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A Stability Analysis of Thermo-Mechanical Flow in the Earth's Mantle

by

© T.J.T. Spanos

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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a dissertation entitled A STABILITY ANALYSIS OF THERMO-MECHANICAL FLOW IN THE EARTH'S MANTLE submitted by Thomas J.T. Spanos in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Physics.

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Supervisor

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External examiner

Date ..October 11....1977

ABSTRACT

A numerical analysis of the equations of thermo-mechanical flow applicable to geodynamics is considered under the assumption of creep. An analytical investigation of these equations suggests the seismic low velocity zone (L.V.Z.) can be interpreted as a region of stationary thermo-mechanical coupling between the tectonic plates at the earth's surface and the mantle convection cells below. With the introduction of a steady state concentration gradient of melt one can obtain a viscosity distribution which fits with the observed shear wave velocity distribution in the L.V.Z. as well as with measurements of seismic attenuation. The introduction of inertial terms in the thermal and mechanical field equations permit an explanation of deep earthquakes as a shear heating instability in the boundary layer at the upper boundary of the subducting plates. The deep earthquakes must occur within this fluid boundary layer in order to produce the observed compressional focal mechanism. This differs from suggestions by others that the events occur in the subducting lithosphere. The predictions of the theory developed for this mechanism of deep earthquakes then exhibit a good correlation with observations.

ACKNOWLEDGEMENTS

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I am deeply grateful to Dr. M. Razavy for many helpful discussions of the mathematics, to Dr. C. Scarfe for explaining many of the geologic phenomena to me and to Dr. W. Israel for showing me the variational principle which I have used in chapter 2.

I would also like to thank Dr. J.D. Cocker for bringing the references on volcanic activity following deep earthquakes to my attention.

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NOTATION

The following symbols are used throughout the thesis. Also given are characteristic values and the references from which these values were taken (the temperatures are given in °C and all other values are in c.g.s. units unless otherwise specified).

	Symbol	Value	Reference
Density	ρ	3	MacKenzie (1970)
Heat Capacitance	c	1.3×10^{7}	MacKenzie (1970)
Thermal Conductivity	k	3×10^{5}	MacKenzie
Activation Energy	E	120 Kcal/mole (Chapter 2) 60 Kcal/mole (Chapter 3)	Griggs & Baker (1969)
Gas Constant		20 Kcal °/mole	
Viscosity	μ	10^{22}	Cathles (1976)
Melting Temperature	T_m	1400 (Low velocity zone)	Ringwood (1975)
Boundary Temperature	T_B	1100 (Lithosphere Boundary)	Ringwood (1975)
Strain Rate	$\dot{\epsilon}$	10^{-14}	Carter (1971)

Only c, ρ and R are well determined. The thermal conductivity k, the viscosity μ and their depencence on temperature are uncertain (MacKenzie 1970). The temperature are magnitudes used only as scaling factors and thus do not affect the general physical outcome. The difference between T_B and T_m however is important and this difference is very uncertain as is the strain rate. The uncertainty in there values is a result of the fact that direct measurement cannot be done and we thus must rely on indirect arguments.

CHAPTER 1

Introduction

1.1 The Geodynamic Problems Under Consideration

The purpose of this dissertation is to construct mathematical models for two specific geodynamic phenomena which are generally assumed to be problems in fluid dynamics. The first is the coupled thermo-mechanical behavior of the upper mantle. This can result from the stress induced by relative motions between tectonic plates and the upper part of mantle convection cells. (Turcotte, D.L. and Oxburgh, E.R. 1972, Torrance, K.E. and Turcotte, D.L. 1971a,b, Schubert G. et al 1976, Parmentier E.M. et al 1976). In this analysis I assume a constant stress. This however is the same as the assumption of a constant velocity boundary condition and creep. Creep means that the time derivatives of velocity are zero and thus from the Navier Stokes equation one can see that this is equivalent to constant stress. The second is the focal mechanism of deep earthquakes (Shaw, H.R. 1969, Griggs, D.T. and Baker, D.W. 1969, Randall M.J. 1972).

Of primary concern in the discussions of these two processes is the stability or the ability of the very viscous fluid which makes up the earth's mantle to withstand the shear stresses imparted to it without melting. In the seismic low velocity zone the temperature and pressure is such that the mantle material (essentially iron magnesium silicates) in this region is close to its solidus. (Ringwood A.E. 1975, MacDonald G.J.F. and Ness N.F. 1961, Wyllie P.J. 1971). In this sense the seismic low velocity zone exists in a marginally stable state (i.e. close to melting) even when unstressed. On the other hand the subducting plate is far from instability in the unstressed state. When the time over which this material can achieve thermal equilibrium is large in comparison to the rate at which heating takes place within the material, we may consider the material to have a low thermal conductivity. P.W. Bridgman (1936) found that this precise circumstance of low thermal conductivity and a high melting point relative to the ambient temperature of the experiment caused materials to exhibit an anomalous behavior when subjected to large shear stress at high pressures. Here high pressure means that the pressure is sufficiently large that brittle fracture (breaking of the material with frictional sliding at the fracture zone) cannot occur. This is probably the case for pressures greater than 30 kb which is the pressure at approximately 100 km in the earth (Griggs D. T. and Baker D. W. 1969).

A series of experiments were performed by Bridgman (1935) in which thin disks were confined between hardened steel parts. The disks were then subjected to normal pressure (50 kb) and torque simultaneously. Bridgman found that most materials deformed smoothly under this condition of large shear stress at high pressure. This appears to be the case in the seismic low velocity zone due to the absence of any observations of sudden stressorelief in that region. However in the case of materials with a low thermal conductivity and high melting point, the application a of large shear stress caused violent snapping. Among the qualitative effects found was that many substances normally stable became unstable and detonated, and conversely combinations of substances normally inert to each other combined explosivly. Here the applied force would suddenly drop to a very small value and then build up at a nearly steady rate until another snap occurred. Bridgman's actual experiments are probably not of much use for geophysical purposes since the strain rates were much to large. Thus the actual conditions in the earth should allow a greater amount of fluid behavior This is important since SiO showed rupture only without flow.

Bridgman (1936) and others (Gruntfest I.J. 1963, Shaw H.R. 1969, Griggs D.T. and Baker D.W. 1969) have interpreted this phenomenon as the mechanism of deep earthquakes. They however encounter problems in the application of this theory to deep earthquakes. It is, generally acknowledged (e.g. Griggs D.T. and Baker D.W. 1969) that the mantle material has very little strength in comparison to the subducted slab. Thus in the above papers it is assumed that the deep earthquakes must occur within the subducted slab. However all attempts to model this fluid instability within the subducted slab have failed because the mantle cannot support sufficient stress to cause instability in the slab. (See section 4.3 of this dissertion)

The mantle must yield to fluid deformation for smaller values of stress and exhibit larger inertial motions (non zero time derivatives in the v ocity field) than the slab because of its lower viscosity. Also the higher temperatures of the mantle cause it to be more ductile than the slab (Ranalli G. 1974). Thus due to the greater strength of the slab it is more realistic to consider the motions of the slab as elastic deformations caused by the viscous resistance of the bounding fluid. One then obtains instability in the fluid at the boundary of the slab relieving the elastic deformation of the slab.

As the slab subducts the surface is heated by thermal conduction and since melting of crustal fractions will occur crustal material must mix with mantle material. Thus the surrounding fluid in which the instability occurs has a composition far more similar to the crustal material than mantle material. This diffusion of crustal material then also causes large viscosity gradients due to the changing chemical composition. This viscosity gradient then induces an inertial shear heating source term in the heat balance equation.

1.2 Fluid Dynamics Content

Fluid dynamics is by its very nature the study of macroscopic phenomena since a fluid is regarded as a continuous medium. One thus considers even in the so called "infinitesimal volume element" of a fluid, the resultant approximate behavior of many molecules assumed to be contained within that volume (Landau L.D. and Lifshitz E.M. 1959).

The description of a classical one-phase fluid is given by the equation of motion for the density (conservation of mass), the equations of motion for the components of velocity (conservation of momentum), an equation of heat transfer (the equation of motion for the entropy) and supplemented by an equation of state relating any three thermodynamic variables for a given amount of substance contained in the system (Putterman S.J. 1974).

Of primary importance in determining if equilibrium solutions exist for one phase fluid flow is a study of the relationship between the heat sources and the heat loss. If the heat input into such a system increases at a sufficient rate with temperature the heat loss from the system may not be sufficient to allow the evolution of an equilibrium state (Landau L.D. and Lifshitz E.M. 1959). This condition will be called a "thermal instability" (Griggs D.T. and Baker D.W. 1969, Gruntfest I.J. 1963).

A limited form of a theory for such a condition was first developed for exothermic combustion reactions and called the "thermal theory of explosions" (Semenov N.N. 1928). A quantitative theory was later worked out for a distribution of exponential temperature dependent heat sources in a material (Frank-Kamenetski D.A. 1955). The first experimental work on viscous flow in this area was done by P.W. Bridgman (1935,1936,1937) as discussed in the previous section. The conditions under which stability (steady state solutions) can be obtained for constant shear stress have been found for both Newtonian and non-Newtonian stress strain-rate relations (Gruntfest I.J. 1963, Griggs D.T. and Baker D.W. 1969). In the applications to deep earthquakes in the above papers only the analysis of the temperature field and its behavior relative to the source term driven by a constant stress has been considered. In Chapter 4 I extend this work by considering the effect of inertial terms in determining the onset of instability.

1.3. Mathematical and Numerical Content

In Chapter 2 a variational principle is used to study the coupled thermal and mechanical field equations with the first order time derivative terms present. Using this variational principle I consider the evolution of a sheared slab under the assumption of creep (the time derivatives in the equations of viscous flow are taken as zero). The values of the physical constants and the dimensions are chosen here to approximate the conditions in the seismic low velocity zone. An interesting result occurs in this analysis. When one assumes a viscosity gradient across the slab one obtains the evolution of a large temperature gradient by the boundary of lower viscosity. One thus obtains the evolution of very low viscosity zone near that boundary. Here it should be noted that one problem which can be encountered in analysing a highly viscous fluid is that of imposed stability by the use of a condition such as constant stress. In using this condition one is assuming inertial motions do not exist which is the very condition which must evolve for instability to occur.

In Chapter 3 I demonstrate that one may obtain exact solutions for all the stress strain-rate relations discussed in that chapter if the physical systems can be described by the stationary field equations. This comes about as a result of the decoupling of the thermo - mechanical field equations in a stationary system.

In Chapter 4 I show that whenever an intertial term occurs in one of the field equations the other must also contain inertial terms and one obtains a coupling of the thermo - mechanical field equations. In these cases exact solutions have not been found.

In the actual physical systems which are discussed I am dealing with non-linear coupled partial differential equations of exponential order. Solving such equations numerically is in general very difficult and in many cases impossible (Bellman R. 1953). The method which I use for the numerical solutions presented here is the variational principle in chapter 2. This method however has many numerical problems. For example, in order to obtain accurate values for the integral and the derivatives a large number of points are needed. However a minimization routine is needed to find the extremes of the functional I(L,t) representing the time integral of the lagrangian. Here an additional variable is obtained for each field equation represented in the lagrangian and for each point at which the minimization is done. Since the amount of error must increase in a minimization routine with the number of points used and also the cost increases dramatically with each additional variable one is faced with severe limitations.

Chapter 2

Shear Heating with the Assumption of Creep

2.1 Physical Assumptions

I define the state of motion in which time derivatives of the velocity are negligible as creep, i.e. the inertial terms in the equation describing the velocity field are negligible. In this chapter I will consider the evolution of a sheared viscous slab under such an assumption.

The response of a viscous layer to a constant shear stress when the viscosity is temperature dependent has been discussed by Gruntfest (1963). A geological discussion and extension of his results for deep earthquake mechanisms has been given by Shaw (1969). A similar analysis of this phenomenon has been given by Griggs and Baker (1969) using a more complex stress strain-rate relation than used by Gruntfest (1963) and Shaw (1969). The Griggs and Baker (1969) stress strain-rate relation will be discussed in chapter 3. Here I will summarize and extend the arguments of Gruntfest (1963).

Consider a material with thermal conductivity k, specific heat c and viscosity

$$\mu = \mu_o \exp[-a(T-T_B)]. \tag{2.1}$$

Here T_B represents a reference temperature (usually the ambient temperature) at which the material has viscosity μ_o and T represents the temperature at any position in the material. The exponential temperature dependence of viscosity is reasonable if the viscosity is controlled by point-defect diffusion and $T/T_B \simeq 1$. Here "a" contains the activation energy. Suppose that a constant shear σ is applied to this material. The rate of heat production due to viscous shear heating is

$$\sigma\dot{\varepsilon} = \sigma^2 \exp[a(T-T_B)]/\mu_o \qquad (2.2)$$

provided a Newtonian stress stain-rate relation is satisfied

$$\sigma = \mu\dot{\varepsilon} \qquad (2.2a)$$

Here $\dot{\varepsilon}$ is the strain rate.

The heat production must act as a source in the heat flow equation. The temperature is therefore governed by

$$\rho c\frac{dT}{dt} - k\frac{\partial^2 T}{\partial x^2} = \frac{\sigma^2}{\mu_o}\exp(a(T-T_B)) \qquad (2.3)$$

where x is a co-ordinate at right angles to the bounding surfaces of the layer and ρ is the density.

If $\sigma = 0$, eq (2.3) is associated with a characteristic cooling time

$$t_c = \rho c\,\frac{\ell^2}{k} \qquad (2.4)$$

where ℓ is the half thickness of the layer.

If $k=0$, that is, if the system is thermally isolated, it is easy to show that T goes to infinity in a time

$$t_M = \frac{\rho c \mu_o}{\sigma^2 a} \tag{2.4a}$$

A real physical system must be between these extremes.

The dimensionless-ratio

$$\lambda = \frac{\sigma^2 \ell^2 a}{\mu_o k}$$

of these two times represents a comparison of the cooling time to the heating time of the system. Instability occurs in the homogeneous plane-shear case for $\lambda > .88$ (Gruntfest I.J. 1963).

Here instability represents the breakdown of the assumption of creep. This has been assumed to be the onset of an explosive instability or shear-melting induced earthquake (Gruntfest I.J. 1963, Shaw H. R. 1969, Griggs T. and Baker D. W. 1969, Nyland E. and Spanos T.J.T. 1976). The validity of this conclusion is not clear and will be explored in more detail in chapters 3 and 4.

2.2 A Variational Principle for the Diffusion Equation

The heat flow equation (2.3) is a diffusion equation with a shear-heating source term. In this section I discuss a variational principle which can be used to find approximate numerical solutions of the diffusion equation and extend it to the full physical system in the following section. When considering a variational principle for a dissipative system one must often part from the traditional methods such as Hamilton's variational principle. (Morse P.M. and Feshbach H. 1953). There one is dealing with a process which is reversible in time. In dealing with a dissipative system without time symmetry, one has an irreversible process, and thus a preferred time direction. This is the fundamental fact used in the construction of this variational technique.

Many authors have studied dissipative systems by variational techniques. Their approaches are of four types:

(1) Some investigators (for example, Morse P.M. and Feshbach H. 1953) introduce a dual absorption system and then couple the two systems in a Hamilton variational principle. By coupling dissipative and absorptive equations one has made the time direction once again indistinguishable. A time reversal simply switches the places of the absorptive and dissipative fields. There is no net dissipation in the total system.

(2) Stability analysis can be formulated as a variational principle (Schechter, R.S. 1967).

(3) A generalization of Hamilton's principle appears possible (Djuk D. and Verjansevic, B.Z. 1971). The argument of the Lagrangian of the system is modified parametrically in this approach.

(4) Other variational techniques exist for irreversible processes (Biot, M.A. 1970). It is possible to remove notions such as a Lagrangian from the mathematics by formulating time-dependent variational principles for dissipative systems. This approach yields solutions at a particular time only. New calculations must be made for other times.

The essential feature of the method presented here is the introduction of a discrete approximation for time derivatives. It is then assumed that the state of the system is known at sometime, t-ε. Thus, the variation over the field variables at this time is zero and one may find the value of the field variables at the time t by demanding that a functional L be stationary. If ε is small this approach allows one to remove the time integration from the variational principle.

This technique (which is discussed in greater detail in Appendix A) is illustrated with a simple example. Consider:

$$\delta L=\delta\int_V\{\frac{\rho C}{2a\varepsilon}[\phi(\vec{x},t)-\phi(\vec{x},t-\varepsilon)]^2+\frac{K}{2a}[\nabla\phi(\vec{x},t)]^2\}dv \qquad (2.5)$$

where $\vec{x}$ is a position vector.

The condition $\delta L = 0$ will generate an approximate form of the diffusion equation provided suitable boundary and initial conditions are chosen.

After applying the boundary condition and $\delta\phi\ (x,t-\epsilon)=0$ one obtains:

$$\delta L=\int_V\{\frac{\rho c}{a\epsilon}[\phi(\vec{x},t)-\phi(\vec{x},t-\epsilon)]-\frac{k}{a}\nabla^2\phi(\vec{x},t)\}\delta\phi(\vec{x},t)dv$$

The term in braces must be zero and in the limit $\epsilon\to 0$ this becomes

$$\frac{c}{a}\frac{\partial\phi}{\partial t}-\frac{K}{a}\nabla^2\phi=0.$$

where $\phi=a(T-T_\circ)$.

2.3 The Coupled Navier Stokes and Heat Flow Equations.

One can find a similar variational principle for the coupled heat flow (eq 2.3) and Navier Stokes equations for viscous flow. The Navier Stokes equations are:

$$\rho\dot{\vec{u}}-\vec{\nabla}\cdot(\mu\vec{\nabla}\vec{u})+\vec{\nabla}P-\rho g\vec{e}_3=0 \qquad (2.6)$$

where $\dot{\vec{u}}$ represents the time derivative of the velocity field, P is the pressure and μ is the temperature - dependent viscosity.

A variational principle which will generate these equations is:

$$\delta L=\delta\int_V\{\frac{\rho c}{2a\epsilon}[\phi(\vec{x},t)-\phi(\vec{x},t-\epsilon)]^2+\frac{K}{2a}[\nabla\phi(\vec{x},t)]^2+\frac{\mu_\circ}{2}\exp(-\phi(\vec{x},t))\dot{\varepsilon}^2_{ij}$$

$$+\frac{\rho}{2\epsilon}[\vec{u}(\vec{x},t)-u(\vec{x},t-\epsilon)]^2+\vec{\nabla}P.\vec{u}(\vec{x},t)-\rho g\vec{e}_3.\vec{u}(\vec{x},t)\}dv=0 \qquad (2.6a)$$

Here all time derivatives are given approximately over an interval . The coefficent of $\delta\phi$ (x,t) becomes the heat flow equation and the coefficient of δu(x,t) becomes the Navier Stokes equation. The variation of any field variable at time t_1 is zero and the field variables and their spatial derivatives are known for all time on the surface S that bounds the slab. If we assume one dimensional creep and a constant pressure across the slab then the last two terms in eqn (2.6) vanish.

2.4 Computation

In the one-dimensional analysis I evaluate particular case of the integral (2.6a) for constant stress (i.e. creep) after interpolating a cubic spline through the integrand. The derivatives w.r.t. x are calculated on this spline. A finite number of values of ϕ and u which will minimize the integrand are determined. The minimization is done using Zangwill's modification of Powell's conjugate direction algorithm (Johnson O.C. and Williams L.L. 1973). This routine was called from the International Mathematical and Statistical Library edition 4, available on the University of Alberta computer. The accuracy of this process was tested by the addition of points until the structure of the velocity and temperature profiles were smooth across the slab. For the perposes of this analysis only approximate values of the temperature and velocity are necessary.

Here $\delta L=0$ does not necessarily imply a unique solution. The functional L contains a term of more than bilinear form. The uniqueness of the solutions here may be tested numerically. This is done by making small perturbations in the initial guesses and changes in the convergence criteria to see if the same solution will result.

2.5 Some Numerical Examples

In figure (2.1) the values of the lagrangian specified in eqn. (2.5) are shown as a function of temperature for different values of the dimensionless constant lambda. Here a constant stress is assumed and the time step is $kt/\rho c\ell^2=1$. In figure (2.2) the solutions of the differential equation (2.3) are illustrated on contour plots of the lagrangian specified in eqn. (2.5) with time steps of $kt/\rho c\ell^2=.25, .5, .75$ and 1. Here temperature is plotted on the vertical axis and lambda on the horizontal axis.

The values of the specific heat c, thermal conductivity k and the activation energy for point - defect diffusion vary widely for geologic materials. In the calculations in this section the values c=.25 cal/g/C°; k=.03 cal/C°/cm/sec; a=.01 /C° are used.

Although these numbers may not fit any particular rock exactly they do not vary by more than a few orders of magnitude for most geologic materials. The density ρ can be taken as 3 g/cm³ within 10% for most geologic materials, at "low" pressures, but the viscosity can vary widely.

If we use a Newtonian stress strain-rate relation with a viscosity of 10^{22} poise and a velocity gradient of 10^{-14}/sec, the assumption that thermal instability occurs leads to a half thickness ℓ in the constant λ of approximately 6×10^{6} cm. This value must be slightly small since the actual value of λ must be reduced due to shear heating. This crude approximation thus yields a value of ℓ which is fairly close to the thickness of the seismic low velocity zone. It is thus possible that the seismic low velocity zone is a marginally stable material. This would seem to fit well with the presence of melt. However if "marginally stable" material is able to exist in the seismic low velocity zone without experiencing any instabilities (observed sudden relief of stress) one must then question the validity of assuming the breakdown of creep to be a representation of coupled thermo - mechanical instability. This problem will be discussed in both chapters 3 and 4.

Consider now some numerical results assuming creep. First assume we have a slab of viscosity 10^{22} poises throughout. The slab has a half thickness of 60 km and the gradient of the velocity field is given by σ/μ throughout the slab. The stress within the slab is assumed to be 100 bars (the values of stress are chosen to give a velocity difference of 10^{-7} cm/sec accross the slab) everywhere and in all cases the boundary temperatures are fixed. I am therefore considering here whether the decrease in viscosity in the seismic low velocity zone can be solely attributed to shear heating. The evolution of the temperature and displacement is shown in figures (2.3 and 2.4) for a time step of 10^{14} seconds. It should be noted that the viscosity does not decrease by the almost 2 orders of magnitude necessary to be in agreement with the observed viscosities in this region (see section 3.1 of this dissertation).

Increasing pressure with depth and the rise of melt towards the surface results in a viscosity which increases with depth. The evolution of a slab under this condition from a constant temperature throughout may be illustrated by $\mu_o \rightarrow \mu_o \exp(2(x+1))$. The results of this evolution of temperature are then given in figure (2.5). The corresponding evolution in velocity is given in figure (2.6). The earth also has a temperature which increases with depth due to cooling at the surface. The evolution of the same slab as in figure (2.5) only from a linear temperature gradient of 300°K across the slab is illustrated in figure (2.7). The corresponding evolution in velocity for this slab is given in figure (2.8). Another example of interest to the analysis in chapter 4 is that of a temperature increase towards the boundary of low viscosity. The evolution of temperature and velocity in this case is illustrated in figures (2.9) and (2.10) respectively. Figure (2.11) illustrates a proposed viscosity depth profile in the seismic low velocity zone.

Figure 2.1: Values of the lagrangian in sec (2.2) are shown as a function of temperature for different values of the dimensionless constant lambda.

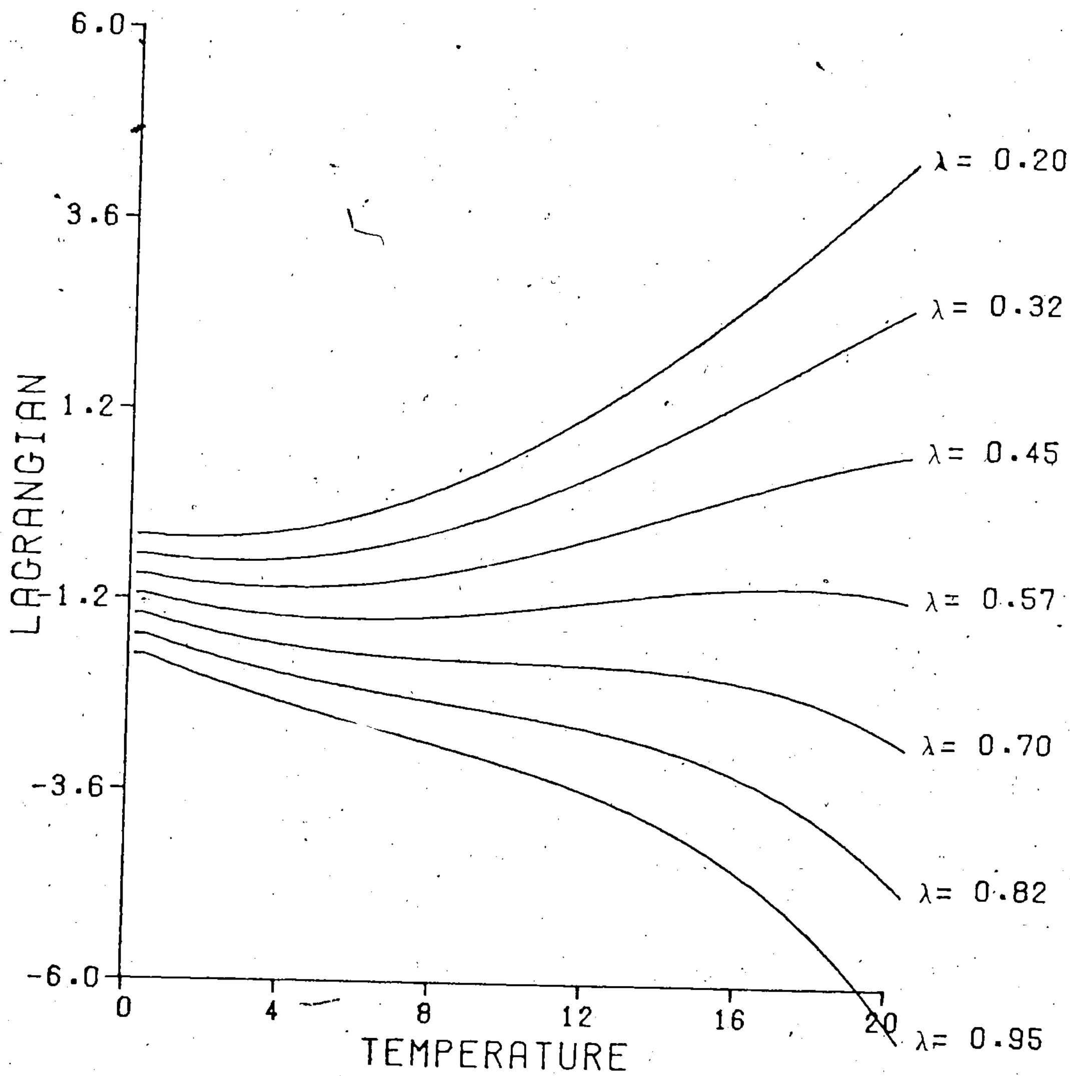

6.0
3.6
1.2
-1.2
-3.6
-6.0
LAGRANGIAN
0
4
8
12
16
20
TEMPERATURE
λ= 0.20
λ= 0.32
λ= 0.45
λ= 0.57
λ= 0.70
λ= 0.82
λ= 0.95

Figure 2.2: Solutions of the variational principle in section 2.2 are illustrated for 4 different sized time steps, .25, .5, .75 and 1.. The initial condition is a constant temperature across the slab.

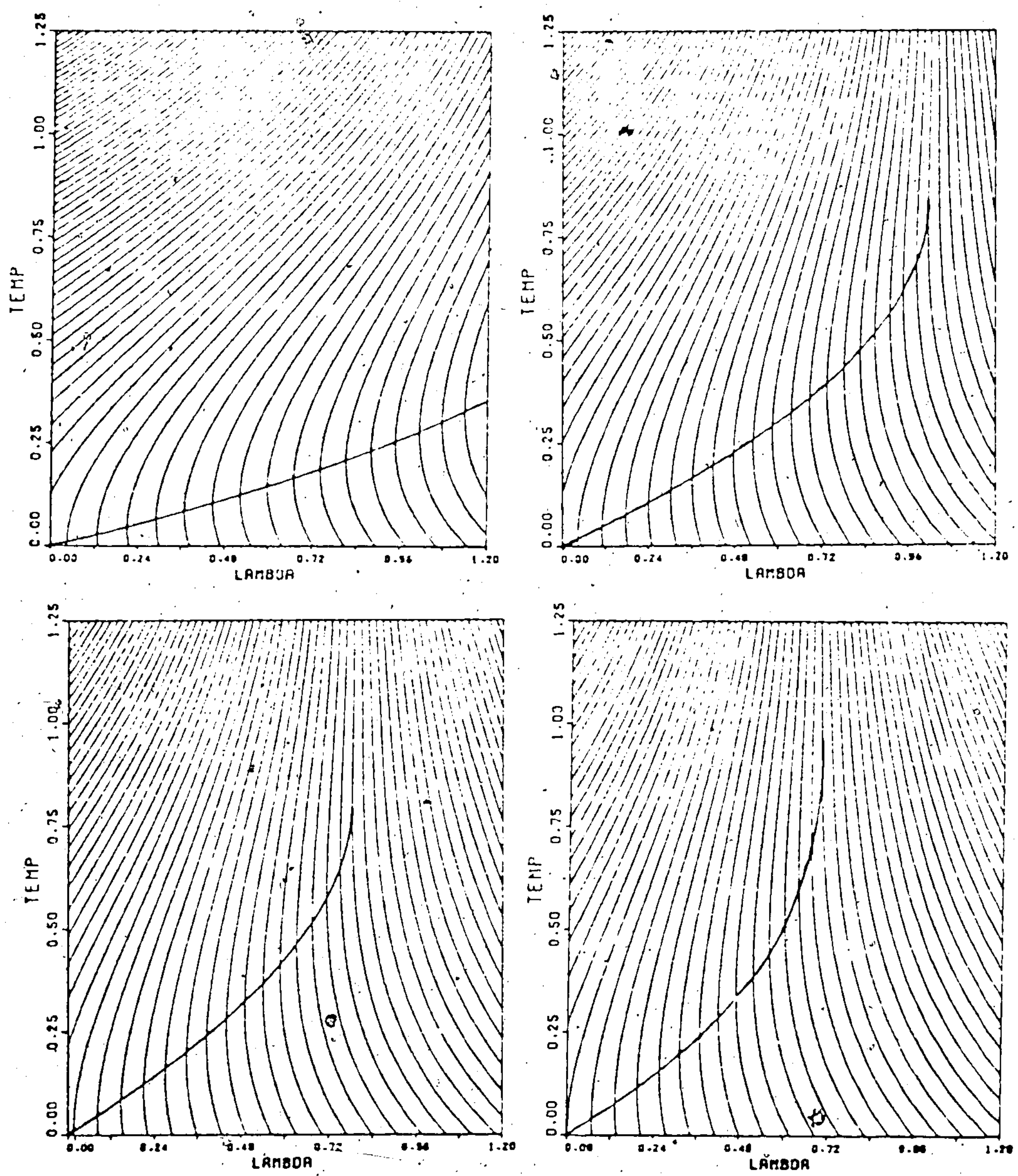
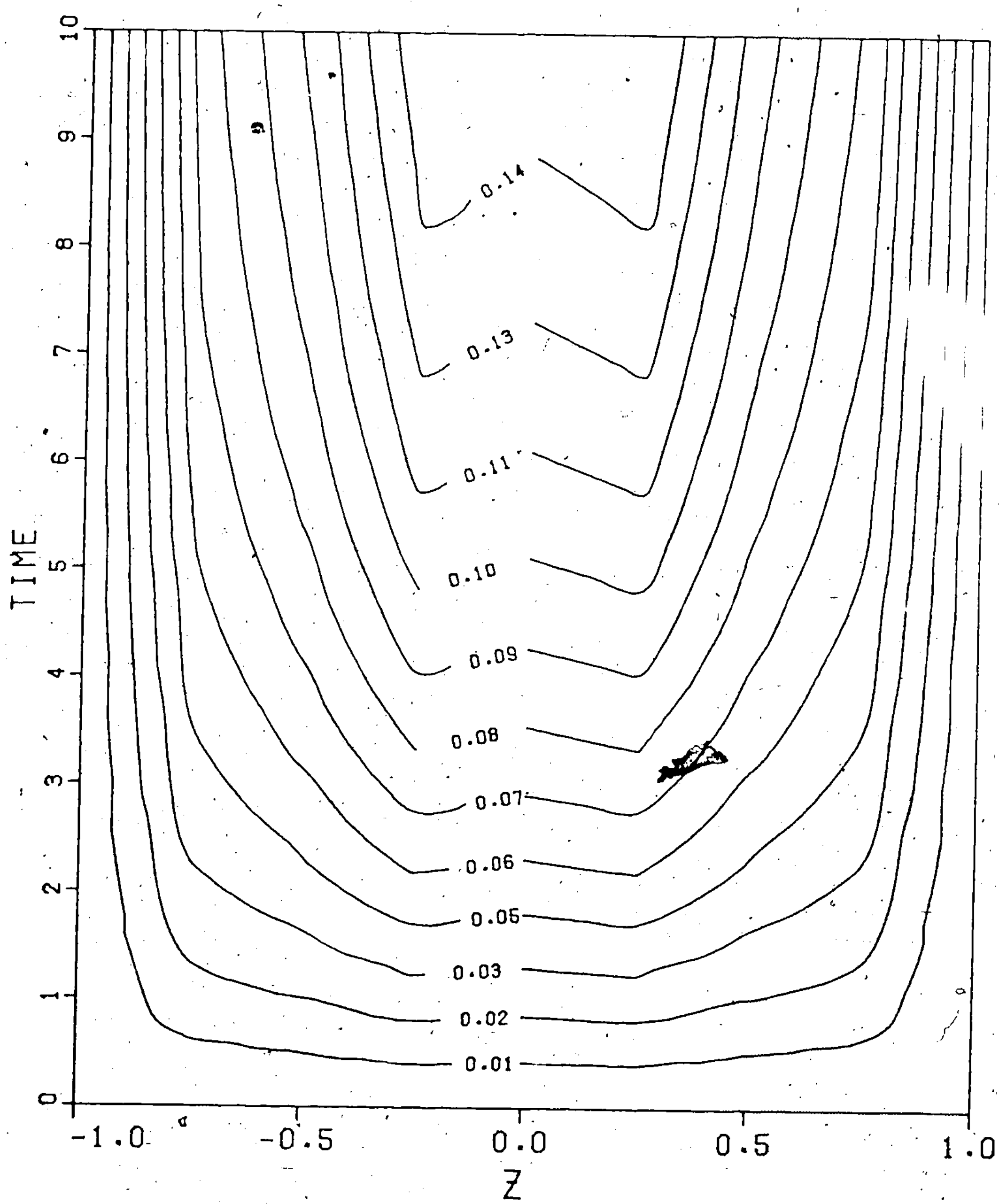

TEMP
LAMBDA
0.00
0.25
0.50
0.75
1.00
1.25
0.00
0.24
0.48
0.72
0.96
1.20

Figure 2.3: The evolution of the temperature field is shown from an initially constant value across the slab. The fluid is homogeneous and 10 time steps of 10^{13} sec (i.e. episilon =.111) are used in the calculation. The temperature change is given in 10^{-2} x °C. Here the stress is 50 bars.



0.14
0.13
0.11
0.10
0.09
0.08
0.07
0.06
0.05
0.03
0.02
0.01
TIME
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
Z

Figure 2.4: The velocity evolution is given corresponding to the temperature evolution of the homogeneous slab in figure (2.3). The values on the contours are the velocities in cm/sec relative to the centre of the slab.

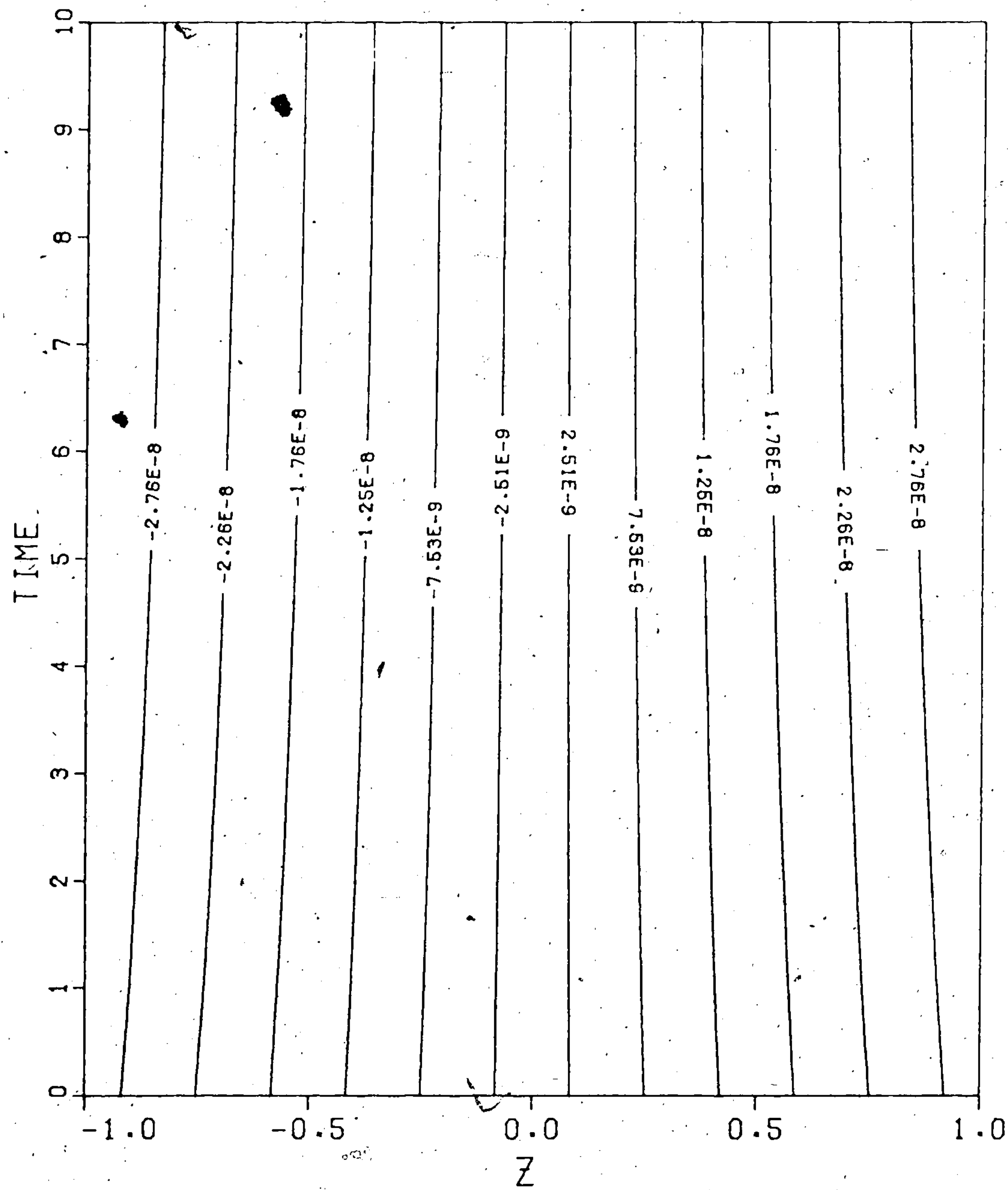
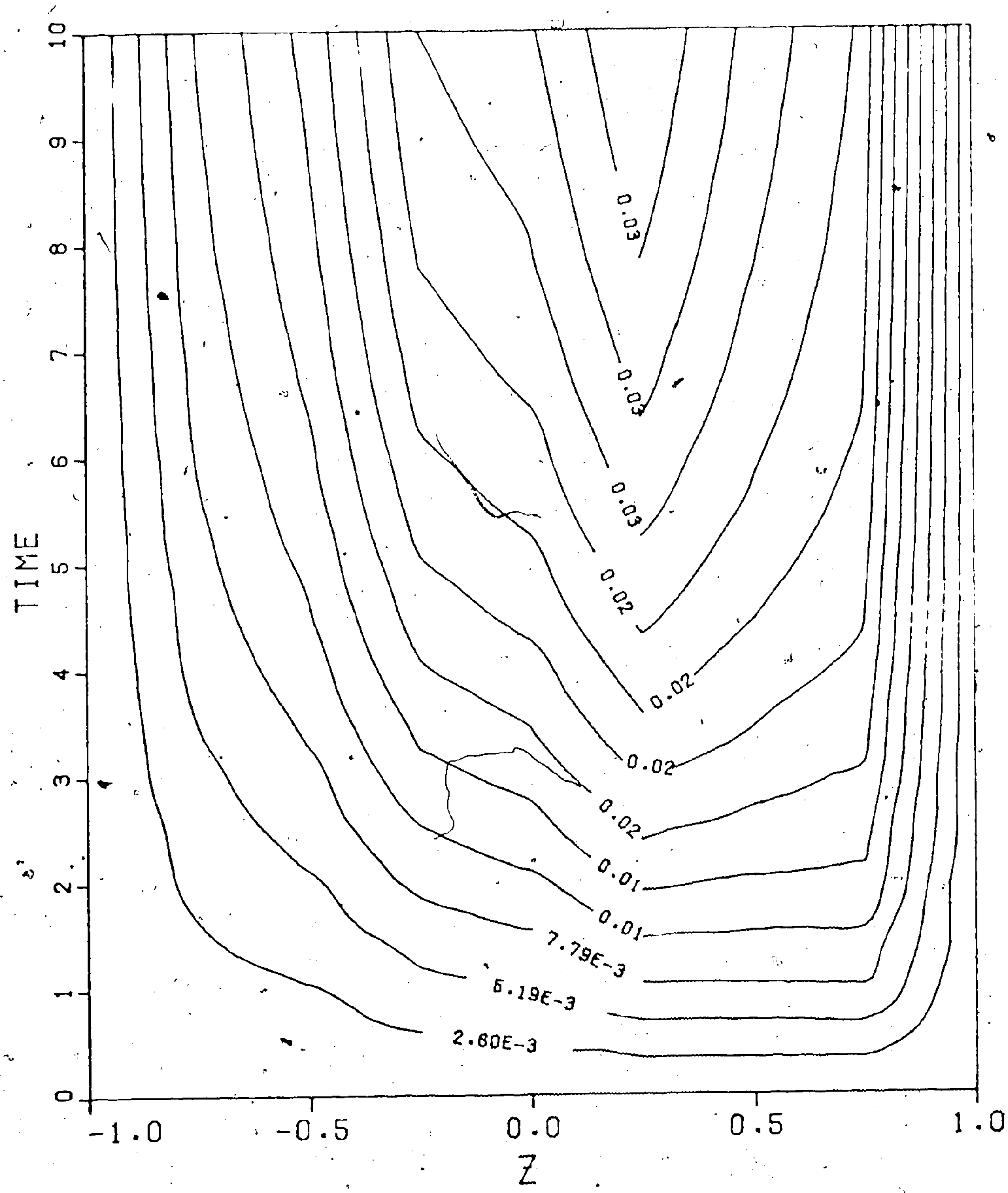

TIME
Z
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
-2.76E-8
-2.26E-8
-1.76E-8
-1.25E-8
-7.53E-9
-2.51E-9
2.51E-9
7.53E-9
1.26E-8
1.76E-8
2.26E-8
2.76E-8

Figure 2.5: The temperature evolution with a viscosity gradient $\mu_O = \mu_O \exp(-3(Z+1)/2)$ is illustrated. Here the stress is 10 bars and ~~the~~ time steps are 10^{13} sec. The initial temperature distribution is constant across the slab.



TIME
Z
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
0.03
0.03
0.03
0.02
0.02
0.02
0.02
0.01
0.01
7.79E-3
5.19E-3
2.60E-3

Figure 2.6: The velocity evolution corresponding to the temperature distribution in figure (2.5) for an inhomogeneous slab is given. The boundary of low viscosity is on the right.

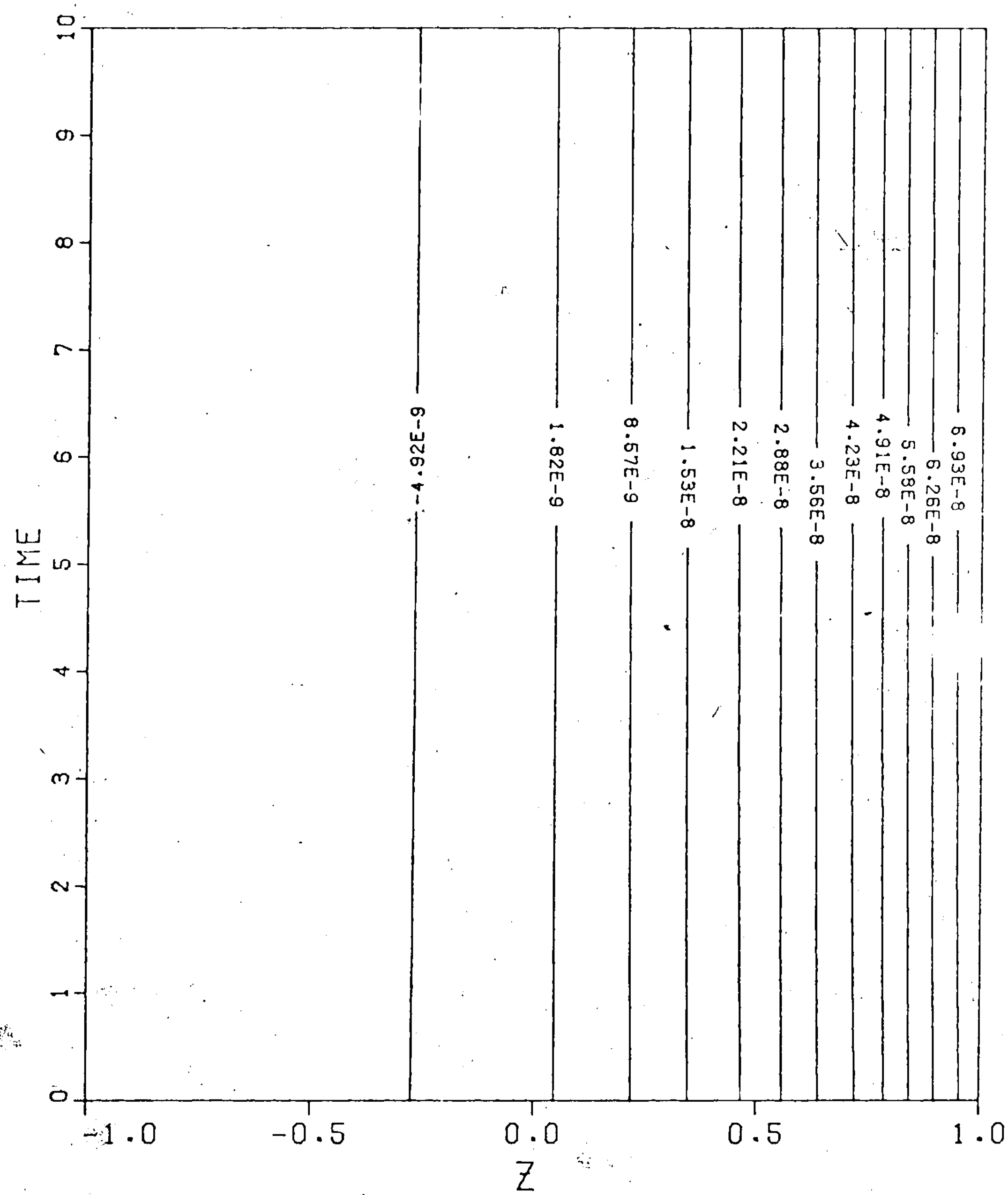

TIME
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
Z
-4.92E-9
1.82E-9
8.57E-9
1.53E-8
2.21E-8
2.88E-8
3.56E-8
4.23E-8
4.91E-8
5.58E-8
6.26E-8
6.93E-8

Figure 2.7: The earth has a known temperature gradient. This figure illustrates the temperature distribution for the slab in figures (2.5) and (2.6) with the temperature of the boundary of high viscosity held at 300°C greater than the boundary of low viscosity. Here the stress is 10 bars.

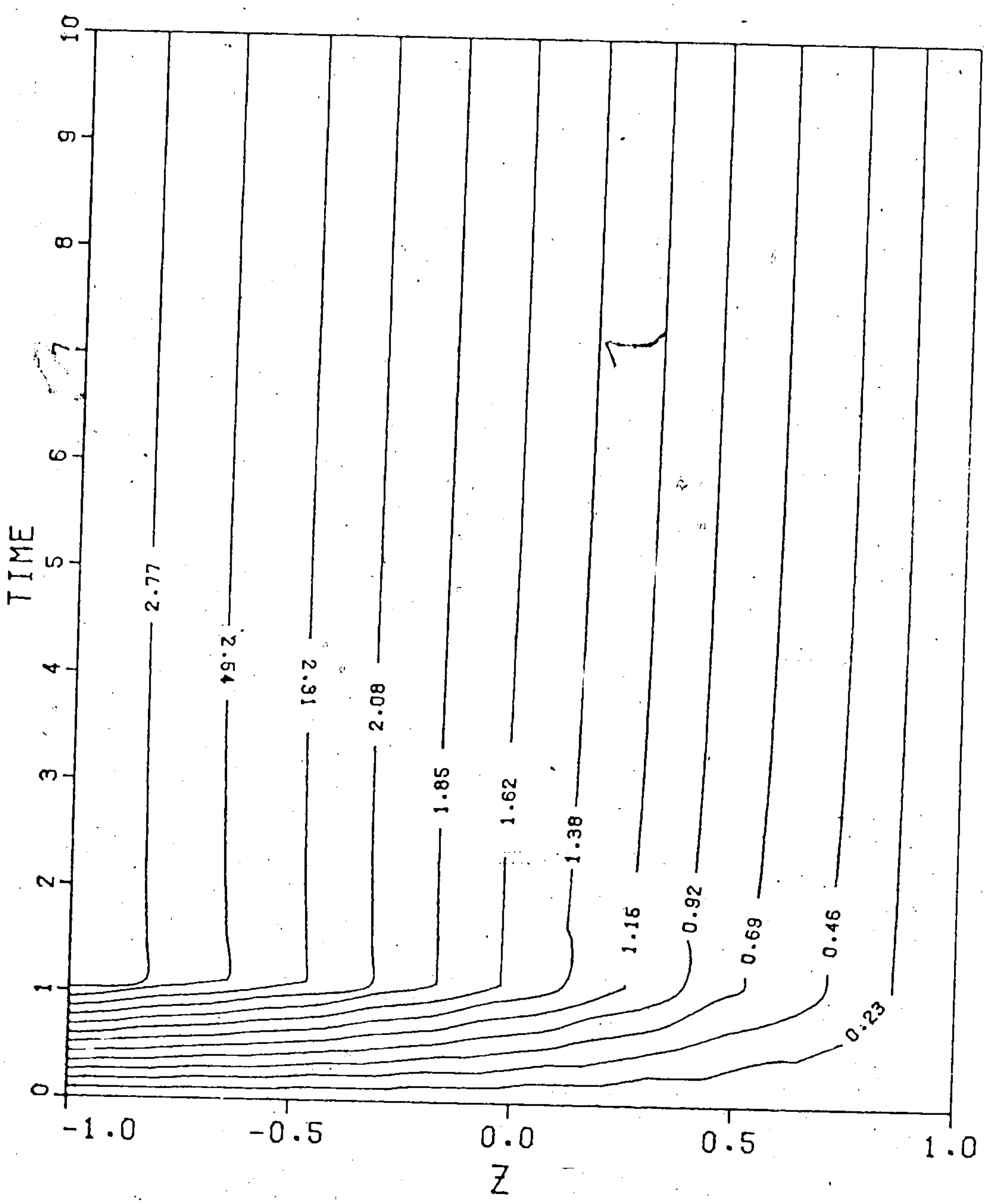

TIME
Z
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
2.77
2.54
2.31
2.08
1.85
1.62
1.38
1.16
0.92
0.69
0.46
0.23

Figure 2.8: The velocity evolution corresponding to the slab described in figure (2.7) is given. Here one obtains a fairly uniform velocity gradient across the slab.

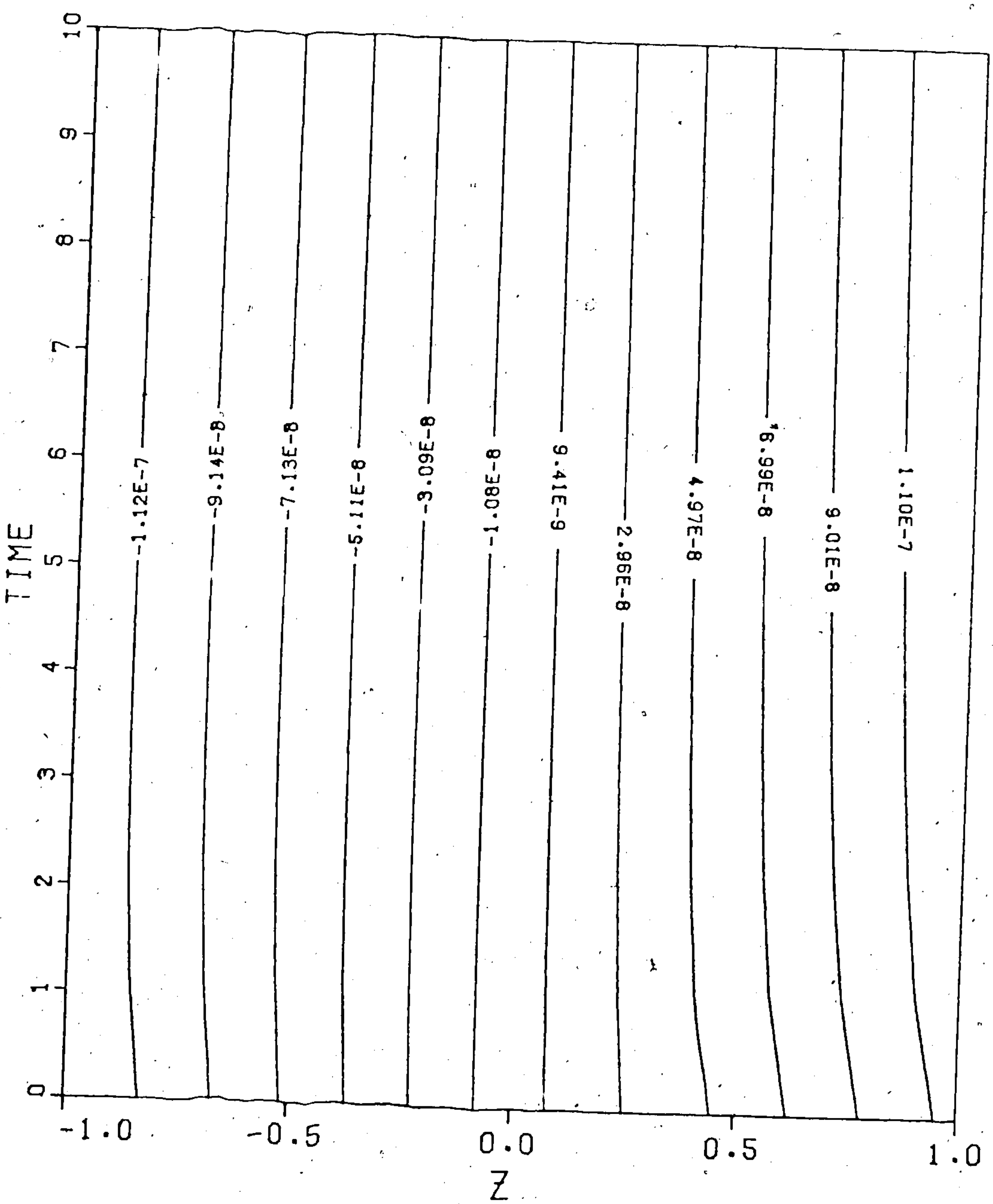

TIME
Z
-1.0
-0.5
0.0
0.5
1.0
0
1
2
3
4
5
6
7
8
9
10
-1.12E-7
-9.14E-8
-7.13E-8
-5.11E-8
-3.09E-8
-1.08E-8
9.41E-9
2.96E-8
4.97E-8
6.99E-8
9.01E-8
1.10E-7

Figure 2.9: A temperature increase of 60°C at the boundary of low velocity over the boundary of high viscosity is considered. This figure illustrates that the shearing of the slab then causes the maximum temperature to move away from the boundary of low viscosity towards the interior. Here the time steps in 10^{13} sec and the stress is 1kb and the slab width is 5 km.

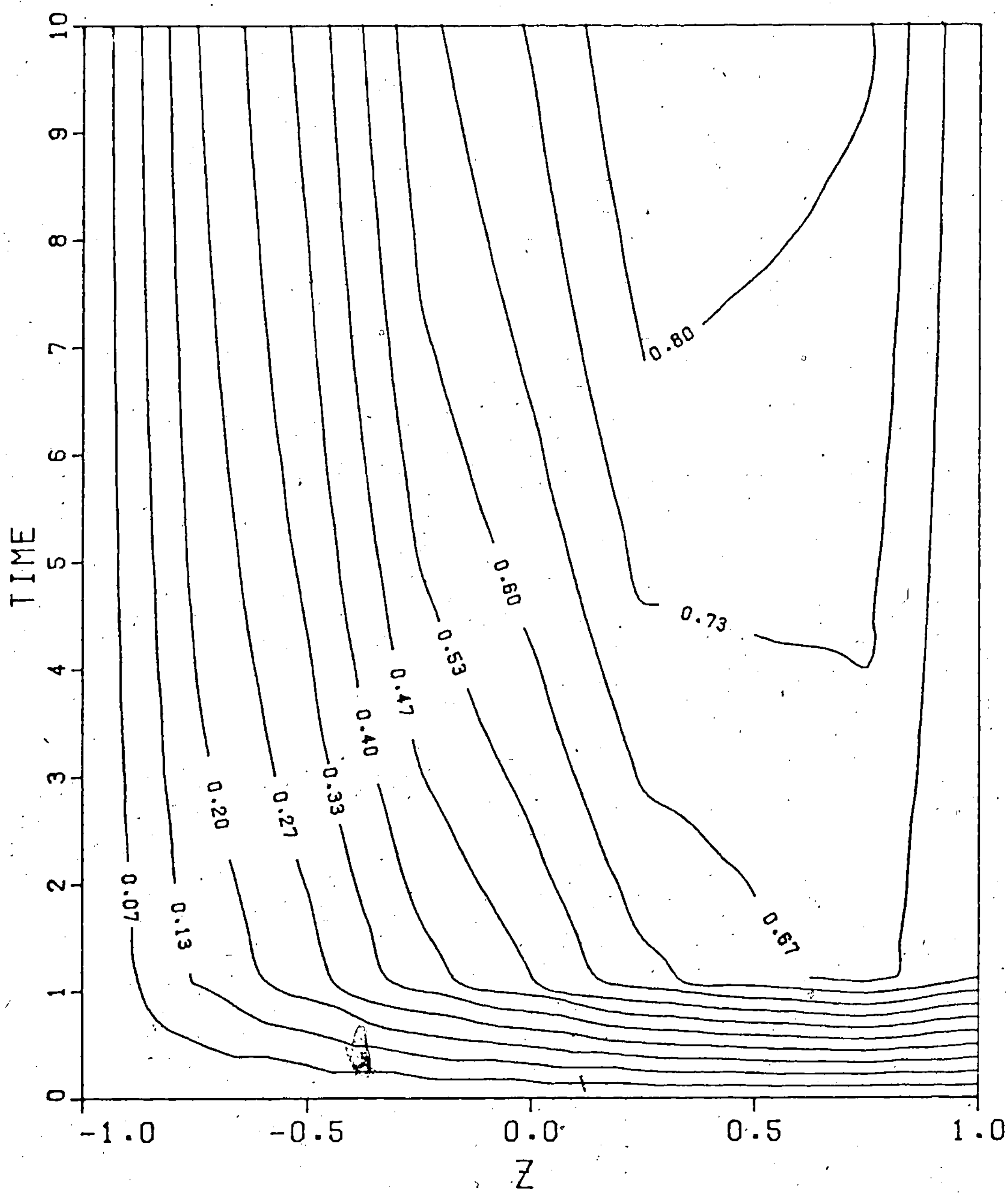

TIME
Z
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
0.07
0.13
0.20
0.27
0.33
0.40
0.47
0.53
0.60
0.67
0.73
0.80

Figure 2.10: The velocity evolution corresponding to the temperature distribution in figure (2.9) is given. Note that the maximum velocity gradients occur in the region of maximum temperature.

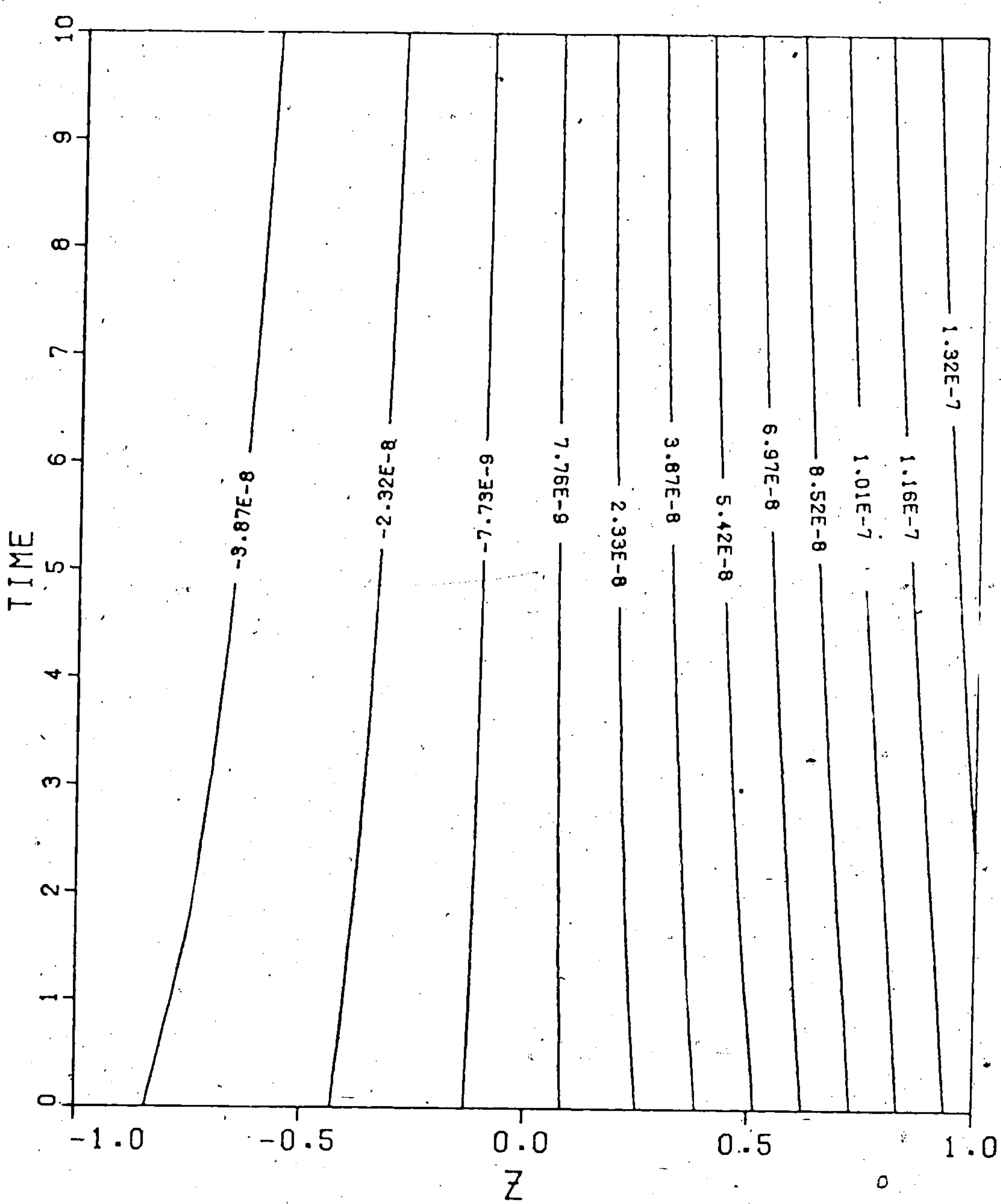
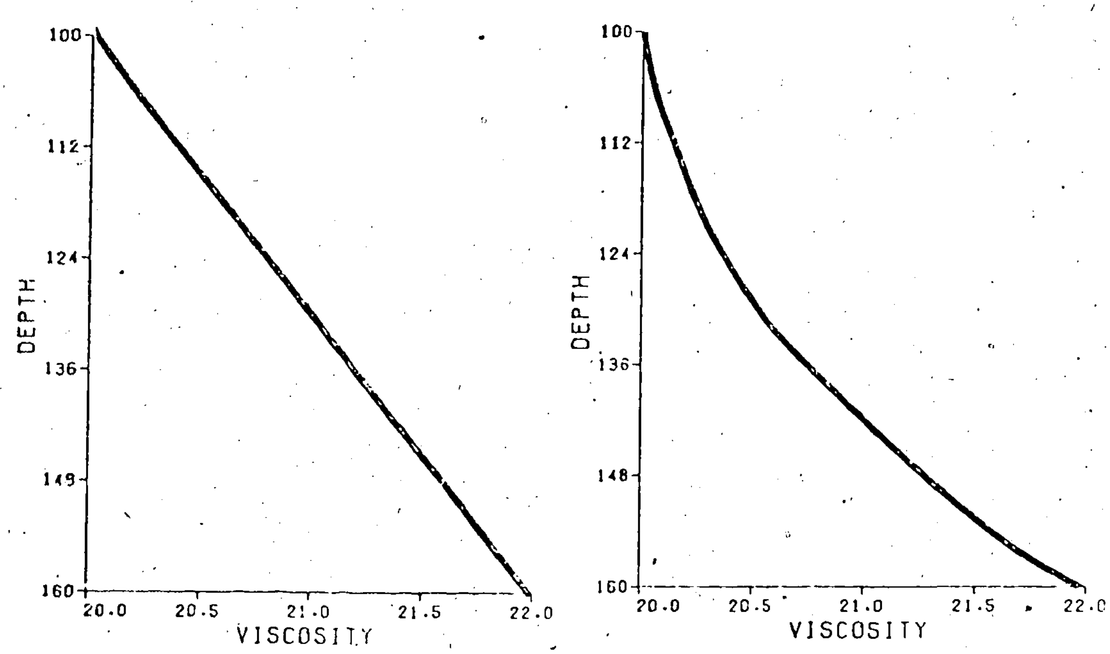

TIME
Z
0
1
2
3
4
5
6
7
8
9
10
-1.0
-0.5
0.0
0.5
1.0
-3.87E-8
-2.32E-8
-7.73E-9
7.76E-9
2.33E-8
3.87E-8
5.42E-8
6.97E-8
8.52E-8
1.01E-7
1.16E-7
1.32E-7

Figure 2.11: Viscosity verses depth is illustrated first without shear heating and secondly with shear heating for a viscosity increasing exponentially with depth. The depth scale is linear and the viscosity scale is logarithmic.


100
112
124
136
148
160
DEPTH
20.0
20.5
21.0
21.5
22.0
VISCOSITY
100
112
124
136
148
160
DEPTH
20.0
20.5
21.0
21.5
22.0
VISCOSITY

Chapter 3

Exact Solutions for Constant Stress and Their Application to the Seismic Low Velocity Zone.

3.1 Physical Discussion

It is generally acknowledged, (Schubert G. et al, 1976) that many geophysical characteristics like topography, lithosphere thickness and heatflow on ocean bottoms are essentially a function of the age of the ocean bottom. Within limits these features can be predicted as a consequence of relatively simple models of heat flow and dynamics in the earth (Turcotte D.L. and Oxburgh E.R., 1967, McKenzie D.P., 1967, Parker R.L. and Oldenburg D.W., 1973 for example). With rare exceptions (Schubert G. and Turcotte D.L., 1972 Schubert G. et al, 1976) none of these models include the thermo-mechanical coupling that must exist if shear-heating contributes to temperature, and viscosity is temperature dependent.

All plausible relations between stress and strain rate in rocks under mantle conditions involve the dissipation of heat in some form. These mechanisms of deformation are controlled by the migration of crystal defects and as such require the generation of heat. This heating modifies the physical properties of the material and as a result can alter its mechanical behaviour. Under most geologic conditions such thermal feedback appears to be a fairly small contribution to geodynamics but, with sufficiently effective heat generation mechanisms, instabilities can arise. The conditions under which the feedback mechanism becomes important to the geodynamic models being considered is of fundamental importance both in this chapter and chapter 4.

Here lithosphere formation will not be dealt with directly. Instead in this chapter I attempt to clarify the part shear heating plays in the existence of the seismic low velocity zone. The results of this analysis are dependent on the many assumptions made about the physical parameters and constants involved. For example it is assumed here that the seismic low velocity zone behaves essentially as a viscous fluid over the time scales of the motion imparted to it. The mantle material below is then able to exhibit solid behavior relative to the relaxation time for the seismic low velocity zone. Yet this material beneath the seismic low velocity zone is able to behave as a fluid itself over longer periods of time. In reality this approximation represents a fluid whose viscosity increases with depth. Highly viscous fluids do have the ability to act as solids or fluids for different rates of application of stress (Kolsky H. 1963). It appears however that this is not a factor in the geodynamic processes being considered. Cathles (1975) claims that there is no evidence for a stress rate dependent viscosity in glacial rebound studies.

Certainly the most controversial physical parameter is in fact the value of viscosity. All of the present kr ledge of the viscosity of the mantle comes from glacial rebound studies. A fairly complete review of this subject and list of references is given by Cathles (1975). It should be noted however that Cathles assumes a Newtonian mantle. This is an assumption which is not generally accepted (e.g. Carter N.L. 1976, Froidevaux C. and Schubert G. 1975). There are differences in the rates of change of viscosity with depth for Newtonian and non-Newtonian flow laws. and the non-Newtonian (power) flow laws give rise to a well defined low viscosity channel in contrast to the linear law (Carter N.L. 1976). Cathles (1975) however claims that the load cycle behavior of earth models that have high viscosity lower mantles, non-Newtonian mantle viscosity or substantial non-adiabatic density gradients are similar to the load cycle behavior of layered Newtonian viscosity models. Thus for the purposes of this analysis I will assume the values of viscosity obtained by Cathles.

The areas of uplift which have been studied are Fennoscandia, the Canadian Shield, Lake Bonneville, Greenland and the Arctic. The area over which uplift occurs is used along with the rate and horizontal variations in the rebound to obtain models for the values of the viscosity to different depths under each region. The general values obtained by Cathles are, a region of relative low viscosity of $4x10^{20}$ poise from the lithosphere-asthenosphere boundary to a depth of approximately 75 km below the plates and a viscosity of approximately 10^{22} poise throughout the rest of the mantle. Although these values are subject to some dispute (eg. MacDonald G.J.H. 1963) they probably cannot be very much in error above the 650 km discontinuity. If the viscosity were much less than this glacial rebound would have to occur at a faster rate than observed. For much larger values of viscosity one would be required to alter the equations used by Cathles governing the dynamics in order to obtain a reasonable rate of rebound. Such an analysis certainly does not seem to be justified since similar values were obtained for all areas which have been studied in spite of the different areal extents and load releases involved.

The values of the other physical constants used in the equations of heat flow and fluid flow, specific heat c, density ρ and thermal conductivity k, seem to be fairly consistent throughout the literature on heat flow in the mantle. (eg McKenzie D.P. 1970, Toksoz N.M. et al 1971). The values of these constants are generally assumed to be $c=10^7$ ergs/°C/gm, $\rho=3$gm/cm^3, $k=4\times10^5$ergs/cm/°C/sec.

3.2 A Simple Newtonian Stress Strain-Rate Relation

Consider now the solutions which "may" evolve for an infinite slab of half thickness ℓ subjected to a shear stress σ . The rate of heat production due to viscous shear heating at any point in the slab is $\sigma\dot{\varepsilon}$ where $\dot{\varepsilon}$ is the strain rate and σ is the stress at that point. The boundaries of the slab are held at temperature T_B. The slab has thermal conductivity k, specific heat c and viscosity $\mu = \mu_o \exp(-a(T-T_B))$. Here μ_o is the viscosity at temperature T_B and a is a material constant. Thus the rate of heat production is given by eq. (2.2) if a newtonian stress strain rate relation is satisfied and the principle of heat balance is once again given by eq. (2.3). Momentum balance eg (2.5) is conveniently derived from the condition

$$\frac{\partial\sigma}{\partial x} = \frac{d(\rho v)}{dt} \qquad (3.1)$$

where v is the velocity in the plane of the slab. The complete solution of a sheared slab requires simultaneous solutions of equations (2.2), (2.3) and (3.1).

The assumption of Gruntfest (1963), Griggs and Baker (1969), Shaw (1969), Schubert et al (1976) and the previous chapter that $(\partial\sigma / \partial x)=0$. But when inertial term exist in the temperature field this is not rigorously valid. In the analysis of the inertial field equations given in Chapter 4 it is shown that the stress is uniform throughout the slab if the inertial terms (derivatives w.r.t. time) are zero in both the heat-flow and Navier Stokes equations. I do no mean to imply that this assumption of creep is not a usefu approximation in many cases. However I do wish to point out that [illegible] analyzing a system changing in time the creep assum[illegible] can [illegible] "mp[illegible]e" stability on highly unstable physical situations and it must always breakdown in the "neighbourhood of instability".

The fashion in which stability can be "imposed" is as follows. It will be shown in chapter 4 that a thermal instability is a localized phenomenon within a fluid. However, the instability criterion derived in chapter 2 depends only on the slab as a whole. Keeping these statements in mind it can be seen from eq (3.1) that any breakdown in the "assumption of creep" can induce a large local velocity gradient without a substantial reduction in stress. This may then produce a local weakness within the slab from which it is not able to recover before the onset of melt. If one were to then take a slab which behaved in this fashion and apply the creep assumptions to it, the velocity field would be averaged out over the slab as a whole and the slab could appear very stable. In the neighbourhood of instability the creep assumptions must breakdown by definition since a sufficient reduction in viscosity must result in the relief of stress by accelerations. The analysis of these phenomena will be the subject of chapter 4.

The rest of this chapter is concerned strictly with analytic solutions to the stationary field equations and the applications discussed in section (3.1) to the seismic low velocity zone.

In general one has the condition

$$\rho c \left(\frac{\partial T}{\partial t} + v_x(x,y) \frac{\partial T}{\partial x} + v_y(x,y) \frac{\partial T}{\partial y} \right) - K\nabla^2 T = \frac{\sigma^2(x,y)}{\mu_o} \exp(a(T-T_B)) \qquad (3.2)$$

for the heat balance in an infinite fluid layer in which flow is not restricted to be laminar. Here x is the co-ordinate across the slab and y is a co-ordinate in the plane of the slab and the direction of the applied shear. If there is no stress in the x direction then $v_x(x,y)=0$. Further, if we assume that the temperature is time independent at any point in the slab then the velocity must also be time independent and must be a function of y only. Equation (3.2) then becomes

$$\rho c v_y(x,y) \frac{\partial T}{\partial y} - K\nabla^2 T = \frac{\sigma^2(y)}{\mu_o} \exp(a(T-T_B))$$

If the stress varies in the horizontal direction one obtains a stress gradient and thus from eqn. (3.1) we have inertial motions. However if the flow is incompressible such a condition of a constant stress across the slab and a variable stress in the plane parallel to the slab is impossible. Also if shear stress causes heating [illegible] in some cases melting it must diffuse at a finite rate which may be zero across 100% melt. Therefore $\partial\sigma/\partial y \neq 0$ implies $\partial\sigma/\partial x \neq 0$. I therefore consider only the fully stationary solution, σ not a function of x, y, or t. The inertial term in the thermal equation must therefore be zero and this implies $\partial T/\partial y = 0$ for $v \neq 0$.

The differential equation for the stable temperature distribution which may evolve is given by

$$k\frac{d^2T}{dx^2} = -\sigma^2 \exp[a(T-T_B)]/\mu_o \qquad (3.3)$$

with boundary conditions $T = T_B$ for $x=\pm \ell$ where 2ℓ is the thickness of the slab.

The solution of equations (2.2, 3.1, 3.3) is similar to one in Landau and Lifshitz (1959, p. 190). In order to reduce the calculation to dimensionless quantities, as in chapter 2, let

$$\phi = a(T-T_B), \text{ and } \xi = x/\ell$$

Then $$\frac{d^2\phi}{d\xi^2} + \lambda\exp(\phi) = 0 \text{ where } \lambda = \frac{\sigma^2\ell^2 a}{\mu_o k} \qquad (3.4)$$

Equation (3.3) has the solution

$$\phi = \phi_c - 2\,\ln\left[\cosh\left(\sqrt{\frac{\lambda}{2}}\exp(\phi_c/2)\,|\xi|\right)\right] \qquad (3.5)$$

where ϕ is the maximum temperature (which must occur at the centre of the slab). I obtain ϕ from the boundary condition $\phi=0$ at $\xi=1$ and observe that solutions of this equation for ϕ exist only if $\lambda \leq .88$ as illustrated in figure(3.1). Figure (3.2) shows graphical presentations of this solution.

From equations (3.1) and (3.3) one finds the nonvanishing component of velocity is given by

$$v = -\frac{k}{\sigma}\frac{dT}{dx} = -\frac{ka}{\sigma\ell}\frac{\partial\phi}{\partial\xi}$$

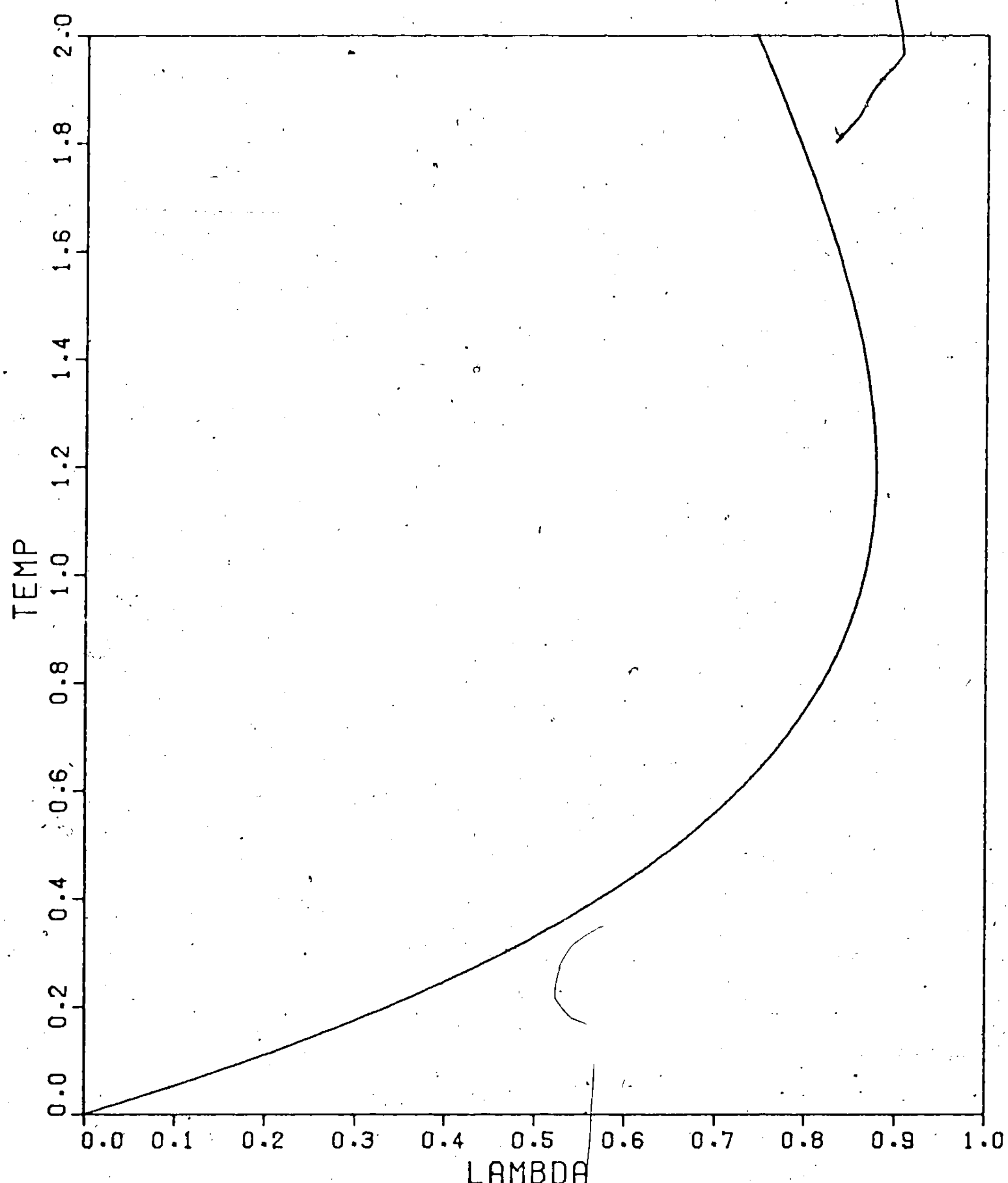
$$= \mathrm{sgn}(\xi)\frac{\sigma\ell}{\mu_o}\sqrt{\frac{2}{\lambda}}\exp(\phi_c/2)\tanh\left(\sqrt{\frac{\lambda}{2}}\exp(\phi_c/2|\xi|)\right)$$

Figure (3.3) show graphical representations of the velocity. Note that the critical parameter is the value of λ . For the particular case solved here the condition that stable solutions may exist is $\lambda<.88$. It is however interesting to consider the case where melting occurs at a finite temperature T_M. Then the time to melting, t_M, as defined in chapter 2 becomes

$$t_M = \frac{\rho c\mu_o}{\sigma^2 a}\;\frac{1}{(1-\exp(-a(T_M-T_B)))}$$

For the low velocity zone $a(T_M-T_B)\simeq 1$ hence $t_c/t_M \simeq .6\lambda$, but from figure (3.1) it can be seen that the value of λ changes very little to .85 for this value of ϕ_c. It should be noted that in this case $t_c/t_M \neq \lambda$ since t_M is the time for the material to melt and λ is defined under the assumption that the material could be heated to an infinite temperature without melting.

Figure 3.1: The central temperature as a function of the dimensionless constant lambda is plotted. Note that the maximum value of lambda for which solutions exist is .88 which corresponds to a temperature of 1.2.


TEMP
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0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
LAMBDA

Figure 3.2: The values of the temperature are given between the middle x=0.0 and the boundary x=1.0 of the class of homogeneous slabs in which a stationary solutions exist. The vertical axis represents different values of the dimensionless constant lambda. The temperatures are given in terms of the dimentionless temperature ϕ (ie. 10^{-2} x ^{o}C).

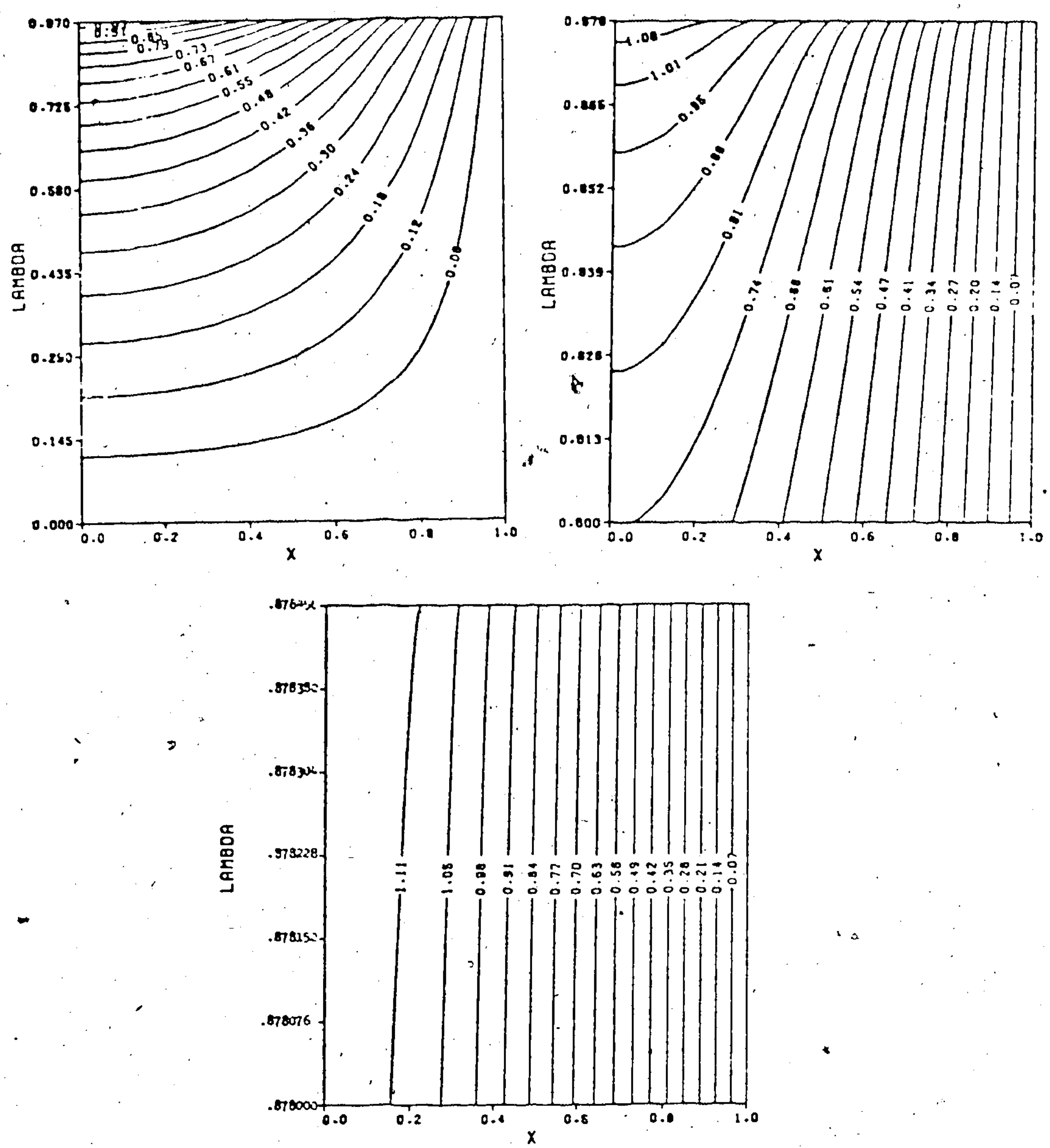
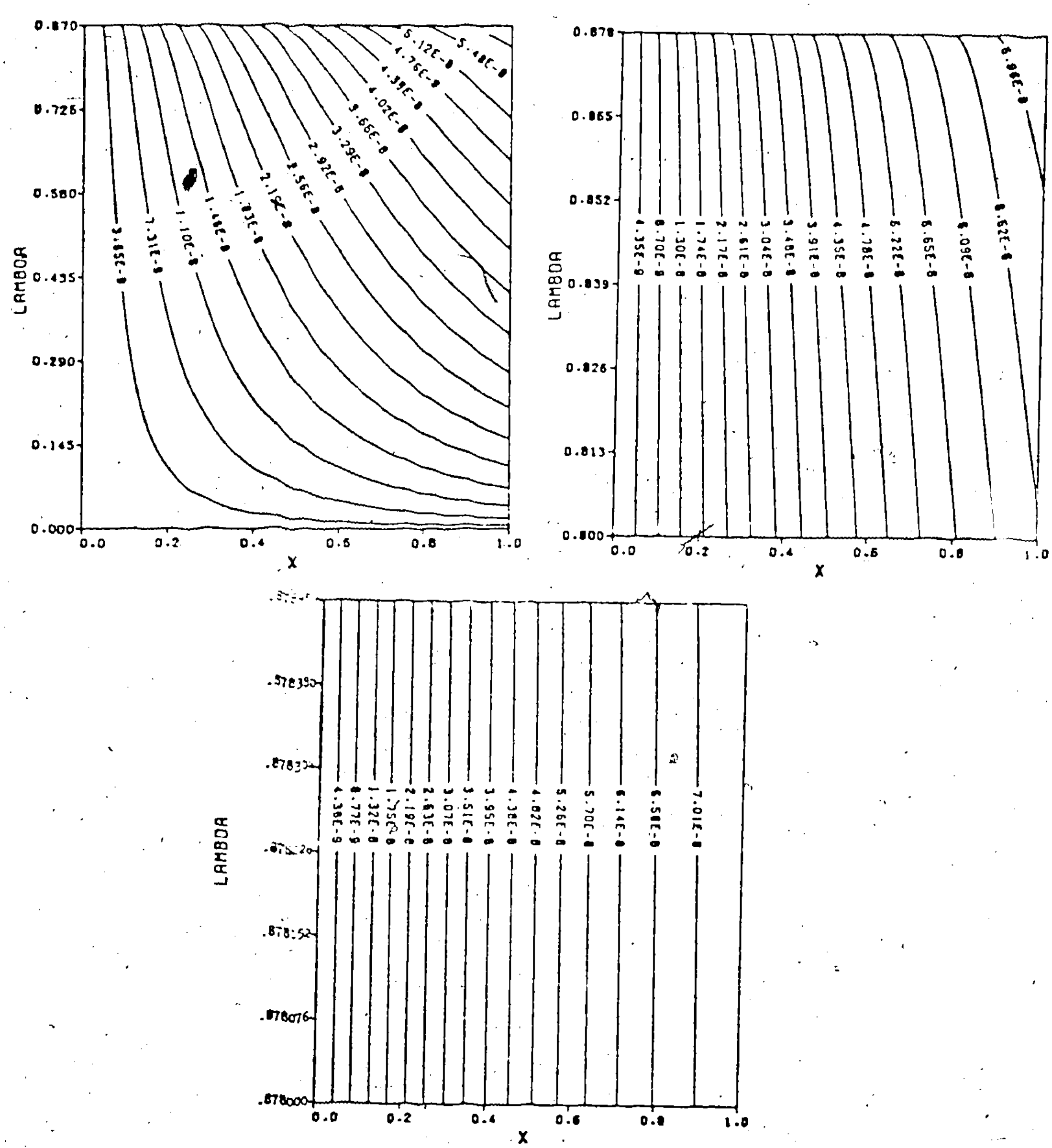

LAMBDA
X
LAMBDA
X
LAMBDA
X

Figure 3.3: The values of the velocity are given for the same slabs as in figure (3.2). The values of the velocity are on the contour lines in cm/sec.


LAMBDA
X
0.870
0.725
0.580
0.435
0.290
0.145
0.000
0.0
0.2
0.4
0.6
0.8
1.0
LAMBDA
X
0.878
0.865
0.852
0.839
0.826
0.813
0.800
LAMBDA
X

3.3 Solutions for more Realistic Stress Strain-rate Relations

A further complication in the geophysical application of these equations is the necessity of using "realistic" stress strain rate relations. This is a subject which has been treated very extensively by Ranalli(1974) and more recently in a review paper by Carter (1976) and thus the justification of the stress strain-rate relation will not be considered here.

Steady state creep rates for metals, ceramics and rocks are related to temperature and stress by a Weertman - type equation of the form (Carter N.L. 1976)

$$\dot{\varepsilon} = A \exp(-(E+PV)/RT) f(\sigma) \qquad (3.6)$$

Here A is a slightly temperature sensitive material constant, E creep activation energy, P is the pressure, V is the activation volume, R is the gas constant, T it the temperature and $f(\sigma)$ is the stress function. In this section I will be concerned mainly with the special case of eqn (3.6),

$$\frac{\dot{\varepsilon}}{\dot{\varepsilon}_o} = \exp\left(\frac{E}{RT_B} - \frac{E}{RT}\right)\left(\frac{\sigma}{\sigma_o}\right)^n \qquad (3.7)$$

which holds for most geologic material in the neighbourhood of instability (Griggs D.T. and Baker D.W. 1969).
Writing this relationship in a form which includes

"viscosity" one has

$$\sigma = \mu \dot{\varepsilon}_o^{1-\frac{1}{n}} \dot{\varepsilon}^{\frac{1}{n}} \qquad \mu = \frac{\sigma_o}{\dot{\varepsilon}_o} \exp\left(\frac{E}{nRT} - \frac{E}{nRT_B}\right).$$

Where n is an experimentally determined constant. Observe that for small temperature changes ($T \simeq T_B$) we can write

$$\exp\left(-\frac{E}{nR}\left(\frac{1}{T_B} - \frac{1}{T}\right)\right) \simeq \exp\left(\frac{E}{nRT_B^2}(T_B - T)\right) \qquad (3.8)$$

so that the constant "a" used previously becomes $\frac{E}{nRT_B^2}$

and μ_o becomes $\sigma_o / \dot{\varepsilon}_o$.

The approximation given by equation (3.8) relies on the condition that at least in the seismic low velocity zone the actual temperature is close to the melting temperature. This is the essence of a stability criterion developed by Robertson (1948). If σ_o is taken as σ the constant stress in the slab, the steady state heat flow becomes

$$\frac{Ek}{R}\frac{d^2\phi}{dx^2} = -\frac{1}{\mu_o}\exp\left(\frac{1}{\phi_B} - \frac{1}{\phi}\right)\sigma^2$$

where $\phi = (R/E)T$. Observe that the source term depends on temperature, on the constant applied stress σ and μ_o which is also constant. This equation is the analogue of equation (3.4). Introducing the dimensionless length $\xi = x/\ell$

$$\frac{d^2\phi}{d\xi^2} = -\frac{R\ell^2}{Ek\mu_o}\exp\left(\frac{1}{\phi_B} - \frac{1}{\phi}\right)\sigma^2$$

which becomes

$$\frac{d^2\phi}{d\xi^2} + \lambda'\exp(-1/\phi) = 0 \qquad (3.9)$$

where

$$\lambda' = \frac{R\ell^2}{Ek\mu_o}\sigma^2 \exp(1/\phi_B) = \lambda\left(\phi_B^2 \exp(1/\phi_B)\right) \qquad (3.10)$$

Here λ was defined in (3.4) and a was derived above for the assumption $T \simeq T_B$. Multiplying both sides of (3.9) by $d\phi/d\xi$ and integrating one obtains

$$\frac{d\phi}{d\xi} = \pm\sqrt{2\lambda'}\left\{\left(-\phi e^{-1/\phi} - Ei\left(-\frac{1}{\phi}\right)\right)\Bigg|_{\phi_c}^{\phi}\right\}^{\frac{1}{2}}$$

where $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t_t}\,dt$, ϕ_c is the non-dimensional temperature at the centre of the slab.

Define function A by

$$A(\phi) = \left\{-\phi e^{-1/\phi} - Ei(-1/\phi)\right\}$$

Then $$\int_{\phi_c}^{\phi_B} \frac{d\phi}{[A(\phi)-A(\phi_c)]^{\frac{1}{2}}} = \pm\sqrt{2\lambda'} \qquad (3.11)$$

where ϕ_B the value of ϕ at $\xi=1$,

determines ϕ_c and

$$\int_{\phi_c}^{\phi} \frac{d\phi}{[A(\phi)-A(\phi_c)]^{\frac{1}{2}}} = \pm\xi\sqrt{2\lambda'} \text{ determines } \phi(\xi) \qquad (3.11a)$$

A plot of $\phi(\lambda, x)$ appears in Figure (3.4). The velocity is given by

$$v = \frac{kR}{E\sigma\ell}\frac{\partial\phi}{\partial\xi} = \frac{\ell\sigma}{\mu_o}\left(2/\lambda\left(A(\phi(\xi)) - A(\phi_C)\right)\right)^{\frac{1}{2}}$$

where the sign on the square root is chosen to correspond to positive or negative ξ. Compare this with equation (3.5).
If $\phi = \phi_C - \varepsilon$ where $\varepsilon/\phi_C \ll 1$ then
$(A(\phi) - A(\phi_C))^{1/2} = (\varepsilon \exp(-1/\phi))^{1/2}$
and thus

$$v = \frac{\ell\sigma}{\mu_o}\sqrt{\frac{2}{\lambda}}\,\phi_B\left(\phi_C - \phi_B\right)^{\frac{1}{2}}\exp\left(\frac{1}{2}\left[\frac{1}{\phi_B} - \frac{1}{\phi_C}\right]\right)$$

We may then use the following approximation in equation (3.5)

$$\tanh\left\{\sqrt{\frac{\lambda}{2}}\exp\left[\left(\phi_C - \phi_B\right)/2\phi_B^{\,2}\right]\xi\right\} \simeq \sqrt{\frac{\lambda}{2}}\exp\left[\frac{\phi_C - \phi_B}{2\phi_B^{\,2}}\right]$$

$$\left[\exp\left(\frac{\phi_C - \phi_B}{\phi_B^{\,2}}\right)\right]^{\frac{1}{2}} \simeq 1$$

to obtain

$$v \simeq \frac{\ell\sigma}{\mu_o}\,\xi\exp\left[\frac{1}{2}\left(\frac{1}{\phi_B} - \frac{1}{\phi_C}\right)\right]$$

The introduction of a melting temperature is now natural. If we argue that $\phi_C < \phi_M$ where ϕ_M is the melting temperature it is easy to use (3.11) to evaluate λ at the condition $\phi_C = \phi_M$. Further it is worth noting that stable solutions exist for all values of λ. Thermal runaways do not occur in the same way as in the previous case. Mechanical instability in the material described by (3.7) comes about only by melting.

Now consider the problem of determining a stability criterion for a slab heated from within by viscous shear heating with the stress strain rate relation (3.7). If we assume zero thermal conductivity and uniform shear throughout the slab then the centre will be heated to the melting temperature in a time

$$t_M = \frac{E\rho c\mu_o}{R\sigma^2}\exp(-1/\phi_B)[\phi_M\exp(1/\phi_M)-\phi_B\exp(1/\phi_B)$$

$$+ Ei(1/\phi_B) - Ei(1/\phi_M)]$$

$$= \frac{E\rho c\mu_o}{R\sigma^2}\exp(-1/\phi_B)f(\phi_M,\phi_B)$$

With no heat sources the slab has a characteristic cooling time

$$t_c = \rho c\frac{\ell^2}{k}$$

By analogy with Gruntfest (1963), let

$$G = \frac{t_c}{t_M} = \lambda'/f(\phi_M,\phi_B)$$

where λ' is defined by equation (3.11) or by equation (3.11a).

If t_M is sufficiently large compared with t_c it is clear that stability exists. The instability occurs when the central temperature in the slab reaches the melting temperature. The corresponding value of λ is the critical value and can be calculated from equation (3.11a). It is easy to show that

$$Q = \frac{1}{2f(\phi_M,\phi_B)} \left\{ \int_{\phi_M}^{\phi_B} \frac{d\phi}{(A(\phi)-A(\phi_M))^{\frac{1}{2}}} \right\}^2$$

is the limiting value of G for stability. When $\phi_B = \phi_M - \varepsilon$

where $\frac{\varepsilon}{\phi_M} << 1$.

$$Q \simeq 2.$$

Another stress strain-rate relation of considerable interest for steady state creep in rocks is (Weertman and Weertman, 1975)

$$\dot{\varepsilon} = C^n \sigma^n \exp(-E/RT)/T$$

Here the value of A from eqn (3.6) has a 1/T dependence. The values of E, C and n are subject to considerable debate, but the analytic properties of flows which couple to thermal behaviour are interesting.

Solving for the heat source term we get for constant stress

$$\sigma\dot{\varepsilon} = C^n \sigma^{n+1} \frac{\exp(-E/RT)}{T}$$

Setting $\phi = \frac{RT}{E_*}$ we get

$$\frac{\partial^2\phi}{\partial\xi^2} + \lambda \frac{\exp(-1/\phi)}{\phi} = 0$$

where

$$\lambda'' = C^n \sigma^{n+1} R^2 \ell^2 / kE^2$$

$$= C'\lambda \text{ where } C' = C^n \mu_0 \sigma_0^{\,n-1} \phi_B^2 \frac{R}{E}$$

from which $\frac{d\phi}{d\xi} = \pm\sqrt{2\lambda''}\ \{Ei(-1/\phi)-Ei(-1/\phi_c)\}^{\frac{1}{2}}$

and $$\int_{\phi_c}^{\phi_B} \frac{d\phi}{(Ei(-1/\phi)-Ei(-1/\phi_c))^{\frac{1}{2}}} = \pm\sqrt{2\lambda''}$$

can be solved numerically for ϕ_c. A stability criterion can be derived exactly as was done for the previous relation.

The preceding arguments involving non-linear rheology are hard to interpret in an immediately useful geophysical way. The success of a simple Newtonian stress strain rate relation in geodynamics suggests we should at least look at the way thermomechanical coupling can modify apparent viscosity. This theoretical development will now be completed with some connections to Newtonian processes in which thermo-mechanical coupling is ignored.

Traditionally experiments on the shear of layers are interpreted in terms of such Newtonian viscosity laws. If shear-heating modifies the viscosity, the observed experimental results are obviously not a good measure of actual viscosity. Calculation of "apparent viscosity" is fairly easy with the results derived above. Here apparent viscosity is the ratio of applied stress to apparent velocity gradient. We can estimate this quantity using equation (3.7) which yields

$$\frac{1}{\bar{\mu}} = \frac{v(1)}{\ell\sigma} = \frac{\exp(-1/n\phi_M)}{\mu_o} \int_o^1 \exp(1/n\phi)d\xi \qquad (3.12)$$

It is generally accepted (e.g. Ringwood A.E. 1975) that the temperature of the seismic low velocity zone is near its solidus. Although there is no well defined melting point for such material, it is still useful to introduce this approximation since it behaves in the presence of partial melt as a material very near instability.

If $\phi = \phi_M - \varepsilon$

then $$\frac{1}{\bar{\mu}} \simeq \frac{1}{\mu_o} \int_o^1 \exp(\varepsilon/n\phi_M^2)d\xi$$

and from (3.11a)

$$\varepsilon \simeq \frac{\xi^2 \lambda'}{2\exp(1/\phi_M)}$$

Let $M = \dfrac{\lambda'}{2n\phi_M{}^2 \exp(1/\phi_M)} \simeq \dfrac{\lambda}{2n}$

Then we have

$$\frac{1}{\bar{\mu}} = \frac{1}{\mu_o} \int_0^1 \exp(M\xi^2)\, d\xi = \frac{1}{\mu_o M^{\frac{1}{2}}} \int_0^{M^{\frac{1}{2}}} \exp(\psi^2)\, d\psi \qquad (3.13)$$

which can be evaluated as a Dawson integral (Abramowitz and Stegun, 1965). Observe that $\bar{\mu}$ depends in a non linear fashion on the stress. In fact, by expanding $\exp(\psi^2)$ and truncating we get

$$\bar{\mu} \simeq \frac{1}{1 + \frac{M}{3} + \frac{M^2}{10}} \simeq \mu_o\left(1 - \frac{M}{3} + \frac{M^2}{90}\right)$$

The stress strain law used to derive (3.13) is more realistic than the assumption that viscosity decays exponentially with temperature. Nevertheless another apparent viscosity can be derived from the high temperature limit of equation (3.7).

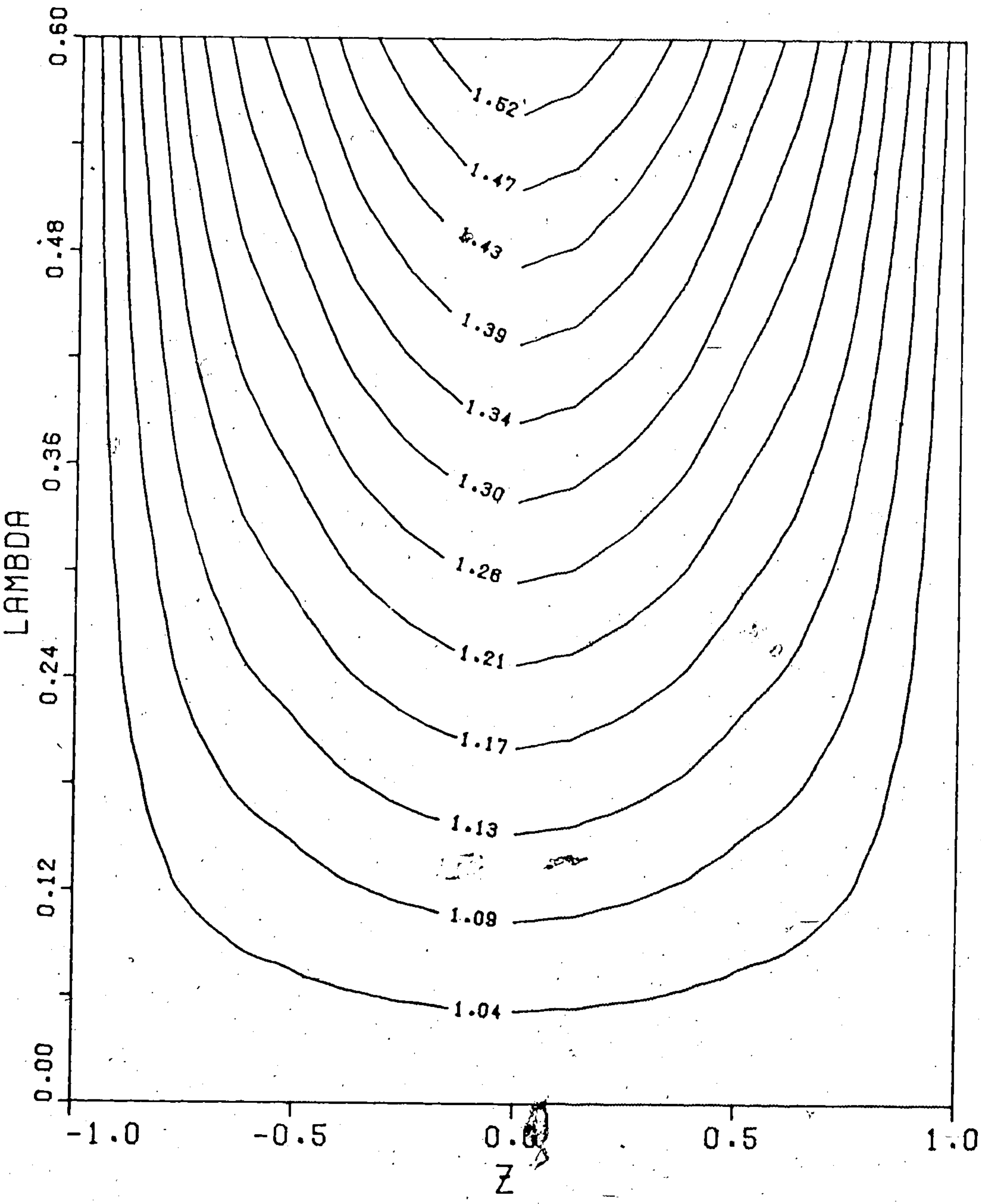
$$\sigma = \frac{\mu_o v}{\ell} \sqrt{\frac{\lambda}{2}}\, \exp(-\phi_c/2)$$

from which it is easy to see

$$\bar{\mu} = \mu_o \sqrt{\frac{\lambda}{2}} \exp(-\phi_c/2)$$

In this case $\bar{\mu}$ varies as σ^2 for high values of ϕ_c.

Figure 3.4: The temperature distribution across a class of stationary slabs with uniform composition is given. Here the stress strain-rate relations is given by equation (3.7) and the temperature is given in term of the dimensionless temperature Φ .


1.52
1.47
1.43
1.39
1.34
1.30
1.26
1.21
1.17
1.13
1.09
1.04
0.60
0.48
0.36
0.24
0.12
0.00
LAMBDA
-1.0
-0.5
0.0
0.5
1.0
Z

3.4 Application to the Seismic Low Velocity Zone

The velocity of seismic waves in general increases with pressure and decreases with increasing temperature. One or the other of these two effects can dominate for certain temperature and pressure gradients depending on the material. In fact many early explanations of the existence of the seismic low velocity zone were soley dependent on this phenomena (Birch F. 1952, Valle P.E. 1956, MacDonald G.J.F. and Ness N.F. 1961). The critical temperature gradients, particularly for S waves are probably exceeded within the outer 150 km so there is little doubt that this explanation is at least partially correct (Ringwood A.E. 1975, Lieberman R.C. and Schrieber E. 1969). It is however pointed out by Ringwood (1975) that chemical and mineralogical heterogeneity must also play a part since the largest vertical temperature gradients in the earth are immediately below the Mohorovic discontinuity where the seismic velocity still increases with depth. I will not consider these factors here as they require a rather detailed petrological study which is given by Ringwood (1975).

The general physical situation which results in plate formation is extremely complex and open to much dispute. The theories in this area vary from the crust and lithosphere being formed by chemical separation (Ringwood A.E. 1975) to a fairly uniform composition throughout with the vertical inhomogenieties being caused by phase changes resulting from the increasing pressure with depth (Kennedy G.C. 1959). Here I simply mention the situation as I envisage it from a fluid dynamics point of view. In the formation of the plates there are a number of factors which must be considered such as the melting point, conductivity and density of the different materials involved. In general the least dense material, the material of lowest melting point and the material of lowest conductivity all tend to rise to the top in lithosphere differentiation. The crust is then formed by material with some or all of these properties depending on the degree to which any of these properties can dominate. The rest of the lithosphere is then formed mainly by material with the opposite properties which has been depleted of the crustal material. Once the plates have been formed they separate the high temperatures of the earth's interior from the much cooler temperatures at the surface. This layer of thermal insulation is very efficient because its high viscosity allows heat to escape only by conduction over the majority of the earth's surface.

Below the plates convection of the earth's mantle then causes smaller temperature gradients. This description however is complicated by the fact that the thickness of the plates varies from less than 50 km near the ridges to 250 km within some of the continents (Isacks B. et al 1968). This causes the temperatures in the continental lithosphere which are at comparable depths to the seismic low velocity zone beneath the oceans to be approximately 200° cooler. As a result the marginal stability beneath the oceans would not be experienced in the mantle material beneath the continents in an unstressed state.

The pressure in the seismic low velocity zone beneath the continental lithosphere is in fact very similar to that of the mantle material beneath the seismic low velocity zone under the oceans. One thus does not get the same condition of an ambient temperature close to the melting temperature because of the effect of pressure on increasing the melting temperature. A region of marginal stability therefore does not exist to the same degree as beneath the oceans allowing a sharp region of decoupling of motion of the plates and the convection cells. As a result the seismic low velocity zone beneath the continents is far more complex than beneath the oceans.

I do not propose to study this region beneath the continents in depth, but wish to simply consider some of its properties. From the present knowledge of plate motions relative to the mantle convection cells it is generally acknowledged that the plates which contain continents move slower than those without. This may be concluded by looking at relative plate motions and considering the hot spots which are assosiated whith uplift (and thus can be considered as originating from depth) as being essentialy fixed (Burke K. et. al. 1973). This observation then seems to imply that the continents impart greater amount of shear stress to the asthenosphere and thus impede plate motions

(Chapple W. M. and Tullis T. E. 1977).

But partial melting is assumed to be the dominant cause of the low viscosities in the seismic low velocity zone beneath the oceans (this has also been speculated by many others e.g. Cathles l.M. 1975). Thus in order to conclude that the shear stress in the seismic low velocity zone beneath the continents is greater one must also conclude that the degree of partial melting is also less. This follows directly from the fact that one must have larger viscosities. This however is about as far as I can argue for the continents in general. Due to the tremendous differences of the tectonic environments of the different continents one finds that for any general theory developed for the existence of the seismic low velocity zone, almost every individual continent is an exception. This is a result of such phenomena as glacial rebound, continental collisions, the subduction of oceanic plates beneath the continents, large variations in the depth of different continents etc. all of which have substantial changes on the physical conditions in the seismic low velocity zone.

Consider now the lithosphere - asthenosphere boundary beneath the ocean. Here one has a slight decrease in density from the lithosphere above (of about 3.35 gm/cm³ to 3.3 gm/cm³) and much larger temperature gradients than beneath the continents. This causes the material to be subjected to relatively high temperatures which are close to the melting point of the material at the pressures at those depths. This material may then be considered as marginally stable meaning that it is either close to partial melting or slightly beyond the solidus of some of the components.

The results of this section will indicate that the thermo-mechanical coupling of the plate motions above to the more viscous mantle below cannot be solely responsible for the observed low viscosities in the seismic low velocity zone. However when an increasing viscosity with depth due to changing physical properties is taken into consideration the shear heating can then result in the observed changes in size and intensity of the seismic low velocity zone. This can occur due to changes in the relative motion of the plates and mantle convection cells. Also it car seen that shear heating must at least be responsible f ncrease in the amount of melt in existence in the seismic low velocity zone. This melting may also be aided by the presence of up to .1% water (Ringwood A. E. 1975). The increasing viscosity with depth is then a result of the pressure gradients becoming greater than the temperature gradients and a changing composition due to a rise of the melting material. I will show at the end of this section that one may construct stationary solutions for the material becoming more viscous with depth due to compositional and pressure changes. The molten material generated in the seismic low velocity zone then rises and cools as part of the lithosphere. This fits theories of lithosphere thickening (e.g. Parker R. L. and Oldenburg D. W. 1973).

The seismic low velocity zone as a region of steady state thermo-mechanical coupling of a marginally stable material would at first sight seem to imply that the low velocity zone may be driven to instability through the applitation of a small amount of shear stress by the moving plates. This however is certainly not the case. Consider the Bridgman (1935, 1936, 1937) experiments once again. They illustrate that materials with low melting points deform smoothly without experiencing a degree of thermo-mechanical coupling which leads to dramatic stress relief. A thermal instability is thus just localized runaway heating which occurs in an ever narrowing region due to the shear heating increasing faster than the ability of the material to conduct heat. Upon examination however one finds that the low velocity zone must be even more stable than a material h low melting point. This is because it is not composed of one material close to melting but numerous materials (olivine, pyroxine, garnet, and perhaps in some regions amphibole (Ringwood A.E. 1975)) which melt over a range of temperature. Thus as the solidus for some of the materials is passed the viscosity drops resulting in a smaller amount of shear stress being imparted by the motion of the plates.

This is simply a consequence of eqn (3.3) or (3.9) The majority of the material then remains unmelted with the melt being considered at present as uniformly distributed across the low velocity zone. The stress being imparted thus is able to be transported at large velocities across th slab as a whole making the concept of a localized instability even more unrealistic.

One possible generalization of the work in this chapter now becomes obvious. That would be to introduce the concept of a concentration field of melt which would allow the material to reach the solidus over a range of temperatures. The equations for this case are shown in appendix B. The problem in this case is that one must consider the behaviour of three coupled equations which makes finding solutions very difficult.

In order to evaluate the results of this chapter the following values which appear to be representative of seismic low velocity zone conditions are used. (Temperatures are in °C, other units are cgs except where noted.)

$c = 1.3 \times 10^7$	$\mu_o = 10^{22}$
$T_B = 1100$	$E = 60$kcal/mole
$T_M = 1400$	$E/R = 3 \times 10^4$
$\rho = 3.5$	$R = 2 \times 10$ kcal/°K/mole
$k = 3 \times 10^5$	$n = 3$

I now calculate the shear stress necessary to maintain this shear coupling zone in a marginally stable state over a distance of 100 km. This zone is assumed to be caused by the shear stress applied by the moving plate at the top and the more viscous material below.

Using the stress strain rate relation (2.2) the stress in this simple stationary thermo-mechanical system is given by

$$\sigma = \sqrt{\frac{.88\, k\, u_o}{a \ell^2}} \qquad (3.13)$$

If the temperature is a monotonically increasing function with depth t ℓ is the thickness of the low velocity zone and is its viscosity at the plate boundary. Eqn. (3.13) then yields an applied stress of approximately 50 bars which in turn yields a strain rate at any position across the slab of

$$\dot{\varepsilon}(x) = 5 \times 10^{-15} \exp(\phi(x))/\text{sec} \qquad (3.14)$$

Here $\phi(x)$ goes from 0 to 1.2 which corresponds to a temperature range of 1100°C to 1220°C.

If one now assumes a plate motion of 5 cm/year, then a velocity of approximately 10^{-7} cm/sec is obtained for the top of the low velocity zone. Also if a much smaller velocity is assumed for the more viscous material below and a constant velocity gradient across the low velocity zone then a strain rate of approximately 10^{-14} /sec results. This result, is almost identical to the strain-rate obtained in eqn(3.14) and depending only on the physical constants and the assumption of marginal stability for the equations in section (3.2).

Now in order to use the stress strain rate relation (3.7) one must first know ϕ_B and ϕ_C so that the stress may be calculated from eqn (3.10). Here a temperature range of 1100°C to 1400°C is taken which is probably more reasonable than the range obtained in the previous example. In this case the stress once again works out to be approximately 50 bars and the strain rate is approximately

$5 \times 10^{-15} \times \exp(9.1 - 1/\phi)$ /sec

Here $1/\phi$ goes from 7.1 to 9.1 and thus the strain-rate once again fits very well with observed plate motions.

An increasing pressure and the rise of molten material both essentially cause the value of μ_o to be an increasing function of depth. Thus I wish to consider the behavior of such a situation in terms of a value $\mu_o(x)$, for which the previous equation may still be solved analytically.

Assume

$$\mu = \mu_o \exp(-\phi) \exp[(1-\xi)\chi]$$

where $\xi = 0$ is the bottom of the low velocity zone, $\mu = 10^{21}$ and χ is a constant such that $\mu_o \times \exp(\chi) = 10^{22}$. We may then write eqn (3.4) in the form

$$\frac{\partial^2 \phi}{\partial \xi^2} = -\lambda \exp(\phi - (1-\xi)\chi)$$

Now let $y = \phi - (1-\xi)\chi$ and we then have

$$\frac{\partial^2 y}{\partial \xi^2} = -\lambda e^y$$

which has the solution

$$\phi = -\chi\xi - 2\ \ln \sqrt{\frac{2\lambda}{\Sigma}}\ \cosh\left\{\cosh^{-1}\left(\frac{\Sigma}{2\lambda}\exp\left(\frac{\chi}{2}\right)\right) - \frac{\Sigma^{\frac{1}{2}}}{2}\,\xi\right\}$$

Here

$$\Sigma = 2\lambda e^{-\chi} + \left(\chi + \frac{\partial\phi}{\partial\xi}\bigg|_b\right)^2$$

and $\partial\phi/\partial\xi\,|_b$ is the temperature gradient evaluated at the boundary of the slab. Thus

$$\frac{\partial\phi}{\partial\xi}\bigg|_b = -\chi + \Sigma^{\frac{1}{2}}\tanh\left\{\cosh^{-1}\left(\sqrt{\frac{\Sigma}{2\lambda}}\exp\left(\frac{\chi}{2}\right)\right)\right\}$$

From the value of ϕ at $\xi = 1$ we can solve for $\partial\phi/\partial\xi\,|_b$. We may then determine the position of the maximum temperature by setting $\partial\phi/\partial\xi = 0$ and solving for ξ. The effect of this assumption of a composition gradient is that it causes the maximum temperature to be shifted towards the boundary of lower viscosity (see figure (2.5)). With the introduction of heating from below (the boundary of higher viscosity) one may get rid of this maximum temperature as illustrated in figure (2.7).

Chapter 4

The Inertial Field Equations and Their Application to Deep Earthquakes

4.1 A Physical Discussion

Studies of intermediate and deep earthquake mechanisms indicate that the principal stresses are developed within the down dipping plate and in the direction of plate motion. (Sykes L.R. 1966, Isacks B.L. and Molnar P. 1969, 1971, Isacks B.L. et al 1969). However the proposed physical [illegible]ions of deep earthquake mechanisms all have [illegible] Consider first a proposal of Isacks and Molnar [illegible]

If one assumes that the slab cannot penetrate the 650 km discontinuity then the upward transmission of the compressional stress could cause deep earthquakes. This mechanism is rejected in great detail by Ringwood (1975). Also it has been shown by Griggs (1972) that such resistance would result in buckling of the subducting plate, for which there is little evidence, and thus could not transmit the necessary stress for hundreds of kilometers upwards in any case.

Another proposal has been made by Ringwood (1975) and others (Bridgman P.W. 1945, Ringwood A.E. 1956, 1967, Evison F. 1963, 1967, Dennis I.C. and Walker C. 1965). Here intermediate and deep focus earthquakes are assumed to be caused by phase transformations associated with sudden volume changes in the subducting slab. Two problems with this mechanism have been suggested by Ringwood (1975). It is not understood how phase transitions can occur quickly enough to cause an earthquake and the analyses of first motions from earthquake seismograms show douple, couple mechanisms exactly as in shear failure rather than monopolar implosion as has been believed to be required in phase change mechanisms. Ringwood (1975) then simply passes the first problem off as due to the large temperature gradients and the second problem as the relief of pre-existing stress due to gravitational body forces. An analysis of the seismic moments of intermediate and deep-forcus earthquakes has been done by McGarr (1977). He claims to get reasonable consistency under the assumption of phase change induced earthquakes in four island arc regions.

I am however not aware of any research which has been done on the dynamics of this mechanism. Part of the reason for this apparent absence of work in this area is certainly due to the fact that the theoretical base consists in applying a static theory to a dynamic system. I would like to suggest that with our present meagre knowledge of the fashion in which the phase changes occur it is just as reasonable to conclude that they inhibit deep earthquakes as cause them. I make this suggestion since a viscosity increase is probably associated with the majority of the phase changes in the subducting slab because of the density increases which occur and I am aware of no evidence that the phase changes can occur as quickly as necessary to induce an earthquake under the conditions present in the subducting slab. Also, almost every subducting plate on earth exhibits an absence of earthquakes in the region where the Olivine-Spinel phase change is thought to occur in the slab.

It has also been suggested that deep earthquakes may be evidence of a fluid instability within the subducting slab, caused by shear stress imparted by the surrounding mantle (Griggs D.T. and Baker D.W. 1969, Shaw H.R. 1969). Although this theory would seem to fit best with observations of focal mechanism of deep earthquakes it is shown in sec. (4.3) that the subducting slab is orders of magnitude away from instability.

I wish to look further into the possibility of fluid instability as a mechanism for deep earthquakes in this chapter. First one should consider carefully where the instability would occur. I show in section (4.3) that the only region which may be realistically driven to instability is the boundary layer between the mantle and upper edge the subducting slab. The interior of the slab is far too stable to exhibit instability while the surrounding mantle is able to deform smoothly. The boundary layer on the other hand has the greatest amount of heat production (Turcotte D. L. and Schubert G. 1973 also see sec. 4.4) and is initially far from melting. These are the precise conditions necessary for the generation of a thermal instability according to the experimental observations made by Bridgman.

4.2 An Analysis of the Equations of Motion

I consider here a discription of the inertial effects of a highly viscous fluid when subjected to a time dependent boundary stress. I start by discussing a fluid contained within an infinite horizontal slab of depth 2ℓ. The analysis I use for the propagation of the velocity field in a highly viscous fluid is similar to the theory for the propagation of a plastic wave in a visco-elastic solid (Taylor G. I. 1946, von Kármán T. and Duwez P. 1950, Rakhmatulin K. A. 1945). In the case of a visco-elastic solid one is dealing with a solid with some fluid properties, i.e. the "fluid like" propagation of a displacement. The difference here is that I am dealing with a fluid with some solid properties, i.e. the "solid like" propagation of a velocity. Both analyses are identical in that they represent the propagation of a stress wave across the material. The difference is simply the method by which stress is propagated in the two cases.

If a uniform shear stress is applied at the boundaries, that stress will be assumed to propagate across the slab at the velocity $\beta(x,t)$. Here x is the vertical co-ordinate and t is the time. $\beta(x,t)$ therefore represents the ratio $\Delta x/\Delta t$ of the distance a velocity perturbation will propagate, orthogonal to the direction of the perturbation, to the time Δt of propagation (i.e. in the case of a fluid of constant viscosity throughout and initially constant velocity throughout $v(x+\Delta x, t+\Delta t) = v(x,t)$). As each part of the slab then becomes subjected to a horizontal force we obtain a corresponding motion. The slab thus undergoes plastic deformation and we obtain a continuous distribution of heat source . This continuum of heat-sources is the shear heating due to the viscous dissipation of the forces responsible for the dynamics. The heating of the fluid then causes a decrease in the viscosity and thus the ability of the fluid to dissipate the shear stress imparted to it (i.e. the plastic shear wave velocity decreases). This causes a change in the dynamics which according to the previous argument once again changes the thermal field. One thus obtains a coupled thermo-mechanical system in which both fields must be solved simultaneously except for the special case of stationary solutions, i.e. solutions which do not contain any inertial motions, which were discussed in Chapter 3.

Now consider the evolution of an instability in such a material. In Appendix C it is shown that one cannot obtain a stable stationary solution for a homogenous half space subjected to constant shear no matter how slowly the shear stress is increased to the desired amount. One should however remember that this is strictly a mathematical result. When using the idealized case of a homogeneous half space, the above result can only be meaningful where the distance from the boundary to the position of the instability, is a distance over which the basic physical assumption can be considered reasonable and the time for an instability to evolve is less than the time for the phenomena being modeled to occur.

It should be noted that all models of deep earthquake mechanisms are very speculative due to the great uncertainties in knowledge of the geologic and physical conditions. Here I attempt to find a simple model which fits with the present experimental evidence and requires a minimum number of hypotheses. I wish to consider the equations of fluid flow in more detail in the remainder of this section and in section (4.3) before presenting a new model for deep earthquakes in section (4.4) based on these equations.

The equation of motion of a horizontally sheared viscous fluid in the absence of convection is given by

$$\frac{\partial(\rho v)}{\partial t} = -\frac{\partial \pi}{\partial x} \qquad (4.1)$$

Here the velocity, v, is in the y direction horizontal to the surfaces, x is perpendicular to the surface, ρ is the density and Π is the momentum flux density. In the case of pure shear (i.e. when the momentum flux due to convection and pressure gradients is zero) one may write

$$\pi = -\sigma$$

where σ is the shear stress and $-\sigma$ represents the momentum flux density due to viscous dissipation for pure shear.

In the absence of phase diffusion the equation of heat transfer is given by

$$\rho c \frac{\partial T}{\partial t} = \sigma \frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left[K \frac{\partial T}{\partial x}\right] \tag{4.2}$$

The right hand side of this equation may also be written in the form

$$\frac{\partial}{\partial x}\left\{K \frac{\partial T}{\partial x} + v\sigma\right\} - v\frac{\partial \sigma}{\partial x}$$

Applying Gauss's theorem to a unit volume of the fluid we find. $k\ \partial T/\partial x$ represents the energy flux density through the surface due to thermal conduction; $v\sigma$ represents the energy flux density through the surface due to non-inertial viscous effects; and $v\partial\sigma/\partial x$ represents the net energy flux density in the unit volume due to the inertial coupling of the thermal and mechanical field equations. The last term may be divided as follows

$$v\frac{\partial \sigma}{\partial x} = \rho v \frac{\partial v}{\partial t}$$

$$= \rho\beta(x,t)\, v\frac{\partial v}{\partial x}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\rho\beta(x,t)v^2\right) - \rho\frac{\partial \beta(x,t)}{\partial x}v^2\right] \tag{4.2a}$$

where the time of propagation for the dynamics resulting from an applied force is

$$\Delta t = \int_{x-\varepsilon}^{x} \frac{1}{\beta(x,t)}\,dx \qquad (4.3)$$

and $\beta(x,t)$ is the velocity of propogation of the dynamics in the direction orthogonal to the applied force. Here $\rho\beta(x,t)v^2$ represents the energy flux density within a unit volume of the fluid due to the inertial interactions of the thermal and mechanical field equations in that volume.

Assuming $\partial T/\partial t=0$ the net energy flux in a unit volume of the fluid becomes zero. This may be written as

$$\beta(x,t)\,\frac{\partial T}{\partial x} = 0 \qquad (4.4)$$

since the inertial heating is strictly a result of a heat source due to the dynamics and thus either $\beta(x,t)=0$ or $\partial T/\partial x=0$. If $\beta(x,t)=0$ then one can see from (4.2a) immediately that $\partial v/\partial t=0$. If $\partial T/\partial x=0$ then the net heat conduction must be zero and the heat production must be the same everywhere. Now assuming a Newtonian stress strain-rate relation we have $\mu(\partial v/\partial x)^2$=constant and since μ is a function of T only μ=constant. Thus $\partial v/\partial x$= constant and since the heat production may also be written as $\sigma\partial v/\partial x$=constant, we have σ=constant. Thus

$$\frac{\partial v}{\partial t} = \frac{1}{\rho}\,\frac{\partial \sigma}{\partial x} = 0$$

Now assume $\partial v/\partial t=0$. Differentiating $\sigma = \mu\, \partial v/\partial x$ with respect to t we obtain

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \mu}{\partial t} \frac{\partial v}{\partial x}$$

which may be written in the form

$$\rho\beta(x,t) \frac{\partial v}{\partial t} = \frac{\partial \mu}{\partial \phi} \frac{\partial \phi}{\partial t} \frac{\partial v}{\partial x}$$

Thus we have

$$0 = \frac{\partial \mu}{\partial \phi} \frac{\partial \phi}{\partial t} \frac{\partial v}{\partial x}$$

but $\partial\mu/\partial\phi \neq 0$ since viscosity is a strong function of temperature in a highly viscous fluid and $\partial v/\partial x \neq 0$ since we have a viscous fluid. Thus any applied stress must result in velocity gradients across the slab. Therefore for a highly viscous fluid undergoing shear stress one obtains the condition

$$\frac{\partial \phi}{\partial t} = 0 \iff \frac{\partial v}{\partial t} = 0$$

We thus have the result that heating can occur only if there are accelerations and accelerations can occur only if there is heating.

4.3 The Transition to the Inertial Field Equations.

Consider now the solutions which evolve for an infinite horizontal slab of half-thickness ℓ when a shear stress $\sigma_b(t)$ is applied at each boundary. The forces responsible for the stresses are assumed to be in opposite directions.

The rate of heat production due to viscous shear heating at any point across the slab is once again given by $\sigma\dot{\varepsilon}$ where σ and $\dot{\varepsilon}$ are the stress and strain rate at that point respectively.

A stress strain-rate relation which holds for most geologic material in the neighbourhood of instability was given by equation (3.7). However I first wish to consider the simpler relationship obtained for $T \simeq T_B$ in eqn (2.2) and the newtonian approximation eqn (2.2a).

The heat balance equation is then given by

$$\frac{\partial\phi}{\partial t'} - \frac{\partial^2\phi}{\partial\xi^2} = \frac{\ell^2 a}{\mu_o K}\,\sigma^2(\xi,t')\exp(\phi)$$

where

$$\phi = a\left(T-T_o\right) , \qquad \xi = x/\ell \quad \text{and} \quad t' = t\frac{K}{\rho c\ell^2}$$

Momentum balance is given by the equation

$$\frac{\partial v}{\partial t'} = \frac{\rho c^2}{K\ell^\sigma}\,\frac{\partial\sigma(\xi,t')}{\partial\xi}$$

Here v is the dimensionless velocity.

In the absence of inertial term it was shown in chapter 3 that analytic solutions may be obtained for $\lambda < .88$. Now in order to illustrate the importance of the inertial terms, the evolution of the system from its initial condition to melting is followed by different processes. First consider a quasi-stationary process. This implies that we study the evolution of the system for an infinitely slow rate of increase of stress at the boundary. The solutions at any time are then given by the stationary theory. Assume the equations yield solutions for values of $\phi_C > \phi_M$ where ϕ_C is the central temperature and ϕ_M is the melting temperature. The maximum temperature occurs at the centre of the slab and it is there that the condition $\phi = \phi_M$ is first encountered and the slab begins to melt. This onset of melting and its attendant reduction in viscosity creates an instability for $\lambda < .88$.

Now consider the same evolution as before with $\phi_C < \phi_M$ for all possible stationary solutions. The position of the instability may be found by considering the slab as three separate slabs
(Slab 1: $-1 \leqslant \xi \leqslant -\varepsilon$, Slab 2: $-\varepsilon \leqslant \xi \leqslant \varepsilon$, Slab 3: $\varepsilon \leqslant \xi \leqslant 1$)

Here ε is an arbitrarily small number. If $\lambda = .88$ for the slab as a whole, slabs 1 and 3 once again yield stationary solutions (see appendix C) capable of transporting a greater amount of stress. Slab 2 has a central maximum temperature of $\phi = 1.2$ which must correspond to $\lambda = .88$ in this slab (see sec. 3.3). Thus any stress increase in this slab must result in an instability. By this analysis one has now obtained the result that any slab which is composed of one material and whose motion and heating is determined by equation (2.2) can reach instability only at the center. The generalization of this argument to the more com stress . strain - rate relation (3.7) is trivial sin stationary solutions exist for any value of the applied stress (sec 3.4) and instability simply occurs when the central maximum temperature reaches the melting temperature.

Note that here one is assuming that one has a perfectly homogeneous fluid composed of a single material and a very well defined melting point. Here one is also considering a constant stress boundary condition and not a constant velocity boundary condition in which case the stress can usually evolve to a sufficiently low value that instability will not occur. This analysis thus does not in any way contradict the work in section (3.4).

Now consider the instantaneous application of stress at the boundary. Here σ_b has a time dependence like $\alpha H(t)$ where

$$H(t) = \begin{cases} 1 & t \geq 1 \\ 0 & t < 0 \end{cases}$$

and α is a constant. Since the stress must propagate into the slab at a finite velocity β we have the condition $\partial\sigma / \partial|x| = \delta(|x| - \ell)$ where $\delta(x)$ is the dirac delta function. This implies that v has a space dependence like $\alpha H(|x| - \ell)$ and $\partial v / \partial x$ has a space dependence like $\delta(|x| - \ell)$ for $-\ell < x < \ell$ at time zero. Since the heat production term is the product of a finite stress and an infinite strain rate, the density of heat sources at the boundary is infinite.

With an infinite density of heat sources the viscosity drops infinitely rapidly. This means the stress cannot propagate into the slab and the velocity gradient must increase without limit. In particular, melting will now occur on the boundary and not at the centre. A true physical system is neither quasi-stationary nor has a zero time interval for the application of stress. With a finite speed of applciation of stress instabilities can be expected at any distance between the boundaries and the center depending on the rate of increase of stress and the magnitude of the stress. This indicates that the characteristic distance to instability is a function of the time rate of change of the boundary force, the material properties and the initial conditions.

I wish to consider now the physical situations which lie between these two very unphysical extremes. Consider an infinite slab of thickness ℓ which is held at a constant temperature at one surface and thermally insulated at the other surface. The thermally insulated surface thus corresponds to the centre of the slabs discussed previously. Assume the slab is subjected to an applied force $\vec{f}(t)$ horizontal to the cooled surface. The traction at the boundary is given by

$$\tilde{\tilde{\sigma}}_b(t) \cdot \tilde{n} = \vec{f}(t)$$

Here the only non zero component of traction yields a horizontal shear stress. This stress propagates to any depth x in the slab in a time

$$t_p(X_o) = \int_{\ell}^{X_o} \frac{1}{\beta(T(x,t))} dx$$

Thus we may write

$$\sigma(x,t) = \sigma_b(t-t_p(x))$$

where $\sigma(x,t)$ represents the shear stress at the position x within the slab at time t. At this point it should be noted that one may obtain further inertial effects, such as those which can result from the partial attenuation of the stress waves, which have not been considered. Neglecting these effects should however give a reasonable upper bound to the depth and time to instability.

The equations of motion may now be written in terms of the boundary stress and shear wave velocity as

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left\{K\frac{\partial T}{\partial x} + v\sigma_b(t-t_p(x)) - \frac{1}{2}\rho\beta(T)v^2\right\} + \frac{1}{2}\rho\frac{\partial\beta(T)}{\partial x}v^2 \qquad (4.5)$$

and

$$\rho\frac{\partial v}{\partial t} = \frac{1}{\beta(T)}\left.\frac{\partial\sigma_b(t')}{\partial t'}\right|_{t'=t-t_p} \qquad (4.6)$$

The velocity of propagation of the dynamics, $\beta(x,t)$

will here be considered to be the shear wave velocity when $\partial\sigma/\partial t$ is large since in this case the material should exhibit only solid properties and the plastic wave velocity when $\partial\sigma/\partial t$ is small since the material should exhibit only fluid properties. It is however such a highly nonlinear function of temperature (especially in the neighbourhood of melting) that it would be foolish to try to solve eq's (4.5 and 4.6) in any generality either analytically or numerically.

One case however which may be calculated, is the time for the stress to propagate to any point in the interior when a stationary solution is subjected to a small perturbation in the applied stress at the boundary. This case represents a reasonable starting point for considering the onset of instability. Since the viscosity decreases exponentially with increasing temperature it is reasonable to assume the velocity of propagation of the dynamics also decreases exponentially with increasing temperature. Viscosity is after all simply a measurement of the ability of a fluid to diffuse momentum. Therefore with the approximation $T \approx T_B$ and the Newtonian stress strain-rate relation one obtains a time

$$t(\xi_o) = \frac{1}{\beta_o}\sqrt{\frac{2}{\lambda}}\,\exp\left(\frac{\phi_c}{2}\right)\tanh\left(\sqrt{\frac{\lambda}{2}}\,\exp\left(\frac{\phi_c}{2}\right)|\xi|\right)\Bigg|_{\xi=\xi_o}^{1}$$

for a small perturbation in the boundary stress to propagate to a distance ξ from the thermally insulated boundary.

Here the shear wave velocity is assumed to be of the form

$$\beta(\phi) = \beta_o \exp(-\phi)$$

where β_o is the velocity of propagation of the dynamics at $\phi=0$. This form for β does not allow the stress to propagate past an instability. It also fits with the natural expectation that β have the same temperature dependence as viscosity. The time of propagation is a highly non linear function of the distance as illustrated in figures (4.1). The propagation time to any point from the boundary here in fact differs from the change in velocity to that point from the boundary by only a constant.

Consider now the case of a plate being subducted into the earth's mantle. The lower part of the plate deforms smoothly and no instabilities occur. Here the lithosphere is a material of lower melting point than the surrounding mantle material (Ringwood A. E. 1973) and has a temperature distribution which is fairly smooth throughout. The initial temperature distribution is simply a result of a solid heated at the lithosphere asthenosphere boundary and cooled at the crust mantle boundary. The temperature and stress at the lower surface of the plate then increase gradually as it subducts. The subducted crustal material however is quite different. It probably has a lower conductivity than the surrounding mantle and is subducted with an extremely low temperature relative to the mantle material. One thus has a large temperature difference at this crust mantle interface.

The time for the slab to reach thermal equilibrium with the mantle when held at a fixed position in the mantle may be estimated from $t = \rho c l^2/k$ to be approximately 10^8 years.

Now consider the mimimum stress required to produce an instability assuming constant stress throughout the slab and an initial temperature equivalent to that of the mantle. Here $\sigma^2 = 10^{34}/t_M$ from eqn (2.3a) where $t_M < 10^{13}$ sec and thus it would take a stress of 10 kilobars approximately 10^6 years to produce an instability within the crust. Here one is also assuming the crust to be thermally insolated from the lithosphere. These figures may not be extremely reliable, but they certainly indicate that it is not reasonable to assume that instability can be produced within the slab by the shearing of the mantle. Here 10 kilobars is far too much shear for the mantle to impart to the slab since the mantle motion induced by the subducting slab must reduce its viscosity substantially. If the stress in the slab is 10 kilobars however then the stress in the mantle must also be 10 kilobars. This fact then yields ~~a~~ very interesting result using the same approximations as previously. One now finds that instability results in the mantle in approximately 10^2 years and within one tenth of a kilometer of the surface of the slab.

The previous analysis certainly indicates that the crust is much more stable than the bounding fluid. It thus seems much more reasonable to consider the crust as undergoing elastic deformation and the fluid instability occurring within some boundary layer in the boundary fluid. This becomes an even more attractive model when one considers that the crustal material is diffusing into the mantle and thus inducing a compositional gradient by the boundary of the slab. Here the lower melting point of the crustal material also means it can support less stress at equivalent temperatures. If one then also considers a shear wave velocity gradient induced by a change in chemical composition as well as by the change in temperature then inertial motions due to stress gradients are by far largest at the slab boundary. The problem is then whether the shear heating induced by these effects is large enough to overcome the ability of the material to conduct heat out of this region. If the velocity of propagation of the stress is many orders of magnitude greater than the fluid velocities it then appears reasonable to assume the stress is constant to some small depth δ initially. Thus $\sigma(\delta) = \sigma_b + \Delta$ where $\Delta << \sigma$ and $T(\delta) - T << T_b$. If the composition in the material being sheared was the same throughout then the temperature distribution could be written as

$$\phi(\xi) = \phi(\delta) - 2\ln\left[\cosh^{-1}\left(\exp(\tfrac{1}{2}\phi(\delta))\right) - \sqrt{\frac{\lambda}{2}}\exp\frac{\phi(\delta)}{2}\,\xi\right]$$

Here the propagation time to a distance ξ from the boundary is given by

$$t_p(\xi) = \frac{1}{\beta_o} \sqrt{\frac{2}{\lambda}} \exp\left(\frac{\phi_c}{2}\right) \tanh\left(\sqrt{\frac{\lambda}{2}} \exp\left(\frac{\phi_c}{2}\right) \tau\right)^{-1} \quad \tau = 1-\xi$$

and the velocity β may then be written as

$$\frac{1}{\beta(\xi)} = \frac{\partial t_p}{\partial \xi}$$

$$= -\frac{1}{\beta_o} \exp(\phi_c) \operatorname{sech}^2\left(\sqrt{\frac{\lambda}{2}} \exp\left(\frac{\phi_c}{2}\right)\left(1-\xi\right)\right)$$

Now assume there exists a depth $\xi\gamma << \xi$ such that

$$\beta_o(\xi) \rightarrow \beta_o \exp[(1-\xi)\gamma]$$

and

$$\mu_o(\xi) \rightarrow \mu_o \exp[(1-\xi)\lambda] \qquad \text{for } \xi < \gamma$$

In this region we can observe how the temperature and veloc y f elds will evolve initially from the class of possible stationary solutions found in section (3.4).

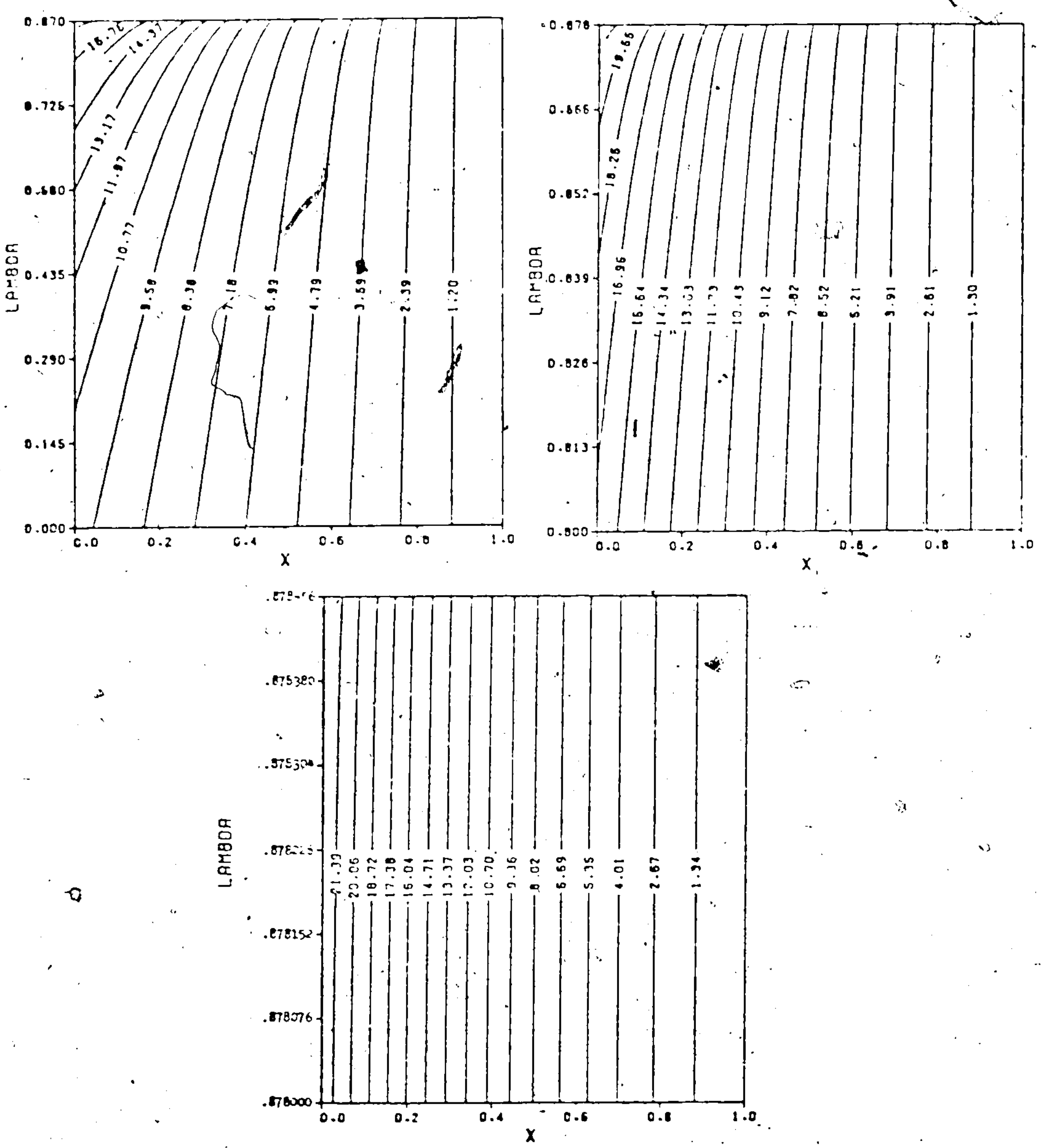
The case of interest in chapter 3 was a temperature increase towards the boundary of high viscosity. In this chapter the opposite case is of intrest, a temperature decrease towards the boundary of high viscosity. When the plate first subducts the heat loss from the mantle material to the upper plate boundary will be greater than the effects of shear heating in that region because of the very large temperature difference between the crust and surrounding mantle material. However as the plate boundary approaches mantle temperature the effects of shear heating are able to dominate. Under these conditions one may obtain a maximum temperature in the boundary layer as illustrated in figures (2.5) and (2.9). In the presence of inertial motions caused by the large shear wave velocity gradients in this region a sharply defined region with the maximum temperature must develop. This is a direct result of the fact that inertial motion must cause inertial heating (see section 4.2).

One is now faced with a problem since for deep earthquakes the failure is not parallel to the subducting plate but at a 30° to 45° angle to plate subduction. One method by which this can be overcome is as follows. The pressure gradient in the earth at 400 km. depth is about 3.8×10^3 gm/cm²/sec² (Stacey F. D. 1969). The temperature variations due to shear heating cause large density variations in the boundary layer. Here the density of the mantle material is 3.8 gm/cm³ (Stacey F. D. 1969), while 4.2 gm/cm³ and 3.5 gm/cm³ are probably reasonable values for the density of the slab and boundary layer at this depth respectively. This yields an approximately zero body force in the mantle, a force per unit volume of 4×10^2 gm/cm²/sec² in the slab directed towards the center of the earth, and a buoyant force per unit volume of 3×10^2 gm/cm²/sec² in the boundary layer. It is interesting to note that these two values are of the same order of magnitude, they are however only approximations. The buoyant force in the boundary layer for example may vary dramatically with the temperature and composition of the material. Now assume that the only forces of importance in the boundary layer are, the force parallel to the plate due to the viscous shearing of the fluid and the buoyant force caused by the heating of the material. The axis of principal shear at any position in the boundary layer is then oriented in the direction of the vector sum of these two forces.

This analysis may then be summarised as follows, as the boundary layer is sheared it is subjected to large decreases in density and is in the neighbourhood of melting. These density changes cause compressional stresses and thus alter the axis of principal shear. The earthquake must therefore be a combination of the relief of the elastic stress built up in the shearing of the subducting slab and the compressional stresses induced by temperature and pressure gradients in subduction.

The occurence of compressional earthquakes thus seems unlikely at shallow depths for a uniformly moving plate due to the dominance of cooling. It therefore seems that such an earthquake would require a large jerk to start it on its way to fluid instability. This however is certainly not a problem as the plate motions are not uniform (Isacks B. and Molnar P. 1971, Spence W. 1977). The frictional sliding of the converging plates at the surface and the tensile earthquakes where plate bending occurs cause large nonuniformities in the motion of the subducting plates. These compressional strain pulses propagate down the slab at velocities of the order of 100 km/year (Spence W. 1977). In the following section I will examine the possibility of these inertial effects leading to instability.

Figure 4.1: The propagation time from the boundary $x=1$ to any position in the interior is given in seconds. These times are probably much smaller than the propagation time for an actual strain-rate pulse as the velocity of propagation is assumed to be the seismic shear wave velocity. The contours values for a more slowly propagating plastic wave however should only differ in magnitude.


LAMBDA
X
0.870
0.725
0.580
0.435
0.290
0.145
0.000
0.0
0.2
0.4
0.6
0.8
1.0
9.58
8.38
7.18
5.99
4.79
3.59
2.39
1.20
0.878
0.865
0.852
0.839
0.826
0.813
0.800
LAMBDA
X

4.4 A Model of Deep Earthquakes.

I wish to consider here the crust as an infinite elastic plate of thickness h shearing a highly viscous fluid. The fluid is composed of crustal material at the interface and its composition changes to that of mantle material within a small boundary layer. The deformation of the plate will be assumed to be homogeneous and only in the direction opposite to the plate motion. Since the strain is constant across the plate for a homogeneous deformation the stress tensor σ_{ik} must also be constant. If the z coordinate is the direction of plate motion and the x coordinate is orthogonal to the plate surface then the only external force on the plate is horizontal shear in the x direction. For small displacements the only non vanishing component of the stress tensor is then σ_{xz}.

The stress for this homogeneous sheared slab is then related to the strain by (Landau L.D. and; Lifshitz E.M. 1959)

$$\sigma_{xz} = \frac{E}{1+P}\,\varepsilon_{xz}$$

Here E is the modulus of extension or Young's modulus and p, the ratio of the transverse compression to the longitudinal extension, is called Poisson's ratio. The displacements are thus related to the stress by

$$U_x = \frac{2(1+P)}{E}\,\sigma\,Z$$

The velocity of slab subduction is of the order 10 cm/year while the velocity of plastic wave propogation into the mantle is orders of magnitude greater. Thus initially one obtains a stress which is essentially constant to great distances into the surrounding mantle. Appendix C however indicates that instability must occur for any constant stress applied to a half space and thus that instability should occur at a great distance from the slab. This is not relevant to the problem at hand as can be seen by a simple calculation. If one assumes instability will result and simply assumes that it may be expressed in terms of the breakdown of the stationary stability criterion one then obtain the following results. Instability at 1000 km will result for an applied stress of approximately 3 bars and a strain rate of $\pi \times 10^{-16} \exp(\phi)$ /sec which yields a plate velocity of 10 cm/year. The time for this instability to evolve assuming no heat conduction is then approximately 10^{10} years which is more than twice the age of the earth. In any case if one is to accept the results of section 3.4 that 50 bars stress is transmitted by the seismic low velocity zone then the mantle material beneath must be capable of imparting an even larger stress on the subducting plate.

By this analysis one finds that for every factor of 10 increase in the stress that the mantle is able to offer, the distance to instability decreases by a factor of 10. The velocity of the plate in all cases remains 10 cm/year but the time to instability decreases in factors of 10^2. The ability of the mantle material to act as a viscous fluid in the neighbourhood of the slab and yet more resemble a solid, beyond some depth ℓ in its ability to resist flow, clearly results from both geologic heterogeneity and the fact that the transition from a highly viscous fluid to a solid is not sharply defined.

The properties of these fluids may be described as follows. When the stress is applied they undergo elastic deformation. If the deformation ceases, shear stresses remain although these are damped in the course of time as they slowly yield to plastic deformation (Landau L.D. and Lifshitz E.M. 1959). The time for stresses to be relieved after the deformation stops is called the "Maxwellian relaxation time". This relaxation time is much longer for the subducting slab than for the surrounding mantle because of its higher viscosity. It is thus the mantle which first yields to plastic deformation as the slab is being subducted into it. However the stress relief of the mantle results in shear heating. Thus since the fluid viscosity decreases as an exponential function of temperature the region of greatest net heat production behaves more and more as fluid subjected to the shearing of elastic solids on each side. This results in a region of shear coupling at the crust-mantle interface similar to the seismic low velocity zone beneath the plates.

Now consider the possibility of constructing stable solutions (i.e. stationary solutions). For a stationary solution to evolve the inertial terms in the stress strain rate relation must vanish (i.e. rate of stress relaxation must balance the rate of stress application). Assume that the subducting slab causes a zone of 10 km thickness to exist in a marginally stable state. This is approximately the depth of boundary layer heating argued by Sydora (1977). If we assume that the normal mantle temperature at the depth under consideration is 2000°K then the stress in the mantle is 50 bars. The temperature at the boundary of this region is then 2120°C or if we use the stress strain-rate relation (3.9) it becomes 2300°C.

Now consider the boundary layer of composition change. Assume this region has a thickness of approximately one kilometer. In this region we can approximate the heating time by $t_\infty = 4x10^{15}$ sec and the cooling time by $t_c = 4x10^{14}$ sec (section 2.1). Using the analysis for a material of uniform composition. Thus if the cooling is dominant over the heating in this approximation one obtains a temperature decrease at the slab boundary. But this does not imply stable solutions result.

stability may evolve in two ways. First the crustal mate... has a lower melting point than the surrounding mantle material (Green T.H. and Ringwood A.E. 1966) and one obtains ...ing of the crust due to thermal conduction from the mantl. The lower thermal conductivity and higher viscosity of the crust will allow this boundary to evolve initially as boundary of thermal insulation. However as the boundary heats it will undergo fluid deformation and due to its low melting point not be able to support the amount of stress that the mantle supports. This follows directly from eqn.(2.4) and the attendant reduction in viscosity which must occur. The fluid then must experience accelerations in order to reduce the stress imparted to it. If this boundary layer is very thin, the onset of melt will result in larger accelerations and thus total melt. This occurs since the applied stress remains greater even in the presence of partial melt, than this thin slab can support.

The second way instability may evolve is by obtaining a maximum temperature away from the boundary of the subducting slab due to shear heating. The viscosity gradients in this region must result in inertial heating. Then as this boundary layer is heated by thermal conduction one obtains changes in the relaxation times of this material. This in turn causes stress gradients and thus accelerations of the fluid in this region (sec 4.2). In this fashion one obtains additional heating due to the applied stress of the subducting slab which is accentuated by the existence of shear wave velocity gradients. Thus once a slab with a concentration gradient undergoes inertial heating in the interior due to an applied stress this in turn may cause inertial motions which themselves become heat sources. This means that for a constant stress at the boundary, inertial motions will occur in an attempt to relieve excess heat production at any position in the slab. These accelerations

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however cause greater velocity gradients at the positions where they occur and thus even greater heat production. At the slab boundary these accelerations are not able to supply a noticeable reduction in stress due to its high viscosity in relation to the mantle. The problem then becomes whether a real instability results in which one obtains snapping (stress relief due to melting in a thin layer) or it is able to deform smoothly as the mantle material does.

The answer to this problem depends on two factors. One is the amount of heating which must take place in order for melting to occur while the other is how well defined the melting point is. If the initial temperature of the material is far from melting, then difference in the rate of heating will cause a very narrow region of melt. Also if the melting point is well defined then the region which first reaches this point exhibits a substantially lower viscosity than the rest of the slab. It then yields to almost the entire deformation which is taking place thus causing total melt and a complete relief of stress.

Now consider the role phase changes in the mantle and and subducting slab may play in this process. If the temperature increases dramatically at the slab boundary, a phase change will occur on both sides of the boundary layer prior to its occurrence within that layer. The layer will thus become narrower with depth. The material on each side will then be able to support more stress and will have a slower relaxation time than it had before undergoing a phase change. The fluid deformation carried by slab subduction will thus be forced to occur almost totally within the boundary layer. The increasing viscosities outside this region will thus prevent a large stress reduction. Thus the attendant reduction in viscosity can easily cause the stability criterion to be exceeded within the boundary layer. For a viscosity of 10^{19} poise, a thickness of 1 km and a stress of 100 bars one obtains instability even for the steady state equation of motion. This however yields a slab velocity of 30 cm/year. If the stress under these conditions is considered to increase to 1kb the stability criterion can be exceeded by a viscosity of 10^{21} poise over the 1 km boundary layer the velocity of slab subduction necessary in this case in the only 3 cm/year. This stress of 1kb. is in fact approximately the stress obtained for slab subduction by Turcotte and Schubert (1973).

Finally consider the evidence that this is in fact the mechanism of deep earthquakes. First the focal mechanism is observed to be the same as in shear fracture. In theory a shear heating instability should be indistinguishable from shear fracture. Second this mechanism should cause volcanic activity following large earthquakes. To test correlations between earthquakes in the upper mantle and volcanic activities Blot (1977) has attempted since 1963 to predict volcanic eruptions in the island arc comprising the New Hebridies, the Solomons and New Zealand. In the period 1970 to 1975 he forcast 25 eruptions of which 20 took place with an accuracy of ±15 days. Other examples of volcanic preditions by the method also exist (Grover J. 1977). A third factor is that one should expect an increase in the number of earthquakes, with depth due to greater mantle viscosities and with the speed of subduction. This is also observed. Fourth the observed compressional earthquakes should define the upper edge of the slab and the orientation should define the direction of subduction. These two phenomena have also been consistently observed (Isacks B. and Molnar P. 1971) However in the past the earthquakes have been assumed to occur within the upper surface of the slab.

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Appendix A

An Analysis of the Variational Principle in Chapter 2

The mathematical approach presents here originates from a technique first used in relativistic particle dynamics (Israel W. and Bailey I., 1975). It is essentially an attempt to insert in the variational formulation the notion of an irreversible system. The approach may be justified rigorously by constructing an action functional simular to the construction in Hamilton's principle. The fundamental difference, however, is that here the action functional is only constructed for an infinitesimal time step.

$$I = \int_{t-\varepsilon}^{t} \int_{V} \mathcal{L}(\Theta(\vec{x},\tau), \Theta(\vec{x},\tau-\varepsilon))$$

$$\simeq \varepsilon \ L$$

now let $\Theta'(\tau) = \Theta(\tau) + d\,\Theta(\tau)$

therefore $\Theta'(\tau-\varepsilon) = \Theta(\tau-\varepsilon) + d\,\Theta(\tau-\varepsilon)$

then vary $\Theta(\vec{x},\tau)$ such that

$$d\,\Theta(\vec{x},\tau) = 0 \qquad \text{for} \qquad \tau \leq t-\varepsilon$$

$$d\,\Theta(\vec{x},\tau) \neq 0 \qquad \text{for} \qquad t \geq \tau > t-\varepsilon$$

One then obtains

$$\delta I = I' - I$$

$$= \int_{t-\varepsilon}^{t} \int_{V} \left.\frac{\delta\mathcal{L}}{\delta\Theta}\right|_{\Theta(\vec{x},\tau-\varepsilon) = \text{const}} d\,\Theta(\vec{x},\tau)\ dV\ d\tau$$

and thus $\lim_{\varepsilon\to 0} \left.\frac{\delta\mathcal{L}}{\delta\Theta}\right|_{\substack{\tau = t \\ \Theta(\vec{x},\tau-\varepsilon) = \text{const}}} = 0$

This principle as applied to a system, with a first order time derivative in the field equations, makes it a property of the variational principle that the system be irreversible. This is done by defining the action of the system for only an infinitesimal time step and constructing the total action as a sum of infinitesimal actions. The direction of the summation may then be performed in only the future pointing time direction since the initial conditions are determined by the final conditions of the previous time step. It is interresting to note that this action functional may also be defined in terms of an integral over a finite time interval which produces the equations of motion at every instant. The direction of evolution of the system is defined by the sign of ε which becomes the differential in this integral. This second type of construction does not lead itself to the numerical evaluation of the integral, which is of primary importance in the geophysical considerations for which it is used here.

Tests on the accuracy of this method have been done numerically by comparing the results obtained by Gruntfest to the solution of the same problem with this variational principle. The solutions obtained were identical.

An analytical test has also been done. Here a semi-infinite slab initially at zero temperature has its face suddenly raised to temperature T_O. It is assumed that T can be represented in the form: $T=T_O(1-x/f(t))^2$ where f(t) is the penetration depth. Solving the variational principle for f(t) I obtain $T=T_o(1-z/3.16)^2$ where $z=x/(kt)^{1/2}$. This is the same result as obtained by Djukic and Vujanovic (1971) for this problem and is a good approximation to the exact solution $T=T_o(1-\mathrm{erf}(z/2))$.

Appendix B

Equations for Multiphase Viscous Flow when Subject to a Shear Stress.

The equation of continuity for the total mass of the fluid is the same as for a one phase fluid.

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \vec{v}) = 0$$

Here the velocity is defined in terms of the total momentum of a unit mass of fluid. The form of the Navier Stokes equation also remains in the same form. It however does depend on the concentrations of the different phases by virtue of the fact that viscosity depends on the relative consentrations.

Diffusion of the phases and the solid body motion of small portions of the fluid both cause changes in the concentrations of the phases at different positions within the fluid. The composition of any given fluid element in the absense of diffusion is given by

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \vec{v} \cdot \vec{\nabla} \eta = 0$$

where η is the ratio of the mass of the iTH phase to the mass of the fluid in a small volume. Using the equation of continuity of total mass and introducing the density of diffusion flux $\vec{i}$ this equation becomes

$$\rho c \left(\frac{\partial \eta}{\partial t} + \vec{v} \cdot \vec{\nabla} \eta\right) = - \vec{\nabla} \cdot \vec{i} = \rho c \frac{d\eta}{dt}$$

The equation of heat transfer than has an additional term due to diffusion and may be written as

$$\frac{\rho c}{a} \frac{d\phi}{dt} - \frac{k}{a} \nabla^2 \phi = \mu \dot{\varepsilon}_{ij}^{\ 2} - \mu_\eta \vec{\nabla} . \vec{i}^{\,2}$$

Here μ_η is the chemical potential of the i'th phase.

If the concentration of the phase we are describing by the equation of continuity is small, as in the case of melt in the asthenosphere, we may express the diffusion flux as

$$\vec{i} = -\rho c D \text{ grad } \eta$$

Here D the diffusion coefficient and gives the diffusion flux in this case.

Appendix C

A Calculation of the Maximum Depth to which a Stationary Solution May Exist in a Homogeneous Fluid Subjected to Constant Shear.

Assume we have an infinite half spare. First we wish to consider the solutions for the temperature field if the stress is assumed constant for all time. In this case the solutions must be stationary.

$$k\frac{d^2T}{dx^2} = -\sigma^2 \exp[a(T-T_B)]/\mu_o$$

$$\frac{\partial\sigma}{\partial x} = 0$$

$$\sigma = \mu_o \; \varepsilon \exp(-\phi)$$

From the temperature equation we obtain

$$\frac{\partial\phi}{\partial x} = \pm\left(\frac{\partial\sigma^2 a}{K\mu_o}\right)^{\frac{1}{2}}\cdot\left(\exp(\phi_m) - \exp(\phi)\right)^{-\frac{1}{2}}$$

$$\phi(o) = 0$$

$$\phi(\infty) = \phi_m$$

$$X = \left(\frac{2K\mu_o}{a\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi_m}{2}\right)\left[\cosh^{-1}\left(\exp(\tfrac{1}{2}\phi_m)\right)\right.$$

$$\left. - \cosh^{-1}\left(\exp(\tfrac{1}{2}(\phi_m-\phi))\right)\right]$$

$$X(\phi_m) = \left(\frac{2K\mu_o}{a\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\phi_m}{2}\right) \cosh^{-1}\left(\exp(\tfrac{1}{2}\phi_m)\right)$$

But the boundary condition at $x=\infty$ cannot be satisfied unless $\sigma=0$. Now assume we start with a zero stress σ and increase it infinitely slowly so that the stress can be considered constant across any thickness. We now wish to find the thickness ℓ over which a stationary solution may result. We thus have the condition

$\Phi(\ell)=\Phi_M=1.2$

The thickness ℓ over which a stationary solution exists is

$$\ell = \left(\frac{2K\mu_o}{a\sigma^2}\right)^{\frac{1}{2}} .66$$

$$= \left(\frac{.88\ K\mu_o}{a\sigma^2}\right)^{\frac{1}{2}}$$

In any real physical situation however the stress is not increased infinitely slowly. Thus the above relation given an upper bound for the length over which a stable solution may exist.