Quantum Gravity:

From Black Holes to Matrix Models

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Abstract

In this thesis we aim to reveal characteristics of quantum gravity mainly from two different perspectives. In the first part we focus on quantum aspects of black holes, in particular, the firewall paradox and nonlocality of quantum gravity. We present an explicit toy qubit transport model for unitary black hole evolution such that the gravitational field is described by nonlocal qubits with the assumption that the radiation still interacts locally with these nonlocal qubits. The model does not have firewalls at the event horizon, yet captures qualitatively what is expected, and it avoids a counterargument raised for subsystem transport models. Furthermore, it fits the set of six physical constraints that Giddings has proposed for unitary models of black hole evaporation. From a different point of view towards quantum gravity, in the second part of the thesis, we next consider supereigenvalue models in the Neveu-Schwarz sector and their recursive structure. We present a formalism that recursively computes all correlation functions of supereigenvalue models by using the Eynard-Orantin topological recursion in conjunction with simple auxiliary Grassmann-valued polynomial equations. Finally, we propose a more general supersymmetric recursive formalism, what we shall call super Airy structures, and discuss a few examples that we expect to have interesting applications to enumerative geometry.

Preface

What is quantum gravity? In order to gain insight into this question, I have been investigating gravitational and quantum theories from various perspectives throughout my doctorate program. In particular, the list of publications that I have or shall have co-authored is the following:

- **Publications**

  [1] V. Bouchard and K. Osuga,
  "Supereigenvalue Models and Topological Recursion,"
  JHEP **1804**, 138 (2018) `arXiv:1802.03536`.

  [2] K. Osuga and D. N. Page,
  "Qubit Transport Model for Unitary Black Hole Evaporation without Firewalls,"
  Phys. Rev. D **97**, no. 6, 066023 (2018) `arXiv:1607.04642`.

  [3] K. Osuga and D. N. Page,
  "A New Way to Derive the Taub-NUT Metric with Positive Cosmological Constant,"
  J. Math. Phys. **58**, no. 8, 082501 (2017) `arXiv:1603.05714`.

- **Future Publications**

  [4] K. Osuga and D. N. Page,
  "Separability of Symmetric Spin States,"
  Work in Progress

  [5] V. Bouchard, P. Ciosmak, L. Hadasz, K. Osuga, B. Ruba, P. Sułkowski,
  "Super Airy Structures,"
  Work in Progress

In this thesis, I will present the above five contributions[1] in detail.

[1] In addition to these five, I have a conference proceeding [6], introduction to topological string theories, which is for a one-hour lecture I gave at the Superschool on Derived Categories and D-branes, 2016.

Dedicated to the memory of my dear grandmother, Chie Taneishi, 1933-2019

Acknowledgments

First of all, I would like to show my deep appreciation to my wonderful supervisors, Don N. Page and Vincent Bouchard, for their magnificent support and uncountable great advice. They have saved me from giving up physics and mathematics due to very expensive tuition and housing cost for graduate schools in Japan. I must not forget to thank them for giving me the opportunity to pursue both physics and mathematics throughout my doctorate program. As each of them has shown me how fascinating physics or mathematics is, I have still not been able to choose which one I would like to focus more, hence, I will continue researching both in the future as much as I can. I could not have accomplished this irregular Ph.D thesis without their invaluable support.

I would like to thank Paweł Ciosmak, Leszek Hadasz, Błaźej Ruba, and Piotr Sułkowski for stimulating and exciting collaborations for which we discuss in this thesis. I also owe special thanks to my former supervisor, Shogo Aoyama, for his beneficial advice regarding my study abroad. I am also grateful to Jens Boos, Nitin K. Chidambaram, Aniket Joshi, Robert Maher, Adrien Ooms, Yasaman Yazdi, and many other friends in physics and mathematics for interesting discussions. Thanks also to many friends from I-House, especially Janita, Jorge, Coleen, Sasha, Hassam, Mike, Samer, and Alex, who have made my life in I-House unforgettable.

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At last, a massive thanks to my parents, who have been always supportive and have respected my decision without doubting. It is truly impressive that they started studying English from scratch after I came to Canada for attending my convocation and more importantly for thanking Don and Vincent by their own words. The time is indeed approaching soon. Their attitudes have shown me that nothing is too late to start learning new things.

Contents

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1 Introduction

A goal of theoretical physics is to construct a consistent unified theory of quantum field theory and general relativity; the former describes physics in a small scale with high precision where gravity is sufficiently weak, while the latter has helped with discovering new phenomena in a large scale where gravity is dominating and quantum effects are negligible. We call such a unified theory *quantum gravity*. We expect that quantum gravity, if it can be constructed, would reduce down to quantum field theory and general relativity in their own regimes, yet it becomes crucially important when we would like to consider physics in a small scale with strong gravity such as the beginning of the Big Bang or evaporation of a black hole. However, quantum gravity is still a deep mystery, and we are still far from grasping even an overall picture of what it is.

In such a challenging stage, a wise approach is probably to calculate quantum effects in a gravitational system within the known physical formalism that we have known, and to observe consequences that would shed light on constructing quantum gravity. From this point of view, one of the greatest hints we have had is, in my opinion, the insightful discovery by Hawking [7] that black holes are not completely black and that quantum field theory makes it possible for black holes to emit radiation to spatial infinity. Shortly after, Hawking noticed a contradiction within a set of fundamental axioms in physics and pointed out that it would require a severe modification from our current understanding of quantum field theory [8]. This puzzle is known as the black hole information paradox, which we will study in detail in Chapter 4. We would like to emphasize here that this contradiction is something we all need to appreciate rather than being annoyed. This is because, putting it the other way around, the Hawking argument suggests that quantum gravity should be equipped with a new property that resolves this puzzle in a natural manner, and his argument is a great starting point to uncover such a property.

Towards this direction, many attempts have been made, and they have brought new creativities to physics. Surprisingly, a notable connection between quantum aspects of black holes and quantum information theory has been recently indicated, in particular, quantum entanglement has become a research trend and we shall review basic definitions and useful theorems of quantum information theory in Chapter 2. Despite the strong attention, however, it is actually not so easy in general to tell whether two systems are entangled or separable when the two systems are in a mixed state. Also, one should be careful about similar terminologies such as entangled states and mixed states, or pure states and separable states. Therefore, it becomes important for physicists to have a solid and clear understanding of quantum information theory.

To grasp a good sense of quantum entanglement, an enhancing fact is that we can interpret some special set of quantum states from a geometric point of view. More precisely, every spin-$n/2$ pure state can be constructed by symmetrizing n qubits, and there is a bijective map between such a state and a polyhedron with n vertices all attached on a unit sphere. See Figure 3.1 for an intuitive picture. Then, an interesting question to ask is: what are the shapes of a polyhedron with n vertices such that the corresponding state has each pair of qubits separable? Note that a qubit is, in general, entangled with the rest of the system, even if each pair of qubits is separable. This might sound counterintuitive, yet it is true, and we explore various examples to answer that question in Chapter 3. The results are planned to be summarized in [4] soon.

There are several more striking developments besides the connection between black hole physics and quantum information theory. Maldacena [9] proposed the well-known AdS/CFT correspondence in the framework of string theory. Even though this is still a conjecture, there are thousands of calculations supporting this correspondence, without any single counterexample. The Hawking argument, the AdS/CFT correspondence, and other inspiring discoveries such as the Ryu-Takayanagi formula [10, 11] have led us to a new idea in physics, the so-called holographic principle. Furthermore, Almheiri *et al* [12, 13] have recently shown another view of black hole paradoxes from a quantum entanglement point of view, the so-called firewall paradox.

Taking account for all progress mentioned above, we have arrived at a potential new characteristic of quantum gravity that is an essential departure from our current understanding of quantum field theory. It is *nonlocality of quantum gravity.* Roughly speaking, we expect quantum gravity to incorporate interactions or some sort of communicational mechanism beyond the classical sense of causality. Then, we immediately encounter a serious problem: how can it be consistent with today's experiments that have confirmed causality with high precision? We unfortunately do not have any rigorous universal answer to this question, but there have been several proposals, including my work with Page [2]. We will develop this idea in Chapter 5.

Quite a bit differently, another curious approach towards the understanding of quantum gravity is to explore quantum field theories in lower dimensions. Such a reduction makes theories simpler than those in four dimensional spacetimes with gravity, and we potentially notice some properties of quantum gravity that are universal to any higher spacetime dimension. This idea might sound too ambitious, yet nobody has ever proved it is impossible, hence it is worth giving a good try. From this perspective, quantum field theories in zero dimensions, so-called matrix models, are the simplest example, and it turns out that matrix models are simple enough to compute many objects of interest and yet have many nontrivial features. In particular, it is known that there is a beautiful relation between matrix models and quantum gravity in two dimensions. See [14–16] and references therein for this type of relationship.

One of the two most crucial aspects of matrix models is a recursive structure. That is, we can define correlation functions of matrix models as we normally do in ordinary quantum field theories in higher dimensions; however, all of them can be computed by a set of few initial data. The mathematical formalism of such a recursive computation is now called topological recursion [17–19]. A bonus of their discovery is that topological recursion has become known as a very powerful tool to compute many important quantities in enumerative geometry. Regardless of a possible uncertainty whether topological recursion can help with understanding quantum gravity, it is fascinating to study the elegant flow from quantum field theories in zero dimensions to enumerative geometry.

Another crucial property of matrix models is that their partition functions obey the so-called Virasoro constraint. Although the details of the Virasoro constraint will be discussed in Chapter 6, since it is known that Virasoro algebras can be generalized to super Virasoro algebras, it is mathematically interesting to see whether we can proceed to a similar story with supersymmetry. Such models themselves are known to exist, so-called supereigenvalue models, but what about their recursive structure? The author with Bouchard investigated the models and showed a formalism in [1] that recursively computes all correlation functions of supereigenvalue models. We present the work in Chapter 8.

More mathematically, topological recursion is now viewed as a special example of a further generalized recursive framework, so-called Airy structures [20, 21]. In short, every Airy structure comes with the associated partition function that can be interpreted as a generating function of some geometric invariants. In this formalism, Lie algebras play a crucial role for the existence and uniqueness of the partition function. Therefore, following the spirit of supersymmetric generalizations of matrix models mentioned above, it is exciting to see whether we can generalize Airy structures by upgrading Lie algebras to super Lie algebras, what we shall call super Airy structure. This is a joint work in progress with V. Bouchard, P. Ciosmak, L. Hadasz, B. Ruba, P. Sułkowski. We explore this idea in Chapter 10.

At last, Page and I also studied a biaxial Bianchi IX model with positive cosmological constant in [3], which is a classical cosmological model. We found a geometrically interesting and elegant way to derive the exact solution for biaxial models where the solution itself was well-known. In short, we consider a dual two-dimensional description of a biaxial Bianchi IX model, a so-called minisuperspace[1], and showed that the minisuperspace admits two nontrivial Killing tensors besides the metric, one of rank 2 and the other of rank 4. These Killing tensors play a crucial role in deriving the exact solution. However, since this work is in classical cosmology whereas the main focus of this thesis is quantum aspects of black holes as well as matrix models, we present [3] in Appendix A.

In summary, this thesis is organized as follows. We review quantum information theory in Chapter 2 and present an in-progress research project in Chapter 3 that is independent of the flow towards quantum gravity. Then, we discuss the black hole information paradox and the firewall paradox in Chapter 4, and several proposals are given in Chapter 5. We change our perspective after that, namely, matrix models and topological recursion are reviewed in Chapter 6 and Chapter 7. Their supersymmetric analogue is developed in Chapter 8, and finally we consider Airy structures and super Airy structure in Chapter 9 and Chapter 10. We show a new way to derive the exact solution for biaxial Bianchi IX models with positive

[1]It has nothing to do with supersymmetry.

cosmological constant in Appexdix A.

Part I

Black Holes

2 Quantum Information Theory

Quantum information theory has become known as a useful tool for black hole physics and other fields. In fact, Figure 2.1 shows how the number of hep-th papers on the arXiv with entanglement in the title has increased for the last 10 years. However, in the author's opinion, quantities in quantum information theory are often explained in a handwaving manner that they are not wrong but not so accurate either. A typical abuse is referring the von Neumann entropy as the entanglement entropy for a mixed state. The author has also noticed confusing explanations about the difference among pure, mixed, separable, and entangled states.

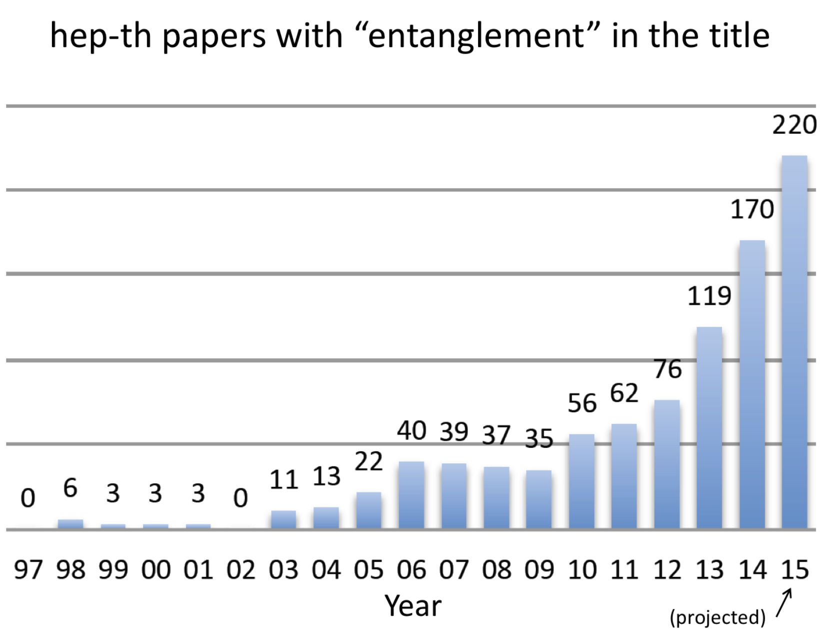


Figure 2.1: The number of hep-th papers on arXiv with entanglement in the title [22].

In this section, therefore, we carefully review definitions and properties of quantum information theory, in particular about quantum entanglement. Discussions below are greatly based on a series of lecture notes by John Preskill [23]. We discuss a few examples for a better understanding of quantum entanglement in the next section in the context of the geometry of the quantum states.

Before jumping into all the details, let us briefly discuss differences between classical and quantum information theory. A fundamental object in classical information theory is a binary system, i.e., a sequence of $n \in \mathbb{Z}_{>0}$ bits such as

$$\begin{aligned} n=1: &\quad \{0,1\}, \\ n=2: &\quad \{00,01,10,11\}, \\ n=3: &\quad \{000,001,010,100,110,101,011,111\}, \\ \vdots & \end{aligned} \tag{2.1}$$

where one can encode $N = 2^n$ messages for a general n. We first discuss the simplest case $n = 1$ for our purpose here. On the other hand, quantum information theory aims to study states consisting of n *qubits* where for $n = 1$, a general state is given by

$$|a, b\rangle = a\,|0\rangle + b\,|1\rangle\,, \tag{2.2}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. $|a|^2$ is the probability of being in the state $|0\rangle$ and $|b|^2$ is the probability of being in the state $|1\rangle$.

Sometimes, this superposition is referred as the difference between classical and quantum information theory. That is, states in classical information theory are only 0 or 1 whereas those in quantum information theory are superpositions of 0 and 1. However, this is not sufficient to characterize their differences[1]. This is because we can consider a classical bit that has the value 0 with a probability p_0 and the value 1 with a probability p_1 where $p_0, p_1 \in \mathbb{R}_{\geq 0}$, $p_0 + p_1 = 1$. What are differences between such a classical bit and a qubit?

The first difference is known as quantum interferences. Let us consider two normalized qubit states $|\phi\rangle\,, |\psi\rangle$ as

$$\begin{aligned} &|\phi\rangle = a\,|0\rangle + b\,|1\rangle\,, \quad |\psi\rangle = c\,|0\rangle + d\,|1\rangle\,, \\ &a, b, c, d \in \mathbb{C}, \quad |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1. \end{aligned} \tag{2.3}$$

[1]A conceptual difference is that when we were to measure a given qubit, we would *project* the qubit onto either $|0\rangle$ or $|1\rangle$, thus, a state after measurement would be indeed different from the given qubit. Measurements in classical information theory, on the other hand, can be done without disturbing the system.

Then, the probability of transition from $|\phi\rangle$ to $|\psi\rangle$ is defined as

$$p^Q(\phi \to \psi) = |\langle\phi|\psi\rangle|^2 = |a|^2|c|^2 + |b|^2|d|^2 + \bar{a}bc\bar{d} + a\bar{b}\bar{c}d, \tag{2.4}$$

where the overline denotes complex conjugate. From classical point of view, there are precisely two ways that this transition occurs. The first process is that $|\phi\rangle$ is mapped to $|0\rangle$ with probability $|a|^2$, and $|0\rangle$ is mapped into $|\psi\rangle$ with probability $|c|^2$ with the probability of this process being $|a|^2|c|^2$. Another process has $|1\rangle$ as the intermediate state, which occurs with the probability $|b|^2|d|^2$. Therefore, the transition probability is simply given by the sum of these two probabilities:

$$p^C(\phi \to \psi) = |\langle\phi|\psi\rangle|^2 = |a|^2|c|^2 + |b|^2|d|^2, \tag{2.5}$$

which differs from (2.4). One may recognize that this difference is analogous to the concept of a double slit experiments in quantum mechanics.

Another way of viewing quantum interference is that the complex numbers a, b, c, d in (2.3) in fact have more information than the probabilities, namely the relative phases of a, b and c, d. If we take out one overall phase for $|\phi\rangle$ and one for $|\psi\rangle$ and reparametrize a, b, c, d as

$$a = \cos\frac{\theta_1}{2}, \quad b = e^{i\varphi_1}\sin\frac{\theta_1}{2}, \quad c = \cos\frac{\theta_2}{2}, \quad d = e^{i\varphi_2}\sin\frac{\theta_2}{2}, \tag{2.6}$$

then $|\phi\rangle$ in (2.3) is in the state $|0\rangle, |1\rangle$ with the probability $\cos^2(\theta_1/2), \sin^2(\theta_1/2)$ respectively. Thus, φ_1 does not appear in these probabilities, and similarly neither does φ_2. Nevertheless, these relative phases still generate observable effects. For example, $p^Q(\phi \to \psi)$ as in (2.4) can be written as

$$p^Q(\phi \to \psi) = p^C(\phi \to \psi) + \frac{\cos(\varphi_1 - \varphi_2)}{2}\sin\theta_1 \sin\theta_2. \tag{2.7}$$

For fixed θ_1, θ_2, the information of quantum interference is encoded in the relative phases φ_1, φ_2.

There is another, deeper concept in which quantum information theory differs from classical one, which we will discuss in Section 2.2. Let us now turn to define concepts rigorously in quantum information theory.

2.1 Definitions

We go through fundamental definitions needed to understand quantum entanglement.

2.1.1 Pure States

We start with the definitions of a Hilbert space and a pure state.

Definition 2.1.1. A *Hilbert space* H is a $\mathbb{C}$-vector space equipped with an bilinear map $\langle\cdot|\cdot\rangle : H \otimes H \to \mathbb{C}$ satisfying:

1. Positivity: $\langle\phi|\phi\rangle > 0,\ \ |\phi\rangle \in H$ where $\langle\phi|\phi\rangle = 0 \Leftrightarrow |\phi\rangle = 0 \in H$.

2. Skew Symmetry: $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}, \quad |\phi\rangle, |\psi\rangle \in H$

Definition 2.1.2. Let H be a Hilbert space. A *pure state* is an equivalence class of vectors in H that differ by multiplication by a nonzero complex number.

We conventionally choose a representative $|\phi\rangle$ of each equivalence class that has the unit norm $\langle\phi|\phi\rangle = 1$. Note that normalization is not enough to uniquely fix a representative since $e^{i\theta}|\phi\rangle$ also has unit norm for any $\theta \in \mathbb{R}$. Hence, the overall phase factor is redundant. It is worth mentioning, however, that relative phases in a superposition of two or more pure states still make observable effects as mentioned in (2.7).

Definition 2.1.3. A *qubit* is a quantum system described by a two-dimensional Hilbert space.

We need two parameters to fix a pure state of a single qubit because it is normalized and also the overall phase can be chosen without loss of generality. A common way of parametrizing a pure state $|\phi\rangle$ is

$$|\phi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \tag{2.8}$$

where $|0\rangle, |1\rangle$ are a basis of a Hilbert space in two dimensions and $0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi$. One can interpret this state as being located on the sphere of radius one, known as the *Bloch sphere*. (See Figure 2.2.)

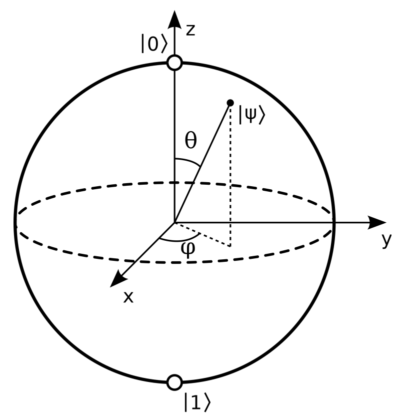


Figure 2.2: A visualization of a qubit on the Bloch sphere. There is a bijective map between a qubit in the form (2.8) and the position on the sphere. The image is retreived from Wikipedia .

Observables in quantum information theory are required to be Hermitian operators, which are self-adjoint operators[2]. This is because the values we actually measure are their eigenvalues, and we expect those to be real numbers in actual experiments. More generally, what we are able to observe in laboratories are expectation values.

Definition 2.1.4. Let H be a Hilbert space and $O : H \to H$ be a Hermitian operator. The *expectation value* of O measured in the state $|\phi\rangle \in H$ is

$$\langle O \rangle_\phi = \langle \phi | O | \phi \rangle \in \mathbb{R}. \tag{2.9}$$

In particular, if H is the eigenspace of O and $\{|i\rangle, a_i\}$ is a set of orthonormal eigenvectors and eigenvalues of O so that

$$O = \sum_{i=1}^{\dim H} a_i |i\rangle \langle i|, \tag{2.10}$$

[2]Mathematically, Hermitian operators are defined to be both self-adjoint and bounded. However in physics, eigenvalues of observables such the Hamiltonian may not be bounded in principle, and we simply define Hermitian operators as self-adjoint in this thesis, though an effective Hamiltonian after cut-off would be bounded.

then for

$$|\phi\rangle = \sum_{i=1}^{\dim H} c_i |i\rangle, \quad c_i \in \mathbb{C}, \quad \sum_{i=1}^{\dim H} |c_i|^2 = 1, \tag{2.11}$$

we have

$$\langle O \rangle_\phi = \sum_{i=1}^{\dim H} |c_i|^2 a_i. \tag{2.12}$$

$|c_i|^2$ denotes the probability that the state $|\phi\rangle$ is in the i-th eigenstate $|i\rangle$. Discussions so far are elementary knowledge in quantum mechanics, but let us now illustrate an equivalent formalism from density operator perspectives.

2.1.2 Density Operators and Mixed States

Definition 2.1.5. Let O be an Hermitian operator and $\{|i\rangle\}$ be an orthogonal basis of a Hilbert space H. Then, a *trace* $\mathrm{Tr} : O \mapsto \mathbb{R}$ of O is defined by

$$\mathrm{Tr}(O) = \sum_{i=1}^{\dim H} \langle i| O |i\rangle. \tag{2.13}$$

Definition 2.1.6. Let H be a Hilbert space and $\{|i\rangle\}$ be an orthogonal basis for it. A *density operator* $\rho : H \to H$ is a positive semi-definite Hermitian operator with $\mathrm{Tr}(\rho) = 1$. If one expands ρ in the $\{|i\rangle\}$ basis as

$$\rho = \sum_{i,j=1}^{\dim H} c_{ij} |i\rangle \langle j|, \quad c_{ij} = \bar{c}_{ji} \in \mathbb{C}, \quad \sum_{i=1}^{\dim H} c_{ii} = 1, \tag{2.14}$$

we call c_{ij} the $i - j$ component of the *density matrix*.

Given a pure state $|\phi\rangle \in H$, an operator $\rho_\phi = |\phi\rangle \langle\phi|$ indeed obeys the definition of a density operator. Notice that an expectation value of an Hermitian operator O (2.9) can be expressed in terms of the density operator ρ_ϕ as

$$\langle O \rangle_\phi = \mathrm{Tr}(O\rho_\phi) = \mathrm{Tr}(\rho_\phi O). \tag{2.15}$$

This shows that a system described by a pure state $|\phi\rangle$ and a system described by its associated density operator ρ_ϕ obey the same physics.

It turns out, however, that the density operator formalism can describe more general physical systems. To understand the reason, let us consider a superposition of pure states $|\phi_k\rangle$,

$$|\Phi\rangle = \sum_{k=1}^{K} |\phi_k\rangle , \tag{2.16}$$

where K can be greater than $\dim H$ because we are not requiring that the different $|\phi_k\rangle$ be orthogonal. This state is still a pure state. On the other hand, a sum of their density operators is not the density operator of $|\Phi\rangle$ in general

$$\rho = \sum_{k=1}^{K} c_k \rho_{\phi_k} = \sum_{k=1}^{K} c_k |\phi_k\rangle \langle\phi_k| \neq |\Phi\rangle \langle\Phi| , \tag{2.17}$$

where K can be greater than $\dim H$. Such a state is actually called a mixed state:

Definition 2.1.7. Let H be a Hilbert space and $|\phi_i\rangle$ be pure states. Given a density operator

$$\rho = \sum_{i} c_{ij} |\phi_i\rangle \langle\phi_j| , \tag{2.18}$$

the system described by ρ is called a pure state if there exists a pure state $|\Psi\rangle \in H$ such that $\rho = |\Psi\rangle \langle\Psi|$. It is called a *mixed state* otherwise. Furthermore, for a Hermitian operator $O : H \to H$, its expectation value is defined as

$$\langle O \rangle_\rho = \mathrm{Tr}(O\rho) = \mathrm{Tr}(\rho O). \tag{2.19}$$

If $c_{11} = 1$ and other $c_{ij} = 0$ in (2.18), ρ is obviously a pure state. Moreover, the density operator can still represent a pure state even if more than one of c_{ij} are nonzero. For example,

$$\rho = |a|^2 |0\rangle \langle 0| + a\bar{b} |0\rangle \langle 1| + \bar{a}b |1\rangle \langle 0| + |b|^2 |1\rangle \langle 1| = (a |0\rangle + b |1\rangle)(\bar{a} \langle 0| + \bar{b} \langle 1|), \tag{2.20}$$

hence, this density operator is a pure state.

One can always express a density matrix as a sum of pure density operators with corresponding probabilities. This is called a *pure state decomposition*. However, this decomposition is not unique in general. Indeed, let H be a Hilbert space and ρ be a density matrix of the system. Then, there exist two sets $\{|\phi_i\rangle, p_i\}$, $\{|\psi_j\rangle, p'_j\}$ of pure states and nonnegative real numbers such that

$$\rho = \sum_{i=1}^{I} p_i |\phi_i\rangle \langle\phi_i| = \sum_{j=1}^{J} p'_j |\psi_j\rangle \langle\psi_j|, \quad \sum_{i=1}^{I} p_i = 1, \quad \sum_{j=1}^{J} p'_j = 1 \tag{2.21}$$

where I, J can be greater than $\dim H$ and they are not necessarily the same because we are not requiring the $\{|\phi_i\rangle\}$ or the $\{|\psi_i\rangle\}$ to be orthogonal. The only way to have a unique decomposition is to expand ρ by its eigenvectors, and even then the decomposition is not unique unless all the eigenvalues of ρ are distinct.

Given a density matrix, how can we tell it is pure or mixed? There are two procedures to check it.

Definition 2.1.8. Let H be a Hilbert space and suppose ρ is the density operator of a system of interest. The *purity* γ of the system is defined by

$$\gamma = \mathrm{Tr}(\rho^2). \tag{2.22}$$

Definition 2.1.9. Let H be a Hilbert space and suppose ρ is the density operator of a system of interest. The *von Neumann entropy* $S(\rho)$ is defined by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho). \tag{2.23}$$

Then it is easy to show the following statement:

Proposition 2.1.10. *Let H be a Hilbert space and suppose ρ is the density operator of a system of interest. Then the purity and the von Neumann entropy of the system have the ranges*

$$\frac{1}{\dim H} \leq \gamma \leq 1, \quad 0 \leq S(\rho) \leq \log \dim H. \tag{2.24}$$

Furthermore, (1) the system is pure, (2) $\gamma = 1$, *and* (3) $S(\rho) = 0$ *are equivalent. Also* $\gamma = 1/\dim H \Leftrightarrow S(\rho) = \log \dim H \Leftrightarrow \rho = I/\dim H$ *holds, where I is the identity operator of dimensions* $\dim H$.

2.1.3 Bipartite Systems and the Schmidt Decomposition

What are differences between pure states and mixed states in physics? It is useful to introduce a few more concepts before diving into such conceptual questions.

A bipartite system consists of a product of two Hilbert spaces. In such a system, one can endow a new operation called a partial trace.

Definition 2.1.11. Let H_A, H_B be Hilbert spaces with orthonormal bases $|a\rangle_A, |\beta\rangle_B$, and let O_{AB} be a Hermitian operator acting on $H_A \otimes H_B$. A *partial trace* Tr_B of O_{AB} over B is defined by

$$\mathrm{Tr}_B(O_{AB}) := \sum_{\beta=1}^{\dim H_B} \langle\beta|_B \, O_{AB} \, |\beta\rangle_B \,. \tag{2.25}$$

In particular, if we choose an Hermitian operator O_{AB} to be a density operator ρ_{AB}, we call $\mathrm{Tr}_B(\rho_{AB})$ a *reduced density operator*, which we denote by ρ_A.

Let us now justify that a reduced density operator ρ_A indeed describes physics of the system A. Suppose a bipartite system $H_A \otimes H_B$ is described by a density operator ρ. We then prepare an Hermitian operator O_A defined only in the system A. If we would like to compute the expectation value of O_A in the bipartite system, we actually need to compute the expectation values of $O_A \otimes 1_B$ where 1_B is the identity operator acting on H_B. Thus, we have

$$\langle O_A \otimes 1_B\rangle = \mathrm{Tr}((O_A \otimes 1_B)\rho) = \mathrm{Tr}_A(O_A\rho_A) = \langle O_A\rangle \,. \tag{2.26}$$

Therefore, one can regard ρ_A as the density matrix acting on H_A in the bipartite system $H_A \otimes H_B$. Note that two distinct bipartite states ρ, ρ' can have the same reduced density matrix, that is, $\rho_A = \rho'_A$ can hold.

Putting it the other way around, a density operator of a mixed state in H can be always viewed as a reduced density operator in a bipartite Hilbert space $H \otimes \tilde{H}$ where $\tilde{H}$ is another Hilbert space. A density operator of a mixed state in H_A is generally written as

$$\rho_A = \sum_{a,a'=1}^{\dim H_A} c_{aa'} \, |a\rangle \langle a'| \,. \tag{2.27}$$

On the other hand, a general pure state $|\phi\rangle_{AB}$ in $H_A \otimes H_B$ is

$$|\phi\rangle_{AB} = \sum_{a=1}^{\dim H_A} \sum_{\beta=1}^{\dim H_B} C_{a\beta} |a\rangle_A |\beta\rangle_B \,. \tag{2.28}$$

Hence, the reduced density operator becomes

$$\rho'_A = \mathrm{Tr}_B(|\phi\rangle_{AB} \langle\phi|_{AB}) = \sum_{a,a'=1}^{\dim H_A} \sum_{\beta=1}^{\dim H_B} C_{a\beta} \bar{C}_{a'\beta} |a\rangle \langle a'| \,. \tag{2.29}$$

Thus, if we choose $\dim H_B$ appropriately, one can always find $C_{a\beta}$ so that $\rho'_A = \rho_A$.

Remark 2.1.12. The process is called *purification*. That is, given a density operator of a mixed state in H_A, we prepare an auxiliary Hilbert space H_B and reconstruct a pure state in the bipartite system $H_A \otimes H_B$. Note that purification is not unique at all, and the auxiliary Hilbert space may not have any physical meaning.

Remark 2.1.13. Any bipartite pure state[3] can be written as

$$|\phi\rangle_{AB} = \sum_{a=1}^{\dim H_A} \sqrt{p_a} |a\rangle_A |a'\rangle_B \,, \qquad \sum_{a=1}^{\dim H_A} p_a = 1, \tag{2.30}$$

where $|a\rangle_A$ is chosen to be an orthonormal basis of H_A, $|a'\rangle_B$ are orthogonal vectors in H_B, and we assume $\dim H_A \leq \dim H_B$. In particular, the reduced density matrix is diagonalized

$$\rho_A = \sum_{a=1}^{\dim H_A} p_a |a\rangle \langle a| \,, \tag{2.31}$$

where p_a is the probability that a state is in the a-th state $|a\rangle \langle a|$. This is called the *Schmidt decomposition*.

Remark 2.1.14. Even though this is referred as *the* Schmidt decomposition, there are some ambiguities. If all p_a are distinct, then the Schmidt decomposition is indeed uniquely determined, up to their phase differences $|a\rangle_A \mapsto e^{i\theta_a} |a\rangle_A \,, |a'\rangle_B \mapsto e^{-i\theta_a} |a'\rangle_B$. However, if a

[3] A bipartite state is, in some literature, called a joint state.

set of n p_a's are equal, then there is the freedom of a $U(n)$ transformation of the corresponding vectors $|a\rangle_A$ and the inverse for the $|a'\rangle_B$ vectors.

At last, we give a proposition on von Neumann entropies of reduced density operators.

Proposition 2.1.15 ([24]). *Let H_A, H_B be Hilbert spaces and ρ be a density operator of their bipartite system $H_A \otimes H_B$. We also let ρ_A, ρ_B be reduced density operators of H_A, H_B respectively. Then, their von Neumann entropies satisfy the following inequality called* sub-additivity

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho) \leq S(\rho_A) + S(\rho_B). \tag{2.32}$$

In particular, if the bipartite state ρ is pure, then we have $S(\rho_A) = S(\rho_B)$.

Subadditivity can be generalized for a product ofnthree systems.

Proposition 2.1.16 ([25]). *Let H_A, H_B, H_C be Hilbert spaces and ρ_{ABC} be a density operator of their tensor product system $H_A \otimes H_B \otimes H_C$. We also let $\rho_{AB}, \rho_{BC}, \rho_B$ be reduced density operators of $H_A \otimes H_B, H_B \otimes H_C, H_B$ respectively. Then, their von Neumann entropies satisfy the following inequality called* strong subadditivity

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}). \tag{2.33}$$

2.1.4 Unitarity

When we apply quantum information theory to physics, we would like to know how states evolve over time. A fundamental assumption of quantum mechanics is that such time evolution is described by a unitary operator. More precisely, suppose a physical system is described by a unitary operator $U(t)$ and the pure initial state $|\phi(0)\rangle$. Then, the state $|\phi(t)\rangle$ at time t is given by

$$|\phi(t)\rangle = U(t)\, |\phi(0)\rangle\,. \tag{2.34}$$

Similarly, if the initial state is a density operator $\rho(o)$, no matter whether it is pure or mixed, the state $\rho(t)$ at time t becomes

$$\rho(t) = U(t)\rho(0)U^{\dagger}(t). \tag{2.35}$$

Imposing unitarity on evolution closely links to the concept of predictability and the probability interpretation. Let us give a set of advantages of unitary evolution.

1. Linearity : The evolution operator $U(t)$ acts linearly on pure states as elements of the Hilbert space.

2. Preservation of the norm : Unit norm states evolve to unit norm states.

3. Invertibility : There is a bijective relation between the past, present, and future states.

4. Purity : Pure states evolve to pure states.

Non-unitary evolution implies that we need to give up at least one of the above, which seems a severe modification from a current understanding of quantum mechanics. We will discuss the perspective more later for the black hole information paradox.

At last, we state an important theorem regarding unitary evolution, the so-called *no-cloning theorem*:

Theorem 2.1.17 (No-cloning Theorem)**.** *Let H be a Hilbert space, and $|\phi\rangle, |\psi\rangle, |e\rangle \in H$ be three arbitrary pure states. Then, there is no unitary operator U_{cl} acting on the bipartite system $H \otimes H$ such that*

$$U_{\mathrm{cl}} |\phi\rangle |e\rangle = |\phi\rangle |\phi\rangle , \quad U_{\mathrm{cl}} |\psi\rangle |e\rangle = |\psi\rangle |\psi\rangle . \tag{2.36}$$

Proof. Suppose such a unitary operator U_{cl} exists. However, this implies

$$\langle\psi|\phi\rangle = \langle\psi| \langle e|e\rangle |\phi\rangle = \langle\psi| \langle e| U_{\mathrm{cl}}^{\dagger} U_{\mathrm{cl}} |e\rangle |\phi\rangle = \langle\psi| \langle\psi|\phi\rangle |\phi\rangle = (\langle\psi|\phi\rangle)^2. \tag{2.37}$$

Thus, $\langle\psi|\phi\rangle$ should be either 0 or 1. This is in contradiction to the assumption that $|\phi\rangle, |\psi\rangle$ are chosen *arbitrary*. This proves the theorem. □

Note that it is still possible to construct a cloning unitary operator (2.37) for a finite number of priori known states. This should be clear from how we have proved the theorem. Indeed, there is no contradiction to have a unitary operator $U(e)$ holding (2.37) for an arbitrarily chosen pure state $|e\rangle \in H$ and a set of known orthonormal pure states $|\phi_i\rangle$ where i ranges from 1 to at most $\dim H$ because the condition $U(e)$ needs to hold is simply (2.37).

2.1.5 Information and Mutual Information

Let us come back to the question about differences between pure states and mixed states. Thanks to Proposition 2.1.10, we can rephrase the question in a more quantitative way: given a density operator ρ, what does the von Neumann entropy tell us? To understand this question better, let us consider a spin system, that is a Hilbert space in two dimensions. Suppose two density operators ρ_1, ρ_2 are given, the expectation values of each spin direction are measured respectively as

$$\langle \sigma_x(\rho_1) \rangle = 1, \quad \langle \sigma_y(\rho_1) \rangle = 0, \quad \langle \sigma_z(\rho_1) \rangle = 0, \tag{2.38}$$

$$\langle \sigma_x(\rho_2) \rangle = 0, \quad \langle \sigma_y(\rho_2) \rangle = 0, \quad \langle \sigma_z(\rho_2) \rangle = 0. \tag{2.39}$$

For ρ_1, we can indeed determine the density operator as

$$\rho_1 = \frac{1}{2}(|\uparrow\rangle_z + |\downarrow\rangle_z)(\langle\uparrow|_z + \langle\downarrow|_z), \tag{2.40}$$

or equivalently,

$$\rho_1 = |\uparrow\rangle_x \langle\uparrow|_x . \tag{2.41}$$

Therefore, we are able to grasp the entire information of the state. Namely, if one measures the spin of state ρ_1 along the x-direction, one observes $|\uparrow\rangle_x$ with probability 100%. Note that if one performs other experiments, such as the spin of the state ρ_1 in the z-direction, one still observes $|\uparrow\rangle_z$ with the probability 50% and $|\downarrow\rangle_z$ with the probability 50%. However, as long as the state given is pure, there exists an experiment such that a certain consequence occurs with the probability 100%.

On the other hand, it may not be possible for a general mixed state to find a nontrivial experiment such that one could expect the consequence with the probability 100%[4]. In particular, we can tell from (2.39) that the density operator is proportional to the identity

[4] Of course if the mixed state is given by the tensor product of a pure state and a smaller mixed state, then it is still possible to expect a certain consequence with 100% by focusing on the pure state part. In the language of linear algebra, the density operator for a pure state can be thought of as a Hermitian matrix of rank 1 whereas the density operator for a mixed state is of rank greater than 1

operator I,

$$\rho_2 = \frac{1}{2} I, \tag{2.42}$$

in any basis. This state is completely mixed and every spin measurement occurs with the same probability. There is no preferred state, and every observation gives a pure state randomly.

By keeping this analysis in mind, let us compute the von Neumann entropies of ρ_1, ρ_2. We have

$$S(\rho_1) = 0, \quad S(\rho_2) = \log 2 = \max. \tag{2.43}$$

Therefore, we can speculate from this example that the greater von Neumann entropy is, the more random the state appears. Indeed, the amount of information stored in the state is defined as follows:

Definition 2.1.18. Let H be a Hilbert space and consider a state described by a density operator ρ. The *information* $I(\rho)$ stored in the state may be defined by

$$I(\rho) = \log \dim H - S(\rho) \geq 0, \tag{2.44}$$

To clarify, the information by this definition measures how random consequences of observation are, but we can still accurately determine ρ by appropriate measurements no matter what the value of $I(\rho)$ is, as one can see in the toy example above. This is just a particular definition of information, but there could be other definitions by which we gain the information by determining ρ. Therefore, it is somewhat misleading to think that we can have the exact information about a pure state whereas we can know only part of the information about a mixed state[5]. One has to be very careful what type of *information* it is if this type of statement is given.

It is also worth noting that Definition 2.1.18 gives the *amount* of information stored in the state but does not specify the explicit form of the state. In particular, it is invariant under any unitary transformation $\rho \mapsto U^\dagger \rho U$, thus, the information about such a unitary transformation is invisible in $I(\rho)$. Also, every pure qubit state gives $I(\rho_1) = \log 2$ but this does not tell whether the spin of the pure state is pointing in the z-direction, or in any other

[5] Another disadvantage of the use of Definition 2.1.18 is that it is not well defined if the Hilbert space is infinite dimensional.

direction.

Let us now consider another scenario in a bipartite system $H_A \otimes H_B$ with $\dim H_A = \dim H_B$. Suppose we start with a pure state $|\phi\rangle = |\psi_A\rangle |\chi_B\rangle$ where $|\psi_A\rangle \in H_A, |\chi_B\rangle \in H_B$. The amounts of information stored in $|\psi_A\rangle$ and $|\chi_B\rangle$ are both $\log \dim H_A$, that is, we can know all the information about each state at the initial moment. We now act a unitary operator U on $|\phi\rangle$ in such a way that the state after the operation has $S(\rho_A) = S(\rho_B) = \max$. Such a unitary operator definitely exists, for example, we can construct a unitary operator sending a product of two spin-1/2 pure states to the singlet Bell state. However, the amount of information observed in each system now becomes absolutely zero. Where did all the initial information go? Is it lost? The answer is no, but rather that information is stored in the correlation between the two systems, which is called mutual information.

Definition 2.1.19. Let H_A, H_B be Hilbert spaces and ρ be a density operator of the bipartite system $H_A \otimes H_B$. Then, the *mutual information* between the system A and B is defined by

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho) \geq 0. \tag{2.45}$$

Note that the mutual information is nonnegative due to the subadditivity (2.32). In the scenario of the above example, the mutual information is initially zero, and all the information is stored in each system. After the unitary operation, all the information is transferred into their correlation, i.e., the mutual information.

2.2 Entanglement

From a physics point of view, a pure state describes the possible physics of a closed system, and an open system generically has a mixed state. This insight follows from purification; a mixed state can be thought of as being correlated with another *auxiliary* system. Then, one may ask: is a mixed state the same as an entangled state? This question is not so well-defined because the answer could be yes in such a sense that for a mixed state, one can always construct an auxiliary system to be entangled with. However, such system is not unique and not necessarily a physical system either. On the other hand, when we say a system is entangled with another system, we normally assume the two systems are both physical. Moreover, it turns out entanglement is a rather vague concept that requires careful consideration.

2.2.1 The Einstein-Podolsky-Rosen Paradox

Entanglement is defined in a bipartite system $H_A \otimes H_B$. Before formally defining entanglement, we try to grasp an underlying idea of entanglement. The best starting point to do so, in my opinion, is the Einstein-Podolsky-Rosen paradox.

Suppose Alice and Bob are observers of the system A and B respectively, and they are so far apart that they have access only to their systems and cannot communicate with each other. How much can Bob be affected by Alice's measurements? Classically, Alice's measurements should have nothing to do with Bob no matter what she does because they cannot communicate by the assumption. However, this does not seem true any more from quantum information theory perspectives if one considers a wave function collapse, and this confusion is called the *Einstein-Podolsky-Rosen paradox.*

We quantitatively explore this quantum effect in a toy model. Let H_A, H_B be two-dimensional Hilbert spaces with basis $\{|0\rangle_A, |1\rangle_A\}, \{|0\rangle_B, |1\rangle_B\}$ respectively. We consider two bipartite pure states

$$|\phi\rangle = (a\,|0\rangle_A + b\,|1\rangle_A)(c\,|0\rangle_B + d\,|1\rangle_B), \tag{2.46}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A\,|1\rangle_B + |1\rangle_A\,|0\rangle_B). \tag{2.47}$$

Suppose Alice makes a measurement of the system A and observes either the value 0 or 1. After the measurement, $|\phi\rangle$ is collapsed onto either

$$|0\rangle_A \langle 0|_A \otimes \mathbf{1}_B\,|\phi\rangle = a\,|0\rangle_A\,(c\,|0\rangle_B + d\,|1\rangle_B) \tag{2.48}$$

or

$$|1\rangle_A \langle 1|_A \otimes \mathbf{1}_B\,|\phi\rangle = b\,|1\rangle_A\,(c\,|0\rangle_B + d\,|1\rangle_B), \tag{2.49}$$

where $\mathbf{1}_B$ is the identity operator acting on H_B. Therefore, Bob has the same state no matter what is the outcome of Alice's measurement.

On the other hand, if Alice and Bob were to observe the state $|\psi\rangle$ in (2.47), Alice's measurement maps $|\psi\rangle$ to either

$$|0\rangle\langle 0|_A \otimes \mathbf{1}_B |\phi\rangle = \frac{1}{\sqrt{2}} |0\rangle_A |1\rangle_B \tag{2.50}$$

or

$$|1\rangle\langle 1|_A \otimes \mathbf{1}_B |\phi\rangle = \frac{1}{\sqrt{2}} |1\rangle_A |0\rangle_B . \tag{2.51}$$

Hence, the state Bob has access to depends on what Alice did. If Alice's measurement results in the value 0, Bob necessarily observes the value 1, and vice versa. Note that Alice's measurement seems to *instantly* affects Bob's state no matter how far they are apart. This is paradoxical from the causality point of view.

Does this actually break causality? Does this imply a communication faster than the speed of light? The answer is no, because Bob himself has no way to instantly know what Alice measured. For Bob, the only thing he can do is simply to measure either the value 0 or the value 1 as he would do without any correlation with Alice. Hence, it has nothing to do with causality.

Let us extend to this scenario a bit more. Suppose it is Charlie who prepares the state (2.47) and gives the qubit A to Alice and the qubit B to Bob. Charlie does not tell Alice and Bob what state he prepares, and he is with Alice when she does her experiment. If she measures the value 0, Charlie instantly knows Bob will observe the value 1, and vice versa. Does it suggests an instantaneous communication between Bob and Charlie so that it breaks the causality? The answer is, again, no. Charlie has already known the correlation between Alice and Bob, hence, it does not mean Charlie has obtained the information of Bob's measurement from Bob.

Note that if Charlie prepares the state (2.46) instead, even Charlie cannot figure out what Bob will observe after Alice's measurement. This is because the qubit A and B are separable in (2.46) whereas they are entangled in (2.47). Quantum entanglement (2.47) makes it possible for Charlie to predict what Bob's qubit gives after Alice's experiment.

We define separable states and entangled states more rigorously below.

2.2.2 Entanglement Entropy

Definition 2.2.1. Let H_A, H_B be Hilbert spaces and $|\phi\rangle \in H_A \otimes H_B$ be a bipartite pure state. We say the systems A and B are *separable* if and only if one of the following equivalent statements is correct.

1. $\exists\, |\psi_A\rangle \in H_A, |\chi_B\rangle \in H_B$ such that $|\phi\rangle = |\psi_A\rangle\, |\chi\rangle_B$.

2. $\rho_A = \mathrm{Tr}_B(|\phi\rangle\, \langle\phi|)$ is pure.

The systems A and B are *entangled* if they are not separable.

Notice that Propostion (2.1.10) implies that the von Neumann entropy $S(\rho_A)$ of the reduced density matrix ρ_A can be used to check whether two systems are entangled. Also, this definition applies only if a bipartite state is pure. We will give a more general definition for mixed bipartite states shortly.

Once we figure out whether the two systems are entangled, a natural question to ask next is how entangled they are. It turns out that the von Neumann entropy $S(\rho_A)$ of a reduced density matrix ρ_A can play a role of a measure of entanglement too. The greater $S(\rho_A)$ is, the more entangled the two systems are. In particular, we define a maximally entangled state as follows.

Definition 2.2.2. Let ρ be a pure density operator of a bipartite system $H_A \otimes H_B$ with $\dim H_A \leq \dim H_B$. Then, we say the system A is *maximally entangled* with the system B if

$$S(\rho_A) = \max = \log \dim H_A. \tag{2.52}$$

Note that even if the system A is maximally entangled with the system B, the system B is not maximally entangled with the system A in general. This is because $\dim H_A \leq \dim H_B$, while $S(\rho_A) = S(\rho_B)$ due to subadditivity (2.32). Hence, $S(\rho_B)$ is not also maximal unless $\dim H_A = \dim H_B$. As the von Neumann entropy of a reduced density operator is very useful in order to measure an entanglement of a bipartite system, we give a special name on it.

Definition 2.2.3. Let ρ be a pure density operator of a bipartite system $H_A \otimes H_B$. Then, the von Neumann entropy $S(\rho_A)$ of the reduced density operator ρ_A is called the *entanglement entropy*.

Remark 2.2.4. $S(\rho_A)$ is called the entanglement entropy *only if* its bipartite state is pure.

If a bipartite state is mixed, it becomes more subtle to define entanglement.

Definition 2.2.5. Let ρ be a mixed density operator of a bipartite system $H_A \otimes H_B$. Then, two systems are *separable* if there exists a set of pure states $\rho_A^{(k)}, \rho_B^{(k)}$ such that ρ can be written as

$$\rho = \sum_{k=1}^{K} p_k \rho_A^{(k)} \otimes \rho_B^{(k)}, \quad \sum_{k=1}^{K} p_k = 1, \tag{2.53}$$

where K can be greater than $\dim(H_A \otimes H_B)$, and p_k are nonnegative real numbers which can be interpreted as probabilities since they sum to 1.

This definition indeed makes sense. It indicates that the bipartite system is described by $\rho_A^{(k)} \otimes \rho_B^{(k)}$ with the probability p_k, and two systems are separable at each event. Note that even though the systems A and B were separable, the reduced density matrix ρ_A (not $\rho_A^{(k)}$) can be mixed, hence, the system A is not pure in general. Therefore, we need to have a clear understanding of differences among pure, mixed, separable and entangled. For instance, one can have a situation that three systems A, B and C are all separable from each other, but A is entangled with BC, B is entangled with AC and C is entangled with AB. Pictorially, this can be, in a sense, visualized by a Borromean ring. Figure 2.3. We will explore this concept with a few examples in Chapter 3.

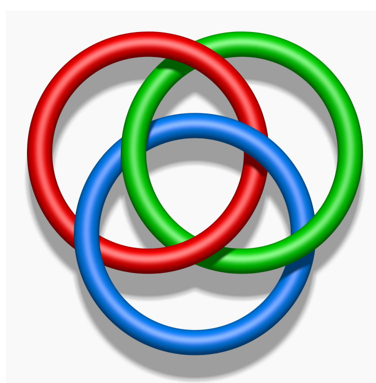


Figure 2.3: The red circle is entangled with a set of the blue and green circle together, while the red and blue circles are separable if we ignore the green circle. The image is retrieved from Wikipedia.

For a bipartite mixed state, the condition $S(\rho_A) = \max$ is not sufficient to conclude that the system A is maximally entangled with the other system B. This is why the von

Neumann entropy is not referred as the entanglement entropy for a bipartite mixed state any more because it does not measure entanglement. In fact, two systems can be separable even if the von Neumann entropies of their reduced density operators are maximized. For example, the following maximally mixed bipartite state,

$$\rho = \frac{1}{\dim H_A \dim H_B} I_A \otimes I_B, \tag{2.54}$$

is clearly separable by Definition 2.2.5 for $K = \dim H_A \dim H_B$. One way of defining maximally entangled states is discussed in [26, 27].

Definition 2.2.6. Let ρ be a mixed density operator of a bipartite system $H_A \otimes H_B$ where $\dim H_A \leq \dim H_B$. The system A is *maximally entangled* with the system B if all of the following three conditions are satisfied:

1. there exists a subspace $H'_B \subset H_B$ where $H_B = H'_B \otimes \overline{H}'_B$ with $\dim H_A = \dim H'_B$
2. there exists a pure state $|\psi\rangle \in H_A \otimes H'_B$ such that $S(\mathrm{Tr}_{B'}(|\psi\rangle\langle\psi|)) = \log \dim H_A$
3. there exists a density operator $\bar{\rho}$ on $\overline{H}_B$ such that $\rho = \bar{\rho} \otimes |\psi\rangle\langle\psi|$.

We have reviewed all elementary definitions and properties in quantum information theory focusing on quantum entanglement in this section. There are of course much more to study, and we refer the readers to Preskill's lecture notes [23] and references therein for further discussions. As mentioned at the beginning, quantities in quantum information theory are more vague and delicate than how they are explained in some literature. Hopefully remarks and examples given above would help the readers with precisely grasping ideas of quantum information theory. To close this section, we give two theorems which will be important to understand the firewall paradox.

Theorem 2.2.7 (Monogamy of entanglement [28, 29])**.** *Let A, B, C be independent systems in which the system A is maximally entangled with the system B. Then, the system A cannot be entangled with the system C at all.*

Theorem 2.2.8 (Average Entropy [30, 31])**.** *Let H_A, H_B be Hilbert spaces and we randomly pick a bipartite pure state ρ in $H_A \otimes H_B$ where $\dim H_A \leq \dim H_B$. Then, the average* *entanglement entropy* $S(\rho_A)$ *between the two systems is*

$$S(\rho_A) = \sum_{k=\dim H_B+1}^{\dim H_A \dim H_B} \frac{1}{k} - \frac{\dim H_A - 1}{2\dim H_B}. \tag{2.55}$$

In particular, for $1 \ll \dim H_A \leq \dim H_B$*, this implies that the system* A *is nearly maximally entangled with the system* B*,*

$$S(\rho_A) \sim \log\dim H_A - \frac{\dim H_A - 1}{2\dim H_B}. \tag{2.56}$$

3 Geometry of Quantum States

Let us consider a few examples to establish a deeper understanding of entanglement, though the discussions below have no direct relation to black hole physics. We shall find that entanglement between qubits turns out to have some geometric interpretation on the Bloch sphere, which is well explained in [37]. The Bloch sphere can serve as the circumsphere of a polyhedron of n vertices such that the shape and orientation of the polyhedron corresponds to a spin-$n/2$ state. See Figure 3.1 below for an intuitive pictorial correspondence, and detailed definitions as well as our methodology are given shortly. What we are interested in is the relation between the shape of polyhedra and the condition that two qubits are separable from each other. This is work in progress with Page [4] and is hoped to be published soon.

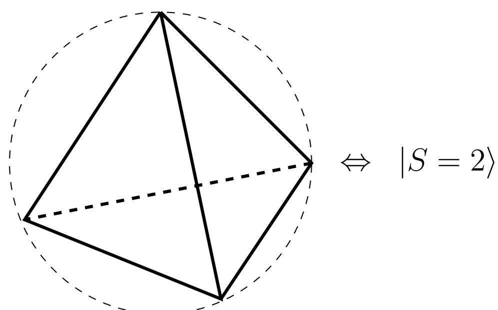


Figure 3.1: A bijective map between a spin-2 state and a tetrahedron attached to the Bloch sphere.

3.1 Entanglement of Formation

As our interest is the separability between two subsystems, we need a mathematical tool to determine whether two systems are separable or entangled. If the bipartite state of such two subsystems is pure, the entanglement entropy becomes useful. However, measurements of entanglement of bipartite mixed states are rather complex ones. The definition we gave in Definition 2.2.6 applies only to maximal entanglement, and as of today, there is no universal measure that is also analytically calculable. Yet, one of the most realistic measures is the so-called entanglement of formation.

Definition 3.1.1. Let H_A, H_B be Hilbert spaces and ρ be a density operator of a bipartite system $H_A \otimes H_B$. We consider all pure state decompositions of ρ

$$\rho = \sum_{k=1}^{K} p_k \left|\phi_k\right\rangle \left\langle\phi_k\right|, \quad \sum_{k=1}^{K} p_k = 1, \tag{3.1}$$

where $|\phi_k\rangle \in H_A \otimes H_B$ are bipartite pure states and K can be greater than $\dim H_A \otimes H_B$. Then, the *entanglement of formation* $E(\rho)$ is defined as

$$E(\rho) = \min \sum_{k=1}^{K} p_k E(\phi_k) \geq 0, \tag{3.2}$$

where it is minimized over all decompositions of ρ, and $E(\phi_k)$ is the entanglement entropy of each pure state $|\phi_k\rangle$

$$E(\phi_k) = -\mathrm{Tr}_A\Big(\mathrm{Tr}_B(\left|\phi_k\right\rangle \left\langle\phi_k\right|) \log \mathrm{Tr}_B(\left|\phi_k\right\rangle \left\langle\phi_k\right|)\Big). \tag{3.3}$$

Then, we immediately notice the following proposition:

Proposition 3.1.2. *Let H_A, H_B be Hilbert spaces and ρ be a density operator of a bipartite system $H_A \otimes H_B$. The systems A and B are separable if and only if the entanglement of formation vanishes, $E(\rho) = 0$.*

Recall that there is no unique pure state decomposition for (3.1) as commented in Section 2.1.2. For example, let us consider the following mixed density operator

$$\rho = \frac{1}{2} \left|00\right\rangle \left\langle 00\right| + \frac{1}{2} \left|11\right\rangle \left\langle 11\right|. \tag{3.4}$$

It is straightforward to check that these two qubits are actually separable by Definition 2.2.5. This decomposition gives

$$\sum_{k=1}^{K} p_k E(\phi_k) = 0, \tag{3.5}$$

which is consistent with Proposition 3.1.2. On the other hand, the density operator (3.4) can be decomposed in another way

$$\rho = \frac{1}{4}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + \frac{1}{4}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|), \tag{3.6}$$

which returns

$$\sum_{k=1}^{K} p_k E(\phi_k) = \frac{1}{2}\log 2. \tag{3.7}$$

This example indicates a reason why we need to take the minimum over all possible pure decompositions in (3.2).

The entanglement of formation is still difficult to compute for general bipartite mixed states. Also, it remains to be seen whether it is useful beyond being a measure of separability. If both H_A, H_B are two-dimensional, however, [32] showed a relatively easy way of computing the entanglement of formation.

Theorem 3.1.3 ([32])**.** *Let H_A, H_B be Hilbert spaces of two dimensions and ρ be a density operator of a bipartite system $H_A \otimes H_B$. For the Pauli matrix σ_y, we define a Hermitian operator $\tilde{\rho}$*

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\bar{\rho}(\sigma_y \otimes \sigma_y), \tag{3.8}$$

where $\bar{\rho}$ is the complex conjugate of ρ. It can be shown that the eigenvalues of the non-Hermitian operator $\rho\tilde{\rho}$ are nonnegative. Let $\lambda_i \in \mathbb{R}_{\geq 0}$ be square roots of such eigenvalues in decreasing order. Then, the entanglement of formation $E(\rho)$ is given by

$$E(\rho) = h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right), \tag{3.9}$$

$$h(x) = -x\log x - (1-x)\log(1-x), \tag{3.10}$$

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{3.11}$$

In particular, $E(\rho) = 0$ if and only if $C(\rho) = 0$.

Remark 3.1.4. When all λ_i's are complicated, it provides an extra computational difficulty to determined what is the greatest one. However, since our focus is only on the separability of states, we may take another equivalent measure $\tilde{C}(\rho)$, instead of $C(\rho)$, defined by

$$\tilde{C}(\rho) = (-\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1-\lambda_2+\lambda_3+\lambda_4)(\lambda_1+\lambda_2-\lambda_3+\lambda_4)(\lambda_1+\lambda_2+\lambda_3-\lambda_4). \tag{3.12}$$

Two qubits H_A, H_B are separable if and only if $\tilde{C}(\rho) \geq 0$. The advantage of $\tilde{C}(\rho)$ is that we do not need to consider the order of the λ_i's because it is symmetrical among them. Meanwhile, $C(\rho)$ is linear in λ_i's while $\tilde{C}$ requires some multiplications, hence, this is sort of a trade-off. We may apply $C(\rho)$ or $\tilde{C}(\rho)$ case by case.

3.2 Separability of Symmetric States

By using the entanglement of formation, we explore interesting relations between geometric symmetry and the separability of two qubits. More precisely, let H be a Hilbert space of two dimensions, i.e., a qubit, and consider n points (θ_i, φ_i) on the Block sphere to fix n qubits $|\phi_i\rangle$

$$|\phi_i\rangle = \cos\frac{\theta_i}{2}|0\rangle + e^{i\varphi_i}\sin\frac{\theta_i}{2}|1\rangle\,. \tag{3.13}$$

Then it is known [33–35] that any spin-$n/2$ pure state in $H^{\otimes n}$ is given, up to normalization, by

$$|\Phi_n\rangle \propto \sum_{\tau\in S_n} |\phi_{\tau(1)}\rangle_1 \cdots |\phi_{\tau(n)}\rangle_n = \sum_{k=0}^{n} c_k |S_{n,k}\rangle\,, \tag{3.14}$$

$$c_k = \sum_{\sigma\in S_n} S_{\sigma(1)}\cdots S_{\sigma(k)} C_{\sigma(k+1)}\cdots C_{\sigma(n)}, \tag{3.15}$$

$$S_{\sigma(i)} = e^{i\varphi_i}\sin\frac{\theta_i}{2}, \quad C_{\sigma(n)} = \cos\frac{\theta_i}{2}, \tag{3.16}$$

where S_n is the symmetric group of permutations of degree n. $|S_{n,k}\rangle$ is the *Dicke state* [36] which is defined as

$$|S_{n,k}\rangle = \frac{1}{(n-k)!k!}\sum_{S_n} |\underbrace{00\cdots0}_{n-k}\underbrace{11\cdots1}_{k}\rangle\,, \tag{3.17}$$

where the sum is taken over all permutations of the basis with $n-k$ qubits being in $|0\rangle$ and k qubits being in $|1\rangle$.[1] For the rest of this section, we refer to $|\Phi_n\rangle$ as a pure spin-$n/2$ state.

Since (3.14) is defined by the permutational symmetrization of n states, there is a bijective map between the state (3.14) and the polyhedron of n vertices whose circumsphere is the Bloch sphere. Note that in principle, two polyhedra in the same shape but different ori-

[1]The Dicke state is often defined with normalization $\langle S_{n,k}|S_{n,k}\rangle = 1$. However, we take the definition without normalization for simplicity.

entations would give two different states[2]. At the same time, (3.14) is symmetric among all n qubits, hence, every qubit is separable from each other qubit if any one such pair is separable. The separability only depends on the shape of the polyhedron but not the orientation. Now an interesting question arises:

What are the shapes of a polyhedron with n vertices such that the corresponding state has each pair of qubits separable?

One may remark that a qubit is, in general, entangled with the rest of the system as a whole, even if each pair of qubits is separable. This statement might sound counterintuitive at first, but we will show below that this is actually possible[3].

3.2.1 $n = 2$ Qubits

This case is the simplest and trivial because the bipartite state is pure. The spherical symmetry allows us to fix one point on the north pole of the Bloch sphere and the other on the xz-plane, that is, $\varphi = 0$:

$$|\phi_1\rangle = |0\rangle\,, \quad |\phi_2\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\,. \tag{3.18}$$

A spin-1 state (3.14) is simply

$$|\Phi_2\rangle \propto |\phi_1\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_1\rangle\,. \tag{3.19}$$

It is straightforward to see that the two qubits are separable if and only if the two states are identical, $\theta = 0$. That is, two points are both on the north pole. Note that the two qubits are maximally entangled if and only if $\theta = \pi$, hence, one is at the north pole and the other is at the south pole on the Bloch sphere.

[2] An obvious example is when $n = 1$. Two distinct points on the Bloch sphere correspond to two distinct state by definition.

[3] [34, 35] also investigate characteristics of maximally entangled symmetric states, but what we focus on is rather separability.

3.2.2 $n = 3$ Qubits

The case $n = 3$ is more much complicated but interesting. Similar to the $n = 2$ case, the spherical symmetry is used to set three qubit states corresponding to three points on the Bloch sphere as[4]

$$|q_1\rangle = |0\rangle\,, \quad |q_2\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\,, \quad |q_3\rangle = \cos\frac{\eta}{2}|0\rangle + e^{i\varphi}\sin\frac{\eta}{2}|1\rangle\,. \tag{3.20}$$

A spin-3/2 state is then given by

$$\begin{aligned}|\Phi_3\rangle \propto\, & 3\cos\frac{\theta}{2}\cos\frac{\eta}{2}|000\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sin\frac{\eta}{2}(|011\rangle + |101\rangle + |110\rangle) \\ & + \left(e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\eta}{2} + \cos\frac{\eta}{2}\sin\frac{\theta}{2}\right)(|001\rangle + |010\rangle + |100\rangle).\end{aligned} \tag{3.21}$$

It is tedious to compute λ_i defined as in Theorem 3.1.3, but thanks to Maple, we know that two of them are zero, and the other two are in the form

$$\lambda_1 = \sqrt{c + \sqrt{D}}, \quad \lambda_2 = \sqrt{c - \sqrt{D}}, \tag{3.22}$$

where both c and D are functions of (θ, η, φ). Thus, Theorem 3.1.3 implies that a pair of two systems is separable if and only if the discriminant $D = 0$.

D is also a complicated function at first glance. It turns out the expression is simplified if we introduce the angle σ between the two points that are not on the north pole in place of φ. The relation between σ and φ is

$$\cos\sigma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos\varphi. \tag{3.23}$$

[4] Since one can always define a plane that supports the triangle, one may suspect it might be simpler to use the spherical symmetry to put the three points on the same latitude. I have also tried such a parametrization, but computations are not so simple as those for the choice given.

Then, it can be shown that the discriminant D as a function of (θ, η, σ) is proportional to

$$\begin{aligned} D \propto &\Big((2\cos\theta - \cos\eta - \cos\sigma)^2 + (2\cos\eta - \cos\sigma - \cos\theta)^2 + (2\cos\sigma - \cos\theta - \cos\eta)^2\Big) \\ &\times \Big((2\cos\theta - \cos\eta - \cos\sigma)^2 + (2\cos\eta - \cos\sigma - \cos\theta)^2 + (2\cos\sigma - \cos\theta - \cos\eta)^2 \\ &+ 9\sin\theta\sin\eta\sin\sigma\Big). \end{aligned} \tag{3.24}$$

Note that the normalization for the discriminant D neither vanishes nor diverges. Since $0 \leq \theta, \eta, \sigma \leq \pi$, D is zero if and only if $\theta = \eta = \sigma$. Thus, we have found the following proposition.

Proposition 3.2.1. *An arbitrary pair of qubits in a pure spin-$3/2$ state in $H^{\otimes 3}$ is separable if and only if the three points on the Block sphere form an equilateral triangle.*

Pictorially, one can interpret Proposition 3.2.1 as analogous to a Borromean ring in Figure 2.3. That is, a qubit is generally entangled with the set of the other two qubits. However, if the three points on the Bloch sphere form an equilateral triangle, each pair of qubits is not entangled with each other. Note that every qubit is maximally entangled with the two others if and only if the three points form an equilateral triangle on a plane through the centre of the Bloch sphere.

3.2.3 $n \geq 4$ Qubits

For a general setting with $n \geq 4$, it has become challenging to obtain the eigenvalues λ_i even with the use of Maple. However, one can still explore examples with a few constraints. Here, we show a few interesting observations.

Proposition 3.2.2. *If r points are equally placed at a ring on the Bloch sphere, then any pair of qubits in a pure spin-$r/2$ state in $H^{\otimes n}$ is always separable.*

Proof. Suppose the ring is located at the angle θ from the z-axis. The k-th qubit state is expressed by

$$|\phi_k\rangle = \cos\frac{\theta}{2}|0\rangle + e^{\frac{2\pi i k}{r}}\sin\frac{\theta}{2}|1\rangle. \tag{3.25}$$

Then, the pure spin-$r/2$ state (3.14) in $H^{\otimes r}$ is

$$|\Phi_r\rangle \propto \cos^r \frac{\theta}{2} |0\rangle^r + (-1)^{r+1} \sin^r \frac{\theta}{2} |1\rangle^r . \tag{3.26}$$

As a result, the density matrix ρ after taking a partial trace over the last $H^{\otimes(r-2)}$ is given by

$$\rho \propto \cos^{2r} \frac{\theta}{2} |00\rangle \langle 00| + \sin^{2r} \frac{\theta}{2} |11\rangle \langle 11| . \tag{3.27}$$

Thus, a pair of qubits is separable by Definition 2.2.5. □

If we further take a partial trace over the second qubit, we obtain the reduced density matrix of the first qubit as

$$\rho_1 = \frac{1}{N} \left(\cos^{2r} \frac{\theta}{2} |0\rangle \langle 0| + \sin^{2r} \frac{\theta}{2} |1\rangle \langle 1| \right), \tag{3.28}$$

$$N = \cos^{2r} \frac{\theta}{2} + \sin^{2r} \frac{\theta}{2}. \tag{3.29}$$

As a consequence, the entanglement entropy between the first system and the rest is given by

$$S(\rho_1) = \log N - \frac{2r}{N} \left(\cos^{2r} \frac{\theta}{2} \log \cos \frac{\theta}{2} + \sin^{2r} \frac{\theta}{2} \log \sin \frac{\theta}{2} \right). \tag{3.30}$$

It can be shown that this entanglement entropy is a monotonically increasing function of θ in the domain $0 \leq \theta \leq \pi/2$ with $S(\rho_1) = 0$ at $\theta = 0$ and $S(\rho_1) = \log 2$ at $\theta = \pi/2$. Therefore, one qubit is generally entangled with the rest, and the entanglement is maximized when the ring is on the equator. However, if one focuses on an arbitrary pair of qubits, they are always separable.

One can generalize Proposition 3.2.2 as follows

Proposition 3.2.3. *If one point is at the north pole with multiplicity n, another point at the south pole with multiplicity s, and $r \geq 3$ points are on a ring at angle θ from the z-axis, then an arbitrary pair of qubits is separable if $(r-1)^2(r+n+s) \geq 4ns$ and if θ satisfies*

$$x_- \leq \tan^{2r} \frac{\theta}{2} \leq x_+, \tag{3.31}$$

where $x_\pm$ are

$$
\begin{aligned}
x_\pm =& \frac{(n+r)!s!}{2n(s+r)(s+r)!n!}\Bigg((n+r)(s+r)(r-1) - sn(r+1) \\
& \pm \sqrt{\Big((n+r+s)(r-1)^2 - 4ns\Big)r^2(n+r+s)}\Bigg).
\end{aligned}
\tag{3.32}
$$

Proof. The proof becomes long and computational so we summarize a few facts here. First of all, a spin-$(r+n+s)/2$ pure state in $H^{\otimes r+n+s}$ becomes

$$
\begin{aligned}
|\Phi_{r+n+s}\rangle \propto \cos^r\frac{\theta}{2} & \sum_{\sigma\in S_{n+s+r}} |0\rangle_{\sigma(1)} \cdots |0\rangle_{\sigma(n+r)} |1\rangle_{\sigma(n+r+1)} \cdots |1\rangle_{\sigma(n+s+r)} \\
& + (-1)^r \sin^r\frac{\theta}{2} \sum_{\sigma\in S_{n+s+r}} |0\rangle_{\sigma(1)} \cdots |0\rangle_{\sigma(n)} |1\rangle_{\sigma(n+1)} \cdots |1\rangle_{\sigma(n+s+r)}.
\end{aligned}
\tag{3.33}
$$

Then, the reduced density matrix becomes diagonal in the orthnormal basis $\{|00\rangle, |11\rangle, |B_\pm\rangle\}$ where

$$
|B_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{3.34}
$$

More precisely, we obtain

$$
\rho \propto \mathrm{diag}(\rho_{00}, \rho_{11}, \rho_{B_+}, 0) \tag{3.35}
$$

$$
\rho_{00} = (n+r)(n+r-1)\frac{(n+r)!}{n!}\cos^{2r}\frac{\theta}{2} + n(n-1)\frac{(s+r)!}{s!}\sin^{2r}\frac{\theta}{2}, \tag{3.36}
$$

$$
\rho_{11} = s(s-1)\frac{(n+r)!}{n!}\cos^{2r}\frac{\theta}{2} + (s+r)(s+r-1)\frac{(s+r)!}{s!}\sin^{2r}\frac{\theta}{2}, \tag{3.37}
$$

$$
\rho_{B_+} = 2s(n+r)\frac{(n+r)!}{n!}\cos^{2r}\frac{\theta}{2} + 2n(s+r)\frac{(s+r)!}{s!}\sin^{2r}\frac{\theta}{2}. \tag{3.38}
$$

Following the definition (3.8), $\tilde{\rho}$ in this basis is given by

$$
\tilde{\rho} \propto \mathrm{diag}(\rho_{11}, \rho_{00}, \rho_{B_+}, 0). \tag{3.39}
$$

Hence, the eigenvalues λ_i are, up to normalization, respectively,

$$
\lambda_1 = \rho_{B_+}, \quad \lambda_2 = \lambda_3 = \sqrt{\rho_{00}\rho_{11}}, \quad \lambda_4 = 0. \tag{3.40}
$$

Then, we notice that a pair of qubits is separable if and only if $\rho_{B_+}^2 - 4\rho_{00}\rho_{11} \leq 0$. The proof is completed by finding the domain of θ for given n, s, r. $\square$

In particular for $s = 0$, where points on the Bloch sphere form a regular pyramid with the apex at the north pole, we have

$$x_- = 0, \quad x_+ = \frac{(n+r)(r-1)}{n} \frac{(n+r)!}{n!r!}. \tag{3.41}$$

Thus, a pair of qubits is separable when the regular pyramid is sufficiently short. In particular, $x_+ \geq 1$ for any $(n \geq 0, r \geq 3)$ so that it is always separable if the regular pyramid is confined to the northern hemisphere. Moreover, for the case $(n, s, r) = (1, 0, 3)$, we find an interesting fact. (3.35) implies that the reduced density operator for a single qubit becomes

$$\rho_1 \propto \left(8\cos^6\frac{\theta}{2} + \sin^6\frac{\theta}{2}\right)|0\rangle\langle 0| + 2\sin^6\frac{\theta}{2}|1\rangle\langle 1|. \tag{3.42}$$

Therefore, a single qubit is maximally entangled with the rest of the system $H^{\otimes n-1}$ if

$$8\cos^6\frac{\theta}{2} + \sin^6\frac{\theta}{2} = 2\sin^6\frac{\theta}{2} \quad \Rightarrow \quad \tan\frac{\theta}{2} = \sqrt{2}. \tag{3.43}$$

This is precisely when the four points on the Bloch sphere form a regular tetrahedron!

Note that $x_+ = 32$, hence a pair of qubits is still separable even if a regular triangular pyramid is slightly taller (or equivalently sharper) than the regular tetrahedron. For a regular pyramid of $n = 1, s = 0$ and $r \geq 3$, let θ_c be the angle where the center of mass of $r + 1$ points[5] is at the origin, θ_m be the angle where each qubit is maximally entangled with the rest, and θ_s be the maximum angle that a pair of qubits is separable. Then, we have

$$\theta_c \leq \theta_m < \theta_s, \tag{3.44}$$

where the first inequality is saturated only for $r = 3$ as we have shown above.

[5] We assume that they are equally weighted.

3.2.4 Two Variable Examples

All the examples above for $n \geq 4$ qubits considered have only one variable for the computational simplicity. Let us investigate cases with two variables. Note that there would be originally 8 variables to specify 4 points on the Bloch sphere. The spherical symmetry reduces the number by 3, hence, the remaining degrees of freedom are 5 in order to fix the state (3.14) for $n = 4$. How shall we impose 3 more constraints to reduce the number to 2 in some geometrically interesting settings?

Perhaps, one such way is to consider a tetrahedron whose center of mass is at the center of the Bloch sphere assuming that each point is equally weighted. This is equivalent to saying that each pair of opposite edges have the same length, and the corresponding state for each point can be respectively parametrized by

$$\begin{aligned} |\phi_1\rangle &= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle, & |\phi_2\rangle &= \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle, \\ |\phi_3\rangle &= \sin\frac{\theta}{2}|0\rangle + e^{i\varphi}\cos\frac{\theta}{2}|1\rangle, & |\phi_4\rangle &= \sin\frac{\theta}{2}|0\rangle - e^{i\varphi}\cos\frac{\theta}{2}|1\rangle. \end{aligned} \tag{3.45}$$

Accordingly, the state (3.14) for $n = 4$ becomes

$$\begin{aligned} |\Phi_4\rangle_O \propto\, & 6\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}|0000\rangle + 6e^{2i\varphi}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}|1111\rangle \\ & - \left(\sin^4\frac{\theta}{2} + e^{2i\varphi}\cos^4\frac{\theta}{2}\right)(|0011\rangle + |1100\rangle + |1010\rangle + |0101\rangle + |1001\rangle + |0110\rangle), \end{aligned} \tag{3.46}$$

where the subscript O denotes that the center of mass is at the origin.

Remark 3.2.4. Every qubit is maximally entangled with the rest in this setting, and this is indeed why we are focusing on this type of questions. That is, our interest is to find shapes of the tetrahedron such that every qubit is maximally entangled with the rest, yet each pair of qubits is separable. We indeed suspect for $n = 4$ that each qubit is maximally entangled with the rest of the system if and only if the center of mass is at the origin, though we have only proved the if-part as mentioned in the above example. We can prove for $n = 3$ that every qubit is maximally entangled with the rest of the system if and only if the center of mass is at

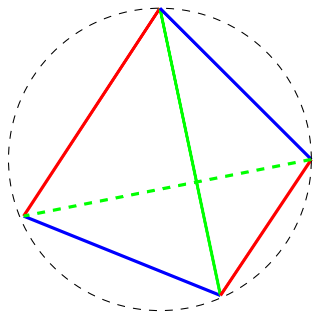


Figure 3.2: A bijective map between a tetrahedron whose center of mass is at the origin and the corresponding spin-2 state (up to isomorphism of the spherical symmetry). The opposite edges have the same length respectively as highlighted in red, blue, and green.

the origin as follows. The if-part is straightforward because the three points should form the equilateral triangle on the equator and we can use the result from (3.30). The only-if-part is shown by computing the expectation values of each spin $\langle S=2|_O\, S_i\, |S=2\rangle_O = 0$, which is equivalent to the maximally entangled condition for permutationally symmetric states (3.20), and show that the equation holds only if the center of mass is at the origin. One can also argue that the entanglement entropy is maximized if and only if the corresponding triangle forms an equilateral triangle on the equator, though the computation would be longer than that for spin directions. Note that the statement does not hold any more for $n \geq 5$ qubits, that is, we know cases where the center of mass is at the origin, yet one qubit is not maximally entangled with the rest.

Returning to the tetrahedron whose center of mass is at the origin, in the basis $\{|00\rangle, |11\rangle, |B_\pm\rangle\}$ where $|B_\pm\rangle$ is given by (3.34), the reduced density operator for the first two qubits has no component for $|B_-\rangle$. The remaining 3×3 part is given by

$$\rho \propto \begin{pmatrix} A & B & 0 \\ \bar{B} & A & 0 \\ 0 & 0 & C \end{pmatrix}. \tag{3.47}$$

where

$$A = 36\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2} + \left|\sin^4\frac{\theta}{2} + e^{2i\varphi}\cos^4\frac{\theta}{2}\right|^2, \tag{3.48}$$

$$B = -6\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\left(\sin^4\frac{\theta}{2} + \cos^4\frac{\theta}{2}\right)(1+e^{-2i\varphi}), \tag{3.49}$$

$$C = 4\left|\sin^4\frac{\theta}{2} + e^{2i\varphi}\cos^4\frac{\theta}{2}\right|^2. \tag{3.50}$$

or equivalently

$$A = \frac{9}{4}\sin^4\theta + \cos^2\theta + \frac{1}{4}\sin^4\theta\cos^2\varphi, \tag{3.51}$$

$$B = -\frac{3}{4}\sin^2\theta(1+\cos^2\theta)(1+e^{-2i\varphi}), \tag{3.52}$$

$$C = 4\cos^2\theta + \sin^4\theta\cos^2\varphi. \tag{3.53}$$

Note that in this case, we have $\rho = \tilde{\rho}$. Then, we have

$$\tilde{\rho}\rho \propto \begin{pmatrix} A^2+|B|^2 & 2AB & 0 \\ 2A\bar{B} & A^2+|B|^2 & 0 \\ 0 & 0 & C^2 \end{pmatrix} \tag{3.54}$$

and the eigenvalues are

$$A^2+|B|^2 \pm 2A\sqrt{|B|^2}, \quad C^2. \tag{3.55}$$

Notice that

$$|B|^2 = \left(\frac{3}{2}\sin^2\theta(1+\cos^2\theta)\cos\varphi\right)^2. \tag{3.56}$$

Thus, the λ_i are

$$0, A \pm |B|, C, \tag{3.57}$$

where one can show $A \geq |B|$.

With the help of Maple, we compute the region where $\tilde{C}(\rho) \geq 0$, and we give the plot below. Note that it is possible analytically to compute the equations defining the boundary of the separability region. The boundary can be determined by quadratic polynomials in $\cos\varphi$, but the coefficients are also quadratic polynomials of $\cos^2\theta$. Thus, the expressions are messy and we omit them here. The plot of $\tilde{C}(\rho) \geq 0$ is shown in Figure 3.3.

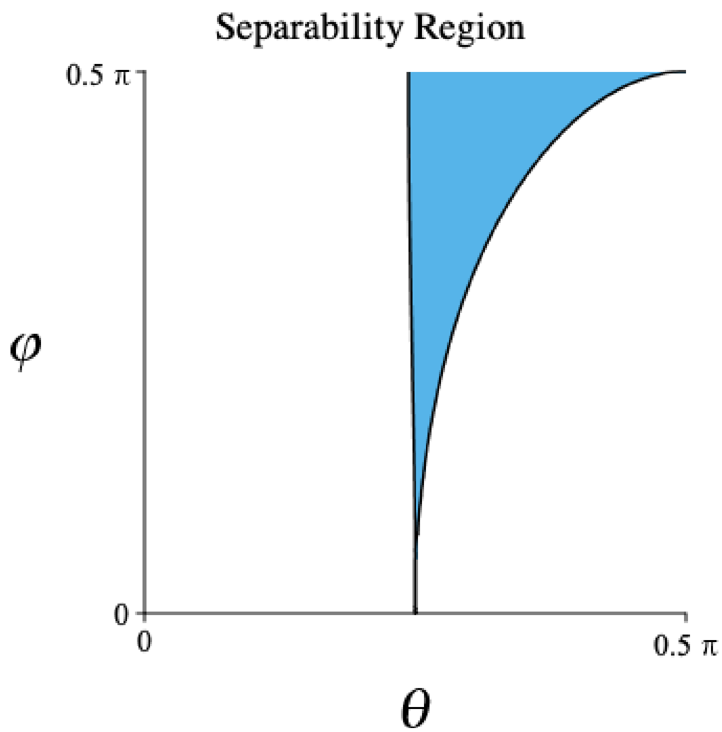


Figure 3.3: The plot of $\tilde{C}(\rho) \geq 0$ region with the help of Maple. A pair of qubits is separable if (θ, φ) are in the blue-highlighted domain. Both axes go from 0 to $\pi/2$. Note that the left boundary is not a straight line, but it is indeed a curve.

We can generalize the above discussion for $2n \geq 6$ qubits. More precisely, we choose two rings at the latitudes $\theta, \pi - \theta$ but twisted relative to each other by the angle φ where the two sets of points are on the same longitude if $\varphi = 2\pi k/n$ for $k \in \mathbb{Z}$. Each point on the Bloch sphere corresponds to one of these states

$$|\phi_k^{(1)}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{\frac{2\pi i k}{n}} \sin\frac{\theta}{2}|1\rangle\,, \quad |\phi_k^{(2)}\rangle = \sin\frac{\theta}{2}|0\rangle - e^{\frac{2\pi i k}{n}} e^{i\varphi}\cos\frac{\theta}{2}|1\rangle \tag{3.58}$$

By multiplying and symmetrizing these qubits, the spin-n state (3.14) becomes

$$|\Phi_{2n}\rangle_O \propto \binom{2n}{n} \cos^n \frac{\theta}{2} \sin^n \frac{\theta}{2} |0\rangle^{2n} + \binom{2n}{n} e^{in\varphi} \cos^n \frac{\theta}{2} \sin^n \frac{\theta}{2} |1\rangle^{2n}$$
$$- (-1)^n \left(\sin^{2n} \frac{\theta}{2} + e^{in\varphi} \cos^{2n} \frac{\theta}{2} \right) |S_{2n,n}\rangle , \tag{3.59}$$
$$\tag{3.60}$$

In the basis $\{|00\rangle , |11\rangle , |B_+\rangle\}$, however, the reduced density matrix is diagonal for $n \geq 3$, unlike the $n = 2$ case,

$$\rho = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{pmatrix}, \tag{3.61}$$

where

$$P = \left(\binom{2n}{n} \cos^n \frac{\theta}{2} \sin^n \frac{\theta}{2} \right)^2 + \binom{2n-2}{n} \left| \sin^{2n} \frac{\theta}{2} + e^{in\varphi} \cos^{2n} \frac{\theta}{2} \right|^2, \tag{3.62}$$
$$Q = 2\binom{2n-2}{n-1} \left| \sin^{2n} \frac{\theta}{2} + e^{in\varphi} \cos^{2n} \frac{\theta}{2} \right|^2, \tag{3.63}$$
$$\left| \sin^{2n} \frac{\theta}{2} + e^{in\varphi} \cos^{2n} \frac{\theta}{2} \right|^2 = \sin^{4n} \frac{\theta}{2} + \cos^{4n} \frac{\theta}{2} + 2 \sin^{2n} \frac{\theta}{2} \cos^{2n} \frac{\theta}{2} \cos n\varphi. \tag{3.64}$$

Then, we can explicitly show $\lambda_i \in \{P, P, Q, 0\}$. Theorem 3.1.3 then implies that a pair of qubits is separable if

$$Q - 2P \leq 0, \tag{3.65}$$

or equivalently,

$$2(2n-1)\binom{2n}{n} - \tan^{2n} \frac{\theta}{2} - (\tan^{2n} \frac{\theta}{2})^{-1} \geq 2 \cos n\varphi. \tag{3.66}$$

For example, for $n = 3$, the separable region is given in Figure 3.4. For $n \geq 4$, the graph would look essentially the same except the left boundary gradually moves towards the right, and reaches $\theta = 2\tan^{-1}(1/2)$ in the limit $n \to \infty$. One should notice this is totally different from the behaviour for the $n = 2$ case.

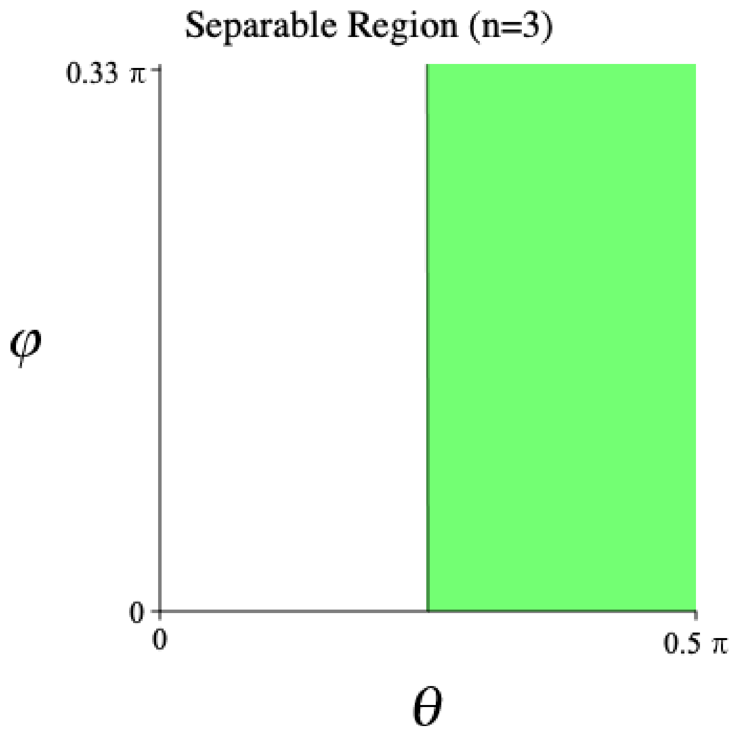


Figure 3.4: The plot of the separable region for $n = 3$. A pair of qubits is separable if (θ, φ) are in the green-highlighted domain. The θ-axis goes from 0 to $\pi/2$ and the φ-axis is from 0 to $\pi/3$. The left boundary is not a straight line, though it becomes fully straightened in the limit $n \to \infty$ at $\theta = 2\tan^{-1}(1/2)$.

3.2.5 Fubini-Study Metric

At last, it is geometrically interesting to compute the area where the corresponding pair of qubits becomes separable in the above setting. Here, what we mean by the area is not the one on the planar graph in Figure 3.3. A natural choice of the metric on the full space of quantum states with $\dim H = 2n + 1$ is a so-called *Fubini-Study metric*, which is a Kähler metric on projective spaces $\mathbf{P}^{2n}$. See [37, 38] for more discussions about relations between quantum states and the Fubini-Study metric.

Let us define a complex variable.

$$z = e^{i\varphi} \tan \frac{\theta}{2}. \tag{3.67}$$

Then, we rewrite the spin-$n \geq 2$ state (3.60) in the $(z, \bar{z})$-coordinates as

$$\begin{aligned}|\Phi_{2n}(z,\bar{z})\rangle_O =& \frac{1}{\sqrt{N_n(z)}}\left(\binom{2n}{n}^{\frac{1}{2}}(z\bar{z})^{\frac{n}{2}}|0\rangle^{2n} + \binom{2n}{n}^{\frac{1}{2}} z^n |1\rangle^{2n}\right.\\ &\left. - (-1)^n\left((z\bar{z})^n + \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}\right)\binom{2n}{n}^{-\frac{1}{2}}|S_{2n,n}\rangle\right), \end{aligned} \tag{3.68}$$

$$N_n(z,\bar{z}) = \left((z\bar{z})^{2n} + 2\binom{2n}{n}(z\bar{z})^n + (z\bar{z})^{\frac{n}{2}}(z^n + \bar{z}^n) + 1\right) \tag{3.69}$$

Note that we have given the correct normalization because it is necessary to obtain an appropriate metric.

The metric we are looking for is not the Fubini-Study metric itself, but the induced one onto the two-dimensional surface describing states (3.69) embedded in $\mathbf{P}^{2n}$ with the Fubini-Study metric. One way of computing such an induced metric is by identifying ds_n^2 with

$$\left|\langle\Phi_{2n}(z,\bar{z})|\Phi_{2n}(z+dz,\bar{z}+d\bar{z})\rangle_O\right|^2 = \cos(ds_n^2) = 1 - \frac{ds_n^2}{2}, \tag{3.70}$$

where we truncated the terms higher than order ds_n^2 by assuming ds_n^2 is infinitesimally small. The computation becomes very long, yet the consequence is summarized in the following relatively simple form:

$$\begin{aligned} ds_n^2 =& \binom{2n}{n}\frac{n^2(z\bar{z})^n}{8z^2N_n(z,\bar{z})^2}\left((z\bar{z})^{2n} + 1 - \binom{2n}{n}(z\bar{z})^n - 2(z\bar{z})^{\frac{n}{2}}(z^n+\bar{z}^n)\right)dzdz\\ &+ \binom{2n}{n}\frac{n^2(z\bar{z})^n}{8\bar{z}^2N_n(z,\bar{z})^2}\left((z\bar{z})^{2n} + 1 - \binom{2n}{n}(z\bar{z})^n - 2(z\bar{z})^{\frac{n}{2}}(z^n+\bar{z}^n)\right)d\bar{z}d\bar{z}\\ &+ \binom{2n}{n}\frac{n^2(z\bar{z})^n}{8z\bar{z}N_n(z,\bar{z})^2}\left(3(z\bar{z})^{2n} + 3 + \binom{2n}{n}(z\bar{z})^n - 2(z\bar{z})^{\frac{n}{2}}(z^n+\bar{z}^n)\right)(dzd\bar{z} + d\bar{z}dz) \end{aligned} \tag{3.71}$$

Then, the determinant $\sqrt{g_n}$ is finally computed as

$$g_n = -\binom{2n}{n}^2\frac{n^4(z\bar{z})^{2n-2}}{8N_n(z,\bar{z})^4}\left(1 + \binom{2n}{n}(z\bar{z})^n + (z\bar{z})^{2n}\right)\cdot\left(1 + (z\bar{z})^{2n} - (z\bar{z})^{\frac{n}{2}}(z^n+\bar{z}^n)\right). \tag{3.72}$$

The area A_n of the separable region of the spin-n state (3.69) is then given by using the induced metric (3.72) as

$$A_n = \int_{\text{separable}} \sqrt{g_n} dz d\bar{z}. \tag{3.73}$$

Note that the separability condition for $n \geq 3$ in the $(z, \bar{z})$-coordinates is

$$2(2n-1)\binom{2n}{n}(z\bar{z})^n - (z\bar{z})^{2n} - 1 - (z\bar{z})^{\frac{n}{2}}(z^n + \bar{z}^n) \geq 0 \tag{3.74}$$

3.2.6 Large n Limit

It is difficult to integrate exactly the area A_n (3.73) for general n, however, we can obtain an approximate result in the limit $n \gg 1$. To do so, let us first transform the complex variables $(z, \bar{z})$ to a set of real variables (L, φ) where φ is the same angle that appeared in (3.67) and L is given by

$$L = (z\bar{z})^n = \tan^{2n}\frac{\theta}{2}. \tag{3.75}$$

Note that $0 \leq \theta \leq \pi/2$, thus we have $0 \leq L \leq 1$. In these coordinates (L, ϕ), $\sqrt{g_n}$ is written as

$$\sqrt{g_n(L,\varphi)} = \binom{2n}{n}\frac{n^2}{2N_n(L,\varphi)^2}\sqrt{2(L^2 - 2L\cos n\varphi + 1)\left(L^2 + \binom{2n}{n}L + 1\right)}, \tag{3.76}$$

where

$$N_n(L,\theta) = L^2 + 2\binom{2n}{n}L + 2L\cos n\varphi + 1. \tag{3.77}$$

The separability region (3.74) reads

$$L^2 + 2L\cos n\varphi - 2(2n-1)\binom{2n}{n}L + 1 \leq 0. \tag{3.78}$$

Since $\sqrt{g_n(L,\varphi)}$ is regular everywhere in the region $0 \leq L \leq 1, 0 \leq \varphi \leq \pi/n$, we can proceed a simple approximation.

First of all, it does not make much sense to compute the area itself because one can alway scale it by an overall constant normalization. Therefore, what we should compute is the ratio

Q_n between the area of the separable region and that of the entangled region:

$$Q_n = \frac{\int_{\text{separable}} \sqrt{g_n(L,\varphi)}dLd\varphi}{\int_{\text{entangled}} \sqrt{g_n(L,\varphi)}dLd\varphi}. \tag{3.79}$$

In particular, we ignore the constant normalization in (3.76).

In the large n limit, the first two terms in the separability condition (3.78) are significantly smaller than the third term because the binomial coefficient becomes very large. Thus, we can approximate (3.78) by

$$-2(2n-1)\binom{2n}{n}L + 1 \leq 0. \tag{3.80}$$

In particular, the boundary between the separable and entangled region is determined approximately by the straight line

$$L = \frac{1}{2(2n-1)\binom{2n}{n}}. \tag{3.81}$$

By the same reason, we can approximate $N_n(L,\varphi)$ as

$$N_n(L,\theta) = 1 + 2\binom{2n}{n}L, \tag{3.82}$$

and the second bracket inside the square root in (3.76) as

$$1 + \binom{2n}{n}L. \tag{3.83}$$

In order to evaluate the first bracket in the square root in (3.76), we use the following inequalities

$$1 + L^2 + 2L \geq 1 + L^2 - 2L\cos n\varphi \geq 1 + L^2 - 2L. \tag{3.84}$$

It follows that we have

$$\mathcal{A}_+ \geq \int \sqrt{g_n(L,\varphi)}dLd\varphi \geq \mathcal{A}_- \tag{3.85}$$

$$\mathcal{A}_\pm = \int dLd\varphi \frac{\sqrt{(1 \pm L)^2\left(1 + \binom{2n}{n}L\right)}}{\left(1 + 2\binom{2n}{n}L\right)^2}, \tag{3.86}$$

where the domain of the integrals are the same for all three and we have already got rid of the constant normalization as we eventually compute the ratio. We then notice that in the higher and lower bound, the integrands are independent of the angle φ, hence the integral over φ is also irrelevant.

In the region $1 \leq L \leq 1$[6], we can explicitly compute the integrals in (3.86) with respect to L. More precisely, they become

$$\mathcal{A}_{\pm}(L,n) = -a\left(\frac{2\pm a}{8\sqrt{2}}\log\frac{\sqrt{2(x+1)}+1}{\sqrt{2(x+1)}-1} + \frac{(2\mp a)\sqrt{x+1}}{4(2x+1)} - \frac{a\sqrt{x+1}}{4}\right) + C, \tag{3.87}$$

$$x = \binom{2n}{n}L, \quad a = \binom{2n}{n}^{-1}, \tag{3.88}$$

where C is some constant. The entangled region is given by

$$\mathcal{A}_{\pm}\left(\frac{a}{2(2n+1)}, n\right) - \mathcal{A}_{\pm}(0,n). \tag{3.89}$$

However, since the value of L given by (3.81) at the boundary of the entangled region is small for large n, we can further approximate it as

$$\begin{aligned}\mathcal{A}_{\pm}\left(\frac{a}{2(2n+1)}, n\right) - \mathcal{A}_{\pm}(0,n) &= \left.\frac{d\mathcal{A}_{\pm}(L,n)}{dL}\right|_{L=0} \cdot \frac{a}{2(2n+1)} + \cdots \\ &= a\left(\frac{1}{2(2n+1)} + \mathcal{O}(n^{-2}).\right)\end{aligned} \tag{3.90}$$

Note that the subleading terms are of order n^{-2}, not $\mathcal{O}(a)$. On the other hand, the total area, namely the area of the entangled region plus the separable region, is simply

$$\mathcal{A}_{\pm}(1,n) - \mathcal{A}_{\pm}(0,n) = a\left(\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2}\right). \tag{3.91}$$

Therefore, the ratio Q_n becomes

$$Q_n = n\left(2\sqrt{2}\log(\sqrt{2}+1)+1\right) + \mathcal{O}(1). \tag{3.92}$$

[6] If L is arbitrary, the answer is almost the same but we need to put the magnitude for the argument in the logarithm.

In summary, the ratio Q_n linearly increases in n, hence, a pair of any two qubits in a typical state in this configuration tends to be separable if n is large. We should contrast this statement with the plane image in Figure 3.4 where the areas of the separable region and entangled region are comparable even in the large n limit.

3.2.7 Useful Technique for Further Investigation

At last, one might be wondering why no example above has all four eigenvalues of $\sqrt{\rho\tilde{\rho}}$ nonzero. It turns out that one of the eigenvalues is always zero for every 2-qubit density operator reduced from a spin-$n/2$ pure state.

Lemma 3.2.5. *Let $|\Phi_n\rangle$ be a spin-$n/2$ pure state defined by* (3.14)*, and ρ be the density operator for two qubits reduced from $|\Phi_n\rangle\langle\Phi_n|$. Then, ρ has at most rank-3.*

Proof. We can compute ρ by

$$\rho \propto \sum_{S_{n-2}} \sum_{j=0}^{n-2} \langle \underbrace{00\cdots0}_{n-j-2}\underbrace{11\cdots1}_{j} |\Phi_n\rangle \langle\Phi_n| \underbrace{00\cdots0}_{n-j-2}\underbrace{11\cdots1}_{j}\rangle\,, \tag{3.93}$$

where we can assume that the sum is taken over the last $n-2$ qubits without loss of generality because $|\Phi_n\rangle$ is symmetric among all qubits. If we expand (3.93) in the Dicke states, we have

$$\begin{aligned}\rho \propto \sum_{S_{n-2}} \sum_{S_n} \sum_{S_n} \sum_{j=0}^{n-2} \sum_{k,k'=0}^{n} \frac{c_k \bar{c}_{k'}}{(n-k)!(n-k')!k!k'!} \langle \underbrace{00\cdots0}_{n-j-2}\underbrace{11\cdots1}_{j} | \underbrace{00\cdots0}_{n-k}\underbrace{11\cdots1}_{k}\rangle \\ \times \langle \underbrace{00\cdots0}_{n-k'}\underbrace{11\cdots1}_{k'} | \underbrace{00\cdots0}_{n-j-2}\underbrace{11\cdots1}_{j}\rangle\,,\end{aligned} \tag{3.94}$$

where $c_k, c_{k'}$ are defined by (3.15). Let us focus on the following factor

$$\langle \underbrace{00\cdots0}_{n-k'}\underbrace{11\cdots1}_{k'} | \underbrace{00\cdots0}_{n-j-2}\underbrace{11\cdots1}_{j}\rangle\,. \tag{3.95}$$

This vanishes unless k' is one of $\{j, j-1, j-2\}$. The resulting factors are respectively

$$\langle 00|\,,\quad \langle 01| + \langle 10|\,,\quad \langle 11|\,, \tag{3.96}$$

This implies that if we represent ρ in the basis $\{|00\rangle, |11\rangle, |B_\pm\rangle\}$, ρ never has nonzero components with $|B_-\rangle$ basis. This proves the lemma. $\square$

Corollary 3.2.6. *One of the eigenvalues of $\sqrt{\rho\tilde{\rho}}$ is always zero. Furthermore, let $\lambda_1, \lambda_2, \lambda_3$ be nontrivially nonzero eigenvalues of $\sqrt{\rho\tilde{\rho}}$. Then, an arbitrary pair of two qubits is separable if the following condition is satisfied:*

$$\begin{aligned}\tilde{C}(\rho) &= (-\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3) \\ &= \Big(Tr(\rho\tilde{\rho})\Big)^2 - 2\,Tr\Big((\rho\tilde{\rho})^2\Big) \geq 0. \end{aligned} \tag{3.97}$$

Proof. The operation $\sigma_y \otimes \sigma_y$ do not mix terms between $|B_-\rangle$ and the other three basis. Indeed,

$$\sigma_y \otimes \sigma_y |B_-\rangle = |B_-\rangle . \tag{3.98}$$

Thus, $\tilde{\rho}$ is also a rank-3 operator so is $\rho\tilde{\rho}$. This proves the corollary. $\square$

Although we did not use the criteria (3.97) in the examples discussed above, this would help us with investigating more complicated examples in the future.

4 The Black Hole Information Paradox

Now we move on to the black hole information paradox and nonlocality of quantum gravity.

Ever since Hawking [7] discovered that quantum effects make it possible for black holes to emit thermal radiation, there have been many discussions whether black hole evaporation is unitary. Although Hawking [8] originally proposed a breakdown of unitarity, several significant pieces of evidence in string theory, particularly the AdS/CFT duality [9], have motivated us to probe the possibility of unitary black hole evaporation. Recently, however, Almheiri *et al* [12] (see also [13]) pointed out another challenge of unitarity from quantum entanglement perspectives, known as the *firewall paradox*. The paradox requires a modification of one of three seemingly fundamental assumptions in physics, namely unitarity, locality and the equivalence principle. Our goal in this section is to explain what the black hole information paradox and the firewall paradox are. We will go through how they are introduced and give perspectives from multiple points of view in order to accurately capture what the issues are.

The first step to understand the black hole information paradox must be to learn what classical black holes are. Let us first review classical aspects of black holes, and then study their quantum aspects and finally introduce the black hole information paradox. Discussions below are mainly based on [39–45].

4.1 Classical Aspects of Black Holes

The most well-known type of black hole is probably a black hole described by the Schwarzschild metric in four dimensions,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{4.1}$$

Here, M is the mass of the black hole and $d\Omega_2^2$ is the metric of of a two-dimensional sphere. The Schwarzschild metric is asymptotically flat in the limit $r \to \infty$, the event horizon is located at $r = 2GM$ which is simply a coordinate singularity, and a spacetime singularity is inside the event horizon at $r = 0$. The role of the event horizon becomes clear in the Penrose diagram in the Kruskal-Szekeres coordinates.

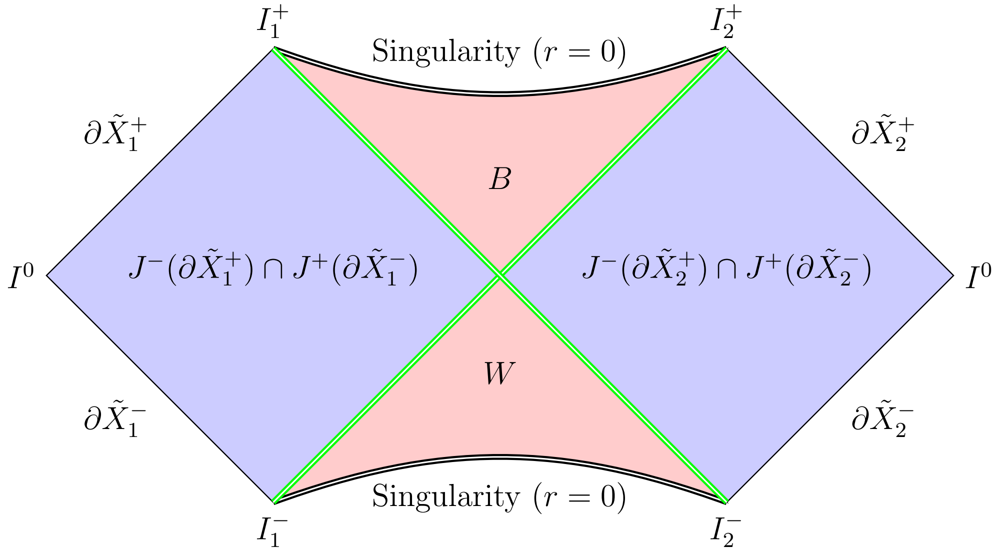


Figure 4.1: Penrose diagram of the Schwarzschild geometry. $\tilde{X}$ is a conformal compactification of the Schwarzschild geometry. The green-highlighted lines denote the event horizon. I^0 denote spatial infinity, $I^{\pm}$ represent future/past timelike infinity, and $\partial X^{\pm}$ are future/past null boundaries. It is clear from this diagram that no causal curve can escape from the black hole region B to reach either future null infinity.

It is probably pedagogical to start with simple black holes such as the Schwarzschild black holes or the Kerr black holes so that we can grasp fundamental properties and characteristics of black holes. However, these are very well explained in almost all standard textbooks on general relativity. Therefore, we rather review how to generally define black holes from a purely geometric perspective. This view helps with understanding that the definition of the event horizon is essentially nonlocal, but rather it is determined by the entire structure of the corresponding spacetime. We closely follow [43] for most of definitions in Section 4.1.

4.1.1 Asymptotically Flat Spaces

We first formally define concepts of conformal transformations and asymptotic spaces. Note that in this section, we call a pseudo-Riemannian manifold X, with metric g that locally looks like Minkowski space $\mathbb{R}^{1,d}$, by a *spacetime* (X, g)[1].

Definition 4.1.1. Two spacetimes (X, g) and $(\tilde{X}, \tilde{g})$ are said to be *conformally equivalent*

[1] M is also commonly used to denote a manifold, but we rather choose X to avoid a confusion because M has been already used for the mass of the black hole in (4.1)

if there exists a C^∞-map $\rho : X \to \tilde{X}$ such that the induced metric $\rho(g)$ on $\tilde{X}$ satisfies $\rho(g) = \Omega(p)^{-2}\tilde{g}$ where $p \in \tilde{X}$ and $\Omega(p)$ is a nonnegative smooth function on $\tilde{X}$.

Definition 4.1.2. Given two conformally equivalent spacetimes $(X, g), (\tilde{X}, \tilde{g})$ and the conformal map ρ where $\rho(g) = \Omega(p)^{-2}\tilde{g}$ and $p \in \tilde{X}$, $\tilde{X}$ is called a *conformal compactification* of X if $\tilde{X}$ is a compact spacetime with boundary $\partial\tilde{X}$, and if $\Omega(p)$ satisfies $\Omega|_{\partial\tilde{X}} = 0$, $\partial_\mu\Omega|_{\partial\tilde{X}} \neq 0$.

For example, a conformal compactification of the Minkowski spacetime in two dimensions is given by

$$U^\pm = \tan^{-1}(t \pm x), \quad -\frac{\pi}{2} \leq U^\pm \leq \frac{\pi}{2}, \tag{4.2}$$

$$\rho(ds^2 = -dt^2 + dx^2) = -\frac{1}{\cos^2 U^+ \cos^2 U^-} dU^+ dU^-, \tag{4.3}$$

where (t, x) are global coordinates of the Minkowski spacetime, while $U^\pm$ are null coordinates of the compactified spacetime. See Figure 4.2 for a pictorial understanding. It is worth mentioning that every conformal compactification $\rho : X \to \tilde{X}$ is a bijective map except on the boundary $\partial\tilde{X}$.

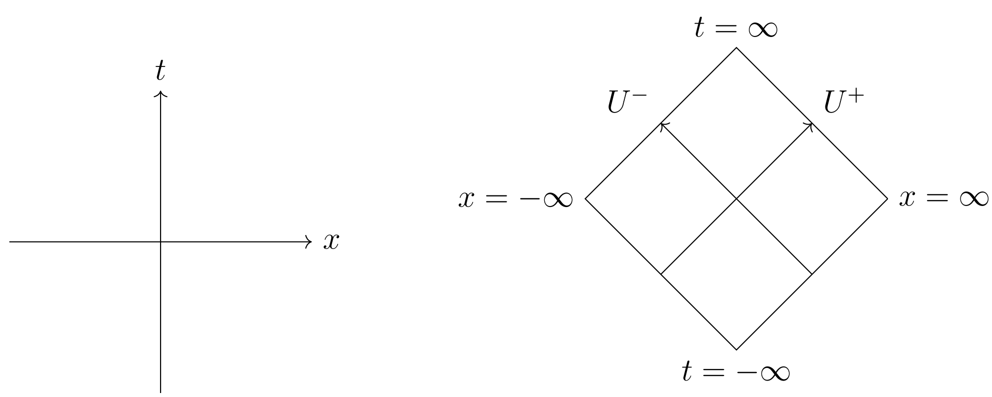


Figure 4.2: A conformal compactification of the Minkowski spacetime in two dimensions. The left diagram denotes the Minkowski spacetime in the glocal coordinates (t, x) whereas the right one is in null coordinates of the compactified spacetime as defined in (4.2).

Geometries of our interest are those that reduce back to the Minkowski spacetime in some appropriate limit, so-called asymptotically flat spacetimes. Black hole geometries are examples of those. To formally define such spaces, we first need to introduce asymptotically simple spacetimes.

Definition 4.1.3. A spacetime (X, g) is said to be asymptotically simple if $(\tilde{X}, \tilde{g})$ is a conformal compactification of (X, g) such that each null geodesic in $\tilde{X}$ begins and ends at $\partial\tilde{X}$.

Black hole geometries are not fully asymptotically simple because of the existence of the event horizon and the singularity. Some geodesics begin and end at $\partial\tilde{X}$ whereas others may hit or come out of the singularity. Such spacetimes are indeed called weakly asymptotically simple.

Definition 4.1.4. A spacetime (X, g) is *weakly asymptotically simple* if its conformal compactification $(\tilde{X}, \tilde{g})$ possesses a subset $\tilde{U} \subset \tilde{X}$ isometric to the neighbourhood of the boundary of $\tilde{Y}$ which is the conformal compactification of some asymptotically simple spacetime Y.

In other words, X is asymptotically simple in the region $\rho^{-1}(\tilde{U})$, and it is not elsewhere. Asymptotically flat spacetimes are asymptotically simple ones with one more condition.

Definition 4.1.5. A weakly asymptotically simple spacetime (X, g) is called *asymptotically flat* if the metric g in the vicinity of the pullback $\partial\tilde{X}$ satisfies the Einstein's equations with an energy-momentum tensor that decreases sufficiently fast.

The definitions given above are needed for, roughly speaking, the regions far from the black hole. One can similarly define asymptotically de Sitter spacetimes and anti-de Sitter spacetimes respectively by adding the term of the positive or negative cosmological constant into the vacuum Einstein equations. Next, we review causality, which is the most important concept of defining black hole geometries.

4.1.2 Causal Structures and Black Holes

Let us remind terminologies of causal structures.

Definition 4.1.6. A causal curve is a curve whose tangent vector is either timelike or null at each point.

Definition 4.1.7. Let Q be a connected set of points in a spacetime (X, g). Then, the *causal future* (*causal past*) $J^+(Q)$ $(J^-(Q))$ of Q is the set of points for each of which there exists a past-directed (future-directed) causal curve that intersects Q.

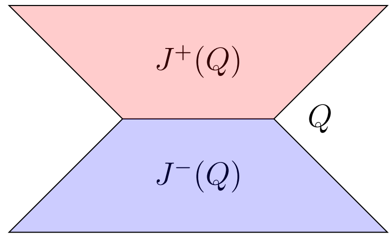


Figure 4.3: A simple visualization of the causal future and past $J^{\pm}(Q)$ of a connected region Q.

Definition 4.1.8. Let Q be a connected set of points in a spacetime (X, g). Then, the *future (past) domain of dependence* $D^+(Q)$ $(D^-(Q))$ of Q is the set of points such that for each of which every past-directed (future-directed) causal curve intersects Q.

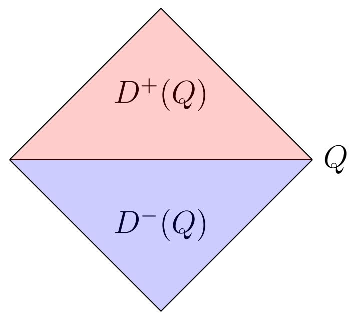


Figure 4.4: A simple visualization of the future and past domain of dependence $D^{\pm}(Q)$ of a connected region Q.

A goal of physics, or science in general, is to uniquely predict the future or retroduce the past from the current data. A mathematical structure that incorporates such a goal is called a globally hyperbolic spacetime.

Definition 4.1.9. A *Cauchy surface* is a non-timelike hypersurface that is intersected by each causal curve exactly once. A *partial Cauchy surface* is a non-timelike hypersurface that is intersected by each causal curve at most once.

Definition 4.1.10. A spacetime (X, g) is *globally hyperbolic* if it admits a Cauchy surface

Roughly speaking, a Cauchy surface makes it possible for us to set a well-defined time slice, and accordingly we can define a bijective map among the past, present and the future. Thus, globally hyperbolic spacetimes equip ideal structures for physics. At the same time, observers at future null infinity in a black hole geometry only have access to a part of messages sent at past null infinity because some will go behind the horizon and hit the singularity. This leads us to define a spacetime whose $J^-(\partial\tilde{X}^+)$ is *well-behaved* but does not include the entire spacetime.

Definition 4.1.11. Let (X, g) be an asymptotically flat spacetime and $(\tilde{X}, \tilde{g})$ be its conformal compactification. (X, g) is said to be *strongly asymptotically predictable* if there exists an open subset $\tilde{V} \subset \tilde{X}$ such that the closure of $X \cap J^-(\partial X^+)$ taken in $\tilde{X}$ is contained in $\tilde{V}$ and that $\tilde{V}$ is globally hyperbolic.

A strongly asymptotically predictable spacetime consists of two subspaces; a space where the observer at the future null infinity could see and a space invisible to the observer. The latter is none other than the black hole region.

Definition 4.1.12. A strongly asymptotically predictable spacetime (X, g) has a *black hole* if its compactification $\tilde{X}$ is not contained in $J^-(\partial\tilde{X}^+)$. The black hole region $B \subset X$ is defined by $B = \rho^{-1}(\tilde{X} \backslash J^-(\partial\tilde{X}^+))$. The boundary of B is called the *future event horizon*.

Note that *strongly asymptotically retrodictable spaces* are similarly defined by replacing $X \cap J^-(\partial X^+)$ with $X \cap J^+(\partial X^-)$. A white hole and the past event horizon are also defined accordingly.

4.1.3 Naked Singularity

How does a spacetime look like if it fails to be strongly asymptotically predictable? Notice that the condition on the closure of $X \cap J^-(\partial X^+)$ guarantees the predictability of the neighbourhood of the event horizon. In other words, the failure of strongly asymptotically predictable implies the existence of a naked singularity where we are not able to predict anything in the neighbourhood.

Definition 4.1.13. An asymptotically flat spacetime (X, g) possess a *naked singularity* if it fails to be strongly asymptotically predictable.

As of today, it is widely believed in physics that the Big-Bang singularity is the only naked singularity in our universe. This is called *strong cosmic censorship*.

4.1.4 Apparent Horizons

There is another notion of a horizon defined locally, the so-called apparent horizon. We first formally define the apparent horizon.

Definition 4.1.14. Let $\Sigma \subset X$ be a Cauchy surface of an asymptotically flat spacetime X, and $B \subset \Sigma$ be a compact spacelike submanifold of co-dimensions 2 with the induced metric γ. Suppose $\theta_\pm$ are two future-directed null vectors normal to B, then the submanifold B is called a *trapped surface* if both $\nabla_i \theta^i_\pm$ are nonpositive everywhere over the submanifold. A region T of Σ inside the trapped surface B is called a *trapped region*, and the boundary of the union of all trapped regions T on Σ is called the *apparent horizon* on Σ.

As B is a compact spacelike submanifold of co-dimension 2, there is a notion of inward- and outward-pointing, which is basically represented by spatial components of $\theta_\pm$ in the above definition. Essentially, $\nabla_i \theta^i_\pm$ tell us whether the trajectories of $\theta_\pm$ are expanding or contracting. The geometric meaning of the apparent horizon becomes the most evident when we consider the time evolution cross section of Σ and a submanifold B. More precisely, let Σ_t be a one parameter family of Cauchy surfaces, and let $B_t \subset \Sigma_t$ be a trapped surface at a given moment t. One can interpret the time evolution of B_t as future null geodesics emitted from B_t. Then, if one computes the change of the cross-sectional area of B_t in time, $\nabla_i \theta^i_\pm < 0$ everywhere implies that the area is indeed contracting in time. In contrast, for example, if one considers such a scenario for an ordinary spherically symmetric surface of co-dimension 2 in Minkowski spacetime, then $\nabla_i \theta^i_+ > 0$ and it is expanding as expected.

Note that the apparent horizon precisely corresponds to the event horizon for stationary black holes. However, their differences appear if we consider, for example, a formation of a black hole by a gravitational collapse of a shock wave whose metric is given by

$$ds^2 = -\left(1 - \frac{2GM\Theta(v - v_0)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^{d-1}, \tag{4.4}$$

where $\Theta(v - v_0)$ is the step function. The region for $v < v^0$ is the flat Minkowski spacetime, while that for $v > v_0$ corresponds to the Schwarzschild metric in the ingoing Eddington-Finkelstein coordinates.

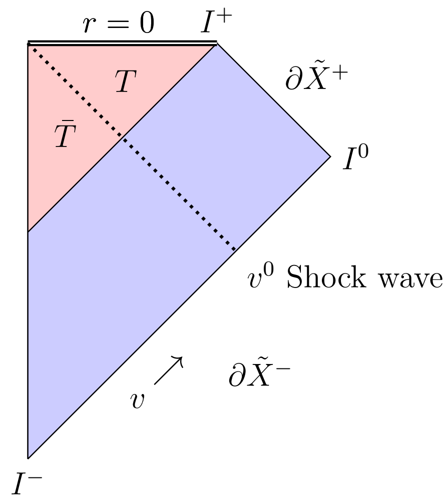


Figure 4.5: Penrose diagram for a black hole formed by spherically symmetric collapse of a null shock wave. The red-shaded region $T \cup \bar{T}$ is the black hole region. The region T is always inside the apparent horizon whereas a part of the region $\bar{T}$ can be outside of the apparent horizon which depends on the choice of a Cauchy surface.

4.2 Quantum Field Theory in a Curved Spacetime

Let us now study quantum aspects of black holes based on quantum field theory in curved spacetimes[2]. We first introduce a few elementary settings of quantum field theory in curved spacetimes, particularly those different from flat spacetimes. Then, we briefly review the Unruh effect and Hawking radiation, which leads us to the black hole information paradox.

The basic idea of quantum field theory in curved spacetimes is essentially the same as that in flat spacetimes, but we need to define inner products, and mode expansion with care. For simplicity, we consider a free massless scalar ϕ in a curved spacetime (X, g) of dimensions $d+1$

$$S = \frac{1}{2}\int_X d^{d+1}x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x). \tag{4.5}$$

The equations of motion for ϕ is simply

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi(x) = 0. \tag{4.6}$$

Let us assume that there exists the complete set of solutions of (4.6) in an open set

[2]This is different from *quantum gravity* because the background spacetime is assumed to be fixed, that is, the metric is not quantized.

$U \subset M$ parametrized by a continuous set of variables $\mathbf{k}$, and we denote the set as $\{f_{\mathbf{k}}(x)\}$. It is easy to show that the following current $j^\mu(f_{\mathbf{k}_1}, f_{\mathbf{k}_2})$ for $f_{\mathbf{k}_1}, f_{\mathbf{k}_2} \in \{f_{\mathbf{k}}\}$ is conserved with the equations of motion (4.6)

$$j^\mu(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = i\sqrt{-g}g^{\mu\nu}(\bar{f}_{\mathbf{k}_1}\partial_\nu f_{\mathbf{k}_2} - f_{\mathbf{k}_2}\partial_\nu \bar{f}_{\mathbf{k}_1}), \quad \nabla_\mu j^\mu(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = 0, \tag{4.7}$$

where $\bar{f}_{\mathbf{k}_1}$ is the complex conjugate of $f_{\mathbf{k}_1}$. Then, we define the conserved charge of this current as the inner product of $f_{\mathbf{k}_1}, f_{\mathbf{k}_2}$:

Definition 4.2.1. Let $U \subset X$ be an open set equipped with a complete set $\{f_{\mathbf{k}}\}$ of the solutions of (4.6), and $V_t \subset U$ be a partial Cauchy surface. The *inner product* of $f_{\mathbf{k}_1}, f_{\mathbf{k}_2} \in \{f_{\mathbf{k}}\}$ is defined by

$$\langle f_{\mathbf{k}_1}, f_{\mathbf{k}_2}\rangle = \int_{V_t} d^d\bar{x}\, j^0 = i\int_{V_t} d^d\bar{x}\, \sqrt{-g}g^{0\nu}(\bar{f}_{\mathbf{k}_1}\partial_\nu f_{\mathbf{k}_2} - f_{\mathbf{k}_2}\partial_\nu \bar{f}_{\mathbf{k}_1}). \tag{4.8}$$

Note that this definition ensures that the inner product is preserved in time because this is the conserved charge of the conserved current (4.7). In particular, if we choose an orthonormal basis, the inner products obey

$$\langle f_{\mathbf{k}_1}, \bar{f}_{\mathbf{k}_1}\rangle = 0, \quad \overline{\langle f_{\mathbf{k}_1}, f_{\mathbf{k}_2}\rangle} = -\langle \bar{f}_{\mathbf{k}_1}, \bar{f}_{\mathbf{k}_2}\rangle = \langle f_{\mathbf{k}_2}, f_{\mathbf{k}_1}\rangle = \delta(\mathbf{k}_1 - \mathbf{k}_2). \tag{4.9}$$

We now proceed with the canonical quantization of ϕ. For an open set $U \subset X$ with coordinates $(x^0, \vec{x})$, let $\pi(x)$ be the conjugate momentum of ϕ defined by

$$\pi(x) = \frac{\delta S}{\delta \partial_0 \phi(x)}. \tag{4.10}$$

Then, we require (ϕ, π) to obey

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{y})] = 0, \quad [\pi(x^0, \vec{x}), \pi(x^0, \vec{y})] = 0, \quad [\pi(x^0, \vec{x}), \phi(x^0, \vec{y})] = i\delta^d(\vec{x} - \vec{y}). \tag{4.11}$$

Unlike the case in flat spacetime, we cannot generally write down the explicit form of $\{f_{\mathbf{k}}(x)\}$ even if we choose the metric. However, one can still, in principle, consider an analogous mode decomposition of $\phi(x)$ into positive frequencies $f_{\mathbf{k}}(x)$ and negative frequencies $\bar{f}_{\mathbf{k}}(x)$ in the same spirit as we do in flat spacetimes.

$$\phi(x)|_U = \int d\mathbf{k}(a_{\mathbf{k}} f_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} \bar{f}_{\mathbf{k}}(x)), \tag{4.12}$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \quad [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0, \quad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}'), \tag{4.13}$$

where $d\mathbf{k}$ includes an appropriate measure[3]. For some asymptotic regions, we can approximate the form of $\{f_{\mathbf{k}}(x)\}$ and explicitly decompose the scalar field into positive and negative frequency modes.

At last, we define a vacuum state $|0\rangle_a$ in terms of the annihilation and creation operators $\{a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}\}$ by

$$a_{\mathbf{k}} |0\rangle = 0. \tag{4.14}$$

All excited states are given by acting creation operators $a^\dagger_{\mathbf{k}}$ on the vacuum state $|0\rangle_a$. The number operator $N_{\mathbf{k}}$ of each mode is given by

$$N^a_{\mathbf{k}} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}}. \tag{4.15}$$

4.2.1 Bogoliubov Transformation

All the discussions above seem essentially the same as those in flat spacetimes besides some computational difficulties due to the non-flat metric. However, the crucial difference between quantum field theory in flat spacetimes and that in curved spacetimes appears in the definition of the vacuum state (4.14). In short, the vacuum depends on the motion of observers and this is known as a consequence of the Bogoliubov transformation.

Let us consider another open set $V \subset X$ with coordinates $(\underline{x}^0, \underline{\vec{x}})$ where $U \cap V \neq \emptyset$. We need to take another complete set $\{g_{\mathbf{k}}(\underline{x})\}$ of solutions in these coordinates for the mode expansion. More precisely, $\phi(\underline{x})|_V$ is given by a set of annihilation and creation operators $A_{\mathbf{k}}, A^\dagger_{\mathbf{k}}$ as

$$\phi(\underline{x})|_V = \int d\mathbf{k}(A_{\mathbf{k}} g_{\mathbf{k}}(\underline{x}) + A^\dagger_{\mathbf{k}} \bar{g}_{\mathbf{k}}(\underline{x})), \tag{4.16}$$

[3]For example for Minkowski spacetimes, $d\mathbf{k} = d^d k / k^{1/2}$

where $A_{\mathbf{k}}, A^{\dagger}_{\mathbf{k}}$ should satisfy exactly the same commutator relations

$$[A_{\mathbf{k}}, A_{\mathbf{k}'}] = 0, \quad [A^{\dagger}_{\mathbf{k}}, A^{\dagger}_{\mathbf{k}'}] = 0, \quad [A_{\mathbf{k}}, A^{\dagger}_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}'). \tag{4.17}$$

Accordingly, we define the vacuum state $|0\rangle_A$ in terms of the modes $A_{\mathbf{k}}, A^{\dagger}_{\mathbf{k}}$ as

$$A_{\mathbf{k}} |0\rangle_A = 0, \tag{4.18}$$

and the number operator $N^A_{\mathbf{k}}$ in terms of these modes is

$$N^A_{\mathbf{k}} = A^{\dagger}_{\mathbf{k}} A_{\mathbf{k}}. \tag{4.19}$$

In the region $U \cap V$, a set of solution $\{g_{\mathbf{k}}(x)\}$ can be given by linear combinations of $\{f_{\mathbf{k}}(x)\}$ as both of them form complete sets. Let us define such linear combinations as

$$g_{\mathbf{k}'} = \int d\mathbf{k}(C_{\mathbf{k}'\mathbf{k}} f_{\mathbf{k}} + D_{\mathbf{k}'\mathbf{k}} \bar{f}_{\mathbf{k}}), \quad \bar{g}_{\mathbf{k}'} = \int d\mathbf{k}(\bar{C}_{\mathbf{k}'\mathbf{k}} \bar{f}_{\mathbf{k}} + \bar{D}_{\mathbf{k}'\mathbf{k}} f_{\mathbf{k}}), \tag{4.20}$$

where the orthonormality of $\{f_{\mathbf{k}}(x)\}$ and $\{g_{\mathbf{k}}(x)\}$ requires $C_{\mathbf{k}'\mathbf{k}}, D_{\mathbf{k}'\mathbf{k}} \in \mathbb{C}$ to satisfy

$$\int d\mathbf{l}(C_{\mathbf{k}\mathbf{l}} \bar{C}_{\mathbf{k}'\mathbf{l}} - D_{\mathbf{k}\mathbf{l}} \bar{D}_{\mathbf{k}'\mathbf{l}}) = \delta(\mathbf{k} - \mathbf{k}'), \tag{4.21}$$

$$\int d\mathbf{l}(C_{\mathbf{k}\mathbf{l}} D_{\mathbf{k}'\mathbf{l}} - C_{\mathbf{k}'\mathbf{l}} D_{\mathbf{k}\mathbf{l}}) = 0. \tag{4.22}$$

Conversely, the modes $a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}}$ and $A_{\mathbf{k}}, A^{\dagger}_{\mathbf{k}}$ are related as

$$a_{\mathbf{k}} = \int d\mathbf{k}'(\bar{C}_{\mathbf{k}'\mathbf{k}} A_{\mathbf{k}'} + D_{\mathbf{k}'\mathbf{k}} A^{\dagger}_{\mathbf{k}'}), \quad A_{\mathbf{k}'} = \int d\mathbf{k}(C_{\mathbf{k}'\mathbf{k}} a_{\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} a^{\dagger}_{\mathbf{k}}). \tag{4.23}$$

Linear transformations that preserve commutator relations are called *Bogoliubov transformations*.

To see whether the vacuum state for an observer in U agrees with one in V, we need to check the following quantity

$$\langle 0|_a N^A_{\mathbf{k}} |0\rangle_a = \int d\mathbf{k}' |D_{\mathbf{k}'\mathbf{k}}|^2. \tag{4.24}$$

If this is zero, the two vacuum states $|0\rangle_a$ and $|0\rangle_A$ are equivalent. If not, they are different. Notably, the vacuum for an observer in U is interpreted as an excited state in terms of an observer in V. In other words, it is generally impossible to define a universal vacuum state[4].

4.2.2 Lorentz Transformation

First of all, let us evaluate (4.24) for two observers in Minkowski spacetimes in arbitrary dimensions where they are related by a Lorentz transformation. This is the case that we normally consider in quantum field theory in flat spacetimes. One way to do so is to explicitly compute the coefficients $C_{\mathbf{k}'\mathbf{k}}, D_{\mathbf{k}'\mathbf{k}}$ to show $D_{\mathbf{k}'\mathbf{k}} = 0$. However, this is actually straightforward without computation. The mode expansion (4.12) or (4.16) is based on the decomposition of the scalar field into positive and negative frequency solution

$$e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})}, \quad e^{i(\omega_k x^0 + \vec{k}\cdot\vec{x})} \tag{4.25}$$

where $\omega_k \geq 0$. Then, we know that every Lorentz transformation preserves the sign of the frequencies $\omega_k \geq 0$. As a consequence, creation and annihilation operators would not be mixed by any Lorentz transformation either; hence, the vacuum state is uniquely determined for all Lorentz observers.

4.2.3 Unruh Effect

The Unruh effect [46, 47] shows that a uniformally accelerating observer feels some temperature even though a free moving observer does not. Strictly speaking, the Unruh effect is not a quantum effect in a curved geometry, but it indicates an ambiguity of the definition of the vacuum state even between two coordinates in Minkowski spacetime.

The Unruh effect computes (4.24) between the global coordinates $(x^0, x^1, x^2, \cdots)$ and the Rindler coordinates $(\psi, \rho, x^2, \cdots)$ where they are related by

$$x^0 = \rho \sinh\psi, \quad x^1 = \pm\rho\cosh\psi, \tag{4.26}$$

[4]If there is an everywhere-timelike Killing vector field, one can still have a universal vacuum by choosing that direction to be the time coordinate.

$$ds^2 = -\rho^2 d\psi^2 + d\rho^2 + (dx^2)^2 + \cdots \tag{4.27}$$

where $\rho > 0$, and $\rho^{-1} > 0$ is the magnitude of the proper acceleration measured by an observer at fixed ρ.

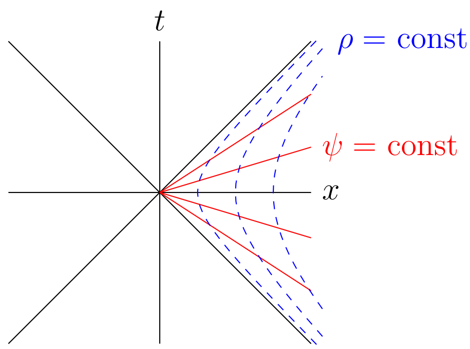


Figure 4.6: The Rindler coordinates in two dimensions. The red lines denote constant τ and the blue ones denote constant ρ.

What we need to do is to write down the equations of motion (4.6) in both coordinates, to obtain a complete set of solutions in respective coordinates, and to find linear coefficients $C_{\mathbf{k'k}}, D_{\mathbf{k'k}}$. We leave the detail computation to [46, 47], and instead simply give the result

$$\langle 0|_M \, N^R_{\mathbf{k}} \, |0\rangle_M \sim \frac{1}{e^{2\pi\omega_k/a} - 1}, \tag{4.28}$$

where $|0\rangle_M$ is the vacuum state defined in the global cooridnates and ω_k is the frequency observed in the Rindler coordinates. This is precisely the same form as the black body radiation of temperature

$$T_U = \frac{a}{2\pi}. \tag{4.29}$$

Therefore, this suggests that the Rindler observer would feel thermal temperature even though the Minkowski observer sees none[5].

Let us emphasize again that the derivation of the Unruh effect has nothing to do with quantum field theory in curved spacetimes. However, the fact that the Schwarzschild metric in the vicinity of the horizon is approximately flat indicates a close link between the Unruh effect and Hawking radiation. Indeed, the Unruh temperature and the Hawking temperature

[5]Strictly speaking, one needs to show that the Green functions have an imaginary time periodicity to conclude that this is a thermal effect.

coincide if we choose the acceleration a to be the surface gravity of the horizon of a black hole.

4.2.4 Hawking Radiation

Hawking [7] considered (4.24) in the scenario of the gravitational collapse of matter. More precisely, he compared a set of solutions at past null infinity ∂X^- before the formation of the black hole with another set of solution at ∂X^+ after the formation. The Penrose diagram of a matter collapse is similar to Figure 4.5. Hawking's original argument took account of the gravitational blue shift as well as gravitational scattering, and analyzed how these effects would contribute to (4.24). However, he ignored interaction between the matter and radiation for simplicity. We leave detailed discussions to [7], or Section 4 in [42], and simply summarize the result here:

$$\langle 0|_- N_\omega^+ |0\rangle_- \sim \frac{1}{e^{8\pi GM\pi\omega} - 1}, \tag{4.30}$$

where $|0\rangle_-$ is the vacuum at the past null infinity and N_ω^+ is the number operator of the frequency ω for the observer at the future null infinity. Therefore, the observer at future null infinity would interpret this equation as if there would be radiation of temperature

$$T_H = \frac{1}{8\pi GM}. \tag{4.31}$$

Accordingly, the Bekenstein-Hawking entropy is determined as

$$S_{\mathrm{BH}} = \frac{A}{4l_p^2}, \tag{4.32}$$

where l_p is the Planck length defined by $l_p^2 = G_N$.

Note that Bekenstein [50] had conjectured that the black hole entropy would be proportional to the area of the horizon based on black hole thermodynamics, but he could not figure out the right coefficient. Then, Hawking successfully determined the coefficient of the black hole temperature, which induced the coefficient of the entropy. At the same time, Hawking's argument is not sufficient enough to show that this entropy has a statistical mechanical meaning, namely, the number of states of a black hole. The first[6] statistical mechanical evidence was shown by Strominger [48] in the context of the AdS/CFT conjecture [9] proposed by Maldacena. More precisely, Strominger compared the BTZ black hole entropy with the statistical mechanical entropy of the corresponding CFT in two dimensions.

Remark 4.2.2. There are criticisms that the Hawking original argument is unphysical because the interaction between the blue-shifted modes and the collapsing matter would be super-Planckian. This point itself is relevant. However, the formula is still widely believed to be true because the foundation of Hawking radiation is the horizon structure, and the derivations from other perspectives have agreed with his original result (4.31). For example, see Section 3 in [44] where Polchinski compared the observer at infinity with an infalling observer at the Schwarzschild geometry in two dimensions and showed the same result.

4.3 The Black Hole Information Paradox

Let us now explain what the black hole information paradox is. We start with a sufficiently large black hole of radius $R \gg l_p$ in four dimensions where l_p is the Planck length. Then, the Hawking temperature formula (4.31) implies that a typical Hawking quantum has energy e_H of order $e_H \sim T_H \sim 1/R$. As the energy of the black hole is of order M, the number of quanta stored in the black hole is approximately $M/e_H \sim R^2/l_p^2$ which agrees with the Bekenstein-Hawking entropy formula. Also, the time for such a quantum to come out from the black hole is of order $t \sim R$ due to the time-energy uncertainty principle.

Since the time scale for Hawking radiation is very slow, $t \sim R$, for a sufficiently large black hole, we can apply the Hawking argument to a dynamical gravitational system of slow evolution. More precisely, suppose a black hole is created due to a gravitational collapse, then Hawking radiation takes the energy out of the black hole little by little and makes it possible to get the black hole to shrink to a smaller size. This scenario seems relevant at least until when the black hole becomes as small as the Planck length where the Hawking analysis would require some modifications. However, let us assume the black hole keeps emitting radiation and completely evaporates away at the end. Would it give some issues?

[6] To the author's best knowledge.

It is important to notice that derivations of Hawking radiation, not necessarily the original one, are given by considering only outside the horizon. Accordingly, the no-hair theorem strongly suggests that the characteristics of radiation are completely independent of the structure of the collapsing matter. This is problematic. Suppose an observer who has stayed outside the horizon during the entire evolution collects all the radiation emitted by the black hole. Then, since the radiation carries no information about the collapsing body, s/he will not be able to reconstruct quantum information about the collapsing body. The information seems lost.

4.3.1 The Unitarity Crisis

One can rephrase the issue in a more quantitative manner from a quantum information theory perspective. Suppose the initial collapsing body is described by a pure state $|\mathrm{C}\rangle$. It is a fundamental assumption in physics that evolution of states is determined by a unitary operator U, hence, any pure state remains to be a pure state during the evolution. In particular, there would exist a unitary operator U_B describing black hole evolution that maps the initial pure state $|\mathrm{C}\rangle$ of the collapsing body to another pure state $|\mathrm{R}\rangle$ of the collection of radiation such that $|\mathrm{R}\rangle = U_B |\mathrm{C}\rangle$.

On the other hand, Hawking radiation is entangled with its Hawking partner, which generates the entanglement between the system inside the horizon and outside. The more the black hole emits radiation, the greater the van Neumann entropy of the outside system becomes. Note that this fact itself is not an issue because the bipartite state of inside and outside can remain to be pure. However, if the black hole completely evaporates away, the outside system becomes the *entire* system whose von Naumann entropy is nonzero. This implies that the initial pure state $|C\rangle$ has evolved into a mixed state, hence, such black hole evaporation cannot be described by any unitary operator.

For this reason, the black hole information paradox is, in some literature, called the *unitarity crisis*. As discussed in Section 2.1.4, however, unitarity seems very fundamental in quantum mechanics. Is there any way of preserving unitarity? One idea is so-called *nonlocality*. In short, nonlocality proposes that Hawking radiation can somehow carry the information from the inside to the outside. In this sense, we violate causality of the quantum level. We will study the idea of nonlocality in length in Chapter 5.

4.3.2 Gravitational Remnants

Before jumping into the firewall paradox, let us mention the *gravitational remnant*. For the process of black hole evaporation, one may claim that it is wrong to assume that a black hole keeps evaporating after reaching the Planck scale, and maybe it stops shrinking because the semiclassical approximation is expected to break down at this scale. In this way, the bipartite system of the remnant and the system outside remains to be pure and there is no issue from this point of view.

However, this raises another issue. Since the initial black hole could have been arbitrarily large, there should be arbitrarily many Planck-size-remnant states. This suggests that remnant states are thermodynamically favoured (higher entropy) due to the second law of thermodynamics, and that they should be produced at high rate if this scenario were correct. However, we have never observed such states on experiments in laboratories as of today. In addition, if we interpret the Bekenstein-Hawking entropy formula in statistical mechanical perspectives, then the number of remnant state would be of order ~ 1, not arbitrarily many.

It might be also worth mentioning here that the firewall issue seems to appear sufficiently before the black hole gets shrunk into the Planck scale, so we would first need to consider resolutions of the firewall paradox anyway.

4.4 Unitary Black Hole Evolution

It would be helpful to model qualitatively how unitary black hole evolution would proceed, starting with a set of plausible assumptions. Note that of course there is no universally accepted process because of the lack of the knowledge of quantum gravity. However, a widely supported scenario was proposed by Page [49], and we shall review his idea below.

4.4.1 The Page Curve and The Page Time

The Page curve and the Page time are potential characteristics of unitary black hole evolution introduced in [49]. His analysis focuses on three types of entropies in time, namely the

Bekenstein-Hawking entropy S_{BH}, the von Neumann entropy S_H of the set of emitted Hawking radiation, and the entanglement entropy S_E between the systems inside and outside the horizon. Here, we are not splitting these two systems sharply on the horizon, but rather, the system outside the horizon can be thought of as the system with, for example, $r > 3R$ where R is the radius of the black hole. This is a relevant approximation because the region $R < r < 3R$ is effectively empty unless there are some incoming particles. Hence, by not considering this region, there is, for the 'entanglement entropy' S_E between the region $r < R$ and the region $r > 3R$, no typical UV divergent of entanglement entropy that would appear between two adjacent systems[7].

Let us consider black hole evaporation such that a sufficiently large black hole is created from gravitational collapse without leaving anything outside the event horizon[8]. We further assume the initial state is pure, but we do not require unitarity for now. At the moment of the creation, the Bekenstein-Hawking entropy S_{BH} is given by (4.32) whereas the von Neumann entropy S_H of the system of Hawking radiation is zero because nothing has been emitted. During the evaporation, S_{BH} decreases while S_H increases due to the entanglement between the Hawking quanta and their partners falling towards the singularity. Eventually, we would have $S_{\mathrm{BH}} = 0$ and $S_H >> 1$ when the evaporation is completed. Schematically, these entropies are represented in the following figure 4.7.

The Page conjecture claims that the 'entanglement entropy' S_E between the systems inside and outside the horizon initially increases in the same rate as $\hat{S}_H$ that is the von Neumann entropy of Hawking radiation for non-unitary evolution[9], i.e., the red curve in Figure 4.7. However, S_E would be maximized around the time where $\hat{S}_H$ and S_{BH} cross, instead of monotonically increasing as $\hat{S}_H$ does. After that, S_E starts decreasing by following

[7]Actually, S_E is half the mutual information of these two regions since their union is entangled with the region $R < r < 3R$, so their joint state is not pure. However, the region $R < r < 3R$ is generally fairly empty except for vacuum fluctuations that do have strong quantum correlations across the boundaries at $r = R$ and $r = 3R$. We generally ignore $R < r < 3R$ and say that $r < R$ and $r > 3R$ effectively form a pure state.

[8]This is just for simplicity. If there are some objects outside the horizon at the moment of the black hole creation, the initial entanglement entropy between the systems inside and outside would be nonzero, though it is expected be much smaller than the Bekenstein-Hawking entropy.

[9]$\hat{S}_H$ can be viewed as the coarse-grained entropy of the Hawking radiation, ignoring quantum correlations between radiation modes and just summing up the von Neumann entropies of the individual modes. It is important to distinguish S_H and $\hat{S}_H$ because the entanglement entropy of Hawking radiation is not expected to monotonically increase in unitary evolution. Rather, The Page conjecture claims $S_H = S_E$ if there is nothing outside the horizon when the black hole is formed.

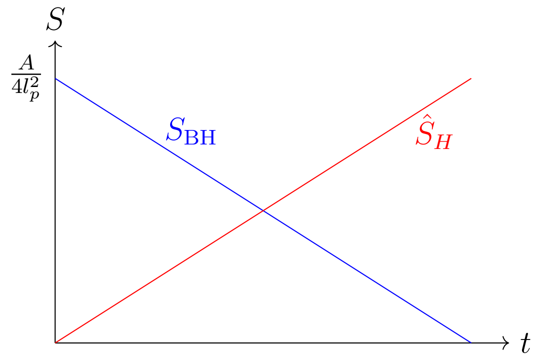


Figure 4.7: The Bekenstein-Hawking entropy S_{BH} and the Hawking radiation entropy $\hat{S}_H$ as functions of time in the semiclassical approximation, i.e., non-unitary evolution.

the curve of S_{BH} (but it is expected to obey $S_E \lesssim S_{\mathrm{BH}}$). Therefore, the system outside the horizon ends up being pure again at the end, and unitarity is preserved. The time where S_E is maximized is called the *Page time* t_P and the curve S_E as a function of t is called the *Page curve*. Pictorially, the Page time and the Page curve are given as follows

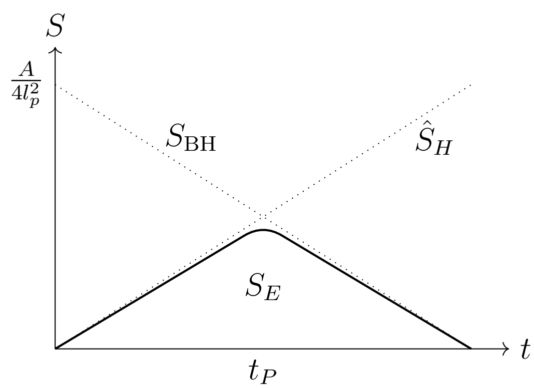


Figure 4.8: The Page curve for the 'entanglement entropy' S_E and the Page time t_P for unitary black hole evolution.

Additionally, S_{BH} is conjectured to serve as the upper bound of S_E from a statistical mechanical point of view. That is, we assume that the Hilbert space describing the system inside the horizon has dimension approximately $e^{S_{\mathrm{BH}}}$. Furthermore, since the Hilbert space of the black hole becomes effectively smaller than that of the rest of the system after the Page time, the average entropy theorem [30] (Theorem 2.2.8) suggests that the black hole system is nearly maximally entangled with the rest of the system, $S_E \sim S_{\mathrm{BH}}$. This approximation is also supported by the concept of black hole fast scrambling [51]

4.4.2 Black Hole Complementarity

Let us consider the following thought experiment in unitary black hole evolution that gives us great insight into black hole evaporation. Alice and Bob are originally located outside the event horizon and Alice will jump through the horizon with some quantum message. Alice will send the message to Bob once she crosses the horizon via some photons. Obviously, Bob will never detect the photons as long as he stays outside the horizon, but he could if he gets inside. Meanwhile, Bob has a machine that can collect any quantum information from Hawking radiation with arbitrarily high accuracy. In particular, he will be able to collect Alice's message through Hawking radiation and will jump in immediately after that. Following this process, Bob will be able to have two identical quantum information, one from Hawking radiation and the other from the message sent by Alice, which seems to violate the no-cloning theorem (Theorem 2.1.17). Is this another challenge to unitary black hole evolution? The figure below shows the thought experiment in the Penrose diagram.

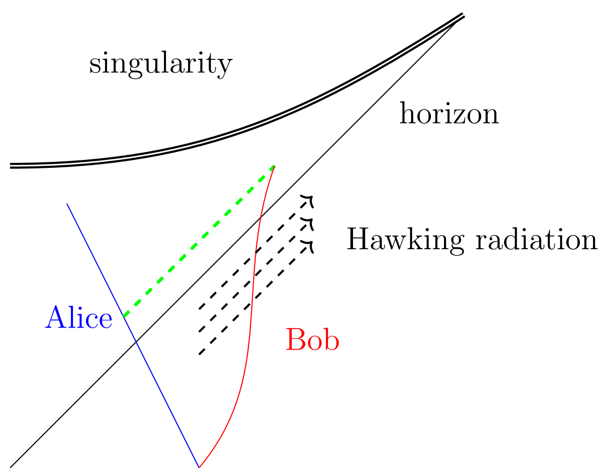


Figure 4.9: The cloning scenario: Bob will pick up the information of the Alice's message through Hawking radiation (dashed arrows), and detect her message inside the horizon (the green-highlighted dashed line).

This is where the concept of *black hole complementarity* [52, 53] comes in. It is important to notice in the above thought experiment that Alice needs to send her message to Bob shortly after she crosses the horizon, otherwise it will hit the singularity. In addition, Bob also needs to collect Alice's information through Hawking radiation quick enough before the message sent by Alice hits the singularity. The time t_B within which Bob has to complete the entire process is estimated to be shorter than of order

$$t_B < R \log \frac{R}{l_p}, \tag{4.33}$$

where R is the radius of the black hole. On the other hand, the shortest time t_S that the Alice's information is carried out through Hawking radiation would be of order [51]

$$t_S \sim R \log \frac{R}{l_p}. \tag{4.34}$$

This suggests that Bob cannot check the violation of no-cloning theorem. This is the spirit of Black hole complementarity, that is, *no single observer* cannot tell whether the no-cloning theorem is violated.

4.5 The Firewall Paradox

Even though the mechanism of unitary black hole evaporation had not yet been known, Page's argument and black hole complementarity gave hope that black hole evaporation could be indeed unitary. However, Almheiri *et al* [12] (see also [13]) recently raises another objection to unitarity, known as the *firewall paradox*.

The firewall paradox shows the mutual inconsistency of the following three assumptions:

- Assumption 1: There exists a unitary operator which describes black hole evaporation from a gravitational collapse.

- Assumption 2: Outside the stretched horizon of a massive black hole, physics can be described to good approximation by a set of semi-classical field equations.

- Assumption 3: A freely falling observer experiences nothing out of the ordinary when crossing the horizon.

We illustrate their argument below.

Unitary black hole evaporation would follow the Page curve (Assumption 1), and we particularly consider the case where the black hole is still sufficiently large after the Page time. Recall that the system describing inside the horizon would be nearly maximally entangled due to the average entropy theorem [30, 31] and the black hole scrambling [51]. Then, a new Hawking quantum coming out of the horizon should be nearly maximally entangled with the system outside because information transfer beyond that obeying causality is forbidden outside the stretched horizon (Assumption 2). Meanwhile, a smooth horizon implies the maximal entanglement of the two modes just inside and outside the horizon, which is equivalent to saying that the modes just inside and outside the horizon are created by a vacuum fluctuation (Assumption 3). However, this contradicts to the monogamy of entanglement (Theorem 2.2.7).

It is crucially important to notice that the inconsistency regarding the monogamy of entanglement can be confirmed by a *single* observer. That is, suppose Alice has stayed outside until the Page time, and she has collected all the quantum information emitted through Hawking radiation earlier. Then, she freely falls towards the black hole and observes the entanglement of a new Hawking quantum where this measurement itself should be done far from the horizon. If she finds that it is entangled with the system outside, she faces her fate that she will be burnt out at the horizon when she reaches there. If she measures no entanglement, then she recognizes that either locality or unitarity should be violated. Therefore, black hole complementarity is not sufficient to avoid the firewall argument. One of the three assumptions should be modified.

4.5.1 What to give up?

As mentioned in Section 2.1.4, unitarity in quantum mechanics is a very fundamental concept in terms of predictability and the probability interpretation. Out of many arguments that support unitarity, the most prominent one is coming from the AdS/CFT conjecture [9] that emerged from string theory. Very simply speaking, it conjectures that a theory with quantized gravity in the asymptotically anti-de Sitter spacetime of dimension $d+1$ is dual to a unitary conformal field theory in dimension d where the latter does not include gravity. If quantum gravity is a universal framework that is applicable to all asymptotically flat, de Sitter, and anti-de Sitter spacetimes, then unitarity in asymptotically anti-de Sitter spacetimes would suggest that unitarity is indeed a fundamental axiom of quantum gravity.

Also, curvature invariants such as the Kretschmann scalar for a sufficiently large black hole is very small around the horizon. Thus, it is very surprising if any severe quantum effect appears near the horizon where classical field equations are expected to work well. Furthermore, if the firewall were located at the event horizon, that would imply that the location is determined nonlocally because the event horizon cannot be defined unless the entire spacetime structure is known as explained in Section 4.1. Therefore, we might encounter a firewall tomorrow without warning. This is, in my opinion, a very radical proposal. [10]

These reasons have led me to explore modifications of Assumption 2. Sometimes, attempts of modifying Assumption 2 are summarized as *nonlocality of quantum gravity*, and we will discuss them in the next section.

[10] "Drama" is sometimes refereed to as the breakdown of the equivalence principle in a sense that a free falling observer would experience something extraordinary. This is, strictly speaking, not equivalent. For example, we would encounter a firewall-like situation if someone in the past sent a strong shock wave towards the future. We have no way to know such a shock wave is coming in prior, yet the equivalence principle is still consistent with this type of scenarios.

5 Nonlocality of Quantum Gravity

In short, an intuitive idea of nonlocality is to allow information transfer beyond the classical understanding of causality. This might sound like a radical idea at first glance because you might imagine particles moving faster than the speed of light. However, what we actually mean by nonlocality is more delicate. Our attitude towards the firewall paradox is to develop a minimal departure from standard local quantum field theory inspired by the idea of *nonviolent nonlocality* [54, 55].

5.1 Nonlocal Qubit Model

Since it is still a great mystery what quantum gravity is, a wise approach is to construct simple models fitting reliable requirements. Locality has been confirmed to be a good approximation in experiments in laboratories where gravity is negligibly small. Therefore, we would first like to confine the nonlocal effects into the gravity sector so that there is no contradiction to today's experiments. Following such a philosophy, Page and I proposed a nonlocal qubit transfer model for unitary black hole evaporation without firewalls [2]. Other papers such as [56–62] also discuss qubit models in a variety of ways, but what's particularly unique in our proposal is how to incorporate nonlocality. We will review our model below.

5.1.1 Settings

Let us consider a black hole of area A when it is created without incoming particles in the future. Interpreting the Bekenstein-Hawking entropy formula from statistical mechanics perspectives, the Hilbert space that describes such a black hole has dimensions

$$d = e^{S_{\rm BH}}, \quad S_{\rm BH} = \frac{A}{4 l_p^{d-2}}. \tag{5.1}$$

Let us assume for simplicity that there exists an integer N that satisfies

$$N = \frac{S_{\rm BH}}{\log 2}. \tag{5.2}$$

Then before the black hole forms, we assume that we have a Hilbert space of dimension 2^N in which each state collapses to form a black hole whose gravitational field can be represented by N qubits.

We assume that we have a pure initial state represented by the set of 2^N amplitudes $A_{q_1 q_2 \ldots q_N}$, where for each i running from 1 to N, the corresponding q_i can be 0 or 1, representing the two basis states of the ith qubit. We label these qubits by a_i, i.e., $|q_i\rangle_{a_i}$. Thus, the initial state is given as

$$|\Psi_0\rangle = \sum_{q_1=0}^{1} \sum_{q_2=0}^{1} \cdots \sum_{q_N=0}^{1} A_{q_1 q_2 \ldots q_N} \prod_{i=1}^{N} |q_i\rangle_{a_i} . \tag{5.3}$$

Let us assume the amplitude is chosen at the moment the black hole forms. We further assume that this state is rapidly scrambled [51] by highly complex unitary transformations during the formation, so that generically a black hole formed by collapse, even if it is initially in a pure state, will have these N qubits highly entangled with each other. That is, if we take a partial trace over all but one qubit, the entanglement entropy will be very close to $\log 2$ which follows Definition (2.2.2). Note that the total state remains to be pure no matter how complex the scrambling unitary operation is. Only the reduced density operator will be mixed by such an operation.

In addition to these N qubits, we introduce another $2N$ qubits that correspond to the smooth horizon state. More specifically, we consider N pairs of the singlet Bell state (3.34)

$$\prod_{j=1}^{N} |B_-\rangle_{b_i c_i} , \tag{5.4}$$

where b_i, c_i denote the system just inside and just outside the event horizon. We define this specific singlet Bell state as the vacuum state as seen by an infalling observer. The maximal entanglement of this Bell state indicates the maximal entanglement between just inside and just outside the horizon. In other words, if the observer sees anything different from $|B_-\rangle$, it is not the vacuum any more. We explicitly assume that infalling observers do not see any drama on the horizon and will only encounter the vacuum state. These states gives no contribution to the Bekenstein-Hawking entropy $S_{\rm BH} = N \log 2$, though we need to have

N such pairs for our model. This is not an essential part, but one can interpret each Bell vacuum state (5.4) as describing the smooth horizon approximately per Planck area, hence there should be N such states. See [63] for an argument for justifying this assumption.

Therefore, we start with the following initial state

$$|\Psi_0\rangle = \sum_{q_1=0}^{1}\sum_{q_2=0}^{1}\cdots\sum_{q_N=0}^{1} A_{q_1 q_2 \dots q_N} \prod_{i=1}^{N} |q_i\rangle_{a_i} \prod_{j=1}^{N} |B_-\rangle_{b_i c_i} . \tag{5.5}$$

Keep in mind that the systems labelled by a_i represent the gravitational degrees of freedom, b_i, c_i denote the system just inside and just outside the event horizon. As we will explain below, however, the systems c_i change their role from ensuring the smooth horizon to describing outgoing Hawking quanta once the black hole starts evaporating. Accordingly, the role of the systems b_i turn to representing infalling Hawking partners.

5.1.2 Subsystem Transfer

From the initial state defined as in (5.5), we now consider when the black hole starts emitting Hawking radiation. Since the rate of Hawking emission is expected to be very slow, it is a good approximation to assume that Hawking quanta are coming out one by one from the horizon to radial infinity. One can model this scenario in our setting as follows.

Pick $|B\rangle_{b_1 c_1}$ that represents the vacuum state localized on the horizon. Then the system c_1 just outside the horizon will start moving towards radial infinity, and the system b_1 starts falling towards the singularity instead. In this view, the systems c_1, b_1 are not localized on the horizon anymore, but rather they represent a localized Hawking mode and its partner respectively. See Figure 5.1 to grasp the transfer of the role of the system b_1, c_1.

In addition, we would like to assume that black hole evaporation is unitary. This requires that Hawking quanta after the transfer should have the information of the initial collapsed matter. Mathematically, this means that one of qubits $|q_1\rangle_{a_1}$ of the gravitational degrees of freedom and the vacuum state $|B_-\rangle_{b_1 c_1}$ satisfy

$$|q_1\rangle_{a_1} |B_-\rangle_{b_1 c_1} \to |B_-\rangle_{a_1 b_1} |q_1\rangle_{c_1} , \tag{5.6}$$

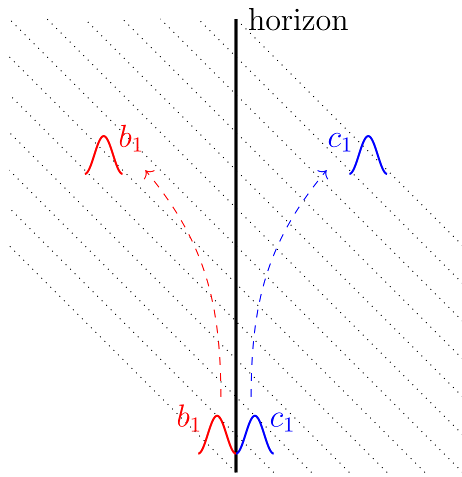


Figure 5.1: Hawking radiation in the ingoing Finkelstein-Eddington coordinates where dotted lines denote ingoing null lines. The red and blue wave represent a negative and positive energy outgoing mode respectively. The system b_1, c_1 at the initial moment describe the superposition of all vacuum fluctuation. After the transfer a physical radiation is labelled by c_1 and its infalling partner is labelled by b_1.

up to some overall phase. This type of swapping of qubits is called *subsystem transfer*.

After the transfer (5.6), the system c_1 encodes the information of the black hole, whereas the system b_1 is confined in the vacuum singlet Bell state with the system a_1. This state can then be omitted from the analysis without any loss of information. In this way we can model the reduction in the size of the black hole as it evaporates by the reduction of the number of black hole qubits. We might say that each such vacuum Bell pair falls into the singularity, but what hits the singularity in this model is a unique quantum state, similar to the proposal of Horowitz and Maldacena [64]

5.1.3 Key Proposal of Nonlocality

The new assumption of this model is that instead of simply saying the interaction for the transfer (5.6) itself is nonlocal, we propose that gravitational degrees of freedom represented by qubits a_i are nonlocal. The area of these nonlocal qubits is called the *zone* in some literature [65, 66] that may or may not be compact but spreads at least of the order of the radius of the black hole. In this assumption, the radiation qubit c_1 that propagates outward interacts *locally* with the qubit a_1 representing the nonlocal black hole gravitational field, in just such a way that when the mode gets out of the zone, the quantum state of that radiation qubit is interchanged with the quantum state of the corresponding black hole gravitational field qubit. This is a purely unitary transformation, not leading to any loss of information.

Let us consider the unitary operator describing such a transfer. We define $P_{a_1c_1} = |B_-\rangle_{a_1c_1} \langle B_-|_{a_1c_1}$. Then, the following unitary operator $U(\theta)$ gives the subsystem transfer (5.6)

$$U(\theta) = \exp(-i\theta P_{a_1c_1}) = I_{a_1c_1} + (e^{i\theta} - 1)P_{a_1c_1}, \tag{5.7}$$

$$U(0)\,|q_1\rangle_{a_1}|B_-\rangle_{b_1c_1} = |q_1\rangle_{a_1}|B_-\rangle_{b_1c_1}, \tag{5.8}$$

$$U(\pi)\,|q_1\rangle_{a_1}|B_-\rangle_{b_1c_1} = -|B_-\rangle_{a_1b_1}|q_1\rangle_{c_1}, \tag{5.9}$$

where I is the identity operator and $0 \leq \theta \leq \pi$. Note that $U(\theta)$ implicitly contains the identity operator acting on the system b_1.

The θ parameter of the unitary transformation (5.7) controls how fast the transfer occurs. We might suppose that as the radiation qubit c_1 moves outward, it is a function of the radius r that changes from 0 at the horizon to π at radial infinity. For example, one could take $\theta(r) = \pi(1 - K/K_h)$, where K is some curvature invariant (such as the Kretschmann invariant, $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$) that decreases monotonically from some positive value at the horizon (where its value is K_h) to zero at infinity.

Note that this transfer should not be completed too fast. This is because if it happens in a short time, quantum mechanically it would suggest that the modes just inside and just outside oscillate very differently. This creates a large gradient of fields and it results in a large energy momentum tensor because $T \sim \partial\phi\partial\phi$, which essentially causes drama on the horizon. Therefore, we assume that outgoing Hawking modes *gradually* and locally pick up the information during the propagation through the nonlocal gravitational degrees of freedom.

Also, we require that nonlocal gravitational qubits a_i do not create firewalls by themselves. That is, even though the vacuum states on the horizon b_i, c_i are in the range of nonlocal effects, they remain to be constrained in the singlet state unless systems c_i are propagating away to infinity as Hawking radiation by (5.6). This is consistent with the above assumption that the parameter $\theta(r)$ is a function of the radius r.

Suppose the unitary operator (5.7) acts on the first Hawking quantum c_1, then the initial state (5.5) turns to

$$|\Psi_1\rangle = \sum_{q_1=0}^{1}\sum_{q_2=0}^{1}\cdots\sum_{q_N=0}^{1} A_{q_1 q_2 \ldots q_N} \prod_{i=2}^{N} |q_i\rangle_{a_i} \prod_{j=2}^{N} |B_-\rangle_{b_i c_i} |q_1\rangle_{c_1} , \tag{5.10}$$

where $|B_-\rangle_{a_1 b_1}$ is dropped out without loss of information. By repeating the $U(\theta)$ operation on each system c_i, the state after n qubit emissions becomes

$$|\Psi_n\rangle = \sum_{q_1=0}^{1}\sum_{q_2=0}^{1}\cdots\sum_{q_N=0}^{1} A_{q_1 q_2 \ldots q_N} \prod_{i=n+1}^{N} |q_i\rangle_{a_i} \prod_{j=n+1}^{N} |B_-\rangle_{b_i c_i} \prod_{k=1}^{n} |q_k\rangle_{c_k} . \tag{5.11}$$

In this model, the Page time corresponds to the moment n reaches $N/2$. After the Page time, the remaining black hole qubits a_i all become nearly maximally entangled with the Hawking radiation qubits c_i, so that the von Neumann entropy of the black hole becomes very nearly $(n - n_r)\ln 2$, which we shall assume is very nearly $A/4l_p^{d-2}$ at that time. If one computes the entanglement entropy S_E between the Hawking qubits emitted earlier $\prod c_{k\leq n}$ and the nonlocal gravitational qubits $\prod a_{i\geq n+1}$, it would actually follow the Page curve Figure 4.8. Note again that the remaining product of the singlet Bell states in (5.11) would not contribute to this entanglement entropy.

Finally, when all N of the original outgoing radiation qubits have left the black hole and propagated to infinity to become Hawking radiation qubits, there are no qubits left for the black hole; hence it has completely evaporated away:

$$|\Psi_N\rangle = \sum_{q_1=0}^{1}\sum_{q_2=0}^{1}\cdots\sum_{q_N=0}^{1} A_{q_1 q_2 \ldots q_N} \prod_{k=1}^{N} |q_k\rangle_{c_k} . \tag{5.12}$$

The N Hawking radiation qubits c_i now form a pure state, just as the original quantum state that formed the black hole was assumed to be. Of course, the unitary scrambling transformation of the black hole qubits means that the pure state of the final Hawking radiation can look quite different from the initial state that formed the black hole, but the two are related by a unitary transformation.

In summary, the net effect is that the emission of one outgoing radiation qubit gives the transfer of the information in one black hole qubit to one Hawking radiation qubit. The black hole qubits itself are always nonlocal, and the outgoing radiation qubit picks up the information in the black hole qubit locally, as it travels outward through the nonlocal gravitational field of the black hole. Therefore, in this picture in which we have separated the quantum field theory qubits of the radiation from the black hole qubits of the gravitational field, we do not need to require any nonlocality for the quantum field theory modes, but only for the gravitational field. In this way, the nonlocality of quantum gravity might not have much observable effect on experiments in the laboratory focussing mainly on local quantum field theory modes.

5.1.4 Mining Issue

AMPSS [13], whose Eq. (3.3) is essentially the same as our (5.6), raised the following issue with subsystem transfer models as resolutions of the firewall paradox, sometimes referred as the *mining issue*. Suppose there exists an ideal mining equipment that can approach arbitrarily close to the horizon without falling into it, and then the equipment interacts with one of systems c_i just outside the horizon. They claim that this can be done without any exchange of energy due to the infinite redshift, and it is assumed that there is no entangling either. For example, the mining equipment can unitarily acts on the system c_i as

$$U_{\mathrm{mine}} : |0\rangle_{c_i} \mapsto e^{i\phi}\, |0\rangle_{c_i}\,, \quad |1\rangle_{c_i} \mapsto e^{-i\phi}\, |1\rangle_{c_i}\,. \tag{5.13}$$

$$U_{\mathrm{mine}} : |B\rangle_{b_i c_i} \mapsto \frac{\cos\phi}{\sqrt{2}}\Big(|0\rangle_{b_i}|1\rangle_{c_i} - |1\rangle_{b_i}|0\rangle_{c_i}\Big) + \frac{i\sin\phi}{\sqrt{2}}\Big(|0\rangle_{b_i}|1\rangle_{c_i} + |1\rangle_{b_i}|0\rangle_{c_i}\Big). \tag{5.14}$$

Thus the system on the horizon has one bit of information after this mining process and is thus no longer in the vacuum state[1]. We give two counterarguments regarding their concerns.

First of all, it seems implausible that such an ideal equipment can be physically realistic. Since the equipment is accelerating in order to stay outside the horizon without falling into the black hole, it has an Unruh temperature (4.29) that becomes very high near the horizon. Then the equipment and the modes it interacts with, c_i in this case, should strongly couple

[1]The author finds this argument analogous to the issue of the state-dependence discussed in Section 5.2.1. That is, one can always create drama on the horizon no matter how strong we assume to be the vacuum, which is essentially the opposite way of thinking the issue of the state-dependence proposal.

and would be expected to be approximately in a thermal state. As a consequence it seems plausible that energy must be transferred between the mining equipment and the modes c_i.

Also, notice that the AMPSS mining argument does not take nonlocality into account. That is, the mining equipment would interact with the nonlocal gravitational degrees of freedom even if it could avoid the objection of the previous paragraph. As the opposite direction of Hawking radiation, interactions with nonlocal gravitational degrees of freedom transfer part of the quantum information of the mining system into the gravitational degrees of freedom as the equipment approaches to the horizon. Note that this dropping-information effect would not be strong for free falling matter in a sufficiently large black hole geometry, but it can be assumed to be sizeable for accelerating matter especially near the horizon. We can think of this transferred part as now being a part of the temporarily enlarged nonlocal gravitational degrees of freedom when the equipment is very near to the horizon. Then in this picture the mining equipment can still produce the phase change Eq. (5.14) on the system just outside the horizon, but this excitation will be eventually absorbed into the enlarged nonlocal gravitational degrees of freedom. This absorption is possible regardless of how old the black hole is, because the nonlocal degrees of freedom are temporarily enlarged by the partially transferred degrees of freedom of the mining equipment. In summary, the AMPSS mining argument is not problematic for our model.

5.1.5 Giddings Constraints

Giddings [57] has proposed a list of physical constraints on models of black hole evaporation following the spirit of nonviloent nonlocality. We shall write each constraint in italics below and then follow that with comments on how our qubit model can satisfy the proposed constraint.

(i) *Evolution is unitary.* Our model explicitly assumes unitary evolution.

(ii) *Energy is conserved.* Our model is consistent with a conserved energy given by the asymptotic behavior of the gravitational field. The unitary transformation $U(\theta(r))$ during the propagation of each radiation qubit can be written in terms of a radially dependent Hamiltonian without any explicit time dependence, so there is nothing in our model that violates energy conservation.

(iii) *The evolution should appear innocuous to an infalling observer crossing the horizon; in this sense the horizon is preserved.* We explicitly assume that the radiation modes are in their vacuum states when they are near the horizon, so there is no firewall or other drama there. In addition, our model avoids the mining issue as discussed in Section 5.1.4.

(iv) *Information escapes the black hole at a rate $dS/dt \sim 1/R$.* Although we only briefly mention the rates of emission above, if one radiation qubit propagates out through some fiducial radius, such as $3R/2$, during a time period comparable to the black hole radius R, since during the early radiation each qubit carries an entropy very nearly $\ln 2$, indeed one would have $dS/dt \sim 1/R$.

(v) *The coarse-grained features of the outgoing radiation are still well-approximated as thermal.* Because of the scrambling of the black hole qubits so that each one is very nearly in a maximally mixed state, when the information is transferred from the black hole qubits to the Hawking radiation qubits, each one of these will also be very nearly in a maximally mixed state, which in the simplified toy model represents thermal radiation. Furthermore, one would expect that any collection of $n' < n/2$ qubits of the Hawking radiation also to be nearly maximally mixed, so all the coarse-grained features of the radiation would be well-approximated as thermal.

(vi) *Evolution of a system $H_A \otimes H_B$ saturates the subadditivity inequality $S_A + S_B \geq S_{AB}$.* Here it is assumed that A and B are subsystems of n_A and n_B qubits respectively of the black hole gravitational field and of the Hawking radiation, not including any of the infalling and outgoing radiation qubits when they are near the horizon, and S_A, S_b, S_{AB} are the von Neumann entropies of $H_A, H_B, H_A \otimes H_B$. Then for $n_A + n_B < n/2$, A, B, and AB are all nearly maximally mixed, so $S_A \sim n_A \ln 2$, $S_B \sim n_B \ln 2$, and $S_{AB} \sim (n_A + n_B) \ln 2$, thus approximately saturating the subadditivity inequality. (Of course, for any model in which the total state of n qubits is pure and any collection of $n' < n/2$ qubits has nearly maximal entropy, $S \sim n' \ln 2$, then if $n_A < n/2$, $n_B < n/2$, but $n_A + n_B > n/2$, then $S_A \sim n_A \ln 2$ and $S_B \sim n_B \ln 2$, but $S_{AB} \sim (n - n_A - n_B) \ln 2$, so $S_A + S_B - S_{AB} \sim 2n_A + 2n_B - n > 0$, so that the subadditivity inequality is generically not saturated in this case.)

5.1.6 Summary of Our Model

We have given a toy qubit model for black hole evaporation that is unitary and does not have firewalls. It does have nonlocal degrees of freedom for the black hole gravitational field, but the quantum field theory radiation modes interact purely locally with the gravitational field, so in some sense the nonlocality is confined to the gravitational sector. The model has no mining issue and also satisfies all of the constraints that Giddings has proposed, though further details would need to be added to give the detailed spectrum of Hawking radiation. The model is in many ways *ad hoc*, such as in the details of the qubit transfer, so one would like a more realistic interaction of the radiation modes with the gravitational field than the simple model sketched here. One would also like to extend the model to include possible ingoing radiation from outside the black hole.

5.2 Selected Other Proposals

At last, I will show selected other proposals as resolutions of the firewall paradox. Note that there are more interesting proposals such as [67–70], but I give only a few of those because of the space limitation.

5.2.1 State-Dependence

We will give a simple toy model to understand the state-dependence proposal here. See [71–74] for more detail. Let us consider unitary black hole evaporation where the state is described by $|\Phi\rangle$ after the Page time. We focus on an outgoing Hawking mode in an effective basis $|n\rangle_H$, then this mode is expected to be nearly maximally entangled with the rest of the system O. Thus, the bipartite pure state $|\Phi\rangle$ is given, up to normalization, by

$$|\Phi\rangle = \sum_n |\phi_n\rangle_O \, |n\rangle_H \,, \tag{5.15}$$

where $|\phi_n\rangle_O$ are appropriately chosen to be orthonormal vectors. On the other hand, if the horizon was smooth, an infalling observer would see the state to be in the following specific form

$$|\Phi\rangle = \sum_n |n\rangle_{\tilde{H}} |n\rangle_H , \tag{5.16}$$

where $|n\rangle_{\tilde{H}}$ is a basis of the system of the infalling Hawking partner. Now, one may propose that we can *define* $|n\rangle_{\tilde{H}} = |\phi_n\rangle_O$ so that two observations do not seem inconsistent.

However, this raises an issue if we proceed the same argument for another state $U|\Phi\rangle$ where the unitary operator U is defined by

$$U = \exp\left(i \sum_n \theta_n |\phi_n\rangle_O \langle\phi_n|_O \right) . \tag{5.17}$$

Now the state would have relative phase factors as

$$U|\Phi\rangle = \sum_n e^{i\theta_n} |\phi_n\rangle_O |n\rangle_H . \tag{5.18}$$

If one follows the definition of the partner Hawking mode to be described by $|n\rangle_{\tilde{H}} = |\phi_n\rangle_O$, the state would be

$$U|\Phi\rangle = \sum_n e^{i\theta_n} |n\rangle_{\tilde{H}} |n\rangle_H . \tag{5.19}$$

Similar to the discussion in Section 5.1.4, the vacuum state should be defined in a very specific form (5.16), and nothing else corresponds to the vacuum state. Therefore, an infalling observer sees some drama in the state (5.19).

Putting another way, we need to define the partner basis as $e^{i\theta_n} |\phi_n\rangle_O = |n\rangle_{\tilde{H}}$, which is different from the previous definition. Thus, the partner basis depends on the state given, and this is where the *state-dependence* comes from. This ambiguity makes it possible that even if we put some firewall on the horizon by hand, we can always regard it as the vacuum state by redefinition, which seems problematic. Furthermore, [75] discusses that the state-dependence suggests a violation of the Born rule in the sense that the inner product of two states with two different observations, a firewall state and no firewall state, is almost one. Two seemingly different states turn out to be very similar. This is a strange consequence in terms of ordinary understanding of quantum mechanics.

5.2.2 ER=EPR *Conjecture*

A naive argument of the ER=EPR conjecture [76] goes as follows. Let us consider the thermal field double state $|\mathrm{TFD}\rangle$ in the context of the AdS/CFT correspondence

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} \left|n\right\rangle_L \left|n\right\rangle_R . \tag{5.20}$$

This state describes an entangled state of two CFT's dual to the left and right side of the maximally extended asymptotically AdS black hole geometry. Now suppose there is a Hawking pair $H\tilde{H}$ near the horizon on the right side. As stated in the firewall paradox, the smooth horizon requires H to be maximally entangled with $\tilde{H}$ whereas unitary evolution suggests H be entangled with some state on the other boundary system L. However, one notices that the system just behind the horizon H can be interpreted as being dependent on the left side system. That is, the state in $\tilde{H}$ is given by a future evolution of some state in H_L which is a subsystem of the left side boundary theory. Therefore, the monogamy of entanglement is not necessarily violated because of the dependence[2]. Note that this proposal does not guarantee there is no firewall, but it shows that it is too early to conclude that there have to be firewalls. One needs to know how the state in H_L evolves and see which one is dominating, the probability of firewalls or the smooth horizon.

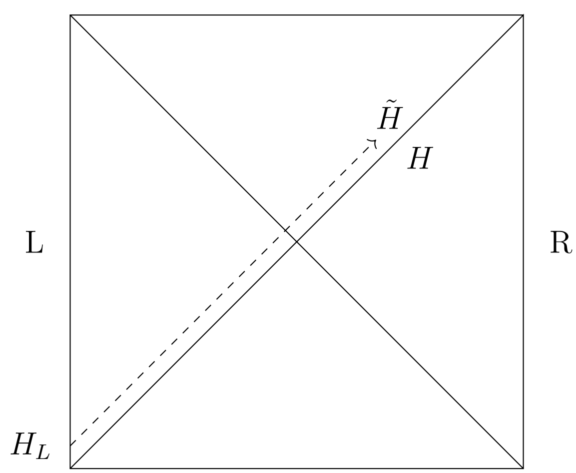


Figure 5.2: The ER=EPR conjecture.

[2]This argument could be extended to some special asymptotically flat spacetimes. For example, [77] argues that one can create two entangled black holes by vacuum pair-creations where they are connected by the wormhole just as the two AdS black hole.

The ER=EPR conjecture is not only saying how to avoid the firewall paradox, but rather it proposes any entangled states are somehow connected in a similar spirit to the property that two black holes are connected through the wormhole. Who knows? This might be a characteristic of quantum gravity. The underlying concept is motivated by the proposal of *emergent spacetime* where spacetimes are, in some sense, secondary objects emerged from quantum mechanics. The AdS/CFT conjecture can be thought of as one example of this idea. Gravity in two dimensions can be also interpreted as being emergent from matrix models that are discussed in depth in the mathematics part of my thesis. See [78] and references therein for more detail.

Part II

Matrix Models

6 Matrix Models

Quantum field theories are complex. Objects of interest such as correlation functions are very difficult to compute. They essentially include divergent nature in computation coming from virtual high energy modes, and more severe problems have become apparent when one tries to unify gravity with quantum field theories as discussed in the previous sections. Besides struggles of directly resolving such issues, an interesting attempt is to simplify theories by reducing the dimension of spacetime, and observing the simplified theory that can potentially be useful for higher dimensions.

Indeed, we have already seen interesting examples of this approach. Although this is in the classical regime, Page [3] and I investigated a four dimensional cosmological model, a so-called Bianchi IX model. (See Appendix A.) In the paper, we considered a so-called minisuperspace in two dimensions, which is an equivalent description of a four dimensional Bianchi IX model. However, since the minisuperspace is in two dimensions, geometric properties are easier to study, and we showed a new way of deriving the exact solution of a Bianchi IX model with a positive cosmological constant.

As an example in the quantum regime, conformal field theories in two dimensions are significantly different from those in higher dimensions because they admit an infinite number of symmetry generators while those in dimensions higher than 2 have only finitely many generators. This difference makes it possible to compute many things in a relatively easy way. Conformal field theory in two dimensions are then used in constructions of string theory whose associated spacetime should be in 26 dimensions, or in 10 dimensions with supersymmetry.

Following this spirit, the simplest example one can think of is probably quantum field theories in *zero* dimensional spacetimes. If one has some trouble to imagine quantum field theories in zero dimensions, imagine an action of a $U(N)$ Yang-Mills theory in a d-dimensional spacetime, and then reduce the dimension $d \to 0$. What remains? All local fields disappear but $U(N)$ matrices since they can exist without the presence of spacetimes. Very interestingly, it turns out such simple quantum field theories in zero dimensions possess nontrivial mathematical structures, and they are now known as *matrix models*.

In this section, we introduce matrix models from two seemingly different points of view: from the path integral formulation and from the Virasoro constraints. These two are indeed equivalent ways of defining matrix models, but we start with the path integral formulation as it is commonly used to define quantum field theories in physics. We then introduce the Virasoro constraints which smoothly lead us to a fascinating mathematical formalism known as topological recursion. Many discussions in Chapter 6-8 are taken from the paper with Bouchard [1].

6.1 Matrix Models from Functional Integrals

As mentioned above, matrix models are quantum gauge field theories in zero dimensions whose fields are matrices. We define the partition function of a formal 1-matrix model from the path integral formulation. We closely follow [79] for the discussions in this section. See also [80] for further reviews.

Definition 6.1.1. Let H_N be the space of $N \times N$ Hermitian matrices and $M \in H_N$. The partition function of a *formal Hermitian 1-matrix model* is given by

$$Z(t, t_3, \cdots, t_d; T_2; N) = \prod_{k=3}^{d} \sum_{n_k \geq 0} \int_{H_N} dM \frac{1}{n_k!} \left(\frac{N}{t} \frac{t_k}{k} \mathrm{Tr}(M^k) \right)^{n_k} e^{-\frac{NT_2}{2t} \mathrm{Tr}(M^2)}, \tag{6.1}$$

where $T_2 \neq 0$ and the measure dM is the $U(N)$ invariant Lebesgue measure on H_N

$$dM = \frac{1}{2^{N/2} (\pi t / N T_2)^{N^2/2}} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re} M_{ij} \, d\mathrm{Im} M_{ij}. \tag{6.2}$$

$(t, t_3, \cdots, t_d)$ are coupling constants and it will become clear in Section 6.2 why we introduce the parameter T_2 here. The normalization is chosen such that $Z = 1$ if $t_3 = \cdots = t_d = 0$.[1] It is crucial in (6.1) that the summation over n_k is outside of the functional integral. In contrast, the partition function of a convergent Hermitian 1-matrix model is defined with the order of summation and integral in (6.1) switched:

[1] This definition can be more generalized. See [79].

Definition 6.1.2. Let H_N be the space of $N \times N$ Hermitian matrices and $M \in H_N$. The partition function of a *convergent 1-matrix model*[2] is given by

$$Z_{\mathrm{conv}}(t, t_3, \cdots, t_d; T_2; N) = \int_{H_N} dM \prod_{k=3}^{d} \sum_{n_k \geq 0} \frac{1}{n_k!} \left(\frac{N}{t} \frac{t_k}{k} \mathrm{Tr}(M^k) \right)^{n_k} e^{-\frac{NT_2}{2t} \mathrm{Tr}(M^2)}$$
$$= \int_{H_N} dM e^{-\frac{N}{t} \mathrm{Tr} V(M)}, \tag{6.3}$$

where

$$V(M) = \frac{T_2}{2} M^2 - \sum_{k=3}^{d} \frac{t_k}{k} M^k \tag{6.4}$$

is called the potential.

Example 6.1.3 ([79])**.** Let us show their difference in a simple example. For $N = 1, T_2 = 1, t_4 = -4$ and all other $t_k = 0$, the convergent model gives

$$Z_{\mathrm{conv}}(t) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2t\pi}} e^{-\frac{1}{2t}x^2 - x^4} = \frac{e^{\frac{1}{32t^2}}}{4t\sqrt{\pi}} B_{II}\left(\frac{1}{4}, \frac{1}{32t^2} \right), \tag{6.5}$$

where B_{II} is the Bessel function of the second kind. It can be shown that $Z_{\mathrm{conv}}(t)$ is a bounded function of $t > 0$ where $Z_{\mathrm{conv}}(t) = 1$ in the limit $t \to 0^+$. On the other hand, the partition function for the formal model is

$$Z_{\mathrm{formal}}(t) = \sum_{n \geq 0} \int_{\mathbb{R}} \frac{dx}{\sqrt{2t\pi}} \frac{(-x^4)^n}{n!} e^{-\frac{x^2}{2t}}$$
$$= \sum_{n \geq 0} \frac{(4n-1)!!}{n!} (-t^2)^n. \tag{6.6}$$

$Z_{\mathrm{formal}}(t)$ diverges for any positive t. Therefore, these two models are different. However, (6.6) corresponds to the asymptotic expansion of (6.5) at $t \to 0^+$.

Our interest in this thesis is only in formal Hermitian 1-matrix models. Indeed, for many applications of matrix models in physics and enumerative geometry, convergence is not really necessary. For simplicity, however, we often omit the arguments $(t, t_k; T_2; N)$, and also denote

[2]This does not guarantee that the partition function in this definition is always convergent.

a formal Hermitian matrix model by

$$Z \stackrel{\text{formal}}{=} \int_{H_N} dM e^{-\frac{N}{t}\mathrm{Tr}V(M)}, \tag{6.7}$$

with the understanding that the summation should be taken outside of the integral.

Hermitian matrix models possess a $U(N)$ gauge symmetry, $M \to U^\dagger M U$, where U is an $N \times N$ unitary matrix. If we fix the gauge freedom such that M is diagonalized, the partition function is given up to normalization by

$$Z \propto \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda)^2 e^{-\frac{N}{t}\sum_{i=1}^{N} V(\lambda_i)}. \tag{6.8}$$

$\Delta(\lambda) = \prod_{i<j}^{N}(\lambda_i - \lambda_j)$ is the Vandermonde determinant, which can be derived by the Fadeev-Popov gauge fixing.

6.1.1 *Free Energy and Correlation Functions*

As usual, the free energy is defined by taking a logarithm of the partition function.

$$F(t, t_3, \cdots, t_d; T_2; N) = \log Z(t, t_3, \cdots, t_d; T_2; N). \tag{6.9}$$

Also, we define the expectation value of a function f by

$$\langle \mathrm{Tr} f(M) \rangle = \frac{1}{Z}\int_{H_N} dM \mathrm{Tr}(f(M)) e^{-\frac{N}{t}\mathrm{Tr}V(M)}, \tag{6.10}$$

and we denote by $\langle \mathrm{Tr} f(M)\rangle_c$ the corresponding connected expectation value[3]. Note that we need to take a trace of f in order to make the expectation value $U(N)$-gauge invariant. We are interested in the expectation values:

$$T_{l_1 \cdots l_n}(t, t_k; T_2; N) = \Big\langle \mathrm{Tr}(M^{l_1}) \cdots \mathrm{Tr}(M^{l_n}) \Big\rangle_c. \tag{6.11}$$

[3]It will be clear what connected means once we consider Feynman diagrams in matrix models.

It turns out to be convenient to collect all $T_{l_1\cdots l_b}(t,t_k;T_2;N)$ for every nonnegative integers $l_1,\cdots,l_n$ in a single expression. We define the following generating functions, known as correlation functions:

$$\begin{aligned} W_n(t,t_k;T_2;N;x_1,\cdots,x_n) &= \sum_{l_1,\cdots,l_n\geq 0} \frac{T_{l_1\cdots l_n}(t,t_k;T_2;N)}{x_1^{l_1+1}\cdots x_n^{l_n+1}}, \\ &= \left\langle \prod_{j=1}^{n} \mathrm{Tr}\left(\frac{1}{x_j - M}\right)\right\rangle_c. \end{aligned} \tag{6.12}$$

The last equality is often used as a definition of the correlation functions; these should be understood as generating series in the variables $1/x_i$.

6.1.2 1/N Expansion

For formal matrix models the free energy and correlation functions have a nice $1/N$ expansion. It follows from the definition of the partition function (6.1) that the free energy (6.9) has an expansion

$$F(t,t_k;T_2;N) = \sum_{g\geq 0}\left(\frac{N}{t}\right)^{2-2g} F_g(t,t_k;T_2), \tag{6.13}$$

where the $F_g(t,t_k;T_2)$ do not depend on N. It can also be shown that the $F_g(t,t_k;T_2)$ are in fact power series in t [79].

A similar $1/N$ expansion also holds for correlation functions:

$$W_n(t,t_k;T_2;N;x_1,\cdots,x_n) = \sum_{g\geq 0}\left(\frac{N}{t}\right)^{2-2g-n} W_{g,n}(t,t_k;T_2;x_1,\cdots,x_n), \tag{6.14}$$

with the $W_{g,n}(t,t_k;T_2;x_1,\cdots,x_n)$ independent of N. Those are also power series in t. For simplicity of notation we will often drop the dependence on t, t_k and T_2.

In fact, the expectation values $T_{l_1\cdots l_n}(t,t_k;T_2;N)$ can be interpreted in terms of ribbon graphs; we briefly review ribbon graphs in Section 6.1.3, but refer the reader to [79] for more details. It follows from the ribbon graph interpretation that they themselves have a $1/N$

expansion of the form:

$$T_{l_1\cdots l_n}(t, t_k; T_2; N) = \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g-n} T^{(g)}_{l_1\cdots l_n}(t, t_k; T_2), \tag{6.15}$$

thus we can write, order by order,

$$W_{g,n}(x_1, \cdots, x_n) = \sum_{l_1,\cdots,l_n\geq 0} \frac{T^{(g)}_{l_1\cdots l_n}(t, t_k; T_2)}{x_1^{l_1+1}\cdots x_n^{l_n+1}}. \tag{6.16}$$

The $T^{(g)}_{l_1\cdots l_n}(t, t_k; T_2)$ are power series in t. Furthermore, it follows from the ribbon graph interpretation discussed below that if we collect the terms in the summation over $l_1, \cdots, l_n$ by powers of t, for each power of t only a finite number of terms are non-zero. In other words, order by order in t, $W_{g,n}(x_1, \cdots, x_n)$ is polynomial in the variables $1/x_i$, $i = 1, \cdots, n$ [79].

6.1.3 Feynman Diagrams

As we do in quantum field theories in higher dimensions, we can consider Feynman diagrams for matrix models called *ribbon graphs*, see Figure 6.1 for a typical ribbon graph. Let $G(v, e, l)$ be a ribbon graph with e edges, l loops, and $v = \sum_k n_k$ vertices where the graph has n_k vertices with k legs. Then, the Feynman rule for matrix models is summarized as follows:

- Multiply t/N at each edges
- Multiply N for each closed loop
- Multiply Nt_k/kt at each vertex of k legs
- Divide by $|\mathrm{Aut}(G)|$ if the graph admits an automorphism

Here, we set $T_2 = 1$ for simplicity. As a consequence, the partition function (6.1) and the free energy (6.9) are given by the sum over all ribbon graphs and connected ones with appropriate weights

$$Z = \sum_{\text{ribbon graphs } G} \frac{t^l}{|\mathrm{Aut}(G)|} \left(\prod_{k=3}^{d} t_k^{n_k}\right) \left(\frac{N}{t}\right)^{\chi(G)}, \tag{6.17}$$

$$F = \sum_{\substack{\text{connected} \\ \text{ribbon graphs } G}} \frac{t^l}{|\mathrm{Aut}(G)|} \left(\prod_{k=3}^{d} t_k^{n_k}\right) \left(\frac{N}{t}\right)^{\chi(G)}, \tag{6.18}$$

where $\chi(G) = v - e + l$ is the Euler characteristic of the ribbon graph G.

The dual description of ribbon graphs is called a *map* which is essentially given by switching the role of loops and vertices, see Figure 6.1. The dual diagrams are the set of polyhedra with l vertices, e edges, and n_k k-gons as faces such that $v = \sum_k n_k$. In this picture, the role of the Euler characteristic $\chi(G)$ becomes easier to intuitively imagine, it is indeed genus of the polyhedron when it is smoothened.

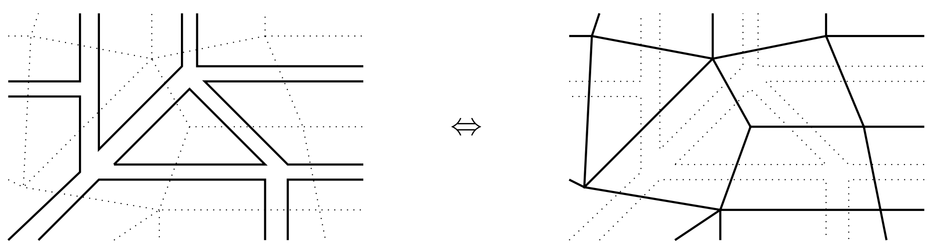


Figure 6.1: Duality between ribbon graphs and maps. Vertices of a ribbon graph correspond to polygons of a map, and loops turn to vertices.

Let $\mathbb{M}_k^{(g)}(v)$ be a set of connected maps of genus g, with k marked boundaries of length at least 1, and v vertices. We further require that each face has at most d edges and we define $\mathbb{M}_1^{(0)}(1)$ to be a point. Then, the free energy (6.9) and the coefficients $T_{l_1 \cdots l_n}(t, t_k; N)$ of correlation functions (6.16) can be written as

$$F_g(t, t_k; N) = \sum_v t^v \sum_{m \in \mathbb{M}_0^{(g)}(v)} \frac{1}{|\mathrm{Aut}(m)|} \prod_{k=3}^{d} t_k^{n_k(m)}, \tag{6.19}$$

$$T^{(g)}_{l_1 \cdots l_n}(t, t_k) = \sum_v t^v \sum_{\substack{m \in \mathbb{M}_b^{(g)}(v) \text{ where } i\text{-th} \\ \text{boundary has length } l_j}} \frac{1}{|\mathrm{Aut}(m)|} \prod_{k=3}^{d} t_k^{n_k(m)}. \tag{6.20}$$

As one can see now, these are actually power series in t. Thus, $W_{g,n}(x_1, \cdots, x_n)$ is a collection of all maps with n marked boundaries of genus g, which can be schematically interpreted as the n-point function of genus g, see Figure 6.2.

$W_{2,4}(x_1, x_2, x_3, x_4) \quad \leftrightarrow$

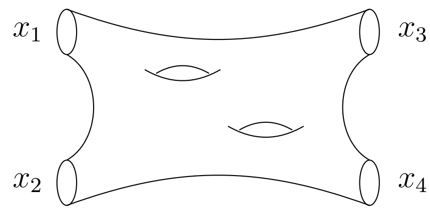


Figure 6.2: A pictorial interpretation of , for example, $W_{2,4}(x_1, x_2, x_3, x_4)$.

6.1.4 Liouville Gravity

Even though our interest in matrix models is in mathematical applications to enumerative geometry, let us briefly mention a striking relation between matrix models and quantum gravity in two dimensions. The author is not an expert on this topic, thus, our purpose for this section is to give only an intuitive hint of their duality. The discussion in this section is mostly taken from [79], but see [14–16] and references therein for further discussions.

As a simple example, let us consider the case where only t_4 is nonzero and all other coupling constants vanish. In this case, (6.19) shows that the free energy of the model becomes

$$F_g(t, t_4; N) = t^{2-2g} \sum_{n_4} (tt_4)^{n_4} \sum_{m \in \mathbb{M}_0^{(g)}(n_4+2-2g)} \frac{1}{|\mathrm{Aut}(m)|}, \tag{6.21}$$

where we used the fact that the Euler characteristic in this model obeys $v = n_4 + 2 - 2g$. The sum is taken over all quadrangulated surfaces and n_4 denotes the number of faces in each surface. In general, this free energy is not an analytical function of t and it diverges at some critical value $t = t_c(t_4)$. Let us investigate the behaviour near the critical point.

As shown in [79], it can be shown that $t_c = 1/12t_4$ and $F_g(t, t_4)$ for $g \geq 2$ in the limit $t \to t_c$ is

$$F_g = \tilde{F}_g t_c^{2-2g} \left(1 - \frac{t}{t_c}\right)^{\frac{5}{4}(2-2g)} + \text{subleading}, \tag{6.22}$$

where $\tilde{F}_g$ is some constant that becomes important later. At the same time, the average number of faces $\langle n_4 \rangle$ of graphs summed for $F_g(t, t_4)$ can be defined by

$$\langle n_4 \rangle = t_4 \frac{\partial \log F_g}{\partial t_4}. \tag{6.23}$$

In particular, in the limit $t \to t_c$, it becomes

$$\langle n_4 \rangle = \frac{5(2-2g)t_c}{4} \frac{1}{t - t_c} + \cdots . \tag{6.24}$$

This suggests that the major contribution to F_g near $t \to t_c$ comes from graphs with many faces, which is essentially a continuous limit of quadrangulated surfaces.

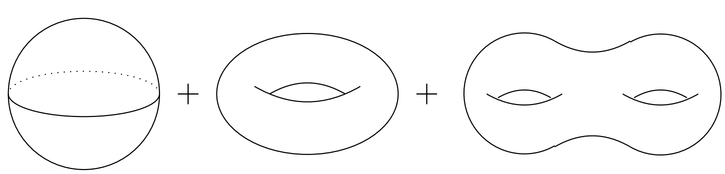
Furthermore, let define a new parameter $\tilde{N}$ as

$$\tilde{N} = N t_c \left(1 - \frac{t}{t_c}\right)^{\frac{5}{4}}. \tag{6.25}$$

Now we take a limit where $N \to \infty, t \to t_c$ yet $\tilde{N}$ to be finite. This is a so-called *double-scaling limit*. Then the rescaled free energy can be written as a formal series in $\tilde{N}$

$$\tilde{F} = \sum_{g \geq 0} \tilde{N}^{2-2g} \tilde{F}_g. \tag{6.26}$$

Note that $\tilde{F}$ is not precisely the same as the original F in the limit $N \to \infty, t \to t_c$ because F_0, F_1 does not obey (6.22), hence, we need to define $\tilde{F}_0, \tilde{F}_1$ differently. $\tilde{F}$ can be viewed as the generating function of the continuous limit of quadrangulated graphs, and we can pictorially represent it as:

$\tilde{F} =$  $+ \cdots$ (6.27)

On the other hand, if one considers gravity in two dimensions potentially coupled with matter preserving conformal symmetry, the path integral over the metric is reduced to the sum of topology of two dimensional surfaces. This is because the Weyl symmetry and diffeomorphism invariance are enough to fix the metric to be flat, at least locally. Then, the free energy of such a theory in two dimensions is given by a sum of smooth surfaces, which is pictorially the same as (6.27).

Then one can ask whether the free energy $\tilde{F}$ of matrix models in a double scaling limit coincides with that for some quantum gravity in two dimensions. The conjectured corresponding gravity theory is called *Liouville gravity*, whose action is the Polyakov worldsheet action coupled to minimal models in conformal field theory. There are a lot more to study about Liouville gravity, and we refer the readers to [14–16, 79] and references therein for rigorous discussions.

6.2 Matrix Models from Virasoro Constraints

Discussions in Section 6.1 are similar to the standard analysis of quantum field theories in higher dimensions. A unique and crucial aspect of matrix models is that we can equivalently define them without the notion of path integral. We review this formulation below. See [81–83] for more general arguments.

6.2.1 Virasoro Constraints

Our first step is to extend the potential (6.4) from a polynomial to a power series

$$V = \frac{T_2}{2}x^2 + \sum_{k\geq 0} g_k x^k. \tag{6.28}$$

We will use this generalized potential to define matrix models from the Virasoro constraints as well as the derivation of the loop equations, but in the end we will set

$$g_k = -\frac{t_k}{k} \;\; (3 \leq k \leq d), \quad g_0 = g_1 = g_2 = g_k = 0 \;\; (k > d), \tag{6.29}$$

to recover a polynomial potential as in (6.4). We now define a sequence of differential operators $\{L_n\}$, for $n \geq -1$:

$$L_n = T_2 \frac{\partial}{\partial g_{n+2}} + \sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{t^2}{N^2}\sum_{j=0}^{n} \frac{\partial}{\partial g_j}\frac{\partial}{\partial g_{n-j}}. \tag{6.30}$$

Note that the third term is defined to be zero if $n = -1$. One can show that these operators are generators for the Virasoro subalgebra:

$$[L_m, L_n] = (m - n)L_{m+n}. \tag{6.31}$$

Now we are ready to define a formal 1-matrix model from the Virasoro constraint.

Definition 6.2.1. Let F_g be a formal series in a set of variables $(t, T_2, g_0, g_1, \cdots)$ and we define Z as

$$Z = \exp\left(\sum_{g\geq 0}\left(\frac{t}{N}\right)^{2-2g} F_g\right). \tag{6.32}$$

Then, the partition function of a formal Hermitian 1-matrix model is defined as the unique solution of the following set of differential equations:

$$L_n Z = 0, \quad n \geq -1, \quad \frac{\partial Z}{\partial T_2} = \frac{1}{2}\frac{\partial Z}{\partial g_2}. \tag{6.33}$$

In this formulation, the partition function Z is simply defined as a formal series, it has nothing to do with path integrals. The first set of conditions $L_n Z = 0$ for $n \geq -1$ is called the *Virasoro constraint*. This becomes a key for the sypersymmetric generalization of 1-matrix model discussed in Chapter 8. Therefore from this perspective, the partition function (6.1) in the path integral formulation can be thought of as an integral representation of the solution of the set of differential equations (6.33). See [82] for discussions on the classification of more general matrix models.

Note that one can obtain a set of equations among $T^{(g)}_{l_1\cdots l_b}(t, t_k)$ with a polynomial potential (6.4) that is equivalent to the Virasoro constraint, called *Tutte equations*. However, it is more convenient to work with the extended power series potential (6.28) for our interests. We set $T_2 = 1$ for simplicity for the remainder of this section.

6.2.2 Loop Insertion Operator

With the generalized potential (6.28), we can obtain the correlation functions $W_{g,n}(x_1, \cdots, x_n)$ from the free energy by acting with the so-called *loop insertion operator*, which is defined by

$$\frac{\partial}{\partial V(x)} = -\sum_{k\geq 0} \frac{1}{x^{k+1}} \frac{\partial}{\partial g_k}. \tag{6.34}$$

Then $W_n(x_1, \cdots, x_n)$ and $W_{g,n}(x_1, \cdots, x_n)$ are obtained by:

$$W_n(x_1, \ldots, x_n) = \left(\frac{N}{t}\right)^{-n} \frac{\partial}{\partial V(x_n)} \cdots \frac{\partial}{\partial V(x_1)} F, \tag{6.35}$$

$$W_{g,n}(x_1, \ldots, x_n) = \frac{\partial}{\partial V(x_n)} \cdots \frac{\partial}{\partial V(x_1)} F_g. \tag{6.36}$$

This technique is analogous to inserting a source term in the action that we normally do in standard quantum field theories to get correlation functions. These correlation functions correspond to those in (6.16) by setting coupling constants as in (6.29) after taking the derivatives with respect to g_k.

6.2.3 Loop Equation

From the Virasoro constraint satisfied by Hermitian matrix models, one can derive a set of relations between correlation functions known as *loop equations*.

We start with the formal series in $1/x$:

$$\begin{aligned} 0 =& \frac{1}{Z} \sum_{n\geq 0} \frac{1}{x^{n+1}} L_{n-1} Z \\ =& \frac{1}{Z} \sum_{n\geq 0} \frac{1}{x^{n+1}} \left(\frac{\partial}{\partial g_{n+1}} + \sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+n-1}} + \frac{t^2}{N^2} \sum_{j=0}^{n-1} \frac{\partial}{\partial g_j} \frac{\partial}{\partial g_{n-j-1}} \right) Z, \end{aligned} \tag{6.37}$$

where the first equality holds due to the Virasoro constraint. We can rewrite (6.37) using the fact that correlation functions can be obtained by acting on the free energy with the loop insertion operator (6.35). After some manipulations[4] we obtain the loop equation

$$-\frac{N}{t}V'(x)W_1(x)+P_1(x)+\Big(W_1(x)\Big)^2+W_2(x,x)=0, \tag{6.38}$$

with

$$P_1(x)=-\frac{\partial}{\partial g_0}F-\sum_{m\geq 0}x^m\sum_{k\geq 0}(m+k+2)g_{m+k+2}\frac{\partial}{\partial g_k}F, \tag{6.39}$$

where $V'(x)$ denotes the derivative of the potential with respect to x.

Further, by acting an arbitrary number of times with the loop insertion operator on the loop equation, one obtains the general loop equation:

$$\begin{aligned}\frac{N}{t}V'(x)W_{n+1}(x,J) &= \sum_{I\subseteq J}W_{|I|+1}(x,I)W_{n-|I|+1}(x,J\backslash I)+W_{n+2}(x,x,J)\\ &\quad+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{W_n(x,J\backslash x_i)-W_n(J)}{x-x_i}+P_{n+1}(x,J),\end{aligned} \tag{6.40}$$

where we introduced the notation $J=(x_1,...,x_n)$, and $P_{n+1}(x;J)$ is defined by

$$P_{n+1}(x,J)=-\frac{t^n}{N^n}\left(\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial g_0}+\sum_{m\geq 0}x^m\sum_{k\geq 0}(m+k+2)g_{m+k+2}\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial g_k}\right)F. \tag{6.41}$$

If we insert the $1/N$ expansion in (6.40), the coefficient of $(N/t)^{2-2g-n}$ gives the expansion of the loop equation:

$$\begin{aligned}V'(x)W_{g,n+1}(x,J) &= \sum_{I\subseteq J}\sum_{h=0}^{g}W_{h,|I|+1}(x,I)W_{g-h,n-|I|+1}(x,J\backslash I)+W_{g-1,n+2}(x,x,J)\\ &\quad+\sum_{i=1}^{|J|}\frac{\partial}{\partial x_i}\frac{W_{g,n}(x,J\backslash x_i)-W_{g,n}(J)}{x-x_i}+P_{g,n+1}(x,J),\end{aligned} \tag{6.42}$$

where $P_{g,n+1}(x,J)$ is defined by

$$P_{g,n+1}(x,J)=-\left(\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial g_0}+\sum_{m\geq 0}x^m\sum_{k\geq 0}(m+k+2)g_{m+k+2}\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial g_k}\right)F_g. \tag{6.43}$$

[4]See Appendix D for the derivation.

If we now set the potential $V(x)$ to be a polynomial of degree d, that is, the coupling constants are chosen as in (6.29), then the $P_{g,n+1}(x,J)$ defined in (6.43) become polynomials in x — note that they are not necessarily polynomials with respect to $x_1,\cdots,x_n$ however. More precisely, $P_{0,1}(x)$ has degree $d-2$, while all other $P_{g,n+1}(x,J)$ are polynomials in x of degree $d-3$. This is because we can rewrite

$$Z = e^{-\frac{N^2 g_0}{t}}\tilde{Z}, \tag{6.44}$$

where $\tilde{Z}$ does not depend on g_0, therefore

$$\frac{\partial F}{\partial g_0} = -\left(\frac{N}{t}\right)^2 t. \tag{6.45}$$

Thus $\frac{\partial F_0}{\partial g_0} = -t$ while

$$\frac{\partial F_g}{\partial g_0} = 0 \qquad \text{for } g \geq 1. \tag{6.46}$$

It then follows that the highest degree term x^{d-2} in $P_{g,n+1}(x,J)$ is only non-vanishing for $(g,n)=(0,1)$.

Remark 6.2.2. Although loop equations (6.42) are derived from the Virasoro constraints, one can obtain precisely the same form of loop equations in the path integral formalism that we studied in the first section. More precisely, one can manipulate the Tutte equations mentioned above for $T^{(g)}_{l_1\cdots l_n}(t,t_k;T_2)$ and rewrite them in terms of correlation functions $W_{g,n+1}(x,J)$ and $P_{g,n+1}(x,J)$. More precisely, we can use Schwinger-Dyson equations, which are a consequence of invariance of the path integral under reparametrization of variables, to show their equivalence. This is of course expected because the Virasoro constraints and the path integral are simply two equivalent ways of defining matrix models, hence, the consequence should be the same. However, the computation from the Virasoro constraints is easier.

6.2.4 Recursion?

The loop equations (6.42) provide a set of relations between correlation functions. However, each relation also depends on a polynomial $P_{g,n+1}(x,J)$, see (6.43), which cannot *a priori* be computed from the matrix model. Then Eynard and Orantin [18, 19] proposed a way to recursively solve the loop equations for the correlation functions $W_{g,n}(x_1, \ldots, x_n)$, without first knowing the polynomials $P_{g,n+1}(x, J)$. We will study their mathematical technique in the next section.

7 Topological Recursion

Even though this recursive method was initially proposed by [18, 19] in the context of formal Hermitian matrix models, it can in fact be generalized beyond matrix models to the broader setup of algebraic geometry [17–19, 84]. The resulting abstract recursive formalism has become known in the literature as the Eynard-Orantin topological recursion, or simply topological recursion. In this section, we briefly review topological recursion, but leave the detail to [79], also see Section 2 in my paper with Bouchard [1].

7.1 Planar Equation

The loop equation (6.42) for $g = 0, n = 1$ is

$$V'(x)W_{0,1}(x) = \big(W_{0,1}(x)\big)^2 + P_{0,1}(x). \tag{7.1}$$

Let us define

$$y(x) = W_{0,1}(x) - \frac{1}{2}V'(x), \tag{7.2}$$

so that (7.1) can be rewritten as

$$y(x)^2 = \frac{1}{4}V'(x)^2 - P_{0,1}(x). \tag{7.3}$$

The loop equation (6.42) can also be rewritten in terms of $y(x)$. We obtain:

$$\begin{aligned} -2y(x)W_{g,n+1}(x,J) \;\; &= \;\; \sum_{I\subseteq J}^{*}\sum_{h=0}^{g} W_{h,|I|+1}(x,I)W_{g-h,n-|I|+1}(x,J\backslash I) + W_{g-1,n+2}(x,x,J) \\ &\quad + \sum_{i=1}^{|J|}\frac{\partial}{\partial x_i}\frac{W_{g,n}(x,J\backslash x_i) - W_{g,n}(J)}{x-x_i} + P_{g,n+1}(x,J), \end{aligned} \tag{7.4}$$

where $\sum_{I\subseteq J}^{*}\sum_{h=0}^{g}$ means that we are excluding the cases $(h,I) = (0,\emptyset)$ and $(h,I) = (g,J)$.

Brown's lemma [15, 85, 86], which applies to formal Hermitian 1-matrix models, implies that (7.3) defines a (potentially singular) genus zero hyperelliptic curve of degree $2d-2$:

Lemma 7.1.1. (Brown's Lemma) *There exists a polynomial $M(x)$ of x of degree $d-2$ whose roots α_i are power series of t, and a pair a, b of power series of $\sqrt{t}$ where $a+b$ and ab are power series of t, such that*

$$y^2 = M(x)^2(x-a)(x-b), \tag{7.5}$$

where

$$a = 2\sqrt{t} + \mathcal{O}(t), \quad b = -2\sqrt{t} + \mathcal{O}(t), \quad \alpha_i = \alpha_i^0 + \mathcal{O}(t), \tag{7.6}$$

with non-zero constants α_i^0.

(7.5) defines an algebraic curve. It is singular if $M(x) \neq 0$, but for any $M(x)$, the associated Riemann surface has genus zero. As we show below, we can parametrize x, y as rational functions of a local coordinate on the Riemann sphere.

A key point here is that while everything so far was defined as formal series in t, in (7.5) all the t-dependence is in a, b and the α_i. In fact, it follows from Brown's lemma that the coefficients of the degree $2d-2$ polynomial in x on the right-hand-side of (7.5) have a well defined power series expansion in t. We can even go further, and "re-sum" the power series; that is, we think of the coefficients as Taylor expansions of actual functions of t. In other words, we think of (7.5) as defining a t-dependent family of (potentially singular) genus zero hyperelliptic curves of degree $2d-2$.

7.1.1 Spectral Curve

This hyperelliptic curve, which is called the *spectral curve* for the general setting for topological recursion, plays a fundamental role for topological recursion. In fact, we will want to interpret the correlation functions $W_{g,n}(x_1, \cdots, x_n)$ as "living" on the spectral curve. Let us be a little more precise.

Since (7.5) has genus zero, we can parameterize it with rational functions:

$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right), \tag{7.7}$$

$$y(z) = M(x(z))\,\frac{a-b}{4}\left(z - \frac{1}{z}\right), \tag{7.8}$$

where z is a coordinate on the Riemann sphere.[1] We can think of $x : \mathbb{C}_\infty \to \mathbb{C}_\infty$ as a branched double covering. Its two simple *ramification points* are at $z = \pm 1$, which are the two simple zeros of the one-form

$$dx(z) = \frac{a-b}{4}\left(1 - \frac{1}{z^2}\right) dz. \tag{7.9}$$

The hyperelliptic involution that exchanges the two sheets of $x : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is given by $z \mapsto \sigma(z) = 1/z$, with

$$x(\sigma(z)) = x(z), \qquad y(\sigma(z)) = -y(z). \tag{7.10}$$

7.2 Pole Structure

We now want to understand the correlation functions as living on the spectral curve. More precisely, we *define* new objects, $\omega_{g,n}(z_1, \ldots, z_n)$, which are multilinear differentials on the Riemann sphere, and functions of t. In other words, they are multilinear differentials on the spectral curve. For $g \geq 0$, $n \geq 1$, and $2g - 2 + n \geq 1$, we define them such that

$$\omega_{g,n}(z_1, \cdots, z_n) = W_{g,n}(x_1, \cdots, x_n) dx_1 \cdots dx_n, \tag{7.11}$$

where we defined $x_i := x(z_i)$. By this equality, we mean that the Taylor expansion near $t = 0$ of the multilinear differential on the left-hand-side recovers the formal series of the correlation functions on the right-hand-side. For the two remaining cases, we define

$$\omega_{0,1}(z) = y(z)dx(z) = \left(W_{0,1}(x(z)) - \frac{1}{2}V'(x(z))\right) dx(z), \tag{7.12}$$

[1] We abuse notation slightly here and use $y(z)$ to define the meromorphic function on the Riemann sphere, while we previously used $y(x)$ to denote its formal t-expansion with polynomial coefficients in $1/x$.

and

$$\omega_{0,2}(z_1,z_2) = \left(W_{0,2}(x(z_1),x(z_2)) + \frac{1}{(x(z_1)-x(z_2))^2}\right) dx(z_1)dx(z_2). \tag{7.13}$$

The $\omega_{g,n}$ are now honest multilinear differentials on the spectral curve, so we can study their properties. As mentioned above, however, we leave all the detailed computations to [1, 19, 79], but rather summarize crucial results below.

7.2.1 $\omega_{0,2}(z_1,z_2)$

For $(g,n) = (0,2)$, after multiplying by $dx(z_1)dx(z_2)$ the loop equation (6.42) with respect to $y(z)$ reduces to

$$\begin{aligned}\omega_{0,2}(z_1,z_2) =& \frac{dx(z_1)dx(z_2)}{2y(z_1)}\left(\frac{d}{dx(z_2)}\frac{2y(z_2)+V'(x(z_1))-V'(x(z_2))}{2(x(z_1)-x(z_2))} + P_{0,2}(x(z_1),x(z_2))\right)\\ &+ \frac{dx(z_1)dx(z_2)}{2(x(z_1)-x(z_2))^2}.\end{aligned} \tag{7.14}$$

Note that it follows

$$\omega_{0,2}(z_1,z_2) + \omega_{0,2}(\sigma(z_1),z_2) = \frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}. \tag{7.15}$$

Let us now consider the pole structure of $\omega_{0,2}(z_1,z_2)$. We start with the zeros of $y(z_1)$ that are roots of $M(x(z_1))$. From (7.14) $\omega_{0,2}(z_1,z_2)$ can have at most poles of the form $1/(x(z_1)-\alpha_i)$ there. But by Brown's lemma, we know that $\alpha_i = \alpha_i^0 + \mathcal{O}(t)$ with a non-zero constant α_i^0. Therefore, if we do a Taylor expansion near $t=0$, the constant term would have the form

$$\frac{1}{x_1-\alpha_i^0} = \sum_{j\geq 0}\frac{(\alpha_i^0)^j}{x_1^{j+1}}, \tag{7.16}$$

where we used $x_1 = x(z_1)$ for clarity. It would then contribute an infinite series in $1/x_1$ for a fixed power of t, which contradicts the statement that $\omega_{0,2}(z_1,z_2)$ should recover a formal expansion in t with coefficients that are polynomials in $1/x_1$. Therefore $\omega_{0,2}(z_1,z_2)$ cannot have poles at the roots of $M(x(z_1))$.

This argument does not work however for the ramification points, which are simple zeros of $y(z_1)$. However, $dx(z_1)$ also has a simple zero there, hence $\omega(z_1,z_2)$ does not have poles at the ramification points.

All that remains are the coinciding points $z_1 = z_2$ and $z_1 = \sigma(z_2)$. As $z_1 \to \sigma(z_2)$, $y(z_1) \to y(\sigma(z_2)) = -y(z_2)$, and the double pole of the first line in (7.14) cancels out with the double pole of the second line. It thus follows that the only pole of $\omega_{0,2}(z_1, z_2)$ is a double pole at $z_1 = z_2$.

In fact, there is a unique bilinear differential on the spectral curve with a double pole at $z_1 = z_2$, no other pole, and that satisfies (7.15):

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{7.17}$$

This is the normalized bilinear differential of the second kind, which can be uniquely defined for Riemann surfaces of arbitrary genus [87]. The normalization is of course trivial here since the Riemann surface has genus zero.

Remark 7.2.1. This is a striking observation. $\omega_{0,2}(z_1, z_2)$ can be uniquely determined as the bilinear differential of the second kind no matter how we choose the potential $V(x)$.

7.2.2 $\omega_{g,n+1}(z_0, J)$ for $2g + n \geq 2$

Let us now study the multilinear differentials $\omega_{g,n+1}$ for $2g - 2 + n \geq 0$. We first show that

$$\omega_{g,n+1}(z, J) + \omega_{g,n+1}(\sigma(z), J) = 0, \tag{7.18}$$

where $J = \{z_1, \cdots, z_n\}$. We will prove this by induction on $2g - 2 + n$. The base cases are $\omega_{0,3}$ and $\omega_{1,1}$ with $2g - 2 + n = 0$.

For $\omega_{1,1}$, the loop equation (7.4) can be rewritten in terms of differentials as

$$-2y(z_0)dx(z_0)\omega_{1,1}(z_0) = -\omega_{0,2}(\sigma(z_0), z_0) + P_{1,1}(x(z_0))dx(z_0)^2. \tag{7.19}$$

The two terms on the right-hand-side are clearly invariant under $z_0 \mapsto \sigma(z_0)$, hence

$$\omega_{1,1}(z_0) + \omega_{1,1}(\sigma(z_0)) = 0. \tag{7.20}$$

As for $\omega_{0,3}$, (7.4) can be rewritten as

$$\begin{aligned}
-2y(z_0)dx(z_0)\omega_{0,3}(z_0,z_1,z_2) &= -\omega_{0,2}(z_0,z_1)\omega_{0,2}(\sigma(z_0),z_2) - \omega_{0,2}(\sigma(z_0),z_1)\omega_{0,2}(z_0,z_2) \\
&+ dx(z_0)^2\Bigg(dx(z_1)\frac{d}{dx(z_1)}\frac{\omega_{0,2}(\sigma(z_1),z_2)}{x(z_0)-x(z_1)} + dx(z_2)\frac{d}{dx(z_2)}\frac{\omega_{0,2}(\sigma(z_1),z_2)}{x(z_0)-x(z_2)} \\
&+ P_{0,3}(x(z_0),x(z_1),x(z_2))dx(z_1)dx(z_2)\Bigg). \qquad (7.21)
\end{aligned}$$

The first two terms on the right-hand-side are exchanged under $z_0 \mapsto \sigma(z_0)$, while the remaining terms on the right-hand-side are invariant. Therefore

$$\omega_{0,3}(z_0,z_1,z_2) + \omega_{0,3}(\sigma(z_0),z_1,z_2) = 0. \qquad (7.22)$$

We now prove (7.18) by induction. Assume that it is true for all (g,n) such that $0 \leq 2g-2+n < k$. We show that it implies that it must be true for $2g-2+n=k$. Assuming the induction hypothesis, for $2g-2+n \geq 1$ we can rewrite (7.4) in terms of differentials as

$$\begin{aligned}
2y(z_0)dx(z_0)\omega_{g,n+1}(z_0,J) &= \sum_{I\subseteq J}^{*}\sum_{h=0}^{g}\omega_{h,|I|+1}(z_0,I)\omega_{g-h,n-|I|+1}(\sigma(z_0),J\backslash I) \\
&+ \omega_{g-1,n+2}(z_0,\sigma(z_0),J) + dx(z_0)^2\Bigg(\sum_{i=1}^{|J|} dx(z_i)\frac{d}{dx(z_i)}\frac{\omega_{g,n}(J)}{x(z_0)-x(z_i)} \\
&- P_{g,n+1}(x(z_0),\cdots,x(z_n))dx(z_1)\cdots dx(z_n)\Bigg), \qquad (7.23)
\end{aligned}$$

with $J=\{z_1,\cdots,z_n\}$. The first summation is invariant under $z_0 \mapsto \sigma(z_0)$, and all other terms on the right-hand-side are also invariant. Therefore

$$\omega_{g,n+1}(z_0,J) + \omega_{g,n+1}(\sigma(z_0),J) = 0, \qquad (7.24)$$

and, by induction, this must hold for all (g,n) such that $2g-2+n \geq 0$.

Let us now study the pole structure for $\omega_{g,n+1}(z_0,J)$ in terms of z_0; since the correlation functions are symmetric the result will hold for all other z_i, $i=1,\cdots,n$ as well. The only possible poles are at zeros of $y(z_0)$, coinciding points $z_0=z_i$ and $z_0=\sigma(z_i)$, $i=1,\cdots,n$, and at poles of $x(z_0)$. First, there is no pole at poles of $x(z_0)$ since $P_{g,n+1}(x(z_0), x(z_1), \cdots, x(z_n))$ has degree $d-3$. Second, there is no pole at coinciding points $z_0 \to z_i$ by the loop equation (7.4) and no pole either at $z_0 \to \sigma(z_i)$ by the anti-symmetric involution relation (7.18). All that remains are zeros of $y(z_0)$. By the same argument as for $\omega_{0,2}(z_1, z_2)$, there cannot be poles at zeros of $M(x(z_0))$, otherwise the $\omega_{g,n+1}$ would have expansions in t with coefficients that are not polynomials in $1/x(z_0)$. The only remaining possible poles are at the ramification points of $x : \mathbb{C}_\infty \to \mathbb{C}_\infty$, that is, $z_0 = \pm 1$. In contrast to $\omega_{0,2}(z_1, z_2)$, these poles can be of higher order, and $dx(z_0)$ is not sufficient to get rid of them.

7.3 Topological Recursion

We are now ready to solve the loop equations recursively to determine all $\omega_{g,n+1}$ from $\omega_{0,1}$ and $\omega_{0,2}$. Let us start with the loop equation (7.23), rewritten as

$$\begin{aligned}\omega_{g,n+1}(z_0, J) = \frac{1}{2\omega_{0,1}(z_0)} \Bigg(\sum_{I \subseteq J}^{*} \sum_{h=0}^{g} \omega_{h,|I|+1}(z_0, I)\omega_{g-h,n-|I|+1}(\sigma(z_0), J \backslash I) \\ + \omega_{g-1,n+2}(z_0, \sigma(z_0), J) \Bigg) \\ + \frac{dx(z_0)}{2y(z_0)} \left(\sum_{i=1}^{|J|} dx(z_i) \frac{d}{dx(z_i)} \frac{\omega_{g,n}(J)}{x(z_0) - x(z_i)} - P_{g,n+1}(x(z_0), \cdots, x(z_n)) dx(z_1) \cdots dx(z_n) \right),\end{aligned} \tag{7.25}$$

with $J = \{z_1, \ldots, z_n\}$. It is clear that the third line of the expression has no pole at the ramification points in z_0. Thus, if we evaluate the residue of the expression on the right-hand-side at the ramification points, the third line does not contribute. We now take advantage of this fact to construct the so-called topological recursion.

Let us introduce the *normalized differential of the third kind* $\omega^{a-b}(z)$, which has simple poles at $z = a$ and $z = b$ with residues $+1$ and -1 respectively. It is given by

$$\omega^{a-b}(z) = \int_{z'=b}^{a} \omega_{0,2}(z', z) = \frac{dz}{z-a} - \frac{dz}{z-b}. \tag{7.26}$$

This object can in fact be defined for Riemann surfaces of arbitrary genus as the integral (in the fundamental domain) of the normalized bilinear differential of the second kind.

Let α be a generic base point on the Riemann sphere, and consider $\omega^{z-\alpha}(z')$. While it is a one-form in z', we can also think of it as a function in z.[2] It then follows that

$$\sum_{a\in\text{all poles}} \operatorname*{Res}_{w=a} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(w,J) = 0. \tag{7.27}$$

Note that this holds because the sum of all residues is zero on any compact Riemann surface. For $2g-2+n \geq 0$, the only poles of the integrand are at $w = z_0$ and at the ramification points $w = \pm 1$. The residue at $w = z_0$ gives

$$\operatorname*{Res}_{w=z_0} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(z_0,J) = -\omega_{g,n+1}(z_0,J). \tag{7.28}$$

It then follows that

$$\omega_{g,n+1}(z_0,J) = \sum_{a=\pm 1} \operatorname*{Res}_{w=a} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(w,J), \tag{7.29}$$

and, substituting (7.25) in the right-hand-side, we obtain *topological recursion*:

$$\begin{aligned}\omega_{g,n+1}(z_0,J) = \sum_{a\in\{-1,1\}} \operatorname*{Res}_{w=a} \frac{\omega^{w-\alpha}(z_0)}{2\omega_{0,1}(w)} \Bigg(\sum_{I\subseteq J}^{*} \sum_{h=0}^{g} \omega_{h,|I|+1}(w,I)\omega_{g-h,n-|I|+1}(\sigma(w),J\backslash I) \\ + \omega_{g-1,n+2}(w,\sigma(w),J)\Bigg). \end{aligned} \tag{7.30}$$

Remark 7.3.1. The last line in (7.25) is dropped because it has no contribution for the residue at the ramification points. In particular, the priori unknown factors $P_{g,n+1}(x,J)$ in (7.25) do not appear in the expression (7.30) as we desired. This is the power of the topological recursion in the context of matrix models.

Remark 7.3.2. It is also possible in (7.26) to have $\omega^{w-\sigma(w)}(z_0)$ instead of $\omega^{w-\alpha}(z_0)$. In this case, the form of the recursion formula (7.30) is slightly different, but it can be shown that it produces the same $\omega_{g,n+1}(z_0,J)$. We use this convention in Theorem 7.4.4 below.

[2]Note however that this is only true on the Riemann sphere, on higher genus Riemann surfaces as a function of z it is only defined in the fundamental domain.

(7.30) is a recursive formula which calculates all $\omega_{g,n+1}(z_0, J)$, $2g-2+n \geq 0$, from the initial data of a genus zero spectral curve

$$y^2 = M(x)(x-a)(x-b), \tag{7.31}$$

a one-form

$$\omega_{0,1}(z) = y(z)dx(z) = \left(W_{0,1}(x(z)) - \frac{1}{2}V'(x(z))\right) dx(z), \tag{7.32}$$

and a bilinear differential

$$\omega_{0,2}(z_1, z_2) = \left(W_{0,2}(x(z_1), x(z_2)) + \frac{1}{(x(z_1)-x(z_2))^2}\right) dx(z_1)dx(z_2). \tag{7.33}$$

7.4 Summary and Applications

We note that the Eynard-Orantin topological recursion is much more general than what we have shown in the previous section. It can be defined for (almost) arbitrary algebraic curves, not just (singular) hyperelliptic genus zero curves [18,19]. Let us give a more general definition of the Eynard-Orantin topological recursion by following Section 9 in [21][3].

Definition 7.4.1. Let Σ be a compact Riemann surface of genus g. A *canonical bilinear differential of the second kind* $\omega_{0,2}(p_1, p_2)$ is the the unique bilinear differential on $\Sigma \times \Sigma$ such that:

- Symmetric: $\omega_{0,2}(p_1, p_2) = \omega_{0,2}(p_2, p_1)$.
- Normalized: let A^I, B_I be a canonical basis of homology cycles of Σ, then for $1 \leq I \leq g$,

$$\oint_{A^I} \omega_{0,2}(\cdot, p) = 0. \tag{7.34}$$

- Double Pole: Its only pole is a double pole at the coinciding point

$$\omega_{0,2}(p_1, p_2) \underset{p_1 \to p_2}{\sim} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{regular}, \tag{7.35}$$

[3] [88,89] have further generalized the Eynard-Orantin topological recursion with arbitrary ramification points, but we stick to this definition because it has crucial connections to Airy structures later in this thesis.

where $z(p)$ is a local coordinate.

Definition 7.4.2. A *spectral curve* is a quadruple $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ where

- Σ is a compact Riemann surface.
- $x : \Sigma \to \Sigma' \subset \mathbb{P}^1$ is a branched covering with simple ramification points.
- $\omega_{0,1}$ is a meromorphic differential on Σ such that it has at most double zeroes at the ramification points.
- $\omega_{0,2}$ is a a canonical bilinear differential of the second kind.

Let $\mathfrak{r}$ be a set of ramification points, zeros of dx[4], and let z be a local coordinate around $r \in \mathfrak{r}$. Since they are simple, we can locally write

$$x(p) = x(r) + \frac{1}{2} z(p)^2. \tag{7.36}$$

Let $\imath : z \mapsto -z$ be a locally well-defined holomorphic involution such that $x(\imath(z)) = x(z)$.

Definition 7.4.3. The *recursion kernel* $K(p, q)$ is defined by

$$K(p, q) = \frac{1}{2} \frac{\int_{\imath(q)}^{q} \omega_{0,2}(\cdot, p)}{\omega_{0,1}(q) - \omega_{0,1}(\imath(q))}. \tag{7.37}$$

Theorem 7.4.4 (Eynard-Orantin Topological Recursion [18,19])**.** *Let $\omega_{g,n+1}(p, J)$ be a multilinear differential on Σ^{n+1} where $2g + n \geq 2$ and $J = (p_1, \cdots, p_2)$. Then, the* topological recursion *is a recursive formalism to compute a sequence of multilinear differential operators $\omega_{g,n+1}(p, J)$ by*

$$\omega_{g,n+1}(p, J) = \sum_{r \in \mathfrak{r}} \operatorname*{Res}_{q \to r} K(p, q) \left(\omega_{g-1,n+2}(q, \imath(q), J) + \sum_{\substack{g_1+g_2=g \\ J_1 \cup J_2 = J}}^{*} \omega_{g_1,|J_1|+1}(q, J_1) \omega_{g_2,|J_2|+1}(\imath(q), J_2) \right), \tag{7.38}$$

where $\sum^*$ means that we exclude terms with $\omega_{0,1}(p)$ from the sum.

[4] The poles of x of degree 2 or higher are also ramification points, but we do not consider such ramification points in the context of topological recursion.

7.4.1 Applications

Starting with a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, one can, in principle, compute an infinite sequence of multilinear differentials $\omega_{g,n+1}(p, J)$ for $2g + n \geq 2$. But why are we interested in these objects? What are they good for? Here are a few examples that show the power of topological recursion.

- Kontsevich-Witten theorem [90,91] states that ψ-classes intersection numbers, denoted by $\langle \tau_{i_1}, \cdots, \tau_{i_n} \rangle$, on moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves hold a recursive structure. The topological recursion for the Airy curve $y^2 = 2x$ generates these intersection numbers $\langle \tau_{i_1}, \cdots, \tau_{i_n} \rangle$.

- Mirzakhani showed in [92, 93] a recursive formula for for the Weil-Petersson volume $\mathcal{V}_{g,n}(L_1, \cdots, L_n)$ of the moduli space of bordered Riemann surfaces of genus g with n boundaries of length $L_1, \cdots, L_n$. It can be shown [94] that Mirzakhani recursion after Laplace transformation is equivalent to the topological recursion for the spectral curve defined by $y = \sin 2\pi\sqrt{2x}$ where this is an example whose spectral curve is not an algebraic curve.

- Topological recursion plays a crucial role for the Bouchard-Klemm-Mariño-Pasquetti theorem [95–97] in the context of mirror symmetry between toric Calabi-Yau 3-folds X, Y. The Eynard-Orantin topological recursion for the mirror curve defined in Y provides a recipe of computing Gromov-Witten invariants of X.

One can find more achievements of topological recursion in Borot's up-to-date notes [98].

8 Supereigenvalue Models

Given that formal Hermitian matrix models can be defined from the Virasoro constraints, it is interesting to see whether a similar story holds if we upgrade the Virasoro constraints into the super-Virasoro constraints. The idea of supereigenvalue models indeed originated in this way. More precisely, we define a partition function as a power series which is annihilated by a set of differential operators that generate a super-Virasoro subalgebra in the Neveu-Schwarz sector. The resulting partition function is not a matrix model, but it can be understood as a supersymmetric generalization of Hermitian matrix models in the eigenvalue formulation (6.8), hence the name supereigenvalue models.

In this section we study supereigenvalue models. Those were introduced in [99] and studied further in, for instance, [100–110]. From the super-Virasoro constraints, one can also derive super-loop equations satisfied by correlation functions. The missing link then is whether there exists a recursive formalism that solves the super-loop equations. Bouchard and I showed [1] that, in fact, the standard Eynard-Orantin topological recursion, combined with simple auxiliary equations, is sufficient to calculate all correlation functions in supereigenvalue models. We review the work in this section.

8.1 Definition and Properties

Let us start by defining supereigenvalue models.

8.1.1 Partition Function and Free Energy

Let $V(x)$ be a power series potential (6.28):

$$V(x) = \frac{T_2}{2}x^2 + \sum_{k\geq 0} g_k x^k, \tag{8.1}$$

and define a fermionic potential $\Psi(x)$ as

$$\Psi(x) = \sum_{k\geq 0} \xi_{k+\frac{1}{2}} x^k, \tag{8.2}$$

where the $\xi_{k+\frac{1}{2}}$ are Grassmann coupling constants.

Definition 8.1.1. We define the partition function of the *formal supereigenvalue model* as

$$\mathcal{Z}(t_s, g_k, \xi_{k+\frac{1}{2}}; T_2; 2N) \stackrel{\text{formal}}{=} \int d\lambda d\theta \Delta(\lambda, \theta) e^{-\frac{2N}{t_s}\sum_{i=1}^{2N}\left(V(\lambda_i)+\Psi(\lambda_i)\theta_i\right)}, \tag{8.3}$$

where the measure is

$$d\lambda = \prod_{i=1}^{2N} d\lambda_i, \quad d\theta = \prod_{i=1}^{2N} d\theta_i, \tag{8.4}$$

with the θ_i Grassmann variables, and $\Delta(\lambda, \theta)$ is

$$\Delta(\lambda, \theta) = \prod_{i<j}^{2N} (\lambda_i - \lambda_j - \theta_i\theta_j). \tag{8.5}$$

One should keep in mind here that this is a formal model, that is, the summation should be understood as being outside the integral. Similar to formal Hermitian matrix models, it can be shown that $\mathcal{Z}$ is given by a formal power series in t_s.

The free energy $\mathcal{F}$ for the supereigenvalue model is defined as usual by

$$\mathcal{F}(t_s, g_k, \xi_{k+\frac{1}{2}}; T_2; 2N) = \log \mathcal{Z}(t_s, g_k, \xi_{k+\frac{1}{2}}; T_2; 2N). \tag{8.6}$$

Remark 8.1.2. We will denote objects in supereigenvalue models, such as partition function, free energy, and correlation functions, with curly letters $\mathcal{Z}$, $\mathcal{F}$ and $\mathcal{W}_n$ to differentiate them from their Hermitian counterparts.

8.1.2 Super-Virasoro Constraints

It is straightforward to show that the partition function is annihilated by a set of differential operators generating a $\mathcal{N} = 1$ superconformal algebra in the Neveu-Schwarz (NS) sector.

That is, we define the super-Virasoro operators L_n, $G_{n+\frac{1}{2}}$ for $n \geq -1$ as

$$L_n = T_2 \frac{\partial}{\partial g_{n+2}} + \sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{1}{2}\left(\frac{t_s}{2N}\right)^2 \sum_{j=0}^{n} \frac{\partial}{\partial g_j} \frac{\partial}{\partial g_{n-j}}$$
$$+ \sum_{k\geq 0} \left(k + \frac{n+1}{2}\right) \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+k+\frac{1}{2}}} + \frac{1}{2}\left(\frac{t_s}{2N}\right)^2 \sum_{j=0}^{n-1} \left(\frac{n-1}{2} - j\right) \frac{\partial}{\partial \xi_{j+\frac{1}{2}}} \frac{\partial}{\partial \xi_{n-j-\frac{1}{2}}}, \quad (8.7)$$
$$G_{n+\frac{1}{2}} = T_2 \frac{\partial}{\partial \xi_{n+\frac{5}{2}}} + \sum_{k\geq 0} \left(k g_k \frac{\partial}{\partial \xi_{n+k+\frac{1}{2}}} + \xi_{k+\frac{1}{2}} \frac{\partial}{\partial g_{k+n+1}}\right) + \left(\frac{t_s}{2N}\right)^2 \sum_{j=0}^{n} \frac{\partial}{\partial \xi_{j+\frac{1}{2}}} \frac{\partial}{\partial g_{n-j}}, \quad (8.8)$$

where $\Sigma_{k=0}^{-1}, \Sigma_{k=0}^{-2}$ are defined to be zero. These operators are generators for the super-Virasoro subalgebra [99] :

$$\begin{aligned} [L_m, L_n] &= (m-n) L_{m+n}, \\ \left[L_m, G_{n+\frac{1}{2}}\right] &= \left(\frac{m-1}{2} - n\right) G_{n+m+\frac{1}{2}}, \\ \left\{G_{m+\frac{1}{2}}, G_{n+\frac{1}{2}}\right\} &= 2L_{n+m+1}. \end{aligned} \quad (8.9)$$

Then one can show that the partition function $\mathcal{Z}$ satisfies the following *super Virasoro constraints*

$$G_{n+\frac{1}{2}} \mathcal{Z} = 0, \qquad L_n \mathcal{Z} = 0, \qquad n \geq -1. \quad (8.10)$$

Putting it the other way around, one can leave $\Delta(\lambda, \theta)$ undetermined in Definition 8.1.1, and impose the super Virasoro constraints (8.10). Then, it is possible to prove that (8.5) is the unique solution, up to normalization, that satisfies the super-Virasoro constraints. Note that the condition $L_n \mathcal{Z} = 0$, $n \geq -1$, is automatically satisfied if $G_{n+\frac{1}{2}} \mathcal{Z} = 0$, $n \geq -1$, by the super-Virasoro algebra (8.9). So we only need to impose the fermionic condition.

Remark 8.1.3. Definition 8.1.1 is, in some sense, a combination of matrix models from functional integrals and matrix models from the Virasoro constraints in the supersymmetric realm. Namely, we started with an integral representation, and uniquely determined $\Delta(\lambda, \theta)$ by the super Virasoro constraints.

Remark 8.1.4. The expressions of $L_n, G_{n+\frac{1}{2}}$ are complicated, and one might be wondering how they were found. This is simply a differential representation of the super conformal field theory of a free boson and a free fermion. In particular, if we represent the bosonic modes a_n and the fermionic modes $\alpha_{n+\frac{1}{2}}$ by a set of formal variables $(g_n, \xi_{n+\frac{1}{2}})$ as

$$a_0 = \frac{\partial}{\partial g_0}, \quad a_{-n} = n g_n + \frac{T_2}{2}\delta_{n,2}, \quad a_n = \frac{\partial}{\partial g_n}, \quad (n > 0), \tag{8.11}$$

$$\alpha_{-n-\frac{1}{2}} = \xi_{n+\frac{1}{2}}, \quad \alpha_{n+\frac{1}{2}} = \frac{\partial}{\partial \xi_{n+\frac{1}{2}}}, \quad (n > 0), \tag{8.12}$$

then we obtain the representation of $L_n, G_{n+\frac{1}{2}}$ as (8.7) and (8.8).

8.1.3 Relation to Hermitian Matrix Models

A remarkable fact, originally proven in [102], see also in Appendix A in our paper [1], is that the free energy $\mathcal{F}$ of the formal supereigenvalue model contains the Grassman coupling constants $\xi_{k+\frac{1}{2}}$ only up to quadratic order. That is highly non-trivial. In the notation above, this means that

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(2)}. \tag{8.13}$$

For completeness, we provide a proof of this truncation for supereigenvalue models in Appendix B.

It turns out that this truncation of the fermionic expansion of $\mathcal{F}$ implies that $\mathcal{F}$ is closely related to the free energy of the formal Hermitian 1-matrix model F. More precisely, setting $t_s = 2t$, we get the following relation, which was proven in [101, 102]:

Proposition 8.1.5.

$$\mathcal{F}(2t, g_k, \xi_{k+\frac{1}{2}}; T_2; 2N) = 2\left(1 - \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_l \partial g_{k+1}}\right) F(t, g_k; T_2; N). \tag{8.14}$$

Note that the free energy on the left-hand-side is for the formal supereigenvalue model, while the free energy on the right-hand-side is for the formal Hermitian model. In other words,

$$\mathcal{F}^{(0)}(2t, g_k; T_2; 2N) = 2F(t, g_k; T_2; N) \tag{8.15}$$

$$\mathcal{F}^{(2)}(2t, g_k, \xi_{k+\frac{1}{2}}; T_2; 2N) = -2\sum_{k,l\geq 0} \xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_l \partial g_{k+1}} F(t, g_k; T_2; N). \tag{8.16}$$

This relation is fundamental. What it says is that the free energy of the formal supereigenvalue model is completely determined in terms of the free energy of the formal Hermitian matrix model. We gave a proof of this proposition in [1] which is different in flavour to the original one in [101]. It is purely algebraic; we show that the relation is a direct consequence of the super-Virasoro constraints. We review the algebraic proof in Appendix C.

Remark 8.1.6. A benefit of our algebraic proof is that we can now define a formal supereigenvalue model without using the integral representation as we defined formal Hermitian matrix models from the Virasoro constraints in Section 6.2. Namely, we can modify Definition 6.2.1 by replacing the Virasoro constraints with the super Virasoro constraints, and add another constraint that the free energy truncates at quadratic order in Grassmann couplings. It is an open question whether some models beyond supereigenvalue models exist if we modify the truncation to higher orders.

We now set $T_2 = 1$ for simplicity. With this under our belt, we can define the $1/N$ expansion of the free energy. Since $t_s = 2t$, it is natural to define the $1/N$ expansion for $\mathcal{F}$ as

$$\mathcal{F}(2t, g_k, \xi_{k+\frac{1}{2}}; 2N) = \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g} \mathcal{F}_g(2t, g_k, \xi_{k+\frac{1}{2}}). \tag{8.17}$$

Then (8.14) implies that

$$\mathcal{F}_g(2t, g_k, \xi_{k+\frac{1}{2}}) = 2\left(1 - \sum_{k,l} \xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}} \frac{\partial}{\partial g_{k+1}}\frac{\partial}{\partial g_l}\right) F_g(t, g_k). \tag{8.18}$$

8.1.4 Correlation Functions

The correlation functions of formal Hermitian 1-matrix models can be obtained by acting with the loop insertion operator (6.34) a number of times on the free energy, as shown in (6.35). We can define correlation functions in supereigenvalue models in a similar way.

We define the following bosonic and fermionic loop insertion operators:

$$\frac{\partial}{\partial V(x)} = -\sum_{k\geq 0} \frac{1}{x^{k+1}}\frac{\partial}{\partial g_k}, \qquad \frac{\partial}{\partial \Psi(X)} = -\sum_{k\geq 0} \frac{1}{X^{k+1}}\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}. \tag{8.19}$$

Correlation functions are then obtained by:

$$\begin{aligned}
\mathcal{W}_{n|m}(J|K) &= \left(\frac{N}{t}\right)^{-n-m} \prod_{j=1}^{n} \frac{\partial}{\partial V(x_j)} \prod_{i=1}^{m} \frac{\partial}{\partial \Psi(X_i)} \mathcal{F}(2t, g_k, \xi_{k+\frac{1}{2}}; 2N) \\
&= \sum_{k_1 \cdots k_m \geq 0} \sum_{l_1 \cdots l_n \geq 0} \sum_{a_1, \cdots a_m = 1}^{2N} \sum_{b_1 \cdots b_n = 1}^{2N} \frac{\left\langle \lambda_{a_1}^{k_1} \cdots \lambda_{a_n}^{k_n} \theta_{b_1} \lambda_{b_1}^{l_1} \cdots \theta_{b_m} \lambda_{b_m}^{l_m} \right\rangle_c}{x_1^{k_1+1} \cdots x_n^{k_n+1} X_1^{l_1+1} \cdots X_m^{l_m+1}},
\end{aligned} \tag{8.20}$$

where $J = \{x_1, \cdots, x_n\}$ and $K = \{X_1, \cdots, X_m\}$. We removed the dependence of the correlation functions on coupling constants for clarity.

As usual, the correlation functions inherit from (8.18) a $1/N$ expansion:

$$\mathcal{W}_{n|m}(J|K) = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g-m-n} \mathcal{W}_{g,n|m}(J|K). \tag{8.21}$$

We can further expand the correlation functions in terms of the fermionic coupling constants $\xi_{k+\frac{1}{2}}$. Since $\mathcal{F}$ is at most quadratic in the Grassmann parameters, *i.e.* $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(2)}$, we see that the only non-vanishing correlation functions have $0 \leq m \leq 2$. Further, we get

$$\mathcal{W}_{g,n|0}(J|) = \mathcal{W}_{g,n|0}^{(0)}(J|) + \mathcal{W}_{g,n|0}^{(2)}(J|), \tag{8.22}$$

$$\mathcal{W}_{g,n|1}(J|X_1) = \mathcal{W}_{g,n|1}^{(1)}(J|X_1), \tag{8.23}$$

$$\mathcal{W}_{g,n|2}(J|X_1, X_2) = \mathcal{W}_{g,n|2}^{(0)}(J|X_1, X_2), \tag{8.24}$$

where, as usual, the superscript denotes the terms of a given order in the Grassmann parameters.

From (8.18) we expect all these correlation functions to be somehow determined in terms of correlation functions of the Hermitian matrix model. Indeed, we showed in [1] the following relations between correlation functions for supereigenvalue models and those in Hermitian matrix models, which can be thought of as a consequence of Proposition 8.1.5 for correlation functions:

Proposition 8.1.7.

$$\begin{aligned}
\mathcal{W}^{(0)}_{g,n|0}(J|) =& 2W_{g,n}(J), && (8.25)\\
\mathcal{W}^{(0)}_{g,n|2}(J|X_1,X_2) =& 2(X_1-X_2)W_{g,n+2}(X_1,X_2,J), && (8.26)\\
\mathcal{W}^{(1)}_{g,n|1}(J|X_1) =& \operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)dX, && (8.27)\\
\mathcal{W}^{(2)}_{g,n|0}(J|) =& \frac{1}{2}\operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(1)}_{g,n|1}(J|X)dX. && (8.28)
\end{aligned}$$

The important point here is that all correlation functions of formal supereigenvalue models are determined in terms of $W_{g,n}(J)$, the correlation functions of formal Hermitian matrix models.

Proof. (8.25) is straightforward from Proposition 8.1.5, hence, we start with $\mathcal{W}^{(0)}_{g,n|2}(J|X_1,X_2)$. We have:

$$\begin{aligned}
\mathcal{W}^{(0)}_{g,n|2}(J|X_1,X_2) =& \prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial\Psi(X_1)}\frac{\partial}{\partial\Psi(X_2)}\mathcal{F}^{(2)}_g(2t,g_k,\xi_{k+\frac{1}{2}})\\
=& -2\sum_{k,l\geq 0}\frac{1}{X_1^{k+1}X_2^{l+1}}\frac{\partial}{\partial\xi_{k+\frac{1}{2}}}\frac{\partial}{\partial\xi_{l+\frac{1}{2}}}\left(\sum_{i,j}\xi_{i+\frac{1}{2}}\xi_{j+\frac{1}{2}}\frac{\partial}{\partial g_{i+1}}\frac{\partial}{\partial g_j}\right)\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}F_g(t,g_k)\\
=& -2\sum_{k,l\geq 0}\frac{1}{X_1^{k+1}X_2^{l+1}}\left(\frac{\partial}{\partial g_{l+1}}\frac{\partial}{\partial g_k}-\frac{\partial}{\partial g_{k+1}}\frac{\partial}{\partial g_l}\right)W_{g,n}(J). && (8.29)
\end{aligned}$$

We can simplify this further. Recall from (6.45) that

$$\frac{\partial F_0}{\partial g_0}=-t,\qquad \frac{\partial F_g}{\partial g_0}=0 \text{ for all } g\geq 1. \tag{8.30}$$

Thus we can rewrite

$$\begin{aligned}
&\sum_{k,l\geq 0}\frac{1}{X_1^{k+1}X_2^{l+1}}\left(\frac{\partial}{\partial g_{l+1}}\frac{\partial}{\partial g_k}-\frac{\partial}{\partial g_{k+1}}\frac{\partial}{\partial g_l}\right)W_{g,n}(J)\\
&=\sum_{k,l\geq 0}\left(\frac{1}{X_1^{k+1}X_2^{l}}-\frac{1}{X_1^{k}X_2^{l+1}}\right)\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_k}W_{g,n}(J)\\
&=(X_2-X_1)\sum_{k,l\geq 0}\frac{1}{X_1^{k+1}X_2^{l+1}}\frac{\partial}{\partial g_l}\frac{\partial}{\partial g_k}W_{g,n}(J)\\
&=(X_2-X_1)\frac{\partial}{\partial V(X_1)}\frac{\partial}{\partial V(X_2)}W_{g,n}(J)\\
&=(X_2-X_1)W_{g,n+2}(X_1,X_2,J).
\end{aligned}\tag{8.31}$$

It thus follows that

$$\mathcal{W}^{(0)}_{g,n|2}(J|X_1,X_2)=2(X_1-X_2)W_{g,n+2}(X_1,X_2,J).\tag{8.32}$$

Let us now turn to $\mathcal{W}^{(1)}_{g,n|1}(J|X_1)$. We do not need to do much work here. We note that

$$\begin{aligned}
\operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)dX =&\operatorname*{Res}_{X=\infty}\Psi(X)\frac{\partial}{\partial\Psi(X)}\mathcal{W}^{(1)}_{g,n|1}(J|X_1)dX\\
=&-\operatorname*{Res}_{X=\infty}\sum_{k\geq 0}\sum_{l\geq 0}\xi_{k+\frac{1}{2}}X^{k-l-1}\frac{\partial}{\partial\xi_{l+\frac{1}{2}}}\mathcal{W}^{(1)}_{g,n|1}(J|X_1)dX\\
=&\sum_{k\geq 0}\xi_{k+\frac{1}{2}}\frac{\partial}{\partial\xi_{k+\frac{1}{2}}}\mathcal{W}^{(1)}_{g,n|1}(J|X_1).
\end{aligned}\tag{8.33}$$

But since $\mathcal{W}^{(1)}_{g,n|1}(J|X_1)$ depends linearly on the Grassmann coupling constants $\xi_{k+\frac{1}{2}}$, the operator $\sum_{k\geq 0}\xi_{k+\frac{1}{2}}\frac{\partial}{\partial\xi_{k+\frac{1}{2}}}$ is the identity operator. Hence, we get

$$\mathcal{W}^{(1)}_{g,n|1}(J|X_1)=\operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)dX.\tag{8.34}$$

For $\mathcal{W}^{(2)}_{g,n|0}(J|)$, we get:

$$\begin{aligned}\mathcal{W}^{(2)}_{g,n|0}(J|) =&\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\mathcal{F}^{(2)}_g(2t,g_k,\xi_{k+\frac{1}{2}})\\ =&-2\sum_{k,l}\xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}}\frac{\partial}{\partial g_{k+1}}\frac{\partial}{\partial g_l}\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}F_g(t,g_k)\\ =&2\sum_{k,l}\xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}}\frac{\partial}{\partial g_{k+1}}\frac{\partial}{\partial g_l}W_{g,n}(J).\end{aligned} \tag{8.35}$$

We can use the same residue trick as for $\mathcal{W}^{(1)}_{g,n|1}(J|X_1)$. It then follows that

$$\operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(1)}_{g,n|1}(J|X)dX = \sum_{k\geq 0}\xi_{k+\frac{1}{2}}\frac{\partial}{\partial\xi_{k+\frac{1}{2}}}\mathcal{W}^{(2)}_{g,n|0}(J|). \tag{8.36}$$

It is easy to see that the right-hand-side is $2\mathcal{W}^{(2)}_{g,n|0}(J|)$, and we obtain

$$\mathcal{W}^{(2)}_{g,n|0}(J|) = \frac{1}{2}\operatorname*{Res}_{X=\infty}\Psi(X)\mathcal{W}^{(1)}_{g,n|1}(X|J)dX. \tag{8.37}$$

$\square$

8.2 Super Loop Equations

Let us now turn to the study of super-loop equations. There are more than one type of loop equations in supereigenvalue models, depending on the order of the Grassmann coupling constants. We call loop equations with an even (resp. odd) dependence on the Grassmann parameters "bosonic" (resp. "fermionic"). We simply give the equations here and leave their derivations to Appendix D.2 and D.3.

8.2.1 Fermionic Loop Equation

The derivation of the fermionic loop equation starts with the following formal series

$$\frac{1}{\mathcal{Z}}\sum_{n\geq 0}\frac{1}{X^{n+1}}G_{n-\frac{1}{2}}\mathcal{Z} = 0. \tag{8.38}$$

After a few manipulations we obtain the fermionic loop equation:

$$-\frac{N}{t}V'(X)\mathcal{W}_{0|1}(|X)-\frac{N}{t}\Psi(X)\mathcal{W}_{1|0}(X|)+\mathcal{W}_{1|1}(X|X)+\mathcal{W}_{1|0}(X|)\mathcal{W}_{0|1}(|X)+\mathcal{P}_{0|1}(|X)=0, \tag{8.39}$$

where

$$\mathcal{P}_{0|1}(|X)=\left(-\frac{\partial}{\partial\xi_{\frac{1}{2}}}-\sum_{k\geq 0}X^k\left(\sum_{l\geq 0}(k+l+2)g_{k+l+2}\frac{\partial}{\partial\xi_{l+\frac{1}{2}}}+\xi_{k+l+\frac{3}{2}}\frac{\partial}{\partial g_l}\right)\right)\mathcal{F}. \tag{8.40}$$

Now we expand the fermionic loop equation (8.39) in terms of $1/N$, and act an arbitrary number of times with the bosonic loop insertion operator on it. Collecting terms order by order in the Grassmann coupling constants, we get the following two fermionic loop equations:

$$\begin{aligned}V'(X)\mathcal{W}^{(1)}_{g,n|1}(J|X)&+\Psi(X)\mathcal{W}^{(0)}_{g,n+1|0}(X,J|)-\mathcal{P}^{(1)}_{g,n|1}(J|X)\\ &=\sum_{I\subseteq J}\sum_{h=0}^{g}\mathcal{W}^{(1)}_{h,m|1}(I|X)\mathcal{W}^{(0)}_{g-h,n-m+1|0}(X,J\backslash I|)+\mathcal{W}^{(1)}_{g-1,n+2}(X|X,J)\\ &\qquad+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\mathcal{W}^{(1)}_{g,n-1|1}(J\backslash x_i|X)-\mathcal{W}^{(1)}_{g,n-1|1}(J\backslash x_i|x_i)}{X-x_i},\end{aligned} \tag{8.41}$$

and

$$\Psi(X)\mathcal{W}^{(2)}_{g,n+1|0}(X,J|)-\mathcal{P}^{(3)}_{g,n|1}(J|X)=\sum_{I\subseteq J}\sum_{h=0}^{g}\mathcal{W}^{(2)}_{h,m+1|0}(X,I|)\mathcal{W}^{(1)}_{g-h,n-m|1}(J\backslash I|X), \tag{8.42}$$

where we defined

$$\mathcal{P}_{g,n|1}(J|X)=\left(-\frac{\partial}{\partial\xi_{\frac{1}{2}}}-\sum_{k\geq 0}X^k\left(\sum_{l\geq 0}(k+l+2)g_{k+l+2}\frac{\partial}{\partial\xi_{l+\frac{1}{2}}}+\xi_{k+l+\frac{3}{2}}\frac{\partial}{\partial g_l}\right)\right)\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\mathcal{F}_g, \tag{8.43}$$

which, by (8.18), has an expansion $\mathcal{P}_{g,n|1}(J|X)=\mathcal{P}^{(1)}_{g,n|1}(J|X)+\mathcal{P}^{(3)}_{g,n|1}(J|X)$.

If we act with the fermionic loop insertion operator on (8.41), we obtain the equation for

$\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)$:

$$\begin{aligned}
&V'(X)\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)-\mathcal{P}^{(0)}_{g,n|2}(J|X,X_1)\\
&\qquad=\sum_{I\subseteq J}\sum_{h=0}^{g}\mathcal{W}^{(0)}_{h,m|2}(I|X,X_1)\mathcal{W}^{(0)}_{g-h,n-m|1}(J\backslash I|X)+\frac{\mathcal{W}^{(0)}_{g,n|1}(J|X)-\mathcal{W}^{(0)}_{g,n|1}(J|X_1)}{X-X_1}\\
&+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\mathcal{W}^{(0)}_{g,n-1|2}(J\backslash x_i|X,X_1)-\mathcal{W}^{(0)}_{g,n-1|2}(J\backslash x_i|x_i,X_1)}{X-x_i}+\mathcal{W}^{(0)}_{g-1,n+1|2}(X,J|X,X_1),
\end{aligned}\tag{8.44}$$

where

$$\mathcal{P}^{(0)}_{g,n|2}(J|X,X_1)=\left(-\frac{\partial}{\partial\xi_{\frac{1}{2}}}-\sum_{k\geq0}X^k\sum_{l\geq0}(k+l+2)g_{k+l+2}\frac{\partial}{\partial\xi_{l+\frac{1}{2}}}\right)\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\frac{\partial}{\partial\Psi(X_1)}\mathcal{F}^{(2)}_g.\tag{8.45}$$

8.2.2 Bosonic Loop Equation

To get the bosonic loop equation, we start with the formal series:

$$\frac{1}{\mathcal{Z}}\sum_{n\geq0}\frac{1}{x^{n+1}}L_{n-1}\mathcal{Z}=0.\tag{8.46}$$

We manipulate the above equation to obtain the bosonic loop equation:

$$\begin{aligned}
-\frac{N}{t}V'(x)\mathcal{W}_{1|0}(x|)+\frac{1}{2}\Big(\mathcal{W}_{1|0}(x|)\Big)^2+\frac{1}{2}\mathcal{W}_{2|0}(x,x|)-\frac{N}{2t}\Psi'(x)\mathcal{W}_{0|1}(|x)+\frac{N}{2t}\Psi(x)\frac{\partial}{\partial x}\mathcal{W}_{0|1}(|x)\\
+\frac{1}{2}\mathcal{W}_{0|1}(|x)\frac{\partial}{\partial x}\mathcal{W}_{0|1}(|x)+\frac{1}{2}\frac{\partial}{\partial x}\left(\mathcal{W}_{0|2}(|x,x')\right)\Big|_{x'=x}+\mathcal{P}_{1|0}(x|)=0,
\end{aligned}\tag{8.47}$$

where we defined

$$\mathcal{P}_{1|0}(x|)=-\frac{\partial\mathcal{F}}{\partial g_0}-\sum_{n\geq0}x^n\left(\sum_{k\geq0}(n+k+2)g_{n+k+2}\frac{\partial\mathcal{F}}{\partial g_k}+\frac{1}{2}\sum_{k\geq0}\xi_{n+k+\frac{5}{2}}\,(n+2k+3)\,\frac{\partial\mathcal{F}}{\partial\xi_{k+\frac{1}{2}}}\right).\tag{8.48}$$

Before we do a $1/N$ expansion, let us study the dependence on the Grassmann parameters. (8.14) implies that the dependence of the bosonic loop equation is at most of order 4. Since

$\mathcal{P}_{1|0}(x|)$ depends on the Grassmann parameters at most quadratically, the order 4 terms in the bosonic loop equation directly yield the condition:

$$\left(\mathcal{W}^{(2)}_{1|0}(x|)\right)^2 = 0. \tag{8.49}$$

Let us now study the bosonic equation at order 2 and 0. We act an arbitrary number of times with the bosonic loop insertion operator on (8.47), and then do a $1/N$-expansion. We also collect terms according to their order in the Grassmann parameters. We obtain the two following equations:

$$\begin{aligned} &V'(x)\mathcal{W}^{(0)}_{g,n+1|0}(x,J|) - \mathcal{P}^{(0)}_{g,n+1|0}(x,J|) \\ &\qquad = \frac{1}{2}\sum_{I\subseteq J}\sum_{h=0}^{g}\mathcal{W}^{(0)}_{h,m+1|0}(x,I|)\mathcal{W}^{(0)}_{g-h,n-m+1|0}(x,J\backslash I|) + \frac{1}{2}\mathcal{W}^{(0)}_{g-1,n+2|0}(x,x,J|) \\ &\qquad\qquad + \frac{1}{2}\frac{\partial}{\partial x}\mathcal{W}^{(0)}_{g-1,n|2}(J|x,x')\Big|_{x'=x} + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\mathcal{W}^{(0)}_{g,n|0}(x,J\backslash x_i|) - \mathcal{W}^{(0)}_{g,n|0}(J|)}{x-x_i}, \end{aligned} \tag{8.50}$$

and

$$\begin{aligned} &V'(x)\mathcal{W}^{(2)}_{g,n+1|0}(x,J|) - \mathcal{P}^{(2)}_{g,n+1|0}(x,J|) \\ &= \sum_{I\subseteq J}\sum_{h=0}^{g}\left(\mathcal{W}^{(2)}_{h,m+1|2}(x,I|)\mathcal{W}^{(0)}_{g-h,n-m+1|0}(x,J\backslash I|) - \frac{1}{2}\mathcal{W}^{(1)}_{h,m|1}(I|x)\frac{\partial}{\partial x}\mathcal{W}^{(1)}_{g-h,n-m|1}(J\backslash I|x)\right) \\ &\qquad + \frac{1}{2}\mathcal{W}^{(2)}_{g-1,n+2|0}(x,x,J|) + \frac{1}{2}\left(\Psi(x)\frac{\partial}{\partial x}\mathcal{W}^{(1)}_{g,n|1}(J|x) - \Psi'(x)\mathcal{W}^{(1)}_{g,n|1}(J|x)\right) \\ &\qquad\qquad + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\mathcal{W}^{(2)}_{g,n|0}(x,J\backslash x_i|) - \mathcal{W}^{(2)}_{g,n|0}(J|)}{x-x_i}, \end{aligned} \tag{8.51}$$

where we defined

$$\mathcal{P}^{(0)}_{g,n+1|0}(x,J|)=\left(-\frac{\partial}{\partial g_0}-\sum_{l\geq 0}x^l\sum_{k\geq 0}(l+k+2)g_{l+k+2}\frac{\partial}{\partial g_k}\right)\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\mathcal{F}^{(0)}_g, \tag{8.52}$$

$$\begin{aligned}\mathcal{P}^{(2)}_{g,n+1|0}(x,J|)=&\Bigg(-\frac{\partial}{\partial g_0}-\sum_{l\geq 0}x^l\Bigg(\sum_{k\geq 0}(l+k+2)g_{l+k+2}\frac{\partial}{\partial g_k}\\&+\frac{1}{2}\sum_{k\geq 0}\xi_{l+k+\frac{5}{2}}(l+2k+3)\frac{\partial}{\partial \xi_{k+\frac{3}{2}}}\Bigg)\Bigg)\prod_{j=1}^{n}\frac{\partial}{\partial V(x_j)}\mathcal{F}^{(2)}_g.\end{aligned} \tag{8.53}$$

To conclude this section, let us show that the bosonic loop equation that is independent of the Grassmann parameters, that is (8.50), is indeed equivalent to the loop equation for Hermitian matrix models (6.42). First of all, (8.26) turns the third term on the right-hand-side in (8.50) into

$$\frac{1}{2}\frac{\partial}{\partial x}\mathcal{W}^{(0)}_{g-1,n|2}(J|x,x')\Big|_{x'=x}=\frac{\partial}{\partial x}\Big((x-x')W_{g-1,n+2}(x,x',J)\Big)\Big|_{x'=x}=W_{g-1,n+2}(x,x,J). \tag{8.54}$$

Also, (8.18) implies that $\mathcal{P}^{(0)}_{g,n+1|0}(x,J|)=2P_{g,n+1}(x,J)$. We further substitute $\mathcal{W}^{(0)}_{h,l|0}(L|)=2W_{h,l}(L)$ for all h,l into (8.50) . We obtain

$$\begin{aligned}&2V'(x)W_{g,n+1}(x,J)-2P_{g,n+1}(x,J)\\&\quad=2\sum_{I\subseteq J}\sum_{h=0}^{g}W_{h,m+1}(x,I)W_{g-h,n-m+1}(x,J\backslash I)+W_{g-1,n+2}(x,x,J)\\&\qquad+W_{g-1,n+2}(x,x,J)+2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{W_{g,n}(x,J\backslash x_i)-W_{g,n}(J)}{x-x_i},\end{aligned} \tag{8.55}$$

which is precisely twice the loop equation for Hermitian matrix models (6.42).

8.3 Topological Recursion

Guessing from the analysis on Hermitian matrix models, one may expect to utilize supersymmetric loop equations derived above in order to recursively compute all correlation functions. However, (8.18) significantly simplifies supereigenvalue models and indeed makes it possible for us to compute all correlation functions using the Eynard-Orantin topological recursion with a single auxiliary Grassmann polynomial equation.

8.3.1 Eynard-Orantin Topological Recursion

Let us recall what we did in Section 7 for Hermitian matrix models. We constructed a sequence of multilinear differentials $\omega_{g,n}(z_1,\ldots,z_n)$ on the Riemann sphere (and functions of t), such that, for $g\geq 0$, $n\geq 1$ and $2g-2+n\geq 1$:

$$\omega_{g,n}(z_1,\cdots,z_n)=W_{g,n}(x_1,\cdots,x_n)dx_1\cdots dx_n, \tag{8.56}$$

where $x_i:=x(z_i)$. By this equality, we meant that the Taylor expansion of the multilinear differential on the left-hand-side near $t=0$ recovers the formal series of the correlation functions on the right-hand-side.

We also defined the two "unstable" cases ($2g-2+n\leq 0$) as:

$$\omega_{0,1}(z)=\left(W_{0,1}(x(z))-\frac{1}{2}V'(x(z))\right)dx(z) \tag{8.57}$$

and

$$\omega_{0,2}(z_1,z_2)=\left(W_{0,2}(x(z_1),x(z_2))+\frac{1}{(x(z_1)-x(z_2))^2}\right)dx(z_1)dx(z_2). \tag{8.58}$$

Then we showed that:

1. There are two meromorphic functions $x(z)$ and $y(z)$ on the Riemann sphere such that $\omega_{0,1}(z)=y(z)dx(z)$ and

$$y^2=M(x)^2(x-a)(x-b), \tag{8.59}$$

with $M(x)$ a polynomial of degree $d-2$. We call this hyperelliptic curve the *spectral curve* of the matrix model. We generally choose the coordinate z on the Riemann sphere as giving by the parameterization

$$\begin{aligned} x(z) &= \frac{a+b}{2}+\frac{a-b}{4}\left(z+\frac{1}{z}\right), \\ y(z) &= M(x(z))\,\frac{a-b}{4}\left(z-\frac{1}{z}\right), \end{aligned} \tag{8.60}$$

of the hyperelliptic curve.

2. $\omega_{0,2}(z_1, z_2)$ takes a very simple form; it is the normalized bilinear differential of the second kind on the Riemann sphere, that is,

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{8.61}$$

3. The multilinear differentials $\omega_{g,n}(z_1, \cdots, z_n)$, for $2g - 2 + n \geq 1$, satisfy the Eynard-Orantin topological recursion (7.30). The initial conditions of the recursion are $\omega_{0,1}(z)$ and $\omega_{0,2}(z_1, z_2)$.

8.3.2 Supereigenvalue Multilinear Differentials

We would like to extend these results to supereigenvalue correlation functions. More precisely, we would like to construct (Grassman-valued) multilinear differentials on the spectral curve (8.59) in a similar way. For $2g - 2 + m + n + p \geq 1$, we define multilinear differentials:

$$\Omega^{(p)}_{g,n|m}(z_1, \cdots, z_n | w_1, \cdots, w_m) = \frac{1}{2} \mathcal{W}^{(p)}_{g,n|m}(x_1, \cdots, x_n | X_1, \cdots, X_m) dx_1 \cdots dx_n dX_1 \cdots dX_m, \tag{8.62}$$

where $x_i := x(z_i)$ and $X_j := x(w_j)$. As for Hermitian matrix models, the equality here means that after Taylor-expanding the left-hand-side near $t = 0$, we recover the formal series of the correlation functions on the right-hand-side. Note that the factor of $1/2$ is there simply for convenience.

For the "unstable" cases ($2g - 2 + m + n + p \leq 0$), we modify the definitions slightly as for Hermitian matrix models. We define:

$$\Omega^{(0)}_{0,1|0}(z|) = \frac{1}{2} \left(\mathcal{W}^{(0)}_{0,1|0}(x(z)|) - V'(x(z)) \right) dx(z) \tag{8.63}$$

$$\Omega^{(0)}_{0,2|0}(z_1, z_2|) = \frac{1}{2} \left(\mathcal{W}^{(0)}_{0,2|0}(x(z_1), x(z_2)|) + \frac{1}{(x(z_1) - x(z_2))^2} \right) dx(z_1) dx(z_2) \tag{8.64}$$

$$\Omega^{(1)}_{0,0|1}(|w) = \frac{1}{2} \left(\mathcal{W}^{(1)}_{0,0|1}(|x(w)) - \Psi(x(w)) \right) dx(w) \tag{8.65}$$

$$\Omega^{(0)}_{0,0|2}(|w_1, w_2) = \frac{1}{2} \left(\mathcal{W}^{(0)}_{0,0|2}(|x(w_1), x(w_2)) + \frac{1}{x(w_1) - x(w_2)} \right) dx(w_1) dx(w_2). \tag{8.66}$$

From (8.25) and (8.26), we immediately obtain the relation between these $\Omega^{(0)}_{g,n|m}(J|K)$ and the differentials $\omega_{g,n}(J)$ computed from the Eynard-Orantin topological recursion:

$$\Omega^{(0)}_{g,n|0}(J|) = \omega_{g,n}(J), \tag{8.67}$$

$$\Omega^{(0)}_{g,n|2}(J|w_1, w_2) = (x(w_1) - x(w_2))\omega_{g,n}(J, w_1, w_2). \tag{8.68}$$

Thus, the remaining step is to re-evaluate (8.27) and (8.28) in terms of supereigenvalue multilinear differentials (8.62), and furthermore, convert the residue at $X = \infty$ to the residue at the ramification points $z = \pm 1$, which we will show below.

8.3.3 Super Spectral Curve

In order to compute the other differentials, it turns out that we need to define a Grassmann-valued meromorphic function $\gamma(z)$ on the Riemann sphere such that $\Omega^{(1)}_{0,0|1}(|w) = \gamma(w)dx(w)$. This $\gamma(z)$ serves as an additional initial condition for recursively computing correlation functions in supereigenvalue models. How is this function related to the spectral curve? Let us consider the fermionic loop equation (8.41) for $g = 0, n = 0$:

$$V'(x)\mathcal{W}^{(1)}_{0,0|1}(|x) - \mathcal{P}^{(1)}_{0,0|1}(|x) + \Psi(x)\mathcal{W}^{(0)}_{0,1|0}(x|) = \mathcal{W}^{(1)}_{0,0|1}(I|x)\mathcal{W}^{(0)}_{0,1|0}(x|). \tag{8.69}$$

By defining $\gamma(z)$ as

$$\gamma(z) = \frac{1}{2}\Big(\mathcal{W}^{(1)}_{0,0|1}(|x) - \Psi(x)\Big), \tag{8.70}$$

we can rewrite (8.69) with $y(z)$ as

$$y(z)\gamma(z) = P^{(1)}(x), \tag{8.71}$$

where

$$P^{(1)}(x) = \frac{1}{4}\Big(V'(x)\Psi(x) - \mathcal{P}^{(1)}_{0,0|1}(|x)\Big) \tag{8.72}$$

is a Grassmann-valued polynomial of degree $\deg V(x) + \deg \Psi(x) - 1$. In particular, note that (8.71) implies that $\gamma(z)$ is odd under the hyperelliptic involution, that is, $\gamma(\sigma(z)) = -\gamma(z)$.

(8.71) can be thought of as a superpartner to the spectral curve (8.59). Together they form a *super spectral curve* — see [108–110] for more on this.

8.3.4 $\Omega^{(1)}_{g,n|1}(J|w_1)$

Let us now consider the recursion formula for $\Omega^{(1)}_{g,n|1}(J|w_1)$ Notice that we can rewrite the third equation in (8.27) as

$$\mathcal{W}^{(1)}_{g,n|1}(J|X_1) = \operatorname*{Res}_{X=\infty}(\Psi(X) - \mathcal{W}^{(1)}_{0,0|1}(|X))\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)dX, \tag{8.73}$$

since $\mathcal{W}^{(1)}_{0,0|1}(|X)\mathcal{W}^{(0)}_{g,n|2}(J|X,X_1)dX$ is regular at $X \to \infty$. Then, in terms of differentials on the Riemann sphere, and using the Grassmann-valued function $\gamma(z)$ introduced earlier, we obtain

$$\Omega^{(1)}_{g,n|1}(J|w_1) = -2\operatorname*{Res}_{x(w)=\infty}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1). \tag{8.74}$$

Remark that the residue here still makes sense. This is because both $\gamma(w)$ and $\Omega^{(0)}_{g,n|2}(J|w,w_1)$ are odd under the hyperelliptic involution $w \to 1/w$, hence the integrand itself is even, hence a well-defined differential form on the base $x(w)$.

We can rewrite this expression as a residue on the Riemann sphere itself:

$$2\operatorname*{Res}_{x(w)=\infty}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1) = \operatorname*{Res}_{w=0}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1) + \operatorname*{Res}_{w=\infty}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1). \tag{8.75}$$

Finally, we notice that (8.68) ensures that the integrand can have poles only at the ramification points of the x-covering (*i.e.* at $w = \pm 1$) and at the poles of $x(w)$ (*i.e.* $w = 0$ and $w = \infty$), since all stable $\omega_{g,n}(J)$ have poles only at the ramification points. Using the fact that the sum of all possible residues of a differential form on the Riemann sphere vanishes, we arrive at:

$$\Omega^{(1)}_{g,n|1}(J|w_1) = \sum_{a\in\{-1,1\}}\operatorname*{Res}_{w=a}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1), \tag{8.76}$$

which holds for $2g + n \geq 1$. In terms of the correlation functions of the Hermitian matrix model, we get

$$\Omega^{(1)}_{g,n|1}(J|w_1) = \sum_{a\in\{-1,1\}}\operatorname*{Res}_{w=a}\gamma(w)(x(w) - x(w_1))\omega_{g,n+2}(w,w_1,J). \tag{8.77}$$

Following the same reasoning, we can turn the fourth equation in (8.28) into a residue formula on the Riemann sphere. We obtain:

$$\Omega^{(2)}_{g,n|0}(J|) = \frac{1}{2} \sum_{a\in\{-1,1\}} \operatorname*{Res}_{w=a} \gamma(w)\Omega^{(1)}_{g,n|1}(J|w), \tag{8.78}$$

which is again valid for $2g+n \geq 1$. In terms of correlation functions of the Hermitian matrix model, we get

$$\Omega^{(2)}_{g,n|0}(J|) = \frac{1}{2} \sum_{a\in\{-1,1\}} \sum_{b\in\{-1,1\}} \operatorname*{Res}_{w=a} \operatorname*{Res}_{z=b} \left(\gamma(w)\gamma(z)(x(z)-x(w))\omega_{g,n+2}(z,w,J)\right). \tag{8.79}$$

Therefore, all correlation functions of supereigenvalue models can be determined using topological recursion on the spectral curve (8.59), in conjunction with auxiliary equations defined in terms of the Grassmann-valued polynomial equation (8.71). To summarize, we get:

Theorem 8.3.1 ([1]). *Starting with the spectral curve* (8.59)*, the Eynard-Orantin topological recursion constructs a sequence of multilinear differentials* $\omega_{g,n}(J)$*. Then the correlation functions of supereigenvalue models are encoded in the following Grassmann-valued multilinear differentials on the spectral curve* (8.59)*.*

The unstable differentials are defined by

$$\begin{aligned} \Omega^{(0)}_{0,1|0}(z_1|) &= y(z_1)dx(z_1), \\ \Omega^{(1)}_{0,1|0}(|z_1) &= \gamma(z_1)dx(z_1), \\ \Omega^{(0)}_{0,2|0}(z_1,z_2|) &= \omega_{0,2}(z_1,z_2), \\ \Omega^{(0)}_{0,0|2}(|w_1,w_2) &= \left(x(w_1)-x(w_2)\right)\omega_{0,2}(w_1,w_2), \end{aligned} \tag{8.80}$$

where $\omega_{0,2}(z_1,z_2) = dz_1dz_2/(z_1-z_2)^2$ *as usual, and the Grassmann-valued meromorphic function* $\gamma(z)$ *on the Riemann sphere is defined by* (8.71)*.*

The stable differentials, with $2g-2+m+n+p\geq 1$, are determined as follows:

$$\begin{aligned}\Omega^{(0)}_{g,n|0}(J|) &= \omega_{g,n}(J),\\ \Omega^{(0)}_{g,n|2}(J|w_1,w_2) &= (x(w_1)-x(w_2))\,\omega_{g,n+2}(w_1,w_2,J),\\ \Omega^{(1)}_{g,n|1}(J|w_1) &= \sum_{a\in\{-1,1\}}\operatorname*{Res}_{w=a}\gamma(w)\Omega^{(0)}_{g,n|2}(J|w,w_1),\\ \Omega^{(2)}_{g,n|0}(J|) &= \frac{1}{2}\sum_{a\in\{-1,1\}}\operatorname*{Res}_{w=a}\gamma(w)\Omega^{(1)}_{g,n|1}(J|w).\end{aligned} \tag{8.81}$$

8.3.5 Super Gaussian Model

As an example, we discuss in [1] the super-Gaussian model, which is the simplest supereigenvalue model whose potentials $V(x),\Psi(x)$ are given respectively by

$$V(x)=\frac{x^2}{2},\qquad \Psi(x)=\xi_{\frac{3}{2}}x+\xi_{\frac{1}{2}}. \tag{8.82}$$

For this special example, one can actually solve the super loop equations explicitly. We verified that the differentials obtained by Theorem 8.3.1 are the same (in the sense of formal expansions in $1/X$) as the correlation functions obtained by the super-loop equations, as it should be.

8.4 Supereigenvalue Models in the Ramond Sector

As an extension of this work, it is natural to ask whether one can define supereigenvalue models whose partition function satisfies the super-Virasoro constraints corresponding to the super-Virasoro subalgebra in the Ramond sector, and then see whether the appropriate correlation functions can be computed recursively. We have found such a supereigenvalue partition function, and derived super-loop equations (see also [110], where such supereigenvalue models are also derived from the point of view of quantum curves).

Most interestingly, we found that the free energy in the Ramond sector is also truncated at the quadratic order as similar to (8.13); however, the relation to Hermitian matrix models (8.14) does not hold any more. Note that there still could be some relation to Hermitian matrix models, but it would not be in the same form as (8.14). The main reason why the

Eynard-Orantin topological recursion is sufficient to calculate all correlation functions is the fundamental equation (8.14), which is not true any more for supereigenvalue models in the Ramond sector. For this reason, one needs a new formalism that goes beyond the Eynard-Orantin topological recursion. I am currently investigating a recursive formalism to solve these super-loop equations [111].

9 Airy Structures

Airy structures were recently introduced by Kontsevich and Soibelman [20], see also [21], as a new mathematical framework of recursive structures. We study this new concept in this section.

9.1 Virasoro Constraints: Revisited

Before formally defining Airy structures, let us recap the recursive structure of Hermitian matrix models, and see from where topological recursion originated. As explained in Section 7, topological recursion can be thought of as a consequence of the loop equations (7.25) in terms of multilinear differentials $\omega_{g,n}(J)$ living on the spectral curve. If we further think back, the loop equations were derived from the Virasoro constraints as described in Section 6.2.3:

$$L_n Z = 0, \tag{9.1}$$

where $n \geq -1$ and L_n generate a Virasoro subalgebra

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{9.2}$$

Therefore, we recognize that the Virasoro constraints are the foundation of topological recursion.

Let us observe this overall recursive picture from a slightly different point of view. We start with a set of differential operators L_i with respect to some variables, to say x^i. Then, we may heuristically expect a formal series Z of x^i as a solution of $L_i Z = 0$ encode some information about interesting enumerative invariants. However, it is not clear whether such a solution exists and even if so, it might not be uniquely determined. For example, we need to impose an additional condition (6.33) on the partition function in order to uniquely define formal Hermitian 1-matrix model. Then a natural question arises: *can we define 'Virasoro-like constraints' such that existence and uniqueness of a solution is guaranteed?* This is the idea behind Airy structures.

9.2 Definitions

The definition of Airy structures by Kontsevich and Soibelman [20] is introduced in an abstract mathematical language. However, I shall describe them in a less abstract manner, which is essentially how [21] defines Airy structures. Note that Kontsevich and Soibelman's approach might be more mathematically elegant, but the approach of [21] is also beneficial because it manifestly captures the importance of Lie algebras.

Let V be a d-dimensional $\mathbb{C}$-vector space, and V^* be its dual space. We choose x^i to be linear coordinates of V. Then, we define tensors A, B, C, D as

$$A \in \mathrm{Hom}(V^{\otimes 3}, \mathbb{C}), \quad B \in \mathrm{Hom}(V^{\otimes 2} \otimes V^*, \mathbb{C}), \quad C \in \mathrm{Hom}(V \otimes V^{*\otimes 2}, \mathbb{C}), \quad D \in \mathrm{Hom}(V, \mathbb{C}). \tag{9.3}$$

Additionally, we require that A is a symmetric tensor and C has a partial symmetry as

$$C(e_i, e^j, e^k) = C_i^{jk} = C_i^{kj}, \tag{9.4}$$

where e_i, e^j are basis of V, V^* respectively. Now we are ready to define a quantum Airy structure.

Definition 9.2.1. A *quantum Airy structure*, or in short, *Airy structure*, is a set of differential operators L_i of the form

$$L_i = \hbar\partial_i - \frac{1}{2}A_{ijk}x^j x^k - \hbar B_{ij}^k x^j \partial_k - \frac{\hbar^2}{2}C_i^{jk}\partial_j\partial_k - \hbar D_i, \tag{9.5}$$

generating a Lie algebra

$$[L_i, L_j] = \hbar f_{ij}^k L_k, \tag{9.6}$$

for some f_{ij}^k and a formal constant $\hbar$.

Note that we use the Einstein summation convention for repeated indices. This setting relates to the notion of quantization of Lagrangian subspaces as outlined in [20, 21] which we also briefly discuss in Section 9.2.3. This is why they are named quantum Airy structures.

However, we often drop *quantum* and simply call them Airy structures. The Lie algebra condition (9.6) imposes a set of constraints on A, B, C, D.

Lemma 9.2.2 ([21]). *A, B, C, D satisfy the following set of equations:*

$$A_{jik} = A_{ijk}, \tag{9.7}$$

$$f_{ij}^k = B_{ij}^k - B_{ji}^k, \tag{9.8}$$

$$B_{ik}^p A_{jpl} + B_{il}^p A_{jpk} + B_{ij}^p A_{pkl} = (i \leftrightarrow j), \tag{9.9}$$

$$B_{ik}^p B_{jp}^l + C_i^{lp} A_{jpk} + B_{ij}^p B_{pk}^l = (i \leftrightarrow j), \tag{9.10}$$

$$C_i^{kp} B_{jp}^l + C_i^{lp} B_{jp}^k + B_{ij}^p C_p^{kl} = (i \leftrightarrow j), \tag{9.11}$$

$$\frac{1}{2} C_i^{pq} A_{jqp} + B_{ij}^p D_p = (i \leftrightarrow j), \tag{9.12}$$

So far we have just prepared a set of differential operators generating a Lie algebra. The reason for considering such operators is that every Airy structure produces a unique power series of x^i that is annihilated by those differential operators. Furthermore, this unique power series generally serves as a generating function of some enumerative invariants. The main theorem given in [20, 21] is summarized as follows:

Theorem 9.2.3 ([20, 21]). *There exists a unique series*

$$Z = \exp\left(\sum_{g\geq 0}\sum_{n\geq 1} \frac{\hbar^{g-1}}{n!} F_{g,n}(i_1, \cdots, i_n) x^{i_1} \cdots x^{i_n}\right), \tag{9.13}$$

where $F_{g,n} \in \mathrm{Hom}(V^{\otimes n}, \mathbb{C})$ are symmetric tensors such that $F_{g,n}(i_1, \cdots, i_n) = F_{g,n}(e_{i_1}, \cdots, e_{i_n})$ and $F_{0,1} = 0, F_{0,2} = 0$, and

$$\forall i \quad L_i \cdot Z = 0. \tag{9.14}$$

Explicitly, $F_{g,n+1}(i,J)$ *for* $2g-2+n \geq 0$ *are recursively given by*

$$\begin{aligned} F_{g,n+1}(i,J) = {} & A_{ii_1i_2}\delta_{g,0}\delta_{n,2} + D_i\delta_{n,0}\delta_{g,1} \\ & + \sum_{k=1}^{n} B^l_{ii_k} F_{g,n}(l, J\backslash i_k) + \frac{1}{2}\sum_{k,l=1}^{d} C^{kl}_i F_{g-1,n+2}(l,k,J) \\ & + \frac{1}{2}\sum_{k,l=1}^{d} C^{kl}_i \sum_{g_1+g_2=g}\sum_{J_1\cup J_2=J} F_{g_1,n_1+1}(k,J_1)F_{g_2,n_2+1}(l,J_2) \end{aligned} \tag{9.15}$$

where $J = \{i_1, \cdots, i_n\}$*.*

Since symmetric tensors $F_{g,n} \in \mathrm{Hom}(V^{\otimes n}, \mathbb{C})$ would have many indices, we denote the components $F_{g,n}(e_{i_1}, \cdots, e_{i_n})$ by $F_{g,n}(i_1, \cdots, i_n)$. We leave the detailed proofs of Lemma 9.2.2 as well as Theorem 9.2.3 to the next section where we construct a supersymmetric version, but let us give a sketch here.

First of all, it is not so difficult to reduce the set of differential equations (9.14) to the recursive equations (9.15), we simply need to collect the coefficients of the same power in x^i order by order in $\hbar$. Note that we use the symmetry of indices of $F_{g,n}(J)$ to derive (9.15) from (9.14). Then, it is also straightforward to see that the solution of (9.14) is uniquely constructed by the recursive equation (9.15) *if* such a solution exists. Therefore, the key is the existence. Notice that the first index i of $F_{g,n+1}(i,J)$ in (9.15) plays a different role from others in J. In particular, indices in J are explicitly symmetric by construction (9.15) while i and J are not symmetric at first glance. This seems problematic because we started from symmetric tensors $F_{g,n}$ but the recursive equations (9.15) may not reconstruct symmetric ones. As explicitly computed in [21], however, it can be proven that the $F_{g,n+1}(i,J)$ constructed from (9.15) become all symmetric if the set of differential operators L_i generates a Lie algebra. Therefore, the requirement of a Lie algebra in the definition of Airy structures is very important.

9.2.1 Airy Equation

Let us consider the simplest example of Airy structures [112]. If the vector space V is one dimensional, an Airy structure us a differential operator of the form

$$L = \hbar\partial_x - \frac{A}{2}x^2 - \hbar B x\partial_x - \frac{\hbar^2}{2}C\partial_x\partial_x - \hbar D, \tag{9.16}$$

where x is a linear coordinate on V. Now we consider an example of the choice $A = B = C = D = 1$. Let us define another coordinate y by

$$y = (2\hbar)^{-\frac{2}{3}}(1 - 2x - \hbar), \tag{9.17}$$

and also the partition function $Z(x)$ by

$$Z(x) = \exp\left(\frac{1}{\hbar}\left(x - \frac{1}{2}x^2\right)\right)\tilde{Z}(y) \tag{9.18}$$

Then, the differential equation $LZ(x) = 0$ can be rewritten in terms of $\tilde{Z}(y)$ as

$$\partial_y^2\tilde{Z} = y\tilde{Z}. \tag{9.19}$$

This is none other than the Airy equation. This is one reason for the name *Airy*.

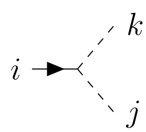
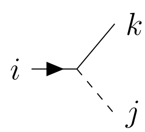
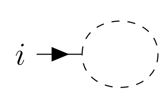
9.2.2 Graphical Interpretation

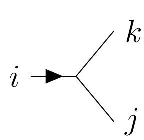
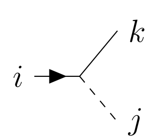
One can interpret the recursive equation (9.15) as a decomposition of trivalent graphs whose vertices have precisely three legs. Trivalent graphs for Airy structures form so-called *trees* where a trivalent graph of $n+1$ external legs have one root and n leaves. We denote leaves by dashed edges, and lines that can potentially be roots by arrowed edges. See (9.20), (9.23), and (9.24) as examples. We now give a detailed definition of graphs for Airy structures following [19].

Definition 9.2.4. For $n, g \geq 0$ such that $2g - 1 + n \geq 1$, we define $\mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$ to be a set of connected trivalent graphs with the following 5 properties:

1. $2g-1+n$ vertices,
2. 1 root labelled by i,
3. n ordered leaves $(i_1, \cdots, i_n)$,
4. $3g+2n-1$ edges,
   - $2g+n-1$ arrowed edges,
   - n non-arrowed edges from a vertex to a leaf,
   - g non-arrowed inner edges where one end is the parent of the other following the arrows along the tree.
5. Arrowed edges form a spanning, planar, binary skeleton tree with the root i. The arrows are oriented from roots towards leaves.
6. If an arrowed edge and a non-arrowed edge come out of a vertex, the arrowed edge is always on the left child.
7. Edges can cross only between two non-arrowed edges.

At each vertex of a graph, we would like to assign some quantities, so-called weights. Vertices incident to an entire loop are assigned D_i, vertices incident to one leaf are B^k_{ij}, vertices incident to two leaves are A_{ijk}, and vertices without leaves are C^{jk}_i. They are pictorially represented by

$A_{ijk} =$  , $B^k_{ij} =$  , $D_i =$  ,

$C^{jk}_i =$  , or $C^{jk}_i =$  , (9.20)

where the standard lines — cannot be connected to leaves except for the root. Then, one can show that $F_{g,n+1}(J)$ for $2g+n \geq 2$ can be given by the sum of all weighted trivalent trees.

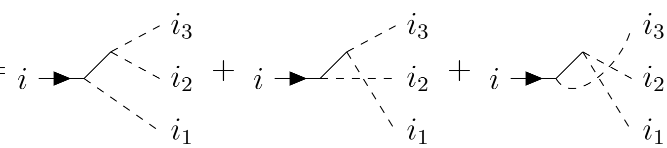
Proposition 9.2.5 ([21, 112])**.** *Let $G \in \mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$ be a graph defined in* Definition 9.2.4*, $w(G)$ be the weight of the graph G given by the assignments* (9.20)*, and $F_{g,n+1}(i, i_1, \cdots, i_n)$ be the free energies computed from the recursive equation* (9.15)*. Then, we have*

$$F_{g,n+1}(i, i_1, \cdots, i_n) = \sum_{G \in \mathbb{G}_{g,n+1}(i,i_1,\cdots,i_n)} \frac{w(G)}{|\mathrm{Aut}(G)|}, \tag{9.21}$$

where $|\mathrm{Aut}(G)|$ is the group of permutations of inner edges of G that preserves the graph structure.

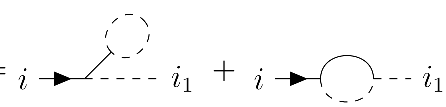
Example 9.2.6. By Proposition 9.2.5, $F_{g,n+1}(i, i_1, \cdots, i_n)$ for $2g + n = 3$ are computed as

$$\sum_{G \in \mathbb{G}_{0,4}(i,i_1,i_2,i_3)} \frac{w(G)}{|\mathrm{Aut}(G)|} = \tag{9.22}$$



$$\begin{aligned} &= B^l_{ii_1} A_{li_2i_3} + B^l_{ii_2} A_{li_1i_3} + B^l_{ii_3} A_{li_1i_2} \\ &= F_{0,4}(i, i_1, i_2, i_3). \end{aligned} \tag{9.23}$$

$$\sum_{G \in \mathbb{G}_{1,2}(i,i_1)} \frac{w(G)}{|\mathrm{Aut}(G)|} =$$



$$\begin{aligned} &= B^l_{ii_1} D_l + \frac{1}{2} C^{kl}_i A_{lki_1} \\ &= F_{1,2}(i, i_1) \end{aligned} \tag{9.24}$$

Note that the 1/2 factor in front of the $C^{kl}_i A_{jkl}$ is from the automorphism of the second graph.

Remark 9.2.7. For a one-dimensional vector space, let us choose all weights A, B, C, D to be 1 as in Section 9.2.1. Since there is only one index, we can omit writing indices for $F_{g,n}(i_1, \cdots, i_n)$ and $\mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$. Then, Proposition 9.2.5 shows that the free energy $F_{g,n}$ is given by

$$F_{g,n} = \sum_{G \in \mathbb{G}_{g,n+1}} \frac{1}{|\mathrm{Aut}(G)|}. \tag{9.25}$$

Thus, $F_{g,n}$ in this simple example essentially compute the number of trivalent graphs.

9.2.3 Quantization of Lagrangian Subspaces

It is always helpful to view things from different perspectives. Here, we give another point of view on Airy structures as quantization of Lagrangian subspaces. Let us define Lagrangian subspaces.

Definition 9.2.8. Let (M, ω) be a symplectic vector space of dimension $2d$ where ω is the symplectic form. A *Lagrangian subspace* L is a submanifold such that $\dim L = d$ and $\omega|_L = 0$.

Definition 9.2.9. Let (M, ω) be a symplectic vector space of dimension $2d$ where ω is the symplectic form, and f, g be smooth functions on M. Then the *Poisson bracket* $\{f, g\}_P$ is defined by

$$\{f, g\}_P = \omega(df, dg) = \omega^{\mu\nu} \partial_\mu f \partial_\nu g. \tag{9.26}$$

One can find coordinates (x^i, y_j) called *Darboux' coordinates* such that ω is written in the canonical form

$$\omega = \sum_{i=1}^{d} dx^i \wedge dy_i. \tag{9.27}$$

Hence, the simplest way of defining a Lagrangian subspace is to set all $y_i = 0$.

Let us now deform such a Lagrangian subspace. That is, we consider a subspace constrained by d equations $H_i = 0$ where every H_i is in the form

$$H_i = y_i - \frac{1}{2} A_{ijk} x^j x^k - B_{ij}^k x^j y_k - \frac{1}{2} C_i^{jk} y_j y_k. \tag{9.28}$$

Note that if $A = B = C = 0$, this condition reduces to the simplest case where $y_i = 0$. A subspace simply constrained by $H_i = 0$ does not always become a Lagrangian subspace because ω does not identically vanish on the subspace. It turns out such a subspace is a Lagrangian submanifold if and only if the Poisson bracket between H_i is closed

$$\{H_i, H_j\}_P = f_{ij}^k H_k. \tag{9.29}$$

Indeed a classical Airy structure is defined as follows.

Definition 9.2.10. Let (V, ω) be a symplectic vector space and (x^i, y_j) be Darboux' coordinates where the symplectic form is written in canonical form. A *classical Airy structure* is a set of quadratic polynomials H_i of (x^i, y_j) of the form

$$H_i = y_i - \frac{1}{2}A_{ijk}x^jx^k - B^k_{ij}x^jy_k - \frac{1}{2}C^{jk}_i y_jy_k, \tag{9.30}$$

such that their Poisson bracket is closed.

Quantization of Airy structures means that we upgrade Poisson brackets $\{\, , \,\}$ to commutators $[\, , \,]$ and coordinates (x^i, y_j) to operators $(\hat{x}^i, \hat{y}_j)$ such that $[\hat{y}_j, \hat{x}^i] = \hbar\delta^i_j$ where $\hbar$ is a formal constant. The canonical choice is

$$\hat{x} = x^i, \quad \hat{y}_i = \hbar\frac{\partial}{\partial x^i}. \tag{9.31}$$

Then, the resulting quantized Airy structure is none other than Definition 9.2.1. Note that the B-terms in (9.30) have an order ambiguity (x^jy_k or y_kx^j) which gives different choices of quantization. Such a quantum effect is encoded in the D term.

9.3 Eynard-Orantin Topological Recursion

As discussed in Theorem 9.2.3, every Airy structures possesses a unique recursively determined partition function Z. Then a natural question is: what is Z computing? Is it a generating function of some interesting enumerative invariants? The answer is yes, and in this section, we give the relation between Airy structures and the Eynard-Orantin topological recursion defined in Theorem 7.4.4. The initial data for the Eynard-Orantin topological recursion is a spectral curve, while the initial data for Airy structures is encoded in a set of tensors A, B, C, D defined in Definition 9.2.1. Therefore, our goal here is to construct tensors A, B, C, D from a spectral curve. The result is summarized below: see [21] for a detailed derivation. We use the same notation as in Section 7.4.

9.3.1 Decomposition

Let $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ be a spectral curve. Since the $\omega_{g,n}(J)$ for $2g+n \geq 3$ obtained from the Eynard-Orantin topological recursion have poles only at the ramification points, we expect that they can be decomposed by a basis of differential forms with poles only at the ramification points. More explicitly, let us consider a point $p \in \Sigma$ in the vicinity of a ramification point $r \in \mathfrak{r}$ where a local coordinate z is defined by (7.36), and construct a locally-defined meromorphic 1-form $\xi_{k,r}(p)$ by

$$\xi_{k,r}(p) = \operatorname*{Res}_{q \to r} \int_r^q \omega_{0,2}(\cdot, p) \frac{(2k+1)dz(q)}{z(q)^{2k+2}}. \tag{9.32}$$

It can be shown [21] that $\xi_{k,r}(p)$ has only one pole of degree $2k+2$ at $p = r$. Then, the following lemma states that we cab rewrite the $\omega_{g,n}(J)$ for $2g+n \geq 3$ by a linear combination of products $\xi_{k,r}(p)$.

Lemma 9.3.1 ([21]). *Let $\omega_{g,n}(J)$ for $2g+n \geq 3$ be a multilinear differential computed from the Eynard-Orantin topological recursion based on a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, and $\{\xi_{k_i,r_i}(p_i)\}$ be a set of meromorphic differential forms defined by (9.32) at each ramification point p_i. Then, there exists a unique decomposition of $\omega_{g,n}(J)$ in the form*

$$\omega_{g,n}(J) = \sum_{\substack{r_1,\cdots,r_n \in \mathfrak{r} \\ k_1,\cdots,k_n \geq 0}} W_{g,n}\begin{bmatrix} r_1 \cdots r_n \\ k_1 \cdots k_n \end{bmatrix} \prod_{i=1}^n \xi_{k_i,r_i}(p_i). \tag{9.33}$$

9.3.2 Correspondence

Next, we define the dual basis $\xi^*_{k,r}(p)$ of $\xi_{k,r}(p)$ by

$$\xi^*_{k,r}(p) = \frac{z(p)^{2k+1}}{2k+1}, \qquad \operatorname*{Res}_{p \to r} \xi_{k,r}(p)\xi^*_{l,r'}(p) = \delta_{k,l}\delta_{r,r'}. \tag{9.34}$$

Furthermore, we define $\theta(p)$ by

$$\theta(p) = -\frac{2}{\omega_{0,1}(p) - \omega_{0,1}(\imath(p))}, \tag{9.35}$$

where $\imath : z(p) \mapsto -z(p)$ is a local holomorphic involution. Since $\omega_{0,1}(p)$ has at most one double zero at $p = r$ by definition, one can expand $\theta(p)$ as

$$\theta(p) = \sum_{m \geq -1} t_{m,r} z(p)^{2m} \frac{1}{dz(p)}. \tag{9.36}$$

At last, we define an expansion of $\omega_{0,2}$ when p_1, p_2 are in the vicinity of $r_1, r_2 \in \mathfrak{r}$ respectively as

$$\omega_{0,2}(p_1, p_2) = \left(\frac{\delta_{r_1,r_2}}{(z(p_1) - z(p_2))^2} + \sum_{l_1, l_2 \geq 0} \varphi_{0,2}\left[\begin{smallmatrix} r_1 & r_2 \\ l_1 & l_2 \end{smallmatrix}\right] z(p_1)^{l_1} z(p_2)^{l_2} \right) dz(p_1) dz(p_2). \tag{9.37}$$

Let us now construct A, B, C, D as

$$A_{(k_1,r_1)(k_2,r_2)(k_3,r_3)} = \operatorname*{Res}_{q \to r_1} (\xi^*_{k_1,r_1}(q) d\xi^*_{k_2,r_2}(q) d\xi^*_{k_3,r_3}(q) \theta(q)), \tag{9.38}$$

$$B^{(k_3,r_3)}_{(k_1,r_1)(k_2,r_2)} = \operatorname*{Res}_{q \to r_1} (\xi^*_{k_1,r_1}(q) d\xi^*_{k_2,r_2}(q) \xi_{k_3,r_3}(q) \theta(q)), \tag{9.39}$$

$$C^{(k_2,r_2)(k_3,r_3)}_{(k_1,r_1)} = \operatorname*{Res}_{q \to r_1} (\xi^*_{k_1,r_1}(q) \xi_{k_2,r_2}(q) \xi_{k_3,r_3}(q) \theta(q)), \tag{9.40}$$

$$D_{k,r} = \delta_{k,0} \left(\frac{1}{2} t_{-1,r} \varphi_{0,2}\left[\begin{smallmatrix} r & r \\ 0 & 0 \end{smallmatrix}\right] + \frac{1}{8} t_{0,r} \right) + \frac{1}{24} \delta_{k,1} t_{-1,r}. \tag{9.41}$$

Then, we have

Proposition 9.3.2 ([20,21]). *Let $F_{g,n}((k_1, r_1), \cdots, (k_n, r_n))$ for $2g+n \geq 3$ be the free energy obtained by the A, B, C, D defined by* (9.38)-(9.41) *as in* Theorem 9.2.3. *Then, we have*

$$F_{g,n}((k_1, r_1), \cdots, (k_n, r_n)) = W_{g,n}\begin{bmatrix} r_1 \cdots r_n \\ k_1 \cdots k_n \end{bmatrix}, \tag{9.42}$$

where $W_{g,n}\begin{bmatrix} r_1 \cdots r_n \\ k_1 \cdots k_n \end{bmatrix}$ are defined in (9.33) *in Lemma 9.3.1*

Remark 9.3.3. As shown above, to any spectral curve we can associate an Airy structure that calculates the same invariants as the Eynard-Orantin topological recursion.

9.3.3 The Airy Curve

Let us give the simplest example in this construction, namely we choose the spectral curve to be the Airy curve $y^2 = 2x$. We parametrize x, y by a local coordinate z as

$$y = -z, \quad x = \frac{1}{2}z^2, \tag{9.43}$$

then, the only ramification point is at $z = 0$. Accordingly, the ξ_k, ξ_k^* basis and θ are given by

$$\xi_k^*(z) = \frac{z^{2k+1}}{2k+1}, \quad \xi_k(z) = \frac{(2k+1)dz}{z^{2k+2}}, \quad \theta(z) = \frac{1}{z^2 dz}. \tag{9.44}$$

In particular, the expansion coefficients $\varphi_{0,2}\left[\begin{smallmatrix} r_1 & r_2 \\ l_1 & l_2 \end{smallmatrix}\right]$ are all zero. If we apply these to the set of formulae (9.38)-(9.41), we get

$$A_{ijk} = \delta_{i=j=k=1}, \tag{9.45}$$

$$B_{ij}^k = \frac{2k-1}{2i-1}\delta_{i+j-2,k}, \tag{9.46}$$

$$C_i^{jk} = \frac{(2j-1)(2k-1)}{2i-1}\delta_{i,j+k+1}, \tag{9.47}$$

$$D_i = \frac{1}{24}\delta_{i,2}. \tag{9.48}$$

We will discuss later a relation between these A, B, C, D and those obtained from a vertex operator algebra. Using this fact, it is straightforward to show that these A, B, C, D satisfy Lemma 9.2.2.

9.4 Examples

We illustrate a few examples of Airy structures that are somewhat related to topics in theoretical physics. In particular, the following examples have supersymmetric analogues in the framework of super Airy structures that we will develop in Chapter 10.

9.4.1 Frobenius Algebras

A simple, yet interesting type of Airy structures is that associated with Frobenius algebras. This is because $F_{g,n}(J)$ in fact compute correlation functions in a topological quantum field theory in two dimensions. See [21, 112] for a detailed discussion.

Let us first define a Frobenius algebra:

Definition 9.4.1. A *Frobenius algebra* $\mathbb{A}$ over $\mathbb{C}$ is a finite-dimensional vector space equipped with a commutative, associative product $\mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$, and a non-degenerate bilinear form $\phi : \mathbb{A} \otimes \mathbb{A} \to \mathbb{C}$.

Note that Frobenius algebras can be defined for arbitrary field $\mathbb{K}$, but we just stick to $\mathbb{C}$ for simplicity. Then we construct a Airy structure as follows.

Lemma 9.4.2 ([21]). *Let (e_i) be a basis of a super-Frobenius algebra $\mathbb{A}$, and (e^j) be the dual basis such that*

$$\phi(e_i, e^j) = \delta_i^j. \tag{9.49}$$

Then for any $\theta_A, \theta_B, \theta_C \in \mathbb{A}$,

$$A_{ijk} = \phi(\theta_A e_i e_j e_k), \quad B_{ij}^k = \phi(\theta_B e_i e_j e^k), \quad C_i^{jk} = \phi(\theta_C e_i e^j e^k), \tag{9.50}$$

and arbitrary D_i define a quantum Airy structure on $V = \mathbb{A}$ with vanishing structure constants $f_{ij}^k = 0$.

Proof. We prove the supersymmetric analogue in the next section which can be specialized to this Lemma. □

On the other hand, topological quantum field theories are theories whose correlation functions are independent of the choice of metric. Well known examples are Chern-Simon theory, or topological A-model and B-model [114]. As discussed in [115,116], it turns out that one can always associates a Frobenius algebra with a topological quantum field theory in two dimensions and vice versa. The precise relation between correlation functions in topological field theories and the free energies of Airy structures associated with Frobenius algebras is summarized as follows:

Proposition 9.4.3 ([21, 112]). *Let $\mathbb{A}$ be a Frobenius algebra. For $2g+n \geq 2$, let $F_{g,n+1} \in \mathrm{Hom}(\mathbb{A}^{\otimes n+1}, \mathbb{C})$ be symmetric tensors computed by* Theorem 9.2.3 *with the Airy structure defined in* Lemma 9.4.2. *Then,*

$$F_{g,n+1} = |G_{g,n+1}|\mathcal{F}(\Sigma_{g,n+1}), \tag{9.51}$$

where $\mathcal{F}(\Sigma_{g,n+1})$ is genus g, $n+1$-point correlation function of the two dimensional topological field theory in two dimensions associated to the Frobenius algebra $\mathbb{A}$, and $|G_{g,n+1}|$ is given by (9.25)*, that is,*

$$|G_{g,n}| = \sum_{G\in\mathbb{G}_{g,n}} \frac{1}{|\mathrm{Aut}(G)|}, \tag{9.52}$$

9.4.2 Vertex Operator Algebras

Vertex operator algebras are algebraic constructions of conformal field theories in two dimensions. It turns out modules for some vertex operator algebras give rise to Airy structures corresponding to the Eynard-Orantin topological recursion as in (9.38)-(9.41). Let us first give a formal definition of vertex operator algebras following [117, 118].

Definition 9.4.4. A vertex operator algebra is a quadruple $(V, Y, \omega, \mathbf{1})$ that follows the axioms below:

- V is a $\mathbb{Z}$-graded vector space

  $$V = \coprod_{k\in\mathbb{Z}} V_k; \quad \text{for } v \in V_k, \quad \text{wt } v = k \tag{9.53}$$

  such that $V_k = 0$ for a sufficiently negative n and $\dim V_k$ is finite for every $k \in \mathbb{Z}$.

- $Y : V \to (\mathrm{End}V)[[x, x^{-1}]]$ is a linear map

  $$Y : v \mapsto Y(v,x) = \sum_{n\mathbb{Z}} v_n x^{-n-1}, \quad v \in V, \quad v_n \in \mathrm{End}V. \tag{9.54}$$

  Furthermore, for every $u, v \in V$, there exists a positive integer N such that

  $$(x-y)^N Y(u,x)Y(v,y) = (x-y)^N Y(v,x)Y(u,y). \tag{9.55}$$

- $\mathbf{1} \in V_0$ obeys $Y(\mathbf{1}, x) = 1_V$.

- For $\omega \in V_2$, let us define

$$Y(\omega, x) = \sum_{n\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\mathbb{Z}} L_n x^{-n-2}, \tag{9.56}$$

then $L_0 v = (\text{wt } v)v$, L_n generate a Virasoro algebra with central charge c

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3 - n), \tag{9.57}$$

and furthermore

$$Y(L_{-1}v, x) = \frac{d}{dx} Y(v, x) \tag{9.58}$$

for every $v \in V$.

In the conformal field theory language, $V, Y, \mathbf{1}, Y(\omega, x)$ are respectively called the Fock space, the state-operator correspondence, the vacuum state, and the (chiral part of the) energy-momentum tensor. v_n play the role of creation/annihilation operators, L_0 tells us the conformal weights by $L_0 v = (\text{wt } v)v$, and (9.55) is a requirement equivalent to the operator product expansion. Therefore, this definition captures all the properties of the chiral part of conformal field theories. Note that vertex algebras are defined without ω.

The Heisenberg vertex operator algebra is the one corresponding to the conformal field theory of a chiral free boson.

Definition 9.4.5. The Heisenberg vertex operator algebra is a vertex operator algebra $(V, Y, \omega, \mathbf{1})$ equipped with:

- $b \in V_1$ such that b_n, defined by

$$Y(b, x) = \sum_{n\mathbb{Z}} b_n x^{-n-1}, \tag{9.59}$$

generate a Heisenberg algebra $[b_n, b_m] = n\delta_{n+m,0}$.

- $Y(v,x)$ is defined for $v = b_{-n_1} \cdots b_{-n_k} \cdot \mathbf{1} \in V$ as

$$Y(v,x) =: \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dx} \right)^{n_1-1} Y(b,x) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dx} \right)^{n_k-1} Y(b,x) \right) :, \tag{9.60}$$

where : : denotes normal ordering, and we define $Y(\mathbf{1},x) = 1_V$ for $k=0$.

- ω is given by

$$\omega = \frac{1}{2} b_{-1} b_{-1} \tag{9.61}$$

One can indeed show that L_n computed by $Y(\omega,x)$ with (9.61) are written as

$$L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} : b_{k-n} b_k :, \tag{9.62}$$

where the central charge is $c = 1$.

Our goal in this section is to find a set of differential operators constructed from the Heisenberg vertex operator algebra. Recall that there are two important features to define Airy structures: differential operators should be in the right form as in (9.5), and they should generate a closed Lie algebra. From the commutator relation (9.57), it is straightforward to see that a closed algebra is generated by a set of Virasoro operators $\{L_{n\geq f}\}$ where the set with $f = -1$ gives the largest subalgebra, and smaller subalgebras are given by sets with $f \geq 0$.

The next step is to represent these L_n in the right form as in (9.5). To do it, we need to consider representations of the Heisenberg vertex operator algebra. One can formally define representations of vertex operator algebras in general, called V-modules. However, for simplicity instead, we shall show two explicit differential ring representations of the Heisenberg vertex operator algebras, twisted and untwisted ones.

Untwisted Modules

We start with untwisted modules. Let g^i for $i \in \mathbb{Z}_{\geq 0}$ be a set of formal variables, then a simple representation of b_n is given by

$$b_0 = \sqrt{2\hbar}\frac{\partial}{\partial x^0}, \quad b_{-k} = \frac{1}{\sqrt{2\hbar}}kx^k, \quad b_k = \sqrt{2\hbar}\frac{\partial}{\partial x^k}, \quad k \geq 1, \tag{9.63}$$

where we abuse notation slightly and denote the representation of the modules b_n by the same symbol. Note that it is straightforward to show that this representation generates the Heisenberg algebra $[b_n, b_m] = n\delta_{n+m,0}$. Then, L_n for $n \geq -1$ can be written as

$$L_n = \sum_{k\geq 0} kx^k \frac{\partial}{\partial x^{k+n}} + \frac{\hbar}{2}\sum_{j=0}^{n} \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^{n-j}}. \tag{9.64}$$

Notice that this is indeed equivalent to the Virasoro operators in (6.30) after shifting $x^2 \to x^2 + T_2/2$ and identifying $x^i = g_i$. However, this Airy structure is not an interesting one due to the absence of A terms and D terms.

It turns out that nontrivial Airy structures can be produced by a smaller subalgebra generated by $\{L_{n\geq f}\}$ for $f \geq 0$. To have the correct linear term as in (9.5) , we define $\hat{L}_i$ for $i \geq 0$ by conjugation as

$$\hat{L}_i = \hbar \exp\left(-\frac{x^f}{\hbar}\right) L_{i+f} \exp\left(\frac{x^f}{\hbar}\right) = \hbar\frac{\partial}{\partial x^i} + L_{i+f} + \frac{1}{2}\delta_{f,i}, \tag{9.65}$$

$$[\hat{L}_i, \hat{L}_j] = \hbar(i-j)\hat{L}_{i+j+f}, \tag{9.66}$$

where L_{i+f} are given by (9.64). This conjugation is sometimes referred as a *Dilaton shift*. Since conjugation does not change the commutation relations, the L_i are still a representation of the Virasoro subalgebra. Note that the last term in (9.65) is not a D term because it has a wrong power of $\hbar$. A key observation is that $\hat{L}_0, \hat{L}_1, \cdots, L_f$ do not appear on the right hand side of the commutation relation (9.66); thus, we can remove the last term of (9.65) and add constant D terms into these $f+1$ generators without changing the commutation relation.

As a consequence, the following set of differential operators $\check{L}$ for a given integer $f \geq 0$

$$\check{L}_i = \hbar\frac{\partial}{\partial x^i} + \hbar\sum_{k\geq 0} kx^k \frac{\partial}{\partial x^{k+i+f}} + \frac{\hbar^2}{2}\sum_{j=0}^{i+f}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^{i+f-j}} + \hbar D_i\delta_{i\leq f}, \tag{9.67}$$

$$[\check{L}_i, \check{L}_j] = \hbar(i-j)\check{L}_{i+j+f}, \tag{9.68}$$

defines an Airy structure whose associated partition function is nontrivial thanks to nonzero D terms. It is worth mentioning that strictly speaking this is not a representation of the Heisenberg vertex operator algebra anymore because we have modified the L_n by adding constant terms by hand.

The Heisenberg algebra is still generated even if we choose $b_0 = 0$ in (9.63), that is, we represent b_n by

$$b_0 = 0, \quad b_{-k} = \frac{1}{\sqrt{2\hbar}}kx^k, \quad b_k = \sqrt{2\hbar}\frac{\partial}{\partial x^k}, \quad k \geq 1, \tag{9.69}$$

Following exactly the same technique given above, we can define an Airy structure $\{\check{L}_i\}$ for $i \geq 1$ and $f \in \{0, 2, 3, 4, \cdots\}$ such that

$$\begin{aligned}\check{L}_i &= \hbar\exp\left(\frac{b_{1-f}}{(1-f)\sqrt{\hbar}}\right)L_{i+f}\exp\left(-\frac{b_{1-f}}{(1-f)\sqrt{\hbar}}\right) - \frac{1}{2}\delta_{i,f-1} + \hbar D_i\delta_{i\leq f+1}\\ &= \hbar\frac{\partial}{\partial x^i} + \hbar\sum_{j\geq 1} jx^j\frac{\partial}{\partial x^{i+j+f-1}} + \frac{\hbar^2}{2}\sum_{j=1}^{i+f-2}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^{i+f-j-1}} + \hbar D_i\delta_{i\leq f+1},\end{aligned} \tag{9.70}$$

$$[\check{L}_i, \check{L}_j] = \hbar(i-j)\check{L}_{i+j+f-1}. \tag{9.71}$$

Note that $f \neq 1$ because we cannot conjugate with b_0 as we chose $b_0 = 0$. Similarly, the partition function associated with this Airy structure becomes nontrivial. As a remark, it is somewhat surprising that we can reconstruct the so-called *topological recursion without branch covers* (see [21]) from this smaller Virasoro subalgebra[1]. Indeed, one can show the following proposition:

Proposition 9.4.6. *Let us construct the Airy structure associated with topological recursion*

[1]It remains to be investigated whether there exists a topological recursion without branch covers corresponding to untwisted modules with nonzero b_0 as in (9.63).

without branched covers that is defined in Section 10 in [21] with spectral curve given by

$$\Sigma = \mathbf{P}^1, \quad \omega_{0,1}(z) = -\frac{dz}{z^f}, \quad \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}, \quad \omega_{1,1}(z) = \sum_{k=1}^{f+1} -D_k \frac{dz}{z^{1+k}}, \tag{9.72}$$

where z is a local coordinate. Then, this Airy structure precisely corresponds to the Airy structure formed by the set of generators (9.70) *of the smaller Virasoro subalgebra.*

Proof. The ξ-basis and $\theta(z)$ in this case are given[2] by

$$\xi_k^*(z) = z^k, \quad \xi_k(z) = \frac{1}{z^{k+1}} dz, \quad \theta(z) = -\frac{z^f}{dz} \tag{9.73}$$

where all indices in this proof are positive integers. According to Section 10 in [21], the tensors A, B, C are computed by (9.38)-(9.40) as

$$A_{ijk} = 0, \quad B_{ij}^k = -j\delta_{i+j-k=1}, \quad C_i^{jk} = -\delta_{i=j+k+1}. \tag{9.74}$$

The definition for D_i is different from (9.41) but it is defined by $\omega_{1,1}(z)$ as

$$\omega(z) = \sum_{k\geq 1} D_k \xi_k(z). \tag{9.75}$$

Thus, we can immediately read all D_i off from the initial condition (9.72). The resulting differential operators are

$$L_i = \hbar \frac{\partial}{\partial x^i} + \hbar \sum_{j\geq 1} j x^j \frac{\partial}{\partial x^{i+j-1}} + \frac{\hbar^2}{2} \sum_{j=1}^{i-2} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^{i-j-1}} + \hbar D \delta_{i,1}. \tag{9.76}$$

They precisely match with (9.70). □

Note, however, that It still remains to be seen whether they are computing some enumerative invariants. Also, note that this smaller Virasoro subalgebra approach does not work for $f = 1$, which is the case where $\omega_{0,1}(z)$ has a residue at the ramification point. Is there any geometric meaning behind this fact? This is worth investigating further.

[2] This definition for the ξ-basis is slightly different from the one given in Section 10 in [21], but this is not an issue because it is just a choice of normalization.

Twisted Modules

We now study twisted modules of b_n generating the Heisenberg algebra. See [119] for a rigorous definition of twisted modules. What computationally differs from untwisted modules is that we shift the indices of representations to be half-integers, yet generating the same algebra

$$[\tilde{b}_n, \tilde{b}_m] = n\delta_{n+m}, \quad n, m \in \mathbb{Z} + \frac{1}{2}, \tag{9.77}$$

where we denote twisted creation/annihilation operators by $\tilde{b}_n$ instead of b_n. For those who are familiar with super conformal field theory, this is essentially analogous to the difference between a fermion in the NS sector and that in Ramond sector. In particular, by using a set of formal variables x^i, $\tilde{b}_k$ can be written as

$$\tilde{b}_{-k} = \frac{1}{\sqrt{2\hbar}} x^{k+\frac{1}{2}}, \quad \tilde{b}_k = k\sqrt{2\hbar} \frac{\partial}{\partial x^{k+\frac{1}{2}}}, \quad k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}. \tag{9.78}$$

The L_n in this representation are then given by

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}+1/2} : \tilde{b}_{k-n} \tilde{b}_k : + \delta_{n,0} \frac{1}{16}. \tag{9.79}$$

Notice that this is different from the expression of L_n (9.62) without twisting. Or in terms of formal variables x^i, they become

$$\tilde{L}_i = \delta_{i,1} \frac{x^1 x^1}{4} + 2 \sum_{j \geq 1} (2i + 2j - 5) x^j \partial_{i+j-2} + \frac{\hbar}{4} \sum_{j=1}^{i-2} (2j-1)(2i-2j-3) \partial_j \partial_{i-j-1} + \delta_{i,2} \frac{1}{16} \tag{9.80}$$

where we simply shifted the indices by $\tilde{L}_i = L_{i-2}$ so that the new L_i are labelled by positive integers $i \in \mathbb{Z}_{>0}$.

To bring the operators $\tilde{L}_i$ to the correct form, we apply a dilaton shift to $\tilde{L}_i$ as

$$\hat{L}_i = \frac{2\hbar}{2i-1} \exp\left(-\frac{\partial}{\partial x^2}\right) \tilde{L}_i \exp\left(\frac{\partial}{\partial x^2}\right), \tag{9.81}$$

where we conventionally multiply by a constant. The resulting differential operators $\hat{L}_i$ are

$$\begin{aligned}\hat{L}_i =&\hbar\partial_i + \delta_{i,1}\frac{x^1x^1}{2} + \hbar\sum_{j\geq 1}\frac{2i+2j-5}{2i-1}x^j\partial_{i+j-2}\\ &+\frac{\hbar^2}{2}\sum_{j=1}^{i-2}\frac{(2j-1)(2i-2j-3)}{2i-1}\partial_j\partial_{i-j-1} + \delta_{i,2}\frac{\hbar}{24}.\end{aligned} \tag{9.82}$$

As one can see from (9.82), $\hat{L}_1$ has a nonzero A term and $\hat{L}_2$ has a nonzero D term, thus, the associated partition function is nontrivial. Note that these $\hat{L}_i$ originated from the biggest Virasoro subalgebra $\{L_{-1}\}$ in contrast to untwisted modules where we started with smaller Virasoro subalgebras. Furthermore interestingly, if we read all the A, B, C, D from (9.82), we get

$$A_{ijk} = \delta_{i=j=k=1}, \tag{9.83}$$

$$B^k_{ij} = \frac{2k-1}{2i-1}\delta_{i+j-2,k}, \tag{9.84}$$

$$C^{jk}_i = \frac{(2j-1)(2k-1)}{2i-1}\delta_{i,j+k+1}, \tag{9.85}$$

$$D_i = \frac{1}{24}\delta_{i,2}. \tag{9.86}$$

They are precisely the same as the A, B, C, D corresponding to the topological recursion for the Airy curve as we showed in Section 9.3.3. Therefore, we have derived the following proposition:

Proposition 9.4.7. *Let $L_{\geq -1}$ be Virasoro operators derived from the twisted module of the Heisenberg vertex operator algebra. Then, a set of differential operators $\hat{L}_i$ for $i \in \mathbb{Z}_{>0}$ defined by*

$$\hat{L}_i = \frac{2\hbar}{2i-1}\exp\left(-\frac{\partial}{\partial x^2}\right)L_{i-2}\exp\left(\frac{\partial}{\partial x^2}\right), \tag{9.87}$$

precisely match with the Airy structure corresponding to the Eynard-Orantin topological recursion (9.38)-(9.41) *whose spectral curve is the Airy curve $y^2 = 2x$.*

In summary, the twisted module gives rise to the Airy structure associated with the Eynard-Orantin topological recursion whose spectral curve is the Airy curve. This means that the $F_{g,n}$ are computing the Kontsevich-Witten intersection numbers on moduli spaces of stable curves. The correspondence between Airy structures and different classes of twisted modules of vertex operator algebras can be generalized further. See [121] for more detail.

Remark 9.4.8. If we construct an Airy structure from a smaller Virasoro subalgebra $\{L_{n\geq 0}\}$, or equivalently $\{\tilde{L}_{i\geq 2}\}$ in (9.80), with a dilaton shift with ∂_1 instead of ∂_2 as in (9.81), it can be shown that the Airy structure corresponds to the one obtained from the Eynard-Orantin topological recursion for the Bessel curve $y^2 = 1/x$ following Proposition 9.3.2. As a consequence, the $F_{g,n}$ compute intersection numbers on moduli spaces of different classes [120].

Remark 9.4.9. These two are the only examples we know from this twisted module (9.78) that correspond to the Eynard topological recursion. If we consider an Airy structure from a further smaller algebra$\{L_f\}$ for $f \geq 1$, or equivalently $\{\tilde{L}_{i\geq f+2}\}$ in (9.80), a Dilaton shift by $x^f/\hbar$ gives a linear term $\hbar\partial_i$ in $\hat{L}_{i\geq 1}$ and a constant term in $\hat{L}_f$ as similar to (9.65). Then we can add constant D terms to the first $f+1$ Virasoro operators such that the associated partition function is nontrivial.

10 Super Airy Structures

In the last section, we shall propose an interesting generalization of Airy structures, called *super Airy structures* [5]. The idea is natural; since an Airy structure is a set of differential operator generating a Lie algebra, can we define a supersymmetric version of Airy structures in terms of Lie superalgebras? Can we show the existence and uniqueness of the corresponding partition function Z? And if so, what is the partition function computing? Is it a generating function of some sort of 'super enumerative invariants'?

In this section we define super Airy structures and prove uniqueness and existence of the partition function. We give examples of super Airy structures in terms of super Frobenius algebra and vertex operator superalgebras.

An interesting new possibility arises. Namely, the number of differential operators can be one less than the dimension of the underlying vector space. Application to enumerative geometry remains to be investigated.

This is a joint work in progress with V. Bouchard, P. Ciosmak, L. Hadasz, K. Osuga, B. Ruba, P. Sułkowski.

10.1 Constructions

Even though we have already seen a super Virasoro algebra in Chapter 8 that is a special example of Lie superalgebras, let us formally define a Lie superalgebra.

Definition 10.1.1. A *super vector space* is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space[1] $V = V_0 \oplus V_1$. Homogeneous elements in V_0 are called *even*, those in V_1 are *odd*. For a homogeneous element $v \in V_i$, we define the *parity* $|v|$ by $|v| = i$.

Definition 10.1.2. A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bilinear product $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous elements $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$, and $c \in \mathfrak{g}_k$,

[1] The vector space is not necessarily over $\mathbb{C}$ in general, but this will be sufficient for the purpose of this thesis.

- $[a, b] \in \mathfrak{g}_{i+j}$,
- $[a, b] = -(-1)^{|a||b|}[b, a]$,
- $(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0$.

It is straightforward to check that a super Virasoro algebra with a super commutator is an example of a Lie superalgebra.

10.1.1 Super Airy Structures

We now turn to the definition of super Airy structures. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $d+1$ dimensional vector space with linear coordinates x^I where higher case indices $I, J, K, \cdots$ have the range $0 \leq I \leq d$ whereas we denote positive integers $1, 2, \cdots, d$ by lower case indices $i, j, k, \cdots$. Hereafter, we often denote the parity for homogeneous elements by their indices, e.g., $|x^I| = |I|$ where $|I| = 0$ if x^I is a coordinate in V_0 and $|I| = 1$ if x^I is a coordinate in V_1. Thus, $x^I x^J = (-1)^{|I||J|} x^J x^I$, and in particular, we choose x^0 to be an odd variable, thus $|x^0| = 1$.

We next prepare zero-graded tensors A, B, C, D as

$$A \in \mathrm{Hom}(V^{\otimes 3}, \mathbb{C}), \quad B \in \mathrm{Hom}(V^{\otimes 2}, V), \quad C \in \mathrm{Hom}(V, V^{\otimes 2}) \quad D \in \mathrm{Hom}(V, \mathbb{C}), \tag{10.1}$$

Note that if T is a rank-(n, m) zero-graded tensor, $T(v_1, \cdots, v_n, w_1, \cdots, w_m) = 0$ for any homogeneous vectors v_i and dual vectors w_j unless $\sum_{i=1}^{n} |v_i| + \sum_{j=1}^{m} |w_j| = 0$. Thus, D has no odd component by definition. We further impose that

$$A_{0JK} = B_{0J}^{K} = C_0^{JK} = 0, \tag{10.2}$$

$$A_{IJK} = (-1)^{|J||K|} A_{IKJ}, \quad A_{ijK} = (-1)^{|i||j|} A_{jiK}, \quad C_i^{JK} = (-1)^{|J||K|} C_i^{KJ}, \tag{10.3}$$

where $A_{IJK}, B_{IJ}^{K}, C_I^{JK}$ are components of tensors A, B, C. We say a rank-$(2, 0)$ tensor T is $\mathbb{Z}_2$-symmetric if $T(v_1, v_2) = (-1)^{|v_1||v_2|} T(v_2, v_1)$ for any homogeneous vectors v_1, v_2, and similarly for higher rank tensors. Thus, A, C are partially $\mathbb{Z}_2$-symmetric tensors.

Remark 10.1.3. One needs to be careful that we do not impose any relation between A_{0iK} and A_{i0K}. We define A_{IJK} to vanish only if the first index is 0, but A_{i0K} can be nonzero.

We are now ready to define super Airy structures:

Definition 10.1.4. *A quantum super Airy structure* on V is a sequence of d differential operators L_i of the form

$$L_i = \hbar\partial_i - \frac{A_{iJK}}{2}x^J x^K - B_{iJ}^K x^J \partial_K - \frac{C_i^{JK}}{2}\partial_J \partial_K - \hbar D_i, \tag{10.4}$$

generating a Lie superalgebra

$$[L_i, L_j]_s := L_i L_j - (-1)^{|i||j|} L_j L_i = \hbar f_{ij}^k L_k, \tag{10.5}$$

for some f_{ij}^k and a formal constant $\hbar$.

We use the Einstein summation convention for repeated indices; if higher case indices are repeated the sum is from 0 to d whereas if lower case indices are repeated the sum is from 1 to d. Note that all L_i may depend on x^0 and the derivative with respect to x^0 in their quadratic term, but x^0 never appears in the linear term in L_i. Accordingly, the dimension of the underlying super vector space is one more than the number of differential operators L_i. For this reason, we call x^0 as an *extra variable.* This should be contrasted with the definition of Airy structures, Definition 9.2.1, where the number of differential operators L_i is the same as the dimension of the underlying vector space.

Remark 10.1.5. If V_1 is empty, Definition 10.1.4 reduces to Definition 9.2.1. Hence, all lemmas, propositions, theorems and their proofs below apply to non-supersymmetric versions discussed in the previous section by setting $\dim V_1 = 0$.

The requirement of a Lie superalgebra requirement (10.5) gives rise to a set of constraints on the tensors A, B, C, D.

Lemma 10.1.6.

$$A_{jiK} = (-1)^{|i||j|} A_{ijK}, , \tag{10.6}$$

$$f_{ij}^k = (-1)^{|i||j|} B_{ij}^k - B_{ji}^k, \quad (-1)^{|i||j|} B_{ij}^0 - B_{ji}^0 = 0, \tag{10.7}$$

$$B_{iK}^P A_{jPL} + (-1)^{|K||L|} B_{iL}^P A_{jPK} + (-1)^{|i||j|} B_{ij}^p A_{pKL} = (-1)^{|i||j|} (i \leftrightarrow j), \tag{10.8}$$

$$B_{iK}^P B_{jP}^L + (-1)^{|K||L|} C_i^{LP} A_{jPK} + (-1)^{|i||j|} B_{ij}^p B_{pK}^L = (-1)^{|i||j|} (i \leftrightarrow j), \tag{10.9}$$

$$C_i^{KP} B_{jP}^L + (-1)^{|K||L|} C_i^{LP} B_{jP}^K + (-1)^{|i||j|} B_{ij}^p C_p^{KL} = (-1)^{|i||j|} (i \leftrightarrow j), \tag{10.10}$$

$$\frac{1}{2} C_i^{PQ} A_{jQP} + (-1)^{|i||j|} B_{ij}^p D_p = (-1)^{|i||j|} (i \leftrightarrow j), \tag{10.11}$$

Proof. We simply expand the super commutator (10.5) and collect terms with respect to x^K and ∂_I as

$$\begin{aligned}
[L_i, L_j]_s =& \Bigg(-\hbar \left(A_{jiK} x^K + \hbar B_{ji}^K \partial_K \right) + \frac{\hbar}{2} x^K x^L \left(B_{iK}^P A_{jPL} + (-1)^{|K||L|} B_{iL}^P A_{jPK} \right) \\
&+ \hbar^2 x^K \partial_l \left(B_{iK}^P B_{jP}^L + (-1)^{|K||L|} C_i^{LP} A_{jPK} \right) + \frac{\hbar^3}{2} \partial_K \partial_L \left(C_i^{KP} B_{jP}^L + (-1)^{|K||L|} C_i^{LP} B_{jP}^K \right) \\
&+ \frac{\hbar^2}{2} C_i^{PQ} A_{jQP} \Bigg) - (-1)^{|i||j|} (i \leftrightarrow j) \\
=& \hbar f_{ij}^k L_k.
\end{aligned} \tag{10.12}$$

Then by comparing both sides, we obtain the set of constraints. □

10.2 Partition Function

It is possible to prove a supersymmetric analogue of Theorem 9.2.3. As the $F_{g,n}$ are symmetric tensors for Airy structures, it is natural to impose that the partition function for a super Airy structure is constructed by $\mathbb{Z}_2$-symmetry tensors $F_{g,n}$. However, the mismatch between the number of differential operators L_i and the number of variables x^I requires extra care. In this section we shall state the generalized theorem and give a proof.

Theorem 10.2.1 ([5]). *There exists a unique $\mathbb{Z}_2$-graded series*

$$Z = \exp \left(\sum_{g \geq 0} \sum_{n \geq 1} \sum_{I_1, \cdots, I_n = 0}^{d} \frac{\hbar^{g-1}}{n!} F_{g,n}(I_1, \cdots, I_n) x^{I_1} \cdots x^{I_n} \right), \tag{10.13}$$

where $F_{g,n}(I_1,\cdots,I_n)$ *are components of* $\mathbb{Z}_2$*-symmetric zero-graded tensors* $F_{g,n}$ *such that* $F_{0,1}=F_{0,2}=0$*, and*

$$\forall i \quad L_i\cdot Z=0. \tag{10.14}$$

Explicitly, $F_{g,n+1}(I,\Phi)$ *for* $2g-2+n\geq 0$ *are recursively given by*

$$\begin{aligned}F_{g,n+1}(i,\Phi) = &A_{iI_1I_2}\delta_{g,0}\delta_{n,2}+D_i\delta_{n,0}\delta_{g,1}\\&+\sum_{k=1}^{n}\sigma_{I_k\subset\Phi}\sum_{P=0}^{d}B^P_{iI_k}F_{g,n}(P,\Phi\backslash I_k)+\frac{1}{2}\sum_{P,Q=0}^{d}C_i^{PQ}F_{g-1,n+2}(Q,P,\Phi)\\&+\frac{1}{2}\sum_{P,Q=0}^{d}\sum_{g_1+g_2=g}\sum_{\Phi_1\cup\Phi_2=J}\sigma_{\Phi_1\subset\Phi}C_i^{PQ}F_{g_1,n_1+1}(P,\Phi_1)F_{g_2,n_2+1}(Q,\Phi_2),\end{aligned}\tag{10.15}$$

$$F_{g,n+1}(0,I_1,\Phi\backslash I_1)=(-1)^{|I_1|}F_{g,n+1}(I_1,0,\Phi\backslash I_1),\tag{10.16}$$

where $\Phi=\{I_1,\cdots,I_n\}$ *and* $\sigma_{I\subset\Phi}$ *is the sign of permutation from* Φ *to* $\{I,\Phi\backslash I\}$*.*

The proof consists of three parts. We first show that the condition $L_iZ=0$ implies the recursive formula for $F_{g,n+1}(i,\Phi)$. However, we independently define (10.16) so that the $F_{g,n}$ are $\mathbb{Z}_2$-symmetric. Then, by using (10.15) and (10.16), we show that the solution is uniquely determined. Finally, the existence is proven in the spirit as in [21] for Airy structures.

Proof. (10.14) $\Rightarrow$ (10.15). It is computationally easier to consider

$$\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}L_iZ\Big|_{x=0}=0,\tag{10.17}$$

and collect terms order by order in $\hbar$. Note that the order of the derivatives is important.

The A term and D term are

$$\begin{aligned}\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}\left(-\frac{A_{iJK}}{2}x^Jx^K-\hbar D_i\right)Z\Big|_{x=0}=&\partial_{I_n}\cdots\partial_{I_1}\left(-\frac{A_{iJK}}{2}x^Jx^K-\hbar D_i\right)\Big|_{x=0}\\=&-\sum_{g\geq 0}\hbar^g\left(A_{iI_1I_2}\delta_{n,2}\delta_{g,0}+\hbar D_i\delta_{n,0}\delta_{g,1}\right).\end{aligned}\tag{10.18}$$

By using the $\mathbb{Z}_2$-symmetry of $F_{g,n}(\Phi)$, it is straightforward to see that the linear term in L_i

contributes to (10.17) by

$$\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}\hbar\partial_i Z\Big|_{x=0}=\sum_{g\geq 0}\hbar^g F_{g,n+1}(i,I_1,\cdots,I_n). \tag{10.19}$$

Similarly, the B term gives

$$\begin{aligned}
&-\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}\hbar B_{iJ}^P x^J\partial_P\cdot Z\Big|_{x=0}\\
&=-\partial_{I_n}\cdots\partial_{I_1}\sum_{g\geq 0}\sum_{m\geq 0}\sum_{P=0}^{d}\frac{\hbar^g}{(m-1)!}B_{iJ}^P F_{g,m}(P,J_1,\cdots,J_{m-1})x^J x^{J_1}\cdots x^{J_{m-1}}\Big|_{x=0}\\
&=-\sum_{g\geq 0}\hbar^g\sum_{k=1}^{n}\sum_{P=0}^{d}\sigma_{I_k\subset\Phi}B_{iI_k}^P F_{g,n}(P,\Phi\backslash I_k).
\end{aligned}\tag{10.20}$$

Finally, the terms involving the C terms would be

$$\begin{aligned}
&-\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}\frac{\hbar^2}{2}C_i^{PQ}\partial_P\partial_Q\cdot Z\Big|_{x=0}\\
&=-\partial_{I_n}\cdots\partial_{I_1}\frac{1}{Z}\frac{\hbar}{2}\sum_{Q=0}^{d}\sum_{P,Q=0}^{d}C_i^{PQ}\partial_P\left(\sum_{g\geq 0}\sum_{m\geq 0}\frac{\hbar^g}{m!}F_{g,m+1}(Q,J_1,\cdots,J_m)x^{J_1}\cdots x^{J_m}Z\right)\Bigg|_{x=0}\\
&=-\partial_{I_n}\cdots\partial_{I_1}\Bigg(\sum_{g\geq 0}\sum_{m\geq 0}\frac{\hbar^{g+1}}{2\cdot m!}\sum_{P,Q=0}^{d}C_i^{PQ}F_{g,m+2}(Q,P,J_1,\cdots,J_m)x^{J_1}\cdots x^{J_m}\\
&\quad+\sum_{g_1,g_2\geq 0}\sum_{m_1,m_2\geq 0}\frac{\hbar^{g_1}\hbar^{g_2}}{2\cdot m_1!m_2!}\sum_{P,Q=0}^{d}C_i^{PQ}F_{g_1,m_1+1}(P,J_1,\cdots,J_{m_1})\\
&\qquad\times F_{g_2,m_2+1}(Q,J_1',\cdots,J_{m_2}')x^{J_1}\cdots x^{J_{m_1}}x^{J_1'}\cdots x^{J_{m_2}'}\Bigg)\Bigg|_{x=0}\\
&=-\sum_{g\geq 0}\frac{\hbar^g}{2}\sum_{P,Q=0}^{d}C_i^{PQ}\Bigg(F_{g-1,n+2}(Q,P,\Phi)\\
&\quad+\sum_{g_1+g_2=g}\sum_{\Phi_1\cup\Phi_2=\Phi}\sigma_{\Phi_1\subset\Phi}F_{g_1,n_1+1}(P,\Phi_1)F_{g_2,n_2+1}(Q,\Phi_2)\Bigg),
\end{aligned}\tag{10.21}$$

If we collect terms order by order in $\hbar$, we obtain the recursive equations (10.15).

Uniqueness. Now suppose there exists a $\mathbb{Z}_2$-symmetric solution to (10.14). Then, by (10.15) and (10.16), we can uniquely determine $F_{g,n+1}(i,\Phi)$ for $2g+n=2$ by

$$F_{0,3}(i,I_1,I_2)=A_{iI_1I_2},\quad F_{0,3}(0,i_1,I_2)=(-1)^{|i_1|}A_{i_10I_2},\quad F_{1,1}(i)=D_i. \tag{10.22}$$

Note that (10.16) guarantees that $F_{g,n+2}(0,0,\Phi)=0$ for any g,n. Let us assume $F_{g',n'+1}(I,\Phi')$ are uniquely determined where $2g'+n'<2g+n$. Then since (10.15) is a recursive equation in terms of $2g+n$, the assumption is sufficient to compute $F_{g,n+1}(i,\Phi)$, and then $F_{g,n+1}(0,\Phi)$ is uniquely fixed by imposing symmetry (10.16). Thus, by induction on $2g+n\geq 2$, this proves uniqueness of $F_{g,n+1}(i,\Phi)$.

Note that uniqueness does not hold any more if x^0 is even. This is because the $F_{g,n}$ with at least two entries being zero are not fixed by (10.15) which would identically vanish if x^0 is odd. Thus, this is a purely supersymmetric feature that never appears in the framework of Airy structures. Also, if there are two extra odd variables x^0,x^d, then there is no unique solution either because any $F_{g,n}$ with 0 and d in its entries is not uniquely determined. This shows that there can be at most one extra variable in order to have a unique partition function, and that this extra variable must be odd.

Existence. To prove existence, we start with the recursive formula (10.15) and (10.16), and we construct the $F_{g,n}$. If those are $\mathbb{Z}_2$-symmetric, then we have constructed a solution to (10.14). Therefore, we need to prove that (10.15) and (10.16) produce $\mathbb{Z}_2$-symmetric $F_{g,n}$. The only nontrivial thing is symmetry between i and the other indices. We thus have to show that

$$F_{g,n+2}(i,j,\Phi)=(-1)^{|i||j|}F_{g,n+2}(j,i,\Phi). \tag{10.23}$$

Let us prove (10.23) by induction on $2g+n\geq 1$. For $2g+n=1$, we have $F_{0,3}(i,j,I_1)=A_{ijI_1}$, hence (10.23) holds because of the condition (10.3). There are $F_{0,4}(i,j,I_1,I_2)$ and

$F_{1,2}(i,j)$ for $2g+n=2$. It follows from (10.15) that $F_{0,4}(i,j,I_1,I_2)$ becomes

$$\begin{aligned}
&F_{0,4}(i,j,I_1,I_2)\\
&=\sum_{P=0}^{d}\left(B_{ij}^{P}F_{0,3}(P,I_1,I_2)+(-1)^{|j||I_1|}B_{iI_1}^{P}F_{0,3}(P,j,I_2)+(-1)^{(|j|+|I_1|)|I_2|}B_{iI_2}^{P}F_{0,3}(P,j,I_1)\right)\\
&=B_{ij}^{0}F_{0,3}(0,I_1,I_2)+(-1)^{|i||j|}\left((-1)^{|i||j|}B_{ij}^{p}A_{pI_1I_2}+B_{iI_1}^{P}A_{jPI_2}+(-1)^{|I_1||I_2|}B_{iI_2}^{P}A_{jPI_1}\right)\\
&=(-1)^{|i||j|}B_{ji}^{0}F_{0,3}(0,I_1,I_2)+\left((-1)^{|i||j|}B_{ji}^{p}A_{pI_1I_2}+B_{jI_1}^{P}A_{iPI_2}+(-1)^{|I_1||I_2|}B_{jI_2}^{P}A_{iPI_1}\right)\\
&=(-1)^{|i||j|}F_{0,4}(j,i,I_1,I_2),
\end{aligned}\tag{10.24}$$

where we used (10.7) and (10.8) for the third equality. Similarly, for $F_{1,2}(i,j)$ we have:

$$\begin{aligned}
F_{1,2}(i,j)=&\sum_{Q=0}^{d}B_{ij}^{Q}F_{1,1}(Q)+\frac{1}{2}\sum_{P,Q=0}^{d}C_{i}^{PQ}F_{0,3}(Q,P,j)\\
=&B_{ij}^{q}D_q+(-1)^{|i||j|}\frac{1}{2}C_{i}^{PQ}A_{jQP}\\
=&(-1)^{|i||j|}\left(\frac{1}{2}C_{i}^{PQ}A_{jQP}+(-1)^{|i||j|}B_{ij}^{q}D_q\right)\\
=&(-1)^{|i||j|}(-1)^{|i||j|}\left(\frac{1}{2}C_{j}^{PQ}A_{iQP}+(-1)^{|i||j|}B_{ji}^{Q}D_Q\right)\\
=&(-1)^{|i||j|}F_{1,2}(j,i),
\end{aligned}\tag{10.25}$$

where we used $F_{1,1}(0)=0$ for the second equality and (10.11) for the fourth equality. Therefore, $F_{g,n+2}(i,j,\Phi)$ are $\mathbb{Z}_2$-symmetric for $2g+n=2$ as well.

Now let us assume $\mathbb{Z}_2$-symmetry for $F_{h,m+2}(i,j,\Phi)$ up to $1\le 2h+m<2g+n$. (10.15) can be rewritten as:

$$
\begin{aligned}
F_{g,n+2}(i,j,\Phi) =& \sum_{Q=0}^{d} B_{ij}^{Q} F_{g,n+1}(Q,\Phi) + \sum_{k=1}^{n} \sigma_{I_k \subset \{j,\Phi\}} \sum_{Q=0}^{d} B_{ii_P}^{Q} F_{g,n+1}(Q,j,\Phi\backslash I_k) \\
&+ \frac{1}{2} \sum_{P,Q=0}^{d} C_i^{PQ} F_{g-1,n+3}(Q,P,j,\Phi) \\
&+ \sum_{P,Q=0}^{d} C_i^{PQ} \sum_{g_1+g_2=g} \sum_{\Phi_1\cup\Phi_2=\Phi} \sigma_{\Phi_1\subset\Phi} F_{g_1,n_1+1}(P,j,\Phi_1) F_{g_2,n_2+1}(Q,\Phi_2) \\
=& B_{ij}^{0} F_{g,n+1}(0,\Phi) + \sum_{q=1}^{d} B_{ij}^{q} F_{g,n+1}(q,\Phi) \\
&+ \sum_{k=1}^{n} (-1)^{|i||j|} \sigma_{I_k\subset\Phi} \sum_{Q=0}^{d} B_{iI_k}^{Q} F_{g,n+1}(j,Q,\Phi\backslash I_k) \\
&+ \frac{1}{2}(-1)^{|i||j|} \sum_{P,Q=0}^{d} C_i^{PQ} F_{g-1,n+3}(j,Q,P,\Phi) \\
&+ \sum_{P,Q=0}^{d} C_i^{PQ} (-1)^{|j||P|} \sum_{g_1+g_2=g} \sum_{\Phi_1\cup\Phi_2=\Phi} \sigma_{\Phi_1\subset\Phi} F_{g_1,n_1+1}(j,P,\Phi_1) F_{g_2,n_2+1}(Q,\Phi_2).
\end{aligned}
\tag{10.26}
$$

The first term in (10.26) is $\mathbb{Z}_2$-symmetric thanks to (10.7). For the second term, we apply (10.15) to $F_{g,n+1}(q,\Phi)$. For the other terms, we substitute (10.15) into $F_{h,m'+1}(j,\Phi')$ for any h,Φ' whenever j is the first index. The computation becomes very tedious; the final result after simplification is summarized in the next page. As one can see, the red-highlighted terms are $\mathbb{Z}_2$-symmetric thanks to Lemma 10.1.6, and the other terms are manifestly $\mathbb{Z}_2$-symmetric. Therefore, we proved (10.23) by induction and this completes the proof of Theorem 10.2.1. □

$$
\begin{aligned}
&F_{g,n+2}(i,j,\Phi)\\
=&B^0_{ij}F_{g,n+1}(0,\Phi)+(-1)^{|i||j|}\left(B^P_{iI_1}A_{jPI_2}+(-1)^{|I_1||I_2|}B^P_{iI_2}A_{jPI_1}+(-1)^{|i||j|}B^l_{ij}A_{lI_1I_2}\right)\delta_{n,2}\delta_{g,0}\\
&+(-1)^{|i||j|}\left(\frac{1}{2}C_i^{PQ}A_{jQP}+(-1)^{|i||j|}B^l_{ij}D_l\right)\delta_{n,0}\delta_{g,1}\\
&+(-1)^{|i||j|}\sum_{k=1}^{n}\sum_{Q=0}^{d}\sigma_{I_k\subset\Phi}F_{g,n}(Q,\Phi\backslash I_k)\left(B^P_{iI_k}B^Q_{jP}+(-1)^{|Q||I_k|}C_i^{QP}A_{jPI_k}+(-1)^{|i||j|}B^p_{ij}B^Q_{pI_k}\right)\\
&+\frac{1}{2}(-1)^{|i||j|}\sum_{R,Q=0}^{d}F_{g-1,n+2}(R,Q,\Phi)\left(C_i^{QP}B^R_{jP}+(-1)^{|Q||R|}C_i^{RP}B^Q_{jP}+(-1)^{|i||j|}B^p_{ij}C_p^{QR}\right)\\
&+\frac{1}{2}(-1)^{|i||j|}\sum_{g_1+g_2=g}\sum_{\Phi_1\cup\Phi_2=\Phi}\sum_{Q,R=0}^{d}\sigma_{\Phi_1\subset\Phi}F_{g_1,n_1+1}(Q,\Phi_1)F_{g_2,n_2+1}(R,\Phi_2)\\
&\quad\times\left(C_i^{QP}B^R_{jP}+(-1)^{|Q||R|}C_i^{RP}B^Q_{jP}+(-1)^{|i||j|}B^p_{ij}C_p^{QR}\right)\\
&+\frac{1}{2}\sum_{k,l=1}^{n}\sum_{P,Q=0}^{d}\sigma_{\{I_k,I_l\}\subset\Phi}(-1)^{|P||I_l|}F_{g,n}(Q,P,\Phi\backslash\{I_k,I_l\})\left((-1)^{|i||j|}B^P_{iI_k}B^Q_{jI_l}+B^P_{jI_k}B^Q_{iI_l}\right)\\
&+\frac{1}{2}\sum_{k=1}^{n}\sum_{g_1+g_2=g}\sum_{\Phi_1\cup\Phi_2=\Phi\backslash I_k}\sum_{P,Q,R=0}^{d}\sigma_{\{I_k,\Phi_1\}\subset\Phi}F_{g_1,n_1+2}(Q,P,\Phi_1)F_{g_2,n_2+1}(R,\Phi_2)\\
&\quad\times\left(B^P_{jI_k}C_i^{QR}+(-1)^{|j||i|}B^P_{iI_k}C_j^{QR}\right)\\
&+\frac{1}{2}\sum_{k=1}^{n}\sum_{P,Q,R=0}^{d}\sigma_{I_k\subset\Phi}F_{g-1,n+2}(P,R,Q,\Phi\backslash I_k)\left(C_i^{QR}B^P_{jI_k}(-1)^{|i||P|}+(-1)^{|j||i|}B^P_{iI_k}C_j^{QR}(-1)^{|j||P|}\right)\\
&+\frac{1}{2}\sum_{g_1+g_2+g_3=g}\sum_{\Phi_1\cup\Phi_2\cup\Phi_3=\Phi}\sum_{P,Q,R,S=0}^{d}\sigma_{\{\Phi_1,\Phi_2\}\subset\Phi}F_{g_1,n_1+2}(P,R,\Phi_1)F_{g_2,n_2+1}(Q,\Phi_2)F_{g_3,n_3+1}(S,\Phi_3)\\
&\quad\times\left(C_i^{RS}C_j^{PQ}(-1)^{|j||S|}+(-1)^{|i||j|}C_j^{RS}C_i^{PQ}(-1)^{|i||S|}\right)\\
&+\sum_{P,Q,R,S=0}^{d}\left((-1)^{|i||j|}C_i^{RS}C_j^{PQ}+C_j^{RS}C_i^{PQ}\right)\times\left(\frac{1}{8}F_{g-2,n+4}(Q,P,S,R,\Phi)\right.\\
&\qquad+\frac{1}{2}\sum_{g_1+g_2=g-1}\sum_{\Phi_1\cup\Phi_2=\Phi}\sigma_{\Phi_1\subset\Phi}F_{g_1,n_1+3}(P,S,R,\Phi_1)F_{g_2,n_2+1}(Q,\Phi_2)\\
&\qquad\left.+\frac{1}{4}\sum_{g_1+g_2=g-1}\sum_{\Phi_1\cup\Phi_2=\Phi}(-1)^{|S||P|}\sigma_{\Phi_1\subset\Phi}F_{g_1,n_1+2}(P,R,\Phi_1)F_{g_2,n_2+2}(Q,S,\Phi_2)\right).
\end{aligned}
\tag{10.27}
$$

10.3 Super Airy Structures Without an Extra Variable

The possibility of having an extra variable x^0 is an interesting feature of super Airy structures that has no analogue in Airy structures. In this section, however, we study super Airy structures without an extra variable.

In Definition 10.1.4 we can replace the $d+1$-dimensional super vector space with a d-dimensional vector space if there is no extra variable. As a result, the dimension of the underlying vector space matches the number of differential operators as in Airy structures. This makes it possible to visualize the recursive formula (10.15) using trivalent graphs. In addition, we can interpret super Airy structures without an extra variable in terms of quantization of super Lagrangian subspaces.

10.3.1 Graphical Interpretation

Almost all definitions and weight assignments in Section 9.2.2 can be applied to construct a graphical interpretation of (10.15). The only thing that one needs to treat carefully is the order of the indices, since they are not simply symmetric but $\mathbb{Z}_2$-symmetric. Therefore, we need to add a few more rules on the weights A, B, C, D to respect $\mathbb{Z}_2$-symmetry. The definition of trivalent graphs $\mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$ is almost the same as Definition 9.2.4:

Definition 10.3.1. For $n, g \geq 0$ such that $2g-1+n \geq 1$, we define $\mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$ to be a set of connected trivalent graphs with the following 5 properties:

1. $2g-1+n$ vertices,
2. 1 root labelled by i,
3. n ordered leaves $(i_1, \cdots, i_n)$ labelled in counterclockwise order,
4. $3g+2n-1$ edges,
   - $2g+n-1$ arrowed edges,
   - n non-arrowed edges from a vertex to a leaf,
   - g non-arrowed inner edges where one end is the parent of the other following the arrows along the tree.

5. Arrowed edges form a spanning, planar, binary skeleton tree with root i. The arrows are oriented from roots towards leaves.

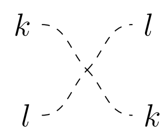
6. If an arrowed edge and a non-arrowed edge come out of a vertex, the arrowed edge is always on the left child.

7. Edges can cross only between two non-arrowed edges.

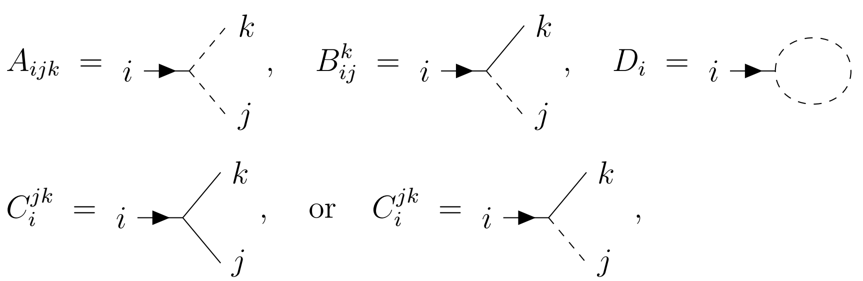
The only difference is red-highlighted, namely, we label indices in counterclockwise order which allows us to assign a sign to the graphs unambiguously. This ordering is implicitly assumed in Definition 9.2.4.

Now we assign the weights A, B, C, D. The most important difference from Airy structures is that whenever two non-arrowed edges (k, l) cross, they give sign $(-1)^{|k||l|}$, which is visualized as.

$$(-1)^{|k||l|} = \tag{10.28}$$



The weights A, B, C, D are defined by



(10.29)

where these are exactly the same as (9.20). The only additional requirement is (10.28). Then, we have:

Proposition 10.3.2 ([5]). *Let $G \in \mathbb{G}_{g,n+1}(i, i_1, \cdots, i_n)$ be a graph as defined in* Definition 10.3.1*, $w(G)$ be the weight of the graph G given by the assignments* (10.29) *with the additional sign structure* (10.28)*, and $F_{g,n+1}(i, i_1, \cdots, i_n)$ be the free energies computed from the recursive equation* (10.15) *without an extra variable. Then, we have*

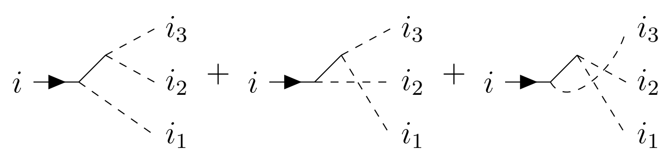
$$F_{g,n+1}(i, i_1, \cdots, i_n) = \sum_{G \in \mathbb{G}_{g,n+1}(i,i_1,\cdots,i_n)} \frac{w(G)}{|\mathrm{Aut}(G)|}, \tag{10.30}$$

where $\mathrm{Aut}(G)$ *is the group of permutations of inner edges of* G *that preserve the graph structure.*

Proof. Since the recursive formula (9.15) for Airy structures and the formula (10.15) for super Airy structures without an extra variable have the same form except the sign, we only need to justify that (10.28) is consistent with the signs that appear in (10.15). As one can see from (10.15), the two sign factors $\sigma_{I_k\subset\Phi}, \sigma_{\Phi_1\subset\Phi}$ only involve non-arrowed edges, orore precisely, leaves. Counting permutations of indices is equivalent to counting the number of crossing lines. Thus, (10.28) is indeed the correct operation, and this proves the proposition. $\square$

Example 10.3.3. Indeed, we can compute $F_{0,4}(i, i_1, i_2, i_3)$ by looking at the corresponding graphs as

$$\sum_{G\in\mathcal{G}_{0,4}(i,i_1,i_2,i_3)} w(G) = \tag{10.31}$$



$$=B^l_{ii_1}A_{li_2i_3} + (-1)^{|i_1||i_2|}B^l_{ii_2}A_{li_1i_3} + (-1)^{(|i_1|+|i_2|)|i_3|}B^l_{ii_3}A_{li_1i_2}$$
$$=F_{0,4}(i, i_1, i_2, i_3). \tag{10.32}$$

Thus, the sign is correctly given by non-arrowed crossing edges.

10.3.2 Super Lagrangian Subspaces

We would like to define a classical analogue to super Airy structures as we did in Section 9.2.3 for Airy structures. We refer the readers to [122–124] and references therein for further details on super Lagrangian subspaces.

Definition 10.3.4. A *super symplectic space* $W = W_0 \oplus W_1$ is a super vector space equipped with a bilinear nondegenerate $\mathbb{Z}_2$-antisymmetric form $\omega : W \otimes W \to \mathbb{C}$.

Definition 10.3.5. Assume that $\dim W_0 = 2d_0, \dim W_1 = 2d_1$. A Lagrangian super subspace $L = L_0 \oplus L_1$ is a subspace of W such that $\dim L_0 = d_0, \dim L_1 = d_1$, and the symplectic form vanishes on L, $\omega|_L = 0$.

Remark 10.3.6. In [122] (Section B.5), the author discusses a definition of Lagrangian subspaces when $\dim W_1$ is odd. This case may be related to super Airy structures with an extra variable, this has not been made precise yet.

Let us consider a special class of symplectic super spaces such that there exist $\mathbb{Z}_2$-graded coordinates x^i, y_j with symplectic form given by:

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_j. \tag{10.33}$$

In contrast to the standard symplectic spaces, such coordinates do not always exist; it depends on the signature of the symplectic form ω. See [122] for more details.

Now every step in Section 9.2.3 follows with supersymmetry. Namely, by using the tensors A, B, C in (10.1) we define a classical super Airy structures as quadratic polynomials of the form

$$L_i = y_i - \frac{A_{ijk}}{2} x^j x^k - B_{ij}^k x^j y_k - \frac{C_i^{jk}}{2} y_j y_k, \tag{10.34}$$

and such that the Poisson bracket is closed. Then, a super Lagrangian subspace is defined by the subspace of W given by the equations $L_i = 0$. Quantization of this classical system produces super Airy structures without an extra variable, as defined in Definition 10.1.4.

10.4 Examples

In this section we present examples of super Airy structures. We do not however have at the moment an enumerative geometric interpretations for their partition function. We postpone the geometric interpretation for future work.

10.4.1 Super Frobenius Algebras

We define super Frobenius algebras following [125], where a definition of the more general G-twisted Frobenius algebras is given. Here we concentrate on the $\mathbb{Z}_2$-graded case.

Definition 10.4.1. A super Frobenius algebra $\mathbb{A}_s = \mathbb{A}_0 \oplus \mathbb{A}_1$ over $\mathbb{C}$ is a finite-dimensional $\mathbb{Z}_2$-graded vector space equipped with a super-commutative, associative product $\mathbb{A}_g \otimes \mathbb{A}_h \to \mathbb{A}_{gh}$

respecting grading, and a non-degenerate bilinear form $\phi : \mathbb{A}_g \otimes \mathbb{A}_h \to \mathbb{C}$ where $\phi = 0$ unless $gh = 1$ and $g, h \in \mathbb{Z}_2$.

It turns out that super Airy structures corresponding to super Frobenius algebras do not involve an extra variable x^0, hence, we can simply forget x^0 and consider a super vector space V of dimension d. We define $A_{ijk}, B^k_{ij}, C^{jk}_i, D_i$ as in Lemma 9.4.2, but simply with $\mathbb{Z}_2$-grading properties.

Lemma 10.4.2. *Let (e_i) be a basis of a super-Frobenius algebra $\mathbb{A}_s$, and (e^j) be the dual basis such that*

$$\phi(e_i, e^j) = (-1)^{|i||j|}\phi(e^j, e_i) = \delta^j_i. \tag{10.35}$$

Then for any $\theta_A, \theta_B, \theta_C \in \mathbb{A}_0$,

$$A_{ijk} = \phi(\theta_A e_i e_j e_k), \quad B^k_{ij} = \phi(\theta_B e_i e_j e^k), \quad C^{jk}_i = \phi(\theta_C e_i e^j e^k), \tag{10.36}$$

and any arbitrary D_i define a quantum super Airy structure on $V = \mathbb{A}_s$ with vanishing structure constants $f^k_{ij} = 0$.

Note that if we included the basis e_0 associated with the extra variable x^0 and if we still applied (10.36) to define tensors A, B, C, then this would imply that it is possible to construct another differential operator L_0 with ∂_0 in the linear term by nonzero $A_{0JK}, B^K_{0J}, C^{JK}_0$. In this case, the number of differential operators with L_0 matches to the dimension of a super vector space, hence, this is effectively the same as ignoring the extra variable x^0. Now we prove Lemma 10.4.2.

Proof. Note that (10.35) implies that every $a \in \mathbb{A}_s$ can be written by $a = \phi(a, e^i)e_i$. This gives

$$\begin{aligned} B^p_{ik}A_{jpl} =& \phi(\theta_B e_i e_k e^p)\phi(\theta_A e_j e_p e_l) = \phi(\theta_A e_j \phi(\theta_B e_i e_k e^p) e_p e_l) \\ =& \phi(\theta_A e_j \theta_B e_i e_k e_l) = \phi(\theta_A \theta_B e_j e_i e_k e_l). \end{aligned} \tag{10.37}$$

Thus, we have

$$\begin{aligned}
&B^p_{ik}A_{jpl} + (-1)^{|k||l|}B^p_{il}A_{jpk} + (-1)^{|i||j|}B^p_{ij}A_{pkl}\\
&= \phi(\theta_A\theta_B e_j e_i e_k e_l) + (-1)^{|k||l|}\phi(\theta_A\theta_B e_j e_i e_l e_k) + (-1)^{|i||j|}\phi(\theta_A\theta_B e_i e_j e_k e_l)\\
&= 3\phi(\theta_A\theta_B e_j e_i e_k e_l) = (-1)^{|i||j|}(i \leftrightarrow j),
\end{aligned} \tag{10.38}$$

and hence the BA relation is satisfied.

Similarly, we find

$$B^p_{ik}B^l_{jp} = \phi(\theta_B^2 e_j e_i e_k e^l), \tag{10.39}$$

$$C^{lp}_i A_{jpk} = \phi(\theta_A\theta_C e_j e_i e^l e_k), \tag{10.40}$$

$$C^{kp}_i B^l_{jp} = \phi(\theta_B\theta_C e_j e_i e^k e^l), \tag{10.41}$$

$$B^p_{ij}C^{kl}_p = \phi(\theta_B\theta_C e_i e_j e^k e^l). \tag{10.42}$$

These ensure that each term in the remaining conditions is $\mathbb{Z}_2$-symmetrical under $i \leftrightarrow j$. □

There a general statement [116] that every G-equivariant topological field theory in two dimensions defines a G-twisted Frobenius algebra from which it can be recovered. Super Frobenius algebras correspond to the case with $G = \mathbb{Z}_2$. Thus, there should be an equivalence, as in Proposition 9.4.3, such that correlation functions of $\mathbb{Z}_2$-graded topological quantum field theories are given by the $F_{g,n}(i_1, \cdots, i_n)$ computed by super Airy structure defined in Lemma 10.4.2. This correspondence should be made precise and investigated further.

10.4.2 Vertex Operator Super Algebras

We now introduce vertex operator super algebras. The content in this section is a generalization of those in Section 9.4.2. Vertex operator super algebras are defined to axiomize super chiral conformal field theories in two dimensions. The formal definition is almost the same as Definition 9.4.4, up to the following modifications:

- The vector space V is not $\mathbb{Z}$-graded, but $\frac{1}{2}\mathbb{Z}$-graded, and elements in $V_{n+1/2}$ have the odd parity.

- The operator product expansion condition respects the sign of the parity of u, v

$$(x-y)^N Y(u,x)Y(v,y) = (-1)^{|u||v|}(x-y)^N Y(v,x)Y(u,y). \tag{10.43}$$

- There exists an element $\tau \in V_{3/2}$ such that

$$Y(\tau, x) = \sum_{r\in\mathbb{Z}+1/2} G_r x^{-r-3/2}, \tag{10.44}$$

  such that G_r, L_n generate a super Virasoro algebra in the Neveu-Schwarz (NS) sector:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{cm(m^2-1)}{12}, \\
[L_n, G_r] &= \left(\frac{n}{2} - r\right) G_{n+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \delta_{r+s,0}\frac{c}{3}\left(r^2 - \frac{1}{4}\right).
\end{aligned} \tag{10.45}$$

If a vertex operator algebra $(V, Y, \omega, \mathbf{1})$ possesses these additional structures, we call it a *vertex operator super algebra*. Note that if such τ exists, ω is a secondary object constructed as $2\omega = G_{1/2}\tau$. Thus, a vertex operator super algebra is sometimes denoted as $(V, Y, \tau, \mathbf{1})$ without ω.

Note that the definition above is, more precisely, for a $N = 1$ vertex operator super algebra. If we include two independent $\tau^{(1,2)}$, we can define a $N = 2$ vertex operator super algebra by requiring that $G_r^{(1,2)}, L_n$ generate a $N = 2$ super Virasoro algebra. Also, it is worth noting that G_r, L_n generate a super Virasoro algebra in the NS sector in the definition, whereas generators in the Ramond sector appear in the context of twisted modules.

To generalize the examples given in Section 9.4.2, we would like to consider modules of vertex operator super algebra with a free boson and a free fermion fields. See [126] and references therein for further details. There are a number of different ways that we can twist fields:

1. untwisted boson and untwisted fermion,

2. untwisted boson and twisted fermion,

3. twisted boson and untwisted fermion,

4. twisted boson and twisted fermion.

Also, super Airy structures may include an extra variable. In addition, there are two different modules for a untwisted boson as in (9.63) or (9.69), that is, whether we choose b_0 to be zero or not. The goal of this section is to construct super Airy structures for each case whose associated partition function is nontrivial.

As we learnt in Section 9.4.2, the procedure to find super Airy structures is the following:

i Pick a (potentially twisted) module for a free boson and a free fermion,

ii Find a set of operators generating a closed super Virasoro subalgebra,

iii Apply a dilaton shift to bring the correct linear term to the operators with taking an extra variable into consideration,

iv Get rid of any wrong term such as the last term in (9.65) if it is possible without the changing the commutation relations,

v Add constant D terms to some operators if it is possible without changing commutation relations.

An extra variable plays an important role to justify what the correct linear term is. We will present below all examples that we can construct by this procedure.

Twisted Boson and Untwisted Fermion

Let us start with a twisted boson and untwisted fermion. Without getting into the details, one can think of their module as being a representation of the form:

$$\phi = \sum_{r\in\mathbb{Z}+1/2} b_r x^{-r-1}, \quad [b_r, b_s] = r\delta_{r,-s}, \tag{10.46}$$

$$\psi = \sum_{r\in\mathbb{Z}+1/2} \alpha_r x^{-n-1/2}, \quad \{\alpha_r, \alpha_s\} = \delta_{r,-s}, \tag{10.47}$$

$$b_{-r}=\frac{1}{\sqrt{2\hbar}}x^{r+\frac{1}{2}},\quad b_r=r\sqrt{2\hbar}\frac{\partial}{\partial x^{r+\frac{1}{2}}},\quad \alpha_{-r}=\frac{1}{\sqrt{\hbar}}\xi^{r-1/2},\quad \alpha_r=\sqrt{\hbar}\frac{\partial}{\partial\xi^{r-1/2}},\quad r\in\mathbb{Z}_{\geq0}+\frac{1}{2},\tag{10.48}$$

where x^i ($i\geq1$) are even variables whereas ξ^k ($k\geq0$) are odd variables. In this case, super Virasoro operators $\{L_n,G_m\}$ are given by

$$L_n=\frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}}:b_{n-k}b_k:+\frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r+\frac{n}{2}\right):\alpha_{-r}\alpha_{r+n}:+\delta_{n,0}\frac{1}{16},\quad G_n=\sum_{r\in\mathbb{Z}+\frac{1}{2}}b_{n-r}\alpha_r,\tag{10.49}$$

and they generate a super Virasoro algebra in the Ramond sector with central charge $c=3/2$.

Since $\{L_n,G_m\}$ are in the Ramond sector, the largest closed subalgebra is generated by $\{L_{n\geq0},G_{m\geq0}\}$. In particular, L_{-1} is not included. Let us consider the largest algebra. If we apply a dilaton shift by $b_{1/2}$ to $\{L_{n\geq0},G_{m\geq0}\}$, we get

$$L'_n=\hbar\exp\left(-2\frac{\partial}{\partial x^1}\right)L_n\exp\left(2\frac{\partial}{\partial x^1}\right)=\hbar\frac{\partial}{\partial x^{n+1}}+\hbar L_n,\tag{10.50}$$

$$G'_m=\frac{\hbar}{\sqrt{2}}\exp\left(-2\frac{\partial}{\partial x^1}\right)G_m\exp\left(2\frac{\partial}{\partial x^1}\right)=\hbar\frac{\partial}{\partial\xi^m}+\frac{\hbar}{\sqrt{2}}G_m.\tag{10.51}$$

Hence the correct linear terms are given. In this case, we need to treat central charge carefully. $\{L_{n\geq0},G_{m\geq0}\}$. Namely, the following anticommutation relation does not accurately fit to a Lie superalgebra requirement due to nonzero central charge:

$$\{G_0,G_0\}=2\left(L_0-\frac{1}{16}\right).\tag{10.52}$$

As L_0 does not appear anywhere else on the right hand side of (10.45), we can redefine L_0 by $L_0\to L_0-1/16$ so that a new set $\{L_{n\geq0},G_{m\geq0}\}$ generates a Lie superalgebra. However, this precisely cancels the last term of L_0 in (10.49). Furthermore, we cannot add a constant D term to any of $L_{n\geq0}$ because they all appear on the right hand side of (10.45). Therefore, the largest super Virasoro subalgebra in this module cannot define a super Airy structure.

Now the importance of an extra variable become apparent. Since we can regard ξ^0 as an extra variable for a super Airy structure, a super Virasoro subalgebra generated by $\{L_{n\geq0},G_{m\geq1}\}$ can define a super Airy structure. In particular, these operators do not involve central charge, and furthermore, we can add a D term to L_0 without changing its commutation relation. Therefore, this super Airy structure gives a nontrivial partition function.

Indeed, this feature holds for smaller subalgebras. A dilaton shift by $x^f/\hbar$ for $f \geq 1$ gives

$$L'_n = \hbar \exp\left(-\frac{x^f}{\hbar}\right) L_n \exp\left(\frac{x^f}{\hbar}\right) = c_f \hbar \frac{\partial}{\partial x^{n-f+1}} + \hbar L_n + \frac{d_f}{2}\delta_{n,2f}, \tag{10.53}$$

$$G'_m = \hbar \exp\left(-\frac{x^f}{\hbar}\right) G_m \exp\left(\frac{x^f}{\hbar}\right) = c_f \hbar \frac{\partial}{\partial \xi^{m-f}} + \hbar G_m, \tag{10.54}$$

where c_f, d_f are some nonzero constants. Both $\{L_{n\geq f}, G_{m\geq f}\}$ and $\{L_{n\geq f}, G_{m\geq f+1}\}$ have the correct linear term where the latter takes ξ^0 as an extra variable. However, since $\{G_f, G_f\} = 2L_{2f}$, the last term in (10.75) cannot be removed without changing the commutation relation. In contrast, for $\{L_{n\geq f}, G_{m\geq f+1}\}$, $L_f, \cdots, L_{2f}$ do not appear on the right hand side of (10.45), hence we can add D terms to these $f+1$ operators. As a consequence, they define a super Airy structure whose associated partition function is nontrivial.

Twisted Boson and Twisted Fermion

There are precisely two nontrivial examples in this case. A twisted module for a free boson and a twisted module for a free fermion are respectively given by

$$\phi = \sum_{r\in\mathbb{Z}+1/2} b_r x^{-r-1}, \quad [b_r, b_s] = r\delta_{r,-s}, \tag{10.55}$$

$$\psi = \sum_{n\in\mathbb{Z}} \alpha_n x^{-n-1/2}, \quad \{\alpha_n, \alpha_m\} = \delta_{m,-n}. \tag{10.56}$$

$$b_{-r} = \frac{1}{\sqrt{2\hbar}} x^{r+\frac{1}{2}}, \quad b_r = r\sqrt{2\hbar}\frac{\partial}{\partial x^{r+\frac{1}{2}}}, \quad r \in \mathbb{Z}_{\geq 0} + \frac{1}{2} \tag{10.57}$$

$$\alpha_0 = \frac{1}{\sqrt{2\hbar}}\left(\xi^0 + \hbar\frac{\partial}{\partial \xi^0}\right), \quad \alpha_{-n} = \frac{1}{\sqrt{\hbar}}\xi^n, \quad \alpha_n = \sqrt{\hbar}\frac{\partial}{\partial \xi^n}, \quad n \in \mathbb{Z}_{>0}, \tag{10.58}$$

In this case, their Virasoro operators L_n, G_r are given by

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} :b_{n-k}b_k: + \frac{1}{2}\sum_{r\in\mathbb{Z}}\left(r+\frac{n}{2}\right):\alpha_{-r}\alpha_{r+n}: + \frac{\delta_{n,0}}{8}, \tag{10.59}$$

$$G_r = \sum_{q\in\mathbb{Z}+\frac{1}{2}} b_q\alpha_{r-q}. \tag{10.60}$$

Note that L_n, G_r generate a super Virasoro algebra in the NS sector with central charge $c = 3/2$, and the largest subalgebra is generated by $L_{n\geq -1}, G_{r\geq -1/2}$. L_0 has a nonzero D-term and $L_{-1}, G_{-1/2}$ have nonzero A_{ijk} terms where L_{-1} and $G_{-1/2}$ are given by

$$L_{-1} = \frac{1}{2}b_{-\frac{1}{2}}b_{-\frac{1}{2}} + \frac{1}{2}\alpha_{-1}\alpha_0 + \cdots, \tag{10.61}$$

$$G_{-\frac{1}{2}} = a_{-\frac{1}{2}}\alpha_0 + \cdots. \tag{10.62}$$

As in Proposition 9.4.7, we conjugate to bring the operators in the right form for a super Airy structure

$$\forall i\in\mathbb{Z}_{>0},\quad \hat{L}_i = \frac{2\hbar}{2i-1}\exp\left(-\frac{\partial}{\partial x^2}\right)L_{i-2}\exp\left(\frac{\partial}{\partial x^2}\right) = \hbar\frac{\partial}{\partial x^i} + \frac{2\hbar}{2i-1}L_{i-2}, \tag{10.63}$$

$$\forall q\in\mathbb{Z}_{\geq 0}+\frac{1}{2},\quad \hat{G}_q = \hbar\exp\left(-\frac{\partial}{\partial x^2}\right)G_{q-1}\exp\left(\frac{\partial}{\partial x^2}\right) = \hbar\frac{\partial}{\partial \xi^{q+\frac{1}{2}}} + \hbar G_{q-1}, \tag{10.64}$$

where ξ^0 is taken as an extra variable which does not contribute to the linear term. Note that we shifted the indices such that $\hat{L}_{i\geq 1}, \hat{G}_{q\geq 1/2}$ generate the closed subalgebra. Therefore, $\hat{L}_{i\geq 1}, \hat{G}_{q\geq 1/2}$ define a super Airy structure with nonzero A and D terms. As a consequence, the associated partition function is nontrivial.

As for its geometric interpretation, the pure bosonic parts of $F_{0,n}$ compute the Kontsevich-Witten intersection numbers on moduli spaces of stable curves, as shown in Proposition 9.4.7. For $g\geq 1$, even pure bosonic parts of $F_{g,n}$ would have fermionic contributions, hence, they should differ from the Kontsevich-Witten intersection numbers. What are they computing? This remains to be a mystery, and needs to be investigated further.

Similar to Remark 9.4.8, if we apply a dilaton shift by ∂_1 instead, and if we consider a smaller subalgebra generated by $\{L_{n\geq 0}, G_{r\geq 1/2}\}$, they define an Airy structure where a nonzero D term is in L_0. Thus, the associated partition function is nontrivial. Note that ξ^0 plays the role of an extra variable in this case too. For this case, we do not know a geometric interpretation either.

More generally, a dilaton shift by $x^f/\hbar$ for $f \geq 1$ brings the correct linear terms to the operators $\{L_{n\geq f}, G_{r\geq f+1/2}\}$ while treating ξ^0 as an extra variable. Then adding constant terms to $L_f, \cdots, L_{2f}$, they define a super Airy structure with $f+1$ nonzero D terms.

Untwisted Boson and Twisted Fermion

We represent the modes for a free boson and free fermion b_n, α_r as

$$n \in \mathbb{Z}_{>0} b^{-n} = \frac{nx^n}{\sqrt{\hbar}}, \qquad b_n = \sqrt{\hbar}\frac{\partial}{\partial x^n}, \quad b_0 = \sqrt{\hbar}\frac{\partial}{\partial x^0} \text{ or } b_0 = 0 \tag{10.65}$$

$$\alpha_{-n} = \frac{1}{\sqrt{\hbar}}\xi^n, \quad \alpha_n = \sqrt{\hbar}\frac{\partial}{\partial \xi^n}, \quad \alpha_0 = \frac{1}{\sqrt{2\hbar}}\left(\xi^0 + \hbar\frac{\partial}{\partial \xi^0}\right). \tag{10.66}$$

Accordingly, the super Virasoro operators L_n, G_r become

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : b_{n-k}b_k : + \frac{1}{2}\sum_{r\in\mathbb{Z}}\left(r + \frac{n}{2}\right) : \alpha_{-r}\alpha_{r+n} : + \frac{1}{16}\delta_{n,0}, \tag{10.67}$$

$$G_m = \sum_{k\in\mathbb{Z}} b_k\alpha_{m-k}, \tag{10.68}$$

where they generate a super Virasoro algebra in the Ramond sector. For $b_0 \neq 0$, a dilaton shift by $x^f/\hbar$ for $f \geq 0$ takes L_n, G_m to

$$L'_n = \hbar \exp\left(-\frac{x^f}{\hbar}\right) L_n \exp\left(\frac{x^f}{\hbar}\right) = \hbar\frac{\partial}{\partial x^{n-f}} + \hbar L_n + \frac{1}{2}\delta_{n,2f}, \tag{10.69}$$

$$G'_m = \hbar \exp\left(-\frac{x^f}{\hbar}\right) G_m \exp\left(\frac{x^f}{\hbar}\right) = \sqrt{\hbar}\alpha_{m-f} + \hbar G_m. \tag{10.70}$$

This implies that we need to consider ξ^0 as an extra variable, and a super Airy structure can be defined by a super Virasoro algebra $\{L_{n\geq f}, G_{m\geq f+1}\}$, where the first $f+1$ operators L_n have nonzero D terms.

Similarly for the case with $b_0 = 0$, a super Airy structure can be defined by a set of operators $\{L_{n\geq f'+1}, G_{m\geq f'+1}\}$ with a dilaton shift by $x^{f'}/\hbar$ for $f' \geq 1$ and with inserting constant terms. ξ^0 plays the role of an extra variable. Note, in contrast, that the first $f'+1$ operators L_n have nonzero D terms due to $\{G_{f'+1}, G_{f'+1}\} = 2L_{2f'+2}$.

Remark 10.4.3. We showed in Section 9.4.2 that a subalgebra $L_{n\geq 0}$ with a dilaton shift by b_1 also supports an Airy structure with a nonzero D term in L_0. However, its supersymmetric analogue becomes trivial due to the anticommutator $\{G_0, G_0\} = 2L_0 - 1/8$.

Untwisted Boson and Untwisted Fermion

All examples with nontrivial partition function include an extra variable in their super Airy structures. We now study examples without extra variables. A module for an untwisted boson and fermion is given by

$$n \in \mathbb{Z}_{>0} \quad : \quad b_n = \hbar^{\frac{1}{2}} \frac{\partial}{\partial x_n}, \quad x^{-n} = \hbar^{-\frac{1}{2}} n x^n, \quad b_0 = \sqrt{\hbar}\frac{\partial}{\partial x^0} \text{ or } b_0 = 0 \tag{10.71}$$

$$r \in \mathbb{Z}_{>0} + \frac{1}{2} \quad : \quad \alpha_r = \hbar^{\frac{1}{2}} \frac{\partial}{\partial \xi^r}, \quad \alpha_{-r} = \hbar^{-\frac{1}{2}} \xi^r, \tag{10.72}$$

and the super Virasoro operators L_n, G_r are

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : b_{n-k} b_k : + \frac{1}{2} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \left(r + \frac{n}{2}\right) : \alpha_{-r}\alpha_{r+n} :, \tag{10.73}$$

$$G_r = \sum_{k\in\mathbb{Z}} b_k \alpha_{r-k}. \tag{10.74}$$

where they generate a super Virasoro constraint in the NS sector. The largest subalgebra for $b_0 \neq 0$ are generated by $L_{n\geq -1}, G_{r\geq -1/2}$ whose representations are the same as (8.7) and (8.8) for supereigenvalue models in the NS sector, except for the T_2 term. However, this super Airy structure is rather boring since there is neither an A term nor a D term[2].

We repeat the same procedure as before. For $b_0 \neq 0$, a dilaton shift by $x^f/\hbar$ for $f \geq 0$

[2]The partition function for supereigenvalue models does not fit to the requirement for the partition function for super Airy structures because the former has nonzero $F_{0,1}$ and $F_{0,2}$. Hence, investigating supereigenvalue models itself is not boring.

transform L_n, G_m to

$$L'_n = \hbar \exp\left(-\frac{x^f}{\hbar}\right) L_n \exp\left(\frac{x^f}{\hbar}\right) = \hbar \frac{\partial}{\partial x^{n-f}} + \hbar L_n + \frac{1}{2}\delta_{n,2f}, \tag{10.75}$$

$$G'_r = \hbar \exp\left(-\frac{x^f}{\hbar}\right) G_r \exp\left(\frac{x^f}{\hbar}\right) = \hbar \frac{\partial}{\partial \xi^{r-f}} + \hbar G_m. \tag{10.76}$$

A subalgebra generated by $\{L_{n\geq f}, G_{r\geq f+1/2}\}$ defines a super Airy structure with nontrivial partition function after adding constant terms into the first $f+1$ terms. Even if we treat $\xi^{1/2}$ as an extra variable and consider a super Airy structure constructed by a set $\{L_{n\geq f}, G_{r\geq f+3/2}\}$, this gives exactly the same partition function.

If we choose $b_0 = 0$, we can consider $\{L_{n\geq f'+1}, G_{r\geq f'+1/2}\}$ and $\{L_{n\geq f'+1}, G_{r\geq f'+1/2}\}$ where we apply a dilaton shift by $x^{f'}/\hbar$ for $f' \geq 1$. The latter subalgebra treats $\xi^{1/2}$ as an extra variable and it can have $f'+2$ nonzero D terms in the operators $L_{f'+1}, \cdots, L_{2f'+2}$ whereas we cannot add D terms to L_{2f+1}, L_{2f+2} for the former case due to $\{, G_{r\geq f'+1/2}, G_{r\geq f'+1/2}\} = 2L_{2f+1}$. Also the same reason as Remark 10.4.3 holds so that the case with a dilaton shift by b_1, which works as an Airy structure, becomes trivial.

We have presented many examples of super Airy structures generating super Virasoro subalgebras. Since super Virasoro subalgebras are a crucial factor of super conformal field theory in two dimensions, it is interesting to see whether there is any geometric interpretation of these examples. The most encouraging example out of what we have shown above is the super Airy structure defined by (10.63) and (10.64) from a twisted boson and a twisted fermion. Can we construct a supersymmetric analogue of Proposition 9.4.7? We leave this challenge to future work.

11 Conclusion and Future Work

In the first part of this thesis (Chapter 2-5), we focused on quantum information theory and black hole physics.

We have researched geometric properties of some special quantum states in Chapter 3 by using entanglement of formation and metrics on the spaces of quantum states. The examples with one variable somehow indicate that the points on the Bloch sphere should be relatively *close* each other in order to have a separable pair of qubits. The above problems we analyzed might be toy examples, yet show interesting characteristics of geometry of quantum states. This section leads to a better understanding of pure, mixed, separable and entangled states. We are hoping to publish this work with Page soon.

Then in Chapter 5, we developed a unitary qubit model without firewalls. It utilizes nonlocal degrees of freedom for the gravitational field, yet we believe it is a realistic proposal, because nonlocality is confined into the gravity sector, hence it is a minimal departure from the current understanding of quantum field theory to resolve the firewall paradox. In addition, our model avoids a counterargument raised by [13] that uses a mining process. At the same time, it is a toy model, hence we would like to consider a more realistic interaction between the radiation modes and the gravitational field than the simple interaction addressed in (5.7).

As an extension of our qubit model, what would be interesting is to compute nonlocality in a more quantitative method. Donnelly, Kinsella, and Giddings have shown in [128, 129] that two gravitationally dressed scalar fields do not commute even if they are separated far apart. They give an analogous result for electromagnetically dressed fields as a comparison, and their commutator in this setting indeed decreases as a function of the distance as expected. It would be worth investigating whether this nonlocal behaviour of dressed fields have some connections to our nonlocal qubit model. Also, their focus was on the flat and Anti-de Sitter background, but the research for the de Sitter background has not been done because of computational difficulties. Yet, it would be interesting to see if a similar story holds in the de Sitter background.

Based on these papers and other prominent ones, Giddings proposed the idea of *quantum* *first gravity* [130,131], which roughly states that the structure of spacetimes is reproduced by quantum mechanics, more precisely the structure of Hilbert space. The idea of understanding local quantum field theory from a Hilbert space perspective is not new. For example, locality of spacetime is encoded in the fact that two operators commute. Then, a question is: what kind of mathematical structure do we need to impose on Hilbert spaces in order to describe gravity such that it matches our current understanding of local quantum field theory in a weak gravity regime? The fact that two dressed scalars far apart from each other do not commute suggests that we do not even know how to define subspaces and locality. More research need to be done.

The second part of this thesis (Chapter 6-10) explored generalizations of important structures in mathematical physics to the supersymmetric realm. After reviewing Hermitian matrix models and the Eynard-Orantin topological recursion, we investigated supereigenvalue models, which are supersymmetric generalizations of Hermitian matrix models, and presented a recursive formalism that computes all correlation functions in supereigenvalue models from simple initial data. More precisely, we showed that the Eynard-Orantin topological recursion, in conjunction with simple Grassmann-valued auxiliary equations, are sufficient to compute all correlation functions of supereigenvalue models recursively.

We are currently developing the theory of supereigenvalue models in the Ramond sector. In contrast to those in the NS sector, we can show that the Eynard-Orantin topological recursion is not sufficient any more to recursively compute correlation functions. Hence, it needs to be generalized. The key difference is that (8.1.5) does not hold for supereigenvalue models in the Ramond sector. Thus, we need to analyze the corresponding super loop equations more carefully to develop a new recursive formalism.

Then, we introduced Airy structures [20, 21] as a new mathematical framework behind topological recursion, and we proposed a supersymmetric generalization, which we called super Airy structures. We discussed the possibility of including an extra Grassmann variable, and proved existence and uniqueness of the partition function for all super Airy structures. We then gave various examples of super Airy structures building up on analogous constructions for Airy structures. At the same time, we do not know what the partition functions of these super Airy structures compute, but we expect them to generalize interesting enumerative invariants. For instance, they could be related to Gromov-Witten invariants with odd cohomology classes, as studied for instance by Okounkov and Pandharipande in [127]. It would be very interesting if we could connect our super Airy structures with their construction.

To conclude this thesis, let us mention an exciting future area of reseasrch. Recently, new relations between black hole physics and matrix models are proposed in [132, 133]. It was shown [134–136] that the Schwarzian theory [137] emerged from the Sachdev-Ye-Kitaev model [138,139] is dual to the Jackiw-Teitelboim gravity [140–142], which is a two-dimensional dilaton gravity theory. Then, [143] argues that the matrix model relating to the Mirzakhani recursion [92–94] also contains the Schwarzian theory. This discovery may connects matrix models, chaotic theory, black hole physics, and enumerative geometry all together. How exciting!

A A Taub-NUT Metric

In this section, I present my first paper with Page [3], in which we investigated a biaxial Bianchi IX model with positive cosmological constant. Even though our original motivation was about the asymptotic behaviour of Bianchi IX models, we found a geometrically interesting and elegant way to derive the exact solution for biaxial models where the solution itself was well-known. We review such a derivation below. Note that in this section we set $c = G = 1$ unlike the body of this thesis since there is not $\hbar$ for this work. Most of the discussions in this section are take from [3].

A.1 Bianchi IX Models

Bianchi IX spacetimes are spatially compact homogeneous anisotropic cosmological models, which can be thought of generalizations of the well-known Friedmann-Lemaı̃tre-Robertson-Walker geometry. The metric for triaxial models is written as

$$ds^2 = -dt^2 + a^2(t)\omega_1^2 + b^2(t)\omega_2^2 + c^2(t)\omega_3^2, \tag{A.1}$$

where (t, a, b, c) have dimension of time or length, and $\{\omega_k\}$ is a set of $\mathbb{S}^3$-invariant one-forms,

$$\begin{aligned} \omega_1 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi, \\ \omega_2 &= \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi, \\ \omega_3 &= d\psi + \cos\theta\, d\phi, \end{aligned} \tag{A.2}$$

obeying

$$d\omega_1 = \omega_2 \wedge \omega_3 \qquad \text{et cyc.}, \tag{A.3}$$

where $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, $0 \le \psi < 4\pi$. The three principal circumferences of the distorted S^3 are then $(4\pi a, 4\pi b, 4\pi c)$.

On the other hand, the Λ-Taub-NUT spacetimes were originally introduced by [144] as spacetimes whose spatial topology is a biaxial S^3 and which satisfy the Einstein equations with positive cosmological constant,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{A.4}$$

The form of the solution depends on the choice of time coordinate; for example it is given by [145, 146] with an arbitrary constant C_0 and arbitrary positive constant D_0 as

$$ds^2 = \frac{3D_0}{\Lambda}\left(-\frac{d\tau^2}{f(\tau)} + \frac{f(\tau)}{4}\,\omega_3^2 + \frac{\tau^2+1}{4}\,(\omega_1^2+\omega_2^2)\right), \tag{A.5}$$

$$f(\tau) = \frac{D_0\tau^4 + 2(3D_0-2)\tau^2 + C_0\tau + 4 - 3D_0}{1+\tau^2}. \tag{A.6}$$

Thus, the Λ-Taub-NUT spacetime is none other than a biaxial Bianchi IX spacetime with positive cosmological constant, and one can take (A.5) to the form (A.1) by an appropriate transformation of the time coordinate τ. In [3], we present a new derivation of (A.5) by considering the minisuperspace defined below.

A.1.1 Minisuperspace

A minisuperspace is an equivalent description of a Bianchi IX spacetime, but from a different point of view. Note that a minisuperspace has nothing to do with superspace in the context of supersymmetry. In short, a particle trajectory in a minisuperspace corresponds to an evolution of $a(t), b(t), c(t)$ in a Bianchi IX spacetime. We will define a minisuperspace more precisely below.

The orthonormal components of the Ricci tensor of a Bianchi IX spacetime are given by [147] as

$$R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}, \tag{A.7}$$

$$R_1^1 = \frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) + \frac{a^4-(b^2-c^2)^2}{2a^2b^2c^2}. \tag{A.8}$$

Here an overdot is a derivative with respect to the proper time t as in the metric (A.1). R_2^2 and R_3^3 are just permutations of R_1^1, and off-diagonal elements of the Ricci tensor are zero. Let us write the dimensional variables (t, a, b, c) in terms of dimensionless Misner variables

$(\zeta, \alpha, \beta_a, \beta_b, \beta_c, \beta, \gamma)$ [148] as

$$t = \sqrt{\frac{3}{\Lambda}}\zeta, \quad a = \sqrt{\frac{3}{\Lambda}}e^{\alpha+\beta_a}, \quad b = \sqrt{\frac{3}{\Lambda}}e^{\alpha+\beta_b}, \quad c = \sqrt{\frac{3}{\Lambda}}e^{\alpha+\beta_c}, \tag{A.9}$$

$$\beta_a = \beta + \sqrt{3}\gamma, \quad \beta_b = \beta - \sqrt{3}\gamma, \quad \beta_c = -2\beta, \tag{A.10}$$

where α tells how spatially large the model is, since the S^3-volume is $16\pi^2(3/\Lambda)^{\frac{3}{2}}e^{3\alpha}$, while β and γ describe how distorted S^3 is.

The scalar curvature of the distorted S^3 at one time is

$$^{(3)}R = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{2a^2b^2c^2}. \tag{A.11}$$

Multiplying the S^3 scalar curvature by a quantity proportional to the two-thirds power of the S^3 volume gives the dimensionless quantity

$$V = \frac{1}{6}(abc)^{\frac{2}{3}}\,{}^{(3)}R = \frac{1}{2\Lambda}e^{2\alpha}\,{}^{(3)}R = \frac{1}{12}\left(4e^{-2\beta}\cosh 2\sqrt{3}\gamma - 4e^{4\beta}\sinh^2 2\sqrt{3}\gamma - e^{-8\beta}\right). \tag{A.12}$$

Now letting an overdot denotes a derivative with respect to the dimensionless time coordinate ζ, the Einstein equations (A.4) give three dimensionless 2nd-order equations

$$\ddot{\alpha} = 3 - 3\dot{\alpha}^2 - 2Ve^{-2\alpha}, \tag{A.13}$$

$$\ddot{\beta} = -3\dot{\alpha}\dot{\beta} + \frac{1}{2}\frac{\partial V}{\partial \beta}e^{-2\alpha}, \tag{A.14}$$

$$\ddot{\gamma} = -3\dot{\alpha}\dot{\gamma} + \frac{1}{2}\frac{\partial V}{\partial \gamma}e^{-2\alpha}, \tag{A.15}$$

and one dimensionless 1st-order constraint equation, which comes from the trace of (A.4),

$$\dot{\alpha}^2 - \dot{\beta}^2 - \dot{\gamma}^2 = 1 - Ve^{-2\alpha}. \tag{A.16}$$

Note that by combining (A.16), its time derivative, and any two of (A.13)-(A.15), one can derive the remaining 2nd-order equation, so only the 1st-order constraint (A.16) and any two of the three 2nd-order equations are independent.

Note also that if we choose $\gamma = \dot{\gamma} = 0$ as an initial condition, then $\partial V/\partial\gamma = 0$ and $\ddot{\gamma} = 0$, so γ remains zero for all time, which is just a biaxial model.

One notices that (A.13)-(A.16) are reproduced by the following action:

$$S = \frac{1}{2}\int d\tau \left(N^{-1}e^{3\alpha}(-\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2) - N\left(e^{3\alpha} - e^{\alpha}V\right)\right), \tag{A.17}$$

where now the dot denotes the derivative with respect to τ, which is not the same τ as in (A.5), and N is a Lagrange multiplier. The relation between ζ and τ is

$$\frac{d}{d\zeta} = \frac{1}{N}\frac{d}{d\tau}. \tag{A.18}$$

If we define

$$\eta = N\left(e^{3\alpha} - Ve^{\alpha}\right), \tag{A.19}$$

then (A.17) becomes

$$S = -\frac{1}{2}\int d\tau \left(\eta^{-1}\left(e^{6\alpha} - Ve^{4\alpha}\right)(-\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2) - \eta\right). \tag{A.20}$$

This is a relativistic point-particle action in three dimensions (α, β, γ) with mass $m = 1$ and the minisuperspace metric

$$ds^2 = \left(e^{6\alpha} - Ve^{4\alpha}\right)(-d\alpha^2 + d\beta^3 + d\gamma^2). \tag{A.21}$$

This three-dimensional (or two-dimensional for the biaxial case) curved space obtained from the four-dimensional Bianchi IX space is an example of a metric on *minisuperspace* whose geodesics give solutions of Einstein equations [149]. Therefore, time evolution of a Bianchi IX space with Λ is equivalent to particle motion along a geodesic curve in this minisuperspace. A more rigorous way to obtain (A.20) is shown for example in [150].

A.2 Geometry of Minisuperspace Associated With a Biaxial Bianchi IX

A number of Killing tensors in a geometry is a crucial measure whether geodesics in the geometry are integrable. Thus, we investigate geometrical properties of the two-dimensional minisuperspace associated with the biaxial Bianchi IX model with $\gamma = 0$, so $a = b = (3/\Lambda)^{1/2} e^{\alpha+\beta}$ and $c = (3/\Lambda)^{1/2} e^{\alpha-2\beta}$. Let us first define null coordinates (u, v) as

$$u = \alpha - \beta + \frac{1}{2}\log 3 - \frac{2}{3}\log 2, \quad v = \alpha + \beta + \frac{1}{2}\log 3. \tag{A.22}$$

Then the minisuperspace metric and the Ricci scalar are respectively

$$ds^2 = -\frac{4}{27} U(u,v)\, dudv, \quad U(u,v) = e^{3u+3v} - e^{3u+v} + e^{6u-2v}, \tag{A.23}$$

$$R = 81 U^{-3} e^{9u} (3e^{-v} - 5e^{u}). \tag{A.24}$$

A.2.1 Nontrivial Killing Tensors

A key discovery shown in [3] is that the minisuperspace does not admit any Killing vector, but it does admit two nontrivial Killing tensors, one rank-2 and the other rank-4. We leave the detailed computations to [3], and instead summarize the result below.

The components of the nontrivial rank-2 Killing tensor are respectively

$$K_{uu} = 0, \quad K_{vv} = 2\, e^{-6u}\, U^2, \quad K_{uv} = 3\, e^{-2v}\, U. \tag{A.25}$$

Note that these $K_{\mu\nu}$ are clearly different from the metric, hence it is a nontrivial rank-2

Killing tensor. The components of the nontrivial rank-4 Killing tensor are

$$K_{uuuu} = 0, \tag{A.26}$$

$$K_{uuuv} = 81\, G_3\, e^{-2v}\, U^3 \tag{A.27}$$

$$K_{uuvv} = \Big(G_0 + 3G_1\, e^{-4v} + 2G_2\, e^{-2v} + G_3 \left(2e^{6v} - 12\, e^{4v} + 18\, e^{2v} + 54e^{6u-4v} - 72e^{3u+v}\right) \Big) U^2, \tag{A.28}$$

$$K_{uvvv} = \Big(3G_1\, e^{-6u-2v} + G_2\, e^{-6u} + G_3 \left(18\, e^{-3u+v} + 27\, e^{-2v} - 6\, e^{-3u+3v}\right) \Big) U^3, \tag{A.29}$$

$$K_{vvvv} = (2G_1\, e^{-12u} + 12G_3\, e^{-6u})U^4, \tag{A.30}$$

with constants G_k. One can indeed check these tensors satisfy Killing equations,

$$\nabla_{(\mu} K_{\nu)} = 0, \quad \nabla_{(\mu} K_{\nu)\rho_1\rho_2\rho_3} = 0. \tag{A.31}$$

Note that if G_0 alone is nonzero, the Killing tensor is proportional to the symmetric product of two metrics; if G_1 alone is nonzero, the Killing tensor is proportional to the symmetric product of $K_{\mu\nu}$ from (A.25) with itself; if G_2 alone is nonzero, the Killing tensor is proportional to the symmetric product of $g_{\mu\nu}$ and $K_{\mu\nu}$; but if $G_3 \neq 0$, one gets a new nontrivial rank-4 Killing tensor. In [3] we set $G_0 = G_1 = G_2 = 0, G_3 = 16$ for computational simplicity.

A.3 Exact Solution

Geodesics in a d-dimensional spacetime are integrable if there are d independent conserved quantities. At the same time, there always exists a conserved quantity in two dimensions, namely the one associated with the metric $g_{\mu\nu}$ However in this case, the minisuperspace admits two additional conserved quantities associated with (3) $K_{\mu\nu}$, and (4) $K_{\nu\rho_1\rho_2\rho_3}$, hence geodesics are integrable. They play a crucial role to analytically determine the exact form of the solution (A.5).

As shown above, the two nontrivial invariants of motion are

$$\begin{aligned} E_1 =& K_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ =& 6e^{-2v}\,U\,\frac{du}{d\tau}\frac{dv}{d\tau} + 2e^{-6u}\,U^2\left(\frac{dv}{d\tau}\right)^2 , \end{aligned} \tag{A.32}$$

$$\begin{aligned} E_2 =& K_{\mu\nu\rho\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \\ =& 2^6\cdot 3^4\,e^{-2v}\,U^3\left(\frac{du}{d\tau}\right)^3\frac{dv}{d\tau} + 2^6\cdot 3\,e^{-6u}\,U^4\left(\frac{dv}{d\tau}\right)^4 \\ &+ 2^6\cdot 3\left(e^{6v} - 6\,e^{4v} + 9\,e^{2v} + 27e^{6u-4v} - 36e^{3u+v}\right)\,U^2\left(\frac{du}{d\tau}\right)^2\left(\frac{dv}{d\tau}\right)^2 \\ &+ 2^6\cdot 3\left(6\,e^{-3u+v} + 9\,e^{-2v} - 2\,e^{-3u+3v}\right)\,U^3\,\frac{du}{d\tau}\left(\frac{dv}{d\tau}\right)^3 , \end{aligned} \tag{A.33}$$

where we are setting the Lagrange multiplier $\eta = 1$ so that τ becomes the proper time along timelike geodesics in the minisuperspace metric, giving

$$\frac{4}{27}\,\frac{du}{d\tau}\frac{dv}{d\tau}U = 1. \tag{A.34}$$

This τ is not to be confused with the Λ-Taub-NUT time coordinate in the metric (A.5). By using (A.34), we can simplify the expressions of E_1 and E_2 to

$$E_1 = \frac{81}{2}e^{-2v} + 2\,e^{-6u}\,U^2\left(\frac{dv}{d\tau}\right)^2 , \tag{A.35}$$

$$\begin{aligned} E_2 =& 3^{13}\,e^{-2v}\left(\frac{dv}{d\tau}\right)^{-2} \\ &+ 4\cdot 3^7\left(e^{6v} - 6\,e^{4v} + 9\,e^{2v} + 27e^{6u-4v} - 36e^{3u+v}\right) \\ &+ 2^4\cdot 3^4\left(6\,e^{-3u+v} + 9\,e^{-2v} - 2\,e^{-3u+3v}\right)\,U^2\left(\frac{dv}{d\tau}\right)^2 \\ &+ 2^6\cdot 3\,e^{-6u}\,U^4\left(\frac{dv}{d\tau}\right)^4 . \end{aligned} \tag{A.36}$$

The right-hand side of (A.35) shows that $E_1 > 0$. The original constraint (A.16) in null coordinates is

$$\frac{du}{d\zeta}\frac{dv}{d\zeta} = e^{-(3u+3v)}\, U. \tag{A.37}$$

Thus by comparing this with (A.34), the relation between $d\zeta$ and $d\tau$ is

$$\frac{d}{d\zeta} = \frac{2U\, e^{-\frac{3}{2}(u+v)}}{3\sqrt{3}}\frac{d}{d\tau}. \tag{A.38}$$

However one can see from the form of (A.35) that it becomes simplified if a new time coordinate T is chosen as

$$\frac{d}{dT} = \frac{2}{9}\, e^{-3u+2v}\, U\,\frac{d}{d\tau} = \frac{1}{\sqrt{3}}\, e^{-\frac{3}{2}u+\frac{7}{2}v}\,\frac{d}{d\zeta} = 2\sqrt{\Lambda}\, a(t)^3\, c(t)^{-1}\frac{d}{dt}. \tag{A.39}$$

Then (A.35) can be rewritten as

$$\begin{aligned} E_1 =& \frac{81}{2}e^{-2v} + \frac{81}{2}e^{-4v}\left(\frac{dv}{dT}\right)^2 \\ =& \frac{81}{2}e^{-2v} + \frac{81}{8}\left(\frac{d}{dT}e^{-2v}\right)^2, \end{aligned} \tag{A.40}$$

or

$$\frac{1}{4}\left(\frac{d}{dT}e^{-2v}\right)^2 = \frac{2}{81}E_1 - e^{-2v}. \tag{A.41}$$

One can obtain the solution as

$$e^{-2v} = C^2 - (T - T_0)^2 \qquad \left(C = \frac{\sqrt{2\,E_1}}{9}\right), \tag{A.42}$$

where the range of T is

$$T_0 - C \le T \le T_0 + C. \tag{A.43}$$

Note that T_0 is just a shift of time, so we choose $T_0 = 0$, and the inequalities above become equalities at past and future infinity for the proper time t of the biaxial Bianchi IX spacetime metric.

The constraint equation (A.34) then gives the solution for u. In terms of the T coordinate, it becomes

$$3\,e^{3u}\,e^{-7v}\,\frac{du}{dT}\frac{dv}{dT}\equiv-\,\frac{1}{2}\,e^{-5v}\,\left(\frac{d}{dT}e^{3u}\right)\left(\frac{d}{dT}e^{-2v}\right)$$
$$=1-e^{-2v}+e^{3u-5v}. \tag{A.44}$$

By using (A.42), one gets

$$\left(\frac{d}{dT}e^{3u}\right)=\frac{e^{3u}+e^{5v}-e^{3v}}{T}, \tag{A.45}$$

which has the solution

$$e^{3u}=B\,T+\frac{(6\,C^2-8)T^4+(12\,C^2-9\,C^4)\,T^2+3\,C^6-3\,C^4}{3\,C^6\,(C^2-T^2)^{\frac{3}{2}}}, \tag{A.46}$$

where B is another constant which is related to E_2 by

$$E_2=4\cdot 3^7\cdot\frac{9\,B^2C^{12}+36\,C^4-96\,C^2+64}{C^6}. \tag{A.47}$$

Since u and v are given as explicit functions of T by (A.42) and (A.46), the biaxial Bianchi IX metric (A.1) can be written explicitly in terms of T and the two parameters as

$$\begin{aligned}
ds^2=&\frac{3}{\Lambda}\left(-\frac{1}{3}\,e^{7v-3u}\,dT^2+\frac{1}{3}\,e^{2v}\,(d\theta^2+\sin^2\theta\;d\phi^2)+\frac{4}{3}\,e^{3u-v}\,(d\psi+\cos\theta\,d\phi)^2\right)\\
=&\frac{3}{\Lambda}\Bigg[-\frac{1}{3}(C^2-T^2)^{-\frac{7}{2}}\left(B\,T+\frac{6\,T^4C^2-9\,T^2C^4+3\,C^6-8\,T^4+12\,T^2C^2-3\,C^4}{3\,C^6\,(C^2-T^2)^{\frac{3}{2}}}\right)^{-1}dT^2\\
&+\frac{1}{3(C^2-T^2)}(d\theta^2+\sin^2\theta\;d\phi^2)\\
&+\frac{4}{3}\sqrt{C^2-T^2}\left(B\,T+\frac{6\,T^4C^2-9\,T^2C^4+3\,C^6-8\,T^4+12\,T^2C^2-3\,C^4}{3\,C^6\,(C^2-T^2)^{\frac{3}{2}}}\right)(d\psi+\cos\theta\,d\phi)^2\Bigg].
\end{aligned} \tag{A.48}$$

One can check that (A.5) and (A.48) coincide with each other by identifying their time coordinates and parameters as follows:

$$\frac{D_0}{4}(\tau^2+1) = \frac{1}{3}\,e^{2v} = \frac{1}{3(C^2-T^2)}, \tag{A.49}$$

$$D_0 = \frac{4}{3C^2}, \qquad C_0 = 4BC^4 \tag{A.50}$$

A.3.1 Exact Solutions for Triaxial Bianchi IX Models?

We have given a new derivation of the Taub-NUT metric with positive cosmological constant by considering the associated minisuperspace in this section. More specifically, we have found that the minisuperspace admits two nontrivial Killing tensors and used them to derive the metric. This method might be useful to obtain the exact solution for triaxial Bianchi IX models as well. If one can find another nontrivial Killing tensors in the minisuperspace of triaxial Bianchi IX models besides the metric and the time-reparametrization, presumably either rank-2 or rank-4 as extensions of those shown in this paper, the system becomes integrable and it would be possible to derive the exact solution.

B Derivation of (8.13)

Here we give a proof of (8.13), which is the statement that the free energy for supereigenvalue models is at most quadratic in the Grassmann parameters. We include a proof of this fact here for completeness; it follows along similar lines as the original proof in [102].

Setting $t_s = 2t$, the partition function (8.3) of supereigenvalue models can be written as

$$\mathcal{Z} \overset{\text{formal}}{=} \int \prod_{i=1}^{2N} d\lambda_i \prod_{i=1}^{2N} d\theta_i \prod_{i<j}^{2N} (\lambda_i - \lambda_j - \theta_i\theta_j)\, e^{-\frac{N}{t}\sum_{l=1}^{2N}(V(\lambda_l)+\Psi(\lambda_l)\theta_l)}. \tag{B.1}$$

We will drop the "formal" superscript in this appendix for clarity.

We now would like to integrate over the $2N$ Grassmann variables θ_i. Recall that Grassmann integrals obey

$$\int d\theta_k = 0, \qquad \int \prod_{i=1}^{2N} d\theta_i \theta_{\sigma(1)} \cdots \theta_{\sigma(2N)} = \operatorname{sgn}(\sigma). \tag{B.2}$$

where $\sigma \in S_{2N}$. The first equation ensures that terms with an odd number of $\xi_{k+\frac{1}{2}}$ vanish, hence the partition function is expanded as

$$\mathcal{Z} = \sum_{K=0}^{N} \mathcal{Z}^{(2K)}, \tag{B.3}$$

where the superscript denotes the order of the Grassmann couplings $\xi_{k+\frac{1}{2}}$. Note that the possible highest order of $\xi_{k+\frac{1}{2}}$ is $2N$ no matter what the degree of the Grassmann potential $\Psi(x)$ is. This is because there are only $2N$ Grassmann variables θ_i to be integrated. More precisely, we have

$$\begin{aligned}\mathcal{Z}^{(2K)} = \left(\frac{N}{t}\right)^{2K} \int \prod_{i=1}^{2N} d\lambda_i \prod_{i<j}^{2N} (\lambda_i - \lambda_j) e^{-\frac{N}{t}\sum_{l=1}^{2N} V(\lambda_l)} \\ \times \left(\frac{1}{(2K)!} \int \prod_{i=1}^{2N} d\theta_i \prod_{i<j} \left(1 + \frac{\theta_i\theta_j}{\lambda_j - \lambda_i}\right) \left(\sum_{l=1}^{2N} \Psi(\lambda_l)\theta_l \right)^{2K} \right).\end{aligned} \tag{B.4}$$

We can now evaluate the integral over the Grassmann variables θ_i. It is not too difficult to see that

$$\frac{1}{(2K)!}\int\prod_{i=1}^{2N}d\theta_i\prod_{i<j}\left(1+\frac{\theta_i\theta_j}{\lambda_j-\lambda_i}\right)\left(\sum_{l=1}^{2N}\Psi(\lambda_l)\theta_l\right)^{2K}$$
$$=\frac{1}{(2K)!2^{N-K}(N-K)!}\sum_{\sigma\in S_{2N}}\operatorname{sgn}(\sigma)\prod_{i=1}^{2K}\Psi(\lambda_{\sigma(i)})\prod_{j=K+1}^{N}\frac{1}{\lambda_{\sigma(2j)}-\lambda_{\sigma(2j-1)}}. \quad \text{(B.5)}$$

Next, the Vandermonde determinant in (B.4) can be expressed as

$$\prod_{i<j}^{2N}(\lambda_i-\lambda_j)=(-1)^N\sum_{\tau\in S_{2N}}\operatorname{sgn}(\tau)\prod_{l=1}^{2N}\lambda_l^{\tau(l)-1}. \quad \text{(B.6)}$$

By plugging this and (B.5) into (B.4), we get

$$\mathcal{Z}^{(2K)}=\left(\frac{N}{t}\right)^{2K}\frac{(-1)^N}{(2K)!2^{N-K}(N-K)!}\sum_{\tau,\sigma\in S_{2N}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$
$$\times\int\prod_{i=1}^{2N}d\lambda_i e^{-\frac{N}{t}\sum_{l=1}^{2N}V(\lambda_l)}\prod_{l=1}^{2N}\lambda_l^{\tau(l)-1}\prod_{i=1}^{2K}\Psi(\lambda_{\sigma(i)})\prod_{j=K+1}^{N}\frac{1}{\lambda_{\sigma(2j)}-\lambda_{\sigma(2j-1)}}. \quad \text{(B.7)}$$

Since every λ_i is integrated, for each permutation $\sigma\in S_n$, we can rename $\lambda_{\sigma(i)}\mapsto\lambda_i$. As a result, each term in the summation over $\sigma\in S_{2n}$ gives the same integral, and we get

$$\mathcal{Z}^{(2K)}=\left(\frac{N}{t}\right)^{2K}\frac{(-1)^N(2N)!}{(2K)!2^{N-K}(N-K)!}\sum_{\tau\in S_{2N}}\operatorname{sgn}(\tau)$$
$$\times\int\prod_{i=1}^{2N}d\lambda_i e^{-\frac{N}{t}\sum_{l=1}^{2N}V(\lambda_l)}\prod_{l=1}^{2N}\lambda_l^{\tau(l)-1}\prod_{i=1}^{2K}\Psi(\lambda_i)\prod_{j=K+1}^{N}\frac{1}{\lambda_{2j}-\lambda_{2j-1}} \quad \text{(B.8)}$$
$$=\left(\frac{N}{t}\right)^{2K}\frac{(-1)^N(2N)!}{(2K)!2^{N-K}(N-K)!}\sum_{\tau\in S_{2N}}\operatorname{sgn}(\tau)\prod_{i=1}^{2K}\int d\lambda_i e^{-\frac{N}{t}V(\lambda_i)}\lambda_i^{\tau(i)-1}\Psi(\lambda_i)$$
$$\times\prod_{j=K+1}^{N}\int d\lambda_{2j-1}d\lambda_{2j}e^{-\frac{N}{t}(V(\lambda_{2j-1})+V(\lambda_{2j}))}\frac{\lambda_{2j}^{\tau(2j)-1}\lambda_{2j-1}^{\tau(2j-1)-1}}{\lambda_{2j}-\lambda_{2j-1}}. \quad \text{(B.9)}$$

We now introduce a $2N\times 2N$ anti-symmetric matrix A and a Grassmann-valued $2N$ vector

ζ with components:

$$A_{ij} = \int d\lambda d\rho e^{-\frac{N}{t}(V(\lambda)+V(\rho))} \frac{\lambda^{i-1}\rho^{j-1}}{\lambda-\rho}, \tag{B.10}$$

$$\zeta_i = \frac{N}{t}\int d\lambda e^{-\frac{N}{t}V(\lambda)}\lambda^{i-1}\Psi(\lambda). \tag{B.11}$$

We can then rewrite $\mathcal{Z}^{(2K)}$ neatly as:

$$\mathcal{Z}^{(2K)} = \frac{(-1)^N(2N)!}{(2K)!2^{N-K}(N-K)!}\sum_{\tau\in S_{2N}}\mathrm{sgn}(\tau)\prod_{i=1}^{2K}\zeta_{\tau(i)}\prod_{j=K+1}^{N}A_{\tau(2j)\tau(2j-1)}. \tag{B.12}$$

Next, recall that the Pfaffian of a $2N\times 2N$ anti-symmetric matrix A is defined by

$$\mathrm{pf}(A) = \frac{(-1)^N}{2^N N!}\sum_{\sigma\in S_{2N}}\mathrm{sgn}(\sigma)\prod_{i=1}^{N}A_{\sigma(2i)\sigma(2i-1)}. \tag{B.13}$$

Thus, for $K=0$, we get directly that

$$\mathcal{Z}^{(0)} = (2N)!\ \mathrm{pf}(A). \tag{B.14}$$

To study the $K>0$ case, we need to say a little more about Gaussian Grassmann integrals. Let M be an $2N\times 2N$ anti-symmetric matrix, and θ be a Grassmann-valued $2N$ vector. Then the Gaussian Grassmann integral can be evaluated as:

$$\int\prod_{i=1}^{2N}d\theta_i e^{-\frac{1}{2}\theta^T M\theta} = (-1)^N\mathrm{pf}(M). \tag{B.15}$$

This follows by expanding the exponential and integrating directly over the Grassmann variables.

Moreover, just as for Gaussian integrals, we can also calculate shifted Gaussian Grassmann integrals. Let M be an $2N\times 2N$ anti-symmetric matrix, θ be a Grassmann-valued $2N$ vector, and η by a Grassmann-valued $2N$ vector. Then

$$\int\prod_{i=1}^{2N}d\theta_i e^{-\frac{1}{2}\theta^T M\theta+\theta^T\eta} = (-1)^N\mathrm{pf}(M)e^{\frac{1}{2}\eta^T M^{-1}\eta}. \tag{B.16}$$

As usual, this can be obtained by completing the square inside the exponential.

With this under our belt, we can finally evaluate $\mathcal{Z}$:

$$\begin{aligned}
\mathcal{Z} =& \sum_{K=0}^{N} \mathcal{Z}^{(2K)} \\
=& (-1)^N (2N)! \sum_{K=0}^{N} \frac{1}{(2K)! 2^{N-K} (N-K)!} \sum_{\tau \in S_{2N}} \mathrm{sgn}(\tau) \prod_{i=1}^{2K} \zeta_{\tau(i)} \prod_{j=K+1}^{N} A_{\tau(2j)\tau(2j-1)} \\
=& (-1)^N (2N)! \int \prod_{i=1}^{2N} d\theta_i \prod_{j,k}^{2N} \left(1 - \frac{1}{2}\theta_j A_{jk} \theta_k\right) \prod_{l}^{2N} (1 + \theta_l \zeta_l) \\
=& (-1)^N (2N)! \int \prod_{i=1}^{2N} d\theta_i e^{-\frac{1}{2}\theta^T A \theta + \theta^T \zeta} \\
=& (2N)! \mathrm{pf}(A) e^{\frac{1}{2}\zeta^T A^{-1} \zeta} \\
=& \mathcal{Z}^{(0)} e^{\frac{1}{2}\zeta^T A^{-1} \zeta}.
\end{aligned} \tag{B.17}$$

In other words, the free energy $\mathcal{F} = \log \mathcal{Z}$ for formal supereigenvalue models takes the form

$$\mathcal{F} = \log \mathcal{Z}^{(0)} + \frac{1}{2}\zeta^T A^{-1} \zeta, \tag{B.18}$$

hence it is at most quadratic in the Grassmann coupling constants $\xi_{k+\frac{1}{2}}$.

C Proof of Proposition 8.1.5

First of all, let us do a power series expansion of $\mathcal{F}$ in the Grassmann coupling constants $\xi_{k+\frac{1}{2}}$. First, we know that only terms with an even number of Grassmann coupling constants will be non-vanishing in the expansion, since $\mathcal{F}$ is a bosonic quantity. We then introduce the notation

$$\mathcal{F} = \sum_{k\geq 0} \mathcal{F}^{(2k)}, \tag{C.1}$$

where $\mathcal{F}^{(2k)}$ denotes the term of order $2k$ in the Grassmann coupling constants. For instance, $\mathcal{F}^{(2)}$ is quadratic in the $\xi_{k+\frac{1}{2}}$.

The condition $G_{n+\frac{1}{2}}\mathcal{Z} = 0$, $n \geq -1$, rewritten in terms of the free energy $\mathcal{F}$, becomes

$$T_2 \frac{\partial \mathcal{F}}{\partial \xi_{n+\frac{5}{2}}} + \sum_{k\geq 0}\left(k g_k \frac{\partial \mathcal{F}}{\partial \xi_{n+k+\frac{1}{2}}} + \xi_{k+\frac{1}{2}} \frac{\partial \mathcal{F}}{\partial g_{k+n+1}} \right) + \left(\frac{t_s}{2N}\right)^2 \sum_{j=0}^{n} \left(\frac{\partial^2 \mathcal{F}}{\partial \xi_{j+\frac{1}{2}} \partial g_{n-j}} + \frac{\partial \mathcal{F}}{\partial \xi_{j+\frac{1}{2}}} \frac{\partial \mathcal{F}}{\partial g_{n-j}} \right) = 0. \tag{C.2}$$

Identifying terms by terms in the expansion in the Grassmann coupling constants, we get the system of equations

$$T_2 \frac{\partial \mathcal{F}^{(2l)}}{\partial \xi_{n+\frac{5}{2}}} + \sum_{k\geq 0}\left(k g_k \frac{\partial \mathcal{F}^{(2l)}}{\partial \xi_{n+k+\frac{1}{2}}} + \xi_{k+\frac{1}{2}} \frac{\partial \mathcal{F}^{(2l-2)}}{\partial g_{k+n+1}} \right) + \left(\frac{t_s}{2N}\right)^2 \sum_{j=0}^{n} \left(\frac{\partial^2 \mathcal{F}^{(2l)}}{\partial \xi_{j+\frac{1}{2}} \partial g_{n-j}} + \sum_{m=1}^{l} \frac{\partial \mathcal{F}^{(2m)}}{\partial \xi_{j+\frac{1}{2}}} \frac{\partial \mathcal{F}^{(2l-2m)}}{\partial g_{n-j}} \right) = 0, \tag{C.3}$$

for $l \geq 1$.

The other Virasoro constraints, $L_n\mathcal{Z}=0$, $n\geq -1$, becomes, in terms of $\mathcal{F}$,

$$T_2\frac{\partial\mathcal{F}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\mathcal{F}}{\partial g_{k+n}}+\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n}\left(\frac{\partial^2\mathcal{F}}{\partial g_j\partial g_{n-j}}+\frac{\partial\mathcal{F}}{\partial g_j}\frac{\partial\mathcal{F}}{\partial g_{n-j}}\right)$$
$$+\sum_{k\geq 0}\left(k+\frac{n+1}{2}\right)\xi_{k+\frac{1}{2}}\frac{\partial\mathcal{F}}{\partial\xi_{n+k+\frac{1}{2}}}$$
$$+\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n-1}\left(\frac{n-1}{2}-j\right)\left(\frac{\partial^2\mathcal{F}}{\partial\xi_{j+\frac{1}{2}}\partial\xi_{n-j-\frac{1}{2}}}+\frac{\partial\mathcal{F}}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial\mathcal{F}}{\partial\xi_{n-j-\frac{1}{2}}}\right)=0. \quad \text{(C.4)}$$

Order by order in the Grassmann coupling constants, we get

$$T_2\frac{\partial\mathcal{F}^{(2l)}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\mathcal{F}^{(2l)}}{\partial g_{k+n}}+\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n}\left(\frac{\partial^2\mathcal{F}^{(2l)}}{\partial g_j\partial g_{n-j}}+\sum_{m=0}^{l}\frac{\partial\mathcal{F}^{(2m)}}{\partial g_j}\frac{\partial\mathcal{F}^{(2l-2m)}}{\partial g_{n-j}}\right)$$
$$+\sum_{k\geq 0}\left(k+\frac{n+1}{2}\right)\xi_{k+\frac{1}{2}}\frac{\partial\mathcal{F}^{(2l)}}{\partial\xi_{n+k+\frac{1}{2}}}$$
$$+\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n-1}\left(\frac{n-1}{2}-j\right)\left(\frac{\partial^2\mathcal{F}^{(2l+2)}}{\partial\xi_{j+\frac{1}{2}}\partial\xi_{n-j-\frac{1}{2}}}+\sum_{m=1}^{l}\frac{\partial\mathcal{F}^{(2m)}}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial\mathcal{F}^{(2l+2-2m)}}{\partial\xi_{n-j-\frac{1}{2}}}\right)=0, \quad \text{(C.5)}$$

for $l\geq 0$.

Now we assume that

$$\mathcal{F}=\mathcal{F}^{(0)}+\mathcal{F}^{(2)}, \quad \text{(C.6)}$$

which is the case for the free energy of formal supereigenvalue models. (C.3) for $l=2$ becomes

$$\sum_{k\geq 0}\xi_{k+\frac{1}{2}}\frac{\partial\mathcal{F}^{(2)}}{\partial g_{k+n+1}}+\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n}\frac{\partial\mathcal{F}^{(2)}}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial\mathcal{F}^{(2)}}{\partial g_{n-j}}=0. \quad \text{(C.7)}$$

For $n=-1$, this is simply

$$\sum_{l\geq 0}\xi_{l+\frac{1}{2}}\frac{\partial\mathcal{F}^{(2)}}{\partial g_l}=0, \quad \text{(C.8)}$$

For $n=0$, we use the fact that $\mathcal{Z}=e^{-\frac{2N^2g_0}{t}}\tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}}$ does not depend on g_0, to see that $\mathcal{F}^{(2)}$ does not depend on g_0. Thus we get

$$\sum_{k\geq 0}\xi_{k+\frac{1}{2}}\frac{\partial\mathcal{F}^{(2)}}{\partial g_{k+1}}=0. \quad \text{(C.9)}$$

On the one hand, (C.8) means that

$$\mathcal{F}^{(2)} = \sum_{l\geq 0} \xi_{l+\frac{1}{2}} \frac{\partial \mathcal{A}^{(1)}}{\partial g_l} \tag{C.10}$$

for some $\mathcal{A}^{(1)}$ which is linear in the Grassmann parameters $\xi_{k+\frac{1}{2}}$. On the other hand, (C.9) says that

$$\mathcal{F}^{(2)} = \sum_{k\geq 0} \xi_{k+\frac{1}{2}} \frac{\partial \tilde{\mathcal{A}}^{(1)}}{\partial g_{k+1}} \tag{C.11}$$

for some $\tilde{\mathcal{A}}^{(1)}$ that is also linear in the $\xi_{k+\frac{1}{2}}$. Therefore

$$\mathcal{F}^{(2)} = \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \frac{\partial^2 \hat{F}^{(0)}}{\partial g_l \partial g_{k+1}}, \tag{C.12}$$

where $\hat{F}^{(0)} = \hat{F}^{(0)}(t, g_k; T_2; N)$ is some unknown function of t, g_k, T_2 and N, which is independent of the Grassmann parameters $\xi_{k+\frac{1}{2}}$.

Let us now consider (C.5) for $l = 1$ and $n = 0$. We get

$$T_2 \frac{\partial \mathcal{F}^{(2)}}{\partial g_2} + \sum_{k\geq 0} k g_k \frac{\partial \mathcal{F}^{(2)}}{\partial g_k} + \sum_{k\geq 0} \left(k + \frac{1}{2}\right) \xi_{k+\frac{1}{2}} \frac{\partial \mathcal{F}^{(2)}}{\partial \xi_{k+\frac{1}{2}}} = 0, \tag{C.13}$$

where we used the fact that $\mathcal{F}^{(2)}$ is independent of g_0. Substituting (C.12), we get

$$\begin{aligned} 0 &= \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \left(T_2 \frac{\partial^3 \hat{F}^{(0)}}{\partial g_l \partial g_{k+1} \partial g_2} + \sum_{m\geq 0} m g_m \frac{\partial^3 \hat{F}^{(0)}}{\partial g_l \partial g_{k+1} \partial g_m} + (k+l+1) \frac{\partial^2 \hat{F}^{(0)}}{\partial g_l \partial g_{k+1}} \right) \\ &= \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_l \partial g_{k+1}} \left(T_2 \frac{\partial \hat{F}^{(0)}}{\partial g_2} + \sum_{m\geq 0} m g_m \frac{\partial \hat{F}^{(0)}}{\partial g_m} \right). \end{aligned} \tag{C.14}$$

We will need this equation soon.

Let us now consider (C.3) for $l = 1$ and $n = -1$. We have

$$T_2 \frac{\partial \mathcal{F}^{(2)}}{\partial \xi_{\frac{3}{2}}} + \sum_{k\geq 0} \left(\xi_{k+\frac{1}{2}} \frac{\partial \mathcal{F}^{(0)}}{\partial g_k} + k g_k \frac{\partial \mathcal{F}^{(2)}}{\partial \xi_{k-\frac{1}{2}}} \right) = 0. \tag{C.15}$$

Substituting (C.12), we get

$$\begin{aligned}
0 &= \sum_{l\geq 0} \xi_{l+\frac{1}{2}} \left(T_2 \frac{\partial^2 \hat{F}^{(0)}}{\partial g_l \partial g_2} - T_2 \frac{\partial^2 \hat{F}^{(0)}}{\partial g_1 \partial g_{l+1}} + \frac{\partial \mathcal{F}^{(0)}}{\partial g_l} + \sum_{m\geq 0} m g_m \left(\frac{\partial^2 \hat{F}^{(0)}}{\partial g_m \partial g_l} - \frac{\partial^2 \hat{F}^{(0)}}{\partial g_{m-1} \partial g_{l+1}} \right) \right) \\
&= \sum_{l\geq 0} \xi_{l+\frac{1}{2}} \left(\frac{\partial}{\partial g_l} \left(T_2 \frac{\partial \hat{F}^{(0)}}{\partial g_2} + \sum_{m\geq 0} m g_m \frac{\partial \hat{F}^{(0)}}{\partial g_m} \right) - \frac{\partial}{\partial g_{l+1}} \left(T_2 \frac{\partial \hat{F}^{(0)}}{\partial g_1} + \sum_{m\geq 0} m g_m \frac{\partial \hat{F}^{(0)}}{\partial g_{m-1}} \right) \right. \\
&\qquad \left. + \frac{\partial}{\partial g_l} (\mathcal{F}^{(0)} + \hat{F}^{(0)}) \right). \qquad \text{(C.16)}
\end{aligned}$$

Let us now multiply by $\xi_{k+\frac{1}{2}}$ on the left, apply $\frac{\partial}{\partial g_{k+1}}$, and sum over k. We get

$$\sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \left(\frac{\partial^2}{\partial g_{k+1} \partial g_l} \left(T_2 \frac{\partial \hat{F}^{(0)}}{\partial g_2} + \sum_{m\geq 0} m g_m \frac{\partial \hat{F}^{(0)}}{\partial g_m} \right) + \frac{\partial^2}{\partial g_{k+1} \partial g_l} (\mathcal{F}^{(0)} + \hat{F}^{(0)}) \right) = 0. \qquad \text{(C.17)}$$

By (C.14), the first term is zero. Therefore, we conclude that

$$\mathcal{F}^{(2)} = \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_{k+1} \partial g_l} \hat{F}^{(0)} = - \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_{k+1} \partial g_l} \mathcal{F}^{(0)}. \qquad \text{(C.18)}$$

In other words, the free energy of the formal supereigenvalue model can be written as

$$\mathcal{F} = \left(1 - \sum_{k,l\geq 0} \xi_{k+\frac{1}{2}} \xi_{l+\frac{1}{2}} \frac{\partial^2}{\partial g_{k+1} \partial g_l} \right) \mathcal{F}^{(0)}. \qquad \text{(C.19)}$$

Note that so far we only used the super-Virasoro constraints for $n = -1$ and $n = 0$.

What remains to be shown is that $\mathcal{F}^{(0)}(2t, g_k; T_2; 2N) = 2F(t, g_k; T_2; N)$, where the right-hand-side is the free energy of the formal Hermitian matrix model. We go back to (C.5) for $l = 0$ and arbitrary n:

$$\begin{aligned}
T_2 \frac{\partial \mathcal{F}^{(0)}}{\partial g_{n+2}} + \sum_{k\geq 0} k g_k \frac{\partial \mathcal{F}^{(0)}}{\partial g_{k+n}} + \frac{1}{2} \left(\frac{t_s}{2N} \right)^2 \sum_{j=0}^{n} \left(\frac{\partial^2 \mathcal{F}^{(0)}}{\partial g_j \partial g_{n-j}} + \frac{\partial \mathcal{F}^{(0)}}{\partial g_j} \frac{\partial \mathcal{F}^{(0)}}{\partial g_{n-j}} \right) \\
+ \frac{1}{2} \left(\frac{t_s}{2N} \right)^2 \sum_{j=0}^{n-1} \left(\frac{n-1}{2} - j \right) \left(\frac{\partial^2 \mathcal{F}^{(2)}}{\partial \xi_{j+\frac{1}{2}} \partial \xi_{n-j-\frac{1}{2}}} \right) = 0, \quad \text{(C.20)}
\end{aligned}$$

We substitute (C.18):

$$T_2\frac{\partial\mathcal{F}^{(0)}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\mathcal{F}^{(0)}}{\partial g_{k+n}}+\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n}\left(\frac{\partial^2\mathcal{F}^{(0)}}{\partial g_j\partial g_{n-j}}+\frac{\partial\mathcal{F}^{(0)}}{\partial g_j}\frac{\partial\mathcal{F}^{(0)}}{\partial g_{n-j}}\right)$$
$$-\frac{1}{2}\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n-1}\left(\frac{n-1}{2}-j\right)\left(\frac{\partial^2\mathcal{F}^{(0)}}{\partial g_{n-j}\partial g_j}-\frac{\partial^2\mathcal{F}^{(0)}}{\partial g_{j+1}\partial g_{n-j-1}}\right)=0, \quad \text{(C.21)}$$

Using the fact that $\frac{\partial\mathcal{F}^{(0)}}{\partial g_0}$ is a constant, this simplifies to

$$T_2\frac{\partial\mathcal{F}^{(0)}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\mathcal{F}^{(0)}}{\partial g_{k+n}}+\left(\frac{t_s}{2N}\right)^2\sum_{j=0}^{n}\left(\frac{\partial^2\mathcal{F}^{(0)}}{\partial g_j\partial g_{n-j}}+\frac{1}{2}\frac{\partial\mathcal{F}^{(0)}}{\partial g_j}\frac{\partial\mathcal{F}^{(0)}}{\partial g_{n-j}}\right)=0. \quad \text{(C.22)}$$

Let us rewrite this equation in terms of $\tilde{F}(t,g_k;T_2;N)=\frac{1}{2}\mathcal{F}^{(0)}(2t,g_k;T_2;2N)$. We get

$$T_2\frac{\partial\tilde{F}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\tilde{F}}{\partial g_{k+n}}+\left(\frac{t}{N}\right)^2\sum_{j=0}^{n}\left(\frac{\partial^2\tilde{F}}{\partial g_j\partial g_{n-j}}+\frac{\partial\tilde{F}}{\partial g_j}\frac{\partial\tilde{F}}{\partial g_{n-j}}\right)=0, \quad \text{(C.23)}$$

or equivalently in terms of $\tilde{Z}=e^{\hat{F}}$

$$T_2\frac{\partial\tilde{Z}}{\partial g_{n+2}}+\sum_{k\geq 0}kg_k\frac{\partial\tilde{Z}}{\partial g_{k+n}}+\left(\frac{t}{N}\right)^2\sum_{j=0}^{n}\frac{\partial^2\tilde{Z}}{\partial g_j\partial g_{n-j}}=0. \quad \text{(C.24)}$$

Furthermore, by the definition of supereigenvalue models (8.3), it is straightforward to obtain

$$\frac{\partial\tilde{Z}}{\partial T_2}=\frac{1}{2}\frac{\partial\tilde{Z}}{\partial g_2}. \quad \text{(C.25)}$$

These two constraints are sufficient to determine that $\tilde{F}(t,g_k;T_2;N)$ is the free energy of formal Hermitian matrix models (see Definition 6.2.1). Thus, we conclude that $\tilde{F}(t,g_k;T_2;N)=F(t,g_k;T_2;N)$, that is, the free energy of the formal supereigenvalue model takes the form

$$\mathcal{F}(2t,g_k,\xi_{k+\frac{1}{2}};T_2;2N)=2\left(1-\sum_{k,l\geq 0}\xi_{k+\frac{1}{2}}\xi_{l+\frac{1}{2}}\frac{\partial^2}{\partial g_l\partial g_{k+1}}\right)F(t,g_k;T_2;N). \quad \text{(C.26)}$$

D Derivation of the Loop and Super-Loop Equations

In this appendix we present a derivation of the loop and super-loop equations from Virasoro and super-Virasoro constraints. An alternative derivation of the super-loop equations in terms of reparameterization of the matrix integral is discussed in [105]. Note that we choose $T_2 = 1$ for simplicity in this section.

D.1 Loop Equation for Hermitian Matrix Models

The derivation of the loop equation starts with the following formal series

$$\begin{aligned} 0 &= \sum_{n\geq 0} \frac{1}{x^{n+1}} L_{n-1} Z \\ &= \frac{1}{Z}\sum_{n\geq 0}\frac{1}{x^{n+1}}\left(\frac{\partial}{\partial g_{n+1}} + \sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+n-1}} + \frac{t^2}{N^2}\sum_{j=0}^{n-1}\frac{\partial}{\partial g_j}\frac{\partial}{\partial g_{n-j-1}}\right) Z, \end{aligned} \tag{D.1}$$

where the equality holds due to the Virasoro constraints. Let us first consider the third term in (D.1). This term vanishes for $n = 0$, hence we can shift indices:

$$\begin{aligned} \frac{1}{Z}\sum_{n\geq 0}\frac{1}{x^{n+1}}\frac{t^2}{N^2}\sum_{j=0}^{n-1}\frac{\partial}{\partial g_j}\frac{\partial}{\partial g_{n-j-1}} Z &= \frac{1}{Z}\sum_{m\geq 0}\frac{1}{x^{m+2}}\frac{t^2}{N^2}\sum_{j=0}^{m}\frac{\partial}{\partial g_j}\frac{\partial}{\partial g_{m-j}} Z \\ &= \frac{1}{Z}\frac{t^2}{N^2}\sum_{k,l\geq 0}\frac{1}{x^{k+1}x^{l+1}}\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_l} Z \\ &= \frac{1}{Z}\frac{t^2}{N^2}\frac{\partial}{\partial V(x)}\frac{\partial}{\partial V(x)} Z \\ &= \Big(W_1(x)\Big)^2 + W_2(x,x). \end{aligned} \tag{D.2}$$

On the other hand, the first two terms can be rewritten as

$$\begin{aligned}
&\frac{1}{Z}\sum_{n\geq 0}\frac{1}{x^{n+1}}\left(\frac{\partial}{\partial g_{n+1}}+\sum_{k\geq 0}kg_k\frac{\partial}{\partial g_{k+n-1}}\right)Z\\
&\qquad\qquad=\left(\sum_{n\geq 0}\frac{1}{x^{n+1}}\frac{\partial}{\partial g_{n+1}}+\sum_{n,k\geq 0}\frac{x^{k-1}}{x^{n+k}}kg_k\frac{\partial}{\partial g_{k+n-1}}\right)F\\
&\qquad\qquad=\left(\sum_{n\geq 0}\frac{1}{x^{n+1}}\frac{\partial}{\partial g_{n+1}}+\sum_{m\geq 0}\frac{1}{x^{m+1}}\sum_{l=0}^{m+1}x^{l-1}lg_l\frac{\partial}{\partial g_m}\right)F\\
&\qquad\qquad=\left(x\sum_{k\geq 1}\frac{1}{x^{k+1}}\frac{\partial}{\partial g_k}+\sum_{l\geq 0}x^{l-1}lg_l\sum_{m\geq 0}\frac{1}{x^{m+1}}\frac{\partial}{\partial g_m}-\sum_{m\geq 0}\sum_{l\geq m+2}x^{l-m-2}lg_l\frac{\partial}{\partial g_m}\right)F\\
&\qquad\qquad=-\frac{N}{t}W_1(x)V'(x)-\frac{\partial}{\partial g_0}F-\sum_{n\geq 0}x^n\sum_{k\geq 0}(n+k+2)g_{n+k+2}\frac{\partial}{\partial g_k}F, \qquad\qquad \text{(D.3)}
\end{aligned}$$

where $V'(x)$ denotes the derivative of the potential with respect to x. Let us denote the last two terms by $P_1(x)$, that is,

$$P_1(x)=-\frac{\partial}{\partial g_0}F-\sum_{n\geq 0}x^n\sum_{k\geq 0}(n+k+2)g_{n+k+2}\frac{\partial}{\partial g_k}F, \tag{D.4}$$

which is a power series in x (and becomes a polynomial in x of degree $d-2$ if we set $g_k=0$ for $k>d$). Putting all this together, we obtain the loop equation (6.38):

$$-\frac{N}{t}V'(x)W_1(x)+P_1(x)+\Big(W_1(x)\Big)^2+W_2(x,x)=0. \tag{D.5}$$

D.2 Fermionic Loop Equation for Supereigenvalue Models

The fermionic loop equation is derived from the following formal series:

$$\begin{aligned}
0=&\frac{1}{\mathcal{Z}}\sum_{n\geq 0}\frac{1}{X^{n+1}}G_{n-\frac{1}{2}}\mathcal{Z}\\
=&\frac{1}{\mathcal{Z}}\sum_{n\geq 0}\frac{1}{X^{n+1}}\left(\frac{\partial}{\partial\xi_{n+\frac{3}{2}}}+\sum_{k\geq 0}\left(kg_k\frac{\partial}{\partial\xi_{n+k-\frac{1}{2}}}+\xi_{k+\frac{1}{2}}\frac{\partial}{\partial g_{k+n}}\right)+\frac{t^2}{N^2}\sum_{j=0}^{n-1}\frac{\partial}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial}{\partial g_{n-j-1}}\right)\mathcal{Z},
\end{aligned}\tag{D.6}$$

where equality holds due to the super-Virasoro constraints. Let us first consider the last term. This term vanishes for $n=0$, hence we can shift indices:

$$\begin{aligned}
\frac{1}{\mathcal{Z}}\sum_{n\geq 0}\frac{1}{X^{n+1}}\frac{t^2}{N^2}\sum_{j=0}^{n-1}\frac{\partial}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial}{\partial g_{n-j-1}}\mathcal{Z} =&\frac{1}{\mathcal{Z}}\frac{t^2}{N^2}\sum_{m\geq 0}\frac{1}{X^{m+2}}\sum_{j=0}^{m}\frac{\partial}{\partial\xi_{j+\frac{1}{2}}}\frac{\partial}{\partial g_{m-j}}\mathcal{Z}\\
=&\frac{1}{\mathcal{Z}}\frac{t^2}{N^2}\sum_{k,l\geq 0}\frac{1}{X^{k+1}X^{l+1}}\frac{\partial}{\partial\xi_{k+\frac{1}{2}}}\frac{\partial}{\partial g_l}\mathcal{Z}\\
=&\frac{1}{\mathcal{Z}}\frac{t^2}{N^2}\frac{\partial}{\partial\Psi(X)}\frac{\partial}{\partial V(X)}\mathcal{Z}\\
=&\mathcal{W}_{1|1}(X|X)+\mathcal{W}_{1|0}(X|)\mathcal{W}_{0|1}(|X).
\end{aligned} \tag{D.7}$$

As for the first three terms, they can be manipulated as follows:

$$\begin{aligned}
&\frac{1}{\mathcal{Z}}\sum_{n\geq 0}\frac{1}{X^{n+1}}\left(\frac{\partial}{\partial\xi_{n+\frac{3}{2}}}+\sum_{k\geq 0}\left(kg_k\frac{\partial}{\partial\xi_{n+k-\frac{1}{2}}}+\xi_{k+\frac{1}{2}}\frac{\partial}{\partial g_{k+n}}\right)\right)\mathcal{Z}\\
&=X\sum_{n\geq 1}\frac{1}{X^{n+1}}\frac{\partial\mathcal{F}}{\partial\xi_{n+\frac{1}{2}}}+\sum_{k,n\geq 0}\frac{1}{X^{n+k+1}}\left(X^k(k+1)g_{k+1}\frac{\partial\mathcal{F}}{\partial\xi_{n+k+\frac{1}{2}}}+X^k\xi_{k+\frac{1}{2}}\frac{\partial\mathcal{F}}{\partial g_{k+n}}\right)\\
&=X\sum_{n\geq 1}\frac{1}{X^{n+1}}\frac{\partial\mathcal{F}}{\partial\xi_{n+\frac{1}{2}}}+\sum_{m\geq 0}\sum_{l=0}^{m}\frac{1}{X^{m+1}}\left(X^l(l+1)g_{l+1}\frac{\partial\mathcal{F}}{\partial\xi_{m+\frac{1}{2}}}+X^l\xi_{l+\frac{1}{2}}\frac{\partial\mathcal{F}}{\partial g_m}\right)\\
&=X\sum_{n\geq 1}\frac{1}{X^{n+1}}\frac{\partial\mathcal{F}}{\partial\xi_{n+\frac{1}{2}}}+\sum_{l\geq 0}X^l(l+1)g_{l+1}\sum_{m\geq 0}\frac{1}{X^{m+1}}\frac{\partial\mathcal{F}}{\partial\xi_{m+\frac{1}{2}}}+\sum_{l\geq 0}X^l\xi_{l+\frac{1}{2}}\sum_{m\geq 0}\frac{1}{X^{m+1}}\frac{\partial\mathcal{F}}{\partial g_m}\\
&\qquad-\sum_{m\geq 0}\sum_{l\geq m+1}X^{l-m-1}\left((l+1)g_{l+1}\frac{\partial\mathcal{F}}{\partial\xi_{m+\frac{1}{2}}}+\xi_{l+\frac{1}{2}}\frac{\partial\mathcal{F}}{\partial g_m}\right)\\
&=-\frac{N}{t}V'(X)\mathcal{W}_{0|1}(|X)-\frac{N}{t}\Psi(X)\mathcal{W}_{1|0}(X|)+\mathcal{P}_{0|1}(|X),
\end{aligned} \tag{D.8}$$

where we defined

$$\mathcal{P}_{0|1}(|X)=-\frac{\partial\mathcal{F}}{\partial\xi_{\frac{1}{2}}}-\sum_{n\geq 0}X^n\sum_{k\geq 0}\left((n+k+2)g_{n+k+2}\frac{\partial\mathcal{F}}{\partial\xi_{k+\frac{1}{2}}}+\xi_{n+k+\frac{3}{2}}\frac{\partial\mathcal{F}}{\partial g_k}\right). \tag{D.9}$$

Putting everything together, we find the fermionic loop equation (8.39):

$$-\frac{N}{t}V'(X)\mathcal{W}_{0|1}(|X)-\frac{N}{t}\Psi(X)\mathcal{W}_{1|0}(X|)+\mathcal{W}_{1|1}(X|X)+\mathcal{W}_{1|0}(X|)\mathcal{W}_{0|1}(|X)+\mathcal{P}_{0|1}(|X)=0. \tag{D.10}$$

D.3 Bosonic Loop Equation for Supereigenvalue Models

The bosonic loop equation is derived starting from the following series:

$$
\begin{aligned}
0 =& \frac{1}{\mathcal{Z}} \sum_{n\geq 0} \frac{1}{x^{n+1}} L_{n-1} \mathcal{Z} \\
=& \frac{1}{\mathcal{Z}} \sum_{n\geq 0} \frac{1}{x^{n+1}} \left(\frac{\partial}{\partial g_{n+1}} + \sum_{k\geq 0} k g_k \frac{\partial}{\partial g_{k+n-1}} + \frac{1}{2}\left(\frac{t}{N}\right)^2 \sum_{j=0}^{n-1} \frac{\partial}{\partial g_j} \frac{\partial}{\partial g_{n-j-1}} \right. \\
& \left. + \sum_{k\geq 0} \left(k + \frac{n}{2}\right) \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+k-\frac{1}{2}}} + \frac{1}{2}\left(\frac{t}{N}\right)^2 \sum_{j=0}^{n-2} \left(\frac{n-2}{2} - j\right) \frac{\partial}{\partial \xi_{j+\frac{1}{2}}} \frac{\partial}{\partial \xi_{n-j-\frac{3}{2}}} \right) \mathcal{Z}.
\end{aligned}
\tag{D.11}
$$

Again, equality holds due to the super-Virasoro constraints. The first line is the same as (D.1) except the 1/2 in the third term. Thus it can be written as

$$
-\frac{N}{t} V'(x) \mathcal{W}_{1|0}(x|) + \frac{1}{2}\Big(\mathcal{W}_{1|0}(x|)\Big)^2 + \frac{1}{2}\mathcal{W}_{2|0}(x,x|) - \frac{\partial \mathcal{F}}{\partial g_0} - \sum_{n\geq 0} x^n \sum_{k\geq 0} (n+k+2) g_{n+k+2} \frac{\partial \mathcal{F}}{\partial g_k}.
\tag{D.12}
$$

We manipulate the first term in the second line of (D.11) to get

$$
\begin{aligned}
& \frac{1}{\mathcal{Z}} \sum_{n\geq 0} \frac{1}{x^{n+1}} \sum_{k\geq 0} \left(k + \frac{n}{2}\right) \xi_{k+\frac{1}{2}} \frac{\partial}{\partial \xi_{n+k-\frac{1}{2}}} \mathcal{Z} \\
&= \sum_{n\geq 0} \sum_{l\geq 0} \frac{x^l}{x^{n+l+1}} (l+1) \xi_{l+\frac{3}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{n+l+\frac{1}{2}}} - \frac{1}{2}\frac{\partial}{\partial x} \sum_{n\geq 0} \sum_{l\geq 0} \frac{x^l}{x^{n+l+1}} \xi_{l+\frac{1}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{n+l+\frac{1}{2}}} \\
&= \sum_{m\geq 0} \frac{1}{x^{m+1}} \sum_{l=0}^{m} x^l (l+1) \xi_{l+\frac{3}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} - \frac{1}{2}\frac{\partial}{\partial x} \sum_{m\geq 0} \frac{1}{x^{m+1}} \sum_{l=0}^{m} x^l \xi_{l+\frac{1}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} \\
&= \sum_{l\geq 0} x^l (l+1) \xi_{l+\frac{3}{2}} \sum_{m\geq 0} \frac{1}{x^{m+1}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} - \frac{1}{2}\frac{\partial}{\partial x} \sum_{l\geq 0} x^l \xi_{l+\frac{1}{2}} \sum_{m\geq 0} \frac{1}{x^{m+1}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} \\
&\qquad - \sum_{m\geq 0} \sum_{l\geq m+1} x^{l-m-1} (l+1) \xi_{l+\frac{3}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} + \frac{1}{2}\frac{\partial}{\partial x} \sum_{m\geq 0} \sum_{l\geq m+1} x^{l-m-1} \xi_{l+\frac{1}{2}} \frac{\partial \mathcal{F}}{\partial \xi_{m+\frac{1}{2}}} \\
&= -\frac{N}{t} \Psi'(x) \mathcal{W}_{0|1}(|x) + \frac{N}{2t} \frac{\partial}{\partial x} \left(\Psi(x) \mathcal{W}_{0|1}(|x)\right) \\
&\qquad - \frac{1}{2} \sum_{n\geq 0} x^n \sum_{k\geq 0} \xi_{n+k+\frac{5}{2}} (n+2k+3) \frac{\partial \mathcal{F}}{\partial \xi_{k+\frac{1}{2}}}.
\end{aligned}
\tag{D.13}
$$

Finally, for the last term in (D.11), terms for $n = 0, 1$ are zero, thus we shift the index to get

$$\begin{aligned}
&\frac{1}{2\mathcal{Z}}\left(\frac{t}{N}\right)^2 \sum_{n\geq 0}\frac{1}{x^{n+1}}\sum_{j=0}^{n-2}\left(\frac{n-2}{2}-j\right)\frac{\partial}{\partial \xi_{j+\frac{1}{2}}}\frac{\partial}{\partial \xi_{n-j-\frac{3}{2}}}\mathcal{Z}\\
&\qquad=\frac{1}{2\mathcal{Z}}\left(\frac{t}{N}\right)^2 \sum_{m\geq 0}\frac{1}{x^{m+3}}\sum_{j=0}^{m}\left(\frac{m}{2}-j\right)\frac{\partial}{\partial \xi_{j+\frac{1}{2}}}\frac{\partial}{\partial \xi_{m-j+\frac{1}{2}}}\mathcal{Z}\\
&\qquad=-\frac{1}{2\mathcal{Z}}\left(\frac{t}{N}\right)^2 \sum_{m\geq 0}\frac{1}{x^{m+3}}\sum_{j=0}^{m}(j+1)\frac{\partial}{\partial \xi_{j+\frac{1}{2}}}\frac{\partial}{\partial \xi_{m-j+\frac{1}{2}}}\mathcal{Z}\\
&\qquad=-\frac{1}{2\mathcal{Z}}\left(\frac{t}{N}\right)^2 \sum_{n\geq 0}\sum_{k\geq 0}\frac{n+1}{x^{n+2}x^{k+1}}\frac{\partial}{\partial \xi_{n+\frac{1}{2}}}\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}\mathcal{Z}\\
&\qquad=\frac{1}{2\mathcal{Z}}\left(\frac{t}{N}\right)^2 \frac{\partial}{\partial x}\left(\sum_{n\geq 0}\sum_{k\geq 0}\frac{1}{x^{n+1}y^{k+1}}\frac{\partial}{\partial \xi_{n+\frac{1}{2}}}\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}\mathcal{Z}\right)\Bigg|_{y=x}\\
&\qquad=\frac{1}{2}\mathcal{W}_{0|1}(|x)\frac{\partial}{\partial x}\mathcal{W}_{0|1}(|x)+\frac{1}{2}\frac{\partial}{\partial x}\left(\mathcal{W}_{0|2}(|x,y)\right)\Big|_{y=x}.
\end{aligned}\tag{D.14}$$

For the second equality we used the fact that

$$\sum_{j=0}^{m}\frac{\partial}{\partial \xi_{j+\frac{1}{2}}}\frac{\partial}{\partial \xi_{m-j+\frac{1}{2}}}\mathcal{Z}=0.\tag{D.15}$$

Putting all this together, we obtain the bosonic loop equation (8.47):

$$\begin{aligned}
-\frac{N}{t}V'(x)\mathcal{W}_{1|0}(x|)+\frac{1}{2}\Big(\mathcal{W}_{1|0}(x|)\Big)^2+\frac{1}{2}\mathcal{W}_{2|0}(x,x|)-\frac{N}{2t}\Psi'(x)\mathcal{W}_{0|1}(|x)+\frac{N}{2t}\Psi(x)\frac{\partial}{\partial x}\mathcal{W}_{0|1}(|x)\\
+\frac{1}{2}\mathcal{W}_{0|1}(|x)\frac{\partial}{\partial x}\mathcal{W}_{0|1}(|x)+\frac{1}{2}\frac{\partial}{\partial x}\left(\mathcal{W}_{0|2}(|x,y)\right)\Big|_{y=x}+\mathcal{P}_{1|0}(x|)=0,
\end{aligned}\tag{D.16}$$

where we defined

$$\mathcal{P}_{1|0}(x|)=-\frac{\partial\mathcal{F}}{\partial g_0}-\sum_{n\geq 0}x^n\left(\sum_{k\geq 0}(n+k+2)g_{n+k+2}\frac{\partial\mathcal{F}}{\partial g_k}+\frac{1}{2}\sum_{k\geq 0}\xi_{n+k+\frac{5}{2}}\left(n+2k+3\right)\frac{\partial\mathcal{F}}{\partial \xi_{k+\frac{1}{2}}}\right).\tag{D.17}$$

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