Figure 4.3: State responses of the reactor controlled by the porposed event-based system

Similar to the previous part, for each specific frequency, the simulations are carried out 10 times and the average number of transmitted samples is provided. For the standard deviation 0.3, as the sampling frequency goes up, the number of data transmission decreases gradually such that for a frequency 10 KHz the number of transmitted points is 40% less than the value corresponding to the traditional mechanism. For the measurement noise with the standard deviation 0.05 a similar trend is observed. However, since, with a high probability, the amplitude of noise samples are too small in this case, for frequencies more than 1 KHz no significant change happens in the calculation of the $\bar{x}_\nu$, and so neither does in the number of data communication points. Fig. 4.3 shows the state trajectories of the proposed event triggering system assuming a sensor sampling rate of 2 KHz. As shown, using this scheme, not only the states of the system are properly stabilized, but also the impact of noise on signals quality has been effectively reduced.

Although, our proposed scheme relies on a Lyapunov-based approach, we have also simulated an optimal-based event triggering scheme, introduced in [51], for the reactor to compare the results. The scheme in [51] is designed for output feedback discrete-time systems, where two separate event detector blocks are considered at plant and controller sides, respectively.

To have a fair comparison, first an exact discrete-time equivalence of the systems dynamics (4.70) is calculated for the sampling time 0.05. To save space, the obtained difference equation is not provided here. Then, since our goal is to design a state feedback scheme with a single event generator block at the plant side, the parameter λ_c (which is the communication price for data transmission from controller to actuator link in the

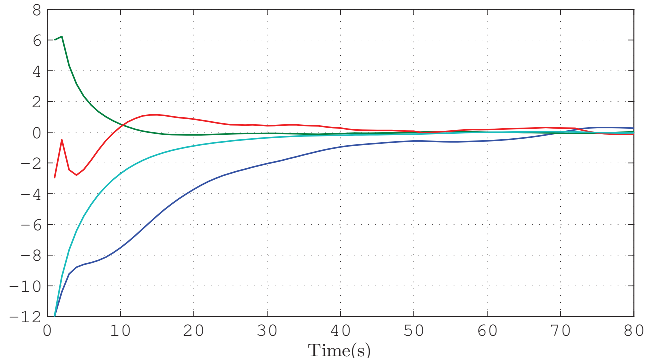


Figure 4.4: State responses of the discrete reactor model, controlled by the given event-based system in (Li & Lemmon, 2011).

objective function) is set to zero. Taking the mentioned steps in [51], the state feedback gain and the triggering condition are achieved as follows, respectively:

$$K_d = \begin{bmatrix} 0.0973 & -0.9805 & -0.2342 & -0.9905 \\ 3.1475 & 0.0711 & 2.1571 & -1.5092 \end{bmatrix},$$

$$e_d^T \begin{bmatrix} 12.1157 & 1.5311 & 7.4443 & -4.0129 \\ 1.5311 & 3.2423 & 1.6684 & 1.1295 \\ 7.4443 & 1.6684 & 6.3916 & -1.9978 \\ -4.0129 & 1.1295 & -1.9978 & 4.5811 \end{bmatrix} e_d \leq 0.034,$$

where, e_d is some measurement error defined in the paper.

Now, a zero white Gaussian Noise process v with variance 0.3^2 is considered as the measurement noise and simulation is carried out for 80 iteration with the same initial condition as the previous parts. The state responses are represented in Fig. 4.4. Based on the obtained results, the triggering condition introduced in [51], triggers 58 data to generate a convergence rate of state trajectories similar to our proposed scheme's. According to Table 4.1, this value is almost equal to the one achieved by our proposed scheme which corresponds to the sampling rate 1 KHz. Note that, using our scheme, this number of information exchange can be effectively reduced by raising the sensor sampling rate.

4.6 Summary

A new event triggering mechanism was proposed for a class of noisy LTI systems. The designed control system is able to significantly attenuate the effect of measurement noise in the event generator module and also retrieve the output performance. Stability analysis and controller parameter design were carried out based on Lyapunov-Krasovskii functionals. Finally, simulations were implemented for an unstable batch reactor system to show the efficiency of the system.

Chapter 5

An Event-Based Observer for Linear Stochastic Systems

In this chapter observer design for LTI systems is studied in the presence of measurement noise. Following the idea given in the last chapter, a novel form of TC is proposed which is robust against noise and help lower data transmission from plant to the observer. The proposed system is modeled in time delay form, similar to the one given in [69], and performance analysis is given using Lyapunov Krasovskii functionals. In addition, hierarchical steps are presented to design the parameters of the system in a proper fashion. Finally simulation results are given to illustrate the efficiency of the introduced system.

The rest of this chapter is organized as follows. In section 5.1, problem statement is given the structure of the proposed observer is presented. Sections 5.2 and 5.3 contain modeling and performance analysis of the system, respectively. Section 5.4 introduces a mechanism to design the parameters of the system. Simulation results are given in section 5.5 and finally, the chapter is summarized in section 5.6.

5.1 Problem Statement

Consider the following class of continuous-time linear systems:

$$\begin{cases} \dot{x} = A_p x + B_p w \\ y(t) = C_p x(t) + \nu(t) \\ z(t) = H_p x(t) \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $w(t) \in \mathbb{R}^m$ is the disturbance input, which is assumed to be in $\mathscr{L}_2[0, +\infty)$. Moreover, $y(t) \in \mathbb{R}^p$ is the measured output contaminated by some stochastic measurement noise $\nu(t)$ and $z(t)$ is the signal to be estimated.

Assumption: The measurement noise $\nu(t)$ is a zero mean white Gaussian noise vector with the covariance vector $\sigma^2 I$.

It is assumed the gain L in the following Luenberger observer has been already designed such that $A_p - LC_p$ is Hurwitz and so the dynamics of estimation error $e_F = x_F - x$ is stable:

$$\begin{cases} \dot{x}_F = A_p x_F + L(y_F(t) - y(t)) \\ y_F = C_p x_F \\ z_F = H_p x_F(t) \end{cases} \tag{5.2}$$

The goal of this paper is to modify the above-mentioned traditional observer by an event-based mechanism, such that the transmission of output samples from plant side to the observer side is effectively reduced, while the signal $z_F(t)$ estimates $z(t)$ and the following H_∞ performance holds:

$$E\{\|e_z\|_{2,T}\} \leq \gamma \|w\|_{2,T} + \varepsilon, \tag{5.3}$$

where ε is some constant value, $e_z = z - z_F$.

In the most Lyapunov-based references, provided for the event-based observation problem in the literature, the TC is basically defined based on the instantaneous sampled values of the output. As an example, in [69], the main structure of the TC is as follows:

$$(y(t) - \hat{y})^T (y(t) - \hat{y}) \leq \beta y^T(t) y(t), \tag{5.4}$$

where $\hat{y}(t)$ is the last output sample, sent to the observer module. Using above TC, the traditional observer (5.2) would turn into the following event-based format:

$$\begin{cases} \dot{x}_F = A_p x_F + L(y_F(t) - \hat{y}(t)) \\ y_F = C_p x_F \\ z_F = H_p x_F(t) \end{cases} \tag{5.5}$$

As seen, if the TC (5.4) is utilized, it would trigger unnecessary samples, due to presence of the stochastic measurement noise $\nu(t)$ on $y(t)$ and also $\hat{y}(t)$. So, although this form of TC is pretty simple to implement, it may not be efficient in practical cases, where stochastic noise is present on the measured output.

In this regard, following the idea given in the last chapter, in the next section, we are going to provide an event-based observer scheme, which is robust against measurement noise and lowers the data communication between the plant and observer efficiently.

5.1.1 Proposed Event Triggering Mechanism and the Observer Scheme

The main structure of the proposed systems is as follows. An event generator block is considered at the plant side, verified periodically with the period of h. Moreover, instead of instant values, an average of measured data over the past time interval T_{int} is fed to the event generator as its input. In the other words, defining $\bar{y}(ih)$ as the input of event generator, we have:

$$\bar{y}(ih) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} y(\alpha) d\alpha \quad \text{for} \quad i \in \mathbb{N}. \tag{5.6}$$

Then, the event generator block transmits an updated value of $\bar{y}$ to the observer just whenever the normalized value of the difference between the latest transmitted value and the new sample exceeds a certain threshold. In the other words, let $\hat{\bar{y}}$ as the last value sent to the observer at the time instant $\hat{t}$. Then, for the sample times ih after $\hat{t}$ (*i.e.* $ih > \hat{t}$), the event generator transmits a new sample of $\bar{y}$ if the following proposed triggering condition is violated:

$$(\bar{y}(i) - \hat{\bar{y}})^T (\bar{y}(i) - \hat{\bar{y}}) - \beta \bar{y}^T(i) \bar{y}(i) \leq 0. \tag{5.7}$$

In the above formulation, for simplicity we have denoted $\bar{y}(ih)$ by $\bar{y}(i)$. Moreover, β is a scalar parameter which should be set to a proper value. This would be explained more in section 5.4. Note that, defining the sequence $\{k_j\}_{j=1}^{\infty}$ as the sample numbers at which data are sent to the observer, then the input to the observer is defined as $\hat{\bar{y}}(t)$:

$$\hat{\bar{y}}(t) := \bar{y}(k_j) \qquad \text{for} \quad t \in [k_j h, k_{j+1} h) \ \ , \ \ j \in \mathbb{N}. \tag{5.8}$$

To show the robustness of our proposed observation paradigm against noise, define

$$\bar{x}(i) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} x(\alpha) d\alpha, \tag{5.9}$$

and

$$\bar{\nu}(i) := \frac{1}{T_{int}} \int_{ih-T_{int}}^{ih} \nu(\alpha) d\alpha. \tag{5.10}$$

Now, using equations (5.1) and (5.6):

$$\bar{y}(i) = C_p \bar{x}(i) + \bar{\nu}(i), \tag{5.11}$$

and so

$$\hat{\bar{y}}(t) = C_p \bar{x}(k_j) + \bar{\nu}(k_j) \qquad \text{for} \quad t \in [k_j h, k_{j+1} h). \tag{5.12}$$

According to equation (5.10), $\bar{\nu}(i)$ can be approximated as

$$\bar{\nu}(i) \approx S_n(i), \tag{5.13}$$

with

$$S_n(i) = \frac{1}{n_s} \sum_{j=1}^{n_s} \nu_{ij}, \tag{5.14}$$

where the noise samples ν_{ij} are some independent random variables with the distribution $\mathcal{N}(0, \sigma^2)$ which are taken over the interval $[ih - T_{int}, T_{int}]$ and their number is n_s.

For large enough number n_s, with a good approximation, the distribution of S_n can be described as $\mathcal{N}(0, \sigma^2/n_s)$ (*Central Limit Theorem* [64]); *i.e.* with a high probability, $\bar{\nu}(i)$ takes the value in a very small region of origin. This issue means that using average of measured states values rather than instant values would effectively reduce the impact of noise on the event-TC.

5.2 Modeling The Event-Based System

Based on the structure, explained in the previous section, the dynamics of the observer is expressed as follows:

$$\begin{cases} \dot{x}_F = A_p x_F + L(y_F(t) - \hat{\bar{y}}(t)) \\ y_F = C_p x_F \\ z_F = H_p x_F(t) \end{cases} \tag{5.15}$$

where, $\hat{\bar{y}}(t)$ is the output of the triggering module and the input to the observer.

Defining $e_F(t) = x(t) - x_F(t)$, using equations (5.1) and (5.15), the error dynamics is obtained as:

$$\begin{aligned} \dot{e}_F &= (A_p - LC_p)e_F + LC_p x(t) - L\hat{\bar{y}}(t) + B_p w, \\ e_z(t) &= H_p e_F(t), \end{aligned} \tag{5.16}$$

where, $e_z = z - z_F$ is the estimation error.

To analyze the dynamics of the system we will use delay system approach, similar to [69]. In this regard, the time interval $[0, +\infty)$ is broken down as

$$[0, +\infty) = \bigcup_{k=1}^{+\infty} [k_j h, k_{j+1} h). \tag{5.17}$$

Using equations (5.8) and (5.12), the error dynamics (5.16) for $t \in [k_j h, k_{j+1} h)$ can be expressed as:

$$\begin{aligned} \dot{e}_F &= (A_p - LC_p)e_F + LC_p x(t) + B_p w - LC_p \bar{x}(k_j) - L\bar{\nu}(k_j), \\ e_z(t) &= H_p e_F(t). \end{aligned} \tag{5.18}$$

Now consider the time instant $t \in [k_j h, k_{j+1} h)$ and denote κ_t as the latest sample number before 't' :

$$\kappa_t = max \left\{ i \in \mathbb{N} : ih \leq t \right\}. \tag{5.19}$$

Define

$$\tau(t) := t - \kappa_t h \qquad \text{with} \quad 0 \leq \tau(t) \leq h, \tag{5.20}$$

and

$$e_t := C_p(\bar{x}(\kappa_t) - \bar{x}(k_j)). \tag{5.21}$$

Adding and subtracting $LC_p \bar{x}(\kappa_t)$ to the dynamics (5.18) and using above definitions, the following time delay equation is concluded:

$$\begin{aligned} \dot{e}_F &= (A_p - LC_p)e_F + LC_p x(t) + B_p w - Le_t - LC_p \bar{x}(t - \tau(t)) - L\bar{\nu}(k_j), \\ e_Z(t) &= H_p e_F(t). \end{aligned} \tag{5.22}$$

Similarly, the event triggering condition is written in time delay form. The steps are similar to the ones given in the last chapter:

According to the equations (5.7) and (5.11), the following inequalities are true for $k_j \leq i < k_{j+1}$:

$$e_t^T e_t + (\bar{\nu}(i) - \bar{\nu}(k_j))^T(\bar{\nu}(i) - \bar{\nu}(k_j)) + 2e_t^T(\bar{\nu}(i) - \bar{\nu}(k_j)) \leq \bar{x}^T(i)C^T\beta C\bar{x}(i) + \bar{\nu}^T(i)\beta\bar{\nu}(i)$$
$$+2\bar{x}^T(i)C^T\beta\bar{\nu}(i). \tag{5.23}$$

Regarding the definition of κ_t, for $t \in [k_j h, k_{j+1}h)$ one can readily find that

$$k_j \leq \kappa_t < k_{j+1}. \tag{5.24}$$

So based on equation (5.23) we have:

$$e_t^T e_t + (\bar{\nu}(\kappa_t) - \bar{\nu}(k_j))^T(\bar{\nu}(\kappa_t) - \bar{\nu}(k_j)) + 2e_t^T(\bar{\nu}(\kappa_t) - \bar{\nu}(k_j)) \leq \bar{x}^T(\kappa_t)C^T\beta C\bar{x}(\kappa_t)$$
$$+\bar{\nu}^T(\kappa_t)\beta\bar{\nu}(\kappa_t) + 2\bar{x}^T(\kappa_t)C^T\beta\bar{\nu}(\kappa_t), \tag{5.25}$$

which, using definition (5.20), can be rewritten as:

$$e_t^T e_t \leq \bar{x}(t-\tau(t))^T C^T\beta C\bar{x}(t-\tau(t)) + \bar{\nu}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) + 2e_t^T(\bar{\nu}(t-\tau(t)) - \bar{\nu}(k_j))$$
$$+2\bar{x}^T(t-\tau(t))C^T\beta\bar{\nu}(t-\tau(t)). \tag{5.26}$$

Applying Lemma 2.1, the above inequality leads to the following formulation:

$$e_t^T D_1 e_t \leq \bar{x}(t-\tau(t))^T C^T\beta C\bar{x}(t-\tau(t)) + \Delta_\nu(t), \tag{5.27}$$

where:

$$\Delta_\nu(t) = \bar{\nu}^T(t-\tau(t))\beta\bar{\nu}(t-\tau(t)) + 2\bar{x}^T(t-\tau(t))C^T\beta\bar{\nu}(t-\tau(t)) - 2e_t^T\bar{\nu}(t-\tau(t))$$
$$+\bar{\nu}(k_j)^T(I-D_1)^{-1}\bar{\nu}(k_j), \tag{5.28}$$

and D_1 is an arbitrary positive definite matrix, satisfying $I - D_1 > 0$.

Having formulated the event-based dynamics and the TC in a time-delay form, in the next section the performance analysis is provided.

5.3 Performance Analysis of the Proposed Event Based System

Before establishing the performance analysis, first it is noted that the integral term $\bar{x}(t-\tau(t))$ is approximated by Trapezoidal rule:

$$\int_{t-\tau(t)-T_{int}}^{t-\tau(t)} x(\alpha)d\alpha \approx \frac{T_{int}}{2}[x(t-\tau(t)-T_{int})+x(t-\tau(t))]. \tag{5.29}$$

It is proved that, the approximation error for this formulation would be [65]:

$$\frac{T_{int}^3}{12}|x^{(2)}(t-\tau(t)-T_{int})|.$$

Since a typical practical integration period is much less than 1, the above formulation provides a trustful approximation, while making the analysis rather simpler. So, the following equation will be used in the analysis of our system:

$$\bar{x}(t-\tau(t)) \approx 1/2(x(t-\tau(t))+x(t-\tau(t)-T_{int})). \tag{5.30}$$

Remark 5.1. *Note that, as mentioned in section 4.2, there are some other alternatives (such as Simpson rule [65]) to the above-mentioned Trapezoidal approximation, which provide more accurate approximation of the integral term. However, exploiting these alternatives would involve more delay terms in our formulation and consequently make the analysis more complicated.*

Since the error dynamics (5.22) contains the terms of $x(t)$, the stability analysis should be carried out by augmenting the plant model. In this regard, defining

$$\mathcal{X}(t) = \begin{bmatrix} x(t) \\ e_F(t) \end{bmatrix},$$

the overall dynamics of the system can be expressed as follows:

$$\begin{aligned} \dot{\mathcal{X}} &= A\mathcal{X}(t) + A_{d1}\mathcal{X}(t-\tau(t)) + A_{d2}\mathcal{X}(t-\tau(t)-T_{int}) + B_e e_t + B_w w + B_\nu \bar{\nu}(k_j), \\ e_z &= He_F, \end{aligned} \tag{5.31}$$

where

$$A = \begin{bmatrix} A_p & 0 \\ LC_p & A_p - LC_p \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0 & 0 \\ -\alpha_1 LCp & 0 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0 \\ -\alpha_2 LC_p & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ L \end{bmatrix},$$
$$B_w = \begin{bmatrix} B_p \\ B_p \end{bmatrix} \qquad B_\nu = \begin{bmatrix} 0 \\ L \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & H_p \end{bmatrix}.$$

Theorem 5.1 provides sufficient conditions in the form of LMIs to guarantee the H_∞ performance of the proposed event-based observer system.

Theorem 5.1. *Consider the event-based observer (5.15) with the proposed TC (5.7). If n_s (i.e. the number of samples over the integration period T_{int}) tends to infinity, the H_∞ performance (5.3) with gain γ is guaranteed for the system, if for given D_1 (satisfying $I - D_1 > 0$) and D_2 there exist matrices $P > 0$, $Q_i > 0$, $R_i > 0$ $(i = 0, 1)$ with appropriate dimensions such that the following LMIs hold:*

$$\mathcal{L}_j^U = \begin{bmatrix} \Gamma_0 + \Pi + \Pi^T & \star & \star & \star & \star \\ \Gamma_1^R & -\Gamma_{W_1} & \star & \star & \star \\ \Gamma_2^R & 0 & -\Gamma_{W_2} & \star & \star \\ \Gamma_2^R & 0 & 0 & -\Gamma_{W_3} & \star \\ \Gamma_j^U & 0 & 0 & 0 & -R_j \end{bmatrix} < 0 \quad \textit{for} \quad j = 0, 1, \tag{5.32}$$

$$\mathcal{L}_j^V = \begin{bmatrix} \Gamma_0 + \Pi + \Pi^T & \star & \star & \star & \star \\ \Gamma_1^R & -\Gamma_{W_1} & \star & \star & \star \\ \Gamma_2^R & 0 & -\Gamma_{W_2} & \star & \star \\ \Gamma_3^R & 0 & 0 & -\Gamma_{W_3} & \star \\ \Gamma_j^V & 0 & 0 & 0 & -R_j \end{bmatrix} < 0 \quad \textit{for} \quad j = 0, 1, \tag{5.33}$$

where:

$$\Gamma_0 = \begin{bmatrix} \Gamma_{01} & PAd_1 & PAd_2 & 0 & 0 & PB_w & PB_e & PB_\nu \\ \star & \Gamma_{02} & \Gamma_{04} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \Gamma_{03} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma^2 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -D_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -D_2 \end{bmatrix},$$

$$\Pi = \begin{bmatrix} V_0 + V_1 & U_0 - V_0 & U_1 - V_1 & -U_0 & -U_1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_j^U = \sqrt{2h + T_{int}} U_j \qquad \textit{for } j = 0, 1,$$

$$\Gamma_j^V = \sqrt{2h + T_{int}} V_j \qquad \textit{for } j = 0, 1,$$

$$\Gamma_j^R = \sqrt{h + jT_{int}} R_j \Lambda \qquad \textit{for } j = 0, 1,$$

$$\Gamma_{01} = A^T P + PA + Q_1 + Q_2 + H^T H,$$

$$\Gamma_{02} = \alpha_1^2 \beta C^T C, \quad \Gamma_{03} = \alpha_2^2 \beta C^T C, \quad \Gamma_{04} = \alpha_1 \alpha_2 \beta C^T C,$$

$$C = \begin{bmatrix} C_p & 0 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} A & Ad_1 & Ad_2 & 0 & 0 & B_w & B_e & B_\nu \end{bmatrix}.$$

Proof. Consider the following Lyapunov Krasovskii functional:

$$V = \sum_{i=1}^{3} V_i, \tag{5.34}$$

where

$$V_1 = \mathcal{X}(t)^T P \mathcal{X}(t), \tag{5.35}$$

$$V_2 = \sum_{j=0}^{1} \int_{t-h-jT_{int}}^{t} \mathcal{X}^T(s) Q_j \mathcal{X}(s) ds, \tag{5.36}$$

$$V_3 = \sum_{j=0}^{1} \int_{0}^{h+jT_{int}} \int_{t-\theta}^{t} \dot{\mathcal{X}}^T(s) R \dot{\mathcal{X}}(s) ds d\theta. \tag{5.37}$$

Define

$$\zeta(t) := \begin{bmatrix} \mathcal{X}(t)^T & \mathcal{X}_\tau^T & \mathcal{X}_M^T & w^T & e_t^T & \bar{\nu}^T \end{bmatrix}^T, \tag{5.38}$$

with

$$\mathcal{X}_\tau^T = \begin{bmatrix} \mathcal{X}^T(t-\tau(t)) & \mathcal{X}^T(t-\tau(t)-T_{int}) \end{bmatrix},$$
$$\mathcal{X}_M^T = \begin{bmatrix} \mathcal{X}^T(t-h) & \mathcal{X}^T(t-h-T_{int}) \end{bmatrix}.$$

Computing the time derivative of V_i along the trajectories of e and x for $t \in [k_j h, k_{j+1}h)$ we have:

$$\begin{aligned}
\dot{V}_1 &= \zeta^T \Lambda^T P \mathcal{X} + \mathcal{X}^T P \Lambda \zeta, \\
\dot{V}_2 &= \sum_{j=0}^{1} \mathcal{X}(t)^T Q_j \mathcal{X}(t) - \sum_{j=0}^{1} \mathcal{X}^T(t-h-jT_{int}) Q_j \mathcal{X}(t-h-jT_{int}), \\
\dot{V}_3 &= \sum_{j=0}^{1} (h+jT_{int})(\zeta^T \Lambda^T R_j \Lambda \zeta) - \sum_{j=0}^{1} \int_0^{h+jT_{int}} \dot{\mathcal{X}}(t-\theta)^T R_j \dot{\mathcal{X}}(t-\theta) d\theta.
\end{aligned} \tag{5.39}$$

Using Leibniz-Newton formula, for any U_j and V_j (for $j = 0, 1$) of proper dimensions we have:

$$\zeta^T U_j(\mathcal{X}(t-\tau(t)-jT_{int}) - \mathcal{X}(t-jT_{int}-h) - \int_{t-jT_{int}-h}^{t-jT_{int}-\tau(t)} \dot{\mathcal{X}}(s)ds) = 0, \tag{5.40}$$

$$\zeta^T V_j(\mathcal{X}(t) - \mathcal{X}(t-jT_{int}-\tau(t)) - \int_{t-\tau(t)-jT_{int}}^{t} \dot{\mathcal{X}}(s)ds) = 0. \tag{5.41}$$

Applying above equations we get:

$$\begin{aligned}
&\dot{V}_3 \le \sum_{j=0}^{1} (h+jT_{int})(\zeta^T \Lambda^T R_j \Lambda \zeta) \\
&+ \sum_{j=0}^{1} [(h-\tau(t))\zeta^T U_j R_j^{-1} U_j^T \zeta + (\tau(t)+jT_{int})\zeta^T V_j R_j^{-1} V_j^T \zeta] + \zeta^T(\Pi + \Pi^T)\zeta.
\end{aligned} \tag{5.42}$$

Now, regarding that the triggering inequality (5.27) holds for all $t \in [k_j h, k_{j+1}h)$, it can be derived that:

$$\begin{aligned}
&\dot{V} - \gamma^2 w^T w + e_z^T e_z \le \zeta^T(\Gamma_0 + \Pi + \Pi^T)\zeta + \sum_{j=0}^{1} (h+jT_{int})\zeta^T \Lambda^T R_j (R_j)^{-1} R_j \Lambda \zeta \\
&+ \sum_{j=0}^{1} [(h-\tau(t))\zeta^T U_j R_j^{-1} U_j^T \zeta + (\tau(t)+jT_{int})\zeta^T V_j R_j^{-1} V_j^T \zeta] + \bar{\nu}(k_j)^T D_2 \bar{\nu}(k_j) + \Delta_\nu(t),
\end{aligned} \tag{5.43}$$

where D_2 is an arbitrary positive definite matrix.

The above inequality can be rewritten as:

$$\dot{V} - \gamma^2 w^T w + e_z^T e_z \le \zeta^T \bar{\Gamma} \zeta + \bar{\nu}(k_j)^T D_2 \bar{\nu}(k_j) + \Delta_\nu(t), \tag{5.44}$$

where :

$$\bar{\Gamma} = \Gamma_0 + \Pi + \Pi^T + \sum_{j=0}^{1} \Gamma_j^{R^T} R_j{}^{-1} \Gamma_j^R + \sum_{j=0}^{1} [\frac{h-\tau(t)}{2h+T_{int}} \Gamma_j^U R_j^{-1} \Gamma_j^{U^T} + \frac{\tau(t)+jT_{int}}{2h+T_{int}} \Gamma_j^V R_j^{-1} \Gamma_j^{V^T}]. \tag{5.45}$$

Since :

$$\sum_{j=0}^{1} [\frac{h-\tau(t)}{2h+T_{int}} + \frac{\tau(t)+jT_{int}}{2h+T_{int}}] = 1, \tag{5.46}$$

equation (5.45) can be expressed as follows:

$$\bar{\Gamma} = \sum_{j=0}^{1} [\frac{h-\tau(t)}{2h+T_{int}} \bar{\Gamma}_j^U + \frac{\tau(t)+jT_{int}}{2h+T_{int}} \bar{\Gamma}_j^V]. \tag{5.47}$$

where:

$$\begin{aligned} \bar{\Gamma}_j^U &= \Gamma_0 + \Gamma_1 + \Gamma_1^T + \Gamma_j^U R_j^{-1} \Gamma_j^{U^T} + \sum_{j=0}^{1} \Gamma_j^{R^T} R_j{}^{-1} \Gamma_j^R, \\ \bar{\Gamma}_j^V &= \Gamma_0 + \Gamma_1 + \Gamma_1^T + \Gamma_j^V R_j^{-1} \Gamma_j^{V^T} + \sum_{j=0}^{1} \Gamma_j^{R^T} R_j{}^{-1} \Gamma_j^R. \end{aligned} \tag{5.48}$$

It follows from equation (5.44) that:

$$\begin{aligned} &E\{\dot{V} - \gamma^2 w^T w + e_z^T e_z\} \le E\{\zeta^T \bar{\Gamma} \zeta\}, \\ &+E\{\bar{\nu}(k_j)^T D_2 \bar{\nu}(k_j)\} + E\{\Delta_\nu(t)\}. \end{aligned} \tag{5.49}$$

Since the elements of the random vector $\bar{\nu}$ are statistically independent, we have:

$$E\{\bar{\nu}(k_j)^T D_2 \bar{\nu}(k_j)\} = tr(D_2)\sigma^2/n_s, \tag{5.50}$$

and for the last term:

$$E\{\Delta_\nu(t)\} = (\beta + tr((I-D_1)^{-1}))\sigma^2/n_s + E\{2\bar{x}^T(t-\tau(t))C^T\beta\bar{\nu}(t-\tau(t)) - 2e_t^T\bar{\nu}(t-\tau(t))\}. \tag{5.51}$$

From the equation (5.1), it can easily get that the plant dynamics does not depend on the noise values. So, the vectors $\bar{x}(t-\tau(t))$ and e_t are independent from $\bar{\nu}(t-\tau(t))$ and so the last term in above equation is zero. Therefore:

$$E\{\Delta_\nu(t)\} + E\{\bar{\nu}(k_j)^T D_2 \bar{\nu}(k_j)\} = c_\nu/n_s, \tag{5.52}$$

with

$$c_\nu = (\beta + tr((I-D_1)^{-1}) + tr(D_2))\sigma^2. \tag{5.53}$$

From (5.46) and (5.47), it is inferred that $\bar{\Gamma}$ is a convex combination $\bar{\Gamma}_0^U$, $\bar{\Gamma}_1^U$, $\bar{\Gamma}_0^V$ and $\bar{\Gamma}_1^V$. So, using Lemma 2.3 and the Schur complement the LMIs (5.32) and (5.33)are equivalent to

$$\bar{\Gamma} < 0. \tag{5.54}$$

Then, from equation (5.49) it is obtained:

$$E\{\dot{V}\} + E\{e_z^T e_z - \gamma^2 w^T w\} \leq -\theta E\{\|x(t)\|^2\} + c_\nu / n_s, \tag{5.55}$$

where

$$-\theta := \max_{0 \leq \tau(t) \leq h} (\sum_{j=0}^{1} [\frac{h - \tau(t)}{2h + T_{int}} \theta_j^U + \frac{\tau(t) + jT_{int}}{2h + T_{int}} \theta_j^V]), \tag{5.56}$$

with

$$\theta_j^U = \lambda_{max}(\bar{\Gamma}_j^U), \quad \theta_j^V = \lambda_{max}(\bar{\Gamma}_j^V). \tag{5.57}$$

To prove the H_∞ performance, integrate both sides of (5.49) from $k_j h$ to $T \in [k_j h, k_{j+1} h)$:

$$E\{V(T)\} - E\{V(0)\} + E\{\int_{k_j h}^{T} e_z(t)^T e_z(t) dt\} \leq \gamma^2 \int_{k_j h}^{T} w(t)^T w(t) dt + c_\nu (T - k_j) / n_s \tag{5.58}$$

Repeating the integration operation over the intervals $[k_i, k_{i+1})$ (for $i = 1, \ldots, j-1$) and since $E\{V(T)\}$ is positive, we get:

$$E\{\|e_z\|_{2,T}^2\} \leq \gamma^2 \|w\|_{2,T}^2 + E\{V(0)\} + T c_\nu / n_s, \tag{5.59}$$

and this concludes the proof.

$\square$

Remark 5.2. *Practically speaking, because of the finiteness of the sample numbers n_s, the term Tc_ν/n_s in (5.59) is not exactly equal to zero. However, this bias term can be made sufficiently small by increasing the number of samples, used for the averaging.*

5.4 Parameters Design

In the introduced event-based scheme, the observer gain L, sampling time h, integration time T_{int}, triggering condition coefficient β and H_∞ gain γ, are the parameters which can be considered as design parameters. h and T_{int} are the parameters which mainly depend on the system properties. The sampling time h represents the minimum possible transmission time between two consecutive data sent to the observer side. So, practically, this parameter has to be chosen based on the network medium properties, such as transmission rate. T_{int} plays a key role in suppressing noise impact on the TC. In the other words, depending on the measurement sensor sampling rate, T_{int} has to be chosen such that n_s (number of samples in the integration interval) is large enough to make the averaging (5.14) more effective.

Next, we will go through two separate steps to design L, β and γ. First, assuming a traditional Luenberger observer, the gain L is found such that eigenvalues of $A - LC$ are located at desired places. Second, β and γ are designed as follows. Based on the definition of the TC in (5.7), one can infer that the larger β, the less data is transmitted through the network channel. However, lower convergence rate of estimation error and less disturbance attenuation level (bigger γ) would be expected. In the other words there is a trade-off in designing β and γ. So, to assign proper values for these variables, one can iteratively solve the following optimization problem for different values of β (starting form zero, to the value which constraints are infeasible) and find the trade-off curve.

$$\begin{aligned} &\min \gamma^2 \\ &\mathcal{L}_0^U < 0, \quad \mathcal{L}_0^V < 0, \\ &\mathcal{L}_1^U < 0, \quad \mathcal{L}_1^V < 0, \end{aligned} \tag{5.60}$$

where, $\mathcal{L}_j^U$ and $\mathcal{L}_j^V$ $(j = 0, 1)$ are defined as (5.32) and (5.33).

5.5 Simulation Results

Consider the following parameters of a quarter-car model with an active suspension, borrowed from [69].

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & 0 & -c_s/m_s & c_s/m_s \\ k_s/m_u & k_u/m_u & c_s/m_u & -c_s/m_u \end{bmatrix}, \quad & B &= \begin{bmatrix} 0 \\ -2\pi q_0 \sqrt{G_0 v} \\ 0 \\ 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, & H &= \begin{bmatrix} 0 & 0 & 0.2 & 0 \end{bmatrix}, \end{aligned} \tag{5.61}$$

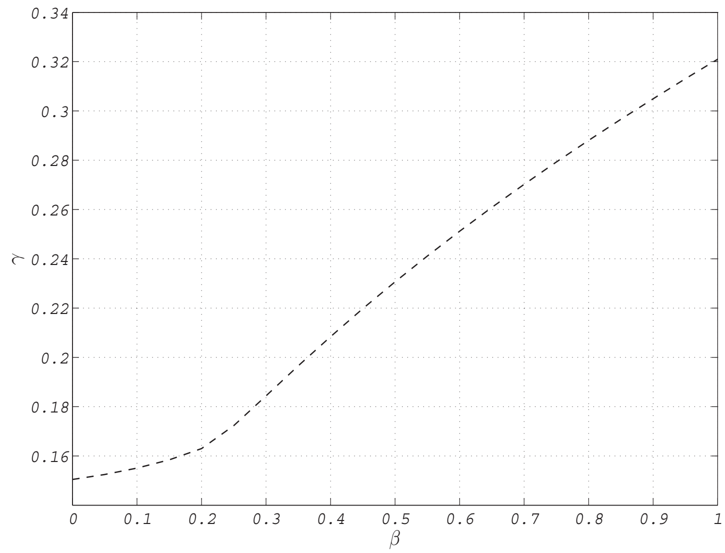


Figure 5.1: Trade-off curve between β and γ

where $m_s = 973$, $k_s = 42720$, $c_s = 3000$, $k_u = 101115$, $m_u = 114$, $G_0 = 512 \times 10^{(-6)}$, $q_0 = 0.1$, $m_u = 114$. A detailed explanation of parameters are given in [70].

We are going to design and implement the proposed event-based observer for this system. To this goal, first, we set the sampling and integration time to 0.05 and 0.02, respectively. In order to have an acceptable convergence rate and based on the approach described in the Parameters Design section, the observer gain is set as $L = \begin{bmatrix} -1.1320 & -0.5994 & 24.6167 & 18.4649 \end{bmatrix}^T$.

Now, the optimization problem (5.60) is implemented for different values of β, starting from 0 to 1 (where the constraints become infeasible). The obtained values for γ versus β are depicted in Fig.5.1. As expected, it is clear from the figure that a higher attenuation level (smaller γ) requires more data transmission (smaller β).

To clarify the efficiency of our described method, the event-based observer is implemented by our proposed TC (5.7) and compared with the case the where the traditional TC (5.4) is in use. It is assumed that the output measurement is contaminated by a white Gaussian noise with a distribution of $\mathcal{N}(0, 0.6^2)$, and the disturbance signal in the plant dynamics is $w(t) = 5e^{-0.1t}sin(4t)$. The initial conditions are set to $x(0) = [-12, 6, 3, 9]^T$ and $x_F(0) = [-3, 3, 0, 0]^T$ and simulations are carried out for 8 seconds, using different sensor sampling rates (to clarify the importance of this issue). The number of data trans-

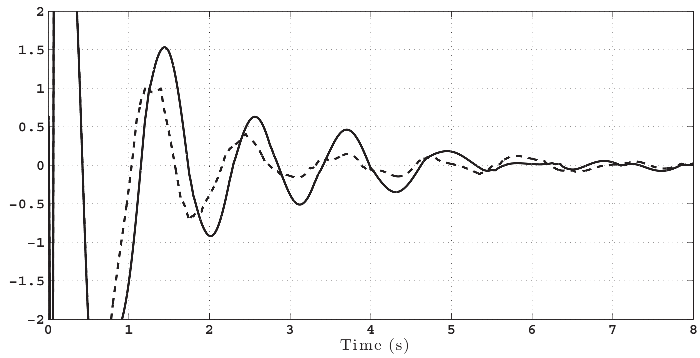


Figure 5.2: Estimation error trajectories for the cases $\beta = 0.7$ (solid) and $\beta = 0.4$ (dashed), while using our proposed event-based observer

mission are given in Table 5.1 and the estimation error trajectories are shown in Fig. 5.2.

Table 5.1: Number of data sent from plant to the observer

	Traditional mechanism	Proposed mechanism with different sensor sampling rate (Hz)			
		200	700	1000	5000
$\beta = 0.4$	97	77	65	62	57
$\beta = 0.7$	81	61	49	42	36

The values given in Table 5.1 represent the number of data transmission from plant side to the observer. As seen, for $\beta = 0.7$, the traditional event-based scheme, sends 81 data samples to the observer. In the other words, the TC is violated at 51% of the verification numbers. However, utilizing our proposed TC (5.7), one is able to reduce this triggering percentage down to 22.5%, depending on the sampling rate chosen for the event generator module. A similar conclusion can be made for other values of β. Note that form the table, increasing β one can reduce the number of triggering samples. However, this would happen at the expense of lowering the estimation error convergence rate and disturbance attenuation level, something which is clear in Fig.5.2.

5.6 Summary

A novel event-based observer scheme was presented for LTI systems, in the presence of measurement noise. The proposed scheme is able to effectively reduce the impact of noise and prevent excessive data communication between pant and the observer. The overall system was modeled in continuous-time form and performance analysis was carried out using Lyapunov-Krasovskii functional. In addition, a mechanism was proposed to design the parameters of the systems and finally simulation were given to show the efficiency of the method.

Chapter 6

Integral-Based Event-Triggered Control for Nonlinear Systems

In this chapter, an integral-based event-driven mechanism is proposed for a general class of nonlinear systems. The proposed scheme is less conservative than earlier work on the subject and achieves asymptotic stability without forcing the derivative of the Lyapunov function to be negative between samples. A rigorous proof is given, showing that the proposed TC is more effective than the corresponding traditional approaches. Simulation results are provided to illustrate the effectiveness of the proposed solution.

The rest of this chapter is organized as follows. Section 6.1 introduces the problem statement. In section 6.2, the proposed event triggering mechanism is introduced and asymptotic stability and existence of minimum inter-event time for the event-based system is proved. Section 6.3 provides a comparison between the IBTC and traditional TC where we show that our proposed scheme is more effective in the sense that it can significantly reduce the data transferred between plant and controller. Section 6.4 presents illustrative examples and Section 6.5 provides summary and final remarks.

6.1 Problem Statement

Consider the following nonlinear systems:

$$\dot{x} = f(x, u), \qquad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{6.1}$$

Throughout the rest of this chapter we assume that this system is controlled using a state feedback law of the form

$$u = k(x), \tag{6.2}$$

so that the closed-loop system

$$\dot{x} = f(x, k(x + e)), \tag{6.3}$$

is ISS with respect to measurement error e and there exist smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, class $\mathcal{K}_\infty$ functions $\bar{\alpha}$, $\underline{\alpha}$, α and γ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|), \tag{6.4}$$

$$\frac{\partial V}{\partial x} f(x, k(x + e)) \leq -\alpha(|x|) + \gamma(|e|). \tag{6.5}$$

As mentioned in the introduction chapter, the primary reference dealing with stability of event-driven systems is [1]. In this reference, Tabuada proposes an event condition to maintain the Lyapunov function V decreasing along the system trajectories. In this chapter we modify the assumptions and use an integral-based triggering mechanism that relaxes the conditions set in [1]. We depart from the conjecture that to guarantee closed-loop convergence to the origin it is enough to ensure that the value of the Lyapunov function decreases from one triggering instance to the next, regardless the sign of $\dot{V}$ in between triggering instants. This condition is enforced provided that the integral of the derivative of Lyapunov function satisfies the following inequality:

$$V(x(t_{i+1})) - V(x(t_i)) = \int_{t_i}^{t_{i+1}} \dot{V}(\tau) d\tau < -\int_{t_i}^{t_{i+1}} \alpha_e(|x(\tau)|) d\tau, \tag{6.6}$$

where t_j represents a triggering instance for each $j \in \mathbb{N}$ and α_e is function belonging to class $\mathcal{K}_\infty$. We look for a TC not only satisfies equation (6.6) but also guarantees

$$V(x(t_i)) > V(x(t)) \qquad \forall t \in (t_i, t_{i+1}]. \tag{6.7}$$

In the other words, regardless of the sign of $\dot{V}(t)$, $V(t)$ remains upper bounded by $V(t_i)$ over the interval $(t_i, t_{i+1}]$.

The following Lemma formalizes this concept and shows how these two conditions ensure asymptotic stability.

Lemma 6.1. *Consider the nonlinear system (6.1) (with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous) along with a control of the form (6.2) (with $k : \mathbb{R}^n \to \mathbb{R}^m$ continuous). Let $T =$*

$\{t_1, t_2, \ldots,\}$ *be an infinite sequence of triggering instants, with* $\min_{i\in\mathbb{N}}\{t_{i+1} - t_i\} > \kappa > 0$. *If there is a Lyapunov function satisfying conditions (6.6) and (6.7), then origin is an asymptotically stable equilibrium point for the closed loop system.*

Proof: Suppose that the system initial condition is $x(t_1)$ at the initial time t_1. To prove the stability of the system, assume that ε_V is given. Based on the equations (6.6) and (6.7), and using (6.4) one can easily obtain:

$$\underline{\alpha}(|x(t)|) \leq \bar{\alpha}(|x(t_1)|) \qquad \forall t > t_1. \tag{6.8}$$

Hence, for any initial condition, satisfying $|x(t_1)| \leq \delta_V = \bar{\alpha}^{-1}(\underline{\alpha}(\varepsilon_V))$, we would have:

$$|x(t)| \leq \varepsilon_V \qquad \forall t > t_1, \tag{6.9}$$

and so, the system is stable.

In the next step the convergence of the state trajectories to the origin is proved:

Condition (6.7) implies that for any $t_i \in T$, there exists $\xi_i > 0$ such that

$$I_i = \{x \mid V(x) + \xi_i \leq V(x(t_i))\} \tag{6.10}$$

is an invariant set containing the origin. Moreover, (6.6) implies that $I_{i+1} \subset I_i \ \forall i \in \mathbb{N}$. It follows that I_i's, so defined, constitute a shrinking sequence of invariant sets, denoted by $I = \{I_i : i \in \mathbb{N}\}$. To prove asymptotic convergence of state trajectories, it suffices to show

$$I_i \to C \ \text{ as } \ i \to \infty, \tag{6.11}$$

where $C = \{0\}$ is the singleton containing the origin. We reason by contradiction and assume that C contains at least a point $\upsilon \neq 0$ and assume that

$$V(x(t_i)) \to V(\upsilon) > 0 \ \text{ as } \ i \to \infty. \tag{6.12}$$

So:

$$V(x(t_{i+1})) - V(x(t_i)) \to 0 \ \text{ as } \ i \to \infty.$$

Then, from condition (6.6) it is easily concluded that :

$$x(t) \to 0 \ \text{ as } \ t \to \infty,$$

which contradicts the assumption. So, x converges asymptotically to the origin.

□

Remark 6.1. *Lemma 6.1 assumes that the sequence T of triggering instants contains an infinite number of terms. To ensure that this is the case, this can be easily enforced adding a simple algorithm to the triggering scheme. The following is an example of such algorithm:*

Let T_s be predefined time index and t_i as the last triggering instant generated by an existing triggering condition. If no triggering occurs after $t_i + T_s$, then a fresh data would be automatically sent from plant side to the controller after each $t = T_s$.

In the next section we propose a new event triggering mechanism that achieves stability requiring fewer samples compared to previously published results.

6.2 Integral Based Event Triggering Mechanism

In this section, we endeavor to construct an event trigger control law based on the ideas of the previous section. We will show that the new law achieves stability while significantly reducing the amount of information sent between plant and controller. To this end, we integrate (6.5) over the interval $[t_i, t)$:

$$V(t) - V(t_i) \leq -\int_{t_i}^{t} \alpha(|x|)d\tau + \int_{t_i}^{t} \gamma(|e|)d\tau, \tag{6.13}$$

and define the Integral-based triggering condition as follows:

$$\int_{t_i}^{t} \gamma(|e|)d\tau \leq \sigma \int_{t_i}^{t} \alpha(|x|)d\tau \qquad t \geq t_i, \tag{6.14}$$

In the above formulation $e = x(t) - x(t_i)$ is the measurement error and $0 < \sigma < 1$ is an arbitrary coefficient. Next execution time ($t_{i+1} \in T$) is the time when above inequality is violated; *i.e.*

$$\int_{t_i}^{t_{i+1}} \gamma(|e|)d\tau = \sigma \int_{t_i}^{t_{i+1}} \alpha(|x|)d\tau. \tag{6.15}$$

In the following theorem, we show that the IBTC (6.14) preserves asymptotic stability of the closed-loop system, while a minimum inter-event time is guaranteed for the scheme.

Theorem 6.1. *Consider the continuous time nonlinear system (6.1) with the pre-defined stable state feedback law (6.2) and assume that the following conditions, introduced in [1], hold:*

1. *$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous on compacts.*

2. *$k : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on compacts.*

3. *There exists an ISS Lyapunov function for the closed-loop system, satisfying (6.4) and (6.5) with α^{-1} and γ Lipschitz continuous on compacts.*

Assume now that instead of continuous information flow from plant to the controller, the control law updates based on an event-based scheme with IBTC (6.14). If $0 < \sigma < 1$, then we have the following properties for the event-based system:

(A) For any compact set $S \subset \mathbb{R}^n$, containing the origin, there exists a lower bound $\tau_{min} \in \mathbb{R}^+$ such that for any initial condition in S we have

$$t_{i+1} - t_i \geq \tau_{min} \qquad \forall\ t_i, t_{i+1} \in T, \tag{6.16}$$

where $T = \{t_i : i \in \mathbb{N}\}$ is a sequence of the triggering instants.

(B) The origin is an asymptotically stable equilibrium point.

Proof.

(A) To show the existence of minimum inter-event time τ_{min}, we introduce an auxiliary system with the same dynamics as (6.3):

$$\dot{\zeta} = f(\zeta, k(\zeta + e')); \tag{6.17}$$

but with the TC proposed in [1]:

$$\gamma(|e'|) \leq \sigma\alpha(|\zeta|). \tag{6.18}$$

Assume now that both systems update their control law at time instant t_i and also have the same state values at this time, *i.e.*:

$$x(t_i) = \zeta(t_i). \tag{6.19}$$

Denote the next execution times of system (6.17) by t'_{i+1}; *i.e.*

$$\gamma(|e'(t'_{i+1})|) = \sigma\alpha(|\zeta(t'_{i+1})|), \tag{6.20}$$

and

$$\gamma(|e'(t)|) < \sigma\alpha(|\zeta(t)|) \qquad \forall t \in [t_i, t'_{i+1}). \tag{6.21}$$

Based on (6.19) we have

$$e'(t) = e(t) \qquad \forall t \in [t_i, t'_{i+1}). \tag{6.22}$$

Integrating (6.21) from t_i to t'_{i+1} and using (6.15), we can easily see that $t_{i+1} > t'_{i+1}$. Since the auxiliary system has lower bound for its execution time, [1], so does the event-based system with IBTC (6.14).

(B) Substituting (6.14) in (6.13) we have

$$V(t) - V(t_i) \leq (\sigma - 1) \int_{t_i}^{t} \alpha(|x|) d\tau, \tag{6.23}$$

and so, for $\sigma < 1$:

$$V(t) < V(t_i) \qquad \forall t \in [t_i, t_{i+1}), \tag{6.24}$$

and asymptotic stability follows from Lemma 6.1.

□

6.2.1 Special Case: Linear Comparison Functions

In this subsection we consider the special case in which the functions α and γ are *linear*. In this special case; namely, when the assumptions of Theorem 6.1, are satisfied with α and γ linear; property B can be proved directly without using any auxiliary system.

If α and γ are linear, then for any $t_s, t_f \in \mathbb{R}$ the following hold:

$$\begin{aligned} \textstyle\int_{t_s}^{t_f} \alpha(|x(\tau)|) d\tau &= \alpha(\textstyle\int_{t_s}^{t_f} |x(\tau)| d\tau), \\ \textstyle\int_{t_s}^{t_f} \gamma(|x(\tau)|) d\tau &= \gamma(\textstyle\int_{t_s}^{t_f} |x(\tau)| d\tau). \end{aligned} \tag{6.25}$$

It then follows that the IBTC (6.14) can be rewritten as:

$$\gamma(\int_{t_i}^{t} |e(\tau)| d\tau) \leq \sigma \alpha(\int_{t_i}^{t} |x(\tau)| d\tau). \tag{6.26}$$

Now, the existence of a lower bound τ_{min} for the execution time intervals is proved similar to the proof, given in [1]:

Given S, select λ large enough such that $S \subset \Omega = \{x \in \mathbb{R}^n : V(x) \leq \lambda\}$. As proved in part A of the theorem, Ω is an invariant set for the event-based system. Define

$$\begin{aligned} \bar{x}(t) &:= \textstyle\int_{t_l}^{t} |x(\tau)| d\tau, \\ \bar{e}(t) &:= \textstyle\int_{t_l}^{t} |e(\tau)| d\tau, \end{aligned} \tag{6.27}$$

where $t_l := \max\{a \in T : a \leq t\}$. Moreover, let $\tau_{max} := \sup\{t_{i+1} - t_i : i \in \mathbb{N}\}$[1] and $r_\Omega := \max\{|x| : x \in \Omega\}$. Then, $B_{\bar{x}} := \{\bar{x} \in \mathbb{R}^n : |\bar{x}| \leq \tau_{max} r_\Omega\}$ is an invariant compact set for $\bar{x}(t)$. Furthermore, define another compact set for $\bar{e}$ as follows:

$$E := \{\bar{e} \in \mathbb{R}^n : \bar{e} \leq \gamma^{-1}(\sigma\alpha(\bar{x})), \bar{x} \in B_{\bar{x}}\},$$

and let L_e to be the Lipschitz constant for the Lipschitz continuous function $\alpha^{-1}(\gamma(\cdot)/\sigma)$ over the set E (containing the origin) so that $\alpha^{-1}(\gamma(\bar{e})/\sigma) \leq L_e \bar{e}$. Therefore, the following conservative TC

$$L_e \bar{e} \leq \bar{x} \tag{6.28}$$

ensures that inequality (6.26) is satisfied. Note that based on Lipschitz continuity of f and k on compacts, one can easily establish the following inequality on set $B_{\bar{x}} \times E$ for some $L_f > 0$:

$$|f(x, k(x+e))| \leq L_f|x| + L_f|e|. \tag{6.29}$$

To find the lower bound τ_{min} on inter-event times, we derive the dynamic of $\omega \triangleq \bar{e}/\bar{x}$ over the interval $[t_i, t_{i+1})$:

$$\begin{aligned} &\dot{\omega} = \frac{d}{dt}\frac{\int_{t_l}^{t} |e(\tau)|d\tau}{\int_{t_l}^{t} |x(\tau)|d\tau} = \frac{|e(t)| \int_{t_l}^{t} |x(\tau)|d\tau - |x(t)| \int_{t_l}^{t} |e(\tau)|d\tau}{(\int_{t_l}^{t} |x(\tau)|d\tau)^2}, \\ &\omega(t_i) = 0. \end{aligned} \tag{6.30}$$

Integrating equation (6.3) from t_i to t and using inequality (6.29):

$$|e(t)| \leq \int_{t_i}^{t} |f(x, k(x+e))|d\tau \leq L_f \left[\bar{x} + \bar{e}\right], \tag{6.31}$$

then, from equation (6.30) we have:

$$\dot{\omega} \leq L_f(1+\omega). \tag{6.32}$$

Based on the comparison lemma [71], the trajectory of ω over $[t_i, t_{i+1})$ is bounded by η, driven by the following dynamic:

$$\dot{\eta} = L_f(1+\eta), \qquad \eta(t_i) = 0. \tag{6.33}$$

In the other words, τ_{min} is lower bounded by the time that η takes to travel from 0 to $1/L_e$, which is $ln(1 + 1/L_e)/L_f$. This concludes the proof.

[1]By Remark 6.1 such a supremum always exists.

Remark 6.2. *Theorem 1 shows that the IBTC relaxes the assumptions in previously established triggering mechanism based on which the derivative of Lyapunov function is kept negative for time. In the next section we study the benefits of the new IBTC in terms of transmission data between plant and controller, compared to the classical approach.*

6.3 Comparison with traditional triggering scheme

In order to make a fair comparison between the proposed and traditional event triggering schemes, in this section we consider two event-based systems with the same dynamic but different triggering strategies and compare the resulting inter-event times. The following theorem is the main result of this section.

Theorem 6.2. *Consider the event-based nonlinear system (6.3) implemented using the IBTC (6.14), and let the infinite sequence* $T = \{t_i : i \in \mathbb{N}\}$ *denote the triggering instants. Consider also the system (6.17) with the same dynamic but implemented using the classical TC (6.18) and let* $T' = \left\{t'_j : j \in \mathbb{N}\right\}$ *represent the triggering instants. Assuming that* α *is Lipschitz continuous on compacts and that the conditions of Theorem 1 are satisfied, then the following properties hold:*

(A) *Zero Triggering-Time State Difference: If* $x(t_m) = \zeta(t'_n)$ *for some* $t_m \in T$ *and* $t'_n \in T'$*, then*

$$t_{m+1} - t_m > t'_{n+1} - t'_n.$$

(B) *Non-Zero Triggering-Time State Difference: For every* $t'_n \in T'$*, there exists* $\epsilon > 0$ *such that if*

$$|x(t_m) - \zeta(t'_n)| < \epsilon \qquad \forall t_m \in T, \tag{6.34}$$

then

$$t_{m+1} - t_m > t'_{n+1} - t'_n. \tag{6.35}$$

Proof.

(A) The system dynamic (6.17) is time invariant over the interval $[t'_n, t'_{n+1})$ in the sense that e' only depends on the initial condition value $\zeta(t'_n)$ and $\zeta(t)$. Therefore, without loss of generality, we assume $t'_n = t_m$ and so $\zeta(t_m) = x(t_m)$ and $e'(t_m) = 0$. Then, to prove this part, it's enough to show $t_{m+1} > t'_{n+1}$, which has been shown in Part (B) of Theorem 1.

(B) Similar to previous part and without loss of generality, we assume $t'_n = t_m$ and will show $t_{m+1} > t'_{n+1}$.

For the sake of simplicity, denote $\zeta(t'_n)$ and $x(t_m)$ by ζ_n and x_m respectively. Since $t'_n = t_m$, we have:

$$\begin{aligned} e' &= \zeta_n - \zeta(t) \quad \forall t \in [t'_n, t_{m+1}), \\ e &= x_m - x(t) \quad \forall t \in [t'_n, t'_{n+1}). \end{aligned} \tag{6.36}$$

Define the following variables:

$$\Delta(t) := x(t) - \zeta(t), \qquad \Delta_0 := x_m - \zeta_n. \tag{6.37}$$

Then, the dynamic of Δ for $t \geq t_m$ is obtained as follows:

$$\dot{\Delta} = f(x, k(x+e)) - f(\zeta, k(\zeta + e')). \tag{6.38}$$

Using (6.36) and (6.37), equation (6.38) can be rewritten as:

$$\dot{\Delta} = f(x, k(x+e)) - f(x - \Delta, k(x + e - \Delta_0)). \tag{6.39}$$

Based on the Lipschitz continuity of f and k on compacts, we have that:

$$|\dot{\Delta}| \leq L'|\Delta| + L'|\Delta_0| \tag{6.40}$$

for some $L' > 0$. By the comparison lemma [71], we have:

$$|\Delta| \leq |\Delta_0|(2e^{L'(t-t'_n)} - 1). \tag{6.41}$$

Now, consider the TC (6.18) over $[t'_n, t'_{n+1}]$, where the following equations hold:

$$\gamma(|e'(t)|) < \sigma\alpha(|\zeta(t)|) \qquad \forall t \in [t'_n, t'_{n+1}), \tag{6.42}$$

$$\gamma(|e'(t'_{n+1})|) = \sigma\alpha(|\zeta(t'_{n+1})|). \tag{6.43}$$

Equation (6.42) can be reformulated as follows:

$$\gamma(|e'(t)|) + \varepsilon(\zeta(t'_n), t) \leq \sigma\alpha(|\zeta(t)|) \qquad \forall t \in [t'_n, t'_{n+1}), \tag{6.44}$$

in which $\varepsilon : \mathbb{R}^n \times [t'_n, t'_{n+1}) \to \mathbb{R}^+$ is some continuous function . Next, By the definitions (6.36) and (6.37) the above inequality can be re-written as follows:

$$\gamma(|e + \Delta - \Delta_0|) + \varepsilon(t) \leq \sigma\alpha(|x - \Delta|), \tag{6.45}$$

where, for simplicity, $\varepsilon(\zeta(t'_n), t)$ is denoted by $\varepsilon(t)$. Since $\gamma(\cdot)$ and $\alpha(\cdot)$ are Lipschitz on compacts, one can establish the following inequalities:

$$\begin{gathered} \gamma(|e|) - L_\gamma(|\Delta - \Delta_0|) \leq \gamma(|e + \Delta - \Delta_0|), \\ \sigma\alpha(|x - \Delta|) \leq \sigma\alpha(|x|) + \sigma L_\alpha|\Delta|, \end{gathered} \tag{6.46}$$

where L_γ and L_α are Lipschitz constants of the functions γ and α respectively. Substituting inequalities (6.46) into (6.45), we obtain:

$$\gamma(|e|) + \varepsilon(t) \leq \sigma\alpha(|x|) + \sigma L_\alpha|\Delta| + L_\gamma|\Delta - \Delta_0|. \tag{6.47}$$

Finally, using the equation (6.41) and integrating the above inequality yields:

$$\textstyle\int_{t'_n}^{t'_{n+1}} \gamma(|e(\tau)|)d\tau + \int_{t'_n}^{t'_{n+1}} \varepsilon(\tau)d\tau \leq \sigma \int_{t'_n}^{t'_{n+1}} \alpha(|x(\tau)|)d\tau + C_{int}|\Delta_0|, \tag{6.48}$$

with

$$C_{int} \triangleq \int_0^{t'_{n+1}-t'_n} (L_\gamma + (\sigma L_\alpha + L_\gamma)(2e^{L'\tau} - 1))d\tau. \tag{6.49}$$

Note that, since the inter-event time $t'_{n+1} - t'_n$ is bounded below, then $C_{int} > 0$. So, selecting

$$|\Delta_0| < \epsilon = C_{int}^{-1} \int_{t'_n}^{t'_{n+1}} \varepsilon(\tau)d\tau$$

from equation (6.48), we have:

$$\int_{t'_n}^{t'_{n+1}} \gamma(|e(\tau)|)d\tau + \varepsilon' < \sigma \int_{t'_n}^{t'_{n+1}} \alpha(|x(\tau)|)d\tau, \tag{6.50}$$

for some $\varepsilon' > 0$. Remembering that t_{m+1} is the time instant when the equality

$$\int_{t_m}^{t_{m+1}} \gamma(|e(\tau)|)d\tau = \sigma \int_{t_m}^{t_{m+1}} \alpha(|x(\tau)|)d\tau \tag{6.51}$$

holds and since $t_m = t'_n$, it is concluded that $t_{m+1} > t'_{n+1}$.

□

Remark 6.3. *The importance of the foregoing theorem becomes clear when we note that the origin is asymptotic stable equilibrium point of both systems. In the other words, after a finite time, both state trajectories ζ and x reach a small neighborhood of each other and stay close from that point on. In this situation, as proved, our proposed event triggering mechanism transmits less data rather than the traditional scheme to guarantee the stability of the system.*

6.4 Simulation Results

In this section the effectiveness of the proposed method is illustrated by simulation.

Example 6.1. *Consider the following LTI system:*

$$\dot{x} = Ax + Bu, \tag{6.52}$$

which is already stabilized using the control law $u = kx$. It can be easily verified that there exists a Lyapunov function $V = x^T Px$, satisfying the following properties [1]:

$$\underline{c}_s|x| \leq V(x) \leq \bar{c}_s|x|, \tag{6.53}$$

$$\frac{\partial V}{\partial x}(Ax + Bk(x+e)) \leq -c_s|x|^2 + c_e|e||x|, \tag{6.54}$$

where $\underline{c}_s, \bar{c}_s, c_s$ and c_e are some positive constant parameters.

Define $\sigma_l := \sigma c_s/c_e$. According to Theorem 6.1, applying the following IBTC:

$$\int_{t_i}^{t} |e(\tau)||x(\tau)|d\tau \leq \sigma_l \int_{t_i}^{t} |x(\tau)|^2 d\tau \quad \forall \quad 0 < \sigma_l < c_s/c_e \tag{6.55}$$

(with $t_i,\ i \in \mathbb{N}$ showing the triggering instants) the system is asymptotically stable.

Note that in [1], the following traditional TC was introduced for the linear systems:

$$|e(t)| \leq \sigma_l|x(t)| \quad \forall \quad 0 < \sigma_l < c_s/c_e. \tag{6.56}$$

Now, to evaluate our event-based scheme numerically, consider a satellite system, discussed in [63], which is comprised of two rigid bodies, connected through a flexible link. Modeling the link as a spring with torque constant k_s and viscous damping f, the motion equations are given as follows:

$$\begin{aligned} J_1\ddot{\theta}_1 + f(\dot{\theta}_1 - \dot{\theta}_2) + k_s(\theta_1 - \theta_2) = u(t), \\ J_2\ddot{\theta}_2 + f(\dot{\theta}_2 - \dot{\theta}_1) + k_s(\theta_2 - \theta_1) = 0, \end{aligned} \tag{6.57}$$

where, J_1 and J_2 represent the moment of inertia of two bodies, θ_1 and θ_2 denote the yaw angle of two bodies and $u(t)$ is the control torque. Now, define the state variables as $x = \begin{bmatrix}\theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2\end{bmatrix}$ and select $J_1 = J_2 = 1$, $f = 0.09$ and $k_s = 0.04$. Then, the state space equation is expressed in the following form:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.3 & 0.3 & -0.004 & 0.004 \\ 0.3 & -0.3 & 0.004 & -0.004 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \tag{6.58}$$

$$\dot{x} = Ax + Bu. \tag{6.59}$$

Suppose that the system is controlled using the state feedback controller:

$$u = \begin{bmatrix} -2.9953 & 2.2955 & -3.1650 & -2.3398 \end{bmatrix} x. \tag{6.60}$$

It can be easily shown the Lyapunov function $V = x^T P x$ with:

$$P = \begin{bmatrix} 3.7236 & 1.0700 & 0.9544 & 8.8164 \\ 1.0700 & 4.1661 & 0.4746 & 5.7729 \\ 0.9544 & 0.4746 & 0.4623 & 2.6293 \\ 8.8164 & 5.7729 & 2.6293 & 32.8624 \end{bmatrix}$$

satisfies the inequalities (6.53) and (6.54) with the following parameters:

$$\begin{aligned} \underline{c}_s = \lambda_{min}(P) = 0.1938, \quad \bar{c}_s = \lambda_{max}(P) = 36.6054, \\ c_s = \lambda_{min}(Q) = 1, \quad c_e = |PBk + k^T B^T P| = 25.0583, \end{aligned} \tag{6.61}$$

where $Q = (A+Bk)^T P + P(A+Bk)$. Now, the initial condition is set to $x_0 = \begin{bmatrix}1 & 0.4 & 0.2 & 0\end{bmatrix}$ and three different controllers are implemented for the system:

- *Classic continuous time state-feedback controller with the feedback law (6.60).*
- *Our proposed event-based scheme with the IBTC (6.55).*
- *Traditional event triggering mechanism proposed in [1] with the TC (6.56).*

Since we must have $0 < \sigma_l < c_s/c_e = 0.0399$ to guarantee the stability of the systems, this coefficient is set to $\sigma_l = 0.039$ for both above triggering conditions. Simulation results

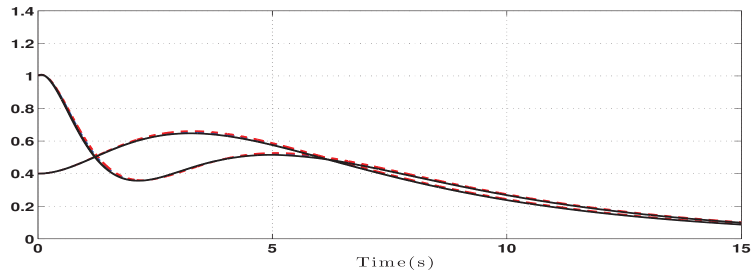


Figure 6.1: x_1 and x_2 trajectories of the plant, controlled by a traditional classic state feedback controller (red dashed), traditional event-based controller (blue dashed) and the proposed event-based controller (black solid).

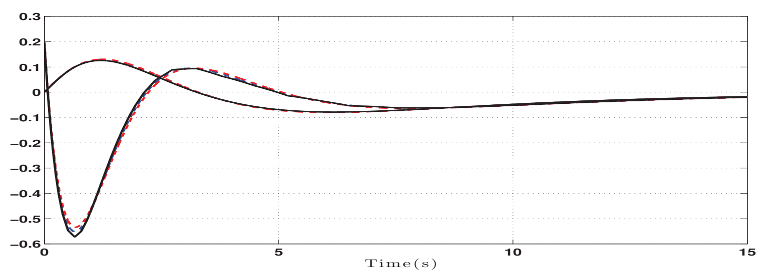


Figure 6.2: x_3 and x_4 trajectories of the plant, controlled by a traditional classic state feedback controller (red dashed), traditional event-based controller (blue dashed) and the proposed event-based controller (black solid).

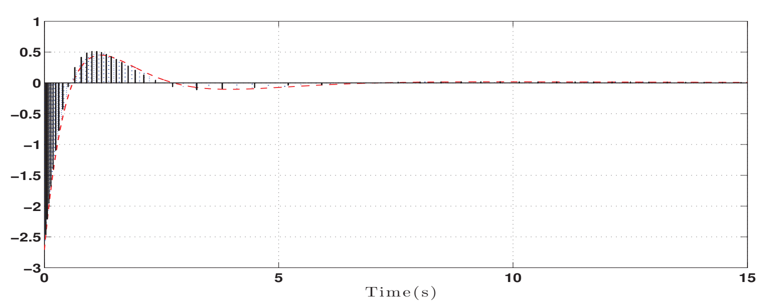


Figure 6.3: Control signal generated by classic state feedback controller (red dashed), traditional event-based controller (blue dashed spikes) and the proposed event-based controller (black solid spikes).

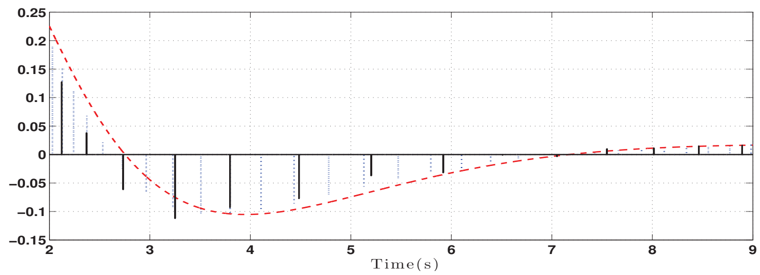


Figure 6.4: Control signals generated by the three controllers over the time interval $[2, 9]$. As seen, for each two samples update made by the traditional event-based controller, averagely just one sample update is made by our proposed controller.

are shown in Figs. 6.1-6.3. As seen in Figs. 6.1 and 6.2, all four states for the three systems asymptotically approach to the origin over the time with similar performances.

Fig. 6.3 represents the controller signals for all three cases where the trends are similar to each other. However, based on the obtained results, using our proposed integral-based event triggering controller, only 88 *samples are sent from plant to the controller over* 30 *sec., while using traditional event-based controller, the controller gets updated* 176 *times over the same time interval. In the other words, to provide a similar performance, our proposed controller requires approximately half as many samples as the traditional event-based controller. Fig. 6.4 validates this issue and confirms our controller effectively reduces the data transmission between plant and controller.*

Example 6.2. *As the second example, the proposed method is implemented for the following unstable plant:*

$$\dot{x} = f(x, u) = \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_1^2 + x_2 + u \end{bmatrix}, \tag{6.62}$$

which is controlled by the following controller

$$u = k(x) = -2x_2. \tag{6.63}$$

Define the Lyapunov function $V = 1/2(x_1^2 + x_2^2)$. Then , it is easily derived that:

$$\frac{\partial V}{\partial x}(f(x, k(x + e))) \leq -|x|^2 + 2|x||e|. \tag{6.64}$$

Setting the parameter σ to 0.8*, simulations are carried out for both traditional and the proposed event-based mechanism. To have a fair evaluation of the system, simulations are* *run 10 times (as a typical value) for different initial conditions, varying in the interval* $|x_0| \leq 3$*, and finally the mean number of triggered samples between plant and controller is calculated. Based on the obtained results, using the traditional mechanism proposed in [1], an average of* 27.1 *data points are exchanged between plant and controller over* 10 *sec.; But, the corresponding value falls to* 10.2*, while the integral-based mechanism is in use.*

6.5 Summary

A new integral-based event triggering condition was developed for a general class of nonlinear systems. Beside the asymptotic stability, it was proved that the proposed event-based system is more efficient and less conservative than the corresponding system, introduced in [1]. At the end, simulations were implemented for both linear and nonlinear dynamics to show the efficiency of the designed system.

Chapter 7

Decentralized Summation-Based Triggering Control for Nonlinear Systems

In this chapter, event-based control for a class of decentralized nonlinear systems is studied. It is assumed that the measurement sensors are geographically distributed and so local event generator modules are employed. Then, a novel periodic triggering condition is proposed for each module which only uses local information to trigger data through the communication channel. The proposed TC can be considered as an extension of IBTC, given in the previous chapter. So, it can potentially reduce the information exchange between subsystems compared to traditional control approaches, while maintaining closed loop asymptotic stability. The TC parameters are designed through a convex optimization problem with LMI constraints. Simulation results are carried out to illustrate the performance of the introduced scheme.

The rest of this chapter is organized as follows. In the next section, problem statement is given and in section 7.2 our proposed TC is introduced. Section 7.3 contains the continuous-time modeling and stability analysis of the event-based system. In section 7.4 event generator design mechanism is provided. Simulation results are represented in section 7.5 for both linear and nonlinear systems. Finally, the main results of the chapter is summarized in 7.6.

7.1 Problem Statement

Consider a class of nonlinear systems modeled by the following dynamics:

$$\dot{x} = Ax + Bu + \varphi(x, u), \tag{7.1}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, satisfying $\varphi(0, u) = 0$, is a locally Lipschitz function in its first argument on the region $\Omega \subset \mathbb{R}^n$ (containing origin in its interior), uniformly in u:

$$\|\varphi(x, u) - \varphi(y, u)\| \leq \ell\|x - y\|, \quad \forall x, y \in \Omega, \tag{7.2}$$

where ℓ is the Lipschitz constant.

We assume that the control law

$$u = Kx \tag{7.3}$$

has been already designed such that the closed-loop systems is asymptotically stable.

The schematic of this control system is such that the state measurement sensors are geographically distributed and grouped into N decentralized nodes. The state vector is written as $x = [x_1^T, x_2^T, \ldots, x_N^T]^T \in \mathbb{R}^n$, where $x_j \in \mathbb{R}^{n_j}$ denotes the state variables belonging to the j^{th} node and $n = \sum\limits_{j=1}^{N} n_j$ is the total number of states. Contrary to the sensor layouts, the controller unit is centralized and uninterruptedly receives state information from the sensor nodes.

In this research, our goal is to design event triggering conditions for each node that guarantee (local) closed-loop asymptotic stability with a large region of attraction. The TCs operate in a decentralized manner such that each event generator module decides when to send new data to the controller based only on local information. The controller, on the other hand, computes the plant input based on the last received data and updates its output whenever new data is received from any of the sensor nodes. In other words, the control law (7.3) is implemented as follows:

$$u = K\hat{x}(t), \tag{7.4}$$

where

$$\hat{x}(t) = [\hat{x}_1^T(t), \hat{x}_2^T(t), \ldots, \hat{x}_N^T(t)]^T, \tag{7.5}$$

and $\hat{x}_j(t)$ (for $j = 1, 2, \ldots, N$) is a piecewise constant signal generated as the outputs of the j^{th} node event generator module.

7.2 Proposed Summation-Based Triggering Condition

As mentioned earlier, in this chapter the TCs are assumed to be clock driven and verified on a periodic basis. We will represent by iT_s the sampling instants, where T_s is the sampling period and i is the index set $i = 0, 1, 2, 3, \ldots$. Denote the triggering sampling instants of the j^{th} node by $\left\{k_l^j\right\}_{l=1}^{\infty}$. For $i > k_l^j$, our proposed summation-based triggering condition (SBTC) for the j^{th} node is defined as follows:

$$f_j(x_j(i), x_j(k_l^j)) := \sum_{s=k_l^j}^{i} (x_j(k_l^j) - x_j(s))^T \beta_j (x_j(k_l^j) - x_j(s)) - \sum_{s=k_l^j}^{i} x_j(s)^T x_j(s) < 0. \quad (7.6)$$

In the other words, the j^{th} node transmits a new pack of data to the controller whenever inequality (7.6) is violated. Note that to simplify our notation we wrote $x(i)$ instead of $x(iT_s)$ in (7.6). The constants $\beta_j > 0$ (for $j = 1, 2, \ldots, N$) are design parameters to be designed.

Considering $\hat{x}_j(t)$ as the piecewise constant signal generated by the output of node j^{th} we have:

$$\hat{x}_j(t) = x_j(k_l^j) \quad \text{for} \quad t \in [k_l^j T_s, k_{l+1}^j T_s). \quad (7.7)$$

Then, the SBTCs (7.6) can be rewritten in the following form:

$$f_j(x_j(i), \hat{x}_j) := \sum_{s=\alpha_j}^{i} (\hat{x}_j - x_j(s))^T \beta_j (\hat{x}_j - x_j(s)) - \sum_{s=\alpha_j}^{i} x_j(s)^T x_j(s) < 0 \quad \text{for } j = 1, \ldots, N, \quad (7.8)$$

where α_j ($j \in \mathbb{N}$) is defined as the last triggering sampling instant before i:

$$\alpha_j := \max\left\{k_l^j : k_l^j \leq i\right\}. \quad (7.9)$$

Based on the definition of the TC, it can be easily inferred that the following inequalities hold at every sampling time:

$$\sum_{s=\alpha_j}^{i} (\hat{x}_j - x_j(s))^T \beta_j (\hat{x}_j - x_j(s)) - \sum_{s=\alpha_j}^{i} x_j(s)^T x_j(s) < 0 \quad \text{for } j = 1, 2, \ldots, N. \quad (7.10)$$

Remark 7.1. *Note that, if one uses the traditional TC form, proposed in [1] (which has been exploited in different papers), then the condition corresponding to the proposed SBTC would be written as :*

$$(\hat{x}_j - x_j(i))^T \beta_j (\hat{x}_j - x_j(i)) - x_j(i)^T x_j(i) < 0. \quad (7.11)$$

Later on in section 7.3 we show that the SBTC proposed here is less conservative than the above traditional TC.

In the next section, the proposed event-based system is modeled in a continuous-time form and the stability analysis is carried out based on Lyapunov-Krasovskii functional.

7.3 Closed-Loop System Modelling and Stability Analysis

Using the event-based controller (7.4), the closed-loop dynamics is given by:

$$\dot{x} = Ax + BK\hat{x}(t) + \varphi(x, K\hat{x}(t)). \tag{7.12}$$

Denoting κ_t the last sampling instant before the time instant 't', *i.e.*:

$$\kappa_t := \max\{i : iT_s \leq t\}, \tag{7.13}$$

and adding and subtracting $BKx(\kappa_t)$ in the right hand side of equation (7.12) we have:

$$\dot{x} = Ax + BK(\hat{x}(t) - x(\kappa_t)) + BKx(\kappa_t) + \varphi(x, K\hat{x}(t)). \tag{7.14}$$

Defining the time-varying delay $\tau(t)$ as:

$$\tau(t) := t - \kappa_t T_s, \quad 0 \leq \tau(t) \leq T_s, \tag{7.15}$$

the dynamics (7.14) can be rewritten as:

$$\dot{x} = Ax + BKe(t) + BKx(t - \tau(t)) + \varphi(x, K\hat{x}(t)), \tag{7.16}$$

where:

$$e = [e_1^T, e_2^T, \ldots, e_N^T]^T, \tag{7.17}$$

with

$$e_j(t) = \hat{x}_j(t) - x_j(t - \tau(t)) \quad \text{for } j = 1, \ldots, N. \tag{7.18}$$

In the following theorem, the conditions in the form of LMIs are given, under which the asymptotic stability of the proposed system is guaranteed

Theorem 7.1. *Consider the nonlinear system (7.1), controlled by the centralized feedback law (7.4), implemented using the decentralized SBTCs (7.8). The closed-loop system is* *asymptotically stable if there exists matrices $P > 0$, $Q > 0$, $R > 0$, $M_1 > 0$, $M_2 > 0$, U_1 and U_2 of proper dimensions such that the following LMIs hold:*

$$\Gamma_i = \begin{bmatrix} \digamma_0 + S^T + S & \star & \star & \star & \star \\ \digamma_i^U & -R & \star & \star & \star \\ \digamma_3 & 0 & -M_1 & \star & \star \\ \digamma_4 & 0 & 0 & -M_2 & \star \\ \digamma_5 & 0 & 0 & 0 & -R \end{bmatrix} < 0 \quad \textit{for} \quad i = 1, 2, \tag{7.19}$$

where:

$$S = \begin{bmatrix} U_2 & U_1 - U_2 & -U_1 & 0 \end{bmatrix}, \tag{7.20}$$

$$\digamma_0 = \begin{bmatrix} \digamma_{01} & PBK & 0 & PBK \\ \star & I & 0 & 0 \\ \star & \star & -Q & 0 \\ \star & \star & \star & -\beta \end{bmatrix}, \tag{7.21}$$

$$\digamma_{01} = A^T P + PA + Q + \ell^2(M_1 + T_s M_2 + T_s R), \tag{7.22}$$

$$\beta = diag(\beta_1, \beta_2, \ldots, \beta_N), \tag{7.23}$$

$$\digamma_1^U = \sqrt{T_s} U_1, \quad \digamma_2^U = \sqrt{T_s} U_2, \tag{7.24}$$

$$\digamma_3 = P, \quad \digamma_4 = \digamma_5 = \sqrt{T_s} R\Lambda, \tag{7.25}$$

$$\Lambda = \begin{bmatrix} A & BK & 0 & BK \end{bmatrix}. \tag{7.26}$$

Proof. Consider the following Lypunov-Krasovskii functional:

$$V = \sum_{i=1}^{3} V_i, \tag{7.27}$$

$$\begin{aligned} V_1 &= x(t)^T P x(t), \\ V_2 &= \textstyle\int_{t-T_s}^{t} x^T(s) Q x(s) ds, \\ V_3 &= \textstyle\int_0^{T_s} \int_{t-\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta. \end{aligned} \tag{7.28}$$

Defining

$$\zeta(t) \triangleq \begin{bmatrix} x(t)^T & x(t-\tau(t))^T & x(t-T_s)^T & e^T \end{bmatrix}^T, \tag{7.29}$$

the time derivative of V_i along the trajectories of x is derived as:

$$\begin{aligned}
\dot{V}_1 &= \zeta^T \Lambda^T P x + x^T P \Lambda \zeta + 2\varphi^T P x \\
\dot{V}_2 &= x(t)^T Q x(t) - x^T(t-T_s) Q x(t-T_s) \\
\dot{V}_3 &= T_s \zeta^T \Lambda^T R \Lambda \zeta + T_s \varphi^T R \varphi + 2T_s \varphi^T R \Lambda \zeta - \textstyle\int_0^{T_s} \dot{x}(t-\theta)^T R \dot{x}(t-\theta) d\theta,
\end{aligned} \tag{7.30}$$

where $\varphi(x,u)$ is abbreviated by φ. Using Leibniz-Newton formula, for any U_1 and U_2 of proper dimensions we have:

$$\zeta^T U_1 (x(t-\tau(t)) - x(t-T_s) - \textstyle\int_{t-T_s}^{t-\tau(t)} \dot{x}(s) ds) = 0, \tag{7.31}$$

$$\zeta^T U_2 (x(t) - x(t-\tau(t)) - \textstyle\int_{t-\tau(t)}^{t} \dot{x}(s) ds) = 0. \tag{7.32}$$

Using above equations and exploiting Lemma 2.1 we have:

$$\begin{aligned}
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 &\leq \zeta^T \Lambda^T P x + x^T P \Lambda \zeta + \varphi^T M_1 \varphi + x^T P M_1^{-1} P x + x(t)^T Q x(t) \\
&- x^T(t-T_s) Q x(t-T_s) + T_s \varphi^T R \varphi + (T_s - \tau(t)) \zeta^T U_1 R^{-1} U_1^T \zeta + \tau(t) \zeta^T U_2 R^{-1} U_2^T \zeta \\
&+ T_s \zeta^T \Lambda^T R M_2^{-1} R \Lambda \zeta + T_s \zeta^T \Lambda^T R \Lambda \zeta + T_s \varphi^T M_2 \varphi + \zeta^T (S^T + S) \zeta,
\end{aligned} \tag{7.33}$$

where M_1 and M_2 are some positive definite matrices.

Using Lipschitz property of the function φ and adding and subtracting the terms $e(t)^T \beta e(t)$ and $x(t-\tau(t))^T x(t-\tau(t))$ on the right hand side of above equation, it is concluded that:

$$\begin{aligned}
\dot{V} \leq\ & \zeta^T (\digamma_0 + S^T + S + \bar{U} + \digamma_3 M_1^{-1} \digamma_3^T + \digamma_4^T M_2^{-1} \digamma_4 + \digamma_5^T R^{-1} \digamma_5) \zeta \\
&+ e(t)^T \beta e(t) - x(t-\tau(t))^T x(t-\tau(t)),
\end{aligned} \tag{7.34}$$

where $\bar{U}$ is a convex combination of two terms $\digamma_1^{U^T} R^{-1} \digamma_1^U$ and $\digamma_2^{U^T} R^{-1} \digamma_2^U$:

$$\bar{U} = \frac{(T_s - \tau(t))}{T_s} \digamma_1^{U^T} R^{-1} \digamma_1^U + \frac{\tau(t)}{T_s} \digamma_2^{U^T} R^{-1} \digamma_2^U. \tag{7.35}$$

So, using Schur complement, the first term on the right hand side of inequality (7.34) is negative definite if the LMIs (7.19) hold.

Define:

$$-\bar{\lambda} := \lambda_{max}(\digamma_0 + S^T + S + \bar{U} + \digamma_3 M_1^{-1} \digamma_3^T + \digamma_4^T M_2^{-1} \digamma_4 + \digamma_5^T R^{-1} \digamma_5). \tag{7.36}$$

Then, from (7.34), it is easy to see that:

$$\dot{V} \leq -\bar{\lambda} \zeta^T \zeta + \sum_{j=1}^{N} e_j^T \beta_j e_j - \sum_{j=1}^{N} x_j(t-\tau(t))^T x_j(t-\tau(t)), \tag{7.37}$$

where e_j and x_j are the values correspond to the j^{th} node. Integrating both sides of equation (7.37), we get:

$$V(t) \leq V(0) - \bar{\lambda}\int_0^t \zeta(s)^T\zeta(s)ds + \sum_{j=1}^{N}\int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds. \tag{7.38}$$

Consider the integral terms on the right hand side. It is broken down at the triggering instants for each node:

$$\begin{array}{c} \int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds = \\ \sum\limits_{l=1}^{m_j-1} \int_{k_{l-1}^j T_s}^{k_l^j T_s} (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds \\ + \int_{k_{m_j}^j T_s}^{t} (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds, \end{array} \tag{7.39}$$

where m_j is the last triggering sampling number of j^{th} node before the time instant t.

Because of the identity of the delay given by equation (7.15) and also the definition of error provided in equation (7.18), it can be inferred that $e_j(t)$ and $x_j(t-\tau(t))$ are constant over each sampling interval. So, the above equation is rewritten as:

$$\begin{array}{c} \int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds = \\ T_s \sum\limits_{l=1}^{m_j-1} \sum\limits_{i=k_{l-1}^j}^{k_l^j} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)) \\ +T_s \sum\limits_{i=k_{m_j}^j}^{l_t-1} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)) + (t - l_t T_s)(e_j(l_t)^T\beta_j e_j(l_t) - x_j(l_t)^T x_j(l_t)), \end{array} \tag{7.40}$$

where l_t is the last sampling time before the time instant 't'.

Now consider the last term on the right hand side. Since $0 < t - l_t T_s \leq T_s$,

- if $e_j(l_t)^T\beta_j e_j(l_t) - x_j(l_t)^T x_j(l_t) \geq 0$, we have:

$$\begin{array}{c} \int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds \leq \\ T_s \sum\limits_{l=1}^{m_j-1} \sum\limits_{i=k_{l-1}^j}^{k_l^j} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)) \\ +T_s \sum\limits_{i=k_{m_j}^j}^{l_t} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)). \end{array} \tag{7.41}$$

- if $e_j(l_t)^T\beta_j e_j(l_t) - x_j(l_t)^T x_j(l_t) < 0$, then:

$$\begin{aligned}
&\textstyle\int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds \leq \\
&\quad T_s \sum_{l=1}^{m_j-1} \sum_{i=k_{l-1}^j}^{k_l^j} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)) \\
&\quad + T_s \sum_{i=k_{m_j}^j}^{l_t-1} (e_j(i)^T\beta_j e_j(i) - x_j(i)^T x_j(i)).
\end{aligned} \tag{7.42}$$

Based on the inequalities (7.10) which are guaranteed by the proposed SBTC, in both cases above, the right hand side terms are negative and consequently:

$$\textstyle\int_0^t (e_j(s)^T\beta_j e_j(s) - x_j(s-\tau(s))^T x_j(s-\tau(s)))ds \leq 0 \quad \text{for } j = 1, \ldots, N. \tag{7.43}$$

Now, substituting above inequality in (7.38), we have:

$$V(t) \leq V(0) - \bar{\lambda} \int_0^t \zeta(s)^T \zeta(s) ds. \tag{7.44}$$

To prove the stability of the system, suppose that ε is given. With respect to the condition (7.2), above proofs are valid as long as the state trajectory remain inside the region Ω so that the Lipschitz condition is satisfied. Hence, to prove the stability of the system, $\delta(\varepsilon)$ should be found such that for initial functions satisfying $\sup\limits_{s\in[-T_s,0)} |x(s)| < \delta$, not only the state vector lies in the region $|x(t)| < \varepsilon$, but also $x(t)$ remains inside Ω for all time instants. In this regard, define $B_r := \{x : |x| \leq r\}$, $r_M = \max\{r : B_r \in \Omega\}$ and let:

$$\varepsilon_{min} = \min\{\varepsilon, r_M\}. \tag{7.45}$$

The Lyapunov function (7.27) is lower bounded as:

$$V(x(t)) \geq \lambda_{min}(P)|x(t)|^2, \tag{7.46}$$

and upper bounded as:

$$V(x(t)) \leq \lambda_{max}(P)|x(t)|^2 + T_s\lambda_{max}(Q) \sup_{s\in[-T_s,0)} |x(t+s)|^2 + \frac{{T_s}^2}{2}\lambda_{max}(R) \sup_{s\in[-T_s,0)} |\dot{x}(t+s)|^2. \tag{7.47}$$

So

$$\begin{aligned}
&\lambda_{min}(P)|x(t)|^2 \leq V(x(t)), \\
&\leq (\lambda_{max}(P) + T_s\lambda_{max}(Q)) \sup_{s\in[-T_s,0)} |x(t+s)|^2 + \tfrac{{T_s}^2}{2}\lambda_{max}(R) \sup_{s\in[-T_s,0)} |\dot{x}(t+s)|^2.
\end{aligned} \tag{7.48}$$

From (7.44) and using (7.48), the following inequality holds:

$$a|x(t)|^2 \leq V(t) \leq V(0) \leq b \sup_{s\in[-T_s,0)} |x(s)|^2, \tag{7.49}$$

where

$$a = \lambda_{min}(P), \quad b = \lambda_{max}(P) + T_s\lambda_{max}(Q) + (2L_{sys} + \sqrt{1/\lambda_{min}(\beta)}L_{sys} + \ell)^2, \tag{7.50}$$

with $L_{sys} = \|[A|BK]\|$.

So, if δ is defined such that $\delta < \sqrt{\frac{a}{b}}\varepsilon_{min}$, then for the initial functions satisfying:

$$\sup_{s\in[-T_s,0)} |x(s)| \leq \delta \tag{7.51}$$

we have:

$$x(t) \in B_\varepsilon \qquad \forall\ t \geq 0, \tag{7.52}$$

and so the system is stable. In addition, the convergence of states is proved based on equation (7.38) and using Barballat's lemma [62]. So the system is asymptotically stable and this concludes the proof.

□

Remark 7.2. *Suppose that instead of the SBTCs (7.8) the event-based system is implemented by the traditional TCs (7.11). In this case, from equation (7.37), we would have:*

$$\dot{V} \leq -\bar{\lambda}\zeta^T\zeta \tag{7.53}$$

the derivative of the Lyapunov function is negative and so the system is asymptotically stable. However, if the SBTCs are in use, based on inequality (7.37), the derivative of the Lyapunov function is not required to be negative at all time instants, while proving asymptotic stability of the nonlinear system. So, our proposed system is potentially less conservative than its corresponding traditional one and triggers less data transmission over the communication channel.

7.4 Event Generator Design and Maximization of the Stability Region

In this section, the SBTC parameters β_j (for $j = 1, \ldots, N$) and the Lipschitz constant ℓ are assumed to be design variables. Then, an optimization-based design tool is provided such that:

1. ℓ is enlarged to expand the operation region of the closed-loop system and
2. β_j's are minimized to reduce the data transmission from sensors to the controller as much as possible.

Theorem 7.2 contains the main results of this section where a multi-objective problem is provided to achieve the mentioned criteria, while the asymptotic stability of the event-based system is ensured.

Theorem 7.2. *Consider the nonlinear system (7.1), controlled by the centralized event-based feedback law (7.4), implemented the decentralized SBTCs (7.8). If β_j's (for $j = 1, \ldots, N$) are obtained through the following optimization problem for some $0 < \gamma < 1$, then, the closed-loop system is asymptotically stable and the maximum admissible Lipschitz constant is obtained as $\ell^\star = \sqrt{\mu(\omega_1 + T_s\omega_2)}^{-1}$.*

$$min \quad \gamma c_l + (1-\gamma)c_\beta \tag{7.54}$$

$$s.t.$$

$$\bar{\Gamma}_i = \begin{bmatrix} \bar{\mathcal{F}}_0 + S^T + S & \star & \star & \star & \star \\ \mathcal{F}_i^U & -R & \star & \star & \star \\ \mathcal{F}_3 & 0 & -\omega_1 I & \star & \star \\ \mathcal{F}_4 & 0 & 0 & -\mathcal{F}_d & \star \\ J_1 & 0 & 0 & 0 & -\mu \end{bmatrix} < 0 \quad for \quad i = 1, 2, \tag{7.55}$$

$$c_l \geq \mu + \omega_1 + T_s\omega_2, \tag{7.56}$$

$$c_\beta I \geq \beta. \tag{7.57}$$

In the above formulation, matrices $P \succ 0$, $Q \succ 0$, $R \succ 0$, β_j ($j = 1, \ldots, N$), U_1, U_2 and the scalars ω_1, ω_2, c_l, c_β and μ are the optimization variables, and

$$\bar{\mathcal{F}}_0 = \begin{bmatrix} A^T P + PA + Q & PBK & 0 & PBK \\ \star & I & 0 & 0 \\ \star & \star & -Q & 0 \\ \star & \star & \star & -\beta \end{bmatrix}, \tag{7.58}$$

$$\digamma_d = R - \omega_2^{-1} R^2, \tag{7.59}$$

$$\beta = diag(\beta_1, \beta_2, \ldots, \beta_N), \tag{7.60}$$

and S, $\digamma_1^U$, $\digamma_2^U$, $\digamma_3$ *and* $\digamma_4$ *are defined as (7.20), (7.24), (7.25), respectively. Moreover,* J_1 *is a transform matrix which satisfies* $x = J_1\zeta$ *with* ζ *defined as (7.29).*

Proof. Consider the same Lyapunov function (7.27), and so the inequality (7.33) is held for $\dot{V}$. Exploiting Lipschitz property of φ, from (7.33) we have:

$$\begin{aligned}\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 &\leq \zeta^T \Lambda^T P x + x^T P \Lambda \zeta + \ell^2 x^T M_1 x + x^T P M_1^{-1} P x + x(t)^T Q x(t) \\ &-x^T(t - T_s) Q x(t - T_s) + T_s \ell^2 x^T R x + (T_s - \tau(t)) \zeta^T U_1 R^{-1} U_1^T \zeta + \tau(t) \zeta^T U_2 R^{-1} U_2^T \zeta \\ &+T_s \zeta^T \Lambda^T R M_2^{-1} R \Lambda \zeta + T_s \zeta^T \Lambda^T R \Lambda \zeta + T_s \ell^2 x^T M_2 x + \zeta^T (S + S^T) \zeta.\end{aligned} \tag{7.61}$$

Define $M_1 = \omega_1 I$, $M_2 = \omega_2 I - R$ and

$$\mu^{-1} := \ell^2(\omega_1 + T_s \omega_2), \tag{7.62}$$

and add and subtract the terms $e(t)^T \beta e(t)$ and $x(t - \tau(t))^T x(t - \tau(t))$ to the right hand side of the equation (7.61). Then:

$$\begin{aligned}\dot{V} \leq &\tfrac{T_s - \tau(t)}{T_s} \zeta^T (\bar{\digamma}_0 + S^T + S + \digamma_1^U R^{-1} \digamma_1^{U^T} + \digamma_3 \omega_1^{-1} \digamma_3^T + \digamma_4^T (M_2^{-1} + R^{-1}) \digamma_4 + \mu^{-1} J_1^T J_1) \zeta \\ &\tfrac{\tau(t)}{T_s} \zeta^T (\bar{\digamma}_0 + S^T + S + \digamma_2^U R^{-1} \digamma_2^{U^T} + \digamma_3 \omega_1^{-1} \digamma_3^T + \digamma_4^T (M_2^{-1} + R^{-1}) \digamma_4 + \mu^{-1} J_1^T J_1) \zeta \\ &+e(t)^T \beta e(t) - x(t - \tau(t))^T x(t - \tau(t)).\end{aligned} \tag{7.63}$$

Note that:

$$\begin{aligned}&R^{-1} + M_2^{-1} = R^{-1} + (\omega_2 I - R)^{-1} = \\ &(\omega_2 I - R)^{-1}((\omega_2 I - R) R^{-1} + I) = (R - \omega_2^{-1} R^2)^{-1} = \digamma_d^{-1}.\end{aligned} \tag{7.64}$$

Using Schur complement, the negativeness of the first two terms on the right hand side of (7.63) is equivalent to the LMIs (7.55). The rest of stability proof is the quite similar to the one, given in the proof of Theorem 7.1 and is thus omitted.

□

Remark 7.3. *As mentioned earlier, Theorem 7.2 tries to enlarge Lipschitz constant and minimize β_j's simultaneously. In this regard, in order to enlarge the Lipschitz constant ℓ, one can minimize the auxiliary variable c_l while satisfying the inequality (7.56). In addition, to minimize β_j(for $j = 1, \ldots, N$), the auxiliary variable c_β is equivalently minimized while satisfying the inequality (7.57). Because of having two optimization criteria the objective function is defined as (7.54), in which γ is a weighting parameter, varying between* 0 *and* 1.

Remark 7.4. *In the above-explained event-based system, if $T_s = 0$ and β_j's tend to infinity, we would deal with a traditional continuous-time control system which, based on what assumed in the Problem Statement section, has been already stabilized by the feedback gain K. Therefore, it is expected that, at least, for small values of T_s the provided LMIs in the previous parts are feasible and the optimization problems have solution.*

7.5 Simulation Results

In this section, the event-based control of previous sections is evaluated through simulations. Since the proposed scheme is valid for linear systems (as a special case of Lipschitz nonlinear systems), we present two examples consisting of (i) a linear system, and (ii) a nonlinear system, respectively.

7.5.1 Linear Case

Consider the following unstable reactor dynamics [68]:

$$\dot{x} = Ax + Bu, \tag{7.65}$$

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix},$$

which is stabilized by the following control gain:

$$K = \begin{bmatrix} 0.1906 & -0.0440 & 0.0411 & -0.2474 \\ 1.0338 & 0.4534 & -0.3662 & 0.7489 \end{bmatrix}. \tag{7.66}$$

Two different event-triggered control schemes are implemented and compared for the system. The first is a *centralized scheme*, where we assume that information from all the sensors is sent to a central event generator module at every sampling time, and a single TC is verified to decide whether a new pack of data is to be sent to the controller or not. The second is a *decentralized scheme*, in which an independent TC is considered for each sensor. In other words, each sensor relies only on its own data to decide whether to sent a fresh data to the controller or not.

Centralized Scheme

Since the number of nodes is 1, only one event generator module is needed which, based on the proposed formulation in (7.6), is defined as:

$$f(x(i),\hat{x}) := \sum_{s=\alpha}^{i}(\hat{x}-x(s))^T\beta(\hat{x}-x(s)) - \sum_{s=\alpha}^{i}x(s)^T x(s) < 0. \tag{7.67}$$

We assume that the TC is verified every $T_s = 0.001$ *sec.* Since there is no nonlinear term in the plant dynamics, the parameter γ is set to 0 and the optimization problem (7.54) is solved, where the parameter β is obtained as follows:

$$\beta = \begin{bmatrix} 20.7784 & 0.9869 & -0.7512 & 0.1979 \\ 0.9869 & 16.7562 & -0.5021 & 1.2560 \\ -0.7512 & -0.5021 & 16.5575 & -1.0660 \\ 0.1979 & 1.2560 & -1.0660 & 19.8208 \end{bmatrix}. \tag{7.68}$$

To have a fair evaluation of the system, ten separate simulations were done using different initial conditions varying in the region $|x(0)| \leq 7$. Based on the obtained results, The average of inter-event times for the proposed TC is 151 ms over 5 s. To show the efficiency of the proposed method, the same set of simulations were implemented using two other traditional event-based methods. In the first one, the TC (7.67) is replaced with the corresponding periodic traditional condition used in previous references such as [45]; namely:

$$h(x(i),\hat{x}) := (\hat{x}-x(i))^T\beta(\hat{x}-x(i)) - x(i)^T x(i) < 0. \tag{7.69}$$

Second, the (continuously verified) TC proposed by Tabuada [1] is applied:

$$g(x(t),\hat{x}) := |\hat{x}-x(t)| - \sigma|x(t)| < 0, \tag{7.70}$$

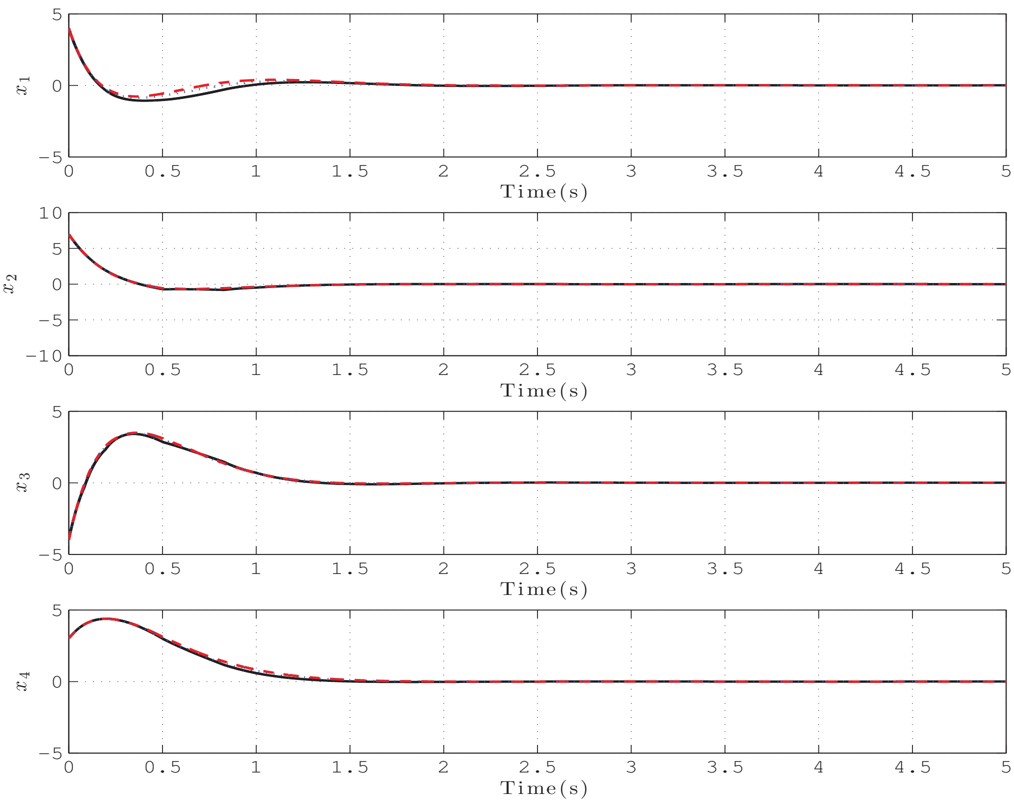


Figure 7.1: State responses of the reactor, controlled in a centralized way, by means of the proposed (solid-black) and the traditional periodic (dashed-red) and continuous (dotted-blue) event-based scheme

where, based on the formulation given in this reference, σ is chosen as $\sigma = 0.57$.

According to the simulation results, using identical conditions as in the previous case, the average inter-event times for the traditional periodic and continuous TCs are 73.5 and 96.2 ms, respectively. These values are remarkably smaller than the value provided using the SBTC (7.67), due to conservatism existing in the traditional approaches. In addition, the state responses for all three systems are shown in Fig. 7.1, where it is clear this data exchange reduction, achieved by the proposed SBTC, does not result in any significant change in system performance, and each state trajectory converges to the origin with similar performance to its corresponding trajectory of the traditional systems.

Decentralized Scheme

In this scheme, a separate, independent TC is considered for each sensor (*i.e.* $N = 4$), and the TCs are defined as:

$$f_j(x_j(i), \hat{x}_j) := \sum_{s=\alpha_j}^{i} (\hat{x}_j - x_j(s))^T \beta_j (\hat{x}_j - x_j(s)) - \sum_{s=\alpha_j}^{i} x_j(s)^T x_j(s) < 0 \quad \text{for } j = 1, \ldots, 4, \tag{7.71}$$

where x_1, x_2, x_3 and x_4 are the states of the system. Similar to the centralized case, the parameters are set as $T_s = 0.001$ and $\gamma = 0$ and the optimization problem (7.54) is solved, resulting in the following TC parameters:

$$\beta_1 = 21.5433, \quad \beta_2 = 21.5414, \quad \beta_3 = 21.5401, \quad \beta_4 = 21.5427. \tag{7.72}$$

The initial conditions are chosen the same as the previous part and a set of ten separate simulations were run for 5 seconds for both our proposed system and also system with the following traditional periodic TCs:

$$h_j(x_j(i), \hat{x}_j) := (\hat{x}_j - x_j(i))^T \beta_j (\hat{x}_j - x_j(i)) - x_j(i)^T x_j(i) < 0 \quad \text{for } j = 1, \ldots, 4. \tag{7.73}$$

The average data transmission for these methods are given in Table 7.1. To have a fair comparison, simulation results of the reference [19] are also represented in the table. As mentioned in the introduction section, in this reference the authors have designed decentralized (continuously verified) TCs and simulated it for the same reactor model.

Based on the numbers in the table, the average of inter-event times for the sensors are $[\tau_1, \tau_2, \tau_3, \tau_4] = [42, 67.6, 52.1, 49.5]$ ms. However, the corresponding values provided by the traditional periodic and continuous mechanisms are $[30.7, 38.5, 32.1, 38.2]$ ms and $[24.9, 27.7, 34.5, 34.2]$ ms, respectively. Fig. 7.2, representing the state trajectories for the initial condition $x(0) = [4, 7, -4, 3]$, shows very similar performances in the state responses of all three mechanisms, despite the larger inter-transmission times for the sensors, obtained by our proposed SBTCs.

Comparing the results of the proposed centralized and decentralized approaches, it is inferred that applying centralized approach the data transmitted from the event generator module to the controller is much less than total data transmitted from the event generators

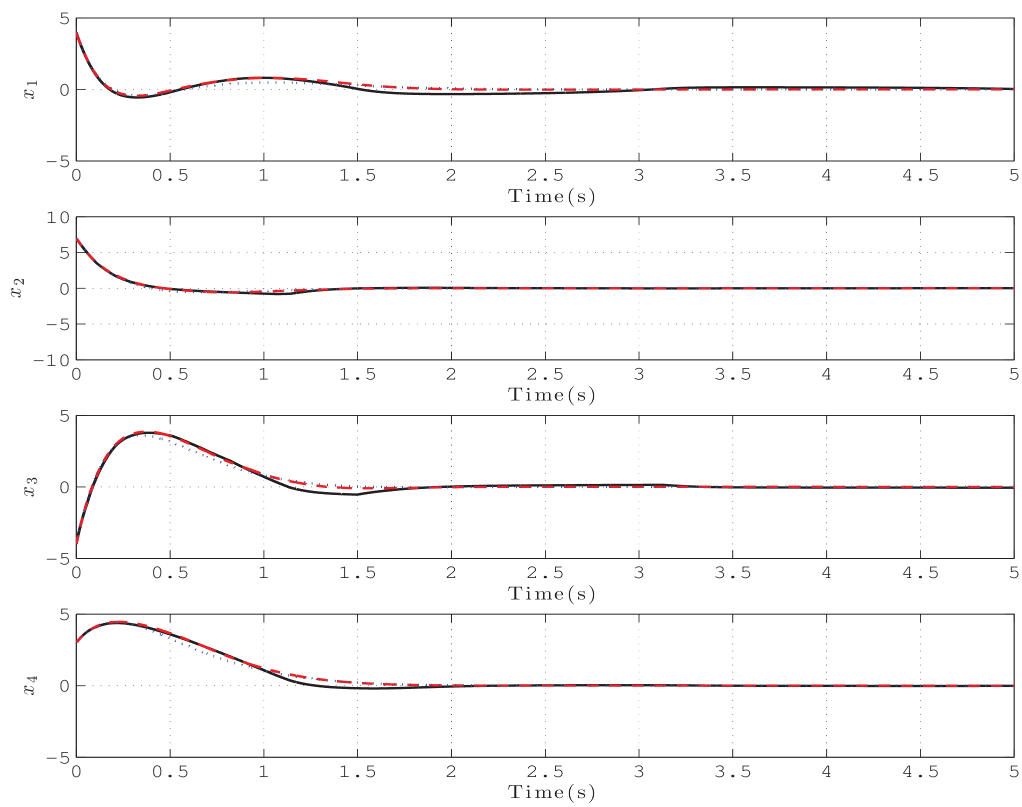


Figure 7.2: State responses of the reactor, controlled in a decentralized way, by means of the proposed (solid-black) and the traditional periodic (dashed-red) and continuous (dotted-blue) event-based scheme

Table 7.1: Number of data exchanged between the event generator modules and the controller, while using different event-based mechanisms.

	Centralized	Decentralized				
		1^{st}	2^{nd}	3^{rd}	4^{th}	sensor
Proposed mechanism	33	119	74	96	101	
Traditional periodic mechanism	68	163	130	156	131	
Traditional continuous mechanism	52	205	180	144	146	

in the decentralized scheme. However, if the sensors are geographically distributed and a single centralized TC is in use, the data of all sensors should be sent to the event generator module at every sampling time to verify the condition, something which can result in a high level of usage of the communication bandwidth. Clearly, whenever sensors are geographically distributed, the decentralized scheme is much more efficient thanks to the ability of each sensor to communicate independently with the control unit.

7.5.2 Nonlinear Case

Consider the following inverted pendulum system dynamic:

$$mr^2\ddot{\theta} - mgr\sin(\theta) = u(t), \tag{7.74}$$

where θ represents pendulum angular displacement from vertical line, $u(t)$ is the control torque, m and r denote the mass and the length of the pendulum, respectively. Defining $x = [x_1, x_2]^T = [\theta, \dot{\theta}]^T$ and setting $m = 1, r = 2, g = 9.8$, the dynamic equations can be written as follows:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 4.9 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4.9(sinx_1 - x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.25 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned} \tag{7.75}$$

Assume that the origin is stabilized using the following state feedback controller:

$$u = \begin{bmatrix} -55.6 & -24 \end{bmatrix} x. \tag{7.76}$$

Similar to the previous example, we now proceed to design centralized and decentralized event-based controllers for this system.

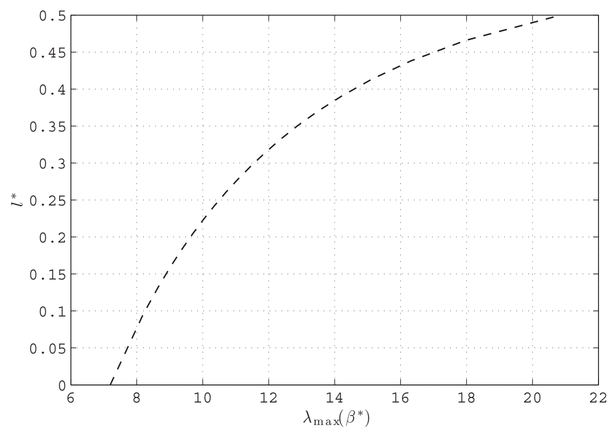


Figure 7.3: Trade-Off curve between ℓ^* and $\lambda_{\max}(\beta^*)$, as the solutions of the optmization problem (7.54) for different values of γ.

Centralized Scheme

We begin considering a single TC of the form (7.67). To design an optimal event-based mechanism, the TC sampling period is set to 0.001 and the optimization problem (7.54) is solved for different values of $0 \leq \gamma \leq 1$. Fig. 7.3 represents the trade-off curve between $\lambda_{max}(\beta^*)$ and ℓ^* where it is clear that the more data is exchanged between plant and controller, the larger the region of stability that we obtain.

Consider the case $\gamma = 0.1$. The following values are derived by solving the optimization problem:

$$\beta = \begin{bmatrix} 15.96 & 4.64 \\ 4.64 & 7.22 \end{bmatrix}, \ell = 0.466. \tag{7.77}$$

Note that with this value of ℓ the theorems are valid as long as $|x_1| = |\theta| < 0.44$. To compare our results with the traditional methods used in the literature (*i.e.* using the TC (7.69)), simulations are carried out for both schemes with the same value of β given in (7.77), and initial condition $x = [0.3, 0.5]$. The state responses are given in Fig. 7.4, where it is clear that both system are asymptotically stabilized. However, the number of data transmission points over 3 seconds in our proposed scheme versus the traditional one are 13 and 24, respectively. In the other words, 46% data transmission reduction is achieved by our method.

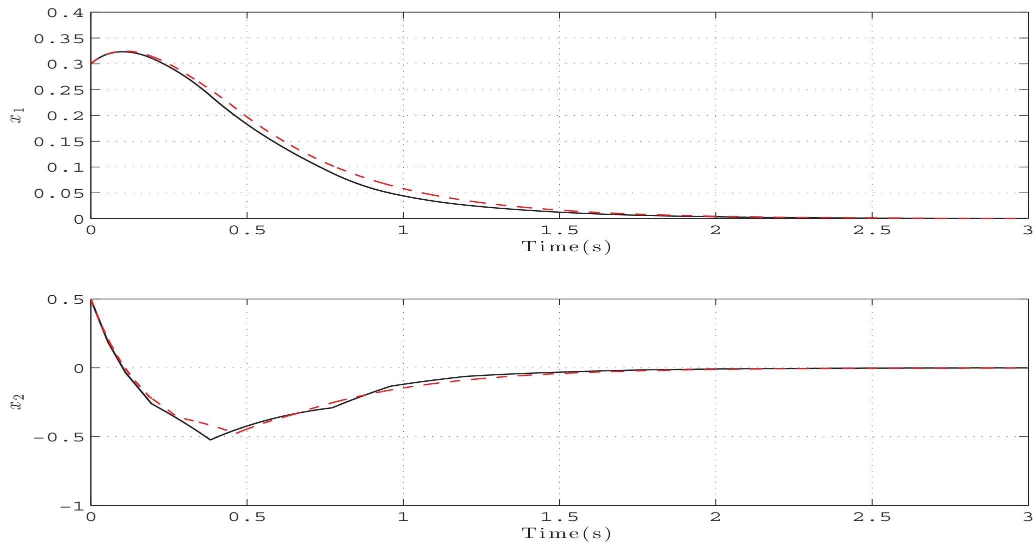


Figure 7.4: State responses of the pendulum, controlled in a centralized way, by means of the proposed (solid-black) and the traditional (dashed-red) event-based scheme

Decentralized Scheme

In this part, separate SBTCs are considered for each of the system states:

$$f_j(x_j(i), \hat{x}_j) := \sum_{s=\alpha_j}^{i} (\hat{x}_j - x_j(s))^T \beta_j (\hat{x}_j - x_j(s)) - \sum_{s=\alpha_j}^{i} x_j(s)^T x_j(s) < 0 \quad \text{for } j = 1, 2. \tag{7.78}$$

Setting $T_s = 0.001$ and having solved the optimization problem (7.54), we obtain a trade-off curve between $\max\{\beta_1^*, \beta_2^*\}$ and ℓ^* curve very similar to Fig. 7.3 and therefore the details are omitted. To simulate the closed-loop, we consider the solution of the optimization problem for the case $\gamma = 0.05$, which are $\beta 1 = \beta_2 = 20.64$ and $l = 0.497$. Again, the event-based system is implemented using both SBTC's (7.78) and also the traditional TC's:

$$h_j(x_j(i), \hat{x}_j) := (\hat{x}_j - x_j(i))^T \beta_j (\hat{x}_j - x_j(i)) - x_j(i)^T x_j(i) < 0 \quad \text{for } j = 1, 2. \tag{7.79}$$

The state responses are given in Fig. 7.5

According to the simulation results, the numbers of transmitted data points by the event-triggering modules in the traditional scheme are 33 and 67. However, the corresponding values in our proposed scheme are 18 and 42, something which confirm its

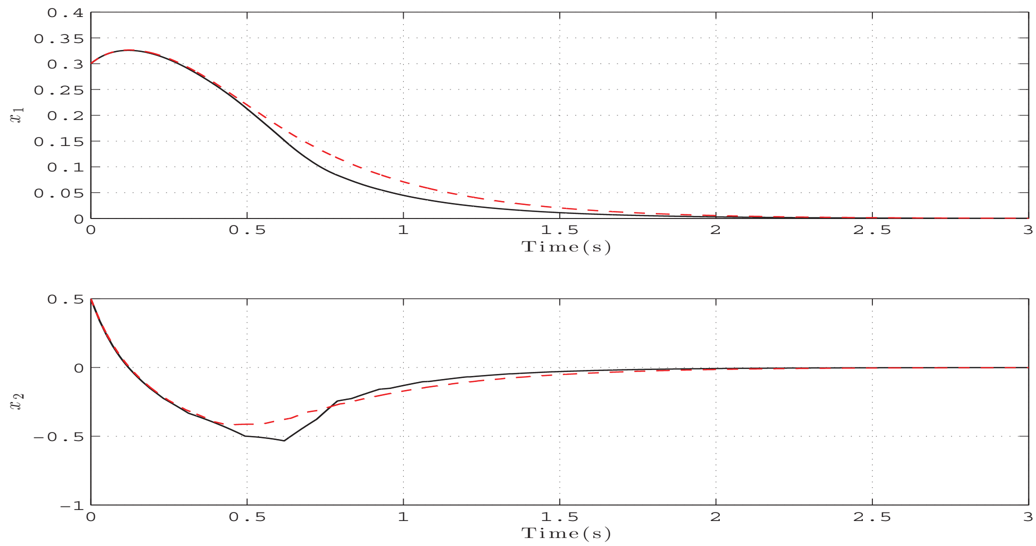


Figure 7.5: State responses of the pendulum, controlled in a decentralized way, by means of the proposed (solid-black) and the traditional (dashed-red) event-based scheme

effectiveness of our methods through 40% reduction in overall data transmission.

7.6 Summary

Decentralized event-triggering mechanism was designed for a class of Lipschitz nonlinear systems, where sensors are assumed to be distributed. The proposed triggering conditions are verified on a periodic basis and rely only on local information to transmit data to the control block. They are also less conservative than the corresponding traditional conditions and can effectively reduce data transmission through the communication channel. The closed-loop system was modeled in a continuous time-delay form and asymptotic stability was proved. In addition, an optimization-based approach was formulated for TCs parameters design. Finally, simulation results for both linear and nonlinear cases were given to illustrate the efficiency of the method.

Chapter 8

Conclusions and Future Works

In this thesis we consider event-based control systems under different scenarios. Event-triggered control is currently and active area of research and a significant body of work has been already produced. The primary goal of this thesis is to contribute to the advancement of the understanding of the virtues and shortcoming of the event-triggered systems in comparison to the better known traditional time-driven systems.

Chapter 2 investigates periodic event-triggered control (PETC) for a class of nonlinear systems. PETC is advantageous, at least from an implementation point of view and so, several contributions have been proposed on this topic in the last few years. However, most of the works are dedicated to LTI systems. In this chapter, PETC was designed for Lipschitz nonlinear systems, which are affected by disturbance. To this end, the system was first modeled in a continuous-time form. Then, a convex optimization problem with LMI constraints was provided to design state feedback gain and TC parameters. The obtained PETC parameters from the optimization problem guarantee stability and H_∞ performance of the closed-loop system. The design is effective in the sense that it lowers data transmission through the network channel and enlarges the admissible Lipschitz constant for the system nonlinear term.

Chapter 3, considers the important case of output feedback LTI systems. Because of the use of a dynamic controller, two separate triggering modules were used at the plant and controller sides. The proposed triggering conditions operate at different sampling rates and consequently trigger in an asynchronous fashion. Similar to the previous chapter, the system was modeled in continuous-time form, using delay-system approach. Then asymptotic stability and H_∞ performance were proved by Lyapunov-Krasovskii functionals and the conditions were provided in form of LMIs. In addition, a procedure was introduced to design the triggering mechanism in an effective way.

Chapter 4 considers the implementation of event-triggered systems in the presence if measurement noise. Basically, the effect of noise on the event-based systems have been widely ignored throughout the literature. As one of the main contributions of this chapter, an event-based mechanism was proposed which is effectively robust against measurement noise. In the other words, the provided system, not only reduces the impact of noise on the control law, but also it prevents excessive triggers, caused by the noise.

Chapter 5 studies event-based observer design for LTI systems. We assume that the system dynamics is affected by some exogenous disturbance and the output measurements contain stochastic noise. Then, using a similar idea to chapter 4, a noise-effective triggering-based observer is proposed. Asymptotic stability and H_∞ performance conditions of estimation error were formulated in form of LMIs and then a procedure was presented to design the observer parameters.

Chapter 6 goes back to the fundamental problem of designing stable event-triggered systems. We argue that most of the work reported so far in the literature, based on Lyapunov theory, is intrinsically conservative and therefore endeavor to reduce the conservatism associated with the existent ideas. Although our approach is also based on Lyapunov approach, we propose an integral-based triggering condition which does not require the Lyapunov function to be monotonous decreasing at all time instants. The result is a more efficient triggering mechanism which reduces data transmission compared to the existing results. A rigorous proof is given to show that the inter-event times in the proposed scheme is larger than in the corresponding scheme given in [1]. Simulation results show that this data transmission reduction does not significantly affect the performance of the states responses.

Finally, in chapter 7, we consider the important case of decentralized systems. We consider nonlinear systems in which sensors are geographically distributed; but their data are transmitted to a central controller. Since each sensor does not have access to other sensor's data, decentralized TCs are considered at each sensor, operating only based on local information. One of the main contributions of this chapter a novel form of TC which is less conservative than previously published results and is more efficient in terms of data transmission. This issue was illustrated through simulation.

The research results, provided in this thesis can be extended and pursued in the following areas:

- **Periodic-event triggered control for nonlinear systems**

  As mentioned earlier, PETC has received more attention in the last few years. In this research we designed this scheme for a class of Lipschitz nonlinear systems. However, there still exist different form of nonlinear systems which this idea can be studied for. Indeed, providing a general framework for PETC design for nonlinear system can be an interesting area of research.

- **Periodic-event triggered observer design for nonlinear systems**

  Going through literature, one can infer that Lyapunov-based event-driven state estimation has not received as much attention as its corresponding control problem. In particular, because of potential advantages of periodic event triggering schemes, they can be utilized and developed for state estimation of nonlinear systems.

- **Performance problem in integral-based event-triggered control systems**

  In this research an integral-based triggering mechanism was proposed which, as proved, is more efficient than its corresponding schemes from a data transmission point of view. However, stability was the only aspect that was studied in this work. In the other words, IBTC idea can be developed for the case disturbance is present in the system and different forms of performance, including $\mathcal{L}_2$ may be studied.

- **Integral-based event-triggered scheme for networked control systems**

  As another area of research, the proposed IBTC can be extended for a networked-control structure. In this case, the constraints imposed by NCS, including time-delay and packet dropouts, should be taken into account and the triggering mechanism has to be redesigned in the way that desired stability and performance criteria are guaranteed.

Bibliography

[1] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control task," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680 – 1685, 2007.

[2] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[3] K. J. Astrom, *Analysis and Design of Nonlinear Control Systems.* Berlin Heidelberg: Springer, 2008, ch. 3, pp. 127–147.

[4] S. P. DeWeerth, L. Nielsen, C. A. Mead, and K. J. Astrom, "A simple neuron servo," *IEEE Trans. Neural Networks*, vol. 2, no. 2, p. 248251, 1991.

[5] Y. Z. Tsypkin, *Relay Control Systems.* Cambridge, US: Cambridge University Press, 1984.

[6] J. E. Miller, "The pulsed integrated pendulous accelerometer(pipa)," *Space Navigation Guidance and Control*, 1996.

[7] R. Obermaisser, *Event-triggered and time-triggered control paradigms.* Springer, 2005.

[8] K. J. Astrom and B. M. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Proc. Conf. Dec. Control*, 2002, pp. 2011–2016.

[9] B. . B. K. J. Astrom, "Comparison of periodic and event based sampling for first-order stochastic systems," in *Proc. of 14th IFAC World Congr*, 1999, pp. 301–306.

[10] K. E. Arzen, "A simple event-based PID controller," in *Proc. of 14th IFAC World Congr*, 1999.

[11] E. Kofman and J. H. Braslavsky, "Level crossing sampling in feedback stabilization under data-rate constraints," in *Proc. Conf. Dec. Control*, December 2006, pp. 4423–4428.

[12] W. P. M. H. Heemels, J. Sandee, and P. P. J. Bosch, "Analysis of event- driven controllers for linear systems," *International Journal of Control*, vol. 81, no. 4, pp. 571–590, 2007.

[13] P. Tabuada and X. Wang, "Preliminary results on state-triggered scheduling of stabilizing control tasks," in *Proc. Conf. Dec. Control*, 2006, pp. 282–287.

[14] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems.* Princeton, New Jersey: Princeton University Press, 2012.

[15] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 586–601, 2011.

[16] ——, "Event-triggering in distributed networked systems with data dropouts and delays," *Hybrid Syst.: Comput. Control*, p. 366380, 2009.

[17] C. D. Persis, R. Sailer, and F. Wirth, "Parsimonious event-triggered distributed control: A zeno free approach," *Automatica*, vol. 49, no. 7, pp. 2116–2124, 2013.

[18] M. Mazo and M. Cao, "Asynchronous decentralized event-triggered control," *http://arxiv.org/pdf/1206.6648.pdf*, 2013.

[19] P. Tallapragada and N. Chopra, "Decentralized event-triggering for control of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 12, pp. 3312–3324, 2014.

[20] M. Mazo and P. Tabuada, "Decentralized event-triggered control over wireless sensor/actuator networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2456–2461, 2011.

[21] R. Postoyan, P. Tabuada, D. Nesic, and A. Anta, "Event-triggered and self-triggered stabilization of distributed networked control systems," in *50th IEEE Conf. Dec. Control and Eur. Control Conf.*, Orlando, FL, December 2011, pp. 2565–2570.

[22] M. C. F. Donkers and W. P. M. H. Heemels, “Output-based event-triggered control with guaranteed L_∞-gain and improved and decentralized event-triggering,” *IEEE Trans. Autom. Control*, vol. 57, no. 6, pp. 1362–1376, 2012.

[23] P. Tallapragada and N. Chopra, “Event-triggered decentralized dynamic output feedback control for LTI systems,” *3rd IFAC Workshop on Distributed Estimation and Control in Networked Systems*, vol. 45, no. 26, pp. 31–36, 2012.

[24] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, “Stabilization of nonlinear systems using event-triggered output feedback controllers,” *IEEE Trans. Autom. Control*, vol. PP, no. 99, pp. 1–1, DOI: 10.1109/TAC.2015.2 502 145, 2015.

[25] R. Olfati-Saber and R. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[26] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, “Distributed event-triggered control for multi-agent systems,” *IEEE Trans. on Automatic Control*, vol. 57, no. 2, pp. 906–909, 2012.

[27] D. V. Dimarogonas and K. H. Johansson, “Event triggered control for multi-agent systems,” in *Proc. of the 48th IEEE Conf. Dec. Control*, Shanghai, 2009, pp. 7131–7136.

[28] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, “Distributed event-triggered control for multi-agent systems,” *IEEE Trans. Autom. Control*, vol. 57, no. 5, p. 12911297, 2012.

[29] Z. Liu, Z. Chen, and Z. Yuan, “Event-triggered averageconsensus of multi-agent systems with weighted and direct topology,” *Journal of Systems Science and Complexity*, vol. 25, no. 2, pp. 845–855, 2012.

[30] E. Garcia, Y. Cao, H. Yu, P. Antsaklis, and D. W. Casbeer, “Decentralized event-triggered cooperative control with limited communication,” *International Journal of Control*, vol. 86, no. 9, pp. 1479–1488, 2013.

[31] C. Nowzari and J. Cortes, “Zeno-free, distributed event-triggered communication and control for multi-agent average consensus,” in *Proc. of American Control Conf.*, Portland, OR, June 2014, pp. 2148–2153.

[32] H. Zhang, R. Yang, H. Yan, and Q. Chen, “Distributed event-triggered control for consensus of multi-agent systems,” *Journal of the Franklin Institute*, vol. 352, no. 9, pp. 3476–3488, 2015.

[33] P. Tallapragada and N. Chopra, “On event triggered trajectory tracking for control affine nonlinear systems,” in *50th IEEE Conf. Dec. Control and Eur. Control Conf.*, Orlando, FL, December 2011, pp. 5377–5382.

[34] ——, “On event triggered tracking for nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 58, no. 9, p. 23432348, 2013.

[35] S. Hu, Y.Zhang, and Z.Du, “Network-based H_∞ traking control with event-triggering sampling scheme,” *IET Control Theory Appl.*, vol. 6, no. 4, pp. 533–544, 2012.

[36] R. Postoyan, M. C. Bragagnolo, E. Galbrun, J. Daafouz, D. Nesic, and E. B. Castelan, “Event-triggered tracking control of unicycle mobile robots,” *Automatica*, vol. 52, pp. 302–308, 2015.

[37] Y. Cheng and V. Ugrinkovskii, “Event-triggered leader-following tracking control for multivariable multi-agent systems,” *Automatica*, vol. 70, pp. 204–210, 2016.

[38] A. Girard, “Dynamic triggering mechanisms for event-triggered control,” *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1992–1997, 2015.

[39] X. Wang and M. Lemmon, “On event design in event-triggered feedback systems,” *Automatica*, vol. 47, no. 9, p. 23192322, 2011.

[40] A. Seuret and C. Prieur, “Event-triggered sampling algorithms based on a lyapunov function,” in *50th IEEE Conf. Dec. Control and Eur. Control Conf.*, Orlando, FL, December 2011, pp. 6128–6133.

[41] W. Heemels, M. C. F. Donkers, and A. R. Teel, “Periodic event-triggered control based on state feedback,” in *50th IEEE Conf. Dec. Control and Eur. Control Conf.*, Oralndo, FL, December 2011, pp. 2571–2576.

[42] ——, “Periodic event-triggered control for linear systems,” *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 847–861, 2013.

[43] X. Meng and T. Chen, “Event detection and control co-design of sampled-data systems,” *International Journal of Control*, vol. 87, no. 4, pp. 777–786, 2014.

[44] R. Postoyan, A. Anta, W. P. M. H. Heemels, P. Tabuada, and D. Nesic, “Periodic event-triggered control for ninlinear systems,” in *Proc. of the 52th IEEE Conf. Dec. Control*, Firenze, December 2013, pp. 7397–7402.

[45] S. Hu, Y.Zhang, and Z.Du, “A delay system method for designing event-triggered controllers of networked control systems,” *IEEE Trans. on Automatic Control*, vol. 58, no. 2, pp. 475–481, 2013.

[46] C. Peng and J. Zhang, “Event-triggered output-feedback H_∞ control for networked control systems with time-varying sampling,” *IET Control Theory & Applications*, vol. 9, no. 9, pp. 1384–1391, 2014.

[47] X. M. Zhang and Q. L. Han, “Event-triggered dynamic output-feedback control for networked control systems,” *IET Control Theory & Applications*, vol. 8, no. 4, pp. 226–234, 2014.

[48] C. Peng and Q. L. Han, “A novel event-triggered transmission scheme and L_2 control co-design for sampled-data control systems,” *IEEE Trans. Autom. Control*, vol. 58, no. 10, pp. 2620–2626, 2013.

[49] O. Demir and J. Lunze, “Optimal and event-based networked control of physically interconnected systems and multi-agent systems,” *International Journal of Control*, vol. 87, no. 1, pp. 169–185, 2014.

[50] X. Meng and T. Chen, “Optimal sampling and performance comparison of periodic and event based impulse control,” *IEEE Trans. Autom. Control*, vol. 57, no. 12, pp. 3252–3259, 2012.

[51] L. Li and M. Lemmon, “Weakly coupled event triggered output feedback control in wireless networked control systems,” in *49th Annual Allerton Conference on Communication, Control, and Computing*, 2011, pp. 572–579.

[52] R. Cogill, "Event-based control using quadratic approximate value function," in *Proceeding of the 48th IEEE Conference on Decision and Control, held jointly with 28th Chinese Control Conference*, 2009, pp. 5883–5888.

[53] Y. Xu and J. P. Hespanha, "Optimal communication logics in networked control systems," in *IEEE Conferenec on Decision and Control*, 2004, pp. 3527–3532.

[54] T. Henningsson, E. Johannesson, and A.Cervin, "Sporadic event-based control of first-order linear stochastic systems," *Automatica*, vol. 44, no. 11, pp. 2890–2895, 2008.

[55] B. Wang, X. Meng, and T. Chen, "Event based pulse-modulated control of linear stochastic systems," *IEEE Trans. Autom. Control*, vol. 59, no. 8, pp. 2144–2150, 2014.

[56] W. Wu, S. Reimann, D. Gorges, and S. Liu, "Suboptimal event-triggered control for time-delayed linear systems," *IEEE Trans. Autom. Control*, vol. 60, no. 5, pp. 1386–1391, 2015.

[57] M. Lemmon, *Networked Control Systems*. Berlin Heidelberg: Springer London, 2010, vol. 406, ch. 9, pp. 293–358.

[58] D. Lehmann and J. Lunze, "Event-based output-feedback control," in *Proceeding of the 19th Mediterranean Conference on Control & Automation (MED)*, June 2011, pp. 982–987.

[59] A. Hu, J. Cao, M. Hu, and L. Guo, "Event-triggered consensus of multi-agent systems with noises," *Journal of the Franklin Institute*, vol. 352, no. 9, pp. 3489–3503, 2015.

[60] M. Abbaszadeh and H. Marquez, "Dynamical robust H_∞ filtering for nonlinear uncertain systems: An LMI approach," *Journal of the Franklin Institute*, vol. 347, pp. 1227–1241, 2010.

[61] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambrige: Cambrige University Press, 2004.

[62] H. J. Marquez, *Nonlinear Control Systems: Analysis and Design*. Wiley, NJ, 2003.

[63] R. M. Biernacki, H. Hwang, and S. P. Battacharyya, "Robust stability with structured real parameter perturbations," *IEEE Trans. Autom. Control*, vol. AC-32, no. 6, pp. 495–506, 1987.

[64] G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*. Oxford: Clarendon Press, 1992.

[65] K. E. Atkinson, *An Introduction to Numerical Analysis*. John Wiley and Sons, 1989.

[66] M. Wua, Y. Hea, J. H. Sheb, and G. P. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, no. 8, pp. 1435–1439, 2004.

[67] W. H. Press, B. P. Flannery, W. T. Vetterling, and S. A. Teukolsky, *Numerical Recipes in Pascal: The Art of Scientific Computing*. Cambridge University Press, 1989.

[68] G. C. Walsh, H. Ye, and L. G. Bushnell, "Stability analysis of networked control systems," *IEEE Transactions on Control System Technology*, vol. 10, pp. 438–446, 2002.

[69] S. Hu and D. Yue, "Event-based H_∞ filtering for networked systems with communication delay," *Signal Processing*, vol. 92, no. 9, pp. 2029–2039, 2012.

[70] J. Wu, X. Chen, and H. Gao, "h_∞ filtering with stochastic sampling," *Signal Processing*, vol. 90, pp. 1131–1145, 2010.

[71] H. Khalil, *Nonlinear Systems, 3rd Edition*. Englewood Cliffs, NJ: Prentice-Hall, 2002.