UNIVERSITY OF ALBERTA

Permutation Summands for Finite Groups

BY

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A Thesis
Submitted to the Faculty of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy

in

Mathematics

DEPARTMENT OF MATHEMATICAL SCIENCES

Edmonton, Alberta
SPRING 1995

ISBN 0-612-01774-5

UNIVERSITY OF ALBERTA

RELEASE FORM

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TITLE OF THESIS: **Permutation Summands for Finite Groups**

DEGREE: Doctor of Philosophy

YEAR THIS DEGREE GRANTED: 1995

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To my parents for their encouragement

To my wife Faith for her understanding and patience

To my daughter Selena

Abstract

A *permutation lattice* for a finite group G over the ring A of integers in a number field is a free A-module with a finite basis which is permuted by G; direct summands of these, as AG-modules, are called *permutation summands* for G over A. Numerical invariants are constructed for these lattices and a class of congruences on the invariants are exhibited. These are used to study the Grothendieck ring $\Omega_A(G)$ of the category of permutation summands in analogy with the Burnside ring of the category of finite G-sets: the nilradical of $\Omega_A(G)$ is in the image of the class group $\mathrm{Cl}(AG)$ and the reduced quotient of $\Omega_A(G)$ is connected and has finite index in the function ring $\mho_A(G)$ of known structure. An induction theorem for $\Omega_A(G)$ is proved along the way and it is used to classify the stable isomorphism classes of permutation lattices. The virtual characters of permutation summands over big number rings are classified through an induction theorem on virtual characters over the maximal unramified extension field of the rational p-adic numbers.

ACKNOWLEDGEMENT

First and foremost, I would like to thank my supervisor Professor A. Weiss for introducing me to the subject of Representation Theory during my studies at the University of Alberta. I am deeply indebted to him for his written comments, stimulating discussions and valuable guidance during my thesis research. His broad and profound knowledge of mathematics has benefited me a great deal, and his style of writing has a strong influence on my exposition.

I owe a debt of gratitude to Professor G. H. Cliff for his interest on my thesis, and the courses he offered on Group Cohomology and Algebraic Number Theory, which are important to my thesis work. I would like to thank Professors H. H. Brungs and R. V. Moody for teaching me mathematical courses on Ring Theory and Representations of Lie Groups. I am grateful to Professor J. You and External Examiner Professor R. Guralnick for their comments on this thesis.

My sincere thanks also go to Dr. H. Xia and Mrs. Vivian Spak for the formatting of this thesis.

Finally, I would like to acknowledge the Summer support from Professor A. Weiss and the Teaching Assistantship from the department of mathematics during my program at the University of Alberta.

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INTRODUCTION

Let G be a finite group and A the ring of integers in a number field K. An AG-lattice is called a *permutation lattice* if it has an A-basis, necessarily finite, which is permuted by the action of G. It will be called a *permutation summand* (for G over A), if it is a direct summand, as AG-module, of a permutation lattice.

Permutation summands for G over A are the objects in this study. In the case of $A = \boldsymbol{Z}$, permutation summands occur in other contexts [Weis], [Swan], [BeLe], where incidentally, permutation summands are usually called invertible [Swan] or permutation projectives [Dre1], [CRII](§81B). This work is done over A, because doing so clarifies the role of the arithmetic of the coefficient ring and the study of characters of permutation summands over large number rings in Chapter 3 advances our understanding on permutation summands.

The study of permutation summands in this work is centered around the Grothendieck group $\Omega_A(G)$ of the category of all of them. $\Omega_A(G)$ is defined as the additive group generated by symbols $[L]$ corresponding to isomorphism classes (L) of permutation summands, with defining relations $(L \oplus L_1) = (L) + (L_1)$. These relations are the same ones given by $(L) = (L_1) + (L_2)$ for each short exact sequence, for if $0 \to L_1 \to L \xrightarrow{\alpha} L_2 \to 0$ is a short exact sequence with permutation summands L_1, L_2 then it splits :

Applying $\mathrm{Hom}_A(L_2, -)$, we have an exact sequence of A modules

$$0 \to \mathrm{Hom}_A(L_2, L_1) \to \mathrm{Hom}_A(L_2, L) \to \mathrm{Hom}_A(L_2, L_2) \to 0.$$

Applying now $\mathrm{Hom}_{AG}(A, -)$ to above sequence and noticing that $\mathrm{Hom}_{AG}(A, X) \simeq X^G = \{x \in X, gx = x \text{ for } g \in G\}$ for any AG-module X, we get a cohomology sequence:

$0 \to \mathrm{Hom}_A(L_2, L_1)^G \to \mathrm{Hom}_A(L_2, L)^G \xrightarrow{\alpha_*} \mathrm{Hom}_A(L_2, L_2)^G \xrightarrow{\partial} H^1(G, \mathrm{Hom}_A(L_2, L_1)) \to \ldots$ Since $\mathrm{Hom}_A(L_2, L_1)$ is here a permutation summand, $H^1(G, \mathrm{Hom}_A(L_2, L_1)) = 0$ follows from Shapiro's lemma. Now α_* is surjective, so $\alpha_*(\beta) = 1$ for some $\beta \in \mathrm{Hom}_A(L_2, L)^G = \mathrm{Hom}_{AG}(L_2, L)$. The β splits the sequence $0 \to L_1 \to L \xrightarrow{\alpha} L_2 \to 0$. This completes the proof of claim.

$\Omega_A(G)$ is made into a commutative ring via the tensor product over A: $[L][L_1] = [L \otimes_A L_1]$ where $L \otimes_A L_1$ is an AG-lattice with the diagonal G-action.

The study of $\Omega_A(G)$ is based on the construction of a sort of numerical character Φ_L of a permutation summand L, which permits the adaptation of the usual character-theoretic methods. The values of Φ_L need not be in A, but a certain cyclotomic extension $\mathbf{Z}'$ is necessary, which is how the arithmetic enters. Here $\mathbf{Z}'$ is the ring of integers in a cyclotomic extension $\mathbf{Q}'/\mathbf{Q}$ large enough that it contains roots of unity of order the exponent of G. Let K' denote the composite field $K\mathbf{Q}'$, and let A' be the integral closure of A in K'. Let Γ_K denote the Galois group of K'/K and $\Gamma_{\mathbf{Q}}$ the Galois group of $\mathbf{Q}'/\mathbf{Q}$. Then restricting K-automorphisms of K' to $\mathbf{Q}'$ gives a group monomorphism $\Gamma_K \to \Gamma_{\mathbf{Q}}$.

By a *triple* of G over A, we mean $(H, b, \mathfrak{p}')$, where $\mathfrak{p}'$ is a prime ideal (always non-zero) of A' so that if p is the unique prime number in $\mathfrak{p}'$ then H is a p-hypoelementary subgroup of G and $H/O_p(H) = \langle b \rangle$. A subgroup H of G is *p-hypoelementary* if $H/O_p(H)$ is cyclic where $O_p(H)$ is the largest normal p-subgroup of H.

Φ_L, associated to each permutation summand L, is a Z'-valued function on triples. The construction of Φ_L in §1.1 can be thought as a globalization of "species" $s_{H,b}$, formulated by Benson and Parker [BePa] [Bens], on permutation summands of G over local rings. It is the foundation of our study on $\Omega_A(G)$ through Φ_L that these $s_{H,b}$, with H varying over all p-hypoelementary subgroups of G and b over generators of $H/O_p(H)$, determine permutation summands over local rings $A_\mathfrak{p}$, $p \in \mathfrak{p}$, up to isomorphism. This result, due to Conlon [Conl], is given a modified proof in §1.2, with the essential use of the Green Correspondence.

The values of Φ_L in Z' are not arbitrary at all. The Galois theoretic properties (1.3) and (1.4) of the values of Φ_L are however, sufficiently restrictive that Φ_L may be viewed as an element of a certain ring of functions $\mho_A(G)$, which is constructed in §1.3 and has a rather transparent structure (Prop. 1.11). Then $L \mapsto \Phi_L$ defines a map $\Phi : \Omega_A(G) \to \mho_A(G)$ which is a ring homomorphism (1.10) and tells us about the ring $\Omega_A(G)$. This formulation is analogous to the known description([TomD],[CRII]§80A) of the Burnside ring, which we briefly recall now. The Burnside ring $\Omega(G)$ is the Grothendieck ring of the category of finite G-sets with sums and products coming from disjoint union and product, with diagonal G-action. More precisely, $\Omega(G)$ is defined to be the quotient F/F_0, where F is the free abelian group generated by symbols (S), one for each isomorphism class of finite G-sets S, and where

F_0 is the subgroup of F generated by expressions $(S \dot{\cup} T) - (S) - (T)$. Then $\Omega(G)$ is an additive abelian group. Define multiplication on generators $(S), (T)$ of F by $(S)(T) = (S \times T)$ where the product $S \times T$ is given the diagonal G -action. F_0 is then an ideal of the commutative ring F. Thus $\Omega(G)$ is a commutative ring with identity.

To build $\mho(G)$ we use the G-set of all subgroups of G under the conjugation action, and, letting G act trivially on Z, we define $\mho(G)$ to be the ring, under point-wise operations, of all G-set maps from the set of subgroups of G to Z. Finally $\Phi : \Omega(G) \to \mho(G)$ is defined by sending a G-set X to the function Φ_X defined by $\Phi_X(H) = \text{card} X^H$. Then Φ is a ring monomorphism with finite cokernel annihilated by $|G|$. In fact, the image of Φ is known to consist of all $f \in \mho(G)$ which satisfy

(Congruence I1.1)
$$\sum_{g \in N_G(H)/H} f(\langle H, g \rangle) \equiv 0 \bmod (N_G(H) : H)$$

for all subgroups H of G.

Combining with (1.9)(b), we have the following commutative diagram

(Diagram I1.2)
$$\begin{array}{ccc} \Omega(G) & \xrightarrow{\Phi} & \mho(G) \\ \downarrow & & \downarrow \\ \Omega_A(G) & \xrightarrow{\Phi} & \mho_A(G), \end{array}$$

where the left vertical map takes the G-set X to its A-linearization $A[X]$, and the right vertical map takes $f \in \mho(G)$ to $\tilde{f} \in \mho_A(G)$, defined by $\tilde{f}(H, b, \mathfrak{p}') = f(H)$.

Theorem A. *There is an exact sequence*

$$Cl(AG) \to \Omega_A(G) \xrightarrow{\Phi} \mho_A(G)$$

and the cokernel of Φ *is annihilated by* $|G|$.

The proof of Theorem A is a sequence of technical results and can be found in [WaWe](§4). We view the locally free class group $\mathrm{Cl}(AG)$ as the kernel of the localization map $K_0(AG) \to \prod_{\mathfrak{p}} K_0(A_{\mathfrak{p}}G)$, where $K_0(AG)$ (resp. $K_0(A_{\mathfrak{p}}G)$) is the Grothendieck group of the category of finitely generated projective left AG-modules (resp. $A_{\mathfrak{p}}G$-modules). Each $x \in \mathrm{Cl}(AG)$ is of the form $x = [F] - [P]$ with F an AG-free and P projective in the same genus. $\mathrm{Cl}(AG)$ is finite by a theorem of Jordan-Zassenhaus [CRII](39.13). It follows from Theorem A that Φ has finite kernel and cokernel, which determines the $\mathbf{Z}$-rank of $\Omega_A(G)$ by that of $\mho_A(G)$, recovering the main result of Dress[Dre1].

In general, the kernel of the map $\mathrm{Cl}(AG) \to \Omega_A(G)$ is contained in the "kernel group" $D(AG)$ of AG, [CRII](49.34). In the case $A = \boldsymbol{Z}$, we even have $\ker(\mathrm{Cl}(\boldsymbol{Z}G) \to \Omega_{\boldsymbol{Z}}(G)) = D(\boldsymbol{Z}G)$, by a theorem of Oliver [Oliv1], in a paper studying the kernel of the map $\Omega(G) \to \Omega_{\boldsymbol{Z}}(G)$ of Diagram (I1.2). However, $D(AG)$ is too large in general [Oliv2].

Theorem A already makes it clear that this formalism enables us to ask more delicate questions on the image of Φ, namely characterizing the image of Φ in $\mho_A(G)$ in analogy with the Burnside Congruence (I1.1) used to characterize the image $\Omega(G) \xrightarrow{\Phi} \mho(G)$. The following result demonstrates a class of congruences on Φ_L

Theorem B(Congruences). *If* $\mathfrak{q}'$ *is a prime ideal of* Z' *containing the prime number* q *then*

$$\Phi_L(H, b, \mathfrak{p}') \equiv \Phi_L(O^q(H), b_{q'}, \mathfrak{p}) \ mod\ \mathfrak{q}'$$

for any p *-triple* $(H, b, \mathfrak{p}')$ *of* G *over* A.

Here $O^q(H)$ is the minimal normal subgroup of H so that $H/O^q(H)$ is a q -group, and $b_{q'}$ is the q'-component of the element b.

Although these congruences are not strong enough to characterize the image of Φ, they are used to study the prime ideal spectrum of $\Omega_A(G)$ and in particuiar to show that the spectrum is connected, in contrary to the result on the prime spectrum of $\Omega(G)$ in [Dre4]. The connectness implies that $\Omega_A(G)$ has no non-trivial idempotents in contrast to results on the Burnside ring $\Omega(G)$ [Glu1] and [Yosh].

Theorem B, together with its application, is proved in Chapter 2 of this thesis. It generalizes the well-known congruences on fixed points of a Q-set X [Ser1]

$$\text{card}X^Q \equiv \text{card}X \text{ mod } q$$

for a q-group Q, in view of the Diagram (I1.2) which has $\Phi_X(Q) = \text{card}(X^Q)$.

In studying $\Omega_A(G)$ through the ring homomorphism $\Phi : \Omega_A(G) \rightarrow \mho_A(G)$, we proved the following induction theorem

Theorem C. *The induction map* $ind: \coprod_E \Omega_A(E) \to \Omega_A(G)$ *is surjective, with* E *varying over the pseudo-elementary subgroups of* G.

Here a group E is called *pseudo-elementary* of type (p,q), with p and q (possibly equal) prime numbers, if it has normal subgroups $E_1 \subseteq E_0$ so that E_1 is a p-group, E_0/E_1 is cyclic, and E/E_0 is a q-group.

Theorem C is related to some induction theorems of Dress [Dre2], [Dre3]. Its proof ([WaWe]§5) in this context is short and direct, by generalizing the proof of Solomon's induction theorem for permutation characters [CRI](15.10).

Theorem C has applications in the study of permutation lattices [GuWe]. It is known there that for a group G, which is not p-hypoelementary for any p, there exist two non-isomorphic G-set X, Y such that the corresponding permutation lattices are isomorphic. The G-sets X, Y in [GuWe] depend on the order of the class group $\mathrm{Cl}(AG)$. The following proposition provides a constructive way for finding such G-sets without any obstruction when G is non-pseudo-elementary. Two permutation summands L, L_1 for G over A are called *stably isomorphic* if $L \oplus A[X] \simeq L_1 \oplus A[X]$ as AG-modules for some G-set X, which is equivalent to $[L] = [L_1]$ in the Grothendieck group $\Omega_A(G)$ [CRII](38.20).

Proposition.

(a) *Two permutation* AG*-lattices* $A[X]$, $A[Y]$ *of* G*-sets* X, Y *are stably isomorphic if*

$$(\ast) \qquad card X^E = card Y^E$$

for all pseudo-elementary subgroups E *of* G;

(b) *Let G be any group which is not pseudo-elementary. Then we can construct two non-isomorphic G-sets X, Y such that $A[X] \simeq A[Y]$ as AG-modules.*

Proof. (a) Since subgroups of E are still pseudo-elementary, conditions $(*)$ imply that the E-sets X, Y (restriction of G to E) are isomorphic by the injectivity of the Burnside map $\Phi : \Omega(E) \to \mho(E)$ for each pseudo-elementary subgroup E of G. It follows that the restrictions $\mathrm{res}_E A[X] \simeq \mathrm{res}_E A[Y]$, as AE-modules, which clearly implies that $[\mathrm{res}_E A[X]] = [\mathrm{res}_E A[Y]]$ in $\Omega_A(E)$. Now Theorem C ensures that the trivial module 1_G^G is expressible as an integral linear combination of induced modules in $\Omega_A(G)$ as $[1_G^G] = \sum_E n_E \mathrm{ind}_E^G[L_E]$ for integers n_E and permutation summands L_E for E over A. Multiplying the above equation by $[A[X]]$ first and applying Frobenius reciprocity, we get $[A[X]] = \sum_E n_E[A[X]] \cdot \mathrm{ind}_E^G[L_E] = \sum_E n_E \mathrm{ind}_E^G[\mathrm{res}_E A[X] \cdot L_E] = \sum_E n_E \mathrm{ind}_E^G[\mathrm{res}_E A[Y] \cdot L_E] = \sum_E n_E[A[Y]] \cdot \mathrm{ind}_E^G[L_E] = [A[Y]]$ as required.

(b) Choose $x \in \Omega(G)$, the Burnside ring, so that $\Phi_x(E) = 0$ for all pseudo-elementary subgroups E of G but $\Phi_x(G) \neq 0$. Rewrite $[x] = [X_1] - [Y_1]$ in $\Omega(G)$ for some G-sets X_1, Y_1, and enlarge X_1 and Y_1 by taking the disjoint unions with a common G-set S (if necessary) so that

(i). Every subgroup of G is the stabilizer of some point of $X_1 \dot\cup S$;

(ii). $X_1 \dot\cup S$ contains two copies of the regular G-set (the one with the trivial point stabilizer).

Letting $X = X_1 \dot\cup S$ and $Y = Y_1 \dot\cup S$, we have $x = [X] - [Y]$ in $\Omega(G)$.

Now $0 = \Phi_x(E) = \mathrm{card} X^E - \mathrm{card} Y^E$ for all pseudo-elementary subgroups E, but $0 \neq \Phi_x(G) = \mathrm{card} X^G - \mathrm{card} Y^G$ It follows that the G-sets X, Y satisfy the conditions $(*)$ of (a) and are not isomorphic G-sets. Thus, by (a), $A[X] \oplus A[Z] \simeq A[X] \oplus A[Z]$ as AG-modules for some G-set Z. Now the permutation lattice $A[Z]$ must be a direct summand of $A[X]^{(k)}$ for some k since X satisfies condition (i). And $A[X]$ is an Eichler lattice by condition (ii), for $KG^{(2)}$ is a direct summand of $K[X]$ as KG-modules and thus the endomorphism algebra $\mathrm{End}_{KG}K[X]$ is Eichler/A ([CRII]§51A). It follows from Jacobinski's Cancellation theorem [CRII](51.28) that $A[X] \simeq A[Y]$ as required.

Chapter 3 of this thesis is devoted to the study of characters of permutation summands in the sense of Grothendieck groups. Let $R_K(G)$ denote the group generated by the characters of the representations of G over K. This is a subring of the ring $A^{\mathrm{cl}(G)}$ of functions from the set $\mathrm{cl}(G)$, of conjugacy classes of G, to A. Letting φ_L denote the character of the KG-module $K \otimes_A L$, then $L \mapsto \varphi_L$ defines a ring homomorphism

$$\varphi : \Omega_A(G) \to R_K(G)$$

Combining with (1.4), we have the following commutative diagram.

(Diagram I1.3)

$$\begin{array}{ccc} \Omega_A(G) & \xrightarrow{\Phi} & \mho_A(G) \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ R_K(G) & \longrightarrow & A^{\mathrm{cl}(G)} \end{array}$$

The right vertical map takes $f \in \mho_A(G)$ to $\tilde{f} \in A^{\mathrm{cl}(G)}$, defined by $\tilde{f}(g) = f(\langle g \rangle)$ for $g \in G$. From this diagram, we can prove (1.4 (a)) that the characters φ_L always take rational values, no matter how large the field K is. Define $\overline{R}_{\mathbf{Q}}(G)$ to be the subring of $A^{\mathrm{cl}(G)}$ generated by all rational valued characters of G. Then the cokernel of $\varphi : \Omega_A(G) \mapsto \overline{R}_{\mathbf{Q}}(G)$ is annihilated by the group order $|G|$: Artin's Induction theorem gives

$$|G| \cdot 1_G = \sum_C n_c \operatorname{ind}_C^G 1 \quad \text{for } n_c \in \mathbf{Z}$$

with C varying over the cyclic subgroups of G. For each $\chi \in \overline{R}_{\mathbf{Q}}(G)$, we have $|G| \cdot \chi = \sum_C n_C \operatorname{ind}_C^G(\operatorname{res}_C \chi)$ by Frobenius Reciprocity. Now each $\operatorname{res}_C \chi$ is a rational valued character of the cyclic group C , hence is a $\mathbf{Z}$-linear combination of permutation characters by [CRII](76.6). Therefore $\operatorname{res}_C \chi$ is in the image of $\varphi : \Omega_A(C) \mapsto \overline{R}_{\mathbf{Q}}(C)$. The transitivity of the induction then implies that $|G| \cdot \chi$ is in the image of $\varphi : \Omega_A(G) \mapsto \overline{R}_{\mathbf{Q}}(G)$ as required.

The image of φ, in the case $A = \mathbf{Z}$, has been studied for instance in [Ben2], in the analogy that permutation lattices correspond to free modules and projectives to permutation summands. It is no easy matter to describe the image in this case because of Schur index problems and very little is known about it [Ben2].

However if we take the ground field K (with the integer ring A) big enough, this is an accessible question, to which the chapter 3 is devoted. This is carried out through the study of characters of permutation summands over local rings.

Let $\mathfrak{K}$ be a finite extension field of $\mathbf{Q}_p$ containing $|G|_{p'}$th roots of unity, and let $\mathfrak{o}$ be the integral closure of the p-adic integers $\mathbf{Z}_p$ in $\mathfrak{K}$. Let $\Omega_{\mathfrak{o}}(G)$ be the Grothendieck group of permutation summands for G over the local ring $\mathfrak{o}$, and $R_{\mathfrak{K}}(G)$ the group generated by the characters of representations of G over $\mathfrak{K}$. Mapping each lattice to its character, we obtain a ring homomorphism as in the global situation above:

$$\varphi \cdot \Omega_o(G) \mapsto R_{\mathfrak{K}}(G)$$

This local image is characterized, in Proposition (3.7), as the subgroup $S_p(G)$ of $R_{\mathfrak{K}}(G)$ generated by induced characters $\mathrm{ind}_H^G \lambda$ of p'-linear characters λ. Here a linear character λ is *p'-linear* if it is a group homomorphism from H to a cyclic p'-subgroup of $\mathfrak{K}^\times$. In the proof in §3.3, we show that $\Omega_o(G)$ has a $\mathbf{Z}$-basis which is related to projective modules and is provided by applying the Green Correspondence to permutation summands. Proposition(3.7) then follows from a characterization in $R_{\mathfrak{K}}(G)$ of virtual characters of projective $\mathfrak{o}G$-modules.

It is clear that $S_p(G)$ is the same for all local p-adic fields $\mathfrak{K}$, whenever $\mathfrak{K}$ contains $|G|_{p'}$th roots of unity. Thus the maximal unramified extension $\mathbf{Q}_p^{nr}$ of the p-adic complete field $\mathbf{Q}_p$ is a common ground for studying the p-local images of φ for all finite groups G. $\mathbf{Q}_p^{nr}$ can be obtained by adjoining to $\mathbf{Q}_p$ all the roots of unity of order prime to p. Letting $R_{\mathbf{Q}_p^{nr}}(G)$ be the group generated by the characters of representations of G over $\mathbf{Q}_p^{nr}$, then $S_p(G)$ is a subgroup of $R_{\mathbf{Q}_p^{nr}}(G)$. Now we have

$$\varphi : \Omega_o(G) \mapsto R_{\mathbf{Q}_p^{nr}}(G).$$

How large are the images $S_p(G)$ in $R_{\mathbf{Q}_p^{nr}}(G)$? Answering this question motivates the following induction theorem on $R_{\mathbf{Q}_p^{nr}}(G)$. The residue field of $\mathbf{Q}_p^{nr}$ is the algebraic closure of the finite field $\mathbf{F}_p$, and the Brauer groups of finite extension fields of $\mathbf{Q}_p^{nr}$ are trivial [Ser2]. The unique faithful absolutely irreducible representation of the quaternion group $Q_8 = \langle a, b : a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ is realizable over $\mathbf{Q}_2^{nr}$ (but not over $\mathbf{Q}$) by

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \omega & \omega^2 \\ \omega^2 & -\omega \end{pmatrix}$$

where ω is a 3rd root of unity so satisfies $\omega^2 + \omega + 1 = 0$.

(Induction) Theorem D. *Every $\mathbf{Q}_p^{nr}$-character of a finite group G is a $\mathbf{Z}$-linear combination of induced characters $\{ind_H^G \phi\}$, where $H \leq G$ and ϕ is one of the following types:*

(i) *ϕ is a p'-linear character,*

(ii) *$p = 2$ and ϕ is the product of a p'-linear character with a $\mathbf{Q}_2^{nr}$-character μ of H such that $H/\ker\mu \simeq Q_8$, and μ is the inflation of the unique faithful irreducible $\mathbf{Q}_2^{nr}$-character of Q_8.*

Theorem D implies $S_p(G) = R_{\mathbf{Q}_p^{nr}}(G)$ for odd prime numbers p, and thus $\varphi : \Omega_{\mathfrak{o}}(G) \mapsto R_{\mathbf{Q}_p^{nr}}(G)$ is surjective for odd p and $\mathfrak{K}$ as above.

The characterization of the global homomorphism $\varphi : \Omega_A(G) \to \overline{R}_{\mathbf{Q}}(G)$ is then obtained, based on the local results, by a technique of gluing (virtual) permutation summands of G over $A_{\mathfrak{p}}$ for all relevant $\mathfrak{p}$ to form a (virtual) permutation summand of G over A. Since the Brauer group of $\mathbf{Q}_p^{nr}$ is trivial, we have $\overline{R}_{\mathbf{Q}}(G) \subset R_{\mathbf{Q}_p^{nr}}(G)$. The main theorem on characters of permutation summands states:

Theorem E. *The image of* $\varphi : \Omega_A(G) \to \overline{R}_{\mathbf{Q}}(G)$ *is always contained in the intersection of* $\overline{R}_{\mathbf{Q}}(G)$ *and* $S_2(G)$ *in* $R_{\mathbf{Q}_2^{nr}}(G)$. *If* K *is big enough this containment is an equality.*

The proof of Theorem E takes place in §3.4 and a direct consequence of it is:

Corollary. *Assume* K *is big enough. Then* $\varphi : \Omega_A(G) \to \overline{R}_{\mathbf{Q}}(G)$ *is surjective whenever* G *has no quaternion section. In general, the cokernel is annihilated by* 2.

This follows from $\overline{R}_{\mathbf{Q}}(G) \subset R_{\mathbf{Q}_2^{nr}}(G) = S_2(G)$, by Theorem D, whenever G has no quaternion section. The annihilator 2 comes from the following observation: if θ is the unique faithful $\mathbf{Q}_2^{nr}$ -irreducible character of Q then it is not virtual permutation, but $2 \cdot \theta$, expressible as $\mathrm{ind}_1^Q 1 - \mathrm{ind}_{C_4}^Q 1$, is a virtual permutation character.

There are further questions which can be asked after Theorem E, namely, given a χ in $R_{\mathbf{Q}_2^{nr}}(G)$, what are the restrictions on χ to have it belong to $S_p(G)$? The first case of this question is that of 2-groups. The last section (§3.5) of this thesis is devoted to this attempt.

The proof of the (Induction) Theorem D is completed in §3.2 by a careful analysis of characters of $(\mathbf{Q}_p^{nr}, q)$-elementary groups, with preparations in §3.1. Theorem D is related to some work in [Fong], where an improvement (Theorem 2) of the corresponding result in [Sol1] states:

Theorem 2 [Fong]. *Let* G *be a finite group of order* $|G| = p^a m$, *where* p *is a rational prime and* $(p, m) = 1$. *Let* $K = \mathbf{Q}(\zeta_m)$ *if* p *is odd, and let*

$K = \mathbf{Q}(\zeta_m, \sqrt[3]{1})$ *if* $p = 2$. *If* χ *is an absolutely irreducible character of* G, *then the Schur index* $m_K(\chi)$ *of* χ *with respect to* K *is* 1.

We can deduce this from Theorem D in the same way as Brauer's Induction Theorem is used to prove that $\mathbf{Q}(\zeta_{|G|})$ is a splitting field for χ [Ser1]: let $\psi = \sum_\sigma \chi^\sigma$, $\sigma \in$ Gal $(K(\chi)/K)$ be the Galois orbit of χ. It suffices to show the representation affording ψ is realizable over K. ψ is K-valued. Since p is unramified in K so K is embeddable into $\mathbf{Q}_p^{nr}$. Thus ψ is a $\mathbf{Q}_p^{nr}$-valued character. Now the Brauer group of $\mathbf{Q}_p^{nr}$ is trivial, so ψ is realizable over $\mathbf{Q}_p^{nr}$. Applying Theorem D to $\psi \in R_{\mathbf{Q}_p^{nr}}(G)$, we can write ψ as an integral linear combination of induced characters $\psi = \sum_i n_i \mathrm{ind}_{H_i}^G \phi_i$. Since p'-linear characters of G and, if $p = 2$, the unique faithful irreducible character θ are realizable over K, all ϕ_i are realizable over K. Thus $\psi \in R_K(G)$, and it follows that ψ is afforded by a K-representation from [Ser1](Prop33).

CHAPTER 1

The Homomorphism $\Phi\colon \Omega_A(G) \to \mho_A(G)$

§1.1 Construction of Φ_L

Given a permutation summand L and a triple $(H, b, \mathfrak{p}')$ for G over A, let $i_{\mathfrak{p}'} : A' \to A'_{\mathfrak{p}'}$ be the inclusion of A' in its completion at $\mathfrak{p}'$ and let $\mathfrak{p} = \mathfrak{p}' \cap A$, Then the completion ring $A_{\mathfrak{p}}$ of A at $\mathfrak{p}$ has the residue characteristic p and $A_{\mathfrak{p}} \otimes L$ is a permutation summand for G over $A_{\mathfrak{p}}$.

We use the notation $M \mid N$ to indicate that $A_{\mathfrak{p}}G$-module M is isomorphic to a direct summand of an $A_{\mathfrak{p}}G$-module N, and denote by M_H the $A_{\mathfrak{p}}H$-module which is the restriction of M to a subgroup H. Let M be an indecomposable $A_{\mathfrak{p}}G$-module. A p-subgroup D of G is called a *vertex* of M if D is a minimal subgroup such that $M \mid \mathrm{ind}_D^G L$ for some $A_{\mathfrak{p}}D$-module L. Here the $A_{\mathfrak{p}}G$-module $\mathrm{ind}_D^G L$ is defined as $A_{\mathfrak{p}}G \otimes_{A_{\mathfrak{p}}D} L$. If D is a vertex of M and L is an indecomposable $A_{\mathfrak{p}}D$-module such that $M \mid \mathrm{ind}_D^G L$, then L is called a *source* of M. The following lemma connects permutation summands to a familiar subject:

(1.1)Lemma. *If M is a permutation summand for G over $A_{\mathfrak{p}}$ then M is a trivial source module. More precisely, if M is indecomposable with vertex D then M is a direct summand of $\mathrm{ind}_D^G(A_{\mathfrak{p}})$.*

Proof. Each $A_{\mathfrak{p}}G$-module is uniquely decomposed into a direct sum of indecomposable modules. Let M be indecomposable with vertex D and source

L. We must show that L is isomorphic to the trivial $A_{\mathfrak{p}}D$-module $A_{\mathfrak{p}}$. Since M is a direct summand of a permutation module and permutation modules are direct sums of transitive ones $\mathrm{ind}_H^G(A_{\mathfrak{p}})$ or $A_{\mathfrak{p}}[G/H]$, it follows that $M \mid \mathrm{ind}_H^G(A_{\mathfrak{p}})$ for a subgroup H of G. $M_D \mid (\mathrm{ind}_H^G(A_{\mathfrak{p}}))_D = \bigoplus_{D\backslash G/H} \mathrm{ind}_{D\cap xHx^{-1}}^D(A_{\mathfrak{p}})$ by Mackey decomposition theorem, and $L \mid M_D$, imply that $L \mid \mathrm{ind}_{D\cap xHx^{-1}}^D(A_{\mathfrak{p}})$ for some coset DxH. Then

$$\mathrm{ind}_D^G L \mid \mathrm{ind}_{D\cap xHx^{-1}}^G(A_{\mathfrak{p}}) \text{ and thus } M \mid \mathrm{ind}_{D\cap xHx^{-1}}^G(A_{\mathfrak{p}}).$$

Now the minimality of D implies $D = D\cap xHx^{-1}$. And $L \mid \mathrm{ind}_{D\cap xHx^{-1}}^D(A_{\mathfrak{p}})$ implies $L \simeq A_{\mathfrak{p}}$ as required.

Trivial source modules have been much studied via the Green correspondence. In particular, it has been known since the work of Conlon [Con1] (see also [Dre2]) that they are distinguished up to isomorphism, in this case, by certain numerical invariants. These invariants were made explicit by Benson and Parker [BePa], [Bens] in the form of "species" $s_{H,b}$. Here H varies over p-hypoelementary subgroups of G, with p the residue field characteristic, and b varies over generators of $H/O_p(H)$.

Our numerical character Φ_L is obtained next by globalization of "species".

Denote $A_{\mathfrak{p}} \otimes_A L$ by M for simplicity in the rest of the section. M is then a trivial source $A_{\mathfrak{p}}G$-module by the above lemma. We decompose the restriction M_H of M to H as

$$M_H \simeq M' \oplus M'',$$

where every indecomposable $A_{\mathfrak{p}}H$-summand of M' has vertex $O_p(H)$, and every indecomposable $A_{\mathfrak{p}}H$-summand of M'' has vertex properly contained in $O_p(H)$, by the Krull-Schmidt decomposition theorem and the fact that $O_p(H)$ is the normal p-Sylow subgroup of the p-hypoelementary subgroup H. Since $O_p(H)$ is a normal subgroup in H, it acts trivially on $\mathrm{ind}_{O_p(H)}^G(A_\mathfrak{p})$, hence trivially on M' by (1.1). M' is then an $A_\mathfrak{p}[H/O_p(H)]$-module, so b, a generator of $H/O_p(H)$, acts $A_\mathfrak{p}$-linearly on the free $A_\mathfrak{p}$-module M'. We define

$$s_{H,b}(M) = \text{ trace of } b \text{ acting on } M'.$$

Now $s_{H,b}(M)$ clearly takes values in $A_\mathfrak{p}$, which differs from the formulation in [Bens], in that there $s_{H,b}(M)$ was claimed to take values in the complex field.

Furthermore, since $|b|$, the order of b, is invertible in $A'_{\mathfrak{p}'}$ and $A'_{\mathfrak{p}'}$ contains $|b|$th roots of unity, the action of b on M' is diagonalizable over $A'_{\mathfrak{p}'}$. Thus the $A'_{\mathfrak{p}'}\langle b\rangle$-module $A'_{\mathfrak{p}'} \otimes_{A_\mathfrak{p}} M'$ has an $A'_{\mathfrak{p}'}$-basis $x_1, ..., x_r$ so that $bx_i = \lambda_i x_i$ for suitable $|b|$th roots of unity λ_i in $A'_{\mathfrak{p}'}$. Then $\lambda_i = i_{\mathfrak{p}'}(\zeta_i)$ for a unique $|b|$th root of unity ζ_i in Z'. Setting

$$\Phi_L(H, b, \mathfrak{p}') = \sum_i \zeta_i$$

completes the construction.

(1.2)Proposition. $i_{\mathfrak{p}'}\Phi_L(H, b, \mathfrak{p}') = s_{H,b}(A_\mathfrak{p} \otimes_A L)$.

Proof. $i_{\mathfrak{p}'}\Phi_L(H, b, \mathfrak{p}') = i_{\mathfrak{p}'}(\sum_i \zeta_i) = \sum_i i_{\mathfrak{p}'}(\zeta_i) = \sum_i \lambda_i = s_{H,b}(M)$. The last equality follows from the above definition of $s_{H,b}(M)$ because λ_i for $i = 1, ..., r$ are just all the eigenvalues of b in $A'_{\mathfrak{p}'}$.

The values of Φ_L in Z' are not at all arbitrary, They satisfy the following Galois-theoretic properties, which are proved in [WaWe] §2.

(1.3)Lemma. *Notation as above, we have*

a) $\Phi_L(H^g, b^g, \mathfrak{p}') = \Phi_L(H, b, \mathfrak{p}')$ *for* $g \in G$

b) $\Phi_L(H, b, \mathfrak{p}')^{\sigma'} = \Phi_L(H, b, \mathfrak{p}'^{\sigma'})$ *for* $\sigma' \in \Gamma_K$

c) $\Phi_L(H, b, \mathfrak{p}')^{\sigma} = \Phi_L(H, b^{j_H(\sigma)}, \mathfrak{p}')$ *for* $\sigma \in \Gamma_{\mathbf{Q}}$, *where* $j_H : \Gamma_{\mathbf{Q}} \to (\mathbf{Z}/|b|\mathbf{Z})^{\times}$ *is defined by* $\zeta_{|b|}^{\sigma} = \zeta_{|b|}^{j_H(\sigma)}$ *for all primitive* $|b|^{\text{th}}$ *roots of unity* $\zeta_{|b|}$ *in* $\mathbf{Q}'$.

There is still a further restriction. The next result is the starting one in the study of characters of permutation summands.

(1.4)Lemma. *Notation as above, let* φ_L *denote the character of* $K \otimes_A L$. *Let* H *be a cyclic subgroup of* G. *Then*

a) $\varphi_L(h)$ *is same for all generators* h *of* H. *We denote this common value, which is in* $\mathbf{Z}$, *by* $\varphi_L(H)$.

b) $\Phi_L(H, b, \mathfrak{p}')$ *is independent of* b *and* $\mathfrak{p}'$. *We denote this common value by* $\Phi_L(H)$. *We have* $\Phi_L(H) = \varphi_L(H)$.

§1.2 Local Results

The references for this section are [Bens], [CRI](§20) and [CRII](§81B). We first recall the Green Correspondence and then apply it to prove a theorem of Conlon on permutation summands for G over the local ring $A_{\mathfrak{p}}$. Denote by $\mathfrak{o}$ the completion ring $A_{\mathfrak{p}}$ at $\mathfrak{p}$ above the rational prime p for simplicity in this section. Fix a p-subgroup D of the group G and a subgroup H so $N_G(D) \subseteq H \subseteq G$. Set

$$\mathcal{X} = \{X \leq G : X \leq D \cap D^g \text{ for some } g \in G - H\}$$

$$\mathcal{Y} = \{Y \leq G : Y \leq H \cap D^g \text{ for some } g \in G - H\}$$

and observe

(i) $\mathcal{X} \subseteq \mathcal{Y}$, $D \notin \mathcal{Y}$

(ii) $Y \in \mathcal{Y} \Longrightarrow D \cap Y \in$ '.

Write $O(\mathcal{X})$ for the class of $\mathfrak{o}G$-modules all of whose indecomposable summands have vertices in $\mathcal{X}$, and similarly $O(\mathcal{Y})$ for the class of $\mathfrak{o}H$-modules.

Theorem(Green Correspondence). *There exist bijections f, g inverse to each other, between $\simeq$ classes of indecomposable $\mathfrak{o}G$-modules with vertex D and $\simeq$ classes of indecomposable $\mathfrak{o}H$-modules with vertex D , characterized by*

i) $V_H \simeq f(V) \oplus E$ *with* $E \in O(\mathcal{Y})$,

ii) $ind_H^G W \simeq g(W) \oplus E$ *with* $E \in O(\mathcal{Y})$.

Applying the theorem to permutation summands for G over $\mathfrak{o}$ in the special case $H = N_G(D)$, we have

(1.5)Theorem. *The correspondence $M \mapsto f(M)$ induces a bijection between the isomorphism classes of indecomposable permutation summands for G over $\mathfrak{o}$ with vertex D and the isomorphism classes of indecomposable projective $\mathfrak{o}[N_G(D)/D]$-modules.*

Proof. Let $\overline{N}_G(D) = N_G(D)/D$ for short in this proof. Restriction and induction take permutation (resp. permutation summand) modules to permutation (resp. permutation summand) modules. And a direct summand of permutation summand is permutation summand. It follows from the Green relations i),ii) above that the bijections f, g take permutation summands to permutation summands.

Now we can identify the isomorphism classes of indecomposable permutation summands for $N_G(D)$ over $\mathfrak{o}$ with vertex D with the isomorphism classes of indecomposable projective $\mathfrak{o}[\overline{N}_G(D)]$-modules as follows: if an indecomposable permutation summand M for $N_G(D)$ over $\mathfrak{o}$ has vertex D, then $M \mid \mathrm{ind}_D^{N_G(D)}\mathfrak{o}$ by Lemma(1.1). $D \lhd N_G(D)$ acts trivially on $\mathrm{ind}_D^{N_G(D)}\mathfrak{o}$ hence trivially on M. M then can be considered as an $\mathfrak{o}[\overline{N}_G(D)]$-module when $M \mid \mathrm{ind}_1^{\overline{N}_G(D)}\mathfrak{o}$ is projective. And vice versa, an indecomposable projective $\mathfrak{o}[\overline{N}_G(D)]$-module, through the inflation $N_G(D) \to \overline{N}_G(D)$, gives an $\mathfrak{o}[N_G(D)]$-module which is an indecomposable summand of $\mathrm{ind}_D^{N_G(D)}\mathfrak{o}$ and thus must have vertex D by [CRII](81.15)(iii). The bijection follows from the Green Correspondence and the above identification.

Let H be a p-hypoelementary subgroup of G with $O_p(H) = P$, hence $H \leq N_G(P)$. Generators b of H/P are p'-elements in $N_G(P)/P$. The "species" $s_{H,b}$ on permutation summands are a generalization of (Brauer) characters on projective modules of G over $\mathfrak{o}$ (modular fields) as the next result show.

(1.6)Lemma. *Let M be an indecomposable permutation summand for G over $\mathfrak{o}$ with vertex D. Decompose the restriction of M to $N_G(P)$ as*

$$M_{N_G(P)} \simeq M' \oplus M''$$

where every indecomposable summand of M' has vertex containing P, and every one of M'' has vertex not containing P.

(1) *If $D = P$ up to conjugacy in G, then M' is the Green correspondent $f(M)$ and $s_{H,b}$ equals the value at b of the character $\varphi_{M'}$ of the projective $\mathfrak{o}[\overline{N}_P(G)]$-module M'.*

(2) *If $D \not\supseteq P$ up to conjugacy in G, then $M' = 0$.*

Proof. (1) It is clear that $M' = f(M)$ by the Green Correspondence. The indecomposable summand of M_H with vertex P coincides with the indecomposable summand of $(M')_H$. For vertices of indecomposable summands can only drop, up to $N_G(P)$-conjugacy, on restricting to H [CRI](19.14). And since P acts trivially on M' by Lemma(1.1) and $P \lhd N_G(P)$, the vertices of $(M')_H$ are all P, also because $(H:P) \not\equiv 0 \bmod p$.

It follows from §1.1 that $s_{H,b} =$ the trace of b acting on $M' = \varphi_{M'}(b)$ as required.

(2) $M \mid \mathrm{ind}_D^G \mathfrak{o}$ by (1.1) so every indecomposable summand of $M_{N_G(P)}$ is a summand of $(\mathrm{ind}_D^G \mathfrak{o})_{N_G(P)}$. Since $(\mathrm{ind}_D^G \mathfrak{o})_{N_G(P)} \simeq \oplus_{D\backslash G/N_G(P)} \mathrm{ind}_{N_G(P)\cap D^g}^{N_G(P)} \mathfrak{o}$, the vertices of whose summands are contained in $N_G(P) \cap D^g$ and thus do not contain P, we have $M' = 0$ as required.

Recall that a projective $\mathfrak{o}[G]$-module Q is determined by the character φ_Q of $K_\mathfrak{p} \otimes_\mathfrak{o} Q$ and this character vanishes on p-singular elements (Swan's theorem in [Ser1]). The next is the main theorem of this section due to Conlon. The proof we adapt is a modification of [Bens].

(1.7) Theorem. *Let M, N be permutation summands for G over $\mathfrak{o}$. Assume that $s_{H,b}(M) = s_{H,b}(N)$ whenever H is a p-hypoelementary subgroup of G and b is a generator of $H/O_p(H)$. Then M, N are isomorphic $\mathfrak{o}G$-modules.*

Proof. Since $\mathfrak{o}$ is a principal ideal domain, permutation summands for G over $\mathfrak{o}$ are free $\mathfrak{o}$-modules. Proof by contradiction. Suppose M, N are counter examples of minimal $\mathfrak{o}$-rank to the Theorem. Then there cannot exist an indecomposable $\mathfrak{o}G$-module which is a common direct summand of M, N. Write $M = \oplus_i M_i$ and $N = \oplus_j N_j$ as sums of indecomposables. Let P be maximal in the set of vertice of $\{M_i, N_j\}$. If $P = 1$, then M, N are both projective. Then equality of the character values

$$\varphi_M(c) = s_{\langle c\rangle,c}(M) = s_{\langle c\rangle,c}(N) = \varphi_N(c)$$

on all p'-elements c in G, by (1.6) (1), implies that $M \simeq N$. So to avoid contradiction we must have $P \neq 1$, and to fix the notation, suppose that M_1 has vertex P. Decompose

$$(M_i)_{N_G(P)} \simeq M_i' \oplus M_i'', \quad (N_i)_{N_G(P)} \simeq N_i' \oplus N_i''$$

as in the above lemma. Then M_i', N_j' are either 0 or projective $\mathfrak{o}[\overline{N}_G(P)]$ - modules, by (1.6), because of the maximality of P. Denote $M' = \oplus_i M_i'$, $M'' = \oplus_i M_i''$ and similarly for N', N''. Then M', N' are both projective $\mathfrak{o}[\overline{N}_G(P)]$-modules and M' is nonzero, because it has a nonzero summand M_1' which is the Green Correspondent of M_1 by (1.6)(i).

For each p'-element b in $\overline{N}_G(P)$, let H be the preimage of $\langle b\rangle$ under $N_G(P) \to N_G(P)/P$. Then H is p- hypoelementary and $H/O_p(H)$ $= \langle b\rangle$. It follows from (1.6) that $s_{H,b}(M) = \varphi_{M'}(b)$ and $s_{H,b}(N) = \varphi_{N'}(b)$. The assumption on M, N implies that $\varphi_{M'}(b) = \varphi_{N'}(b)$ on all p'-elements of $\overline{N}_G(P)$. Swan's theorem quoted above implies that projective $\mathfrak{o}\overline{N}_G(P)$ - modules M', N' are isomorphic. Thus N' has an indecomposable summand, say N_1', isomorphic to M_1'. It follows that the Green Correspondents $g(M_1') = M_1, g(N_1') = N_1$ are isomorphic. We find that M, N do share a common indecomposable summand, contradiction.

Two AG-lattices L, L_1 are in same *genus* if $A_\mathfrak{p} \otimes_A L \simeq A_\mathfrak{p} \otimes_A L_1$ as $A_\mathfrak{p}G$-modules for each maximal ideal $\mathfrak{p}$ of A.

(1.8)Corollary. *If* L, L_1 *are permutation summands for* G *over* A *such that* $\Phi_L = \Phi_{L_1}$ *on all triples, then* L, L_1 *are in the same genus.*

proof. At each maximal ideal $\mathfrak{p}$ of A, we have $s_{H,b}(A_\mathfrak{p} \otimes_A L) = s_{H,b}(A_\mathfrak{p} \otimes_A L_1)$ by (1.2), for all p-hypoelementary subgroups H and all generators b of $H/O_p(H)$. $A_\mathfrak{p} \otimes_A L \simeq A_\mathfrak{p} \otimes_A L_1$ as $A_\mathfrak{p}G$-modules then follows from Theorem (1.7) for each $\mathfrak{p}$, hence L, L_1 are in the same genus.

We conclude this section with an induction formula for calculating Φ_L of an induced lattice L. This character-theoretic formula is used in proving the technical part of Theorem A. Its proof can be derived directly by (1.2) from the corresponding formula on "species" (see [WaWe](1.5),(1.6))

(1.9)Proposition. *Notations as in §1.1, we have*

(a) *If* $L = ind_{G'}^{G} L_1$ *is an induced lattice, where* G' *is a subgroup of* G *and* L_1 *is a permutation summand for* G' *over* A, *we have the induction formula* $\Phi_L(H, b, \mathfrak{p}') = \sum_{xG' \in G/G' \text{ so } H^x \subseteq G'} \Phi_{L_1}(H^x, b^x, \mathfrak{p}')$.

(b) *If* $L = A[X]$ *is a permutation lattice for some finite* G*-set* X, *then* $\Phi_L(H, b, \mathfrak{p}') =$ *card* X^H, *which is* $\Phi_X(H)$.

§1.3 Construction of $\Phi : \Omega_A(G) \to \mho_A(G)$

Proofs of the results in this section are in [WaWe](§3). Define $G_K = (G \times \Gamma_\mathbf{Q} \times \Gamma_K)$ and make G_K act on triples (of G over A) by

$$(H, b, \mathfrak{p}')^{(g,\sigma,\sigma')} = (H^g, (b^g)^{j_H(\sigma)}, (\mathfrak{p}')^{\sigma'})$$

with $j_H : \Gamma \to (\mathbf{Z}/|b|\mathbf{Z})^\times$ given by (1.3)c). To take account of (1.4), we define an equivalence relation on triples by taking $(H_1, b_1, \mathfrak{p}'_1) \sim (H, b, \mathfrak{p}')$ if and only if $H = H_1$ is cyclic. Equivalence classes of triples are called *Triples* and still form a G_K-set.

Now letting G_K act on Z' by

$$x^{(g,\sigma,\sigma')} = x^{\sigma\sigma'},$$

we define $\mho_A(G)$ to be the ring, under pointwise operations, of all G_K-maps from the set of Triples (of G over A) to Z'. Φ_L, for a permutation summand L, is in $\mho_A(G)$ by (1.3),(1.4). Therefore $L \mapsto \Phi_L$ defines a map $\Phi : \Omega_A(G) \to \mho_A(G)$.

(1.10)Proposition. $\Phi : \Omega_A(G) \to \mho_A(G)$ *by* $L \mapsto \Phi_L$ *is a ring homomorphism.*

To understand the ring $\Omega_A(G)$ through the homomorphism Φ, we need to know the structure of $\mho_A(G)$. This is guided by the following general mechanism: suppose Π is a finite group, X is a finite Π-set, B is a commutative ring on which Π acts. Let Π_x be the stabilizer of $x \in X$ and B^{Π_x} be the subring of elements in B fixed under Π_x. Then the ring $\mathrm{Hom}_\Pi(X, B)$, of all Π-maps from X to B, is isomorphic to the product $\prod_x B^{\Pi_x}$ with x ranging over a set of Π-orbit representatives of X, by sending $f \in \mathrm{Hom}_\Pi(X, B)$ to $(..., f(x), ...)$.

A triple $(H, b, \mathfrak{p}')$ is called a p *-triple* if H is p-hypoelementary but not cyclic, and is called a *cyclic triple* if H is cyclic; note that this use of p is unambiguous for if a group H is hypoelementary for different primes p, q then it is cyclic: this is because H is abelian with all Sylow subgroups cyclic, as follows from the embedding of H into (cyclic $\times$cyclic) $H/O_p(H)\times H/O_q(H)$. The same distinction applies to Triples, and we denote the cyclic Triple associated to the cyclic triple $(H, b, \mathfrak{p}')$ simply by (H).

Clarifying G_K-orbits of Triples takes some formalism. By a *pair* of G over A, we mean $(H, \mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of A and H is a p-hypoelementary subgroup of G with p the unique prime number in $\mathfrak{p}$. A pair $(H, \mathfrak{p})$ is a *p-pair* if H is p-hypoelementary but not cyclic, and is a *cyclic pair* if H is cyclic. Again we define $(H_1, \mathfrak{p}_1) \sim (H, \mathfrak{p})$ if and only if $H_1 = H$ is cyclic, and call an equivalence class of pairs a *Pair*. We let G_K act on pairs by

$$(H, \mathfrak{p})^{(g,\sigma,\sigma')} = (H^g, \mathfrak{p})$$

and consequently on Pairs. The Pairs above contain exact information to name orbits of Triples.

Let $(H, \mathfrak{p})$ be a p-pair, and choose a generator b of $H/O_p(H)$. Define $\tau_H : N_G(H) \to (\mathbf{Z}/|b|\mathbf{Z})^\times$ by $b^g = b^{\tau_H(g)}$, and set $N(\mathfrak{p}) = \text{card}A/\mathfrak{p}$, the absolute norm of $\mathfrak{p}$. Define $\Gamma_{(H,\mathfrak{p})}$ to be the preimage, under j_H : $\Gamma \to (\mathbf{Z}/|b|\mathbf{Z})^\times$, of the subgroup $\langle \text{im}(\tau_H), N(\mathfrak{p})\rangle$ of $(\mathbf{Z}/|b|\mathbf{Z})^\times$ generated by $\text{im}(\tau_H)$ and $N(\mathfrak{p})$. Finally let $A_{(H,\mathfrak{p})}$ be the subring of Z' which is fixed by $\Gamma_{(H,\mathfrak{p})}$. Note that the above construction is independent of the choice of b, since $|b| = (H : O_p(H))$.

(1.11)Proposition.

(a) $(H, b, \mathfrak{p}') \mapsto (H, \mathfrak{p})$, *with* $\mathfrak{p} = \mathfrak{p}' \cap A$, *defines a* G_K*-map from triples to pairs of* G *over* A, *called the type map. The preimage of a* G_K*-orbit of pairs, under the type map, is a single* G_K*-orbit of triples. Similarly for Type.*

(b) *There is a ring isomorphism*

$$\mho_A(G) \simeq \prod_{(H)} \boldsymbol{Z} \times \prod_{(H,\mathfrak{p})} A_{(H,\mathfrak{p})},$$

with H *ranging, in the first product, over a set of* G_K*-orbit representatives of cyclic Pairs, and* $(H,\mathfrak{p})$ *ranging, in the second product, over a set of* G_K*-orbit representatives of* p*-Pairs, for the possible primes* p *(necessarily dividing* $|G|$*).*

With the above result on the structure of $\mho_A(G)$, we obtain results on the ring structure of $\Omega_A(G)$ as a consequence of Theorem A. We know that $\ker\Phi$ is a finite ideal of $\Omega_A(G)$ and that $\Omega_A(G)/\ker\Phi$ has finite index in the ring $\mho_A(G)$. In particular, $\mho_A(G)$ has no nilpotent element from (1.11) so $\ker\Phi$ is the nil radical of $\Omega_A(G)$ by the next result.

(1.12) Proposition. *Let* $\varphi : \Omega_A(G) \to R_K(G)$ *be the ring homomorphism in Diagram(I1.3), obtained by letting* φ_L = *character of* $K \otimes_A L$*. Then in* $\Omega_A(G)$ *we have the ideal equation* $(\ker\varphi)(\ker\Phi) = 0$*. In particular,* $(\ker\Phi)^2 = 0$.

Proof. $\ker\Phi \subseteq \ker\varphi$ by Diagram (I1.3) in the introduction, since $R_K(G) \to A^{\mathrm{cl}(G)}$ is inclusion, so we only need to show the first assertion.

This follows from $\mathrm{Cl}(AG)$ being a Frobenius module over the Frobenius functor $R_K(G)$, [CRII](49.47). In particular, $[V] \in R_K(G)$ acts by

$$[V] \cdot ([P] - [Q]) = [L \otimes_A P] - [L \otimes_A Q],$$

where L is any AG -lattice on V. Tensoring with K gives a ring homomorphism $\Omega_A(G) \to R_K(G)$ through which $\mathrm{Cl}(AG)$ becomes a $\Omega_A(G)$ -module, and the map $\mathrm{Cl}(AG) \to \ker\Phi$ is an $\Omega_A(G)$-module homomorphism.

Since $\ker\varphi$ annihilates $\mathrm{Cl}(AG)$, by definition, it also annihilates its image $\ker\Phi$.

CHAPTER 2
Congruences

§2.1 Proof of Theorem B

Notations are the same as those used in §1.1. L is a permutation summand for G over A. Let $i_{\mathfrak{p}'} : A' \to A'_{\mathfrak{p}'}$ be the inclusion of A' in its completion at $\mathfrak{p}'$. If $\mathfrak{p} = \mathfrak{p}' \cap A$, then the completion ring $A_{\mathfrak{p}}$ of A at $\mathfrak{p}$ has the residue characteristic p, and $A_{\mathfrak{p}} \otimes_A L$, denoted by M for simplicity in this section, is an $A_{\mathfrak{p}}G$-module with trivial sources by Lemma(1.1). $(H, b, \mathfrak{p}')$ is a p-triple of G over A. H is a hypoelementary subgroup of G and so are any subgroups of H. $O_p(H)$, denoted by P in this section, is the largest normal p-subgroup of H; if q is a prime number, $O^q(H)$ is defined as the minimal normal subgroup of H so that $H/O^q(H)$ is a q-group. We obtain a p-triple $(O^q(H), b_{q'}, \mathfrak{p}')$ from the p-triple $(H, b, \mathfrak{p}')$ with $b_{q'}$ denoting the q'-part of the element b. Let $\mathfrak{q}'$ be a prime ideal of $\boldsymbol{Z}'$ containing the prime number q.

The proof of the congruence

$$\Phi_L(H, b, \mathfrak{p}') \equiv \Phi_L(O^q(H), b_{q'}, \mathfrak{p}) \bmod \mathfrak{q}'$$

of Theorem B is based on the construction of Φ_L in §1.1 and it is divided into two cases according to whether the prime numbers p, q are equal or not.

Case 1: $p \neq q$ $\quad O_p(O^q(H))$ is clearly equal to $O_p(H) = P$ in this case. Decompose the restriction M_H of M to H as

$$M_H \simeq M' \oplus M'',$$

where every indecomposable $A_{\mathfrak{p}}H$-summand of M' has vertex P, and every indecomposable $A_{\mathfrak{p}}H$-summand of M'' has vertex properly contained in P. Then it follows from the construction of Φ_L in §1.1 that

$$i_{\mathfrak{p}'}\Phi_L(H, b, \mathfrak{p}') = s_{H,b}(M) = \text{ trace of } b \text{ acting on } M'.$$

On the other hand, by the transitivity of the restriction, the restriction of M to $O^q(H)$ has the decomposition

$$M_{O^q(H)} \simeq M'_{O^q(H)} \oplus M''_{O^q(H)}$$

from the above decomposition of M_H. Since every indecomposable $A_{\mathfrak{p}}H$ -summand of M' is an $A_{\mathfrak{p}}H$-summand of $\mathrm{ind}_P^H(A_{\mathfrak{p}})$ by (1.1), every indecomposable $A_{\mathfrak{p}}O^q(H)$-summand of the restriction $M'_{O^q(H)}$ is an $A_{\mathfrak{p}}O^q(H)$ -summand of the restriction $(\mathrm{ind}_P^H(A_{\mathfrak{p}}))_{O^q(H)} \simeq \oplus_{P\backslash H/O^q(H)}\mathrm{ind}_P^{O^q(H)}(A_{\mathfrak{p}})$ from Mackey decomposition, hence has vertex P; since the vertex can only drop after the restriction, every indecomposable $A_{\mathfrak{p}}O^q(H)$-summand of the restriction $M''_{O^q(H)}$ has vertex properly contained in P. It follows from the construction of Φ_L in §1.1 applied to the triple $(O^q(H), b_{q'}, \mathfrak{p}')$ that

$$i_{\mathfrak{p}'}\Phi_L(O^q(H), b_{q'}, \mathfrak{p}') = s_{O^q(H), b_{q'}}(M) = \text{ trace of } b_{q'} \text{ acting on } M'.$$

for any generator $b_{q'}$ of $O^q(H)/P$.

Since $b_{q'}$ is the q'-component of the element b, $b^Q = b_{q'}^Q$ for some integer Q which is a power of q. If the action of b on M' has eigenvalues $\lambda_1, ..., \lambda_r$ in $A'_{\mathfrak{p}'}$, then $b^Q = b_{q'}^Q$ on M' has eigenvalues $\lambda_1^Q, ..., \lambda_r^Q$. Letting ξ_i be the preimage of λ_i under $i_{\mathfrak{p}'} : A' \to A'_{\mathfrak{p}'}$, then $\Phi_L(H, b, \mathfrak{p}'_) = \sum_i \xi_i$ and $\Phi_L(O^q(H), b_{q'}^Q, \mathfrak{p}') = \sum_i \xi_i^Q$ from the definitions in §1.1. Therefore

$$\Phi_L(H, b, \mathfrak{p}')^Q = (\sum_i \xi_i)^Q \equiv \sum_i \xi_i^Q = \Phi_L(O^q(H), b_{q'}^Q, \mathfrak{p}') \text{ mod } \mathfrak{q}'$$

where the middle congruence above follows from the residue field $Z'/\mathfrak{q}'$ having characteristic q.

Since $b_{q'}$, $b_{q'}^Q$ both are generators of $O^q(H)/P$, the same argument as above by using eigenvalues of the actions of $b_{q'}$, $b_{q'}^Q$ on M' will give the congruence

$$\Phi_L(O^q(H), b_{q'}, \mathfrak{p}')^Q \equiv \Phi_L(O^q(H), b_{q'}^Q, \mathfrak{p}') \text{ mod } \mathfrak{q}'.$$

Combining the above congruences, we obtain

$$\Phi_L(H, b, \mathfrak{p}')^Q \equiv \Phi_L(O^q(H), b_{q'}, \mathfrak{p}')^Q \text{ mod } \mathfrak{q}'.$$

Since the field $Z'/\mathfrak{q}'$ has characteristic q and Q is a power of q, we have the following required congruence

$$\Phi_L(H, b, \mathfrak{p}') \equiv \Phi_L(O^q(H), b_{q'}, \mathfrak{p}') \text{ mod } \mathfrak{q}',$$

which proves the Theorem B in the case $p \neq q$.

To complete the proof of Theorem B, we need a structure theorem on permutation summands for nilpotent groups over the local ring $A_{\mathfrak{p}}$ from [Weis], where permutation summands are called permutation projective.

(2.1)Proposition. *Let* M *be an indecomposable permutation summand for* G *over* $A_\mathfrak{p}$, *and let* G *be a nilpotent group written as* $G = G_pG_{p'}$, *where* G_p, $G_{p'}$ *is the largest normal* p, p'*-subgroup of* G *respectively. Then there exists a subgroup* D *of* G_p *and an irreducible* $A_\mathfrak{p}G_{p'}$*-lattice* N *so that*

$$M \simeq N \otimes_{A_\mathfrak{p}} ind_D^{G_p}(A_\mathfrak{p})$$

Here N, $ind_D^{G_p}(A_\mathfrak{p})$ *are considered as* G*-modules by inflating through* $G \to G_{p'}$, $G \to G_p$ *respectively, and the tensor product is given the diagonal* G*-action.*

Proof. The proofs in [Weis]§2 work for $A_\mathfrak{p}$ in place of $\mathbf{Z}_p$.

Case 2: (proof of Theorem B) $p = q$

We start by analyzing the group structure of $O^p(H)$. Since H/P is a cyclic p'-group, P has a cyclic p'-complement C in H and thus $H = PC$. Denoting $P \cap O^p(H)$ by Q, then Q is a normal p-subgroup of H because $P = O_p(H)$ and $O^p(H)$ are normal in H. The injection of $O^p(H)/Q$ into p'-group $H/P \simeq C$ implies that Q is the maximal normal p-subgroup of $O^p(H)$ and $|O^p(H)| \leq |Q| \cdot |C|$. On the other hand, the homomorphism $C \to H \to H/O^p(H)$ of the p'-group C to the p-group $H/O^p(H)$ implies that $C \subseteq O^p(H)$. It follows from above that $|O^p(H)| = |Q| \cdot |C|$ and therefore $O^p(H) = QC$.

Now P/Q is the normal Sylow p-subgroup of H/Q, and $O^p(H)/Q(\simeq C)$ is a normal cyclic p'-subgroup of H/Q. This implies that $H/Q \simeq P/Q \times O^p(H)/Q$ and, in particular, that H/Q is nilpotent.

We decompose, this time, the restriction of M to H as

$$M_H \simeq M' \oplus M_1'' \oplus M_2''$$

where every indecomposable $A_{\mathfrak{p}}H$-summand of M' has vertex P as before; every indecomposable $A_{\mathfrak{p}}H$-summand of M_1'' and M_2'' has vertex properly contained in P with vertices of summands of M_1'' containing Q while vertices of summands of M_2'' do not contain Q. From the construction in §1.1, we have

$$(*) \qquad i_{\mathfrak{p}'}\Phi_L(H,b,\mathfrak{p}') = s_{H,b}(M) = \text{ trace of } b \text{ acting on } M'.$$

On the other hand, by the transitivity of the restriction, the restriction of M to $O^p(H)$ has the decomposition

$$M_{O^p(H)} \simeq (\, M'_{O^p(H)} \oplus (M_1'')_{O^p(H)}\,) \;\oplus\; (M_2'')_{O^p(H)}.$$

Every indecomposable $A_{\mathfrak{p}}H$-summand of M' is an $A_{\mathfrak{p}}H$-summand of $\mathrm{ind}_P^H(A_{\mathfrak{p}})$ by (1.1). Thus every indecomposable $A_{\mathfrak{p}}O^p(H)$-summand of the restriction $M'_{O^p(H)}$ is an $A_{\mathfrak{p}}O^p(H)$-summand of the restriction $(\mathrm{ind}_P^H(A_{\mathfrak{p}}))_{O^p(H)} \simeq \oplus_{P\backslash H/O^p(H)}\mathrm{ind}_Q^{O^p(H)}(A_{\mathfrak{p}})$ by Mackey decomposition as P is normal in H and $P \cap O^p(H) = Q$, hence has vertex Q. The same argument above will imply that every indecomposable $A_{\mathfrak{p}}O^p(H)$-summand of the restriction $(M_1'')_{O^p(H)}$ has vertex Q because vertices of $A_{\mathfrak{p}}H$-summands of M_1'' contain Q. Since the vertex can only drop after the restriction, every indecomposable $A_{\mathfrak{p}}O^p(H)$ -summand of the restriction $(M_2'')_{O^p(H)}$ has vertex properly contained in Q. It follows from the construction of Φ_L in §1.1 applied to the triple $(O^p(H), b, \mathfrak{p}')$ that

$$(**) \quad i_{\mathfrak{p}'}\Phi_L(O^p(H),b,\mathfrak{p}') = s_{O^q(H),b}(M) = \text{ trace of } b \text{ acting on } (M' \oplus M_1'').$$

After cancellation of the trace of b acting on M' from the right hand sides of equations $(*).(**)$, the proof of Theorem B in this case amounts to establishing the

Claim: The preimage under $i_{\mathfrak{p}}$ of the sum of eigenvalues of the action of b on M_1'' is in $\mathbf{Z}'$ and is $\equiv 0 \mod \mathfrak{q}'$.

Assume $M_1'' \neq 0$ so $Q \subsetneq P$. Since $Q \lhd H$ acts trivially on M_1'' by Lemma(1.1), M_1'' can be considered as $A_{\mathfrak{p}}[H/Q]$-module. Applying Proposition(2.1) to $A_{\mathfrak{p}}[H/Q]$-module M_1'', we have

$$M_1'' \simeq \sum_i N_i \otimes_{A_{\mathfrak{p}}} \mathrm{ind}_{D_i}^{P/Q}(A_{\mathfrak{p}})$$

for some $A_{\mathfrak{p}}\langle b\rangle$ -lattices N_i and some p-subgroups D_i of P/Q. From the proofs in [Weis], these D_i are vertices of indecomposable $A_{\mathfrak{p}}H/Q$ -summands of M_1''. But vertices of $A_{\mathfrak{p}}H$-summands of M_1'' are properly contained in P by the definition of M_1'', so D_i must be properly contained in P/Q. We have $|P/Q : D_i| \equiv 0 \mod p$.

If the action of b on N_i has eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$ in $A'_{\mathfrak{p}'}$, then the action of b on $N_i \otimes_{A_{\mathfrak{p}}} \mathrm{ind}_{D_i}^{P/Q}(A_{\mathfrak{p}})$ has eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$ with $|P/Q : D_i|$ repetitions for each λ_{i_k}. This is because N_i, $\mathrm{ind}_{D_i}^{H/Q}(A_{\mathfrak{p}})$ are considered as H/Q-modules by inflating through $H/Q \to \langle b\rangle$, $H/Q \to P/Q$ respectively, and the tensor product is given the diagonal H/Q-action.

Letting ξ_{i_k} be the preimage in A' of λ_{i_k} under $i_{\mathfrak{p}'} : A' \to A'_{\mathfrak{p}'}$, then the preimage of the trace of b on $N_i \otimes_{A_{\mathfrak{p}}} \mathrm{ind}_{D_i}^{P/Q}(A_{\mathfrak{p}})$ is $|P/Q : D_i|(\xi_{i_1} + \ldots + \xi_{i_r})$. Therefore the preimage under $i_{\mathfrak{p}}$ of the sum of eigenvalues of the action of b on $M_1'' \simeq \sum_i N_i \otimes_{A_{\mathfrak{p}}} \mathrm{ind}_{D_i}^{P/Q}(A_{\mathfrak{p}})$ is

$$\sum_i |P/Q : D_i|(\xi_{i_1} + \ldots + \xi_{i_r}) \equiv 0 \mod p\mathbf{Z}'.$$

This proves the claim and completes the proof of Theorem B.

§2.2 The Spectrum of $\Omega_A(G)$

Recall that if C is a commutative ring, then the *spectrum* of C, denoted $\mathrm{Spec}(C)$, is the set of prime ideals of C made into a topological space by declaring the closed subsets to be $V(S)$ = the set of prime ideals of C containing S for any subset S of C [Bour].

We want to examine the spectrum of the ring $\Omega_A(G)$ and show that $\mathrm{Spec}(\Omega_A(G))$ is connected by applying the congruences of Theorem B.

Let $T_G(A)$ be the set of the Triples for G over A defined in §1.3. The ring $(\boldsymbol{Z}')^{T_G(A)}$ (copies of $\boldsymbol{Z}'$) can be identified with the ring of all maps on Triples with values in $\boldsymbol{Z}'$. Then $\mho_A(G)$ is the subring of $(\boldsymbol{Z}')^{T_G(A)}$ consisting of all G_K-maps (§1.3).

Since the ring homomorphism $\Phi : \Omega_A(G) \to \mho_A(G)$ has a nilpotent finite kernel from Proposition (1.12), it induces the homoeomorphism Φ^{-1} : $\mathrm{Spec}(\mathrm{im}\,\Phi) \to \mathrm{Spec}(\Omega_A(G))$. And since $(\boldsymbol{Z}')^{T_G(A)}$ has finite $\boldsymbol{Z}$-rank, the ring inclusions $\mathrm{im}\,\Phi \subset \mho_A(G) \subset (\boldsymbol{Z}')^{T_G(A)}$ are integral extensions. Therefore

$$\mathrm{Spec}((\boldsymbol{Z}')^{T_G(A)}) \xrightarrow{\text{going-down}} \mathrm{Spec}(\mathrm{im}\,\Phi) \xrightarrow{\Phi^{-1}} \mathrm{Spec}(\Omega_A(G))$$

is surjective ([Bour]chII§4).

On the other hand, we know $\mathrm{Spec}(\boldsymbol{Z}')$ consists of the ideal 0 and the maximal ideals of $\boldsymbol{Z}'$. Moreover, if M is maximal in $\boldsymbol{Z}'$, the field $\boldsymbol{Z}'/M$ is finite; its characteristic is called the *residue characteristic* of M.

The spectrum of $(Z')^{T_G(A)}$ can be identified with $T_G(A) \times \mathrm{Spec}(Z')$: with each $T \in T_G(A)$ and each $M \in \mathrm{Spec}(Z')$ we associate the prime ideal M_T consisting of those $f \in (Z')^{T_G(A)}$ such that $f(T) \in M$. The image of M_T in $\mathrm{Spec}(\Omega_A(G))$ is the prime ideal $P_{M,T}$ corresponding to the prime ideal $M_T \cap \mathrm{im}\,\Phi$ in $\mathrm{im}\,\Phi$ by Φ^{-1}, i.e.

$$P_{M,T} = \{x \in \Omega_A(G) : \Phi_x(T) \in M\}.$$

Because $\mathrm{Spec}((Z')^{T_G(A)}) \to \mathrm{Spec}(\Omega_A(G))$ is surjective, each prime ideal of $\Omega_A(G)$ is of the form $P_{M,T}$.

Recall that if X is a topological space, then the closure $\overline{\{x\}}$ of a single point x is always connected. Moreover, if E, F are two connected closed subsets of X with non-empty intersection then the union $E \cup F$ is connected. We will prove next the connectness of $\mathrm{Spec}(\Omega_A(G))$.

(2.2)Proposition. *With above notation, then*

(1) $P_{0,T} \subset P_{M,T}$;

(2) *If* M *is a maximal ideal of* Z' *with the residue characteristic* q *and* $T = (H, b, \mathfrak{p}')$ *is a Triple, we denote the Triple* $(O^q(H), b_{q'}, \mathfrak{p}')$ *(cf. § 1.1) by* T^q. *Then* $P_{M,T} = P_{M,T^q}$;

(3) $\mathrm{Spec}(\Omega_A(G))$ *is connected.*

Proof. (1) $P_{0,T} = \{x \in \Omega_A(G) : \Phi_x(T) = 0\} \subset \{x \in \Omega_A(G) : \Phi_x(T) \in M\} = P_{M,T}$;

(2) follows from the congruence $\Phi_x(T) \equiv \Phi_x(T^q) \mathrm{mod} M$ for $x \in \Omega_A(G)$ of Theorem B: $x \in P_{M,T} \longleftrightarrow \Phi_x(T) \in M \longleftrightarrow \Phi_x(T^q) \in M \longleftrightarrow x \in P_{M,T^q}$.

(3) Let C be the connected component of the point $P_{0,(1)}$ in $\mathrm{Spec}(\Omega_A(G))$ where (1) is the cyclic Triple of the trivial subgroup. It suffices to show that any point $P_{M,T}$ in $\mathrm{Spec}(\Omega_A(G))$ is contained in C.

We proceed by induction on the order of the subgroup H appearing in the Triple $T = (H, b, \mathfrak{p}')$.

In the case $|H| = 1$, the closure $\overline{\{P_{0,(1)}\}}$ is the set of prime ideals of $\Omega_A(G)$ that contain $P_{0,(1)}$. It follows from (1) that $P_{M,(1)} \in \overline{\{P_{0,(1)}\}} \subset C$.

Suppose now $T = (H, b, \mathfrak{p}')$ and H is nontrivial. Since H is solvable, there exists a prime number p_1 such that $O^{p_1}(H) \subsetneq H$. Let M_1 be a prime ideal of Z' containing p_1. From (2) above, $P_{M_1,T} = P_{M_1,T^{p_1}}$; and $P_{M_1,T} = P_{M_1,T^{p_1}} \in \overline{\{P_{0,T}\}} \cap \overline{\{P_{0,T^{p_1}}\}}$ follows from (1). Therefore $\overline{\{P_{0,T}\}} \cup \overline{\{P_{0,T^{p_1}}\}}$ is connected. The induction hypothesis on $|O^{p_1}(H)|$ of the triple $T^{p_1} = (O^{p_1}(H), b_{p_1}, \mathfrak{p}')$ implies that $P_{M_1,T^{p_1}} \in C$. Since the connected component C is the maximal connected subset of $\mathrm{Spec}(\Omega_A(G))$ containing any point in it, the connected subset $\overline{\{P_{0,T}\}} \cup \overline{\{P_{0,T^{p_1}}\}}$, which contains $P_{M_1,T^{p_1}}$ $(\in C)$, must be contained in C. Therefore, $P_{M,T} \in \overline{\{P_{0,T}\}} \subset C$, which completes the proof.

CHAPTER 3

Characters of Permutation Summands Over Big Number Rings

§3.1 Q_p^{nr}-characters of Small Groups

We first give some notations concerning twisted group algebras. If F is a field and G is a group, we use FG or $F[G]$ to denote the group algebra. If A is a normal abelian subgroup of G with quotient H, letting $\{u_h : h \in H\}$ be a set of preimages of H in G, then relations $u_{h_1}u_{h_2} = f(h_1, h_2)u_{h_1h_2}$ in G define a factor set $f : H \times H \to A$. We shall define an algebra $\sum_{h\in H}(FA)u_h$ having a FA-basis $\{u_h : h \in H\}$. The operations are to be manipulated according to formulas

$$u_h \cdot a = (hah^{-1})u_h, \quad u_h \cdot u_{h'} = f(h,h')u_{hh'}, \quad a \in A, \quad h,h' \in H.$$

If we denote the algebra constructed above by $FA \circ H$, then there is an algebra isomorphism

$$FG \simeq FA \circ (G/H).$$

When the group extension G of H is split, we choose a trivial factor set f.

(3.1) Lemma. *If $G = C \rtimes D$ with C cyclic of order p^n $(n \geq 1)$ and D acts faithfully on C then*

(a) *C has unique subgroup C_p of order p;*

(b) *G has a unique $\mathbf{Q}_p^{nr}$-irreducible character θ on which C_p acts nontrivially;*

(c) *$\theta|_C$ is the unique faithful $\mathbf{Q}_p^{nr}$-irreducible character of C,* $\theta(1) = p^{n-1}(p-1)$;

(d) *θ is a virtual permutation character.*

Proof. a) clear.

b) $\mathbf{Q}_p^{nr}[C] \simeq \frac{\mathbf{Q}_p^{nr}[X]}{(X^{p^n}-1)} \simeq \frac{\mathbf{Q}_p^{nr}[X]}{(X^{p^{n-1}}-1)} \oplus \frac{\mathbf{Q}_p^{nr}[X]}{(\Phi_{p^n}(X))}$ by the Chinese Remainder Theorem. The cyclotomic polynomial $\Phi_{p^n}(X)$ is irreducible in $\mathbf{Q}_p^{nr}[X]$ by the Eisenstein criterion, so $\frac{\mathbf{Q}_p^{nr}[X]}{(\Phi_{p^n}(X))} \simeq \mathbf{Q}_p^{nr}[\zeta_{p^n}]$. Thus

$$\mathbf{Q}_p^{nr}[C] \simeq \mathbf{Q}_p^{nr}\left[\frac{C}{C_p}\right] \times \mathbf{Q}_p^{nr}(\zeta_{p^n}).$$

The group algebra $\mathbf{Q}_p^{nr}[G]$ is then expressed via twisted group algebras in the notation above as

$$\mathbf{Q}_p^{nr}[G] \simeq \mathbf{Q}_p^{nr}[C] \circ D \simeq \left(\mathbf{Q}_p^{nr}\left[\frac{C}{C_p}\right] \times \mathbf{Q}_p^{nr}(\zeta_{p^n})\right) \circ D = \mathbf{Q}_p^{nr}\left[\frac{G}{C_p}\right] \times \mathbf{Q}_p^{nr}(\zeta_{p^n}) \circ D$$

The $\mathbf{Q}_p^{nr}[G]$-irreducible module with non-trivial C_p action are the $\mathbf{Q}_p^{nr}(\zeta_{p^n})\circ D$-modules. D is embedded in $\mathrm{Aut}\, C \simeq (Z/p^n)^\times \simeq \mathrm{Gal}\,(\mathbf{Q}_p^{nr}(\zeta_{p^n})/\mathbf{Q}_p^{nr})$ so it acts on $\mathbf{Q}_p^{nr}(\zeta_{p^n})$ by Galois action. If F is the subfield of $\mathbf{Q}_p^{nr}(\zeta_{p^n})$ fixed by D, then $\mathbf{Q}_p^{nr}(\zeta_{p^n}) \circ D \simeq (\mathbf{Q}_p^{nr}(\zeta_{p^n})/F,\ 1)$, the cross-product algebra with trivial factor set ([Rein] (29.1)) coming from the split extension $G = C \rtimes D$. $\mathbf{Q}_p^{nr}(\zeta_{p^n}) \circ D$ is then a simple algebra([Rein] (29.8)). We can make $\mathbf{Q}_p^{nr}(\zeta_{p^n})$ into a $\mathbf{Q}_p^{nr}(\zeta_{p^n}) \circ D$-module by letting $\mathbf{Q}_p^{nr}(\zeta_{p^n})$ act by multiplication and D by Galois action. This module is simple because $\mathbf{Q}_p^{nr}(\zeta_{p^n})$ is a field.

(c) $\theta(1) = \deg_{\mathbf{Q}_p^{nr}} \mathbf{Q}_p^{nr}(\zeta_{p^n}) = p^{n-1}(p-1)$ and $\theta|_C$ is afforded by $\mathbf{Q}_p^{nr}(\zeta_{p^n})$, which is the unique faithful $\mathbf{Q}_p^{nr}$-irreducible module of C.

(d) Write $\mathrm{ind}_D^{C_p \rtimes D} 1 = 1_{C_p \rtimes D} + \alpha$, with α a proper character and consider $\mathrm{ind}_{C_p \rtimes D}^G \alpha$. Now $\mathrm{ind}_D^G 1 = \mathrm{ind}_{C_p \rtimes D}^G 1 + \mathrm{ind}_{C_p \rtimes D}^G \alpha$ and C_p acts non-trivially on $\mathrm{ind}_D^G 1$, trivially on $\mathrm{ind}_{C_p \rtimes D}^G 1$, hence non-trivially on $\mathrm{ind}_{C_p \rtimes D}^G \alpha$, so θ is a $\mathbf{Q}_p^{nr}$-constituent of $\mathrm{ind}_{C_p \rtimes D}^G \alpha$. But $(\mathrm{ind}_{C_p \rtimes D}^G \alpha)(1) =$ $[G : C_p \rtimes D]\alpha(1) = p^{n-1} \cdot (p-1) = \theta(1)$. Hence $\theta = \mathrm{ind}_{C_p \rtimes D}^G \alpha$ is a difference of two transitive permutation characters.

(3.2) Remark. *Much of (3.1) holds for any field of characteristic zero which has discrete valuation with prime element p (so the Eisenstein criterion applies), even when the group extension $C \rightarrowtail G \twoheadrightarrow D$ is not split. Thus (a),(b) are true as stated and (c) can be replaced by the inequality $\theta(1) \geq p^{n-1}(p-1)$. This is because $\mathbf{Q}_p^{nr}(\zeta_{p^n}) \circ D$ is still a simple algebra and its simple module, being acted on by the field $\mathbf{Q}_p^{nr}(\zeta_{p^n})$, has at least the dimension $dim_{\mathbf{Q}_p^{nr}} \mathbf{Q}_p^{nr}(\zeta_{p^n})$. The equality actually holds for $\mathbf{Q}_p^{nr}$ if we use the fact that the Brauer of $\mathbf{Q}_p^{nr}$ is trivial. But for the latter use, the inequality is sufficient.*

(3.3) Lemma. *Each $\mathbf{Q}_p^{nr}$-irreducible character χ of $G_1 \times G_2$ is a product of $\mathbf{Q}_p^{nr}$-irreducibles χ_1 of G_1 with χ_2 of G_2, whenever $gcd\,(|G_1|, |G_2|) = 1$.*

Proof. Suppose $\mathbf{Q}_p^{nr}[G_i] \simeq \oplus \; M_{n_i}(D_i)$ for $i = 1,2$ are the Wedderburn decompositions and χ_i is the $\mathbf{Q}_p^{nr}$-irreducible character corresponding to the simple component $M_{n_i}(D_i)$ for $i = 1, 2$. We have

$$\mathbf{Q}_p^{nr}[G_1 \times G_2] \simeq \mathbf{Q}_p^{nr}[G_1] \otimes \mathbf{Q}_p^{nr}[G_2] \simeq \bigoplus_{n_1, n_2} \Big(M_{n_1}(D_1) \otimes M_{n_2}(D_2) \Big).$$

It suffices for us to show that each $M_{n_1}(D_1)\otimes M_{n_2}(D_2)$ is a simple ring, for the above then gives the Wedderburn decomposition of $\mathbf{Q}_p^{nr}[G_1\times G_2]$ and each $\mathbf{Q}_p^{nr}$-irreducible character χ corresponding to a simple component of the form $M_{n_1}(D_1)\otimes M_{n_2}(D_2)$ must be the product $\chi_1\chi_2$.

Since the Brauer group of any finite extension of $\mathbf{Q}_p^{nr}$ is trivial, D_1, D_2 must be the centres of $M_{n_1}(D_1), M_{n_2}(D_2)$ and thus D_i is the character field $K_i=\mathbf{Q}_p^{nr}(\chi_i)$ [CR] for $i=1,2$. Now $\gcd(|G_1|,|G_2|)=1$ implies that

$$\mathbf{Q}_p^{nr}\subseteq K_1\cap K_2\subseteq \mathbf{Q}_p^{nr}(\zeta_{|G_1|})\cap\mathbf{Q}_p^{nr}(\zeta_{|G_2|})=\mathbf{Q}_p^{nr}.$$

Let K_1K_2 be a subfield generated by K_1, K_2 in a fixed algebraic closure of $\mathbf{Q}_p^{nr}$. If $\{v_1,\dots,v_t\}$ is a basis of the vector space K_2 over $\mathbf{Q}_p^{nr}$, then it is a basis of K_1K_2 over K_1 because $\mathrm{Gal}\,(K_1K_2/K_1)\simeq\mathrm{Gal}\,(K_2/K_1\cap K_2)\simeq\mathrm{Gal}\,(K_2/\mathbf{Q}_p^{nr})$ implies $\dim_{\mathbf{Q}_p^{nr}}K_2=\dim_{K_1}K_1K_2$. Now $K_1\otimes_{\mathbf{Q}_p^{nr}}K_2\to K_1K_2$ by $\sum_{i=1}^{t}c_i\otimes v_i\mapsto\sum_{i=1}^{t}c_iv_i$ is an isomorphism. Hence $K_1\otimes K_2$ is a field, which implies

$$M_{n_1}(D_1)\otimes M_{n_2}(D_2)\simeq M_{n_1n_2}(D_1\otimes D_2)=M_{n_1n_2}(K_1\otimes K_2)$$

is a simple ring as it is wanted to be.

Remark. The proof of (3.3) holds for any characteristic 0 field with trivial Brauer group.

The rational characters of p-groups have been studied in [Feit], [Ras1], [Ford]. All are based on a result of Roquette [Feit] (14.3). The following result is not new, and we give it a self-contained proof for the sake of completeness. Recall that $R_{\mathbf{Q}_p^{nr}}(G)$ is the group generated by all $\mathbf{Q}_p^{nr}$ – characters of G.

(3.4) Proposition. *If* G *is a* p*-group then* $R_{\mathbf{Q}_p^{nr}}(G)$ *is spanned by*

(a) *permutation characters; and*

(b) *if* $p = 2$, *all induced characters of the form* $\mathrm{ind}_H^G \mu$ *with* μ *inflating the unique faithful* $\mathbf{Q}_2^{nr}$*-irreducible character* θ *of* $H/\ker \mu = Q_8$, *the quaternion group of order* 8.

Proof. For each $\mathbf{Q}_p^{nr}$-irreducible character ψ, write $\psi = \mathrm{ind}_H^G \mu$ so that μ is $\mathbf{Q}_p^{nr}$*-primitive* (i.e. μ is not induced from a $\mathbf{Q}_p^{nr}$-character of a proper subgroup of H). Then $H/\ker \mu$ has a faithful $\mathbf{Q}_p^{nr}$-irreducible character μ, which is $\mathbf{Q}_p^{nr}$-primitive so the following **claim** applies. If $H/\ker \mu$ is the cyclic group C_p of order p, then its faithful $\mathbf{Q}_p^{nr}$-character is expressible as $\mathrm{ind}_1^{C_p} 1 - \mathrm{ind}_{C_p}^{C_p} 1$, hence $\mu = \mathrm{ind}_{\ker \mu}^H 1 - \mathrm{ind}_H^H 1$. It follows that $\psi = \mathrm{ind}_H^G \mu = \mathrm{ind}_{\ker \mu}^G 1 - \mathrm{ind}_H^G 1$. Otherwise we have $p = 2$, $H/\ker \mu = Q_8$ and μ is the inflation of θ. We are left to prove:

Claim. *Suppose* G *is a* p*-group and has a faithful irreducible* $\mathbf{Q}_p^{nr}$*-character* χ *which is* $\mathbf{Q}_p^{nr}$*-primitive. Then* G *is either cyclic of order* p *or* $p = 2$ *and* G *is the quaternion group* Q_8 *of order* 8.

Proof of Claim. Let A be an abelian normal subgroup of G, and let η be an irreducible $\mathbf{Q}_p^{nr}$-constituent of $\mathrm{res}_A^G \chi$. Then η is G-stable because χ is primitive, and χ is a constituent of $\mathrm{ind}_A^G \eta$ [Isaa] (6.11). Now $\ker \eta \lhd G$, by η G-stable, so $\ker \eta$ acts trivially on $\mathrm{ind}_A^G \eta$, hence on χ. Then $\ker \eta = 1$, by χ faithful, so η is faithful on abelian group A. Thus A is cyclic.

We have just shown that every abelian normal subgroup of p-group of G is cyclic. By group theory ([Gore] Thm5.4.10) either G is cyclic or $p = 2$ and

G is dihedral, semidihedral, quaternion. We are now going to analyze χ case by case.

If G is cyclic of order p^n then the $\mathbf{Q}_p^{nr}$-irreducible character χ on which G_p (the cyclic subgroup of order p) acts nontrivially, is unique and has degree $p^{n-1}(p-1)$. If ξ is this character of degree $p-1$ for G_p then G_p acts nontrivially on the induced character $\mathrm{ind}_{G_p}^G \xi$. So χ is a constituent of $\mathrm{ind}_{G_p}^G \xi$. Comparing degrees gives $\chi = \mathrm{ind}_{G_p}^G \xi$. Since χ is primitive, it follows that $G = G_p$ is cyclic of order p, which is the first possibility the claim names.

Thus $p = 2$. If G is dihedral or semidihedral then $G = C \rtimes \langle x \rangle$ with $x^2 = 1$, C cyclic so the earlier Lemma(3.1) applies and shows that χ is the unique $\mathbf{Q}_p^{nr}$-character on which C_2 acts nontrivially and χ has degree $\frac{1}{4}|G|$. But G has a subgroup $H = C_2 \times \langle x \rangle$ which has the degree 1 character α which is non-trivial on C_2 and has $\alpha(x) = 1$. Hence C_2 acts nontrivially on $\mathrm{ind}_H^G \alpha$. Then χ is a constituent of $\mathrm{ind}_H^G \alpha$ and comparing degrees again gives $\chi = \mathrm{ind}_H^G \alpha$. Since χ is primitive, we have $G = H = C_2 \times \langle x \rangle$ and $\chi = \alpha$. Now χ is faithful, so $G = C_2$ is the cyclic group of order 2.

Finally G is a quaternion group. Now $C \rightarrowtail G \twoheadrightarrow \langle y \rangle$ with $y^2 = 1$ is non-split but the earlier Remark(3.2) still gives the uniqueness of χ and $\chi(1) \geq \frac{1}{4}|G|$. Moreover G has the quaternion subgroup Q_8. If θ is the unique faithful $\mathbf{Q}_p^{nr}$-irreducible character of Q_8, then C_2 , the order 2 subgroup of C , acts nontrivially on $\mathrm{ind}_{Q_8}^G \theta$. Therefore χ is a constituent of $\mathrm{ind}_{Q_8}^G \theta$. Comparing degrees gives $\chi = \mathrm{ind}_{Q_8}^G \theta$. Primitivity of χ again implies $G = Q_8$ and $\chi = \theta$, which proves the claim.

§3.2 Proof of (Induction) Theorem D

With preparations in §3.1, we complete the proof of the following theorem.

Theorem D. *Every* $\mathbf{Q}_p^{nr}$*-character of a finite group* G *is a* $\boldsymbol{Z}$*-linear combination of induced characters* $\{ind_H^G\phi\}$, *where* H *is a subgroup of* G *and* ϕ *is one of the following type*

(i) ϕ *is a* p'*-linear* $\mathbf{Q}_p^{nr}$*-character,*

(ii) $p=2$, ϕ *is the product of a* $2'$*-linear* $\boldsymbol{Q}_2^{nr}$*-character with a* $\boldsymbol{Q}_2^{nr}$*-character* μ *of* H *such that* $H/\ker\mu \simeq Q_8$, *the quaternion group of order* 8, *and* μ *is the inflation of the unique faithful irreducible* $\boldsymbol{Q}_2^{nr}$*-character of* Q_8.

Proof. The Witt-Berman Induction Theorem [CRI] (21.6), applied to the character ring $R_{\mathbf{Q}_p^{nr}}(G)$, asserts that every virtual character in $R_{\mathbf{Q}_p^{nr}}(G)$ is a $\boldsymbol{Z}$-linear combination of induced characters of the form $\mathrm{ind}_{G_1}^G\mu$, where G_1 is a $(\mathbf{Q}_p^{nr},q)$-elementary subgroup of G for some rational prime q, and μ is a $\mathbf{Q}_p^{nr}$-character afforded by a simple $\mathbf{Q}_p^{nr}G_1$-module. By transitivity of induction, Theorem D follows once we establish it for all $(\mathbf{Q}_p^{nr},q)$-elementary groups.

The $(\mathbf{Q}_p^{nr},q)$-elementary groups are of the form $\langle x\rangle \rtimes Q$ where $\langle x\rangle$ is a cyclic q'-group, Q is a q-group and Q acts on $\langle x\rangle$ as follows: there is a monomorphism $j:\ \mathrm{Gal}\left(\mathbf{Q}_p^{nr}(\zeta_{|x|})/\mathbf{Q}_p^{nr}\right) \to (\boldsymbol{Z}/|x|\boldsymbol{Z})^\times$ defined by $\zeta_{|x|}^\sigma = \zeta_{|x|}^{j(\sigma)}$ with $\zeta_{|x|}$ a primitive $|x|$th-root of unity, and the action of Q on $\langle x\rangle$ is such that for each $u\in Q$, $uxu^{-1}=x^{j(\sigma)}$ for some $\sigma\in \mathrm{Gal}\left(\mathbf{Q}_p^{nr}(\zeta_{|x|})/\mathbf{Q}_p^{nr}\right)$.

The $(\mathbf{Q}_p^{nr},q)$-elementary groups are split into two cases according to whether q is equal to p or not.

Case 1. $q = p$ $(\mathbf{Q}_p^{nr}, p)$-elementary groups must be elementary $\langle x\rangle \times P$, because $\mathbf{Q}_p^{nr}$ contains all p'-roots of unity and thus triviality of Gal $(\mathbf{Q}_p^{nr}(\chi_{|x|})/\mathbf{Q}_p^{nr})$ forces a trivial action of P on $\langle x\rangle$. If χ is an irreducible $\mathbf{Q}_p^{nr}$-character of $G = \langle x\rangle \times P$, then χ is a product of irreducible $\mathbf{Q}_p^{nr}$-characters χ_1 of $\langle x\rangle$ with χ_2 of the p-group P by Lemma (3.3). χ_1 is necessarily a p'-linear $\mathbf{Q}_p^{nr}$-character of G. For χ_2, from (3.4), it is a $\mathbf{Z}$-linear combination of induced characters $\mathrm{ind}_{P_i}^{P}\psi$, where ψ is a trivial character or equal to the character μ which inflates the unique faithful $\mathbf{Q}_2^{nr}$-irreducible character of Q_8. It follows that χ is a $\mathbf{Z}$-linear combination of characters $\chi_1 \cdot \mathrm{ind}_{P_i}^{P}\psi = \mathrm{ind}_{P_i}^{P}(\mathrm{res}\chi_1 \cdot \psi)$. So Theorem D is established for $(\mathbf{Q}_p^{nr}, p)$-elementary groups.

Case 2. $q \neq p$. Using the decomposition $x = x_{p'}x_p$ of elements of G into p-, p'-parts, we can write the $(\mathbf{Q}_p^{nr}, q)$-elementary group as $\langle x\rangle \rtimes Q = (\langle x_{p'}\rangle \cdot \langle x_p\rangle) \rtimes Q$. Since $\mathbf{Q}_p^{nr}$ contains all p'-roots of unity, Q must act trivially on $\langle x_{p'}\rangle$. Therefore

$$\langle x\rangle \rtimes Q \simeq \langle x_{p'}\rangle \times (\langle x_p\rangle \rtimes Q).$$

An irreducible $\mathbf{Q}_p^{nr}$-character χ of $\langle x_{p'}\rangle \times (\langle x_p\rangle \rtimes Q)$ is then a product $\chi_1\chi_2$ by Lemma (3.3). χ_1 is a p'-linear character of $\langle x_{p'}\rangle$, and χ_2, by the following Lemma (3.5), is a $\mathbf{Z}$-linear combination of induced characters $\mathrm{ind}_H^{\langle x_p\rangle \rtimes Q}\mu$ with p'-linear $\mathbf{Q}_p^{nr}$-characters μ. Thus χ is the $\mathbf{Z}$-linear combination of induced characters $\chi_1 \cdot \mathrm{ind}_H^{\langle x_p\rangle \rtimes Q}\mu = \mathrm{ind}_H^{\langle x_p\rangle \rtimes Q}(\mathrm{res}\chi_1 \cdot \mu)$, with p'-linear characters $\mathrm{res}\chi_1 \cdot \mu$.

To complete the proof of Theorem D, we are left to show that:

(3.5) Lemma. *Suppose* $G = C \rtimes D$ *with a cyclic* p*-group* C *and a* p'*-group* D. *Then every irreducible* $\mathbf{Q}_p^{nr}$*-character* χ *of* G *is a* Z*-linear combination of induced characters* $ind_H^G \phi$ *of* p'*-linear* $\mathbf{Q}_p^{nr}$*-characters* ϕ.

The proof of Lemma (3.5), given below, is based on a Theorem of Clifford and a result on extension of characters, which are stated next for the reader's conveniences.

Clifford's Theorem. *Let* $H \lhd G$, $\eta \in Irr_{\mathbf{Q}_p^{nr}}(H)$, *and* $T = I_G(\eta)$. *Then* $\psi \mapsto ind_T^G \psi$ *induces a bijection from the set* $\{\psi \in Irr_{\mathbf{Q}_p^{nr}}(T) \mid (res_H \psi, \eta) \neq 0\}$ *to the set* $\{\chi \in Irr_{\mathbf{Q}_p^{nr}}(G) \mid (res_H \chi, \eta) \neq 0\}$.

The proof in [Isaa] (6.11) works for characters over any field of characteristic zero.

Extension Theorem ([Isaa] (11.22)). *Let* $N \lhd G$ *with* G/N *cyclic and let* η *be an absolutely irreducible character which is invariant in* G. *Then* η *is extendible to* G.

Proof of (3.5). If χ has nontrivial kernel K, then it gives a faithful and irreducible $\mathbf{Q}_p^{nr}$-character $\bar{\chi}$ of G/K. The lemma follows once we establish the result for $\bar{\chi}$. So we may assume that χ is faithful.

If C is trivial, the lemma follows from Brauer's Induction Theorem as $\mathbf{Q}_p^{nr}$ contains all p'th roots of unity. Let $|C| = p^n$, $n \geq 1$. The kernel of the homomorphism $D \to \text{Aut } C$ is $C_D(C)$, and the image of D is necessarily a p'-subgroup of $\text{Aut } C$, hence is cyclic.

Denote $C_D(C)$ by D_0 and let $H = C \times D_0$. Then H is normal and $G/H \simeq D/D_0$ is a cyclic p'-group from the last paragraph. Letting η be a $\mathbf{Q}_p^{nr}$-constituent of $\text{res}_H \chi$, then $\eta = \xi\mu$ with $\xi \in \text{Irr}_{\mathbf{Q}_p^{nr}}(C)$ and

$\mu \in \mathrm{Irr}_{\mathbf{Q}_p^{nr}}(D_0)$ by Lemma (3.2). As χ is a $\mathbf{Q}_p^{nr}$-constituent of $\mathrm{ind}_H^G \eta$, by Frobenius reciprocity, so $\ker\ \xi$ acts trivially on χ, for $\ker\ \xi$ is normal in G and therefore acts trivially on $\mathrm{ind}_H^G \eta$. Since χ is faithful, we have $\ker\ \xi = 1$. Then ξ is the unique faithful $\mathbf{Q}_p^{nr}$-irreducible character of C, hence its inertia group is G. Let $D_1 = I_D(\mu) = \{t \in D : \mu^t = \mu\}$. Then the inertia group T of $\eta = \xi \cdot \mu$ is

$$T = I_G(\eta) = I_G(\xi) \cap I_G(\mu) = G \cap (C \rtimes D_1) = C \rtimes D_1.$$

Applying Lemma (3.1) to $C \rtimes (D_1/D_0)$, we obtain the unique faithful character θ. This is an extension of ξ and is a virtual permutation character. Letting $\widetilde{\xi}$ be the inflation of θ through $C \rtimes D_1 \to C \rtimes (D_1/D_0)$, then the $\mathbf{Q}_p^{nr}$-character $\widetilde{\xi}$ of T is an extension of ξ and is a virtual permutation character. On the other hand, since D_1/D_0 is cyclic p'-group and $\mathbf{Q}_p^{nr}$ contains all p'th-roots of unity, the Extension Theorem applied to μ and $D_0 \lhd D_1$ asserts that μ has an extension $\widetilde{\mu}$ in $\mathrm{Irr}_{\mathbf{Q}_p^{nr}}(D_1)$. Denote the inflation of $\widetilde{\mu}$ through $C \rtimes D_1 \to D_1$ still by $\widetilde{\mu}$. Then $\widetilde{\mu} \in \mathrm{Irr}_{\mathbf{Q}_p^{nr}}(T)$ is an extension of μ. Combining the above, $\eta = \zeta \cdot \mu$ has an extension $\widetilde{\xi} \cdot \widetilde{\mu}$, denoted by $\widetilde{\eta}$, to its inertia group T.

Frobenius reciprocity gives $\mathrm{ind}_H^T \eta = \widetilde{\eta} \cdot \mathrm{ind}_H^T 1$ because $\mathrm{res}_H\ \widetilde{\eta} = \eta$. Let $\mathrm{ind}_H^T 1 = \Sigma_i \lambda_i$ be the decomposition into $\mathbf{Q}_p^{nr}$-irreducibles from the corresponding decomposition of $\mathrm{ind}_1^{T/H} 1$. Since T/H is a cyclic p'-group and $\mathbf{Q}_p^{nr}$ contains all p'th-roots of unity, these λ_i are necessarily p'-linear characters. Products $\widetilde{\eta} \cdot \lambda_i$ must be $\mathbf{Q}_p^{nr}$-irreducible because λ_i is one-dimensional and $\widetilde{\eta}$ is $\mathbf{Q}_p^{nr}$-irreducible. Therefore

$$\mathrm{ind}_H^T \eta = \widetilde{\eta} \cdot \mathrm{ind}_H^T 1 = \Sigma_i \widetilde{\eta} \lambda_i$$

is the decomposition of $\mathrm{ind}_H^T \eta$ into $\mathbf{Q}_p^{nr}$-irreducibles.

Now each $\psi \in \{\psi \in \mathrm{Irr}_{\mathbf{Q}_p^{nr}}(T) \mid (\eta, \mathrm{res}_H \psi) \neq 0\}$ is a $\mathbf{Q}_p^{nr}$-constituent of $\mathrm{ind}_H^T \eta$ by Frobenius reciprocity and thus is one of the $\widetilde{\eta}\lambda_i$ from the last paragraph. The quoted Theorem of Clifford, applied to χ and η, gives that $\chi = \mathrm{ind}_T^G \psi$ for a ψ in $\{\psi \in \mathrm{Irr}_{\mathbf{Q}_p^{nr}}(T) \mid (\eta, \mathrm{res}_H \psi) \neq 0\}$. Therefore

$$\chi = \mathrm{ind}_T^G(\widetilde{\eta}\lambda_i) = \mathrm{ind}_T^G(\widetilde{\xi}\widetilde{\mu}\lambda_i) = \mathrm{ind}_T^G(\widetilde{\xi} \cdot \widetilde{\mu}\lambda_i)$$

where $\widetilde{\xi}$ is a virtual permutation character, λ_i is a p'-linear character and $\widetilde{\mu}$ is an inflation of a $\mathbf{Q}_p^{nr}$-character of the p'-group D_1 and thus is a $\mathbf{Z}$-linear combination of induced characters of p'-linear characters by Brauer's Induction Theorem. The lemma then follows from Frobenius reciprocity as in the first paragraph of Case 1. The proof of Lemma (3.5) is completed.

§3.3 Image of $\varphi : \Omega_{\mathfrak{o}}(G) \to R_{\mathfrak{K}}(G)$

Let $\mathfrak{K}$ be a finite extension field of $\mathbf{Q}_p$, and let $\mathfrak{o}$ be the integral closure of the p-adic integers $\mathbf{Z}_p$ in $\mathfrak{K}$. *In this section, we always assume that $\mathfrak{K}$ contains the $|G|_{p'}th$ roots of unity.*

Let $P_{\mathfrak{o}}(G)$ be the Grothendieck group of the category of finitely generated projective $\mathfrak{o}G$-modules. The homomorphism $e : P_{\mathfrak{o}}(G) \to R_{\mathfrak{K}}(G)$ is defined by sending each projective to its $\mathfrak{K}$-character [Ser1]. The following lemma on the image of e is due to Brauer and the proof given below is basically the same as that of in Lemma 1 of [Fong].

(3.6) Lemma. *The image of* $e : P_{\mathfrak{o}}(G) \to R_{\mathfrak{K}}(G)$ *is the subgroup generated by induced characters* $\mathrm{ind}_{P'}^{G} \lambda$ *of linear* $\mathfrak{K}$*-characters* λ *of* p'*-groups* P'.

Proof. It is clear that each $\mathrm{ind}_{P'}^{G} \lambda$ is in the image of e, as P' is a p'-group. Let $\mathfrak{K}'$, with the integer ring $\mathfrak{o}'$, be an extension field of $\mathfrak{K}$ obtained by adjointing the $|G|_p$th roots of unity. Then $1 = \Sigma_i n_i \, \mathrm{ind}_{E_i}^{G} \lambda_i$ in $R_{\mathfrak{K}'}(G)$, by Brauer's Induction Theorem, with elementary groups E_i. Assuming that χ is in the image of e, then it can be expressed as $\chi = \Sigma n_i \, \mathrm{ind}_{E_i}^{G}(\lambda_i \cdot \mathrm{res}_{E_i} \chi)$, and each $\lambda_i \cdot \mathrm{res}_{E_i} \chi$ is in the image of $e : P_{\mathfrak{o}'}(E_i) \to R_{\mathfrak{K}'}(E_i)$, because the tensor product of an $\mathfrak{o}' E_i$-lattice with a projective $\mathfrak{o}' E_i$-module is projective.

Let E denote any one of the elementary groups E_i . Write $E = P \times P'$ as a product of a p-group P and a p'-group P'. Each projective $\mathfrak{o}'[E]$-module is isomorphic to $\mathfrak{o}'[P] \otimes W$ by [Ser1] (15.7) for some $\mathfrak{o}'[P']$-lattice W; and $A'_{\mathfrak{p}}[P] \otimes W \simeq \mathrm{ind}_{P'}^{E} W$ by Frobenius reciprocity. Its character is $\mathrm{ind}_{P'}^{E} \chi_W$. Now $\chi_W \in R_{\mathfrak{K}'}(P') = R_{\mathfrak{K}}(P')$, because $\mathfrak{K}$ contains $|G|_{p'}$th-roots of unity. Therefore, χ_W is a Z-linear combination of induced characters of linear $\mathfrak{K}$-characters of subgroups of P', by Brauer's Induction Theorem [Ser1] (Theorem 20). The required result on $\mathrm{ind}_{P'}^{E} \chi_W$ follows from the transitivity of induction.

(3.7) Proposition. *The image of* $\varphi : \Omega_{\mathfrak{o}}(G) \to R_{\mathfrak{K}}(G)$ *is the subgroup* $S_p(G)$ *generated by induced characters* $\mathrm{ind}_H^G \phi$ *of* p'*-linear characters* ϕ *of subgroups* H *of* G.

Proof. We first exhibit a Z-basis of $\Omega_{\mathfrak{o}}(G)$ and show that their characters are sums of induced characters $\mathrm{ind}_H^G \phi$. The Grothendieck group $\Omega_{\mathfrak{o}}(G)$ of the category of permutation summand $A_{\mathfrak{p}} G$-lattices has a Z-basis, by Krull-Schmidt and vertex theory, parametrized by pairs (P, V), where P is a p-subgroup (determined upto conjugacy) and V is an indecomposable permutation summand $\mathfrak{o}G$-lattice with vertex P. The Green correspondent $f_P(V)$ is an indecomposable $\mathfrak{o}[N_G(P)]$-module with vertex P. Since P acts trivially on $f_P(V)$ by (1.1) in §1.1, $f_P(V)$ can be considered as an indecomposable projective $\mathfrak{o}[N_G(P)/P]$-module M; and, vice versa, an indecomposable projective $\mathfrak{o}[N_G(P)/P]$ -module M will give, from the inflation inf M, an indecomposable $\mathfrak{o}[N_G(P)]$-module of vertex P [CRII] (81.15)(iii). Then $\mathrm{ind}_{N_G(P)}^G(\inf M)$, parametrized by (P, M), is a second $\boldsymbol{Z}$-basis of $\Omega_{\mathfrak{o}}(G)$ because the Green relations

$$\mathrm{ind}_{N_G(P)}^G(\inf M) = V \oplus V', \quad \mathrm{vtx}\,(V') \subsetneq P$$

provide a transition matrix which is upper triangular with 1's on the main diagonal.

Denote $N_G(P)/P$ by $\overline{N}_G(P)$ for simplicity. The character χ_M, in the image of $\epsilon : P_{\mathfrak{o}}(\overline{N}_G(P)) \to R_{\mathfrak{o}}(\overline{N}_G(P))$, is expressible as $\chi_M = \Sigma_i n_i\, \mathrm{ind}_{P_i'}^{\overline{N}_G(P)} \lambda_i$ by (3.6). Thus its inflation is $\inf \chi_M = \Sigma_i n_i\, \mathrm{ind}_{H_i}^{N_G(P)} \phi_i$, where each H_i is the preimage of P_i', ϕ_i is the inflation of λ_i and thus is a p'-linear character of H_i. The images of the basis $\{\mathrm{ind}_{N_G(P)}^G(\inf M) \mid (P, M)\}$ in $R_{\mathfrak{K}}(G)$ are $\{\mathrm{ind}_{N_G(P)}^G(\inf \chi_M)\}$ and $\mathrm{ind}_{N_G(P)}^G \inf \chi_M = \Sigma_i n_i\, \mathrm{ind}_{H_i}^G \phi_i$ from above, as required.

On the other hand, we need to show $\mathrm{ind}_H^G \phi$ must be in the image of φ. It suffices to construct a permutation summand $\mathfrak{o}H$-lattice with the given p'-linear character ϕ as character. The image $\phi(H)$ is a cyclic p'-group C in $\mathfrak{K}$, actually in $\mathfrak{o}$. Letting M be the rank 1 $\mathfrak{o}[C]$-lattice on which C acts via ϕ, then M is a projective $\mathfrak{o}[C \simeq H/\ker \varphi]$-lattice. The inflation inf M is a permutation summand $\mathfrak{o}H$-lattice and its character is ϕ. This completes the proof of (3.7).

$S_p(G)$ in the above Proposition is a subgroup of $R_{\mathbf{Q}_p^{nr}}(G)$, because $\mathbf{Q}_p^{nr}$ contains all p'th roots of unity. Therefore we have the image of φ in $R_{\mathbf{Q}_p^{nr}}(G)$. Combining (3.7) with the (Induction) Theorem D, we prove that

(3.8) Theorem. *The homomorph* $: \Omega_o(G) \to R_{\mathbf{Q}_p^{nr}}(G)$ *are surjective for odd prime numbers* p; *if* $p = 2$, *the image of* φ *is* $S_2(G)$ *and the cokernel is annihilated by* 2.

Proof. The first follows from $R_{\mathbf{Q}_p^{nr}}(G) = S_p(G)$ of Theorem D. For the second, $R_{\mathbf{Q}_2^{nr}}(G)$ is generated by $S_2(G)$ and the characters $\mathrm{ind}_H^G(\psi \cdot \mu)$, from ii) of Theorem D, where ψ is p'-linear and μ is the inflation of the unique faithful irreducible character θ of $H/\ker \mu \simeq Q_8$, the quaternion group of order 8. Now 2θ is a virtual permutation character given by $2\theta = \mathrm{ind}_1^Q 1 - \mathrm{ind}_{C_4}^Q 1$, hence so is 2μ. Therefore

$$2\,\mathrm{ind}_H^G(\psi \cdot \mu) = \mathrm{ind}_H^G(\psi \cdot 2\mu) \in S_2(G)$$

follows from Frobenius reciprocity.

§3.4 Proof of Theorem E

We begin with a different proof to the following lemma [WaWe](2.4). The proof is based on Swan's theorem on characters of projective AG-modules ([Ser1]Theorem 36).

(3.9) Lemma. *Given a permutation summand* L *of* G *over* A. *Let* φ_L *denote the character of* $K \otimes_A L$. *Then the value* $\varphi_L(x)$ *is in* $\mathbf{Z}$ *for each element* x *in* G.

Proof. We may assume G is cyclic of order n, generated by x. For each prime divisor p of n , then there exist subgroups P and E in G such that $G = E \times P$, where P is a p-group and E is of order $n_{p'}$ prime to p.

Let $K' = K(\sqrt[n_{p'}]{1})$, $A' =$ integral closure of A in K', and $S = A'_{\mathfrak{p}'}$, for some maximal ideal $\mathfrak{p}'$ in A' containing p. The SG-lattice $S \otimes_A L$ is still a permutation summand and we may compute $\varphi_L(x)$ by finding the value of the character of $K'_{\mathfrak{p}'} \otimes_S (S \otimes_A L)$ at x.

Claim: $\varphi_L(x)$ is a sum of $n_{p'}$th roots of unity.

For the purpose of proving the above claim, we may replace $S \otimes_A L$ by an indecomposable SG-summand$M$. Clearly M is a permutation summand for G over S. If D is the vertex of M, then M is a direct summand of $\mathrm{ind}_D^G(S)$ as SG-modules by Lemma 1.1. Since D is normal in G, D acts trivially on $\mathrm{ind}_D^G(S)$, hence it acts trivially on M. Considering $\mathrm{ind}_D^G(S)$ and M as $S[\frac{G}{D}]$-modules we have that M is projective and $\varphi_M(x)$ is the value of the character of projective $S[\frac{G}{D}]$-modules at xD. If $D \subsetneq P$, then xD is p-singular in G/D, hence $\varphi_M(x) = 0$ by Swan's theorem. Otherwise, xD is of order $n_{p'}$, so the character value at xD is a sum of $n_{p'}$th roots of unity. The Claim is established.

We have shown that $\varphi_L(x) \in \mathbf{Q}(\zeta_{n_{p'}})$ for each $p|n$. Therefore $\gcd\{n_{p'} : p|n\} = 1$ implies

$$\varphi_L(x) \in \bigcap_{p|n} \mathbf{Q}(\zeta_{n_{p'}}) = \mathbf{Q}$$

as required.

The images of $\varphi : \Omega_o(G) \to R_{\mathbf{Q}_p^{nr}}(G)$ are characterized in (3.8) on all local rings $\mathfrak{o}$, whenever $\mathfrak{o}$ contain $|G|_{p'}$th roots of unity. To prove the Theorem E on the global ring A, we use a technique of gluing permutation summand lattices over local rings $A_\mathfrak{p}$ at all $\mathfrak{p}$, to form a permutation summand lattice over A. The foundation of this gluing construction is the following known result.

(3.10) Lemma. *Given a KG-module V, and for each $\mathfrak{p}$ above a rational prime divisor of $|G|$, let there be given a permutation summand $Y(\mathfrak{p})$ of G over $A_\mathfrak{p}$, such that $K_\mathfrak{p} \otimes_{A_\mathfrak{p}} Y(\mathfrak{p}) = K_\mathfrak{p} V$. Then there exists a permutation summand L of G over A, such that*

$$KL = V, \quad A_\mathfrak{p} \otimes_A L \simeq Y(\mathfrak{p}) \text{ for all such } \mathfrak{p}.$$

Proof. Let M be a G-stable A-submodule in V such that $KM = V$. Denote by S the set of prime ideals of A lying above rational prime divisors of the group order $|G|$. Define

$$L = V \cap \{\bigcap_{\mathfrak{p}\in S} Y(\mathfrak{p})\} \cap \{\bigcap_{\mathfrak{p}\notin S} (A_\mathfrak{p} \otimes_A M)\}$$

where the intersection is taken over all prime ideal $\mathfrak{p}$ of A. Then $KL = V$, and $A_\mathfrak{p} \otimes_A L \simeq Y(\mathfrak{p})$ for $\mathfrak{p} \in S$, follow from [Rein] (5.3)(ii). L is a permutation summand of G over A by Lemmas 1, 2 in [ClWe], on replacing $\mathbf{Z}$ by A.

Since the Brauer group of $\mathbf{Q}_p^{nr}$ is trivial, we have $\overline{R}_\mathbf{Q}(G) \subset R_{\mathbf{Q}_p^{nr}}(G)$. We are going to prove the main theorem on characters of permutation summands.

Theorem E. *The image of* $\varphi : \Omega_A(G) \to \overline{R}_{\mathbf{Q}}(G)$ *is always contained in the intersection of* $\overline{R}_{\mathbf{Q}}(G)$ *and* $S_2(G)$ *in* $R_{\mathbf{Q}_2^{nr}}(G)$. *If* K *is big enough this containment is an equality.*

Proof. Given a permutation summand L of G over A, its image φ_L is in $\overline{R}_{\mathbf{Q}}(G)$ by (3.9).

For each prime ideal $\mathfrak{p}$ of A above a prime number p which divides the group order $|G|$, we obtain the $\mathfrak{p}$-adic completion field $K_{\mathfrak{p}}$ with the ring of integers $A_{\mathfrak{p}}$. Let $\mathfrak{K}$ be an extension field of $K_{\mathfrak{p}}$ with $|G|_{p'}$th roots of unity adjointed, and let $\mathfrak{o}$ be the integer ring of $\mathfrak{K}$. Now the local field $\mathfrak{K}$ contains $|G|_{p'}$th roots of unity, so Theorem (3.8) applies. The first part of Theorem E follows from the commutative diagram below:

$$\begin{array}{ccc} \Omega_A(G) & \xrightarrow{\Phi} & \overline{R}_{\mathbf{Q}}(G) \\ {\scriptstyle \mathfrak{o}\otimes_A}\downarrow & & \downarrow \\ \Omega_{\mathfrak{o}}(G) & \xrightarrow{\Phi} & R_{\mathbf{Q}_p^{nr}}(G) \end{array}$$

For the second part, a number field K is called *big enough* (with respect to G) if it satisfies the following two conditions:

(1) The completion field $K_{\mathfrak{p}}$ contains $|G|_{p'}$th roots of unity for each $\mathfrak{p}$ above a prime divisor p of $|G|$;

(2) All rational valued characters are realizable over K.

The field $\mathbf{Q}(\zeta_{|G|})$, for instance, is one example of a big enough fields. Alternatively we can arrange that $K/\mathbf{Q}$ is unramified at all prime divisors of $|G|$ by a theorem of Grunwald-Wang.

The second part of the Theorem for big enough K amounts to: given a virtual character χ in $\overline{R}_{\mathbf{Q}}(G) \cap S_2(G)$, we want to construct a (virtual) permutation summand x in $\Omega_A(G)$ such that the K-character of x is χ. Since K is big enough, we have $\chi \in \overline{R}_{\mathbf{Q}}(G) \subset R_K(G)$, and the local fields $K_{\mathfrak{p}}$ satisfy the requirement of §3.3 so Theorem (3.8) applies. From Theorem (3.8), for each odd rational prime p which divides $|G|$, $\chi \in \overline{R}_{\mathbf{Q}}(G) \subset R_{\mathbf{Q}_p^{nr}}(G)$ is in the image of $\varphi : \Omega_{A_{\mathfrak{p}}}(G) \to R_{\mathbf{Q}_p^{nr}}(G)$, hence $\chi = \varphi_{x(\mathfrak{p})}$ for some $x(\mathfrak{p}) \in \Omega_{A_{\mathfrak{p}}}(G)$; if $2|\ |G|$, the condition $\chi \in S_2(G)$ ensures there exists $x(\mathfrak{p}) \in \Omega_{A_{\mathfrak{p}}}(G)$ such that $\varphi_{x(\mathfrak{p})} = \chi$ for each $\mathfrak{p}$ above 2.

Next, for each $\mathfrak{p} \in S =$ the set of primes of K above rational prime divisors of $|G|$, write $x(\mathfrak{p}) = [M_1] - [M_2]$ as a difference of permutation summands for G over $A_{\mathfrak{p}}$. Then $M_2 \oplus M_2'' \simeq A_{\mathfrak{p}}[S(\mathfrak{p})]$ for some G-set $S(\mathfrak{p})$ and $A_{\mathfrak{p}}G$-lattice M_2'', so, setting $X(\mathfrak{p}) = M_1 \oplus M_2''$, we have $x(\mathfrak{p}) = [X(\mathfrak{p})] - [A_{\mathfrak{p}}[S(\mathfrak{p})]]$ in $\Omega_{A_{\mathfrak{p}}}(G)$.

Then $S = \dot{\bigcup}_{\mathfrak{p} \in S} S(\mathfrak{p})$ is a G-set, so on setting $Y(\mathfrak{p}) = X(\mathfrak{p}) \oplus A_{\mathfrak{p}}[S \backslash S(\mathfrak{p})]$, we have $x(\mathfrak{p}) = (Y(\mathfrak{p})) - (A_{\mathfrak{p}}[S])$ in $\Omega_{A_{\mathfrak{p}}}(G)$ for each $\mathfrak{p} \in S$. Let the character of $A_{\mathfrak{p}}[S]$ be φ_S, which is determined by the G-set S and is independent of $\mathfrak{p}$. Since $x(\mathfrak{p})$ has character χ by construction, the character of $K_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} Y(\mathfrak{p})$ is the same as the character $\chi + \varphi_S$. It follows that the virtual character $\chi + \varphi_S \in R_K(G)$ is indeed a K-character afforded by a KG-module V ([Serl], Prop. 33). Applying now (3.10) to V, $Y(\mathfrak{p})$, we have a permutation summand L for G over A, such that $\varphi_L = \chi + \varphi_S$. Setting $x = [L] - [A[S]]$ in $\Omega_A(G)$, then $\varphi_x = \varphi_L - \varphi_S = \chi$ as desired.

Now Theorem E is proved. There is a further question on the image of φ, namely that of finding conditions on characters χ in $R_{\mathbf{Q}_2^{nr}}(G)$ that restrict χ to be in $S_2(G)$. The next proposition makes a reduction of this question to a small class of groups.

(3.11) Proposition. *Let* χ *be a virtual character in* $R_{\mathbf{Q}_2^{nr}}(G)$. *Then* $\chi \in S_2(G)$ *if and only if to* $res_H \chi \in S_2(H)$ *for all 2-hyperelementary subgroups* H *of* G.

Proof. Solomon's Induction Theorem ([CRI] (15.10)) gives that

$$1_G = \sum_H n_H \ \mathrm{ind}_H^G 1_H$$

with H varying over the p-hyperelementary subgroups of H for all p. It follows that $\chi = \sum_H n_H \ \mathrm{ind}_H^G(\mathrm{res}_H \chi)$ by Frobenius Reciprocity. Thus $\chi \in S_2(G)$ is equivalent to $\mathrm{res}_H \chi \in S_2(H)$ for all H above.

If H is a p-hyperelementary group with $p \neq 2$, then $H = \langle x \rangle \rtimes P$ with p-group P and p'-element x. Decomposing $\langle x \rangle$ into the product of 2-part and $2'$-part $\langle x \rangle = \langle x_2 \rangle \times \langle x_{2'} \rangle$, we see that P acts trivially on the cyclic 2-group $\langle x_2 \rangle$ because P is a $2'$-group and thus its image in $P \to \mathrm{Aut}\ (\langle x_2 \rangle)$ is trivial. Now $H = \langle x_2 \rangle \times (\langle x_{2'} \rangle \times P)$ is a product of the cyclic 2-group with a $2'$-group. It follows that $\mathrm{res}_H \chi$ is always in $S_2(H)$ by Lemma (3.2), since $\mathbf{Q}_2^{nr}$-characters of cyclic 2-groups are virtual permutation characters and $\mathbf{Q}_2^{nr}$-characters of $2'$-groups are sums of induced characters of $2'$-linear characters by Brauer's Induction Theorem.

The remaining condition for $\chi \in S_2(G)$ is that $\chi \in S_2(H)$ for all 2-hyperelementary subgroups H.

§3.5 $\mathbf{Q}_2^{nr}$-characters of 2-groups

Let G be a 2-group. By a *critical* character of G we mean an irreducible $\mathbf{Q}_2^{nr}$ character which is not a virtual permutation character, i.e. not in

$S_2(G)$. Putting $\mathcal{R}(G) = R_{\mathbf{Q}_2^{nr}}(G)/S_2(G)$, it follows that $\mathcal{R}(G)$ is generated by critical characters.

By a *quaternion section* of G we mean a pair $H = (H_0, H_1)$ of subgroups $H_0 \subseteq H_1$ of G with $H_0 \lhd H_1$ and $H_1/H_0 \simeq Q_{2^n} = \langle x, y : x^{2^{n-2}} = y^2,\ yxy^{-1} = x^{-1} \rangle$ a quaternion group of order 2^n with $n \geq 3$. Each quaternion section H defines a unique $\mathbf{Q}_2^{nr}$-character of G : for H_1/H_0 has a unique faithful $\mathbf{Q}_2^{nr}$-irreducible character θ, which we call the *quaternion character* of Q, and which can be inflated to H_1 and then induced to G. We say this character *comes from* the quaternion section H.

We know from Proposition (3.3) (rather, the claim appearing in its proof) that every critical character of G comes from a quaternion section. Since 2θ is a virtual permutation character of $H_1/H_0 \simeq Q_{2^n}$ by $2\theta = \operatorname{ind}_1^{Q_{2^n}} 1 - \operatorname{ind}_{\langle x \rangle}^{Q_{2^n}} 1$, it follows that $\mathcal{R}(G)$ is a vector space over $\mathbb{F}_2$ spanned by critical characters.

We call a quaternion section $H = (H_0, H_1)$ *big* if $H_1 = N_G(H_0)$, i.e. a quaternion section is big if enlarging H_1 will not give a section. Now G will have fewer big quaternion sections e.g. if G is itself a quaternion group then its only big quaternion section is $(1, G)$: for if $H_0 = 1$ then $H_1 = N_G(1) = G$ while $H_0 \neq 1$ then H_0 contains the centre Z of G hence H_1/H_0 is a section of the dihedral group G/Z which has no quaternion section.

(3.12) Proposition. *Every critical character comes from a big quaternion section.*

Proof. Let the critical character χ come from the quaternion section $H = (H_0, H_1)$ and choose H so that $(H_1 : H_0)$ is maximal for this property. We show that H is then big by contradiction, so suppose $H_1 \subsetneq N_G(H_0)$: then taking successive normalizers of H_1/H_0 in the 2-group $N_G(H_0)/H_0$ will terminate in $N_G(H_0)/H_0$, hence there exists $K \subseteq N_G(H_0)/H_0$ so $(K : H_1/H_0) = 2$. Set $Q = H_1/H_0$ and let θ be its quaternion character; we now consider $\mathrm{ind}_Q^K \theta$ and note that if $\widehat{K} =$ preimage of K under $N_G(H_0) \to N_G(H_0)/H_0$ then $\chi = \mathrm{ind}_{\widehat{K}}^G\ (\inf_{\widehat{K} \to K}(\mathrm{ind}_Q^K \theta))$ (because this is $\mathrm{ind}_{\widehat{K}}^G(\mathrm{ind}_{H_1}^{\widehat{K}}(\inf_{H_1 \to Q}\theta)) = \mathrm{ind}_{H_1}^G(\inf_{H_1 \to Q}\theta)$. Everything now follows from the

Claim. One of the following happens

a) $\mathrm{ind}_Q^K \theta$ is a virtual permutation character,

b) $\mathrm{ind}_Q^K \theta$ is reducible,

c) K is a quaternion group.

In case a) we get a contradiction to χ critical, and in case b) again (but now to irreducibility of χ). And in case c) we contradict the maximality of H : for $(H_0, \widehat{K})$ is now a quaternion section with $(\widehat{K} : H_0) = 2(H : H_0)$ and χ comes from $(H_0, \widehat{K})$ (because ind_Q^K is the quaternion character of K. So we are reduced to the

Proof of Claim. Write $Q = \langle x, y :\ x^{2^{n-2}} = y^2,\ yxy^{-1} = x^{-1} \rangle$ hence $Z(Q) = \langle y^2 \rangle$. We will try to apply the following

Criterion. *Suppose* K *contains an element* h *so that*

i) $h^2 = 1$, $K = Q \rtimes \langle h \rangle$, iii) h *is* K*-conjugate to* $y^2 h$.

Then $\mathrm{ind}_Q^K \theta$ *is a virtual permutation character, i.e case a).*

Proof of Criterion. Use $(\mathrm{ind}_A^K 1)(k) = \frac{1}{|A|}$ card $\{x \in K : x^{-1}kx \in A\}$. Then

$$(\mathrm{ind}_{\langle h \rangle}^K 1)(k) = \begin{cases} 2^n, & k = 1 \\ \frac{1}{2}|C_K(h)|, & k \underset{K}{\sim} h \\ 0, & \text{else.} \end{cases}$$

Similarly to calculate $\operatorname{ind}_{\langle y^2,h\rangle}^{K} 1$ note that h commutes with y^2, since $\langle y^2\rangle = Z(Q)$ and that $\{1\}$, $\{y^2\}$, $\{h, y^2h\}$ are the intersections of $\langle y^2, h\rangle$ with the conjuga[illegible]asses of K, because of i), iii). It follows that

$$(\operatorname{ind}_{\langle y^2,h\rangle}^{K} 1)(k) = \begin{cases} 2^{n-1}, & k = 1 \\ \frac{2^{n+1}}{4}, & k = y^2 \\ \frac{2|C_k(h)|}{4}, & k \underset{K}{\sim} h \\ 0, & \text{else.} \end{cases}$$

Subtracting these two chara[illegible]es

$$(\operatorname{ind}_{\langle k\rangle}^{K} 1 - \operatorname{ind}_{\langle y^2,h\rangle}^{K} 1)(k) = \begin{cases} 2^{n-1}, & k = 1 \\ -2^{n-1}, & k = y^2 \\ 0 & \text{else} \end{cases}$$

$$= (\operatorname{ind}_{Q}^{K} \theta)(k)$$

(for $(\operatorname{ind}_Q^K \theta)(k) = 0$ unless $k \in Q$ when it is $\theta(k) + \theta(k^h) = 2\theta(k)$ because $\theta^h = \theta$ by uniqueness of θ). This establishes the Criterion.

Let $a \in K$ generate K/Q; we will modify a by multiplying by elements of Q to arrange various things. First conjugation by a permutes the subgroups of index 2 in Q. If $n \geq 4$ there is only one such, namely $\langle x\rangle$. If $n = 3$ there are 3 of them, so conjugation by a (of order 2) must stabilize one of them, which can take to be $\langle x\rangle$ by renaming the elements (for Q has an automorphism of order 3 which rearranges the names). So we may assume $a\langle x\rangle a^{-1} = \langle x\rangle$ and can write $axa^{-1} = x^r$ with $r \in (\mathbf{Z}/2^{n-1}\mathbf{Z})^\times$. Since $yxy^{-1} = x^{-1}$ replacing a by ya, if necessary, allows us to assume $r \equiv 1 \bmod 4$.

From $a^2xa^{-2} = x^{r^2}$ and $a^2 \in Q$ we get $x^{r^2} = x^{\pm 1}$ hence $r^2 = \pm 1 \bmod 2^{n-1}$ when $r \equiv 1 \bmod 4$ implies $r^2 \equiv 1 \bmod 2^{n-1}$.

Now conjugation induces $K \to \text{Aut}\,\langle x\rangle$ sending $a,\ y \mapsto r, -1$ and having kernel $C_K(\langle x\rangle) \supseteq \langle x\rangle$. Since $(K : \langle x\rangle) = 4$ we get

$$(C_K(\langle x\rangle) : \langle x\rangle) = \begin{cases} 2, & r \equiv 1 \bmod 2^{n-1} \\ 1, & r \not\equiv 1 \bmod 2^{n-1}. \end{cases}$$

We must now consider various cases and first dispose of the

Special case. Assume $C_K(\langle x\rangle)$ is cyclic of order 2^n. Now $r \equiv 1 \bmod 2^{n-1}$ and $C_K(\langle x\rangle) = \langle a, x\rangle$ with x of order 2^{n-1} so multiplying by a by a power of x allows us to assume $a^2 = x$. Then $yay^{-1} = a^i$ with $i \in (\mathbf{Z}/2^n\mathbf{Z})^\times$, because $C_K(\langle x\rangle) \lhd K$, implies $x^{-1} = yxy^{-1} = x^i$ and thus $i \equiv -1 \bmod 2^{n-1}$. Since $a^{2^{n-1}} = x^{2^{n-2}} = y^2$ this means that $yay^{-1} = a^{-1}$ or $a^{-1}y^2$.

If $yay^{-1} = a^{-1}$ then $K = \langle a, y : a^{2^{n-1}} = y^2,\ yay^{-1} = a^{-1}\rangle$ is the quaternion group of order 2^{n+1}, i.e. this is case c).

So assume $yay^{-1} = a^{-1}y^2$ (when K is semidihedral of order 2^{n+1}). We show that the criterion is satisfied with $h = ay$ (so are in case a)). For $h^2 = ayay^{-1}y^2 = aa^{-1}y^2y^2 = 1$ and $h \notin Q$ gives i), ii) while $h = ay_{\bar{K}}x^{2^{n-3}}(ay)x^{-2^{n-3}} = ax^{2^{n-3}}yx^{-2^{n-3}}y^{-1}y = ax^{2^{n-3}}x^{2^{n-3}}y = x^{2^{n-2}}ay = y^2h$ gives iii). This takes care of the special case.

In all other cases we look at the group extension

$$1 \to \langle x\rangle \to \langle x, a\rangle \to \langle x, a\rangle/\langle x\rangle \to 1$$

with $\langle x, a\rangle/\langle x\rangle$ cyclic of order 2 generated by the image of a. If $r \equiv 1 \bmod 2^{n-1}$ then $C_G(\langle x\rangle) = \langle x, a\rangle$ non-cyclic means this group extension splits. And if $r \not\equiv 1 \bmod 2^{n-1}$ then it again splits by $n \geq 4$ and $r \equiv 1 \bmod 4$: for then $r \equiv 1 + 2^{n-2} \bmod 2^{n-1}$ implies

$$H^2(\langle x, a\rangle/\langle x\rangle, \langle x\rangle) = \overline{H}^0(\langle x, a\rangle/\langle x\rangle, \langle x\rangle) = 0.$$

Multiplying a by a power of x we may thus assume $a^2 = 1$.

Now $\langle x\rangle \subseteq \langle a, x\rangle$ are both normal in K so $yay^{-1} = x^i a$ with $i \in \boldsymbol{Z}/2^{n-1}\boldsymbol{Z}$. Then $1 = ya^2y^{-1} = (yay^{-1})^2 = x^i a x^i a = x^i x^{ir} = x^{i(1+r)}$ implies $i(1+r) \equiv 0 \bmod 2^{n-1}$ when $1 + r \equiv 2 \bmod 4$ yields $i \equiv 0 \bmod 2^{n-2}$. Since $x^{2^{n-2}} = y^2$ we conclude $yay^{-1} = y^{2j}a$ with $j \in \boldsymbol{Z}/2\boldsymbol{Z}$.

Suppose $j = 1$; we verify the Criterion with $h = a$. For $a^2 = 1$, $a \notin Q$ and $h = a_{\widetilde{K}}\, yay^{-1} = y^2 a = y^2 h$.

So $j = 0$ from now on. If now $r \not\equiv 1 \bmod 2^{n-1}$ then from $r \equiv 1 \bmod 4$ and $r^2 \equiv 1 \bmod 2^{n-1}$ we get $n \geq 4$ and $r \equiv 1 + 2^{n-2} \bmod 2^{n-1}$. Again we verify the Criterion with $h = a$; for $a^2 = 1$, $a \notin Q$ and $axa = x^{1+2^{n-2}} = xy^2$ implies $xax^{-1} = y^2 a$ i.e. $h_{\widetilde{K}}\, y^2 h$.

Thus $j = 0$ and $r \equiv 1 \bmod 2^{n-1}$ from now on. But now our relations say $a^2 = 1$ and $a \notin Q$ commutes with Q i.e. $K = Q \times \langle a\rangle$. This time the Criterion fails but we will show that we are in case b). Let $\widetilde{\theta}$ be the inflation of θ under $K \twoheadrightarrow Q$, hence $\widetilde{\theta}(a) = 1$ and $\mathrm{res}_Q \widetilde{\theta} = \theta$. By Frobenius reciprocity we have $\mathrm{ind}_Q^K \theta = \widetilde{\theta}\ \mathrm{ind}_Q^K 1 = \widetilde{\theta}(1 + \alpha)$ where α is the unique nontrivial character of K/Q inflated to K. But then $\mathrm{ind}_Q^K \theta = \widetilde{\theta} + \widetilde{\theta}\alpha$ is indeed reducible and the Proposition is completely proved.

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