Option Pricing and Logarithmic Euler-Maruyama Convergence of Stochastic Delay Equations driven by Lévy process

by

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Abstract

In this thesis we study the product formula for finitely many multiple Itô-Wiener integrals of Lévy process, option pricing formula where the stock price is modeled by stochastic delay differential equation (SDDE) driven by Lévy process and logarithmic Euler-Maruyama scheme for the SDDE. In the first part we derive a product formula for finitely many multiple Itô-Wiener integrals of Lévy process, expressed in terms of the associated Poisson random measure. The formula is compact and the proof is short and uses the exponential vectors and polarization techniques. In the second part of the thesis we discuss the option pricing when the underlying model follows SDDE. In this part, we obtain the existence, uniqueness, and positivity of the solution to SDDE with jumps. This equation is then applied to model the price movement of the risky asset in a financial market and the Black-Scholes formula for the price of European option is obtained together with the hedging portfolios. The option price is evaluated analytically at the last delayed period by using the Fourier transformation technique. However, in general, there is no analytical expression for the option price. To evaluate the price numerically, we then use the Monte-Carlo method. To this end, we need to simulate the delayed stochastic differential equations with jumps. We propose a logarithmic Euler-Maruyama scheme to approximate the equation and prove that all the approximations remain positive and the rate of convergence of the scheme is proved to be 0.5. Finally in the last part of the thesis we discuss logarithmic Euler-Maruyama scheme and convergence of logarithmic Euler-Maruyama scheme for a multi-dimensional SDDE's.

Preface

This thesis is based on two published papers and one completed work. In particular

- Chapter 2 of this thesis is a joint work with Prof. Yaozhong Hu, Ms. Neha Sharma and has been published as "*General Product Formula of Multiple Integrals of Lévy Process*" in the journal of Stochastic Analysis.
- Chapter 4 of this thesis is a joint work with Prof. Yaozhong Hu and has been published as "*Jump Models with Delay-Option Pricing and Logarithmic Euler-Maruyama Scheme*" in the journal Mathematics by MDPI.
- Chapter 5 of this thesis is a joint work with Prof. Yaozhong Hu and is new work as per my knowledge. As of writing this thesis, the work in chapter 5 is complete and will soon be submitted for publication.

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(PhD Candidate)

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Contents

Chapter 0

Summary of the the work

This dissertation concerns with topics related to Poisson random measures, stochastic differential equations, logarithmic Euler-Maruyama scheme and option pricing. It consists of two published research articles and one completed work which will be submitted soon. The three works are listed below.

1. *General product formula of multiple Integrals of Lévy process*, with Yaozhong Hu and Neha Sharma, Journal of Stochastic Analysis: Vol. 1 : No. 3 , Article 3.

2. *Jump models with delay-option pricing and logarithmic Euler-Maruyama scheme* with Yaozhong Hu, Mathematics 2020, 8(11).

3. Logarithmic Euler-Maruyama scheme for system of SDDE driven by Lévy process with Yaozhong Hu, completed and can be found here [2].

Work related to 1 is presented in chapter 2. In this chapter we derive a product formula for finitely many multiple Itô-Wiener integrals of Lévy process, expressed in terms of the linear combination of n-fold iterated intergrals with respect to Poisson random measure. The formula is compact. The proof is short and uses the exponential vectors and polarization techniques.

Work related to 2 is the content in chapter 4. In this chapter we obtain the existence, uniqueness, and positivity of the solution to stochastic delayed differential equations with jumps. This equation is then applied to model the price movement of the risky asset in a financial market and the Black-Scholes formula for the price of European option is obtained together with the corresponding hedging portfolios. The option pricing formula is evaluated analytically at the last delayed period by using the Fourier transformation technique. However in general there is no analytical expression for the option price. To evaluate the price numerically, we then use the Monte-Carlo method. To this end, we need to simulate the delayed stochastic differential equations with jumps. We propose a logarithmic Euler-Maruyama scheme to approximate the equation and prove that all the approximated solutions remain positive and the rate of convergence of the scheme is proved to be 0.5.

Work related to 3 is in chapter 5. In this chapter we propose a logarithmic Euler-Maruyama scheme to approximate the system of SDDE's driven by Lévy process and prove that all the approximations remain positive and converge in L^p sense for $p \geq 2$.

The following three subsections provide brief summary of each of the works.

0.1 Summary of work on General Product Formula

For any $f \in \hat{L}^{2,n}$, the finitely many multiple Wiener-Itô integral of Lévy process (multiple n-fold iterated integrals) is

$$I_n(f) := \int_{\mathbb{T}^n} f(t_1, z_1, \cdots, t_n, z_n)\tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \tag{1.1}$$

where

$$\hat{L}^{2,n} := \left(L^2(\mathbb{T}, \lambda \times \nu)\right)^{\otimes n} \subseteq L^2\left(\mathbb{T}^n, (\lambda \times \nu)^n\right)$$

be the space of symmetric, deterministic real functions f and $\tilde{N}$ is the compensated Poisson random measure. We discuss $\prod_{k=1}^{m} I_{q_k}(f_k)$ and try to express this product of multiple integrals as linear combinations of some other multiple integrals. Our main result of this work can be summed up in the following theorem,

Theorem Let q_k be a positive integer, let $f_k \in (L^2([0,T] \times \mathbb{R}_0, dt \otimes \nu(dz)))^{\hat{\otimes} q_k}$, $k = 1, \cdots, m$. Then

$$\prod_{k=1}^{m} I_{q_k}(f_k) = \sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\le q_1 \\ \cdots \\ \chi(m,\vec{l},\vec{n})\le q_m}} \frac{\prod_{k=1}^{m} q_k!}{\prod_{\alpha=1}^{\kappa_m} l_{\mathbf{i}_\alpha}! \prod_{\beta=1}^{\kappa_m} \mu_{\mathbf{j}_\beta}! \prod_{k=1}^{m} (q_k - \chi(k,\vec{l},\vec{n}))!}$$
$$I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,n_{j_{\kappa_m}}}(f_1,\cdots,f_m)), \tag{1.2}$$

where we recall $|q| = q_1 + \cdots + q_m$ and $|\chi(\vec{l},\vec{n})| = \chi(1,\vec{l},\vec{n}) + \cdots + \chi(m,\vec{l},\vec{n})$. For notations and further details please refer chapter 2.

0.2 Summary of work on Option pricing formula and Euler-Maruyama Convergence

We consider the following delayed stochastic differential equation driven by compound Poisson process $Z(t)$:

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t), & t \ge 0\,, \\ S(t) = \phi(t)\,, \quad t \in [-b,0]\,. \end{cases} \tag{2.3}$$

To study the above stochastic differential equation, we introduce the Poisson random measure associated with Lévy process $Z(t)$ and write

$$Z(t) = \int_{[0,t]\times\mathcal{J}} zN(ds,dz) \quad \text{or} \quad dZ(t) = \int_{\mathcal{J}} zN(dt,dz)$$

and hence write (2.3) as

$$\begin{aligned} dS(t) &= \left[f(S(t-b)) + g(S(t-b))\int_{\mathcal{J}} z\nu(dz)\right] S(t)dt \\ &\quad + g(S(t-b))S(t-)\int_{\mathcal{J}} z\tilde{N}(dt,dz)\,. \end{aligned}$$

Then for the above equation we show that the stochastic differential delay equation (2.3) admits a unique pathwise solution with the property that if $\phi(0) > 0$, then for all $t > 0$, $S(t) > 0$ almost surely.

We then also discuss the logarthmic Euler-Maruyama scheme in which we consider Z_t as a compound Poisson process i.e $Z_t = \sum_{i=1}^{N_t} Y_i$ and for the process

$$S(t) = \phi(0)\exp\Big(\int_0^t f(X(u-b))du + \sum_{0\le u\le t, \Delta Z(u)\neq 0} \ln(1+g(X(u-b))Y_{N(u)})\Big) \quad (2.4)$$

we propose a logarithmic Euler-Maruyama scheme to approximate (2.3) as follows:

$$\begin{aligned} S^\pi(t_{k+1}) &= S^\pi(t_k)\exp\Big(f(S^\pi(t_k-b))\Delta\Big) \\ &\quad \cdot \exp\Big(\ln(1+g(S^\pi(t_k-b))\Delta Z_k)\Big)\,, \quad k = 0,1,2,...,n-1 \end{aligned}$$

with $S^\pi(t) = \phi(t)$ for all $t \in [-b, 0]$. Where π is a partition of the time interval $[0, T]$. We assume Lipschitz continuity of f, g, ϕ and show that for some constant $K_{p,T}$, independent of π we will have

$$\mathbb{E}\Big[\sup_{0\le t\le T} |S(t) - S^\pi(t)|^p\Big] \le K_{p,T}\Delta^{p/2}\,. \quad (2.5)$$

The logarithmic Euler-Maruyama scheme helps in simulating the paths realised as stock prices. Using this scheme we have generated the paths and applied Monte-Carlo technique to obtain the price of European call option. We have shown this in the numerical attempt section in the chapter 4 and the MATLAB codes are given in appendix section.

We have also developed the formula to price European call option under risk neutral measure where price of the risky asset is given by

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t), & t \geq 0\,, \\ S(t) = \phi(t)\,, \qquad t \in [-b, 0]\,, \end{cases} \tag{2.6}$$

and the price of risk-free asset is given by

$$dB(t) = rB(t)dt\,, \quad \text{or} \quad B(t) = e^{rt}\,, t \geq 0\,.$$

We find the risk neutral measure $\mathbb{Q}$ under stated assumptions (in the chapter) and we can write (2.6) as

$$d\tilde{S}(t) = \tilde{S}(t-)\int_{\mathcal{J}} zg(S(t-b))\tilde{N}_{\mathbb{Q}}(dt, dz).$$

Using the martingale representation theorem we have also derived the hedging portfolio. We finally discuss the formula of European call option on the interval $[T-b, T]$ where T is the maturity and b is the delay factor. We state our formula below. The detailed result has been discussed in chapter 4.

Theorem When $t \in [T-b, T]$, price for the European Call option is given by

$$\begin{aligned} V(t) \;=\;& e^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A(t)\cdot\tilde{S}(t)\exp\left\{\int_t^T\int_{\mathcal{J}}\Big((1+zg(S(u-b)))^{(1-i\xi)}\right. \\ & \left. -\,(1-i\xi)\ln(1+zg(S(u-b))) - 1\Big)\nu_{\mathbb{Q}}(dz)du\right\} \\ & -Ke^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A(t)\cdot\tilde{S}(t)\exp\left\{\int_t^T\int_{\mathcal{J}}\Big((1+zg(S(u-b)))^{-i\xi}\right. \\ & \left. +\,i\xi\ln(1+zg(S(u-b))) - 1\Big)\nu_{\mathbb{Q}}(dz)du\right\}\,, \end{aligned} \tag{2.7}$$

where $w = \ln(K/A) - rT$ and

$$A(t) = \exp\Big(\int_t^T \int_{\mathcal{J}} \{\ln(1 + zg(S(u-b))) - zg(S(u-b))\nu_{\mathbb{Q}}(dz)du\Big). \tag{2.8}$$

The difference of this work from the work discussed in [22] is that we do not include diffusion component in our model and the expression for risk neutral measure found in [22] doesn't allow the zero coefficient of the diffusion component of the model discussed in [22]. The work we have discussed also differs from the work in [23], since the convergence of Euler-Maruyama scheme is shown in L^2 norm and we have shown in L^p norm. Further we have also taken care that our scheme is always positive since we also simulate stock price with the same scheme.

0.3 Euler-Maruyama scheme and convergence for a system of equations

In this work we discuss the logarithmic Euler-Maruyama scheme and its convergence for system of equations. We consider the following system of delayed stochastic differential equations driven by compound Poisson process:

$$\begin{cases} dS_i(t) = \sum_{j=1}^{d} f_{ij}(S(t-b))S_j(t)dt \\ \qquad\qquad + S_i(t-)\sum_{j=1}^{d} g_{ij}(S(t-b))dZ_j(t),, \quad i = 1, \cdots, d, \\ S_i(t) = \phi_i(t), \quad t \in [-b, 0], \ i = 1, \cdots, d, \end{cases} \tag{3.9}$$

where $S(t) = (S_1(t), \cdots, S_d(t))^T$, $Z_j(t) = \sum_{l=1}^{N_j(t)} Y_{j,l}$ where $Y_{j,l}$ are i.i.d and for each j, $Y_{j,l}$ and $N_j(t)$ are independent for all l where $N_j(t)$ is a poisson process. Motivated by application to finance we are interested in under what conditions the solution $S_i(t)$ are all positive and find numerical solution which remain positive. We shall decompose equation (3.9) into the following system:

$$\begin{cases} dX_i(t) = f_{ii}((S(t-b)))X_i(t)dt + X_i(t-)\sum_{j=1}^{d} g_{ij}(S(t-b))dZ_j(t) & (3.10a) \\ dp_i(t) = \sum_{j=1, j\neq i}^{d} f_{ij}((S(t-b)))p_j(t)dt, & (3.10b) \\ S_i(t) = p_i(t)\cdot X_i(t)\,, \quad i = 1, 2, \cdots, d\,. & (3.10c) \end{cases}$$

Here

$$dp_i(t) = \sum_{j=1, j\neq i}^{d} f_{ij}((S(t-b)))p_j(t)dt, \quad i = 1, 2, \cdots, d \tag{3.11}$$

and we write

$$\frac{dp(t)}{dt} = F((S(t-b)))p(t),$$

where $F(S(t-b))$ is a $d \times d$ matrix consisting of entries $f_{ij}(S(t-b))$ when $i \neq j$ and diagonal enteries as 0. We consider a finite time interval $[0, T]$ for some fixed $T > 0$. We consider the partition π of the time interval $[0, T]$. We then propose the following logarithmic scheme to approximate the solution:

$$\begin{cases} X_i^\pi(t) = X_i^\pi(t_k)\exp\Big(f_{ii}(S^\pi(t_k-b))(t-t_k) \\ \qquad\qquad + \sum_{j=1}^{d} \ln\Big(1 + g_{ij}(S^\pi(t_k-b))(Z_j(t) - Z_j(t_k)\Big)\Big)\,, & (3.12a) \\ p^\pi(t) = \Big[F(S^\pi(t_k-b)))(t-t_k) + I\Big]p^\pi(t_k), & (3.12b) \\ S_i^\pi(t) = p_i^\pi(t)X_i^\pi(t)\,, & (3.12c) \\ X_i^\pi(0) = \phi_i(0)\,, \quad p^\pi(0) = \mathbf{1}\,, \qquad t_k \le t \le t_{k+1}\,, \qquad k = 1, 2, \cdots, n-1\,. & (3.12d) \end{cases}$$

In this work we assume Lipschitz continuity of f_{ij}, g_{ij}, ϕ_i and show that for some constant $K_{pd,T} > 0$, independent of π, Δ we have

$$\mathbb{E}\left[\sup_{0\le t\le T}\left[|S(t) - S^{\pi}(t)|^p\right]\right] \le K_{pd,T}\Delta^{p/2}.$$

Chapter 1

Lévy process and Jump Diffusion models

1.1 Lévy Process

1.1.1 Lévy Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1.1.1. *A one- dimensional Lévy process is a stochastic process* $\{\eta = \eta(t), t \geq 0\}$

$$\eta(t) = \eta(t, \omega), \qquad \omega \in \Omega$$

with the following properties

1. $\eta(0) = 0$ $\mathbb{P} - a.s.$

2. η *has independent and stationary increments.*

3. *It is stochastically continuous, i.e for every* $t \geq 0$ *and* $\epsilon > 0$

$$\lim_{s \to t} \mathbb{P}\{|\eta(t) - \eta(s)| > \epsilon\} = 0$$

.

4. η has càdlàg paths, that is the trajectories are right continuous with left limits.

The jump of the process η at time t is defined by

$$\Delta\eta(t) := \eta(t) - \eta(t-)$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the σ- algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$, where $\bar{U}$ is the closure of U. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we then define the *Poisson random measure*, $N : [0,T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{N} \cup \{0\}$, associated with η by

$$N(t,U) := \sum_{0\leq s\leq t} \chi_U(\Delta\eta(s)), \tag{1.1}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of η is defined by

$$\nu(U) := \mathbb{E}[N(1,U)] \tag{1.2}$$

and compensated jump measure $\tilde{N}$ is defined by

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt \tag{1.3}$$

where ν satisfies

$$\int \min\{1,x^2\}\nu(dx) < \infty$$

For further discussion on Lévy process see [9], [13], [14].

1.1.2 Stochastic Calculus

In this section we discuss Lévy-Ito decomposition, Ito formula and stochastic differential equation. We state a few results without proofs and for proof and further discussion please refer to [4], [13].

Theorem 1. Lévy-Ito decomposition *Let η be a Lévy process. Then $\eta = \eta(t), t \geq 0$ admits the following integral representation*

$$\eta(t) = a_1 t + \sigma W(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz) \qquad (1.4)$$

for some constants $a_1, \sigma \in \mathbb{R}$. Here $W = W(t), t \geq 0$ is a standard Wiener process.

(1.4) above can be written as

$$\eta(t) = at + \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

if

$$\int_{|z|\geq 1} |z|^2 \nu(dz) < \infty.$$

Motivated by above we consider process $X = X(t), t \geq 0$ admitting stochastic integral representation in the form

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz)$$

where $\alpha(t), \beta(t)$ are adapted process and $\gamma(t, z)$ is predictable processes with respect to filtration generated by $W(t)$ and $\tilde{N}$ such that, for all $t > 0$, $z \in \mathbb{R}_0$

$$\int_0^t [|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s, z)\nu(dz)]ds < \infty.$$

The above condition implies that the stochastic integrals are well-defined and are local martingales. The above process is called Itô-Lévy process. The Lévy-Itô decomposition entails that for every Lévy process there exist a vector γ, a positive definite matrix A and a positive measure ν that uniquely determine its distribution. The triplet (A, ν, γ) is called characteristic triplet or Lévy triplet of the process X_t. Let us first define Lévy process in R^d.

Definition 1.1.2. *An $\mathbb{R}^d$ valued Lévy process is a stochastic process $\{\eta = \eta(t), t \geq 0\}$*

$$\eta(t) = \eta(t, \omega), \qquad \omega \in \Omega$$

with the following properties

1. *$\eta(0) = 0$ $\mathbb{P} - a.s.$*

2. *η has independent and stationary increments.*

3. *It is stochastically continuous, i.e for every $t \geq 0$ and $\epsilon > 0$*

$$\lim_{s \to t} \mathbb{P}\{|\eta(t) - \eta(s)| > \epsilon\} = 0$$

.

4. *η has càdlàg paths, that is the trajectories are right continuous with left limits.*

We also note the Itô formula for Itô-Lévy process.

Theorem 2. Itô formula *Let $X = X(t), t \geq 0$, be the Itô-Lévy process and let $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0, \infty) \times \mathbb{R})$ and define*

$$Y(t) := f(t, X(t)), \qquad t \geq 0.$$

Then the process $Y = Y(t), t \geq 0$, is also an Itô-Lévy process and its differential form is given by

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))\alpha(t)dt + \frac{\partial f}{\partial x}(t,X(t))\beta(t)dW(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\beta^2(t)dt + \int_{\mathbb{R}_0}[f(t,X(t)+\gamma(t,z)) - f(t,X(t)) \\ &\quad - \frac{\partial f}{\partial x}(t,X(t))\gamma(t,z)]\nu(dz)dt \\ &\quad + \int_{\mathbb{R}_0}[f(t,X(t^-)+\gamma(t,z)) - f(t,X(t^-))]\tilde{N}(dt,dz). \end{aligned}$$

Theorem 3. ***Characteristic function of a Lévy process*** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. Then there exists a continuous function $\psi : \mathbb{R}^d \to \mathbb{C}$ called the characteristic exponent of X, such that:*

$$\mathbb{E}[e^{iz.X_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d.$$

Theorem 4 (Lévy-Khinchin representation)**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (A, ν, γ). Then*

$$E[e^{izX_t}] = e^{t\psi(z)}, \;\; z \in \mathbb{R}^d \tag{1.5}$$

with

$$\psi(z) = -\frac{1}{2}z.Az + i\gamma.z \tag{1.6}$$

$$+ \int_{\mathbb{R}^d}(e^{izx} - 1 - izx\mathbb{1}_{|x|\leq 1})\nu(dx). \tag{1.7}$$

1.1.3 Stochastic Differential Equation (SDE)

Proposition 1. *Assume that λ is a positive constant and μ, σ, ϕ are functions* $: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *which satisfy*

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| + |\phi(t,x) - \phi(t,y)| \le C|x-y|, \ \forall t, x, y$$
$$|\mu(t,0)| + |\sigma(t,0)| + |\phi(t,0)| \le C, \ \ \forall t.$$

Then the SDE

$$\begin{aligned} dX_t &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t + \phi(t,X_{t-})dM_t, \\ X_0 &= x_0 \end{aligned}$$

admits a unique (pathwise) solution where M is a compensated martingale associated with a Poisson process N with intensity λ.

Proof For details one may refer to [24] chapter 10. ■

For comprehensive study we refer to [39] or [25].

1.1.4 Doléans-Dade Exponential

Proposition 2. *Let X be a real valued (σ^2, ν, γ) Lévy process and Z the Doléans-Dade exponential of X, i.e., the solution of*

$$dZ_t = Z_{t-}dX_t, \ \ Z_0 = 1.$$

Then

$$Z_t = e^{X_t - \sigma^2 t/2} \prod_{0<s\le t} (1 + \Delta X_s) e^{-\Delta X_s} := \mathcal{E}(X)_t.$$

We also note here that $\mathcal{E}(X)_t$ is a multiplicative Lévy process.

For further discussion on Doléans-Dade exponential we refer to [4],[13].

1.1.5 Compound Poisson process

Definition 1.1.3. *Let $(\tau_i)_{(i\geq 1)}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t, t \geq 0)$ defined by*

$$N_t = \sum_{n\geq 1} \mathbb{1}_{t\geq T_n}$$

is called a Poisson process with intensity λ.

The Poisson process is therefore defined as a counting process as it counts number of random times (T_n) which occur between 0 and t.

Definition 1.1.4. *A compound Poisson process on $\mathbb{R}^d$ with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where jumps sizes Y_i are i.i.d. with distribution f and (N_t) is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$.

We can deduce following properties of compound Poisson process from the definition.

- The sample paths of X are càdlàg piecewise constant functions.
- The jump times $(T_i)_{i\geq 1}$ have the same law as the jump times of the Poisson process N_t: they can be expressed as partial sums of independent exponential random variables with parameter λ.
- The jump sizes $(Y_i)_{i\geq 1}$ are independent and identically distributed with law f.

We shall state without proofs a few results about compound Poisson process. Interested readers can refer to ([13] or [14]) for further details.

Theorem 5. ***Characteristic function of a Compound Poisson process*** *Let $(X_t)_{t\geq 0}$ be a compound Poisson process on $\mathbb{R}^d$. Its characteristic function has the following representation:*

$$\mathbb{E}[\exp iu.X_t] = \exp\left[t\lambda \int_{\mathbb{R}^d} (e^{iu.x} - 1) f(dx)\right] \quad \forall u \in \mathbb{R}^d,$$

where λ denotes the jump intensity and f the jump size distribution.

We now define the jump measure for compound Poisson process.

Definition 1.1.5. *As discussed above, to every càdlàg process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ one can associate a random measure on $[0,\infty) \times \mathbb{R}^d$ describing the jumps of X: for any measurable set $B \subset [0,\infty) \times \mathbb{R}^d$, $N_X(B) = \#\{(t, X_t - X_{t}-) \in B\}$. For every measurable set $A \subset \mathbb{R}^d$, $N_X([t_1, t_2] \times A)$ counts the number of jump times of X between t_1 and t_2 such that their jump sizes are in A. Following result shows that N_X is Poisson random measure in the sense of 1.1.4.*

Theorem 6. ***Jump measure of a compound Poisson process****. Let $(X_t)_{t\geq 0}$ be a compound Poisson process with intensity λ and jump size distribution f. Its jump measure N_X is a Poisson random measure on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$.*

Definition 1.1.6. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. The measure ν on $\mathbb{R}^d$ defined by:*

$$\nu(A) = \mathbb{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d) \tag{1.8}$$

is called the Lévy measure of X: $\nu(A)$ is the expected number per unit time of jumps whose size belongs to A.

1.2 Jump diffusion process

The Lévy process that we consider are of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dZ_t \tag{2.9}$$

where $Z_t = \sum_{i=1}^{N_t} Y_i$, Y_i are i.i.d, $\{N_t, t \geq 0\}$ is a Poisson process with rate λ. Y_i is the size of ith jump and Y_i are independent of N_t.

Theorem 7. *(Itô formula for jump diffusion processes) Let X be defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i \tag{2.10}$$

where b_t and σ_t are continuous non-anticipating processes with

$$\mathbb{E}[\int_0^T \sigma_t^2 dt] < \infty.$$

Then for any $C^{1,2}$ function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ the process $Y_t = f(t, X_t)$ can be represented as:

$$\begin{aligned} f(t,X_t) - f(t,X_0) &= \int_0^t \Big[\frac{\partial f}{\partial s}(s,X_s) + b_s \frac{\partial f}{\partial x}(s,X_s)\Big] ds \\ &+ \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)\sigma_s dW_s \\ &+ \sum_{i\geq 1, T_i \leq t} \Big[f(X_{T_i-} + \Delta X_i) - f(X_{T_i-}) - \Delta X_i f(X_{T_i-})\Big]. \end{aligned} \tag{2.11}$$

For more discussion on jump diffusion process see [13], [24]. We now discuss different distributions for Y_i in (2.9).

- (2.9) becomes an asymmetric double exponential jump diffusion model (DEJP) if Y_i has the distribution of the 'jump size' given by

$$f_Y(x) \quad = \quad p.\eta_1 e^{-\eta_1 x}\chi_{\{x\geq 0\}} + q.\eta_2 e^{\eta_2 x}\chi_{\{x<0\}} \tag{2.12}$$

with $\eta_i > 0$ with $p + q = 1$.

- (2.9) becomes a hyper exponential jump diffusion model (HEM) if Y_i has the distribution of the 'jump size' given by

$$f_Y(x) = \sum_{i=1}^{m} p_i\eta_i e^{-\eta_i x}\chi_{\{x\geq 0\}} + \sum_{j=1}^{n} q_i\theta_j e^{\theta_j x}\chi_{\{x<0\}}$$

with $\eta_i > 1, \theta > 0, p_i, q_i > 0$ with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$.

- (2.9) becomes a mixed exponential jump diffusion model (MEJP) if Y_i has the distribution of the 'jump size' given by

$$f_Y(x) = p_u\sum_{i=1}^{m} p_i\eta_i e^{-\eta_i x}\chi_{\{x\geq 0\}} + q_d\sum_{j=1}^{n} q_i\theta_j e^{\theta_j x}\chi_{\{x<0\}}$$

where $p_u \geq 0$, $q_d = 1-p_u \geq 0$. With $p_i \in (-\infty, \infty)$, $\forall i = 1, 2, 3, ..., m-1, m$ with $\sum_{i=1}^{m} p_i = 1$ and $q_j, p_j \in (-\infty, \infty)$, $\forall j = 1, 2, 3, ..., n-1, n$ with $\sum_{i=1}^{n} q_j = 1$. We would also want $p_i > 0, q_i > 0$ and $\sum_{i=1}^{m} p_i\eta_i \geq 0, \sum_{i=1}^{n} q_j\theta_j \geq 0$ for f to remain a density function.

Class of hyper-exponential distribution is rich enough to approximate many heavy tailed distribution, power tailed distribution in the sense of weak distribution. HEM is flexible enough to incorporate the uncertainty of the heaviness of the asset return tails therefore can capture the leptokurtic feature. (leptokurtic feature = fat tails + kurtosis). For further discussion on HEM, MEJP, DEJP please see [10], [11], [29].

1.3 Option pricing

Below we discuss results of option pricing in the jump diffusion model when the underlying process is asymmetric double exponential process or hyper exponential process. In this section we discuss the works of Steven Kou, Hui Wang and Ning Cai which can be found in [29], [28], [31], [10]. They have worked extensively on jump diffusion models.

1.3.1 Pricing European call option under DEJP

We consider the model described by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) + d\Big(\sum_{i=1}^{N_t}(V_i - 1)\Big) \tag{3.13}$$

where $W(t)$ is a standard Brownian motion, N_t is a Poisson process with rate λ, and V_i is a sequence of independent identically distributed (i.i.d.) positive random variables such that $Y = log(V)$ has an asymmetric double exponential distribution with the density given by (2.12) where $p, q \geq 0,\ p+q = 1$, represent the probabilities of upward and downward jumps. Here $N(t), W(t), Y(t)$ are assumed to be independent. The coefficients are assumed to be constants. Solving above equation using the Itô formula we get

$$S(t) = S(0)\exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right)\prod_{i=1}^{N(t)} V_i \tag{3.14}$$

with $\mathbb{E}(Y) = \frac{p}{\eta_1} - \frac{q}{\eta_2}$, $Var(Y) = pq(\frac{1}{\eta_1} + \frac{1}{\eta})^2 + (\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2})$ and

$$\begin{aligned}\mathbb{E}(V) &= \mathbb{E}(\exp(Y)) \\ &= q\frac{\eta_2}{\eta_2+1} + p\frac{\eta_1}{\eta_1 - 1}, \quad \eta_1 > 1, \eta_2 > 0.\end{aligned}$$

The requirement $\eta_1 > 1$ is needed to ensure that $\mathbb{E}(V) < \infty$ which basically means that the average upward jump cannot exceed 100%. To discuss the price of European call option for the above model we introduce

$$\Gamma(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) = \mathbb{P}[Z(T) \geq a]$$

where $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i$ where Y has an asymmetric double exponential distribution described by (2.12), $N(t)$ is a Poisson process with rate λ and

$$\begin{aligned}
\Gamma(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) &= \frac{\exp((\sigma\eta_1)^2 T/2)}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k}(\sigma\sqrt{T}\eta_1)^k \\
&\quad \times I_{k-1}(a - \mu T; -\eta_1, -\frac{1}{\sigma\sqrt{T}}, -\sigma\eta_1\sqrt{T}) \\
&\quad + \frac{\exp((\sigma\eta_2)^2 T/2)}{\sigma\sqrt{2\pi T}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k}(\sigma\sqrt{T}\eta_2)^k \\
&\quad \times I_{k-1}(a - \mu T; \eta_2, -\frac{1}{\sigma\sqrt{T}}, -\sigma\eta_2\sqrt{T}) \\
&\quad + \pi_0 \Phi(-\frac{a - \mu T}{\sigma\sqrt{T}})
\end{aligned} \tag{3.15}$$

where $P_{n,k}, Q_{n,k}$ are given by

$$P_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} . \Big(\frac{\eta_1}{\eta_1 + \eta_2}\Big)^{i-k} \Big(\frac{\eta_2}{\eta_1 + \eta_2}\Big)^{i-k} p^i q^{n-i}$$

$$Q_{n,k} = \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} . \Big(\frac{\eta_1}{\eta_1 + \eta_2}\Big)^{n-i} \Big(\frac{\eta_2}{\eta_1 + \eta_2}\Big)^{i-k} p^i q^{n-i}$$

and for $\beta > 0$ and $\alpha \neq 0$, I_n is given by

$$\begin{aligned}
I_n(c; \alpha, \beta, \delta) &= -\frac{e^{ac}}{\alpha} \sum_{i=0}^{n} \Big(\frac{\beta}{\alpha}\Big)^{n-i} Hh_i(\beta c - \delta) \\
&\quad + \Big(\frac{\beta}{\alpha}\Big) \frac{\sqrt{2\pi}}{\beta} \exp\Big(\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}\Big) \Phi\Big(-\beta c + \delta + \frac{\alpha}{\beta}\Big)
\end{aligned}$$

and for $\beta < 0$ and $\alpha < 0$, I_n is given by

$$\begin{aligned} I_n(c;\alpha,\beta,\delta) &= -\frac{e^{ac}}{\alpha}\sum_{i=0}^{n}\Big(\frac{\beta}{\alpha}\Big)^{n-i} Hh_i(\beta c-\delta) \\ &\quad -\Big(\frac{\beta}{\alpha}\Big)\frac{\sqrt{2\pi}}{\beta}\exp\Big(\frac{\alpha\delta}{\beta}+\frac{\alpha^2}{2\beta^2}\Big)\Phi\Big(\beta c-\delta-\frac{\alpha}{\beta}\Big) \end{aligned}$$

where

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(y)dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-t^2/2}dt, \qquad n = 0, 1, 2, \cdots, .$$

We now state the main result to compute the price of European call option under DEJP. For proof and further discussion see [31], [29].

Theorem 8. *The price of European call option, $V_c(0)$ for the model* (3.13) *with jumps* (2.12) *is given by*

$$\begin{aligned} &V(0) = S(0)\Gamma(r+\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta_1},\tilde{\eta_2};\log(K/S(0)),T) \\ &-K\exp(-rT)\Gamma(r-\frac{1}{2}\sigma^2-\lambda\zeta,\sigma,\lambda,p,\eta_1,\eta_2;\log(K/S(0)),T) \end{aligned}$$

where

$$\tilde{p} = \frac{p}{1+\zeta}\cdot\frac{\eta_1}{\eta_1-1}, \quad \tilde{\eta_1} = \eta_1 - 1$$

$$\tilde{\eta_2} = \eta_2+1, \quad \tilde{\lambda} = \lambda(\zeta+1), \quad \zeta = \frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1.$$

1.3.2 Pricing Asian options under Black-Scholes model (BSM)

In this subsection we price Asian option under BSM via Laplace transform. We first briefly discuss infinitesimal generator and Lévy exponent.

- In BSM under risk neutral measure the return process modelled by $\{X(t) = \log(\frac{S(t)}{S(0)}) : t \geq 0\}$ is given by

$$X(t) = (r - \frac{\sigma^2}{2})t + \sigma W(t), \;\; X(0) = 0$$

where r is the risk free rate, σ is the volatility, W_t is the standard Brownian Motion. The infinitesimal generator of $S(t)$ is

$$Lf(s) = \frac{\sigma^2}{2}s^2 f''(s) + rsf'(s).$$

- The Lévy exponent of X_t is

$$G(x) = \frac{\mathbb{E}[e^{xX(t)}]}{t} = \frac{\sigma^2 x^2}{2} + (r - \frac{\sigma^2}{2})x$$

- Let α_1, α_2 be the roots of $G(x) = \mu$ in BSM then $\alpha_1, \alpha_2 = \frac{-\bar{\nu} \mp \sqrt{\bar{\nu}^2 + 2\bar{\mu}}}{2}$ with $\alpha_1 > 0, \alpha_2 < 0$ where $\bar{\mu} = \frac{4\mu}{\sigma^2}$, and $\bar{\nu} = \frac{2r}{\sigma^2} - 1$.

- We consider the following non-homogeneous ODE

$$Ly(s) = (s + \mu)y(s) - \mu, \quad s \geq 0. \tag{3.16}$$

(3.16) has infinitely many solutions as it has two singularities $0, \infty$. If solutions are bounded then it is unique.

We now state a few results without proofs which helps us in pricing the Asian option under BSM. For further details and discussion see [10].

Theorem 9. *A bounded solution to* (3.16)*, if exists must be unique. More precisely let* $a(s)$ *solve the ODE* (3.16) *and* $\sup_{s \in [0,\infty)} |a(s)| \leq C < \infty$ *for some* *positive C. Then we must have*

$$a(s) = \mathbb{E}[\exp(-sA_{T_\mu})] \tag{3.17}$$

where $A_{T_\mu} = \int_0^{T_\mu} e^{X(s)} ds.$

- Theorem 9 implies that if we can find a particular bounded solution to the ODE (3.16), it must have the stochastic representation in (3.17). To find such a one, consider a difference equation for a function $H(\nu)$ defined on $(-1, \alpha_1)$

$$\begin{aligned} & h(\nu)H(\nu) = \nu H(\nu - 1) \\ & \text{with } h(\nu) = \mu - G(\nu) = \frac{-\sigma^2}{2}(\nu - \alpha_1)(\nu - \alpha_2) \end{aligned} \tag{3.18}$$

Theorem 10. *If there exist a non-negative random variable X such that* $H(\nu) = \mathbb{E}[X^\nu]$ *satisfies* (3.18) *then the Laplace transform of X i.e* $\mathbb{E}[e^{-sX}]$*, solves the non-homogeneous ODE* (3.16)*.*

Theorem 11. *Under BSM we have*

$$A_{T_\mu} =_d \frac{2Z(1, -\alpha_2)}{\sigma^2 Z(\alpha_1)}$$

and therefore

$$\mathbb{E}[A_{T_\mu}^\nu] = \frac{2}{\sigma^2} \frac{\Gamma(\nu+1)\Gamma(1-\alpha_1)\Gamma(\alpha_1-\nu)}{\Gamma(-\alpha+\nu+1)\Gamma(\alpha_1)}, \quad \forall \nu \in (-1, \alpha_1)$$

here $Z(a, b)$ *denotes a beta random variable and* $Z(a)$ *is a gamma random variable with scale 1 and parameter 'a' and* $=_d$ *denotes equality in distribution.* $\Gamma()$ *is the gamma function. Moreover* $Z(1, -\alpha_2), Z(\alpha_1)$ *are independent.*

We now finally state the main result of this subsection where we price Asian option under BSM.

Theorem 12. *Under the BSM, for every μ, ν such that $\mu > 0$ and $\nu \in (0, \alpha_1 - 1)$ the double Laplace transform of $X\mathbb{E}(\frac{S_0}{X}A_t - e^{-k})^+$ with respect to t, k is given by*

$$\mathcal{L}(\mu, \nu) = \tfrac{X}{\mu\nu(\nu+1)}\left(\tfrac{2S_0}{X\sigma^2}\right)^{\nu+1}\tfrac{\Gamma(\nu+2)\Gamma(\alpha_1-\nu-1)\Gamma(1-\alpha_2)}{\Gamma(-\alpha+\nu+2)\Gamma(\alpha_1)}$$

Therefore the Asian option price is equal to

$$P(t, k) = \frac{e^{-rt}}{t}\mathcal{L}^{-1}(\mathcal{L}(\mu, \nu))\Big|_{k=\ln(\frac{X}{Kt})}$$

where K is the strike price and $\mathcal{L}^{-1}$ a function of t, k is the inverse Laplace of $\mathcal{L}$. Furthermore we can also find the common greeks.

1.3.3 Pricing Asian options under Hyper Exponential Jump diffusion model (HEM)

Let asset return process $\{X_t, t > 0\}$ under the risk neutral measure is given by

$$X(t) = (r - \frac{\sigma^2}{2} - \lambda\zeta)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i, \quad X(0) = 0$$

where r is the risk free rate σ the volatility $\zeta = \mathbb{E}[e^{Y_1}] - 1 = \sum_{i=1}^{m} \frac{p_i\eta_i}{\eta_i - 1} + \sum_{j=1}^{n} \frac{q_j\theta_j}{\theta_j + 1} - 1$ with $\{W(t) : t \geq 0\}$ the standard Brownian Motion, $\{N(t) : t \geq 0\}$ a Poisson process with rate λ and $\{Y_i : i \in \mathbb{N}\}$ are i.i.d. with density

$$f_Y(x) = \sum_{i=1}^{m} p_i\eta_i e^{-\eta_i x}\chi_{\{x\geq 0\}} + \sum_{j=1}^{n} q_j\theta_j e^{\theta_j x}\chi_{\{x<0\}}$$

where $\eta_i > 1, \theta > 0, p, q > 0$ with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$.

Due to jumps the risk neutral measure is not unique. We assume the risk neutral measure is chosen within the rational expectations equilibrium setting such that equilibrium option of a price of an option is given by expectation under $\mathbb{P}$ of the discounted option payoff.

- The Lévy Exponent of $\{X_t\}$ is given by

$$\begin{aligned} G(x) &= \frac{\mathbb{E}[e^{xX(t)}]}{t} \\ &= \frac{\sigma^2 x^2}{2} + (r - \frac{\sigma^2}{2} - \lambda\zeta)x + \lambda\Big(\sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1\Big) \end{aligned}$$

  for any $x \in (-\theta, \eta_1)$ and for $G(x) = \mu$ we have exactly (m+n+2) roots.

- The infinitesimal generator is given by

$$Lf(x) = \frac{\sigma^2}{2} s^2 f''(s) + (r - \lambda\zeta) s f'(s) + \lambda \int_{-\infty}^{\infty} [f(se^u) - f(s)] f_Y(u) du \quad (3.19)$$

- We consider the Ordinary Integro-Differential Equation (OIDE)

$$Ly(s) = (s + \mu) y(s) - \mu \quad (3.20)$$

  where L is as given above in (3.19).

We now state a few results without proofs which helps us in pricing the Asian option under HEM. For further details and discussion see [10]. We now describe the distribution of A_{T_μ}

Theorem 13. *A bounded solution to* (3.20), *if exists must be unique. More precisely let $a(s)$ solve the ODE* (3.20) *and* $\sup_{s\in[0,\infty)} |a(s)| \leq C < \infty$ *for some positive C. Then we must have*

$$a(s) = \mathbb{E}[\exp(-sA_{T_\mu})].$$

Theorem 14. *If there exist a non-negative random variable X such that $H(\nu) = \mathbb{E}[X^{\nu}]$ satisfies* (3.19) *then the Laplace transform of X i.e $\mathbb{E}[e^{-sX}]$, solves the non-homogeneous ODE* (3.20)*.*

Theorem 15. *Under HEM we have*

$$A_{T_\mu} =_d \frac{2Z(1, -\gamma_1) \prod_{j=1}^{n} Z(\theta_j + 1, -\gamma_{j+1} - \theta_j)}{\sigma^2 Z(\beta_{m+1}) \prod_{i=1}^{n} Z(\beta_i, \eta_i - \beta_i)}$$

where all the gamma and beta random variable on RHS are independent and therefore $\nu \in (-1, \beta_1)$

$$\mathbb{E}[A_{T_\mu}^{\nu}] = \Big(\frac{2}{\sigma^2}\Big)^{\nu} \frac{\Gamma(\nu+1)\Gamma(1-\gamma_1)}{\Gamma(-\gamma+\nu+1)} . \prod_{j=1}^{n} \Big[\frac{\Gamma(\theta_j+1+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(-\gamma_{j+1}+1+\nu)\Gamma(1+\theta_j)}\Big]$$
$$. \prod_{i=1}^{m} \Big[\frac{\Gamma(\beta_i-\nu)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu)\Gamma(\beta_i)}\Big] . \frac{\Gamma(\beta_{m+1}-\nu)}{\Gamma(\beta_{m+1})} . \tag{3.21}$$

We now finally state the main result of this subsection where we price Asian option under HEM.

Theorem 16. *Under the HEM, for every μ, ν such that $\mu > 0$ and $\nu \in (0, \beta_1 - 1)$ the double Laplace transform of $X\mathbb{E}(\frac{S_0}{X}A_t - e^{-k})^+$ with respect to t, k is given by*

$$\mathcal{L}(\mu, \nu) = \frac{X}{\mu\nu(\nu+1)} \frac{\Gamma(\nu+2)\Gamma(1-\gamma_1)}{\Gamma(-\gamma+\nu+2)} . \prod_{j=1}^{n} \Big[\frac{\Gamma(\theta_j+2+\nu)\Gamma(1-\gamma_{j+1})}{\Gamma(-\gamma_{j+1}+2+\nu)\Gamma(1+\theta_j)}\Big]$$
$$. \prod_{i=1}^{m} \Big[\frac{\Gamma(\beta_i-\nu-1)\Gamma(\eta_i)}{\Gamma(\eta_i-\nu-1)\Gamma(\beta_i)}\Big] . \frac{\Gamma(\beta_{m+1}-\nu-1)}{\Gamma(\beta_{m+1})} .$$

Therefore the Asian option price is equal to

$$P(t, k) = \frac{e^{-rt}}{t} \mathcal{L}^{-1}(\mathcal{L}(\mu, \nu))\Big|_{k=\ln(\frac{X}{Kt})}$$

where k is the strike price and $\mathcal{L}^{-1}$ a function of t, k is the inverse Laplace of $\mathcal{L}$. Furthermore we can also find the common greeks.

Chapter 2

General Product formula of Multiple Integrals of Lévy Process

2.1 Introduction

Stochastic analysis of nonlinear functionals of Lévy processes (including Brownian motion and Poisson process) have been studied extensively and found many applications. There have been already many standard books on this topic [4, 40, 41]. In the analysis of Brownian nonlinear functional the Wiener-Itô chaos expansion to expand a nonlinear functional of Brownian motion into the sum of multiple Wiener-Itô integrals is a fundamental contribution to the field. The product formula to express the product of two (or more) multiple integrals as linear combinations of some other multiple integrals is one of the important tools ([20]). It plays an important role in stochastic analysis, e.g. Malliavin calculus ([20, 38]).

The product formula for two multiple integrals of Brownian motion is known since the work of [42, Section 4] and the general product formula can be found for instance in [20, chapter 5]. In this chapter we give a general formula for the product of m multiple integrals of the Poisson random measure associated with (purely jump) Lévy process. The formula is in a compact form and it reduced to the Shigekawa's formula when $m = 2$ and when the Lévy process is reduced to Brownian motion.

When $m = 2$ a similar formula was obtained in [34], where the multiple integrals is with respect to the Lévy process itself (ours is with respect to the associated Poisson random measure which has a better properties). To obtain their formula in [34] Lee and Shih use white noise analysis framework. Here, we have only used the classical framework.

The product formula for multiple Wiener-Itô formula of Brownian motion plays an important role in many applications such as U-statistics [35]. We hope similar things may happen. But we are not pursuing this goal in the current chapter. Our formula is for purely jump Lévy process. It can be combined with the classical formulas [20, 35, 38, 42] to general case.

This chapter is organized as follows. In Section 2.2, we give some preliminaries on Lévy process, the associated Poisson random measure, multiple integrals. We also state our main result in this section. In Section 2.3, we give the proof of the formula.

2.2 Preliminary and main results

Let $T > 0$ be a positive number and let $\{\eta(t) = \eta(t, \omega), 0 \le t \le T\}$ be a Lévy process on some probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, 0 \le t \le T\}$ satisfying the usual condition. This means that $\{\eta(t)\}$ has independent and stationary increment and the sample path is right continuous with left limit. Without loss of generality, we assume $\eta(0) = 0$. If the process $\eta(t)$ has all moments for any time index t, then presumably, one can use the polynomials of the process to approximate any nonlinear functional of the process $\{\eta(t), 0 \le t \le T\}$. However, it is more convenient to use the associated Poisson random measure to carry out the stochastic analysis of these nonlinear functionals.

The jump of the process η at time t is defined by

$$\Delta\eta(t) := \eta(t) - \eta(t-) \quad \text{if } \Delta\eta(t) \neq 0\,.$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we then define the *Poisson random measure*, $N : [0,T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with η by

$$N(t,U) := \sum_{0\leq s\leq t} \chi_U(\Delta\eta(s))\,, \tag{2.1}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of η is defined by

$$\nu(U) := \mathbb{E}[N(1,U)] \tag{2.2}$$

and compensated jump measure $\tilde{N}$ is defined by

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt\,. \tag{2.3}$$

The stochastic integral $\int_{\mathbb{T}} f(s,z)\tilde{N}(ds,dz)$ is well-defined for a predictable process $f(s,z)$ such that $\int_{\mathbb{T}} \mathbb{E}|f(s,z)|^2\nu(dz)ds < \infty$, where and throughout this chapter we use $\mathbb{T}$ to represent the domain $[0,T] \times \mathbb{R}_0$ to simplify notation.

Let

$$\hat{L}^{2,n} := \left(L^2(\mathbb{T}, \lambda \times \nu)\right)^{\otimes n} \subseteq L^2\left(\mathbb{T}^n, (\lambda \times \nu)^n\right)$$

be the space of symmetric, deterministic real functions f such that

$$\|f\|^2_{\hat{L}^{2,n}} = \int_{\mathbb{T}^n} f^2(t_1,z_1,\cdots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n) < \infty\,,$$

where $\lambda(dt) = dt$ is the Lebesgue measure. In the above the symmetry means that

$$f(t_1, z_1, \cdots, t_i, z_i, \cdots, t_j, z_j, \cdots, t_n, z_n) = f(t_1, z_1, \cdots, t_j, z_j, \cdots, t_i, z_i, \cdots, t_n, z_n)$$

for all $1 \le i < j \le n$. For any $f \in \hat{L}^{2,n}$ the multiple Wiener-Itô integral

$$I_n(f) := \int_{\mathbb{T}^n} f(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \tag{2.4}$$

is well-defined. The importance of the introduction of the associated Poisson measure and the multiple Wiener-Itô integrals are in the following theorem which means any nonlinear functional F of the Lévy process η can be expanded as multiple Wiener-Itô integrals.

We now state without proof result of Wiener-Itô chaos expansion for Lévy process. For proof and related examples please see [14].

Theorem 17 (`Wiener-Itô chaos expansion for Lévy process`). *Let $\mathcal{F}_T = \sigma(\eta(t), 0 \le t \le T)$ be σ - algebra generated by the **Lévy** process η.*

Let $F \in L^2(\Omega, \mathcal{F}_T, P)$ be an $\mathcal{F}_T$ measurable square integrable random variable. Then F admits the following chaos expansion:

$$F = \sum_{n=0}^{\infty} I_n(f_n)\,, \tag{2.5}$$

where $f_n \in \hat{L}^{2,n}, n = 1, 2, \cdots$ and where we denote $I_0(f_0) := f_0 = \mathbb{E}(F)$. Moreover, we have

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n\,!\|f_n\|^2_{\hat{L}^{2,n}}\,. \tag{2.6}$$

This chaos expansion theorem is one of the fundamental result in stochastic analysis of Lévy processes. It has been widely studied in particular when η is the Brownian motion (Wiener process). We refer to [20], [38], [40] and references therein for further reading.

To state our main result of this chapter, we need some notation. Fix a positive integer $m \geq 2$. Denote

$$\Upsilon = \Upsilon_m = \{\mathbf{i} = (i_1, \cdots, i_\alpha),\ \alpha = 2, \cdots, m\ 1 \leq i_1 < \cdots < i_\alpha \leq m\} \tag{2.7}$$

where $\alpha = |\mathbf{i}|$ is the length of the multi-index $\mathbf{i}$ (we shall use α, β to denote a natural number). It is easy to see that the cardinality of Υ is $\kappa_m := 2^m - 1 - m$. Denote $\vec{\mathbf{i}} = (\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m})$, which is unordered list of the elements of Υ, where $\mathbf{i}_\beta \in \Upsilon$. We use $\vec{l} = (l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}})$ to denote a multi-index of length κ_m associated with Υ, where $l_{\mathbf{i}_\alpha}$, $1 \leq \alpha \leq \kappa_m$ are nonnegative integers. $\vec{l}$ can be regarded as a function from Υ to $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. Denote

$$\begin{cases} \Omega = \left\{\vec{l}, \vec{n} : \Upsilon \to \mathbb{Z}_+\right\} \qquad \text{and for any } \vec{l},\, \vec{n} \in \Omega\,, \\ \chi(k, \vec{l}, \vec{n}) = \sum_{1 \leq \alpha \leq \kappa_m} \left[l_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}} + n_{\mathbf{i}_\alpha} \chi_{\{\mathbf{i}_\alpha \text{ contains } k\}} \right] . \end{cases} \tag{2.8}$$

Above χ on left side refers to the indicator function. The conventional notations such as $|\vec{l}| = l_{\mathbf{i}_1} + \cdots + l_{\mathbf{i}_{\kappa_m}}$; $\vec{l}! = l_{\mathbf{i}_1}! \cdots l_{\mathbf{i}_{\kappa_m}}!$ and so on are in use. Notice that we use $l_{\mathbf{i}_1}$ instead of l_1 to emphasize that the $l_{\mathbf{i}_1}$ corresponds to $\mathbf{i}_1$. For $\mathbf{i} = (i_1, \cdots, i_\alpha), \mathbf{j} = (j_1, \cdots, j_\beta) \in \Upsilon$, and non negative integers μ and ν denote

$$\begin{aligned} \hat{\otimes}_{\mathbf{i}}^{\mu}(f_1, \cdots, f_m) \;=\; & \int_{([0,T]\times\mathbb{R}_0)^\mu} f_{i_1}((s_1, z_1), \cdots, (s_\mu, z_\mu), \cdots) \hat{\otimes} \cdots \\ & \hat{\otimes} f_{i_\alpha}((s_1, z_1), \cdots, (s_\mu, z_\mu), \cdots) ds_1 \nu(dz_1) \cdots \\ & ds_\mu \nu(dz_\mu)\ f_1 \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \hat{f}_{i_\alpha} \cdots \hat{\otimes} f_m\,, \end{aligned} \tag{2.9}$$

and

$$V_{\mathbf{j}}^{\nu}(f_1,\cdots,f_m) = f_{j_1}((t_1,z_1),\cdots,(t_\nu,z_\nu),\cdots)\hat{\otimes}\cdots$$
$$\hat{\otimes} f_{j_\beta}((t_1,z_1),\cdots,(t_\nu,z_\nu),\cdots) f_1 \hat{\otimes}\cdots\hat{\otimes}\hat{f}_{j_1}\hat{\otimes}\cdots\hat{\otimes}\hat{f}_{j_\beta}\cdots\hat{\otimes} f_m\,, \tag{2.10}$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\hat{f}_{j_1}$ means that the function f_{j_1} is removed from the list. Let us emphasize that both $\hat{\otimes}_{\mathbf{i}}^{\mu}$ and $V_{\mathbf{j}}^{\nu}$ are well-defined when the lengths of $\mathbf{i}$ and $\mathbf{j}$ are one. However, we shall not use $\hat{\otimes}_{\mathbf{i}}^{\mu}$ when $|\mathbf{i}|=1$ and when $|\mathbf{j}|=1$, $V_{\mathbf{j}}^{\nu}(f_1,\cdots,f_m) = f_1\hat{\otimes}\cdots\hat{\otimes} f_m$ (namely, the identity operator). For any two elements $\vec{l}=(l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}})$ and $\vec{n}=(\mu_{\mathbf{j}_1},\cdots,\mu_{\mathbf{j}_{\kappa_m}})$ in Ω, denote

$$\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}} = \hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}} = \hat{\otimes}_{\mathbf{i}_1}^{l_{\mathbf{i}_1}}\cdots\hat{\otimes}_{\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_{\kappa_m}}}\,,\quad V_{\vec{\mathbf{j}}}^{\vec{n}} = V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,n_{j_{\kappa_m}}} = V_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}}\hat{\otimes}\cdots\hat{\otimes} V_{\mathbf{j}_{\kappa_m}}^{n_{\mathbf{j}_{\kappa_m}}}\,. \tag{2.11}$$

Now we can state the main result of the chapter.

Theorem 18. *Let* $f_k \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes} q_k}$, $k=1,\cdots,m$. *Then*

$$\prod_{k=1}^m I_{q_k}(f_k) = \sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\le q_1 \\ \cdots \\ \chi(m,\vec{l},\vec{n})\le q_m}} \frac{\prod_{k=1}^m q_k!}{\prod_{\alpha=1}^{\kappa_m} l_{\mathbf{i}_\alpha}!\prod_{\beta=1}^{\kappa_m}\mu_{\mathbf{j}_\beta}!\prod_{k=1}^m(q_k-\chi(k,\vec{l},\vec{n}))!}$$
$$I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}(\hat{\otimes}_{\mathbf{i}_1,\cdots,\mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1},\cdots,l_{\mathbf{i}_{\kappa_m}}}\hat{\otimes} V_{\mathbf{j}_1,\cdots,\mathbf{j}_{\kappa_m}}^{\mu_{\mathbf{j}_1},\cdots,n_{j_{\kappa_m}}}(f_1,\cdots,f_m))\,, \tag{2.12}$$

where we recall $|q| = q_1+\cdots+q_m$ *and* $|\chi(\vec{l},\vec{n})| = \chi(1,\vec{l},\vec{n})+\cdots+\chi(m,\vec{l},\vec{n})$.

If $m=2$, then $\kappa_m=1$. To shorten the notations we can write $q_1=n$, $q_2=m$, $f_1=f_n$, $f_2=g_m$, $l_{\alpha_1}=l$, $n_{\beta_1}=k$. Thus, $\chi(1,\vec{l},\vec{n})=\chi(2,\vec{l},\vec{n})=l+k$ and $|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})| = n+m+k-2(l+k) = n+m-2l-k$. Hence we have;

if $f_n \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes}n}$, $g_m \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes}m}$. Then

$$I_n(f_n)I_m(g_m) = \sum_{\substack{k,l\in\mathbb{Z}_+\\ k+l\le m\wedge n}} \frac{n!m!}{l!k!(n-k-l)!(m-k-l)!} I_{n+m-2l-k}\Big(f_n \hat{\otimes}_{k,l} g_m\Big),$$

where $\mathbb{Z}_+$ denotes the set of non negative integers and

$$\begin{aligned}
&f_n \hat{\otimes}_{k,l} g_m(s_1, z_1, \cdots, s_{n+m-k-2l}, z_{n+m-k-2l})\\
&= \text{symmetrization of } \int_{\mathbb{T}^l} f_n(s_1, z_1, \cdots, s_{n-l}, z_{n-l}, t_1, y_1, \cdots, t_l, y_l)\\
&\qquad g_m(s_1, z_1, \cdots, s_k, z_k, s_{n-l+1}, \cdots, z_{n-l+1}, \cdots,\\
&\qquad s_{n+m-k-2l}, \cdots, z_{n+m-k-2l}, t_1, z_1, \cdots, t_l, z_l) dt_1\nu(dz_1)\cdots dt_l\nu(dz_l).
\end{aligned} \tag{2.13}$$

If $m = 3$, then $\kappa_m = 4$. To shorten the notations we can write, We write the product formula for $f_1 \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes}q_1}$, $f_2 \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes}q_2}$, $f_3 \in (L^2([0,T]\times\mathbb{R}_0, dt\otimes\nu(dz)))^{\hat{\otimes}q_3}$.

Lets write using the notions discussed

$l_{\mathbf{i}_1} = l_{12}$, $l_{\mathbf{i}_2} = l_{23}$, $l_{\mathbf{i}_3} = l_{13}$, $l_{\mathbf{i}_4} = l_{123}$ and $\vec{l}! = l_{12}!l_{23}!l_{13}!l_{123}!$

$\mu_{\mathbf{j}_1} = k_{12}$, $\mu_{\mathbf{j}_2} = k_{23}$, $\mu_{\mathbf{j}_3} = k_{13}$, $\mu_{\mathbf{j}_4} = k_{123}$ and $\vec{n}! = k_{12}!k_{23}!k_{13}!k_{123}!$.

Thus,

$\chi(1,\vec{l},\vec{n}) = l_{12} + l_{13} + l_{123} + k_{12} + k_{13} + k_{123}$,

$\chi(2,\vec{l},\vec{n}) = l_{12} + l_{23} + l_{123} + k_{12} + k_{23} + k_{123}$,

$\chi(3,\vec{l},\vec{n}) = l_{13} + l_{23} + l_{123} + k_{13} + k_{23} + k_{123}$ and

$$\begin{aligned}
|q| + |\vec{n}| - |\chi(\vec{l},\vec{n})| &= q_1 + q_2 + q_3 - 2l_{12} - 2l_{23} - 2l_{13} - 3l_{123}\\
&\quad -k_{12} + -k_{23} - k_{13} - 2k_{123}.
\end{aligned}$$

Hence we have;

$$I_{q_1}(f_1)I_{q_2}(f_2)I_{q_3}(f_3) = \sum_{\substack{\vec{l},\vec{n}\in\Omega \\ \chi(1,\vec{l},\vec{n})\leq q_1 \\ \chi(2,\vec{l},\vec{n})\leq q_2 \\ \chi(3,\vec{l},\vec{n})\leq q_3}} \frac{q_1!q_2!q_3}{\vec{l}!\vec{n}!\prod_{r=1}^{3}(q_i-\chi(i,\vec{l},\vec{n}))!} \cdot I_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}\left((\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\ \hat{\otimes}\ V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3))\right),$$

$$\begin{aligned}
&\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}}\ \hat{\otimes}\ V_{\vec{\mathbf{j}}}^{\vec{n}}(f_1,f_2,f_3)(s_1,z_1,\cdots,s_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|},z_{|q|+|\vec{n}|-|\chi(\vec{l},\vec{n})|}) \\
&= \text{symmetrization of } \int_{\mathbb{T}^{|\vec{l}|}} f_1(s_1,z_1,\cdots,s_{(q_1-l_{12}+l_{13}+l_{123})},z_{(q_1-l_{12}+l_{13}+l_{123})}, \\
&t_1,y_1,\cdots,t_{l_{12}},y_{l_{12}}t_{l_{12}+1},y_{l_{12}+1}\cdots t_{l_{12}+l_{13}},y_{l_{12}+l_{13}},t_{l_{12}+l_{13}+1},y_{l_{12}+l_{13}+1}, \\
&\cdots t_{l_{12}+l_{13}+l_{123}},y_{l_{12}+l_{13}+l_{123}})\cdot f_2(s_1,z_1,\cdots,s_{k_{12}},z_{k_{12}},s_{k_{12}+1},z_{k_{12}+1}, \\
&\cdots s_{k_{12}+k_{123}},z_{k_{12}+k_{123}},s_{q_1+k_{12}+k_{123}+1},z_{q_1+k_{12}+k_{123}+1}, \\
&\cdots s_{q_1+k_{12}+k_{123}+k_{23}},z_{q_1+k_{12}+k_{123}+k_{23}},s_{q_1+k_{12}+k_{123}+k_{23}+1},z_{q_1+k_{12}+k_{123}+k_{23}+1}, \\
&\cdots,\cdots,s_{q_1+q_2-l_{12}-l_{23}-l_{123}},z_{q_1+q_2-l_{12}-l_{23}-l_{123}},t_1,z_1, \\
&\cdots,t_{l_{123}},z_{l_{123}},t_{q_1+q_2+1},z_{q_1+q_2+1},\cdot,t_{q_1+q_2+l_{23}},z_{q_1+q_2+l_{23}}) \\
&\cdot f_3(s_{k_{12}+1},z_{k_{12}+1},\cdots s_{k_{12}+k_{13}},z_{k_{12}+k_{13}},s_{k_{12}+k_{13}+1},z_{k_{12}+k_{13}+1}, \\
&\cdots s_{k_{12}+k_{123}+k_{13}},z_{k_{12}+k_{123}+k_{13}},s_{q_1+q_2+k_{12}+k_{123}+1}, \\
&z_{q_1+q_2+k_{12}+k_{123}+1},\cdots s_{q_1+q_2+k_{12}+k_{123}+k_{23}},z_{q_1+q_2+k_{12}+k_{123}+k_{23}}, \\
&s_{q_1+q_2+k_{12}+k_{123}+k_{23}+1},z_{q_1+q_2+k_{12}+k_{123}+k_{23}+1},\cdots s_{q_1+q_2+q_3-l_{23}-l_{123}-l_{13}}, \\
&z_{q_1+q_2+q_3-l_{23}-l_{123}-l_{13}},\cdots t_{l_{12}+1},y_{l_{12}+1}t_{l_{12}+l_{13}},y_{l_{12}+l_{13}},t_{l_{12}+l_{13}+1},y_{l_{12}+l_{13}+1}, \\
&\cdots t_{l_{12}+l_{13}+l_{123}},y_{l_{12}+l_{13}+l_{123}},t_{q_1+q_2+1},z_{q_1+q_2+1}, \\
&\cdots t_{q_1+q_2+l_{23}},z_{q_1+q_2+l_{23}})dt_1\nu(dz_1)\cdots dt_{l_{12}}\nu(dz_{l_{12}}),dt_{l_{12}+1}\cdots\nu(dz_{l_{12}+1})dt_{l_{12}+l_{13}} \\
&\nu(dz_{l_{12}+l_{13}})dt_{l_{12}+l_{13}+1}\nu(dz_{l_{12}+l_{13}+1})\cdots dt_{l_{12}+l_{13}+l_{123}}\nu(dz_{l_{12}+l_{13}+l_{123}}) \\
&dt_{q_1+q_2+1},\nu(dz_{q_1+q_2+1})\cdots dt_{q_1+q_2+l_{23}},\nu(dz_{q_1+q_2+l_{23}}).
\end{aligned} \tag{2.14}$$

Remark 2.1. *(1) When η is the Brownian motion, the product formula* (2.13) *is known since [42] (see e.g. [20, Theorem 5.6] for a formula of the general form* (2.12)*) and is given by*

$$I_n(f_n)I_m(g_m) = \sum_{l=0}^{n\wedge m} \frac{n!m!}{l!(n-l)!(m-l)!} I_{n+m-2l}\Big(f_n \hat{\otimes}_l g_m\Big) . \tag{2.15}$$

It is a "special case" of (2.13) *when* $k = 0$.

2.3 Proof of Theorem 18

We shall prove the main result (Theorem 18). We shall prove this by using the polarization technique (see [20, Section 5.2]). First, let us find the Wiener-Itô chaos expansion for the *exponential functional* (random variable) of the form

$$\begin{aligned} Y(T) &= \mathcal{E}(\rho(s,z)) \\ &:= \exp\left\{\int_{\mathbb{T}} \rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}} \Big(e^{\rho(s,z)} - 1 - \rho(s,z)\Big)\nu(dz)ds\right\} \end{aligned} \tag{3.16}$$

where $\rho(s,z) \in \hat{L}^2 := \hat{L}^{2,1} = L^2(\mathbb{T}, \nu(dz) \otimes \lambda(dt))$. An application of Itô formula (see e.g. [40]) yields

$$Y(T) = 1 + \int_0^T \int_{\mathbb{R}_0} Y(s-)\Big[\exp\left(\rho(s,z)\right) - 1\Big]\tilde{N}(ds,dz) .$$

Repeatedly using this formula, we obtain the chaos expansion of $Y(T)$ as follows.

$$\begin{aligned}\mathcal{E}(\rho(s,z)) &= \exp\left\{\int_{\mathbb{T}}\rho(s,z)\tilde{N}(dz,ds) - \int_{\mathbb{T}}\left(e^{\rho(s,z)} - 1 - \rho(s,z)\right)\nu(dz)ds\right\} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}I_n(f_n)\,, \end{aligned} \tag{3.17}$$

where the convergence is in $L^2(\Omega, \mathcal{F}_T, P)$ and

$$f_n = f_n(s_1, z_1, \cdots, s_n, z_n) = (e^{\rho} - 1)^{\hat{\otimes}n} = \prod_{i=1}^{n}\left(e^{\rho(s_i,z_i)} - 1\right). \tag{3.18}$$

We shall first make critical application of the above expansion formula (3.17)-(3.18). For any functions $p_k(s,z) \in \hat{L}^2$ (in what follows when we write k we always mean $k = 1, 2, \cdots, m$ and we shall omit $k = 1, 2, \cdots, m$), we denote

$$\rho_k(u_k, s, z) = \log(1 + u_k p_k(s,z))\,, \tag{3.19}$$

From (3.17)-(3.18), we have (consider u_k as fixed real numbers)

$$\mathcal{E}(\rho_k(u_k, s, z)) = \sum_{n=0}^{\infty}\frac{1}{n!}u_k^n I_n(f_{k,n})\,, \tag{3.20}$$

where

$$f_{k,n} = \frac{1}{u_k^n}\prod_{i=0}^{n}(e^{\rho_k(u_k,s_i,z_i)} - 1) = p_k^{\otimes n} = \prod_{i=1}^{n}p_k(s_i, z_i) \tag{3.21}$$

It is clear that

$$\prod_{k=1}^{m}\mathcal{E}(\rho_k(u_k, s, z)) = \sum_{q_1,\cdots,q_m=0}^{\infty}\frac{1}{q_1!\cdots q_m!}u_1^{q_1}\cdots u_m^{q_m}I_{q_1}(f_{1,q_1})\cdots I_{q_m}(f_{m,q_m}) \tag{3.22}$$

where f_{k,q_k}, $k = 1, \cdots, m$ are defined by (3.21). On the other hand, from the definition of the exponential functional (3.16), we have

$$
\begin{aligned}
&\prod_{k=1}^{m} \mathcal{E}(\rho_k(u_k, s, z)) \\
&= \prod_{k=1}^{m} \exp\Big\{ \int_{\mathbb{T}} \rho_k(u_k, s, z)\tilde{N}(dz, ds) - \int_{\mathbb{T}} \Big(e^{\rho_k(u_k,s,z)} - 1 - \rho_k(u_k, s, z)\Big)\nu(dz)ds\Big\} \\
&= \exp\Big\{ \int_{\mathbb{T}} \sum_{k=1}^{m} \rho_k(u_k, s, z)\tilde{N}(dz, ds) \\
&- \int_{\mathbb{T}} \Big(e^{\sum_{k=1}^{m} \rho_k(u_k,s,z)} - 1 - \sum_{k=1}^{m} \rho_k(u_k, s, z)\Big)\nu(dz)ds\Big\} \\
&\cdot \exp\Big\{ \int_{\mathbb{T}} e^{\sum_{k=1}^{m} \rho_k(u_k,s,z)} - \sum_{k=1}^{m} e^{\rho_k(u_k,s,z)} + m - 1\Big)\nu(dz)ds\Big\} \\
&=: A \cdot B
\end{aligned}
\tag{3.23}
$$

where A and B denote the above first and second exponential factors.

The first exponential factor A is an exponential functional of the form (3.16). Thus, again by the chaos expansion formula (3.17)-(3.18), we have

$$
A = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(u_1, \cdots, u_m))\,, \tag{3.24}
$$

where

$$
h_n(u_1, \cdots, u_m) = \prod_{i=0}^{n} (e^{\sum_{k=1}^{m} \rho_k(u_k,s_i,z_i)} - 1)\,. \tag{3.25}
$$

By the definition of ρ_k, we have

$$
\sum_{k=1}^{m} \rho_k(u_k, s_i, z_i) \quad = \quad \log \prod_{k=1}^{m} (1 + u_k p_k(s_i, z_i))\,.
$$

Or

$$
\begin{aligned}
h_n(u_1, \cdots, u_m) &= \left(\left[\prod_{k=1}^{m} (1 + u_k p_k) - 1 \right] \right)^{\hat{\otimes} n} \\
&= \mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)} \prod_{i=1}^{n} \left[\prod_{k=1}^{m} (1 + u_k p_k(s_i, z_i)) - 1 \right] ,
\end{aligned}
$$

where $\hat{\otimes}$ denotes the symmetric tensor product and $\mathrm{Sym}_{(s_1,z_1),\cdots,(s_n,z_n)}$ denotes the symmetriization with respect to $(s_1, z_1), \cdots, (s_n, z_n)$. Define

$$
S = \{ \mathbf{j} = (j_1, \cdots, j_\beta),\ \beta = 1, \cdots, m,\ 1 \le j_1 < \cdots < j_\beta \le m \} .
$$

The cardinality of S is $|S| = \tilde{\kappa}_m := 2^m - 1$. We shall freely use the notations introduced in Section 2. Denote also

$$
u_{\mathbf{j}} = u_{j_1} \cdots u_{j_\beta} , \quad p_{\mathbf{j}}(s,z) = p_{j_1}(s,z) \cdots p_{j_\beta}(s,z) \quad (\text{for } \mathbf{j} = (j_1, \cdots, j_\beta) \in S) .
$$

We have

$$
\begin{aligned}
h_n(u_1, \cdots, u_m) &= \left(\sum_{\mathbf{j} \in S} u_{\mathbf{j}} p_{\mathbf{j}} \right)^{\hat{\otimes} n} = \sum_{|\vec{\mu}| = n} \frac{|\vec{\mu}|!}{\vec{\mu}!} u_{\vec{\mathbf{j}}}^{\vec{\mu}} p_{\vec{\mathbf{j}}}^{\hat{\otimes} \vec{\mu}} \\
&= \sum_{\mu_{\mathbf{j}_1} + \cdots + \mu_{\mathbf{j}_{\tilde{\kappa}_m}} = n} \frac{n!}{\mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} p_{\mathbf{j}_1}^{\hat{\otimes} \mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes} \mu_{\mathbf{j}_{\tilde{\kappa}_m}}} ,
\end{aligned}
$$

where $\vec{\mu} : S \to \mathbb{Z}_+$ is a multi-index and we used the notation $u_{\vec{\mathbf{j}}}^{\vec{\mu}} = u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}}$; and $p_{\vec{\mathbf{j}}}^{\hat{\otimes} \vec{\mu}} = p_{\mathbf{j}_1}^{\hat{\otimes} \mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes} \mu_{\mathbf{j}_{\tilde{\kappa}_m}}}$. Inserting the above expression into (3.24) we have

$$
A = \sum_{n=0}^{\infty} \sum_{\mu_{\mathbf{j}_1} + \cdots + \mu_{\mathbf{j}_{\tilde{\kappa}_m}} = n} \frac{1}{\mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_{\tilde{\kappa}_m}}} I_n(p_{\mathbf{j}_1}^{\hat{\otimes} \mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa}_m}}^{\hat{\otimes} \mu_{\mathbf{j}_{\tilde{\kappa}_m}}}) \tag{3.26}
$$

Now we consider the second exponential factor in (3.23):

$$\begin{aligned} B &= \exp\Big\{ \int_{\mathbb{T}} \Big(e^{\sum_{k=1}^m \rho_k(u_k,s,z)} - \sum_{k=1}^m e^{\rho_k(u_k,s,z)} + m - 1 \Big) \nu(dz)ds \Big\} \\ &= \exp\Big\{ \sum_{\mathbf{i}\in\Upsilon} u_{\mathbf{i}} \int_{\mathbb{T}} p_{\mathbf{i}}(s,z)\nu(dz)ds \Big\}, \end{aligned}$$

where Υ is defined by (2.7) (which is a subset of S such that $|\mathbf{j}| \ge 2$). Thus,

$$\begin{aligned} B &= \sum_{n=0}^\infty \frac{1}{n!} \left(\sum_{\mathbf{i}\in\Upsilon} u_{\mathbf{i}} \int_{\mathbb{T}} p_{\mathbf{i}}(s,z)\nu(dz)ds \right)^n \\ &= \sum_{n=0}^\infty \sum_{l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa m}}=n} \frac{1}{l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa m}}!} u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}} \cdots u_{\mathbf{i}_{\kappa m}}^{l_{\mathbf{i}_{\kappa m}}} \left(\int_{\mathbb{T}} p_{\mathbf{i}_1}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_1}} \\ &\qquad \cdots \left(\int_{\mathbb{T}} p_{\mathbf{i}_{\kappa m}}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_{\kappa m}}}, \end{aligned} \tag{3.27}$$

where $\vec{l} \in \Omega$ is a multi-index. Combining (3.26)-(3.27), we have

$$\begin{aligned} AB = \sum_{n,\tilde{n}=0}^\infty & \sum_{\substack{\mu_{\mathbf{j}_1}+\cdots+\mu_{\mathbf{j}_{\tilde{\kappa} m}}=n \\ l_{\mathbf{i}_1}+\cdots+l_{\mathbf{i}_{\kappa m}}=\tilde{n}}} \frac{1}{\mu_{\mathbf{j}_1}!\cdots \mu_{\mathbf{j}_{\tilde{\kappa} m}}! l_{\mathbf{i}_1}!\cdots l_{\mathbf{i}_{\kappa m}}!} u_{\mathbf{j}_1}^{\mu_{\mathbf{j}_1}} \cdots u_{\mathbf{j}_{\tilde{\kappa} m}}^{\mu_{\mathbf{j}_{\tilde{\kappa} m}}} \\ & u_{\mathbf{i}_1}^{l_{\mathbf{i}_1}} \cdots u_{\mathbf{i}_{\kappa m}}^{l_{\mathbf{i}_{\kappa m}}} B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}}, \qquad \text{where} \end{aligned} \tag{3.28}$$

$$\begin{aligned} B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}} := & \left(\int_{\mathbb{T}} p_{\mathbf{i}_1}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_1}} \cdots \\ & \left(\int_{\mathbb{T}} p_{\mathbf{i}_{\kappa m}}(s,z)\nu(dz)ds \right)^{l_{\mathbf{i}_{\kappa m}}} I_n(p_{\mathbf{j}_1}^{\hat{\otimes}\mu_{\mathbf{j}_1}} \hat{\otimes} \cdots \hat{\otimes} p_{\mathbf{j}_{\tilde{\kappa} m}}^{\hat{\otimes}\mu_{\mathbf{j}_{\tilde{\kappa} m}}}). \end{aligned} \tag{3.29}$$

To get an expression for $B_{\mathbf{i},\mathbf{j},l_{\mathbf{i}},\mu_{\mathbf{j}}}$ we use the notations (2.9)-(2.10) and (2.11). Then

$$B_{\mathbf{j},\tilde{\mathbf{j}},n_{\mathbf{j}},\tilde{n}_{\mathbf{j}}} = I_n(\hat{\otimes}_{\vec{\mathbf{i}}}^{\vec{l}} \hat{\otimes} V_{\vec{\mathbf{j}}}^{\vec{\mu}}(p_1^{\otimes n_{\mathbf{i}_1}}, \cdots, p_m^{\otimes n_m})). \tag{3.30}$$

To compare the coefficients of $u_1^{n_{\mathbf{i}_1}} \cdots u_m^{n_m}$, we need to express the right hand side of (3.28) as a power series of $u_1, \cdots, u_m$. For $k = 1, \cdots, m$ denote

$$\tilde{\chi}(k, \vec{l}, \vec{\mu}) = \sum_{1 \le \alpha \le \kappa_m} l_{\mathbf{i}_\alpha} I_{\{\mathbf{i}_\alpha \text{ contains } k\}} + \sum_{1 \le \beta \le \tilde{\kappa}_m} \mu_{\mathbf{j}_\beta} I_{\{\mathbf{j}_\beta \text{ contains } k\}} . \tag{3.31}$$

Combining (3.23), (3.28) and (3.30), we have

$$\begin{aligned} &\sum_{q_1, \cdots, q_m = 0}^{\infty} \frac{u_1^{q_1} \cdots u_m^{q_m}}{q_1! \cdots q_m!} I_{q_1}(p_1^{\otimes q_1}) \cdots I_{q_m}(p_m^{\otimes q_m}) \\ &= \sum_{n, \tilde{n} = 0}^{\infty} \sum_{\substack{\mu_{\mathbf{j}_1} + \cdots + \mu_{\mathbf{j}_{\tilde{\kappa}_m}} = n \\ l_{\mathbf{i}_1} + \cdots + l_{\kappa_m} = \tilde{n} \\ \tilde{\chi}(k, \vec{l}, \vec{\mu}) = q_k, k = 1, \ldots, m}} \frac{u_1^{q_1} \cdots u_m^{q_m}}{l_{\mathbf{i}_1}! \cdots l_{\mathbf{i}_{\kappa_m}}! \mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} \\ &\qquad I_n(\hat{\otimes}_{\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1, \cdots, \mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_1}, \cdots, \mu_{\mathbf{j}_{\tilde{\kappa}_m}}}(p_1^{\otimes q_1}, \cdots, p_m^{\otimes q_m})) . \end{aligned} \tag{3.32}$$

Comparing the coefficient of $u_1^{q_1} \cdots u_m^{q_m}$, we have

$$\begin{aligned} \prod_{k=1}^{m} I_{q_k}(p_k^{\otimes q_k}) = &\sum_{\substack{\mathbf{j}_1, \cdots, \mathbf{j}_{\tilde{\kappa}_m} \in S \\ \mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m} \in \Upsilon}} \sum_{\tilde{\chi}(k, \vec{l}, \vec{\mu}) = q_k, k = 1, \ldots, m} \frac{q_1! \cdots q_m!}{l_{\mathbf{i}_1}! \cdots l_{\mathbf{i}_{\kappa_m}}! \mu_{\mathbf{j}_1}! \cdots \mu_{\mathbf{j}_{\tilde{\kappa}_m}}!} \\ &I_n(\hat{\otimes}_{\mathbf{i}_1, \cdots, \mathbf{i}_{\kappa_m}}^{l_{\mathbf{i}_1}, \cdots, l_{\mathbf{i}_{\kappa_m}}} \hat{\otimes} V_{\mathbf{j}_1, \cdots, \mathbf{j}_{\tilde{\kappa}_m}}^{\mu_{\mathbf{j}_1}, \cdots, \mu_{\mathbf{j}_{\tilde{\kappa}_m}}}(p_1^{\otimes q_1}, \cdots, p_m^{\otimes q_m})) . \end{aligned} \tag{3.33}$$

Notice that when $|\mathbf{j}| = 1$, namely, $\mathbf{j} = (k), k = 1, \cdots, m$, then $V_{\mathbf{j}}^{\mu}(f_1, \cdots, f_m) = f_1 \hat{\otimes} \cdots \hat{\otimes} f_m$. We can separate these terms from the remaining ones, which will satisfy $|\mathbf{j}| \ge 2$. Thus, the remaining multi-indices $\mathbf{j}$'s consists of the set Υ. We can write a multi-index $\vec{\mu} : S \to \mathbb{Z}_+$ as $\vec{\mu} = (n_{(1)}, \cdots, n_{(m)}, \vec{n})$, where $\vec{n} \in \Upsilon$. We also observe $q_k = \tilde{\chi}(k, \vec{l}, \vec{\mu}) = n_{(k)} + \chi(k, \vec{l}, \vec{n})$. After replacing $\vec{\mu}$ by $\vec{n}$, (3.33) gives (2.12). This proves Theorem 18 for $f_k = p_k^{\otimes q_k}$, $k = 1, \cdots, m$. By polarization technique (see e.g. [20, Section 5.2]), we also know the identity (2.12) holds true for $f_k = p_{k,1} \otimes \cdots \otimes p_{k,q_k}$, $p_{k,q_k} \in L^2([0,T] \times \mathbb{R}_0, ds \times \nu(dz))$, $k = 1, \cdots, m$. Because both sides of (2.12) are multi-linear with respect to f_k, we know (2.12) holds true for

$$f_k = \sum_{\ell=1}^{\nu_k} c_{k,\ell} p_{k,1,\ell} \otimes \cdots \otimes p_{k,q_k,\ell}\,, \qquad k = 1, \cdots, m\,,$$

where $c_{k,\ell}$ are constants, $p_{k,k',\ell} \in L^2([0,T] \times \mathbb{R}_0, ds \times \nu(dz))$, $k = 1, \cdots, m, k' = 1, \cdots, q_k$ and $\ell = 1, \cdots, \nu_k$. Finally, the identity (2.12) is proved by a routine limiting argument.

Chapter 3

Option Pricing and Euler-Maruyama scheme for SDDE driven by Brownian Motion

3.1 Introduction

In this chapter we discuss the option pricing formula and Euler-Maruyama scheme convergence on stochastic delay differential equation (SDDE). In this chapter we discuss the work in [5] where the authors came up with a formula in the last delayed period for pricing European options where the underlying model is driven by Brownian motion. We also discuss the work in [22] where the authors derived a formula for pricing European option on jump diffusion model in the last delayed period. Finally we also discuss the Euler-Maruyama scheme convergence for SDDE driven by Brownian motion and Poisson random measure.

3.2 Delayed Black-Scholes formula.

In this section we discuss the work in [5]. We discuss about the explicit formula for pricing European options when the underlying stock price follows nonlinear stochastic functional differential equations with fixed delays. The model maintains the no-arbitrage property and the completeness of the market. The derivation of the option-pricing formula is based on an equivalent local martingale measure. Here the model considers the effect of the past in the determination of the fair price of a call option. In particular, the stock price satisfies a stochastic functional differential equation (SFDE) with delay and pricing of European option is considered.

3.2.1 Stochastic delay model

Consider a stock whose price at time t is given by a stochastic process $S(t)$ satisfying the following SFDE:

$$\begin{cases} dS(t) = \mu S(t-a)S(t)dt + g(S(t-b))S(t)dW(t), & t \geq 0\,, \\ S(t) = \phi(t)\,, \qquad t \in [-L, 0]\,, \end{cases} \tag{2.1}$$

on $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{\{0 \leq t \leq T\}}$ satisfying usual conditions where $L = \max\{a, b\}$ with $a > 0, b > 0$. In the above, L, b and T are positive constants with $L \geq b$. The drift coefficient $\mu > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous. The initial process $\phi : \Omega \to C([-L, 0], \mathbb{R})$ is $\mathcal{F}_0$ measurable with respect to the Borel σ-algebra of $C([-L, 0], \mathbb{R})$ where the space $C([-L, 0], \mathbb{R})$ of all continuous functions $\eta : [-L, 0] \to \mathbb{R}$ is a Banach space with the sup norm. The process W is a one-dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{\{0 \leq t \leq T\}}$.

We now state without proof that the SDDE admits a pathwise-unique positive solution. For further details we refer to [5]

Theorem 19. *Given an $\mathcal{F}_0$ measurable initial process $\phi(t)$, the stochastic delay differential equation given by* (2.1) *with* $\mu > 0$, $g : \mathbb{R} \to \mathbb{R}$ *admits a unique pathwise solution with the property that* $S(t) \geq 0$ *whenever* $\phi(0) \geq 0$ *for all* $t \geq 0$ *almost surely. If in addition* $\phi(0) > 0$ *a.s, then* $S(t) > 0$ *for all* $t \geq 0$ *a.s.*

3.2.2 Delayed option pricing formula

Consider a market consisting of a riskless asset (e.g., a bond or bank account) $B(t)$ with rate of return $r \geq 0$ (i.e., $B(t) = e^{rt}$) and a single stock whose price $S(t)$ at time t satisfies the SDDE (2.1) where $\phi(0) > 0$ *a.s.*. Consider a European option, written on the stock, with maturity at some future time $T > t$ and a strike price K. To price the European option we first look at the following results (without proofs) discussing the martingale measure and these are essential in obtaining the price of the option. For proofs please see [5].

Let

$$\sum(u) = \frac{\mu S(u-a) - r}{g(S(u-b))}, \qquad u \in [0,T]. \tag{2.2}$$

Theorem 20. *Let* $W(t)$, $t \in [0,T]$, *be a standard Wiener process on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $\sum$ *be a adapted process such that* $\int_0^T \left|\sum(u)\right|^2 du < \infty$ *a.s., and let*

$$\zeta_t = \exp\left(\int_0^t \sum(u) dW(u) - \frac{1}{2}\int_0^t \left|\sum(u)\right|^2 du\right) \; t \in [0,T]. \tag{2.3}$$

Suppose that $\mathbb{E}_{\mathbb{P}}(\zeta_T) = 1$, *where* $\mathbb{E}_{\mathbb{P}}$ *denotes expectation with respect to the probability measure* $\mathbb{P}$. *Define the probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *by* $d\mathbb{Q} = \zeta_T d\mathbb{P}$. *Then the process*

$$\hat{W}(t) := W(t) - \int_0^t \sum(u) du, \;\; t \in [0,T]$$

is a standard Wiener process under the measure $\mathbb{Q}$.

Using the risk neutral measure obtained above and martingale representation theorem we can also find the hedging portfolio.

$$S(t) = \phi(0)\exp\Big(\int_0^t g(S(u-b))dW(u) + \mu\int_0^t S(u-a)du - \frac{1}{2}\int_0^t g(S(u-b))^2 du\Big)$$

$a.s$ for $t \in [0,T]$. Hence using Theorem (20) from above we have

$$\tilde{S}(t) = \phi(0)\exp\Big(\int_0^t g(S(u-b))d\hat{W}(u) - \frac{1}{2}\int_0^t g(S(u-b))^2 du\Big) \tag{2.4}$$

where

$$\tilde{S}(t) = \frac{S(t)}{B(t)} = e^{-rt}S(t).$$

Hence we have

$$d\tilde{S}(t) = \tilde{S(t)}\Big[\big(\mu S(t-a) - r\big)dt + g(S(t-b))dW(t)\Big] \tag{2.5}$$

and therefore we will have

$$d\tilde{S} = \tilde{S}(t)g(S(u-b))d\hat{W}. \tag{2.6}$$

We now discuss that how we can find the hedging portfolio for a contingent claim X. Consider the $\mathbb{Q}$ martingale

$$M(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X|\mathcal{F}_t^S) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X|\mathcal{F}_t^S)$$

where $\mathcal{F}_t^S = \mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{\hat{W}} = \mathcal{F}_t^W$. Then using martingale representation theorem there exists a $\mathcal{F}_t^{\hat{W}}$ adapted process $h_0(t), t \in [0,T]$ such that

$$\int_0^T h_0(u)^2 du < \infty$$

and

$$M(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X) + \int_0^t h_0(u)d\hat{W}(u), \ \ t \in [0,T].$$

Define

$$\pi_S(t) := \frac{h_0(t)}{\tilde{S}(t)g(S(t-b))}, \ \ \pi_B(t) := M(t) - \pi_S(t)\tilde{S}(t), \ ,t \in [0,T]. \tag{2.7}$$

Consider the strategy $\{\pi_B(t), \pi_S(t) : t \in [0,T]\}$ which consists of holding $\pi_S(t)$ units of the stock and $\pi_B(t)$ units of the bond at time t. The value $V(t)$ of the portfolio at any time $t \in [0,T]$ is given by

$$V(t) = \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$

Therefore, by the product rule and the definition of the strategy $\{\pi_B(t), \pi_S(t) : t \in [0,T]\}$, it follows that

$$dV(t) = e^{rt}dM(t) + M(t)de^{rt} = \pi_B(t)de^{rt} + \pi_S(t)dS(t), \ \ t \in [0,T].$$

Consequently, $\{\pi_B(t), \pi_S(t) : t \in [0,T]\}$ is a self-financing strategy. Moreover, $V(T) = e^{rT}M(T) = X$ a.s. Hence the contingent claim X is attainable. This shows that the market $\{B(t), S(t) : t \in [0,T]\}$ is complete, since every contingent claim is attainable. Moreover, in order for the augmented market $\{B(t), S(t), X : t \in [0,T]\}$ to satisfy the no-arbitrage property, the price of the claim X must be

$$V(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t^S) \tag{2.8}$$

at each $t \in [0, T]$ *a.s.*.

We can summarize above as

Theorem 21. *Suppose that the stock price S is given by the SDDE (2.1), where $\phi(0) > 0$ and $a, b > 0$. Let T be the maturity time of an option (contingent claim) on the stock with payoff function X, i.e., X is an $\mathcal{F}_T^S$ measurable non-negative integrable random variable. Then at any time $t \in [0, T]$, the fair price $V(t)$ of the option is given by the formula*

$$V(t) \quad = \quad e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t^S) \tag{2.9}$$

where $\mathbb{Q}$ denotes the probability measure on $(\Omega, \mathcal{F})$ defined by $dQ = \zeta_T dP$ with

$$\zeta_t = \exp\left(\int_0^t \frac{\mu S(u-a) - r}{g(S(u-b))} dW(u) - \frac{1}{2}\int_0^t \left|\frac{\mu S(u-a) - r}{g(S(u-b))}\right|^2 du\right) \ t \in [0, T]. \tag{2.10}$$

The measure $\mathbb{Q}$ is a local martingale measure and the market is complete. Moreover, there is an adapted and square integrable process $h_0(u), u \in [0, T]$ such that

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}X|\mathcal{F}_t^S) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}X) + \int_0^t h_0(u) d\hat{W}(u), \ \ t \in [0, T]$$

where $\hat{W}$ is a standard $\mathbb{Q}$ Wiener process given by

$$\hat{W}(t) := W(t) + \int_0^t \frac{\mu S(u-a) - r}{g(S(u-b))} du, \ \ t \in [0, T]. \tag{2.11}$$

The hedging strategy is given by

$$\pi_S(t) := \frac{h_0(t)}{\tilde{S}(t) g(S(t-b))}, \ \ \pi_B(t) := M(t) - \pi_S(t)\tilde{S}(t), \ , t \in [0, T]. \tag{2.12}$$

The following result is a consequence of Theorem 21. It gives a Black-Scholes type formula for the value of a European option on the stock at any time prior to maturity.

Theorem 22. *Assume the conditions of Theorem 21. Let $V(t)$ be the fair price of a European call option written on the stock S with exercise price K and maturity time T. Let Φ denote the distribution function of the standard normal law, i.e.,*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du, \; x \in \mathbb{R}. \tag{2.13}$$

Then for all $t \in [T-l, T]$ where l:=min$\{a,b\}$, $V(t)$ is given by

$$V(t) = S(t)\Phi(\beta_+(t)) - Ke^{-r(T-t)}\Phi(\beta_-(t)) \tag{2.14}$$

where

$$\beta_\pm := \frac{\log \frac{S(t)}{K} + \int_t^T (r \pm \frac{1}{2} g(S(u-b))^2) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}. \tag{2.15}$$

If $T > l$ and $t < T - l$, then

$$V(t) = e^{rt}\mathbb{E}_{\mathbb{Q}}\left(H\left(\tilde{S}(T-l), -\frac{1}{2}\int_{T-l}^{T} g(S(u-b))^2 du, \int_{T-l}^{T} g(S(u-b))^2 du \right) \Bigg| \mathcal{F}_t \right) \tag{2.16}$$

where H is given by

$$H(x, m, \sigma^2) := xe^{m+\sigma^2/2}\Phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\Phi(\alpha_2(x, m, \sigma)) \tag{2.17}$$

and

$$\alpha_1(x,m,\sigma) = \frac{1}{\sigma}\Big[\log\Big(\frac{x}{K}\Big) + rT + m + \sigma^2\Big] \tag{2.18}$$
$$\alpha_2(x,m,\sigma) = \frac{1}{\sigma}\Big[\log\Big(\frac{x}{K}\Big) + rT + m\Big] \tag{2.19}$$

for $\sigma, x \in \mathbb{R}^+, m \in \mathbb{R}$. *The hedging strategy is given by*

$$\pi_S(t) = \Phi(\beta_+(t)),\ \ \pi_B(t) = -Ke^{-rT}\Phi(\beta_-(t)),\ \ t \in [T-l, T]. \tag{2.20}$$

3.3 Euler-Maruyama scheme

3.3.1 Introduction

In this section we discuss the work of [23]. The work is on convergence of Euler-Maruyama scheme for SDDE driven by Brownian motion and Poisson Random measure.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions. Let A' denote the transpose of a vector or a matrix A. Let $B(t)$ be a m-dimensional Brownian motion and $N(t,z)$ be a n-dimensional Poisson process and denote the compensated Poisson process by

$$\begin{aligned}\tilde{N}(dt,dz) &= \big(\tilde{N}_1(dt,dz), \cdots, \tilde{N}_n(dt,dz_n)\big)' \\ &= \big(N_1(dt,dz) - \nu_1(dz_1)dt, \cdots, N_n(dt,dz_n) - \nu_n(dz_n)dt\big)'\end{aligned}$$

where $N_j, j = 1, \cdots, n$ are Poisson random measures with Lévy measure $\nu_j, j = 1, \cdots, n$, coming from n independent 1-dimensional Poisson point processes. Here we assume that $N(t,z)$ and $B(t)$ are independent. Let $|\cdot|$ denote the Euclidean norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^d)$ denote the family of continuous function $\phi : [-\tau, 0] \to \mathbb{R}^d$ with the norm $||\phi|| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$ the family of all bounded, $\mathcal{F}_0$ measurable $C([-\tau, 0]; \mathbb{R}^d)$ valued random variables. Here we consider d dimensional stochastic delay equation with jumps and diffusion component.

$$\begin{aligned} dX(t) &= \alpha(X(t), X(\delta(t)))dt + \sigma(X(t), X(\delta(t)))dB(t) \\ &\quad + \int_{\mathbb{R}^n} \gamma(X(t^-), X(\delta^-), z)\tilde{N}(dt, dz) \end{aligned} \tag{3.21}$$

on $t \in [0, T]$ with initial data

$$\{X(t) : -\tau \leq t \leq 0\} = \{\zeta(t) : -\tau \leq t \leq 0\} \in C^b_{\mathcal{F}_0} \tag{3.22}$$

$\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, $\gamma : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{d\times n}$. We note here that the right hand side of (3.21) is a shorthand matrix expression. Using the notation from above we can write

$$\begin{aligned} dX_i(t) &= \alpha_i(X(t), X(\delta(t)))dt + \sum_{j=1}^{m} \sigma_{i,j}(X(t), X(\delta(t)))dB_j(t) \\ &\quad + \sum_{j=1}^{n} \int_{\mathbb{R}} \gamma_{ij}(X(t^-), X(\delta^-(t)), z_j)\tilde{N}_j(dt, dz_j). \end{aligned} \tag{3.23}$$

It is assumed that α, σ and γ are sufficiently smooth so that (3.21) has a unique solution. We refer to [40] for detials. The following assumptions are always made

(A1) The Lipschitz continuous function $\delta : [0, \infty] \to \mathbb{R}$ stands for the time delay and satisfies

$$-\tau \leq \delta(t) \leq t \ \text{ and } \ |\delta(t) - \delta(s)| \leq \rho|t - s|, \ \forall t, s \geq 0.$$

(A2) α, σ and γ are sufficiently smooth so that (3.21) has a unique solution on $[-\tau, T]$.

(A3) There exist constants $K_1 > 0$ and $\beta \in (0,1]$ such that for all $-\tau \le s < t \le 0$,

$$\mathbb{E}|\zeta(t) - \zeta(s)| \le K_0 |t-s|^{\beta}.$$

(A4) The measures $\nu = (\nu_1, \cdots, \nu_n)'$ are bounded Lévy measures, i.e, $\nu(\mathbb{R}^n) < \infty$ and $\nu(A) = \nu(-A)$ for all Borel set $A \in \mathbb{R}^n$.

3.3.2 The Euler-Maruyama (EM) method

Let step size $\Delta \in (0,1)$ be a fraction of τ, that is $\Delta = \frac{\tau}{N}$ for some sufficiently large integer N. Then the scheme can be defined by

$$\begin{aligned} Y((k+1)\Delta) &= Y(k\Delta) + \alpha(Y(k\Delta), Y(I_\Delta[\delta(k\Delta)]\Delta))\Delta \\ &\quad + \sigma(Y(k\Delta), Y(I_\Delta\delta(k\Delta)))\Delta B_k \\ &\quad + \iint_{\mathbb{R}^n} \gamma(Y(k\Delta), Y(I_\Delta[\delta(k\Delta)])\Delta \tilde{N}(dz) \end{aligned} \tag{3.24}$$

with $Y(0) = \zeta(0)$ on $-\tau \le t \le 0$ where $k = 1, 2, \cdots$ and $I_\Delta[\delta(k\Delta)]$ denotes the integer part of $\delta(k\Delta)/\Delta$, $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$ and $\Delta\tilde{N}_k(dz) = \tilde{N}((0,(k+1)\Delta], dz) - \tilde{N}((0,k\Delta], dz)$. We note that

$$-\tau \le I_\Delta[\delta(k\Delta)]\Delta \qquad \forall k \ge 0. \tag{3.25}$$

We then have

$$-N = -\frac{\tau}{\Delta} \le \frac{\delta(k\Delta)}{\Delta} \le k$$

i.e

$$-N \le I_\Delta[\delta(k\Delta)] \le k.$$

We now define the continuous interpolation by introducing the two step process. The two step process can be written as

$$y_1(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[k\Delta,(k+1)\Delta)}(t) Y(k\Delta) \tag{3.26}$$

$$y_2(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[k\Delta,(k+1)\Delta)}(t) Y(I_\Delta[\delta(k\Delta)]\Delta) \tag{3.27}$$

Then the continuous EM numerical solution is defined by

$$\bar{Y}(t) = \begin{cases} \zeta(0) \\ \zeta(0) + \int_0^t \alpha(y_1(s), y_2(s))ds + \int_0^t \sigma(y_1(s), y_2(s))dB(s) \\ + \int_0^t \int_{\mathbb{R}^n} \gamma(y_1(s^-), y_2(s^-), z)\tilde{N}(ds, dz). \end{cases}$$

We discuss a few results which would be helpful in stating the main result on convergence of EM scheme.

Lemma 3.1. *Assume that α, σ, γ satisfy the linear growth condition:*

(LG) There exists a constant $h > 0$ such that

$$|\sigma(x, y)|^2 + |\alpha(x, y)|^2 \leq h(1 + |x|^2 + |y|^2) \quad \forall x, y \in \mathbb{R}^d$$

and

$$\int_{\mathbb{R}} \sum_{k=1}^{n} |\gamma^{(k)}(x, y, z_k)|^2 \nu_k(dz_k) \leq h(1 + |x|^2 + |y|^2) \quad \forall x, y \in \mathbb{R}^d. \tag{3.28}$$

Then there is a constant K_1, which dependents only on T, h, ζ but is independent of Δ, such that the exact solution and the EM numerical solution to the SDDE (3.21) *satisfy*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X(t)|^2\Big] \vee \mathbb{E}\Big[\sup_{0\leq t\leq T} |\bar{Y}(t)|^2\Big] \leq K_1. \tag{3.29}$$

In order to estimate the p-th moment the following assumption is also required.

(A5) Assume that α, σ and γ satisfy (3.28) and

$$\int_{\mathbb{R}} \sum_{k=1}^{n} |\gamma^{(k)}(x, y, z_k)|^p \nu_k(dz_k) \leq h(1 + |x|^p + |y|^p) \quad \forall x, y \in \mathbb{R}^d \tag{3.30}$$

Lemma 3.2. *Under the assumption (A5), for any $p > 2$, there is a positive constant K_p which depends only on p, ν, T, h but is independent of Δ such that*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |X(t)|^p\Big] \vee \mathbb{E}\Big[\sup_{0 \leq t \leq T} |\bar{Y}(t)|^p\Big] \leq K_p. \tag{3.31}$$

We state the following global Lipschitz condition which are used in the results which follow.

[(GL)] There exists a constant $L > 0$ such that

$$\begin{aligned} &|\sigma(x, y) - \sigma(\bar{x}, \bar{y})|^2 + |\alpha(x, y) - \alpha(\bar{x}, \bar{y})|^2 \\ &+ \sum_{k=1}^{n} \int_{\mathbb{R}} |\gamma^{(k)}(x, y, z_k) - \gamma^{(k)}(\bar{x}, \bar{y}, z_k)|^2 \nu_k(dz_k) \\ &\leq L(|x - \bar{x}|^2 + |y - \bar{y}|^2) \quad x, y, \bar{x}, \bar{y} \in \mathbb{R}^d. \end{aligned} \tag{3.32}$$

We now discuss some of the results which are required for the proof of the main convergence result.

Lemma 3.3. *Under the linear growth condition, one has*

$$\mathbb{E}|\bar{Y}(t) - y_1(t)|^2 \leq K_2 \Delta.$$

Lemma 3.4. *Under the (A1),(A3) and the linear growth condition, if the stepsize satisfies* $(\rho+1)\Delta \leq 1$ *one has*

$$\mathbb{E}|\bar{Y}(\delta(t)) - y_2(t)|^2 \leq K_3\Delta \quad \forall t \in [0, T]$$

where K_3 *is a constant independent of* Δ*.*

We now state the main convergence result under global Lipschitz condition.

Theorem 23. *Under the global Lipschitz condition, we have*

$$\lim_{\Delta\to 0} \mathbb{E}\Big[\sup |X(t) - \bar{Y}(t)|^2\Big] = 0. \tag{3.33}$$

3.4 Pricing European options with SDDE driven by Brownian motion and by compound Poisson process.

In this section we will briefly discuss the work of [22] which shows how to price European call option when underlying model is stochastic delay differential equations driven by Brownian motion and compound Poisson process.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}\}_{t\geq 0}$ which satisfies the usual conditions (i.e., $\mathcal{F}$ is right continuous and $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$). Let $W(t)$ be standard Brownian motion and $N(t)$ be a Poisson process with intensity λ. Let $Y_1, Y_2, Y_3, \cdots$, be independent and identically distributed random variables with

$$\mathbb{E}(Y_j) = \alpha, \quad j = 1, 2, 3, \cdots$$

Here we also assume Y_j are independent of Poisson process $N(t)$. The compound Poisson process $Q(t)$ can be defined as

$$Q(t) = \sum_{j=1}^{N(t)} Y_j, \quad t \geq 0.$$

We consider a stock model as

$$\begin{cases} dS(t) = \mu S(t-a)S(t)dt + f(S(t-a))S(t)dW(t) + g(S(t-a))S(t-)dL(t), & t \geq 0\,, \\ S(t) = \psi(t)\,, \qquad t \in [-a, 0]\,, \end{cases} \tag{4.34}$$

where μ and a are positive constants with $g > -1$. Also $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions and initial process $\psi : \Omega \to C([-a, 0], \mathbb{R})$ is $\mathcal{F}_0$ measurable with respect to the Borel sigma algebra of $C([-a, 0], \mathbb{R})$. The process W is a one dimensional Brownian motion which is adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ and $L(t) = Q(t) - \alpha\lambda t$ is a compensated compound Poisson process with intensity $\alpha\lambda t$. We also assume that $W(t)$ and $L(t)$ are independent from one another. Similar to Theorem 19 we also discuss pathwise uniqueness and the property that $S(t) \geq 0$ whenever $\psi(0) \geq 0$ for all $t \geq 0$ almost surely.

Theorem 24. *Given an $\mathcal{F}_0$ measurable initial process $\psi(t)$, the stochastic differential delay equation with jumps given by* (4.34) *admits a unique pathwise solution with the property that $S(t) \geq 0$ whenever $\psi(0) \geq 0$ for all $t \geq 0$ almost surely. If in addition $\psi(0) > 0$ a.s, then $S(t) > 0$ for all $t \geq 0$ a.s.*

3.4.1 Pricing of European Option

We first discuss the risk neutral measure required to price the European option.

We assume that price $B(t)$ of risk free asset is given by

$$B(t) = B(0)e^{rt} \quad \forall t \in [0, T] \tag{4.35}$$

where $r > 0$ is the risk free rate of return and $S(t)$ is given by (4.34). For a non-dividend paying stock the discounted price $\tilde{S}(t)$ of the stock $S(t)$ is

$$\tilde{S}(t) = \frac{S(t)}{B(t)} = e^{-rt}S(t). \tag{4.36}$$

We find the risk neutral measure $\mathbb{Q}$ which will make the discounted stock price into a martingale. For this, let $N_1, N_2, \cdots, N_K$ be independent Poisson processes with $N_m(t)$ having intensity λ_m. Let $\lambda_1, \lambda_2, \cdots, \lambda_K$ be positive numbers then we can define

$$Z_1(t) = \exp\Big\{ -\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t |\Theta(u)|^2 du\Big\}, \tag{4.37}$$

$$Z_2(t) = \prod_{i=1}^{K} e^{(\lambda_i - \bar{\lambda}_i)t}, \tag{4.38}$$

$$Z(t) = Z_1(t)Z_2(t) \tag{4.39}$$

where

$$\Theta(u) = \frac{\mu S(u-a) - r}{f(S(u-a))}, \quad u \in [0, T]. \tag{4.40}$$

Then the process $Z(t)$ is a martingale.

Lemma 3.5. *The Process Z(t) of* (4.39) *is a martingale. In particular* $\mathbb{E}(Z(t)) = 1$ *for* $t \geq 0$.

We can now define the risk neutral measure $\mathbb{Q}$ by Radon Nikodyn density

$$\mathbb{Q}(A) = \int_A Z(T)\mathbb{P}(A) \tag{4.41}$$

Let $\tilde{p}(y_k)$ be the probability that the jump is of size y_k.

Theorem 25. *Let the risk neutral measure $\mathbb{Q}$ be as (4.41) defined above. Under $\mathbb{Q}$, the process*

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du \tag{4.42}$$

is a Brownian motion, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda} = \sum_{i=1}^K \tilde{\lambda}_i$ and i.i.d jump sizes satisfying $\mathbb{Q}(Y_i = y_k) = \tilde{p}(y_k)$ for all i and $k = 1, \cdots, K$ and the process $\tilde{W}(t)$ and $Q(t)$ are independent.

3.4.2 Pricing of European option

We now consider the pricing of European call option in this model where the price process follows (4.34). We assume risk neutral measure given by (4.41). Then the price process under risk neutral measure is given by

$$\begin{cases} dS(t) = rS(t)dt + f(S(t-a))S(t)d\tilde{W}(t) + g(S(t-a))S(t)d\tilde{L}(t), & t \geq 0\,, \\ S(t) = \psi(t)\,, \qquad t \in [-a, 0]\,, \end{cases} \tag{4.43}$$

where $\tilde{L}(t) = Q(t) - \tilde{\alpha}\lambda t$ and

$$\begin{aligned} S(t) \;&=\; S(0)\exp\Big\{\int_0^t [r - \tilde{\alpha}\tilde{\lambda} g(S((s-a))]ds + \int_0^t f(S(s-a))d\tilde{W}(s) \\ &\quad -\frac{1}{2}\int_0^t f(S(s-a))^2 ds\Big\}\prod_{i=1}^{N(t)}[1 + g(S(t-a))Y_i] \end{aligned}$$

so that the discounted price

$$\tilde{S}(t) \;=\; \frac{S(t)}{B(t)} \tag{4.44}$$

is a martingale under $\mathbb{Q}$. We consider a European call option with strike price K and maturity $T > 0$. The payoff of such an option is given by $(S(T) - K)^+$. Let $V(t)$ denote the price of such an option at any time $0 \leq t \leq T$. Then

$$V(T) = (S(T) - K)^+.$$

Let $\tau = T - t$, then at any time $0 \leq t \leq T$, we have

$$V(t) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V(T)|\mathcal{F}(t)] = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[(S(T) - K)^+|\mathcal{F}(t)] \tag{4.45}$$

where

$$\begin{aligned} S(T) = S(t) \exp\Big\{ & \int_t^T [r - \tilde{\alpha}\tilde{\lambda}g(S(s-a))]ds \\ & \int_t^T f(S(s-a))d\tilde{W}(s) - \frac{1}{2}\int_t^T f(S(s-a))^2 ds\Big\} \\ & \times \prod_{i=N(t)+1}^{N(T)} [1 + g(S(t-a))Y_i]. \end{aligned} \tag{4.46}$$

If we assume that $t \in [T - a, T]$ then we are able to obtain an explicit formulae for European call (and put) options. For this, we can see that $S(t)$ is measurable with respect to $\mathcal{F}(t)$. We also note that

1. $S(t)$ is $\mathcal{F}_t$ measurable.
2. $\int_t^T [r - \tilde{\alpha}\tilde{\lambda}g(S(s-a)) - \frac{1}{2}f(S(s-a))^2]ds$ is also $\mathcal{F}_t$ measurable in the interval $[T - a, T]$.
3. For any $z \in \mathbb{R}$, $\prod_{i=N(t)+1}^{N(T)}[1 + zY_i]$ is independent of the $\mathcal{F}_t$.
4. $\int_t^T f(S(s-a))d\tilde{W}(s)$ has the same distribution as under $\mathbb{Q}$ as ηX where X is a standard Gaussian random variable and $\eta^2 = \int_t^T f(S(s-a))^2 ds$.

Based on the discussion above, (3.5) and Theorem 25 we state the following result to calculate the price of European call option.

Theorem 26. *Let $V(t)$ be the fair price of a European call option written on the stock S following the model (4.43), with strike price K and maturity time T. Let Φ denote the distribution function of the standard normal law, i.e.,*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du, \; x \in \mathbb{R}. \tag{4.47}$$

Then for all $t \in [T-a, T]$, $V(t)$ is given by

$$V(t) = S(t)e^{m+\frac{\alpha^2}{2}}\Phi(d_+(S(t), m, \alpha)) - ke^{-r\tau}\Phi(d_-(S(t), m, \alpha)) \tag{4.48}$$

where

$$d_+ := \frac{1}{\sigma}\Big[\log\frac{S(t)}{K} + rT + m + \alpha^2\Big] \tag{4.49}$$

$$d_- := \frac{1}{\sigma}\Big[\log\frac{S(t)}{K} + rT + m\Big]. \tag{4.50}$$

If $T > a$ and $t < T - a$, then

$$\begin{aligned} V(t) &= \mathbb{E}_{\mathbb{Q}}\Big[G\Big(S(T-a), g(S(T-a)), \int_{T-a}^T \Big[r - \frac{1}{2}f(S(s-a))^2\Big]ds, \\ &\quad \int_{T-a}^T \tilde{\alpha}\tilde{\lambda}g(S(s-a))ds, \int_{T-a}^T f(S(s-a))^2 ds\Big)\Big|\mathcal{F}_t\Big] \end{aligned} \tag{4.51}$$

where

$$G(y, z, m, n, \eta^2) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)}\frac{(\tilde{\lambda}(T-t))^j}{j!}\mathbb{E}_{\mathbb{Q}}\Big[\zeta(\tau, ye^{-n}\prod_{i=1}^{j}(1+zY_i), m, \eta^2)\Big]$$

and $\zeta(\tau, y, m, \alpha^2) = e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[(ye^{m+\alpha Y} - K)^+]$.

Chapter 4

Option Pricing and Euler-Maruyama scheme for SDDE driven by Lévy process

4.1 Introduction

The risky asset in the classical Black-Scholes market is described by the geometric Brownian motion given by the stochastic differential equation driven by standard Brownian motion:

$$dS(t) = S(t)\left[rdt + \sigma dW(t)\right], \tag{1.1}$$

where r and σ are two positive constants and $W(t)$ is the standard Brownian motion. Ever since the seminal work of Black, Scholes and Merton there have been many research works to extend the Black-Scholes-Merton's theory of option pricing from the original Black-Scholes market to more sophisticated models.

One of these extensions is the delayed stochastic differential equation (SDDE) driven by the standard Brownian motion (e.g. [5], see also [36, 44]). In these works the risky asset is described by the following stochastic delay differential equation

$$dS(t) = S(t)\left[f(t, S_t)dt + g(t, S_t)dW(t)\right] ,$$

where $S_t = \{S(s)\,, t - b \le s \le t\}$ or $S_t = S(t - b)$ for some constant $b > 0$. .

On the other hand, there have been some recent discovery (see e.g. [28, 31, 11, 30]) that to better fit some risky assets it is more desirable to use the hyper-exponential jump process along with the classical Brownian motion:

$$dS(t) = S(t)\left[rdt + \sigma dW(t) + \beta dZ(t)\right] ,$$

where $Z(t)$ is a hyper-exponential jump process (see the definition in the next section).

Let $N(dt, dz)$ be the Poisson random measure associated with a jump process which includes the hyper-exponential jump process as a special case and let $\tilde{N}(dt, dz)$ denote its compensated Poisson random measure. Then the above equation with $\sigma = 0$ is a special case of the following equation

$$dS(t) = S(t)\Big(rdt + \beta \int_{[0,T]\times\mathbb{R}_0} z\tilde{N}(dz, ds)\Big) \tag{1.2}$$

and it has been argued in (eg. [6, 15, 13]) that the equation (1.2) is a better model for stock prices than (1.1).

In this chapter, we propose a new model to describe the risky asset by combining the hyper-exponential process with delay. More precisely, we propose the following stochastic differential equation as a model for the price movement of the risky asset:

$$dS(t) = S(t-)\left[f(t, S(t-b))dt + g(t, S(t-b))dZ(t)\right] , \tag{1.3}$$

where f and g are two given functions, and $Z(t)$ is a Lévy process which include the hyper-exponential jump processes as a special case. The above model along with the Brownian motion component can be found in [22], where the coefficient of Brownian motion cannot be allowed to be zero. In this chapter, we let the coefficient of the Brownian motion to be zero and we use the Girsanov formula for the jump process to address the issue of completeness of the market and hedging portfolio missed in [22].

With the introduction of this new market model, the first question is that whether the equation has a unique solution or not and if the unique solution exists whether the solution is positive or not (since the price of an asset is always positive). We shall first answer these questions in Section 4.3, where we prove the existence, uniqueness and positivity of the solutions to a larger class of equations than (1.3). To guarantee that the solution is positive, we need to assume that the jump part $g(t, S(t-b))dZ(t)$ of the equation is bounded from below by some constant (see the assumption (A3) in the next section for the precise meaning). The class of the equations our results can be applied is larger in the following two aspects: The first one is that $Z(t)$ can be replaced by a more general Lévy process or more general Poisson random measure and the second one is that the equation can be multi-dimensional.

Following the Black-Scholes-Merton's principle we then obtain a formula for the fair price for the European option and the corresponding replica hedging portfolio is also given. To evaluate this formula during the last delay period, we propose a Fourier transformation method. This method appears more explicit than the partial differential equation method in the literature and is more closed to the original Black-Scholes formula in spirit. This is done in Section 4.5.

Due to the involvement of $f(S(t-b))$ and $g(S(t-b))$ the above analytical expression for the fair option price formula is only valid in the last delay period. Then how do we perform the evaluation by using this option price formula? We propose to use Monte-Carlo method to get the numerical value approximately. For this reason we need to simulate the equation (1.3) numerically. We observe that there have been a lot of works (eg. [32, 16, 45]) on Euler-Maruyama convergence scheme for SDDE models. There has already been study on the Euler-Maruyama scheme for SDDE models with jumps (e.g. [23]). However, in general the Euler-Maruyama scheme cannot preserve the positivity of the solution. Since the solution to the equation (1.3) is positive (when the initial condition is positive), we wish all of our approximations of the solution is also positive. To this end and motivated by the similar work in the Brownian motion case (see e.g. [46]) we introduce a logarithmic Euler-Maruyama scheme, a variant of the Euler-Maruyama scheme for (1.3). With this scheme all the approximate solutions are positive and the rate of the convergence of this scheme is also 0.5. This rate is optimal even in the Brownian motion case (e.g. [12]). Let us point out that the 0.5 rate of the usual Euler-Maruyama scheme for SDDE with jumps studied in [23] is only obtained in the L^2 sense. Not only our logarithmic Euler-Maruyama scheme preserves the positivity, its rate is 0.5 in L^p for any $p \geq 2$. This is done in Section 4.4.

Finally in Section 4.6 we present some numerical attempts and compared that with the classical Black-Scholes price formula against the market price for some famous call options in the real financial market.

4.2 Delayed stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{\{t\geq 0\}}$ satisfying the usual conditions. On $(\Omega, \mathcal{F}, \mathbb{P})$ let $W(t)$ be a brownian motion adapted to the filtration $\mathcal{F}_t$. We shall consider the following delayed stochastic differential equation driven by the Brownian Motion $W(t)$:

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + g(S(t-b))S(t)dW(t), & t \geq 0\,, \\ S(t) = \phi(t)\,, \qquad t \in [-b, 0]\,, \end{cases} \tag{2.4}$$

where

(i) $f, g : \mathbb{R} \to \mathbb{R}$ are some given bounded measurable functions;

(ii) $b > 0$ is a given number representing the delay of the equation;

(iii) $\phi : [-b, 0] \to \mathbb{R}$ is a (deterministic) measurable function.

4.3 Delayed stochastic differential equations with Jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{\{t\geq 0\}}$ satisfying the usual conditions. On $(\Omega, \mathcal{F}, \mathbb{P})$ let $Z(t)$ be a Lévy process adapted to the filtration $\mathcal{F}_t$. We shall consider the following delayed stochastic differential equation driven by the Lévy process $Z(t)$:

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t), & t \geq 0\,, \\ S(t) = \phi(t)\,, \qquad t \in [-b, 0]\,, \end{cases} \tag{3.5}$$

where

(i) $f, g : \mathbb{R} \to \mathbb{R}$ are some given bounded measurable functions;

(ii) $b > 0$ is a given number representing the delay of the equation;

(iii) $\phi : [-b, 0] \to \mathbb{R}$ is a (deterministic) measurable function.

To study the above stochastic differential equation, it is common to introduce the Poisson random measure associated with this Lévy process $Z(t)$ (see e.g. [4, 13, 14, 40] and references therein). First, we write the jump of the process Z at time t by

$$\Delta Z(t) := Z(t) - Z(t-) \quad \text{if } \Delta Z(t) \neq 0\,.$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. For any $t > 0$ and for any $U \in \mathcal{B}(\mathbb{R}_0)$ we define the *Poisson random measure*, $N : [0,T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with the Lévy process Z by

$$N(t,U) := \sum_{0\le s\le t,\ \Delta Z_s \neq 0} \chi_U(\Delta Z(s))\,, \tag{3.6}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of the Lévy process Z is given by

$$\nu(U) := \mathbb{E}[N(1,U)] \tag{3.7}$$

and the compensated Poisson random measure $\tilde{N}$ associated with the Lévy process $Z(t)$ is defined by

$$\tilde{N}(dt,dz) := N(dt,dz) - \mathbb{E}\left[N(dt,dz)\right] = N(dt,dz) - \nu(dz)dt\,. \tag{3.8}$$

For some technical reason, we shall assume that the process $Z(t)$ has bounded negative jumps and positive jumps to guarantee that the solution $S(t)$ to (1.1) is positive. This means that there is an interval $\mathcal{J} = [-R, \infty)$ bounded from the left such that $\Delta Z(t) \in \mathcal{J}$ for all $t > 0$. With these notations, we can write

$$Z(t) = \int_{[0,t]\times\mathcal{J}} zN(ds,dz) \quad \text{or} \quad dZ(t) = \int_{\mathcal{J}} zN(dt,dz)$$

and the equation (1.1) becomes

$$\begin{aligned} dS(t) &= \left[f(S(t-b)) + g(S(t-b))\int_{\mathcal{J}} z\nu(dz)\right] S(t)dt \\ &\quad + g(S(t-b))S(t-)\int_{\mathcal{J}} z\tilde{N}(dt,dz)\,. \end{aligned}$$

It is a special case of the following equation:

$$dS(t) = f(S(t-b))S(t)dt + \int_{\mathcal{J}} g(z, S(t-b))S(t-)\tilde{N}(dt, dz)\,. \tag{3.9}$$

Theorem 27. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g(z, x) \geq \alpha_0 > -1$ for all $z \in \mathcal{J}$ and for all $x \in \mathbb{R}$, where $\mathcal{J}$ is the supporting set of the Poisson measure $N(t, dz)$. Then, the stochastic differential delay equation* (1.6) *admits a unique pathwise solution with the property that if $\phi(0) > 0$, then for all $t > 0$, the random variable $X(t) > 0$ almost surely.*

Proof First, let us consider the interval $[0, b]$. When t is in this interval $f(X(t-b)) = f(\phi(t-b))$ and $g(z; X(t-b)) = g(z; \phi(t-b))$ are known given functions of t (and z). Thus, (1.6) is a linear equation driven by Poisson random measure. The standard theory (see e.g. [4, 40]) can be used to show that the equation has a unique solution. Moreover, it is also well-known (see the above mentioned books or [3]) that by Itô's formula the solution to (1.6) can be written as

$$\begin{aligned} X(t) \;=\; & \phi(0) \exp\Bigg\{ \int_0^t f(\phi(s-b))ds + \int_{[0,t]\times\mathcal{J}} \log\left[1 + g(z, \phi(s-b))\right] \tilde{N}(ds, dz) \\ & + \int_{[0,t]\times\mathcal{J}} \Big(\log\left[1 + g(z, \phi(s-b))\right] - g(z, \phi(s-b)) \Big) ds\nu(dz) \Bigg\}. \end{aligned}$$

From this formula we see that if $\phi(0) > 0$, then the random variable $X(t) > 0$ almost surely for every $t \in [0, b]$.

In similar way, we can consider the equation (1.6) on $t \in [kb, (k+1)b]$ recursively for $k = 1, 2, 3, \cdots$, and obtain the same statements on this interval from previous results on the interval $t \in [-b, kb]$. ■

Since (1.1) is a special case of (1.6), we can write down a corresponding result of the above theorem for (1.1).

Corollary 4.1. *Let the Lévy process $Z(t)$ have bounded negative jumps (e.g. $\Delta Z(t) \in \mathcal{J} \subseteq [-R, \infty)$). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g(x) \leq \frac{\alpha_0}{R}$ for all $x \in \mathbb{R}$. Then, the stochastic differential delay equation* (1.1) *admits a unique pathwise solution with the property that if $\phi(0) > 0$, then for all $t > 0$ the random variable $X(t) > 0$ almost surely.*

Proof Equation (1.1) is a special case of (1.6) with $g(z, x) = zg(x)$. The condition $g(x) \leq \frac{\alpha_0}{R}$ implies $g(z, x) \geq \alpha_0 > -1$ for all $z \in \mathcal{J}$ and for all $x \in \mathbb{R}$. Thus, Theorem 32 can be applied. ■

Example One example of the Lévy process $Z(t)$ we have in mind which is used in finance is the hyper-exponential jump process, which we explain below. Let $Y_i, i = 1, 2, \cdots$ be independent and identically distributed random variables with the probability distribution given by

$$f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{\{x < 0\}} \, ,$$

where

$$\eta_i > 0, \; p_i \geq 0, \quad \theta_j > 0, \; q_j \geq 0 \, , \quad i = 1, \cdots, m, \; j = 1, \cdots, n$$

with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. Let N_t be a Poisson process with intensity λ. Then

$$Z(t) = \sum_{i=1}^{N_t} Y_i$$

is a Lévy process. If $m = 1, n = 1$ then $Z(t)$ is called a double exponential process. The assumption on the boundedness of the negative jumps can be made possible by requiring that $q_j = 0$ for all $j = 1, \cdots, n$ or by replacing the negative exponential distribution by truncated negative exponential distributions, namely,

$$f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x\geq 0\}} + \sum_{j=1}^{n} q_j \frac{\theta_j}{1-e^{-\theta_j R_j}} e^{\theta_j x} I_{\{-R_j<x<0\}}\,,$$

where

$$\eta_i > 0,\ p_i \geq 0, \quad \theta_j > 0, R_j > 0,\ q_j \geq 0\,, \quad i = 1, \cdots, m,\ j = 1, \cdots, n$$

with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. For this truncated hyper-exponential process, we can take $\mathcal{J} = [-R, \infty)$ with $R = \max\{R_1, \cdots, R_n\}$. Although in this chapter we will mainly concern with the one dimensional delayed stochastic differential equation (1.6) or (1.1) it is interesting to extend Theorem 32 to more than one dimension.

Let $\tilde{N}_j(ds, dz)$, $j = 1, \cdots, d$ be independent compensated Poisson random measures. Consider the following system of delayed stochastic differential equations driven by Poisson random measures:

$$\begin{aligned} dS_i(t) &= \sum_{j=1}^{d} f_{ij}(S(t-b))S_j(t)dt \\ &\qquad + S_i(t-) \sum_{j=1}^{d} \int_{\mathcal{J}} g_{ij}(z, S(t-b)) \tilde{N}_j(dt, dz)\,, \quad i = 1, \cdots, d\,, \\ S_i(t) &= \phi_i(t)\,, \quad t \in [-b, 0]\,,\ i = 1, \cdots, d\,, \end{aligned} \tag{3.10}$$

where $S(t) = (S_1(t), \cdots, S_d(t))^T$.

Theorem 28. *Suppose that $f_{ij} : \mathbb{R}^d \to \mathbb{R}$ and $g_{ij} : \mathcal{J} \times \mathbb{R}^d \to \mathbb{R}\,,\ 1 \leq i, j \leq d$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g_{ij}(z, x) \geq \alpha_0 > -1$ for all $1 \leq i, j \leq d$, for all $z \in \mathcal{J}$ and for all $x \in \mathbb{R}^d$, where $\mathcal{J}$ is the common supporting set of the Poisson measures $\tilde{N}_j(t, dz), j = 1, \cdots, d$. If for all $i \neq j$, $f_{ij}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $\phi_i(0) \geq 0\,,\ i = 1, \cdots, d$, then, the stochastic* *differential delay equation* (1.7) *admits a unique pathwise solution with the property that for all* $i = 1, \cdots, d$ *and for all* $t > 0$*, the random variable* $S_i(t) \geq 0$ *almost surely.*

Proof We can follow the argument as in the proof of Theorem 32 to show that the system of delayed stochastic differential equations (1.7) has a unique solution $S(t) = (S_1(t), \cdots, S_d(t))^T$. We shall modify slightly the method of [19] to show the positivity of the solution. Denote $\tilde{g}_{ij}(t,z) = g_{ij}(z, S(t-b))$. Let $Y_i(t)$ be the solution to the stochastic differential equation

$$dY_i(t) = Y_i(t-) \sum_{j=1}^{d} \int_{\mathcal{J}} \tilde{g}_{ij}(t,z) \tilde{N}_j(dt, dz)$$

with initial conditions $Y_i(0) = \phi_i(0)$. Since this is a scalar equation for $Y_i(t)$, its explicit solution can be represented

$$\begin{aligned} Y_i(t) &= \phi_i(0) \exp\Bigg\{ \sum_{j=1}^{d} \log\left[1 + \tilde{g}_{ij}(s,z)\right] \tilde{N}_j(ds, dz) \\ &\quad + \sum_{j=1}^{d} \int_{[0,t]\times\mathcal{J}} \Big(\log\left[1 + \tilde{g}_{ij}(s,z)\right] - \tilde{g}_{ij}(s,z) \Big) ds \nu_j(dz) \Bigg\}, \end{aligned}$$

where ν_j is the associated Lévy measure for $\tilde{N}_j(ds, dz)$. Denote $\tilde{f}_{ij}(t) = f_{ij}(S(t-b))$ and let $p_i(t)$ be the solution to the following system of equations

$$dp_i(t) = \sum_{j=1}^{d} \tilde{f}_{ij}(t) p_j(t) dt, \quad p_i(0) = 1, \quad i = 1, \cdots, d.$$

By the assumption on f we have that when $i \neq j$, $\tilde{f}_{ij}(t) \geq 0$ almost surely. By a theorem in [7, p.173] we see that $p_i(t) \geq 0$ for all $t \geq 0$ almost surely. Now it is easy to check by the Itô formula that $\tilde{S}_i(t) = p_i(t) Y_i(t)$ is the solution to (1.7) which satisfies that $\tilde{S}_i(t) \geq 0$ almost surely. By the uniqueness of the solution we see that $S_i(t) = \tilde{S}_i(t)$ for $i = 1, \cdots, d$. The theorem is then proved. ■

4.4 Logarithmic Euler-Maruyama scheme

The equation (1.1) or (1.6) is used in Section 4.5 to model the price of a risky asset in a financial market and its the solution is proved to be positive as in Theorem 32. As it is well-known the usual Euler-Maruyama scheme cannot preserve the positivity of the solution (e.g. [46] and references therein). Motivated by the work [46], we propose in this section a variant of the Euler-Maruyama scheme (which we call logarithmic Euler-Maruyama scheme) to approximate the solution so that all approximations are always non-negative. For the convenience of the future simulation, we shall consider only the equation (1.1), which we rewrite here:

$$dS(t) \quad = \quad f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t)\,, \tag{4.11}$$

where $Z(t) = \sum_{i=1}^{N_t} Y_i$ is a Lévy process. Here N_t is a Poisson process with intensity λ and $Y_1, Y_2, \cdots$, are iid random variables.

The solution to the above equation can be written as

$$S(t) = \phi(0)\exp\Big(\int_0^t f(S(u-b))du + \sum_{0\le u\le t, \Delta Z(u)\neq 0} \ln(1+g(S(u-b))Y_{N(u)})\Big)\,. \tag{4.12}$$

We shall consider a finite time interval $[0, T]$ for some fixed $T > 0$. Let $\Delta = \frac{T}{n} > 0$ be a time step size for some positive integer $n \in \mathbb{N}$. For any nonnegative integer $k \geq 0$, denote $t_k = k\Delta$. We consider the partition π of the time interval $[0, T]$:

$$\pi : 0 = t_0 < t_1 < \cdots < t_n = T\,.$$

On the subinterval $[t_k, t_{k+1}]$ the solution (4.12) can also be written as

$$\begin{aligned} S(t) &= S(t_k)\exp\Big(\int_{t_k}^{t} f(S(u-b))du \\ &\quad + \sum_{t_k\le u\le t, \Delta Z(u)\neq 0} \ln(1+g(S(u-b))Y_{N(u)})\Big), t\in[t_k, t_{k+1}]. \end{aligned} \quad (4.13)$$

Motivated by the formula (4.13), we propose a logarithmic Euler-Maruyama scheme to approximate (1.1) as follows.

$$\begin{aligned} S^{\pi}(t_{k+1}) &= S^{\pi}(t_k)\exp\Big(f(S^{\pi}(t_k-b))\Delta\Big) \\ &\quad \cdot\exp\Big(\ln(1+g(S^{\pi}(t_k-b))\Delta Z_k)\Big), \quad k=0,1,2,...,n-1 \end{aligned} \quad (4.14)$$

with $S^{\pi}(t)=\phi(t)$ for all $t\in[-b,0]$. It is clear that if $\phi(0)>0$, then $S^{\pi}(t_k)>0$ almost surely for all $k=0,1,2,...,n$. Then our approximations $S^{\pi}(t_k)$ are always positive. Notice that the approximations from usual Euler-Maruyama scheme is always not positive preserving (see e.g. [46] and references therein).

We shall prove the convergence and find the rate of convergence for the above scheme. For the convergence of the usual Euler-Maruyama scheme of jump equation with delay, we refer to [23]. To study the convergence of the above logarithmic Euler-Maruyama scheme, we make the following assumptions.

(A1) The initial data $\phi(0)>0$ and it is Hölder continuous i.e there exist constant $\rho>0$ and $\gamma\in[1/2,1)$ such that for $t,s\in[-b,0]$

$$|\phi(t)-\phi(s)|\le\rho|t-s|^{\gamma}. \quad (4.15)$$

(A2) f is bounded. f and g are global Lipschitz. This means that there exists a constant $\rho > 0$ such that

$$\begin{cases} \Big|g(x_1) - g(x_2)\Big| \le \rho|x_1 - x_2|\,; \\ \Big|f(x_1) - f(x_2)\Big| \le \rho|x_1 - x_2|\ , \quad \forall\ x_, x_2 \in \mathbb{R}\,; \\ \big|f(x)\big| \le \rho\,, \quad \forall x \in \mathbb{R} \end{cases}$$

(A3) The support $\mathcal{J}$ of the Poisson random measure N is contained in $[-R, \infty)$ for some $R > 0$ and there are constants $\alpha_0 > 1$ and $\rho > 0$ satisfying $-\rho \le g(x) \le \frac{\alpha_0}{R}$ for all $x \in \mathbb{R}$.

(A4) For any $q > 1$ there is a $\rho_q > 0$

$$\int_{\mathcal{J}} (1 + |z|)^q \nu(dz) \le \rho_q\,. \tag{4.16}$$

For notational simplicity we introduce two step processes

$$\begin{cases} v_1(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[t_k, t_{k+1})}(t) S^\pi(t_k) \\ v_2(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[t_k, t_{k+1})}(t) S^\pi(t_k - b). \end{cases}$$

Define the continuous interpolation of the logarithmic Euler-Maruyama approximate solution on the whole interval $[-b, T]$ (not only on $t_k, k = 0, \cdots, n$) as follows:

$$S^\pi(t) = \begin{cases} \phi(t) & t \in [-b, 0] \\ \phi(0) \exp\Big(\displaystyle\int_0^t f(v_2(u)) du & \\ \qquad + \sum_{0 \le u \le t, \Delta Z(u) \ne 0} \ln(1 + g(v_2(u)) Y_{N(u)}) \Big) & t \in [0, T]. \end{cases} \tag{4.17}$$

With this interpolation, we see that $S^\pi(t) > 0$ almost surely for all $t \ge 0$.

Lemma 4.1. *Let the assumptions (A1)-(A4) be satisfied. Then for any $q \geq 1$ there exists K_q, independent of the partition π, such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|S(t)|^q\Big] \vee \mathbb{E}\Big[\sup_{0\leq t\leq T}|S^\pi(t)|^q\Big] \leq K_q.$$

Proof We can assume that $q > 2$. First, let us prove $\mathbb{E}\Big[\sup_{0\leq t\leq T}|S^\pi(t)|^q\Big] \leq K_q$. From (4.17) it follows

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq t\leq T}|S^\pi(t)|^q\Big] &\leq |\phi(0)|^q\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\int_0^t f(v_2(u))du \\ &\quad +q\sum_{0\leq u\leq t,\Delta Z(u)\neq 0}\ln(1+g(v_2(u))Y_{N(u)})\Big)\Big].\end{aligned}$$

Since $|f(t)| \leq \rho$ we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq t\leq T}|S^\pi(t)|^q\Big] \\ &\leq \phi(0)^q e^{q\rho T}\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\sum_{0\leq u\leq t,\Delta Z(u)\neq 0}\ln(1+g(v_2(u))Y_{N(u)})\Big)\Big] \\ &= \phi(0)^q e^{q\rho T}\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\int_{\mathbb{T}}\ln(1+zg(v_2(u)))N(du,dz)\Big)\Big], \qquad (4.18)\end{aligned}$$

where and throughout the remaining part of this chapter, we denote $\mathbb{T} = [0,t] \times \mathcal{J}$. Now we are going to handle the factor

$$I := \mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\int_{\mathbb{T}}\ln(1+zg(v_2(u)))N(du,dz)\Big)\Big].$$

Let $h = ((1 + zg(v_2(u))^{2q} - 1))/z$. Then

$$\begin{aligned}
I &= \mathbb{E}\Big[\sup_{0\le t\le T} \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \ln(1+zh)N(du,dz)\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T} \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \ln(1+zh)\tilde{N}(du,dz) + \frac{1}{2}\int_{\mathbb{T}} \ln(1+zh)\nu(dz)du\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T} \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \ln(1+zh)\tilde{N}(du,dz) + \frac{1}{2}\int_{\mathbb{T}} \left[\ln(1+zh) - zh\right]\nu(dz)du\Big)\Big] \\
&\qquad \sup_{0\le t\le T} \exp\Big(-\frac{1}{2}\int_{\mathbb{T}} (1+zg(v_2(u))^{2q} - 1)\ \nu(dz)du\Big)\Big] \\
&\le C_q\mathbb{E}\Big[\sup_{0\le t\le T} \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \ln(1+zh)\tilde{N}(du,dz) + \frac{1}{2}\int_{\mathbb{T}} \left[\ln(1+zh) - zh\right]\nu(dz)du\Big)\Big],
\end{aligned}$$

where we used boundedness of g and the assumption (A4). Now an application of the Cauchy-Schwartz inequality yields

$$I \le C_q \Big\{\mathbb{E}\Big[\sup_{0\le t\le T} M_t\Big]\Big\}^{1/2},$$

where

$$M_t := \exp\Big(\int_{\mathbb{T}} \ln(1+zh)\tilde{N}(du,dz) + \int_{\mathbb{T}} \left[\ln(1+zh) - zh\right]\nu(dz)du\Big).$$

But $(M_t, 0 \le t \le T)$ is an exponential martingale. Thus,

$$\mathbb{E}\Big[\sup_{0\le t\le T} M_t\Big] \le 2\mathbb{E}\Big[M_T\Big] = 2.$$

Inserting this estimate of I into (4.18) proves $\mathbb{E}\Big[\sup_{0\le t\le T} |S^{\pi}(t)|^q\Big] \le K_q < \infty$. In the same way we can show $\mathbb{E}\Big[\sup_{0\le t\le T} |S(t)|^q\Big] \le K_q < \infty$. This completes the proof of the lemma. ■

Lemma 4.2. *Assume (A1)-(A4). Then there is a constant* $K > 0$*, independent of* π*, such that*

$$\mathbb{E}\Big|S^\pi(t) - v_2(t)\Big|^p \leq K\Delta^{p/2}, \quad \forall\ t \in [0,T]\,.$$

Proof Let $t \in [t_j, t_{j+1})$ for some j. Using $|e^x - e^y| \leq (e^x + e^y)|x - y|$ we can write

$$\begin{aligned}\Big|S^\pi(t) - v_2(t)\Big| &= \Big|S^\pi(t) - S^\pi(t_j)\Big| \\ &\leq \Big|S^\pi(t) + S^\pi(t_j)\Big| \cdot \Big|\int_{t_j}^t f(v_2(s))ds + \sum_{t_j \leq s \leq t} \ln(1 + g(v_2(s))Y_{N(s)})\Big|.\end{aligned}$$

An application of the Hölder inequality yields that for any $p > 1$,

$$\begin{aligned}\mathbb{E}\left[\Big|S^\pi(t) - v_2(t)\Big|^p\right] &\leq \left\{\mathbb{E}\left[\Big|S^\pi(t) + S^\pi(t_j)\Big|\right]^{2p}\right\}^{1/2} \\ &\quad \cdot \left\{\mathbb{E}\left|\int_{t_j}^t f(v_2(s))ds + \sum_{t_j \leq s \leq t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2} \\ &\leq K_p\left\{\mathbb{E}\left|\int_{t_j}^t f(v_2(s))ds\right|^{2p} + \mathbb{E}\left|\sum_{t_j \leq s \leq t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2} \\ &\leq K_p\left\{\Delta^{2p} + \mathbb{E}\left|\sum_{t_j \leq s \leq t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2}. \qquad (4.19)\end{aligned}$$

Now we want to bound

$$I := \mathbb{E}\left|\sum_{t_j \leq s \leq t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}.$$

(we use the same notation I to denote different quantities in different occasions and this will not cause ambiguity). We write the above sum as an integral:

$$\begin{aligned} I &= \mathbb{E}\Big|\int_{\mathcal{J}}\int_{t_j}^{t} \ln(1+zg(v_2(s)))N(ds,dz)\Big|^{2p} \\ &= \mathbb{E}\Big|\int_{\mathcal{J}}\int_{t_j}^{t} \ln(1+zg(v_2(s)))\tilde{N}(ds,dz) \\ &\quad + \int_{\mathcal{J}}\int_{t_j}^{t} \ln(1+zg(v_2(s)))\nu(dz)ds\Big|^{2p} \\ &\le C_p\left(\Delta^{2p} + \mathbb{E}\Big|\int_{\mathcal{J}}\int_{t_j}^{t} \ln(1+zg(v_2(s)))\tilde{N}(ds,dz)\Big|^{2p}\right). \end{aligned}$$

By the isometry condition, we have

$$\begin{aligned} &\mathbb{E}\Big|\int_{\mathcal{J}}\int_{t_j}^{t} \ln(1+zg(v_2(s)))\tilde{N}(ds,dz)\Big|^{2p} \\ &\qquad = \mathbb{E}\left(\int_{\mathcal{J}}\int_{t_j}^{t} \Big|\ln(1+zg(v_2(s)))\Big|^2 \nu(dz)ds\right)^p \\ &\qquad \le K_p\Delta^p . \end{aligned}$$

Thus, we have

$$I \le K_{p,T}\Delta^p .$$

Inserting this bound into (4.19) yields the lemma. ■

Our next objective is to obtain the rate of convergence of our logarithmic Euler-Maruyama approximation $S^\pi(t)$ to the true solution $S(t)$.

Theorem 29. *Assume (A1)-(A4). Let $S^\pi(t)$ be the solution to* (4.14) *and let $S(t)$ be the solution to* (4.11)*. Then there is a constant $K_{p,T}$, independent of π such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|S(t)-S^\pi(t)|^p\Big] \le K_{p,T}\Delta^{p/2} . \tag{4.20}$$

Proof We write $S(t) = \phi(0)\exp(X(t))$ and $S^\pi(t) = \phi(0)\exp(p(t))$. Then

$$\Big|S(t) - S^\pi(t)\Big|^p \le \Big|S(t) + S^\pi(t)\Big|^p \Big|X(t) - p(t)\Big|^p .$$

Hence by Lemma 4.1 we have for any $r \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le r} |S(t) - S^\pi(t)|^p\Big] \\
&\le \mathbb{E}\Big[\sup_{0\le t\le r}\Big|S(t) + S^\pi(t)\Big|^{2p}\Big]^{1/2} \mathbb{E}\Big[\sup_{0\le t\le r}\Big|X(t) - p(t)\Big|^{2p}\Big]^{1/2} \\
&\le 2^{2p-1}\Big(\mathbb{E}\Big[\sup_{0\le t\le r}\Big|S(t)\Big|^{2p}\Big] + \mathbb{E}\Big[\sup_{0\le t\le r}\Big|S^\pi(t)\Big|^{2p}\Big]\Big)^{1/2}\Big[\mathbb{E}\sup_{0\le t\le r}\Big|X(t) - p(t)\Big|^{2p}\Big]^{1/2} \\
&\le K_p\Big[\mathbb{E}\sup_{0\le t\le r}\Big|X(t) - p(t)\Big|^{2p}\Big]^{1/2} = K_p I^{1/2} . \qquad (4.21)
\end{aligned}$$

Thus we need only to bound the above expectation I, which is given by the following.

$$\begin{aligned}
I &= \mathbb{E}\Big[\sup_{0\le t\le r} |X(t) - p(t)|^{2p}\Big] \\
&\le \mathbb{E}\sup_{0\le t\le r}\Big|\int_0^t (f(S(u-b)) - f(v_2(u)))du \qquad (4.22) \\
&+ \sum_{0\le u\le t, \Delta Z(u)\ne 0} \ln(1 + g(S(u-b))Y_{N(u)}) - \ln(1 + g(v_2(u))Y_{N(u)})\Big|^{2p} .
\end{aligned}$$

By the Lipschitz conditions we have

$$\begin{aligned}
I \quad &\le K_p\mathbb{E}\int_0^r \Big|S(u-b) - v_2(u)\Big|^{2p} du \\
&+ K_p\mathbb{E}\sup_{0\le t\le r}\Big|\sum_{0\le u\le t, \Delta Z(u)\ne 0} \ln(1 + g(S(u-b))Y_{N(u)}) - \ln(1 + g(v_2(u))Y_{N(u)})\Big|^{2p} \\
&\le K_p\Big[\mathbb{E}\int_0^r \Big|S(u-b) - S^\pi(u-b)\Big|^{2p} du + \mathbb{E}\int_0^r \Big|S^\pi(u-b) - v_2(u)\Big|^{2p} du\Big] \\
&+ K_p\mathbb{E}\sup_{0\le t\le r}\Big|\sum_{0\le u\le t, \Delta Z(u)\ne 0} \ln(1 + g(S(u-b))Y_{N(u)}) - \ln(1 + g(v_2(u))Y_{N(u)})\Big|^{2p} \\
&= I_1 + I_2 + I_3 . \qquad (4.23)
\end{aligned}$$

By Lemma 4.2 and by the assumption (A1) about the Hölder continuity of the initial data ϕ we have

$$I_2 \leq K_{p,T}\Delta^p \,. \tag{4.24}$$

We write the above sum I_3 with jumps as a stochastic integral:

$$\begin{aligned}
I_3 &= \mathbb{E}\sup_{0\leq t\leq r}\Big|\sum_{0\leq u\leq t,\Delta Z(u)\neq 0} \ln(1+g(S(u-b))Y_{N(u)}) - \ln(1+g(v_2(u))Y_{N(u)})\Big|^{2p} \\
&= \mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}}\int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\,\tilde{N}(du,dz) \\
&\qquad + \int_{\mathcal{J}}\int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\,\nu(dz)du\Big|^{2p} \\
&= 4^p\mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}}\int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\,\tilde{N}(du,dz)\Big|^{2p} \\
&\qquad + 4^p\mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}}\int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\,\nu(dz)du\Big|^{2p} \\
&=: I_{31} + I_{32}\,.
\end{aligned}$$

Using the Lipschitz condition on g and (A3), we have

$$\begin{aligned}
I_{32} &\leq K_p\mathbb{E}\Big(\int_0^r \Big|g(S(u-b)) - g(v_2(u))\Big|du\Big)^{2p} \\
&\leq K_{p,T}\mathbb{E}\sup_{0\leq t\leq r}|S(t-b)) - S^{\pi}(t-b)|^{2p}\,.
\end{aligned}$$

Using the theorem 2.13 of [17] we have

$$I_{31} \leq K_p\mathbb{E}\Big(\int_{\mathcal{J}}\int_0^r \Big|\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))\Big|^{2p}\nu(dz)du\Big)\,.$$

Similar to the bound for I_{32}, we have

$$I_{31} \leq K_{p,T}\mathbb{E}\sup_{0\leq t\leq r}|S(t-b)) - S^{\pi}(t-b)|^{2p}\,.$$

Combining the estimates for I_{31} and $_{32}$, we see

$$I_3 \leq K_{p,T}\mathbb{E}\sup_{0\leq t\leq r}|S(t-b)) - S^{\pi}(t-b)|^{2p} \,. \tag{4.25}$$

It is easy to verify

$$I_1 \leq K_{p,T}\mathbb{E}\sup_{0\leq t\leq r}|S(t-b)) - S^{\pi}(t-b)|^{2p} \,. \tag{4.26}$$

Inserting the bounds obtained in (4.24)-(4.27) into (4.23), we see that

$$I \leq K_{p,T}\mathbb{E}\sup_{0\leq t\leq r}|S(t-b)) - S^{\pi}(t-b)|^{2p} + K_{P,T}\Delta^p \,. \tag{4.27}$$

Combining this estimate with (4.21), we see

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq t\leq r}|S(t) - S^{\pi}(t)|^p\Big] \\ &\qquad \leq K_{p,T}\left[\mathbb{E}\sup_{0\leq t\leq r}|S(t-b) - S^{\pi}(t-b)|^{2p}\right]^{1/2} + K_{P,T}\Delta^{p/2} \end{aligned} \tag{4.28}$$

for any $p \geq 2$ and for any $r \in [0,T]$. Now we shall use (4.28) to prove the theorem on the interval $[0,kb]$ recursively for $k = 1, 2, \cdots, [\frac{T}{b}] + 1$. Since $S^{\pi}(t) = S(t) = \phi(t)$ for $t \in [-b, 0]$. Taking $r = b$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq b}|S(t) - S^{\pi}(t)|^p\Big] \leq K_{p,T}\Delta^{p/2} \tag{4.29}$$

for any $p \geq 2$. Now taking $r = 2b$ in (4.28), we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq t\leq 2b} |S(t) - S^{\pi}(t)|^p \Big] \\ \leq K_{p,T} \left[\mathbb{E} \sup_{-b\leq t\leq b} |S(t)) - S^{\pi}(t)|^{2p} \right]^{1/2} + K_{P,T}\Delta^{p/2} \\ \leq K_{p,T} \left[K_{2p,T}\Delta^{p}\right]^{1/2} + K_{P,T}\Delta^{p/2} \leq K_{p,T}\Delta^{p/2} . \end{aligned} \tag{4.30}$$

Continuing this way we obtain for any positive integer $k \in \mathbb{N}$,

$$\mathbb{E}\Big[\sup_{0\leq t\leq kb} |S(t) - S^{\pi}(t)|^p \Big] \leq K_{k,p,T}\Delta^{p/2} . \tag{4.31}$$

Now since T is finite, we can choose a k such that $(k-1)b < T \leq kb$. This completes the proof of the theorem. ■

4.5 Option Pricing in Delayed Black-Scholes market with jumps

In this section we consider the problem of option pricing in a delayed Black-Scholes market which consists of two assets. One is risk free, whose price is described by

$$dB(t) = rB(t)dt \, , \quad \text{or} \quad B(t) = e^{rt} \, , t \geq 0 \, . \tag{5.32}$$

Another asset is a risky one, whose price is described by the delayed equation (1.1) or (4.11), namely,

$$dS(t) \;\; = \;\; f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t) \, , \tag{5.33}$$

where $Z(t) = \sum_{i=1}^{N_t} Y_i$ is a Lévy process, N_t is a Poisson process with intensity λ, and $Y_1, Y_2, \cdots$, are iid random variables. As in Section 2, we introduce the Poisson random measure $N(dt, dz)$ and its compensator $\tilde{N}(dt, dz)$. The above delayed equation can be written as

$$\begin{aligned} dS(t) &= \left[f(S(t-b)) + g(S(t-b)) \int_{\mathcal{J}} z\nu(dz) \right] S(t)dt \\ &\qquad + g(S(t-b))S(t-) \int_{\mathcal{J}} z\tilde{N}(dt, dz)\,. \end{aligned}$$

Denote

$$L = \int_{\mathcal{J}} z f_Y(z) dz\,, \tag{5.34}$$

where f_Y is the probability density of Y_i (whose support is $\mathcal{J}$). Then

$$\int_{\mathcal{J}} z\nu(dz) = \lambda L\,.$$

Set

$$\tilde{S}(t) = \frac{S(t)}{B(t)}\,.$$

Then by Itô's formula we have

$$d\tilde{S}(t) = \tilde{S}(t-)g(S(t-b)) \Big(\int_{\mathcal{J}} z \big[\theta(t)\nu(dz)dt + \tilde{N}(dt, dz) \big] \Big)\,, \tag{5.35}$$

where $\theta(t) = \frac{f(S(t-b)) + g(S(t-b)) - r}{\lambda L g(S(t-b))}$. We shall keep the assumptions (A1)-(A4) made in previous section and we need to make an additional assumption:

(A5) There is a constant $\alpha_1 \in (1, \infty)$ such that $\int_{\mathcal{J}} \nu(dz) \geq \alpha_1 \Big| \frac{f(s) + g(s) - r}{g(t)} \Big|$ $\forall\, s, t \in [0, \infty)$

To find the risk neutral probability measure we apply Girsanov theorem for Lévy process (see [14, Theorem 12.21]). The $\theta(t)$ is predictable for $t \in [0, T]$. From the assumptions above we also have that $0 < \theta(s) \leq \frac{1}{\alpha_1}$. Thus,

$$\int_{[0,T]\times\mathcal{J}} \Big(|\log(1+\theta(s))| + \theta^2(s)\Big)\nu(dz)ds \leq K < \infty\,.$$

Now define

$$\begin{aligned} S^\theta(t) \quad := \quad & \exp\Big(\int_{[0,t]} \{\log\big(1-\theta(s)\big) + \theta(s)\}\nu(dx)ds \\ & + \int_{[0,t]} \log\big(1-\theta(s)\big)\tilde{N}(dx,ds)\Big)\,. \end{aligned}$$

In order for us to obtain an equivalent martingale measure we need to verify the following Novikov condition:

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_{[0,T]\times\mathcal{J}} \{(1-\theta(s))\log(1-\theta(s)) + \theta(s)\}\nu(dz)ds\Big)\Big] < \infty \qquad (5.36)$$

This is a consequence of our assumption (A5). In fact, we have first

$$|\theta(s)| \quad = \quad \frac{|f(S(t-b)) - r|}{\lambda L g(S(t-b))} \leq \frac{1}{\alpha_1} < 1\,.$$

Hence we have

$$\int_{[0,T]} \{(1-\theta(s))\log(1-\theta(s)) + \theta(s)\}ds \quad < \quad \infty\,.$$

But $\nu(dz) = \lambda f_Y(z)dz$, we have

$$\int_{\mathcal{J}} \nu(dz) \quad = \quad \int_{\mathcal{J}} \lambda f_Y(z)dz < \infty\,.$$

Thus, we have (5.36).

Now since we have verified the Novikov condition (5.36) we have then $\mathbb{E}[S^\theta(T)] = 1$. Define an equivalent probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q} := S^\theta(T)d\mathbb{P}\,. \tag{5.37}$$

On the new probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$ (new probability $\mathbb{Q}$) the random measure

$$\tilde{N}_{\mathbb{Q}}(dz, ds) = \theta(t)\nu(dz)ds + \tilde{N}(dz, ds)\,, \tag{5.38}$$

is a compensated Poisson random measure. With this new Poisson random measure we can write (5.35) as

$$d\tilde{S}(t) = \tilde{S}(t-)\int_{\mathcal{J}} zg(S(t-b))\tilde{N}_{\mathbb{Q}}(dt, dz)\,. \tag{5.39}$$

The following result gives the fair price formula for the European call option as well as the corresponding hedging portfolio.

Theorem 30. *Let the market be given by* (5.32) *and* (5.33)*, where the coefficients f and g satisfy the assumptions (A1)-(A5). Then the market is complete. Let T be the maturity time of the European call option on the stock with payoff function given by $X = (S_T - K)^+$. Then at any time $t \in [0, T]$, the fair price V(t) of the option is given by the formula*

$$V(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big((S_T - K)^+|\mathcal{F}_t\Big) \tag{5.40}$$

where $\mathbb{Q}$ is the martingale measure on $(\Omega, \mathcal{F}_T)$ given by (5.37)*.*
Moreover, if $\int_{\mathcal{J}} z^j \nu_{\mathbb{Q}}(dz) < \infty$, $\int_{\mathbb{R}_+} g(t)^j dt < \infty$ for $j = 1, 2, 3, 4$, there is an adapted

and square integrable process $\psi(z,t) \in \mathcal{L}^2(\mathcal{J} \times [0,T])$ *such that*

$$\mathbb{E}_{\mathbb{Q}}\Big(e^{-rT}(S_T - K)^+|\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-rT}(S_T - K)^+\Big) + \int_{[0,t]\times\mathcal{J}} \psi(z,s)\tilde{N}_{\mathbb{Q}}((dz,ds)$$

and the hedging strategy is given by

$$\pi_S(t) := \frac{\displaystyle\int_{\mathcal{J}} \psi(z,t)\tilde{N}_{\mathbb{Q}}(dz,t)}{\tilde{S}(t)g(S(t-b))}, \quad \pi_B(t) := U(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0,T]\,, \tag{5.41}$$

where $U(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+|\mathcal{F}_t)$.

Proof Applying the Itô formula to (5.39) we get

$$\begin{aligned}\tilde{S}(T) = \exp\Big(&\int_{[0,T]\times\mathcal{J}} \{\ln(1 + zg(S(t-b))) - zg(S(t-b)\}\nu_{\mathbb{Q}}(dz)dt \\ &+ \int_{[0,T]\times\mathcal{J}} \ln(1 + zg(S(t-b)))\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\end{aligned} \tag{5.42}$$

Denote $X = (S_T - K)^+$ and consider

$$U(t) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}X|\mathcal{F}_t)\,.$$

In order to apply martingale representation theorem for Lévy process (see e.g. [4, Theorem 5.3.5]) we shall first show that $U_t \in \mathcal{L}^2$, which is implied by $\mathbb{E}_{\mathbb{Q}}[S_T^2] < \infty$.

Write $h = g(S(t-b))$. Then we can write

$$\begin{aligned}\tilde{S}_T^2 = \exp\Big(&\int_{[0,T]\times\mathcal{J}} \{\ln(1 + zh)^2 - 2zh\}\nu_{\mathbb{Q}}(dz)dt \\ &+ \int_{[0,T]\times\mathcal{J}} \ln(1 + zh)^2\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\,.\end{aligned} \tag{5.43}$$

Denoting $\mathbb{T} = [0, T] \times \mathcal{J}$ and taking $\tilde{h} = \frac{(1+zh)^4-1}{z}$ we have

$$\begin{aligned}\tilde{S}_T^2 &= \exp\Big(\frac{1}{2}\int_{\mathbb{T}}\{\ln(1+z\tilde{h}) - z\tilde{h}\}\nu_{\mathbb{Q}}(dz)dt + \frac{1}{2}\int_{\mathbb{T}}\ln(1+z\tilde{h})\tilde{N}_{\mathbb{Q}}(dt,dz)\Big). \\ &\qquad \exp\Big(\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big).\end{aligned}$$

Applying the Hölder inequality we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\big[\tilde{S}_T^2\big] \\ &\leq \Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(\int_{\mathbb{T}}\{\ln(1+z\tilde{h}) - z\tilde{h}\}\nu_{\mathbb{Q}}(dz)dt + \int_{\mathbb{T}}\ln(1+z\tilde{h})\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\Big]^{1/2} \\ &\qquad \cdot\Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(2\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{1/2} \\ &= \Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(2\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{1/2}.\end{aligned}$$

From the definition of $\tilde{h}$, we have $z\tilde{h} = (1+zh)^4 - 1$. Then

$$z\tilde{h} - 2zh = (1+zh)^4 - 1 - 2zh = z^4h^4 + 4z^3h^3 + 6z^2h^2 + 2zh\,.$$

Thus,

$$\mathbb{E}_{\mathbb{Q}}\big[\tilde{S}_T^2\big] \leq \exp\Big(\int_{\mathbb{T}}\Big(z^4h^4 + 4z^3h^3 + 6z^2h^2 + 2zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)$$

which is finite by the assumptions of the theorem.

From the martingale representation theorem (see e.g. [4, theorem 5.3.5]) there exists a square integrable predictable mapping $\psi : \mathbb{T} \times \Omega \to \mathbb{R}$ such that

$$U(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+) + \int_0^t\int_{\mathcal{J}}\psi(s,z)\tilde{N}(ds,dz).$$

Define

$$\begin{aligned}\pi_S(t) &:= \frac{\int_{\mathcal{J}} \psi(z,t)\tilde{N}_{\mathbb{Q}}(dz,t)}{\tilde{S}(t)g(S(t-b))} \\ &= \frac{\int_{\mathcal{J}} \psi(z,t)\tilde{S}(t)g(S(t-b))d\tilde{S}(t)}{\tilde{S}(t)g(S(t-b))}, \\ \pi_B(t) &:= U(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0,T]\,.\end{aligned}$$

Consider the strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0,T]\}$ to invest $\pi_B(t)$ units in the riskyless asset $B(t)$ and $\pi_S(t)$ units in the risky asset $S(t)$ at time t. Then the value of the portfolio at time t is given by

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}U(t)$$

By the definition of the strategy we see that

$$dV(t) = \pi_B(t)de^{rt} + \pi_S(t)dS(t) = e^{rt}dU(t) + U(t)de^{rt}\,.$$

Hence the strategy is self-financing. Moreover, we have

$$V(T) = e^{rT}U(T) = (S_T - K)^+.$$

Hence the claim (referring to the European call option) is attainable stand therefore the market $\{S(t), B(t) : t \in [0,T]\}$ is complete. ■

The pricing formula (5.40) is hard to evaluate analytically and we shall use a general Monte-Carlo method to find the approximate values. But when the time fall in the last delay period, namely, when $t \in [T-b, T]$ we have the following analytic expression for the price.

Theorem 31. *Assume the conditions of Theorem 30. When $t \in [T-b, T]$, then price for the European Call option is given by*

$$\begin{aligned} V(t) &= e^{rt} \lim_{v\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi}) A(t) \cdot \tilde{S}(t) \exp\Big\{ \int_t^T \int_{\mathcal{J}} \Big((1 + zg(S(u-b)))^{(1-i\xi)} \\ &\quad - (1-i\xi)\ln(1 + zg(S(u-b))) - 1 \Big) \nu_{\mathbb{Q}}(dz)du \Big\} \\ &\quad - Ke^{rt} \lim_{v\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi}) A(t) \cdot \tilde{S}(t) \exp\Big\{ \int_t^T \int_{\mathcal{J}} \Big((1 + zg(S(u-b)))^{-i\xi} \\ &\quad + i\xi \ln(1 + zg(S(u-b))) - 1 \Big) \nu_{\mathbb{Q}}(dz)du \Big\}, \end{aligned} \tag{5.44}$$

where $w = \ln(K/A) - rT$ and

$$A(t) = \exp\Big(\int_t^T \int_{\mathcal{J}} \{\ln(1 + zg(S(u-b))) - zg(S(u-b)) \nu_{\mathbb{Q}}(dz)du \Big). \tag{5.45}$$

Proof By (5.40) for any time $t \in [0, T]$ we have

$$\begin{aligned} V(t) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\Big((S(T) - K)^+ \mid \mathcal{F}_t \Big) \\ &= e^{rt} \mathbb{E}_{\mathbb{Q}}\Big((\tilde{S}(T) - Ke^{-rT})^+ \mid \mathcal{F}_t \Big) \\ &= e^{rt} \mathbb{E}_{\mathbb{Q}}\Big(\tilde{S}(T) \mathbb{1}_{\{\tilde{S}(T) \ge Ke^{-rT}\}} \mid \mathcal{F}_t \Big) - Ke^{rt} \mathbb{Q}(\tilde{S}(T) \ge Ke^{-rT}) \\ &=: V_1(t) - V_2(t). \end{aligned} \tag{5.46}$$

First, let us compute $V_1(t)$ and $V_2(t)$ can be computed similarly. The solution $\tilde{S}(t)$ is given by (5.42), which we rewrite here:

$$\begin{aligned} \tilde{S}(T) &= \tilde{S}(t) \exp\Big\{ \int_t^T \int_{\mathcal{J}} \{\ln(1 + zg(S(u-b))) - zg(S(u-b))\} \nu_{\mathbb{Q}}(dz)du \\ &\quad + \int_t^T \int_{\mathcal{J}} \ln(1 + zg(S(u-b))) \tilde{N}_{\mathbb{Q}}(dz, du) \Big\}. \end{aligned} \tag{5.47}$$

When $u \in [t, T]$ and $t \in [T-b, T]$, we see that $S(u-b)$ is $\mathcal{F}_t$-measurable. Hence while computing the conditional expectation of $h(\tilde{S}(T))$ with respect to $\mathcal{F}_t$, we can consider the integrands $\ln(1 + zg(S(u-b)))$ and $\ln(1 + zg(S(u-b))) - zg(S(u-b))$ as "deterministic" functions. Thus, the analytic expression for the conditional expectation is possible. But it is still complicated. To find the exact expression and to simplify the presentation, let us use the notation (5.45) and introduce

$$Y = \int_t^T \int_{\mathcal{J}} \ln\left(1 + zg(S(u-b))\right)\tilde{N}_{\mathbb{Q}}(dz, du)\,.$$

With these notation we have

$$\tilde{S}(T) = \tilde{S}(t) A \exp Y\,.$$

To calculate $\mathbb{E}_{\mathbb{Q}}\Big(e^Y \mathbb{1}_{\{v \geq Y \geq w\}}\Big)$ we first express $\mathbb{1}_{[w,v]}$ as the (inverse) Fourier transform of exponential function because $\mathbb{E}(e^{i\xi Y})$ is computable. Since the Fourier transform of $\mathbb{1}_{\{w,v\}}$ is

$$\int_{-\infty}^{\infty} e^{ix\xi}\mathbb{1}_{[w,v]}dx = \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})$$

we can write

$$\mathbb{1}_{[w,v]}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{i[v-x]\xi} - e^{i[w-x]\xi})d\xi\,.$$

Therefore we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(e^Y \mathbb{1}_{\{v \geq Y \geq w\}} \mid \mathcal{F}_t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{E}_{\mathbb{Q}}\Big(\frac{1}{i\xi}(e^{i[v-Y]\xi + Y} - e^{i[w-Y]\xi+Y}) \mid \mathcal{F}_t\Big)d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})\mathbb{E}_{\mathbb{Q}}(e^{Y(1-i\xi)} \mid \mathcal{F}_t)d\xi\,.
\end{aligned}$$

Denote $\mathbb{T}_t = [t, T] \times \mathcal{J}$. Then we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(e^{Y-iY\xi}) &= \mathbb{E}_{\mathbb{Q}}\Big(\exp \int_{\mathbb{T}_t} (1-i\xi)\ln\left(1+zg(S(u-b))\right)\tilde{N}(dz,du) \mid \mathcal{F}_t\Big) \\
&= \mathbb{E}_{\mathbb{Q}}\Big(\exp \int_{\mathbb{T}_t} (1-i\xi)\ln\left(1+zg(S(u-b))\right)\tilde{N}(dz,du)\Big) \\
&= \exp\Big(\int_{\mathbb{T}_t}\{e^{(1-i\xi)\ln(1+zg(S(u-b)))} \\
&\qquad -(1-i\xi)\ln(1+zg(S(u-b)))-1\}\nu_{\mathbb{Q}}(dz)du\Big) \\
&= \exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\
&\qquad -\ln(1+zg(S(u-b)))^{(1-i\xi)}-1\}\nu_{\mathbb{Q}}(dz)du\Big)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(e^{Y}\mathbb{1}_{\{v\geq Y\geq w\}} \mid \mathcal{F}_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi}-e^{iw\xi})\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\
-\ln(1+zg(S(u-b)))^{(1-i\xi)}-1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi\,.
\end{aligned}
$$

Taking $w = \ln(K/A) - rT$, $v \to \infty$ in the above formula we can evaluate (5.46) as follows.

$$
\begin{aligned}
V_1(t) &= e^{rt}\mathbb{E}_{\mathbb{Q}}\Big(\tilde{S}(T)\mathbb{1}_{\{\tilde{S}(T)\geq Ke^{-rT}\}} \mid \mathcal{F}_t\Big) \\
&= e^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi}-e^{iw\xi})A\cdot\tilde{S}(t)\cdot\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\
&\quad -\ln(1+zg(S(u-b)))^{(1-i\xi)}-1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi \\
&= e^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi}-e^{iw\xi})A\cdot\tilde{S}(t).\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\
&\quad -\ln(1+zg(S(u-b)))^{(1-i\xi)}-1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi\,.
\end{aligned}
$$

Exactly in the same way (and now without the factor e^Y), we have

$$V_2(t) \quad = Ke^{rt} \lim_{v\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A \cdot \tilde{S}(t). \exp\Big(\int_{\mathbb{T}_t} \{(1 + zg(S(u-b)))^{-i\xi} - \ln(1 + zg(S(u-b)))^{-i\xi} - 1\}\nu_{\mathbb{Q}}(dz)du\Big) d\xi.$$

This gives (5.44). ■

4.6 Numerical attempt

In this section we make an attempt to carry out some numerical computations of our formula (3.39) against the American call options Microsoft stock traded in Questrade platform. To apply our model in the financial market, we need to estimate all the parameters including the delay factor b from the real data. To the best of our knowledge the theory on the parameter estimation is still unavailable even in the case of the classical model of [3]. Motivated by the work of [19], we try our best guess of the parameters in the model (3.31)-(3.32).

The real market option prices we consider is for the American call option on Microsoft stock. The data we use is from Questrade trading/investment platform on October 5, 2020 at 12:25 PM (EDT). We take T to be one, three and six months active trading period respectively. The real prices of the options of different strike prices are listed in the last column of the three tables below.

The readers may wonder that since the option pricing formulas for both our model and the classical Black-Scholes model are for the European call option, why we use the market price for the American option. The reason is that we can only find the market price for the American option. On the other hand, as stated in [33, p.251] "There is no advantage to exercise an American call prematurely when the asset received upon early exercise does not pay dividends. The early exercise right is rendered worthless when the underlying asset does not pay dividends, so in this case the American call has the same value as that of its European counterpart". See also [26, p.61, Theorem 6.1]. This justifies our use of the market price for the American option.

Using Monte-Carlo simulation we calculate the prices of European option given by (5.40) and the analogous Black-Scholes formula obtained from the model: $dS(t) = S(t)[\alpha dt + \sigma dW(t)]$. We simulate 2000 paths of the solutions to both equations using the logarithmic Euler-Maruyama scheme [for Black-Scholes model the logarithmic Euler-Maruyama scheme is the same by replacing the jump process by Brownian motion]. In the simulations we take the time step Δ to be the trading unit minute. So when $T = 1$ month, there are

$$n = \text{trading hours} \times 60 \times \text{trading days} = 6.5 \times 60 \times 22 = 8580$$

minutes. So $\Delta = \frac{1}{8580}$. We do the same for $T = 3$ and $T = 6$.

In our calculation for the delayed jump model we use the double exponential jump process as our Y_i's with parameters $p = .60, q = 1 - p = .40, \eta = 12.8, \theta = 8.40$ with the intensity $\lambda = .03$. The interest rate $r = .01$ is the risk free rate. The delay factor was taken to be one day which is $b = \frac{6.5 \times 60}{8580}$ because there are trading 6.5 hours in a trading day. The function $f(x)$ was taken to be a fixed constant $f(x) = .1$, $g(x) = .15 * \sin(x/209.11)$ and $\phi(x) = \exp(\alpha x/n)$ with $\alpha = .11$. We choose $\alpha = .11$ since the initial price we have taken is 209.11 and the predicted average price target of Microsoft stock for next one year (around 12 months from October 5, 2020) is 230 which is 11%.

For the simulation of the Black-Scholes model, based on stock prices for the year 2019 we take volatility of the Microsoft stock as $\sigma = 15\%$ to calculate Black-Scholes price. We have taken $r = 1\%$ since in the last one year the range of 10 year treasury rate has been between .52% to 1.92%. In the following tables the computations have

Call Option price comparison for $T = 1$ month for Microsoft stock			
Strike Price	Black-Scholes option price (European) with 1 month expiration (no delay)	Option price of jump model (European) with 1 month expiration	Market Price of American option with expiration 1 month
195	16.27	16.08	18.3
200	11.41	11.05	15.15
205	7.65	6.91	12
210	4.54	3.62	9.43
215	2.05	1.48	7
220	.83	.61	5.15

Table 4.1: T=1 month

Call Option price comparison for $T = 3$ month for Microsoft stock			
Strike Price	Black-Scholes option price (European) with 3 month expiration (no delay)	Option price of jump model (European) with 3 month expiration	Market Price of American option with expiration 3 months
195	21.37	21.27	24.40
200	16.72	16.99	21.35
205	13.08	14.50	18.55
210	9.65	11.43	15.95
215	6.35	8.58	13.65
220	4.31	7.51	11.55

Table 4.2: T=3 month

been summarized. Notice an interesting phenomenon that the price we obtain by using our formula is comparable to the Black-Scholes price for shorter maturities and is more closer to the real market price for longer maturity. This may be because of our choice of the parameters by guessing.

Call Option price comparison for $T = 6$ month for Microsoft stock			
Strike Price	Black-Scholes option price (European) with 6 month expiration (no delay)	Option price of jump model (European) with 6 month expiration	Market Price of American option with expiration 6 months
195	28.41	29.53	29.00
200	23.85	26.11	26.15
205	19.49	24.44	23.50
210	16.24	21.15	21.05
215	12.83	18.39	18.80
220	10.58	17.97	16.70

Table 4.3: T=6 month

Chapter 5

Logarithmic Euler-Maruyama scheme for multi-dimensional SDDE driven by Lévy process

In [1] we introduced a logarithmic Euler-Maruyama scheme for a single stochastic delay equation, which preserve positivity if the solution to the original equation is positive. The convergence rate was also obtained for such scheme. This scheme is important for simulation of the paths of the equation. It plays important role in option pricing for example since we often cannot obtain the explicit pricing value and we need to use Monte-Carlo to complete the evaluation. Naturally our next question would be what will be the analogous scheme for a system of stochastic delay equations and if such schemes converges. This type of problems is very important since there is always more than one stock in the real market. Now in more than one dimension, the problem of positive solution and the numerical schemes which preserve the positivity are much more complicated. In this chapter we shall extend our work in [1] to a system of stochastic delay differential equations. The problems of existence and uniqueness of a positive are solved. The multi-dimensional logarithmic Euler-Maruyama scheme are constructed which preserve the positivity of the approximate solutions. The scheme is proved to be convergent with rate 0.5.

5.1 Positivity

We consider the following system of stochastic delayed differential equations:

$$\begin{cases} dS_i(t) = \sum_{j=1}^{d} f_{ij}(S(t-b))S_j(t)dt \\ \qquad\qquad + S_i(t-)\sum_{j=1}^{d} g_{ij}(S(t-b))dZ_j(t), , \quad i = 1, \cdots, d, \\ S_i(t) = \phi_i(t), \quad t \in [-b, 0], \ i = 1, \cdots, d, \end{cases} \tag{1.1}$$

where $S(t) = (S_1(t), \cdots, S_d(t))^T$ and

(i) $f_{ij}, g_{ij} : \mathbb{R}^d \to \mathbb{R}$ are some given bounded measurable functions for all $0 \le i, j \le d$ with $f_{ij} \ge 0$ for $i \ne j$.

(ii) $b > 0$ is a given number representing the delay of the equation.

(iii) $\phi_i : [-b, 0] \to \mathbb{R}$ is a (deterministic) measurable function for all $0 \le i \le d$.

(iv) $Z_j(t) = \sum_{k=1}^{N_j(t)} Y_{j,k}$ are Lévy processes, where $Y_{j,k}$ are i.i.d random variables, $N_\ell(t)$ are independent Poisson random processes which are also independent of $Y_{j,k}$ for $j, \ell, = 1, 2, \cdots, d$, $k = 1, 2, \cdots$

Let $|\cdot|$ Euclidean norm in $\mathbb{R}^d$. If A is $d \times m$ matrix, we denote

$$|A| = \sup_{|x| \le 1} |Ax| .$$

For example, if $A = I + M$ is a $d \times d$ matrix, where $M = (m_{ij})_{1 \le i,j \le d}$ is a matrix, then we can bound the norm of A as follows. Let $0 \le \lambda_1 \le \cdots \le \lambda_d$ be eigenvalues of

M^TM (since M^TM is a positive definite matrix, we can assume that its eigenvalues are all positive). Then

$$\begin{aligned} |I+M| &= \sup_{|x|\le 1}\sqrt{|x|^2+x^TM^TMx} \le \sqrt{1+\max_{1\le i\le d}\lambda_i}|x| \\ &\le \sqrt{1+\sum_{i=1}^d \lambda_i}|x|\,. \end{aligned}$$

But $\sum_{i=1}^d \lambda_i = \operatorname{Tr}(M^TM)$. Thus we have

$$|I+M| \le \sqrt{1+\operatorname{Tr}(M^TM)}|x|\,. \tag{1.2}$$

To study the above stochastic differential equation, it is common to introduce the Poisson random measure associated with the Lévy process $Z_j(t)$. We write the jumps of the process Z_j at time t by

$$\Delta Z_j(t) := Z_j(t) - Z_j(t-) \quad \text{if } \Delta Z_j(t) \neq 0 \quad j = 1, 2, \cdots, d\,.$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. For any $t > 0$ and for any $U \in \mathcal{B}(\mathbb{R}_0)$ we define the *Poisson random measure*, $N_j : [0,T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$ (without confusion we use the same notation N), associated with the Lévy process $Z_j(t)$ by

$$N_j(t,U) := \sum_{0\le s\le t,\ \Delta Z_j(s)\neq 0} \chi_U(\Delta Z_j(s)), \quad j = 1, 2, \cdots, d, \tag{1.3}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of the Lévy process Z_j is given by

$$\nu_j(U) := \mathbb{E}[N_j(1,U)] \qquad j = 1, 2, \cdots, d. \tag{1.4}$$

We now define the compensated Poisson random measure $\tilde{N}_j$ associated with the Lévy process $Z_j(t)$ by

$$\tilde{N}_j(dt,dz) := N_j(dt,dz) - \mathbb{E}\left[N_j(dt,dz)\right] = N_j(dt,dz) - \nu_j(dz)dt\,. \tag{1.5}$$

We assume that the process $Z_j(t)$ has only bounded negative jumps to guarantee that the solution $S(t)$ to (1.1) is positive. This means that there is an interval $\mathcal{J} = [-R,\infty)$ bounded from the left such that $\Delta Z_j(t) \in \mathcal{J}$ for all $t > 0$ and for all $j = 1, 2, \cdots d$.

With these notations, we can write

$$Z_j(t) = \int_{[0,t]\times\mathcal{J}} zN_j(ds,dz) \quad \text{or} \quad dZ_j(t) = \int_{\mathcal{J}} zN_j(dt,dz)$$

and write (1.1) as

$$\begin{cases} dS_i(t) = \displaystyle\sum_{j=1}^d f_{ij}(S(t-b))S_j(t)dt + S_i(t-)\sum_{j=1}^d \int_{\mathcal{J}} zg_{ij}(S(t-b))\nu_j(dz)dt \\ \qquad\qquad + S_i(t-)\displaystyle\sum_{j=1}^d \int_{\mathcal{J}} zg_{ij}(S(t-b))\tilde{N}_j(dz,dt)\,, \\ S_i(t) = \phi_i(t)\,, \quad t\in[-b,0]\,,\ i = 1,\cdots,d\,. \end{cases} \tag{1.6}$$

In fact we can consider a slightly more general version of system of equations than (1.6):

$$\begin{cases} dS_i(t) = \displaystyle\sum_{j=1}^d f_{ij}(S(t-b))S_j(t)dt \\ \qquad\qquad + S_i(t-)\displaystyle\sum_{j=1}^d \int_{\mathcal{J}} g_{ij}(z,S(t-b))\tilde{N}_j(dz,dt), \quad i = 1,\cdots,d\,, \\ S_i(t) = \phi_i(t)\,, \quad t\in[-b,0]\,,\ i = 1,\cdots,d\,. \end{cases} \tag{1.7}$$

First, we discuss the existence, uniqueness and positivity of (1.7).

Theorem 32. *Suppose that* $f_{ij}:\mathbb{R}^d\to\mathbb{R}$ *and* $g_{ij}:\mathcal{J}\times\mathbb{R}^d\to\mathbb{R}\,,\ 1\le i,j\le d$ *are bounded measurable functions such that there is a constant* $\alpha_0>1$ *satisfying* $g_{ij}(z,x)\ge\alpha_0>-1$ *for all* $1\le i,j\le d$*, for all* $z\in\mathcal{J}$ *and for all* $x\in\mathbb{R}$*, where* $\mathcal{J}=[-R,\infty)$ *is the common supporting set of the Poisson measures* $\tilde N_j(t,dz), j=1,\cdots,d$*. If for all* $i\ne j$*,* $f_{ij}(x)\ge 0$ *for all* $x\in\mathbb{R}$*, and* $\phi_i(0)\ge 0\,,\ i=1,\cdots,d$*, then, the stochastic differential delay equation* (1.7) *admits a unique pathwise solution such that* $S_i(t)\ge 0$ *almost surely for all* $i=1,\cdots,d$ *and for all* $t>0$*.*

Proof The theorem is stated and proved in [1, Theorem 1] following the method of [19] (where the case of Brownian motion was dealt with). In fact, the existence and uniqueness are routine and easy. The main point is to show the positivity of the solution. The idea in [1] was to decompose the solution to (1.7) as product of some nonnegative processes. Here we give a slightly different decomposition which will prove the positivity and will be very useful in our numerical scheme.

Denote $\tilde f_{ij}(t)=f_{ij}(S(t-b))$ and $\tilde g_{ij}(t,z)=g_{ij}(z,S(t-b))$. Let $Y_i(t)$ be the solution to the stochastic differential equation

$$dY_i(t)=\tilde f_{ii}(t)Y_i(t)dt+Y_i(t-)\sum_{j=1}^d\int_{\mathcal{J}}\tilde g_{ij}(t,z)\tilde N_j(dt,dz)$$

with initial conditions $Y_i(0)=\phi_i(0)$. Since this is a scalar equation for $Y_i(t)$, its explicit solution can be represented

$$\begin{aligned}Y_i(t)\ &=\ \phi_i(0)\exp\Big\{\sum_{j=1}^d\log\left[1+\tilde g_{ij}(s,z)\right]\tilde N_j(ds,dz)+\int_0^t\tilde f_{ii}(s)ds\\ &\quad+\sum_{j=1}^d\int_{[0,t]\times\mathcal{J}}\Big(\log\left[1+\tilde g_{ij}(s,z)\right]-\tilde g_{ij}(s,z)\Big)ds\nu_j(dz)\Big\}\,,\qquad(1.8)\end{aligned}$$

where ν_j is the associated Lévy measure for $\tilde{N}_j(ds, dz)$. Let $p_i(t)$ be the solution to the following system of equations

$$dp_i(t) = \sum_{j=1, j\neq i}^{d} \tilde{f}_{ij}(t) p_j(t) dt\,, \quad p_i(0) = 1\,, \quad i = 1, \cdots, d\,.$$

Since by the assumption that $\tilde{f}_{ij}(t) \geq 0$ almost surely for all $i \neq j$, Theorem [8, p.173] implies that $p_i(t) \geq 0$ for all $t \geq 0$ almost surely. Now it is easy to check by the Itô formula that $\tilde{S}_i(t) = p_i(t) Y_i(t)$ satisfies (1.7) and $\tilde{S}_i(t) \geq 0$ almost surely. By the uniqueness of the solution we see that $S_i(t) = \tilde{S}_i(t)$ for $i = 1, \cdots, d$. The theorem is then proved. ■

5.2 Convergence rate of logarithmic Euler-Maruyama scheme

In this section we construct numerical scheme to approximate (1.1) by positive value processes.

Motivated by the proof of Theorem 32 we shall decompose equation (1.1) into the following system:

$$\begin{cases} dX_i(t) = f_{ii}((S(t-b)))X_i(t)dt + X_i(t-)\sum_{j=1}^{d} g_{ij}(S(t-b))dZ_j(t) & (2.9a) \\ dp_i(t) = \sum_{j=1, j\neq i}^{d} f_{ij}((S(t-b)))p_j(t)dt, & (2.9b) \\ S_i(t) = p_i(t) \cdot X_i(t)\,, \quad i = 1, 2, \cdots, d\,. & (2.9c) \end{cases}$$

The reason is, as in the proof of Theorem 32, that $X_i(t)$ and $p_i(t)$ are all positive.

Consider a finite time interval $[0, T]$ for some fixed $T > 0$ and let π be a partition of the time interval $[0, T]$:

$$\pi : 0 = t_0 < t_1 < \cdots < t_n = T\,.$$

Let $\Delta_k = t_{k+1} - t_k$ and $\Delta = \max_{0\le k\le n-1}(t_{k+1} - t_k)$ and assume $\Delta < b$.

We shall now construct explicit logarithmic Euler-Marauyama recursive scheme to numerically solve (2.9a)-(2.9c). By the expression (1.8) the solution X on $[t_k, t_{k+1}]$ to Equation (2.9a) is given

by

$$X_i(t) \quad = \quad X_i(t_k) \exp\left\{ \int_{t_k}^t f_{ii}(S(s-b))ds + \sum_{j=1}^d \int_{t_k}^t \log\left[1 + g_{ij}(S(u-b))dZ_j(s)\right] \right\},$$

where $Z_j(t) := \sum_{k=1}^{N_j(t)} Y_{j,k}$. If we denote by $F(x)$ the $d \times d$ matrix whose diagonal elements are all zero and whose off diagonal entries are $f_{ij}(x)$, namely,

$$F_{ij}(x) = \begin{cases} 0 & \text{when } i = j \\ f_{ij}(x) & \text{when } i \neq j\,. \end{cases}$$

With this notation we can write (2.9b) as a matrix form:

$$\frac{dp(t)}{dt} = F((S(t-b)))p(t)\,, \quad p(t) = (p_1(t), \cdots, p_d(t))^T\,, \tag{2.10}$$

and its solution on the sub-interval $[t_k, t_{k+1}]$ is given by

$$p(t) = \exp\left(\tilde{F}(S(t-b))\right)p(t_k)\,, \quad t \in [t_k, t_{k+1}]\,, \tag{2.11}$$

where the exponential of a matrix is in the usual sense: $e^A = \sum_{k=0}^{\infty} A^k/k!$, the integral of a matrix is entry-wise. Here due to the noncommutativity $\tilde{F}(S(t-b))$ is complicated to determine and we give the following formula for the sake of completeness:

$$\begin{aligned}\tilde{F}(S(t-b)) &= \sum_{r=1}^{\infty}\sum_{\sigma\in P_r}\left(\frac{(-1)^{e(\sigma)}}{r^2\binom{r-1}{e(\sigma)}}\right)\int_{T_r(t)} \\ &\quad\times[[\cdots[F(S(u_{\sigma(1)}-b)F(S(u_{\sigma(2)}-b)]\cdots]F(S(u_{\sigma(r)}-b)]du_1\cdots du_r\end{aligned} \tag{2.12}$$

is given by the Campbell-Baker-Hausdorff-Dynkin Formula (see e.g. [21], [43]), where P_r is the set of all permutations of $\{1,2,\cdots,r\}$, $e(\sigma)$ is the number of errors in ordering consecutive terms in $\{\sigma(1),\cdots,\sigma(r)\}$, $[AB]=AB-BA$ denotes the commutator of the matrices, and $T_r(t)=\{0<u_1<\cdots<u_r<t\}$.

Analogously to [1] we propose the following logarithmic scheme to approximate the solution:

$$\begin{cases} X_i^\pi(t) = X_i^\pi(t_k)\exp\Big(f_{ii}(S^\pi(t_k-b))(t-t_k) \\ \qquad\qquad + \sum_{j=1}^{d}\ln\Big(1+g_{ij}(S^\pi(t_k-b))(Z_j(t)-Z_j(t_k))\Big)\Big)\,, & \text{(2.13a)} \\ p^\pi(t) = \Big[F(S^\pi(t_k-b)))(t-t_k)+I\Big]p^\pi(t_k), & \text{(2.13b)} \\ S_i^\pi(t) = p_i^\pi(t)X_i^\pi(t)\,, & \text{(2.13c)} \\ X_i^\pi(0)=\phi_i(0)\,,\quad p^\pi(0)=\mathbf{1}\,,\qquad t_k\le t\le t_{k+1}\,,\qquad k=1,2,\cdots,n-1\,. & \text{(2.13d)} \end{cases}$$

We introduce step processes

$$\begin{cases} v_1(t)=\sum_{k=0}^{\infty}\mathbb{1}_{[t_k,t_{k+1})}(t)S^\pi(t_k) \\ v_2(t)=\sum_{k=0}^{\infty}\mathbb{1}_{[t_k,t_{k+1})}(t)S^\pi(t_k-b). \end{cases}$$

Using the above step process we can write the continuous interpolation for X_i as

$$X_i^\pi(t) = \exp\Big(\int_0^t f_{ii}(v_2(u))du + \sum_{j=1}^{d} \sum_{\substack{0\le u\le t\\ \Delta Z(u)\neq 0}} \ln\Big(1+g_{ij}(v_2(u))Y_{j,N_j(u)}\Big)\Big) \tag{2.14}$$

Denote $\lfloor t \rfloor = \max\{k, t_k < t\}$. From (2.13b) we have

$$p^\pi(t) \;=\; \Big[\int_{t_{\lfloor t\rfloor}}^{t} F(v_2(u))du + I\Big] \prod_{k=1}^{\lfloor t\rfloor} \Big[\int_{t_{k-1}}^{t_k} F(v_2(u))du + I\Big]. \tag{2.15}$$

We first show that $p^\pi(t_k) \geq 0$.

Lemma 5.1. *If $\phi(0) \geq 0$ a.s., then $p^\pi(t_k) \geq 0$ a.s. with $p^\pi(t) = \phi(t)$ for all $t \in [-b, 0]$.*

Proof This can be seen from (2.13b) and by induction. Assume $p^\pi(t_k) \geq 0$ a.s. Since by our definition of $F(S^\pi(t_k - b))$ we know all of its components are positive, we see from (2.13b) that $p^\pi(t) \geq 0$ a.s. for all $t_k \leq t \leq t_{k+1}$. ■

Similarly we will have

Lemma 5.2. *If $\phi(0) \geq 0$ a.s., then $X^\pi(t) \geq 0$ a.s. , hence $S^\pi(t) \geq 0$ a.s. for all $0 \leq t \leq T$.*

To obtain the convergence of the logarithmic Euler–Maruyama scheme (2.13a)-(2.13d), we make the following assumptions:

(A1) The initial data $\phi_i(0) > 0$ and it is Hölder continuous, i.e. there exist constant $\rho > 0$ and $\gamma \in [1/2, 1)$ such that for $t, s \in [-b, 0]$

$$|\phi_i(t) - \phi_i(s)| \leq \rho |t-s|^\gamma. \;\; i = 1, 2, \cdots, d. \tag{2.16}$$

(A2) f_{ij} and g_{ij} are global Lipschitz for $i, j = 1, 2, \cdots, d$. This means that there exists a constant $\rho > 0$ such that

$$\begin{cases} \Big|g_{ij}(x_1) - g_{ij}(x_2)\Big| \le \rho|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbb{R}^d \,; \\ \Big|f_{ij}(x_1) - f_{ij}(x_2)\Big| \le \rho|x_1 - x_2| \,, \quad \forall\, x_1, x_2 \in \mathbb{R}^d \,; \\ \big|f_{ij}(x)\big| \le \rho \,, \quad \forall x \in \mathbb{R}^d. \end{cases}$$

(A3) The support $\mathcal{J}$ of the Poisson random measure N_j (associated with Z) is contained in $[-R, \infty)$ for each $j = 1, 2, \cdots, d$ for some $R > 0$ and there are constants $\alpha_0 > 1$ and $\rho > 0$ satisfying $-\rho \le g_{ij}(x) \le \frac{\alpha_0}{R}$ for all $x \in \mathbb{R}^d$ and for all $i, j = 1, 2, \cdots, d$.

(A4) For any $q > 1$ there is a $\rho_q > 0$

$$\int_{\mathcal{J}} (1 + |z|)^q \nu_i(dz) \le \rho_q \,, \quad i = 1, 2, \cdots, d. \tag{2.17}$$

Lemma 5.3. *Let Assumptions (A1)–(A4) be satisfied. Then, for any $q \ge 1$, there exists K_q, independent of the partition π, such that*

$$\mathbb{E}\Big[\sup_{1\le i\le d}\sup_{0\le t\le T} |X_i(t)|^q\Big] \vee \mathbb{E}\Big[\sup_{1\le i\le d}\sup_{0\le t\le T} |X_i^\pi(t)|^q\Big] \le K_q.$$

Proof From our definition of X_i^π and boundedness of f_{ij} for all i,j we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|X_i^\pi(t)|^q\Big] &= \mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(q\int_0^t f_{ii}(v_2(u))du \\
&\quad +q\sum_{j=1}^{d}\sum_{\substack{0\le u\le t\\ \Delta Z(u)\neq 0}}\ln(1+g_{ij}(v_2(u))Y_{j,N_j(u)})\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(q\int_0^t f_{ii}(v_2(u))du \qquad (2.18)\\
&\quad +q\sum_{j=1}^{d}\int_{\mathbb{T}}\ln(1+z_jg_{ij}(v_2(u)))N_j(du,dz)\Big)\Big] \\
&\le K\mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(q\sum_{j=1}^{d}\int_{\mathbb{T}}\ln(1+z_jg_{ij}(v_2(u)))N_j(du,dz)\Big)\Big] \\
&=: KI\,, \qquad (2.19)
\end{aligned}$$

where $\mathbb{T}=[0,t]\times\mathcal{J}$. Denote $h_j=((1+z_jg_{i,j}(v_2(u))^{2q}-1))/z_j$. Then,

$$\begin{aligned}
I &= \mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(\frac{1}{2}\sum_{j=1}^{d}\int_{\mathbb{T}_t}\ln(1+z_jh_j)N_j(du,dz_j)\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(\sum_{j=1}^{d}\Big(\frac{1}{2}\int_{\mathbb{T}_t}\ln(1+z_jh_j)\tilde{N}_j(du,dz_j) \\
&\quad +\frac{1}{2}\int_{\mathbb{T}_t}\ln(1+z_jh_j)\nu_j(dz_j)du\Big)\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(\sum_{j=1}^{d}\Big(\frac{1}{2}\int_{\mathbb{T}_t}\ln(1+z_jh_j)\tilde{N}_j(du,dz_j) \\
&\quad +\frac{1}{2}\int_{\mathbb{T}_t}\left[\ln(1+z_jh_j)-z_jh_j\right]\nu_j(dz_j)du\Big)\Big)\Big] \\
&\qquad \sup_{0\le t\le T}\exp\Big(\sum_{j=1}^{d}-\frac{1}{2}\int_{\mathbb{T}_t}(1+z_jg_{ij}(v_2(u))^{2q}-1)\ \nu_j(dz_j)du\Big)\Big] \\
&\le C_q\mathbb{E}\Big[\sup_{0\le t\le T}\exp\Big(\sum_{j=1}^{d}\Big(\frac{1}{2}\int_{\mathbb{T}_t}\ln(1+z_jh_j)\tilde{N}_j(du,dz_j) \\
&\quad +\frac{1}{2}\int_{\mathbb{T}_t}\left[\ln(1+z_jh_j)-z_jh_j\right]\nu_j(dz_j)du\Big)\Big)\Big]\,,
\end{aligned}$$

where we used Assumption (A4) and the boundedness of g_{ij}. Write for $k = 1, 2, \cdots, d$

$$M_{k,t} := \exp\Big(\int_{\mathbb{T}_t} \ln(1 + z_k h_k)\tilde{N}_k(du, dz_k) + \int_{\mathbb{T}_t} [\ln(1 + z_k h_k) - z_k h_k]\, \nu_k(dz_k)du\Big).$$

Then $(M_{k,t}, 0 \leq t \leq T)$ is an exponential martingale. Now an application of the Cauchy–Schwartz inequality yield

$$I \;\leq\; C_q \Big\{ \mathbb{E}\Big[\sup_{0\leq t\leq T} M_{1,t} \Big] \Big\}^{d/2},$$

which proves

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_i^\pi(t)|^q \Big] \leq K_q < \infty.$$

In the same way, we can show $\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_i(t)|^q \Big] \leq K_q < \infty$. This completes the proof of the lemma. ■

Lemma 5.4. *Assume Assumptions (A1)–(A4). Then for $\Delta < 1$, there is a constant $K > 0$, independent of π, such that*

$$\mathbb{E} \sup_{0\leq t\leq T} \Big| S^\pi(t) - v_2(t) \Big|^p \leq K\Delta^{p/2}.$$

Proof Let $t_k = \lfloor t \rfloor$ if $t \in [t_k, t_{k+1})$ for some k. We have $v_2 = (v_{21}, v_{22}, \cdots, v_{2d})$ for which we write in short $v_2 = (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_d)$. For any $i = 1, \cdots, d$,

$$\begin{aligned}
&\mathbb{E} \sup_{0\leq t\leq T} \Big| S_i^\pi(t) - \bar{v}_i(t) \Big|^p = \mathbb{E} \sup_{0\leq t\leq T} \Big| p_i^\pi(t) X_i^\pi(t) - p_i^\pi(\lfloor t \rfloor) X_i^\pi(\lfloor t \rfloor) \Big|^p \\
&= \mathbb{E} \sup_{0\leq t\leq T} \Big| p_i^\pi(t) X_i^\pi(t) - p_i^\pi(\lfloor t \rfloor) X_i^\pi(t) + p_i^\pi(\lfloor t \rfloor) X_i^\pi(t) - p_i^\pi(\lfloor t \rfloor) X_i^\pi(\lfloor t \rfloor) \Big|^p \\
&\leq C\Big(\mathbb{E} \sup_{0\leq t\leq T} \Big| p_i^\pi(t) - p_i^\pi(\lfloor t \rfloor) \Big|^{2p} \Big)^{1/2} \Big(\mathbb{E} \sup_{0\leq t\leq T} \Big| X_i^\pi(t) \Big|^{2p} \Big)^{1/2} \qquad (2.20) \\
&\qquad + C\Big(\mathbb{E} \sup_{0\leq t\leq T} \Big| X_i^\pi(t) - X_i^\pi(\lfloor t \rfloor) \Big|^{2p} \Big)^{1/2} \Big(\mathbb{E} \sup_{0\leq t\leq T} \Big| p_i^\pi(\lfloor t \rfloor) \Big|^{2p} \Big)^{1/2}. \qquad (2.21)
\end{aligned}$$

By Assumption 2 we can bound $\mathbb{E}\sup_{0\le t\le T}\left|p_i^\pi(\lfloor t\rfloor)\right|^{2p}$ and by lemma (5.3) we can bound $\mathbb{E}\sup_{0\le t\le T}\left|X_i^\pi(t)\right|^{2p}$. We now bound the other two components.

$$\mathbb{E}\sup_{0\le t\le T}\left|p_i^\pi(t)-p_i^\pi(\lfloor t\rfloor)\right|^{2p}\le\sum_{j,j\neq i}^{d}\mathbb{E}\sup_{0\le t\le T}\Big|\int_{\lfloor t\rfloor}^{t}f_{ij}(v_2(u))du\Big|^{2p}. \qquad (2.22)$$

By Assumption 2 it is easy to see that for some constant C_1

$$\mathbb{E}\sup_{0\le t\le T}\left|p_i^\pi(t)-p_i^\pi(\lfloor t\rfloor)\right|^{2p}\le C_1\Delta^{2p}. \qquad (2.23)$$

For $\mathbb{E}\sup_{0\le t\le T}\left|X_i^\pi(t)-X_i^\pi(\lfloor t\rfloor)\right|^{2p}$ we use the expression for $X_i^\pi(t)$, boundedness of f_{ij} for all i,j and use $|e^x-e^y|\le|e^x+e^y||x-y|$ to obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\left|X_i^\pi(t)-X_i^\pi(\lfloor t\rfloor)\right|^{2p} \le\ & \left\{\mathbb{E}\sup_{0\le t\le T}\left|X_i^\pi(t)+X_i^\pi(\lfloor t\rfloor)\right|^{2p}\right\}^{1/2}\\ & \cdot K\left\{\mathbb{E}\sup_{0\le t\le T}\left[\Big|\sum_{j=1}^{d}\sum_{\lfloor t\rfloor\le s<t}\ln(1+g_{ij}(v_2(s))Y_{j,N(s)})\Big|\right]^{2p}\right\}^{1/2}.\end{aligned}$$

The first factor is bounded and now, we want to bound the second factor:

$$I:=\mathbb{E}\sup_{0\le t\le T}\left|\sum_{j=1}^{d}\sum_{\lfloor t\rfloor\le s\le t}\ln(1+g_{i,j}(v_2(s))Y_{j,N_j(s)})\right|^{2p}.$$

(We use the same notation I to denote different quantities in different occasions and this does not cause ambiguity). We write the above sum as an integral:

$$\begin{aligned} I &= \mathbb{E}\sup_{0\le t\le T}\Big|\sum_{j=1}^{d}\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\ln(1+z_j g_{ij}(v_2(s)))N_j(ds,dz_j)\Big|^{2p} \\ &= \mathbb{E}\sup_{0\le t\le T}\Big|\sum_{j=1}^{d}\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\ln(1+z_j g_{ij}(v_2(s)))\tilde{N}_j(ds,dz_j) \\ &\quad +\sum_{j=1}^{d}\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\ln(1+z_j g_{ij}(v_2(s)))\nu_j(dz_j)ds\Big|^{2p} \\ &\le C_p\left(\Delta^{2p}+\mathbb{E}\sup_{0\le t\le T}\Big|\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\ln(1+z_j g_{ij}(v_2(s)))\tilde{N}_j(ds,dz_j)\Big|^{2p}\right). \end{aligned}$$

By the theorem 2.13 of [17], we have

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T}\Big|\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\ln(1+z_j g_{ij}(v_2(s)))\tilde{N}_j(ds,dz_j)\Big|^{2p} \\ &\qquad\le \mathbb{E}\left(\int_{\mathcal{J}}\int_{\lfloor t\rfloor}^{t}\Big|\ln(1+z_j g_{ij}(v_2(s)))\Big|^{2p}\nu_j(dz_j)ds\right) \\ &\qquad\le K_p\Delta^{2p}. \end{aligned} \tag{2.24}$$

Plugging above, (2.23), in (2.21) we get for some $K, K_1, K_2 > 0$

$$\mathbb{E}\sup_{0\le t\le T}\Big|S_i^{\pi}(t)-v_i(t)\Big|^{p}\le K_1\Delta^{p}+K_2\Delta^{p}<K\Delta^{p/2}. \tag{2.25}$$

This proves the lemma. ■

Theorem 33. *Assume that Assumptions (A1)–(A4) are true. Then, there is a constant $K_{pd,T}$, independent of π such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\Big[|S(t)-S^{\pi}(t)|^{p}\Big]\right]\le K_{pd,T}\Delta^{p/2}. \tag{2.26}$$

Proof First, we we want to bound

$$I_1 := \mathbb{E}\Big(\sup_{0\le t\le r} |p(t) - p^\pi(t)|^p\Big)\,. \tag{2.27}$$

From (2.12), we see that when $t \in [t_k, t_{k+1}]$,

$$\tilde{F}(S(t-b)) = \int_{t_k}^{t} F(S(u-b))du + O(\Delta^2)\,.$$

Thus

$$\exp\Big(\tilde{F}(S(t-b))\Big) = I + \int_{t_k}^{t} F(S(u-b))du + O(\Delta^2)\,.$$

Thus we have a formula for $p(t)$ which is analogous to the one for $p^\pi(t)$ (Equation (2.15)):

$$\begin{aligned} p(t) &= \Big[I + \int_{\lfloor t\rfloor}^{t} F(S(u-b))du + O(\Delta^2)\Big]\prod_{k=0}^{\lfloor t\rfloor}\Big[I + \int_{t_k}^{t_{k+1}} F(S(u-b))du + O(\Delta^2)\Big] \\ &= \rho(\lfloor t\rfloor, t)\prod_{k=0}^{\lfloor t\rfloor}\rho(t_k, t_{k+1})\,, \end{aligned} \tag{2.28}$$

where

$$\rho(r,s) = I + \int_{r}^{s} F(S(u-b))du + O(\Delta^2)\,.$$

We can also write

$$p^\pi(t) = \rho^\pi(\lfloor t\rfloor, t)\prod_{k=0}^{\lfloor t\rfloor}\rho^\pi(t_k, t_{k+1})\,, \tag{2.29}$$

where

$$\rho^\pi(r,s) = I + F(S^\pi(s-b))(s-r)\,.$$

When $r, s \in [t_k, t_{k+1}], r < s$, we have by the Lipschitz condition

$$\begin{aligned} |\rho(r,s) - \rho^\pi(r,s)| &\leq |F(S(t_k - b)) - F(S^\pi(t_k - b))|(s-r) \\ &\quad + \int_r^s |F(S(u-b)) - F(S^\pi(t_k - b))|du + O(\Delta^2) \\ &\leq C|S(t_k - b) - S^\pi(t_k - b)| + O(\Delta^{3/2})\,. \end{aligned} \tag{2.30}$$

We also have

$$|\rho^\pi(r,s)| = |I + F(S^\pi(s-b))(s-r) \leq |I + C(s-r)| \leq e^{C(s-r)}\,. \tag{2.31}$$

In the same way we have

$$|\rho(r,s)| \leq e^{C(s-r)}\,. \tag{2.32}$$

Thus

$$\begin{aligned} |p^\pi(t) - p(t)| &\leq |\rho(\lfloor t\rfloor, t) - \rho^\pi(\lfloor t\rfloor, t)| \prod_{k=0}^{\lfloor t\rfloor} \rho^\pi(t_k, t_{k+1}) \\ &\quad + \sum_{\ell=0}^{\lfloor t\rfloor} |\rho(t_\ell, t_{\ell+1}) - \rho^\pi(t_\ell, t_{\ell+1})|\, \rho(\lfloor t\rfloor, t) \prod_{k=0, k\neq\ell}^{\lfloor t\rfloor} \rho^\pi(t_k, t_{k+1}) \\ &\leq \left[C\,|S(t_k - b) - S^\pi(t_k - b)| + O(\Delta^{3/2})\right] \prod_{k=0}^{\lfloor t\rfloor} e^{C(t_{k+1} - t_k)} \\ &\quad + \sum_{\ell=0}^{\lfloor t\rfloor} \left[C|S(t_\ell - b) - S^\pi(t_\ell - b)| + O(\Delta^{3/2})\right] \rho(\lfloor t\rfloor, t) \prod_{k=0, k\neq\ell}^{\lfloor t\rfloor} e^{C(t_{\ell+1} - t_\ell)} \end{aligned} \tag{2.33}$$

Thus we have for some $C > 0$

$$I_1 \leq C\mathbb{E} \sup_{0\leq t\leq r} \left|S(t-b) - S^\pi(t-b)\right|^p + K_1 \mathbb{E} \sup_{0\leq t\leq r} \left|v_2(u) - S^\pi(t-b)\right|^p.$$

Then by lemma 5.4 we have

$$I_1 \leq C\mathbb{E} \sup_{0\leq t\leq r} \Big| S(t-b) - S^{\pi}(t-b) \Big|^p + C\Delta^{p/2}. \tag{2.34}$$

We now bound $\mathbb{E}\sup_{0\leq t\leq r} |X(t) - X^{\pi}(t)|^p$. Denote

$$\begin{aligned} A_{i,t} &= \sum_{j=1}^{d} \sum_{\substack{0\leq u\leq t \\ \Delta Z(u)\neq 0}} \ln\Big(1 + g_{ij}(S(u-b))Y_{j,N_j(u)}\Big) \\ A^{\pi}_{i,t} &= \sum_{j=1}^{d} \sum_{\substack{0\leq u\leq t \\ \Delta Z(u)\neq 0}} \ln\Big(1 + g_{ij}(v_2(u))Y_{j,N_j(u)}\Big) \end{aligned} \tag{2.35}$$

and denote $I_2 = \mathbb{E}\Big(\sup_{0\leq t\leq r} |X(t) - X^{\pi}(t)|^p\Big)$. Then,

$$\begin{aligned} I_2 &= \mathbb{E}\Big(\sup_{0\leq t\leq r} |X(t) - X^{\pi}(t)|^p\Big) \\ &\leq \Big(\mathbb{E} \sup_{0\leq t\leq r} \sum_{i=1}^{d} \Big| \sum_{\substack{0\leq u\leq t \\ \Delta Z(u)\neq 0}} \sum_{j=1}^{d} \big[\ln(1 + g_{ij}(S(u-b))Y_{j,N_j(u)}) \\ &\qquad\qquad - \ln(1 + g_{ij}(v_2(u))Y_{j,N(u)}) \\ &\qquad + \int_0^t (f_{ii}(S(u-b)) - f_{ii}(v_2(u)))du)\big] \Big|^{2p}\Big)\Big)^{1/2} \Big(\mathbb{E}\Big(|\exp(A_{i,t}) + \exp(A^{\pi}_{i,t})|^{2p}\Big)\Big)^{1/2} \\ &= \Big(\Big(\sum_{i=1}^{d} \mathbb{E} \sup_{0\leq t\leq r} \Big| \int_{\mathcal{J}\times[0,t]} \sum_{j=1}^{d} [\ln(1 + z_j g_{ij}(S(u-b))) \\ &\qquad\qquad - \ln(1 + z_j g_{ij}(v_2(u)))]\tilde{N}_j(du, dz) \\ &+ \int_{\mathcal{J}\times[0,t]} \sum_{j=1}^{d} [\ln(1 + z_j g_{ij}(S(u-b))) - \ln(1 + z_j g_{ij}(v_2(u)))]\,\nu_j(dz)du \\ &+ \int_0^t (f_{ii}(S(u-b)) - f_{ii}(v_2(u)))du \Big|^{2p}\Big)\Big)^{1/2} \cdot \Big(\mathbb{E}\Big(|\exp(A_{i,t}) + \exp(A^{\pi}_{i,t})|^{2p}\Big)\Big)^{1/2}. \end{aligned}$$

Then for some $C_1 > 0$ we have

$$\begin{aligned} I_2 \quad &\leq \Big[\Big(C_1\mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}\times[0,t]}\sum_{j=1}^{d}[\ln(1+z_jg_{1j}(S(u-b))) \\ &\qquad\qquad - \ln(1+z_jg_{1j}(v_2(u)))]\tilde{N}_j(du,dz_j)\Big|^{2p}\Big)^{1/2} \\ &+\Big(C_1\mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}\times[0,t]}\sum_{j=1}^{d}[\ln(1+z_jg_{1j}(S(u-b))) \\ &\qquad\qquad - \ln(1+z_jg_{1j}(v_2(u)))]\nu_j(dz_j)du\Big|^{2p}\Big)^{1/2} \\ &+\Big(C_1\mathbb{E}\sup_{0\leq t\leq r}\Big|\int_0^t(f_{ii}(S(u-b))-f_{ii}(v_2(u)))du\Big|^{2p}\Big)^{1/2}\Big] \\ &\cdot\Big(\mathbb{E}\Big(|\exp(A_{1,t})+\exp(A^{\pi}_{1,t})|^{2p}\Big)\Big)^{1/2} \\ &=: C_1(I_{21}^{1/2}+I_{22}^{1/2}+I_{23}^{1/2})\cdot\Big(\mathbb{E}\Big(|\exp(A_{1,t})+\exp(A^{\pi}_{1,t})|^{2p}\Big)\Big)^{1/2}. \end{aligned}$$

Using the Lipschitz condition on g_{ij}, $\int_{\mathcal{J}} z_j\nu_j(dz_j) = K_\nu < \infty$ for $j = 1,2\cdots,d$, Lemma 5.4 and Assumption 3 we have

$$\begin{aligned} I_{22} &\leq \mathbb{E}\sup_{0\leq t\leq r}\Big|\int_{\mathcal{J}\times[0,t]}\sum_{j=1}^{d}\left[\ln(1+z_jg_{ij}(S(u-b)))-\ln(1+z_jg_{ij}(v_2(u)))\right]\nu_j(dz_j)du\Big|^{2p}\Big) \\ &\leq C\mathbb{E}\sup_{0\leq t\leq r}\Big|S(t-b)-S^{\pi}(t-b)\Big|^{2p}+C\mathbb{E}\sup_{0\leq t\leq r}\Big|v_2(u)-S^{\pi}(t-b)\Big|^{2p} \\ &=: C\mathbb{E}\sup_{0\leq t\leq r}\Big|S(t-b)-S^{\pi}(t-b)\Big|^{2p}+C\Delta^p. \end{aligned}$$

Using the theorem 2.13 from [17] we have

$$\begin{aligned} &I_{21} \\ &\leq \sum_{i=1}^{d}\mathbb{E}\Big(\int_{\mathcal{J}}\int_0^t\sum_{j=1}^{d}\Big|\ln(1+z_jg_{ij}(S(u-b)))-\ln(1+z_jg_{ij}(v(u-b)))\Big|^{2p}\nu_j(dz)du\Big). \end{aligned}$$

Similar to the bound for I_{22} we have

$$I_{21} \leq C\mathbb{E} \sup_{0\leq t\leq r} \Big|S(t-b) - S^{\pi}(t-b)\Big|^{2p} + C\Delta^p.$$

Similar to the bound for I_{22} using assumption (A2) we have

$$I_{23} \leq C\mathbb{E} \sup_{0\leq t\leq r} \Big|S(t-b) - S^{\pi}(t-b)\Big|^{2p} + C\Delta^p.$$

Combining the bounds for I_{21}, I_{22}, I_{23} with the help of lemma (5.3), we get for some $K_2 > 0$

$$I_2 \leq K_2\Big(\mathbb{E} \sup_{0\leq t\leq r} \Big|S(t-b) - S^{\pi}(t-b)\Big|^{2p}\Big)^{1/2} + K_2\Delta^{p/2}. \tag{2.36}$$

We write $I_3 = \mathbb{E}\Big(\sup_{0\leq t\leq r} |S(t) - S^{\pi}(t)|^p\Big)$. Then we have

$$\begin{aligned}
&I_3 = \mathbb{E}\Big(\sup_{0\leq t\leq r} |S(t) - S^{\pi}(t)|^p\Big)\\
&\leq \mathbb{E}\Big(\sup_{0\leq t\leq r} \Big|(p(t) - p^{\pi}(t))X(t) - (X(t) - X^{\pi}(t))p^{\pi}(t)\Big|^p\Big)\\
&\leq 2^{p-1}\mathbb{E}\Big(\sup_{0\leq t\leq r} \Big|(p(t) - p^{\pi}(t))X(t)\Big|^p\Big) + 2^{p-1}\mathbb{E}\Big(\sup_{0\leq t\leq r} \Big|p^{\pi}(t)(X(t) - X^{\pi}(t))\Big|^p\Big).\\
&=: C(I_{31} + I_{32}).
\end{aligned}$$

We now bound I_{31}, I_{32}

$$I_{31} \leq C\Big(\mathbb{E}\Big(\sup_{0\leq t\leq r} \Big|X(t)\Big|^{2p}\Big)\Big)^{1/2}\Big(\mathbb{E}\Big(\sup_{0\leq t\leq r} \Big|(p(t) - p^{\pi}(t))\Big|^{2p}\Big)\Big)^{1/2}. \tag{2.37}$$

Using the Lemmas 5.3 and 2.34 we will have f

$$I_{31} \leq C\Big(\mathbb{E} \sup_{0\leq t\leq r} \Big|S(t-b) - S^{\pi}(t-b)\Big|^{2p} + \Delta^p\Big)^{1/2}.$$

Using assumption 2 we can show that p^π is bounded, hence we can write using (2.36)

$$I_{32} \;\leq\; C\Big(\Big(\mathbb{E}\sup_{0\leq t\leq r}\Big|S(t-b)-S^\pi(t-b)\Big|^{4p}\Big)^{1/2}+\Delta^p\Big)^{1/2}. \tag{2.38}$$

Hence we have for some $K_3>0$

$$I_3 \leq K_3\Big(\mathbb{E}\sup_{0\leq t\leq r}\Big|S(t-b)-S^\pi(t-b)\Big|^{2p}\Big)^{1/2}+K_3\Delta^{p/2}. \tag{2.39}$$

Therefore we get

$$\begin{aligned}&\mathbb{E}\Bigg[\sup_{0\leq t\leq r}\Big[|S(t)-S^\pi(t)|^p\Big]\Bigg]\\ &\leq C\Big(\mathbb{E}\sup_{0\leq t\leq r}\Big|S(t-b)-S^\pi(t-b)\Big|^{2p}\Big)^{1/2}+K\Delta^{p/2}.\end{aligned} \tag{2.40}$$

Taking $r=b$, we have

$$\mathbb{E}\Bigg[\sup_{0\leq t\leq b}\Big[|S(t)-S^\pi(t)|^p\Big]\Bigg]\leq C\Delta^{p/2} \tag{2.41}$$

for any $p\geq 2$. Now, taking $r=2b$ in (2.40), we have

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{0\leq t\leq 2b}\Big[|S(t)-S^\pi(t)|^p\Big]\Bigg] &\leq C\left[\mathbb{E}\sup_{-b\leq t\leq b}|S(t))-S^\pi(t)|^{2p}\right]^{1/2}+K\Delta^{p/2}\\ &\leq C\left[K\Delta^p\right]^{1/2}+K\Delta^{p/2}\leq C\Delta^{p/2}\,.\end{aligned} \tag{2.42}$$

Continuing this way, we obtain for any positive integer $k\in\mathbb{N}$,

$$I_{0\leq t\leq kb}\leq C_{p,k,d,T}\Delta^{p/2}\,. \tag{2.43}$$

Now, since T is finite, we can choose a k such that $(k-1)b<T\leq kb$. This completes the proof of the theorem. ■

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Appendix A

Matlab codes to jump model used in chapter 4

Formula used in first code

$$S(t_{k+1}) = S(t_k) * (1 + g * Y_i) * exp((\alpha - f\lambda)\Delta) \tag{0.1}$$

```
function
   path=jump_delay(p,q,eta,phi,Nsteps,lambda,Npaths,T,S0,delay_fac,alpha)
n=zeros(Nsteps+1,Npaths);
for k=1:Npaths
   for i=1:Nsteps+1
      n(i,k)=poissrnd(lambda*i*(T/Nsteps));
   end
end
function y = sigma(S0,x)
      y=.15*sin(x/S0 );% .0224-.0222*x/S0);
   end
```

```
path=zeros(1,Npaths);
path(1,:)=S0;
    if delay_fac>Nsteps
        delay_fac=0;
    else
        delay_fac=delay_fac;
    end
for k = 1:Npaths
    for j=1:Nsteps
        if j<=(delay_fac)
            path(j+1,k)=path(j,k)*(1 +
                .15*doubleexpo1(p,q,eta,phi,n(j+1,k)))*exp((alpha-lambda*.2)
            *T/Nsteps);
        else
            path(j+1,k)= path(j,k)*(1 +
                sigma(S0,path(j-floor(delay_fac),k))
            *doubleexpo1(p,q,eta,phi,n(j+1,k)))
            *exp((alpha-lambda*sigma(S0,path(j-floor(delay_fac),k)))
            *T/Nsteps);
        end
    end
end

plot(0:T/Nsteps:T,path)
end
```

Function for generating double exponential process

```
function sum= doubleexpo1(p,q,eta,phi,Nsamples)
```

```
r=rand(1,Nsamples);
Y=zeros(1,Nsamples);
sum=0;
for i=1:Nsamples
    if r(i)<q
        Y(i)=(1/phi)*log(r(i)/q);
    elseif r(i)==q
        Y(i)=0;
    else
        Y(i)=(1/eta)*log(p/(1-r(i)));

    end
    sum=sum+Y(i);
end
%plot(Y)
```

We also write the program to simulate geometric brownian motion

```
function Ssample = GeoBMPaths2(S0,nu,sigma,T,Nsteps,Npaths)


s = sigma*(T/Nsteps)^.5;
n = nu*T/Nsteps;
incr(1,1:Npaths) = S0;
incr(2:Nsteps+1,:) = exp(n+s*randn(Nsteps,Npaths));
Ssample = cumprod(incr);
plot(0:T/Nsteps:T,Ssample)
title('sample paths of geometric Brownian motion','fontsize',14);
ylabel('value of sampled geometric Brownian motion','fontsize',14);
set(gca,'fontsize',14,'FontWeight','bold');
```

```
xlabel('time','fontsize',14);
```

Using below we can compare the call option prices among models considered

```
K=210.0:5.0:215.0;
prices=zeros(length(K),3);
%prices(length(K),1)=K;
for j = 1:length(K)
    paths=GeoBMPaths2(209.11,.11,.15,6/12,6.5*60*22*6,200);
    GeoBMPaths2(S0,nu,sigma,T,Nsteps,Npaths)
    prices(j,1)=priceCall1(paths,K(j),.01,6/12);
    prices(j,1)=Call_p_j_no_bm(0.60,0.40,12.8,8.40,2*6.5*60*
    *22*6,.03,150,6/12,209.11,60*6.5,.11,.01,K(j));
    prices(j,2)=Call_p_j_no_bm(0.60,0.40,12.8,8.40,4*6.5*60
    *22*6,.03,150,6/12,209.11,60*6.5,.11,.01,K(j));
    prices(j,3)=Call_p_j_no_bm(0.60,0.40,12.8,8.40,8*6.5*60
    *22*6,.03,150,6/12,209.11,60*6.5,.11,.01,K(j));
    Call_p_j_no_bm(p,q,eta,phi,Nsteps,lambda,Npaths,T,S0,delay_fac,
    alpha,rho,K)
end
prices
```