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INELASTIC LATERAL BUCKLING OF STEEL BEAM-COLUMNS

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Inelastic Lateral Buckling of Steel Beam-Columns

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INELASTIC LATERAL BUCKLING OF STEEL BEAM-COLUMNS

SUMMARY

A series of investigations of the inelastic lateral buckling behaviour of steel beam-columns is described. These originate from many previous studies of the elastic lateral buckling of beams. It was found that present methods of predicting the effects of moment gradient in elastic beam-columns are unnecessarily conservative, and it was concluded that many practical continuous beam-columns will have significant warping restraints.

Fourteen inelastic lateral buckling tests were carried out on 9 continuous steel beam-columns. The results of these tests were compared with predictions made by a new and improved finite element computer method of analysing inelastic buckling, and very good agreement was found. The analytical method was then used to develop a simple approximation for predicting the inelastic buckling of isolated beam-columns with unequal end moments, and a design method was proposed.

KEYWORDS: Beams, buckling, columns, flexure, residual stresses, steel, structural design, structural engineering, torsion.

1. INTRODUCTION

When a steel beam-column which is bent about its major axis is insufficiently braced laterally, then it may fail by deflecting laterally out of the plane of bending and twisting. For beam-columns of intermediate slenderness, the in-plane actions cause yielding which reduces the resistance to lateral buckling. This paper describes a series of investigations of the inelastic lateral buckling of steel beam-columns.

The investigations have their origin in many previous studies of the elastic lateral buckling of beams. In these, the effects of cross-section, slenderness, support, moment gradient, load height, and restraint have been thoroughly researched. Extensions to the inelastic buckling of beams have shown the importance of residual stresses, moment gradient, and the location of yield regions in both simply supported and continuous beams, and have led to methods of incorporating these effects into design procedures.

The initial studies of beam-columns concentrated on elastic members, and it was first found that a more accurate method was required for predicting the effects of moment gradient than those of present design procedures. Following this the restraining effects caused by concentrated moments in continuous members were investigated.

The second phase of the investigations was experimental, and 14 tests were conducted on 9 continuous steel beam-columns which buckled inelastically. At the same time, a new and improved analytical method was developed for predicting the inelastic lateral buckling of continuous beam-columns, and tested against the results of previous analytical studies. The predictions of the new method were then compared with the experimental results, and a very high degree of correlation was obtained.

The next phase of the investigations involved the use of the new analytical method to undertake systematic research into inelastic lateral buckling. Already the effects of moment gradient on isolated beam-columns have been studied, and a significantly improved design method has been developed. Future work planned includes the lateral buckling of beam-columns which sway in the plane of loading, and the buckling interactions between adjacent segments of continuous beam-columns.

2. LATERAL BUCKLING OF BEAMS

2.1 Elastic Buckling

The elastic flexural-torsional buckling of beams has been studied by many investigators, and there are a number of research summaries (5,11,16-18). For simply supported beams of length L bent in uniform bending in a plane of symmetry as shown in Fig. 1, the elastic buckling moment M is given by

$$[1] \qquad \frac{M}{M_{yz}} = \sqrt{\left\{1 + \left(\frac{\beta_x P_y}{2M_{yz}}\right)^2\right\}} + \left(\frac{\beta_x P_y}{2M_{yz}}\right)$$

where M_{yz} is the buckling moment for a doubly symmetric beam

$$[2] \qquad M_{yz} = \sqrt{\left\{P_y\left(GJ + \pi^2 EI_w/L^2\right)\right\}}$$

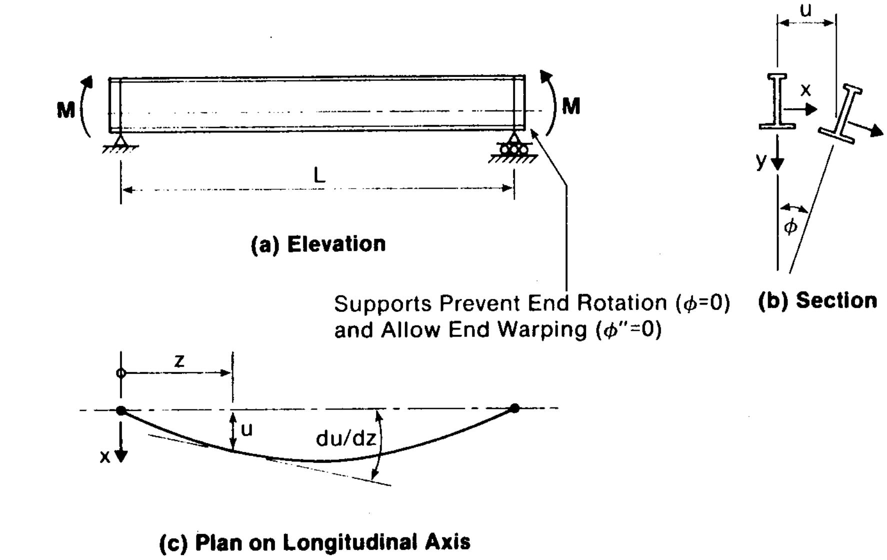


Fig. 1. Monosymmetric Beam in Uniform Bending

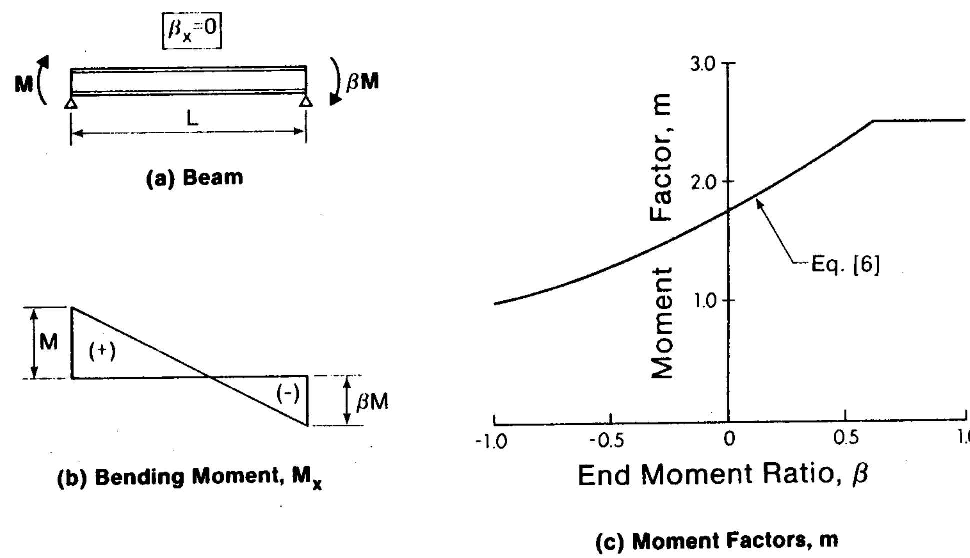


Fig. 2. Doubly Symmetric Beams Under Moment Gradient

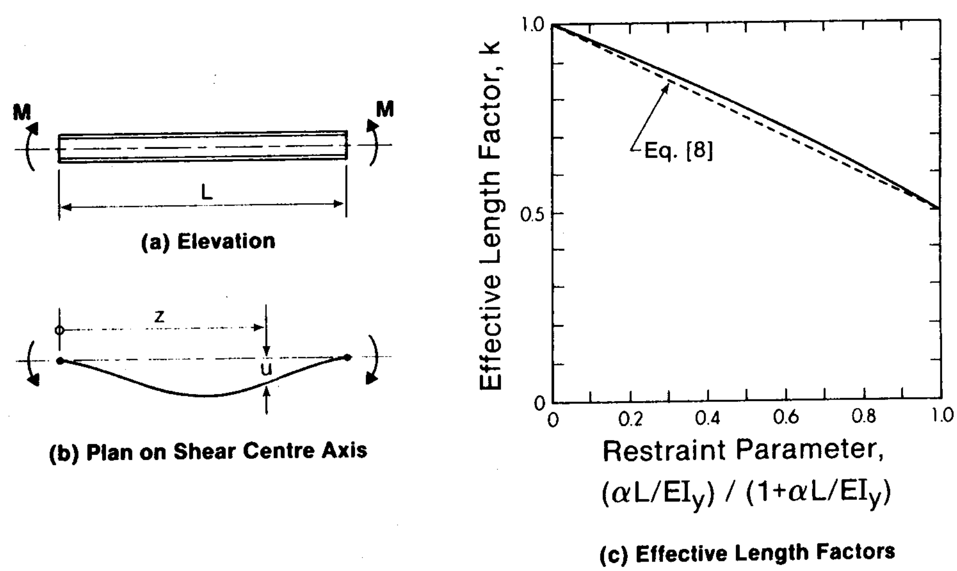


Fig. 3. Elastic Buckling of End Restrained Beams

P_y is the flexural buckling load of a column

[3] $$P_y = \pi^2 EI_y/L^2$$

β_x is a monosymmetry property of the cross-section

[4] $$\beta_x = (1/I_x) \int_A (x^2y + y^3)\, dA - 2y_o$$

y_o is the shear centre coordinate, and EI_y, GJ, EI_w are the flexural, torsional, and warping rigidities of the cross-section.

For doubly symmetric beams bent by unequal end moments M, βM as shown in Fig. 2, the maximum moment at elastic buckling can be expressed as

[5] $$M_E = mM_{yz}$$

where m is approximated by

[6] $$m = 1.75 + 1.05\beta + 0.3\beta^2 \not> 2.5$$

The elastic buckling resistance of a beam may be significantly increased by end restraints (22). For doubly symmetric beams in uniform bending with equal flange end restraints as shown in Fig. 3, the elastic buckling moment can be obtained from Equations [2] and [3] by substituting the effective length

[7] $$\ell = kL$$

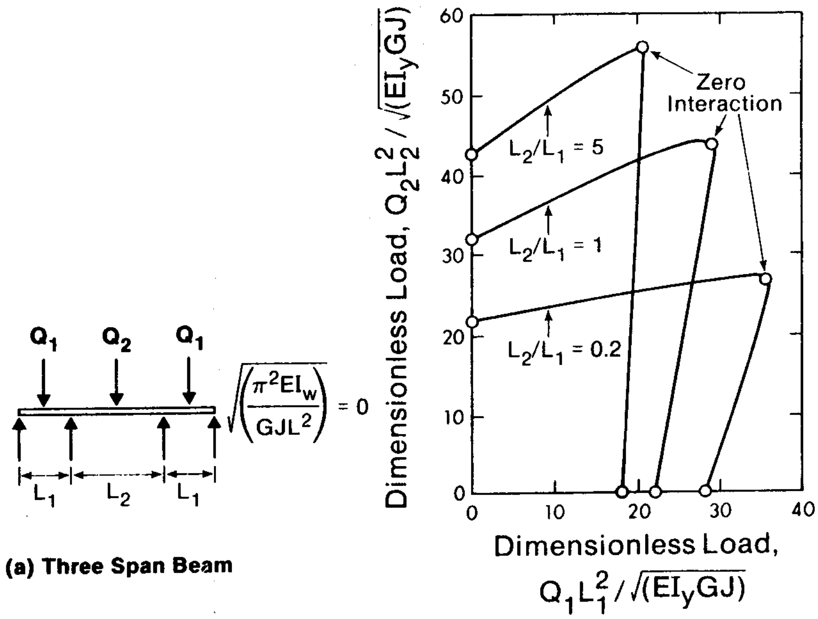
for the actual length L, in which the effective length factor k is approximated by

[8] $$k = \frac{2 + \alpha L/EI_y}{2 + 2\alpha L/EI_y}$$

where α is the moment-rotation stiffness of each of the four flange end restraints.

The restraining actions between adjacent segments of braced or continuous beams are more difficult to assess, since there are a number of different restraining modes possible, as shown in Fig. 4. These include the easily analysed zero interaction case, in which each segment buckles as if independent of the adjacent segments. An approximate method has been developed for more general analysis (12,17), in which a lower bound is first produced by assuming that each segment buckles independently, and by determining the most critical segment. The restraining actions of the adjacent segments are then approximated and used to obtain an improved estimate of the buckling load of the critical segment.

A recent study (8) has considered the elastic buckling of continuous beams with concentrated moments acting at the support points, as shown in Fig. 5. It was found that the jump discontinuities in the bending moment caused unexpected restraint effects, which might be approximated as equivalent end warping restraints. Recognition of the fact that concentrated moments will often require significant web stiffening of the beam (6), which will produce



(a) Three Span Beam

(b) Critical Load Combinations

Fig. 4. Interaction Buckling of Symmetrical Three Span Beams

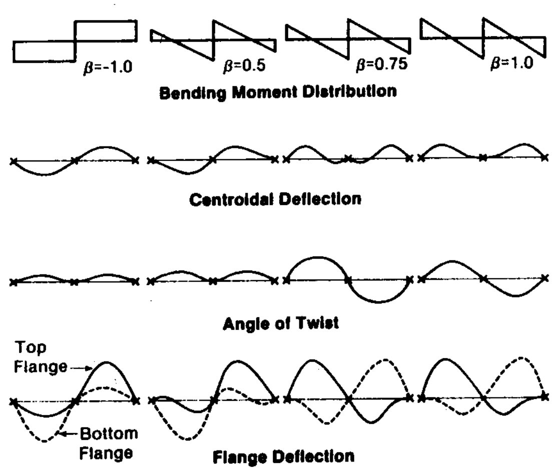


Fig. 5. Buckled Shapes of Beams with Concentrated Moments

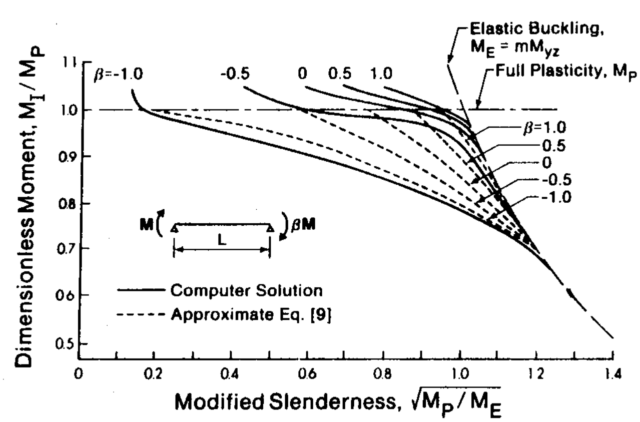


Fig. 6. Hot-Rolled Beams with Unequal End Moments

further warping restraints, suggests that it is not unreasonable to assume that end warping is effectively prevented in continuous members with concentrated moments.

2.2 Inelastic Buckling

The buckling resistance of an intermediate length steel beam is reduced by yielding caused by a combination of the effects of the applied loads and the residual stresses left in the beam after manufacture. A tangent modulus theory of inelastic buckling has been developed (19,21), in which the initial elastic moduli E, G are replaced by the strain-hardened values E_s, G_s for all yielded and strain-hardened regions of the beam, and which accounts for the non-uniform, monosymmetric nature of the beam after partial yielding.

For a simply supported steel beam under moment gradient, the inelastic buckling resistance M_I may be approximated (13) as shown in Fig. 6 by

[9] $$\frac{M_I}{M_P} = 0.7 + \frac{0.3\,(1 - 0.7M_P/M_E)}{(0.61 - 0.3\beta + 0.07\beta^2)} < 1.0$$

in which M_P is the fully plastic moment capacity. It can be seen that the moment distribution is very important, as there are very substantial reductions in buckling resistance for uniform bending ($\beta = -1$), when all the beam is yielded. On the other hand, the reductions are quite small for double curvature bending ($\beta = 1$), for which yielding is concentrated near the supports.

For simply supported beams with central concentrated loads (Fig. 7), the resistance is a little higher than for uniform bending, because while yielding occurs in the mid-span region of the beam, it is limited in its extent. A similar conclusion can be drawn for continuous beams (23), except in the special cases where yielding first occurs at the supports, in which case the inelastic buckling resistance is much higher, as indicated by the results for $Q_1/Q_2 = 1.56$ shown in Fig. 7.

3. ELASTIC BUCKLING OF BEAM-COLUMNS

The elastic flexural-torsional buckling of a simply supported beam-column in uniform bending ($\beta = -1$) is approximated by

[10] $$(M + Py_o)^2 = (P_y - P)\,\{(r_o^2 + y_o^2)\,(P_z - P) + M\beta_x\}$$

in which P_z is the torsional buckling load of a column

[11] $$P_z = \frac{GJ + \pi^2 EI_w/L^2}{r_o^2 + y_o^2}$$

and r_o is the polar radius of gyration given by

[12] $$r_o^2 = (I_x + I_y)/A$$

For doubly symmetric sections, $y_o = 0$ and $\beta_x = 0$, and a more accurate solution (17) is obtained from

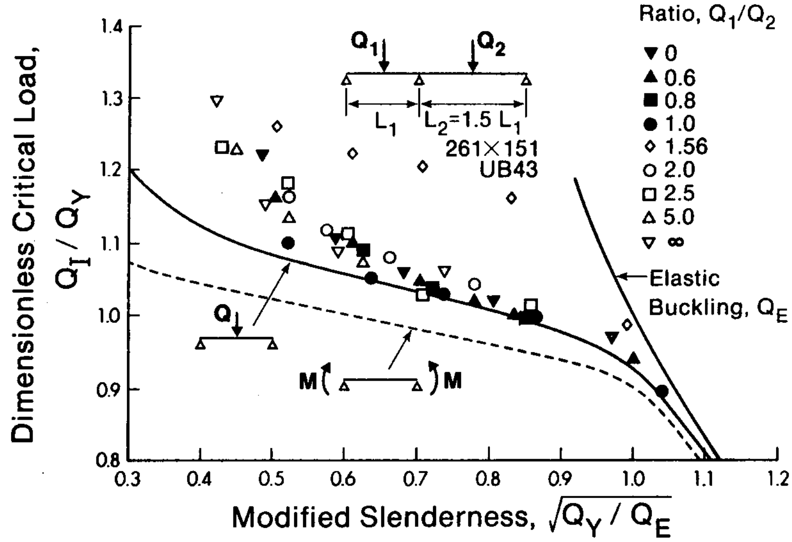


Fig. 7. Inelastic Buckling Predictions for Continuous Beams

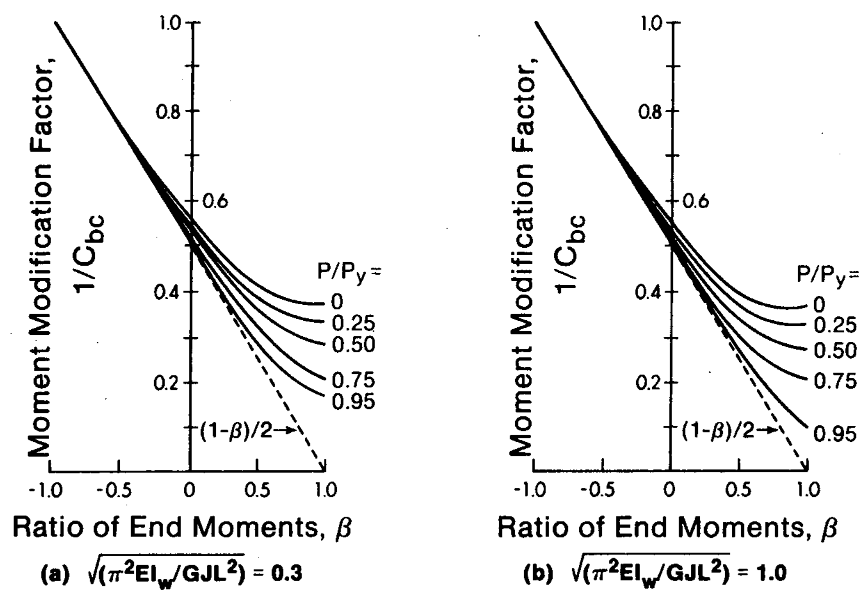


Fig. 8. Beam-Column Factors for Unequal End Moments

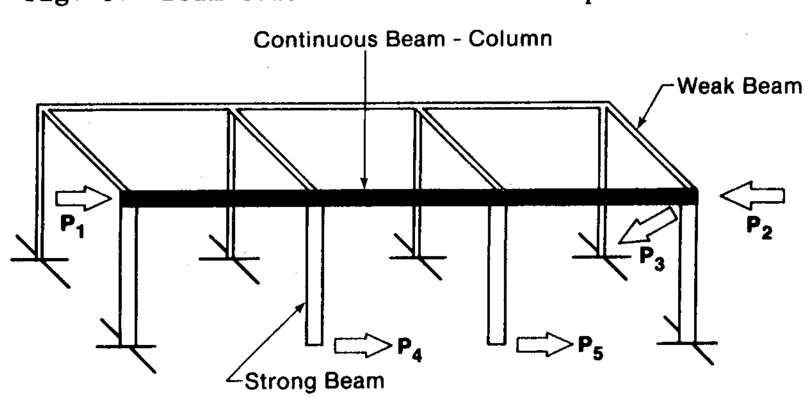


Fig. 9. Arrangement of 3-Span Beam-Column

[13] $$(M/M_{yz})^2 = (1 - P/P_x)\,(1 - P/P_y)\,(1 - P/P_z)$$

in which P_x is the in-plane column buckling load

[14] $$P_x = \pi^2 EI_x/L^2$$

The term $(1 - P/P_x)$ in Equation [13] is often close to unity.

For beam-columns with end moments, M, βM, the elastic buckling resistance may be approximated by

[15] $$(M/mM_{yz})^2 = (1 - P/P_y)\,(1 - P/P_z)$$

in which the moment distribution factor m is given by Equation [6]. However this approximation is often conservative (7), as indicated in Fig. 8, and more accurate predictions may be obtained (7,2) by using

[16] $$\frac{1}{C_{bc}} \doteq \left(\frac{1 - \beta}{2}\right) + \left(0.4 - 0.23\,\frac{P}{P_y}\right)\left(\frac{1 + \beta}{2}\right)^3$$

for 1/m in Equation [15].

4. ANALYSIS OF INELASTIC BUCKLING

4.1 Pre-Buckling Analysis of In-Plane Bending

Before a prediction can be made of inelastic lateral buckling, the in-plane bending must be analysed so that the distributions of the elastic, yielded, and strain-hardened regions throughout the member can be determined. The effective out-of-plane rigidities which contribute to the inelastic buckling resistance can be evaluated using these distributions.

When the member is statically determinate, the in-plane analysis can be made in two separate stages. First the variation of the axial force and bending moment along the member can be determined from statics. Following this, the locations of the boundaries of the elastic, yielded, and strain-hardened regions within selected cross-sections can be determined using the cross-section geometry, material properties, residual stresses, and the axial force and bending moment.

When an elastic member is statically indeterminate, the two stages cannot be separated, because the material non-linearity closes the chain of dependence of yielding on stress resultants, on redundant actions, on deflections, on stiffnesses, on yielding. In addition, it may be necessary to consider the effects of geometric non-linearity, as for example when the term $(1 - P/P_x)$ is not close to unity.

A finite element computer method of analysing the in-plane behaviour of steel frames is discussed in Reference 9. This method, which allows for the effects of residual stresses, yielding, strain-hardening, and finite deflections, is used to determine the yielded and strain-hardened boundaries.

4.2 Analysis of Out-of-Plane Buckling

Finite element methods of analysing elastic flexural-torsional buckling

(10) may be simply adapted for the analysis of inelastic buckling. For members with equal flanges, it is easiest to specify the buckling displacements in terms of the lateral displacement u and twist ϕ of the elastic centroidal axis, in which case the strain energy stored in an element can be expressed as (1)

[17] $$U = \frac{1}{2} \int_0^L \{\varepsilon_u\}^T [D_u] \{\varepsilon_u\} \, dz$$

where the generalised strain vector is

[18] $$\{\varepsilon_u\}^T = \{u'', \phi', \phi''\}$$

[19] $$[D_u] = \begin{bmatrix} (EI_T + EI_B)_t & 0 & (EI_T - EI_B)_t \\ 0 & (GJ)_t & 0 \\ (EI_T - EI_B)_t & 0 & (EI_T + EI_B)_t \, h^2/4 \end{bmatrix}$$

and the subscript t denotes the tangent modulus values of GJ and the top and bottom flange rigidities EI_T, EI_B, and the dash indicates differentiation with respect to z.

The work done by the forces acting on the element during buckling can be expressed as

[20] $$V = \frac{1}{2} \int_0^L \{\varepsilon_v\}^T [D_v] \{\varepsilon_v\} \, dz$$

where the generalised stability strain vector is

[21] $$\{\varepsilon_v\}^T = \{u', \phi, \phi'\}$$

[22] $$[D_v] = \begin{bmatrix} S_1 & S_2' & S_2 \\ S_2' & S_2'' y_q & 0 \\ S_2 & 0 & (S_3 + S_4) \end{bmatrix}$$

[23] $$S_1 = \int_A f \, dA$$
$$S_2 = \int_A f \, y \, dA$$
$$S_3 = \int_A f \, y^2 \, dA$$
$$S_4 = \int_A f \, x^2 \, dA$$

f is the total normal longitudinal stress, and y_q is the distance below the centroidal axis at which the distributed load $- S_2''$ acts.

The element stiffness and stability matrices may be formed from Equations [17] and [20], and these may be transformed and assembled into the global matrices [K], [G] in

[24] $$\frac{1}{2} \{u\}^T [K + G] \{u\} = 0$$

in which {u} is the vector of the global nodal deformations. In inelastic buckling problems, [K] and [G] must be recalculated for each load level on the structure, and so the usual eigenvalue methods used for elastic buckling problems lose their efficiency. Instead, a series of calculations are made at increasing load levels until an approximately zero determinant is obtained from [K + G], which determines the buckling load. Some care must be taken to ensure that the lowest buckling load is not missed.

5. INELASTIC BUCKLING PREDICTIONS

5.1 Tests on Continuous Beam-Columns

Reference 6 describes a series of 14 tests on 9 beam-columns which were continuous over three spans, as shown in Fig. 9. The hot-rolled I-section members were loaded by end forces P_1, P_2 and concentrated in-plane moments developed by the forces P_4, P_5, and were restrained against in-plane sway by the bracing force P_3. These forces caused significant yielding of the beam-columns, reducing their resistances to out-of-plane buckling. Because of this, the restraining out-of-plane actions developed by weak axis beams played important roles in increasing the member strengths.

The purpose of the tests was to obtain experimental data which could be used to evaluate inelastic buckling theories. A comparison of the experimental failure loads P_F with the predictions P_I obtained (3) from the theory developed in Reference 1 is shown in Fig. 10, which indicates extremely close agreement.

5.2 Isolated Beam-Columns Under Moment Gradient

Inelastic buckling predictions (2) of isolated hot-rolled beam-columns with end moments M, βM have been compared with approximations obtained from the linear interaction equation

[25] $$\frac{P}{P_{Iy}} + \frac{C_M}{(1 - P/P_x)} \frac{M}{M_{Io}} \leqslant 1$$

in which

[26] $$C_m = 0.6 - 0.4\beta \geqslant 0.4$$

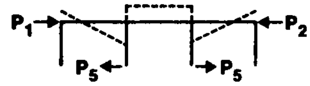
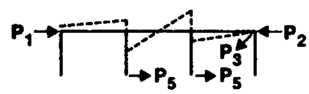
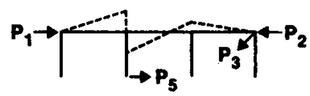
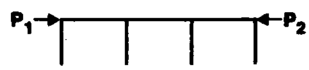
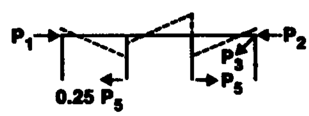
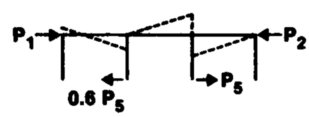
These equations are similar to those used in present design codes, such as Reference 4, except that P_{Iy} is the inelastic out-of-plane flexural buckling load of a simply supported column and M_{Io} is the uniform buckling moment of a simply supported inelastic beam. Approximations for P_{Iy}, M_{Io} were developed (2) from inelastic buckling analyses of a wide range of hot-rolled I-section members as

[27] $$P_{Iy}/P_Y = 1.035 - 0.181 \sqrt{(P_Y/P_y)} - 0.128 P_Y/P_y \leqslant 1.0$$

[28] $$M_{Io}/M_P = 1.008 - 0.245\, M_P/M_{yz} \leqslant 1.0$$

in which $P_Y = AF_Y$ is the squash load.

It was found that the approximations calculated from Equation [25] were generally conservative, and especially so for high moment gradients ($\beta >$ 0.5). This conservatism was attributed to the use of a linear interaction

Load Set	Nominal Load Configuration	Specimen Number	$(P_5/P_1)_n$	(P_I/P_F)
1		1 2	0.082 0.221	0.96 0.96
2		3 5A	0.066 0.032	0.95 1.03
3		4 6A 7A	0.116 0.045 0.045	1.00 1.00 0.99
4		3A 4A 5	0.0 0.0 0.0	1.06 1.00 0.98
5		6 7	0.122 0.048	1.01 1.03
6		8 9	0.055 0.136	1.02 0.95

Note: 'A' indicates specimen previously tested to failure

Fig. 10. Experimental Failure Conditions

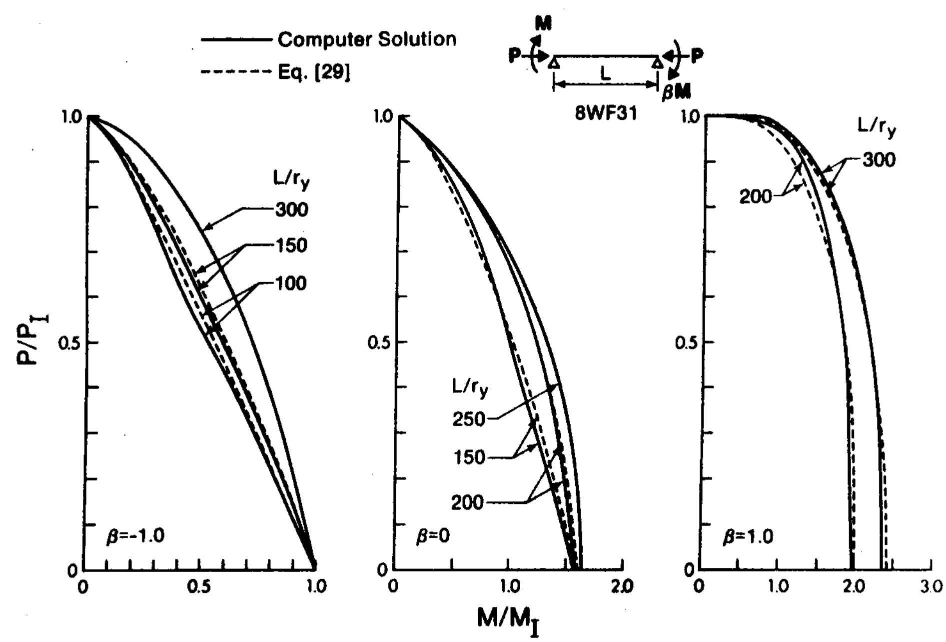


Fig. 11. Improved Buckling Interaction Equation

equation, instead of a parabolic one similar to Equations [13] and [15], and to the use of a C_m factor for non-uniform bending which was independent of the axial load P, instead of varying with P as in Equation [16].

Because of this it was decided to modify the elastic parabolic interaction equation and the non-uniform bending factor to

[29] $$\left(\frac{M}{C_{bc} M_{Io}}\right)^2 = \left(1 - \frac{P}{P_{Iy}}\right)\left(1 - \frac{P}{P_z}\right)$$

[30] $$\frac{1}{C_{bc}} = \left(\frac{1-\beta}{2}\right) + \left(0.4 - 0.23\,\frac{P}{P_{Iy}}\right)\left(\frac{1+\beta}{2}\right)^3$$

These equations proved to be of high accuracy, as is demonstrated in Fig. 11.

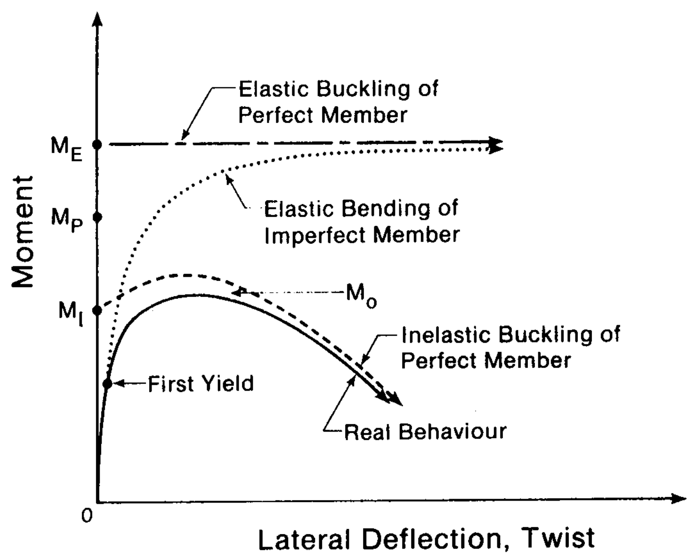


Fig. 12. Behaviour of Real Members

6. APPLICATION TO DESIGN

Beams and beam-columns which fail by flexural-torsional buckling must be almost perfectly straight and untwisted before loading, and the applied loads must initially cause deflections only in the plane of loading. Real members have initial curvatures and twists, and their loads are applied eccentrically and with components which cause out-of-plane bending and torsion immediately, as shown in Fig. 12.

Because of this, the strengths of real members are reduced below their buckling resistances. Design rules usually allow for this by modifying the buckling predictions. For example, for simply supported columns (15), the column strength P_o may be approximated by

[31] $$\frac{P_o}{P_Y} = \left(\frac{1+\eta+P_Y/P_y}{2\,P_Y/P_y}\right)\left\{1 - \left[1 - \frac{P_y}{P_Y}\left(\frac{2\,P_Y/P_y}{1+\eta+P_Y/P_y}\right)^2\right]^{1/2}\right\}$$

in which η is an imperfection parameter given by

[32] $$\eta = 0.293\,\{\sqrt{(P_Y/P_y)} - 0.15\} \geqslant 0$$

while the strength M_o of a beam in uniform bending may be approximated by (20)

$$[33] \qquad \frac{M_o}{M_p} = 0.6 \left\{ \left[\left(\frac{M_p}{M_{yz}} \right)^2 + 3 \right]^{1/2} - \frac{M_p}{M_{yz}} \right\} \leq 1$$

At present, the flexural-torsional design strengths of beam-columns are approximated by using equations similar to Equations [25] and [26], but with P_{Iy}, M_{Io} replaced by equations similar to Equations [31] and [32]. The unsatisfactory nature of Equations [25] and [26] for inelastic flexural-torsional buckling has been noted above, as has the marked improvement provided by Equations [28] and [29]. It seems logical therefore to propose that this should be extended to estimate the out-of-plane design strengths of beam-columns from

$$[34] \qquad \left(\frac{M}{C_{bc} M_o} \right)^2 = \left(1 - \frac{P}{P_o} \right) \left(1 - \frac{P}{P_z} \right)$$

in which C_{bc}, M_o, and P_o are given by Equations [30], [31], and [33].

Thus the design of beam-columns will require three conditions to be satisfied: -

(1) Cross-section capacity,
(2) In-plane member strength, and
(3) Out-of-plane member strength (Equation [34])

for which the present methods may be retained for assessing the first two conditions of cross-section capacity and in-plane strength.

7. CONCLUSIONS

Previous studies of the elastic flexural-torsional buckling of beams and beam-columns have demonstrated the importance of the bending moment distribution and of end restraints. The inelastic buckling of beams has also been studied, including the effects of residual stresses and the yield distribution.

Recent research studies have extended this work to the inelastic flexural-torsional buckling of steel beam-columns, and have led to the development of a general computer method of predicting inelastic buckling, and this has received experimental confirmation. The computer method has been used to study the inelastic buckling of beam-columns with unequal end moments, and to develop comparatively simple equations for predicting their inelastic buckling resistances. This has allowed the formulation of an improved method of estimating their design out-of-plane member strengths. Thus a beam-column bent in-plane would be checked for cross-section capacity and in-plane member strength as at present, and for out-of-plane strength by using the new formulation.

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10. NOTATION

A	Cross-sectional area
C_{bc}, C_m	Beam-column factors for unequal end moments
$[D_u], [D_v]$	Element matrices (Equations [19] and [22])
E	Young's modulus of elasticity
E_s	Strain-hardening modulus
f	Total longitudinal stress
F_Y	Yield stress
$[G]$	Global stability matrix
G	Shear modulus of elasticity
G_s	Strain-hardening shear modulus
h	Distance between flange centroids
I_B, I_T	Second moments of area of bottom and top flanges
I_w	Warping section constant
I_x, I_y	Second moments of area about x, y axes
J	Torsion section constant
k	Effective length factor
$[K]$	Global stiffness matrix
ℓ	Effective length
L	Length of member or element
m	Beam factor for moment gradient
M	Moment
M_E	Elastic buckling moment
M_I	Inelastic buckling moment
M_{Io}	Value of M_I for uniform bending
M_o	Uniform bending strength
M_p	Full plastic moment
M_{yz}	Value of M_E for uniform bending
P	Axial load
P_1-P_5	Forces on beam-column
P_F	Force at failure
P_I	Inelastic buckling load
P_{Iy}	Value of P_I for flexural buckling
P_o	Out-of-plane column strength

P_x, P_y	Elastic buckling loads for flexure about x, y axes
P_Y	Squash load
P_z	Elastic torsional buckling load
Q_1, Q_2	Transverse loads
r_o	Polar radius of gyration
S_1-S_4	Stress resultants (Equation [23])
u	Lateral deflection of shear centre
$\{u\}$	Vector of global nodal displacements
U	Strain energy stored in element
V	Work done on element
x,y	Principal axes of cross-section
y_o	Shear centre coordinate
y_q	Distance below centroid of distributed load - S_2"
z	Longitudinal axis through centroid
α	Stiffness of flange end restraints
β	End moment ratio
β_x	Monosymmetry section constant
$\{\varepsilon_u\}, \{\varepsilon_v\}$	Strain vectors
ϕ	Angle of twist rotation
η	Imperfection parameter (Equation [32])