UNIVERSITY OF ALBERTA

BENDIXSON CRITERIA FOR DIFFERENCE EQUATIONS

by

C. CONNELL MCCLUSKEY

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of MASTER of SCIENCE

in

APPLIED MATHEMATICS

DEPARTMENT OF MATHEMATICAL SCIENCES

Edmonton, Alberta

FALL 1996

ISBN 0-612-18296-7

University of Alberta

Library Release Form

Name of Author: C. Connell McCluskey

Title of Thesis: Bendixson Criteria for Difference Equations

Degree: Master of Science

Year this Degree Granted: 1996

C. Connell McCluskey

61 Birchdale Drive

Sydney River, N. S.

June 20, 1996

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

UNIVERSITY OF ALBERTA

Faculty of Graduated Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Bendixson Criteria for Difference Equations** submitted by **C. Connell McCluskey** in partial fulfillment of the requirements for the degree of **Master of Science** in **Applied Mathematics**.

J. S. Muldowney

H. I. Freedman

J. A. Tuszynski

ABSTRACT

Work has been done by Bendixson [1], Dulac [3], and others related to finding conditions which preclude the existence of non-trivial periodic solutions to autonomous differential equations. These conditions have stability implications related to limit sets. This is explored in Chapter 1.

In Chapter 2, analogous conditions are given which imply the non existence of periodic solutions to autonomous difference equations in one dimension.

Some of these results are generalized to higher dimensional difference equations in Chapter 3.

I would like to thank Dr. James Muldowney for his assistence in the research I have done as well as for his aid in the preparation of this thesis.

I would also like to thank my parents for giving me more support than they realize.

TABLE OF CONTENTS

		Page
INTRODUCTION		1
CHAPTER 1	Autonomous Differential Equations	3
CHAPTER 2	One Dimensional Difference Equations	13
CHAPTER 3	Difference Equations in Higher Dimensions	20
BIBLIOGRAPHY		29

LIST OF FIGURES

	Page
Figure 1	8
Figure 2	9

INTRODUCTION

A dynamical system or a semidynamical system is a system which changes over time where the change is a function of the state of the system. Let I denote either the integers or real numbers or the non-negative integers or real numbers and let X be a metric space.

Definition. *Suppose the function* $(t, x) \mapsto \varphi_t(x)$, $t \in I$, $x \in X$ *is continuous and satisfies*

1. $\varphi_o(x) = x$ *for all* $x \in X$

2. $\varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$ *for all* $t, s \in I$ *and all* $x \in X$

The function is a dynamical system on X *if* I *is the integers or real numbers and is a semidynamical system if* I *is the non-negative integers or real numbers.*

The set $\{\varphi_t(x) : t \in I\}$ is the *orbit* of x and $\{\varphi_t(x) : t \in I, t \geq 0\}$ is the *positive semiorbit* of x. If there exists some $\omega > 0$ such that $\varphi_{t+\omega}(x_o) = \varphi_t(x_o)$ for all $t \in I$ then we say that the orbit is *ω-periodic* with period ω. If ω is the smallest positive number such that an orbit is ω-periodic then we say that the orbit is a *proper ω-periodic orbit.* If $\varphi_t(x_o) = x_o$ for all $t \in I$ then x_o is called an *equilibrium* and is said to be a trivial periodic orbit.

When I is the set of integers or non-negative integers, then a proper ω-periodic orbit is a finite set containing ω points in X. When I is the set of real numbers or non-negative real numbers, then a proper ω-periodic orbit is a simple closed curve in X.

Definition. *The alpha and omega limit sets,* $A(x_o)$, $\Omega(x_o)$ *are defined as follows:* $x \in A(x_o)$ $[x \in \Omega(x_o)]$ *if* $\lim_{n\to\infty} \varphi_{t_n}(x_o) = x$ *for some sequence* $t_n \to -\infty$ $[t_n \to \infty]$.

If $f : X \to X$ is continuous, consider the difference equation

$$x_{t+1} = f(x_t)$$

The solution which satisfies $x_o = x$ is $x_t = f^t(x)$, where $f^{t+1} = f(f^t(x))$ and f^{-1} denotes the inverse of f when it exists. Then $\varphi_t(x) = f^t(x)$ defines a semidynamical system on X. It defines a dynamical system on X if f is one-to-one and $f(X) = X$. It is clear that every dynamical system on X, when I is the set of integers corresponds to a difference equation with $f = \varphi_1$.

Consider the autonomous differential equation

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continously differentiable function. Let $\varphi(t, x)$ denote the solution such that $\varphi(0, x) = x$. Then $\varphi_t(x) = \varphi(t, x)$ is a semidynamical system if, for each $x \in \mathbb{R}^n$ exists for each $t \geq 0$ and is a dynamical system if $\varphi(t, x)$ exists for all t.

Some examples of semidynamical systems where I is the non-negative integers are annual census figures, digital clock readouts and some random number generators for computers. Some examples of dynamical systems where I is taken to be the real numbers are the stock market, a simple pendulum, and projectile motion.

This thesis is primarily concerned with conditions which guarantee the non-existence of non-trivial periodic orbits which are contained in a given set.

CHAPTER 1

AUTONOMOUS DIFFERENTIAL EQUATIONS

In autonomous planar differential equations, Bendixson and Dulac's criteria [1], [3] give conditions which preclude the existence of non trivial periodic solutions to a given differential equation. Consider the equation

$$\dot{x} = f(x) \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

The absence of non-trivial periodic solutions to an autonomous differential equation has strong stability implications. For example, the Poincare Bendixson Theorem for 2-dimensional systems shows that every bounded solution either has a periodic orbit as its alpha or omega limit set or this set contains an equilibrium [1]. Therefore, any condition which precludes the existence of non trivial periodic orbits implies that every bounded solution has an equilibrium in its Ω-limit set. The Bendixson-Dulac criteria have even stronger implications, in that they imply that the limit set consists of a single equilibrium as we will show later in this chapter. Bendixson's Criterion may be stated as follows. As usual, $\operatorname{div} f$ denotes the divergence of f; in particular, $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ when $n = 2$.

Theorem 1.1 - Bendixson's Criterion. *For $n = 2$, let D be a simply connected subset of $\mathbb{R}^2$ such that*

$$\operatorname{div} f \neq 0 \tag{1.2}$$

on D. Then there is no non-trivial periodic solution of (1.1) whose orbit lies entirely in D.

Proof. Suppose that $x(t) = \big(x_1(t), x_2(t)\big)$ is a periodic solution with least period $\omega > 0$ and orbit $C = \{x(t) : 0 \leq t \leq \omega\} \subset D$. It may be assumed without loss of generality that C is positively oriented by the parameterization $t \mapsto x(t)$. Then Green's Theorem implies

$$\iint_K \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\, dx_1\, dx_2 = \int_C f_1\, dx_2 - f_2\, dx_1 \tag{1.3}$$

where K is the region bounded by C.

Since f is continuously differentiable and K is connected, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \operatorname{div} f \neq 0$ implies $\operatorname{div} f$ is of one sign on K and hence that the left-hand side of (1.3) is non-zero. However, the right-hand side is

$$\begin{aligned}\int_C f_1\, dx_2 - f_2\, dx_1 &= \int_0^\omega \Big[f_1\big(x(t)\big)\dot{x}_2(t) - f_2\big(x(t)\big)\dot{x}_1(t) \Big]\, dt \\ &= \int_0^\omega \Big[f_1\big(x(t)\big)\, f_2\big(x(t)\big) - f_2\big(x(t)\big)\, f_1\big(x(t)\big) \Big]\, dt \\ &= 0\end{aligned}$$

This is a contradiction, so there can be no such periodic orbit. □

Bendixson's Criterion is stronger than is stated in the above theorem. The orbit C, used in the proof, can be replaced with any simple closed curve which is a union of orbits of (1.1). Thus, Theorem 1.1 rules out the existence of any simple closed curve which is invariant under the flow described by (1.1). This means there can be no homoclinic orbits since a homoclinic orbit and its associated equilibrium form an invariant simple closed curve. This also applies to Dulac's Criterion and Lloyd's Theorem [5] which follow.

Theorem 1.2 - Dulac's Criterion. *For $n = 2$, let D be a simply connected subset of $\mathbb{R}^2$. If there exists a differentiable real valued function α such that*

$$\operatorname{div}(\alpha f) \neq 0 \tag{1.4}$$

on D. then there is no periodic soluion of (1.1) whose orbit lies entirely in D.

Proof. Let C be a non-trivial periodic orbit of (1.1) in D. Without loss of generality, we may assume that C is positively oriented. Let K be the region bounded by C. By Green's Theorem, we have

$$\iint_K \left[\frac{\partial(\alpha f_1)}{\partial x_1} + \frac{\partial(\alpha f_2)}{\partial x_2}\right] dx_1\, dx_2 = \int_C (\alpha f_1)\, dx_2 - (\alpha f_2)\, dx_1 \qquad (1.5)$$

Since $\operatorname{div}(\alpha f) \neq 0$, the left hand side of (1.5) is non-zero. As was seen in the proof of Theorem 1.1, the right hand side of (1.5) is zero since C is a periodic orbit of (1.1). This is a contradiction so there can be no such periodic orbit. □

Lloyd's Theorem is a generalization of Dulac's Criterion in which the assumption of simply connectedness is dropped. Because of this relaxation of assumptions, the existence of periodic orbits can no longer be precluded. Instead, an upper bound on the number of possible periodic orbits is found based on the number of holes in the region D.

Theorem 1.3 - Lloyd's Theorem. *Let $n = 2$. Suppose that $D \subseteq \mathbb{R}^2$ is open and connected. Suppose further that there is a differentiable function $\alpha : D \to \mathbb{R}$ such that*

$$\operatorname{div}(\alpha f) \neq 0$$

on D. If the complement of D has k bounded components then there are at most k periodic solutions to (1.1) which lie entirely in D.

Definition. *Let C be a periodic orbit of (1.1). Let $B \subset \mathbb{R}^2$ be a connected set. C is said to be adjacent to B if B lies in the interior of C but is encircled by no other such orbit.*

Proof of Theorem. Let D_i for $i = 1, ..., k$ be the bounded components of the complement of D. Each non-trivial periodic orbit of (1.1) must be adjacent to at least one of the D_i. This is proven by contradiction. Suppose C is a non-trivial periodic orbit of (1.1) which encircles $D_1, \dots, D_r$ for some $r \le k$. Suppose $C_1, \dots, C_s$ are periodic orbits lying in the interior of C so that each D_i is encircled by some C_{j_i} for $1 \le j_i \le s$ and $i = 1, \dots, r$. Note that the C_j can be chosen so that no one encircles another. Let

$$R = C^o \setminus \bigcup_{j=1}^{s} (C_j \cup C_j^o)$$

where the superscript (^{o}) denotes the interior of the given curve. Then R is an open, connected subset of D. By Green's Theorem,

$$\iint_R div(\alpha f)\, dx_1\, dx_2 = \int_C (-\alpha f_2\, dx_1 + \alpha f_1\, dx_2) - \sum_{j=1}^{s} \int_{C_j} (-\alpha f_2\, dx_1 + \alpha f_1\, dx_2). \tag{1.6}$$

(This assumes that C and each of the C_j are oriented counterclockwise. For any of the orbits which are oriented clockwise, a minus sign should be introduced.) Since $\mathrm{div}(\alpha f) \neq 0$, the left hand side of (1.6) is non-zero. Each term on the right hand side of (1.6) is zero because C and C_j ($j = 1, \dots, s$) are orbits of (1.1). Contradiction.

Since each periodic orbit of (1.1) must be adjacent to at least one of the D_i and no two periodic orbits can be adjacent to the same D_i, there can be at most k periodic orbits. □

As was mentioned earlier, the Bendixson-Dulac criteria have very strong stability implications. In order to see this we need the next result about Ω-limit sets in the plane.

Proposition 1.4. *For $n = 2$, if (1.2) holds on an open, simply connected subset $D \subseteq \mathbb{R}^2$ then every semi-orbit of (1.1) whose topological closure is a compact subset of D has a limit set which consists entirely of equilibria.*

Proof. Suppose $C^+(y_o)$, the positive semi-trajectory through y_o, is bounded. Then the omega limit set of y_o is a non-empty compact subset of D. Suppose the omega limit set, $\Omega(y_o)$, contains a non-equilibrium point $\bar{y}$. Let $T \subset D$ be a transversal through $\bar{y}$ and let $\{y_n\}$ be the sequence of successive intersections of $C^+(y_o)$ with T. Then $\{y_n\}$ is monotone on T and $\lim_{n\to\infty} y_n = y$. Let S_n be the segment of T joining y_n and y_{n+1}. Let C_n be the segment of $C^+(y_o)$ joining y_n and y_{n+1}. Let D_n be the region bounded by S_n and C_n.

The D_n are nested. In other words, we have either $D_1 \subseteq D_2 \subseteq \dots$ or $D_1 \supseteq D_2 \supseteq \dots$ Assume, for now that we have the former. See Figure 1.1. Define the positive constant K by

$$K = \left| \int_{D_1} \text{div} f \, dx_1 \, dx_2 \right|.$$

Since $D_1 \subseteq D_n$ for all n and $\text{div} f$ has constant sign, we have

$$K \leq \lim_{n\to\infty} \left| \int_{D_n} \text{div} f \, dx_1 \, dx_2 \right|$$

Using Green's Theorem to convert the right hand side to a line integral gives

$$K \leq \lim_{n\to\infty} \left| \int_{S_n} f_2 \, dx_1 - f_1 \, dx_2 \right| + \left| \int_{C_n} f_2 \, dx_1 - f_1 \, dx_2 \right| \tag{1.7}$$

As in the proof of Theorem 1.1, the second term on the right-hand side of (1.7) is equal to zero. Since f is continuous, we have $||f|| \leq M < \infty$ on the segment of T joining y_1 and y for some real number M. Therefore, if ds represents integration with respect to arclength, then

$$\begin{aligned} K &\leq \lim_{n\to\infty} \left| \int_{S_n} 2M \, ds \right| \\ &= 2M \lim_{n\to\infty} ||y_n - y_{n+1}|| \\ &= 0 \end{aligned}$$

This contradicts the fact that K is positive.

Consider the case where $D_1 \supseteq D_2 \supseteq \dots$ See Figure 1.2. There exists $z \in \mathbb{R}^2$ and $\epsilon > 0$ such that $B(z, \epsilon) \subseteq D_n$ for all n, where $B(z, \epsilon)$ is the open disc with centre z and radius ϵ. To see that this is so, consider the following. The transversal

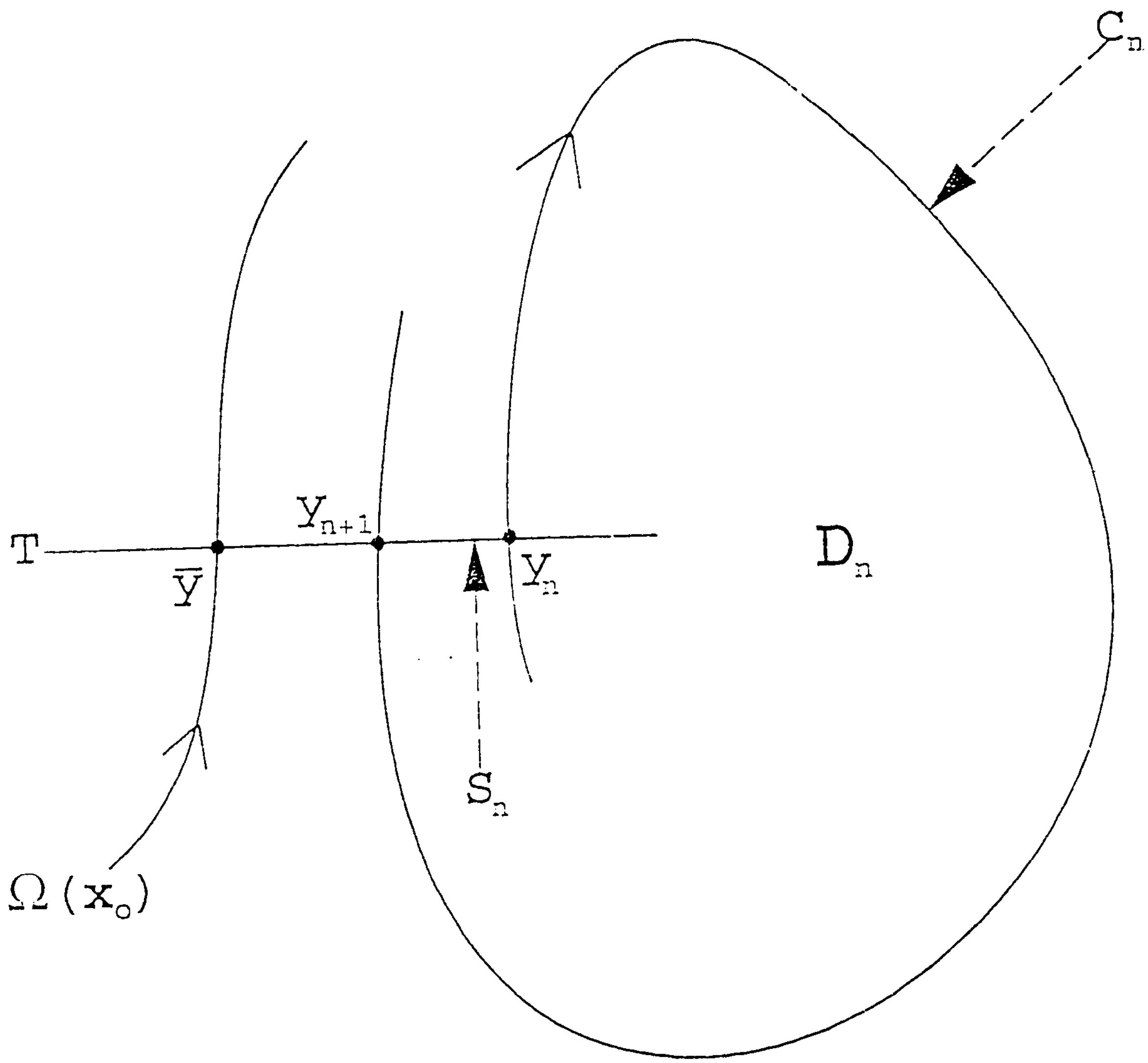

Figure 1.1

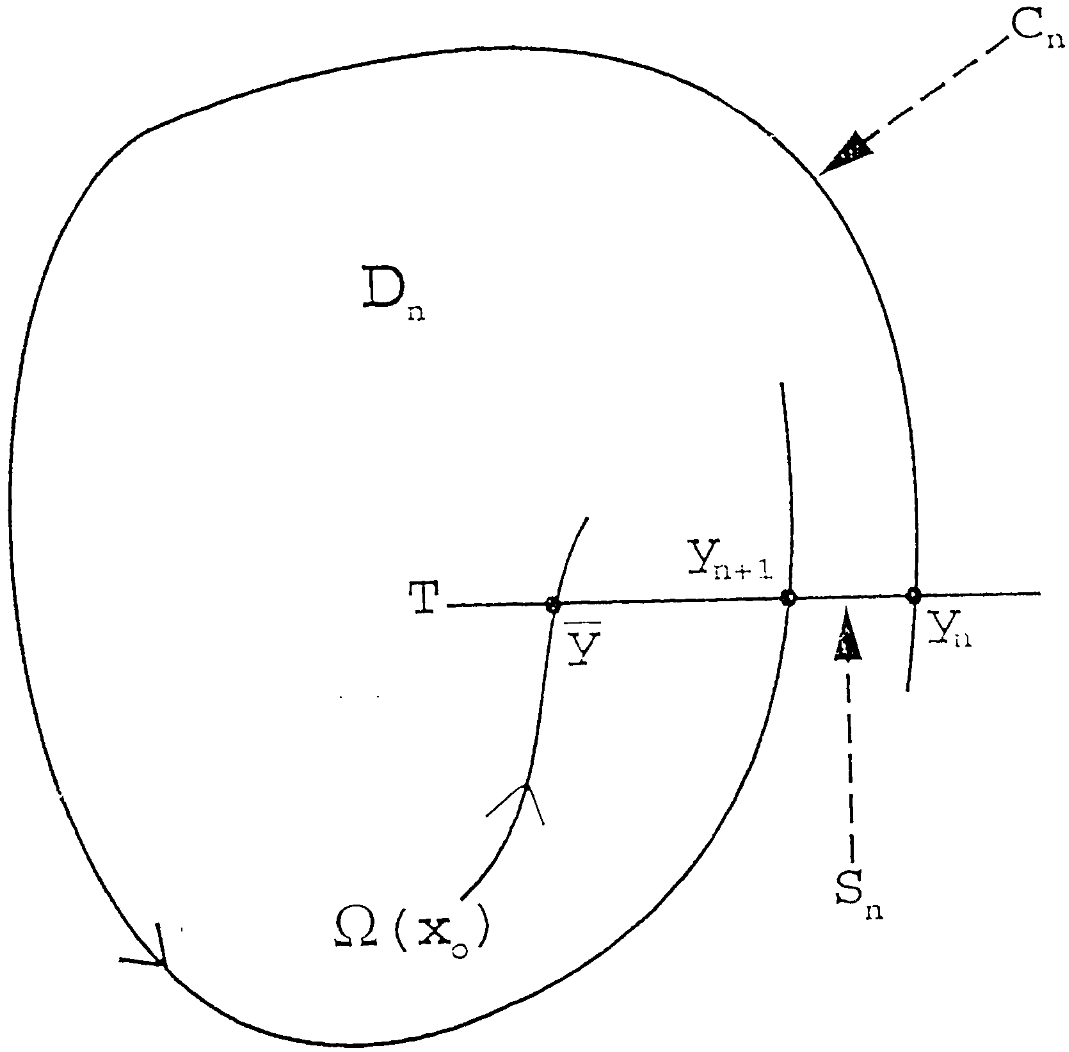

Figure 1.2

T is divided into two components by $\bar{y}$. One of these components T_1 contains the sequence $\{y_n\}$. The other component T_2 still has the property that solutions cross it in the same direction as solutions cross T_1. For all n, we have $T_2 \subset D_n$. Since each D_n is positively invariant, solutions which intersect T_2 lie in each D_n for all positive time. Let I be an open ended subsegment of T_2. Let $B = \{\varphi_t(y) : t \in (0,1), y \in I\}$. Then B is an open set contained in D_n for all n. Choose z and ϵ such that $B(z,\epsilon) \subset B$.

Defining K' by

$$K' = \left| \int_{B(z,\epsilon)} \text{div} f \; dx_1 \, dx_2 \right| > 0$$

we can use an argument similar to the one used above to get a contradiction.

Thus we see that $\text{div} f \neq 0$ implies that the omega limit set of a bounded orbit consists entirely of equilibria. The treatment of alpha limit sets is similar. □

Theorem 1.5. *For $n = 2$, if (1.2) holds on an open, simply connected subset D of $\mathbb{R}^2$, then every semi-orbit of (1.1) whose topological closure is a compact subset of D must limit to an equilibrium.*

Proof. Proposition 1.4 implies that the limit set of a bounded semi-orbit in D consists entirely of equilibria.

Let C be a bounded semi-orbit in D with an omega limit set containing more than a single point. The omega limit set of C must be connected, so it must contain a continuum of equilibria. Let P be an omega limit point of C. There are equilibria arbitrarily close to P, so P must have a center manifold of dimension at least one.

Let λ_1 and λ_2 be the eigenvalues of the linearization of (1.1) about P. Note that

$$\lambda_1 + \lambda_2 = \text{Tr} \frac{\partial f}{\partial x} = \text{div} f \neq 0.$$

Since P has a center manifold of dimension at least one, we must have one of the λ_i equal to zero, which means the other eigenvalue must be non-zero. Thus, the dimension of the center manifold of P is exactly one. Depending on the sign of the non-zero eigenvalue, P has either an unstable or a stable manifold of dimension one. In either case, the equilibria in a neighbourhood of P will have associated with them a manifold of similar type. Thus we have either a neighbourhood of P which no orbit enters or a neighbourhood of P which no orbit leaves. In the first case, it would be impossible for P to be an omega limit point. In the second case, once an orbit entered the neighbourhood, it would be on the stable manifold of one of the equilibria and so would limit to only that equilibrium. Thus, that equilibrium would be the only Ω-limit point. Either way, we have a contradiction. □

In n dimensions, (1.2) is no longer sufficient to preclude the existence of non trivial periodic solutions of (1.1). To see this, consider the following system.

$$\dot{x} = -y + (1 - x^2 - y^2)x$$

$$\dot{y} = x + (1 - x^2 - y^2)y$$

$$\dot{z} = 4z(x^2 + y^2)$$

We get $\mathrm{div} f = 2$, but the curve given by $\big(x(t), y(t), z(t)\big) = (\cos t, \sin t, 0)$ is a periodic solution.

An alternative proof of Bendixson's Criterion may be obtained by considering its implications for the evolution of areas under the flow of (1.1). In two dimensions, $\mathrm{div} f \neq 0$ means that areas are either increasing or decreasing in time, depending on the sign of the expression. Because solutions cannot cross, orbits which lie in the interior of an invariant closed curve at some time must remain there for all time. Thus, the total area inside an invariant closed curve must be constant. One way to generalize Bendixson's Criterion to higher dimensions is to find a condition on f which implies that two dimensional areas are either all increasing or all decreasing with time.

Any closed curve in $\mathbb{R}^n$ can be thought of as the boundary of a family of surfaces. If that family has an element S of minimum area, then we can consider how S changes in time. If areas are all decreasing (increasing) then S must flow into a surface with less area for any positive (negative) time. Therefore the boundary of S must not be fixed in time. In other words, there can be no invariant closed curve with a surface of minimum area if f satisfies a condition which implies areas are either all increasing or all decreasing. This line of reasoning was used by Li and Muldowney in [4] to formulate a higher dimensional generalization of Bendixson's Criterion. For example, they obtain a new proof of a theorem of Smith [7].

Theorem 1.6. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\frac{1}{2}((\partial f/\partial x)^* + \partial f/\partial x)$ where $\partial f/\partial x$ is the Jacobian of f and the asterisk denotes transposition. A simple closed rectifiable curve which is invariant with respect to (1.1) cannot exist if one of the following is satisified on $\mathbb{R}^n$:*

$$(i) \quad \lambda_1 + \lambda_2 < 0$$

$$(ii) \; \lambda_{n-1} + \lambda_n > 0$$

More generally, by considering measures other than the usual surface area, Li and Muldowney obtain many results which reduce to the Bendixson and Dulac criteria when $n = 2$. For example, if instead of (i) or (ii), it is assumed that

$$\frac{\partial f_i}{\partial x_i} + \frac{\partial f_j}{\partial x_j} + \sum_{r \neq i,j} \left|\frac{\partial f_r}{\partial x_i}\right| + \left|\frac{\partial f_r}{\partial x_j}\right| < 0, \qquad 1 \leq i < j \leq n$$

or that

$$\frac{\partial f_i}{\partial x_i} + \frac{\partial f_j}{\partial x_j} - \sum_{r \neq i,j} \left|\frac{\partial f_r}{\partial x_i}\right| + \left|\frac{\partial f_r}{\partial x_j}\right| > 0, \qquad 1 \leq i < j \leq n$$

then the conclusion of Theorem 1.6 still holds. The conclusion also holds if either

$$\frac{\partial f_i}{\partial x_i} + \frac{\partial f_j}{\partial x_j} + \sum_{r \neq i,j} \left|\frac{\partial f_i}{\partial x_r}\right| + \left|\frac{\partial f_j}{\partial x_r}\right| < 0, \qquad 1 \leq i < j \leq n$$

or

$$\frac{\partial f_i}{\partial x_i} + \frac{\partial f_j}{\partial x_j} - \sum_{r \neq i,j} \left| \frac{\partial f_i}{\partial x_r} \right| + \left| \frac{\partial f_j}{\partial x_r} \right| > 0, \qquad 1 \leq i < j \leq n.$$

CHAPTER 2

ONE DIMENSIONAL DIFFERENCE EQUATIONS

In obtaining Bendixson's Criterion, a 2-form evaluated over a region was converted to a 1-form evaluated on the boundary of the region. A condition was then found which implied that the boundary could not be the orbit of a solution to the given differential equation. For difference equations, a similar approach can be taken.

Consider the difference equation

$$x_{t+1} = f(x_t) \qquad \text{for } t = 0, 1, 2, \ldots \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Proposition 2.1. *If $f'(x) \neq -1$ on an interval $I \subseteq \mathbb{R}$ then (2.1) has no proper 2-periodic orbits in I.*

Proof. Let $a \in I$ be a non-fixed point of f with $f(a) \in I$. Then

$$\int_a^{f(a)} \left[f'(x) + 1\right] dx = f^2(a) - a \tag{2.2}$$

The left hand side of (2.2) is non-zero by the assumption made in the proposition. Therefore $f^2(a) \neq a$ and it can seen that there are no non-trivial 2-periodic solutions to (2.1) which lie in I. □

In a similar manner, a condition can be found which will preclude the existence of non-trivial periodic orbits of other periods.

Proposition 2.2. *Let I be an interval on the real line. Suppose*

$$\frac{d}{dx}[f^{\omega-1}(x) + \ldots + f(x) + x] \neq 0 \tag{2.3}$$

on I. Then (2.1) has no non-trivial ω-periodic orbits with two succesive points lying in I.

Proof. Let $a \in I$ be a non-fixed point of f with $f(a) \in I$. Then

$$\int_a^{f(a)} \frac{d}{dx}[f^{\omega-1}(x) + \ldots + f(x) + x]\,dx = f^{\omega}(a) - a \tag{2.4}$$

The left hand side of (2.4) is non-zero by (2.3). Therefore $f^{\omega}(a) \neq a$, and it can seen that f has no non-trivial ω-periodic orbits for which two successive points lie in I. □

Remark. The non-existence of non-trivial ω-periodic orbits also rules out k-periodic orbits where k divides ω, $k \neq 1$. Also, when $\omega = 2$, this result reduces exactly to Proposition 2.1.

Consider the quadratic map $f : [0,1] \to [0,1]$ given by $f(x) = \mu x(1-x)$. This is known to have no 2-periodic orbits for $\mu \in (1,3)$. However, the condition of Proposition 2.2 is not satisfied. Attempting to apply the proposition gives

$$\frac{d}{dx}[f(x) + x] = \mu + 1 - 2\mu x$$

This is zero at $\bar{x} = \frac{\mu+1}{2\mu}$ and $\bar{x} \in [0,1]$ for all $\mu \in (1,3)$. So, this is a case in which Proposition 2.2 does not apply, and yet there are no 2-periodic orbits. This means that (2.3) is a sufficient condition for the non-existence of ω-periodic orbits, but not a necessary condition. It is possible to find a more general condition which will be applicable to more cases.

If we consider Proposition 2.2, we can see what made the proof work. The essential characteristic was that $[f^{\omega-1}(x) + \ldots + f(x) + x]$ took on the same value at a and $f(a)$ if a was an ω-periodic point. Using this observation and the next definition, we can replace (2.3) with a more general condition to obtain a broader result.

Definition. *Any* $\mathbb{R}$ *valued function on* $\mathbb{R}^\omega$, $(y_1, ..., y_\omega) \mapsto \Phi(y_1, ..., y_\omega)$ *will be called* ω*-cyclic if*

$$\Phi(y_1, ..., y_\omega) = \Phi(y_\omega, y_1, ..., y_{\omega-1}) \qquad \text{for all } y_i \in \mathbb{R}. \quad i = 1, ..., \omega. \tag{2.5}$$

With any ω*-cyclic function* Φ *and map* $f : \mathbb{R} \to \mathbb{R}$, *the function* $F : \mathbb{R} \to \mathbb{R}$ *given by* $F(x) = \Phi(x, f(x), ..., f^{\omega-1}(x))$ *is associated.*

Examples. The following are ω-cyclic functions from $\mathbb{R}^\omega$ to $\mathbb{R}$

$$\Phi(y_1, ..., y_\omega) = y_1 + ... + y_\omega$$

$$\Phi(y_1, ..., y_\omega) = y_1 y_2 + ... + y_{\omega-1} y_\omega + y_\omega y_1$$

The corresponding functions $F : \mathbb{R} \to \mathbb{R}$ are:

$$F(x) = x + f(x) + ... + f^{\omega-1}(x)$$

$$F(x) = xf(x) + .. + f^{\omega-1}(x) f^\omega(x) + f^\omega(x) x$$

Theorem 2.3. *If there exists an* ω*-cyclic continuously differentiable function* $\Phi : \mathbb{R}^\omega \to \mathbb{R}$ *with corresponding* F *such that*

$$\frac{d}{dx} F(x) \neq 0 \tag{2.6}$$

on an interval $I \subseteq \mathbb{R}$ *then (2.1) has no non-trivial* ω*-periodic solutions with two succesive points lying in* I.

Proof. Let $a \in I$ be a non-fixed point of f with $f(a) \in I$. Then

$$\int_a^{f(a)} \frac{d}{dx} F(x)\, dx = \Phi\big(f^\omega(a), f(a), ..., f^{\omega-1}(a)\big) - \Phi\big(a, f(a), ..., f^{\omega-1}(a)\big) \tag{2.7}$$

The left hand side of (2.7) is non-zero by (2.6). Therefore $f^\omega(a) \neq a$ and f has no non-trivial ω-periodic orbits for which two successive points lie in I. □

It should be noted that (2.6) can be replaced with the condition $\frac{d}{dx}F(x) \geq 0$ with strict inequality when $f(x) \neq x$. This ensures that the left hand side of (2.7) is non-zero since there is a neighbourhood about a in which $f(x) \neq x$. This condition will be used later in Theorem 2.8.

Corollary 2.4. *Let I be an interval on the real line. If there exists a differentiable function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\frac{d}{dx}[\alpha \circ f^{\omega-1}(x) + \ldots + \alpha \circ f(x) + \alpha(x)] \neq 0$ on I then (2.1) has no non-trivial ω-periodic solutions with two succesive points lying in I.*

Proof. Apply Theorem 2.3 with the ω-cyclic function Φ given by

$$\Phi(y_1, \ldots, y_\omega) = \alpha(y_1) + \ldots + \alpha(y_\omega).$$ □

Corollary 2.5. *If there exists an ω-cyclic continuously differentiable function $\Phi : \mathbb{R}^\omega \to \mathbb{R}$ and associated function $F : \mathbb{R} \to \mathbb{R}$ such that $\frac{d}{dx}F(x) \neq 0$ on $\mathbb{R}$ then (2.1) has no non-trivial ω-periodic solutions.*

The next step is to associate the work so far with known results for difference equations on the real line. Towards this end, the contrapositive of Sarkovskii's Theorem [2] is quite useful.

Sarkovskii's ordering of the natural numbers is as follows. The usual order of the odd integers greater than 1 is reversed:

$$3 \rhd 5 \rhd 7 \rhd \ldots$$

All even multiples of these are added to the order by

$$2^n 3 \rhd 2^n 5 \rhd 2^n 7 \rhd \ldots \quad \text{and} \quad 2^n(2i+1) \rhd 2^m(2j+1)$$

if $m > n$ and $i, j > 0$. Finally, all powers of 2 are added to the order in decreasing powers so that finally we have:

$$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \ldots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \ldots$$

$$\ldots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

Formally, we have the following definition.

Definition - Sarkovskii's Ordering. *The relation* $\triangleright$ *is defined on the positive integers so that* $a \triangleright b, b \triangleright c$ *implies* $a \triangleright c$. *It is defined as follows.*

1. $2^n 3 \triangleright 2^n 5 \triangleright 2^n 7 \triangleright \dots$ *for* $n = 0, 1, 2, \dots$
2. $2^n(2i+1) \triangleright 2^m(2j+1)$ *for* $m > n$ *and* $i, j > 0$
3. $2^n(2i+1) \triangleright 2^m$ *for* $n, m \geq 0$ *and* $i > 0$
4. $2^n \triangleright 2^m$ *for* $n > m \geq 0$

Theorem 2.6 - Sarkovskii's Theorem. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous. Further, suppose* f *has a periodic point of least period* k. *If* $k \triangleright l$ *then* f *also has a point of least period* l.

A proof of this can be found in [2, p 63].

Note that the existence of any non-trivial orbit implies the existence of 2-periodic orbits. Therefore, if it is shown that no 2-periodic orbits exist then it can be concluded that there are no non-trivial periodic orbits. Sarkovskii's Theorem is now combined with Corollary 2.5 to get the following.

Theorem 2.7. *Suppose there exists an* ω*-cyclic continuously differentiable function* $\Phi : \mathbb{R}^\omega \to \mathbb{R}$ *with corresponding* F *such that* $\frac{d}{dx}F(x) \neq 0$ *on* $\mathbb{R}$.

1. If ω *is even then* f *has no non-trivial periodic orbits.*

2. If ω *is odd then* f *has no non-trivial* k*-periodic orbits where* k *is odd and* $1 < k \leq \omega$.

Proof. By Corollary 2.5, f has no non-trivial ω-periodic orbits. If ω is even then f has no non-trivial 2-periodic orbits since they would also be ω-periodic. By Sarkovskii's Theorem, f must not have any non-trivial periodic points since they would imply the existence of 2-periodic points.

If ω is odd then by Sarkovskii's Theorem, f has no k-periodic orbits with $1 < k \leq \omega$ where k is odd since such an orbit would imply the existence of ω periodic points. □

A necessary and sufficient condition is now given for the case in which f is a diffeomorphism.

Theorem 2.8. *Suppose $f'(x) \neq 0$ for $x \in I$, an open interval. Then there are no non-trivial periodic orbits of (2.1) in I if and only if there exists a continuously differentiable real-valued function α on $I \cup f(I)$ such that*

$$\frac{d}{dx}\left[\alpha \circ f(x) + \alpha(x)\right] \geq 0 \tag{2.8}$$

with strict inequality if x is not a fixed point of f.

Proof. Theorem 2.3 and the discussion following the theorem show that (2.8) is sufficient for the non-existence of non-trivial 2-periodic orbits which by Sarkovskii's Theorem precludes the existence of all non-trivial periodic orbits.

Next we show that the existence of α satisfying (2.8) is also a necessary condition. Suppose that (2.1) has no non-trivial periodic orbits. If $f'(x) > 0$ for all $x \in I$ then $\alpha(x) = x$ satisfies (2.8) with strict inequality for all x. The only case left to discuss is when $f'(x) < 0$ for all $x \in I$. If there is no equilibrium, then either $f(x) > x$ or $f(x) < x$ for all $x \in I$. Since f is decreasing, this means $I \cap f(I) = \phi$. Thus, for an arbitrary choice of α on I, α can be chosen on $f(I)$ to get $\frac{d}{dx}[\alpha \circ f(x) + \alpha(x)] = 1$ for all $x \in I$, and so (2.8) is satisfied.

Suppose c is an equilibrium of f. Since f is decreasing, c must be the only equilibrium. If $x_o < c$ is sufficiently close to c so that $f(x_o)$ and $f^2(x_o)$ both exist then either $x_o < f^2(x_o) < c < f(x_o)$ or $f^2(x_o) < x_o < c < f(x_o)$. The first of these possibilities will be dealt with first. For any $x < c$ such that $f^2(x)$ exists we must have $x < f^2(x) < c$. Otherwise, the Intermediate Value Theorem would imply the existence of a non-trivial 2-periodic point. So for $x < c$, the sequence $f^{2t}(x)$ is increasing and the sequence $f^{2t+1}(x)$ is decreasing. For $i \in \{0, 1\}$, $\lim_{t\to\infty} f^{2t+i}(x) = d \neq c$ implies d is a non-trivial 2-periodic point. So we must have $\lim_{t\to\infty} f^t(x) = c$. Similarly, for $c < x$, we also get $\lim_{t\to\infty} f^t(x) = c$.

Choose $M \in \mathbb{R}$ such that $M > 1$ and $M > \sup\{-f'(x) : x \in [x_o, f(x_o)]\}$. Define $\alpha'(f^t(x_o)) = M^{-t}$ for $t = 0, 1, \ldots$. For $x \in (x_o, f^2(x_o))$, define $\alpha'(x) = 1 + s(M^{-2} - 1)$ where $x = x_o + s(f^2(x_o) - x_o)$. Let $\alpha'(f^t(x)) = M^{-t}\alpha'(x)$ for $x \in [x_o, f^2(x_o)]$, $t = 0, 1, \ldots$. Finally, define $\alpha'(c) = 0$.

Thus, α' is a continuous function on $[x_o, f(x_o)]$ and we have

$$\begin{aligned} \frac{d}{dx}\left[\alpha \circ f(x) + \alpha(x)\right] &= \alpha'(f(x))\, f'(x) + \alpha'(x) \\ &= \alpha'(x)\left(M^{-1} f'(x) + 1\right) \end{aligned}$$

By our choice of M, this is positive for $x \neq c$. At c, it is zero.

We now extend α' to I. Let $\beta : I \to \mathbb{R}$ be continuous with $\beta(x) > 0$ if $x \neq c$ and $\beta(x) = \alpha'(f(x))\, f'(x) + \alpha'(x)$ if $x \in [x_o, f(x_o)]$. Iteration of the formula

$$\alpha'(x) = \beta(x) - \alpha'(f(x))\, f'(x)$$

extends α' continuously to $J = \{x \in I : f(x) \in I\}$. An arbitrary continuous extension of α' to $f(I \setminus J)$ finishes the process. Finding an antiderivative of α' gives the desired function α to satisfy (2.8).

Finally, the case $x_o < f^2(x_o) < c < f(x_o)$ may be treated as in the preceding discussion if f is replaced by its inverse f^{-1}. $\square$

CHAPTER 3

DIFFERENCE EQUATIONS IN HIGHER DIMENSIONS

Many of the one dimensional results found in Chapter 2 can be generalized to higher dimensions. One result that does not hold in higher dimensions is Sarkovskii's Theorem. To see this, consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by simply rotating the entire plane 120 degrees about the origin. Obviously, every point, aside from the origin, has least period three. Sarkovskii's Theorem would imply the existence of points of every least period. This is not the case so Sarkovskii's Theorem cannot be used in higher dimensions.

A more general definition of ω-cyclic functions is needed.

Definition. *A function $\Phi : \mathbb{R}^{n\omega} \to \mathbb{R}^m$ will be called ω-cyclic if*

$$\Phi(y_1, ..., y_\omega) = \Phi(y_\omega, y_1, ..., y_{\omega-1}) \tag{3.1}$$

for any $y_i \in \mathbb{R}^n$, $i = 1, ..., \omega$. With any ω-cyclic function Φ and map $f : \mathbb{R}^n \to \mathbb{R}^n$, the function $F : \mathbb{R}^n \to \mathbb{R}^m$ given by $F(x) = \Phi(x, f(x), ..., f^{\omega-1}(x))$ is associated.

Examples. The following are ω-cyclic functions from $\mathbb{R}^{n\omega}$ to $\mathbb{R}$.

$$\Phi_1(y_1, ..., y_\omega) = \sum_{i=1}^{\omega}\sum_{k=1}^{\omega} y_i \cdot y_k$$

$$\Phi_2(y_1, ..., y_\omega) = u \cdot \sum_{i=1}^{\omega} y_i \quad \text{for some fixed } u \in \mathbb{R}^n$$

$$\Phi_3(y_1, ..., y_\omega) = ||y_1|| + ... + ||y_\omega||$$

The corresponding functions $F : \mathbb{R}^n \to \mathbb{R}$ are:

$$F_1(x) = \sum_{i=0}^{\omega-1}\sum_{i=0}^{\omega-1} f^i(x) \cdot f^k(x)$$

$$F_2(x) = u \cdot \sum_{i=0}^{\omega-1} f^i(x)$$

$$F_3(x) = ||x|| + ||f(x)||... + ||f^{\omega-1}(x)||$$

Theorem 2.3 is now generalized to higher dimensions. Consider the difference equation

$$x_{t+1} = f(x_t) \qquad \text{for } t = 0, 1, 2, \ldots \tag{3.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

Theorem 3.1. *Suppose $\{f^t(a) : t = 0, 1, 2, \ldots\}$ is a non-trivial ω-periodic orbit of (3.2). Then for each smooth curve γ with endpoints at a and $f(a)$ and each ω-cyclic continuously differentiable function Φ with corresponding F, there exists $c \in \gamma$ such that $\operatorname{grad} F(c) \cdot v = 0$ if v is tangent to γ at c.*

Proof. Parameterize γ such that $\gamma(0) = a$ and $\gamma(1) = f(a)$. Then

$$\begin{aligned} 0 &= F(f(a)) - F(a) && \text{since } a \text{ is } \omega\text{-periodic} \\ &= \int_0^1 \frac{d}{dt} F(\gamma(t))\, dt \\ &= \int_0^1 \operatorname{grad} F(\gamma(t)) \cdot \gamma'(t)\, dt \end{aligned}$$

By Rolle's Theorem, there is some $t_o \in (0, 1)$ such that $\operatorname{grad} F(\gamma(t_o)) \cdot \gamma'(t_o) = 0$. Let $c = \gamma(t_o)$. Then v is a scalar multiple of $\gamma'(t_o)$ and we get $\operatorname{grad} F(c) \cdot v = 0$. □

Corollary 3.2. *Suppose that, for every, $a \in \mathbb{R}$ [illegible]ists a smooth curve γ with endpoints at a and $f(a)$ and an ω-cyclic conti[illegible]ifferentiable function Φ with corresponding F such that*

$$\operatorname{grad} F(x) \cdot v(x) \neq 0$$

for all $x \in \gamma$, where $v(x)$ is tangent to γ at x. Then (3.2) has no non-trivial ω-periodic solutions.

Applying Theorem 3.1 to maps in the plane gives some readily applicable results.

Corollary 3.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable. Suppose there exists a real valued ω-cyclic continuously differentiable function Φ with corresponding F such that $\text{grad}\, F(x) \in \mathbb{R}^2_+$ (or $\mathbb{R}^2_-$) for all $x \in \mathbb{R}^2$. Then for any non-trivial ω-periodic point a of f, the straight line joining a and $f(a)$ must have negative slope.*

Note. $\mathbb{R}^2_+$ ($\mathbb{R}^2_-$) is the set of all ordered pairs with both numbers greater than (less than) zero.

Proof. Let a be a non-trivial ω-periodic point and let $v = f(a) - a$. Then somewhere on the line segment joining a and $f(a)$, we must have $\text{grad}\, F(c) \cdot v = 0$. Since the components of $\text{grad}\, F(c)$ have the same sign, the components of v must have opposite signs. This means that the line through a and $f(a)$ must have negative slope. □

Corollary 3.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable. Suppose there exists a real valued ω-cyclic continuously differentiable function Φ with corresponding F such that $\text{grad}\, F(x)$ is in the interior of the second or fourth quadrant for all $x \in \mathbb{R}^2$. Then, for any non-trivial ω-periodic point a of f, the straight line joining a and $f(a)$ must have positive slope.*

Corollary 3.5. *Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (g(x,y), kx)$ where $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $k \in \mathbb{R}$. Then any proper 2-periodic orbit consists of points which lie on a straight line of slope $-k$.*

Proof. Define the 2-cyclic function Φ by

$$\Phi(\,(x_1, y_1),\, (x_2, y_2)\,) = y_1 + y_2.$$

This gives $F(x,y) = y + kx$ and $\text{grad}\, F = (k, 1)$ Let $(p, q) = f(a) - a$ where a is a proper 2-periodic point. Then we get:

$$\begin{aligned} 0 &= (k,1)\cdot(p,q) \\ &= kp+q \\ -k &= \frac{q}{p} \end{aligned}$$

and so the line through a and $f(a)$ has slope $-k$. □

Consider the Hénon map $f:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$f(x,y) = (a - by - x^2, x) \tag{3.3}$$

This is the simplest known map that appears to have a strange attractor. Since very little is known about strange attractors, the Hénon map is studied intently. The attractor is only present for certain values of the parameters a and b. The well known Hénon attractor comes from a map which is topologically equivalent to the Hénon map with $a=\frac{7}{5}$ and $b=-\frac{10}{3}$. The two periodic orbits of the Hénon map can be calculated directly but the three periodic orbits are difficult to find in general. Thus, information about the location of three periodic orbits is potentially useful. The following will be demonstrated.

1. Each 2-periodic orbit lies on a line of slope -1.
2. For a 2-periodic orbit {p,q}, p and q lie on different sides of the line $x=\frac{1+b}{2}$.
3. A region is found which contains no proper 3-periodic points for $a=\frac{7}{5}$ and $b=-\frac{10}{3}$.

The first statement follows from Corollary 3.5. To see that the second statement is true, consider the 2-cyclic function, $\Phi:\mathbb{R}^4\to\mathbb{R}$

$$\Phi\big((x_1,y_1)\ ,\ (x_2,y_2)\big) = x_1+x_2.$$

This defines the corresponding function $F:\mathbb{R}^2\to\mathbb{R}$ given by $F(x,y)=x+a-by-x^2$. This gives $\operatorname{grad} F=(1-2x,-b)$. Suppose (x_o,y_o) is a 2-periodic point of the

Hénon map. Applying Theorem 3.1, where γ is the straight line segment joining (x_o, y_o) and $f(x_o, y_o)$ with slope of -1, we see that we must have $1 - 2x + b = 0$ at some point in γ. Thus, (x_o, y_o) and $f(x_o, y_o)$ lie on different sides of the line $x = \frac{1+b}{2}$.

To fulfill the third statement, define the 3-cyclic functions $\Phi, \Psi : \mathbb{R}^6 \to \mathbb{R}$

$$\Phi((x_1, y_1), (x_2, y_2), (x_3, y_3)) = y_1 + y_2 + y_3$$

$$\Psi((x_1, y_1), (x_2, y_2), (x_3, y_3)) = (x_1 + y_1^2) + (x_2 + y_2^2) + (x_3 + y_3^2)$$

These define, respectively, the corresponding functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x, y) = y + x + (a - by - x^2)$$

$$G(x, y) = (x + y^2) + (a - by) + (a - bx)$$

and we have $\operatorname{grad} F(x, y) = (1 - 2x, 1 - b)$ and $\operatorname{grad} G(x, y) = (1 - b, 2y - b)$. Suppose the point (x_o, y_o) is a 3-periodic point. Then by Corollary 3.2, there are points (x_1, y_1) and (x_2, y_2) on the line segment joining (x_o, y_o) and $f(x_o, y_o)$ such that

$$\begin{aligned} 0 &= \operatorname{grad} F(x_1, y_1) \cdot \big(f(x_o, y_o) - (x_o, y_o)\big) \\ &= \big(1 - 2x_1,\ 1 - b\big) \cdot \big(a - by_o - x_o^2 - x_o,\ x_o - y_o\big) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} 0 &= \operatorname{grad} G(x_2, y_2) \cdot \big(f(x_o, y_o) - (x_o, y_o)\big) \\ &= \big(1 - b,\ 2y_2 - b\big) \cdot \big(a - by_o - x_o^2 - x_o,\ x_o - y_o\big) \end{aligned} \tag{3.5}$$

Substituting the values $a = \frac{7}{5}$ and $b = -\frac{10}{3}$ into (3.4) and (3.5) and dropping the subscripts on x_1 and y_2 gives

$$0 = (1 - 2x)\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \frac{13}{3}(x_o - y_o) \tag{3.6}$$

and

$$0 = \frac{13}{3}\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \left(2y + \frac{10}{3}\right)(x_o - y_o) \tag{3.7}$$

for some $x \in [x_o, f_1(x_o, y_o)]$ and $y \in [x_o, y_o]$ where $f_1(x_o, y_o)$ is the first coordinate of $f(x_o, y_o)$.

So, given a point $(x_o, y_o) \in \mathbb{R}^2$, if there is no $x \in [x_o, f_1(x_o, y_o)]$ which satisfies (3.6) or there is no $y \in [x_o, y_o]$ which satifies (3.7) then (x_o, y_o) cannot be a non-trivial 3-periodic point of (3.3). Since (3.7) is linear in y, if the right hand side takes on the same sign when y equals each of x_o and y_o, then the equation has no solution on the interval $[x_o, y_o]$. Thus, any point lying in the region A described by

$$0 < \left[\frac{13}{3}\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \left(2x_o + \frac{10}{3}\right)(x_o - y_o)\right] \times \left[\frac{13}{3}\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \left(2y_o + \frac{10}{3}\right)(x_o - y_o)\right]$$

is not non-trivially 3-periodic. Similarly, if evaluating the right hand side of (3.6) at x equals x_o and $f_1(x_o, y_o)$, and multiplying the results together gives a positive value then the point (x_o, y_o) is not non-trivially 3-periodic. This defines the region B given by

$$0 < \left[(1 - 2x_o)\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \frac{13}{3}(x_o - y_o)\right] \times \left[(-\frac{9}{5} - \frac{20}{3}y_o + 2x_o^2))\left(\frac{7}{5} + \frac{10}{3}y_o - x_o^2 - x_o\right) + \frac{13}{3}(x_o - y_o)\right].$$

No point lying in $A \cup B$ can be non-trivially 3-periodic.

Example. Consider a Hawk-Dove interaction between males of some bird species competing for mating priveleges. The behavior of a male can be likened to that of a hawk or a dove. Every time a female is ready to mate, two males compete for the privelege. If two hawks meet, they fight to the death. If two doves are competing, one of them retreats. If a hawk meets a dove, then the dove retreats. Assuming that hawks and doves are identical in all other aspects, the following difference equation gives a primitive model of how the population will develop in time. The numbers of hawks and doves in the population at time t are denoted by H_t and D_t respectively.

$$\begin{aligned} H_{t+1} &= H_t\left[q + \frac{p-q}{2}\frac{H_t}{H_t+D_t} + p\frac{D_t}{H_t+D_t}\right] \\ D_{t+1} &= D_t\left[q + \frac{p}{2}\frac{D_t}{H_t+D_t}\right] \end{aligned} \tag{3.8}$$

where q is the probability that a male will live to the next mating season and p is the expected number of offspring from a mating. It is assumed that H_t and D_t are non-negative and that $H_t + D_t$ and $p+q$ are postive.

Using the 2-cyclic function

$$\Phi\big((H_1, D_1),\ (H_2, D_2)\big) = 2(H_1 + \frac{q}{p}D_1) + 2(H_2 + \frac{q}{p}D_2)$$

and the corresponding function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned} F(H,D) &= 2(H + \frac{q}{p}D) \\ &+ 2\bigg(H\left[q + \frac{p-q}{2}\frac{H}{H+D} + p\frac{D}{H+D}\right] + \frac{q}{p}D\left[q + \frac{p}{2}\frac{D}{H+D}\right]\bigg) \end{aligned}$$

we can investigate the possible existence of 2-periodic solutions of (3.8). Then $\operatorname{grad} F = (2+p+q,\ 2+p+3q+\frac{2q}{p}+\frac{2q^2}{p})$, a constant. Choosing γ to be the straight line segment joining (H, D) to $f(H, D)$ with tangent $v = f(H,D) - (H,D)$, we find

$$\begin{aligned} \operatorname{grad} F \cdot v \;=\; & \frac{1}{2(H+D)}\Bigg[\Big((p+q+2)(p+q-2)\Big)H^2 \\ & + \bigg((p+q+2)(p+2q-2) + 2q(q-1)(1+\frac{1}{p}+\frac{q}{p})\bigg)2HD \\ & + \bigg((p+q+2)(p+2q-2) + 2q(1+\frac{1}{p}+\frac{q}{p})(p+2q-2)\bigg)D^2\Bigg]. \end{aligned}$$

For $p + q > 2$, the above expression is positive for all H and D. So, by Corollary 3.2, we see that (3.8) has no non-trivial 2-periodic orbits when $p + q > 2$.

Theorem 3.6. *Suppose that D is an open convex subset of the domain of f and that $\Phi : \mathbb{R}^{n\omega} \to \mathbb{R}^n$ is an ω-cyclic continuously differentiable function with corresponding $F : \mathbb{R}^n \to \mathbb{R}^n$. If for each non-zero $v \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^n$ such that*

$$u^* \frac{\partial F}{\partial x}(x) v > 0 \tag{3.9}$$

for all $x \in D$, then there is no nontrivial ω-periodic point x_o of (3.1) such that $x_o,\ f(x_o) \in D$.

Proof. This follows from Theorem 3.1, where given a non-equilibrium $x_o \in D$, γ is taken to be the straight line joining x_o and $f(x_o)$. Choose u to correspond with $v = f(x_o) - x_o$. Using the ω-cyclic function $\Psi = u^*\Phi$ with corresponding $G(x) = u^*F(x)$. We get the result that x_o can not be ω-periodic point if $x_o,\ f(x_o) \in D$. □

A $n \times n$ real matrix $A = [a_{ij}]$ is positive definite if $u^*Au \geq 0$ for each non-zero $u \in \mathbb{R}^n$. A is diagonally dominant by rows if

$$|a_{ii}| - \sum_{j \neq i} |a_{ij}| > 0, \qquad i = 1, \ldots, n.$$

A is diagonally dominant by columns if

$$|a_{jj}| - \sum_{i \neq j} |a_{ij}| > 0, \qquad j = 1, \ldots, n.$$

Concrete conditions which ensure (3.9) may be expressed in these terms as follows:

A. $\frac{\partial F}{\partial x}(x)$ is positive definite for each $x \in D$.

B. $\frac{\partial F}{\partial x}(x)$ is diagonally dominant by rows for each $x \in D$.

C. $\frac{\partial F}{\partial x}(x)$ is diagonally dominant by columns for each $x \in D$.

Corollary 3.7. *Let D and F be defined as in Theorem 3.6. If one of conditions A, B, and C is satisfied, then (3.2) has no non-trivial ω-periodic point x_o such that $x_o, f(x_o) \in D$.*

Proof. If condition A is satisfied, then the choice $u = v$ satisfies (3.9). If condition B holds and $0 \neq v \in \mathbb{R}^n$, choose i such that $|v_i| \geq |v_j|$ for $j = 1, \ldots, n$. Let $u_j = 0$ for $j \neq i$ and choose u_i such that $u_i \frac{\partial F_i}{\partial x_i} v_i = |\frac{\partial F_i}{\partial x_i} v_i|$. Then

$$u^* \frac{\partial F}{\partial x} v = u_i \left(\frac{\partial F_i}{\partial x_i} v_i + \sum_{j \neq i} \frac{\partial F_i}{\partial x_j} v_j \right) \geq \left(\left| \frac{\partial F_i}{\partial x_i} \right| - \sum_{j \neq i} \left| \frac{\partial F_i}{\partial x_j} \right| \right) |v_i| > 0$$

and we see that B implies (3.9). Similarly, if condition C is satisfied, choose u_j such that $u_j \frac{\partial F_j}{\partial x_j} v_j = |\frac{\partial F_j}{\partial x_j} v_j|$ for $j = 1, \ldots, n$. Then

$$u^* \frac{\partial F}{\partial x} v = \sum_{j=1}^{n} \sum_{i=1}^{n} u_i \frac{\partial F_i}{\partial x_j} v_j \geq \sum_{j=1}^{n} \left(\left| \frac{\partial F_j}{\partial x_j} \right| - \sum_{i \neq j} \left| \frac{\partial F_i}{\partial x_j} \right| \right) |v_j| > 0$$

Again, (3.9) is satisfied. In each of these cases, since (3.9) is satisfied, there are no non-trivial ω-periodic orbits of (3.2) with two consecutive points lying in D. □

Bibliography

1. I. Bendixson, *Sur les courbes définiés par des équations différentielles*, Acta Math. **24** (1901), 1–88.

2. R. L. Devaney, An Introduction to Chaotic Dynamical Systems, The Benjamin/Cummings Publishing Co. Inc., Menlo Park, 1986.

3. H. Dulac, *Recherche des cycles limites*, C. R. Acad. Sci. Paris **204** (1937), 1703–1706.

4. Y. Li and J. S. Muldowney, *On Bendixson's Criterion*, J. Diff. Eqn. **106** (1993), 27–39.

5. N. G. Lloyd, *A note on the number of limit cycles in certain two-dimensional systems*, J. London Math. Soc. (2) **20** (1979), 1–88.

6. E. Rasmusen, Games and Information, Cambridge University Press, Cambridge, 1989.

7. R. A. Smith, *Some applications of Hausdorff dimension inequalities for ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sec. A **104** (1986), 235–259.

8. F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Epsilon Uitgaven, Utrecht, 1985.