Quantum Loop Algebras, Yangians and their Representations

by

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Abstract

Among representation theorists, it is well known that Yangians can be realized as some type of degenerate form of quantum loop algebras. What is not well known is precisely how this degeneration takes place. In the first part of this thesis, we will demonstrate explicitly the process by which certain quantum loop algebras related to the Lie algebras $\mathfrak{gl}_N$, $\mathfrak{o}_N$ and $\mathfrak{sp}_N$ degenerate into an associated Yangian. In the second part, we will prove a theorem which classifies all of the finite dimensional irreducible representations of Yangians over complex semisimple Lie algebras.

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Contents

Introduction

We begin in Chapter 1 by precisely defining certain Yangians and quantum loop algebras associated to the Lie algebras $\mathfrak{gl}_N$, $\mathfrak{o}_N$ and $\mathfrak{sp}_N$, and we will give a brief overview some of their most important properties. The Yangian and quantum loop algebra associated to $\mathfrak{gl}_N$ are examples of quantized enveloping algebras, which are quantum groups that can be attached to certain finite or infinite dimensional Lie algebras. In particular, they are Hopf algebras, which contain as coideal subalgebras the twisted Yangians and twisted quantum loop algebras over $\mathfrak{o}_N$ and $\mathfrak{sp}_N$ discussed below. These objects have been of importance in mathematical physics over the last 30 years or so. As we shall see, they share many properties in common with their associated Lie algebras, particularly with regard to their representation theories.

After drawing some of these connections, we will move on to Chapter 2, where we give an outline of some recent important papers which prove a statement by Drinfeld about how Yangians can be realized as some kind of limit form of quantum loop algebras. The fact that quantum loop algebras degenerate into Yangians was already well known in some vague sense prior to the publication of these papers, but the precise details were unknown except possibly to a few experts. The particular algebras treated in those papers differ from those discussed here, but we shall see that similar ideas can be used to prove analogous results in our case: we will construct an explicit isomorphism for the $\mathfrak{gl}_N$ case in Chapter 3, and then show how this same isomorphism can be used to treat the twisted orthogonal and symplectic cases in Chapters 4 and 5, respectively.

Finally, in Chapter 6 we move on to a topic which is independent of the previous chapters. Namely, we will provide a complete statement and proof of a theorem which classifies all of the finite dimensional irreducible representations of Yangians over complex semisimple Lie algebras. This result is very reminiscent of the classification theorem for the Lie algebra itself; it asserts that such representations are parametrized by monic polynomials over $\mathbb{C}$, and these polynomials keep track of certain 'weights' with respect to the action of some commutative subalgebra. Given the above relationship between Yangians and quantum loop algebras, it is unsurprising that a similar result is true for the latter. A proof of the classification of finite dimensional irreducible representations of quantum loop algebras was published in the 1990's, but a complete proof has never appeared for Yangians. This chapter therefore serves to fill a gap in the literature.

Throughout this thesis, it is assumed that the reader has a modest understanding of the theory of complex semisimple Lie algebras and their representations. One can refer to the books [9] and [10] for this theory.

Chapter 1

Preliminaries

The contents of this chapter come primarily from [11] and [12]. Full justification for all the results that follow may be found therein, but we state the majority of it without proof.

1.1 Yangian for $\mathfrak{gl}_N$

Definition 1.1 *The Yangian for $\mathfrak{gl}_N$ is the unital associative algebra over $\mathbb{C}$ generated by $\{t_{ij}^{(r)} \mid 1 \le i, j \le N,\ r \in \mathbb{Z}_+\}$ where $t_{ij}^{(0)} = \delta_{ij}$, with defining relations given by*

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}. \tag{1.1}$$

We denote this algebra by $Y(\mathfrak{gl}_N)$. More generally, the Yangian can be defined as an algebra over $\mathbb{C}[h]$ by introducing a factor of h on the right hand side of (1.1). In this case, we denote the Yangian by $Y_h(\mathfrak{gl}_N)$. The following proposition illustrates that these two definitions are equivalent.

Proposition 1.1 *Given any nonzero $a \in \mathbb{C}$, the map $t_{ij}^{(r)} \mapsto a^r t_{ij}^{(r)}$ yields an isomorphism $Y_h(\mathfrak{gl}_N)/(h-a) \xrightarrow{\sim} Y_h(\mathfrak{gl}_N)/(h-1) = Y(\mathfrak{gl}_N)$.*

Note that we can express the defining relations (1.1) in a more compact way by using formal power series. For each i, j, let $t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \in Y(\mathfrak{gl}_N)[[u^{-1}]]$. Then (1.1) is equivalent to the relation

$$(u - v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u). \tag{1.2}$$

To recover (1.1), just compare the coefficients of $u^{-r}v^{-s}$ on each side of (1.2). Here, u and v are formal variables which commute with each other and also with every element of $Y(\mathfrak{gl}_N)$.

If we multiply each side of (1.2) by the series $\sum_{p=0}^{\infty} u^{-p-1} v^p$, we deduce that we can also express (1.1) in the following way:

Proposition 1.2 *Relation* (1.1) *is equivalent to the system of equations given by*

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min\{r,s\}} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \right). \tag{1.3}$$

Let us now outline some notation which will be useful throughout. Suppose that $\mathcal{A}$ is some associative algebra, and let $T(u)$ be any element of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N) \otimes \mathcal{A}[[u^{-1}]]$. We can write $T(u)$ in the form

$$T(u) = \sum_{i,j} E_{ij} \otimes X_{ij}(u)$$

for some $X_{ij}(u) \in \mathcal{A}[[u^{-1}]]$, where E_{ij} is the usual elementary matrix. If m is any fixed positive integer, then for any $a \in \{1, \ldots, m\}$, we denote by $T_a(u)$ the element of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes m} \otimes \mathcal{A}[[u^{-1}]]$ which corresponds to $T(u)$ with the E_{ij} terms occupying the a'th copy of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$; that is,

$$T_a(u) = \sum_{i,j} 1^{\otimes(a-1)} \otimes E_{ij} \otimes 1^{\otimes(m-a)} \otimes X_{ij}(u)$$

where 1 is the $N \times N$ identity matrix. Similarly, for any $C \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$, we may write

$$C = \sum_{i,j,k,l} c_{ijkl} E_{ij} \otimes E_{kl}$$

for some $c_{ijkl} \in \mathbb{C}$, and for any $a, b \in \{1, \ldots, m\}$ with $a < b$, we define the element $C_{ab} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes m}$ by

$$C_{ab} = \sum_{i,j,k,l} c_{ijkl} 1^{\otimes(a-1)} \otimes E_{ij} \otimes 1^{\otimes(b-a-1)} \otimes E_{kl} \otimes 1^{\otimes(m-b)}.$$

When N is even, we shall use a prime to denote the involution on the set of indices $\{1, \ldots, N\}$ given by

$$i' = \begin{cases} i-1 & i \text{ even} \\ i+1 & i \text{ odd.} \end{cases}$$

Finally, if $a(u) \in \mathcal{A}[[u^{-1}]]$ and $b(u) \in \mathcal{B}[[u^{-1}]]$ where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary associative algebras, then assignments of the form

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{B} \\ a(u) &\mapsto b(u) \end{aligned} \tag{1.4}$$

will be understood as the map which sends each coefficient from $a(u)$ to the corresponding coefficient from $b(u)$. Meanwhile, if $A(u) = [A_{ij}(u)] \in \mathrm{M}_n(\mathcal{A})[[u^{-1}]]$ and $B(u) = [B_{ij}(u)] \in \mathrm{M}_n(\mathcal{B})[[u^{-1}]]$, then the assignment

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{B} \\ A(u) &\mapsto B(u) \end{aligned} \tag{1.5}$$

means that for each i, j, $A_{ij}(u) \mapsto B_{ij}(u)$ in accordance with (1.4).

The defining relation (1.1) is commonly presented in terms of a matrix equation. Let

$$T(u) = \sum_{i,j=1}^{N} E_{ij} \otimes t_{ij}(u) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]].$$

Note that we can regard $T(u)$ as the $n \times n$ matrix whose ij'th entry is $t_{ij}(u)$. In particular, we can break up this matrix into a power series in u^{-1}, the coefficient of u^{-r} being the matrix whose ij'th entry is $t_{ij}^{(r)}$. Observe that the constant term of this power series is the identity matrix, hence this series has an inverse which we denote by $T^{-1}(u)$; see (1.12) below.

Define the *Yang R-matrix*

$$R(u) = 1 - h \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji} u^{-1} \tag{1.6}$$

where we have used 1 as shorthand for $1 \otimes 1 \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$. This matrix is the simplest nontrivial solution to the *Yang-Baxter Equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{1.7}$$

This is an important equation in mathematical physics, and generating its solutions was the main motivation behind the discovery of Yangians, quantum loop algebras and other so-called quantum groups.

It will also be useful to introduce the 'transposed' Yangian R-matrix

$$R^t(u) := 1 - h \sum_{i,j=1}^{N} E_{ij} \otimes E_{ij} u^{-1}. \tag{1.8}$$

Proposition 1.3 *The defining relation* (1.1) *is equivalent to the matrix equation*

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \tag{1.9}$$

where we have identified $R(u-v)$ with the element $R(u-v) \otimes 1 \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes 2} \otimes Y(\mathfrak{gl}_N)(u,v)$.

Equation (1.9) is called the *RTT relation* for $Y(\mathfrak{gl}_N)$. To obtain (1.1) from (1.9), simply apply the left and right hand sides to the elements $e_j \otimes e_l \otimes 1 \in \mathbb{C}^N \otimes \mathbb{C}^N \otimes Y(\mathfrak{gl}_N)$ and then compare $Y(\mathfrak{gl}_N)$-coefficients. But it is often more convenient to work with the RTT relation and other such matrix equations for the Yangians, especially for studying their representations and in particular when working with maps of the form (1.5).

The representation theory of $Y(\mathfrak{gl}_N)$ is closely related to that of the Lie algebra $\mathfrak{gl}_N$, and one indication of this is the fact that relation (1.1) can actually be found within the universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}_N)$ (after replacing the $t_{ij}^{(r)}$ by certain elements from the enveloping algebra); in fact, we can view $\mathfrak{U}(\mathfrak{gl}_N)$ as a subalgebra of $Y(\mathfrak{gl}_N)$ via the embedding $E_{ij} \hookrightarrow t_{ij}^{(1)}$. Moreover, the map

$$\pi_N : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1} \tag{1.10}$$

is a surjective algebra homomorphism from $Y(\mathfrak{gl}_N)$ onto $\mathfrak{U}(\mathfrak{gl}_N)$, called the *evaluation homomorphism.* This enables us to pull back representations of $\mathfrak{gl}_N$ and view them as modules over the Yangian. By the surjectivity of π_N, a $Y(\mathfrak{gl}_N)$-invariant subspace must also be invariant under the action of $\mathfrak{U}(\mathfrak{gl}_N)$, hence the pullback of an irreducible representation of $\mathfrak{gl}_N$ remains irreducible over $Y(\mathfrak{gl}_N)$.

This connection is used extensively in the classification of finite dimensional irreducible representations of $Y(\mathfrak{gl}_N)$; as we shall see in Chapter 6, a key role in the proof of the classification theorem is played by the modules obtained by taking tensor products of so-called *evaluation modules*, which are simply the pullback to $Y(\mathfrak{gl}_N)$ of irreducible highest weight modules over $\mathfrak{gl}_N$.

Note that $Y(\mathfrak{gl}_N)$ is a Hopf algebra with comultiplication Δ, antipode S and counit ε given by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u) \tag{1.11}$$

$$S : T(u) \mapsto T^{-1}(u) \tag{1.12}$$

$$\varepsilon : T(u) \mapsto 1. \tag{1.13}$$

The coassociativity of Δ guarantees that the tensor product of the evaluation modules is well defined.

We see one more similarity of the representation theories of $Y(\mathfrak{gl}_N)$ and $\mathfrak{U}(\mathfrak{gl}_N)$ in the existence of a PBW basis:

Theorem 1.1 (Poincaré Birkhoff-Witt) *Given any total ordering on the collection of generators $t_{ij}^{(r)}$, a basis of $Y(\mathfrak{gl}_N)$ is provided by the set of all ordered monomials in these generators.*

1.2 Orthogonal and Symplectic Twisted Yangians

We will now define in terms of generators and relations the twisted Yangians corresponding to the classical Lie algebras of orthogonal and symplectic type. We will see that the twisted Yangians actually embed into the Yangian for $\mathfrak{gl}_N$, and we will present the twisted analogue of most of the results from the previous section.

Let $G = [g_{ij}]$ be the matrix associated to some nondegenerate bilinear form on $\mathbb{C}^N$ which is either symmetric or alternating (note that in the alternating case, the nondegeneracy of G implies that N must be even). Following the notation of [11], whenever we use the symbols $\pm$ or $\mp$, the sign on the top will correspond to the symmetric case, while the sign on the bottom corresponds to the alternating case.

Let $\mathfrak{g}_N$ be the orthogonal Lie algebra $\mathfrak{o}_N$ if G is symmetric; otherwise, if G is alternating, let $\mathfrak{g}_N$ be the symplectic Lie algebra $\mathfrak{sp}_N$. Then $\mathfrak{g}_N$ is isomorphic to the Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements

$$F_{ij} = \sum_{k=1}^{N} (E_{ik} g_{kj} \mp E_{jk} g_{ki}).$$

Definition 1.2 *The twisted Yangian for $\mathfrak{g}_N$ is the unital associative algebra over $\mathbb{C}[h]$ generated by $\{s_{ij}^{(r)} \mid 1 \le i, j \le N, r \in \mathbb{Z}_+\}$ where $s_{ij}^{(0)} = g_{ij}$, with defining relations given by the matrix equations*

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v) \tag{1.14}$$

$$S^t(-u) = \pm S(u) + h\frac{S(u) - S(-u)}{2u} \tag{1.15}$$

where

$$S(u) = \sum_{i,j=1}^{N} E_{ij} \otimes s_{ij}(u) \tag{1.16}$$

with $s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)} u^{-r}$. $S^t(u)$ is the transposition of $S(u)$ in one of its two factors:

$$S^t(u) = \sum_{i,j=1}^{N} E_{ij} \otimes s_{ji}(u).$$

We shall denote this algebra by $Y_h^{tw}(\mathfrak{g}_N)$. One might think that our notation should depend somehow on G, but it turns out that the twisted Yangian is independent of the choice of bilinear form:

Proposition 1.4 *Let G and G' be the matrices associated to any two nondegenerate bilinear forms on $\mathbb{C}^n$ which are either both symmetric or both alternating. Then the twisted Yangians corresponding with G and G' are isomorphic to each other.*

Let us therefore fix for the remainder of our discussions that G is the identity matrix in the symmetric case, and the matrix $\sum_{k=1}^{N/2} E_{2k-1,2k} - E_{2k,2k-1}$ in the alternating case.

The twisted Yangian is a deformation of the universal enveloping algebra of the twisted current algebra $\mathfrak{g}_N^{tw}[s]$ which is defined in the following way:

Definition 1.3 *Let σ be the automorphism of $\mathfrak{gl}_N$ given by*

$$\sigma(E_{ij}) = -E_{ji} \tag{1.17}$$

if G is symmetric, while

$$\sigma(E_{ij}) = (-1)^{i+j-1} E_{j'i'} \tag{1.18}$$

if G is alternating. The twisted current algebra is the subalgebra of $\mathfrak{gl}_N[s]$ given by

$$\mathfrak{g}_N^{tw}[s] = \{A(s) \in \mathfrak{gl}_N[s] : \sigma(A(s)) = A(-s)\}. \tag{1.19}$$

A basis of $\mathfrak{o}_N^{tw}[s]$ is provided by all the elements $(E_{ij} - E_{ji})s^r$ with $i < j$ when r is even and $(E_{ij} + E_{ji})s^r$ with $i \leq j$ when r is odd.

Meanwhile, a basis of $\mathfrak{sp}_N^{tw}[s]$ is provided by all the elements $((-1)^{j-1}E_{ij'} + (-1)^{i-1}E_{ji'})s^r$ with $i \leq j$ when r is even, and $((-1)^{j-1}E_{ij'} - (-1)^{i-1}E_{ji'})s^r$ with $i < j$ when r is odd.

The twisted Yangians can be regarded as subalgebras of the Yangian for $\mathfrak{gl}_N$:

Proposition 1.5 *The map $S(u) \mapsto T(u)GT^t(-u)$ provides an embedding of the twisted Yangian $Y^{tw}(\mathfrak{g}_N) = Y_h^{tw}(\mathfrak{g}_n)/(h-1)$ into $Y(\mathfrak{gl}_N)$.*

We can lift this map to an embedding $Y_h^{tw}(\mathfrak{g}_N) \hookrightarrow Y_h(\mathfrak{gl}_N)$. Explicitly, this embedding is given by

$$s_{ij}^{(r)} \mapsto \sum_{k=1}^{N} \left(g_{kj} t_{ik}^{(r)} + (-1)^r g_{ik} t_{jk}^{(r)} \right) + h \sum_{k,l=1}^{N} \sum_{p=1}^{r-1} (-1)^{r-p} g_{kl} t_{ik}^{(p)} t_{jl}^{(r-p)}. \tag{1.20}$$

In particular, taking G to be the identity matrix, we will make the identification

$$s_{ij}^{(r)} = t_{ij}^{(r)} + (-1)^r t_{ji}^{(r)} + h \sum_{k=1}^{N} \sum_{p=1}^{r-1} (-1)^{r-p} t_{ik}^{(p)} t_{jk}^{(r-p)} \in Y_h^{tw}(\mathfrak{o}_N) \subset Y_h(\mathfrak{gl}_N).$$

Taking $G = \sum_{k=1}^{N/2} E_{2k-1,2k} - E_{2k,2k-1}$, we will make the identification

$$\begin{aligned} s_{ij}^{(r)} &= (-1)^j t_{i,j'}^{(r)} - (-1)^{r+i} t_{j,i'}^{(r)} \\ &+ h \sum_{k=1}^{N/2} \sum_{p=1}^{r-1} (-1)^{r-p} (t_{i,2k-1}^{(p)} t_{j,2k}^{(r-p)} - t_{i,2k}^{(p)} t_{j,2k-1}^{(r-p)}) \in Y_h^{tw}(\mathfrak{sp}_N) \subset Y_h(\mathfrak{gl}_N). \end{aligned}$$

We can draw connections between the representation theory of the twisted Yangian and that of the Lie algebra $\mathfrak{g}_N$ with the same techniques we used for $\mathfrak{gl}_N$. Namely, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_N)$ can be viewed as a subalgebra of $Y_h^{tw}(\mathfrak{g}_N)$ via the embedding $F_{ij} \hookrightarrow s_{ij}^{(1)}$. Furthermore, the map given by

$$\varrho_N : s_{ij}(u) \mapsto g_{ij} + F_{ij} \left(u \pm \frac{1}{2} \right)^{-1} \tag{1.21}$$

is a surjective $\mathbb{C}$-algebra homomorphism from $Y_h^{tw}(\mathfrak{g}_N)/(h-1)$ onto $\mathfrak{U}(\mathfrak{g}_N)$. This again allows us to pull back representations of $\mathfrak{g}_N$ and view them as modules over $Y_h^{tw}(\mathfrak{g}_N)$, and an irreducible representation of $\mathfrak{g}_N$ remains irreducible over $Y_h^{tw}(\mathfrak{g}_N)$.

We conclude this section by stating the PBW theorem for the twisted Yangian.

Theorem 1.2 *Given any total ordering on the collection of generators $s_{ij}^{(r)}$ with $i \geq j$ if $r > 0$ is even and with $i > j$ if r is odd, a basis of $Y_h^{tw}(\mathfrak{o}_N)$ is provided by the set of all ordered monomials in these generators.*

Similarly, given any total ordering on the collection of generators $s_{ij}^{(r)}$ with $i > j$ if $r > 0$ is even and with $i \geq j$ if r is odd, a basis of $Y_h^{tw}(\mathfrak{sp}_N)$ is provided by the set of all ordered monomials in these generators.

1.3 Quantum Loop Algebras

We will now introduce the quantum loop algebra associated to $\mathfrak{gl}_N$. Towards this end, we need to define the quantum analogue of the Yang R-matrix:

Definition 1.4 *Let q be a nonzero complex parameter. The quantum affine R-matrix is the element $R_q(u,v) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[u,v]$ given by*

$$R_q(u,v) = \sum_{i,j=1}^{N} (uq^{-\delta_{ij}} - vq^{\delta_{ij}}) E_{ii} \otimes E_{jj} - (q-q^{-1})u \sum_{\substack{i,j=1 \\ i>j}}^{N} E_{ij} \otimes E_{ji}$$
$$- (q-q^{-1})v \sum_{\substack{i,j=1 \\ i<j}}^{N} E_{ij} \otimes E_{ji}. \tag{1.22}$$

For the twisted quantum loop algebras, we will also need the 'transposed' quantum affine R-matrix

$$R_q^t(u,v) := \sum_{i,j=1}^{N} (uq^{-\delta_{ij}} - vq^{\delta_{ij}}) E_{ii} \otimes E_{jj} - (q-q^{-1})u \sum_{\substack{i,j=1 \\ i>j}}^{N} E_{ji} \otimes E_{ji}$$
$$- (q-q^{-1})v \sum_{\substack{i,j=1 \\ i<j}}^{N} E_{ji} \otimes E_{ji}. \tag{1.23}$$

Definition 1.5 *The quantum loop algebra $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$ is the unital associative algebra over $\mathbb{C}(q)$ generated by $\{T_{ij}^{(r)}, \overline{T}_{ij}^{(r)} \mid 1 \leq i,j \leq N,\ r \in \mathbb{Z}_+\}$, with defining relations given by*

$$R_q(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_q(u,v) \tag{1.24}$$
$$R_q(u,v)\overline{T}_1(u)\overline{T}_2(v) = \overline{T}_2(v)\overline{T}_1(u)R_q(u,v) \tag{1.25}$$
$$R_q(u,v)\overline{T}_1(u)T_2(v) = T_2(v)\overline{T}_1(u)R_q(u,v) \tag{1.26}$$
$$T_{ij}^{(0)} = 0 = \overline{T}_{ji}^{(0)} \ \text{if} \ 1 \leq i < j \leq N \tag{1.27}$$
$$T_{ii}^{(0)}\overline{T}_{ii}^{(0)} = 1 = \overline{T}_{ii}^{(0)}T_{ii}^{(0)} \ \forall\ 1 \leq i \leq N \tag{1.28}$$

where

$$T(u) = \sum_{i,j=1}^{N} E_{ij} \otimes T_{ij}(u), \qquad \overline{T}(u) = \sum_{i,j=1}^{N} E_{ij} \otimes \overline{T}_{ij}(u) \tag{1.29}$$

with $T_{ij}(u) = \sum_{r=0}^{\infty} T_{ij}^{(r)} u^{-r}$ and $\overline{T}_{ij}(u) = \sum_{r=0}^{\infty} \overline{T}_{ij}^{(r)} u^{r}$.

We can state the relations (1.24), (1.25) and (1.26) in a more explicit way; by applying each side of these equations to the elements $e_j \otimes e_l \otimes 1 \in \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$ and comparing $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$-coefficients, we deduce the following result.

Proposition 1.6 *Relations* (1.24)*,* (1.25) *and* (1.26) *are respectively equivalent to the relations*

$$\begin{aligned}&(q^{-\delta_{ik}}T_{ij}^{(r+1)}T_{kl}^{(s)} - q^{\delta_{ik}}T_{ij}^{(r)}T_{kl}^{(s+1)}) - (q^{-\delta_{jl}}T_{kl}^{(s)}T_{ij}^{(r+1)} - q^{\delta_{jl}}T_{kl}^{(s+1)}T_{ij}^{(r)})\\ &= (q-q^{-1})(\delta_{i>k}T_{kj}^{(r+1)}T_{il}^{(s)} + \delta_{i<k}T_{kj}^{(r)}T_{il}^{(s+1)}) - (q-q^{-1})(\delta_{l>j}T_{kj}^{(s)}T_{il}^{(r+1)} + \delta_{l<j}T_{kj}^{(s+1)}T_{il}^{(r)})\end{aligned} \tag{1.30}$$

$$\begin{aligned}&(q^{-\delta_{ik}}\overline{T}_{ij}^{(r-1)}\overline{T}_{kl}^{(s)} - q^{\delta_{ik}}\overline{T}_{ij}^{(r)}\overline{T}_{kl}^{(s-1)}) - (q^{-\delta_{jl}}\overline{T}_{kl}^{(s)}\overline{T}_{ij}^{(r-1)} - q^{\delta_{jl}}\overline{T}_{kl}^{(s-1)}\overline{T}_{ij}^{(r)})\\ &= (q-q^{-1})(\delta_{i>k}\overline{T}_{kj}^{(r-1)}\overline{T}_{il}^{(s)} + \delta_{i<k}\overline{T}_{kj}^{(r)}\overline{T}_{il}^{(s-1)}) - (q-q^{-1})(\delta_{l>j}\overline{T}_{kj}^{(s)}\overline{T}_{il}^{(r-1)} + \delta_{l<j}\overline{T}_{kj}^{(s-1)}\overline{T}_{il}^{(r)})\end{aligned} \tag{1.31}$$

$$\begin{aligned}&(q^{-\delta_{ik}}\overline{T}_{ij}^{(r-1)}T_{kl}^{(s)} - q^{\delta_{ik}}\overline{T}_{ij}^{(r)}T_{kl}^{(s+1)}) - (q^{-\delta_{jl}}T_{kl}^{(s)}\overline{T}_{ij}^{(r-1)} - q^{\delta_{jl}}T_{kl}^{(s+1)}\overline{T}_{ij}^{(r)})\\ &= (q-q^{-1})(\delta_{i>k}\overline{T}_{kj}^{(r-1)}T_{il}^{(s)} + \delta_{i<k}\overline{T}_{kj}^{(r)}T_{il}^{(s+1)}) - (q-q^{-1})(\delta_{l>j}T_{kj}^{(s)}\overline{T}_{il}^{(r-1)} + \delta_{l<j}T_{kj}^{(s+1)}\overline{T}_{il}^{(r)})\end{aligned} \tag{1.32}$$

The quantum loop algebra $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$ contains as subalgebras the twisted quantum loop algebras associated to $\mathfrak{o}_N$ and $\mathfrak{sp}_N$. We shall first define these algebras independently in terms of generators and relations, and then show how they embed into $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$.

Definition 1.6 *The twisted quantum loop algebra* $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ *is the unital associative algebra over* $\mathbb{C}(q)$ *generated by* $\{S_{ij}^{(r)} \mid 1 \le i,j \le N,\ r \in \mathbb{Z}_+\}$*, with defining relations given by*

$$S_{ij}^{(0)} = 0 \text{ if } i<j \tag{1.33}$$

$$S_{ii}^{(0)} = 1 \ \forall\ 1 \le i \le N \tag{1.34}$$

$$R_q(u,v)S_1(u)R_q^t(u^{-1},v)S_2(v) = S_2(v)R_q^t(u^{-1},v)S_1(u)R_q(u,v) \tag{1.35}$$

where

$$S(u) = \sum_{i,j=1}^{N} E_{ij} \otimes S_{ij}(u) \tag{1.36}$$

with $S_{ij}(u) = \sum_{r=0}^{\infty} S_{ij}^{(r)} u^{-r}$*.*

The algebra $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ is a deformation of the enveloping algebra of the twisted loop algebra $\mathfrak{o}_N^{tw}[s,s^{-1}]$ which is defined in the following way.

Definition 1.7 *The twisted loop algebra* $\mathfrak{o}_N^{tw}[s,s^{-1}]$ *is the Lie subalgebra of* $\mathcal{L}(\mathfrak{gl}_N) = \mathfrak{gl}_N[s,s^{-1}]$ *given by*

$$\mathfrak{o}_N^{tw}[s,s^{-1}] = \{A(s) \in \mathcal{L}(\mathfrak{gl}_N) : \sigma(A(s)) = A(s^{-1})\} \tag{1.37}$$

where σ *is the automorphism* (1.17)*. This algebra is also denoted by* $\mathcal{L}^{tw}(\mathfrak{o}_N)$*.*

A basis for $\mathfrak{o}_N^{tw}[s,s^{-1}]$ is provided by all the elements $E_{ij}s^r - E_{ji}s^{-r}$ with $1 \le i,j \le N$ and $r \ge 0$ except that, when $r=0$, only $E_{ij} - E_{ji}$ with $i<j$ should be included to obtain a basis.

Proposition 1.7 *The assignment $S(u) \mapsto T(u)\overline{T}(u^{-1})^t$ extends to an embedding of the twisted quantum loop algebra $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ into $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$.*

This embedding can be written more explicitly as

$$S_{ij}^{(r)} \mapsto \sum_{k=1}^{N}\sum_{p=0}^{r} T_{ik}^{(p)}\overline{T}_{jk}^{(r-p)}.$$

We will also use the notation $S_{ij}^{(r)}$ for the generators of the twisted quantum loop algebra for $\mathfrak{sp}_N$. It shall always be clear whether we are working in the orthogonal or symplectic case.

Definition 1.8 *Suppose N is even. The twisted quantum loop algebra $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ is the unital associative algebra over $\mathbb{C}(q)$ generated by $\{S_{ij}^{(r)} \mid 1 \le i,j \le N,\ r \in \mathbb{Z}_+\}$ and $\{S_{ii'}^{(0)^{-1}} \mid i = 1,3,\ldots,N-1\}$, with defining relations given by*

$$S_{ij}^{(0)} = 0 \text{ whenever } i<j \text{ and } j \ne i' \tag{1.38}$$

$$S_{i'i'}^{(0)}S_{ii}^{(0)} - q^2 S_{i'i}^{(0)}S_{ii'}^{(0)} = q^3 \ \forall\ i = 1,3,\ldots,N-1 \tag{1.39}$$

$$S_{ii'}^{(0)}S_{ii'}^{(0)^{-1}} = S_{ii'}^{(0)^{-1}}S_{ii'}^{(0)} = 1 \ \forall\ i = 1,3,\ldots,N-1 \tag{1.40}$$

$$R_q(u,v)S_1(u)R_q^t(u^{-1},v)S_2(v) = S_2(v)R_q^t(u^{-1},v)S_1(u)R_q(u,v) \tag{1.41}$$

where

$$S(u) = \sum_{i,j=1}^{N} E_{ij} \otimes S_{ij}(u) \tag{1.42}$$

with $S_{ij}(u) = \sum_{r=0}^{\infty} S_{ij}^{(r)}u^{-r}$.

The algebra $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ is a deformation of the enveloping algebra of the twisted loop algebra $\mathfrak{sp}_N^{tw}[s,s^{-1}]$ which is defined in the following way.

Definition 1.9 *The twisted loop algebra $\mathfrak{sp}_N^{tw}[s,s^{-1}]$ is the Lie subalgebra of $\mathcal{L}(\mathfrak{gl}_N) = \mathfrak{gl}_N[s,s^{-1}]$ given by*

$$\mathfrak{sp}_N^{tw}[s,s^{-1}] = \{A(s) \in \mathcal{L}(\mathfrak{gl}_N) : \sigma(A(s)) = A(s^{-1})\} \tag{1.43}$$

where σ is the automorphism (1.18)*. This algebra is also denoted by $\mathcal{L}^{tw}(\mathfrak{sp}_N)$.*

A basis for $\mathfrak{sp}_N^{tw}[s,s^{-1}]$ is provided by all the elements $(-1)^j E_{ij'}s^r + (-1)^i E_{ji'}s^{-r}$ with $1 \le i,j \le N$ and $r \ge 0$ except that, when $r=0$, only $(-1)^j E_{ij'} + (-1)^i E_{ji'}$ with $i \le j$ should be included to obtain a basis.

Proposition 1.8 *The assignment* $S(u) \mapsto T(u)B\overline{T}(u^{-1})^t$*, where*

$$B = \sum_{k=1}^{N/2} qE_{2k-1,2k} - E_{2k,2k-1} \tag{1.44}$$

extends to an embedding of the twisted quantum loop algebra $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ *into* $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$.

This embedding can be written more explicitly as

$$S_{ij}^{(r)} \mapsto \sum_{k=1}^{N/2} \sum_{p=0}^{r} (qT_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(r-p)} - T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(r-p)}).$$

Chapter 2

Motivation

In this chapter, we will begin by defining the Yangians and quantum loop algebras associated to an arbitrary complex semisimple Lie algebra. While it is well known that these Yangians are limit forms of the associated quantum loop algebra, a precise statement of this fact only appeared recently in a paper of Drinfeld. A complete proof was published recently in both [7] and [8], although the results of [7] were a bit stronger. We give a summary of these two papers as motivation for the chapters that follow.

Throughout this chapter, let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan matrix $C = (c_{ij})_{i,j\in I}$ where I indexes a basis of simple roots in $\mathfrak{g}$. Then there exists a set of coprime positive integers $\{d_i\}_{i\in I}$ such that the matrix $(d_i c_{ij})_{i,j\in I}$ is symmetric. We begin by defining the Yangians and quantum loop algebras associated to $\mathfrak{g}$.

Definition 2.1 *The Yangian $Y(\mathfrak{g})$ is the unital associative algebra over $\mathbb{C}$ generated by $\{X^{\pm}_{i,r},\ H_{i,r} \mid i \in I,\ r \in \mathbb{Z}_+\}$, with defining relations given by*

$$[H_{i,r}, H_{j,s}] = 0, \quad [H_{i,0}, X^{\pm}_{j,s}] = \pm d_i c_{ij} X^{\pm}_{j,s}; \tag{2.1}$$

$$[H_{i,r+1}, X^{\pm}_{j,s}] - [H_{i,r}, X^{\pm}_{j,s+1}] = \pm\frac{d_i c_{ij}}{2}(H_{i,r}X^{\pm}_{j,s} + X^{\pm}_{j,s}H_{i,r}); \tag{2.2}$$

$$[X^{+}_{i,r}, X^{-}_{j,s}] = \delta_{ij} H_{i,r+s}; \tag{2.3}$$

$$[X^{\pm}_{i,r+1}, X^{\pm}_{j,s}] - [X^{\pm}_{i,r}, X^{\pm}_{j,s+1}] = \pm\frac{d_i c_{ij}}{2}(X^{\pm}_{i,r}X^{\pm}_{j,s} + X^{\pm}_{j,s}X^{\pm}_{i,r}); \tag{2.4}$$

$$\sum_{\pi\in S_m}\Big[X^{\pm}_{i,r_{\pi(1)}}, \Big[\ldots, [X^{\pm}_{i,r_{\pi(m)}}, X^{\pm}_{j,s}]\ldots\Big]\Big] = 0 \ \forall r_1\ldots, r_m, s \geq 0 \ \textit{if}\ \ i \neq j \tag{2.5}$$

where $m = 1 - c_{ij}$.

$Y(\mathfrak{g})$ can also be defined as an algebra over the polynomial ring $\mathbb{C}[h]$ by introducing a factor of h on the right hand side of relations (2.2) and (2.4). In this case, we denote the Yangian by $Y_h(\mathfrak{g})$.

We need some more notation in order to define the quantum loop algebra. Suppose that q and h are formal variables related via the equation $q^2 = e^h$. When k and n are nonnegative integers with $k \leq n$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$
$$[n]_q! = [n]_q[n-1]_q \dots [1]_q, \qquad \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Finally, for each $i \in I$, let $q_i = q^{d_i}$.

Definition 2.2 *The quantum loop algebra $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))$ is the unital associative algebra topologically generated over $\mathbb{C}[[h]]$ by $\{\mathcal{X}^{\pm}_{i,r}, \mathcal{H}_{i,r} \mid i \in I, \ r \in \mathbb{Z}\}$, with defining relations given by*

$$[\mathcal{H}_{i,r}, \mathcal{H}_{i,s}] = 0, \qquad [\mathcal{H}_{i,0}, \mathcal{X}^{\pm}_{j,s}] = \pm c_{ij}\mathcal{X}^{\pm}_{j,s};$$
$$[\mathcal{H}_{i,r}, \mathcal{X}^{\pm}_{j,s}] = \pm\frac{[ra_{ij}]_{q_i}}{r}\mathcal{X}^{\pm}_{j,r+s}, \ r \neq 0;$$
$$\mathcal{X}^{\pm}_{i,r+1}\mathcal{X}^{\pm}_{j,s} - q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r+1} = q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{i,r}\mathcal{X}^{\pm}_{j,s+1} - \mathcal{X}^{\pm}_{j,s+1}\mathcal{X}^{\pm}_{i,r};$$
$$[\mathcal{X}^{+}_{i,r}, \mathcal{X}^{-}_{j,s}] = \delta_{ij}\frac{\Psi^{+}_{i,r+s} - \Psi^{-}_{i,r+s}}{q_i - q_i^{-1}};$$
$$\sum_{\pi \in S_m}\sum_{k=0}^{m}(-1)^k \left[\begin{array}{c} m \\ k \end{array}\right]_{q_i} \mathcal{X}^{\pm}_{i,r_{\pi(1)}} \cdots \mathcal{X}^{\pm}_{i,r_{\pi(k)}}\mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r_{\pi(k+1)}} \cdots \mathcal{X}^{\pm}_{i,r_{\pi(m)}} = 0$$
$$\forall r_1, \dots, r_m, s \in \mathbb{Z} \text{ if } i \neq j$$

where $m = 1 - c_{ij}$, and the elements $\Psi^{\pm}_{i,r}$ are defined by the equation

$$\sum_{r=0}^{\infty}\Psi^{\pm}_{i,\pm r}z^{-r} = \exp\left(\pm\frac{hd_i}{2}\mathcal{H}_{i,0}\right)\exp\left(\pm(q_i - q_i^{-1})\sum_{s=1}^{\infty}\mathcal{H}_{i,\pm s}z^{-s}\right)$$

and $\Psi^{\pm}_{i,r} = 0$ when $\mp r > 0$.

Below, we denote by $\mathfrak{U}(\mathcal{L}(\mathfrak{g}))$ the universal enveloping algebra of the loop algebra $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}[s, s^{-1}]$:

Proposition 2.1 (Prop. 1.1 of [8]) $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))/h\mathfrak{U}_h(\mathcal{L}(\mathfrak{g})) \cong \mathfrak{U}(\mathcal{L}(\mathfrak{g}))$, *and $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))$ is isomorphic to $\mathfrak{U}(\mathcal{L}(\mathfrak{g}))[[h]]$ as $\mathbb{C}[[h]]$-modules.*

This proposition is used by Guay and Ma to prove the following theorem, which details precisely how $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))$ degenerates into $Y_h(\mathfrak{g})$:

Theorem 2.1 (Theorem 2.2 of [8]) *Let $\mathbf{K}$ be the kernel of the composite of algebra homomorphisms given by*

$$\mathfrak{U}_h(\mathcal{L}(\mathfrak{g})) \twoheadrightarrow \mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))/h\mathfrak{U}(\mathcal{L}(\mathfrak{g})) \xrightarrow{\sim} \mathfrak{U}(\mathcal{L}(\mathfrak{g})) \overset{s\mapsto 1}{\twoheadrightarrow} \mathfrak{U}(\mathfrak{g}). \tag{2.6}$$

Then there exists an isomorphism

$$Y_h(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathbf{K}^n/\mathbf{K}^{n+1} \tag{2.7}$$

where $\mathbf{K}^0 = \mathfrak{U}_h(\mathcal{L}(\mathfrak{g}))$.

This is actually a specific case of the more general theorem in [8], which shows that the twisted quantum loop algebra $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g})^\sigma)$ associated to a Dynkin diagram automorphism σ degenerates into the 'twisted' Yangian $Y_h(\mathfrak{g}, \sigma)$ via the same process (the case treated here corresponds to the case then σ is trivial; see [8] for all the precise definitions). This depends on the assumption that Proposition 2.1 is also true in the twisted case - an assumption which the authors believe is correct, although they could not find a reference for it.

Observe that $Y_h(\mathfrak{g})$ has a natural $\mathbb{N}$-grading given by assigning $\deg(H_{i,r}) = \deg(X^{\pm}_{i,r}) = r$ and $\deg(h) = 1$. We can form the completion $\widehat{Y_h(\mathfrak{g})}$ with respect to this grading; that is,

$$\widehat{Y_h(\mathfrak{g})} = \prod_{n=0}^{\infty} Y_h(\mathfrak{g})_n$$

where $Y_h(\mathfrak{g})_n$ is the span of all the homogeneous elements in $Y_h(\mathfrak{g})$ of degree n.

In the paper [7], the degeneration isomorphism of Theorem 2.1 is constructed explicitly as the inverse of a homomorphism of associated graded rings induced by a map $\mathfrak{U}_h(\mathcal{L}(\mathfrak{g})) \to \widehat{Y_h(\mathfrak{g})}$. The procedure is as follows:

For each $i \in I$ and $r \in \mathbb{Z}_+$, define $t_{i,r} \in Y_h(\mathfrak{g})$ by comparing coefficients in the equation

$$h \sum_{r \geq 0} t_{i,r} u^{-r-1} = \log \left(1 + h \sum_{r \geq 0} H_{i,r} u^{-r-1} \right).$$

Let $G(v) = \log \left(\frac{v}{e^{v/2} - e^{-v/2}} \right) \in v\mathbb{Q}[[v]]$, and let

$$\gamma_i(v) = h \sum_{r \geq 0} \frac{t_{i,r}}{r!} \left(-\frac{d}{dv} \right)^{r+1} G(v).$$

For each $i \in I$ and $r \in \mathbb{Z}_+$, define $g^{\pm}_{i,m} \in \widehat{Y_h(\mathfrak{g})}$ by comparing coefficients in the equation

$$\sum_{m=0}^{\infty} g^{\pm}_{i,m} v^m = \left(\frac{h}{q_i - q_i^{-1}} \right)^{1/2} \exp \left(\frac{\gamma_i(v)}{2} \right).$$

Finally, let $Y_h(\mathfrak{b}^{\pm}) \subset Y_h(\mathfrak{g})$ be the subalgebra generated by all the $H_{i,r}$ and the $X^{\pm}_{i,r}$, and for each $i \in I$, let $\sigma_i^{\pm}$ be the endomorphism of $Y_h(\mathfrak{b}^{\pm})$ given by

$$X^{\pm}_{j,r} \mapsto X^{\pm}_{j,r+\delta_{ij}} \qquad H_{j,r} \mapsto H_{j,r}.$$

Then the assignment

$$\Phi(\mathcal{H}_{i,0}) = d_i^{-1} t_{i,0} \qquad \Phi(\mathcal{H}_{i,r}) = \frac{h}{q-q^{-1}} \sum_{k\geq 0} t_{i,k} \frac{r^k}{k!}$$

$$\Phi(\mathcal{X}_{i,r}^{\pm}) = e^{r\sigma_i^{\pm}} \sum_{m=0}^{\infty} g_{i,m}^{\pm} X_{i,m}^{\pm}$$

defines an algebra homomorphism $\Phi : \mathfrak{U}_h(\mathcal{L}(\mathfrak{g})) \to \widehat{Y_h(\mathfrak{g})}$. Moreover, Φ maps $\mathbf{K}$ to the ideal $\widehat{Y}_+ = \prod_{n\geq 1} Y_h(\mathfrak{g})_n$, according to Theorem 6.2 in [7]. It follows that Φ induces a natural homomorphism of associated graded rings

$$\mathrm{gr}(\Phi) : \bigoplus_{n=0}^{\infty} \mathbf{K}^n/\mathbf{K}^{n+1} \to \bigoplus_{n=0}^{\infty} \widehat{Y}_+^n/\widehat{Y}_+^{n+1} = Y_h(\mathfrak{g}).$$

Finally, Proposition 6.5 in [7] asserts that $\mathrm{gr}(\Phi)$ is the inverse of the degeneration isomorphism of Theorem 2.1.

They then go on to show that the same ideas can be used to realize $Y(\mathfrak{gl}_N)$ as a degenerate form of $U_q(\mathcal{L}(\mathfrak{gl}_N))$ in terms of their so called 'Drinfeld presentations'. This leaves open the question of how the degeneration works in terms of the 'RTT presentations' which we discussed in Chapter 1; we will explore this question in Chapter 3.

It is natural to expect similar results for the twisted Yangians and quantum loop algebras, and this will be the subject of Chapters 4 and 5. The twisted case treated here is not the same as that which is discussed in [8]. The main difference is that the algebras $U_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ and $U_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ discussed here in Chapter 1 are deformations of the enveloping algebra of the Lie subalgebra of $\mathcal{L}(\mathfrak{gl}_N)$ spanned by $A(s)$ with the property that $\sigma(A(s)) = A(s^{-1})$, where σ is an involution of $\mathfrak{gl}_N$ (see (1.37) and (1.43)); on the other hand, the twisted quantum loop algebra in [8] is related to the Lie subalgebra of $\mathfrak{g}[s, s^{-1}]$ spanned by $A(s)$ such that $\sigma(A(s)) = A(-s)$, where σ is an automorphism of $\mathfrak{g}$ which comes from a Dynkin diagram automorphism.

Chapter 3

$Y(\mathfrak{gl}_N)$ as a degenerate form of $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$

In this chapter, we will demonstrate how to realize the Yangian for $\mathfrak{gl}_N$ as a limit form of the quantum loop algebra $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$ using the RTT presentation given in Chapter 1.

Let $\mathcal{A}$ be the localization of $\mathbb{C}[q, q^{-1}]$ at the ideal $(q-1)$. Let $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$ generated by the elements $\tau_{ij}^{(r)}, \overline{\tau}_{ij}^{(r)}$ given by

$$\tau_{ij}^{(r)} = \frac{T_{ij}^{(r)}}{q-q^{-1}}, \quad \overline{\tau}_{ij}^{(r)} = \frac{\overline{T}_{ij}^{(r)}}{q-q^{-1}} \text{ for } r \geq 0, 1 \leq i,j \leq N,$$

except that, when $r=0$ and $i=j$, we set

$$\tau_{ii}^{(0)} = \frac{T_{ii}^{(0)}-1}{q-1}, \quad \overline{\tau}_{ii}^{(0)} = \frac{\overline{T}_{ii}^{(0)}-1}{q-1}.$$

Theorem 3.1 (Section 3 of [12]) *The assignment $E_{ij}s^r \mapsto \tau_{ij}^{(r)}$ $\forall\ r \geq 0, 1 \leq i,j \leq N$ except if $r=0$ and $1 \leq i < j \leq N$, $-E_{ij}s^{-r} \mapsto \overline{\tau}_{ij}^{(r)}$ $\forall\ r \geq 0, 1 \leq i,j \leq N$ except if $r=0$ and $1 \leq j < i \leq N$, induces an isomorphism $\mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \otimes_{\mathcal{A}} \mathbb{C}$ where $\mathbb{C}$ is viewed as an $\mathcal{A}$-module via $\mathcal{A}/(q-1) \xrightarrow{\sim} \mathbb{C}$.*

We have the following composite of algebra homomorphisms:

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N)) \overset{s\mapsto 1}{\twoheadrightarrow} \mathfrak{U}(\mathfrak{gl}_N). \tag{3.1}$$

For $m \geq 0$, denote by K_m the Lie ideal of $\mathcal{L}(\mathfrak{gl}_N)$ spanned by $X \cdot s^r(s-1)^m$ for all $r \in \mathbb{Z}$ and $X \in \mathfrak{gl}_N$. Let U be the subspace of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))$ spanned over $\mathbb{C}$ by all the generators $\tau_{ij}^{(r)}, \overline{\tau}_{ij}^{(r)}$, and observe that

$$U \cap (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) = U \cap \mathrm{Ker}(\psi)$$

where ψ is the composite

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N)).$$

By definition, for any $X \in U$, we may write

$$X = \sum_{i,j=1}^{N} X_{ij}$$

where

$$X_{ij} = \sum_{r=0}^{n_{ij}} \left(a_{ij}^{(r)} \tau_{ij}^{(r)} + b_{ij}^{(r)} \overline{\tau}_{ij}^{(r)} \right).$$

Then clearly $\psi(X) = 0$ if and only if each $a_{ii}^{(0)} = b_{ii}^{(0)}$ and $a_{ij}^{(r)} = b_{ij}^{(r)} = 0$ when $i \neq j$ or $r \geq 1$. Therefore,

$$U \cap (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) = \mathrm{span}_{\mathbb{C}}\{\tau_{ii}^{(0)} + \overline{\tau}_{ii}^{(0)} \mid i = 1, \ldots, N\}. \tag{3.2}$$

Let $\mathbf{K}_0 = \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))$, and for $m \geq 1$ let $\mathbf{K}_m$ be the two-sided ideal of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))$ generated by

$$(q - q^{-1})^{m_0} \mathbb{K}_{m_1} \cdots \mathbb{K}_{m_k}$$

with $m_0 + m_1 + \cdots + m_k \geq m$, where $\mathbb{K}_m = \psi^{-1}(\mathsf{K}_m) \cap U$. This is slightly different from the definition of the analogous ideals $\mathbf{K}_m$ in [8] in the case of $\mathfrak{sl}_N$, because for $\mathfrak{gl}_N$, $\mathbf{K}_1^m$ is strictly smaller than $\mathbf{K}_m$. This difference can be explained by the fact that $[\mathcal{L}(\mathfrak{sl}_N), \mathcal{L}(\mathfrak{sl}_N)] = \mathcal{L}(\mathfrak{sl}_N)$ but $[\mathcal{L}(\mathfrak{gl}_N), \mathcal{L}(\mathfrak{gl}_N)] = \mathcal{L}(\mathfrak{sl}_N) \subsetneq \mathcal{L}(\mathfrak{gl}_N)$.

Let $\widetilde{Y}(\mathfrak{gl}_N)$ be the $\mathbb{C}$-algebra

$$\widetilde{Y}(\mathfrak{gl}_N) = \bigoplus_{m=0}^{\infty} \mathbf{K}_m / \mathbf{K}_{m+1}.$$

$\widetilde{Y}(\mathfrak{gl}_N)$ can be viewed as a $\mathbb{C}[h]$-algebra if we set $h = \overline{q - q^{-1}} \in \mathbf{K}_1/\mathbf{K}_2$. In this case, we denote it by $\widetilde{Y}_h(\mathfrak{gl}_N)$.

Theorem 3.2 *$\widetilde{Y}_h(\mathfrak{gl}_N)$ is isomorphic to $Y_h(\mathfrak{gl}_N)$.*

For $r, m \geq 0$, define recursively elements $T_{ij}^{(r,m)}$ in the following way:

$$T_{ij}^{(r,0)} = \tau_{ij}^{(r)} \text{ and } T_{ij}^{(r,m+1)} = T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)},$$

except that, if $i < j$, $T_{ij}^{(0,0)} = -\overline{\tau}_{ij}^{(0)}$.

Set $\xi_{ij}^{(r,m)} = \overline{T_{ij}^{(r,m)}} \in \mathbf{K}_m/\mathbf{K}_{m+1}$, which makes sense since $T_{ij}^{(r,m)} \in \mathbf{K}_m$ (one can easily check by induction on m that for every r, $\psi(T_{ij}^{(r,m)}) = E_{ij} s^r (s-1)^m \in \mathsf{K}_m$, and $T_{ij}^{(r,m)}$ is in U by definition).

Proof of theorem 3.2. We will prove that an isomorphism $\varphi : Y_h(\mathfrak{gl}_N) \xrightarrow{\sim} \widetilde{Y}_h(\mathfrak{gl}_N)$ is given by $t_{ij}^{(m+1)} \mapsto \xi_{ij}^{(0,m)}$ for $m \geq 0$.

By Proposition 1.6 we have

$$\begin{aligned}
q^{-\delta_{ik}}(T_{ij}^{(r+1)} - T_{ij}^{(r)})T_{kl}^{(s)} - T_{ij}^{(r)}(q^{\delta_{ik}}T_{kl}^{(s+1)} - q^{-\delta_{ik}}T_{kl}^{(s)})&\\
- (q^{-\delta_{jl}}T_{kl}^{(s)}(T_{ij}^{(r+1)} - T_{ij}^{(r)}) - (q^{\delta_{jl}}T_{kl}^{(s+1)} - q^{-\delta_{jl}}T_{kl}^{(s)})T_{ij}^{(r)})&\\
= (q-q^{-1})(\delta_{i>k}T_{kj}^{(r+1)}T_{il}^{(s)} + \delta_{i<k}T_{kj}^{(r)}T_{il}^{(s+1)})&\\
- (q-q^{-1})(\delta_{l>j}T_{kj}^{(s)}T_{il}^{(r+1)} + \delta_{l<j}T_{kj}^{(s+1)}T_{il}^{(r)}).&
\end{aligned}$$

If $r, s \geq 1$ then, after rearranging and dividing both sides by $(q-q^{-1})^2$, we get

$$\begin{aligned}
q^{-\delta_{ik}}(T_{ij}^{(r,1)}T_{kl}^{(s,0)} - T_{ij}^{(r,0)}T_{kl}^{(s,1)}) - (q^{\delta_{ik}} - q^{-\delta_{ik}})T_{ij}^{(r,0)}T_{kl}^{(s+1,0)}&\\
- q^{-\delta_{jl}}(T_{kl}^{(s,0)}T_{ij}^{(r,1)} - T_{kl}^{(s,1)}T_{ij}^{(r,0)}) + (q^{\delta_{jl}} - q^{-\delta_{jl}})T_{kl}^{(s+1,0)}T_{ij}^{(r,0)}&\\
= (q-q^{-1})(\delta_{i>k}T_{kj}^{(r+1,0)}T_{il}^{(s,0)} + \delta_{i<k}T_{kj}^{(r,0)}T_{il}^{(s+1,0)})&\\
- (q-q^{-1})(\delta_{l>j}T_{kj}^{(s,0)}T_{il}^{(r+1,0)} + \delta_{l<j}T_{kj}^{(s+1,0)}T_{il}^{(r,0)}).&
\end{aligned}$$

Using $T_{ij}^{(r+1,m)} - T_{ij}^{(r,m)} = T_{ij}^{(r,m+1)}$ and $T_{kl}^{(s+1,n)} - T_{kl}^{(s,n)} = T_{kl}^{(s,n+1)}$, we deduce by induction on m and n that, for all $r, s \geq 1$ and all $m, n \geq 0$,

$$\begin{aligned}
q^{-\delta_{ik}}(T_{ij}^{(r,m+1)}T_{kl}^{(s,n)} - T_{ij}^{(r,m)}T_{kl}^{(s,n+1)}) - (q^{\delta_{ik}} - q^{-\delta_{ik}})T_{ij}^{(r,m)}T_{kl}^{(s+1,n)}&\\
- q^{-\delta_{jl}}(T_{kl}^{(s,n)}T_{ij}^{(r,m+1)} - T_{kl}^{(s,n+1)}T_{ij}^{(r,m)}) + (q^{\delta_{jl}} - q^{-\delta_{jl}})T_{kl}^{(s+1,n)}T_{ij}^{(r,m)}&\\
= (q-q^{-1})(\delta_{i>k}T_{kj}^{(r+1,m)}T_{il}^{(s,n)} + \delta_{i<k}T_{kj}^{(r,m)}T_{il}^{(s+1,n)})&\\
- (q-q^{-1})(\delta_{l>j}T_{kj}^{(s,n)}T_{il}^{(r+1,m)} + \delta_{l<j}T_{kj}^{(s+1,n)}T_{il}^{(r,m)}).& \quad (3.3)
\end{aligned}$$

Consider the case $r = s = 1$ in (3.3):

$$\begin{aligned}
q^{-\delta_{ik}}(T_{ij}^{(1,m+1)}T_{kl}^{(1,n)} - T_{ij}^{(1,m)}T_{kl}^{(1,n+1)}) - (q^{\delta_{ik}} - q^{-\delta_{ik}})T_{ij}^{(1,m)}T_{kl}^{(2,n)}&\\
- q^{-\delta_{jl}}(T_{kl}^{(1,n)}T_{ij}^{(1,m+1)} - T_{kl}^{(1,n+1)}T_{ij}^{(1,m)}) + (q^{\delta_{jl}} - q^{-\delta_{jl}})T_{kl}^{(2,n)}T_{ij}^{(1,m)}&\\
= (q-q^{-1})(\delta_{i>k}T_{kj}^{(2,m)}T_{il}^{(1,n)} + \delta_{i<k}T_{kj}^{(1,m)}T_{il}^{(2,n)})&\\
- (q-q^{-1})(\delta_{l>j}T_{kj}^{(1,n)}T_{il}^{(2,m)} + \delta_{l<j}T_{kj}^{(2,n)}T_{il}^{(1,m)}).& \quad (3.4)
\end{aligned}$$

Using $T_{ij}^{(1,m)} = T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)}$ and $T_{kl}^{(1,n)} = T_{kl}^{(0,n+1)} + T_{kl}^{(0,n)}$ we obtain, for all $m, n, \geq 0$:

$$\begin{aligned}
&q^{-\delta_{ik}}(T_{ij}^{(0,m+2)} + T_{ij}^{(0,m+1)})T_{kl}^{(0,n+1)} - q^{-\delta_{ik}}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})T_{kl}^{(0,n+2)}\\
&- (q^{\delta_{ik}} - q^{-\delta_{ik}})(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})T_{kl}^{(1,n+1)} + q^{-\delta_{ik}}(T_{ij}^{(0,m+2)} + T_{ij}^{(0,m+1)})T_{kl}^{(0,n)}\\
&- q^{-\delta_{ik}}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})T_{kl}^{(0,n+1)} - (q^{\delta_{ik}} - q^{-\delta_{ik}})(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})T_{kl}^{(1,n)}\\
&- q^{-\delta_{jl}}T_{kl}^{(0,n+1)}(T_{ij}^{(0,m+2)} + T_{ij}^{(0,m+1)}) + q^{-\delta_{jl}}T_{kl}^{(0,n+2)}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})\\
&+ (q^{\delta_{jl}} - q^{-\delta_{jl}})T_{kl}^{(1,n+1)}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)}) - q^{-\delta_{jl}}T_{kl}^{(0,n)}(T_{ij}^{(0,m+2)} + T_{ij}^{(0,m+1)})\\
&+ q^{-\delta_{jl}}T_{kl}^{(0,n+1)}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)}) + (q^{\delta_{jl}} - q^{-\delta_{jl}})T_{kl}^{(1,n)}(T_{ij}^{(0,m+1)} + T_{ij}^{(0,m)})
\end{aligned}$$

$$
\begin{aligned}
&= (q-q^{-1})(\delta_{i>k}(T_{kj}^{(1,m+1)}+T_{kj}^{(1,m)})T_{il}^{(0,n+1)}+\delta_{i<k}(T_{kj}^{(0,m+1)}+T_{kj}^{(0,m)})T_{il}^{(1,n+1)})\\
&\quad + (q-q^{-1})(\delta_{i>k}(T_{kj}^{(1,m+1)}+T_{kj}^{(1,m)})T_{il}^{(0,n)}+\delta_{i<k}(T_{kj}^{(0,m+1)}+T_{kj}^{(0,m)})T_{il}^{(1,n)})\\
&\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n+1)}(T_{il}^{(1,m+1)}+T_{il}^{(1,m)})+\delta_{l<j}T_{kj}^{(1,n+1)}(T_{il}^{(0,m+1)}+T_{il}^{(0,m)}))\\
&\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n)}(T_{il}^{(1,m+1)}+T_{il}^{(1,m)})+\delta_{l<j}T_{kj}^{(1,n)}(T_{il}^{(0,m+1)}+T_{il}^{(0,m)})).
\end{aligned}
$$

We now expand the previous expression:

$$
\begin{aligned}
&q^{-\delta_{ik}}(T_{ij}^{(0,m+2)}T_{kl}^{(0,n+1)}-T_{ij}^{(0,m+1)}T_{kl}^{(0,n+2)})-(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m+1)}T_{kl}^{(1,n+1)}\\
&+q^{-\delta_{ik}}(T_{ij}^{(0,m+1)}T_{kl}^{(0,n+1)}-T_{ij}^{(0,m)}T_{kl}^{(0,n+2)})-(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m)}T_{kl}^{(1,n+1)}\\
&+q^{-\delta_{ik}}(T_{ij}^{(0,m+2)}T_{kl}^{(0,n)}-T_{ij}^{(0,m+1)}T_{kl}^{(0,n+1)})-(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m+1)}T_{kl}^{(1,n)}\\
&+q^{-\delta_{ik}}(T_{ij}^{(0,m+1)}T_{kl}^{(0,n)}-T_{ij}^{(0,m)}T_{kl}^{(0,n+1)})-(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m)}T_{kl}^{(1,n)}\\
&-q^{-\delta_{jl}}(T_{kl}^{(0,n+1)}T_{ij}^{(0,m+2)}-T_{kl}^{(0,n+2)}T_{ij}^{(0,m+1)})+(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n+1)}T_{ij}^{(0,m+1)}\\
&-q^{-\delta_{jl}}(T_{kl}^{(0,n+1)}T_{ij}^{(0,m+1)}-T_{kl}^{(0,n+2)}T_{ij}^{(0,m)})+(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n+1)}T_{ij}^{(0,m)}\\
&-q^{-\delta_{jl}}(T_{kl}^{(0,n)}T_{ij}^{(0,m+2)}-T_{kl}^{(0,n+1)}T_{ij}^{(0,m+1)})+(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n)}T_{ij}^{(0,m+1)}\\
&-q^{-\delta_{jl}}(T_{kl}^{(0,n)}T_{ij}^{(0,m+1)}-T_{kl}^{(0,n+1)}T_{ij}^{(0,m)})+(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n)}T_{ij}^{(0,m)}\\
&\quad =(q-q^{-1})(\delta_{i>k}T_{kj}^{(1,m+1)}T_{il}^{(0,n+1)}+\delta_{i<k}T_{kj}^{(0,m+1)}T_{il}^{(1,n+1)})\\
&\quad\quad + (q-q^{-1})(\delta_{i>k}T_{kj}^{(1,m)}T_{il}^{(0,n+1)}+\delta_{i<k}T_{kj}^{(0,m)}T_{il}^{(1,n+1)})\\
&\quad\quad + (q-q^{-1})(\delta_{i>k}T_{kj}^{(1,m+1)}T_{il}^{(0,n)}+\delta_{i<k}T_{kj}^{(0,m+1)}T_{il}^{(1,n)})\\
&\quad\quad + (q-q^{-1})(\delta_{i>k}T_{kj}^{(1,m)}T_{il}^{(0,n)}+\delta_{i<k}T_{kj}^{(0,m)}T_{il}^{(1,n)})\\
&\quad\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n+1)}T_{il}^{(1,m+1)}+\delta_{l<j}T_{kj}^{(1,n+1)}T_{il}^{(0,m+1)})\\
&\quad\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n+1)}T_{il}^{(1,m)}+\delta_{l<j}T_{kj}^{(1,n+1)}T_{il}^{(0,m)})\\
&\quad\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n)}T_{il}^{(1,m+1)}+\delta_{l<j}T_{kj}^{(1,n)}T_{il}^{(0,m+1)})\\
&\quad\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n)}T_{il}^{(1,m)}+\delta_{l<j}T_{kj}^{(1,n)}T_{il}^{(0,m)}).
\end{aligned}
$$

Notice that both sides of this last equality are in $\mathbf{K}_{m+n+1}$ (and some of the terms are in $\mathbf{K}_{m+n+2}$ or in $\mathbf{K}_{m+n+3}$). Modulo $\mathbf{K}_{m+n+2}$, we obtain the congruence:

$$
\begin{aligned}
&q^{-\delta_{ik}}(T_{ij}^{(0,m+1)}T_{kl}^{(0,n)}-T_{ij}^{(0,m)}T_{kl}^{(0,n+1)})-(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m)}T_{kl}^{(1,n)}\\
&\quad - q^{-\delta_{jl}}(T_{kl}^{(0,n)}T_{ij}^{(0,m+1)}-T_{kl}^{(0,n+1)}T_{ij}^{(0,m)})+(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n)}T_{ij}^{(0,m)}\\
&\quad\quad \equiv (q-q^{-1})(\delta_{i>k}T_{kj}^{(1,m)}T_{il}^{(0,n)}+\delta_{i<k}T_{kj}^{(0,m)}T_{il}^{(1,n)})\\
&\quad\quad\quad - (q-q^{-1})(\delta_{l>j}T_{kj}^{(0,n)}T_{il}^{(1,m)}+\delta_{l<j}T_{kj}^{(1,n)}T_{il}^{(0,m)}).
\end{aligned}
$$

Moreover, modulo $\mathbf{K}_{m+n+2}$, we also have:

$$
(q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m)}T_{kl}^{(1,n)} \equiv (q^{\delta_{ik}}-q^{-\delta_{ik}})T_{ij}^{(0,m)}T_{kl}^{(0,n)},
$$

$$
(q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(1,n)}T_{ij}^{(0,m)} \equiv (q^{\delta_{jl}}-q^{-\delta_{jl}})T_{kl}^{(0,n)}T_{ij}^{(0,m)}
$$

$$(q-q^{-1})T_{kj}^{(1,m)}T_{il}^{(0,n)} \equiv (q-q^{-1})T_{kj}^{(0,m)}T_{il}^{(0,n)}, \;\; (q-q^{-1})T_{kj}^{(0,m)}T_{il}^{(1,n)} \equiv (q-q^{-1})T_{kj}^{(0,m)}T_{il}^{(0,n)}$$
$$(q-q^{-1})T_{kj}^{(0,n)}T_{il}^{(1,m)} \equiv (q-q^{-1})T_{kj}^{(0,n)}T_{il}^{(0,m)}, \;\; (q-q^{-1})T_{kj}^{(1,n)}T_{il}^{(0,m)} \equiv (q-q^{-1})T_{kj}^{(0,n)}T_{il}^{(0,m)}.$$

Therefore, passing to the quotient $\mathbf{K}_{m+n+1}/\mathbf{K}_{m+n+2}$, we obtain:

$$\begin{aligned}(\xi_{ij}^{(0,m+1)}\xi_{kl}^{(0,n)} - \xi_{ij}^{(0,m)}\xi_{kl}^{(0,n+1)}) - \delta_{ik}h\xi_{ij}^{(0,m)}\xi_{kl}^{(0,n)} \\ - (\xi_{kl}^{(0,n)}\xi_{ij}^{(0,m+1)} - \xi_{kl}^{(0,n+1)}\xi_{ij}^{(0,m)}) + \delta_{jl}h\xi_{kl}^{(0,n)}\xi_{ij}^{(0,m)} \\ = h(\delta_{i>k}\xi_{kj}^{(0,m)}\xi_{il}^{(0,n)} + \delta_{i<k}\xi_{kj}^{(0,m)}\xi_{il}^{(0,n)}) \\ - h(\delta_{l>j}\xi_{kj}^{(0,n)}\xi_{il}^{(0,m)} + \delta_{l<j}\xi_{kj}^{(0,n)}\xi_{il}^{(0,m)}).\end{aligned}$$

This last relation is equivalent to:

$$[\xi_{ij}^{(0,m+1)}, \xi_{kl}^{(0,n)}] - [\xi_{ij}^{(0,m)}, \xi_{kl}^{(0,n+1)}] = h(\xi_{kj}^{(0,m)}\xi_{il}^{(0,n)} - \xi_{kj}^{(0,n)}\xi_{il}^{(0,m)}).$$

This holds for all $m, n \geq 0$.

Remark 3.1 *We could have obtained this relation more directly by starting with Proposition 1.6 and taking $r = s = 0$, but this would have required considering many different cases depending on how i, j, k, l all compare with each other.*

All the previous computations prove that $\varphi : Y_h(\mathfrak{gl}_N) \longrightarrow \widetilde{Y}_h(\mathfrak{gl}_N)$ given by $\varphi(t_{ij}^{(m+1)}) = \xi_{ij}^{(0,m)}$ for $m \geq 0$ is an algebra homomorphism. We still have to show that φ is bijective.

We will first demonstrate surjectivity. Towards this end, we define elements $\overline{T}_{ij}^{(r,m)}$ as follows. Let $\overline{T}_{ij}^{(r,0)} = \overline{\tau}_{ij}^{(r)}$, except that $\overline{T}_{ij}^{(0,0)} = -\tau_{ij}^{(0)}$ when $i \geq j$. Then, for each $m \geq 0$, let

$$\overline{T}_{ij}^{(r,m+1)} = \overline{T}_{ij}^{(r+1,m)} - \overline{T}_{ij}^{(r,m)}.$$

Also for each $m \geq 0$, let $\widetilde{T}_{ij}^{(0,m)} = \overline{T}_{ij}^{(0,m)}$ and $\widetilde{T}_{ij}^{(m,m)} = (-1)^{m+1}T_{ij}^{(0,m)}$, and for $1 \leq r \leq m$ define recursively

$$\widetilde{T}_{ij}^{(r,m+1)} = \widetilde{T}_{ij}^{(r-1,m)} - \widetilde{T}_{ij}^{(r,m)}.$$

Induction on m shows that the elements $T_{ij}^{(r,m)}$, $\overline{T}_{ij}^{(r,m)}$ and $\widetilde{T}_{ij}^{(r,m)}$ respectively map via ψ to the elements $E_{ij}s^r(s-1)^m$, $(-1)^{m+1}E_{ij}s^{-(m+r)}(s-1)^m$ and $(-1)^{m+1}E_{ij}s^{-(m-r)}(s-1)^m$ in $\mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N))$. It follows that for fixed m, the images of these elements under ψ span K_m. Moreover, all these elements are in U by definition.

Note that for any fixed $X \in \mathbb{K}_m = \psi^{-1}(\mathsf{K}_m) \cap U$, there exists some element Y in

$$\mathrm{span}_{\mathbb{C}}\{T_{ij}^{(r,m)}, \overline{T}_{ij}^{(r,m)}, \widetilde{T}_{ij}^{(r,m)} \mid i,j = 1,\ldots,N, \;\; r \in \mathbb{Z}_+\}$$

for which $X - Y \in (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))$, because the map

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N))$$

is an isomorphism. Since $X - Y$ is also in U, by (3.2) we have

$$X - Y \in \operatorname{span}_{\mathbb{C}}\{\tau_{ii}^{(0)} + \overline{\tau}_{ii}^{(0)} \mid i = 1, \ldots, N\} \subset \mathbf{K}_\ell \ \forall \ell \geq 0.$$

That $\tau_{ii}^{(0)} + \overline{\tau}_{ii}^{(0)}$ is in $\mathbf{K}_\ell$ for all $\ell \geq 0$ follows from the fact that $\psi(\tau_{ii}^{(0)} + \overline{\tau}_{ii}^{(0)}) = 0 \in \mathsf{K}_\ell$.

It follows that any element of $\mathbf{K}_m$ is congruent modulo $\mathbf{K}_{m+1}$ to a sum monomials of the form $f(q)(q-q^{-1})^{m_0}\mathcal{M}$ where $f(q) \in \mathcal{A}$ is not divisible by $q-1$, and $\mathcal{M} = \tau_{i_1j_1}^{(r_1,m_1)} \ldots \tau_{i_kj_k}^{(r_k,m_k)}$ with

$$\tau_{i_dj_d}^{(r_d,m_d)} \in \{T_{i_dj_d}^{(r_d,m_d)}, \overline{T}_{i_dj_d}^{(r_d,m_d)}, \widetilde{T}_{i_dj_d}^{(r_d,m_d)}\}$$

and $m_0 + \ldots + m_k \geq m$. Moreover, since $T_{i_dj_d}^{(r_d,m_d)} - T_{i_dj_d}^{(r_d-1,m_d)} = T_{i_dj_d}^{(r_d-1,m_d+1)} \in \mathbb{K}_{m_d+1}$ (and similarly for the $\overline{T}_{i_dj_d}^{(r_d,m_d)}$ and $\widetilde{T}_{i_dj_d}^{(r_d,m_d)}$), we can reduce modulo $\mathbf{K}_{m+1}$ to the case when $r_d = 0$ for each $d = 1, \ldots, k$.

Observe that modulo $\mathbf{K}_{m+1}$, we have

$$\overline{T}_{ij}^{(0,m)} = (-1)^{m+1}T_{ij}^{(0,m)}, \qquad \widetilde{T}_{ij}^{(0,m)} = (-1)^{m+1}T_{ij}^{(0,m)}.$$

To see this, just take the difference of the elements on each side (this difference is in U by definition) and apply ψ. Finally, observe that we can replace $f(q)$ by $f(1)$ modulo $\mathbf{K}_{m+1}$. Indeed, if $f(q) = \frac{a(q)}{b(q)}$, we can take the Laurent expansion of $a(q)$ and $b(q)$ about $q = 1$; then, note that

$$f(q) - f(1) = \frac{b(1)a(q) - a(1)b(q)}{b(1)b(q)}$$

and the constant term in the numerator is annihilated, so that every remaining term is divisible by $q - 1$.

In summary, we have shown that each of these monomials $f(q)(q - q^{-1})^{m_0}\mathcal{M}$ in $\mathbf{K}_m$ is congruent modulo $\mathbf{K}_{m+1}$ to

$$f(1)(q - q^{-1})^{m_0}T_{i_1j_1}^{(0,m_1)} \ldots T_{i_k,j_k}^{(0,m_k)}$$

at least up to a sign. The image modulo $\mathbf{K}_{m+1}$ of this element is

$$f(1)h^{m_0}\xi_{i_1j_1}^{(0,m_1)} \ldots \xi_{i_kj_k}^{(0,m_k)}$$

and this is in the image of φ by definition. This completes the proof that φ is surjective.

To prove that φ is injective, it is enough to show that the basis of $Y(\mathfrak{gl}_N)$ given by ordered monomials in the generators $t_{ij}^{(m)}$ is mapped via φ to some linearly independent set in $\widetilde{Y}_h(\mathfrak{gl}_N)$.

By definition, any two ordered monomials $t_{i_1j_1}^{(m_1+1)} \ldots t_{i_aj_a}^{(m_a+1)}$ and $t_{k_1l_1}^{(n_1+1)} \ldots t_{k_bl_b}^{(n_b+1)}$ with each $m_d, n_d \geq 0$ and $m_1 + \ldots + m_a \neq n_1 + \ldots n_b$ are mapped via φ to distinct graded pieces in $\widetilde{Y}_h(\mathfrak{gl}_N)$. It therefore suffices to show that for each fixed m, the images under φ of all the ordered monomials $t_{i_1j_1}^{(m_1+1)} \ldots t_{i_aj_a}^{(m_a+1)}$ with $m_1 + \ldots + m_a = m$ are linearly independent in $\mathbf{K}_m/\mathbf{K}_{m+1}$.

Consider any linear sum over $\mathbb{C}$ of ordered monomials of the form

$$\xi^{(0,m_1)}_{i_1j_1}\dots\xi^{(0,m_a)}_{i_aj_a}$$

with $m_1+\dots+m_a=m$, and suppose that this sum is zero in $\mathbf{K}_m/\mathbf{K}_{m+1}$. Then we have a linear sum S of ordered monomials $T^{(0,m_1)}_{i_1j_1}\dots T^{(0,m_a)}_{i_aj_a}$ which is not just in $\mathbf{K}_m$, but also in $\mathbf{K}_{m+1}$.

By definition, $\psi(S)$ is a sum of ordered monomials

$$E_{i_1j_1}(s-1)^{m_1}\dots E_{i_aj_a}(s-1)^{m_a}.$$

On the other hand, since $S\in\mathbf{K}_{m+1}$, $\psi(S)$ is a sum of monomials of the form

$$E_{k_1l_1}s^{r_1}(s-1)^{n_1}\dots E_{k_bl_b}s^{r_b}(s-1)^{n_b}$$

with $r_1,\dots,r_b\in\mathbb{Z}$ and $n_1+\dots+n_b\geq m+1$.

For each $r\geq 1$, we have a composite of algebra homomorphisms

$$\mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N))\xrightarrow{\Delta}\mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N))^{\otimes r}\xrightarrow{f^{\otimes r}}\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes r}\otimes\mathbb{C}[x_1^{\pm1},x_2^{\pm1},\dots,x_r^{\pm1}]$$

where Δ is given by the coproduct $\Delta(X)=1\otimes X+X\otimes 1$ and f is given by

$$\begin{gathered}f:\mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N))\to\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)\otimes\mathbb{C}[x,x^{-1}]\\ E_{ij}x^t\mapsto E_{ij}\otimes x^t\end{gathered}$$

We also have for each choice of r nonnegative integers $\alpha_1,\dots,\alpha_r$ a differential operator

$$\partial_{\alpha_1,\dots,\alpha_r}:\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes r}\otimes\mathbb{C}[x_1^{\pm1},x_2^{\pm1},\dots,x_r^{\pm1}]\to\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes r}$$

given by

$$\partial_{\alpha_1,\dots,\alpha_r}=\left.\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_r}}{\partial x_r^{\alpha_r}}\right|_{x_1,\dots,x_r=1}.$$

Take $r\geq\max\{a,b\}$ where the maximum is taken over all the monomials in $\psi(S)$ and note that for any choice of $\alpha_1,\dots,\alpha_r$ with $\alpha_1+\dots+\alpha_r=m$, $\psi(S)$ is in the kernel of the composite $\partial_{\alpha_1,\dots,\alpha_r}\circ f^{\otimes r}\circ\Delta$, because $n_1+\dots+n_b\geq m+1$.

On the other hand, if S is nonzero, then we can find some $\alpha_1,\dots,\alpha_r$ such that $\alpha_1+\dots+\alpha_r=m$ and $\psi(S)$ is not in the kernel of $\partial_{\alpha_1,\dots,\alpha_r}\circ f^{\otimes r}\circ\Delta$: just choose any of the ordered monomials

$$E_{i_1j_1}(s-1)^{m_1}\dots E_{i_aj_a}(s-1)^{m_a}.$$

and set $\alpha_1=m_1,\ \dots,\ \alpha_a=m_a$ and $\alpha_d=0$ for $d>a$.

This is a contradiction, so $S=0$ and the linear sum of ordered monomials

$$\xi^{(0,m_1)}_{i_1j_1}\dots\xi^{(0,m_a)}_{i_aj_a}$$

must in fact be trivial, as desired. ■

Chapter 4

$Y^{tw}(\mathfrak{o}_N)$ as a degenerate form of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$

We will now show how the isomorphism φ of Theorem 3.2 can also be used to construct the twisted Yangian $Y_h^{tw}(\mathfrak{o}_N)$ as a degenerate form of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$. Throughout this chapter, we will view $Y_h^{tw}(\mathfrak{o}_N)$ and $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ as subalgebras of $Y_h(\mathfrak{gl}_N)$ and $\mathfrak{U}_q(\mathcal{L}(\mathfrak{gl}_N))$, respectively, via the embeddings of Propositions 1.5 and 1.7.

Let $S_{ij}^{(r,0)} := \frac{S_{ij}^{(r)}}{q-q^{-1}}$, except that $S_{ii}^{(0,0)} = \frac{S_{ii}^{(0)}-1}{q-1}$ and $S_{ij}^{(0,0)} = -\frac{S_{ji}^{(0)}}{q-q^{-1}}$ when $i < j$. For each $m \geq 0$, define inductively $S_{ij}^{(r,m+1)} := S_{ij}^{(r+1,m)} - S_{ij}^{(r,m)}$. Let $\zeta_{ij}^{(r,m)}$ be the image of $S_{ij}^{(r,m)}$ in the quotient $\mathbf{K}_m/\mathbf{K}_{m+1}$. It is easy to check by induction on m that for each $r, m \geq 0$, we have

$$S_{ij}^{(r,m)} = \sum_{n=r}^{m+r} (-1)^{m-n+r} \binom{m}{n-r} S_{ij}^{(n,0)}.$$

Theorem 4.1 *If $i > j$, then*

$$s_{ij}^{(m+1)} \overset{\varphi}{\mapsto} \zeta_{ij}^{(0,m)}$$

where $\varphi : Y_h(\mathfrak{gl}_N) \xrightarrow{\sim} \widetilde{Y}_h(\mathfrak{gl}_N)$ is the isomorphism given by $t_{ij}^{(m+1)} \mapsto \xi_{ij}^{(0,m)}$; if $i \leq j$, then $s_{ij}^{(m+1)} \overset{\varphi}{\mapsto} \zeta_{ij}^{(1,m)}$.

We can obtain an analogue of Theorem 3.2. From Theorem 3.1 and Proposition 1.7, we can deduce that the enveloping algebra of $\mathfrak{o}_N^{tw}[s]$ is the limit when $q \mapsto 1$ of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ in the following precise sense: if we let $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{o}_N))$ generated by the $S_{ij}^{(r,0)}$, then $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$ is isomorphic to $\mathfrak{U}(\mathfrak{o}_N^{tw}[s, s^{-1}])$ (see the proof of Theorem 3.3 in [12]). The following diagram is commutative:

$$\begin{tikzcd}
 & \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) \arrow[r] \arrow[d] & \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \arrow[d] & \\
\mathfrak{U}(\mathcal{L}^{tw}(\mathfrak{o}_N)) \arrow[r, equal] & \frac{\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))}{(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))} \arrow[r] & \frac{\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))}{(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))} \arrow[r, equal] & \mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N)).
\end{tikzcd}$$

We can define an algebra $\widetilde{Y}^{tw}(\mathfrak{o}_N)$ similarly to how we defined $\widetilde{Y}(\mathfrak{gl}_N)$. For $m \geq 0$, denote by K_m^{tw} the Lie ideal of $\mathfrak{o}_N^{tw}[s, s^{-1}]$ spanned by

$$E_{ij}s^r(s-1)^m - E_{ji}s^{-r}(s^{-1}-1)^m$$

for all $r \in \mathbb{Z}$. Let U^{tw} be the subspace of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$ spanned over $\mathbb{C}$ by all the generators $S_{ij}^{(r,0)}$, and observe that $U^{tw} \cap (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) = \{0\}$. Set $\mathbb{K}_m^{tw} = \psi^{-1}(\mathsf{K}_m^{tw}) \cap U^{tw}$ where ψ this time denotes the composite

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{o}_N^{tw}[s, s^{-1}]).$$

Let $\mathbf{K}_m^{tw}$ be the two-sided ideal of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$ generated by $(q-q^{-1})^{m_0}\mathbb{K}_{m_1}^{tw} \cdots \mathbb{K}_{m_k}^{tw}$ with $m_0 + m_1 + \cdots + m_k \geq m$.

Let $\widetilde{Y}^{tw}(\mathfrak{o}_N)$ be the $\mathbb{C}$-algebra

$$\widetilde{Y}^{tw}(\mathfrak{o}_N) = \bigoplus_{m=0}^{\infty} \mathbf{K}_m^{tw}/\mathbf{K}_{m+1}^{tw}$$

where $\mathbf{K}_0^{tw} = \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$. We also view $\widetilde{Y}^{tw}(\mathfrak{o}_N)$ as an algebra over $\mathbb{C}[h]$ by setting $h = \overline{q - q^{-1}} \in \mathbf{K}_1^{tw}/\mathbf{K}_2^{tw}$. In this case, we denote it by $\widetilde{Y}_h^{tw}(\mathfrak{o}_N)$.

Corollary 4.1 *$Y_h^{tw}(\mathfrak{o}_N)$ is isomorphic to $\widetilde{Y}_h^{tw}(\mathfrak{o}_N)$ via the function φ^{tw} that sends $s_{ij}^{(m)}$ to $\overline{S_{ij}^{(1,m-1)}} \in \mathbf{K}_{m-1}^{tw}/\mathbf{K}_m^{tw}$ for $m \geq 1$.*

Proof. Theorem 4.1 implies that the following diagram is commutative:

$$\begin{tikzcd}
Y_h^{tw}(\mathfrak{o}_N) \arrow[r] \arrow[d, "\varphi^{tw}"'] & Y_h(\mathfrak{gl}_N) \arrow[d, "\varphi"] \\
\widetilde{Y}_h^{tw}(\mathfrak{o}_N) \arrow[r] & \widetilde{Y}_h(\mathfrak{gl}_N).
\end{tikzcd}$$

In this diagram, the top horizontal arrow is the embedding of proposition 1.5 and the bottom horizontal arrow is the one induced from the embedding of proposition 1.7. The injectivity of φ^{tw} now follows from the fact that φ provides an isomorphism between $Y_h(\mathfrak{gl}_N)$ and $\widetilde{Y}_h(\mathfrak{gl}_N)$: see Theorem 3.2.

We need to see that φ^{tw} is surjective. Define elements $\widetilde{S}_{ij}^{(r,m)}$ with $0 \leq r \leq m$ as follows: for each $m \geq 0$, let $\widetilde{S}_{ij}^{(0,m)} = S_{ji}^{(0,m)}$ and $\widetilde{S}_{ij}^{(m,m)} = (-1)^{m+1}S_{ij}^{(0,m)}$, and for $1 \leq r \leq m$ let

$$\widetilde{S}_{ij}^{(r,m+1)} = \widetilde{S}_{ij}^{(r-1,m)} - \widetilde{S}_{ij}^{(r,m)}.$$

Then induction on m shows that

$$\begin{aligned}
\psi(S_{ij}^{(r,m)}) &= E_{ij}s^r(s-1)^m - E_{ji}s^{-r}(s^{-1}-1)^m, \\
\psi(S_{ji}^{(r,m)}) &= (-1)^{m+1}(E_{ij}s^{-(m+r)}(s-1)^m - E_{ji}s^{m+r}(s^{-1}-1)^m), \\
\psi(\widetilde{S}_{ij}^{(r,m)}) &= (-1)^{m+1}(E_{ij}s^{-(m-r)}(s-1)^m - E_{ji}s^{m-r}(s^{-1}-1)^m).
\end{aligned}$$

It follows that for any fixed m, the images of these elements under ψ span K_m^{tw}, and they are all in U^{tw} by definition. Now note that for any element $X \in \mathbb{K}^{tw} = \psi^{-1}(\mathsf{K}_m^{tw}) \cap U^{tw}$, there is some element Y in

$$\mathrm{span}_{\mathbb{C}}\{S_{ij}^{(r,m)}, \widetilde{S}_{ij}^{(r,m)} \mid i,j = 1,\dots,N,\ r \in \mathbb{Z}_+\}$$

for which $X - Y \in (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))$. This follows from the fact that

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{o}_N^{tw}[s,s^{-1}])$$

is an isomorphism. Since $X - Y$ is also in U^{tw} and since $U^{tw} \cap (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{o}_N)) = \{0\}$, we see that $X = Y$, hence

$$\mathbb{K}_m^{tw} = \mathrm{span}_{\mathbb{C}}\{S_{ij}^{(r,m)}, \widetilde{S}_{ij}^{(r,m)} \mid i,j = 1,\dots,N,\ r \in \mathbb{Z}_+\}.$$

Therefore, any element of $\mathbf{K}_m^{tw}$ is a sum of monomials $f(q)(q-q^{-1})^{m_0}\mathcal{M}$ where $f(q) \in \mathcal{A}$ is not divisible by $q-1$ and $\mathcal{M} = \sigma_{i_1j_1}^{(r_1,m_1)} \dots \sigma_{i_kj_k}^{(r_k,m_k)}$ with

$$\sigma_{i_dj_d}^{(r_d,m_d)} \in \{S_{i_dj_d}^{(r_d,m_d)}, \widetilde{S}_{i_dj_d}^{(r_d,m_d)}\}$$

and $m_0 + \ldots + m_k \geq m$. Following the same argument as the $\mathfrak{gl}_N$ case, this is congruent modulo $\mathbf{K}_{m+1}^{tw}$ to

$$f(1)(q-q^{-1})^{m_0} S_{i_1j_1}^{(1,m_1)} \dots S_{i_kj_k}^{(1,m_k)}$$

up to a sign. The image modulo $\mathbf{K}_{m+1}^{tw}$ of this element is

$$f(1)h^{m_0}\overline{S_{i_1j_1}^{(1,m_1)}} \dots \overline{S_{i_kj_k}^{(1,m_k)}}$$

and this is in the image of φ^{tw}, which proves that φ^{tw} is surjective. ■

Proof of Theorem 4.1. First note that for any $k,l \in \{1,\dots,N\}$ we have the following relation:

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n} T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)} \\
&\quad= \sum_{c+d=m}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right)\sum_{b=0}^{d}\left((-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}\right) \qquad (4.1)\\
&\qquad+ \sum_{c+d=m-1}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right)\sum_{b=0}^{d}\left((-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}\right).
\end{aligned}$$

To verify this equality, we need to check that the coefficients of $T_{ik}^{(a)}\overline{T}_{jl}^{(b)}$ are the same on both sides. On the left-hand side, $n - a = b$, so $n = a + b$; on the right-hand side, in the first sum, $c + d = m$, so $d = m - c$ and c can take any value from a to m. On the right-hand side, in the second sum, $c+d=m-1$, so $d=m-1-c$ and c can take any value from a to $m-1$. We thus have to see why the following equality holds:

$$\begin{aligned}(-1)^{m-a-b}\binom{m}{a+b} &= \sum_{c=a}^{m}(-1)^{c-a}(-1)^{m-c-b}\binom{c}{a}\binom{m-c}{b}\\ &\quad -\sum_{c=a}^{m-1}(-1)^{c-a}(-1)^{m-c-b}\binom{c}{a}\binom{m-1-c}{b}.\end{aligned}$$

Because $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, this is equivalent to:

$$\binom{m}{a+b}=\sum_{c=a}^{m}\binom{c}{a}\binom{m-c}{b}-\sum_{c=a}^{m-1}\binom{c}{a}\binom{m-1-c}{b}=\sum_{c=a}^{m-1}\binom{c}{a}\binom{m-c-1}{b-1}$$

when $b\geq 1$. Note that when $b=0$, it is trivial to check that

$$\binom{m}{a+b}=\sum_{c=a}^{m}\binom{c}{a}\binom{m-c}{b}-\sum_{c=a}^{m-1}\binom{c}{a}\binom{m-1-c}{b}.$$

Here is an explanation why the equality of binomial coefficients

$$\binom{m}{a+b}=\sum_{c=a}^{m-1}\binom{c}{a}\binom{m-c-1}{b-1}$$

holds when $b\geq 1$. The number of ways to pick up $a+b$ objects out of a set of m identical objects, all ordered in a row, and such that the $(a+1)^{st}$ selected object is in position $c+1$ is $\binom{c}{a}\binom{m-c-1}{b-1}$: this is because if the object in position $c+1$ is selected and if there are a objects preceding it, then there must be $b-1$ objects among those labelled $c+2,\ldots,m$. There are $\binom{c}{a}$ ways to chose a objects among the first c and there are $\binom{m-c-1}{b-1}$ ways to chose $b-1$ objects among those labelled $c+2,\ldots,m$. Summing over all possibilities for c, and noting that different values of c give different ways of selecting objects, we obtain the equality

$$\binom{m}{a+b}=\sum_{c=a}^{m-1}\binom{c}{a}\binom{m-c-1}{b-1}.$$

Equation (4.1) will be particularly useful when $j<k<i$. We can deduce some more useful relations by rewriting (4.1) as follows:

$$\begin{aligned}&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}\\ &=\sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)}\\ &+\sum_{c=0}^{m-1}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right)\\ &\qquad\cdot\left(\overline{T}_{jl}^{(m-c)}+\sum_{b=0}^{m-c-1}(-1)^{m-c-b}\left(\binom{m-c}{b}-\binom{m-c-1}{b}\right)\overline{T}_{jl}^{(b)}\right).\end{aligned}$$

Observe that the coefficient of $\overline{T}_{jl}^{(b)}$ on the fourth line is zero when $b = 0$. Therefore, we can replace these $\overline{T}_{jl}^{(0)}$ by $-\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)} - \delta_{jl})$ and deduce that the following formula is also true for any k, l:

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)} \\
&\quad = \sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)} \\
&\qquad + \sum_{\substack{c+d=m \\ c\neq m}}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)} - \delta_{jl})\right) \\
&\qquad + \sum_{c+d=m-1}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)} - \delta_{jl})\right).
\end{aligned} \tag{4.2}$$

This formula will be useful for the case when $k \leq j$. Using the same trick, we can replace the $T_{ik}^{(a)}$ by $-\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(a)} - \delta_{ik})$ when $d \neq m$ and $a = 0$ in (4.1) to deduce another formula, which will be useful for the case when $k \geq i$:

$$\begin{aligned}
&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)} \\
&\quad = T_{ik}^{(0)}\sum_{b=0}^{m}\left((-1)^{m-b}\binom{m}{b}\overline{T}_{jl}^{(b)}\right) \\
&\qquad + \sum_{\substack{c+d=m \\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)} - \delta_{ik})\right) \\
&\qquad\qquad \cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}\right) \\
&\qquad + \sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)} - \delta_{ik})\right) \\
&\qquad\qquad \cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}\right).
\end{aligned} \tag{4.3}$$

Recall that if $j < k$, then modulo $\mathbf{K}_{d+1}$, we have

$$\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\frac{\overline{T}_{jk}^{(b)}}{q - q^{-1}} = (-1)^{d+1}T_{jk}^{(0,d)}. \tag{4.4}$$

Similarly, if $k \leq j$ then modulo $\mathbf{K}_{d+1}$ we have

$$\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\frac{\overline{T}_{jk}^{(b)}}{q-q^{-1}}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}\frac{T_{jk}^{(0)}-\delta_{jk}}{q-q^{-1}}=(-1)^{d+1}T_{jk}^{(0,d)}. \tag{4.5}$$

Equipped with all these formulas, we can now prove the claim that $s_{ij}^{(m+1)} \overset{\varphi}{\mapsto} \zeta_{ij}^{(0,m)}$ when $i > j$. First we must compute $S_{ij}^{(0,m)}$ and then see what is its image in the quotient $\mathbf{K}_m/\mathbf{K}_{m+1}$. By definition,

$$(q-q^{-1})S_{ij}^{(0,m)}=\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}S_{ij}^{(n)}=\sum_{k=1}^{N}\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jk}^{(n-a)}.$$

Let us split this summation into the cases $k \leq j$, $j < k < i$ and $k \geq i$; using our formulas (4.1), (4.2), (4.3), (4.4), (4.5) and recalling that $T_{ik}^{(0)} = 0$ if $k > i$ and $\overline{T}_{jk}^{(0)} = 0$ if $k < j$, we have

$$\begin{aligned}
(q-q^{-1})S_{ij}^{(0,m)} = {} & \sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ij}^{(a)}\right)\left(\overline{T}_{jj}^{(0)}-1\right)+\sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ij}^{(a)}\right) \\
& + \sum_{k\leq j}\sum_{\substack{c+d=m\\ c\neq m}}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
& \qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
& + \sum_{k\leq j}\sum_{c+d=m-1}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
& \qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
& + \sum_{j<k<i}\sum_{c+d=m}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right)\sum_{b=0}^{d}\left((-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right) \\
& + \sum_{j<k<i}\sum_{c+d=m-1}\sum_{a=0}^{c}\left((-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right)\sum_{b=0}^{d}\left((-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right) \\
& + (T_{ii}^{(0)}-1)\sum_{b=0}^{m}\left((-1)^{m-b}\binom{m}{b}\overline{T}_{ji}^{(b)}\right)+\sum_{b=0}^{m}\left((-1)^{m-b}\binom{m}{b}\overline{T}_{ji}^{(b)}\right) \\
& + \sum_{k\geq i}\sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right) \\
& \qquad \cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right)
\end{aligned}$$

$$+\sum_{k\geq i}\sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)$$
$$\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right).$$

Most of these terms vanish after dividing by $q-q^{-1}$ and passing to $\mathbf{K}_m/\mathbf{K}_{m+1}$; what remains is:

$$\begin{aligned}\zeta_{ij}^{(0,m)}&=\xi_{ij}^{(0,m)}+(-1)^{m+1}\xi_{ji}^{(0,m)}+h\sum_{k=1}^{N}\sum_{c+d=m-1}(-1)^{d+1}\xi_{ik}^{(0,c)}\xi_{jk}^{(0,d)}\\&=\xi_{ij}^{(0,m)}+(-1)^{m+1}\xi_{ji}^{(0,m)}+h\sum_{k=1}^{N}\sum_{c=1}^{m}(-1)^{m+1-c}\xi_{ik}^{(0,c-1)}\xi_{jk}^{(0,m-c)}\end{aligned}$$

and the right hand side is precisely $\varphi(s_{ij}^{(m+1)})$.

Now suppose $i\leq j$. We will show that $s_{ij}^{(m+1)}\mapsto\zeta_{ij}^{(1,m)}$. To achieve this, we first rewrite equation (4.2) in an equivalent form:

$$\begin{aligned}&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}\\&=\sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)}\\&+T_{ik}^{(0)}\left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\&-\left(\sum_{a=0}^{m-1}(-1)^{m-1-a}\binom{m-1}{a}T_{ik}^{(a)}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\&+\sum_{d=1}^{m-1}\left(T_{ik}^{(m-d)}+\sum_{a=0}^{m-1-d}(-1)^{m-d-a}\left(\binom{m-d}{a}-\binom{m-1-d}{a}\right)T_{ik}^{(a)}\right)\\&\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}-\delta_{jl})\right).\end{aligned}$$

Observe that the coefficient of $T_{ik}^{(a)}$ is zero when $a=0$ and $d\neq 0,m$ (see the fourth line). We can therefore replace these $T_{ik}^{(0)}$ by $-\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})$ and deduce the following formula, which will be useful for the case when $i\leq k\leq j$:

$$
\begin{aligned}
&\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)} \\
&\quad=\sum_{a=0}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)} \\
&\quad+T_{ik}^{(0)}\left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{jl}^{(b)}-(-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\quad-\left(\sum_{a=0}^{m-1}(-1)^{m-1-a}\binom{m-1}{a}T_{ik}^{(a)}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\quad+\sum_{\substack{c+d=m\\ c,d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right) \\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\quad+\sum_{\substack{c+d=m-1\\ c\neq m-1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right) \\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{n=0}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}+\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)} \\
&\quad=\sum_{a=1}^{m+1}\left((-1)^{m+1-a}\binom{m+1}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)}+\sum_{a=1}^{m}\left((-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\overline{T}_{jl}^{(0)} \\
&\qquad+\left((-1)^{m+1}+(-1)^{m}\right)T_{ik}^{(0)}\overline{T}_{jl}^{(0)} \\
&\quad+T_{ik}^{(0)}\left(\sum_{b=1}^{m+1}(-1)^{m+1-b}\binom{m+1}{b}\overline{T}_{jl}^{(b)}-(-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\qquad+T_{ik}^{(0)}\left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{jl}^{(b)}-(-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\quad-\left(\sum_{a=1}^{m}(-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\qquad-\left(\sum_{a=1}^{m-1}(-1)^{m-1-a}\binom{m-1}{a}T_{ik}^{(a)}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right) \\
&\quad-\left((-1)^{m}+(-1)^{m-1}\right)T_{ik}^{(0)}\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}\left(\overline{T}_{jl}^{(0)}-\delta_{jl}\right)\right)
\end{aligned}
$$

$$+\sum_{\substack{c+d=m+1\\ c,d\neq m+1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)$$

$$\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)$$

$$+\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)$$

$$\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)$$

$$+\sum_{\substack{c+d=m\\ c,d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)$$

$$\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)$$

$$+\sum_{\substack{c+d=m-1\\ c\neq m-1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)$$

$$\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right).$$

Since

$$\left((-1)^{m+1}+(-1)^{m}\right)T_{ik}^{(0)}\overline{T}_{jl}^{(0)}=-\left((-1)^{m+1}+(-1)^{m}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{ik}}\left(\overline{T}_{ik}^{(0)}-\delta_{ik}\right)\right)\overline{T}_{jl}^{(0)}$$

and

$$\begin{aligned}&\left((-1)^{m}+(-1)^{m-1}\right)T_{ik}^{(0)}\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}\left(\overline{T}_{jl}^{(0)}-\delta_{jl}\right)\right)\\&=-\left((-1)^{m}+(-1)^{m-1}\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{ik}}\left(\overline{T}_{ik}^{(0)}-\delta_{ik}\right)\right)\left(\left(\frac{q+1}{q}\right)^{\delta_{jl}}\left(\overline{T}_{jl}^{(0)}-\delta_{jl}\right)\right)\end{aligned}$$

we deduce that

$$\begin{aligned}&\sum_{n=0}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}+\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}\\&\quad=\left(\sum_{a=1}^{m+1}(-1)^{m+1-a}\binom{m+1}{a}T_{ik}^{(a)}-(-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\overline{T}_{jl}^{(0)}\\&\qquad+\left(\sum_{a=1}^{m}(-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}-(-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\overline{T}_{jl}^{(0)}\end{aligned}$$

$$
\begin{aligned}
&+ T_{ik}^{(0)} \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{jl}^{(b)} - (-1)^{m+1} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&\qquad + T_{ik}^{(0)} \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{jl}^{(b)} - (-1)^{m} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&- \left(\sum_{a=1}^{m} (-1)^{m-a} \binom{m}{a} T_{ik}^{(a)} - (-1)^{m} \left(\frac{q+1}{q}\right)^{\delta_{ik}} \left(\overline{T}_{ik}^{(0)} - \delta_{ik}\right) \right) \\
&\qquad\qquad \cdot \left(\left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&- \left(\sum_{a=1}^{m-1} (-1)^{m-1-a} \binom{m-1}{a} T_{ik}^{(a)} - (-1)^{m-1} \left(\frac{q+1}{q}\right)^{\delta_{ik}} \left(\overline{T}_{ik}^{(0)} - \delta_{ik}\right) \right) \\
&\qquad\qquad \cdot \left(\left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&+ \sum_{\substack{c+d=m+1 \\ c,d\neq m+1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{ik}^{(a)} - (-1)^{c} \left(\frac{q+1}{q}\right)^{\delta_{ik}} (\overline{T}_{ik}^{(0)} - \delta_{ik}) \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{jl}^{(b)} - (-1)^{d} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&+ \sum_{\substack{c+d=m \\ c\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{ik}^{(a)} - (-1)^{c} \left(\frac{q+1}{q}\right)^{\delta_{ik}} (\overline{T}_{ik}^{(0)} - \delta_{ik}) \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{jl}^{(b)} - (-1)^{d} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&+ \sum_{\substack{c+d=m \\ c,d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{ik}^{(a)} - (-1)^{c} \left(\frac{q+1}{q}\right)^{\delta_{ik}} (\overline{T}_{ik}^{(0)} - \delta_{ik}) \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{jl}^{(b)} - (-1)^{d} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right) \\
&+ \sum_{\substack{c+d=m-1 \\ c\neq m-1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{ik}^{(a)} - (-1)^{c} \left(\frac{q+1}{q}\right)^{\delta_{ik}} (\overline{T}_{ik}^{(0)} - \delta_{ik}) \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{jl}^{(b)} - (-1)^{d} \left(\frac{q+1}{q}\right)^{\delta_{jl}} (T_{jl}^{(0)} - \delta_{jl}) \right).
\end{aligned}
$$

After collecting terms, we obtain the formula:

$$
\begin{aligned}
&\sum_{n=0}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}+\sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jl}^{(n-a)}\\
&=\left(\sum_{a=1}^{m+1}(-1)^{m+1-a}\binom{m+1}{a}T_{ik}^{(a)}-(-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\overline{T}_{jl}^{(0)}\\
&\qquad+\left(\sum_{a=1}^{m}(-1)^{m-a}\binom{m}{a}T_{ik}^{(a)}-(-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\overline{T}_{jl}^{(0)}\\
&+T_{ik}^{(0)}\left(\sum_{b=1}^{m+1}(-1)^{m+1-b}\binom{m+1}{b}\overline{T}_{jl}^{(b)}-(-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\
&\qquad+T_{ik}^{(0)}\left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{jl}^{(b)}-(-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\
&+\sum_{\substack{c+d=m+1\\ c,d\neq m+1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\
&+\sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right)\\
&+\sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jl}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jl}}(T_{jl}^{(0)}-\delta_{jl})\right). \qquad (4.6)
\end{aligned}
$$

From this together with (4.2) and (4.3), and recalling that $\overline{T}_{jk}^{(0)}=0$ if $k<j$, it follows that:

$$
\begin{aligned}
&(q-q^{-1})S_{ij}^{(1,m)} = (q-q^{-1})\left(S_{ij}^{(0,m+1)} + S_{ij}^{(0,m)}\right) \\
&= \sum_{n=0}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}S_{ij}^{(n)} + \sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}S_{ij}^{(n)} \\
&= \sum_{k=1}^{N}\left(\sum_{n=0}^{m+1}(-1)^{m+1-n}\binom{m+1}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jk}^{(n-a)} + \sum_{n=0}^{m}(-1)^{m-n}\binom{m}{n}\sum_{a=0}^{n}T_{ik}^{(a)}\overline{T}_{jk}^{(n-a)}\right) \\
&= \sum_{k<i}\sum_{\substack{c+d=m+1\\ c\neq m+1}}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
&+\sum_{k<i}\sum_{c+d=m}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
&+\sum_{k<i}\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
&+\sum_{k<i}\sum_{c+d=m-1}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}\right) \\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right) \\
&+\left(\sum_{a=1}^{m+1}(-1)^{m+1-a}\binom{m+1}{a}T_{jk}^{(a)} - (-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ij}}(\overline{T}_{ij}^{(0)}-\delta_{ij})\right)\left(\overline{T}_{jj}^{(0)}-1\right) \\
&\qquad +\left(\sum_{a=1}^{m+1}(-1)^{m+1-a}\binom{m+1}{a}T_{jk}^{(a)} - (-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ij}}(\overline{T}_{ij}^{(0)}-\delta_{ij})\right) \\
&+\left(\sum_{a=1}^{m}(-1)^{m-a}\binom{m}{a}T_{ij}^{(a)} - (-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ij}}(\overline{T}_{ij}^{(0)}-\delta_{ij})\right)\left(\overline{T}_{jj}^{(0)}-1\right) \\
&\qquad +\left(\sum_{a=1}^{m}(-1)^{m-a}\binom{m}{a}T_{ij}^{(a)} - (-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ij}}(\overline{T}_{ij}^{(0)}-\delta_{ij})\right) \\
&+\left(T_{ii}^{(0)}-1\right)\left(\sum_{b=1}^{m+1}(-1)^{m+1-b}\binom{m+1}{b}\overline{T}_{ji}^{(b)} - (-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ji}}(T_{ji}^{(0)}-\delta_{ji})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\sum_{b=1}^{m+1}(-1)^{m+1-b}\binom{m+1}{b}\overline{T}_{ji}^{(b)} - (-1)^{m+1}\left(\frac{q+1}{q}\right)^{\delta_{ji}}(T_{ji}^{(0)}-\delta_{ji})\right)\\
&\qquad + \left(T_{ii}^{(0)}-1\right)\left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{ji}^{(b)} - (-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ji}}(T_{ji}^{(0)}-\delta_{ji})\right)\\
&+ \left(\sum_{b=1}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{ji}^{(b)} - (-1)^{m}\left(\frac{q+1}{q}\right)^{\delta_{ji}}(T_{ji}^{(0)}-\delta_{ji})\right)\\
&+ \sum_{i\le k\le j}\ \sum_{\substack{c+d=m+1\\ c,d\neq m+1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right)\\
&+ \sum_{i\le k\le j}\ \sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right)\\
&+ \sum_{i\le k\le j}\ \sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i\le k\le j}\ \sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad \cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{jk}}(T_{jk}^{(0)}-\delta_{jk})\right)\\
&+ \sum_{k>j}\ \sum_{\substack{c+d=m+1\\ d\neq m+1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\qquad \cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right)\\
&+ \sum_{k>j}\ \sum_{c+d=m}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\qquad \cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k>j}\sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right)\\
&+\sum_{k>j}\sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{ik}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{ik}}(\overline{T}_{ik}^{(0)}-\delta_{ik})\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{jk}^{(b)}\right).
\end{aligned}
$$

After dividing by $q-q^{-1}$ and passing to $\mathbf{K}_m/\mathbf{K}_{m+1}$, we are left with

$$
\begin{aligned}
\zeta_{ij}^{(1,m)} &= \xi_{ij}^{(0,m)}+(-1)^{m+1}\xi_{ji}^{(0,m)}-h\sum_{k=1}^{N}\sum_{c+d=m-1}(-1)^{d}\xi_{ik}^{(0,c)}\xi_{jk}^{(0,d)}\\
&= \xi_{ij}^{(0,m)}+(-1)^{m+1}\xi_{ji}^{(0,m)}+h\sum_{k=1}^{N}\sum_{c=0}^{m-1}(-1)^{m-c}\xi_{ik}^{(0,c)}\xi_{jk}^{(0,m-1-c)}
\end{aligned}
$$

$$
\begin{aligned}
&= \xi_{ij}^{(0,m)}+(-1)^{m+1}\xi_{ji}^{(0,m)}+h\sum_{k=1}^{N}\sum_{c=1}^{m}(-1)^{m+1-c}\xi_{ik}^{(0,c-1)}\xi_{jk}^{(0,m-c)}\\
&= \varphi(s_{ij}^{(m+1)}).
\end{aligned}
$$

■

Chapter 5

$Y^{tw}(\mathfrak{sp}_N)$ as a degenerate form of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$

In the same spirit as the previous two chapters, we will now realize the twisted Yangian $Y^{tw}(\mathfrak{sp}_N)$ as a degenerate form of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$. We will regard these two algebras as subalgebras of $Y(\mathfrak{gl}_N)$ and $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ respectively, via the embeddings of Propositions 1.5 and 1.8. The proof amounts to a computation which is very similar to the one seen Chapter 4. Recall that for this case, N must be even.

Let $S_{ij}^{(r,0)} = \frac{S_{ij}^{(r)}}{q-q^{-1}}$ when $r > 0$, and for $i \geq j$ let $S_{ij}^{(0,0)} = \frac{S_{ij}^{(0)}-b_{ij}}{q-q^{-1}}$ where $B = (b_{ij})$ is given by (1.44). For $i < j$, let $S_{ij}^{(0,0)} = -\frac{S_{ji}^{(0)}}{q-q^{-1}}$. For each $m \geq 0$, define inductively $S_{ij}^{(r,m+1)} = S_{ij}^{(r+1,m)} - S_{ij}^{(r,m)}$. Let $\zeta_{ij}^{(r,m)}$ be the image of $S_{ij}^{(r,m)}$ in the quotient $\mathbf{K}_m/\mathbf{K}_{m+1}$. Once again we have

$$S_{ij}^{(r,m)} = \sum_{n=r}^{m+r} (-1)^{m-n+r} \binom{m}{n-r} S_{ij}^{(n,0)}$$

for all $r, m \geq 0$.

Theorem 5.1 *If $i > j+1$, then*

$$s_{ij}^{(m+1)} \overset{\varphi}{\mapsto} \zeta_{ij}^{(0,m)}$$

where $\varphi : Y_h(\mathfrak{gl}_N) \xrightarrow{\sim} \widetilde{Y}_h(\mathfrak{gl}_N)$ is the isomorphism given by $t_{ij}^{(m+1)} \mapsto \xi_{ij}^{(0,m)}$; if $i \leq j+1$, then $s_{ij}^{(m+1)} \overset{\varphi}{\mapsto} \zeta_{ij}^{(1,m)}$.

We can obtain an analogue of Theorem 3.2. From Theorem 3.1 and Proposition 1.8, we can deduce that the enveloping algebra of $\mathfrak{sp}_N^{tw}[s,s^{-1}]$ is the limit when $q \mapsto 1$ of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ in the following precise sense: if we let $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ be the subalgebra of $\mathfrak{U}_q(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ generated by all the $S_{ij}^{(r,0)}$, then $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ is isomorphic to $\mathfrak{U}(\mathfrak{sp}_N^{tw}[s,s^{-1}])$ (see the proof of Theorem 3.10 in [12]). The following diagram is commutative:

$$\begin{tikzcd}
& \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N)) \arrow[r] \arrow[d] & \mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N)) \arrow[d] & \\
\mathfrak{U}(\mathfrak{sp}_N^{tw}[s,s^{-1}]) \arrow[r, equal] & \frac{\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))}{(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))} \arrow[r] & \frac{\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))}{(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}(\mathfrak{gl}_N))} \arrow[r, equal] & \mathfrak{U}(\mathcal{L}(\mathfrak{gl}_N)).
\end{tikzcd}$$

We can define an algebra $\widetilde{Y}^{tw}(\mathfrak{sp}_N)$ similarly to how we defined $\widetilde{Y}^{tw}(\mathfrak{o}_N)$. For $m \geq 0$, denote by K_m^{tw} the Lie ideal of $\mathfrak{sp}_N^{tw}[s,s^{-1}]$ spanned by

$$(-1)^j E_{ij'} s^r (s-1)^m + (-1)^i E_{ji'} s^{-r} (s^{-1}-1)^m$$

for all $r \in \mathbb{Z}$. Let U^{tw} be the subspace of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ spanned over $\mathbb{C}$ all the generators $S_{ij}^{(r,0)}$, and observe that $U^{tw} \cap (q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N)) = \{0\}$. Set $\mathbb{K}_m^{tw} = \psi^{-1}(\mathsf{K}_m^{tw}) \cap U^{tw}$ where ψ this time denotes the composite

$$\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N)) \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))/(q-1)\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N)) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{sp}_N^{tw}[s,s^{-1}]).$$

Let $\mathbf{K}_m^{tw}$ be the two-sided ideal of $\mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))$ generated by $(q-q^{-1})^{m_0}\mathbb{K}_{m_1}^{tw} \cdots \mathbb{K}_{m_k}^{tw}$ with $m_0 + m_1 + \cdots + m_k \geq m$.

Let $\widetilde{Y}^{tw}(\mathfrak{sp}_N)$ be the $\mathbb{C}$-algebra

$$\widetilde{Y}^{tw}(\mathfrak{sp}_N) = \bigoplus_{m=0}^{\infty} \mathbf{K}_m^{tw}/\mathbf{K}_{m+1}^{tw}$$

where $\mathbf{K}_0^{tw} = \mathfrak{U}_{\mathcal{A}}(\mathcal{L}^{tw}(\mathfrak{sp}_N))$. We also view $\widetilde{Y}^{tw}(\mathfrak{sp}_N)$ as an algebra over $\mathbb{C}[h]$ by setting $h = \overline{q-q^{-1}} \in \mathbf{K}_1^{tw}/\mathbf{K}_2^{tw}$, and then denote it by $\widetilde{Y}_h^{tw}(\mathfrak{sp}_N)$.

Corollary 5.1 *$Y_h^{tw}(\mathfrak{sp}_N)$ is isomorphic to $\widetilde{Y}_h^{tw}(\mathfrak{sp}_N)$ via the function φ^{tw} that sends $s_{ij}^{(m)}$ to $\overline{S_{ij}^{(1,m-1)}} \in \mathbf{K}_{m-1}^{tw}/\mathbf{K}_m^{tw}$ for $m \geq 1$.*

Proof. Theorem 5.1 implies that the following diagram is commutative:

$$\begin{tikzcd}
Y_h^{tw}(\mathfrak{sp}_N) \arrow[r] \arrow[d, "\varphi^{tw}"'] & Y_h(\mathfrak{gl}_N) \arrow[d, "\varphi"] \\
\widetilde{Y}_h^{tw}(\mathfrak{sp}_N) \arrow[r] & \widetilde{Y}_h(\mathfrak{gl}_N).
\end{tikzcd}$$

In this diagram, the top horizontal arrow is the embedding of proposition 1.5 and the bottom horizontal arrow is the one induced from the embedding of proposition 1.8. The injectivity of φ^{tw} now follows from the fact that φ provides an isomorphism between $Y_h(\mathfrak{gl}_N)$ and $\widetilde{Y}_h(\mathfrak{gl}_N)$: see Theorem 3.2. The proof that this map is surjective is the same as in the orthogonal case. ■

Proof of Theorem 5.1. Suppose that $j < i-1$. We need to show that

$$\zeta_{ij}^{(0,m)} = (-1)^j \xi_{i,j'}^{(0,m)} + (-1)^{m+i} \xi_{j,i'}^{(0,m)} - h \sum_{k=1}^{N/2} \sum_{p=1}^{m} (-1)^{m-p} (\xi_{i,2k-1}^{(0,p-1)} \xi_{j,2k}^{(0,m-p)} - \xi_{i,2k}^{(0,p-1)} \xi_{j,2k-1}^{(0,m-p)}).$$

Using the formulas (4.1), (4.2) and (4.3), we find that

$$
\begin{aligned}
&(q-q^{-1}) S_{ij}^{(0,m)} = \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} S_{ij}^{(n)} \\
&= \sum_{k=1}^{N/2} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} (q T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} - T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)}) \\
&= \sum_{2k \le j} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} \left((q-1) T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} + T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} \right) \\
&\qquad - \sum_{2k \le j+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)} \\
&+ \sum_{j<2k<i+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} \left((q-1) T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} + T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} \right) \\
&\qquad - \sum_{j+1<2k<i} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)} \\
&+ \sum_{2k \ge i+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} \left((q-1) T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} + T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} \right) \\
&\qquad - \sum_{2k \ge i} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)} \\
&= (q-1) \sum_{2k \le j} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k-1}^{(a)} \right) \overline{T}_{j,2k}^{(0)} \\
&+ (q-1) \sum_{2k \le j} \sum_{\substack{c+d=m \\ c \ne m}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^d \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ (q-1) \sum_{2k \le j} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^d \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{2k\le j}\left(\sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a}T^{(a)}_{i,2k-1}\right)(\overline{T}^{(0)}_{j,2k}-\delta_{j,2k})\\
&+\sum_{2k\le j}\delta_{j,2k}\left(\sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a}T^{(a)}_{i,2k-1}\right)\\
&+\sum_{2k\le j}\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k-1}\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k}}(T^{(0)}_{j,2k}-\delta_{j,2k})\right)\\
&+\sum_{2k\le j}\sum_{c+d=m-1}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k-1}\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k}}(T^{(0)}_{j,2k}-\delta_{j,2k})\right)\\
&-\sum_{2k\le j+1}\left(\sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a}T^{(a)}_{i,2k}\right)(\overline{T}^{(0)}_{j,2k-1}-\delta_{j,2k-1})\\
&-\sum_{2k\le j+1}\delta_{j,2k-1}\left(\sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a}T^{(a)}_{i,2k}\right)\\
&-\sum_{2k\le j+1}\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k}\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k-1}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}}(T^{(0)}_{j,2k-1}-\delta_{j,2k-1})\right)\\
&-\sum_{2k\le j+1}\sum_{c+d=m-1}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k}\right)\\
&\qquad\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k-1}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}}(T^{(0)}_{j,2k-1}-\delta_{j,2k-1})\right)\\
&+(q-1)\sum_{j<2k<i+1}\sum_{c+d=m}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k-1}\right)\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right)\\
&+(q-1)\sum_{j<2k<i+1}\sum_{c+d=m-1}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k-1}\right)\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right)\\
&+\sum_{j<2k<i+1}\sum_{c+d=m}\left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a}T^{(a)}_{i,2k-1}\right)\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j<2k<i+1} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&- \sum_{j+1<2k<i} \sum_{c+d=m} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&- \sum_{j+1<2k<i} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&+ (q-1) \sum_{2k\geq i+1} T_{i,2k-1}^{(0)} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ (q-1) \\
&\qquad \cdot \sum_{2k\geq i+1} \sum_{\substack{c+d=m \\ d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ (q-1) \\
&\qquad \cdot \sum_{2k\geq i+1} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ \sum_{2k\geq i+1} (T_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ \sum_{2k\geq i+1} \delta_{i,2k-1} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ \sum_{2k\geq i+1} \sum_{\substack{c+d=m \\ d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ \sum_{2k\geq i+1} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&- \sum_{2k\geq i} (T_{i,2k}^{(0)} - \delta_{i,2k}) \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k-1}^{(b)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{2k\geq i} \delta_{i,2k} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k-1} \right) \\
&- \sum_{2k\geq i} \sum_{\substack{c+d=m \\ d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^c \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} \right) \\
&- \sum_{2k\geq i} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^c \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} \right).
\end{aligned}
$$

After dividing by $q - q^{-1}$ and passing to $\mathbf{K}_m/\mathbf{K}_{m+1}$ (always bearing in mind that $T^{(0)}_{kl} = 0$ when $k < l$ and $\overline{T}^{(0)}_{kl} = 0$ when $k > l$), we find, as desired, that

$$
\begin{aligned}
\zeta^{(0,m)}_{ij} &= (-1)^j \xi^{(0,m)}_{ij'} + (-1)^{m+i} \xi^{(0,m)}_{ji'} - h \sum_{k=1}^{N/2} \sum_{c+d=m-1} (-1)^d \left(\xi^{(0,c)}_{i,2k-1} \xi^{(0,d)}_{j,2k} - \xi^{(c)}_{i,2k} \xi^{(d)}_{j,2k-1} \right) \\
&= (-1)^j \xi^{(0,m)}_{ij'} + (-1)^{m+i} \xi^{(0,m)}_{ji'} \\
&\qquad - h \sum_{k=1}^{N/2} \sum_{c=0}^{m-1} (-1)^{(m-1-c)} \left(\xi^{(0,c)}_{i,2k-1} \xi^{(0,m-1-c)}_{j,2k} - \xi^{(0,c)}_{i,2k} \xi^{(0,m-1-c)}_{j,2k-1} \right) \\
&= (-1)^j \xi^{(0,m)}_{ij'} + (-1)^{m+i} \xi^{(0,m)}_{ji'} \\
&\qquad - h \sum_{k=1}^{N/2} \sum_{c=1}^{m} (-1)^{(m-c)} \left(\xi^{(0,c-1)}_{i,2k-1} \xi^{(0,m-c)}_{j,2k} - \xi^{(0,c-1)}_{i,2k} \xi^{(0,m-c)}_{j,2k-1} \right).
\end{aligned}
$$

Now suppose that $i \leq j$. We will show that $s^{(m+1)}_{ij} \overset{\varphi}{\mapsto} \zeta^{(1,m)}_{ij}$ using (4.2), (4.3) and (4.6):

$$
\begin{aligned}
(q-q^{-1}) S^{(1,m)}_{ij} &= (q-q^{-1}) \left(S^{(0,m+1)}_{ij} + S^{(0,m)}_{ij} \right) \\
&= \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} S^{(n)}_{ij} + \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} S^{(n)}_{ij} \\
&= \sum_{k=1}^{N/2} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} (q T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} - T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1}) \\
&\qquad + \sum_{k=1}^{N/2} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} (q T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} - T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1})
\end{aligned}
$$

$$
\begin{aligned}
&= q \sum_{2k<i+1} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&\qquad + q \sum_{2k<i+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&- \sum_{2k<i} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&\qquad - \sum_{2k<i} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&+ q \sum_{i+1\leq 2k\leq j} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&\qquad + q \sum_{i+1\leq 2k\leq j} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{s} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&- \sum_{i\leq 2k\leq j+1} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&\qquad - \sum_{i\leq 2k\leq j+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&+ q \sum_{2k>j} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&\qquad + q \sum_{2k>j} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k-1} \overline{T}^{(n-p)}_{j,2k} \\
&- \sum_{2k>j+1} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&\qquad - \sum_{2k>j+1} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} T^{(p)}_{i,2k} \overline{T}^{(n-p)}_{j,2k-1} \\
&= q \sum_{2k<i+1} \left(\sum_{a=0}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T^{(a)}_{i,2k-1} \right) \left(\overline{T}^{(0)}_{j,2k} - \delta_{j,2k} \right) \\
&\qquad + q \sum_{2k<i+1} \delta_{j,2k} \left(\sum_{a=0}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T^{(a)}_{i,2k-1} \right) \\
&+ q \sum_{2k<i+1} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T^{(a)}_{i,2k-1} \right) \left(\overline{T}^{(0)}_{j,2k} - \delta_{j,2k} \right) \\
&\qquad + q \sum_{2k<i+1} \delta_{j,2k} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T^{(a)}_{i,2k-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q \sum_{2k<i+1} \sum_{\substack{c+d=m+1 \\ c\neq m+1}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i+1} \sum_{c+d=m} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i+1} \sum_{\substack{c+d=m \\ c\neq m}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i+1} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} \right) \\
&\qquad\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&- \sum_{2k<i} \left(\sum_{a=0}^{m+1} (-1)^{m-a} \binom{m+1}{a} T_{i,2k}^{(a)} \right) \overline{T}_{j,2k-1}^{(0)} - \sum_{2k<i} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k}^{(a)} \right) \overline{T}_{j,2k-1}^{(0)} \\
&- \sum_{2k<i} \sum_{\substack{c+d=m+1 \\ c\neq m+1}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k<i} \sum_{c+d=m} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k<i} \sum_{\substack{c+d=m \\ c\neq m}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{2k<i} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&+ q \\
&\cdot \sum_{i+1 \le 2k \le j} \left(\sum_{a=1}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k-1}^{(a)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad \cdot \left(\overline{T}_{j,2k}^{(0)} - \delta_{j,2k} \right) \\
&+ q \sum_{i+1 \le 2k \le j} \delta_{j,2k} \\
&\quad \cdot \left(\sum_{a=1}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k-1}^{(a)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&+ q \sum_{i+1 \le 2k \le j} \left(\sum_{a=1}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k-1}^{(a)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad \cdot \left(\overline{T}_{j,2k}^{(0)} - \delta_{j,2k} \right) \\
&+ q \sum_{i+1 \le 2k \le j} \delta_{j,2k} \left(\sum_{a=1}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k-1}^{(a)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&+ q \sum_{i+1 \le 2k \le j} \left(T_{i,2k-1}^{(0)} - \delta_{i,2k-1} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{i+1 \le 2k \le j} \delta_{i,2k-1} \\
&\quad \cdot \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{i+1 \le 2k \le j} \left(T_{i,2k-1}^{(0)} - \delta_{i,2k-1} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right) \\
&+ q \sum_{i+1 \le 2k \le j} \delta_{i,2k-1} \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T_{j,2k}^{(0)} - \delta_{j,2k}) \right)
\end{aligned}
$$

$$+ q \sum_{i+1\le 2k\le j} \sum_{\substack{c+d=m+1\\ c,d\neq m+1}} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^c \left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} (\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1})\right)$$

$$\cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^d \left(\frac{q+1}{q}\right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k})\right)$$

$$+ q \sum_{i+1\le 2k\le j} \sum_{\substack{c+d=m\\ c\neq m}} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^c \left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} (\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1})\right)$$

$$\cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^d \left(\frac{q+1}{q}\right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k})\right)$$

$$+ q \sum_{i+1\le 2k\le j} \sum_{\substack{c+d=m\\ d\neq m}} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^c \left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} (\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1})\right)$$

$$\cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^d \left(\frac{q+1}{q}\right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k})\right)$$

$$+ q \sum_{i+1\le 2k\le j} \sum_{c+d=m-1} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^c \left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} (\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1})\right)$$

$$\cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^d \left(\frac{q+1}{q}\right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k})\right)$$

$$- \sum_{i\le 2k\le j+1} \sum_{a=1}^{m+1} \left((-1)^{m+1-a}\binom{m+1}{a} T^{(a)}_{i,2k} - (-1)^{m+1} \left(\frac{q+1}{q}\right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k})\right)$$

$$\cdot \left(\overline{T}^{(0)}_{j,2k-1} - \delta_{j,2k-1}\right)$$

$$- \sum_{i\le 2k\le j+1} \delta_{j,2k-1} \sum_{a=1}^{m+1} \left((-1)^{m+1-a}\binom{m+1}{a} T^{(a)}_{i,2k} - (-1)^{m+1} \left(\frac{q+1}{q}\right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k})\right)$$

$$- \sum_{i\le 2k\le j+1} \sum_{a=1}^{m} \left((-1)^{m-a}\binom{m}{a} T^{(a)}_{i,2k} - (-1)^{m} \left(\frac{q+1}{q}\right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k})\right)$$

$$\cdot \left(\overline{T}^{(0)}_{j,2k-1} - \delta_{j,2k-1}\right)$$

$$- \sum_{i\le 2k\le j+1} \delta_{j,2k-1} \sum_{a=1}^{m} \left((-1)^{m-a}\binom{m}{a} T^{(a)}_{i,2k} - (-1)^{m} \left(\frac{q+1}{q}\right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k})\right)$$

$$- \sum_{i\le 2k\le j+1} \left(T^{(0)}_{i,2k} - \delta_{i,2k}\right)$$

$$\cdot \left(\sum_{b=1}^{m+1}(-1)^{m+1-b}\binom{m+1}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{m+1} \left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1})\right)$$

$$
\begin{aligned}
&- \sum_{i \le 2k \le j+1} \delta_{i,2k} \\
&\qquad \cdot \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \left(T^{(0)}_{i,2k} - \delta_{i,2k} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \delta_{i,2k} \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \sum_{\substack{c+d=m+1 \\ c,d \neq m+1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \sum_{\substack{c+d=m \\ c \neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \sum_{\substack{c+d=m \\ d \neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&- \sum_{i \le 2k \le j+1} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}^{(0)}_{i,2k} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k-1} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T^{(0)}_{j,2k-1} - \delta_{j,2k-1}) \right) \\
&+ q \sum_{2k > j} \left(T^{(0)}_{i,2k-1} - \delta_{i,2k-1} \right) \left(\sum_{b=0}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k > j} \delta_{i,2k-1} \left(\sum_{b=0}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k > j} \left(T^{(0)}_{i,2k-1} - \delta_{i,2k-1} \right) \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q \sum_{2k>j} \delta_{i,2k-1} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ q \sum_{2k>j} \sum_{\substack{c+d=m+1 \\ d\neq m+1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ q \sum_{2k>j} \sum_{c+d=m} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ q \sum_{2k>j} \sum_{\substack{c+d=m \\ d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&+ q \sum_{2k>j} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k-1}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k-1}} (\overline{T}_{i,2k-1}^{(0)} - \delta_{i,2k-1}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k}^{(b)} \right) \\
&- \sum_{2k>j+1} T_{i,2k}^{(0)} \left(\sum_{b=0}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&- \sum_{2k>j+1} T_{i,2k}^{(0)} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&- \sum_{2k>j+1} \sum_{\substack{c+d=m+1 \\ d\neq m+1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&- \sum_{2k>j+1} \sum_{c+d=m} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{2k>j+1} \sum_{\substack{c+d=m \\ d\neq m}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right) \\
&- \sum_{2k>j+1} \sum_{c+d=m-1} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} \right).
\end{aligned}
$$

After dividing by $q - q^{-1}$ and passing to $\mathbf{K}_m/\mathbf{K}_{m+1}$, what remains is

$$
\begin{aligned}
\zeta_{ij}^{(1,m)} &= (-1)^{j} \xi_{i,j'}^{(0,m)} + (-1)^{m+i} \xi_{j,i'}^{(0,m)} + h \sum_{k=1}^{N/2} \sum_{c+d=m-1} (-1)^{d+1} \left(\xi_{i,2k-1}^{(0,c)} \xi_{j,2k}^{(0,d)} - \xi_{i,2k}^{(0,c)} \xi_{j,2k-1}^{(0,d)} \right) \\
&= (-1)^{j} \xi_{i,j'}^{(0,m)} + (-1)^{m+i} \xi_{j,i'}^{(0,m)} \\
&\qquad - h \sum_{k=1}^{N/2} \sum_{c=1}^{m} (-1)^{m-c} \left(\xi_{i,2k-1}^{(0,c-1)} \xi_{j,2k}^{(0,m-c)} - \xi_{i,2k}^{(0,c-1)} \xi_{j,2k-1}^{(0,m-c)} \right) \\
&= \varphi(s_{ij}^{(m+1)}).
\end{aligned}
$$

Finally, suppose that $j = i - 1$. Using (4.1), (4.2), (4.3) and (4.6) we have

$$
\begin{aligned}
&(q - q^{-1}) S_{ij}^{(1,m)} = (q - q^{-1}) \left(S_{ij}^{(0,m+1)} + S_{ij}^{(0,m)} \right) \\
&= \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{s} S_{ij}^{(n)} + \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} S_{ij}^{(n)} \\
&= \sum_{k=1}^{N/2} \sum_{n=0}^{m+1} (-1)^{m+1-n} \binom{m+1}{n} \sum_{p=0}^{n} (q T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} - T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)}) \\
&\qquad + \sum_{k=1}^{N/2} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sum_{p=0}^{n} (q T_{i,2k-1}^{(p)} \overline{T}_{j,2k}^{(n-p)} - T_{i,2k}^{(p)} \overline{T}_{j,2k-1}^{(n-p)}) \\
&= q \sum_{2k<i} \left(\sum_{a=0}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k-1}^{(a)} \right) \left(\overline{T}_{j,2k}^{(0)} - \delta_{j,2k} \right) \\
&+ q \sum_{2k<i} \delta_{j,2k} \left(\sum_{a=0}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k-1}^{(a)} \right) \\
&+ q \sum_{2k<i} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k-1}^{(a)} \right) \left(\overline{T}_{j,2k}^{(0)} - \delta_{j,2k} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q \sum_{2k<i} \delta_{j,2k} \left(\sum_{a=0}^{m} (-1)^{m-a} \binom{m}{a} T^{(a)}_{i,2k-1} \right) \\
&+ q \sum_{2k<i} \sum_{\substack{c+d=m+1 \\ c\neq m+1}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i} \sum_{c+d=m} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i} \sum_{\substack{c+d=m \\ c\neq m}} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k<i} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \\
&\qquad\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k}} (T^{(0)}_{j,2k} - \delta_{j,2k}) \right) \\
&+ q \sum_{2k=i} \sum_{c+d=m+1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ 2q \sum_{2k=i} \sum_{c+d=m} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k=i} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T^{(a)}_{i,2k-1} \right) \left(\sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k>i} \left(T^{(0)}_{i,2k-1} - \delta_{i,2k-1} \right) \left(\sum_{b=0}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k>i} \delta_{i,2k-1} \left(\sum_{b=0}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k>i} \left(T^{(0)}_{i,2k-1} - \delta_{i,2k-1} \right) \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k} \right) \\
&+ q \sum_{2k>i} \delta_{i,2k-1} \left(\sum_{b=0}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}^{(b)}_{j,2k} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q \sum_{2k>i} \sum_{\substack{c+d=m+1\\ d\neq m+1}} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} \left(\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1}\right)\right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right) \\
&+ q \sum_{2k>i} \sum_{c+d=m} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} \left(\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1}\right)\right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right) \\
&+ q \sum_{2k>i} \sum_{\substack{c+d=m\\ d\neq m}} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} \left(\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1}\right)\right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right) \\
&+ q \sum_{2k>i} \sum_{c+d=m-1} \left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k-1} - (-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k-1}} \left(\overline{T}^{(0)}_{i,2k-1} - \delta_{i,2k-1}\right)\right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k}\right) \\
&- \sum_{2k<i} \left(\sum_{a=0}^{m+1}(-1)^{m+1-a}\binom{m+1}{a} T^{(a)}_{i,2k}\right)\overline{T}^{(0)}_{j,2k-1} - \sum_{2k<i} \left(\sum_{a=0}^{m}(-1)^{m-a}\binom{m}{a} T^{(a)}_{i,2k}\right)\overline{T}^{(0)}_{j,2k-1} \\
&- \sum_{2k<i} \sum_{\substack{c+d=m+1\\ c\neq m+1}} \left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k}\right) \\
&\qquad \cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k-1} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}} \left(T^{(0)}_{j.2k-1} - \delta_{j,2k-1}\right)\right) \\
&- \sum_{2k<i} \sum_{c+d=m} \left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k}\right) \\
&\qquad \cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k-1} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}} \left(T^{(0)}_{j,2k-1} - \delta_{j,2k-1}\right)\right) \\
&- \sum_{2k<i} \sum_{\substack{c+d=m\\ c\neq m}} \left(\sum_{a=0}^{c}(-1)^{c-a}\binom{c}{a} T^{(a)}_{i,2k}\right) \\
&\qquad \cdot \left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}^{(b)}_{j,2k-1} - (-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}} \left(T^{(0)}_{j.2k-1} - \delta_{j,2k-1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{2k<i} \sum_{c+d=m-1} \left(\sum_{a=0}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k=i} \left(\sum_{a=1}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k}^{(a)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\overline{T}_{j,2k-1}^{(0)} - \delta_{j,2k-1} \right) \\
&- \sum_{2k=i} \delta_{j,2k-1} \left(\sum_{a=1}^{m+1} (-1)^{m+1-a} \binom{m+1}{a} T_{i,2k}^{(a)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad - \sum_{2k=i} \left(\sum_{a=1}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k}^{(a)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\overline{T}_{j,2k-1}^{(0)} - \delta_{j,2k-1} \right) \\
&- \sum_{2k=i} \delta_{j,2k-1} \left(\sum_{a=1}^{m} (-1)^{m-a} \binom{m}{a} T_{i,2k}^{(a)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&- \sum_{2k=i} \left(T_{i,2k}^{(0)} - \delta_{i,2k} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k=i} \delta_{i,2k} \\
&\qquad \cdot \left(\sum_{b=1}^{m+1} (-1)^{m+1-b} \binom{m+1}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{m+1} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k=i} \left(T_{i,2k}^{(0)} - \delta_{i,2k} \right) \\
&\qquad \cdot \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k=i} \delta_{i,2k} \left(\sum_{b=1}^{m} (-1)^{m-b} \binom{m}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{m} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right) \\
&- \sum_{2k=i} \sum_{\substack{c+d=m+1 \\ c,d\neq m+1}} \left(\sum_{a=1}^{c} (-1)^{c-a} \binom{c}{a} T_{i,2k}^{(a)} - (-1)^{c} \left(\frac{q+1}{q} \right)^{\delta_{i,2k}} (\overline{T}_{i,2k}^{(0)} - \delta_{i,2k}) \right) \\
&\qquad \cdot \left(\sum_{b=1}^{d} (-1)^{d-b} \binom{d}{b} \overline{T}_{j,2k-1}^{(b)} - (-1)^{d} \left(\frac{q+1}{q} \right)^{\delta_{j,2k-1}} (T_{j,2k-1}^{(0)} - \delta_{j,2k-1}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{2k=i}\sum_{\substack{c+d=m\\ c\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}}(T_{j,2k-1}^{(0)}-\delta_{j,2k-1})\right)\\
&-\sum_{2k=i}\sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}}(T_{j,2k-1}^{(0)}-\delta_{j,2k-1})\right)\\
&-\sum_{2k=i}\sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=1}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}-(-1)^{d}\left(\frac{q+1}{q}\right)^{\delta_{j,2k-1}}(T_{j,2k-1}^{(0)}-\delta_{j,2k-1})\right)\\
&-\sum_{2k>i}T_{i,2k}^{(0)}\left(\sum_{b=0}^{m+1}(-1)^{m+1-b}\binom{m+1}{b}\overline{T}_{j,2k-1}^{(b)}\right)-\sum_{2k>i}T_{i,2k}^{(0)}\left(\sum_{b=0}^{m}(-1)^{m-b}\binom{m}{b}\overline{T}_{j,2k-1}^{(b)}\right)\\
&-\sum_{2k>i}\sum_{\substack{c+d=m+1\\ d\neq m+1}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{2k>i}\sum_{c+d=m}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}\right)\\
&-\sum_{2k>i}\sum_{\substack{c+d=m\\ d\neq m}}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}\right)\\
&-\sum_{2k>i}\sum_{c+d=m-1}\left(\sum_{a=1}^{c}(-1)^{c-a}\binom{c}{a}T_{i,2k}^{(a)}-(-1)^{c}\left(\frac{q+1}{q}\right)^{\delta_{i,2k}}(\overline{T}_{i,2k}^{(0)}-\delta_{i,2k})\right)\\
&\qquad\cdot\left(\sum_{b=0}^{d}(-1)^{d-b}\binom{d}{b}\overline{T}_{j,2k-1}^{(b)}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\zeta_{ij}^{(1,m)} &= (-1)^j \xi_{i,j'}^{(0,m)} + (-1)^{m+i} \xi_{j,i'}^{(0,m)} + h \sum_{k=1}^{N/2} \sum_{c+d=m-1} (-1)^{d+1} \left(\xi_{i,2k-1}^{(0,c)} \xi_{j,2k}^{(0,d)} - \xi_{i,2k}^{(0,c)} \xi_{j,2k-1}^{(0,d)} \right) \\
&= (-1)^j \xi_{i,j'}^{(0,m)} + (-1)^{m+i} \xi_{j,i'}^{(0,m)} \\
&\qquad - h \sum_{k=1}^{N/2} \sum_{c=1}^{m} (-1)^{m-c} \left(\xi_{i,2k-1}^{(0,c-1)} \xi_{j,2k}^{(0,m-c)} - \xi_{i,2k}^{(0,c-1)} \xi_{j,2k-1}^{(0,m-c)} \right) \\
&= \varphi(s_{ij}^{(m+1)}).
\end{aligned}
$$

■

Chapter 6

Classification of Finite Dimensional Irreducible Representations of $Y(\mathfrak{g})$

Throughout this section, we fix a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, and a basis of simple roots $\{\alpha_i\}_{i\in I}$. We begin by recalling some facts about representations of $\mathfrak{g}$ (equivalently, representations of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$).

6.1 Preliminaries and Definitions

Let V be a representation of $\mathfrak{U}(\mathfrak{g})$. We say that $\lambda \in \mathfrak{h}^*$ is a *weight* of V if the simultaneous eigenspace

$$V_\lambda = \{v \in V \mid Hv = \lambda(H)v \ \forall \ H \in \mathfrak{h}\}$$

is nonzero. If V is finite dimensional and irreducible, then there exists a weight space decomposition

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda.$$

A nonzero vector $v \in V_\lambda$ is called a *highest weight vector* if $\mathfrak{U}(\mathfrak{n}^+)v = 0$ where $\mathfrak{n}^+$ is given by the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. V is a *highest weight representation* with highest weight λ if V is generated by a highest weight vector $v \in V_\lambda$ for some $\lambda \in \mathfrak{h}^*$.

Proposition 6.1
(a) *Every finite dimensional representation of* $\mathfrak{U}(\mathfrak{g})$ *is completely reducible.*
(b) *Every finite dimensional irreducible representation of* $\mathfrak{U}(\mathfrak{g})$ *admits a weight space decomposition.*
(c) *Every finite dimensional irreducible representation of* $\mathfrak{U}(\mathfrak{g})$ *is highest weight. Its highest weight* λ *is unique, and the weight space* V_λ *is one dimensional.*
(d) *An irreducible representation of* $\mathfrak{U}(\mathfrak{g})$ *is finite dimensional if and only if its highest weight* λ *belongs to* $P^+ = \{\lambda \in \mathfrak{h}^* \mid 2\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \in \mathbb{Z}_+ \ \forall \ i \in I\}$.
(e) *Every weight of a finite dimensional irreducible representation of* $\mathfrak{U}(\mathfrak{g})$ *is of the form*

$\lambda - \eta$ with $\eta \in Q^+$, where λ is the highest weight and

$$Q^+ = \sum_{i \in I} \mathbb{Z}_+ \alpha_i.$$

Point (e) above motivates the introduction of the following partial ordering on $\mathfrak{h}^*$: we shall say that $\alpha \leq \beta$ if $\beta - \alpha \in Q^+$. Then (e) means precisely that all the weights of a finite dimensional irreducible representation of $\mathfrak{U}(\mathfrak{g})$ are comparable with respect to this ordering, and the highest weight is maximal among them.

In this chapter, we will prove some analogous results for the Yangian $Y(\mathfrak{g})$. We will first need to introduce the notion of highest weight in this case. The appropriate definitions were given in section 12.1 of [4]. A statement of Theorem 6.3 below can also be found therein, but a full proof was not given, because it should be similar to the proof of the classification of finite dimensional irreducible representations of the quantum loop algebra for $\mathfrak{g}$. The proof of the quantum loop case for $\mathfrak{sl}_2$ can be found in [3], and a proof the quantum loop case in general can be found in [5] and section 12.2 of [4]. The majority of the proof of Theorem 6.3 below was obtained by adapting those arguments for Yangians.

Let V be a representation of $Y(\mathfrak{g})$. We will call a nonzero vector $v \in V$ a *weight vector* if, for all $i \in I$ and $r \in \mathbb{Z}_+$, $H_{i,r}v = \Phi_{i,r}v$ for some complex numbers $\Phi_{i,r}$. The $I \times \mathbb{Z}_+$-tuple $\Phi = (\Phi_{i,r})_{i \in I, r \in \mathbb{Z}_+}$ is then called the *weight* of v. A weight vector v is *highest weight* if, in addition, $X_{i,r}^+ v = 0$ for every $i \in I$ and $r \in \mathbb{Z}_+$.

V is a *highest weight representation* if it is generated by some highest weight vector v, and in this case the weight Φ of v is called the *highest weight* of V.

Theorem 6.1 (PBW Theorem for $Y(\mathfrak{g})$; Cor. 12.1.9 of [4]) *Given any total ordering on the set of generators $\{X_{i,r}^{\pm}, H_{i,r} \mid i \in I,\ r \in \mathbb{Z}_+\}$, a basis for $Y(\mathfrak{g})$ is provided by the collection of all ordered monomials in these generators. In particular, if we choose this ordering so that each $X_{i,r}^-$ precedes each $H_{i,r}$, which in turn precedes each $H_{i,r}^+$, then we obtain an isomorphism of vector spaces*

$$Y(\mathfrak{g}) \cong Y^- \otimes Y^0 \otimes Y^+$$

where $Y^{\pm}$ (respectively Y^0) is the subalgebra of $Y(\mathfrak{g})$ generated by all the $X_{i,r}^{\pm}$ (respectively $H_{i,r}$).

It is easy to check that the assignment

$$e_i \mapsto d_i^{-1} X_{i,0}^+ \qquad f_i \mapsto X_{i,0}^- \qquad h_i \mapsto d_i^{-1} H_{i,0} \tag{6.1}$$

defines a natural homomorphism from $\mathfrak{U}(\mathfrak{g})$ into $Y(\mathfrak{g})$. Consequently, we can view any representation V of $Y(\mathfrak{g})$ as a module over $\mathfrak{U}(\mathfrak{g})$. Then, we can investigate the structure of V by making use of the familiar representation theory of $\mathfrak{g}$.

Theorem 6.2 *Let V be an irreducible highest weight representation of $Y(\mathfrak{g})$ with highest weight $\Phi = (\Phi_{i,r})_{i\in I, r\in\mathbb{Z}_+}$. Then the highest weight vectors in V span a one dimensional subspace. In particular, the highest weight Φ is unique.*

Proof. Since V is highest weight, it is generated by some highest weight vector $v \in V$ of weight Φ. Then according to the PBW theorem and the definition of a highest weight vector, V is spanned by vectors of the form $X^-_{i_1,r_1}\dots X^-_{i_k,r_k}v$. If we view V as a representation of $\mathfrak{g}$, then by definition the weight λ of v with respect to the Cartan subalgebra $\mathfrak{h}$ is given by $\lambda(h_i) = d_i^{-1}\Phi_{i,0}$, and each vector of the form $X^-_{i_1,r_1}\dots X^-_{i_k,r_k}v$ has weight $\lambda-\alpha_{i_1}-\dots-\alpha_{i_k}$. We therefore have a weight space decomposition

$$V = \bigoplus_{\eta\in Q^+} V_{\lambda-\eta}.$$

Let $\tilde{v} \in V$ be any other highest weight vector. Then $\tilde{v}$ is in particular a simultaneous eigenvector under the action of all the $H_{i,0}$, so it belongs to one of the weight spaces $V_{\lambda-\eta}$. By irreducibility, $\tilde{v}$ generates V, but by the PBW theorem, this is impossible unless $\eta = 0$ because $\tilde{v}$ is highest weight. It follows that $\tilde{v} \in V_\lambda$. On the other hand, V_λ is spanned by v, because $X^-_{i_1,r_1}\dots X^-_{i_k,r_k}v$ has weight $\lambda-\alpha_{i_1}-\dots-\alpha_{i_k}$. ■

Proposition 6.2 *Every finite dimensional irreducible representation of $Y(\mathfrak{g})$ is highest weight.*

Proof. Let V be a finite dimensional irreducible representation of $Y(\mathfrak{g})$, and let

$$V^0 = \{v \in V \mid X^+_{i,r}v = 0 \;\forall\; i \in I,\; r \in \mathbb{Z}_+\}.$$

First, we will show that $V^0 \neq \{0\}$.

Assume that this is not the case, so that $V^0 = \{0\}$. Choose a nonzero simultaneous eigenvector $w \in V$ with respect to the action of all the $H_{i,r}$ (this exists because the $H_{i,r}$ commute with each other). Note that by definition, w is a weight vector with some weight $\Phi = (\Phi_{i,r})_{i\in I, r\in\mathbb{Z}_+}$. Since w is nonzero, it is not in V^0; therefore, there is some $X^+_{i_1,r_1}$ such that $X_{i_1,r_1}w \neq 0$. We can repeat this argument inductively to obtain an infinite sequence $w, X^+_{i_1,r_1}w, X^+_{i_2,r_2}X^+_{i_1,r_1}w, \dots$ of nonzero vectors in V. On the other hand, these vectors are all linearly independent, because they each have different weights when we view V as a representation of $\mathfrak{g}$ via the homomorphism (6.1); indeed, the weight of $X^+_{i_k,r_k}\dots X^+_{i_1,r_1}w$ is $\lambda+\alpha_{i_1}+\alpha_{i_2}+\dots+\alpha_{i_k}$, where $\lambda(h_i) = d_i^{-1}\Phi_{i,0}$.

This contradicts the assumption that V is finite dimensional, proving that $V^0 \neq \{0\}$. Any nonzero element of V^0 which is a simultaneous eigenvector under the action of all the $H_{i,r}$ (if it exists) will be highest weight by definition, and by irreducibility it must generate all of V; it therefore only remains to show that that V^0 is stable with respect to the action of all the $H_{i,r}$.

Suppose $v \in V^0$, and fix any $i \in I$. We will show by induction on r that $X^+_{j,s}H_{i,r}v = 0$ for every $j \in I$ and $r, s \in \mathbb{Z}_+$:

The base case $r = 0$ follows from the second relation in $Y(\mathfrak{g})$, and the fact that $v \in V^0$:

$$X^+_{j,s}H_{i,0}v = (X^+_{j,s}H_{i,0} - H_{i,0}X^+_{j,s})v = -[H_{i,0}, X^+_{j,s}]v = -d_i c_{ij} X^+_{j,s}v = 0.$$

Now suppose for some $r \in \mathbb{Z}_+$ that $X^+_{j,s}H_{i,r}v = 0$ for every $j \in I$ and $s \in \mathbb{Z}_+$. Then since $v \in V^0$, relation (2.2) gives us

$$\begin{aligned} X^+_{j,s}H_{i,r+1}v &= (X^+_{j,s}H_{i,r+1} - H_{i,r+1}X^+_{j,s})v + (H_{i,r}X^+_{j,s+1} - X^+_{j,s+1}H_{i,r})v \\ &= -([H_{i,r+1}, X^+_{j,s}] - [H_{i,r}, X^+_{j,s+1}])v \\ &= -\frac{d_i c_{ij}}{2}(H_{i,r}X^+_{j,s} + X^+_{j,s}H_{i,r})v \\ &= 0. \end{aligned}$$

■

Given any $I \times \mathbb{Z}_+$-tuple $\Phi = (\Phi_{i,r})_{i \in I, r \in \mathbb{Z}}$, let $M(\Phi)$ be the quotient of $Y(\mathfrak{g})$ by the left ideal generated by $\{X^+_{i,r},\ H_{i,r} - \Phi_{i,r} \cdot 1\}$. Then clearly, $M(\Phi)$ is a highest weight representation of $Y(\mathfrak{g})$ with highest weight Φ, and the highest weight vector is the image 1_Φ in the quotient of the element $1 \in Y(\mathfrak{g})$. Note that the weight λ of 1_Φ is given by $\lambda(h_i) = d_i^{-1}\Phi_{i,0}$ when we view $M(\Phi)$ as a representation of $\mathfrak{g}$. The weight space $M(\Phi)_\lambda$ is one dimensional, because any element of $M(\Phi)$ is a sum of elements of the form $X^-_{i_k,r_k} \ldots X^-_{i_1,r_1}1_\Phi$, but the weight of such an element is $\lambda - \alpha_{i_1} - \alpha_{i_2} - \ldots - \alpha_{i_k}$.

Consequently, any proper subrepresentation of $M(\Phi)$ must have trivial intersection with $M(\Phi)_\lambda$, and it follows that the sum of all of the proper subrepresentations is the unique proper maximal subrepresentation of $M(\Phi)$.

Definition 6.1 *The irreducible highest weight representation of $Y(\mathfrak{g})$ with weight Φ is the quotient of $M(\Phi)$ by its unique proper maximal subrepresentation. We denote this representation by $L(\Phi)$.*

Now suppose that V is any finite dimensional irreducible representation of $Y(\mathfrak{g})$. By Proposition 6.2, V is a highest weight representation with some weight Φ. If $v \in V$ is a highest weight vector, then the assignment $1_\Phi \mapsto v$ defines a surjective $Y(\mathfrak{g})$-module homomorphism $M(\Phi) \to V$. It follows that V is isomorphic to the quotient of $M(\Phi)$ by the kernel K of this homomorphism. On the other hand, since V is irreducible, this means that the quotient $M(\Phi)/K$ must also be irreducible; therefore, K must coincide with the unique maximal proper submodule of $M(\Phi)$, hence $V \cong L(\Phi)$.

In light of this fact, if we want to classify all of the finite dimensional irreducible representations of $Y(\mathfrak{g})$, we only need to find some necessary and sufficient condition on the weight Φ which determines whether or not $L(\Phi)$ is finite dimensional. We will prove the following result:

Theorem 6.3 *Let $\Phi = (\Phi_{i,r})_{i\in I, r\in\mathbb{Z}_+}$. Then $L(\Phi)$ is finite dimensional if and only if for each $i \in I$, there exists a polynomial $P_i \in \mathbb{C}[u]$ such that*

$$\frac{P_i(u+d_i)}{P_i(u)} = 1 + \sum_{r=0}^{\infty} \Phi_{i,r} u^{-r-1} \tag{6.2}$$

in the sense that the right hand side is the Laurent expansion of the left hand side about $u = \infty$.

Note that if P_i satisfies condition (6.2), then so does any nonzero scalar multiple of P_i, so we may as well take P_i to be monic. Then all the P_i are uniquely determined by Φ; indeed, if P_i and Q_i are both polynomials satisfying (6.2), then

$$\frac{P_i(u+d_i)}{P_i(u)} = \frac{Q_i(u+d_i)}{Q_i(u)}.$$

Equivalently,

$$\frac{P_i(u+d_i)}{Q_i(u+d_i)} = \frac{P_i(u)}{Q_i(u)}.$$

In particular, the rational function $\frac{P_i(u)}{Q_i(u)}$ is periodic in u, which is impossible unless $P_i(u)$ is some scalar multiple of $Q_i(u)$. If $P_i(u)$ and $Q_i(u)$ are both monic, it follows that $P_i(u) = Q_i(u)$.

Consequently, we may identify Φ with the sequence of monic polynomials $P = (P_i)_{i\in I}$, and use the notation $L(P)$ in place of $L(\Phi)$. Accordingly, we will simply call P the highest weight of this representation.

Definition 6.2 *$L(P)$ is called a fundamental representation if, for some $i \in I$, $P_i(u)$ has degree 1, while $P_j(u) = 1$ for all $j \neq i$.*

It will be useful to consider tensor products of the fundamental representations, but we first remark that this is a well defined notion because $Y(\mathfrak{g})$ is a Hopf algebra (so we can use its comultiplication Δ to define the action of $Y(\mathfrak{g})$ on a tensor product of representations). While an explicit formula for Δ in terms of the presentation for $Y(\mathfrak{g})$ given in Chapter 2 has been found for the case when $\mathfrak{g} = \mathfrak{sl}_n$, there is no known formula that works for arbitrary $\mathfrak{g}$. However, it is at least known (cf. p385, [4]) that if $N^+ = \sum_{i,r} X^+_{i,r} Y^+$, then modulo $Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) N^+$, we have

$$\Delta(X^+_{i,r}) \equiv X^+_{i,r} \otimes 1 + 1 \otimes X^+_{i,r} + \sum_{s=1}^{r} H_{i,s-1} \otimes X^+_{i,r-s} \tag{6.3}$$

$$\Delta(H_{i,r}) \equiv H_{i,r} \otimes 1 + 1 \otimes H_{i,r} + \sum_{s=1}^{r} H_{i,s-1} \otimes H_{i,r-s}. \tag{6.4}$$

Proposition 6.3 (Proposition 12.1.12 of [4]) *Let v and w be highest weight vectors of $L(P)$ and $L(Q)$, respectively. Then the submodule of $L(P) \otimes L(Q)$ generated by $v \otimes w$ is a highest weight representation of $Y(\mathfrak{g})$ with highest weight $P \otimes Q = (P_iQ_i)_{i\in I}$.*

Proof. It is immediate from (6.3) that $v \otimes w$ is annihilated by all the $X_{i,r}^+$. Let $\Phi^P = (\Phi_{i,r}^P)_{i\in I, r\in\mathbb{Z}_+}$ (respectively $\Phi^Q = (\Phi_{i,r}^Q)_{i\in I, r\in\mathbb{Z}_+}$) be related to P (respectively Q) as in equation (6.2). Then according to equation (6.4), we have

$$H_{i,r}(v\otimes w) = \left(\Phi_{i,r}^P + \Phi_{i,r}^Q + \sum_{s=1}^{r} \Phi_{i,s-1}^P \Phi_{i,r-s}^Q\right)(v\otimes w).$$

It follows that $Y(\mathfrak{g})(v\otimes w)$ is a highest weight representation of weight $\Phi = (\Phi_{i,r})_{i\in I, r\in\mathbb{Z}_+}$, where $\Phi_{i,r} = \Phi_{i,r}^P + \Phi_{i,r}^Q + \sum_{s=1}^r \Phi_{i,s-1}^P \Phi_{i,r-s}^Q$. In order to conclude the proof, we only need to show that Φ is related to the polynomials P_iQ_i by equation (6.2). For each $i \in I$, we have

$$\begin{aligned}\frac{P_iQ_i(u+d_i)}{P_iQ_i(u)} &= \left(1 + \sum_{k=0}^{\infty} \Phi_{i,k}^P u^{-k-1}\right)\left(1 + \sum_{l=0}^{\infty} \Phi_{i,l}^Q u^{-l-1}\right) \\ &= 1 + \sum_{k,l=0}^{\infty} \left[(\Phi_{i,k}^P + \Phi_{i,k}^Q)u^{-k-1} + \Phi_{i,k}^P\Phi_{i,l}^Q u^{-(k+l)-2}\right].\end{aligned}$$

The coefficient of u^{-r-1} in this expression is

$$\begin{aligned}\Phi_{i,r}^P + \Phi_{i,r}^Q + \sum_{k+l=r-1} \Phi_{i,k}^P\Phi_{i,l}^Q &= \Phi_{i,r}^P + \Phi_{i,r}^Q + \sum_{k=0}^{r-1} \Phi_{i,k}^P \Phi_{i,r-1-k}^Q \\ &= \Phi_{i,r}^P + \Phi_{i,r}^Q + \sum_{k=1}^{r} \Phi_{i,k-1}^P \Phi_{i,r-k}^Q = \Phi_{i,r}\end{aligned}$$

as desired. ■

Given any sequence $P = (P_i)_{i\in I}$ of polynomials, we can repeatedly apply the above proposition to construct a tensor product of fundamental representations containing a highest weight subrepresentation of weight P. Then, $L(P)$ is isomorphic to the irreducible quotient of this submodule. The next result follows.

Corollary 6.1 *For any $P = (P_i)_{i\in I}$, $L(P)$ is isomorphic to a subquotient of some tensor product of fundamental representations.*

Accordingly, if we are to prove that every $L(P)$ is finite dimensional, it will suffice to only consider the case where $L(P)$ is fundamental. Let us first consider the case when $\mathfrak{g} = \mathfrak{sl}_2$.

6.2 Proof of Theorem 6.3 for $\mathfrak{g} = \mathfrak{sl}_2$

Recall that the finite dimensional irreducible representations of $\mathfrak{sl}_2$ are parametrized by nonnegative integers; namely, for each $r \in \mathbb{Z}_+$, there is a unique representation $V(r)$ of $\mathfrak{sl}_2$

of dimension $r+1$, and there is a basis $\{v_0, \dots, v_r\}$ of $V(r)$ such that the action of $\mathfrak{sl}_2$ is given by

$$ev_s = (r-s+1)v_{s-1} \qquad fv_s = (s+1)v_{s+1} \qquad hv_s = (r-2s)v_s$$

where $v_{-1} = v_{r+1} = 0$.

For each $a \in \mathbb{C}$, there is a surjective *evaluation homomorphism*

$$\mathrm{ev}_a : Y(\mathfrak{sl}_2) \to \mathfrak{U}(\mathfrak{sl}_2)$$

which acts as the identity on $\mathfrak{U}(\mathfrak{sl}_2) \subset Y(\mathfrak{sl}_2)$. We can view $V(r)$ as a representation of $Y(\mathfrak{sl}_2)$ via this homomorphism; we denote the resulting $Y(\mathfrak{sl}_2)$-module by $V(r)_a$ and call it an *evaluation module.* The precise definition of ev_a is given in [4], Proposition 12.1.15. The action of $Y(\mathfrak{sl}_2)$ on $V(r)_a$ is also given in [4] page 389, namely

$$\begin{aligned}
X^+_{1,k}v_s &= \left(a + \frac{1}{2}r - s + \frac{1}{2}\right)^k (r-s+1)v_{s-1}\\
X^-_{1,k}v_s &= \left(a + \frac{1}{2}r - s - \frac{1}{2}\right)^k (s+1)v_{s+1}\\
H_{1,k}v_s &= \left((a + \frac{1}{2}r - s - \frac{1}{2})^k (r-s)(s+1) - (a + \frac{1}{2}r - s + \frac{1}{2})^k (r-s+1)s\right) v_s.
\end{aligned}$$

Note that since $V(r)$ is irreducible and ev_a is surjective, $V(r)_a$ is also irreducible. It follows that the vector v_0 generates $V(r)_a$, and according to the above equations, it is also a highest weight vector with highest weight $\Phi = (\Phi_{1,k})_{k\in\mathbb{Z}_+}$ given by

$$\Phi_{1,k} = \left(a + \frac{1}{2}r - \frac{1}{2}\right)^k r.$$

In particular, $V(r)_a$ is isomorphic to $L(\Phi)$. Consider the case when $r = 1$, so that $\Phi_{1,k} = a^k$. Let $L(P_1)$ be some fundamental representation, so that $P_1(u) = u - a$ for some $a \in \mathbb{C}$. Since $d_1 = 1$ for $\mathfrak{g} = \mathfrak{sl}_2$, the Laurent expansion of $\frac{P_1(u+d_1)}{P_1(u)}$ about $u = \infty$ is

$$\frac{u+1-a}{u-a} = \frac{u-a}{u-a} + \frac{1}{u-a} = 1 + \frac{u^{-1}}{1-au^{-1}} = 1 + \sum_{k=0}^{\infty} a^k u^{-k-1} = 1 + \sum_{k=0}^{\infty} \Phi_{i,k} u^{-k-1}.$$

It follows that $L(P_1) = L(\Phi) \cong V(1)_a$, hence every fundamental representation is finite dimensional (in fact 2 dimensional) because $V(1)_a$ is. This proves the "if" part of Theorem 6.3 for $\mathfrak{g} = \mathfrak{sl}_2$.

For the converse, it will be more convenient to work with a different presentation of $Y(\mathfrak{sl}_2)$. It is given by taking an appropriate quotient of $Y(\mathfrak{gl}_2)$ in its RTT presentation; see [1].

Theorem 6.4 *For $r \in \mathbb{Z}_+$, let $e^{(r)}, f^{(r)}, h^{(r)} \in Y(\mathfrak{gl}_2)$ be given by the equations*

$$e(u) = \sum_{r=0}^{\infty} e^{(r)} u^{-r-1} = t_{22}(u)^{-1} t_{12}(u)$$
$$f(u) = \sum_{r=0}^{\infty} f^{(r)} u^{-r-1} = t_{21}(u) t_{22}(u)^{-1}$$
$$h(u) = 1 + \sum_{r=0}^{\infty} h^{(r)} u^{-r-1} = t_{11}(u) t_{22}(u)^{-1} - t_{21}(u) t_{22}(u)^{-1} t_{12}(u) t_{22}(u)^{-1}.$$

Let $X^{\pm}(u) = \sum_{r=0}^{\infty} X^{\pm}_{1,r} u^{-r-1}$, $H(u) = 1 + \sum_{r=0}^{\infty} H_{1,r} u^{-r-1} \in Y(\mathfrak{sl}_2)[[u^{-1}]]$. Then the assignment

$$X^+(u) \mapsto e(u), \qquad X^-(u) \mapsto f(u), \qquad H(u) \mapsto h(u)$$

extends to an isomorphism $Y(\mathfrak{sl}_2) \xrightarrow{\sim} Y(\mathfrak{gl}_2)/(\partial(u) - 1)$, where

$$\partial(u) = t_{11}(u) t_{22}(u-1) - t_{21}(u) t_{12}(u-1).$$

Remark 6.1 *We can rewrite $h(u)$ as $h(u) = t_{22}(u)^{-1} t_{22}(u-1)^{-1} \partial(u)$, cf. [1].*

Assume that $L(\Phi)$ is finite dimensional. We can use the isomorphism of Theorem 6.4 to lift $L(\Phi)$ to a representation of $Y(\mathfrak{gl}_2)$. Since the composition $Y(\mathfrak{gl}_2) \twoheadrightarrow Y(\mathfrak{gl}_2)/(\partial(u) - 1) \xrightarrow{\sim} Y(\mathfrak{sl}_2)$ is surjective, $L(\Phi)$ remains irreducible as a module over $Y(\mathfrak{gl}_2)$. The classification of finite dimensional irreducible representations of $Y(\mathfrak{gl}_2)$ is well known (see [11], Chapter 3); they are parametrized by their 'highest weights' with respect to the actions of $t_{11}(u)$ and $t_{22}(u)$. We will use this theory to investigate the properties of $L(\Phi)$. Let us begin by defining the notion of a highest weight module over $Y(\mathfrak{gl}_2)$:

Definition 6.3 (Proposition 3.2.2, [11]) *A representation L of $Y(\mathfrak{gl}_2)$ is called highest weight if it is generated by some vector ζ such that*

$$t_{12}(u)\zeta = 0, \text{ and}$$
$$t_{ii}(u)\zeta = \lambda_i(u)\zeta \text{ for } i = 1, 2$$

for some $\lambda_i(u) = \sum_{r=0}^{\infty} \lambda_i^{(r)} u^{-1} \in \mathbb{C}[[u^{-1}]]$. In this case, the pair $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ is called the highest weight, and ζ is the highest weight vector.

It is easy to construct a universal highest weight representation of $Y(\mathfrak{gl}_2)$.

Definition 6.4 (Definition 3.2.3, [11]) *Let $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ be any pair of formal series as above. The Verma module $M(\lambda(u))$ is the quotient of $Y(\mathfrak{gl}_2)$ by the left ideal generated by $t_{12}^{(r)}$ and $t_{ii}^{(r)} - \lambda_i^{(r)}$ with $i = 1, 2$, for all $r \geq 0$.*

The same arguments used for the Yangian $Y(\mathfrak{g})$ can be used to show that $M(\lambda(u))$ has a unique irreducible quotient $L(\lambda(u))$, and additionally that any finite dimensional irreducible representation of $Y(\mathfrak{gl}_2)$ has a unique one dimensional subspace of highest weight vectors, and is isomorphic to some $L(\lambda(u))$; see [11] for complete details. The collection of all finite dimensional irreducible representations of $Y(\mathfrak{gl}_2)$ is therefore made up from precisely those $L(\lambda(u))$ which are finite dimensional. In the next section, we will prove the following result:

Theorem 6.5 (Theorem 3.3.3, [11]) *If $L(\lambda(u))$ is finite dimensional, then there exists some monic polynomial $P(u)$ such that*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

Remark 6.2 *The converse of this theorem is also true, but we only need this weaker result for our purposes.*

We eventually want to apply this theorem to $L(\Phi)$, but this only makes sense once we have proven that $L(\Phi)$ is isomorphic to some $L(\lambda(u))$:

Lemma 6.1 *$L(\Phi)$ is an irreducible highest weight representation of $Y(\mathfrak{gl}_2)$.*

Proof. It is clear that $L(\Phi)$ remains irreducible when viewed as a module over $Y(\mathfrak{gl}_2)$, and that it is generated by some vector v_Φ which is highest weight with respect to the action of $Y(\mathfrak{sl}_2)$. We need to show that v_Φ is still a highest weight vector under the action of $Y(\mathfrak{gl}_2)$. By Theorem 6.2, the subspace of highest weight vectors in $L(\Phi)$ is one dimensional. In order to show that the $t_{ii}(u)$ act by scalars on v_Φ, it therefore suffices to prove that $t_{ii}^{(r)}v_\Phi$ is a highest weight vector for each $r \in \mathbb{Z}_+$.

According to the proof of Theorem 6.2, we have an $\mathfrak{sl}_2$-weight decomposition

$$L(\Phi) = \bigoplus_{k \geq 0} L(\Phi)_{\Phi_{i,0}-2k}$$

and the one dimensional space of highest weight vectors is precisely $L(\Phi)_{\Phi_{i,0}}$. Observe that

$$h_1 t_{ii}^{(r)} v_\Phi = H_{1,0} t_{ii}^{(r)} v_\Phi = h^{(1)} t_{ii}^{(r)} v_\Phi = (t_{11}^{(1)} - t_{22}^{(1)}) t_{ii}^{(r)} v_\Phi.$$

According to relation (1.1), we have $[t_{jj}^{(1)}, t_{ii}^{(r)}] = 0$. It follows that $h_1 t_{ii}^{(r)} v_\Phi = t_{ii}^{(r)} h_1 v_\Phi = \Phi_{i,0} t_{ii}^{(r)} v_\Phi$, hence $t_{ii}^{(r)} v_\Phi \in L(\Phi)_{\Phi_{i,0}}$, as desired. Finally, we see also that $t_{12}(u)v_\Phi = t_{22}(u)e(u)v_\Phi = t_{22}(u)X^+(u)v_\Phi = 0$. ■

According to this lemma, $L(\Phi)$ is isomorphic to $L(\lambda(u))$ for some $\lambda(u) = (\lambda_1(u), \lambda_2(u))$. We have assumed that $L(\Phi)$ is finite dimensional, so by Theorem 6.5, there exists some monic polynomial $P(u)$ such that

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

We also know that if v_Φ is the highest weight vector, then since $\partial(u) = 1$ in $Y(\mathfrak{sl}_2)$, we have $v_\Phi = \partial(u)v_\Phi = (t_{11}(u)t_{22}(u-1) - t_{21}(u)t_{12}(u-1))v_\Phi = \lambda_1(u)\lambda_2(u-1)v_\Phi$, hence

$$\lambda_1(u)\lambda_2(u-1) = 1.$$

On the other hand, we have

$$\begin{aligned}\left(1 + \sum_{r=0}^{\infty} \Phi_{1,r} u^{-r-1}\right) v_\Phi &= H(u)v_\Phi \\ &= h(u)v_\Phi = t_{22}(u)^{-1}t_{22}(u-1)^{-1}\partial(u)v_\Phi = \lambda_2(u)^{-1}\lambda_2(u-1)^{-1}v_\Phi.\end{aligned}$$

It follows that

$$1 + \sum_{r=0}^{\infty} \Phi_{1,r} u^{-r-1} = \lambda_2(u)^{-1}\lambda_2(u-1)^{-1} = \frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

This completes the proof of Theorem 6.3 for $\mathfrak{g} = \mathfrak{sl}_2$. Before moving on to prove the general case, we will first turn to the proof of Theorem 6.5.

6.3 Proof of Theorem 6.5

A full proof of this theorem is found in [11], Chapter 3. The details are provided below, for the convenience of the reader.

We begin by explaining how a highest weight representation L of $Y(\mathfrak{gl}_2)$ with highest weight $\lambda(u) = (\lambda_1(u), \lambda_2(u))$ decomposes into weight spaces with respect to the action of the diagonal Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}_2$:

Let ζ be a highest weight vector in L. If we choose an ordering on the set of generators of $Y(\mathfrak{gl}_2)$ in a way that every $t_{21}^{(r)}$ precedes every $t_{11}^{(r)}$ and $t_{22}^{(r)}$, which in turn precede every $t_{12}^{(r)}$, then we see from the PBW theorem for $Y(\mathfrak{gl}_2)$ that L is spanned by all the vectors of the form $t_{21}^{(r_1)} \dots t_{21}^{(r_k)}\zeta$.

Let $\alpha = \epsilon_1 - \epsilon_2 \in \mathfrak{h}^*$, where the ϵ_i are the basis elements of $\mathfrak{h}^*$ which are dual to the E_{jj}; that is, $\epsilon_i(E_{jj}) = \delta_{ij}$. Observe that if $X \in L$ is any vector with $\mathfrak{gl}_2$-weight $\mu \in \mathfrak{h}^*$ (so that $E_{ii}X = \mu(E_{ii})X$ for $i = 1, 2$), then for any r, $t_{21}^{(r)}X$ has weight $\mu - \alpha$. Indeed, if we identify E_{ii} with $t_{ii}^{(1)} \in Y(\mathfrak{gl}_2)$, then relation (1.1) gives us the equation

$$[E_{ii}, t_{21}^{(r)}] = (\delta_{i2} - \delta_{i1})t_{21}^{(r)},$$

hence $E_{ii}t_{21}^{(r)}X = t_{21}^{(r)}E_{ii}X + (\delta_{i2} - \delta_{i1})t_{21}^{(r)}X = (\mu - \alpha)(E_{ii})t_{21}^{(r)}X$. Note that a similar argument shows that $t_{12}^{(r)}X$ has weight $\mu + \alpha$.

By definition, ζ has weight $\lambda \in \mathfrak{h}^*$ given by $\lambda(E_{ii}) = \lambda_i^{(1)}$ where $\lambda_i(u) = \sum_{r=0}^{\infty} \lambda_i^{(r)} u^{-r}$. It follows by induction that an element in L of the form $t_{21}^{(r_1)} \dots t_{21}^{(r_k)}\zeta$ has weight $\lambda - k\alpha$, hence we get a decomposition into weight spaces:

$$L = \bigoplus_{k \geq 0} L_{\lambda - k\alpha} \tag{6.5}$$

where $L_{\lambda-k\alpha} = \{X \in L \mid E_{ii}X = (\lambda - k\alpha)(E_{ii})X,\ i = 1,2\}$.

We will also need to define universal highest weight modules over $\mathfrak{gl}_2$.

Definition 6.5 *Let $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the Verma module $M(\lambda_1, \lambda_2)$ is the quotient of $\mathfrak{U}(\mathfrak{gl}_2)$ by the left ideal generated by E_{12} and $E_{ii} - \lambda_i,\ i = 1,2$.*

The standard arguments again show that $M(\lambda_1, \lambda_2)$ has a unique irreducible quotient $L(\lambda_1, \lambda_2)$.

Observe that $L(\lambda_1, \lambda_2)$ remains irreducible as a module over $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$, because $\mathfrak{gl}_2 = \mathfrak{sl}_2 \oplus Z(\mathfrak{gl}_2)$, and $Z(\mathfrak{gl}_2)$ only consists of scalar multiples of the identity matrix. Moreover, if ζ is the image of $1 \in \mathfrak{U}(\mathfrak{gl}_2)$ in the irreducible quotient $L(\lambda_1, \lambda_2)$, then we have by definition $E_{12}\zeta = 0$ and $(E_{11} - E_{22})\zeta = (\lambda_1 - \lambda_2)\zeta$. It follows that $L(\lambda_1, \lambda_2)$ is isomorphic to the irreducible highest weight module over $\mathfrak{sl}_2$ with highest weight $\lambda_1 - \lambda_2$ and ζ is its highest weight vector, hence a basis of $L(\lambda_1, \lambda_2)$ is given by $E_{21}^r\zeta$ with $r = 0, 1, \ldots, \lambda_1 - \lambda_2$ if $\lambda_1 - \lambda_2 \in \mathbb{Z}_+$, and with r running over all nonnegative integers if $\lambda_1 - \lambda_2 \notin \mathbb{Z}_+$.

We can also view $L(\lambda_1, \lambda_2)$ as a module over $Y(\mathfrak{gl}_2)$ via the evaluation homomorphism (1.10). We call the resulting $Y(\mathfrak{gl}_2)$ module an *evaluation module* (similar to the evaluation modules over $Y(\mathfrak{sl}_2)$). We shall abuse notation by denoting this evaluation module by $L(\lambda_1, \lambda_2)$, but it will always be clear from context whether we are viewing $L(\lambda_1, \lambda_2)$ as a module over $\mathfrak{gl}_2$ or over $Y(\mathfrak{gl}_2)$. Note that the evaluation module $L(\lambda_1, \lambda_2)$ is an irreducible highest weight module over $Y(\mathfrak{gl}_2)$:

It is irreducible because the evaluation homomorphism is surjective, hence it is generated by the vector ζ. We have $t_{12}(u)\zeta = (E_{12}u^{-1})\zeta = 0$, and $t_{ii}(u)\zeta = (1 + E_{ii}u^{-1})\zeta = (1 + \lambda_i u^{-1})\zeta$. It follows that $L(\lambda_1, \lambda_2)$ is isomorphic as $Y(\mathfrak{gl}_2)$-modules to $L(\lambda_1(u), \lambda_2(u))$ where

$$\lambda_i(u) = (1 + \lambda_i u^{-1}). \tag{6.6}$$

Recall that we can take tensor products of evaluation modules and equip them with a well defined $Y(\mathfrak{gl}_2)$-module structure via the comultiplication Δ (see (1.11)).

Proposition 6.4 (Proposition 3.2.9, [11]) *Let $\lambda_i^{(r)} \in \mathbb{C}$ with $i = 1,2$ and $r = 1, \ldots, k$. Let L be the tensor product of evaluation modules*

$$L = L(\lambda_1^{(1)}, \lambda_2^{(1)}) \otimes \ldots \otimes L(\lambda_1^{(k)}, \lambda_2^{(k)}). \tag{6.7}$$

Let ζ_i be the highest weight vector of $L(\lambda_1^{(i)}, \lambda_2^{(i)})$, and let $\zeta = \zeta_1 \otimes \ldots \otimes \zeta_k$. Then the submodule $Y(\mathfrak{gl}_2)\zeta$ is a highest weight representation with highest weight $\lambda(u) = (\lambda_1(u), \lambda_2(u))$, where

$$\lambda_i(u) = (1 + \lambda_i^{(1)} u^{-1}) \ldots (1 + \lambda_i^{(k)} u^{-1})$$

and ζ is its highest weight vector.

Proof. The submodule $Y(\mathfrak{gl}_2)\zeta$ is generated by ζ by definition, so we only need to check that $t_{12}(u)\zeta = 0$ and $t_{ii}(u)\zeta = \lambda_i(u)\zeta$. We proceed by induction on k. If $k = 1$ this is trivial, so assume $k > 1$. By definition of Δ, we have

$$t_{12}(u)\zeta = t_{11}(u)(\zeta_1 \otimes \ldots \otimes \zeta_{k-1}) \otimes t_{12}(u)\zeta_k + t_{12}(u)(\zeta_1 \otimes \ldots \otimes \zeta_{k-1}) \otimes t_{22}(u)\zeta_k.$$

The first term is zero because ζ_k is a highest weight vector, and the second term is zero by induction. Finally, note that

$$t_{ii}(u)\zeta = t_{i1}(u)(\zeta_1 \otimes \ldots \otimes \zeta_{k-1}) \otimes t_{1i}(u)\zeta_k + t_{i2}(u)(\zeta_1 \otimes \ldots \otimes \zeta_{k-1}) \otimes t_{2i}(u)\zeta_k.$$

If $i = 1$, then by induction the second term is zero and (in view of (6.6)) we find that

$$t_{11}(u)\zeta = (1 + \lambda_1^{(1)}) \ldots (1 + \lambda_1^{(k)}).$$

The same argument works for $i = 2$. ■

Proposition 6.5 (Proposition 3.2.11, [11]) *For each $i, j = 1, 2$, the action of $t_{ij}(u)$ on any element of the $Y(\mathfrak{gl}_2)$ module* (6.7) *is a polynomial in u^{-1} with degree no more than k.*

Proof. It suffices to consider the action of $t_{ij}(u)$ on a simple tensor $\eta = \eta_1 \otimes \ldots \otimes \eta_k \in L$. We proceed by induction on k. If $k = 1$, the result follows because $t_{ij}(u)$ acts as $1 + E_{ij}u^{-1}$ by definition of the evaluation homomorphism.

Suppose $k > 1$. Then by induction, $t_{ab}(u)(\eta_1 \otimes \ldots \otimes \eta_{k-1})$ is a polynomial in u^{-1} with degree no more than $k - 1$, for any a, b. It follows that

$$t_{ij}(u)\eta = t_{i1}(u)(\eta_1 \otimes \ldots \otimes \eta_{k-1}) \otimes t_{1j}(u)\zeta_k + t_{i2}(u)(\eta_1 \otimes \ldots \otimes \eta_{k-1}) \otimes t_{2j}(u)\zeta_k.$$

The last factor is a polynomial of degree at most 1 by definition of the evaluation homomorphism, so by induction $t_{ij}(u)\eta$ is a polynomial of degree at most k. ■

In view of (6.5), every evaluation module has a $\mathfrak{gl}_2$-weight space decomposition, and each of these weight spaces is finite dimensional (in fact one dimensional, because each one is spanned by one of the basis vectors $E_{21}^r\zeta$). To such a representation, we can define a $Y(\mathfrak{gl}_2)$ module structure on the restricted dual space. More precisely, if L is any representation of $Y(\mathfrak{gl}_2)$ with a decomposition into finite dimensional $\mathfrak{gl}_2$-weight spaces $L = \bigoplus_{\mu \in \mathfrak{h}^*} L_\mu$ then the restricted dual space is

$$L^* = \bigoplus_{\mu \in \mathfrak{h}^*} L_\mu^*.$$

Then L^* becomes a $Y(\mathfrak{gl}_2)$ module under the action

$$(y\omega)(\eta) = \omega(\rho(y)\eta), \; y \in Y(\mathfrak{gl}_2), \; \omega \in L^*, \; \eta \in L$$

where ρ is the antiautomorphism of $Y(\mathfrak{gl}_2)$ given by $\rho(t_{ij}(u)) = t_{2-i+1,2-j+1}(-u)$.

For any $Y(\mathfrak{gl}_2)$-submodule $K \subset L$, we can form the subspace

$$\mathrm{Ann}(K) = \{\omega \in L^* \mid \omega(\eta) = 0 \; \forall \; \eta \in K\}.$$

In fact $\mathrm{Ann}(K)$ is actually a submodule of L^*: if $\omega \in \mathrm{Ann}(K)$ and $y \in Y(\mathfrak{gl}_2)$, then for any $\eta \in K$ we have

$$(y\omega)(\eta) = \omega(\rho(y)\eta) = 0$$

because $\rho(y)\eta \in K$, hence $y\omega \in \mathrm{Ann}(K)$. Similarly, if $M \subset L^*$ is any $Y(\mathfrak{gl}_2)$-submodule, then the subspace

$$\mathrm{Ker}(M) = \{\eta \in L \mid \omega(\eta) = 0 \ \forall \ \omega \in M\}$$

is a submodule of L: if $y \in Y(\mathfrak{gl}_2)$ and $\eta \in \mathrm{Ker}(M)$, then for any $\omega \in M^*$, since ρ is involutive we have

$$\omega(y\eta) = \omega(\rho(\rho(y))\eta) = (\rho(y)\omega)(\eta) = 0$$

because $\rho(y)\omega \in M$.

Proposition 6.6 (Proposition 3.2.12, [11]) *Let L be the tensor product of evaluation modules* (6.7)*. Then*

$$L^* \cong L(-\lambda_2^{(1)}, -\lambda_1^{(1)}) \otimes \ldots \otimes L(-\lambda_2^{(k)}, -\lambda_1^{(k)}).$$

Proof. The proof is by induction on k.

Suppose $k = 1$. Then $L = L(\lambda_1^{(1)}, \lambda_2^{(2)})$. We will first show that L^* is irreducible. Towards this end, suppose there exists a nonzero proper submodule $M \subset L^*$. Then we know that $\mathrm{Ker}(M)$ is a submodule in L. Since L is an evaluation module, it is irreducible, so $\mathrm{Ker}(M) = L$ or $\mathrm{Ker}(M) = \{0\}$. Clearly if $\mathrm{Ker}(M) = L$, then $M = \{0\}$. But M was assumed nonzero, so let us assume instead that $\mathrm{Ker}(M) = 0$. Since M is a nonzero proper submodule, we can choose any basis of M and extend it to a basis B of L^*, and the basis of L which is dual to B by definition contains a (nonzero) vector v which is annihilated by all the elements in $B \cap M$. Since these elements span M, it follows that $v \in \mathrm{Ker}(M)$, hence $\mathrm{Ker}(M) \neq 0$. This is a contradiction, so L^* must be irreducible.

Next we show that L^* is a highest weight module whose highest weight vector ζ^* is dual to the highest weight vector $\zeta \in L$; that is, $\zeta^*(\zeta) = 1$ and $\zeta^*(E_{21}^r\zeta) = 0$ for every $r > 0$. Indeed, we have for all $\eta \in L$,

$$\begin{aligned}(t_{12}(u)\zeta^*)(\eta) &= \zeta^*(t_{21}(-u)\eta) \\ &= \zeta^*((-E_{21}u^{-1})\eta) = 0\end{aligned}$$

because $E_{21}\eta$ has no weight component proportional to ζ. Additionally, for $i = 1, 2$ we have

$$\begin{aligned}(t_{ii}(u)\zeta^*)(\eta) &= \zeta^*(t_{2-i+1,2-i+1}(-u)\eta) \\ &= \zeta^*((1 - E_{2-i+1,2-i+1}u^{-1})\eta) = (1 - \lambda_{2-i+1}^{(1)}u^{-1})\zeta^*(\eta),\end{aligned}$$

hence ζ^* is a highest weight vector with highest weight $\lambda(u) = (1 - \lambda_2^{(1)}u^{-1}, 1 - \lambda_1^{(1)}u^{-1})$. In summary, L^* is an irreducible highest weight module over $Y(\mathfrak{gl}_2)$ with the same weight as the evaluation module $L(-\lambda_2^{(1)}, -\lambda_1^{(1)})$, hence these modules are isomorphic. This proves the base case $k = 1$.

Suppose $k > 1$. Let $L_1 = L(\lambda_1^{(1)}, \lambda_2^{(1)}) \otimes \ldots \otimes L(\lambda_1^{(k-1)}, \lambda_2^{(k-1)})$ and $L_2 = L(\lambda_1^{(k)}, \lambda_2^{(k)})$, so that $L = L_1 \otimes L_2$. By induction, $L_1^* \cong L(-\lambda_2^{(1)}, -\lambda_1^{(1)}) \otimes \ldots L(-\lambda_2^{(k-1)}, -\lambda_1^{(k-1)})$, and $L_2^* \cong L(-\lambda_2^{(k)}, -\lambda_1^{(k)})$. We know that $L^* \cong L_1^* \otimes L_2^*$ as vector spaces, so we only need to check that the action of $Y(\mathfrak{gl}_2)$ on L^* agrees with the action on $L_1^* \otimes L_2^*$.

If $y \in Y(\mathfrak{gl}_2)$, $\omega \in L^*$ and $\eta = \eta_1 \otimes \eta_2 \in L = L_1 \otimes L_2$, then

$$y\omega(\eta) = \omega(\rho(y)\eta) = \omega[(\Delta \circ \rho)(y)(\eta_1 \otimes \eta_2)] = \omega[(\Delta \circ \rho)(y)\eta].$$

On the other hand, if $\omega = \omega_1 \otimes \omega_2 \in L_1^* \otimes L_2^*$, we have

$$y\omega(\eta) = \Delta(y)(\omega_1 \otimes \omega_2)(\eta) = (\omega_1 \otimes \omega_2)[((\rho \otimes \rho) \circ \Delta)(y)\eta] = \omega[((\rho \otimes \rho) \circ \Delta)(y)\eta].$$

It is easy to check directly that $\Delta \circ \rho = (\rho \otimes \rho) \circ \Delta$, and this completes the proof. ■

Proposition 6.7 (Proposition 3.3.1, [11]) *Let $L(\lambda(u))$ be any irreducible highest weight module over $Y(\mathfrak{gl}_2)$ with highest weight $\lambda(u) = (\lambda_1(u), \lambda_2(u))$. Suppose $L(\lambda(u))$ is finite dimensional. Then there exists a formal power series $f(u) \in \mathbb{C}[[u^{-1}]]$ such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in u^{-1}.*

Proof. First we note that the map map $T(u) \mapsto \lambda_2(u)^{-1}T(u)$ is an automorphism of $Y(\mathfrak{gl}_2)$ (it is bijective because $\lambda_2(u)$ is invertible, and one can easily check that it preserves the RTT relation (1.9)). It is immediate that if we pull back the action of $Y(\mathfrak{gl}_2)$ on $L(\lambda(u))$ via this automorphism, we then the resulting module is isomorphic to $L(\nu(u))$ where $\nu(u) = (\lambda_1(u)/\lambda_2(u), 1)$. We will use the module $L(\nu(u))$ to show that there is some $g(u) \in \mathbb{C}[u^{-1}]$ such that $g(u)(\lambda_1(u)/\lambda_2(u))$ is a polynomial in u^{-1}; then, by setting $f(u) = g(u)\lambda_2^{-1}(u)$ obtain the desired result.

Since $L(\lambda_1(u), \lambda_2(u))$ is finite dimensional by assumption, so is $L(\nu(u))$. It follows that if ζ is the highest weight vector of $L(\nu(u))$, then the vectors $t_{21}^{(r)}\zeta$ with $r > 0$ are linearly dependent. We therefore have some nontrivial linear combination

$$\sum_{i=1}^{m} c_i t_{21}^{(i)} \zeta = 0$$

where $c_m \neq 0$. By definition of $L(\nu(u))$, this means precisely that if $1_{\nu(u)}$ is the image of $1 \in Y(\mathfrak{gl}_2)$ after passing to the quotient $M(\nu(u))$, then the vector

$$\xi = \sum_{i=1}^{m} c_i t_{21}^{(i)} 1_{\nu(u)}$$

is in the kernel of the natural projection map $M(\nu(u)) \to L(\nu(u))$ (i.e. $1_{\nu(u)} \mapsto \zeta$). This means that ξ belongs to the unique maximal proper submodule K in $M(\nu(u))$.

Note that by definition of $M(\nu(u))$, $t_{12}(u)1_{\nu(u)} = 0$, and also $t_{22}^{(r)}1_{\nu(u)} = 0$ for any $r > 0$, because $\nu(u) = (\lambda_1(u)/\lambda_2(u), 1)$. It therefore follows from relation (1.3) that for each $r \geq 1$ and $i = 1, \dots, m$ we have

$$t_{12}^{(r)}t_{21}^{(i)}1_{\nu(u)} = \left[t_{21}^{(i)}t_{12}^{(r)} + \sum_{a=1}^{\min\{r,i\}} \left(t_{22}^{(a-1)}t_{11}^{(r+i-a)} - t_{22}^{(r+i-a)}t_{11}^{(a-1)}\right)\right]1_{\nu(u)} = \nu^{(r+i-1)}1_{\nu(u)} \tag{6.8}$$

where $\lambda_1(u)/\lambda_2(u) = \sum_{j=0}^{\infty} \nu^{(j)}u^{-j}$, $\nu^{(j)} \in \mathbb{C}$.

In particular, this shows that $t_{12}^{(r)}\xi$ is a scalar multiple of $1_{\nu(u)}$ for every $r \geq 1$. On the other hand these elements are all in the submodule K which has trivial intersection with the one dimensional highest weight space spanned by $1_{\nu(u)}$. This means that $t_{12}^{(r)}\xi = 0$ for every $r \geq 1$, hence by equation (6.8) we have

$$\sum_{i=1}^{m} c_i\nu^{(r+i-1)} = 0. \tag{6.9}$$

Let $c(u) = \sum_{i=1}^{m} c_iu^{i-1}$. Then

$$\frac{\lambda_1(u)}{\lambda_2(u)}c(u) = \sum_{i=1}^{m}\sum_{j=0}^{\infty} c_i\nu^{(j)}u^{i-j-1}.$$

The coefficient of u^{-r} for each $r \geq 1$ is $\sum_{i=1}^{m} c_i\nu^{(r+i-1)} = 0$ in light of (6.9). It follows that $c(u)\frac{\lambda_1(u)}{\lambda_2(u)}$ is a polynomial in u of degree $m-1$. Then setting $g(u) = u^{-(m-1)}c(u)$, we see that $g(u)$ and $g(u)\frac{\lambda_1(u)}{\lambda_2(u)}$ are polynomials in u^{-1}, as desired. ∎

Remark 6.3 *In the above proof, we actually showed that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ have the same degree $m-1$. Additionally, we can ensure both these polynomials have a constant term of 1 if we scale $g(u)$ by a factor of c_m^{-1}.*

According to this proposition, we can investigate some properties of $L(\lambda_1(u), \lambda_2(u))$ in the case that $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials with the same degree, and then go back to the general case by twisting the action of $Y(\mathfrak{gl}_2)$ by an appropriate automorphism.

Proposition 6.8 (Proposition 3.3.2, [11]) *Suppose $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1}, and their factorization over $\mathbb{C}$ is given by*

$$\lambda_1(u) = (1 + \alpha_1u^{-1})\dots(1 + \alpha_ku^{-1}),$$
$$\lambda_2(u) = (1 + \beta_1u^{-1})\dots(1 + \beta_ku^{-1}).$$

Assume in addition for every $i = 1, \dots, k$ that the following condition holds: if the multiset

$$\{\alpha_p - \beta_q \mid i \leq p, q \leq k\}$$

contains any nonnegative integers, then $\alpha_i - \beta_i$ is the smallest one among them. Then $L(\lambda_1(u), \lambda_2(u))$ is isomorphic to a tensor product L of evaluation modules given by

$$L = L(\alpha_1, \beta_1) \otimes \dots \otimes L(\alpha_k, \beta_k).$$

Proof. According to proposition 6.4, $L(\lambda_1(u), \lambda_2(u))$ is isomorphic to the irreducible quotient of the submodule $Y(\mathfrak{gl}_2)\zeta$ of L, where $\zeta = \zeta_1 \otimes \ldots \otimes \zeta_k$ is the tensor product of the highest weight vector in each factor. It is enough to show that L is irreducible; indeed, in this case $Y(\mathfrak{gl}_2)\zeta$ is also irreducible and we have $L = Y(\mathfrak{gl}_2)\zeta \cong L(\lambda_1(u), \lambda_2(u))$.

We begin by proving that any nonzero vector $\xi \in L$ with the property that $t_{12}(u)\xi = 0$ must be proportional to ζ, and we proceed by induction on k.

If $k = 1$ this is immediate, because if ξ has any nonzero weight component ω which is not proportional to ζ, then $t_{12}(u)\omega$ is nonzero (we know at least that $t_{12}^{(1)}\omega = E_{12}\omega \neq 0$ because $L = L(\alpha_1, \beta_1)$ is irreducible over $\mathfrak{gl}_2$, hence ω generates all of L).

Suppose $k > 1$. In this case, we may decompose ξ as a sum

$$\xi = \sum_{r=0}^{p} E_{21}^r \zeta_1 \otimes \xi_r$$

where $\xi_r \in L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$ and $E_{21}^p\zeta_1, \xi_p \neq 0$. Then by assumption,

$$\begin{aligned} t_{12}(u)\xi &= \sum_{r=0}^{p} t_{11}(u)E_{21}^r\zeta_1 \otimes t_{12}(u)\xi_r + t_{12}(u)E_{21}^r\zeta_1 \otimes t_{22}(u)\xi_r \\ &= \sum_{r=0}^{p}(1 + E_{11}u^{-1})E_{21}^r\zeta_1 \otimes t_{12}(u)\xi_r + (E_{12}u^{-1})E_{21}^r\zeta_1 \otimes t_{22}(u)\zeta_r \\ &= \sum_{r=0}^{p}(1 + (\alpha_1 - r)u^{-1})E_{21}^r\zeta_1 \otimes t_{12}(u)\xi_r + u^{-1}r(\alpha_1 - \beta_1 - r + 1)E_{21}^{r-1}\zeta_1 \otimes t_{22}(u)\xi_r \qquad (6.10) \\ &= 0. \end{aligned}$$

By considering the coefficient of E_{21}^p in (6.10), we see that

$$(1 + (\alpha_1 - p)u^{-1})t_{12}(u)\xi_p = 0,$$

hence $t_{12}(u)\xi_p = 0$. By induction, it follows that ξ_p is proportional to $\zeta_2 \otimes \ldots \otimes \zeta_k$. With this in mind, if we want to show that ξ is proportional to ζ, we only need to prove that $p = 0$. So suppose on the contrary that $p \geq 1$.

Considering the coefficient of E_{21}^{p-1} in (6.10), we have

$$(1 + (\alpha_1 - p + 1)u^{-1})t_{12}(u)\xi_{p-1} + u^{-1}p(\alpha_1 - \beta_1 - p + 1)t_{22}(u)\xi_p = 0. \qquad (6.11)$$

Note that since ξ_p is proportional to $\zeta_2 \otimes \ldots \otimes \zeta_k$, we have

$$t_{22}(u)\xi_p = (1 + \beta_2 u^{-1}) \ldots (1 + \beta_k u^{-1})\xi_p,$$

hence multiplying (6.11) by u^k, we get

$$(u + \alpha_1 - p + 1)u^{k-1}t_{12}(u)\xi_{p-1} + p(\alpha_1 - \beta_1 - p + 1)(u + \beta_2) \ldots (u + \beta_k)\xi_p = 0.$$

Now recall that by Proposition 6.5, $t_{12}(u)\xi_{p-1}$ is a polynomial in u^{-1} of degree no more than $k-1$, hence $u^{k-1}t_{12}(u)\xi_{p-1}$ is a polynomial in u. We may therefore evaluate the above equation at $u = -\alpha_1 + p - 1$ and find that

$$p(\alpha_1 - \beta_1 - p + 1)(\alpha_1 - \beta_2 - p + 1)\dots(\alpha_1 - \beta_k - p + 1) = 0.$$

Since $p \neq 0$, it follows that $\alpha_1 - \beta_j = p - 1 \in \mathbb{Z}_+$ for some j. There are two possible cases. If $\alpha_1 - \beta_1 \notin \mathbb{Z}_+$, then by assumption neither is $\alpha_1 - \beta_j$ for any j. If $\alpha_1 - \beta_1 \in \mathbb{Z}_+$ then we recall that a basis of $L(\alpha_1, \beta_1)$ is given by $E_{21}^r\zeta_1$ with $r = 0, \dots, \alpha_1 - \beta_1$. From $\mathfrak{sl}_2$ theory, we know that $E_{21}^r = 0$ for $r > \alpha_1 - \beta_1$. Since $E_{21}^p\zeta_1$ was assumed nonzero, this means that $\alpha_1 - \beta_1 \geq p$. But then by assumption, any $\alpha_1 - \beta_j$ which is a nonnegative integer is also at least p, so $\alpha_1 - \beta_j \neq p - 1$ for any j. In either case this is a contradiction, so p must be zero, completing the proof that ξ is proportional to ζ.

We are now prepared to show that L is irreducible. Let $M \subset L$ be any nonzero $Y(\mathfrak{gl}_2)$-submodule. We will show that $M = L$. Note that repeated application of $t_{12}(u)$ to any nonzero vector in M eventually produces a nonzero vector $\xi \in M$ such that $t_{12}(u)\xi = 0$ (because the weights of each factor of L are bounded above). This proves that M contains ζ, hence $Y(\mathfrak{gl}_2)\zeta \subset M$. It remains to show that $Y(\mathfrak{gl}_2)\zeta = L$.

Suppose on the contrary that $Y(\mathfrak{gl}_2)\zeta$ is a proper submodule of L. Then its annihilator $\mathrm{Ann}(Y(\mathfrak{gl}_2)\zeta)$ is a nonzero submodule in L^*. On the other hand, by Proposition 6.6, L^* is isomorphic to

$$L(-\beta_1, -\alpha_1) \otimes \dots \otimes L(-\beta_k, -\alpha_k), \tag{6.12}$$

and the proof of that proposition indicates that the highest weight vector in each factor $L(-\beta_j, -\alpha_j) \cong L(\alpha_j, \beta_j)^*$ can be identified with the linear functional ζ_j^* which sends the highest weight vector ζ_j to 1, and all other weight vectors in $L(\alpha_j, \beta_j)$ to zero.

The tensor product of highest weight vectors in the module (6.12) can therefore be identified with $\zeta^* = \zeta_1^* \otimes \dots \otimes \zeta_k^*$. Note that $\zeta^* \notin \mathrm{Ann}(Y(\mathfrak{gl}_2)\zeta)$, because $\zeta^*(\zeta) = 1$.

In summary, the module (6.12) contains a nonzero submodule which does not contain the tensor product of highest weight vectors. On the other hand, the conditions on α_i and β_j in this proposition are still met if we replace each α_i by $-\beta_i$ and each β_i by $-\alpha_i$, so this contradicts the beginning of the proof. ■

We finally have all the ingredients we need to complete the proof of Theorem 6.5:

Proof of Theorem 6.5. Suppose that $L(\lambda_1(u), \lambda_2(u))$ is finite dimensional. Then by Proposition 6.7, we can find some $f(u) \in \mathbb{C}[[u^{-1}]]$ such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in u^{-1} with the same degree k and with constant term 1. Factorize these polynomials over $\mathbb{C}$ as

$$f(u)\lambda_1(u) = (1 + \alpha_1 u^{-1})\dots(1 + \alpha_k u^{-1}),$$
$$f(u)\lambda_2(u) = (1 + \beta_1 u^{-1})\dots(1 + \beta_k u^{-1}).$$

After appropriate renumeration of the α_i and β_j, we may assume that they meet the conditions of Proposition 6.8, so that $L(f(u)\lambda_1(u), f(u)\lambda_2(u))$ is isomorphic to

$$L = L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_k, \beta_k).$$

On the other hand, $L(f(u)\lambda_1(u), f(u)\lambda_2(u))$ is isomorphic to the module obtained by pulling back the action of $Y(\mathfrak{gl}_2)$ on $L(\lambda_1(u), \lambda_2(u))$ via the automorphism $T(u) \mapsto f(u)T(u)$. In particular, this means that $L(f(u)\lambda_1(u), f(u)\lambda_2(u))$ is finite dimensional, hence all of the tensor factors of L must be finite dimensional. It follows that $\alpha_i - \beta_i \in \mathbb{Z}_+$ for each $i = 1, \ldots, k$.

It therefore makes sense to define the polynomial

$$P(u) = \prod_{i=1}^{k} (u + \beta_i)(u + \beta_i + 1) \ldots (u + \alpha_i - 1).$$

Then

$$\frac{P(u+1)}{P(u)} = \prod_{i=1}^{k} \frac{u + \alpha_i}{u + \beta_i} = \frac{f(u)\lambda_1(u)}{f(u)\lambda_2(u)} = \frac{\lambda_1(u)}{\lambda_2(u)}$$

and this completes the proof. ■

6.4 Proof of Theorem 6.3: General Case

Let $P = (P_i)_{i \in I}$; we will show that $L(P)$ is finite dimensional. Let $\Phi = (\Phi_{i,r})_{i \in I, r \in \mathbb{Z}_+}$ be related to P as in equation (6.2), and let v_P be a nonzero vector in $L(P)$ of weight Φ. If we view $L(P)$ as a representation of $\mathfrak{g}$ via (6.1), then v_P has weight λ, where $\lambda(h_i) = d_i^{-1}\Phi_{i,0}$. On the other hand, we can check directly that $\Phi_{i,0} = d_i \deg(P_i)$:

Let $P_i(u) = (u - a_1) \cdots (u - a_d)$ be the factorization of $P_i(u)$ over $\mathbb{C}$, so that $d = \deg(P_i)$. Then

$$\begin{aligned}
\frac{P_i(u + d_i)}{P_i(u)} &= \frac{u + d_i - a_1}{u - a_1} \cdots \frac{u + d_i - a_d}{u - a_d} = \left(1 + \frac{d_i}{u - a_1}\right) \cdots \left(1 + \frac{d_i}{u - a_d}\right) \\
&= \left(1 + d_i u^{-1} \frac{1}{1 - a_1 u^{-1}}\right) \cdots \left(1 + d_i u^{-1} \frac{1}{1 - a_d u^{-1}}\right) \\
&= \left(1 + d_i \sum_{k_1=0}^{\infty} a_1^{k_1} u^{-k_1 - 1}\right) \cdots \left(1 + d_i \sum_{k_d=0}^{\infty} a_d^{k_d} u^{-k_d - 1}\right) \\
&= \left(\sum_{k_1=0}^{\infty} b_{1,k_1} u^{-k_1}\right) \cdots \left(\sum_{k_d=0}^{\infty} b_{d,k_d} u^{-k_d}\right) \\
&= \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_d = k} b_{1,k_1} \cdots b_{d,k_d} u^{-k}.
\end{aligned}$$

where $b_{j,k_j} = 1$ if $k_j = 0$ and $b_{j,k_j} = d_i a_j^{k_j-1}$ for $k_j > 0$. The coefficient of u^{-1} in this expression is by definition $\Phi_{i,0}$, hence

$$\begin{aligned}\Phi_{i,0} = \sum_{k_1+\cdots+k_d=1} b_{1,k_1}\cdots b_{d,k_d} &= \sum_{j=1}^{d} b_{1,0}\cdots b_{j,1}\cdots b_{d,0}\\ &= \sum_{j=1}^{d} b_{j,1} = \sum_{j=1}^{d} d_i = dd_i = d_i \deg(P_i).\end{aligned}$$

Thus we have shown that for each $i \in I$, $\lambda(h_i) = d_i^{-1}\Phi_{i,0} = \deg(P_i)$, hence $\lambda \in P^+$. Furthermore, by the PBW theorem for $Y(\mathfrak{g})$, we have a decomposition

$$L(P) = \bigoplus_{\eta \in Q^+} L(P)_{\lambda-\eta}$$

because $X^-_{i_k,r_k}\cdots X^-_{i_1,r_1}v_P$ has $\mathfrak{g}$-weight $\lambda - \alpha_{i_1} - \ldots - \alpha_{i_k}$. To prove that $L(P)$ is finite dimensional, it is therefore enough to show that:

a) $\mathbf{L(P)_{\lambda-\eta} = 0}$ **for all but finitely many** $\eta \in \mathbf{Q}^+$

and

b) $\mathbf{L(P)_{\lambda-\eta}}$ **is finite dimensional for any** $\eta \in \mathbf{Q}^+$.

Let us first prove (a):

Let $\mu = \lambda - \eta$ for some $\eta \in Q^+$, and suppose $L(P)_\mu \neq 0$. Choose any nonzero $v \in L(P)_\mu$, and for each $i \in I$, let $L_i = Y_i v$, where Y_i is the subalgebra of $Y(\mathfrak{g})$ generated by $X^{\pm}_{i,0}$ and $H_{i,0}$.

Let $\mathfrak{U}_i$ be the subalgebra of $\mathfrak{U}(\mathfrak{g})$ generated by e_i, f_i and h_i. We can view L_i as a representation of $\mathfrak{U}_i \cong \mathfrak{U}(\mathfrak{sl}_2)$ via (6.1) (L_i is a representation of Y_i, which is the image of $\mathfrak{U}_i$ under this homomorphism).

Assume that L_i is finite dimensional for each $i \in I$. Then its set of weights under the action of $\mathfrak{U}_i \subset \mathfrak{U}(\mathfrak{g})$ is stable with respect to the Weyl group of $\mathfrak{U}_i$, hence in particular under the action of the fundamental reflection s_i. Therefore, $L(P)$ contains nonzero vectors of weight $s_i(\mu)$ for each $i \in I$; since the Weyl group W of $\mathfrak{g}$ is generated by the s_i, it follows by induction that for any $w \in W$, $L(P)_{w(\mu)} \neq 0$ whenever $L(P)_\mu \neq 0$. In particular, we can choose w so that $w(\mu) \in P^+$ (because W acts transitively on the set of Weyl chambers of $\mathfrak{g}$). Then since $L(P)_{w(\mu)} \neq 0$, we have $w(\mu) \leq \lambda$ with $w(\mu) \in P^+$ and, of course, $L(P)_\mu = L(P)_{w^{-1}(w(\mu))}$. This shows that if $L(P)_\mu \neq 0$, then μ belongs to the finite set $W \cdot \{\nu \in P^+ \mid \nu \leq \lambda\}$, proving (a).

We are therefore reduced to justifying the assumption that L_i is finite dimensional. This will follow from the fact that if $L(P)_\mu \neq 0$, then there is some $N > 0$ such that $L(P)_{\mu - r\alpha_i} = L(P)_{\mu + r\alpha_i} = 0$ for $r > N$; indeed, in this case L_i is spanned by the finite set $\{(X^{\pm}_{i,0})^r v \mid 0 \leq r \leq N\}$. Observe that $\mu + r\alpha_i \leq \lambda$ for only finitely many r, so it is clear that $L(P)_{\mu + r\alpha_i} = 0$ for r sufficiently large. We will show on the other hand that $L(P)_{\mu - r\alpha_i} = 0$ when $r > 3h + \lambda(h_i)$, where h is the height of $\lambda - \mu$.

To prove this, note that if $\lambda - \mu = \alpha_{i_1} + \cdots + \alpha_{i_h}$, then for any $r > 0$, $L(P)_{\mu - r\alpha_i}$ is spanned by vectors of the form

$$X_1^- X_{i_1,k_1}^- X_2^- X_{i_2,k_2}^- \cdots X_h^- X_{i_h,k_h}^- X_{h+1}^- v_P \tag{6.13}$$

where $k_1, \ldots, k_h \in \mathbb{Z}_+$, and for each $1 \leq p \leq h+1$, X_p^- is some product of the form

$$X_{i,l_{1,p}}^- X_{i,l_{2,p}}^- \cdots X_{i,l_{r_p,p}}^-$$

for some $l_{1,p}, \ldots, l_{r_p,p} \in \mathbb{Z}_+$ and with $r_1 + r_2 + \cdots + r_{h+1} = r$.

Let us refer to elements of the form (6.13) as *spanning vectors*. We will call these elements *admissible* if

$$r_1, \ldots, r_h \leq 3. \tag{6.14}$$

We will show that $L(P)_{\mu - r\alpha_i}$ is actually spanned by the admissible vectors. For this it is enough to see that any spanning vector (which may not itself be admissible) can be expressed as a linear combination of admissible elements. The proof is by induction on h:

Note that if $h = 0$ then any spanning vector is by definition admissible. Assume for some fixed height $h \geq 0$ that any spanning vector is a linear combination of admissible elements. Let v be any spanning vector for the case when $\lambda - \mu$ has height $h+1$; that is,

$$v = X_1^- X_{i_1,k_1}^- X_2^- X_{i_2,k_2}^- \cdots X_{h+1}^- X_{i_{h+1},k_{h+1}}^- X_{h+2}^- v_P.$$

By induction, $X_2^- X_{i_2,k_2}^- \cdots X_{h+1}^- X_{i_{h+1},k_{h+1}}^- X_{h+2}^- v_P$ is a linear combination of admissible elements; if $r_1 \leq 3$ then v itself is also a linear combination of admissible elements, so it only remains to consider the case when $r_1 > 3$.

Let $m = 1 - c_{i,i_1}$ where $C = (c_{ij})$ is the Cartan matrix of $\mathfrak{g}$ (so $m \leq 4$, hence $r_1 \geq m$). We may as well assume that $i \neq i_1$, because if $i = i_1$ then we can just relabel the first factor of X_1^- as X_{i_1,k_1}^- (so that what was previously called X_{i_1,k_1}^-, as well as all the other factors in X_1^- get absorbed into X_2^-), and then we are back to the case when $r_1 \leq 3$ (in fact $r_1 = 0$). Let $X_{i,l_1}^- \cdots X_{i,l_m}^-$ be the last m factors of X_1^-. We will show by induction on $l_1 + \ldots + l_m$ that the product $X_{i,l_1}^- \cdots X_{i,l_m}^- X_{i_1,k_1}^-$ can be expressed as a linear combination of elements of the same form but with the X_{i_1,k_1} moved to the left (and with the second index on each factor possibly shifted):

If $l_1 + \ldots + l_m = 0$, then $l_1 = l_2 = \ldots = l_m = 0$; a single application of (2.5) therefore expresses $X_{i,l_1}^- \cdots X_{i,l_m}^- X_{i_1,k_1}^-$ in the desired form.

Suppose $l_1 + \ldots + l_m > 0$. If $l_m > 0$, then we apply relation (2.4):

$$\begin{aligned}
&X_{i,l_1}^- \cdots X_{i,l_m}^- X_{i_1,k_1}^- \\
&= X_{i,l_1}^- \cdots X_{i,l_{m-1}}^- \left(X_{i_1,k_1}^- X_{i,l_m}^- + X_{i,l_m-1}^- X_{i_1,k_1+1}^- - X_{i_1,k_1+1}^- X_{i,l_m-1}^- \right) \\
&\quad - \frac{d_i c_{i,i_1}}{2} X_{i,l_1}^- \cdots X_{i,l_{m-1}}^- \left(X_{i,l_m-1}^- X_{i_1,k_1}^- + X_{i_1,k_1}^- X_{i,l_m-1}^- \right)
\end{aligned}$$

The X_{i_1,k_1} has been moved to the left in the first, third and fifth terms; for the others we can move it to the left by induction, as $l_1 + \cdots + l_m - 1 < l_1 + \cdots + l_m$. We are therefore reduced to the case when $l_m = 0$.

For this case, suppose that $l_{m-1} > 0$. Apply relation (2.4) with $i = j$:

$$\begin{aligned}
&X^-_{i,l_1} \cdots X^-_{i,l_{m-1}} X^-_{i,0} X^-_{i_1,k_1} \\
&= X^-_{i,l_1} \cdots X^-_{i,l_{m-2}} \left(X^-_{i,0} X^-_{i,l_{m-1}} + X^-_{i,l_{m-1}-1} X^-_{i,1} - X^-_{i,1} X^-_{i,l_{m-1}-1} \right) X^-_{i_1,k_1} \\
&\quad - d_i X^-_{i,l_1} \cdots X^-_{i,l_{m-2}} \left(X^-_{i,l_{m-1}-1} X^-_{i,0} + X^-_{i,0} X^-_{i,l_{m-1}-1} \right) X^-_{i_1,k_1}.
\end{aligned}$$

The last two terms are taken care of by induction. With the first two terms we are back to the case when $l_m > 0$, and also with the third term if $l_{m-1} - 1 > 0$. If $l_{m-1} - 1 = 0$, then we may use relation (2.4) again to deal with this term:

$$\begin{aligned}
&X^-_{i,l_1} \cdots X^-_{i,l_{m-2}} X^-_{i,1} X^-_{i,0} X^-_{i_1,k_1} \\
&= X^-_{i,l_1} \cdots X^-_{i,l_{m-2}} \left(X^-_{i,0} X^-_{i,1} - d_i X^-_{i,0} X^-_{i,0} \right) X^-_{i_1,k_1}.
\end{aligned}$$

The first term here is back to the case when $l_m > 0$, and the second term is again dealt with by induction. We are then reduced to the case when $l_{m-1} = l_m = 0$. We can repeat this argument to reduce to the case when $l_{m-2} = l_{m-1} = l_m = 0$, and so on until $l_2 = \ldots = l_m = 0$ and $l_1 > 0$. In this case, a single application of relation (2.5) expresses $X^-_{i,l_1} X^-_{i,0} \ldots X^-_{i,0} X^-_{i_1,k_1}$ as a linear combination of terms that can be dealt with by the previous cases.

We can iterate this process to keep moving the $X^-_{i_1,k_1}$ further to the left, until it has no more than 3 factors preceding it in each term. Each of these terms is therefore back to the case when $r_1 \leq 3$, which finally proves that v is a linear combination of admissible elements.

We will also need the following lemma to show that $L(P)_{\mu - r\alpha_i} = 0$ if $r > 3h + \lambda(h_i)$, completing the proof of (a). For each fixed $i \in I$, let $\hat{Y}_i$ be the subalgebra of $Y(\mathfrak{g})$ generated by all the $X^\pm_{i,r}$ and the $H_{i,r}$, and let $\hat{L}_i = \hat{Y}_i v_P$. It is clear that the assignment

$$\widetilde{X_{1,r}}^+ \mapsto d_i^{-r-1} X^+_{i,r}, \qquad \widetilde{X_{1,r}}^- \mapsto d_i^{-r} X^-_{i,r}, \qquad \widetilde{H_{1,r}} \mapsto d_i^{-r-1} H_{i,r}, \tag{6.15}$$

defines an isomorphism $Y(\mathfrak{sl}_2) \xrightarrow{\sim} \hat{Y}_i$ (here, we have marked the generators of $Y(\mathfrak{sl}_2)$ with a tilde, so as not to confuse them with the generators of $Y(\mathfrak{g})$). We can therefore view $\hat{L}_i$ as a representation of $Y(\mathfrak{sl}_2)$.

Lemma 6.2 *Let $Q_i(u) = d_i^{-\deg(P_i)} P_i(d_i u)$. Then $\hat{L}_i$ is isomorphic to the $Y(\mathfrak{sl}_2)$ module $L(Q_i)$.*

Proof. Note that the vector v_P generates $\hat{L}_i$ as a $Y(\mathfrak{sl}_2)$-module by definition, and for each $r \in \mathbb{Z}_+$, we have $\widetilde{X^+_{1,r}} v_P = d^{-r-1} X^+_{i,r} v_P = 0$. Moreover, $\widetilde{H_{1,r}} v_P = d_i^{-r-1} H_{i,r} v_P = d_i^{-r-1} \Phi_{i,r} v_P$. It follows that $\hat{L}_i$ is a highest weight module over $Y(\mathfrak{sl}_2)$ with highest weight vector v_P, and highest weight given by $\hat{\Phi} = (\hat{\Phi}_{1,r})_{r \in \mathbb{Z}_+}$, where $\hat{\Phi}_{1,r} = d_i^{-r-1} \Phi_{i,r}$.

Now, observe that

$$\frac{Q_i(u+1)}{Q_i(u)} = \frac{P_i(d_i(u+1))}{P_i(d_iu)} = \frac{P_i((d_iu)+d_i)}{P_i(d_iu)} = 1 + \sum_{r=0}^{\infty} \Phi_{i,r}(d_iu)^{-r-1} = 1 + \sum_{r=0}^{\infty} \hat{\Phi}_{1,r}u^{-r-1}.$$

It remains to show that $\hat{L}_i$ is irreducible as a module over $Y(\mathfrak{sl}_2)$. According to the PBW theorem for $Y(\mathfrak{g})$, $\hat{L}_i$ is spanned by all the vectors in $L(P)$ of the form $X^-_{i,r_1} \dots X^-_{i,r_k} v_P$. It follows that $\hat{L}_i = \bigoplus_{k \geq 0} L(P)_{\lambda - k\alpha_i}$.

Let W be a nonzero irreducible $Y(\mathfrak{sl}_2)$-submodule of $\hat{L}_i$, and let $\hat{W} \subset W$ be the subspace of elements $w \in W$ with the property that $X^+_{i,r}w = 0$ for every $r \in \mathbb{Z}_+$. Observe that $\hat{W}$ is nonzero, because one can take any nonzero element of W and obtain from it a nonzero element of $\hat{W}$ by repeated application of the $X^+_{i,r}$ (because the weights of $\hat{L}_i$ are bounded from above).

Let us show by induction on s that $\hat{W}$ is stable under the action of $H_{j,s}$ for every $j \in I$ and $s \in \mathbb{Z}_+$.

For the case $s = 0$, let $w \in \hat{W}$ and decompose w into its components with respect to the weight space decomposition of $\hat{L}_i$; we have $w = \sum_k w_k$ where $w_k \in L(P)_{\lambda - k\alpha_i}$. Since $w \in W$, we have $X^+_{i,r}w = 0$, hence $X^+_{i,r}w_k = 0$ for each k. Therefore, for each $j \in I$, we have

$$X^+_{i,r}H_{j,0}w = X^+_{i,r}\sum_k H_{j,0}w_k = X^+_{i,r}\sum_k d_jh_jw_k = d_j\sum_k(\lambda - k\alpha_i)(h_j)X^+_{i,r}w_k = 0.$$

It follows that $H_{j,0}w \in \hat{W}$.

Now suppose for some s that $\hat{W}$ is stable under the action of all the $H_{j,s}$, and let $w \in \hat{W}$. Then by relation (2.2), we have

$$X^+_{i,r}H_{j,s+1}w = H_{j,s+1}X^+_{i,r}w - H_{j,s}X^+_{i,r+1}w + X_{i,r+1}H_{j,s}w - \frac{d_jc_{ji}}{2}\left(H_{j,s}X^+_{i,r} + X^+_{i,r}H_{j,s}\right)w.$$

The first, second and fourth terms on the right hand side are annihilated because $w \in \hat{W}$, while the third and fifth are annihilated because $H_{j,s}w \in \hat{W}$ by induction hypothesis. It follows that $H_{j,s+1}w \in \hat{W}$, hence $\hat{W}$ is stable with respect to $H_{j,s}$ for every $j \in I$ and $s \in \mathbb{Z}_+$, as claimed.

Since the $H_{j,s}$ are pairwise commuting operators on $\hat{W}$, it follows that there is some nonzero simultaneous eigenvector $w \in \hat{W}$; that is, $H_{j,s}w = \tilde{\Phi}_{j,s}w$ for some scalars $\tilde{\Phi}_{j,s} \in \mathbb{C}$. In particular, w is a simultaneous eigenvector under the action of all the $h_j \in \mathfrak{g}$, so it must be contained in one of the weight spaces $L(P)_{\lambda - k\alpha_i}$. But then it follows that $X^+_{j,r}w = 0$ for every $j \in I$ and $r \in \mathbb{Z}_+$; this is true for $j = i$ by definition, and for $j \neq i$, $X^+_{j,r}w$ must be zero because otherwise, it is a weight vector of weight $\lambda - k\alpha_i + \alpha_j$, but all the weights of $L(P)$ are of the form $\lambda - \eta$ for $\eta \in Q^+$.

In summary, we have shown that w is a highest weight vector in $L(P)$, hence it must be contained in the one dimensional subspace $L(P)_\lambda$; in particular, w is a nonzero scalar multiple of the highest weight vector v_P. Since v_P generates $\hat{L}_i$, it follows that $W = \hat{L}_i$, hence $\hat{L}_i$ is irreducible. ■

Since we have already proven Theorem 6.3 for $\mathfrak{g} = \mathfrak{sl}_2$, the above lemma tells us that $\hat{L}_i$ is finite dimensional. It is also a representation of $\mathfrak{sl}_2$ via the embedding $\mathfrak{U}(\mathfrak{sl}_2) \hookrightarrow Y(\mathfrak{sl}_2) \cong \hat{Y}_i$. The $\mathfrak{sl}_2$-weights of $\hat{L}_i$ are given as follows: if $v \in L(P)_{\lambda - k\alpha_i}$, then $h_1 v = \widetilde{H_{1,0}} v = d_i^{-1} H_{i,0} v = h_i v = (\lambda - k\alpha_i)(h_i) v = (\lambda(h_i) - 2k)v$. Then since $\hat{L}_i = \bigoplus_{k \geq 0} L(P)_{\lambda - k\alpha_i}$, the weight space decomposition of $\hat{L}_i$ as a representation of $\mathfrak{sl}_2$ is

$$\hat{L}_i = \bigoplus_{k \geq 0} (\hat{L}_i)_{\lambda(h_i) - 2k}$$

where $(\hat{L}_i)_{\lambda(h_i)-2k} = \{v \in \hat{L}_i \mid h_1 v = (\lambda(h_1) - 2k)v\}$. Furthermore, the above argument shows that $L(P)_{\lambda - k\alpha_i} = (\hat{L}_i)_{\lambda(h_i)-2k}$. On the other hand, the symmetry of weights for finite dimensional representations of $\mathfrak{sl}_2$ tells us that $\lambda(h_i) - 2k$ is not a weight when $k > \lambda(h_i)$. Therefore, for $k > \lambda(h_i)$, we have

$$L(P)_{\lambda - k\alpha_i} = (\hat{L}_i)_{\lambda - k\alpha_i} = (\hat{L}_i)_{\lambda(h_i) - 2k} = 0,$$

hence $L(P)_{\lambda - k\alpha_i} = 0$. Recall that $L(P)_{\mu - r\alpha_i}$ is spanned by admissible elements, and for an admissible element, we have $r_{h+1} = r - r_1 - \ldots - r_h \geq r - 3h$. If $r > 3h + \lambda(h_i)$, then $r_{h+1} > \lambda(h_i)$, so that $L(P)_{\lambda - r_{h+1}\alpha_i} = 0$. Since $X_{h+1}^- v_p \in L(P)_{\lambda - r_{h+1}\alpha_i}$, it follows that $L(P)_{\mu - r\alpha_i} = 0$ when $r > 3h + \lambda(h_i)$, because all the admissible elements are zero. The proof of (a) is now complete.

We will prove (b) by induction on the height h of η.

If $h = 0$, then $\lambda - \eta = \lambda$, and we know that $L(P)_\lambda$ is one dimensional.

If $h = 1$, then $\eta = \alpha_i$ for some $i \in I$, hence $L(P)_{\lambda - \mu} = L(P)_{\lambda - \alpha_i} \subset \hat{L}_i$, and we showed already that $\hat{L}_i$ is finite dimensional.

Assume $h \geq 2$ and that (b) has been proven for all η with height $< h$. We know that $L(P)_{\lambda - \eta}$ is spanned by all the vectors in $L(P)$ of the form

$$X_{i_1, r_1}^- \ldots X_{i_h, r_h}^- v_P \tag{6.16}$$

where $\eta = \alpha_{i_1} + \ldots + \alpha_{i_h}$. Moreover, for each $i \in \{i_1, \ldots, i_h\}$, $L(P)_{\lambda - \eta + \alpha_i}$ is finite dimensional by induction hypothesis, hence it is spanned by some set of vectors of the form

$$X_{j_2, s_2}^- \ldots X_{j_h, s_h}^- v_P$$

where $\eta - \alpha_i = \alpha_{j_2} + \ldots + \alpha_{j_h}$ and $s_2, \ldots, s_h \leq M_i$ for some $M_i \in \mathbb{Z}_+$.

Let $M = \max_{i \in I}\{M_i\}$. By induction hypothesis, the subspace

$$V = \sum_{r_2 = 0}^{M+1} X_{i_2, r_2}^- V(P)_{\lambda - \eta + \alpha_{i_2}} + X_{i_1, 0}^- V(P)_{\lambda - \eta + \alpha_{i_1}}$$

is finite dimensional. To demonstrate that $L(P)_{\lambda - \eta}$ is also finite dimensional, we will show that it is contained in V. It is enough to show that every vector of the form (6.16) is in V. We proceed by induction on r_1:

If $r_1 = 0$, then the vector (6.16) is in V by definition. For the inductive step, suppose that $r_1 > 0$. Since $X^-_{i_2,r_2} \dots X^-_{i_h,r_h} v_P \in L(P)_{\lambda-\eta+\alpha_1}$, we can assume that $r_2, \dots, r_h \le M$. Since $h \ge 2$, we can apply relation (2.4) to the first two factors of (6.16):

$$
\begin{aligned}
&X^-_{i_1,r_1} X^-_{i_2,r_2} X^-_{i_3,r_3} \dots X^-_{i_h,r_h} v_P \\
&\qquad = \left(X^-_{i_2,r_2} X^-_{i_1,r_1} + X^-_{i_1,r_1-1} X^-_{i_2,r_2+1} - X^-_{i_2,r_2+1} X^-_{i_1,r_1-1} \right) X^-_{i_3,r_3} \dots X^-_{i_h,r_h} v_P \\
&\qquad\qquad - \frac{d_{i_1} c_{i_1,i_2}}{2} \left(X^-_{i_1,r_1-1} X^-_{i_2,r_2} + X^-_{i_2,r_2} X^-_{i_1,r_1-1} \right) X^-_{i_3,r_3} \dots X^-_{i_h,r_h} v_P .
\end{aligned}
$$

The first, third and fifth terms are in V by definition; the second and fourth terms are in V by induction hypothesis. This concludes the proof of the "if" part of Theorem 6.3. The "only if" part follows immediately from the $Y(\mathfrak{sl}_2)$ case:

Suppose $L(\Phi)$ is finite dimensional. Then for each $i \in I$, $\hat{L}_i$ is also finite dimensional. Moreover, the proof of Lemma 6.2 indicates that $\hat{L}_i$ is isomorphic to the irreducible highest weight module $L(\hat{\Phi})$ over $Y(\mathfrak{sl}_2)$, where $\hat{\Phi} = (\hat{\Phi}_{1,r})_{r\in\mathbb{Z}_+}$ is given by $\hat{\Phi}_{1,r} = d_i^{-r-1}\Phi_{i,r}$. It follows from the $Y(\mathfrak{sl}_2)$ case of Theorem 6.3 that there exists a monic polynomial $Q_i(u)$ such that

$$\frac{Q_i(u+1)}{Q_i(u)} = 1 + \sum_{r=0}^{\infty} \hat{\Phi}_{1,r} u^{-r-1}.$$

Set $P_i(u) = d_i^{\deg(Q_i)} Q(d_i^{-1}u)$. Then we have

$$\frac{P_i(u+d_i)}{P_i(u)} = \frac{Q_i(d_i^{-1}u+1)}{Q_i(d_i^{-1}u)} = 1 + \sum_{r=0}^{\infty} \hat{\Phi}_{1,r} (d_i^{-1}u)^{-r-1} = 1 + \sum_{r=0}^{\infty} \Phi_{i,r} u^{-r-1}$$

as desired.

Bibliography

[1] Y. Billig, V. Futorny, and A. Molev. Verma modules for Yangians. *Lett. Math. Phys.*, 78(1):1–16, 2006.

[2] V. Chari and A. Pressley. Fundamental representations of Yangians and singularities of R-matrices. *J. Reine Angew. Math.*, 417:87–128, 1991.

[3] V. Chari and A. Pressley. Quantum affine algebras. *Comm. Math. Phys.*, 142(2):261–283, 1991.

[4] V. Chari and A. Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original.

[5] V. Chari and A. Pressley. Quantum affine algebras and their representations. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 59–78. Amer. Math. Soc., Providence, RI, 1995.

[6] V. Chari and A. Pressley. Yangians: their representations and characters. *Acta Appl. Math.*, 44(1-2):39–58, 1996.

[7] S. Gautam and V. Toledano Laredo. Yangians and quantum loop algebras. *Selecta Math. (N.S.)*, 19(2):271–336, 2013.

[8] N. Guay and X. Ma. From quantum loop algebras to Yangians. *J. Lond. Math. Soc. (2)*, 86(3):683–700, 2012.

[9] J. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York-Berlin, 1972. Graduate Texts in Mathematics, Vol. 9.

[10] A. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1996.

[11] A. Molev. *Yangians and classical Lie algebras*, volume 143 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2007.

[12] A. Molev, E. Ragoucy, and P. Sorba. Coideal subalgebras in quantum affine algebras. *Rev. Math. Phys.*, 15(8):789–822, 2003.