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Nonlinear Robust Observers for Simultaneous State and Fault Estimation

by

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To my family,
Vida, Masoud and Roozbeh
for their unconditional love and support

Abstract

A fault in the system operation is deemed to occur when the system practically experiences an abnormal condition, such as a malfunction in the actuators/sensors. This situation is happening very often in many control process such as: Chemical processes, Jet engine control, flight control, Robotics, temperature control, and etc. Faults can cause catastrophic damages to control systems. Therefore, reliability is one of the key requirements for process industries. Since many process control loops are utilized, the fault-free operation of these control loops is strictly required. For this purpose, effective model based Fault Detection and Isolation (FDI) has to be developed. On the other hand, effective control and monitoring of a system requires accurate information of internal behavior of the system. This internal behavior can be analyzed by *system's states*. Practically, in many real systems, state space variables are not fully available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all these state space variables. Thus, one is faced with the problem of estimating system's state space variables. This can be done by constructing another dynamical system called *state observer*.

The two critical problems stated above have motivated significant research work in the area of robust state and fault estimation. Fault reconstruction and estimation is regarded as a stronger extension to FDI since accurate fault estimation automatically implies fault detection. Fault reconstruction is excellent for directly detecting and isolating the malfunctions within a system by reviling which sensor or actuator is faulty and is useful for diagnosing incipient and small faults. Moreover, Fault reconstruction finds solid applications in Fault Tolerant Control Systems (FTC). Therefore, in this PhD thesis, we restrict our attention to design observers (esti-

mators) that can simultaneously estimate the system states and faults. It is worth mentioning that both faults and disturbances considerably affect the state observation (estimation) and designing a robust observer which is insensitive to faults and disturbances is of great interest for achieving an accurate state estimation.

It is well known that two promising control strategies to cope with vastly uncertain control processes are $\mathcal{H}_\infty$ Control and Sliding Mode Control. The robustness and simple implementation of these two control theories introduce them as strong practical control methods. Sliding mode observers are very successful to deal with uncertain faulty systems. Furthermore, in Robust FDI, the main objective is to design residuals that can distinguish faults from disturbances/uncertainties by reducing the effect of disturbances. However, in this thesis, we go one step further and we propose Robust Fault Reconstruction (RFR) by integrating $\mathcal{H}_\infty$ filtering and Sliding Mode Control. It is also shown how adaptive control can improve the robustness of the observer based RFR by assuming that there is no information on the bound of a fault and nevertheless the observer can still reconstruct the fault effectively.

Another open problem in the context of FDI and RFR is due to systems with multiple faults at different system's components since it is often the case where actuators and also sensors suffer from faults during the course of the system's operation. Both actuators and sensors can suffer from faults either alone, at separate times or simultaneously. In this case, detection and reconstruction of all faults is highly important. The co-existence of unknown fault at both some sensor(s) and actuator(s) has not been addressed in any earlier design of fault reconstruction schemes. Thus, in this Thesis, inspired by the theory of singular systems, we aim at solving this open problem. Unknown Input Observers (UIOs) for estimation of unknown input and sensor fault are also studied by proposing a new UIO structure. The application of the proposed UIO for chaotic communication is also addressed. The class of system which will be considered throughout this thesis is *Lipschitz* nonlinear systems with fault and uncertainty. The reason behind focusing on Lipschitz system is that Lipschitz systems constitute a very important and wide class, since any nonlinear system with continuously differentiable nonlinearities can be locally expressed in this form. As a conclusion, we design novel observers (estimators) which benefits from the following main features:

- are robust and insensitive to faults
- minimize the effect of disturbances on the state and fault estimation
- are able of detecting unknown behavior-type faults via adaptive gain adjustment
- can simultaneously estimate sensor(s) and actuator(s) fault
- have sensor fault reconstruction ability via the use of a reduced-order UIO

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Chapter 1

Introduction

It is often the case when dealing with complex systems requiring safe operation, that some form of supervisory function is needed to indicate undesired process states or *faults*. Faulty signals can exist in actuators, sensors and process components that can deteriorate normal operation or even lead to instability. Taking immediate and appropriate actions in order to preserve safe operation while avoiding the possibly of catastrophic damages is crucial. In addition to safety concerns, fault detection is critical from an economical perspective by preventing any unexpected total failure that can potentially causes lose of revenues. From an environmental perspective, incorporation of fault detection in industrial process can prevent catastrophic damage to the environment. Thus, fault detection and isolation (FDI) is of significant technical, economical and environmental importance. The nonlinear behavior exhibited by most industrial processes, the presence of constraints on the operating conditions, the presence of modeling uncertainty and disturbances, the possibility of faults in a system's components, and the unavailability of all the process states have motivated significant research work in the area of nonlinear state and fault estimation. Fault reconstruction and estimation (abbreviated as FRE hereafter) may be regarded as an extension to fault detection since accurate fault reconstruction immediately imply fault detection and isolation (FDI). Furthermore, Fault reconstruction and FDI are also very important in Fault -Tolerant Control (FTC). A FTC is designed to preserve the control and stability of a system in the event of a set of possible faults. FDI techniques automatically reconfigure the controller once a malfunction has been detected. However, as a great advantage of FRE, the corrupted measured signals and actuator signals are corrected by FRE before being used by the controller. Hence, there is no need of control reconfiguration and a relatively simple and effective control method would still work by retaining its structure.

As mentioned above, effective controller design methods and monitoring of a system requires accurate information of internal behavior of the system. This internal behavior can be analyzed by *system's states*. Practically, in many real systems, state space variables are not fully available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all these state space variables. Thus, one is faced with the problem of estimating system's state space variables. Therefore, there is a need to design filters to estimate both **states** and **faults** of a system coincidentally during the course of system's operation. Now, we briefly study the history of FDI and FRE.

1.1 Fault Detection and Isolation

Safety and reliability are two major concerns of modern control systems. Fault and malfunction in a system's components is the main reason behind the safety issues of control systems. This problem can be dealt with if a supervisory action is taken to monitor a system and detect faults, locate them and then isolate the faulty component. Hence, Fault Detection and Isolation (FDI) is of great importance in a wide variety of applications of control systems. There are two main approaches in FDI: (i) signal-based FDI, (ii) model-based FDI. In signal-based FDI, some statistical operations are employed on the measurements, or some artificial intelligent network is trained using measurements to extract the information about the fault. For instance, neural network has been the center of attention for signal-based FDI in research [20],[48],[61],[73]. In model-based FDI, since its beginning in 1970s, the objective is generally obtained by comparing the actual system's behavior with the corresponding behavior of it's *mathematical model*. The difference of these behaviors, referred to as *residuals* are sensitive to any fault. There are a large number of different approaches developed through the years for model-based FDI [9]. However, due to the high sensitivity of model-based FDI to the corresponding mathematical models, a major downside is the need for the perfectly and idealistically accurate mathematical models which is not often easy to derive. That is due to model uncertainties, process complexities, noise and disturbances. Any discrepancies between the actual process and its model cause misleading alarms of fault and therefore, fault detection may become ineffective and useless. This problem has encouraged a lot of control engineering research towards *Robust FDI*. A number of methods proposed for robust FDI can be found in [21], [22], [50], [51]. With regard to the fact that many systems and processes are highly nonlinear, it is of great interest to develop method of nonlinear FDI. Regarding the inherent nonlinearity of many control systems and also the incorporation of mathematical models for FDI, an excellent tool has been traditionally the use of *nonlinear observers* [9]. In observer-based FDI, which is a subcategory of model-based FDI, the residual is constructed as a weighted difference between the measured output and the observer-based estimated output [9]. Due to the importance of robustness of FDI and also the nonlinearity of systems, in this thesis, *nonlinear robust observer* design is performed to deal with FDI.

1.2 Fault Reconstruction

Instead of generating residuals, fault reconstruction and estimation (FRE) scheme attempts to reconstruct the fault. FRE is different from the majority of FDI methods described previously in the sense that it not only detects and isolates the fault, but provides an estimate of the fault. This approach is very useful for incipient faults and slow drifts, which are very difficult to detect. Also, this approach is very useful for FTC systems in the sense that instead of reconfiguration of the control system, the faulty sensors or actuators can be corrected and the simple control method can still be effectively used. Motivated by these useful features of FRE, in this thesis, we are interested in performing *observer-based* FRE schemes for nonlinear systems which finds applications in FDI systems.

1.3 Overview and Statement of Contributions

In this thesis, we restrict attention to design novel filters to estimate both systems's states and also sensor and actuator faults simultaneously. It is very important to mention that FRE schemes inherently require state estimation since the state estimates play an important role in fault reconstruction. Due to the importance and practicality of Lipschitz nonlinear systems, we consider this class of nonlinear system throughout this thesis. Detailed discussions on the background of each problem and the literature surveys may be found in the introduction of each chapter. The chapters are written in a self-sufficient format. We believe this improves the readability of the thesis and greatly facilitates the partial readings which are most probable for long documents such as PhD dissertations. The rest of the thesis is organized as follows:

- **Chapter 2**: In this chapter, we survey the currently established results in the context of Linear and Lipschitz Observer design problems. Next, we introduce the principle of sliding mode control since this theory is one of the main tools used extensively throughout this thesis. In the context of robust observation, classical sliding mode observers are discussed in details. Finally, singular system theory is studied since we will use this concept in Chapter 5.

- **Chapter 3**: This chapter presents a scheme to design robust sliding mode observers (SMO) with $\mathcal{H}_\infty$ performance for uncertain Lipschitz nonlinear systems where both faults and disturbances are considered. We study the necessary conditions to achieve insensitivity of the proposed sliding mode observer to the unknown input (fault). The objective is to derive a sufficient condition using LMI optimization for minimizing the $\mathcal{H}_\infty$ gain between the estimation error and disturbances, whilst at the same time the design method guarantees that the solution satisfies the so-called structural matching condition. The sliding motion affects only a part of the system through a novel reduced-order switching gain. The structure of the observer gain is new in the sense that it facilitates the design. Furthermore, the so-called equivalent control concept is discussed for fault estimation. Finally, a numerical example of

MCK chaos demonstrates the high performance of the results compared to a pure SMO.

- **Chapter 4**: In Chapter 3, we design the robust $\mathcal{H}_\infty$ SMO using a change of coordinates. Unlike Chapter 3, to further simplify the design, in this chapter we develop a new parametrization based method to avoid any change of coordinates. This chapter presents an adaptive sliding mode observer based approach for nonlinear fault reconstruction using LMIs. A Nonlinear Lipschitz uncertain system is considered where the uncertainty (or disturbance) is assumed to have a constant upper bound. The novelty of the proposed approach lies in the simplicity of the design since there is no need for any single or multiple coordinate transformations. This interesting feature is made possible due to introducing a new solution for the so-called matching condition on the original system. The upper bound of the fault signal is allowed to be unknown since the variable structure gain adaptively adjusts itself to maintain the ideal sliding motion on the defined sliding surface. The reconstruction signal can approximate the fault to some degree of accuracy depending on the size of the disturbances. Finally, a simulation study shows the effectiveness of this approach.

- **Chapter 5**: The new SMOs presented in Chapters 3 and 4 can accomplish only actuator fault reconstruction and they do not have the ability to solve the fault estimation problem for the case of the coincidence of faulty sensors and actuators. In this Chapter, a new filter for state and fault estimation in a class of nonlinear systems is presented. The estimator benefits from both sliding mode control and singular systems theory. The novelty of this approach is based upon dealing with systems prone to faults at sensors and actuators during the course of the system's operation coincidentally. Conditions and proofs of conversion for the proposed estimator are presented. A noticeable feature of the proposed approach is that the state trajectories do not leave the sliding manifold even in presence of sensor/actuator faults. This allows for actuator faults to be reconstructed based upon information retrieved from the equivalent output error injection signal. Due to employing a generalized state space form (singular system theory), the sensor faults are also estimated. A simulation example based on a flexible joint robot arm demonstrates the effectiveness of the proposed estimator.

- **Chapter 6**: Inspired by the concept of unknown input observers (UIOs), this chapter considers the problem of chaos secure communication. A new scheme for chaos communication is presented which does not require any feedback of the message signal into the chaotic dynamics. By introducing a low pass filter, the message signal can be regarded as the unknown input to be recovered. The chaos attractor acts as the transmitter and the new proposed reduced-order UIO synchronizes the slave chaotic system at the receiver end. Necessary and sufficient conditions for the existence and asymptotical stability of the proposed UIO were established. The theoretical analysis and a numerical simulation on Chua's circuit verify the effectiveness of the proposed scheme. Next, we extend the robust design of the proposed reduced-order UIO for systems with disturbances using $\mathcal{H}_\infty$ method. The proposed $\mathcal{H}_\infty$-UIO is employed to robustly estimate and detect sensor faults. The $\mathcal{H}_\infty$-UIO exists if an LMI optimization problem is solvable. The configuration of the proposed sensor fault estimation and diagnosis is novel.

- **Chapter 7**: In this Chapter, concluding remarks are presented and future research topics are proposed.

Chapter 2

Background and Mathematical Framework

2.1 State Observation

Practically, in many real systems, state space variables are either not fully available for measurements, or it is not practical to measure all of them. In order to be able to apply full state feedback control to a system, all of its state space variables should be available at all times. Also, in some control system applications, one is interested in having information about system state space variables at any time instant for monitoring purposes. Thus, one is faced with the problem of estimating system state space variables. This can be done by constructing another dynamical system called observer or estimator. Thus the problem of estimating the state of a dynamical system from outputs and inputs, (commonly known as observing the state) is a very important problem in the theory of control systems.

For linear systems it has been extensively studied, and has proven very useful, especially for control methods such as observer-based control design. For nonlinear systems, the theory of observers is not nearly as complete or successful as is the case for linear systems. Applying linear observer theory to nonlinear problems has had some success as exemplified by Luenburger Observer [42] and the extended Kalman filter (EKF) [60], but has by no means closed the book on nonlinear observer design. Instead, attempts continue to be made to construct nonlinear observers using tools from nonlinear control system. In the following subsection, we briefly introduce the concept of observability.

2.1.1 Observability

Before studying the state observer design, we start with a brief introduction to *observability* which is a essential issue on state observer design. It is worthy to note that the concept of observability is dual to that of controllability. Consider the following system

$$\begin{cases} \dot{x} = Ax + Bu, x \in \mathbf{R}^n, \ u \in \mathbf{R}^m \\ y = Cx + Du, \ y \in \mathbf{R}^p \end{cases} \tag{2.1}$$

First we introduce the definition of observability.

Definition:[8] *The state equation (2.1) is said to be observable if for any unknown initial state $x(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input u and the output y over $[0, t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable.*

It is well-known that the response of the above linear system excited by initial state x_0 and the input $u(t)$ is derived as

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2.2}$$

In the study of observability, the output y and the input u are assumed to be known; the initial state is only unknown. Thus the above definition can be modified as follows:

Definition:[8] *Equation (2.1) is observable if and only if the initial state $x(0)$ can be determined uniquely from its zero-input, i.e.*

$$\bar{y}(t) = Ce^{At}x(0)$$

where

$$\bar{y}(t) := y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

is a known function. Thus the observability problem reduces to solving $x(0)$. If $u \equiv 0$, then $\bar{y}(t)$ reduces to zero-input response $Ce^{At}x(0)$.

The following well-known theorem introduces the observability of the pair $\{A, C\}$ mathematically.

Theorem 2.1. [8] *The pair {A,C} of the system (2.1) is observable if and only if one of the following statements holds:*

- *The $n \times n$ matrix $W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ is nonsingular for any $t > 0$.*
- *The $np \times n$ observability matrix*

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

  has full column rank (rank n).
- *The $(n+p) \times n$ matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalues, λ, of A.*

2.1.2 Linear Observer Design

Consider the following uncontrolled linear system:

$$\begin{cases} \dot{x} = Ax, x \in \mathbf{R}^n \\ y = Cx, y \in \mathbf{R}^p \end{cases} \tag{2.3}$$

The problem of generating an estimate of the state, x , in (2.3) is usually referred to as the problem of designing an observer for (2.3). In the early 1960's, Luenberger [42] gave the following result for the construction of an observer for (2.3):

" Let $\hat{x}$ be our estimate of the true state, x , and assume $\hat{x}$ obeys the following dynamics.

$$\dot{\hat{x}} = A\hat{x} + L(\hat{y} - y), \hat{x} \in \mathbf{R}^n \tag{2.4}$$

$$\hat{y} = C\hat{x}, \hat{y} \in \mathbf{R}^p$$

where $L \in \mathbf{R}^{n\times p}$. Then if (A, C) is an observable pair, L may be chosen such that $\hat{x}$ will converge to x arbitrarily exponentially fast. To see that this is the case, we consider the dynamics of the error between the state estimate $\hat{x}$ and the true state x. Denoting this error by $e = \hat{x} - x$, we obtain

$$\dot{e} = (A + LC)\, e \tag{2.5}$$

The error dynamics is linear. Due to this, it is clear that if (A, C) is an observable pair, then the eigenvalues of $A + LC$ can be arbitrarily assigned, and hence placed as far into the left half plane as desired, causing the error to decay to zero at any desired exponential rate."

At this point it is important to note that the observer structure and (2.3) will reveal that observer dynamics are exactly those of the true system (2.3) except with an additional linear function, L, of the difference between the estimated and true output, $\hat{y} - y$, injected into the dynamics. This is a standard structure used throughout observer design problem, and is commonly referred to as *output injection* [42].

2.1.3 Lipschitz Observer Design

Consider the class of Lipschitz nonlinear systems of the form

$$\begin{cases} \dot{x} = Ax + \Phi\,(x\ , u, t), x \in \mathbf{R}^n \\ y = Cx, y \in \mathbf{R}^p \end{cases} \tag{2.6}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, $A \in \mathbf{R}^{n\times n}$, and $C \in \mathbf{R}^{p\times n}$. We also assume that the pair (C, A) is observable. This allows us to find an $L \in \mathbf{R}^{p\times n}$ such that the eigenvalues of $A + LC$ are in the open left half plane. The known nonlinear function $\Phi(x, u, t)$ satisfies a Lipschitz condition *locally* on a set $\mathcal{D} \subset \mathbf{R}^n$ in which

$$\|\Phi(x_1, u, t) - \Phi(x_2, u, t)\| \leq \mathcal{L}_\Phi \, \|(x_1 - x_2)\| \tag{2.7}$$

where $x_1, x_2 \in \mathcal{D}$ and $\mathcal{L}_\Phi \in \mathbf{R}^+$ is a known positive constant called *Lipschitz gain* or *Lipschitz constant* [43]. If $\mathcal{D} = \mathbf{R}^n$, the function Φ is said to be *globally* Lipschitz. Throughout this thesis, we assume that the system to be addressed is at least locally Lipschitz on a set $\mathcal{D}$. This class of systems represents a very important category of nonlinear systems in general. The reason is that any nonlinear system $\dot{x} = g(x, u)$ can be expressed in the form of (2.6) if the nonlinear function $g(x, u)$ is continuously differentiable with respect to x, at least in a locality. Observer design for this class of systems first introduced by Thau [68]. First, to have a better picture of a possible observer design method for Lipschitz systems, we briefly describe Thau's design. Next, we study the well-known results on Lipschitz observers which came out later to address difficulties of Thau's design framework. In most of the literature, the Lipschitz observer is built as

$$\left\{ \begin{array}{l} \dot{\hat{x}} = A\hat{x} + \Phi\left(\hat{x}\ , u, t\right) + L\left(\hat{y} - y\right) \\ \hat{y} = C\hat{x} \end{array} \right. \tag{2.8}$$

Once again let e denote that error between the true state and our estimated state, $e = \hat{x} - x$, thus e satisfies

$$\dot{e} = \left(A + LC\right)e + \Phi\left(\hat{x}\ , u, t\right) - \Phi(x, u, t). \tag{2.9}$$

The following theorem presents Thau's results.

Theorem 2.2 [68] *If the gain L is chosen such that*

$$\frac{\lambda_{\min}\left(Q\right)}{2\lambda_{\max}\left(P\right)} > \mathcal{L}_{\Phi} \tag{2.10}$$

with the Lyapunov equation $\left(A + LC\right)^T P + P\left(A + LC\right) = -Q$, then the estimation error is asymptotically stable.

Theorem 2.2 provides an important sufficient condition for the asymptotic stability of the error dynamics. However, employing this condition in observer design is not trivial and no design algorithm was proposed in [68]. Raghavan in [54] proposed an iterative binary search procedure over finding a $\varepsilon > 0$ for the following Algebraic Riccati Equation (ARE)

$$AP + PA^T + P(\mathcal{L}_{\Phi}^2 I - \varepsilon^{-1}C^T C)P + (1 + \varepsilon)I = 0,$$

where the observer gain would be $L = (2\varepsilon)^{-1}PC^T$ and P is a s.p.d solution for the ARE. Unfortunately, this methods fails even for systems with an observable $\{A, C\}$

pair and does not provide necessary conditions on $(A-LC)$ to ensure the observer's error stability [55]. Rajamani, in his well-known article [55], derived a condition on $(A - LC)$ to get around the problem stated above. His result is as follows:

Theorem 2.3. [55] *The observer gain L stabilizes the error dynamics in (2.9) if L is chosen so as to ensure that $(A - LC)$ is stable and such that*

$$\min_{\omega \in \mathbf{R}_+} \sigma_{\min}(A + LC - j) > \mathcal{L}_\Phi. \tag{2.11}$$

Interestingly, the observer design can be framed as an $\mathcal{H}_\infty$ problem by rewriting the above condition as [55]

$$\|[sI - (A + LC)]^{-1}\|_\infty < \mathcal{L}_\Phi^{-1}.$$

Since all the above results consider a static observer approach for Lipschitz systems, such an observer may not always exist numerically. To relax this restriction, recently in [53] a dynamical approach was presented. The Observer has extra dynamics, namely a dynamical compensator, which brings additional degree of freedom in the design. All the regularity assumptions of $\mathcal{H}_\infty$ filtering hold and thus the observer gain is computed in this framework straightforwardly. Due to the lengthy technical background and formulation of the dynamical observer approach, readers are referred to [53] for details. Most recently, based on convex optimization and LMIs, [1] presented a method to tackle Lipschitz observer design problem. In this method, for a given Lipschitz gain, an LMI feasibility problem must be solved to compute the observer gain. Moreover, using LMI optimization, the admissible Lipschitz gain can be maximized and hence the observer can tolerate more Lipschitzian nonlinearities [1]. We employ this latter concept to cope with Lipschitz systems since it fits in the LMI optimization context. Therefore, we skip the details for now. Details are fully provided in Chapters 4.

2.2 Sliding Mode Control

Variable structure systems (VSS) theory was first proposed in the early 1950s and has been extensively developed since then [71] and [70]. VSS provides a systematic solution to the problem of maintaining stability and consistent performance in the face of bounded disturbances. The most popular operation regime associated with VSS is known as sliding mode control, which is a nonlinear deterministic control with a high speed, nonlinear feedback that switches discontinuously in time on a prescribed sliding surface. Many practical applications of sliding mode control have been reported in the control literature including flight control, robotic manipulators and servo systems ([70], [71] and [18]).

The reason for this popularity is the attractive properties of sliding mode control; it is robust and in some cases insensitive to external disturbances and parameter variations. It also provides fast error convergence characteristics by emulating a prescribed reduced order system. In particular, it is of interest to explore the possibilities of using mode control for robust state reconstruction. Considerable attention has been paid to robust non-linear observer design problem (See [18] and references there in). In general, the design of a sliding mode controller (SMC) involves the determination of a sliding surface that represents the desired stable dynamics and the description of a controller that guarantees the reaching condition of sliding mode. The trajectories, starting from a given initial condition, tend towards the sliding surface. Excellent robustness of sliding mode control systems motivate us to design robust nonlinear observers based upon this control strategy. Now, we illustrate the principles of sliding mode control through an example. Consider the double integrator given by [1]

$$\ddot{y}(t) = u(t). \tag{2.12}$$

Consider the switching control law

$$u(t) = \begin{cases} -1 & if \;\; s(y, \dot{y}) \geq 0 \\ 1 & if \;\; s(y, \dot{y}) < 0 \end{cases} \tag{2.13}$$

Where the *switching function* is defined by

$$s(y, \dot{y}) = my + \dot{y} \tag{2.14}$$

and m is a positive design scalar. The term *switching control* is commonly used since the function given in equation (2.12) switches between two control structure at any point $(y, \dot{y})$ in the phase plane. The expression in equation (2.13) is usually written more concisely in the following form

$$u(t) = -sgn(s(t)) \tag{2.15}$$

where sgn(.) is the sign function. Controller (2.15) represents a classic variable structure controller. For values of $\dot{y}$ satisfying the inequality $m\,|\dot{y}| < 1$, it follows that

$$s\dot{s} = s(m\dot{y} + \ddot{y}) = s(m\dot{y} - sgn(s)) < |s|(m\,|\dot{y}| - 1) < 0.$$

Consequently, we can write

$$\underset{s\to 0^+}{Lim}\,\dot{s} < 0 \;\; and \;\; \underset{s\to 0^-}{Lim}\,\dot{s} > 0. \tag{2.16}$$

It should be pointed out that the conditions in (2.16) are usually written as

$$s\dot{s} < 0 \tag{2.17}$$

which is referred to as the *reachability condition*. When $m\,|\dot{y}| < 1$ the system trajectories on either side of the line

$$L_s = \{(y, \dot{y}) : s(y, \dot{y}) = 0\} \tag{2.18}$$

move towards the line. The motion when confined to the line L_s satisfies the differential equation obtained from rearranging $s(y, \dot{y}) = 0$, namely

$$\dot{y}(t) = -my(t). \tag{2.19}$$

This represents a first order decay and the trajectories will *slide* along the line L_s to the origin. Such dynamical behavior is described as an *ideal sliding mode* or an *ideal sliding motion* and the line L_s is termed the *sliding surface* [1]. Reachability condition guarantees that the trajectories will slide onto the sliding surface and stay there after a finite time. In this case, as argued earlier, the reachability condition is only satisfied in a domain of the phase plane $\Omega = (\{y, \dot{y}\} : m\,|\dot{y}| < 1)$. In the next section, we restrict attention to the generalization of sliding mode control for robust observer design.

2.3 Sliding Mode Observers

The objective of this section is to address well-known classical sliding mode robust observer design methods proposed by Utkin [71] and Walcott and Żak [72]. The latter design does not necessitate exact knowledge of the system nonlinearities. The aim is accomplished by employing the sliding mode approach. The downside of these classical designs are also briefly discussed.

2.3.1 Walcott-Żak SMO

Consider the class of systems modeled by:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x, u) \\ y(t) = Cx(t) \end{cases} \tag{2.20}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$, the matrices A and C are of appropriate dimension, and the matrix C is of full rank. The function $f(t, x, u)$ can be construed as the uncertainties or disturbances in the plant. For existence purposes, we require that $f(t, x, u)$ be continuous in x . The problem is to design an observer with inputs y and u whose output $\hat{x}$ will converge to x (i.e., $\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0$). Consider the following three assumptions for the system denoted in (2.20).

A1: The pair $\{A, C\}$ is detectable which implies that we can find a matrix $K \in \mathbf{R}^{n \times p}$ such that $\lambda[A_0] \subseteq \mathbf{C}_-$ where $A_0 = A - KC$.

A2: There exists a symmetric, positive definite matrix $Q \in \mathbf{R}^{n \times n}$, and function h where $h(\cdot, \cdot, \cdot) : R_+^1 \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$ such that

$$f(t, x, u) = P^{-1} C^T F^T h(t, x, u)$$

where P is the unique, positive definite solution to the Lyapunov equation $A_0^T P + PA_0 = -Q$.

A3: There exists a positive scalar valued function, ρ such that$\|h(t,x,u)\| \leq \rho(t,u)$ for all $t \in R_+$ and $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$.

Remark: It should be pointed out the the above model is a perturbed linear system which can be represented as $\dot{x}(t) = Ax(t) + Bh(t,x,u)$. Furthermore, the structural condition $B^T P = FC$ is known as *matching condition* [72].

Let the error difference between the observer estimate and the true state be denoted by

$$e(t) = \hat{x}(t) - x(t)$$

Consider the following nonlinear observer dynamical equation:

$$\dot{\hat{x}} = A_0\hat{x} + S(\hat{x}, y, \rho) + Ky \tag{2.21}$$

where

$$S(\hat{x}, y, \rho) = \begin{cases} \frac{-P^{-1}C^T F^T FCe}{\|FCe\|}\rho(t,u) & : e \notin N \\ 0 & : e \in N \end{cases}$$

and $N = \{e : FCe = 0\}$ is the sliding surface. Note that this observer design incorporates only the bound of the nonlinearities and/or uncertainties, $\rho(t,u)$, and does not require exact knowledge of the plant's uncertainties and disturbances except that they satisfy Assumption **A2**. We now state the following theorem.

Theorem 2.4. [72] *Given system (2.20) and the observer governed by (2.21), if Assumptions **A1-A3** are valid, then* $\lim_{t\to\infty}(\hat{x}(t) - x(t)) = \lim_{t\to\infty} e(t) = 0$.

Proof. The error difference between the output of the observer and the true state obeys the following differential equation:

$$\dot{e} = A_0 e - \frac{P^{-1}C^T F^T FCe}{\|FCe\|}\rho(t,u) - P^{-1}C^T F^T h \tag{2.22}$$

Consider the following positive definite Lyapunov function candidate

$$V(e) = e^T P e \tag{2.23}$$

where P is defined in Assumption **A2**. The time derivative of this Lyapunov function candidate is given by

$$\dot{V}(e) = e^T \left(A_0^T P + P A_0\right) e - 2\frac{e^T P \left(P^{-1} C^T F^T F C e\right)}{\|FCe\|}\rho - 2e^T P P^{-1} C^T F^T h(t,x) \tag{2.24}$$

which simplifies to

$$\dot{V}(e) = -e^T Q e - 2\|FCe\|\rho - 2e^T C^T F^T h(t,x). \tag{2.25}$$

Taking the Euclidean norm of the last term and noting the Assumption **A3** yield

$$\dot{V}(e) \leq -e^T Q e - 2\|FCe\|\rho + 2\|FCe\|\rho < 0 \tag{2.26}$$

Therefore, the $\lim_{t\to\infty} e(t) = 0$. Theorem 2.4 shows that error difference between the estimate and the true state asymptotically tends to zero. However, it is desirable to know the rate at which the estimate converges since, if the time response of the observer is of the same order or greater than the system's response time, the observer is of little use in an observer-controller configuration. We have

$$\frac{-\dot{V}(e)}{V(e)} \geq \frac{e^T Q e}{e^T P e} \tag{2.27}$$

or

$$V(e(t)) \leq V(e_0,(t_0))\, e^{-\eta(t-t0)} \tag{2.28}$$

where η is the minimum eigenvalues of $P^{-1}Q$. Thus, if we consider $e^T P e$ to be a measure of the magnitude of the error, then the error will approach zero in magnitude exponentially, with a rate of decay that is at least as fast as $e^{-\eta t}$. However, The major difficulty in designing the Walcott and Żak SMO is the computing of the matrix pair $\{P, F\}$ such that both the Lyapunov equation and the matching condition are satisfied. Despite simple and novel structure of The Walcott and Żak SMO and the appealing property of insensitivity of the observer to the matched uncertainties, symbolically finding proper $\{P, F\}$ is not practical for high order systems. From a computational point of view, this is due to the manipulation and solution of the associated constrained Lyapunov problem.

2.3.2 Utkin SMO

Consider the linear system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.29}$$

Where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times n}$ and $p \geq m$. Assume that the matrices B and C are of full rank and the pair $\{A, C\}$ is observable. Consider the transformation $x \to Tx$ where

$$T = \begin{bmatrix} \Theta^T \\ C \end{bmatrix}. \tag{2.30}$$

Where the columns of $\Theta \in \mathbf{R}^{n\times(n-p)}$ span the null space of C. The output distribution matrix in the new coordinates is given by $CT^{-1} = [0 \quad I_p]$ and also we obtain

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \;\; TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Then the nominal system can be written as

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}y(t) + B_1u(t) \tag{2.31}$$

$$\dot{y}(t) = A_{21}x_1(t) + A_{22}y(t) + B_2u(t) \tag{2.32}$$

Where $Tx = \begin{bmatrix} x_1 \\ y \end{bmatrix}$. The observer proposed in [70] has the following from

$$\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\hat{y}(t) + B_1u(t) + L\nu \tag{2.33}$$

$$\dot{\hat{y}}(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{y}(t) + B_2u(t) - \nu \tag{2.34}$$

Where $(\hat{x}_1, \hat{y})$ represent the state estimates for (x_1, y), $L \in \mathbf{R}^{(n-p)\times p}$is a static gain and the discontinues vector ν is componentwise given by

$$\nu_i = Msgn(\hat{y}_i - y_i) \tag{2.35}$$

Where $M \in \mathbf{R}_+$. Let $e_1 = \hat{x}_1 = x_1$and $e_y = \hat{y} - y$. Therefore

$$\dot{e}_1(t) = A_{11}e_1(t) + A_{12}e_y(t) - L\nu \tag{2.36}$$

$$\dot{e}_y(t) = A_{21}e_1(t) + \mathrm{A}_{22}e_y(t) - \nu \tag{2.37}$$

From observability of the pair $\{A, C\}$ we can conclude the observability of the pair $\{A_{11}, A_{21}\}$. As a consequence, L can be chosen to assign the eigenvalues of $A_{11} + LA_{21}$ in $\mathbf{C}_-$. By Defining another change of coordinates and after some manipulations (See [18] for details) it follows that inside a domain Ω [18] the reachability condition

$$e_y^T \dot{e}_y < -\eta \left\| e_y \right\| \tag{2.38}$$

is satisfied. Consequently, an ideal sliding motion will take place on the surface

$$S_o = \{(e_1, e_y) : e_y = 0\} \tag{2.39}$$

After some finite time t_s , for all subsequent time, $e_y = 0$ and $\dot{e}_y = 0$. It follows that

$$\dot{\tilde{e}}_1(t) = (A_{11} + LA_{21})\tilde{e}_1(t) \tag{2.40}$$

where $\tilde{e}_1 = e_1 + Ly$ and consequently, $\hat{x}_1 \to x_1$ as $t \to \infty$ by a proposer choice of stabilizing gain L. The Utkin SMO does not have a static observer gain in its structure and instead, the switching gain $L\nu$ plays the role of stabilizing the error dynamics. However, the disadvantage of this sliding observer structure reveals itself when there exist uncertainties and disturbances. In this case, the observer can only estimate the states with a bounded error and not asymptotically. Therefore, The walcott-Żak SMO offers much more appealing features. Thus, throughout this thesis, despite of proposing a new SMO structure to get around the major difficulties of walcott-Żak SMO, we also aim at achieving the same appealing features of it.

2.4 Singular Systems

Consider the system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.41}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$. When matrix E is nonsingular, system (2.41) becomes

$$\dot{x}(t) = E^{-1}Ax(t) + E^{-1}Bu(t)$$

This system is referred to as *normal* system. However, if E is singular, i.e., $\text{rank}(E) = q < n$, then this class of systems called singular (descriptor) systems [15]. In practical system analysis and control system design, many systems may be established in the form of (2.41), while they could not be modelled as a normal system. In many articles, singular systems are called descriptor variable systems, generalized state space systems, semi-state systems, differential-algebraic systems, constrained systems (See [15] and references there in). Singular systems theory includes many systems, such as engineering systems (for example: power system, electrical networks), social economic systems, network analysis. To provide a touchable example, we consider the following economical discrete-time system [15]:

Example. The fundamental dynamic Leontief model of economic is:

$$x(k) = \mathcal{A}x(k) + \mathcal{B}[x(k+1) - x(k)] + D(k) \tag{2.42}$$

where $x(k)$ represents the n dimensional production vector of n sectors; $\mathcal{A}$ input-output (or production) matrix; $\mathcal{A}x(k)$ the fraction of production required as input for the current production, $\mathcal{B}$ the capital coefficient matrix, and $\mathcal{B}[x(k+1) - x(k)]$ is the amount for capacity expansion. $D(k)$ is the vector that include demand or consumption. Equation (2.42) may be rewritten as

$$\mathcal{B}x(k+1) = (I - \mathcal{A} + \mathcal{B})x(k) - D(k).$$

Most of the elements in $\mathcal{B}$ are zero expect for a few of them. The reason is twofold: (i) In multi-sector economic, production in one sector often doesn't require the investment from all other sectors, and (ii) in practical cases only a few sectors can invest in other sectors. Hence $\mathcal{B}$ is very often *singular*. In this sense the system (2.42) is a typical discrete-time singular system in economy. Now, some useful and basic concepts of singular (descriptor) systems, needed in Chapter 5, are presented [15], [33].

System (2.41) is called *regular* if

$$\det(sE - A) \neq 0, s \in \mathbf{C} \tag{2.43}$$

which ensures that the plant (2.41) is solvable by possessing a unique solution for any given initial value. The pair $\{E, A\}$ (or equivalently the system (2.41)) is internally stable provided that

$$\text{rank}(sE - A) = n, \forall s \in \mathbf{C}_+ \tag{2.44}$$

The pair $\{E, A\}$ is impulse-free if

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}(E). \tag{2.45}$$

This ensures that the system (2.41) does not exhibit any impulsive behavior. Now, we present the observability concepts for singular systems.

Finite-Observability: The triple $\{E, A, C\}$ is called finite-observable if

$$\text{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \forall s \in \mathbf{C}. \tag{2.46}$$

Impulsive-Observability: The triple $\{E, A, C\}$ is called impulsive-observable if

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \\ C & 0 \end{bmatrix} = n + \text{rank}(E), \quad \forall s \in \mathbf{C}. \tag{2.47}$$

Finally, the singular system (2.41) is called *observable* if it is finite-observable and impulsive-observable. These definitions are crucial when we design observers in a singular system framework.

2.5 Notation

The notation used throughout the thesis is fairly standard. $\mathbf{R}_+$ represents the set of nonnegative real numbers, $\mathbf{C}_-$ ($\mathbf{C}_+$) the set of complex numbers with negative (positive) real parts, $\mathbf{C}$ the set of all complex numbers.When A is square, $A > 0 (\geq 0)$ denotes a symmetric positive definite (semi-definite) matrix, and $\lambda(A)$ denotes the eigenvalues of A. The symbol I_n represents the *n*th order unit matrix. The subscripts "T" and "-1" stand for matrix transposition and matrix inverse, respectively; $\mathbf{R}^n$ denotes n-dimensional Euclidean space. The space of square-integrable vector functions over $[0, \infty)$ is denoted by $\mathcal{L}_2[0, \infty)$, and for $\omega = \{\omega(t)\} \in \mathcal{L}_2[0, \infty)$, its norm is given by $\|\omega\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \omega^T(t)\omega(t)dt}$. Finally, $\|.\|$ denotes the Euclidean norm.

Chapter 3

$\mathcal{H}_\infty$ Sliding Mode Observers[1]

3.1 Background Results

In recent years the sliding mode discontinuous approach to control design of perturbed systems has attracted significant attention ([70], [71], [18]). A dual problem is *state reconstruction* or *observer design*. In a typical control problem involving systems described by state space realizations, one rarely has access to the full state since that would require one sensor for each state variable. Observers are then used to reconstruct the state from available measurements.

One of the principal sources of error in state reconstruction is the unavoidable deviation between the trajectories of a real system and the predictions by its mathematical model. Moreover, disturbance signals exist in virtually all control applications and the combined effect of these two problems generates the need to study observer design techniques that are *robust* to disturbances and model uncertainties. Sliding mode theory has been recognized as a promising robust control approach to confront uncertain or perturbed systems ([70], [71], [18]). When used in observer design, the sliding mode observer (SMO) forces the trajectories of the error dynamics to stay on a sliding surface (in the error space) despite the existence of perturba-

[1]The results in this chapter have been accepted for publication in the article: R Raoufi, H. J. Marquez and A. S. I. Zinober, "$\mathcal{H}_\infty$ Sliding Mode Observers for Uncertain Nonlinear Lipschitz Systems with Fault Estimation Synthesis", *International Journal of Robust and Nonlinear Control*, John Wiley and Sons, Inc., 2009.

tions, thus enabling the observer to reject disturbances when certain conditions are satisfied.

[72] introduced the use of the sliding mode approach in observer design and used Lyapunov theory to prove stability. [71] proposed an alternative approach to the design of sliding mode observers using a discontinuous sliding term fed back through a suitable gain. SMOs for linear unknown input systems were studied in [32]. [66] proposed a canonical sliding observer form design for linear systems in which a sufficient condition for stability based on linear matrix inequalities (LMIs) was derived. An LMI based SMO design method was proposed in [11] for a class of multivariable linear uncertain systems with matched uncertainties. Observation of linear systems with unknown inputs via high-order sliding-mode was addressed in [27] and [5]. More recently development of sliding mode observers for unknown input systems was proposed in [25]. For linear systems, necessary existence conditions in conjunction with design methods for SMOs have been addressed in [32], [66] and [11]. A more complex design procedure for SMO with less restrictive conditions using a supplementary SMO can be found in [65].

It is worth pointing out that the previous work referred to above consider only linear systems. For nonlinear systems, the synthesis and computation of the proper static and switching gains involved in the sliding mode observers are more challenging and complex. Sliding mode observers for Lipschitz nonlinear systems with bounded disturbance were addressed in [37]. An SMO for nonlinear models with unbounded noise and measurement uncertainties was studied by [75]. An adaptive sliding mode observer with a boundary layer sliding term was also proposed by [75]. In [24] the so-called super twisting step-by-step algorithm was employed to design a sliding mode observer for nonlinear systems with unknown inputs.

In this chapter we introduce a new SMO structure to tackle matched disturbances ([18] and [72]) modelled as a fault for Lipschitz nonlinear systems. This formulation commonly finds application in fault detection and estimation. Disturbances, that may not be matched, are also considered and a prescribed $\mathcal{H}_\infty$ disturbance attenuation level is integrated into the SMO design using LMI optimization.

[37] considered this problem without any disturbance and proposed a robust observer design technique for Lipschitz nonlinear systems. The approach in [37] is simple and direct, and provides an important step towards robust observers for Lipschitz systems. One shortcoming of this approach, however, is some level of conservatism and complexity that originates in simultaneously solving the so-called matching condition and the Lyapunov equation. One way to overcome this problem is to introduce the use of coordinate transformations, first introduced in [13] to facilitate this process, and is a key feature of the proposed SMO. Indeed, the observer proposed in this chapter has the characteristic that, in the new coordinates, the so-called structural matching condition is already satisfied by the novel structure of the switching gain proposed. A new structure for the observer static gain is proposed to facilitate the sliding mode observer design and guarantee the stability of the observer error dynamics. By extending the previous results regarding linear systems, we derive the same necessary existence conditions for the nonlinear case.

The main contribution of our work is the following: Since we consider disturbances without any structural or geometric conditions, the disturbances drastically corrupt the state estimation although the fault can be rejected by SMO theory. However, in recent papers in [53], [1], [2], the authors proposed $\mathcal{H}_\infty$ observers for Lipschitz nonlinear systems. In this article we extend the $\mathcal{H}_\infty$ observer principle to the sliding mode observer design. We obtain an LMI sufficient condition using convex optimization to compute the observer static gain. We satisfy the so-called matching condition and simultaneously maximize the disturbance attenuation level. Therefore, the LMI optimization problem structure is multiobjective. This feature considerably improves the state estimation.

Another important contribution of the proposed SMO is that the structure of the variable structure gain is of reduced order when compared to the number of measurements. In other words, only a subspace of the measurement output is required in the structure of the gain. Under the same assumptions, this novel feature gives us the possibility to tackle a class of disturbances at the output where, unlike [64], it is no imperative to use any state augmentation or low pass filtering.

After solving the observer design problem, we focus our attention on the problem of fault reconstruction. It is well known that sliding mode observers are capable of reconstructing unknown input or matched disturbances (For example [32], [64] and [74]). Therefore, by analyzing the error dynamics in the sliding mode and the notion of the equivalent control in the sliding mode, the potential of SMOs to reconstruct the unknown input is investigated. The proposed observer is more robust compared to the previous results addressed in [18], [32], [64] and [74], since the proposed $\mathcal{H}_\infty$-SMO is not only capable of estimating the states and the fault signal but also successful in reaching a prescribed $\mathcal{H}_\infty$ gain minimization. As a consequence it follows that the sliding mode observer can endure both faults (modelled as matched disturbances) and disturbances. In general, the accuracy of fault reconstruction directly depends on the state estimation. Thus, in the presence of disturbances, the shape of a reconstructed fault is drastically distorted. However, as a consequence of proposed $\mathcal{H}_\infty$ filtering integrated into SMO, fault estimation is much more robust against the disturbances and can preserve the fault signal shape effectively.

The remainder of this chapter is organized as follows; Section 3.2 provides some preliminaries and assumptions on the class of nonlinear system addressed. Some preliminary lemmas are introduced in Section 3.3. The design of the robust SMO and the analysis of the stability of the error dynamics are given in Section 3.4. In Section 3.5 the synthesis of the error system in the sliding mode is discussed. Fault estimation is studied in Section 3.6. The effectiveness of the proposed SMO is studied with an example in Section 3.7. Finally some concluding remarks are presented in Section 3.8.

3.2 Preliminaries and Assumptions

Consider a dynamical systems of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + \Phi(x,t) + Ef(t) + \Delta\xi(t) \\ y(t) = Cx(t) + D\omega(t) \end{cases} \tag{3.1}$$

where $x \in \mathbf{R}^n$ represents the system state, $u \in \mathbf{R}^m$ the control input, $y \in \mathbf{R}^p$ the measured system output and $t \in \mathbf{R}^+$. (A, B, C, E, Δ, D) is the set of real constant known matrices of appropriate dimensions where $D \in \mathbf{R}^{p\times(p-q)}$. $f(t) : \mathbf{R}^+ \to \mathbf{R}^q$ denotes the fault (unknown input) that is bounded in the Euclidean norm

$$\|f(t)\| \leq \rho < \infty. \tag{3.2}$$

The signal $\xi(t) : \mathbf{R}^+ \to \mathbf{R}^r \in \mathcal{L}_2[0,\infty)$ models the uncertainties and disturbances where Δ is the corresponding distribution matrix. $\omega(t) : \mathbf{R}^+ \to \mathbf{R}^{p-q} \in \mathcal{L}_2[0,\infty)$ represents the output disturbances where D is the corresponding distribution matrix with full columns rank. Therefore, without lose of generality, we can assume the following geometric condition associated with D:

$$D = \left[\begin{array}{c} 0 \\ \hline D_2 \end{array} \right] \tag{3.3}$$

where $D_2 \in \mathbb{R}^{(p-q)\times(p-q)}$ and is invertible. The known nonlinearity $\Phi(x,t)$ satisfies a locally Lipshitz condition as in (2.7). We make the following two well-known assumptions:

Minimum Phase Condition: For every complex number s with nonnegative real part

$$\text{rank} \begin{bmatrix} sI_n - A & E \\ C & 0 \end{bmatrix} = n + \text{rank}(E). \tag{3.4}$$

Matching Condition: Assume that there exists an arbitrary matrix $F \in \mathbf{R}^{q\times p}$ and $P = P^T > 0 \in \mathbf{R}^{n\times n}$ satisfying

$$E^T P = FC. \tag{3.5}$$

3.3 Some Preliminary Lemmas

Lemma 3.1. [13] *Given the system (1), we have*

$$rank(CE) = rank\ (E) = q \tag{3.6}$$

if and only if there exist nonsingular transformation matrices T and S such that

$$\begin{aligned} TAT^{-1} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad TE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \\ SCT^{-1} &= \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}, \quad T\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \end{aligned} \tag{3.7}$$

where $A_1 \in \mathbf{R}^{q\times q}$, $A_4 \in \mathbf{R}^{(n-q)\times(n-q)}$, $C_1 \in \mathbf{R}^{q\times q}$, $C_4 \in \mathbf{R}^{(p-q)\times(n-q)}$, rank $(E_1) = q$ and C_1 is invertible.

Remark. Assume that $p = q$ so that $C_4 \in \{\emptyset\}$. Then, from Lemma 3.1, it follows that the minimum phase condition (3.4) holds for all s such that $Re(s) \geq 0$ if and only if the matrix A_4 is asymptotically stable.

3.3.1 Computation of T and S:

In [32] and [13], for the case $D = 0$, different methods for computation of T and S for linear systems were reported based on the Q-R decomposition and the singular value decomposition respectively. Those method can be used for system (3.1) when $D = 0$. Here, for the more general case where $D \neq 0$, we present another method of computing T and S that is straightforward. By assumption we have rank $(E) = q$. Therefore, without loss of generality, we partition the matrix E as

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{3.8}$$

where $E_1 \in \mathbf{R}^{q\times q}$ with rank $(E_1) = q$. Now, introduce a nonsingular coordinate transformation

$$T_1 = \begin{bmatrix} I_q & 0 \\ -E_2E_1^{-1} & I_{n-q} \end{bmatrix} \tag{3.9}$$

then

$$T_1E = \begin{bmatrix} I_q & 0 \\ -E_2E_1^{-1} & I_{n-q} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ -E_2E_1^{-1}E_1 + E_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tag{3.10}$$

where $E_1 \in \mathbf{R}^{q\times q}$ is nonsingular. Now we partition CT_1^{-1} as $CT_1^{-1} = (\bar{C}_1 \;\; \bar{C}_4)$. Therefore

$$CE = CT_1^{-1}T_1E = (\bar{C}_1 \;\; \bar{C}_4)\begin{bmatrix} E_1 \\ 0 \end{bmatrix} = \bar{C}_1E_1. \tag{3.11}$$

Consequently, using Assumption rank$(CE) = q$ and the nonsingularity of the matrix E_1, we conclude directly that

$$\text{rank}(\bar{C}_1) = q. \tag{3.12}$$

Therefore, without loss of any generality, we partition $\bar{C}_1$ as follows

$$\bar{C}_1 = \begin{bmatrix} C_1 \\ C_{21} \end{bmatrix}, \tag{3.13}$$

where $C_1 \in \mathbf{R}^{q\times q}$ and

$$\text{rank}(C_1) = q. \tag{3.14}$$

Consequently det $(C_1) \neq 0$. Let

$$S = \begin{bmatrix} I_q & 0 \\ -C_{21}C_1^{-1} & I_{p-q} \end{bmatrix} \tag{3.15}$$

which yields

$$S\bar{C}_1 = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}. \tag{3.16}$$

Therefore

$$SCT_1^{-1} = \begin{bmatrix} C_1 & C_{12} \\ 0 & C_4 \end{bmatrix}. \tag{3.17}$$

Let

$$T_2^{-1} = \begin{bmatrix} I_q & -C_1^{-1}C_{12} \\ 0 & I_{n-q}. \end{bmatrix} \tag{3.18}$$

Then one obtains

$$SCT_1^{-1}T_2^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}. \tag{3.19}$$

Finally, we obtain T from $T = T_2T_1$.

Lemma 3.2. [32] *Consider the system (3.1) and assume that* rank $(CE) =$ rank (E). *Then the pair (A_4, C_4) is detectable if and only if*

$$\text{rank}\begin{bmatrix} sI_n - A & E \\ C & 0 \end{bmatrix} = n + q \tag{3.20}$$

for all s such that $Re(s) \geq 0$.

Lemma 3.3. *Consider the system (1). There exists a solution $P = P^T > 0$ such that $E^T P = FC$ if and only if*

$$\text{rank}(CE) = \text{rank }(E) \tag{3.21}$$

Proof:

(proof of necessity)

Define the change of coordinate $\bar{T} \in \mathbf{R}^{n \times n}$ as

$$\bar{T} := \begin{bmatrix} C_\perp^T P \\ C \end{bmatrix} \tag{3.22}$$

where $C_\perp \in \mathbf{R}^{n \times (n-p)}$ is any full rank matrix whose columns span the null space of C. This transformation is nonsingular. Then, with respect to the structural condition (3.5), it follows that

$$\bar{T}E = \left[\frac{C_\perp^T P(P^{-1}C^T F^T)}{CE}\right] = \left[\frac{0}{CE}\right]. \tag{3.23}$$

Note that we use the property $CC_\perp = 0$. Since rank$(E) = q$ and the nonsingular transformation $\bar{T}$ preserves the rank of $\bar{T}E$, rank$(CE) = m$.

(proof of sufficiency)

Since rank$(CE) =$ rank (E), using Lemma 3.1 it follows that there exist nonsingular similarity transformation matrices T, S satisfying

$$\tilde{E} := TE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad \tilde{C} := SCT^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}$$

and E_1 is full row rank. Thus $E = T^{-1}\tilde{E}$ and $C = S^{-1}\tilde{C}T$. Letting $P = T^T \tilde{P} T$ and $F = \tilde{F}S$ where $\tilde{P}$ is symmetric, substituting P, F, E, C into the matching condition (3.5) yields $\tilde{E}^T \tilde{P} = \tilde{F}\tilde{C}$. We select

$$\tilde{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad \tilde{F} = \begin{bmatrix} {E_1}^T P_1 C_1^{-1} & 0 \end{bmatrix} \tag{3.24}$$

Then, considering the structure of $\tilde{E}$ and $\tilde{C}$, the matrix equality $\tilde{E}^T \tilde{P} = \tilde{F}\tilde{C}$ always holds. This completes the proof.

Lemma 3.4. *Under change of coordinate S defined in (3.15) we have $SD = D$.*

Proof.

$$SD = \begin{bmatrix} I_q & 0 \\ -C_{21}C_1^{-1} & I_{p-q} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} = D. \tag{3.25}$$

3.4 Sliding Mode Observer Design

In this section, we propose a theorem to design the sliding mode observer. We employ the nonsingular state transformations introduced in Lemma 3.1 as the key to deal with the problem of $\mathcal{H}_\infty$ SMO design. Based on Lemma 3.1, system (3.1) in the new coordinates $\tilde{x} := (x_1^T, x_2^T)^T = Tx$ and $\tilde{y} := (y_1^T, y_2^T)^T = Sy$ is

$$\begin{cases} \dot{x}_1 = A_1x_1 + A_2x_2 + B_1u + \Phi_1(T^{-1}\tilde{x}, t) + E_1f(t) + \Delta_1\xi(t) \\ y_1 = C_1x_1 \quad (x_1 \in \mathbf{R}^q) \end{cases} \tag{3.26}$$

$$\begin{cases} \dot{x}_2 = A_3x_1 + A_4x_2 + B_2u + \Phi_2(T^{-1}\tilde{x}, t) + \Delta_2\xi(t) \\ y_2 = C_4x_2 + D_2\omega(t) \quad (x_2 \in \mathbf{R}^{n-q}) \end{cases} \tag{3.27}$$

where $T\Phi(x,t) := (\Phi_1^T, \Phi_2^T)^T$. Also partition S as

$$S = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{bmatrix}, \bar{S}_1 \in \mathbf{R}^{q\times p}, \bar{S}_2 \in \mathbf{R}^{(p-q)\times p}.$$

So that the variable x_1 can be obtained from the measured output y by

$$x_1 = C_1^{-1}\bar{S}_1y(t) \tag{3.28}$$

We will employ the above system structure in our observer design. Consider the following sliding mode observer structure

$$\begin{cases} \dot{\hat{x}}_1 = A_1\hat{x}_1 + A_2\hat{x}_2 + B_1u + \Phi_1(T^{-1}\hat{\tilde{x}}, t) \\ \qquad +L_1(y_1 - \hat{y}_1) + E_1\nu \\ \hat{y}_1 = C_1\hat{x}_1 \end{cases} \tag{3.29}$$

$$\begin{cases} \dot{\hat{x}}_2 = A_4\hat{x}_2 + B_2u + \Phi_2(T^{-1}\hat{\tilde{x}}, t) \\ \qquad +L_3y_1 + L_4(y_2 - \hat{y}_2) \\ \hat{y}_2 = C_4\hat{x}_2 \end{cases} \tag{3.30}$$

where the novel reduced-order sliding mode gain $\nu(t)$ and the observer gain $\tilde{L}$ are respectively

$$\nu(t) = \begin{cases} (\rho + \rho_0)\frac{E_1^TP_1(C_1^{-1}\bar{S}_1y - \hat{x}_1)}{\|E_1^TP_1(C_1^{-1}\bar{S}_1y - \hat{x}_1)\|} & : \ C_1^{-1}\bar{S}_1y - \hat{x}_1 \neq 0 \\ 0 & : \text{otherwise} \end{cases} \tag{3.31}$$

$$\tilde{L} := \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} := \begin{bmatrix} \bar{A}_1 C_1^{-1} & 0 \\ A_3 C_1^{-1} & P_2^{-1} K \end{bmatrix} \tag{3.32}$$

where ρ_0 is some positive scalar. P_1, P_2 and K will be determined through the stability proof and $\bar{A}_1 = A_1 - A_1^s$ where A_1^s represents a stable design matrix. Furthermore, suppose that

$$z(t) = H \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix} \tag{3.33}$$

is the controlled output for the error system where H is a full rank design matrix (See [1] and [2] and references therein) having the following structure

$$H := \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{3.34}$$

Consider the standard induced $\mathcal{L}_2$ gain between z and disturbances $\varpi = \begin{bmatrix} \xi \\ \omega \end{bmatrix}$ (also known as $\mathcal{H}_\infty$ gain)

$$\|\mathcal{H}\|_\infty^2 = \gamma = \sup_{\|\varpi\|_{\mathcal{L}_2} \neq 0} \frac{\|z\|_{\mathcal{L}_2}^2}{\|\varpi\|_{\mathcal{L}_2}^2}. \tag{3.35}$$

We now present Theorem 3.1 which is the main result of this section. The importance of this theorem is that it establishes sufficient conditions for the existence of a sliding mode observer with a prescribed $\mathcal{H}_\infty$ performance for system (3.1) and outlines a constructive design procedure.

Theorem 3.1. *Given the nonlinear uncertain system (3.1) with assumptions (2.7), (3.2), (3.4) and(3.5), consider the SMO structure (3.29)-(3.32). The observer error dynamics is asymptotically stable for the case $\varpi = 0$ with an $\mathcal{H}_\infty$ disturbance attenuation level $\sqrt{\gamma} > 0$ subject to $\|\mathcal{H}\|_\infty \leq \sqrt{\gamma}$ if there exist matrices K, $P_1^T = P_1 > 0$ and $P_2^T = P_2 > 0$ such that the following LMI optimization problem has a solution:*

minimize γ subject to $P_1 > 0$, $P_2 > 0$ and

$$\begin{bmatrix} \Pi_{11} & P_1 A_2 & P_1 \Delta_1 & 0 & P_1 & 0 \\ A_2^T P_1 & \Pi_{22} & P_2 \Delta_2 & K D_2 & 0 & P_2 \\ \Delta_1^T P_1 & \Delta_2^T P_2 & -\gamma I & 0 & 0 & 0 \\ 0 & D_2^T K^T & 0 & -\gamma I & 0 & 0 \\ P_1 & 0 & 0 & 0 & -I & 0 \\ 0 & P_2 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{3.36}$$

where

$$\Pi_{11} = P_1 A_1^s + A_1^{sT} P_1 + \tilde{\mathcal{L}}_\Phi^2 I_q + H_1^T H_1 \tag{3.37}$$

$$\Pi_{22} = {A_4}^T P_2 + P_2 A_4 - (K C_4 + {C_4}^T K^T) + \tilde{\mathcal{L}}_\Phi^2 I_{n-q} + H_2^T H_2 \tag{3.38}$$

Proof. Using the system structure in (3.26) and (3.27), and the observer equations described by (3.29) and(3.30) along with the observer gain (3.32), it is observed that the error dynamics in the new coordinate is

$$\dot{\tilde{e}} = \tilde{A}_0 \tilde{e} + T(\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)) + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} (f(t) - \nu) + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ L_4 D_2 \end{bmatrix} \omega \tag{3.39}$$

where

$$\tilde{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix}, \ \tilde{A}_0 = \begin{bmatrix} A_1^s & A_2 \\ 0 & A_4 - L_4 C_4 \end{bmatrix}.$$

Therefore in the new coordinates, it can be easily verified that $\lambda(\tilde{A}_0) = \lambda(A_1^s) \cup \lambda(A_4 - L_4 C_4)$. Consider the Lyapunov function $V(\tilde{e}) = \tilde{e}^T(t) \tilde{P} \tilde{e}$ where

$$\tilde{P} := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, P_1 \in \mathbf{R}^{q \times q}, \ P_2 \in \mathbf{R}^{(n-q) \times (n-q)} \tag{3.40}$$

with symmetric $P_1 > 0, P_2 > 0$ are yet to be determined. $\tilde{e} = \tilde{x} - \hat{\tilde{x}}$ is defined as the state estimation error. The derivative of $V(\tilde{e})$ is

$$\begin{aligned} \dot{V} &= \tilde{e}^T ({\tilde{A}_0}^T \tilde{P} + \tilde{P} \tilde{A}_0) \tilde{e} + \tilde{e}^T \tilde{P} T (\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)) \\ &+ (\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t))^T T^T \tilde{P} \tilde{e} \\ &+ 2\tilde{e}^T \tilde{P} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} (f(t) - \nu) + \tilde{e}^T \tilde{P} \begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix} \varpi + \varpi^T \begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix}^T \tilde{P} \tilde{e} \end{aligned} \tag{3.41}$$

where $\varpi^T = \begin{bmatrix} \xi^T & \omega^T \end{bmatrix}$. From (3.40) we have

$$\tilde{e}^T \tilde{P} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} = e_1^T P_1 E_1. \tag{3.42}$$

From (3.28) it follows that $e_1 = C_1^{-1} \bar{S}_1 y - \hat{x}_1$. Then using the switching gain (3.31) and (3.2) we obtain

$$\begin{aligned} \tilde{e}^T \tilde{P} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} (f(t) - \nu) = e_1^T P_1 E_1 f(t) - (\rho + \rho_0) \tfrac{\|e_1^T P_1 E_1\|^2}{\|e_1^T P_1 E_1\|} \leq \\ \rho \|e_1^T P_1 E_1\| - (\rho + \rho_0) \|e_1^T P_1 E_1\| = -\rho_0 \|e_1^T P_1 E_1\| < 0 \end{aligned} \tag{3.43}$$

Moreover, using the Lipschitz condition (2.7) we have

$$\begin{aligned}
&\tilde{e}^T \tilde{P} T(\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)) + (\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t))^T T^T \tilde{P}\tilde{e} \\
&\leq \tilde{e}^T \tilde{P}^2 \tilde{e} + (\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t))^T T^T T(\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)) \\
&= \tilde{e}^T \tilde{P}^2 \tilde{e} + \left\| T(\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)) \right\|^2 \\
&\leq \tilde{e}^T \tilde{P}^2 \tilde{e} + \|T\|^2 \mathcal{L}_\Phi^2 \|e\|^2 \leq \tilde{e}^T \tilde{P}^2 \tilde{e} + \tilde{\mathcal{L}}_\Phi^2 \|\tilde{e}\|^2
\end{aligned} \tag{3.44}$$

where $\tilde{\mathcal{L}}_\Phi := ||T||.||T^{-1}||\mathcal{L}_\Phi$. Thus, substituting (3.43) and (3.44) into (3.41), it follows that

$$\dot{V} \leq \tilde{e}^T(\tilde{A}_0^{\,T}\tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{P}^2 + \tilde{\mathcal{L}}_\Phi^2 I)\tilde{e} + \tilde{e}^T\tilde{P}\begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix}\varpi + \varpi^T\begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix}^T \tilde{P}\tilde{e}$$

To attain robustness to the disturbances in $\mathcal{L}_2$ sense, we impose the following constraint on our stability criteria

$$\dot{V} + z^T(t)z(t) - \gamma\varpi^T(t)\varpi(t) \leq 0. \tag{3.45}$$

Integration of both sides of the above condition with respect to t over the time period $[0, \infty]$ gives

$$V(\infty) - V(0) + \int_0^\infty \left[z^T(t)z(t) - \gamma\varpi^T(t)\varpi(t)\right] dt \leq 0. \tag{3.46}$$

Since $V(\infty) \geq 0$, with zero initial condition $V(0) = 0$, one obtains

$$\sqrt{\int_0^\infty z^T(t)z(t)dt} \leq \sqrt{\gamma}\sqrt{\int_0^\infty \varpi^T(t)\varpi(t)dt} : \qquad \forall t > 0 \tag{3.47}$$

therefore we have

$$\frac{\sqrt{\int_0^\infty z^T(t)z(t)dt}}{\sqrt{\int_0^\infty \varpi^T(t)\varpi(t)dt}} \leq \sqrt{\gamma} : \qquad \forall t > 0. \tag{3.48}$$

Thus the definition of the induced $\mathcal{L}_2$ norm and (3.45) yields

$$\|\mathcal{H}\|_\infty \leq \sqrt{\gamma} \tag{3.49}$$

In other words, (3.45) enforces the minimization of the worst case effect of the disturbance on the estimation error. Let

$$\tilde{A}_0^{\,T}\tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{P}^2 + \tilde{\mathcal{L}}_\Phi^2 I = -\tilde{Q}. \tag{3.50}$$

Then from (3.45), we have

$$\begin{aligned}
&\dot{V} + z^T(t)z(t) - \gamma\varpi^T(t)\varpi(t) \\
&\leq \tilde{e}^T(-\tilde{Q} + H^T H)\tilde{e} + \tilde{e}^T\tilde{P}\begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix}\varpi \\
&+\varpi^T\begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & L_4 D_2 \end{bmatrix}^T \tilde{P}\tilde{e} - \gamma\varpi^T(t)\varpi(t)
\end{aligned}$$

Let

$$\tilde{Q} := \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}. \tag{3.51}$$

From the decomposed structure of $\tilde{A}_0$ and $\tilde{P}$ in (3.40), after some algebraic manipulation and simplifications, it can be verified that

$$\begin{aligned}
&-\tilde{Q} + H^T H = \begin{bmatrix} A_1^{sT} & 0 \\ A_2^T & A_4^T - C_4^T L_4^T \end{bmatrix}\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\
&+\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} A_1^s & A_2 \\ 0 & A_4 - L_4 C_4 \end{bmatrix} \\
&+\begin{bmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{bmatrix} + \begin{bmatrix} \tilde{\mathcal{L}}_\Phi^2 I_q & 0 \\ 0 & \tilde{\mathcal{L}}_\Phi^2 I_{(n-q)} \end{bmatrix} \\
&+H^T H = \begin{bmatrix} P_1 A_1^s + A_1^{sT} P_1 + P_1^2 + \tilde{\mathcal{L}}_\Phi^2 I_q + H_1^T H_1 & P_1 A_2 \\ A_2^T P_1 & -Q_2 + H_2^T H_2 \end{bmatrix}
\end{aligned} \tag{3.52}$$

where

$$(A_4 - L_4 C_4)^T P_2 + P_2(A_4 - L_4 C_4) + P_2^2 + \tilde{\mathcal{L}}_\Phi^2 I = -Q_2. \tag{3.53}$$

Thus we obtain

$$\begin{aligned}
&\dot{V} + z^T(t)z(t) - \gamma\varpi^T(t)\varpi(t) \leq \begin{bmatrix} e_1 \\ e_2 \\ \varpi \end{bmatrix}^T \\
&\times \begin{bmatrix} P_1 A_1^s + A_1^{sT} P_1 + P_1^2 + \tilde{\mathcal{L}}_\Phi^2 I_q + H_1^T H_1 & P_1 A_2 & P_1\Delta_1 & 0 \\ A_2^T P_1 & -Q_2 + H_2^T H_2 & P_2\Delta_2 & P_2 L_4 D_2 \\ \Delta_1^T P_1 & \Delta_2^T P_2 & -\gamma I & 0 \\ 0 & D_2^T L_4^T P_2 & 0 & -\gamma I \end{bmatrix} \\
&\times \begin{bmatrix} e_1 \\ e_2 \\ \varpi \end{bmatrix}.
\end{aligned} \tag{3.54}$$

If the inner matrix of the righthand side of the above inequality is negative definite, then $\dot{V} + z^T(t)z(t) - \gamma\varpi^T(t)\varpi(t) \leq 0$. The asymptotical stability of the observation error when $\varpi = 0$ is a direct result of the above optimization problem by setting $\Delta_1 = \Delta_2 = D_2 = 0$. Thus, the observer error dynamics is asymptotically stable with the prescribed $\mathcal{H}_\infty$ attenuation level $\sqrt{\gamma}$. Notice that the above matrix inequality is nonlinear. From the structure of the observer gain $\tilde{L}$ in (3.32), it is observed that $L_4 = P_2^{-1}K$ and consequently, by using the Schur complement [6], we can express it in the Linear Matrix Inequality (LMI) form

$$\begin{bmatrix} \Pi_{11} & P_1A_2 & P_1\Delta_1 & 0 & P_1 & 0 \\ A_2^TP_1 & \Pi_{22} & P_2\Delta_2 & KD_2 & 0 & P_2 \\ \Delta_1^TP_1 & \Delta_2^TP_2 & -\gamma I & 0 & 0 & 0 \\ 0 & D_2^TK^T & 0 & -\gamma I & 0 & 0 \\ P_1 & 0 & 0 & 0 & -I & 0 \\ 0 & P_2 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{3.55}$$

where

$$\Pi_{11} = P_1A_1^s + A_1^{sT}P_1 + \tilde{\mathcal{L}}_\Phi^2 I_q + H_1^TH_1 \tag{3.56}$$

$$\Pi_{22} = A_4{}^TP_2 + P_2A_4 - (KC_4 + C_4{}^TK^T) + \tilde{\mathcal{L}}_\Phi^2 I_{n-q} + H_2^TH_2 \tag{3.57}$$

This completes the proof.

Notice that the necessary condition for the existence of any solution of the LMI (3.36) is that the pair (A_4, C_4) must be detectable . According to Lemma 3.2, the detectability of (A_4, C_4) has been guaranteed by Assumption (3.4). Furthermore, the LMI optimization problem derived here seeks two main objectives. Technically speaking, the first objective is the computation of the proper matrices P_2 and K; while the second objective is boosting the robustness of the sliding mode observer against disturbances $\varpi(t)$ by minimizing the $\mathcal{H}_\infty$ gain between the controlled output observation error $z(t)$ and $\varpi(t)$. It is clear that the smaller the computed γ is, the more robust the sliding mode observer becomes.

Remark. Notice that in the error dynamics of e_2, we have $A_3x_1 - L_3y_1 = A_3x_1 - A_3C_1^{-1}(C_1x_1) = 0$. Therefore, we are able to cancel the appearance of the terms including A_3 by design.

Remark. In the case that there is no disturbance at the output of the original system (3.1), i.e $\omega(t) = 0$, we have the following simplified LMI optimization which is a direct result of Theorem 3.1 by setting $D_2 = 0$.

minimize γ subject to $P_1 > 0$, $P_2 > 0$ and

$$\begin{bmatrix} \Pi_{11} & P_1 A_2 & P_1 \Delta_1 & P_1 & 0 \\ A_2^T P_1 & \Pi_{22} & P_2 \Delta_2 & 0 & P_2 \\ \Delta_1^T P_1 & \Delta_2^T P_2 & -\gamma I & 0 & 0 \\ P_1 & 0 & 0 & -I & 0 \\ 0 & P_2 & 0 & 0 & -I \end{bmatrix} < 0 \tag{3.58}$$

$$\Pi_{11} = P_1 A_1^s + A_1^{sT} P_1 + \tilde{\mathcal{L}}_\Phi^2 I_q + H_1^T H_1 \tag{3.59}$$

$$\Pi_{22} = A_4^T P_2 + P_2 A_4 - (K C_4 + C_4^T K^T) + \tilde{\mathcal{L}}_\Phi^2 I_{n-q} + H_2^T H_2 \tag{3.60}$$

Remark. Recently, the sliding mode observer for Lipschitz nonlinear systems has been studied in [74], where both faults and disturbances are assumed as matched uncertainties. This assumption either can be very restrictive or, due to the structure of disturbances, may fail. In comparison, in the proposed $\mathcal{H}_\infty$-SMO, disturbances ξ can be unmatched and $\mathcal{H}_\infty$ filtering was introduced to cope with this problem.

Remark. According to the structure of the switching gain ν in (3.31), it follows that $\nu \in \mathbf{R}^q$. We define

$$\mathcal{Y} = \{y \in \mathbf{R}^p |\ y = Cx\} \tag{3.61}$$

$$\mathcal{Y}_1 = \{y_1 \in \mathbf{R}^q |\ y_1 = \bar{S}_1 y\} \tag{3.62}$$

$$\mathcal{Y}_2 = \{y_2 \in \mathbf{R}^{p-q} |\ y_2 = \bar{S}_2 y\}. \tag{3.63}$$

Thus in the case $q < p$, $\mathcal{Y}_1 \subset \mathcal{Y}$ and the variable structure gain ν needs only the components of the output measurement y_1. Therefore, any disturbances at the output y_2 are acceptable to exist since they are not involved in the structure of the sliding mode controller ν. To further elaborate on this feature, in [18], [32], [66], [37], [64] and [74], the sliding mode controller has the following general form

$$\nu = (\rho + \rho_0)\frac{P_0(y - C\hat{x})}{||P_0(y - C\hat{x})||}$$

with order p. As mentioned earlier, the proposed sliding mode gain (3.31) is of reduced order q.

Remark. References [18], [32], [66], [37], [64] and [74] studied the design of pure sliding mode observers. Without integrating the $\mathcal{H}_\infty$ filtering into the SMO design, a pure SMO can also be designed based on the observer structure proposed in this article. In such case, the observer error dynamics is ultimately bounded if the following standard LMI feasibility problem has a solution:

For given A_1^s and $\tilde{\mathcal{L}}_\Phi > 0$, find matrices $P_1 > 0$, $P_2 > 0$ and K such that

$$\begin{bmatrix} \Pi_{11} & P_1A_2 & P_1 & 0 \\ A_2^TP_1 & \Pi_{22} & 0 & P_2 \\ P_1 & 0 & -I & 0 \\ 0 & P_2 & 0 & -I \end{bmatrix} < 0 \tag{3.64}$$

$$\Pi_{11} = P_1A_1^s + A_1^{sT}P_1 + \tilde{\mathcal{L}}_\Phi^2 I_q \tag{3.65}$$

$$\Pi_{22} = A_4^TP_2 + P_2A_4 - (KC_4 + C_4{}^TK^T) + \tilde{\mathcal{L}}_\Phi^2 I_{n-q} \tag{3.66}$$

Using the above LMIs, the pure SMO performance will be compared to the proposed $\mathcal{H}_\infty$-SMO in the simulation example.

Remark. With regard to (3.40), the same Lyapunov quadratic form has been used in previous results such as references [18], [32], [13], [64], and [74] for stability analysis. Note that $\tilde{P}$ in the new coordinates is diagonal and since

$$P = T^T\tilde{P}T, \quad \tilde{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

the Lyapunov function in the original coordinates would be $V = e^TPe$ where the matrix P is not necessarily diagonal.

3.5 Synthesis of the Sliding Motion

It is well-known that, in order to confine an stable motion of a dynamical system onto a sliding surface $\mathcal{S}$, it is necessary to use a switching gain which is discontinuous about the surface $\mathcal{S}$ [18]. Therefore, due to the structure of the switching gain (3.31) and the fact that $\mathcal{N}(E_1) = \{\emptyset\}$, it follows that

$$\mathcal{S} = \{t \in \mathbf{R}^+ : s(t) = 0 \mid s(t) = C_1^{-1}\bar{S}_1 y - \hat{x}_1\}. \tag{3.67}$$

Suppose that disturbances $\xi(t), \omega(t)$ are bounded subject to $\|\xi(t)\| \leq \xi_0 < \infty$ and $\|\omega(t)\| \leq \omega_0 < \infty$ respectively . The main result of this section will now be proved.

Theorem 3.2. *Given the system (3.1) with assumptions (3.2)-(3.5) and the observer (3.29)-(3.32), an ideal sliding motion takes place after some finite time on the hyperplane $\mathcal{S}$ if the convex optimization problem stated in Theorem 3.1 is solvable.*

Proof. The error system with respect to the new coordinates can be written as

$$\left\{ \begin{array}{l} \dot{e}_1 = (A_1 - \bar{A}_1)e_1 + A_2 e_2 + e_{\Phi_1} + E_1(f(t) - \nu) + \Delta_1 \xi \\ \dot{e}_2 = (A_4 - L_4 C_4)e_2 + e_{\Phi_2} + \Delta_2 \xi + L_4 D_2 \omega \end{array} \right. \tag{3.68}$$

where for simplicity we defined

$$\begin{bmatrix} e_{\Phi_1} \\ e_{\Phi_2} \end{bmatrix} := T(\Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t))$$

If the optimization problem in Theorem 3.1 is solvable, then it implies that the error dynamics is asymptotically stable with prescribed $\mathcal{H}_\infty$ filtering attenuation γ. Thus for some small $\varepsilon_\infty > 0$, we have $\|\tilde{e}\| \leq \varepsilon_\infty$. We obtain

$$\begin{array}{l} s^T \dot{s} = e_1^T((A_1 - \bar{A}_1)e_1 + A_2 e_2 + e_{\Phi_1} + E_1(f(t) - \nu) + \Delta_1 \xi) \\ \leq \|e_1\|(\|A_1 - \bar{A}_1\|\|e_1\| + \|A_2 e_2\| + \xi_0\|\Delta_1\| + \tilde{\mathcal{L}}_\Phi \varepsilon_\infty) - \rho_0\|E_1^T e_1\| \\ \leq \|E_1^{-T}\|\|E_1^T e_1\|(\|A_1 - \bar{A}_1\|\|e_1\| + \|A_2 e_2\| + \tilde{\mathcal{L}}_\Phi \varepsilon_\infty + \xi_0\|\Delta_1\| - \rho_0) \end{array} \tag{3.69}$$

Therefore it follows that in the domain

$$\Omega = \{(e_1, e_2) : \ \|A_1^s\|\|e_1\| + \|A_2 e_2\| + \tilde{\mathcal{L}}_\Phi \varepsilon_\infty + \xi_0\|\Delta_1\| < \rho_o - \tilde{\rho}\} \tag{3.70}$$

where $\tilde{\rho} < \rho_0$ is some small positive constant, the well-known reachability condition [18]

$$s^T \dot{s} < -\tilde{\rho}\|s\| \tag{3.71}$$

is satisfied. As a consequence, an ideal sliding motion will take place on the surface $\mathcal{S}$ and after some finite time t_s

$$e_1 = \dot{e}_1 = 0, \ \forall t > t_s. \tag{3.72}$$

3.6 Robust Fault Estimation

In this section the objective is to reconstruct the system fault $f(t)$ by using the proposed observer and the output information.

Corollary 3.1. *Let all conditions of Theorem 3.1 hold and $|H| \neq 0$, then*

$$\|f(t) - v_{eq}\|_{\mathcal{L}_2} \leq \beta \|\varpi\|_{\mathcal{L}_2}, \tag{3.73}$$

where $\varpi = [\xi^T \ \ \omega^T]^T$, and

$$\begin{array}{l} \beta = \sqrt{\gamma}[\sigma_{max}(E_1^{-1}A_2) + \sigma_{max}(E_1^{-1}\tilde{\mathcal{L}}_\Phi)]\sigma_{max}(H^{-1}) \\ +\sigma_{max}(E_1^{-1}\Delta_1) + \sigma_{max}(E_1^{-1}L_4 D_2). \end{array} \tag{3.74}$$

Consequently, the unknown input (or fault) $f(t)$ can be approximately estimated by

$$\hat{f}(t) = (\rho + \rho_0)\frac{E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\| + \delta}. \tag{3.75}$$

Proof. If all conditions of Theorem 3.1 are satisfied and the LMI optimization is solved, then

$$\|z\|^2_{\mathcal{L}_2} \leq \gamma \|\varpi\|^2_{\mathcal{L}_2} \tag{3.76}$$

for some $\gamma > 0$ and $z = H\tilde{e}, \ \tilde{e} = [{e_1}^T \ \ {e_2}^T]^T$. Next, according to Theroem 2 the ideal sliding motion takes place on $\mathcal{S}$ in finite time and $\dot{e}_1 = e_1 = 0$. Consequently, the error dynamics for e_1 in sliding mode is given by

$$0 = A_2 e_2 + e_{\Phi_1} + E_1(f(t) - \nu_{eq}) + \Delta_1 \xi + L_4 D_2 \omega, \tag{3.77}$$

where, in the sliding mode, the discontinuous signal ν in (3.31) must takes on the average ν_{eq} (referred to as the equivalent output error injection [71]) to preserve the sliding motion. Then

$$f(t) - \nu_{eq} = -E_1^{-1}[A_2 e_2 + e_{\Phi_1} + \Delta_1 \xi + L_4 D_2 \omega], \tag{3.78}$$

and

$$\begin{aligned} &\|f(t) - v_{eq}\|_{\mathcal{L}_2} \leq \|E_1^{-1}[A_2 e_2 + e_{\Phi_1} + \Delta_1 \xi + L_4 D_2 \omega]\|_{\mathcal{L}_2} \\ &\leq \sigma_{max}(E_1^{-1} A_2)\|e_2\|_{\mathcal{L}_2} + \sigma_{max}(E_1^{-1})\tilde{\mathcal{L}}_\Phi \|e_1\|_{\mathcal{L}_2} \\ &+\sigma_{max}(E_1^{-1}\Delta_1)\|\xi\|_{\mathcal{L}_2} + \sigma_{max}(E_1^{-1} L_4 D_2)\|\omega\|_{\mathcal{L}_2} \\ &\leq [\sigma_{max}(E_1^{-1} A_2) + \sigma_{max}(E_1^{-1}\tilde{\mathcal{L}}_\Phi)]\|\tilde{e}\|_{\mathcal{L}_2} \\ &+[\sigma_{max}(E_1^{-1}\Delta_1) + \sigma_{max}(E_1^{-1} L_4 D_2)]\|\varpi\|_{\mathcal{L}_2}, \end{aligned} \tag{3.79}$$

where $\sigma_{max}(A)$ for a matrix A denotes the maximum singular value of the matrix, i.e.

$$\sigma_{max}(A) = \sqrt{\max\{\lambda(A^T A)\}}.$$

The result (3.73) follows by substituting (3.76) into (3.79) and taking in mind that $\|\tilde{e}\|_{\mathcal{L}_2} \leq \sigma_{max}(H^{-1})\|z\|_{\mathcal{L}_2}$. From the corollary, the $\mathcal{L}_2$ norm of the error $f(t) - \nu_{eq}$ is proportional to the corresponding norm of the disturbance ϖ. Thus, the following upper bound for the $\mathcal{L}_2$ gain of the fault estimation error is obtained

$$\|\nu_{eq} - f(t)\|_{\mathcal{L}_2} \leq \epsilon \tag{3.80}$$

where

$$\epsilon := \beta\|\varpi\|_{\mathcal{L}_2}. \tag{3.81}$$

Therefore, approximately, for some small ϵ

$$\nu_{eq} \approx f(t). \tag{3.82}$$

And based on the concept of equivalent output error injection [71], the signal ν_{eq} can be approximated to any degree of accuracy by

$$\nu_{eq} \approx (\rho + \rho_0)\frac{E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\| + \delta} \tag{3.83}$$

where δ is a small positive scalar to smooth out the signal ν [18]. Therefore

$$\hat{f}(t) = (\rho + \rho_0)\frac{E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|E_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\| + \delta}. \tag{3.84}$$

This completes the proof.

Remark: The prescribed $\mathcal{H}_\infty$ performance of the SMO provides robustness against the disturbances by minimizing the $\mathcal{H}_\infty$ gain between the signals $z = H\tilde{e}$ and ϖ and, as a direct consequence, β drops significantly and the SMO can reconstruct/estimate the fault $f(t)$ with more accuracy and robustness against $\varpi(t)$. Notice that the size of the reconstruction error in (3.81) is directly related to disturbances by the term $\sigma_{max}(E_1^{-1}\Delta_1) + \sigma_{max}(E_1^{-1}L_4D_2)$. This shows that, due to the disturbances $\xi(t)$ and $\omega(t)$, precise fault reconstruction is not possible. However, in the case that the size of the error bound ϵ is considerably smaller than the fault signal $f(t)$, the sliding mode observer reconstruction scheme can still preserve the fault signal shape effectively.

Remark: It is well known that the nonlinear control injection of a variable structure control law is not smooth because of the intrinsic discontinuity. To obtain a continuous sliding gain capable of estimating an unknown input, we need to smooth out the signal ν. To cope with the discontinuity we can employ an arbitrary accurate approximation of the discontinuous switching function with the small smoothing term $\delta > 0$. Using $0 < \delta \ll 1$ a smooth sigmoid-like continuous approximation of the signum function can be obtained [18]. This approximation is consistent with the concept of equivalent output error injection and is essential for fault estimation by the sliding mode observer.

Remark: Suppose the case that the geometric condition Im$(E)\cap$ Im$(\Delta) = \{0\}$ holds. Then by using the nonsingular transformation T in (3.7) we have Im$(TE)\cap$ Im$(T\Delta) = \{0\} \Leftrightarrow$ Im$(\begin{bmatrix} E_1 \\ 0 \end{bmatrix})\cap$ Im$(\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}) = \{0\}$. From the nonsingularity of E_1 it follows that $\Delta_1 = 0$ and the new upper bound for the fault estimation error reduces since $\beta = \sqrt{\gamma}[\sigma_{max}(E_1^{-1}A_2)+\sigma_{max}(E_1^{-1}\tilde{\mathcal{L}}_\Phi)]\sigma_{max}(H^{-1})+\sigma_{max}(E_1^{-1}L_4D_2)$ which is clearly smaller than β in (3.74).

3.7 Design Example

Consider the MCK system [63] which behaves chaotically. In the form of system (3.1) MCK is presented by with the following system matrices:

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 1.5 & 0 \end{bmatrix}, \ B = 0, \ E = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \ D = 0$$

and the Lipschitz nonlinear term $\Phi(x)$ is

$$\Phi(x) = \begin{bmatrix} -1 \\ 0 \\ 10 \\ 0 \end{bmatrix} \begin{cases} -0.2 + 3(x_1 - x_3 - 1) & : \ x_1 - x_3 > 1 \\ 0.2(x_1 - x_3) & : \ -1 \le x_1 - x_3 \le 1 \\ -0.2 + 3(x_1 - x_3 + 1) & : \ x_1 - x_3 < -1 \end{cases}$$

This system satisfies the rank condition (3.21) and Assumption (3.4). Therefore the sliding mode observer exists. Nonsingular transformation matrices T and S exist and we compute them by using the Q-R decomposition method given by [32] as follows:

$$T = \begin{bmatrix} 0 & -1.4142 & 0 & -1.4142 \\ -0.7071 & 0.5 & -0.5 & 0 \\ -0.7071 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The new state space realization is

$$\tilde{x} = \left[\begin{array}{c} -1.4142(x_2 + x_4) \\ \hline -0.7071x_1 + 0.5(x_2 - x_3) \\ -0.7071x_1 - 0.5(x_2 - x_3) \\ x_4 \end{array} \right]$$

We now transform the system into the new coordinates. The matrices engaged in the SMO structure are

$$C_1 = -0.7071, \ \ C_4 = \begin{bmatrix} -0.7071 & -0.7071 & 0 \end{bmatrix}, \ \ \bar{S}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$TAT^{-1} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{c|ccc} 2.2000 & 3.1213 & -1.1213 & 3.1113 \\ \hline -0.7475 & -0.3536 & -0.3536 & 3.9429 \\ -0.2525 & 0.3536 & 0.3536 & -5.3571 \\ -1.0607 & -1.5000 & 1.5000 & -1.5000 \end{array}\right]$$

$$\left[\begin{array}{c} \Delta_1 \\ \hline \Delta_2 \end{array}\right] = \left[\begin{array}{c} 0.0000 \\ \hline -1.2071 \\ -0.2071 \\ 0.0000 \end{array}\right]$$

with C_1 invertible. In particular, in the new coordinates the unknown input distribution matrix E is decomposed into

$$TE = \left[\begin{array}{c} E_1 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} -1.4142 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

We note that the pair (A_4, C_4) is detectable (Lemma 3.2). Letting $\tilde{\mathcal{L}}_\Phi = 0.1$, $\bar{A}_1 = 50$ and

$$H = \begin{bmatrix} 0.2I_1 & 0 \\ 0 & 0.5I_3 \end{bmatrix},$$

the MATLAB LMI Toolbox solver, after 44 iterations, gives

$$P_1 = 1.1000,\ P_2 = \begin{bmatrix} 0.6632 & 0.5619 & 0.0770 \\ 0.5619 & 1.0547 & -0.0925 \\ 0.0770 & -0.0925 & 1.0645 \end{bmatrix},\quad K = \begin{bmatrix} -21.6182 \\ -21.9350 \\ 1.4796 \end{bmatrix}$$

and $\gamma = 0.0730$. Therefore, the guaranteed disturbance attenuation level is $\|\mathcal{H}\|_\infty \le \sqrt{\gamma} = 0.2701$. Using (3.32), we obtain the observer gain $\tilde{L}$ as

$$\tilde{L} = \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array}\right] = \left[\begin{array}{c|c} -70.7107 & 0.0000 \\ \hline 1.0571 & -28.3298 \\ 0.3571 & -5.4453 \\ 1.5000 & 2.9658 \end{array}\right].$$

In the corresponding simulation, the constants in the expression ν (3.31) have been selected to be $\delta = .05$ and $\rho_0 = 20$. Thus the observer design is complete. The simulation was carried out with the disturbance ξ, assumed to be noise with variance of 10, applied to the system from $t = 0$. The fault $f(t)$ is a ramp signal applied from $t = 2$ sec to $t = 6$ sec, with a positive slope from $t = 2$ sec to $t = 4sec$ and a negative slope from $t = 4$ sec until it settles to zero at $t = 6$ sec. It can be verified that the eigenvalues of A_0 are $\{-0.7819 \pm 3.8575i, -47.8000, -23.8188\}$, hence it is stable. For the pure SMO, the LMI feasibility problem (3.64) gives us the following solution

$$P_1 = 0.0211,\ P_2 = \begin{bmatrix} 0.2816 & 0.0641 & 0.1004 \\ 0.0641 & 0.2621 & -0.0091 \\ 0.1004 & -0.0091 & 0.5709 \end{bmatrix},\quad K = \begin{bmatrix} -0.1542 \\ -0.5557 \\ 0.4234 \end{bmatrix}$$

$$\tilde{L} = \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array}\right] = \left[\begin{array}{c|c} -70.7107 & 0.0000 \\ \hline 1.0571 & -0.3679 \\ 0.3571 & -2.0035 \\ 1.5000 & 0.7745 \end{array}\right].$$

Figs. 3.1 to 3.4 show the actual states (blue line) and their estimates by $\mathcal{H}_\infty$-SMO (black line) and pure SMO (red line) respectively. Fig. 3.5 depicts the signal $s(t)$ versus time indicating that an ideal sliding motion is taking place after a finite time and remains on the surface $\mathcal{S}$ afterwards. Fig. 3.6 is concerned with the fault reconstruction. It shows that despite the presence of disturbances $\xi(t)$, the proposed sliding mode observer with $\mathcal{H}_\infty$ performance can still reconstruct the fault signal with relatively high accuracy (black line) compared to pure SMO (red line). It can be seen that the reconstruction error bounds are small compared to the fault when we use $\mathcal{H}_\infty$-SMO. Therefore fault detection is easily achievable by setting appropriate thresholds. It is important to mention that the sliding mode observer design must satisfy the so-called matching condition (3.5). The LMI convex optimization problems, stated in Theorem 3.1 and 3.2, guarantee a valid solution for (3.5). However, to further examine it, from the LMI solution for the $\mathcal{H}_\infty$-SMO, it can be verified that

$$P = T^T \tilde{P} T = \begin{bmatrix} 1.4208 & 0.1384 & -0.1384 & 0.0110 \\ 0.1384 & 2.1485 & -0.1485 & 2.0848 \\ -0.1384 & -0.1485 & 0.1485 & -0.0848 \\ 0.0110 & 2.0848 & -0.0848 & 3.0645 \end{bmatrix}$$

and as a consequence, $E^T P = FC = [0.0000 \quad 2.0000 \quad 0.0000 \quad 2.0000]$ in the original coordinates.

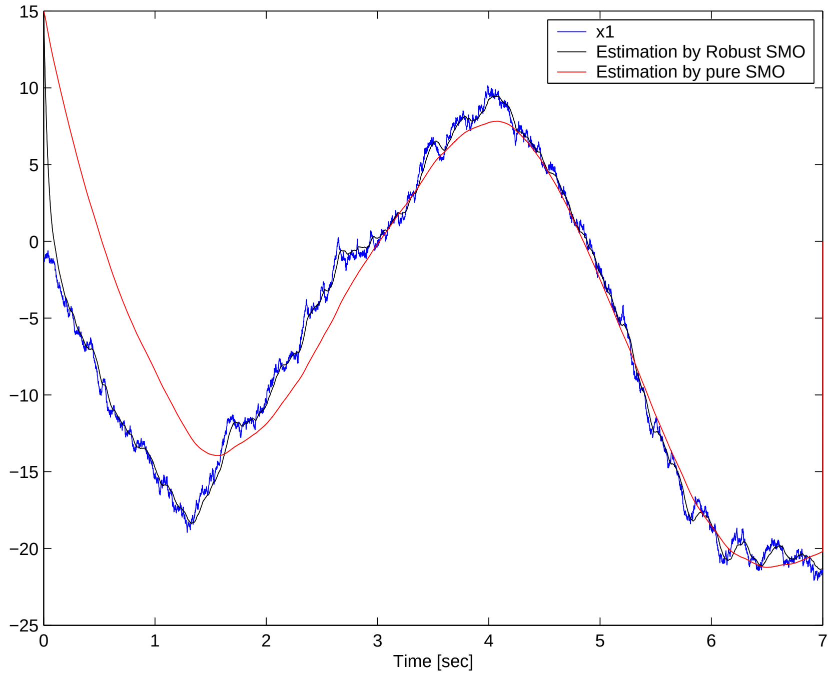

Figure 3.1: First state and its estimate

3.8 Summary

This chapter presents a new robust SMO with $\mathcal{H}_\infty$ performance methodology for uncertain Lipschitz nonlinear systems with unknown inputs. Our work generalizes the known results of linear systems to Lipschitz nonlinear systems by using Lyapunov stability theory and LMIs. A novel switching gain has been proposed to satisfy automatically the matching condition in the new coordinates. The derived LMI optimization problem results in calculating the maximum disturbance attenuation level so that the observer can tackle hard disturbances. It is demonstrated that with the same necessary existence conditions for linear systems, one can build SMO for Lipschitz nonlinear systems if the LMI optimization problem (3.36) is feasible. The reconstruction of the fault (unknown input) is also addressed.

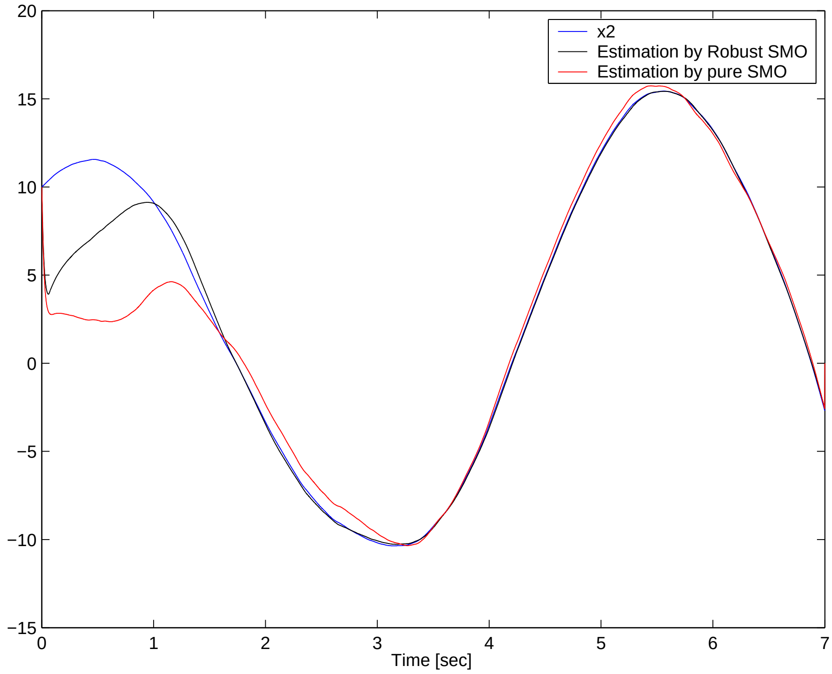

Figure 3.2: Second state and its estimate

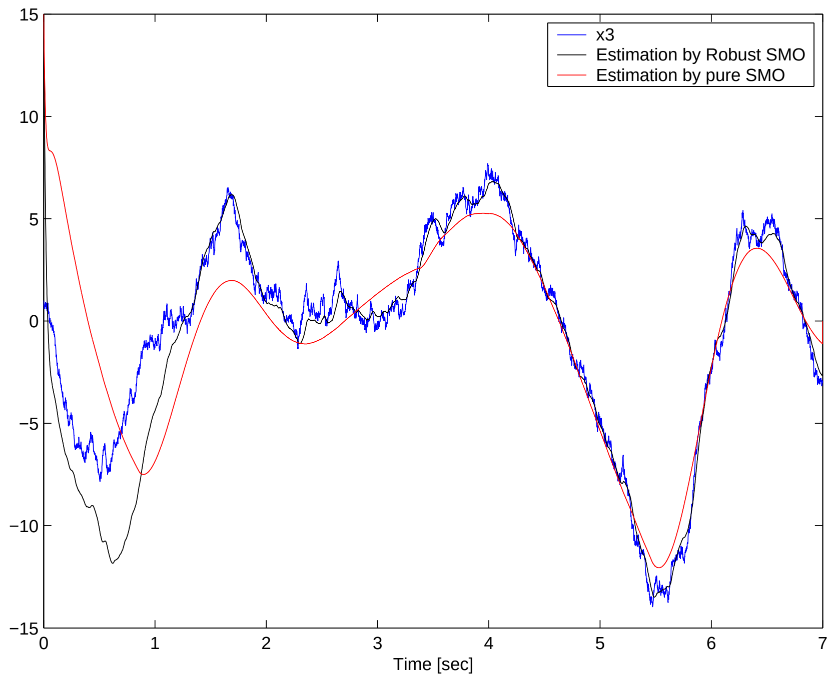

Figure 3.3: Third state and its estimate

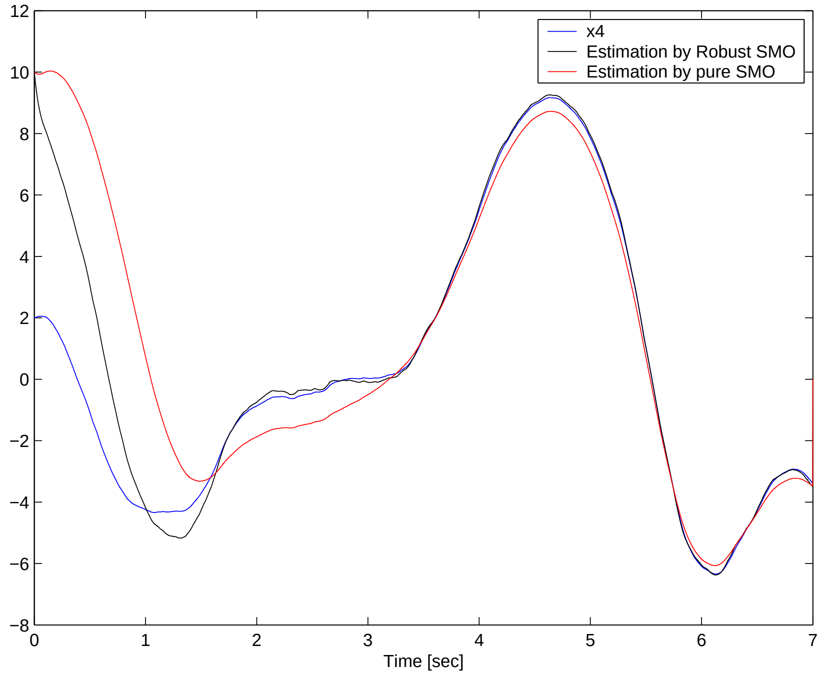

Figure 3.4: Forth state and its estimate

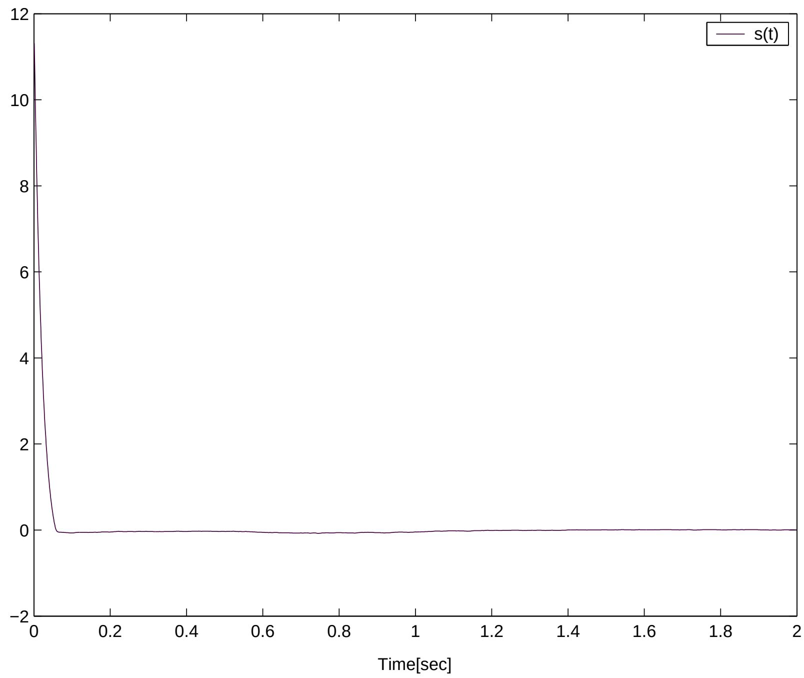

Figure 3.5: sliding motion s(t)

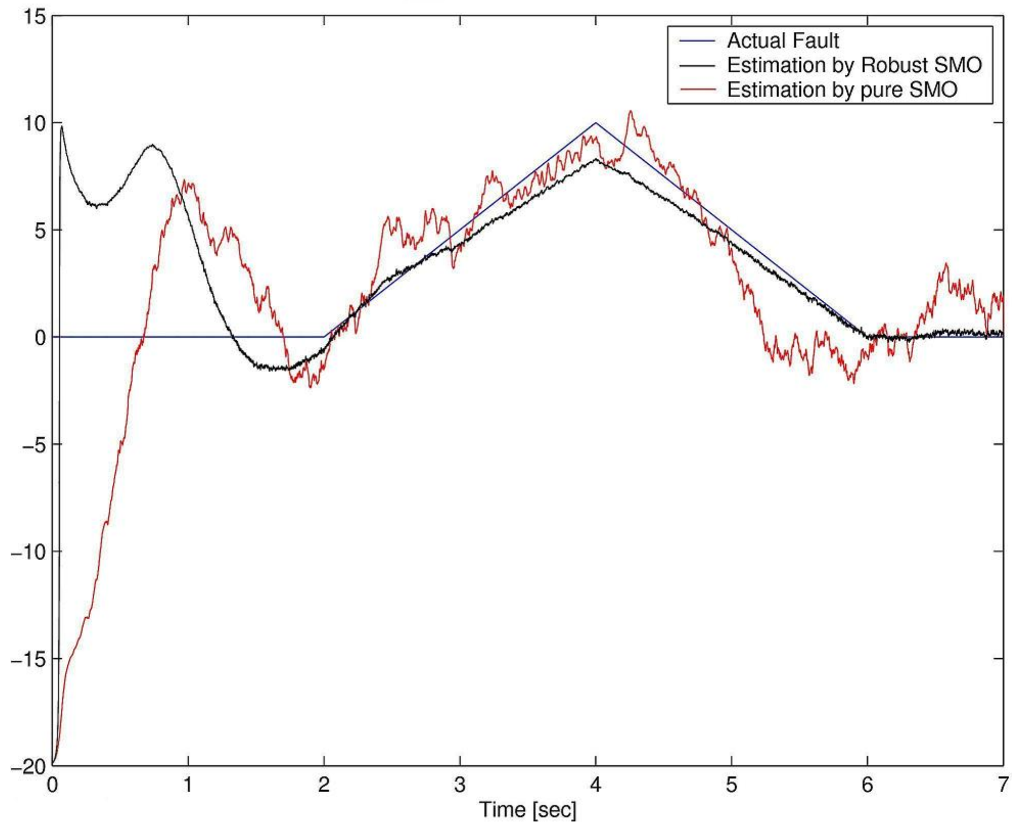

Figure 3.6: Unknown input $f(t)$ (blue line) and its reconstructions by $\nu(t)$ (black and red lines)

Chapter 4

A Parametrization Design for Adaptive Sliding Mode Observers[1]

4.1 Background Results

The main focus of this chapter is in the application of new adaptive SMOs to the problem of fault reconstruction and FDI. In essence, the SMO approach consists of first defining a sliding surface and then using a variable structure control law, forcing the error system trajectories to the sliding surface in finite time. Recently, [17] and [18] proposed that using the variable structure control law of the SMO and the concept of equivalent output injection, a fault can be reconstructed to any required accuracy for linear systems. These two references consider the case of perfect modeling without uncertainty or unmatched disturbance, hence precise fault reconstruction is feasible. Sensor fault reconstruction using SMO was studied by [64], further extending the results for linear systems with disturbance and uncertainty. For nonlinear Lipschitz systems, [74] addressed SMO based fault reconstruction by assuming that disturbances are matched and can be lumped into the so-called matching condition. A robust fault detection method for nonlinear systems with disturbances was studied by [23] where strict geometric conditions where exploited. Applications of SMO for fault tolerant control of linear systems were addressed by [19] and [3].

[1]The results in this chapter have been submitted for publication in the article: R Raoufi and H. J. Marquez "A New Parametrization for Sliding Mode Observer Design with Fault Estimation and Diagnosis", submitted to *IEEE Transaction on Automatic Control*, November 2009.

In this chapter, nonlinear Lipschitz systems with fault and disturbances are considered where the fault satisfies the so-called matching condition and the corresponding rank condition. Firstly, it is shown that this rank condition is necessary and sufficient for the existence of a solution for the so-called matching condition. The main objective of this chapter is to design the sliding mode observer for fault reconstruction without using any change of coordinates. It should be pointed out that in articles by [19], [74], [64] and [3] multiple coordinate transformations must be utilized to design the SMO which adds to the complexity and conservativism in the approach. To avoid coordinate changes we introduce a novel solution for the structural matching condition based on pseudo-inverse of the fault distribution matrix. Next, a linear matrix inequality (LMI) convex optimization problem is exploited for the original system. The solution of LMIs guarantees the stability of the error dynamics and in addition, provides a valid solution for the matching condition. The above solution guarantees an ideal sliding motion on the sliding surface in finite time and there is no need of any multiple change of coordinates.

An important feature of our solution is the following: the solution presented in references [19], [74], [64] and [3] assumes the upper bound of the fault to be known. Technically speaking, this is undesirable since faults are inherently unknown and the fault may exceed the assumed upper bound. In this case the sliding motion breaks down and SMO is incapable of any fault reconstruction, resulting in instantaneous failure of fault reconstruction and diagnosis. We propose a new approach and modify the formulation assuming that the fault is *bounded*, however the bound is *unknown*. To cope with the unknown upper bound, an adaptive algorithm is employed to maintain the sliding motion on the sliding surface against any rapid and unexpected increase in the fault. The SMO based fault reconstruction is implementable in real systems, since only state estimates and available output measurement are used.

The remainder of this chapter is organized as follows; Section 4.2 provides some preliminaries and assumptions on the class of nonlinear system addressed. Some preliminary lemmas are introduced in Section 4.3. The design of the robust adaptive SMO and the analysis of the stability of the error dynamics are given in Section 4.4. In Section 4.5 the stability of the sliding motion is discussed. Adaptive fault estimation is studied in Section 4.6. A generalization of the proposed method for sensor fault reconstruction is presented in Section 4.7. The effectiveness of the proposed Adaptive SMO based fault reconstruction is studied with an example in Section 4.8. Finally some concluding remarks are presented in Section 4.9.

4.2 Preliminaries and Assumptions

Consider a dynamical systems of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + f(t)) + \Phi(x,t) + \Delta\xi(t) \\ y(t) = Cx(t) \end{cases} \tag{4.1}$$

where $x \in \mathbf{R}^n$ represents the system state, $u \in \mathbf{R}^m$ the control input, $y \in \mathbf{R}^p$ the measured system output and $t \in \mathbf{R}^+$. $f(t) : \mathbf{R}^+ \to \mathbf{R}^m$ is the unknown input (actuator faults). $\xi(t) : \mathbf{R}^+ \to \mathbf{R}^d$ denotes disturbances subject to $||\xi|| \leq \beta < \infty$. (A, B, C, Δ) is the set of real constant known matrices of appropriate dimensions. $B \in \mathbf{R}^{n\times m}$ is a full column rank matrix and $C \in \mathbf{R}^{p\times n}$ is a full row rank matrix. The known nonlinearity $\Phi(x,t)$ satisfies a Lipcshitz condition (2.7). The unknown input $f(t)$ is bounded in the Euclidean norm with an unknown upper bound subject to (3.2). We also assume that the conditions (3.4) and (3.5)are satisfied. Furthermore, we have the following assumptions:

$$\text{rank}(CB) = \text{rank}\ (B). \tag{4.2}$$

4.3 A Parametrization Lemma

We now present the following lemmas which are essential for the SMO design approach proposed in Section 4.4.

Lemma 4.1. *Assume that rank* $(CB) =$ *rank* (B)*. A solution* $P = P^T > 0$ *for the matching condition* $B^T P = FC$ *exists if and only if*

$$P = \Theta X_1 \Theta + C^T X_2 C \tag{4.3}$$

$$F = B^T C^T X_2 \tag{4.4}$$

$$\Theta = I_n - BB^+, \Theta \in \mathbf{R}^{n\times n} \tag{4.5}$$

where $B^+ = (B^T B)^{-1} B^T$ and $X_1 = X_1^T, X_1 \in \mathbf{R}^{n\times n}$, $X_2 = X_2^T, X_2 \in \mathbf{R}^{p\times p}$ are arbitrary weight matrices.

Proof.

(proof of necessity)

Define a matrix $\bar{\Theta} \in \mathbf{R}^{(n+m)\times n}$ as

$$\bar{\Theta} = \begin{pmatrix} B^+ \\ \Theta \end{pmatrix}. \tag{4.6}$$

Considering that

$$\text{rank}(\bar{\Theta}) = \text{rank}\begin{pmatrix} I_m & 0_{m\times n} \\ B & I_n \end{pmatrix} . \begin{pmatrix} B^+ \\ \Theta \end{pmatrix} = \text{rank}\begin{pmatrix} B^+ \\ I_n \end{pmatrix} = n$$

it follows that $\text{rank}(\bar{\Theta}) = n$. We can always find a matrix $\Lambda \in \mathbf{R}^{(n-p)\times n}$ such that

$$\text{rank}\begin{pmatrix} C \\ \Lambda \end{pmatrix} = n \tag{4.7}$$

and, therefore, is nonsingular. Thus

$$\text{rank}\left\{\begin{pmatrix} C \\ \Lambda \end{pmatrix} .\bar{\Theta}^T\right\} = \text{rank}\begin{pmatrix} C(B^+)^T & C\Theta \\ \Lambda(B^+)^T & \Lambda\Theta \end{pmatrix} = n\ . \tag{4.8}$$

By definition of B^+, we know $C(B^+)^T = CB(B^T B)^{-1}$. Since we assumed that $\text{rank}(CB) = m$, it readily follows that $\text{rank}(C(B^+)^T) = m$. As a consequence, the matrix $\begin{pmatrix} C(B^+)^T \\ \Lambda(B^+)^T \end{pmatrix} \in \mathbf{R}^{(n\times m)}$ is full column rank with m independent columns. Hence, from (4.8), we have

$$n - m \leq \text{rank}\begin{pmatrix} C\Theta \\ \Lambda\Theta \end{pmatrix} \leq n. \tag{4.9}$$

Assumption $\text{rank}(CB) = m$ gives $m \leq \min\{\text{rank}(B), \text{rank}(C)\} = \min\{m, p\}$. Therefore, we obtain $m \leq p$ and $n - p$ independent columns of $\begin{pmatrix} C\Theta \\ \Lambda\Theta \end{pmatrix}$ can be pulled out to construct the matrix $\Lambda_1 \in \mathbf{R}^{n\times(n-p)}$. In other words, one can always obtain the following partitioned structure

$$\Theta \begin{pmatrix} C \\ \Lambda \end{pmatrix}^T := (\Lambda_1 \mid \Lambda_2)\,. \tag{4.10}$$

Also, the matrix Λ_1 can be expressed by

$$\Lambda_1 = \Theta \begin{pmatrix} C \\ \Lambda \end{pmatrix}^T \overline{\Lambda},\ \overline{\Lambda} \in \mathbf{R}^{n\times(n-p)}. \tag{4.11}$$

Since Λ_1 is of rank $n-p$, we can introduce the nonsingular transformation matrix $T \in \mathbf{R}^{n\times n}$ as

$$T := \begin{pmatrix} \Lambda_1^T \\ C \end{pmatrix}. \tag{4.12}$$

Let $\bar{P} := \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$. Since any symmetric positive definite matrix P can be expressed by

$$P = T^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} T \Rightarrow P = T^T \bar{P} T \tag{4.13}$$

therefore, using (4.12), it follows that

$$P = \Lambda_1 P_1 \Lambda_1^T + C^T P_2 C\ . \tag{4.14}$$

From the matching condition (3.5) and the derived structure of P in (4.14), we obtain

$$B^T \Lambda_1 P_1 \Lambda_1^T + B^T C^T P_2 C - FC = 0$$

and consequently

$$\begin{pmatrix} B^T \Lambda_1 P_1 & B^T C^T P_2 - F \end{pmatrix} T = 0. \tag{4.15}$$

Since T is nonsingular and $B^T\Theta = 0 (\Rightarrow\ B^T\Lambda_1 = 0)$, it indicates that $F = B^T C^T P_2$. Substituting the structure of Λ_1 from (4.11) into (4.14) yields

$$P = \Theta \left(C^T \Lambda^T\right) \overline{\Lambda} P_1 \overline{\Lambda}^T \begin{pmatrix} C \\ \Lambda \end{pmatrix} \Theta + C^T P_2 C. \tag{4.16}$$

Letting

$$X_1 = \left(C^T \, \Lambda^T \right) \overline{\Lambda} \; P_1 \; \overline{\Lambda}^T \begin{pmatrix} C \\ \Lambda \end{pmatrix} \tag{4.17}$$

$$X_2 = P_2 \tag{4.18}$$

leads to the structure of P and F in (4.3)-(4.5). This completes the proof of necessity.

(proof of sufficiency)

From the definition of Moore-Penrose pseudo-inverse matrix ,B^+, we know that the following criteria are satisfied

$$BB^+B = B \tag{4.19}$$

$$(BB^+)^T = BB^+ \tag{4.20}$$

therefore

$$B^T \Theta = 0. \tag{4.21}$$

By substituting (4.3), (4.4) and (4.5) into matching condition (3.5), it simply follows that (3.5) holds. This completes the proof of sufficiency.

Remark. In reference [11], a different parametrization of the Lyapunov matrix P was also considered which satisfy the matching condition. This proposed parametrization is of the form:

$$P = B_\perp Y_1 B_\perp^T + C^T Y_2 C$$

where Y_1 and Y_2 are arbitrary symmetric matrices with appropriate dimensions and $B_\perp$ is any permissable full rank matrix whose columns are from the basis of the null space of the matrix B^T, i.e. $B_\perp$ is the orthogonal complement of B. In our proposed parametrization of P, we have $P = \Theta X_1 \Theta + C^T X_2 C$ where the definition of Θ is given in (4.5). Thus, these solutions are different. Notice that the dimension of X_1 and Y_1 are also different. Two disadvantages of the proposed structure given by [11] are as the following.

First, while Θ is uniquely defined by formula (4.5), $B_\perp$ of B is not a unique matrix and may cause confusion on how to choose it for high order systems. Secondly and more importantly, $Y_1 \in \mathbf{R}^{(n-m)\times(n-m)}$ while $X_1 \in \mathbf{R}^{n\times n}$. Therefore X_1 brings more additional freedom from a parametrization perspective. Notice that the level of parametrization plays a very important role when an LMI feasibility problem must be solved on the matrix P with the associated constraint attached to it. To make this point more clear, lets consider the following example.

Example. Assume that $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}$. Then, we obtain

$$B_\perp = \begin{bmatrix} -3\alpha \\ \alpha \\ -2\alpha \end{bmatrix}, \ \ \forall \alpha \in \mathbf{R}, \ \ \Theta = \begin{bmatrix} 0.6429 & -0.2143 & 0.4286 \\ -0.2143 & 0.0714 & -0.1429 \\ 0.4286 & -0.1429 & 0.2857 \end{bmatrix}.$$

Consequently, $X_1 \in \mathbf{R}^{3\times 3}$ and $Y_1 \in \mathbf{R}$ are respectively parameterized as

$$X_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}, \ \ Y_1 = y_{11}$$

Clearly, the parametrization of X_1 is more flexible to solve an LMI numerically. Summing up the parameterizations given in this chapter and [11], one can directly conclude that the following Lemma also provides an acceptable solution for matrix equation (3.5).

Lemma 4.2. *Assume that rank* $(CB) =$ *rank* (B). *A solution* $P = P^T > 0$ *for the matching condition* $B^T P = FC$ *exists if and only if*

$$P = \Theta X_1 \Theta + C^T X_2 C + B_\perp X_3 B_\perp^T \tag{4.22}$$

$$F = B^T C^T X_2 \tag{4.23}$$

$$\Theta = I_n - BB^+, \Theta \in \mathbf{R}^{n\times n} \tag{4.24}$$

where $B^+ = (B^T B)^{-1} B^T$, $B_\perp$ is the orthogonal complement of matrix B^T and $X_1 = X_1^T, X_1 \in \mathbf{R}^{n\times n}$, $X_2 = X_2^T, X_2 \in \mathbf{R}^{p\times p}$, $X_3 = X_3^T, X_3 \in \mathbf{R}^{(n-m)\times(n-m)}$ are arbitrary weight matrices.

Lemma 4.1 established a specific and new solution for matrix equation (3.5). We will employ this solution to build up the new robust sliding mode filter. Now, we put forward the following lemma which links the existence of any solution for the matching condition (3.5) with the rank condition (4.2). This Lemma 4.is a well-known result in the literature of sliding mode observer design [18],[19],[74],[25].

Lemma 4.3. *Consider the system (1). There exists a solution $P = P^T > 0$ such that $B^T P = FC$ if and only if*

$$\text{rank}(CB) = \text{rank}\ (B) \tag{4.25}$$

Proof. See the proof of Lemma 3.3.

4.4 Adaptive Sliding Mode Observer Design

In this section, a new robust adaptive sliding mode observer is proposed. The term $\xi(t)$ encapsulates all the disturbances. Consider the following adaptive sliding mode observer structure

$$\begin{array}{l} \dot{\hat{x}} = A\hat{x} + Bu + \Phi(\hat{x}, t) + L(y - C\hat{x}) + \nu(t) \\ \hat{y} = C\hat{x}. \end{array} \tag{4.26}$$

Let $s : \mathbf{R}^n \to \mathbf{R}^m$ be a linear function represented as

$$s(t) = C(x - \hat{x}). \tag{4.27}$$

Next, let $\mathcal{S}$ be the hyperplane defined by

$$\mathcal{S} = \{t \in \mathbf{R}^+ : s(t) = 0\} \tag{4.28}$$

then the switching gain $\nu(t)$ is of the form

$$\nu(t) = \begin{cases} (\hat{\rho}(t) + \rho_0) P^{-1} C^T \|B^T C^T X_2\| \frac{s(t)}{\|s(t)\|} & : \ s(t) \neq 0 \\ 0 & : \text{otherwise} \end{cases} \tag{4.29}$$

and the gain $\hat{\rho}(t)$ is updated by the following adaptive algorithm

$$\frac{d\hat{\rho}(t)}{dt} = \eta||B^T C^T X_2||||s(t)|| \tag{4.30}$$

where $\eta, \rho_0 > 0$ and $X_2 \in \mathbf{R}^{p\times p}$ is a design symmetric wight matrix as defined in Lemma 4.1. The following theorem states a sufficient condition using LMIs to guarantee the stability of the proposed observer error dynamics in conjunction with the adaptive performance of the switching gain.

Theorem 4.1. *Given the nonlinear uncertain system (4.1) with the associated assumptions, consider the observer (4.26)-(4.29). The observer error dynamics is stable with maximized Lipschitz gain $1/\sqrt{\gamma}$ if there exist matrices $X_1 = X_1^T$, $X_2 = X_2^T$ and $\breve{L}$ such that the following LMI optimization problem has a solution:*

minimize γ subject to

$$X_2 > 0 \tag{4.31}$$

$$\Theta X_1 \Theta + C^T X_2 C > I \tag{4.32}$$

$$\begin{bmatrix} M + M^T & \Theta X_1 \Theta + C^T X_2 C & I_n \\ \Theta X_1 \Theta + C^T X_2 C & -I & 0 \\ I_n & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.33}$$

where $M = A^T \Theta X_1 \Theta + A^T C^T X_2 C - \breve{L}C$. Then, the observer gain is

$$L = (\Theta X_1 \Theta + C^T X_2 C)^{-1} \breve{L}. \tag{4.34}$$

Proof. Regarding (4.1) and (4.26), the error dynamical system is

$$\dot{e} = A_0 e + e_\Phi + Bf(t) - \nu(t) + \Delta\xi(t)$$

where $e = x - \hat{x}$ is the state estimation error, $\tilde{\rho} = \rho - \hat{\rho}$ and $A_0 = A - LC$. From (3.4), it follows that the pair (A, C) is observable. For simplicity, we define $e_\Phi = \Phi(x, t) - \Phi(\hat{x}, t)$. Consider the Lyapunov function $V = e^T P e + \eta^{-1}\tilde{\rho}^2$. The derivative of $V(e)$ is

$$\begin{aligned} \dot{V} = {} & (A_0 e + e_\Phi + Bf_a(t) - \nu(t) + \Delta\xi(t))^T P e \\ & + e^T P (A_0 e + e_\Phi + Bf(t) - \nu(t) + \Delta\xi(t)) \\ & + 2\eta^{-1}\tilde{\rho}(-\dot{\hat{\rho}}). \end{aligned} \tag{4.35}$$

From (4.29), (4.30),(3.2) and (3.5) it yields

$$\begin{array}{l} e^T P(Bf(t) - \nu(t)) + \eta^{-1}\tilde{\rho}(-\dot{\hat{\rho}}) = e^T C^T F^T f(t) \\ -e^T P P^{-1} C^T \|B^T C^T X_2\|(\hat{\rho}(t) + \rho_0)\frac{s(t)}{\|s(t)\|} \\ +\eta^{-1}\tilde{\rho}(-\eta||B^T C^T X_2||||s(t)||) \\ \leq \rho||F||||Ce|| - (\rho + \rho_0)\|Ce\|\|B^T C^T X_2\| \\ = -\rho_0\|Ce\|\|B^T C^T X_2\| < 0. \end{array} \tag{4.36}$$

Therefore, using Assumption (2.7) it follows that

$$\dot{V} \leq (A_0 e + e_\Phi + \Delta\xi)^T P e + e^T P(A_0 e + e_\Phi + \Delta\xi) - e_\Phi^T e_\Phi + \mathcal{L}_\Phi^2 e^T e \ . \tag{4.37}$$

Suppose that disturbance $\xi(t)$ is bounded subject to $\|\xi(t)\| \leq \beta < \infty$. Hence,

$$\dot{V} \leq \begin{pmatrix} e \\ e_\Phi \end{pmatrix}^T \underbrace{\begin{pmatrix} A_0^T P + P A_0 + \gamma^{-1} I & P \\ P & -I \end{pmatrix}}_{-\overline{Q}} \begin{pmatrix} e \\ e_\Phi \end{pmatrix} + 2\beta \|P\Delta\| \left\| \begin{pmatrix} e \\ e_\Phi \end{pmatrix} \right\| \tag{4.38}$$

where $\mathcal{L}_\Phi^{-2} = \gamma$. Therefore, if $-\overline{Q} < 0$

$$\dot{V} \leq \left\| \begin{pmatrix} e \\ e_\Phi \end{pmatrix} \right\| \left(-\lambda_{min}\left(\overline{Q}\right) \left\| \begin{pmatrix} e \\ e_\Phi \end{pmatrix} \right\| + 2\beta \|P\Delta\| \right). \tag{4.39}$$

Thus, for $\left\| \begin{pmatrix} e \\ e_\Phi \end{pmatrix} \right\| > \frac{2\beta\|P\Delta\|}{\lambda_{min}\left(\overline{Q}\right)}$, we obtain $\dot{V} < 0$ which guarantees that the magnitude of the error is ultimately bounded with respect to the set

$$\Omega_\varepsilon = \left\{ \begin{pmatrix} e \\ e_\Phi \end{pmatrix} : \left\| \begin{pmatrix} e \\ e_\Phi \end{pmatrix} \right\| < \frac{2\beta \|P\Delta\|}{\lambda_{min}\left(\overline{Q}\right)} + \varepsilon, \varepsilon > 0 \right\}. \tag{4.40}$$

On the other hand, matrix $P > 0$ must also satisfy the structural condition (3.5) as well. From the solution (4.3)-(4.4) for P and F in Lemma 4.1 and using Schur complement ([6]), the above inner matrix $-\overline{Q}$ can be expressed by the LMI in (4.33) (which any solution for $P > 0$, obtained from the LMIs, satisfies the matching condition (3.5)) and the observer gain is defined in (4.34). Consequently, the LMI optimization problem of Theorem 4.1 guarantees the ultimate boundedness of the error dynamics and the minimization of γ implies that the observer would be tolerant against the Lipschitz nonlinearity up to max $(\mathcal{L}_\Phi) = \frac{1}{\sqrt{\gamma}}$. This completes the proof.

Technically speaking, the concept of maximizing the Lipschitz gain for observer design was proposed by [1]. It is shown that if the optimal solution is larger than the actual Lipschitz gain, the observer remains stable in the presence of Lipschitz nonlinear uncertainties (see [1] for details). The first objective of our LMI optimization problem is the computation of the proper matrices X_1, X_2 and $\breve{L}$. However, the second objective is to boost up the tolerance of the sliding mode observer against the system Lipschitz nonlinearities. Note that we assumed the Lipschitz condition on $\Phi(x,t)$ with a known Lipschitz constant $\mathcal{L}_\Phi$. The calculation of the maximum of $\mathcal{L}_\Phi$ is obtained by solving the LMI optimization problem subject to (4.31)-(4.33). If $\frac{1}{\sqrt{\gamma}} \geq \mathcal{L}_\Phi$, then the sliding mode observer is capable of tackling the nonlinear uncertain system (1).

4.5 Synthesis of the Sliding Motion

It is well-known that in order to confine a stable motion of a dynamical system onto a sliding surface $\mathcal{S}$, it is necessary to use a switching gain which is discontinuous about the surface $\mathcal{S}$ ([18]). Let

$$\mathcal{S} = \{t \in \mathbf{R}^+ : s(t) = 0 \mid s(t) = Ce\} \tag{4.41}$$

then the employed structure of switching gain (4.29) meets the above requirement. We now present Theorem 4.2, which is the main result of this section. To simplify our notation, we define

$$\varepsilon^* := \frac{2\beta \left\| P\Delta \right\|}{\lambda_{min}\left(\overline{Q}\right)} + \varepsilon. \tag{4.42}$$

Theorem 4.2. *Choose the gain ρ_0 to satisfy*

$$\rho_0 \geq \frac{\varepsilon^*(2||A_0|| + \gamma||C||) + \beta||\Delta|| + \tilde{\eta}}{||F||||\mathcal{P}||^{-1}}, \quad \tilde{\eta} > 0. \tag{4.43}$$

Then given system (4.1), satisfying the assumptions outlined in Section 2, an ideal sliding motion takes place in finite time on the hyperplane $\mathcal{S}$, defined in (4.41), if the convex optimization problem (4.31)-(4.33) is solvable.

Proof. The error dynamics is

$$\dot{e} = A_0 e + e_\Phi + Bf(t) - \nu + \Delta\xi(t). \tag{4.44}$$

Define the change of coordinate $\bar{T} \in \mathbf{R}^{n\times n}$ as

$$\bar{T} := \begin{pmatrix} C_\perp^T P \\ C \end{pmatrix} \tag{4.45}$$

where $C_\perp \in \mathbf{R}^{n\times(n-p)}$ is any full rank matrix whose columns are from the basis of the null space of the matrix C, Thus $C_\perp$ is an orthogonal complement of the matrix C^T. The state estimation error in the new coordinate system is

$$\dot{\tilde{e}} = \bar{T}A_0\bar{T}^{-1}\tilde{e} + \bar{T}(e_\Phi + Bf - \nu + \Delta\xi) \tag{4.46}$$

where

$$\tilde{e} = \begin{pmatrix} e_1 \\ e_y \end{pmatrix} = \bar{T}e.$$

From the definition of $\bar{T}$ and the property $CC_\perp = 0$, it follows that

$$\bar{T}A_0\bar{T}^{-1} := \begin{pmatrix} A_{11}^o & A_{12}^o \\ A_{21}^o & A_{22}^o \end{pmatrix} \quad , \quad \bar{T}\Delta := \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \tag{4.47}$$

$$\bar{T}\nu = \left[\frac{0}{(\hat{\rho}(t) + \rho_o)CP^{-1}C^T \|F\| \frac{e_y}{\|e_y\|}}\right] := \left[\frac{0}{\bar{\nu}}\right] \tag{4.48}$$

$$\bar{T}B = \left[\frac{0}{CP^{-1}C^TF^T}\right] \tag{4.49}$$

$$\bar{T}e_\Phi := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{4.50}$$

Hence, partitioning the error dynamics in the new coordinate yields

$$\begin{aligned} \dot{e}_1 &= A_{11}^o\, e_1 + A_{12}^o e_y + \phi_1 + \Delta_1\xi \\ \dot{e}_y &= A_{21}^o\, e_1 + A_{22}^o\, e_y + \phi_2 + CP^{-1}C^TF^Tf(t) - \bar{\nu}(t) + \Delta_2\xi. \end{aligned} \tag{4.51}$$

Now consider the Lyapunov function $V_s = \frac{1}{2}(\, e_y^T\mathcal{P}e_y + \eta^{-1}\tilde{\rho}^2), \mathcal{P} \in \mathbf{R}^{p\times p}, \mathcal{P} > 0$. The derivative along the trajectory is

$$\begin{aligned} \dot{V}_s = e_y^T(&\mathcal{P}A_{21}^o\, e_1 + \mathcal{P}\, A_{22}^o\, e_y + \mathcal{P}\,\phi_2 \\ &+\mathcal{P}\, CP^{-1}C^TF^Tf - \mathcal{P}\,\bar{\nu}(t) + \mathcal{P}\Delta_2\xi) + \eta^{-1}\tilde{\rho}(-\dot{\hat{\rho}}). \end{aligned} \tag{4.52}$$

By assumption, the matrix C is full row rank . Therefore let

$$\mathcal{P} = (CP^{-1}C^T)^{-1} \tag{4.53}$$

which is nonsingular. Thus (4.52) becomes

$$\begin{aligned}\dot{V}_s &= e_y^T \mathcal{P} A_{21}^o e_1 + e_y^T \mathcal{P} A_{22}^o e_y + e_y^T \mathcal{P} \Delta_2 \xi \\ &+ e_y^T F^T f(t) - e_y^T (\hat{\rho}(t) + \rho_o) \|F\| \tfrac{e_y}{\|e_y\|} + e_y^T \mathcal{P} \phi_2 - \tilde{\rho} ||F|| ||s(t)||.\end{aligned} \tag{4.54}$$

Then

$$\begin{aligned}\dot{V}_s &\le e_y^T \mathcal{P} A_{21}^o\, e_1 + e_y^T \mathcal{P}\, A_{22}^o e_y + e_y^T \mathcal{P}\, \Delta_2 \xi \\ &+ e_y^T \mathcal{P} \phi_2 + \|F\|\, \rho\, \|e_y\| - (\hat{\rho}(t) + \rho_o)\, \|F\|\, \|e_y\| - \tilde{\rho} ||F|| ||s(t)||.\end{aligned} \tag{4.55}$$

Using the adaptation law (4.30)

$$\begin{aligned}\dot{V}_s &\le \|\mathcal{P}\|\, \|e_y\|\, (\|A_{21}^o e_1\| + \|A_{22}^o\|\, \|e_y\| \\ &+ \|\Delta_2\|\, \beta - \rho_o\, \|F\|\, \|\mathcal{P}\|^{-1} + \|C\|\, \gamma\, \left\|\bar{T}^{-1}\right\|\, \|\tilde{e}\|).\end{aligned} \tag{4.56}$$

From Theorem 4.1, in finite time $e(t) \in \Omega$ which implies that $||e|| < \varepsilon^*$. And, from the definition of ρ_0 in (4.43)

$$\dot{V}_s \le -\tilde{\eta}\, \|\mathcal{P}\|\, \|e_y\| \Rightarrow \dot{V}_s < 0$$

Therefore, an ideal sliding motion will take place on the surface $\mathcal{S}$ after some finite time t_s and consequently, for all subsequent time $e_y = \dot{e}_y = 0$. This completes the proof.

Now, we present the following important proposition which shows the necessity of the minimum phase condition (3.4) for the stability of the reduced order system in sliding mode (the stability of the poles of the sliding motion).

Proposition 4.1. *If the system triple $\{A, B, C\}$ is minimum phase and rank$(CB) =$ rank(B), then the reduced-order system in sliding mode is stable.*

Proof. When ideal sliding mode takes place, we have $e_y = \dot{e}_y = 0$ and the reduced order system in sliding mode is given by

$$\dot{e}_1 = A_{11}^o\, e_1 + \phi_1 + \Delta_1 \xi. \tag{4.57}$$

Since the matrix A_{11}^o appears in the reduced order system above, its eigenvalues are the poles of the sliding motion. Moreover, from nonsingular change of coordinate

$\bar{T}$, we obtain

$$\begin{aligned}&\bar{T}A_0\bar{T}^{-1} := \begin{pmatrix} A_{11}^o & A_{12}^o \\ A_{21}^o & A_{22}^o \end{pmatrix} = \bar{T}A\bar{T}^{-1} - \bar{T}LC\bar{T}^{-1} \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - \begin{pmatrix} C_\perp^T PL \\ CL \end{pmatrix} (0 \ \ I_p) = \begin{pmatrix} A_{11} & A_{12} - C_\perp^T PL \\ A_{21} & A_{22} - CL \end{pmatrix}.\end{aligned} \tag{4.58}$$

It follows that $A_{11}^o = A_{11}$ and the poles of the sliding motion are independent of the linear observer gain L, Thus A_{11} must be stable. From the nonsingularity of $\mathcal{P}$ and X_2, we have

$$\text{rank}(CP^{-1}C^TF^T) = \text{rank}(\mathcal{P}^{-1}X_2CB) = \text{rank}(CB) = m \ , \quad (CP^{-1}C^TF^T \in \mathbf{R}^{p\times m}) \tag{4.59}$$

then from (3.4) it follows that for every complex number s with nonnegative real part

$$\begin{aligned}n+m &= \text{rank}\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} \bar{T} & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sI-A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \bar{T}^{-1} & 0 \\ 0 & I_m \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} sI_n - \bar{T}A\bar{T}^{-1} & \bar{T}B \\ C\bar{T}^{-1} & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} sI_{n-p} - A_{11} & -A_{12} & 0 \\ -A_{21} & sI_p - A_{22} & CP^{-1}C^TF^T \\ 0 & I_p & 0 \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} sI_{n-p} - A_{11} & 0 \\ -A_{21} & CP^{-1}C^TF^T \end{bmatrix} + p := \mathcal{Q}(s) + p\end{aligned} \tag{4.60}$$

and $\mathcal{Q}(s) = n+m-p$. For $m=p$, we have

$$n = \mathcal{Q}(s) = \text{rank}\begin{bmatrix} sI_{n-p} - A_{11} \end{bmatrix} + m \tag{4.61}$$

Thus, it is guaranteed that for every complex number s with nonnegative real part we have

$$\text{rank}\begin{bmatrix} sI_{n-p} - A_{11} \end{bmatrix} = n - p \tag{4.62}$$

Hence A_{11} is always stable. For the case $m < p$, define a further change of coordinates, dependent on design matrix $\bar{L}$, by

$$\tilde{T} = \begin{bmatrix} I_{n-p} & \bar{L} \\ 0 & I_p \end{bmatrix} \tag{4.63}$$

where

$$\bar{L} = [\bar{L}_1 \quad 0_{(n-p)\times m}] \tag{4.64}$$

Let $\tilde{e}_1 = e_1 + \bar{L}e_y$. The error system with respect to the new coordinates can be written as

$$\begin{aligned} \dot{\tilde{e}}_1 &= (A_{11} + \bar{L}A_{21})e_1 + (A_{12} + \bar{L}A_{22} - (A_{11} + \bar{L}A_{21})\bar{L})e_y \\ &+(\phi_1 + \bar{L}\phi_2) + (\Delta_1 + \bar{L}\Delta_2)\xi + \bar{L}(CP^{-1}C^T F^T f - \bar{\nu}) \\ \dot{e}_y &= A_{21}^o\, e_1 + A_{22}^o\, e_y + \phi_2 + CP^{-1}C^T F^T f(t) - \bar{\nu}(t) + \Delta_2 \xi \end{aligned} \tag{4.65}$$

In sliding mode, we have $e_y = \dot{e}_y = 0$ and the reduced order system in sliding mode is given by

$$\dot{\tilde{e}}_1 = (A_{11} + \bar{L}A_{21})e_1 + (\phi_1 + \bar{L}\phi_2) + (\Delta_1 + \bar{L}\Delta_2)\xi + \bar{L}(CP^{-1}C^T F^T f - \bar{\nu}). \tag{4.66}$$

Therefore, in this case, for a stable sliding motion, we need $A_{11} + \bar{L}A_{21}$ to be a stable matrix. From assumption (3.4), it follows that for every complex number s with nonnegative real part

$$\begin{aligned} n + m &= \text{rank} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \tilde{T}\bar{T} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{T}^{-1}\tilde{T}^{-1} & 0 \\ 0 & I_m \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} sI_{n-p} - (A_{11} + \bar{L}A_{21}) & -\tilde{A}_{12} & \bar{L}CP^{-1}C^T F^T \\ -A_{21} & sI_p - A_{22} & CP^{-1}C^T F^T \\ 0 & I_p & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} sI_{n-p} - (A_{11} + \bar{L}A_{21}) & \bar{L}CP^{-1}C^T F^T \\ -A_{21} & CP^{-1}C^T F^T \end{bmatrix} + p := \tilde{\mathcal{Q}}(s) + p. \end{aligned} \tag{4.67}$$

We partition A_{21} as

$$A_{21} = \left[\begin{array}{c} A_{211} \\ \hline A_{212} \end{array} \right] \begin{array}{l} \updownarrow p - m \\ \updownarrow m \end{array} \tag{4.68}$$

Since rank$(CP^{-1}C^T F^T) = m$, therefore, without loss of generality, we partition

$$CP^{-1}C^T F^T = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \tag{4.69}$$

where $\tilde{B}_2 \in \mathbf{R}^{m \times m}$ and invertible. Thus

$$\begin{aligned} \tilde{\mathcal{Q}}(s) &= \text{rank} \begin{bmatrix} sI_{n-p} - (A_{11} + \bar{L}_1 A_{211}) & \bar{L}CP^{-1}C^T F^T \\ -A_{211} & \tilde{B}_1 \\ -A_{212} & \tilde{B}_2 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} sI_{n-p} - (A_{11} + \bar{L}_1 A_{211}) \\ -A_{211} \end{bmatrix} + m \\ &= \text{rank} \begin{bmatrix} I_{n-p} & -\bar{L}_1 \\ 0 & -I_{p-m} \end{bmatrix} \begin{bmatrix} sI_{n-p} - A_{11} \\ A_{211} \end{bmatrix} + m = \text{rank} \begin{bmatrix} sI_{n-p} - A_{11} \\ A_{211} \end{bmatrix} + m. \end{aligned} \tag{4.70}$$

Therefore

$$\text{rank}\begin{bmatrix} sI_{n-p} - A_{11} \\ A_{211} \end{bmatrix} = n - p \tag{4.71}$$

and consequently, the pair (A_{11}, A_{211}) is detectable and since $A_{11} + \bar{L}_1 A_{211} = A_{11} + \bar{L}A_{21}$, the matrix $(A_{11} + \bar{L}A_{21})$ is stable by a proper choice of $\bar{L}$. This completes the proof.

Remark. It is important to note that the conditions stated in proposition for the stability of the sliding motion are consistent with the conditions for the existence of SMOs presented by [18],[19],[74],[64] and [25].

4.6 Robust Adaptive Fault Reconstruction

In this section we consider the problem of reconstructing the system fault (unknown input), by using the adaptive sliding mode observer and the output information. It is shown that the reconstruction is carried out online and is dependent only on available information.

Corollary 4.1. *Given system (4.1) and observer (4.26)-(4.29), the fault $f(t)$ can be approximately reconstructed adaptively by*

$$\hat{f}_a \approx (\hat{\rho}(t) + \rho_0)(CB)^{\dagger}\mathcal{P}^{-1}\|B^T C^T X_2\|\frac{s(t)}{||s(t)|| + \delta} \tag{4.72}$$

where $\hat{\rho}(t) = \eta\|B^T C^T X_2\| \int_0^t ||s(\tau)||d\tau$, if the following optimization problem is solvable:

minimize γ subject to (4.31)-(4.33).

Proof. The error dynamics for e_y is given by

$$\begin{aligned} \dot{e}_y = {} & A_{21}^o\, e_1 + A_{22}^o\, e_y + \phi_2 + CP^{-1}C^T F^T f(t) \\ & -\bar{\nu}(t) + \Delta_2\xi. \end{aligned} \tag{4.73}$$

If the LMI optimization problem is solvable, then from Theorem 4.2, an ideal sliding motion takes place on $\mathcal{S}$ in finite time. Thus

$$e_y = \dot{e}_y = 0 \tag{4.74}$$

which yields

$$C\nu_{eq} - CBf = A_{21}^o e_1 + \phi_2 + \Delta_2 \xi \tag{4.75}$$

where in the sliding mode the discontinuous signal ν in (4.29) takes on the average, ν_{eq}, (referred to as the equivalent output error injection [71]) to preserve the sliding motion. In addition, notice that Theorem 4.1 guarantees the error dynamics evolution enters the domain Ω and therefore $||e|| < \varepsilon^*$. As a consequence, one obtains

$$\|C\nu_{eq} - CBf\| \leq (||A_{21}^o|| + ||C||\mathcal{L}_\Phi)\varepsilon^* + \beta||\Delta_2||. \tag{4.76}$$

Therefore, we have the following upper bound for the fault reconstruction error

$$\|C\nu_{eq} - CBf\| \leq \epsilon \tag{4.77}$$

where $\epsilon := (||A_{21}^o|| + ||C||\mathcal{L}_\Phi)\varepsilon^* + \beta||\Delta_2||$. For some small ϵ

$$C\nu_{eq} \approx CBf(t). \tag{4.78}$$

Notice the size of the error bound is directly related to the disturbances $\xi(t)$, indicating that precise fault reconstruction is not possible. However, in the case that the size of the error bound ϵ is smaller compared to the fault signal $f(t)$, the sliding mode observer reconstruction scheme can still preserve the fault shape effectively. As we assumed that $\text{rank}(CB) = m$, its pseudo-inverse is $(CB)^\dagger = ((CB)^T CB)^{-1}(CB)^T$. Thus

$$f \approx (CB)^\dagger C\nu_{eq}. \tag{4.79}$$

Based on the concept of equivalent output error injection [71], the signal ν_{eq} can be approximated to any degree of accuracy by

$$\nu_{eq} = (\hat{\rho}(t) + \rho_0)P^{-1}C^T\|B^TC^TX_2\|\frac{s(t)}{||s(t)|| + \delta} \tag{4.80}$$

where δ is a small positive scalar to smooth out the signal ν. Therefore

$$f \approx (\hat{\rho}(t) + \rho_0)(CB)^\dagger \mathcal{P}^{-1} \|B^T C^T X_2\| \frac{s(t)}{||s(t)|| + \delta}. \tag{4.81}$$

Remark. It should be pointed out that the signal ν_{eq} can be computed online by available information, namely system output y and state estimates $\hat{x}$.

Remark. In the case of no disturbances ($\xi = 0$), we obtain $\beta = \varepsilon = 0$ which provides the asymptotical stability of the error dynamics. It follows that $\epsilon = 0$ and the precise fault reconstruction is possible by

$$f = (CB)^\dagger C \nu_{eq} \tag{4.82}$$

4.7 A Generalization to Sensor Fault Reconstruction

In this section we generalize the concept proposed in this chapter for sensor fault reconstruction and detection. We introduce an Integral Adaptive SMO scheme when the fault occurs at the measurement output. Consider a dynamical system of the form:

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) + \Phi_1(x, t) + \Delta_1 \xi(t) \\ y_1(t) = C_1 x(t) + D_1 f_s(t) \end{cases} \tag{4.83}$$

where $x \in \mathbf{R}^{n_1}$ represents the system state, $u \in \mathbf{R}^{m_1}$ is the control input, $y \in \mathbf{R}^{p_1}$ the measured system output and $t \in \mathbf{R}^+$. $f_s(t) : \mathbf{R}^+ \to \mathbf{R}^m$ represents the sensor faults. $\xi(t) : \mathbf{R}^+ \to \mathbf{R}^{d_1}$ denotes disturbances subject to $||\xi|| \leq \beta < \infty$. $(A_1, B_1, C_1, D_1, \Delta_1)$ is the set of real constant known matrices of appropriate dimensions. $D_1 \in \mathbf{R}^{p_1 \times m}$ is a full column rank matrix

$$\text{rank}(D_1) = m \tag{4.84}$$

and $C_1 \in \mathbf{R}^{p_1 \times n_1}$ is a full row rank matrix. The known nonlinearity $\Phi_1(x_1, t)$ satisfies a Lipcshitz condition

$$\|\Phi_1(x_1, t) - \Phi_1(x_2, t)\| \leq \mathcal{L}_\Phi \|(x_1 - x_2)\| \tag{4.85}$$

where $x_1, x_2 \in \mathbf{R}^{n_1}$ and $\mathcal{L}_\Phi \in \mathbf{R}^+$ is a known positive constant. The sensor fault $f_s(t)$ is bounded in the Euclidean norm with an unknown upper bound subject to

$$\|f_s(t)\| \leq \rho < \infty \tag{4.86}$$

The main idea here is to consider the integral variable

$$x_2 = \int_0^t y_1(\tau)d\tau \tag{4.87}$$

as the new output and thus $\dot{x}_2 = y_1(t)$. The augmented system is given by

$$\begin{aligned} \tfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ C_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \Phi_1(x_1, t) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} \Delta_1 \\ 0 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_1 \end{bmatrix} f_s \end{aligned} \tag{4.88}$$

Let $n = n_1 + p_1$ and

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} A_1 & 0 \\ C_1 & 0 \end{bmatrix}, \ \Phi(x, t) = \begin{bmatrix} \Phi_1(x_1, t) \\ 0 \end{bmatrix} \\ B_u &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_1 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ D_1 \end{bmatrix}, C = [0 \ \ I_{p_1}] \end{aligned} \tag{4.89}$$

then system (4.83) can be rewritten in the form of system (4.1) with Bu replaced by $B_u u$. Since in observer design, the input signal u is not involved, it can be ignored by setting $u = 0$. Therefore, augmented system above and system (4.1) are identical. Consequently, if the LMI optimization problem stated in Theorem 4.1 is solvable, then the adaptive sliding mode observer (4.26)-(4.29) exists for the system (4.83) . And according to Therorem 2, the ideal sliding motion will take place on $\mathcal{S}$ in finite time. Notice that the matching condition (3.5) must be satisfied for system (4.83). As proved in Lemma 4.3, the rank condition (4.2) is necessary to satisfy (3.5). Assuming rank$(D_1) = m$, we have

$$\text{rank}(CB) = \text{rank}\ ([0 \ \ I_{p_1}].\begin{bmatrix} 0 \\ D_1 \end{bmatrix}) = \text{rank}(D_1) = m \tag{4.90}$$

Thus the rank condition (4.2) always holds and the adaptive SMO for sensor fault case exists. From Corollary in Section 6 and the structure of the augmented system above, it follows that the sensor fault f_s can be reconstructed by

$$f_s \approx (\hat{\rho}(t) + \rho_0)(D_1)^\dagger \mathcal{P}^{-1}\|D_1^T X_2\|\frac{s(t)}{||s(t)|| + \delta}. \tag{4.91}$$

In the case that $\xi = 0$, we have precise sensor fault reconstruction via

$$f_s = (D_1)^{\dagger} C \nu_{eq}. \tag{4.92}$$

In such case, due to the integrator in series with the system output, the sliding surface is given by

$$\mathcal{S} = \{t \in \mathbf{R}^+ : s(t) = 0 \mid s(t) = \int_0^t y_1(\tau)d\tau - \hat{x}_2(t)\}. \tag{4.93}$$

4.8 Design Example

Consider the MCK system which behave chaotically ([63]). In the form of system (4.1) MCK is presented by with the following system matrices:

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 1.5 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$
$$\Delta = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and nonlinear function $\Phi(x)$ is

$$\Phi(x) = \begin{bmatrix} -1 \\ 0 \\ 10 \\ 0 \end{bmatrix} \begin{cases} -0.2 + 3(x_1 - x_3 - 1) & : \ x_1 - x_3 > 1 \\ 0.2(x_1 - x_3) & : \ -1 \leq x_1 - x_3 \leq 1 \\ -0.2 + 3(x_1 - x_3 + 1) & : \ x_1 - x_3 < -1 \end{cases}$$

This system satisfies the rank condition (4.2). Therefore, Lemma 4.1 can be applied to determine a solution for the structural condition (3.5) where

$$\Theta = I_4 - BB^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solving the LMI optimization problem (4.31)-(4.33), provides the maximum Lipschitz gain $\max \mathcal{L}_\Phi = 0.5429$ and

$$X_1 = \begin{bmatrix} 3.0419 & -0.0001 & 0.0001 & -0.9993 \\ -0.0001 & 0.1061 & -0.2123 & 0.1201 \\ 0.0001 & -0.2123 & 0.1061 & -0.1201 \\ -0.9993 & 0.1201 & -0.1201 & 1.1838 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 3.0419 & 0.2261 \\ 0.2261 & 2.9541 \end{bmatrix}, \; \breve{L} = \begin{bmatrix} 52.1194 & -2.0590 \\ -0.5183 & 32.2650 \\ -1.2592 & 3.6157 \\ -1.5406 & 35.5010 \end{bmatrix}.$$

Hence, the observer gain (4.34) is

$$L = (\Theta X_1 \Theta + C^T X_2 C)^{-1} \breve{L} = \begin{bmatrix} 10.0141 & -0.51641 \\ -9.8694 & 10.5098 \\ -11.3728 & 34.4861 \\ 8.5014 & 1.6758 \end{bmatrix}$$

In the corresponding simulation, the constants associated with ν have been chosen to be $\eta = 0.1$, $\delta = .02$ and $\rho_0 = 10$ which satisfies (4.43). Thus, the observer design is complete. The simulation was carried out with the disturbance $\xi = 0.04sin(8t) + 0.05cos(40t)$ applied to the system from $t = 0$. The fault $f(t)$ is a ramp signal applied from $t = 2$ sec to $t = 6$ sec, with a positive slope from $t = 2$ sec to $t = 4$ sec and declining with a negative slope from $t = 4$ sec until it settles to zero at $t = 6$ sec. It can be verified that the eigenvalues of A_0 are $\{-8.4847 - 4.4766, -4.2692 \pm 4.8315i\}$, hence, A_0 is stable. Fig. 4.1 to Fig. 4.4 show the actual states (dash line) and their estimates (solid line). Fig. 4.5 depicts the signal $\|s(t)\|$ versus time, indicating an ideal sliding motion is taking place in finite time and remains on the surface $\mathcal{S}$ afterwards. Fig. 4.6 is concerned with the fault reconstruction, showing that, despite the presence of disturbances $\xi(t)$, the proposed sliding mode observer can still reconstruct the fault signal with a very good accuracy and the reconstruction error bounds are small comparing to the fault. Therefore, fault detection is easily achievable by setting appropriate thresholds. The adaptive gain $\hat{\rho}(t)$ is shown in Fig. 4.7. It is important to mention that the sliding mode observer design must satisfy the so-called matching condition (3.5). To further examine (3.5), from the LMI solution above and Lemma 4.1, it can be verified that

$$P = \Theta X_1 \Theta + C^T X_2 C = \begin{bmatrix} 6.0839 & 0.2260 & 0.0001 & -0.7733 \\ 0.2260 & 3.1133 & -0.1592 & 3.0742 \\ 0.0001 & -0.1592 & 0.1592 & -0.1201 \\ -0.7733 & 3.0742 & -0.1201 & 4.1379 \end{bmatrix}$$

and $F = B^T C^T X_2 = \begin{bmatrix} 0.2261 & 2.9541 \end{bmatrix}$. Thus

$$B^T P = FC = \begin{bmatrix} 0.2261 & 2.9541 & 0 & 2.9541 \end{bmatrix}$$

which shows the the structural condition (3.5) holds and the designed sliding mode observer is valid.

4.9 Summary

A new scheme for nonlinear robust fault reconstruction has been studied based on a sliding mode observer with an adaptive variable structure gain. The approach does not need any change of coordinates and is quite straightforward. An LMI optimization problem sufficiently guarantees the stability of the error dynamics. Unlike the predecessor methods reported in the literature, the adaptive mechanism allows the fault to have an unknown upper bound. Since there is no constraint on the disturbances, an approximation of fault can be reconstructed. The accuracy of the reconstruction directly depends upon the size of the disturbance. It is important to note that in the absence of uncertainty or disturbance, precise fault reconstruction is guaranteed by the approach. Simulations based on a nonlinear chaotic system called MCK have shown effective performance of the proposed method. Based on the approach presented here, design of fault tolerant control schemes are worth for future study.

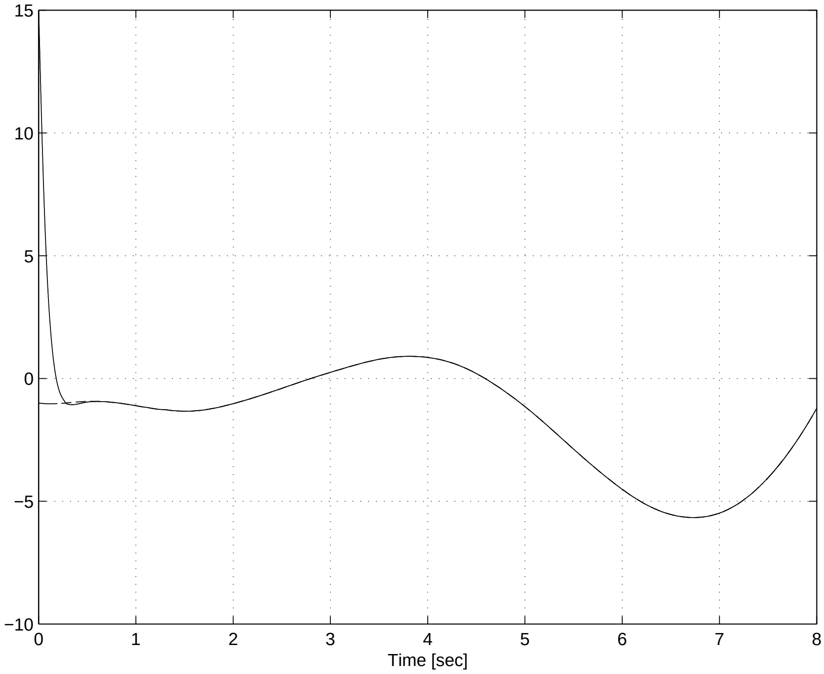

Figure 4.1: First state and its estimate

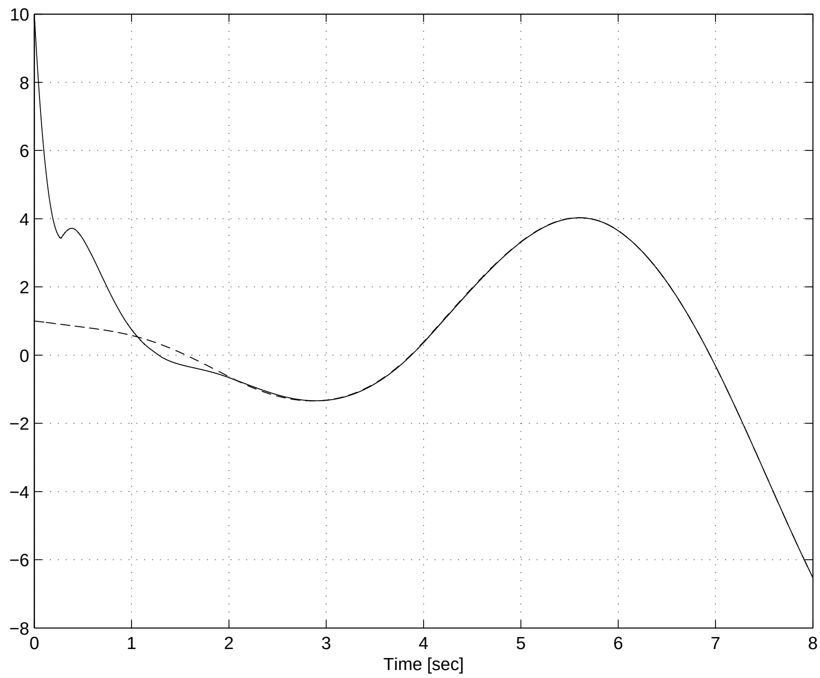

Figure 4.2: Second state and its estimate

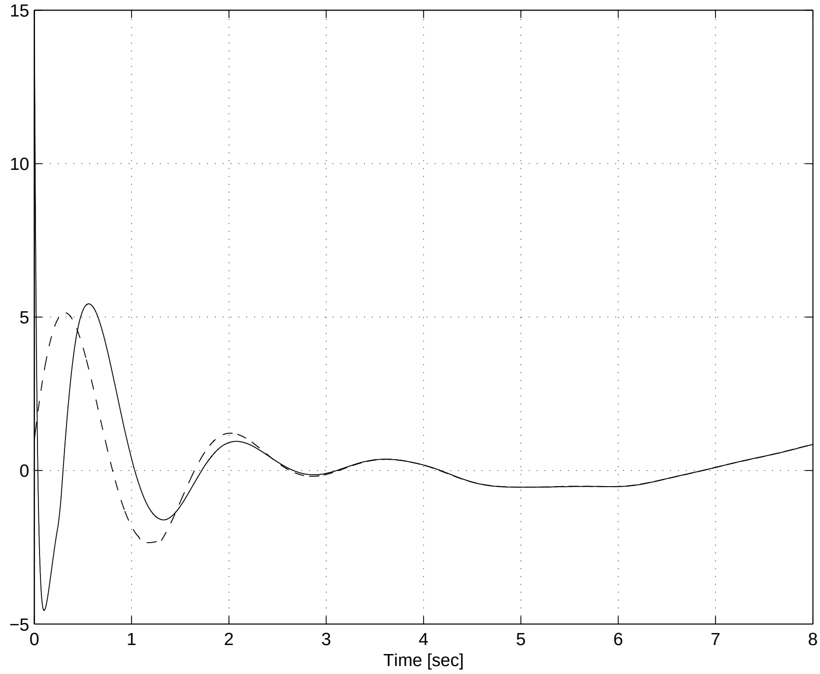

Figure 4.3: Third state and its estimate

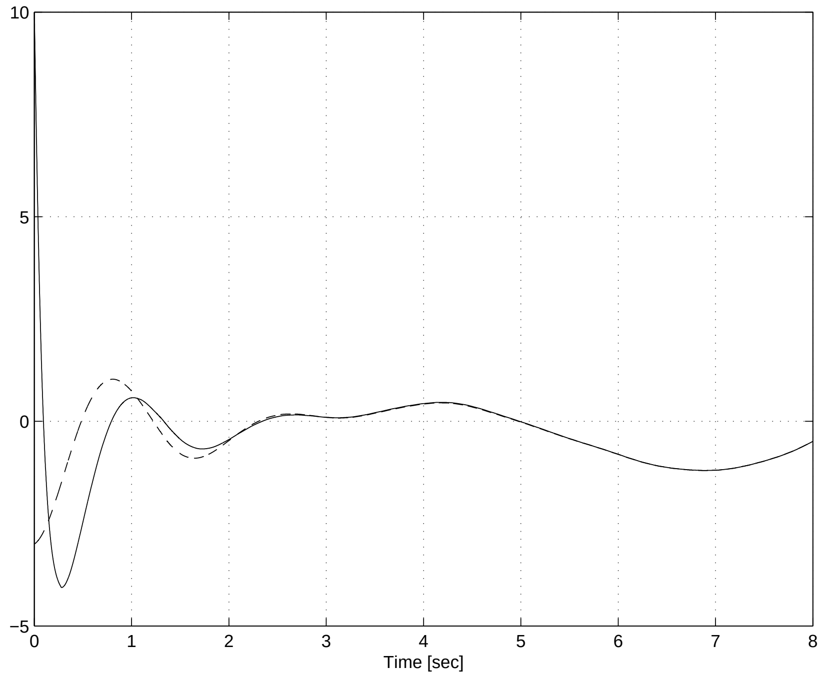

Figure 4.4: Forth state and its estimate

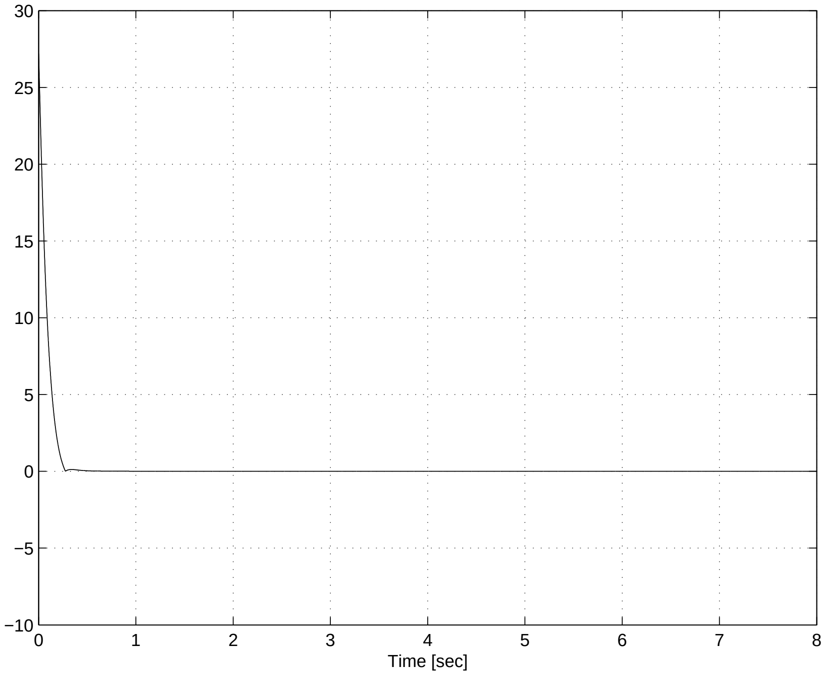

Figure 4.5: The sliding mode quantity $\|s(t)\|$

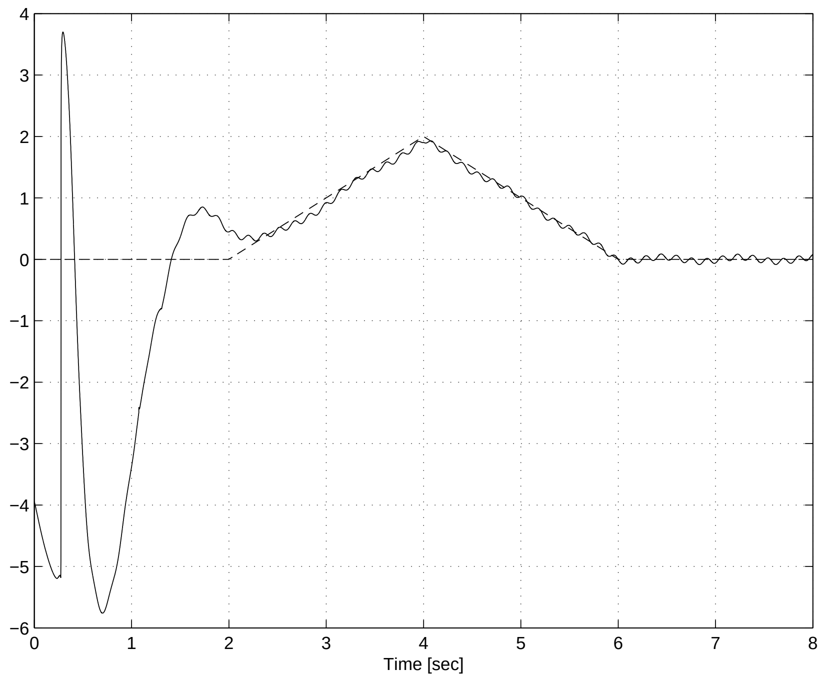

Figure 4.6: Unknown input $f(t)$ (dot line) and its reconstruction by $\nu(t)$ (solid line)

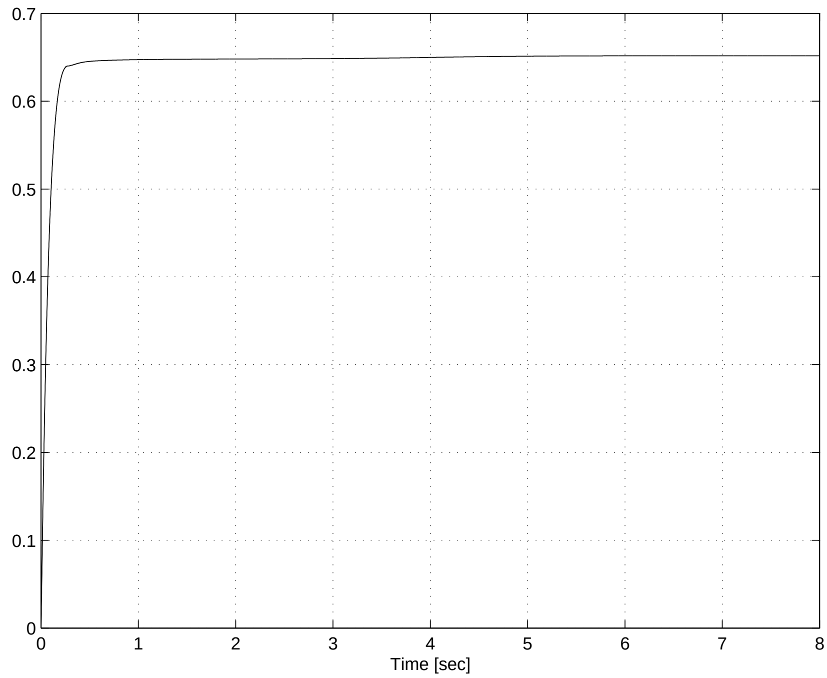

Figure 4.7: The adaptive gain $\hat{\rho}(t)$

Chapter 5

Generalized Sliding Mode Observers[1]

5.1 Background Results

This chapter deals with observer design for fault detection and estimation using *sliding mode observers* (SMOs) in a generalized state space form inspired by singular system theory. Due to their ability to cope with model uncertainties, SMOs offer great potential in fault detection applications [18].

The main focus of this chapter is to explain the design of sliding mode observers for the problem of fault reconstruction and FDI. In essence, the SMO approach consists of first defining a sliding surface and then using a variable structure control law to force the error system trajectories to the sliding surface in finite time. Recently, [17] and [18] proposed that using the variable structure control law of the SMO and the concept of equivalent output injection, a fault can be reconstructed to any required accuracy for linear systems. These two references consider the case of perfect modeling without uncertainty or unmatched disturbance, hence precise fault reconstruction is feasible. Sensor fault reconstruction using SMO was studied by [64], further extending the results for linear systems with disturbance and

[1]The results in this chapter have been submitted for publication in the article: R. Raoufi and H. J. Marquez, "Generalized Sliding Mode Estimator Design for State and Sensor/Actuator Fault Estimation", submitted to *Automatica*, August 2009, and also in the article: R. Raoufi and H. J. Marquez , "Generalized Sliding Mode Observers for State and Sensor/Actuator Fault Estimation", *submitted to American Control Conference*, IEEE, 2010.

uncertainty. For nonlinear Lipschitz systems, [74] addressed SMO based fault reconstruction by assuming that disturbances are matched and can be lumped into the so-called matching condition.

On the other hand, inspired by the theory of singular systems, [7] proposed a new generalized state-space observer design to estimate unknown signals for a class of nonlinear systems. In [28] an interesting method to design descriptor observers for systems with measurement noise and application to sensor fault diagnosis was proposed. State/noise estimator for descriptor systems with application to sensor fault diagnosis was also studied by [29]. The approaches of these references play an important role in inspiring our observer design.

Fault reconstruction is excellent for directly isolating the flaws within a system by revealing which sensor or actuator is faulty and is useful for diagnosing incipient and small faults. The detailed knowledge of the fault's shape, obtained from fault reconstruction, can highly facilitate the fault tolerant control design. However, in practical systems, it is often the case where actuators and also sensors suffer from faults during the course of the system's operation. Both Actuators and Sensors can suffer from faults either alone, at separate times or simultaneously. In this case, detection and reconstruction of all faults is highly important. The co-existence of unknown fault at both some sensor(s) and actuator(s) has not been addressed in any earlier design of the sliding mode observers or other fault reconstruction schemes. Clearly the sensor fault corrupts measurement, therefore making it harder to reconstruct the actuator faults. On the other hand, reconstruction of the sensor faults also remain challenging and unsolved due to faulty actuators. Thus, fault reconstruction/identification brings important benefits to the system and thus in this chapter, we aim at reconstructing the faults when they coexist at sensors and actuators during the system operation.

What is unique about our approach is that it involves a new design of a robust sliding mode observer in a generalized state space form when faults occur at *both* *sensors and actuators* coincidentally. To cope with the sensor faults, inspired by [7], a generalized state space form is employed such that the augmentation results in a descriptor system form. This singular (descriptor) formulation, provides the possibility for sensor fault estimation. In this singular form, we define a sliding surface and the proposed filter forces the trajectories of the estimation error to approach the sliding surface and remain there afterwards. The actuator fault satisfies the so-called matching condition and is targeted by the sliding mode controller for reconstruction. These features allow for actuator faults to be reconstructed based on information retrieved from the equivalent output error injection signal. Thus, not only the actuator faults but also sensor faults are reconstructed. In addition, the states of the system are also estimated by the proposed robust observer which is very important due to the corruption of the measurements by sensor faults. As a result, the proposed observer will be called the generalized sliding mode observer (GSMO) here on after.

Another important feature of our solution is the following: the solution presented in references [19], [74], [64] and [3] assumes all the actuator fault/unknown inputs are classified as matched disturbances. In these references, any unmatched disturbance can not be estimated. However, the proposed robust GSMO is novel due to the ability of estimating a class of disturbances which are not *matched*. The class of disturbance that can be dealt with is described in detail and has a common behavior with output disturbances (or sensor faults). The estimation of this class of unmatched disturbances also benefits from the singular system model that arises in our formulation. Research is underway to adopt this novel observer, GSMO, for simultaneous senosr/actuator fault tolerant control. For a different approach on robust state observers compared to SMOs see [44], [45].

The remainder of this chapter is organized as follows; Section 5.2 provides some preliminaries and assumptions on the class of nonlinear system addressed. Some preliminary lemmas are introduced in Section 5.3. The design of the robust GSMO and the analysis of the stability of the error dynamics are given in Section 5.4. In Section 5.5 the stability of the sliding motion is discussed. Fault estimation is studied in Section 5.6. The effectiveness of the proposed GSMO based fault reconstruction is studied with an example in Section 5.7. Finally some concluding remarks are presented in Section 5.8.

5.2 Preliminaries and Assumptions

Consider a dynamical system affected by sensor and actuator faults of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + f(t)) + B_{\Phi}\Phi(x,t) + \Delta\omega(t) \\ y(t) = Cx(t) + D\omega(t) \end{cases} \tag{5.1}$$

where $x \in \mathbf{R}^n$ represents the system state, $u \in \mathbf{R}^m$ the control input, $y \in \mathbf{R}^p$ the measured system output and $t \in \mathbf{R}^+$. Throughout this chapter we assume that $p > m$. The set (A, B, C, D) is of real constant known matrices of appropriate dimensions where $D \in \mathbf{R}^{p\times(p-m)}$ and (A, C) is an observable pair. $f(t) : \mathbf{R}^+ \to \mathbf{R}^m$ denotes the fault (unknown input) that is bounded in the Euclidean norm as in (3.2). The function $\omega(t) : \mathbf{R}^+ \to \mathbf{R}^{p-m}$ is the output disturbances (or sensor fault) where D is the corresponding distribution matrix with full columns rank. Therefore, without lose of generality, we can assume there exist a nonsingular change of coordinate S_0 (yet to be designed) which provides the following geometric condition associated with D:

$$S_0 D = \left[\begin{array}{c} 0 \\ \hline D_2 \end{array}\right] \tag{5.2}$$

where $D_2 \neq 0,\ D_2 \in \mathbf{R}^{(p-m)\times(p-m)}$ and is invertible. This assumption implies that certain sensors are prone to fault and not all of them. The known nonlinearity $\Phi(x, t)$ satisfies a Lipshitz-like condition (2.7). We also assume that the minimum phase condition (3.4) for the triple $\{A, B, C\}$ and matching condition (3.5) for matrices $\{B, P\}$ hold. Suppose that

$$\mathrm{Im}(B) \cap \mathrm{Im}(\Delta) = \{\emptyset\} \tag{5.3}$$

with

$$\mathrm{rank}\ (C\Delta) < \mathrm{rank}\ (\Delta). \tag{5.4}$$

5.3 Some Preliminary Lemmas

Lemma 5.1. [13] *Consider the system (5.1). There exists a solution $P = P^T > 0$ such that $B^T P = FC$ if and only if*

$$\text{rank}(CB) = \text{rank}\ (B) \tag{5.5}$$

Remark: Due to Lemma 5.1, the inequality

$$\text{rank}\ (C\Delta) < \text{rank}\ (\Delta)$$

implies that Δ does not satisfy a matching condition of the form $\Delta^T P = F_\Delta C$, hence the term $\Delta\omega(t)$ is categorized as class of *unmatched uncertainties*.

Lemma 5.2. *Given the system (5.1) with rank(CB) = rank (B) and associated assumption (3.4), there exist nonsingular transformation matrices T and S such that*

$$\begin{array}{c} TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\ T\Delta = \begin{bmatrix} 0 \\ \Delta_2 \end{bmatrix}, \\ TB_\Phi = \begin{bmatrix} B_{\Phi 1} \\ B_{\Phi 2} \end{bmatrix},\ SCT^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix},\ SD = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \end{array} \tag{5.6}$$

$A_1 \in \mathbf{R}^{m\times m}$, $A_4 \in \mathbf{R}^{(n-m)\times(n-m)}$, $C_1 \in \mathbf{R}^{m\times m}$, $C_4 \in \mathbf{R}^{(p-m)\times(n-m)}$, $\Delta_2 \in \mathbf{R}^{(n-m)\times(p-m)}$, B_1, C_1 and D_2 are invertible.

Proof. We have rank $(B) = m$, therefore without loss of generality, by using a nonsingular transformation T_0, we partition the matrix B as

$$T_0 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{5.7}$$

where $B_1 \in \mathbf{R}^{m\times m}$ with rank $(B_1) = m$. Now, introduce a nonsingular coordinate transformation T_1 as

$$T_1 = \begin{bmatrix} I_m & 0 \\ -B_2 B_1^{-1} & I_{n-m} \end{bmatrix} \tag{5.8}$$

then

$$T_1 T_0 B = \begin{bmatrix} I_m & 0 \\ -B_2 B_1^{-1} & I_{n-m} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{5.9}$$

where $B_1 \in \mathbf{R}^{m\times m}$ is nonsingular. Now we partition $CT_0^{-1}T_1^{-1}$ as $CT_0^{-1}T_1^{-1} = (\bar{C}_1 \;\; \bar{C}_4)$. Therefore

$$CB = CT_0^{-1}T_1^{-1}T_1T_0B = (\bar{C}_1 \;\; \bar{C}_4)\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \bar{C}_1B_1 \tag{5.10}$$

and consequently, using rank$(CB) =$ rank (B) we have

$$\text{rank}(\bar{C}_1B_1) = \text{rank}(CB) = \text{rank}(B_1),$$

then from the nonsingularity of the matrix B_1, we directly conclude that

$$\text{rank}(\bar{C}_1) = m. \tag{5.11}$$

Without loss of any generality, using the proper nonsingular change of coordinate S_0, we partition $\bar{C}_1$ as follows

$$S_0\bar{C}_1 = \begin{bmatrix} C_1 \\ C_{21} \end{bmatrix} \tag{5.12}$$

where $C_1 \in \mathbf{R}^{m\times m}$ and

$$\text{rank}(C_1) = m. \tag{5.13}$$

Consequently det $(C_1) \neq 0$. Let

$$S_1 = \begin{bmatrix} I_m & 0 \\ -C_{21}C_1^{-1} & I_{p-m} \end{bmatrix} \tag{5.14}$$

which yields

$$S_1S_0\bar{C}_1 = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}. \tag{5.15}$$

Letting $S = S_1S_0$, we obtain

$$SCT_0^{-1}T_1^{-1} = \begin{bmatrix} C_1 & C_{12} \\ 0 & C_4 \end{bmatrix}. \tag{5.16}$$

Let

$$T_2^{-1} = \begin{bmatrix} I_m & -C_1^{-1}C_{12} \\ 0 & I_{n-m} \end{bmatrix} \tag{5.17}$$

and $T = T_2T_1T_0$, then it is easy to obtain

$$TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\; SCT_0^{-1}T_1^{-1}T_2^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix} \tag{5.18}$$

$$SD = \begin{bmatrix} I_m & 0 \\ -C_{21}C_{11}^{-1} & I_{p-m} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ D_2 \end{bmatrix}. \tag{5.19}$$

Suppose that $T\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$. With regard to Assumption (5.3) and the fact that T is nonsingular, it follows that

$$\text{Im}(TB) \cap \text{Im}(T\Delta) = \text{Im}(\begin{bmatrix} B_1 \\ 0 \end{bmatrix}) \cap \text{Im}(\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}) = \{\emptyset\}$$

which implies that $\Delta_1 = 0$. In the new coordinate, the matrices A and B_Φ are transformed as in (5.6). This completes the proof.

Lemma 5.3. [32] *Consider the system (5.1) and assume that* rank $(CB) =$ rank (B). *Then the pair (A_4, C_4) is detectable if and only if*

$$\text{rank} \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} = n + m \tag{5.20}$$

for all s such that $Re(s) \geq 0$.

Lemma 5.4. *Consider the system (5.1) and assume that rank $(CB) =$ rank (B), then the matching condition (3.5) always holds in the new coordinates of Lemma 5.1.*

Proof. See the proof of Lemma 3.3.

5.4 SMO Design: A Descriptor Approach

In this section, we propose a theorem to design the new GSMO. We employ the nonsingular state transformations introduced in Lemma 5.2 as the key to deal of our design. Due to Lemma 5.2, system (5.1) in the new coordinates $\tilde{x} := (x_1^T, x_2^T)^T = Tx$ and $\tilde{y} := (y_1^T, y_2^T)^T = Sy$ is

$$\begin{cases} \dot{x}_1 = A_1x_1 + A_2x_2 + B_1(u + f(t)) + B_{\Phi 1}\Phi_1(T^{-1}\tilde{x}, t) \\ y_1 = C_1x_1 \quad (x_1 \in \mathbf{R}^m) \end{cases} \tag{5.21}$$

$$\begin{cases} \dot{x}_2 = A_3x_1 + A_4x_2 + B_{\Phi 2}\Phi_2(T^{-1}\tilde{x}, t) \\ y_2 = C_4x_2 + D_2\omega(t) \quad (x_2 \in \mathbf{R}^{n-m}) \end{cases} \tag{5.22}$$

where $T\Phi(x,t) := (\Phi_1^T, \Phi_2^T)^T$. Also partition S as

$$S = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{bmatrix}, \bar{S}_1 \in \mathbf{R}^{m\times p}, \bar{S}_2 \in \mathbf{R}^{(p-m)\times p}$$

So that the variable x_1 can be obtain from the measured output y by

$$x_1 = C_1^{-1}\bar{S}_1 y(t). \tag{5.23}$$

First, we employ following state augmentation $x_3 = \begin{bmatrix} x_2 \\ \omega \end{bmatrix}$, $x_3 \in \mathbf{R}^{n+p-2m}$ which leads to the following descriptor plant with singular $E \in \mathbf{R}^{(n+p-2m)\times(n+p-2m)}$ (rank$(E) = n - m$)

$$\begin{cases} E\dot{x}_3 = \mathcal{A}_4 x_3 + \mathcal{A}_3 x_1 + \mathcal{B}_{\Phi 2}\Phi(x_1, x_2) + M\omega \\ y_2 = \mathcal{C}_4 x_3 \end{cases} \tag{5.24}$$

where

$$E = \begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \end{bmatrix}, \ \mathcal{A}_4 = \begin{bmatrix} A_4 & \Delta_2 \\ 0 & -I_{p-m} \end{bmatrix}, \ \mathcal{A}_3 = \begin{bmatrix} A_3 \\ 0 \end{bmatrix}$$

$$\mathcal{B}_{\Phi 2} = \begin{bmatrix} B_{\Phi_2} \\ 0 \end{bmatrix}, M = \begin{bmatrix} 0 \\ I_{p-m} \end{bmatrix}, \ \mathcal{C}_4 = \begin{pmatrix} C_4 & D_2 \end{pmatrix}.$$

Subsystem (5.21) is rewritten as

$$\dot{x}_1 = A_1 x_1 + \mathcal{A}_2 x_3 + B_1(u(t) + f(t)) + B_{\Phi_1}\Phi(x_1, x_3, t) \tag{5.25}$$

where $\mathcal{A}_2 = \begin{pmatrix} A_2 & 0_{m\times(n-m)} \end{pmatrix}$.

To design an observer for the above subsystems, we first need to check the finite-observability and impulsive-observability of the singular system . Regarding the observability definitions in (2.46) and (2.47) for singular systems, we present the following Lemma.

Lemma 5.5. The system (5.4) is observable, i.e,

$$\text{rank}\begin{bmatrix} E & 0 \\ \mathcal{A}_4 & E \\ \mathcal{C}_4 & 0 \end{bmatrix} = (n + p - 2m) + \text{rank}(E), \ \ \forall s \in \mathbf{C}. \tag{5.26}$$

and

$$\text{rank}\begin{bmatrix} sE - \mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} = n + p - 2m, \ \ \forall s \in \mathbf{C}. \tag{5.27}$$

Proof. We have rank$(E) = n - m$ and therefore

$$\text{rank}\begin{bmatrix} E & 0 \\ \mathcal{A}_4 & E \\ \mathcal{C}_4 & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} I_{n-m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_4 & \Delta_2 & I_{n-m} & 0 \\ 0 & -I_{p-m} & 0 & 0 \\ C_4 & D_2 & 0 & 0 \end{bmatrix}$$
$$= 2(n-m) + (p-m) = (n+p-2m) + \text{rank}(E).$$

Thus according to (2.47), system (5.4) is impulsive-observable. In addition, we have

$$\text{rank}\begin{bmatrix} sE - \mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} = \text{rank}\begin{bmatrix} sI_{n-m} - A_4 & -\Delta_2 \\ 0 & I_{p-m} \\ C_4 & D_2 \end{bmatrix} = p - m + \text{rank}\begin{bmatrix} sI_{n-m} - A_4 \\ C_4 \end{bmatrix}.$$

From Lemma 5.3, we can conclude that if (A_4, C_4) is an observable pair, then

$$\text{rank}\begin{bmatrix} sI_{n-m} - A_4 \\ C_4 \end{bmatrix} = n - m, \quad \forall s \in \mathbf{C}, \tag{5.28}$$

and according to the definition (2.46), the system (5.4) is finite-observable. Hence, we conclude observability. This completes the proof.

We now put forward the following important Lemma.

Lemma 5.6. *The inverse $(E + \mathcal{K}\mathcal{C}_4)^{-1}$ exists for some gain $\mathcal{K} \in \mathbf{R}^{(n+p-2m)\times(p-m)}$.*

Proof. We have

$$\begin{aligned} &\text{rank}\begin{bmatrix} E \\ \mathcal{C}_4 \end{bmatrix} = \text{rank}\begin{bmatrix} I_{n-m} & 0 \\ 0 & 0 \\ C_4 & D_2 \end{bmatrix} = n - m + \text{rank}(D_2) \\ &= n + p - 2m \end{aligned} \tag{5.29}$$

then there exists gain $\mathcal{K}$ of appropriate dimension such that rank$(E + \mathcal{K}\mathcal{C}_4) = n + p - 2m$, hence $(E + \mathcal{K}\mathcal{C}_4)$ is invertible.

By design, we adopt the following structure for gain $\mathcal{K}$

$$\mathcal{K} = \left[\frac{0_{(n-m)\times(p-m)}}{\mathcal{K}_2}\right] \tag{5.30}$$

where $\mathcal{K}_2 \in \mathbf{R}^{(p-m)\times(p-m)}$ and full rank by design. Due to the structure of $\mathcal{K}$,it follows that

$$(E+\mathcal{K}\mathcal{C}_4)^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -C_4 & D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix} \tag{5.31}$$

$$(E+\mathcal{K}\mathcal{C}_4)^{-1}M = \begin{bmatrix} 0 \\ D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix} \tag{5.32}$$

$$\begin{array}{l} \mathcal{C}_4(E+\mathcal{K}\mathcal{C}_4)^{-1}\mathcal{K} = \\ (C_4 \;\; D_2) \begin{bmatrix} I_{n-m} & 0 \\ -C_4 & D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ k_2 \end{bmatrix} = I_{n-m}. \end{array} \tag{5.33}$$

We will employ the system structures (5.21) and (5.4) in the new observer design. Consider the following generalized sliding mode observer structure

$$\begin{array}{c} (E+\mathcal{K}\mathcal{C}_4)\dot{z} = \mathcal{A}_3\hat{x}_1 + (\mathcal{A}_4 - L_4\mathcal{C}_4)z + L_3(y_1 - C_1\hat{x}_1) \\ +\mathcal{A}_4(E+\mathcal{K}\mathcal{C}_4)^{-1}\mathcal{K}y_2 + \mathcal{B}_{\Phi_2}\Phi(\hat{x}_1,\hat{x}_3) \end{array} \tag{5.34}$$

$$\hat{x}_3 = z + (E+\mathcal{K}\mathcal{C}_4)^{-1}\mathcal{K}y_2 \tag{5.35}$$

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + \mathcal{A}_2\hat{x}_3 + L_1(y_1 - C_1\hat{x}_1) + B_1\nu(t) + B_{\Phi_1}\Phi(\hat{x}_1,\hat{x}_3) \tag{5.36}$$

where the novel reduced-order sliding gain structure $\nu \in \mathbf{R}^m$ and the observer gain $\tilde{L}$ are respectively

$$\nu(t) = \begin{cases} (\rho+\rho_0)\frac{B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\|} & : \; C_1^{-1}\bar{S}_1 y - \hat{x}_1 \neq 0 \\ 0 & : \text{otherwise} \end{cases} \tag{5.37}$$

$$\tilde{L} := \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} := \begin{bmatrix} \bar{A}_1 C_1^{-1} & 0 \\ \mathcal{A}_3 C_1^{-1} & (E+\mathcal{K}\mathcal{C}_4)P_3^{-1}K \end{bmatrix}, \quad \lambda > 0 \tag{5.38}$$

where ρ_0 is some positive scalar. P_1, P_3 and K will be determined through the stability proof and $\bar{A}_1 = A_1 - A_1^s$ where A_1^s represents a stable design matrix.

Remark. The novelty if the proposed sliding mode controller ν is that under the same assumptions used in [64] and references there in, the controller requires just some components of the output $y(t)$. Consequently its order is equal to $m < p$. In [74] and [64], the gain ν has the following general form:

$$\nu = (\rho+\rho_0)\frac{P_0(y - C\hat{x})}{||P_0(y - C\hat{x})||}$$

with order p. In our proposed GSMO, the controller ν is of order $m < p$. Interestingly, due to this reduced-order structure of ν it is possible to tackle a class of disturbances/faults at the output.

We now present Theorem 5.1 which is the main result of this section.

Theorem 5.1. *Given the nonlinear uncertain system (5.1) with assumptions (3.2)-(5.4), consider the GSMO structure (5.34)-(5.38). The observer error dynamics is ultimately bounded with an arbitrary small upper bound if there exist matrices K, $P_1^T = P_1 > 0$ and $P_3^T = P_3 > 0$ such that the following LMI feasibility problem has a solution:*

$P_1 > 0$, $P_3 > 0$ *and*

$$\begin{bmatrix} A_1^{s^T} P_1 + P_1 A_1^s + \bar{\mathcal{L}} I_m & P_1 \mathcal{A}_2 & P_1 B_{\Phi 1} \\ \mathcal{A}_2^T P_1 & M_{22} + \bar{\mathcal{L}} I_{n+p-2m} & P_3 \Lambda \mathcal{B}_{\Phi 2} \\ B_{\Phi 1}^T P_1 & \mathcal{B}_{\Phi 2}^T \Lambda^T P_3 & -I \end{bmatrix} < 0 \tag{5.39}$$

where

$$\bar{\mathcal{L}} = {\mathcal{L}_\Phi}^2 \left\| T^{-1} \right\|^2 \tag{5.40}$$

$$\Lambda := (E + \mathcal{K} \mathcal{C}_4)^{-1} \tag{5.41}$$

$$M_{22} = P_3 \Lambda \mathcal{A}_4 - K \mathcal{C}_4 - \mathcal{A}_4^T \Lambda^T P_3 - \mathcal{C}_4^T K^T \tag{5.42}$$

Proof. We define $e_1 = x_1 - \hat{x}_1, e_3 = x_3 - \hat{x}_3$ and $e_\Phi = \Phi(x_1, x_3) - \Phi(\hat{x}_1, \hat{x}_3)$. From (5.35), substituting $z = \hat{x}_3 - (E + \mathcal{K}\mathcal{C}_4)^{-1} \mathcal{K} y_2$ into (5.34) yields

$$\begin{gathered} \Lambda^{-1}(\dot{\hat{x}}_3 - \Lambda \mathcal{K} \dot{y}_2) = \mathcal{A}_3 \hat{x}_1 + (\mathcal{A}_4 - L_4 \mathcal{C}_4)(\hat{x}_3 - \Lambda K y_2) \\ + L_3 C_1 e_1 + \mathcal{A}_4 (E + \mathcal{K}\mathcal{C}_4)^{-1} \mathcal{K} y_2 + \mathcal{B}_{\Phi_2} \Phi(\hat{x}_1, \hat{x}_3). \end{gathered} \tag{5.43}$$

Adding $\mathcal{K}\dot{y}$ to both sides of (5.4) and then subtracting it from the above equation, one obtains

$$\Lambda^{-1} \dot{e}_3 = \mathcal{A}_3 e_1 - L_3 C_1 e_1 + (\mathcal{A}_4 - L_4 \mathcal{C}_4) e_3 + M\omega + \mathcal{B}_{\Phi 2} e_\Phi \tag{5.44}$$

By choosing $L_3 = \mathcal{A}_3 C_1^{-1}$, it follows that

$$\dot{e}_3 = \Lambda \mathcal{A}_4^o e_3 + \Lambda M \omega + \Lambda \mathcal{B}_{\Phi 2} e_\Phi \tag{5.45}$$

where $\mathcal{A}_4^o := (\mathcal{A}_4 - L_4\mathcal{C}_4)$. From (5.32) the disturbance term is expressed as

$$\Lambda M\omega(t) = \begin{bmatrix} 0 \\ D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix}$$

consequently, by design, we can choose a high-gain $\mathcal{K}_2$ to reduce the amplification of the bounded noise and disturbances $\omega(t)$ to any arbitrary low magnitude. Furthermore, notice that $\forall s \in C_+$,

$$\begin{aligned}
&\text{rank}\begin{bmatrix} sI_{n+p-2m} - \Lambda\mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I_{p-m} \end{bmatrix}\begin{bmatrix} sI_{n+p-2m} - \Lambda\mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} s\Lambda^{-1} - \mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} = \text{rank}\begin{bmatrix} I_{n+p-2m} & sK \\ 0 & I_{p-m} \end{bmatrix}\begin{bmatrix} sE - \mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} sE - \mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} = \text{rank}\begin{bmatrix} sI_{n-m} - A_4 & -\Delta_2 \\ 0 & I_{p-m} \\ C_4 & D_2 \end{bmatrix} \\
&= p - m + \text{rank}\begin{bmatrix} sI_{n-m} - A_4 \\ C_4 \end{bmatrix}.
\end{aligned}$$

From Lemma 5.3, we know the pair (A_4, C_4) is a detectable pair. Thus

$$\text{rank}\begin{bmatrix} sI_{n+p-2m} - \Lambda\mathcal{A}_4 \\ \mathcal{C}_4 \end{bmatrix} = n + p - 2m, \quad \forall s \in C_+, \tag{5.46}$$

which means that the pair $(\Lambda\mathcal{A}_4, \mathcal{C}_4)$ is detectable and we can choose a matrix $\bar{L}_4$ such that $(\Lambda\mathcal{A}_4 - \bar{L}_4\mathcal{C}_4)$ is a stable matrix. Let $L_4 = \Lambda^{-1}\bar{L}_4$, we obtain the stable matrix $\Lambda\mathcal{A}_4^o$. From (5.21) and (5.36), we obtain

$$\dot{e}_1 = (A_1 - L_1C_1)e_1 + \mathcal{A}_2e_3 + B_1f - B_1\nu(t) + B_{\Phi 1}\Phi(e_1, e_3) \tag{5.47}$$

Let $L_1 = (A_1 - A_1^s)C_1^{-1}$, then

$$\dot{e}_1 = A_1^se_1 + \mathcal{A}_2e_3 + B_1(f - \nu(t)) + B_{\Phi 1}\Phi(e_1, e_3) \tag{5.48}$$

Consider $V_1 = e_1^TP_1e_1$ and $V_2 = e_2^TP_2e_2$. We define

$$V = V_1 + V_2.$$

Then the derivative of V_1 and V_2 are given by:

$$\begin{array}{l}\dot{V}_1 = e_1^T(A_1^{S^T}P_1 + P_1A_1^s)e_1 + e_1^TP_1\mathcal{A}_2e_3 + e_3^TA_2^TP_1e_1 \\ +e_1^TP_1\bar{B}_{\Phi_1}e_{\Phi 1} + e_{\Phi}{}_1^T\bar{B}_{\Phi 1}^TP_1e_1 \\ +e_1^TP_1B_1(f-\nu) + (f-\nu)^TB_1^TP_1^Te_1\end{array} \tag{5.49}$$

$$\begin{array}{l}\dot{V}_2 = e_3^T(\mathcal{A}_4^{0^T}\Lambda^TP_3 + P_3\Lambda\mathcal{A}_4^0)e_3 + e_3^TP_3\Lambda M\omega(t) \\ +\omega^TM^T\Lambda^TP_3e_3 + e_3^TP_3\Lambda\bar{B}_{\Phi_2}e_{\Phi 2} + e_{\Phi}{}_2^T\bar{B}_{\Phi 2}^T\Lambda^TP_3e_3\end{array} \tag{5.50}$$

From (5.23) it follows that $e_1 = C_1^{-1}\bar{S}_1y - \hat{x}_1$. Then using the switching gain (5.37) and (3.2) we obtain

$$\begin{array}{c}e_1^TP_1B_1(f(t)-\nu(t)) = \\ e_1^TP_1B_1f(t) - (\rho+\rho_0)\frac{\|e_1^TP_1B_1\|^2}{\|e_1^TP_1B_1\|} \leq \\ \rho\|e_1^TP_1B_1\| - (\rho+\rho_0)\|e_1^TP_1B_1\| = -\rho_0\|e_1^TP_1B_1\| < 0\end{array} \tag{5.51}$$

Consequently, by choosing $\bar{L}_4 = \Lambda^{-1}P_3^{-1}\mathcal{K}$ and with regard to (5.49), (5.50) and (5.51), the stability criteria $\dot{V} < 0$ is equivalent to the following inequality

$$\begin{array}{c}\begin{bmatrix} e_1 \\ e_3 \\ e_\Phi \end{bmatrix}^T \begin{bmatrix} A_1^{s^T}P_1 + P_1A_1^s + \bar{\mathcal{L}}I_m & P_1\mathcal{A}_2 & P_1B_{\Phi 1} \\ \mathcal{A}_2^TP_1 & M_{22} + \bar{\mathcal{L}}I_{n+p-2m} & P_3\Lambda\mathcal{B}_{\Phi 2} \\ B_{\Phi 1}^TP_1 & \mathcal{B}_{\Phi 2}^T\Lambda^TP_3 & -I \end{bmatrix} \\ \times \begin{bmatrix} e_1 \\ e_3 \\ e_\Phi \end{bmatrix} + e_3^TP_3\Lambda M\omega + \omega^TM^T\Lambda^TP_3e_3 < 0\end{array}$$

Thus if

$$-Q := \begin{bmatrix} A_1^{s^T}P_1 + P_1A_1^s + \bar{\mathcal{L}}I_m & P_1\mathcal{A}_2 & P_1B_{\Phi 1} \\ \mathcal{A}_2^TP_1 & M_{22} + \bar{\mathcal{L}}I_{n+p-2m} & P_3\Lambda\mathcal{B}_{\Phi 2} \\ B_{\Phi 1}^TP_1 & \mathcal{B}_{\Phi 2}^T\Lambda^TP_3 & -I \end{bmatrix} < 0 \tag{5.52}$$

then

$$\dot{V} \leq -\left\| \begin{matrix} e_1 \\ e_3 \\ e_\Phi \end{matrix} \right\| \left(\lambda_{min}(Q) \left\| \begin{matrix} e_1 \\ e_3 \\ e_\Phi \end{matrix} \right\| - 2\|P_3\Lambda M\|\omega_0 \right)$$

which guarantees that the magnitude of the error is ultimately bounded with respect to the set:

$$\begin{array}{l}\Omega_\varepsilon = \\ \left\{ \begin{pmatrix} e_1 \\ e_3 \\ e_\Phi \end{pmatrix} : \left\| \begin{pmatrix} e_1 \\ e_3 \\ e_\Phi \end{pmatrix} \right\| < \frac{2\|P_3\Lambda M\|\omega_0}{\lambda_{min}(Q)} + \varepsilon, \varepsilon > 0 \right\}\end{array} \tag{5.53}$$

Using (5.32), we have

$$\Omega_\varepsilon = \left\{ \begin{pmatrix} e_1 \\ e_3 \\ e_\Phi \end{pmatrix} : \left\| \begin{pmatrix} e_1 \\ e_3 \\ e_\Phi \end{pmatrix} \right\| < 2\omega_0 \frac{\left\| P_3 \begin{bmatrix} 0 \\ D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix} \right\|}{\lambda_{min}(Q)} + \varepsilon, \varepsilon > 0 \right\}. \quad (5.54)$$

Thus, similar to the approach given by [28], the upper bound of the error can be significantly dropped by arbitrarily choosing a high-gain $\mathcal{K}_2$. This completes the proof.

5.5 Ideal Sliding Motion and Fault Reconstruction

It is well-known that, in order to confine an stable motion of a dynamical system onto a sliding surface $\mathcal{S}$, it is necessary to use a switching gain which is discontinuous about the surface $\mathcal{S}$ [18]. Therefore, due to the structure of the switching gain (5.37) and the fact that $\mathcal{N}(B_1) = \{\emptyset\}$, it follows that

$$\mathcal{S} = \{t \in \mathbf{R}^+ : s(t) = 0 \mid s(t) = C_1^{-1}\bar{S}_1 y - \hat{x}_1\}. \quad (5.55)$$

The error system with respect to the new coordinates can be written as in (5.48) and (5.45). If the optimization problem in Theorem 5.1 is solvable, then it implies that the error dynamics is ultimately bounded with arbitrary small upper bound subject to (5.54). For simplicity, define

$$\epsilon := 2\omega_0 \frac{\left\| P_3 \begin{bmatrix} 0 \\ D_2^{-1}\mathcal{K}_2^{-1} \end{bmatrix} \right\|}{\lambda_{min}(Q)} + \varepsilon. \quad (5.56)$$

Consider the Lyapunov function $V_s = \frac{1}{2}s^T P_1 s$. We obtain

$$\begin{aligned} \dot{V}_s &= s^T P_1 \dot{s} \\ &= e_1^T P_1 (A_1^s e_1 + \mathcal{A}_2 e_3 + B_{\Phi 1} e_\Phi + B_1(f(t) - \nu)) \\ &\leq \|e_1\|(\|A_1^s\|\|e_1\| + \|\mathcal{A}_2 e_3\| + B_{\Phi 1} e_\Phi) - \rho_0 \|B_1^T e_1\| \\ &\leq \|B_1^{-T}\|\|B_1^T e_1\|((\|A_1^s\| + \|\mathcal{A}_2\| + B_{\Phi 1})\epsilon - \rho_0). \end{aligned} \quad (5.57)$$

Choose the gain ρ_0 to satisfy

$$\rho_0 \geq (\|A_1^s\| + \|\mathcal{A}_2\| + \|B_{\Phi 1}\|)\epsilon + \eta_0, \quad \eta_0 > 0 \quad (5.58)$$

Therefore it follows that $\dot{V}_s < 0$ and the well-known reachability condition [18] is satisfied. As a consequence, an ideal sliding motion will take place on the surface $\mathcal{S}$ and after some finite time t_s

$$e_1 = \dot{e}_1 = 0, \ \forall t > t_s. \tag{5.59}$$

The subsystem (5.48)in sliding mode is given by

$$0 = \mathcal{A}_2 e_3 + B_1(f - \nu_{eq}) + B_{\Phi 1}\Phi(e_1, e_3) \tag{5.60}$$

where in the sliding mode the discontinuous signal ν in (5.37) must takes on the average ν_{eq} (referred to as the equivalent output error injection [71]) to preserve the sliding motion. Thus

$$\|\nu_{eq} - f(t)\| \leq \kappa \tag{5.61}$$

where

$$\kappa = \|B_1^{-1}\mathcal{A}_2 + B_1^{-1}B_{\Phi 1}\|\epsilon \tag{5.62}$$

Therefore, approximately, for some small κ

$$\nu_{eq} \approx f(t). \tag{5.63}$$

And based on the concept of equivalent output error injection [71], the signal ν_{eq} can be approximated to any degree of accuracy by

$$\nu_{eq} \approx (\rho + \rho_0)\frac{B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\| + \delta} \tag{5.64}$$

where δ is a small positive scalar to smooth out the signal ν [18]. Therefore

$$\hat{f}(t) = (\rho + \rho_0)\frac{B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)}{\|B_1^T P_1(C_1^{-1}\bar{S}_1 y - \hat{x}_1)\| + \delta}. \tag{5.65}$$

Next, From (5.54) and (5.56), it follows that $\|x_3 - \hat{x}_3\| \leq \epsilon$. Thus

$$\|\omega - \hat{\omega}\| \leq \epsilon \tag{5.66}$$

and for some small ϵ, we can directly conclude that

$$\hat{\omega}(t) \approx \omega(t). \tag{5.67}$$

It should be pointed out that upper bound of the estimation error ϵ is arbitrarily reduced by the choice of a high-gain $\mathcal{K}_2$.

5.6 Sensor and Actuator Fault Estimation and Diagnosis

Now, we present the important feature of the proposed observer to deal with faulty systems when the faults are prone in both sensor(s) and actuator(s) during the course of the system's operation. Consider system (5.1) with $\omega(t) = Wf_s(t)$ and $\Delta = 0$ where W is a known constant matrix of appropriate dimension and full column rank. Thus, we have

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + f(t)) + B_\Phi \Phi(x,t) \\ y(t) = Cx(t) + DWf_s(t) \end{cases} \tag{5.68}$$

where $f(t)$ and $f_s(t)$ represent actuator and sensor faults respectively. Using the proposed GSMO and with regard to Section 5.4, one can estimate the faults as follows:

Actuator Fault Estimation:

$$\hat{f}(t) = (\rho + \rho_0) \frac{B_1^T P_1 (C_1^{-1} \bar{S}_1 y - \hat{x}_1)}{\| B_1^T P_1 (C_1^{-1} \bar{S}_1 y - \hat{x}_1) \| + \delta} \tag{5.69}$$

Sensor Fault Estimation:

$$\hat{f}_s(t) = (W^T W)^{-1} W^T [0 \quad I_{p-m}] \hat{x}_3(t). \tag{5.70}$$

Remark. In the case $\Delta \neq 0$, the disturbance $\omega(t)$ which represents a class of unmatched disturbances is estimated by $\hat{\omega}(t) = [0 \quad I_{p-m}]\hat{x}_3(t)$. This feature is also unique about the proposed GSMO since the SMOs recently addressed in articles are not able to estimate any unmatched disturbance (See [74], [64] and references there in).

5.7 Design Example

The nonlinear model of a single link flexible joint robot arm (Shown in Figure 5.1) is described by following set of equations [56].

$$\begin{aligned}
&\dot{\theta}_m = \omega_m \\
&\dot{\omega}_m = \tfrac{k}{J_m}(\theta_1 - \theta_m) - \tfrac{B_R}{J_m}\omega_m + \tfrac{K_\tau}{J_m}(u + f(t)) \\
&\dot{\theta}_l = \omega_l \\
&\dot{\omega}_l = -\tfrac{k}{J_l}(\theta_1 - \theta_m) - \tfrac{mgh}{J_l} sin(\theta_l)
\end{aligned}$$

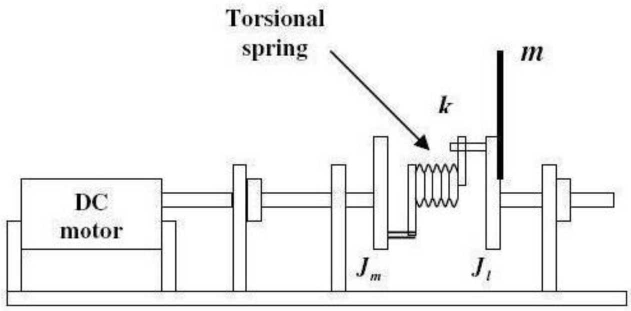

Figure 5.1: Single link flexible joint robot schematic

where, θ_m and ω_m are, respectively, the position and angular velocity of the DC motor and θ_l and ω_l represent those of the link. The DC motor is excited with $u(t)$ being the excitation signal. The variable $f(t)$ denotes an actuator fault signal. It is assumed that the motor position, motor velocity and the link position are measured. Although, the sensors for motor position and link position are prone to faults, f_{s1} and f_{s2}, in some time intervals. Thus, we can model the output, y, by

$$y = \begin{bmatrix} \theta_m + f_{s1} \\ \omega_m \\ \theta_l + f_{s2} \end{bmatrix}.$$

The moment of inertia of the DC motor is denoted by J_m while that of the controlled link is denoted by J_l . Parameter k symbolizes the torsional spring constant. The length of the link is given by h while B_R stands for the viscous friction in the motor bearing. The values of these parameters are given in table I. The aim of this study is to simulate faults in the excitation signal of the DC motor, $u(t)$, and measurement output, $y(t)$, and then reconstruct the faults . Furthermore states are also estimated for control/monitoring. For convenience, the flexible joint robot arm system can be described in the form of (5.1) with system states $x^T = [x_1 \ x_2 \ x_3 \ x_4] := [\theta_m, \omega_m, \theta_l, \omega_l]$, system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}$$

$$B_\Phi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ \Delta = 0$$

and the Lipschitz nonlinear term is $\Phi(x) = \sin(x_3)$. We apply input $u(t) = \sin(t)$ from $t = 0$. The actuator fault $f(t)$ and sensor faults $f_s(t) = \begin{bmatrix} f_{s1} \\ f_{s2} \end{bmatrix}$ respectively are

$$f(t) = \begin{cases} 0.1t - 1, & 10 \leq t \leq 20 \\ -0.1t + 3, & 20 < t \leq 30 \end{cases}$$

$$f_{s1} = \sin(0.5t) + .2\sin(5t), \quad 10 \leq t \leq 30$$

$$f_{s2} = \begin{cases} 0.2t - 4, & 20 \leq t \leq 25 \\ -0.2t + 6, & 25 < t \leq 30 \end{cases}$$

Consequently, with regard to the distribution matrix D, the first and the third sensors are prone to fault while the second sensor is assumed to be fault-free. Furthermore, it is assumed that $W = I_2$. Introduce transformations $\tilde{x} = Tx$ and $\tilde{y} = Sy$ with T and S computed by the method given in Lemma 5.2 as follows

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that $\bar{S}_1 = [0 \ \ 1 \ \ 0]$ and

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{c|ccc} -1.2500 & -48.6 & 48.6 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0.3536 & 0.3536 & -5.3571 \\ 0 & 19.5000 & -19.5000 & 0 \end{array}\right]$$

$$\left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 21.6 \\ \hline 0 \\ 0 \\ 0.0000 \end{array}\right], \quad \left[\begin{array}{c} B_{\Phi 1} \\ \hline B_{\Phi 2} \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.3300 \end{bmatrix}, \quad \Delta_2 = 0$$

$$\left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_4 \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad \left[\begin{array}{c} 0 \\ \hline D_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

and therefore

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{rank}(E) = 3 < n(=5)$$

$$\mathcal{A}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 19.5 & -19.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \mathcal{C}_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

To design the proposed GSMO, choose

$$\mathcal{K} = \left[\begin{array}{c} 0 \\ \hline \mathcal{K}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 40 & 0 \\ 0 & 2.8 \end{array}\right]$$

such that $(E + \mathcal{K}\mathcal{C}_4)$ is nonsingular. Let $\bar{A}_1 = 150$ and $\mathcal{L}_\Phi = 0.5$. From the LMI synthesis, after 10 iterations, we obtain

$$P_1 = 0.7984,$$

$$P_3 = \begin{bmatrix} 292.8041 & -19.5106 & -0.7537 & -36.6898 & -19.2042 \\ -19.5106 & 369.2266 & 0.1833 & 20.8496 & 405.1866 \\ -0.7537 & 0.1833 & 0.1879 & 0.4543 & 0.5946 \\ -36.6898 & 20.8496 & 0.4543 & 331.2106 & 20.0682 \\ -19.2042 & 405.1866 & 0.5946 & 20.0682 & 445.7535 \end{bmatrix}$$

$$K = \begin{bmatrix} 145.5688 & 11.9180 \\ 1.5400 & 170.8633 \\ 1.7497 & -39.3617 \\ 146.6352 & -5.5142 \\ 4.8638 & 14.7479 \end{bmatrix}.$$

Consequently, using (5.38), it follows that

$$\tilde{L} = \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array}\right] = \left[\begin{array}{c|cc} 150 & 0.0000 & 0.0000 \\ \hline 1 & 0.6371 & 4.8105 \\ 0 & 4.7302 & 487.4907 \\ 0 & 19.8012 & 748.8055 \\ 0 & 43.4589 & 19.8193 \\ 0 & 1.1822 & 122.6429 \end{array}\right].$$

Notice that this system satisfies necessary conditions (3.4) and (5.5). In the corresponding simulation, the constants in the expression ν (5.37) have been selected to be $\rho = 1$, $\delta = .05$ and $\rho_0 = 5$. Thus the observer design is complete. For the simulation study, we assume that system is initially (time $t = 0$ sec) at rest and fault free. At this time the GSMO is switched on with different initial condition than that of the system. The simulation was carried out for 50 seconds. It can be verified that the eigenvalues of $\Lambda\mathcal{A}4_0$ are $\{-40.0304, -0.0417, -1.0928, -2.0524 \pm 4.8834i\}$, hence it is stable. Also, it can be verified that the above LMI solution satisfy the structural matching condition (3.5), resulting that the numerical solution is valid to maintain a stable sliding motion on the sliding surface $\mathcal{S}$. Figs. 5.2 shows estimation error of the actual states by GSMO. Fig. 5.3 is concerned with the actuator fault reconstruction. It shows that despite the presence of disturbances at the output (sensor faults), the proposed generalized sliding mode observer can still reconstruct the fault effectively via the average of the sliding mode controller $\nu(t)$. Fig. 5.4 and Fig. 5.5 depict the estimation of the senors faults using the GSMO. As it was discussed before, by choosing a high gain $\mathcal{K}_2$, the upper bound of the sensor fault reconstruction error becomes relatively very small compared to the magnitude of the faults. Hence, sensor fault estimation is effectively accomplished. Notice that the actuator fault $f(t)$ and sensor fault f_{s1} have been elaborately chosen to take place in the same time interval, $t \in [10, 30]$ sec, to show that the proposed GSMO can still effectively estimate both faults as it was expected from Theorem 5.1.

5.8 Summary

This chapter presents a generalized sliding mode observer (observer) based fault estimation approach for nonlinear Lipschitz systems when both sensor and actuator faults exist coincidentally during the course of the system's operation. The ap-

SYSTEM PARAMETERS (Units)	**VALUE**
Motor inertia, J_m (Kg m^2)	3.7 $\times 10^{-3}$
Link inertia, J_l (Kg m^2)	9.3 $\times 10^{-3}$
Pointer mass, m (Kg)	2.1 $\times 10^{-1}$
Link length, h (m)	3.0 $\times 10^{-1}$
Torsional spring constant, k (Nm rad^{-1})	1.8 $\times 10^{-1}$
Viscous friction coefficient, B_R (Nm V^{-1})	4.6 $\times 10^{-2}$
Amplifier gain, K_τ (Nm V^{-1})	8 $\times 10^{-2}$

Table 5.1: Model parameters for the single link flexible joint robot arm

proach utilizes a sliding mode observer in a new generalized state space form. The conditions for the stability of convergence are derived. Interestingly, distinguishable from its sliding mode observer predecessors, it was studied that the proposed GSMO can estimate a class of disturbances which are not *matched*. The technique is successfully implemented on a faulty single link flexible joint robot system subject to simultaneous actuator and sensor faults. It is shown that the presented approach can be easily used to design the GSMO for the detection and reconstruction of the actuator and sensor faults and estimation of system states simultaneously.

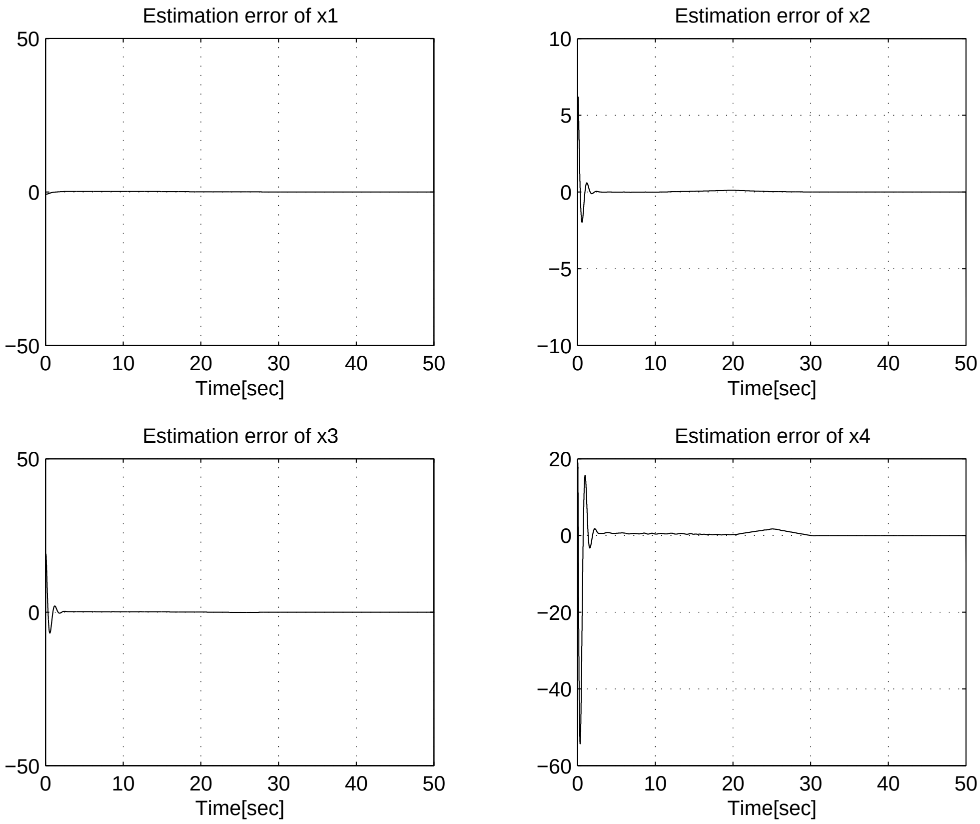

Figure 5.2: State estimation errors

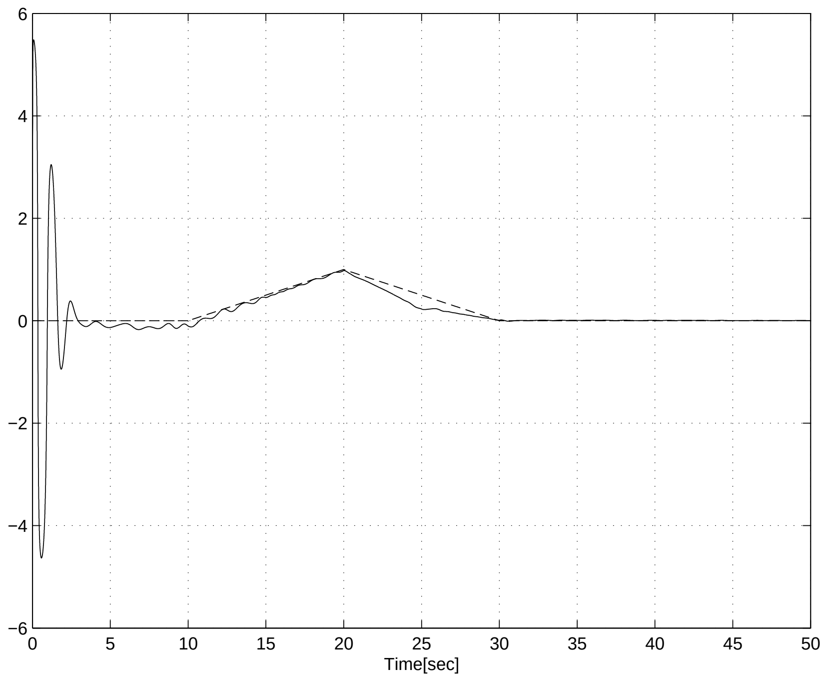

Figure 5.3: Actuator fault $f(t)$ (dash line) and its reconstruction by equivalent sliding mode controller ν_{eq} (solid line)

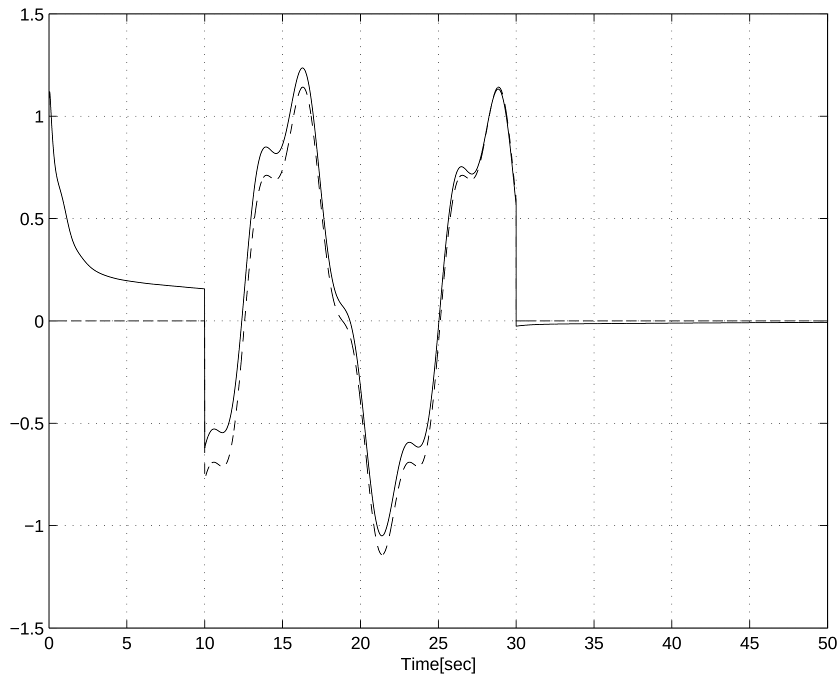

Figure 5.4: Sensor fault f_{s1} (dash line) and its estimation by GSMO (solid line)

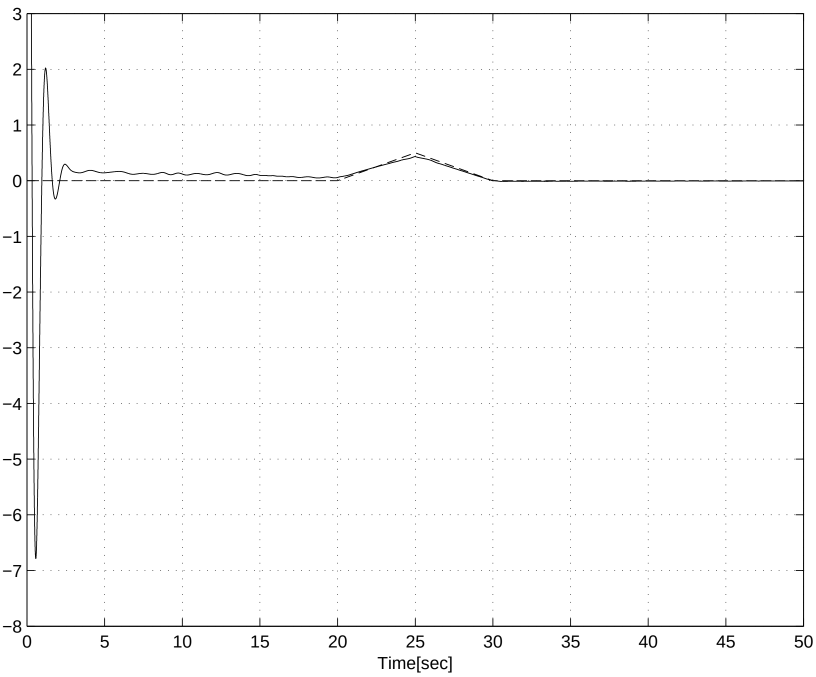

Figure 5.5: Sensor fault f_{s2} (dash line) and its estimation by GSMO (solid line)

Chapter 6

Reduced-Order Unknown Input Observers[1]

6.1 Background Results

For the past two decades there has been significant interest in the use of chaotic dynamics to realize secure communications (See [35],[36], [40],[34], [57] and references therein). There are several features of chaotic signals, the trajectories of chaotic dynamical systems, which make them attractive for use in secure communication systems. Chaotic trajectories are deterministic, noise-like signals with broad bandwidth and aperiodic behavior. Another attractive feature of chaotic signals is their high dependence on initial conditions; small changes can lead to dramatically different behaviour over a short time interval. Therefore, long-term prediction becomes virtually impossible.

Early work on the synchronization of chaotic systems by Pecora and Carroll [52], enforced trajectories of the so-called “slave” chaotic system, to track those of the “master” system. Most of the work in this area is focused on synchronization of chaotic systems to recover information signals [67]-[31]. In a typical chaotic synchronization communication scheme the information to be transmitted is carried

[1]The results in this chapter have been submitted for publication in the article: R Raoufi and H. J. Marquez, ”A New Chaotic Communication Scheme Using Reduced-order Unknown Input Observers”, submitted to *International Journal of Bifurcation and Chaos*, World Scientific Publisher, August 2009, and also in the article: R. Raoufi and H. J. Marquez , ”A New Reduced-order Unknown Input Observer-based Chaotic Communication”, *Proceeding of American Control Conference*, IEEE, 2010.

from the transmitter to the receiver by a chaotic signal through an analog channel and decoding of the information can be carried out by means of synchronizing the chaotic trajectories of both systems [31]-[34]. Recently, different observer-based methods have been proposed to synchronize chaotic systems in the presence of a message signal. Reference [7] proposed a generalized state-space observer design method for chaotic communication which guarantees the synchronization of chaotic dynamic. In [62] an adaptive chaotic communication scheme was used to cope with the effect of channel noise. In reference [41] an adaptive observer was designed for chaotic masking secure communication schemes. A sliding mode observer based robust chaotic communication scheme was proposed in [57]. A new chaotic synchronization method based on gain scheduling was studied in [38]-[39].

The chaotic masking methods recently given in [40] and [41] for the Lorenz attractor take the following form:

$$\begin{cases} \dot{x} = \begin{bmatrix} -p_1 & p_1 & 0 \\ p_2 & -1 & 0 \\ 0 & 0 & -p_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ -yx_3 \\ yx_2 \end{bmatrix} + \mathbf{Ks(t)} \\ y = Cx + s(t) \end{cases}$$

where the first equation consists of the Lorenz attractor with the addition of the information signal $s(t)$ through the gain K. This approach consists of two steps. First, the observer stabilizing gain is computed for a message-free chaotic synchronization. Next, the information signal is added to the output and also injected into the chaotic dynamics through the observer gain (Fig. 6.1). Despite of the attractiveness of this approach and its well established stability criteria, an inherent source of limitation and deficiency in this approach is the following: chaos is a steady state phenomena and, in general, there are no guarantees that the chaotic attractor used in the implementation will remain chaotic after injecting the term $\mathbf{Ks(t)}$ which may corrupt the chaotic attractor.

On the other hand, a powerful tool to tackle the problem of chaotic synchronization in the presence of a message signal is the use of Unknown input observers (UIOs) [10], [69]. The UIO synchronizes itself whit the master chaos while it treats the message signal as the unknown input (See [10] and [69] and reference there in). Necessary and sufficient conditions for the stability can be found in [10] and [16]. The UIO based approach in [10] can cope with either differential message signal or it needs the derivative of the estimation for message recovery. However, this problem has been completely solved in our proposed method by virtue of a novel UIO design and a different recovery algorithm.

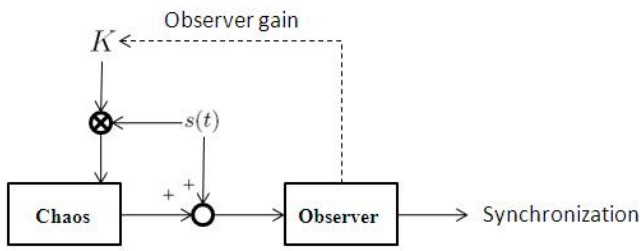

Figure 6.1: The existing chaotic masking scheme

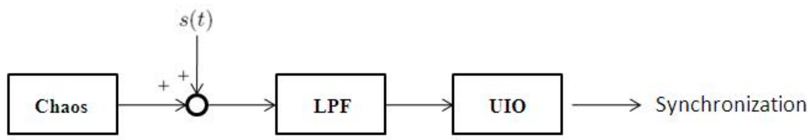

Figure 6.2: The proposed chaotic masking scheme

Motivated by the problem described earlier in the approach given in [40] and [41] and also inspired by UIOs for chaotic synchronization [10], [69], the primary goal of this work is to develop a new chaotic communication scheme which preserves the structure of the chaotic attractor without any additional inputs. We proceed towards this goal inspired by the concept of unknown input observers (UIOs). In a master-slave configuration, we introduce a low pass filter to create an augmented system in which the information signal can be regarded as an “unknown” input. At the receiver end our task is to design a new UIO (slave) to recover the “unknown” message. Fig. 6.2 shows the proposed scheme. It is observed that the chaotic transmitter is *free of any injection of the term $Ks(t)$ into the chaos* and the novel receiver is composed of two main blocks: a low pass filter and a UIO. The new proposed UIO is of reduced-order compared to the augmented system and exists under the same existence conditions as the other UIOs (see, for example, [10]). The message signal is recovered using only the chaotic masking output and the state estimate by the UIO. It should be pointed out that the reduced-order UIO is designed in a new coordinates and hence its structure is novel. Furthermore, we investigate the ability of the proposed UIO for sensor fault estimation and diagnosis. Using static $\mathcal{H}_\infty$ filtering, we develop a robust $\mathcal{H}_\infty$-UIO and we study the ability of this kind of filter for sensor fault estimation in the presence of process disturbance. Interestingly, the configuration of the proposed sensor fault estimation scheme and chaotic communication scheme are similar, hence, the same UIO structure successes to deal with both of these problems.

The remainder of this chapter is organized as follows; Section 6.2 provides some preliminaries, assumptions on the class of nonlinear chaotic system to be considered. Section 6.3 introduces an important Lemma. In Section 6.4 we consider the design of the new reduced-order UIO. In Section 6.5 details the process of recovering the information signal. Section 6.6 considers an illustrative example. Section 6.7 studies the use of robust UIO for sensor fault estimation. Section 6.8 summarizes this chapter.

6.2 Problem Formulation

Consider the following Chaotic system acting as the transmitter:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}_\Phi \Phi(\bar{x}, t) \tag{6.1}$$

where $\bar{x} \in \mathbf{R}^{\bar{n}}$ represents the system state and $t \in \mathbf{R}^+$. The set of $(\bar{A}, \bar{B}_\Phi, \bar{C})$ is known matrices of appropriate dimension. $\bar{C}$ is a full row rank matrix. $\Phi(t)$:

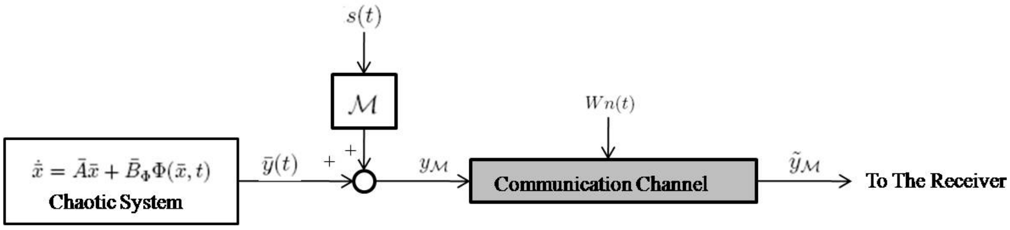

Figure 6.3: Chaotic Transmitter

$\mathbf{R}^{\bar{n}} \times \mathbf{R}^{+} \to \mathbf{R}^{r}$ represents the system nonlinearities which satisfies a Lipshitz condition (2.7) locally. We perform the following chaotic masking on the information signal $s(t) : \mathbf{R}^{+} \to \mathbf{R}^{q}$

$$y_{\mathcal{M}} = \bar{y}(t) + \mathcal{M}s(t) \tag{6.2}$$

where $\bar{y} = \bar{C}\bar{x} \in \mathbf{R}^{p}$ is the measured output, $y_{\mathcal{M}} \in \mathbf{R}^{p}$ and $\mathcal{M} \in \mathbf{R}^{p\times q}$ is a full column rank design matrix where $q \leq p$, therefore,

$$\text{rank}(\mathcal{M}) = q. \tag{6.3}$$

Assume now that for every complex number s with nonnegative real part

$$\text{rank} \begin{bmatrix} sI_n - \bar{A} & 0 \\ \bar{C} & \mathcal{M} \end{bmatrix} = n + \text{rank}(\mathcal{M}). \tag{6.4}$$

Fig. (6.3) depicts the chaotic masking transmitter and $Wn(t)$ represents the channel noise. Next, We employ a low pass filter to theoretically modify the formulation of the chaotic transmitter in a certain way which is explored now. We introduce the filter $\mathcal{F}$ with $x_f \in \mathbf{R}^{p}$ as

$$\mathcal{F} : \quad \dot{x}_f = -A_f x_f + B_f u_f \tag{6.5}$$

where $A_f \in \mathbf{R}^{p\times p}$ is a stable filter matrix and $B_f \in \mathbf{R}^{p\times p}$ invertible. We define the new augmented state as $x := \begin{bmatrix} \bar{x} \\ x_f \end{bmatrix} \in \mathbf{R}^{\bar{n}+p}$ and let $u_f = y_{\mathcal{M}}$. Therefore, the augmented system is described by

$$\begin{cases} \dot{x}(t) = \underbrace{\begin{bmatrix} \bar{A} & 0 \\ B_f\bar{C} & -A_f \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \bar{B}_{\phi} \\ 0 \end{bmatrix}}_{B_{\phi}} \Phi(x, t) + \underbrace{\begin{bmatrix} 0 \\ B_f\mathcal{M} \end{bmatrix}}_{D} s(t) \\ y(t) = \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{C} x \end{cases} \tag{6.6}$$

which it can be written as

$$
\begin{array}{l}
\dot{x} = Ax + B_{\Phi}\Phi(x,t) + Ds(t) \\
y = Cx
\end{array}
\tag{6.7}
$$

Thus, the new augmented system is an unknown input system. We now present the following Lemma which plays a key role in the structure of the observer-based receiver (synchronizer).

Remark. The primary reason for using the filter $\mathcal{F}$ is to restructure the original system as an unknown input augmented system for which the message $s(t)$ is the unknown input. This will be elaborated in details later on in Section V.

6.3 An Important Lemma

Lemma 6.1. *Given the system (6.6) with $rank(\mathcal{M}) = q$ and associated assumption (6.4), there exist nonsingular transformation matrices T and S such that*

$$
TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ TD = \begin{bmatrix} D_1 \\ 0 \end{bmatrix},\ TB_{\Phi} = \begin{bmatrix} B_{\Phi 1} \\ B_{\Phi 2} \end{bmatrix},\ SCT^{-1} = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}
\tag{6.8}
$$

where $n = \bar{n} + p$, $A_1 \in \mathbf{R}^{q\times q}$, $A_4 \in \mathbf{R}^{(n-q)\times(n-q)}$, $C_1 \in \mathbf{R}^{q\times q}$, $C_4 \in \mathbf{R}^{(p-q)\times(n-q)}$, D_1 and C_1 are invertible and the pair (A_4, C_4) is detectable.

Proof. we have $\operatorname{rank}(D) = q$, therefore without loss of generality, by using a nonsingular transformation T_0, we partition the matrix D as

$$
T_0 D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}
\tag{6.9}
$$

where $D_1 \in \mathbf{R}^{q\times q}$ with $\operatorname{rank}(D_1) = q$. Introducing a nonsingular coordinate transformation T_1 as

$$
T_1 = \begin{bmatrix} I_q & 0 \\ -D_2 D_1^{-1} & I_{n-q} \end{bmatrix}
\tag{6.10}
$$

we have that

$$
T_1 T_0 D = \begin{bmatrix} I_q & 0 \\ -D_2 D_1^{-1} & I_{n-q} \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ -D_2 D_1^{-1} D_1 + D_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}
\tag{6.11}
$$

where $D_1 \in \mathbf{R}^{q\times q}$ is nonsingular. Now we partition $CT_0^{-1}T_1^{-1}$ as $CT_0^{-1}T_1^{-1} = (\bar{C}_1 \ \ \bar{C}_4)$. Therefore

$$CD = CT_0^{-1}T_1^{-1}T_1T_0D = (\bar{C}_1 \ \ \bar{C}_4)\begin{bmatrix} D_1 \\ 0 \end{bmatrix} = \bar{C}_1D_1 \tag{6.12}$$

and consequently, using $CD = B_f\mathcal{M}$ we have

$$\text{rank}(\bar{C}_1D_1) = \text{rank}(CD) = \text{rank}(B_f\mathcal{M}) = \text{rank}(\mathcal{M})$$

From this point, similar to the proof of Lemma 5.2, we can find nonsingular transformations $T = T_2T_1T_0$ and $S = S_1S_0$ to transform matrices A D, C and B_Φ as in (6.8). On the other hand, using Assumption (6.4), for any $s \in \mathbf{C}^+$ we have

$$\begin{aligned}
\bar{n}+q &= \text{rank}\begin{bmatrix} sI_{\bar{n}-\bar{A}} & 0 \\ \bar{C} & \mathcal{M} \end{bmatrix} = \text{rank}(\begin{bmatrix} I_{\bar{n}} & 0 \\ 0 & B_f \end{bmatrix}\begin{bmatrix} sI-\bar{A} & 0 \\ \bar{C} & \mathcal{M} \end{bmatrix}) \\
&= \text{rank}\begin{bmatrix} sI_{\bar{n}-\bar{A}} & 0 \\ B_f\bar{C} & B_f\mathcal{M} \end{bmatrix} = \text{rank}\begin{bmatrix} sI_{\bar{n}}-\bar{A} & 0 & 0 \\ B_f\bar{C} & sI_p+A_f & B_f\mathcal{M} \\ 0 & I_p & 0 \end{bmatrix} - p \\
&= \text{rank}\begin{bmatrix} sI_n-A & D \\ C & 0 \end{bmatrix} - p
\end{aligned}.$$

Thus

$$\begin{aligned}
\bar{n}+q+p &= \text{rank}\begin{bmatrix} sI_n-A & D \\ C & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} sI_n-A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & I_q \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} sI_n-TAT^{-1} & TD \\ SCT^{-1} & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} sI_q-A_1 & -A_2 & D_1 \\ -A_3 & sI_{n-q}-A_4 & 0 \\ C_1 & 0 & 0 \\ 0 & C_4 & 0 \end{bmatrix} \\
&= \text{rank}\begin{bmatrix} sI_{n-q}-A_4 \\ C_4 \end{bmatrix} + 2q
\end{aligned}$$

consequently, it follows that

$$\text{rank}\begin{bmatrix} sI_{n-q}-A_4 \\ C_4 \end{bmatrix} = n-q \tag{6.13}$$

which implies that the pair (A_4, C_4) is detectable. This completes the proof.

6.4 Reduced-Order UIO design for Chaos Synchronization

Define the new coordinates as $\tilde{x} = Tx$ and $\tilde{y} = Sy$, then system (6.6) can be described as follows:

$$\begin{array}{l} \dot{\tilde{x}} = TAT^{-1}\tilde{x} + TBs(t) + TB_{\Phi}\Phi(T^{-1}\tilde{x}, t) \\ \tilde{y} = SCT^{-1}\tilde{x} \end{array} \tag{6.14}$$

and in the new coordinate, the sets of partitioned states are

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $x_1 \in \mathbf{R}^q$, $x_2 \in \mathbf{R}^{n-q}$, $y_1 \in \mathbf{R}^q$ and $y_2 \in \mathbf{R}^{p-q}$, and the corresponding transformed subsystems are

$$\begin{cases} \dot{x}_1 = A_1x_1 + A_2x_2 + D_1s(t) + B_{\Phi 1}\Phi(T^{-1}\tilde{x}, t) \\ y_1 = C_1x_1 \end{cases} \tag{6.15}$$

$$\begin{cases} \dot{x}_2 = A_3x_1 + A_4x_2 + B_{\Phi 2}\Phi(T^{-1}\tilde{x}, t) \\ y_2 = C_4x_2 \end{cases} \tag{6.16}$$

Also let us consider the partition of S as

$$S = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{bmatrix}, \bar{S}_1 \in \mathbf{R}^{q\times p}, \bar{S}_2 \in \mathbf{R}^{(p-q)\times p}.$$

From $y_1 = \bar{S}_1y$, the variable x_1 can be directly measured from the filtered output $y(t) = x_f(t)$ by

$$x_1(t) = C_1^{-1}\bar{S}_1y(t) \tag{6.17}$$

Hence x_1 is fully measurable and there is no need to estimate x_1. Notice that the information signal $s(t)$ only appears at the first subsystem. By Considering the information signal $s(t)$ as an unknown input of the augmented system (6.6), this interesting feature facilitates the design of a reduced-order unknown input observer. Thus, the problem reduces to design a state observer for only the subsystem (6.16). Consider the following reduced-order dynamical observer of order $n - q$

$$\dot{\hat{x}}_2 = A_3C_1^{-1}\bar{S}_1y + A_4\hat{x}_2 + B_{\Phi 2}\Phi(T^{-1}\hat{\tilde{x}}, t) + L(y_2 - C_4\hat{x}_2) \tag{6.18}$$

Let $e_2 = x_2 - \hat{x}_2$. Then the observer error dynamics of the system is

$$\dot{e}_2 = A_4^o e_2 + B_{\Phi 2}\phi \tag{6.19}$$

where $A_4^o = A_4 - LC_4$ and $\phi = \Phi(T^{-1}\tilde{x}, t) - \Phi(T^{-1}\hat{\tilde{x}}, t)$. Consider the Lyapunov function $V = {e_2}^T P_2 e_2$ where $P_2 = P_2^T > 0, P_2 \in \mathbf{R}^{n-q}$. The derivative of V is

$$\dot{V} \leq \begin{bmatrix} e_2 \\ \phi \end{bmatrix}^T \begin{bmatrix} P_2 A_4^o + A_4^{oT} P_2 + \mathcal{L}_\Phi^2 I & P_2 B_{\Phi 2} \\ {B_{\Phi 2}}^T P_2 & -I \end{bmatrix} \begin{bmatrix} e_2 \\ \phi \end{bmatrix}. \tag{6.20}$$

We now sum up the above analysis and design in the form of the following theorem.

Theorem 6.1: *Given the class of system (6.6) with associated assumptions, the asymptotic reduced-order UIO (6.18) exists if there is a solution for the following standard LMI feasibility problem:*

For given $\mathcal{L}_\Phi$, find symmetric $P_2 > 0$ and K such that

$$\begin{bmatrix} P_2 A_4 + {A_4}^T P_2 - KC_4 - C_4^T K^T + \mathcal{L}_\Phi^2 I & P_2 B_{\Phi 2} \\ {B_{\Phi 2}}^T P_2 & -I \end{bmatrix} < 0 \tag{6.21}$$

Once the problem is solved the UIO gain is $L = P_2^{-1} K$.

Remarks:

- From Lemma 6.1 we know that the pair (A_4, C_4) is detectable and therefore, there exists stabilizing gain L to assign the eigenvalues of A_4^o to the open left half complex plane.

- Assume that $p = q$ so that $C_4 \in \{\emptyset\}$. Then, from Lemma 6.1, it follows that the minimum phase condition (6.4) holds for all s such that $Re(s) \geq 0$ if and only if the matrix A_4 is asymptotically stable.

- One can estimate the original state using

$$\hat{x} = T^{-1} \begin{bmatrix} C_1^{-1} \bar{S}_1 y \\ \hat{x}_2 \end{bmatrix} \tag{6.22}$$

where only $\hat{x}_2$ is estimated by the reduced-order UIO. Furthermore, with regard to the structure of T_1 in (6.10), it can be verified that

$$T_1^{-1} = \begin{bmatrix} I_q & 0 \\ D_2 D_1^{-1} & I_{n-q} \end{bmatrix} \tag{6.23}$$

thus

$$
\begin{aligned}
\hat{x} &= T_0^{-1}T_1^{-1}T_2^{-1}\begin{bmatrix} C_1^{-1}\bar{S}_1 y \\ \hat{x}_2 \end{bmatrix} \\
&= T_0^{-1}\begin{bmatrix} I_q & 0 \\ D_2D_1^{-1} & I_{n-q} \end{bmatrix}\cdot\begin{bmatrix} I_q & -C_{11}^{-1}C_{12} \\ 0 & I_{n-q} \end{bmatrix}\begin{bmatrix} C_1^{-1}\bar{S}_1 y \\ \hat{x}_2 \end{bmatrix} \\
&= T_0^{-1}\begin{bmatrix} I_q & -C_1^{-1}C_{12} \\ D_2D_1^{-1} & -D_2D_1^{-1}C_1^{-1}C_{12} + I_{n-q} \end{bmatrix}\begin{bmatrix} C_1^{-1}\bar{S}_1 y \\ \hat{x}_2 \end{bmatrix} \\
\Rightarrow \hat{x} &= T_0^{-1}\begin{bmatrix} C_1^{-1}\bar{S}_1 y - C_1^{-1}C_{12}\hat{x}_2 \\ D_2D_1^{-1}C_1^{-1}\bar{S}_1 y + \mathcal{W}\hat{x}_2 \end{bmatrix}
\end{aligned}
\tag{6.24}
$$

where $\mathcal{W} = -D_2D_1^{-1}C_1^{-1}C_{12} + I_{n-q}$.

- From a practical standpoint, noise always exists in communication channels, therefore, we consider additive channel noise $n(t)$ of the following form:

  $$\tilde{y}_{\mathcal{M}} = \bar{y}(t) + \mathcal{M}s(t) + Wn(t) \tag{6.25}$$

  where W is the distribution of noise of appropriate dimension. It is important to note that a secondary benefit of introducing the low pass filter $\mathcal{F}$ is the attenuation of high frequency noise $n(t)$ of the communication channel, so that the UIO can access a filtered version of the output for state estimation at the receiver side. The bandwidth of the low pass filter (LPF) $\mathcal{F}$ should be designed according to the bandwidth of the chaotic masking modulation $y_{\mathcal{M}}(t)$. In other words, LPF filter allows the necessary frequencies that contain the information signal and chaotic output to pass through and cuts off the high frequency noise.

6.5 The Recovery of information signal $s(t)$

The asymptotic reduced-order UIO acts as the synchronizer at the receiver side. Let $\tilde{C} = [\bar{C}\ \ 0]$ and $\mathcal{M}^+ = (\mathcal{M}^T\mathcal{M})^{-1}\mathcal{M}^T$. From (6.2) we have $s(t) = \mathcal{M}^+(y_M(t) - \bar{C}\bar{x})$ As a consequence, we can recover the information signal using

$$\hat{s}(t) = \mathcal{M}^+(y_M(t) - \tilde{C}\hat{x}). \tag{6.26}$$

Therefore, by using output information $y(t)$ and state estimate $\hat{x}_2(t)$, we can directly reconstruct the message via

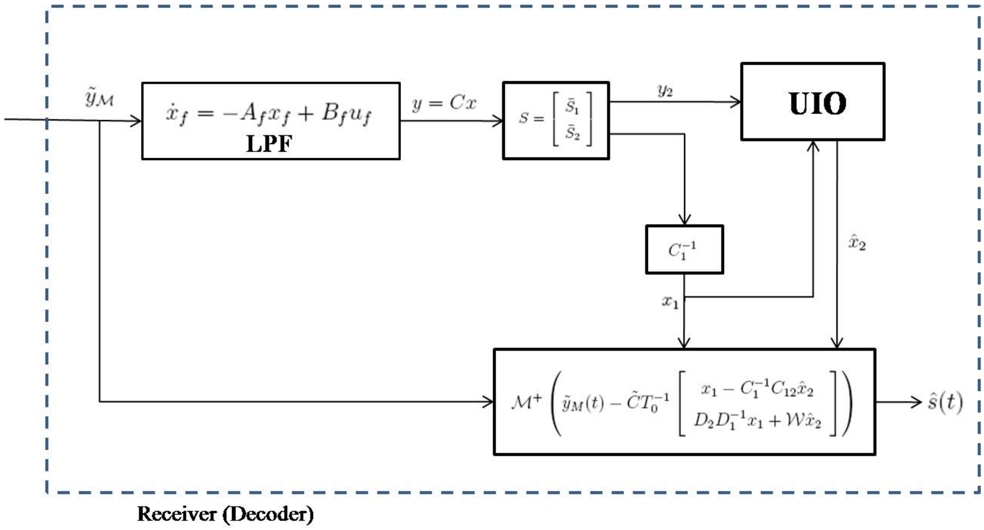

Figure 6.4: Receiver (Decoder)

$$\hat{s}(t) = \mathcal{M}^+ \left(y_M(t) - \tilde{C}T_0^{-1} \begin{bmatrix} C_1^{-1}\bar{S}_1 y - C_1^{-1}C_{12}\hat{x}_2 \\ D_2 D_1^{-1} C_1^{-1} \bar{S}_1 y + \mathcal{W}\hat{x}_2 \end{bmatrix} \right) \tag{6.27}$$

where $\mathcal{W} = -D_2 D_1^{-1} C_1^{-1} C_{12} + I_{n-q}$. And equivalently, when chancel noise is considered, we have:

$$\hat{s}(t) = \mathcal{M}^+ \left(\tilde{y}_M(t) - \tilde{C}T_0^{-1} \begin{bmatrix} x_1 - C_1^{-1}C_{12}\hat{x}_2 \\ D_2 D_1^{-1} x_1 + \mathcal{W}\hat{x}_2 \end{bmatrix} \right) \tag{6.28}$$

Figure 6.4 depicts the structure of the receiver (Decoder). As shown in the configuration of the receiver, the low pass filter $\mathcal{F}$ and the UIO in (6.18) play the main roles. Due to the augmentation of filter $\mathcal{F}$ in (6.5) with the chaotic transmitter shown in figure (6.3), the cascaded chaos-LPF system is seen as an unknown input system for UIO in (6.18). Notice that the signal $s(t)$ is the unknown. Then, the reduced order UIO estimates x_2 and using demodulation method given in (6.27), the message $s(t)$ is recovered.

Remarks: Recently in [10], a UIO filter was designed for a chaotic masking modulation scheme. In the sense that, in [10] there is no requirement to inject the message signal into the chaos, it is similar to our approach. However, in that method, the message signal either have to be differentiable (which is very restrictive) or if it

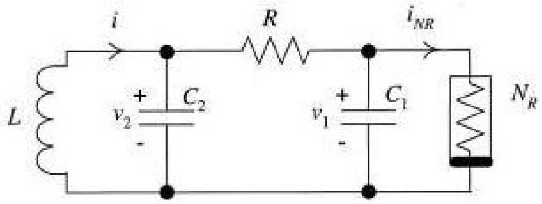

Figure 6.5: Schematics of Chua's circuit

is non-differentiable, the estimation of a signal and its derivative both are needed to recover the original message. Interestingly, the approach presented here works for non-differentiable message signals and there is no need to use any derivative. This important feature is due to the novel structure of the reduced-order UIO in the new coordinate.

6.6 Design Example

In this section, the numerical example is based on Chua's circuit, which exhibits a family of chaotic attractors and can be easily implemented in hardware [12]. As shown in Fig 6.5, Chua's circuit consists of a linear inductor L, a linear resistor R, two linear capacitors C_1 and C_2 and a nonlinear resistor (the Chua's diode) N_R. The state equations for Chua's circuit are given by [12]

$$\begin{cases} \frac{di}{dt} = -L^{-1}v_2 \\ \frac{dv_2}{dt} = C_2^{-1}(i - G(v_2 - v_1)) \\ \frac{dv_1}{dt} = C_1^{-1}(G(v_2 - v_1) - i_{NR}) \end{cases} \tag{6.29}$$

where v_1, v_2 and i are the voltage across C_1, the voltage across C_2 and the current through L, respectively and $G = R^{-1}$. The term i_{NR} , the piece-wise linear $v - i$ characteristic of the Chuas diode, is given by

$$i_{NR} = G_b v_1 + 0.5(G_a - G_b)(|v_1 + E| - |v_1 - E|) \tag{6.30}$$

where i_{NR} is characterized by a slope equal to G_a in the inner region and G_b in the Outer region and E is the breakpoint voltage of the Chuas diode. For the simulation results we have used the following change of variables ([12])

$$\bar{x}_1 = \frac{v_1}{E}, \bar{x}_2 = \frac{v_2}{E}, \bar{x}_3 = \frac{i}{GE}, \alpha = \frac{C_2}{C_1}, \beta = \frac{C_2}{LG^2}$$

$$a = \frac{G_a}{G}, b = \frac{G_b}{G}, t \to \frac{t}{RC_2}$$

which gives the following model in the form of system (6.1):

$$\bar{A} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \ \bar{B}_\Phi = \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where we assumed that the voltage across C_1 and the voltage across C_2 are measured and the nonlinear term $\Phi(x)$ corresponding to the Chua's diode is given by

$$\Phi(\bar{x}) = b\bar{x}_1 + 0.5(a-b)(|\bar{x}_1 + E| - |\bar{x}_1 - E|).$$

We choose $\mathcal{M} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which it can be shown that the triple $(\bar{A}, \mathcal{M}, \bar{C})$ satisfies rank condition (6.4). The filter matrices from (6.5) are chosen as $A_f = B_f = 20I_2$ and the Chua's parameters are $\alpha = 8.3, \beta = 15.5811, a = -1.9, b = -0.68, E = 1$. With regard to Lemma 6.1 and the computation method given in its proof, the nonsingular change of coordinates T and S are

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and consequently the augmented system in the new coordinates is described by

$$TAT^{-1} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{c|cccc} -20 & 0 & 0 & 20 & 0 \\ \hline 0 & -1.0000 & 1.0000 & 1.0000 & 0 \\ 0 & -15.5811 & 0 & 0 & 0 \\ 0 & 8.3000 & 0 & -8.3000 & 0 \\ 0 & 20 & 0 & 0 & -20 \end{array}\right]$$

$$TB_\Phi = \left[\begin{array}{c} B_{\Phi 1} \\ \hline B_{\Phi 2} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ -8.3 \\ 0 \end{array}\right], \ TD = \left[\begin{array}{c} D_1 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 20 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$SCT^{-1} = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_4 \end{array}\right] = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where C_1 and D_1 are invertible (as proved in Lemma 6.1). Furthermore, from S, we have $\bar{S}_1 = [1 \;\; 0]$ and we can directly measure x_1 from $y(t)$ by $x_1 = C_1^{-1}\bar{S}_1 y(t) = [1 \;\; 0]y(t)$. Thus, only x_2 needs to be estimated. It can be verified that the pair (A_4, C_4) is observable as rank condition (6.4) holds (See Lemma 6.1). Letting $\mathcal{L}_\Phi = 0.7$, the MATLAB LMI Toolbox solver, after 6 iterations, gives the following solution for LMI feasibility problem (6.21):

$$P_2 = \begin{bmatrix} 7.1141 & -0.7217 & -0.2513 & -1.0112 \\ -0.7217 & 2.5053 & 0.0563 & 1.6732 \\ -0.2513 & 0.0563 & 0.1814 & -0.2547 \\ -1.0112 & 1.6732 & -0.2547 & 3.2849 \end{bmatrix}, \quad K = \begin{bmatrix} 45.7942 \\ -34.6047 \\ 3.7808 \\ -36.8922 \end{bmatrix}.$$

Therefore

$$L = \begin{bmatrix} 6.2758 \\ -12.4815 \\ 32.8685 \\ -0.3929 \end{bmatrix}.$$

The simulation was carried out for 20 second with the channel noise $n(t)$, assumed to be noise with variance of 0.05, applied to the system from $t = 0$ and $W = [1 \;\; 0.5]^T$. The information signal is chosen to be$s(t) = 5\sin(0.5t)$. It can be verified that the eigenvalues of A_4^0 are $\{-15.9047, -6.3726 \pm 8.5297i, -0.2572\}$, hence it is stable and the observer (6.18) is well-defined. State estimates of the original system are obtained from Eq. (6.22). Fig. 6.6 shows the chaotic orbits of Chua's circuit. Actual states (dash line) and their estimates (solid line) are depicted in Fig. 6.7. Fig. 6.8 shows the encrypted message signal $y_{\mathcal{M}1}$ passes through the communication channel. Finally, Fig 6.9. is concerned with the recovery of information signal $s(t)$ using (6.26). The simulation validates the effectiveness of the proposed UIO based chaotic communication scheme.

6.7 $\mathcal{H}_\infty$-UIOs for Sensor Fault Reconstruction

In this section, we discuss the sensor fault estimation, detection and diagnosis problem by using the same designed UIO in this chapter. Compared to system (6.6), we replace the message signal $s(t)$ by sensor fault $f_s(t)$. However, in this case, due to

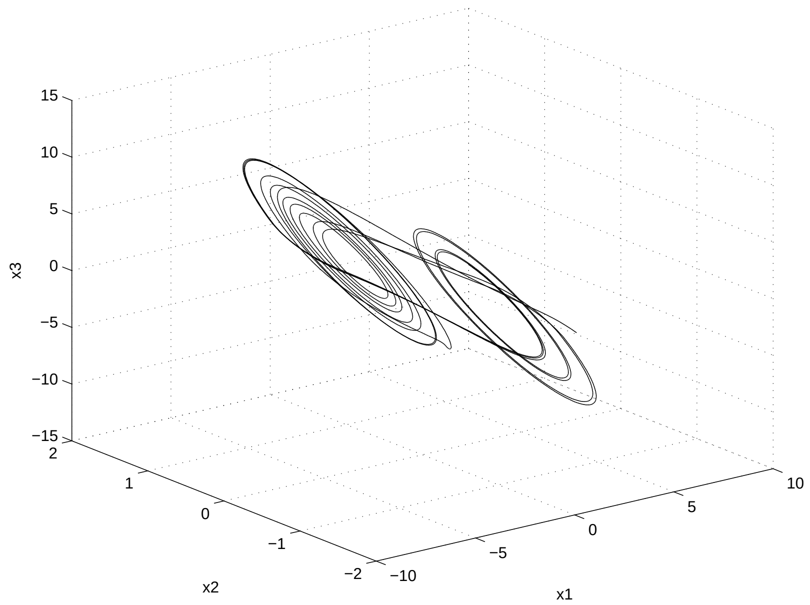

Figure 6.6: Chaotic orbits

disturbances $\xi(t)$ in the system we integrate static $\mathcal{H}_\infty$ filtering into the UIO design. Consider the plant with the sensor fault and disturbance as follows

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}_u u(t) + \bar{B}_\Phi \Phi(\bar{x}, t) + \bar{\Delta}\xi(t) \\ \bar{y} &= \bar{C}\bar{x} + \mathcal{M} f_s(t) \end{aligned} \tag{6.31}$$

where $\bar{x} \in \mathbf{R}^{\bar{n}}$ represents the system state, $u \in \mathbf{R}^{m_u}$ the system input, $\bar{y} \in \mathbf{R}^p$ the measured output. $f_s(t) : \mathbf{R}^+ \to \mathbf{R}^m$ denotes the sensor fault. The set of $(\bar{A}, \bar{B}_u, \bar{B}_\Phi, \bar{\Delta}, \bar{C}, \mathcal{M})$ is known matrices of appropriate dimension. $\bar{C}$ is a full row rank matrix and $\mathcal{M} \in \mathbf{R}^p \times \mathbf{R}^q$ is a full column rank matrix where $q \leq p$. The signal $\xi(t) : \mathbf{R}^+ \to \mathbf{R}^q \in \mathcal{L}_2[0, \infty)$ models the uncertainties and disturbances where $\bar{\Delta}$ is the corresponding distribution matrix. $\Phi(t) : \mathbf{R}^{\bar{n}} \times \mathbf{R}^+ \to \mathbf{R}^r$ represents the system nonlinearities which satisfies a Lipschitz-like condition (2.7) locally and also the condition (6.4) holds. We again introduce the filter $\mathcal{F}$ with $x_f \in \mathbf{R}^p$ as

$$\mathcal{F} : \quad \dot{x}_f = -A_f x_f + B_f u_f \tag{6.32}$$

where $A_f \in \mathbf{R}^{p\times p}$ is a stable filter matrix and $B_f \in \mathbf{R}^{p\times p}$ invertible. We define the new augmented state as $x := \begin{bmatrix} \bar{x} \\ x_f \end{bmatrix} \in \mathbf{R}^{\bar{n}+p}$ and let $u_f = \bar{y}$. Therefore, the augmented system of order $n = \bar{n} + p$ is described by

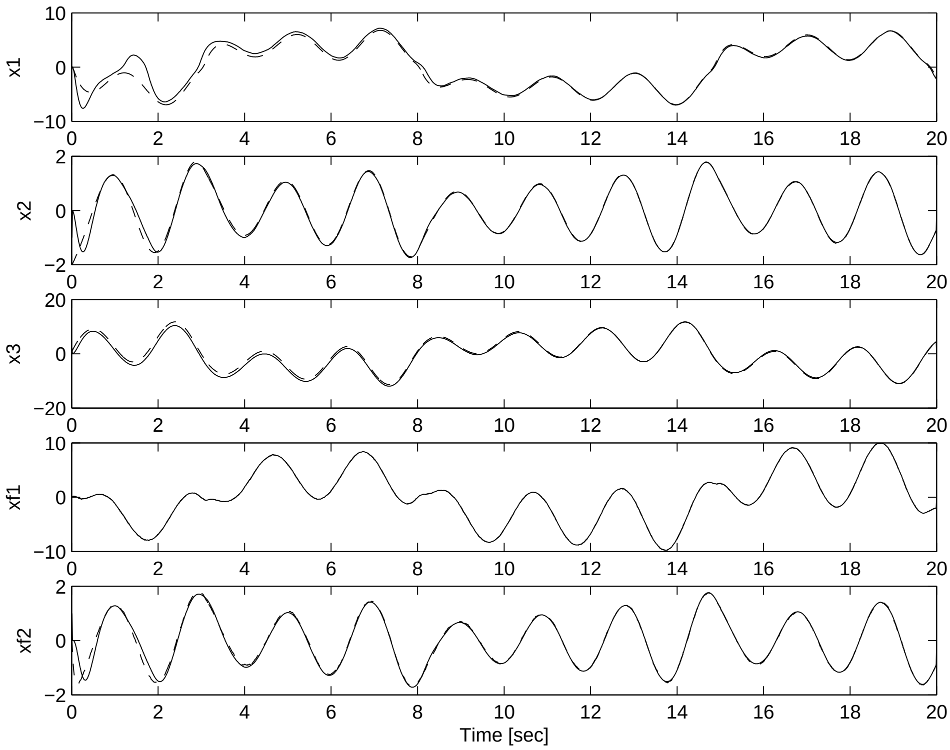

Figure 6.7: Actual state (dash line) and their estimate (solid line)

$$
\left\{
\begin{aligned}
\dot{x}(t) &= \underbrace{\begin{bmatrix} \bar{A} & 0 \\ B_f\bar{C} & -A_f \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \bar{B}_\phi \\ 0 \end{bmatrix}}_{B_\phi} \Phi(x,t) + \underbrace{\begin{bmatrix} 0 \\ B_f\mathcal{M} \end{bmatrix}}_{B} f_s(t) \\
&+ \underbrace{\begin{bmatrix} \bar{B}_u \\ 0 \end{bmatrix}}_{B_u} u(t) + \underbrace{\begin{bmatrix} \bar{\Delta} \\ 0 \end{bmatrix}}_{\Delta} \xi(t) \\
y(t) &= \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{C} x
\end{aligned}
\right. \tag{6.33}
$$

Thus we can rewrite

$$
\begin{aligned}
\dot{x} &= Ax + B_\Phi\Phi(x,t) + B_u u(t) + Bf_s(t) + \Delta\xi(t) \\
y &= Cx
\end{aligned} \tag{6.34}
$$

Notice that (6.33)represents a system with an unknown input $f_s(t)$ so a $\mathcal{H}_\infty$-UIO driven by the signal x_f can be designed. Using Lemma 6.1, we obtain the subsystems (6.15) and (6.16) we an additional disturbance terms $\Delta_1\xi$ and $\Delta_2\xi$ in the state space dynamics. Once again Consider the proposed reduced-order UIO (6.18).

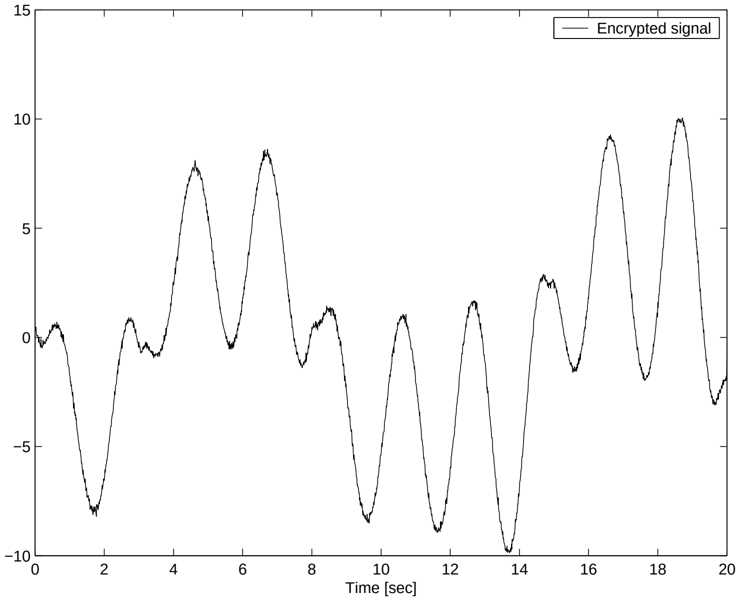

Figure 6.8: Encryption of information signal $s(t)$

Similar to the approach described for the recovery of the information signal, we can directly state the following Theorem.

Theorem 6.2: *Given the class of system (6.33) with associated assumptions, the robust reduced-order robust UIO (6.18) with the prescribed disturbance attenuation level $\sqrt{\gamma}$ subject to $\|\mathcal{H}_{\xi z}\|_\infty \leq \sqrt{\gamma}$ exists if there is a solution for the following LMI optimization problem:*

min γ

subject to $P_2 > 0$, K and

$$\begin{bmatrix} P_2A_4 + A_4{}^TP_2 - KC_4 - C_4^TK^T + \mathcal{L}_\Phi^2 I + H^TH & P_2B_{\Phi 2} & P_2\Delta_2 \\ B_{\Phi 2}{}^TP_2 & -I & 0 \\ \Delta_2{}^TP_2 & 0 & -\gamma I \end{bmatrix} < 0 \tag{6.35}$$

Once the optimization problem is solved the observer gain is $L = P_2^{-1}K$ and the original state and sensor fault estimates are respectively

$$\hat{x}(t) = T_0^{-1}\begin{bmatrix} C_1^{-1}\bar{S}_1y - C_1^{-1}C_{12}\hat{x}_2 \\ B_2B_1^{-1}C_1^{-1}\bar{S}_1y + \mathcal{M}\hat{x}_2 \end{bmatrix}, \ \mathcal{M} = -B_2B_1^{-1}C_1^{-1}C_{12} + I_{n-m}. \tag{6.36}$$

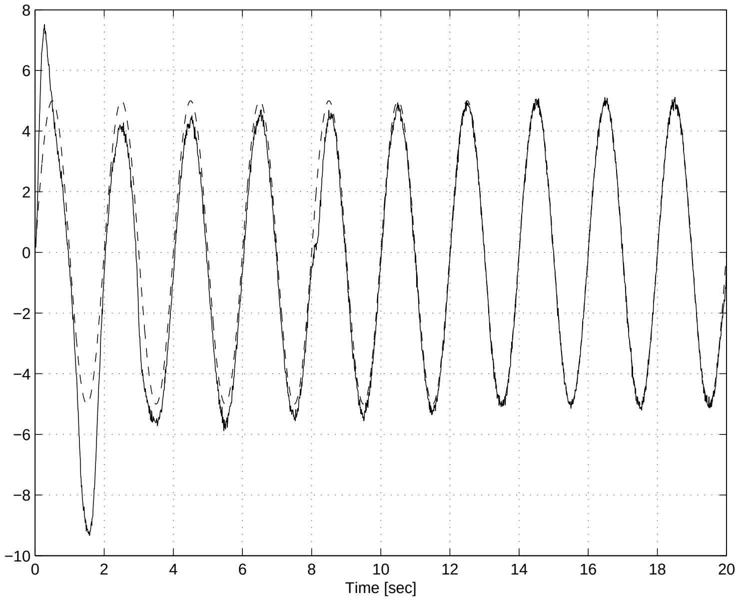

Figure 6.9: Actual $s(t)$ (dash line) and Recovered $s(t)$ (solid line)

$$\hat{f}_s(t) = \mathcal{M}^+(\bar{y}(t) - \tilde{C}\hat{x}) \tag{6.37}$$

where $\tilde{C} = [\bar{C} \;\; 0_{p\times p}]$ *and* $\mathcal{M}^+ = (\mathcal{M}^T\mathcal{M})^{-1}\mathcal{M}^T$.

It should be noted that the extension of using $\mathcal{H}_\infty$ prescribed disturbance attenuation level for the UIO is similar to the static $\mathcal{H}_\infty$ filtering approach employed in Chapter 3. The method to reconstruct the sensor fault $f_S(t)$ is also similar to the recovery of the message signal $s(t)$. Hence, we skip the proof. For the purpose of further elaboration, Figure 6.10 is included to depict the sensor fault reconstruction scheme using the proposed $\mathcal{H}_\infty$-UIO.

6.8 Summary

In this chapter, we addressed the problem of chaos secure communication by a new master-slave scheme which does not require the injection of the message signal through the observer gain into the chaotic system. A new reduced-order UIO was proposed in new coordinates and necessary conditions for the exitance of the pro-

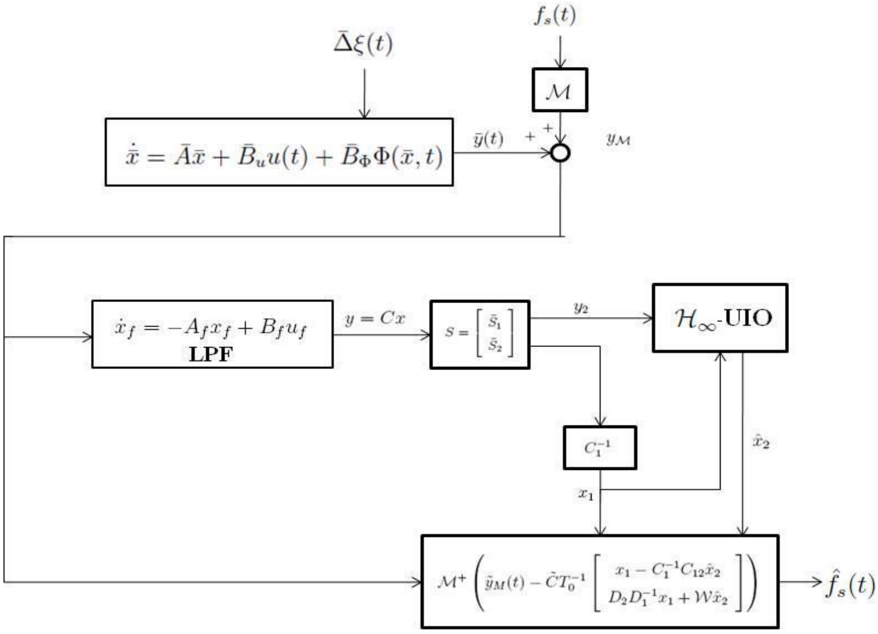

Figure 6.10: Sensor fault reconstruction scheme

posed chaos communication scheme was exploited. These necessary conditions are satisfied by a proper choice of the chaotic masking matrix. The UIO stabilizing gain was computed using an LMI feasibility problem in the new coordinate. It was theoretically shown how the augmentation of a filter with the chaos can eliminate the existing downside of chaotic masking schemes by removing the injection of $Ks(t)$ into the chaos. Hence, one can be sure that the chaos attractor will remain chaotic regardless of the frequency and amplitude of the message signal and also the amplification of the observer gain. As it was shown the combination of the LPF and the new proposed UIO brings this important advantage to our design. A numerical simulation based on Chua's circuit was carried out to verify the effectiveness of the proposed method. Using static $\mathcal{H}_\infty$ filtering, we extend the use of the proposed reduced-order UIO for sensor fault estimation.

Chapter 7

Conclusions

This thesis explored the nonlinear observer-based approach for robust state estimation and also fault estimation and diagnosis problem in a class of nonlinear control systems, known as Lipschitz systems. The observer-based approach adopted in this thesis falls in the category of sliding mode controllers. Mathematical modelling, state estimation, $\mathcal{H}_\infty$ filtering and singular system theory are employed for development of new Robust Fault Reconstruction (RFR) schemes. The major advantages of this approach is that it can be implemented online in software on any process control only using the available information.

7.1 Summary of Contributions

The major contributions of this thesis are as follows:

1. A new robust SMO with $\mathcal{H}_\infty$ performance methodology for uncertain Lipschitz nonlinear systems with unknown inputs was introduced. Our work generalized the known results of linear systems to Lipschitz nonlinear systems by using Lyapunov stability theory and LMIs. We studied the necessary conditions to achieve insensitivity of the proposed sliding mode observer to the unknown input (fault). The objective was to derive a sufficient condition using LMI optimization for minimizing the $\mathcal{H}_\infty$ gain between the estimation error and disturbances, whilst at the same time the design method fulfilled that the solution satisfies the so-called structural matching condition. The sliding motion affected only a part of the system through a novel reduced-order switching gain. A numerical example of MCK chaos demonstrated the high performance of the results compared to a pure SMO.

2. An adaptive sliding mode observer based approach for nonlinear fault reconstruction using LMIs was introduced. Importantly, the approach did not require any change of coordinates and was quite straightforward due to introducing a new parametrization of the Lyapunov matrix. The advantage of this parametrization compared to a previously published result was also discussed. The upper bound of the fault signal was allowed to be unknown since the variable structure gain adaptively adjusts itself to maintain the ideal sliding motion on the defined sliding surface. Hence, the adaptiveness of the proposed filtered enhanced the robustness of the fault reconstruction scheme. The reconstruction signal approximated the fault to some degree of accuracy depending on the size of the disturbances. Finally, a simulation study showed the effectiveness of this approach.

3. It is often the case that both actuator fault and sensor fault may happen during the course of the system's operation. Hence, an estimator to estimate both of the faults simultaneously wold be of great interest in many control applications. A new combined sliding mode and descriptor filter for state and fault estimation in a class of nonlinear systems was presented. The proposed observer performance for simultaneous estimation of sensor and actuator faults was studied. The approach utilizes a sliding mode observer in a new generalized state space form. The conditions for the stability of convergence are derived. Interestingly, distinguishable from its sliding mode observer predecessors, it was studied that the proposed GSMO can estimate a class of disturbances which are not *matched*. The technique is successfully implemented on a faulty single link flexible joint robot system subject to simultaneous actuator and sensor faults.

4. A new reduced order unknown input observer (UIO) was introduced. Applications of the UIO for a new chaotic communication and a novel sensor fault reconstruction were discussed. First, we addressed the problem of chaos secure communication by a new master-slave scheme which does not require the injection of the message signal through the observer gain into the chaotic system. The UIO stabilizing gain was computed using an LMI feasibility problem in the new coordinate. It was theoretically shown how the augmentation of a filter with the chaos can eliminate the existing downside of chaotic masking schemes by removing the injection of $Ks(t)$ into the chaos. Hence, one can be sure that the chaos attractor will remain chaotic regardless of the frequency and amplitude of the message signal and also the amplification of the observer gain. Next, Using static $\mathcal{H}_\infty$ filtering, we extended the use of the proposed reduced-order UIO for sensor fault estimation of Lipschitz systems.

7.2 Future Work

The following are suggested areas that could be pursued in future research.

- **$\mathcal{H}_\infty$-SMO design for networked control systems with channel time delay and quantization error:**

  Networked systems have many advantageous over traditional systems, such as efficiency in maintenance, practicality , energy consumption, installation, cost and etc. From the estimation standpoint, since the channels between the output of the system and the input of the filters (estimators) are not perfect, the network system physically imposes some limitations such as channel time delay and quantization error. The development of $\mathcal{H}_\infty$-SMO for network control systems with limited communication capacity is of great interest when the networked system is prone to faults. The $\mathcal{H}_\infty$-SMO brings insensitivity to faults of the network and also robustness against disturbances. The network faults can be reconstructed despite of the channel time delay and quantization errors. However, little attention has been paid to the estimation problem of faulty networked control systems and we believe the estimators designed in the thesis have the potential to be extended for networked systems in future.

- **SMOs for fault-tolerant output tracking controllers:**

Fault-Tolerant Control (FTC) is designed to preserve the control and stability of a system in the event of a set of possible faults. Clearly, Fault Reconstruction and Estimation (FRE) scheme proposed in this thesis are very useful to deal with faulty systems. The corrupted measured signals and actuator signals can be corrected by FRE before being used by the controller. Hence, there is no need of control reconfiguration and a relatively simple and effective control method would still work by retaining its structure. Therefore, a potential future work would be FRE based FTC systems. The proposed $\mathcal{H}_\infty$-SMO in Chapter 3 provides an excellent filter to be used in FTC schemes due to its robustness to disturbances and insensitivity to faults. Furthermore, the adaptive scheme studied in Chapter 4 is useful to handle FTC systems when the fault is bounded with an unknown upper bound. The generalized SMO proposed in Chapter 5 can deal with FTC systems where both actuators and sensors are faulty and control reconfiguration is required for both of these components.

- **Sliding mode observer design for descriptor systems with application to fault estimation and diagnosis:**

  Singular systems are very important since many real systems can be mathematically modeled in singular state space form. It is not possible to model some systems as a normal system. Therefore, estimation for singular systems has long been an interesting problem. Consider a system which suffers from faults and is singular as the following:

$$\left\{ \begin{array}{l} E\dot{x}(t) = Ax(t) + B(u(t) + f(t)) + \Delta\xi(t) \\ y(t) = Cx(t) \end{array} \right. \tag{7.1}$$

  where $x \in \mathbf{R}^n$ represents the system state, $u \in \mathbf{R}^m$ the control input, $y \in \mathbf{R}^p$ the measured system output and $t \in \mathbf{R}^+$. (A, B, C, E, Δ) is the set of real constant known matrices of appropriate dimensions. Function $f(t) : \mathbf{R}^+ \to \mathbf{R}^q$ denotes the fault (unknown input) that is bounded in the Euclidean norm and E is singular

$$\text{rank}(E) = s < n$$

If we employ sliding mode control to design an observer for this singular system we face the following constraint Generalized Lyapunov Equation (GLE):

$$(A - LC)^T P + P(A - LC) = -Q$$

$$E^T P = P^T E \geq 0$$

$$B^T P = FC$$

where $Q = Q^T, Q > 0$. The matrix L is the static observer gain. The second matrix equation appears due to using the generalized Lyapunov stability theory with the following Lyapunov function

$$V = e^T P^T E e$$

where e is the estimation error. The third matrix equation is the matching condition for SMO design. An interesting and challenging future research topic could be solving this problem using LMIs.

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