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NAME OF AUTHOR..A.B.D.E.L.-F.A.T.T.A.H...A.B.D.E.L.-L.A.T.I.F SELIM

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..Faculty...of Science
Republic of Egypt.

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NONLINEAR EFFECTS IN PLASMAS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled NONLINEAR EFFECTS IN PLASMAS submitted by Abdel-Fattah Abdel-Latif Sayed Selim in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

....
D. Rankin, Supervisor

....
S. Krishan, Co-Supervisor

....
A.B. Bhatia

....
J.A. Jacobs

....
G.L. Cumming

....
C.R. James

....
W.B. Thompson, External Examiner

Date.....................

ABSTRACT

A quantum mechanical method is used to investigate the linear and nonlinear collisional processes in an unmagnetized plasma, which arise out of the scattering of two particles through an effective Coulomb field via emission or absorption of one and two plasmons.

The validity of Kadomtsev's assumption (1965) concerning the ion nonlinear Landau damping has been questioned (Sloan and Drummond, 1970), inasmuch as the electron nonlinear wave-particle effects make the wave further destabilizing. The nonlinear dielectric function and its effect on wave-particle coupling constant is investigated and it is shown that a reduction in the growth rate occurs and that the amplitude dependent frequency shift causes stabilization.

The four-wave interactions and the amplitude dependent frequency shift for electromagnetic waves (light, whistler and Alfvén) propagating parallel to an external magnetic field are studied by employing the diagrammatic method of field theory.

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Finally, I would like to acknowledge the contribution of Miss Mila Oliva Flores for proof-reading the final manuscript of this thesis.

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INTRODUCTION

The linear theory in plasma physics had been adequate for the understanding of small amplitude plasma oscillations and indeed it was believed that the problem of the stable confinement of plasma could be solved within this linear theory. More recent theoretical development and experimental measurements have demonstrated a whole chain of new plasma instabilities and in fact the presence of instabilities is the most characteristic attribute of this state of matter.

The fundamental question as to how the instabilities develop and whether or not they play a significant role in plasma phenomena can only be answered by nonlinear theory. In the past few years, the application of nonlinear theory (Sturrock, 1957; Ginzburg and Zhelemiakov, 1958; Kadomtsev, 1965 and Vedenov, 1965) has undergone such vigorous development as to result in the formulation of some clear physical concepts regarding these nonlinear mechanisms. However while the achievements of the above authors were notable, the range of nonlinear interactions was by no means completely understood nor was a systematic methodology developed. In addition to the class of problems in which the amplitude of the wave fields is of primary concern, nonlinear theory can be used to explain mode conversion. This process is of fundamental importance since mode conversion leads to electromagnetic radiation by which interstellar and solar plasmas, for example, can be studied.

When considering the problem of nonlinear interactions, the quantum mechanical calculations are more straightforward and less difficult than the corresponding classical calculations. One of the most successful early methods of attacking this problem, which is still used extensively today, is the canonical transformation technique. However, one would prefer a more systematic method of obtaining system properties and it was shown in a series of revolutionary papers (Vedenov, et al., 1962, 1963; Pines and Schrieffer, 1962) that quantum field theory, previously restricted to elementary particle physics, provides such a systematic method. An essential role in the field theoretic treatment of the many body problem is played by the Green's function or propagator of the system. The propagators can be calculated relatively more easily by using the well known Feynman diagram technique which has the additional advantage of providing a pictorial representation corresponding to each term in the perturbation expansion.

From the above point of view, the waves in a plasma can be thought as being composed of quasiparticles (quanta of waves). These quasiparticles interact with the particles of the plasma and with each other and their interactions are described in terms of an interaction Hamiltonian. The interaction Hamiltonians for particles and plasmons (quanta of plasma oscillations) and particles and phonons (quanta of ion sound waves) were obtained by Pines and Schrieffer (1962). They used these together with the "Fermi Golden Rule" to write the quasilinear equations. Pines and Schrieffer, (1962) and Vedenov, et al. (1962) pointed out that the Landau damping or growth process which forms the basis of quasilinear theory, can be described as the competition between absorption and stimulated emission of quantized plasma waves or plasmons. Similarly, the quantum form of the equation describing nonlinear Landau damping is discussed by Rosenbluth, et al., (1969); Ross, (1969); Harris, (1969) and Krishan and Fukai, (1971).

The rate of change of particle distribution is derived by Wyld and Pines (1962). They assumed that the matrix element for a Coulomb collision, $4\pi e^2/q^2$ must be modified by the factor $\varepsilon^{-1}(\vec{q},\omega)$ (Thompson, 1962) where ε is the dielectric function of the plasma, $\hbar\vec{q}$ is the momentum transfer, and $\hbar\omega$ is the energy transfer in the collision. The equation of Wyld and Pines reduces to the well known Boltzman equation in the classical limit. Quantum mechanical calculations for four-plasmons (Langmuir waves) in an unmagnetized plasma have been carried out by Zakharov (1967) and for three plasmon interactions for longitudinal and transverse waves by Krishan (1968) and by Krishan and Selim (1968). Attention should also be directed to the paper of Gailitis, et al. (1966) in which, although the calculations are classical, the language is quantum mechanical. This deals with the interaction of plasmons, phonons and photons in an isotropic plasma. A quantum mechanical theory of nonlinear phenomena in a very strong magnetic field was developed by Walters and Harris (1967, 1968). The three plasmon interaction of Walters and Harris had previously been derived classically by Aamodt and Drummond (1964).

In chapter II, the occupation number formalism (second quantization) and the motion of charged particles in a uniform magnetic field are introduced. The electromagnetic field in a plasma is quantized as has previously been done by Kihara, Aono, and Dodo, (1962); Alekseev and Nikitin (1966) and Harris (1969). The wave-particle and particle-particle interaction Hamiltonians are derived and represented by first order Feynman diagrams. The theory of the scattering matrix and its representation by Feynman graphs are developed and represented in a suitable form for studying the nonlinear plasma interaction. General rules for the calculations of the matrix elements (coupling constants) for higher order diagrams are established.

Chapter III, deals with the transition matrix elements for the scattering of two particles through effective Coulomb field and the emission (absorption) of one and two plasmons (waves) in an unmagnetized plasma. Using these matrix elements and the "Fermi Golden Rule" one obtains the linear and nonlinear collisional damping for longitudinal plasma waves.

In chapter IV, the dielectric function and linear Landau growth rate for ion acoustic waves are derived quantum mechanically, in such a way as to emphasize the similarity in the classical and quantum mechanical derivations. Using the same method, the nonlinear dielectric function ε_{NL} is derived and its effect on the wave-particle coupling constant is discussed. Solving $\varepsilon_{NL} = 0$, the frequency shift and its stabilization influence on the unstable waves is discussed. Using the modified coupling constant and the Fermi Golden Rule, the quasilinear equations for the plasma particles are obtained. These equations yield the ratio of the heating rates for the two plasma species. A comparison is made between the first order correction to the linear growth rate obtained in this chapter and electron nonlinear Landau growth estimated by Sloan and Drummond (1970).

Chapter V deals with the problem of electromagnetic waves propagating parallel to the external magnetic field. The theory of chapter II is extended to study four wave interactions for light, whistlers and Alfvén modes. A general formula for the amplitude dependent frequency shift is derived. As a special case, the frequency shift for the above waves in a cold plasma is obtained.

CHAPTER II

THEORY

2.1 Occupation Number Formalism (Second Quantization)

In the quantum mechanical investigation of systems composed of large numbers of particles, it is beneficial to use the occupation number formalism or 'second quantisation' as it is often called. In this method, one seeks a mathematical formalism in which the occupation number of the states (and not the coordinates of the particles) play the part of the independent variables.

a. Bose Statistics

Consider a system of N non-interacting particles (bosons) with wave functions $\xi_1(\vec{x})$, $\xi_2(\vec{x})$, ..., which then form a complete set of orthogonal and normalised wave functions. The total wave function of this system is a symmetrised sum of products of the functions $\xi_i(\vec{x})$

$$\Psi_{N_1,N_2\ldots}(\vec{x}_1,\ldots,\vec{x}_N)=\left[\frac{N_1!N_2!\ldots}{N!}\right]^{1/2}\sum\xi_{p_1}(\vec{x}_1)\xi_{p_2}(\vec{x}_2)\ldots\xi_{p_N}(\vec{x}_N) \tag{1}$$

Here N_i are the number of particles in the ith state, $p_1,p_2,\ldots p_N$ are the ordinal numbers of the states in which the individual particles are, and the sum is taken over all permutations of those suffixes, which are different. The constant factor is chosen so that the function $\Psi_{N_1, N_2, \ldots}$ is normalized.

Let f_α be the operator of some physical quantity acting only on a function of $\vec{x}_\alpha$. Consider the operator

$$F = \sum_\alpha f_\alpha \qquad (2)$$

which is symmetrical with respect to all the particles and whose matrix elements with respect to the wave functions (1) will be determined. These matrix elements will be different from zero for transitions which leave the numbers $N_1, N_2, \ldots,$ unchanged corresponding to the diagonal elements, or, since each of the operators f_α acts only on one function in the product (1), its matrix element can be different from zero for transition whereby the state of a single particle is changed, which implies that the number of particles in one state is diminished by unity, while the number in another state is correspondingly increased. Simple calculations yield the non-diagonal elements

$$(F)^{N_i, N_k - 1}_{N_i - 1, N_k} = f_{ik}\sqrt{N_i N_k} \qquad (3)$$

where f_{ik} is the matrix element

$$f_{ik} = \int \xi_i(\vec{x}) f \xi_k(\vec{x}) d^3x \qquad (4)$$

The diagonal matrix elements of F are the mean values of the quantity F in the states $\Psi_{N_1 N_2} \cdots$ and are given by

$$\bar{F} = \sum_i f_{ii} N_i \tag{5}$$

The operators F can be pictured as acting on the occupation numbers N_i if one introduces operators a_i, which decrease by one the number of particles in the ith state and possess the matrix elements

$$(a_i)_{N_i}^{N_i - 1} = \sqrt{N_i} \tag{6}$$

The Hermitian conjugate operators a_i^+ obviously have the matrix elements

$$(a_i^+)_{N_i - 1}^{N_i} = \left[(a_i)_{N_i - 1}^{N_i} \right]^+ = \sqrt{N_i} \tag{7}$$

i.e. they increase the number of particles by one. It is easily shown that the operator F can be written as

$$F = \sum_{i,k} f_{ik} a_i^+ a_k \tag{8}$$

The matrix elements of this operator are the same as those of (3) and in fact this is the expression for F in the second quantisation form. Thus, one had been able to express an ordinary operator (of the form (2)) acting on functions of the coordinates, in the form of another operator acting on functions of new variables, the occupation number N_i. This result is easily generalized to operators pertaining to more than one particle at once.

From (6) and (7), one can prove the following commutation relations

$$[a_i, a_k^+] = a_i a_k^+ - a_k^+ a_i = \delta_{ik}$$
$$[a_i, a_k] = [a_i^+, a_k^+] = 0 \tag{9}$$

The operators a_i and a_i^+ are known as annihilation and creation operators.

Finally, it remains to express, in terms of the operators a_i and a_i^+, the Hamiltonian H of the physical system of N identical interacting particles that is actually being considered. In the nonrelativistic approximation H can be represented in general form as follows

$$H = \sum_{\alpha} H_\alpha + \sum_{\alpha>\beta} U(x_\alpha, x_\beta) + \sum U(x_\alpha, x_\beta, x_\gamma) + \ldots \tag{10}$$

Here H_α is the part of the Hamiltonian which depends on the coordinates of the αth particle only $H_\alpha = -\frac{\hbar^2}{2m}\nabla_\alpha^2 + U(x_\alpha)$, where $U(x_\alpha)$ is the potential energy of a single particle in an external field. The remaining terms in (10) correspond to the mutual interaction energy of the particles. In the second quantisation form, equation (10) becomes

$$H = \sum_{i,k} H_{ik}a_i^+a_k + \sum_{ik\ell m} [U]^{ik}_{\ell m}a_i^+a_k^+a_\ell a_m + \ldots \qquad (11)$$

This gives the required expression for the Hamiltonian in the form of an operator acting on functions of the occupation numbers. For a system of non-interacting particles, equation (11) becomes

$$H = \sum_{i,k} H_{ik}a_i^+a_k \qquad (12)$$

If the function $\xi_i(\vec{x})$ is taken to be the eigenfunction of the Hamiltonian of an individual particle, the matrix H_{ik} is diagonal, and its diagonal elements are the eigenvalues E_i of the energy of the particle. Thus equation (12) yields

$$E = \sum_i E_iN_i \qquad (13)$$

This is the expression for the energy levels of the system.

The formalism which has been developed can be put in a somewhat more compact form by introducing the operators

$$\Psi(x) = \sum_i a_i\xi_i(\vec{x}), \Psi^+(x) = \sum_i a_i^+\xi_i^+(\vec{x}) \qquad (14)$$

With the properties of a_i and a_i^+, it is clear that the operator $\Psi(x)$ decreases the total number of particles in the system by one, while $\Psi^+(x)$ increases it by unity. One can easily prove that these operators satisfy the commutation rules

$$[\Psi(x),\Psi^+(x')] = \delta(x-x')$$
$$[\Psi(x),\Psi(x')] = [\Psi^+(x),\Psi^+(x')] = 0 \tag{15}$$

The expression (8) for the operator (2) can be written, using the operators(14), in the form

$$\begin{aligned} F &= \int \Psi^+(x) f \Psi(x) d^3x \\ &= \sum_{i,k} a_i^+ a_k \int \Psi_i^+ f \Psi_k d^3x \\ &= \sum_{i,k} f_{ik} a_i^+ a_k \end{aligned} \tag{16}$$

which is the same as (8). Thus, the Hamiltonian H can be written, using (16) and similar expressions, as follows

$$H = \int d^3x \Psi^+(x) H(x) \Psi(x) + \iint d^3x d^3x' \Psi^+(x) \Psi^+(x') U(x,x') \Psi(x) \Psi(x') + \ldots \tag{17}$$

To clarify equation (17), suppose each particle of the given system is described (at a given instant) by the same wave function $\xi(\vec{x})$ which is normalised so that $\frac{1}{N}\int |\xi(\vec{x})|^2 d^3x = 1$.

Then, it is immediately evident that, if one replaces the operators $\Psi(x)$ in (17) by $\xi(\vec{x})$, this expression leads to the mean energy of the system in the state considered. This gives the following rule for deriving the Hamiltonian in the second quantisation formalism. The expression for the mean energy is written in terms of the wave function of an individual particle, and this function is then replaced by the operators $\Psi(x)$.

b. Fermi Statistics

The basic theory of the method of second quantisation remains unchanged for systems of identical particles obeying Fermi statistics. However the actual formulae for the matrix elements of quantities and for the operators a_i are different, because unlike the case of Bose statistics the wave functions are antisymmetric.

The matrix element of an operator F of the type (2) are in the present case

$$\bar{F} = \sum_i f_{ii} n_i \tag{18}$$

for the diagonal elements, and

$$(F)^{1_i 0_k}_{0_i 1_k} = f_{ik} (-1)^{\sum(i+1,k-1)} \tag{19}$$

for the off-diagonal elements. The factor (-1) appears to a power equal to the sum of the occupation numbers of all states between the ith and the kth $(i<k)$, $\sum(i+1,k-1) = \sum_{r=i+1}^{k-1} n_r$ and $0_i, 1_i$ refer to $n_i = 0$, $n_i = 1$ (n_i being the number of fermions in the ith state).

In order to represent the operator F in the form (8), the operators a_i which will be replaced by C_i for fermions must be defined as matrices whose elements are

$$(C_i)^{0_i}_{1_i} = (C_i^+)^{1_i}_{0_i} = (-1)^{\sum(1,i-1)} \tag{20}$$

These operators satisfy the following anticommutation rules:

$$\{C_i, C_k^+\} = C_i C_k^+ + C_k^+ C_i = \delta_{ik}$$
$$\{C_i, C_k\} = \{C_i^+, C_k^+\} = 0 \tag{21}$$

From the definition (20), it is obvious that the result of the action of the operators C_i depends not only on the number n_i itself (as in the case of Bose statistics), but also on the occupation numbers of all the preceding states.

With these properties of the operators C_i and C_i^+, the formulae (8) and (11) remain valid. The formulae (16) and (17) which express the operators of physical quantities in terms of the operators $\Psi(x)$ and $\Psi^+(x)$ defined by (14) also hold good. The commutation rules (15), however changes into the anticommutation relations

$$\{\Psi(x),\Psi^{+}(x')\} = \delta(x-x')$$

$$\{\Psi(\vec{x}),\Psi(\vec{x}')\} = \{\Psi^{+}(\vec{x}),\Psi^{+}(\vec{x}')\} = 0 \qquad (22)$$

2.2 Motion of a Charged Particle in a Uniform Magnetic Field

The non-relativistic Hamiltonian for a particle of species j in a uniform magnetic field is given by

$$H_j = \frac{1}{2m_j}\left[\vec{P}_j + \frac{e_j}{c}\vec{A}(x)\right]^2 + e_j\phi(x) \qquad (23)$$

where $\vec{P}_j$ is the canonical momentum of the jth particle and $\vec{A}(x)$ and $\phi(x)$ are the vector and scalar potentials of the field which can be split into time independent zeroth order term and a time dependent perturbation term which in this case will be due to collective excitation of waves.

$$\vec{A}(x) = \vec{A}_0 + \vec{A}_1(x) \qquad (24)$$

$$\phi(x) = \phi_0 + \phi_1(x) \qquad (25)$$

Here $\vec{A}_0$ is the vector potential due to a uniform external magnetic field ($\phi_0 = 0$); $\vec{A}_1(x)$ and $\phi_1(x)$ are the vector and scalar potential due to collective fields of transverse and longitudinal waves respectively.

Substituting (24) and (25) into (23) yields

$$H_j = H_j^{(0)} + H_j^{(1)} + H_j^{(2)} \tag{26}$$

where

$$H_j^{(0)} = \frac{1}{2m_j} (\vec{P}_j + \frac{e_j}{c}\vec{A}_0)^2 \tag{26.a}$$

$$H_j^{(1)} = \frac{e_j}{2m_j c} \left\{ (\vec{P}_j + \frac{e_j}{c}\vec{A}_0)\cdot\vec{A}_1(x) + \vec{A}_1(x)\cdot(\vec{P}_j + \frac{e_j}{c}\vec{A}_0) \right\} + e_j\phi_1(x) \tag{26.b}$$

$$H_j^{(2)} = \frac{e_j^2}{2m_j c^2}\vec{A}_1^2(x), \tag{26.c}$$

and $H_j^{(0)}$ is the unperturbed Hamiltonian. The first and second terms in equation (26.b) are responsible for the linear interaction of particles of species j with the electromagnetic waves and the third term is responsible for the linear interaction of particles with longitudinal fields. $H_j^{(2)}$, equation (26.c), will be responsible for the nonlinear wave-particle interaction (electromagnetic waves).

The vector and scalar potentials $\vec{A}_1(x)$ and $\phi_1(x)$ can be expanded in Fourier series in a large box of volume V, assuming the usual periodic boundary conditions (Harris, 1969)

$$\vec{A}(x) = \sum_{\vec{q}} \left[\frac{2\pi\hbar c^2}{V\omega_T(\vec{q})F_T(\vec{q})}\right]^{1/2} \vec{e}_q \Big\{ A_q e^{i[\vec{q}\cdot\vec{x}-\omega_T(\vec{q})t]}$$

$$+ A_q^+ e^{-i[\vec{q}\cdot\vec{x}-\omega_T(\vec{q})t]} \Big\} \tag{27}$$

$$\phi_1(x) = \sum_{\vec{q}} \left[\frac{4\pi\hbar\omega_L(\vec{q})}{Vq^2F_L(q)}\right]^{1/2} \Big\{ a_q e^{i[\vec{q}\cdot\vec{x}-\omega_L(\vec{q})t]}$$

$$+ a_q^+ e^{-i[\vec{q}\cdot\vec{x}-\omega_L(\vec{q})t]} \Big\} \tag{28}$$

The reason for the factors in square brackets is discussed later. The functions $F_T(\vec{q})$ and $F_L(\vec{q})$ are given by

$$F_T(\vec{q}) = \left|\frac{1}{2\omega}\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon_T(\omega,\vec{q})\right)\right|_{\omega=\omega_T(\vec{q})} , \tag{29}$$

$$F_L(\vec{q}) = \left|\frac{\partial}{\partial\omega}\left(\omega\varepsilon_L(\omega,\vec{q})\right)\right|_{\omega=\omega_L(\vec{q})} , \tag{30}$$

$\omega_L(\vec{q})$, $\omega_T(\vec{q})$ are the longitudinal and transverse frequencies respectively; $\varepsilon_L(\vec{q},\omega)$, $\varepsilon_T(\vec{q},\omega)$ are called the longitudinal and transverse dielectric functions and a_q, A_q are Fourier coefficients; a_q^+ and A_q^+ are their complex conjugates. Finally, $\vec{e}_q$ is a polarization vector.

The transition from classical to quantum mechanics (quantisation of electromagnetic fields) is made by reinterpreting a_q and A_q as annihilation operators for longitudinal and transverse fields (bosons) of momentum $\hbar\vec{q}$ and energy $\hbar\omega_q$; a_q^+ and A_q^+ are the corresponding creation operators. These operators satisfy the commutation relations (9).

The energy in the electromagnetic field (transverse waves) is given

$$H_T = \frac{1}{8\pi} \int d^3x \left\{ |E_T|^2 + |B|^2 \right\} \tag{31}$$

This is not the total energy associated with the wave; in so far as the particles of the medium move in response to the wave, their energy must be properly included in the total energy. Landau and Lifschitz (1960) have shown that in such a dielectric medium the total energy is

$$\begin{aligned} U_T &= \frac{1}{8\pi} \int d^3x \left\{ |E_T|^2 \left| \frac{\partial}{\partial\omega} (\omega\epsilon_T(\omega)) \right|_{\omega=\omega_T} + |B|^2 \right\} \\ &= \frac{1}{4\pi c^2} \int d^3x \left| \frac{\partial\vec{A}}{\partial t} \right|^2 \left| \frac{1}{2\omega} \frac{\partial}{\partial\omega} (\omega^2\epsilon_T(\omega)) \right|_{\omega=\omega_T} \end{aligned} \tag{32}$$

This shows that the energy density which a wave would have in a vacuum must be corrected by the factor $\left|\frac{1}{2\omega} \frac{\partial}{\partial\omega}(\omega^2\epsilon_T(\omega))\right|_{\omega=\omega_T}$ $= F_T(\vec{q})$ when it moves in a medium of a dielectric constant $\epsilon_T(\omega)$. Substituting equation (27) into (32) and performing the integration, one obtains

$$U_T = \sum_{\vec{q}} \hbar\omega_T(\vec{q}) A_q^+ A_q \tag{33}$$

Similarly

$$U_L = \frac{1}{8\pi} \int d^3x \left|E_L\right|^2 \left|\frac{\partial}{\partial\omega}(\omega\varepsilon_L(\omega))\right|_{\omega=\omega_L}$$

$$= \frac{1}{8\pi} \int d^3x \left|\nabla\phi\right|^2 \left|\frac{\partial}{\partial\omega}(\omega\varepsilon_L(\omega))\right|_{\omega=\omega_L} \tag{34}$$

is the total energy in the electric field of longitudinal waves. Substituting (28) into (34) and performing the integration yields

$$U_L = \sum_{\vec{q}} \hbar\omega_L(q) a^{+}_{\vec{q}} a_{\vec{q}} \tag{35}$$

It is obvious that the terms in the square brackets of equations (27) and (28) provide the correct form for equations (33) and (35).

2.3 Particle-Quasiparticle Interaction

According to the theory developed in sec. 1, the total interaction Hamiltonian between particles of species j and longitudinal quasiparticles can be written in the second quantisation formalism as

$$H_{int} = \int d^3x \Psi^{+}(x) e_j \phi_l(x) \Psi(x) \tag{36}$$

Here $\Psi(x)$ and $\Psi^{+}(x)$ are the field operators which satisfy the anticommutation relations (22) and $\phi_l(x)$ is the electrostatic potential given by (28). The field operators Ψ's can be expanded in the annihilation and creation operators $C_{\vec{k}}$

and $C^{+}_{\vec{k}}$ which satisfy (21)

$$\Psi(x) = \sum_{\vec{k}} C_{\vec{k}} \xi_{\vec{k}}(\vec{x}) \tag{37}$$

$$\Psi^{+}(x) = \sum_{\vec{k}} C^{+}_{\vec{k}} \xi^{+}_{\vec{k}}(\vec{x}) \tag{38}$$

where $\xi_{\vec{k}}(\vec{x})$ is the solution of the Schrõdinger equation

$$(H_0 - E_{\vec{k}})\xi_{\vec{k}}(\vec{x}) = 0 \tag{39}$$

and H_0 is the unperturbed Hamiltonian. In unmagnetized plasma ($\vec{A}_0 = 0$) the unperturbed Hamiltonian is given by $H_0 = -\frac{\hbar^2}{2m_j}\nabla^2$ and (39) yields

$$\xi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \tag{40}$$

Substituting (28), (37), (38) and (40) into (36) and carrying out the integration, one obtains

$$H_{int} = \sum_{\vec{k},\vec{q}} M_0(q) C^{+}_{\vec{k}+\vec{q}} C_{\vec{k}} a_{\vec{q}} + M^{*}_0(q) C_{\vec{k}+\vec{q}} C^{+}_{\vec{k}} a^{+}_{\vec{q}} \tag{41}$$

where

$$M_0(q) = M^{*}_0(q) = e_j \left[\frac{4\pi\hbar\omega_L(q)}{Vq^2F_L(q)}\right]^{1/2} \tag{42}$$

The terms in the interaction Hamiltonian (41) can be represented by Feynman diagrams as follows: the first term which contains $c^{+}_{\vec{k}+\vec{q}}c_{\vec{k}}a_{\vec{q}}$ describes the process in which a particle in the state $\vec{k}$ is destroyed, one is created in the state $\vec{k}+\vec{q}$ and a quasiparticle of momentum $\hbar\vec{q}$ is destroyed. The term containing $c_{\vec{k}+\vec{q}}c^{+}_{\vec{k}}a^{+}_{\vec{q}}$ describes the inverse process. These processes are represented by the diagrams in Fig. 1 and yield the so called 'linear Landau damping'.

Fig. 1 Linear Landau damping

The interaction Hamiltonian for a magnetized plasma undergoing electromagnetic wave-wave and wave-particle interactions and its representation in Feynman diagrams will be developed later in chapter V.

2.4 Scattering of Particles

Consider a test particle of charge e_j and mass m_j interacting with a plasma, the interaction Hamiltonian is given by

$$H_{int}(\vec{x}) = \sum_s \sum_i \frac{e_j e_s}{|\vec{x}-\vec{x}_{si}|} = V(\vec{x}) \tag{43}$$

where $V(\vec{x})$ is the electrostatic potential energy at the position $\vec{x}$ of the test particle, and $\vec{x}_{si}$ is the position of the ith particle of species s in the plasma. Only Coulomb interactions are taken into account. The potential $V(\vec{x})$ can be expanded in a Fourier series in a box of volume V as follows

$$V(\vec{x}) = \sum_q V(\vec{q}) e^{i\vec{q}\cdot\vec{x}} \tag{44}$$

where $V(\vec{q})$ is given by

$$V(\vec{q}) = \int \frac{d^3x}{V} e^{-i\vec{q}\cdot\vec{x}} V(\vec{x})$$

$$= \sum_s \sum_i \frac{4\pi e_j e_s}{Vq^2} e^{-i\vec{q}\cdot\vec{x}_{si}} \tag{45}$$

Thus, substituting (44) and (45) into (43) yields

$$H_{int}(\vec{x}) = \sum_{\vec{q}} \sum_s \sum_i \frac{4\pi e_j e_s}{Vq^2} e^{i\vec{q}\cdot(\vec{x}-\vec{x}_{si})} \tag{46}$$

The total interaction Hamiltonian in the second quantization formalism is given by

$$H_{int} = \int d^3x d^3x' \Psi^+(x)\Psi^+(x') H_{int}(\vec{x}) \Psi(x)\Psi(x') \tag{47}$$

Substituting the field operators (37), (38) and the interaction Hamiltonian (46) into (47) and performing the integrations, one obtains

$$H_{int} = \sum_{\vec{k},\vec{k}',\vec{q}} V_q C^+_{\vec{k}} C^+_{\vec{k}'} C_{\vec{k}'-\vec{q}} C_{\vec{k}+\vec{q}} \tag{48}$$

where the coupling constant V_q is given by

$$V_q = \frac{4\pi e_j e_s}{Vq^2} \tag{49}$$

The interaction Hamiltonian (48), can be represented by Feynman graphs as follows: a particle in the state $\vec{k}+\vec{q}$ is scattered into a state $\vec{k}$ by a particle initially in the state $\vec{k}'-\vec{q}$ which is scattered into a state $\vec{k}'$. Momentum is conserved in the process. This is shown in Fig. 2(a).

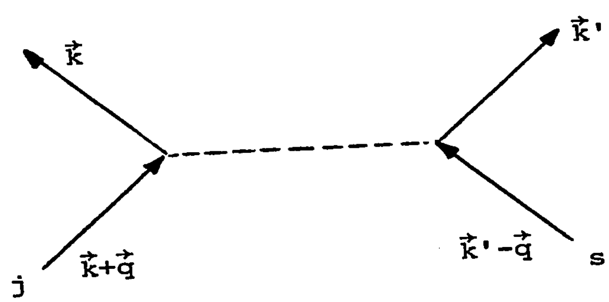
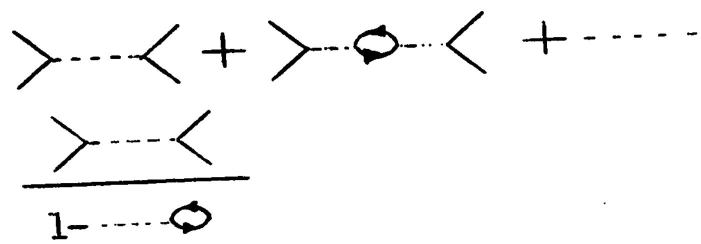


Fig. 2(a) Coulomb interaction

Fig. 2(a) is simply a scattering of a particle of species j with a particle of species s. In a many body problem, this process represents only the first term in an infinite series which represent the total scattering that takes place. If the sum of the infinite series is represented by V_{eff} where the first term is V_q, one obtains in the random phase approximation

V_{eff} =



$$= \frac{4\pi e_s e_j}{Vq^2 \varepsilon(\vec{q},\omega)} \qquad (50)$$

where

$\varepsilon(\vec{q},\omega) = 1$  (51)

is obviously the dielectric function of the plasma medium. The effective field (50) is represented diagrammatically in Fig. 2(b).

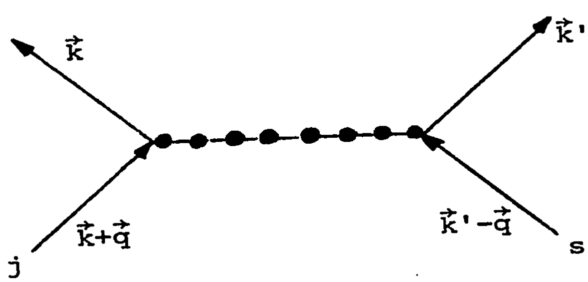


Fig. 2(b) Effective Coulomb interaction

2.5 The Scattering Matrix

The theory of the scattering matrix (S-matrix) can be developed by introducing the interaction picture (I.P.) in which the state vector $\Psi(t)$ is given by

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H_{int}\Psi(t) \tag{52}$$

Thus, in the I.P. the state vectors have the time-dependence of Schrödinger picture with the interaction Hamiltonian H_{int} instead of the total Hamiltonian. It is useful to obtain a solution of this equation particularly suited to describing scattering processes, with the initial state of the system at $t = -\infty$

$$\Psi(-\infty) = \Psi_i = |i\rangle \tag{53}$$

This state vector will completely specify the particles present initially (i.e. long before scattering occurs when all particles are still far apart). Equation (52) then tells how the state vector (53) changes with time. In particular, it gives the final state $\Psi(+\infty) = \Psi_f = |f\rangle$ at time $t = \infty$, long after scattering is over and all particles are far apart again. The S-matrix is defined as the operator which transforms $\Psi(-\infty)$ into $\Psi(+\infty)$:

$$\Psi(+\infty) = S\Psi(-\infty) \tag{54}$$

so that finding the S-matrix is equivalent to solving equation (52) which can be transformed into an integral equation as

$$\Psi(t) = \Psi(-\infty) + \frac{(-i)}{\hbar} \int_{-\infty}^{t} dt_1 H_{int}(t_1)\Psi(t_1) \qquad (55)$$

Solving this equation by iteration, one obtains the S-matrix

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!\hbar^n} \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dt_1 \ldots dt_n P\{H_I(t_1)H_I(t_2)\ldots H_I(t_n)\} \qquad (56)$$

Here the Dyson chronological product P{...} of n factors means that the factors are not to be taken in the order in which they are written in the P-bracket; but, operators with later times stand to the left of operators with earlier times. The equivalence of equation (56) holds for each term n separately and is easily verified.

The S-matrix expansion (56) will effect many complicated transitions. However, only certain terms will contribute to a given transition $|i\rangle \to |f\rangle$ as they must contain just the right absorption operators to destroy the particles present in $|i\rangle$ and that they must contain just the right creation operators to emit the particles present in $|f\rangle$.

It is convenient to introduce what is called the normal product (N-product) and the Wick's chronological product (T-product). An operator which is a product of creation and annihilation operators is called a normal product if all creation operators stand to the left of all annihilation operators. For example

$$N\left[\psi^+_1\psi_1\psi^+_2\psi_2\right] = -\psi^+_1\psi^+_2\psi_1\psi_2 = \psi^+_1\psi^+_2\psi_2\psi_1 \qquad (57)$$

The minus sign resulting from the interchange $\psi_1 \leftrightarrow \psi^+_2$. Such
an operator first absorbs a certain number of particles and
then emits some particles. It does not cause emission and
reabsorption of intermediate particles. Thus, one writes
each term in the S-matrix expansion (56) as a sum of normal
products. Each of these will effect a particular transition
$|i\rangle \rightarrow |f\rangle$, which can be represented by Feynman graphs.

The T-product is defined as follows:

(i) For two boson fields, the T-product is equal to the P-product defined by

$$P\{\phi(x_1)\phi(x_2)\} = \begin{cases} \phi(x_1)\phi(x_2), & \text{if } t_1>t_2 \\ \phi(x_2)\phi(x_1), & \text{if } t_2>t_1 \end{cases} \qquad (58)$$

(ii) If $A(x_1)$ is one of the operators $\Psi(x_1)$ or $\Psi^+(x_1)$ and
$B(x_2)$ is one of the operators $\Psi(x_2)$ or $\Psi^+(x_2)$ then the P-product is defined as (58) but the T-product is defined by

$$T\{A(x_1)B(x_2)\} = \begin{cases} A(x_1)B(x_2), & \text{if } t_1>t_2 \\ -B(x_2)A(x_1), & \text{if } t_2>t_1 \end{cases} \qquad (59)$$

Because of the anticommuting nature of fermion operators,
the T-product, which allows for the change of sign when

interchanging two fermion fields, is more appropriate. Thus, equation (59) can be written in general as

$$T\{A(x_1)B(x_2)\ldots\} = (-1)^P P\{A(x_1)B(x_2)\ldots\} \tag{60}$$

where the exponent P is the number of interchanges of pairs of fermion fields required to change from $\{A(x_1)B(x_2)\ldots\}$ to the chronological order.

For theories in which the interaction Hamiltonian is bilinear in the fermion field, one can simply replace the P-product in the S-matrix expansion (56) by T-products, since only even numbers of interchanges of pairs of fermion fields are involved. Thus, equation (56) becomes

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!\hbar^n} \int \ldots \int dt_1 \ldots dt_n T\left\{H_I(t_1)H_I(t_2)\ldots H_I(t_n)\right\} \tag{61}$$

From equation (57), it is obvious that the N-product of any set of operators has zero expectation value in the vacuum state. For this reason, one can introduce the contraction or chronological pairing of two operators as follows:

$$\overline{A(x_1)B}(x_2) = T\left\{A(x_1)B(x_2)\right\} - N\left[A(x_1)B(x_2)\right] \tag{62}$$

Then the expectation value of the T-product in the vacuum state will be

$$\overline{A(x_1)B}(x_2) = \langle 0|T\left\{A(x_1)B(x_2)\right\}|0\rangle \qquad (63)$$

This is just a function that picks out cases of non-vanishing commutators or anticommutators between the operators in the product.

All one needs now is a general procedure for reducing any more complicated T-products to N-products and pairings. This comes from Wick's theorem which states that "any chronological product is equal to the sum of all possible N-products that can be formed with all possible pairings." This can be seen from the following example

$$\begin{aligned} T\{ABCD\} = {} & N[ABCD] + N[A^{\bullet}B^{\bullet}CD] + N[A^{\bullet}BC^{\bullet}D] \\ & + N[A^{\bullet}BCD^{\bullet}] + N[AB^{\bullet}C^{\bullet}D] + N[AB^{\bullet}CD^{\bullet}] \\ & + N[ABC^{\bullet}D^{\bullet}] + N[A^{\bullet}B^{\bullet}C^{\circ}D^{\circ}] + N[A^{\bullet}B^{\circ}C^{\bullet}D^{\circ}] \\ & + N[A^{\bullet}B^{\circ}C^{\circ}D^{\bullet}] \end{aligned} \qquad (64)$$

The generalization of this equation has been proven by Wick (1950). Thus, one can write the mixed T-products occuring in the S-matrix (61) as a sum of N-products. Each of these products corresponds to a definite process characterized by the operators which are not contracted. In the next section the contractions, as interpreted in terms of virtual particles in intermediate states, will be discussed.

2.6 Feynman Graphs

The expansion of the S-matrix into normal products is quite complex, even for terms of low order in n. The interpretations of these various terms are greatly facilitated by the use of Feynman graphs, introduced in the previous sections (figs. 1 and 2). A qualitative discussion of the basic ideas of Feynman graphs for the second order S-matrix is given in this section. For n=2, equation (61) yields

$$S^{(2)} = \frac{(-i)^2}{2!\hbar^2} \iint dt_1 dt_2 T\left\{H_{int}(t_1) H_{int}(t_2)\right\}$$

$$= \frac{(-i)^2}{2!\hbar^2} \iint d^4x_1 d^4x_2 T\left\{H_{int}(x_1) H_{int}(x_2)\right\} \qquad (65)$$

Here $x \equiv (\vec{x},t)$, $d^4x = dtd^3x$ and $H_{int}(x)$ is the Hamiltonian density. Substituting $H_{int}(x) = \psi^+(x) e_j \phi(x) \psi(x)$ into (65) one obtains

$$S^{(2)} = \frac{(-i)^2 e^2}{2!\hbar^2} \iint d^4x_1 d^4x_2 T\left\{\psi^+(x_1)\phi(x_1)\psi(x_1)\psi^+(x_2)\phi(x_2)\psi(x_2)\right\} \qquad (66)$$

Here $e\phi$ is the electrostatic potential energy, ψ's are the field operators and the chronological T-product is given by

$$T\left\{\psi_1^+\phi_1\psi_1\psi_2^+\phi_2\psi_2\right\} = N\left[\psi_1\phi_1\psi_1\psi_2\phi_2\psi_2\right] + N\left[\overbrace{\psi_1^+\phi_1\psi_1\psi_2^+\phi_2\psi_2}\right]$$

$$+ N\left[\psi_1^+\phi_1\overbrace{\psi_1\psi_2^+}\phi_2\psi_2\right] + N\left[\psi_1^+\overbrace{\phi_1\psi_1\psi_2^+\phi_2}\psi_2\right] + N\left[\overbrace{\psi_1^+\overbrace{\phi_1\psi_1\psi_2^+\phi_2}\psi_2}\right]$$

$$+ N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] + N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] + N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] \tag{67}$$

where $\Psi_i \equiv \Psi(x_i)$ and $\phi_i \equiv \phi(x_i)$. Equation (66) can be written as follows

$$S^{(2)} = S_a^{(2)} + S_b^{(2)} + S_c^{(2)} + S_d^{(2)} + S_e^{(2)} + S_f^{(2)} \tag{68}$$

where

$$S_a^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] \tag{68.a}$$

$$S_b^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 \Big\{N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] + N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right]\Big\} \tag{68.b}$$

$$S_c^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] \tag{68.c}$$

$$S_d^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 \Big\{N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] + N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right]\Big\} \tag{68.d}$$

$$S_e^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] \tag{68.e}$$

$$S_f^{(2)} = \frac{(-i)^2e^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 N\left[\Psi_1^+\phi_1\Psi_1\Psi_2^+\phi_2\Psi_2\right] \tag{68.f}$$

and contractions over equal times operators have been omitted as well as contractions which obviously vanished such as

$$\overline{\Psi_1\Psi_2} = \overline{\Psi^+_1\Psi^+_2} = \overline{\Psi_1\phi_2} = \overline{\Psi^+_1\phi_2} = \overline{\Psi_2\phi_1} = \overline{\Psi^+_2\phi_1} = 0 \qquad (69)$$

As shown in Fig. 1 the interaction Hamiltonian H_{int} is represented by a graph consisting of a vertex with two fermion lines and one boson line. The second order terms $S^{(2)}$ will be described by graphs involving two vertices, suitably joined together. Those transitions which a particular term $S_i^{(2)}$ can effect, depend on the external fields (that is, those which are not contracted) in this operator, for it is those external fields which are responsible for absorbing initial and creating final particles.

The term $S_a^{(2)}$, equation (68.a) involves only external lines corresponding to two processes of the types in fig. 1 operating independently of each other and has no physical meaning.

The operators $S_b^{(2)}$ describes Compton scattering as well as other processes. Compton scattering is represented by Feynman graphs as shown in fig. 3(a) and (b). The contraction $\overline{\Psi^+_1\Psi_2}$ is described by the line from the vertex t_1 to t_2. For $t_1<t_2$ it describes the emission of a virtual fermion at t_1, its propagation to t_2 and its absorption there. Thus, $\overline{\Psi^+_1\Psi_2}$ is called a fermion propagator, and it describes a virtual intermediate state. The other two processes describe the creation and annihilation of two bosons. This is known as nonlinear Landau damping (growth) and respresented dia-

grammatically by fig. 4(a) and (b). The factor 2! in the denominator of equation (68.b) will be cancelled due to

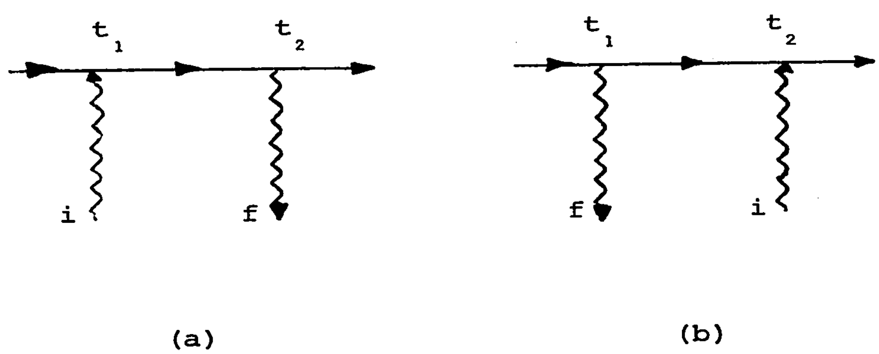


Fig. 3 Compton scattering

the equality of the two terms. In equation (68.c) there are four external fermion lines. The bosons occur only in the

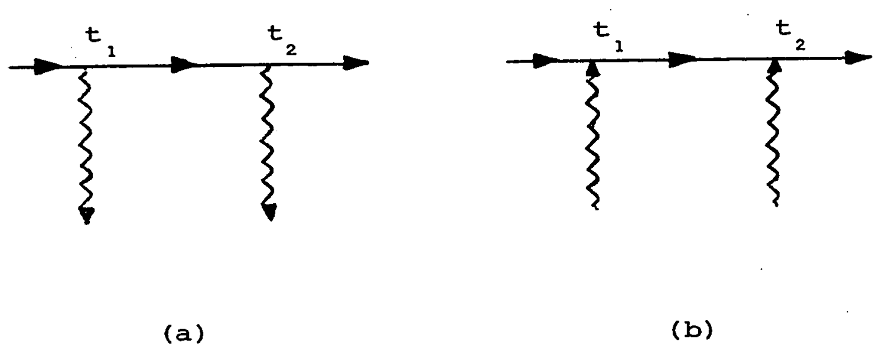


Fig. 4 Nonlinear Landau damping (growth)

contraction $\overbrace{\phi_1 \phi_2}$ which is interpreted as a boson propagator. The Feynman graph for this process is shown in fig. 5 and is known as particle-particle scattering.

Equation (68.d) is represented diagrammatically in fig. 6. This leads to the self energy of the fermion. Sim-

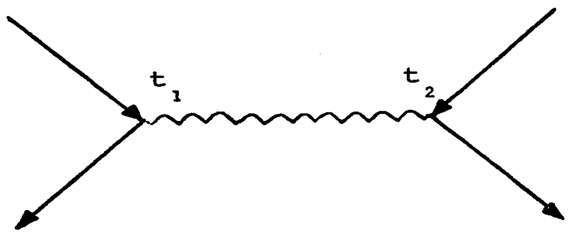


Fig. 5 Particle-particle scattering

ilar to (68.b), the two terms are equal by changing $x_1 \rightleftarrows x_2$ in the second term.

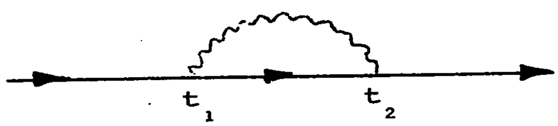


Fig. 6 Fermion self-energy

Equation (68.e) represented in fig. 7, similarly describes a boson self energy, that is a modification of

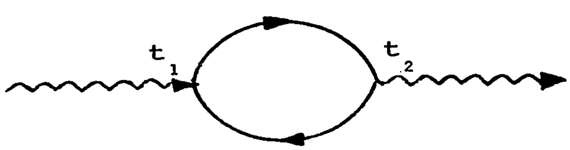


Fig. 7 Boson self-energy

boson energy due to its interaction with the fermion field. This interaction enables the boson to create a virtual fermion pair which subsequently absorbed again.

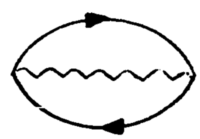


Fig. 8

Finally, fig. 8 shows the graph representing equation (68.f). This has no external lines at all and consequently does not cause any transitions.

This completes the analysis of the second order term $S^{(2)}$ into normal products and the interpretation in terms of Feynman graphs. The extension to a decomposition into higher order contributions although, of course, more complicated, presents no essential difficulty and is described in the rest of this section.

The Feynman graphs for three and four wave interactions can be obtained from the third and fourth order terms of the S-matrix. For n = 3, equation (61) gives

$$S^{(3)} = \frac{(-i)^3}{3!\hbar^3} \iiint d^4x_1 d^4x_2 d^4x_3 T\left\{H_{int}(x_1)H_{int}(x_2)H_{int}(x_3)\right\} \quad (70)$$

where

$$T\left\{H_{int}(x_1)H_{int}(x_2)H_{int}(x_3)\right\} = e_j^3 T\left\{(\Psi_1^+\phi_1\Psi_1)(\Psi_2^+\phi_2\Psi_2)(\Psi_3^+\phi_3\Psi_3)\right\}$$

(71)

and the terms which yield the three wave interactions are

$$T\{\ldots\ldots\}\Rightarrow 6e_j^3\left\{N\left[\psi_1^+\phi_1\psi_1\psi_2^+\phi_2\psi_2\psi_3^+\phi_3\psi_3\right]+N\left[\psi_1^+\phi_1\psi_1\psi_2^+\phi_2\psi_2\psi_3^+\phi_3\psi_3\right]\right\} \tag{72}$$

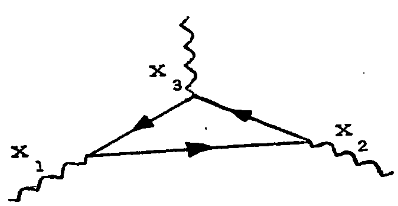
where the factor 6 is due to the summation over all possible fully contracted products. Equation (72) can be written in the following form after taking the N-product of the contractions

$$T\{\ldots\ldots\}\Rightarrow 6e_j^3\left[\psi_1^+\psi_2\,\psi_2^+\psi_3\,\psi_3^+\psi_1 + \psi_1^+\psi_3\,\psi_3^+\psi_2\,\psi_2^+\psi_1\right]N\left\{\phi_1\phi_2\phi_3\right\} \tag{72.a}$$

Substituting (72.a) into (70) yields

$$S^{(3)}=\frac{(-i)^3e_j^3}{\hbar^3}\iint d^4x_1d^4x_2\left[\psi_1^+\psi_2\,\psi_2^+\psi_3\,\psi_3^+\psi_1+\psi_1^+\psi_3\,\psi_3^+\psi_2\,\psi_2^+\psi_1\right]N\left\{\phi_1\phi_2\phi_3\right\} \tag{73}$$

In equation (73) there are three internal fermion lines (three propagator) and three external bosons. The Feynman graphs for these three wave interactions are shown in fig. 9.



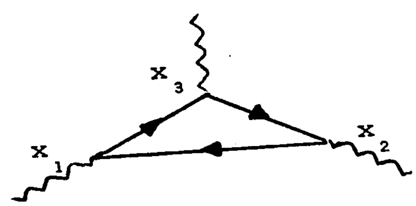


Fig. 9 Three wave interactions

Similarly, for n = 4, equation (61) yields

$$S^{(4)} = \frac{(-i)^4}{4!\hbar^4} \int \ldots \int d^4x \ldots d^4x \; T\left\{H_I(x_1)H_I(x_2)H_I(x_3)H_I(x_4)\right\} \quad (74)$$

where the terms in the T-product which are responsible for the four wave interactions are

$$\begin{aligned} T\{\ldots\ldots\} \Rightarrow (-1)\,24e_j^4 \Big[& \overline{\psi^+_1\psi_2}\;\overline{\psi^+_2\psi_3}\;\overline{\psi^+_3\psi_4}\;\overline{\psi^+_4\psi_1} + \overline{\psi^+_1\psi_3}\;\overline{\psi^+_3\psi_4}\;\overline{\psi^+_4\psi_2}\;\overline{\psi^+_2\psi_1} \\ & + \overline{\psi^+_1\psi_4}\;\overline{\psi^+_4\psi_3}\;\overline{\psi^+_3\psi_2}\;\overline{\psi^+_2\psi_1} + \overline{\psi^+_1\psi_2}\;\overline{\psi^+_2\psi_4}\;\overline{\psi^+_4\psi_3}\;\overline{\psi^+_3\psi_1} + \overline{\psi^+_1\psi_3}\;\overline{\psi^+_3\psi_2}\;\overline{\psi^+_2\psi_4}\;\overline{\psi^+_4\psi_1} \\ & + \overline{\psi^+_1\psi_4}\;\overline{\psi^+_4\psi_2}\;\overline{\psi^+_2\psi_3}\;\overline{\psi^+_3\psi_1} \Big] N\Big[\phi_1\phi_2\phi_3\phi_4\Big] \end{aligned} \quad (75)$$

In this process, there are four fermion propagators and four external bosons. As shown in equation (75) there are six possible nonequivalent diagrams. Each of these has 4! equivalent diagrams. Equation (75) is represented diagrammatically in fig. 10. The choice of the direction of the bosons (figs. 9 and 10) depends on the problem under investigation.

2.7 General Rules For the Calculations of the Scattering Matrix Element

The scattering matrix element for the three wave interactions will be calculated as an example from which the general rules for more complicated diagrams are established. The scattering matrix for this process is given by (73) and the fermion Propagator $\overline{\psi^+_1\psi_2}$ is calculated as follows:

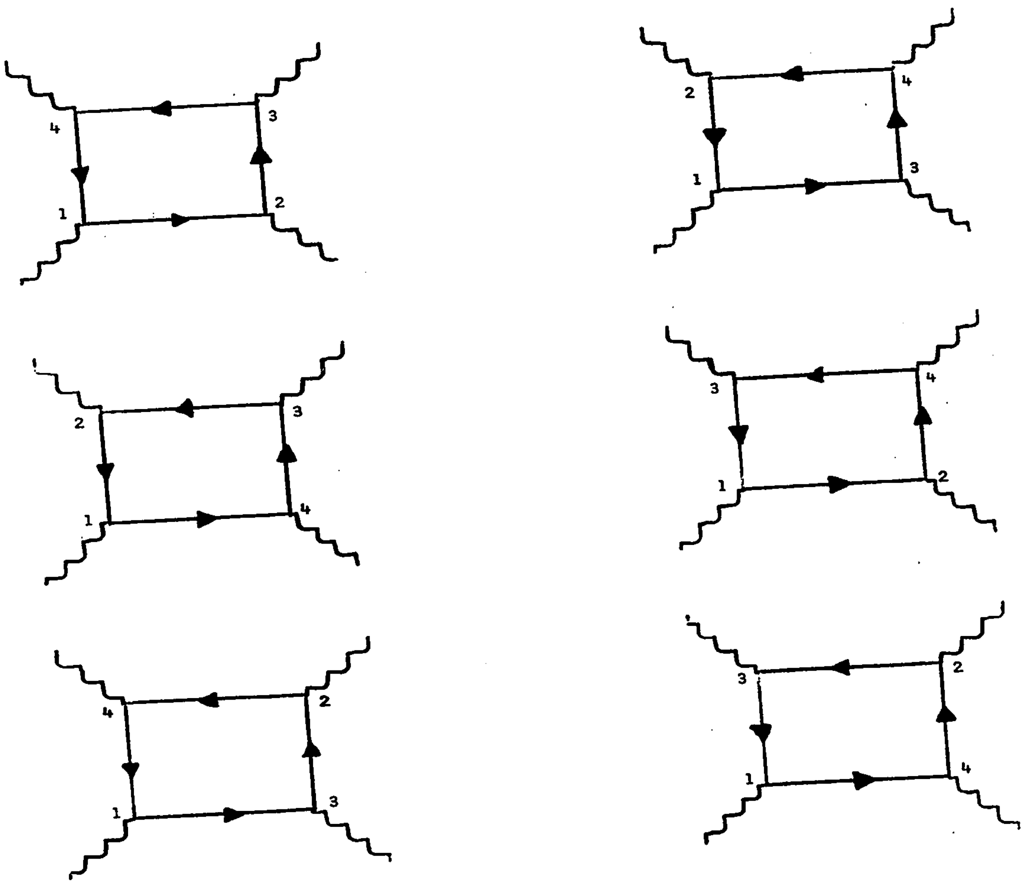


Fig. 10 Four wave interaction

$$\overline{\Psi_1^+\Psi_2} = \langle 0|T\{\Psi_1^+\Psi_2\}|0\rangle = 0 \quad \text{if } t_1>t_2$$

$$= -\langle 0|\Psi_2\Psi_1^+|0\rangle \quad \text{if } t_1<t_2$$

$$= -\frac{1}{V}\sum_{\vec{k}} e^{-iE_{\vec{k}}T} e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} \tag{76}$$

where the field operator $\Psi(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} C_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}-E_{\vec{k}}t)}$ has been used and $T = t_2-t_1$.

At this point, one needs an analytical expression for a function with the following properties (Ziman, 1969)

$$g(E_{\vec{k}}) = \begin{cases} -e^{-iE_{\vec{k}}T} & T>0 \\ 0 & T<0 \end{cases} \tag{77}$$

This can be constructed by a contour integration

$$g(E_{\vec{k}}) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dEe^{-iET}}{E-E_{\vec{k}}+i\delta} \tag{78}$$

where the integral is along the real axis in the plane of the complex variable E, and the small positive quantity δ is simply a convergence factor. When (78) is substituted into (76), it yields the following formula for the fermion propagator

$$\overline{\psi_1^+\psi_2} = -\frac{1}{2\pi V}\sum_{\vec{k}}\int_{-\infty}^{+\infty} dE\, iG(E,E_{\vec{k}})\, e^{-i(\vec{k}\cdot\vec{x}_1 - Et_1)}\, e^{i(\vec{k}\cdot\vec{x}_2 - Et_2)} \qquad (79)$$

and

$$G(E,E_{\vec{k}}) = \frac{1}{E-E_{\vec{k}}+i\delta} \qquad (79.a)$$

Similarly, one can write expressions for $\overline{\psi_2^+\psi_3}$ and $\overline{\psi_2^+\psi_1}$.

For the specific example under investigation, one is interested only in the contribution of (73) for the specified initial and final state configuration. This is the annihilation of two bosons at x_1, x_2 and the creation of another at x_3 (fig. 9). Hence, from the factor $\phi_1\phi_2\phi_3$ one chooses the term that has this property. Since ϕ_1^- and ϕ_2^- annihilate bosons and ϕ_3^+ creates another, one retains only $\phi_1^-\phi_2^-\phi_3^+$ where $\phi_1 = \phi_1^- + \phi_1^+$, etc. Therefore, the N-product in (73) yields

$$e_j^3 N\left[\phi_1^-\phi_2^-\phi_3^+\right] = \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3} M_0(q_1)M_0(q_2)M_0(q_3)a^+_{\vec{q}_3}a_{\vec{q}_1}a_{\vec{q}_2} \times$$

$$e^{-i\left[(\vec{q}_3\cdot\vec{x}_3-\omega_{\vec{q}_3}t_3)-(\vec{q}_1\cdot\vec{x}_1-\omega_{\vec{q}_1}t_1)-(\vec{q}_2,\vec{x}_2-\omega_{\vec{q}_2}t_2)\right]} \qquad (80)$$

where $M_0(\vec{q})$ is given by (42).

Substituting the similar expressions for $\overline{\psi_n^+\psi_m}$ (n,m = 1,2,3 and n ≠ m), from (79) and (80) into (73), one obtains

$$S^{(3)} = \frac{(-i)^3}{\hbar^3}\sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3}(-1)M_0(q_1)M_0(q_2)M_0(q_3)a^+_{\vec{q}_3}a_{\vec{q}_1}a_{\vec{q}_2} \times$$

$$\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} \int dE_1 dE_2 dE_3\, iG(E_1,E_{\vec{k}_1})\, iG(E_2,E_{\vec{k}_2})\, iG(E_3,E_{\vec{k}_3})$$

$$\left[\frac{(2\pi)^3}{V}\right]^3 \Big[\delta(\vec{k}_1-\vec{k}_3-\vec{q}_1)\,\delta(\vec{k}_2-\vec{k}_1-\vec{q}_2)\,\delta(\vec{k}_3-\vec{k}_2+\vec{q}_3)\ \times$$

$$\delta(E_3-E_1+\omega_{\vec{q}_1})\,\delta(E_1-E_2+\omega_{\vec{q}_2})\,\delta(E_2-E_3-\omega_{\vec{q}_3})$$

$$+\ \delta(\vec{k}_2-\vec{k}_1-\vec{q}_1)\,\delta(\vec{k}_1-\vec{k}_3-\vec{q}_2)\,\delta(\vec{k}_3-\vec{k}_2+\vec{q}_3)\ \times$$

$$\delta(E_1-E_2+\omega_{\vec{q}_1})\,\delta(E_3-E_1+\omega_{\vec{q}_2})\,\delta(E_2-E_3-\omega_{\vec{q}_3})\Big] \qquad (81)$$

The δ-functions imply the conservation of momentum and energy parameters at each vertex. Alternatively, (81) gives after performing the $\vec{k}_2, \vec{k}_3, E_2$ and E_3 integrations,

$$S^{(3)} = \frac{(-i)^3}{\hbar^3} \sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3} (-1) M_0(q_1) M_0(q_2) M_0(q_3)\, a^+_{\vec{q}_3} a_{\vec{q}_1} a_{\vec{q}_2}\ \times$$

$$\sum_{\vec{k}} \int dE \Big[iG(E,E_{\vec{k}})\, iG(E+\omega_2, E_{\vec{k}+\vec{q}_2})\, iG(E-\omega_1, E_{\vec{k}-\vec{q}_1})$$

$$+\ iG(E,E_k)\, iG(E+\omega_{\vec{q}_1}, E_{\vec{k}+\vec{q}_1})\, iG(E-\omega_{\vec{q}_2}, E_{\vec{k}-\vec{q}_2})\Big]\ \times$$

$$\frac{(2\pi)^3}{V}\, \delta(\vec{q}_1+\vec{q}_2-\vec{q}_3)\, \delta(\omega_{\vec{q}_1}+\omega_{\vec{q}_2}-\omega_{\vec{q}_3}), \qquad (82)$$

where again the δ-functions ensure the conservation of momentum and energy of the initial and final states.

The scattering matrix element $S_{fi}^{(3)}$ is given by

$$S_{fi}^{(3)} = \langle f|S^{(3)}|i\rangle$$

$$= \frac{(-i)^3}{\hbar^3}(-1)M_0(q_1)M_0(q_2)M_0(q_3) \times$$

$$\sum_{\vec{k}} \int dE \Big[iG(E,E_{\vec{k}})\, iG(E+\omega_{\vec{q}_2}, E_{\vec{k}+\vec{q}_2})\, iG(E-\omega_{\vec{q}_1}, E_{\vec{k}-\vec{q}_1})$$

$$+ iG(E,E_{\vec{k}})\, iG(E+\omega_{\vec{q}_1}, E_{\vec{k}+\vec{q}_1})\, iG(E-\omega_{\vec{q}_2}, E_{\vec{k}-\vec{q}_2}) \Big] \delta(E_i - E_f) \qquad (83)$$

where

$$\frac{(2\pi)^3}{V}\delta(\vec{q}_1+\vec{q}_2-\vec{q}_3) = \delta_{\vec{q}_1+\vec{q}_2,\vec{q}_3} \qquad (84)$$

and

$$\sum_{\vec{q}_1,\vec{q}_2,\vec{q}_3} \langle f|a^+_{\vec{q}_3}\, a_{\vec{q}_2}\, a_{\vec{q}_1}|i\rangle = \delta_{\vec{q}_1,\vec{q}'}\,\delta_{\vec{q}_2,\vec{q}''}\,\delta_{\vec{q}_3,\vec{q}'''} \qquad (85)$$

have been used.

From the preceding calculations, one can easily establish general rules to obtain the scattering matrix element (83) for the three wave interactions.

1. Construct all possible nonequivalent diagrams (fig. 11) with initial $\vec{q}_1, \vec{q}_2$ and final $\vec{q}_3$ plasmons.
2. Conserve momentum and energy parameters at each vertex.
3. The matrix element S_{fi} will be the multiplication of
   a. $(\frac{-i}{\hbar})$ for each vertex and (-1) for each fermion loop.
   b. a factor $M_0(q)$ for each vertex,

c. a fermion propagator $iG(E,E_{\vec{k}}) = i/(E-E_{\vec{k}}+i\delta)$ for each intermediate state (internal line),

d. $\delta(E_i-E_f)$ due to the energy conservation of initial and final state,

and the sum over the internal variable $\vec{k}$ and the integration over the energy parameter E.

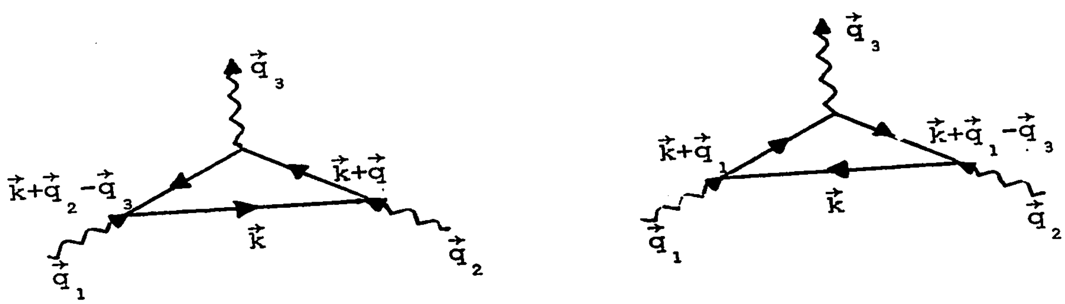


Fig. 11 Three wave interaction example: annihilation of two plasmons and creation of another.

These rules are general for any diagrams which can include any number of external waves.

One can generalize these rules to include virtual bosons by writing a boson propagator similar to that of (3.c) $(iD(\omega,\omega_q) = i/\omega-\omega_q+i\delta)$ if the internal lines consist of bosons.

The relation between the scattering matrix element and the transition matrix element M_{fi} is given by (Davydov, 1969)

$$S_{fi} = \delta_{fi} - \left(\frac{2\pi i}{\hbar}\right) M_{fi}\,\delta(E_i - E_f) \tag{86}$$

from (83) and (86), one obtains after performing the E-integration

$$M_{fi} = \frac{M_0(\vec{q}_1)M_0(\vec{q}_2)M_0(\vec{q}_3)}{\hbar^2} \sum_{\vec{k}} \left[\frac{f(\vec{k})}{(E_{\rightarrow}+\omega_{\vec{q}_2}-E_{\vec{k}+\vec{q}_2})(E_{\vec{k}}-\omega_{\vec{q}_1}-E_{\vec{k}-\vec{q}_1})} \right.$$

$$+ \frac{f(\vec{k}+\vec{q}_2)}{(E_{\vec{k}+\vec{q}_2}-\omega_{\vec{q}_2}-E_{\vec{k}})(E_{\vec{k}+\vec{q}_2}-\omega_{\vec{q}_2}-\omega_{\vec{q}_1}-E_{\vec{k}-\vec{q}_1})}$$

$$+ \frac{f(\vec{k}-\vec{q}_1)}{(E_{\vec{k}-\vec{q}_1}+\omega_{\vec{q}_1}-E_{\vec{k}})(E_{\vec{k}-\vec{q}_1}+\omega_{\vec{q}_1}+\omega_{\vec{q}_2}-E_{\vec{k}+\vec{q}_2})}$$

$$+ \frac{f(\vec{k})}{(E_{\vec{k}}+\omega_{\vec{q}_1}-E_{\vec{k}+\vec{q}_1})(E_{\vec{k}}-\omega_{\vec{q}_2}-E_{\vec{k}-\vec{q}_2})}$$

$$+ \frac{f(\vec{k}+\vec{q}_1)}{(E_{\vec{k}+\vec{q}_1}-\omega_{\vec{q}_1}-E_{\vec{k}})(E_{\vec{k}+\vec{q}_1}-\omega_{\vec{q}_1}-\omega_{\vec{q}_2}-E_{\vec{k}-\vec{q}_2})}$$

$$\left. + \frac{f(\vec{k}-\vec{q}_2)}{(E_{\vec{k}-\vec{q}_2}+\omega_{\vec{q}_2}-E_{\vec{k}})(E_{\vec{k}-\vec{q}_2}+\omega_{\vec{q}_2}+\omega_{\vec{q}_1}-E_{\vec{k}+\vec{q}_1})} \right] \tag{87}$$

To obtain a common factor $f(\vec{k})$, one makes the following transformation $\vec{k}+\vec{q}_2 \to \vec{k}$ in the 2nd term, $\vec{k}-\vec{q}_1 \to \vec{k}$ in the third term, etc. then after taking the classical limit

$$\sum_{\vec{k}} f(\vec{k}) \to V\int f(\vec{v})$$

$$E_{\vec{k}} = \frac{hk^2}{2m_j} \quad \text{and} \quad \hbar\vec{k} = m_j\vec{v}$$

one obtains upon expansion of the denominators up to the order of $\hbar^2$ in equation (87)

$$M = \frac{M_0(\vec{q}_1)M_0(\vec{q}_2)M_0(\vec{q}_3)}{m_j{}^2} \; V \int \frac{d^3vf_j(\vec{v})}{(\omega_{\vec{q}_1} - \vec{q}_1\cdot\vec{v})(\omega_{\vec{q}_2} - \vec{q}_2\cdot\vec{v})(\omega_{\vec{q}_3} - \vec{q}_3\cdot\vec{v})}$$

$$\left[\frac{(\vec{q}_1\cdot\vec{q}_2)q_3^2}{(\omega_{\vec{q}_3} - \vec{q}_3\cdot\vec{v})} + \frac{(\vec{q}_2\cdot\vec{q}_3)q_1^2}{(\omega_{\vec{q}_1} - \vec{q}_1\cdot\vec{v})} + \frac{(\vec{q}_3\cdot\vec{q}_1)q_2^2}{(\omega_{\vec{q}_2} - \vec{q}_2\cdot\vec{v})}\right] \qquad (88)$$

This is the same result obtained by Harris (1969). These calculations will be used later in chapter IV to find the nonlinear dielectric function for the ion acoustic waves.

CHAPTER III

LINEAR AND NONLINEAR COLLISIONAL DAMPING OF PLASMA WAVES

3.1 Introduction

The problem of electron oscillations in a plasma neglecting interparticle collisions has been studied amongst others by Landau (1946), Bohm and Gross (1949), Van Kampen (1955) and Bernstein, Green and Kruskal (1957). The qualitative importance of collisional damping was pointed out by Bohm and Gross, where they noted that since Landau damping was negligible for $\vec{q}$ (wave vector) $\rightarrow 0$, collisional effects must dominate in this limit. Bhatnagar, Gross and Krook (1954) have shown that in the presence of collisions, plasma oscillations decay with a rate proportional to the collisional frequency, however, a complete calculation of this effect was not carried out until Ichikawa (1960) examined the effect of binary and tertiary corrections in high temperature plasma on the basis of the so-called BBGKY equation for a system of charged particles. Similar calculations based on the same equation have been carried out by Willis (1962). A formula for the dielectric function including the effects of the binary Coulomb collisions has been derived by many authors during the sixties. Among these, Comisar (1962) investigated the attenuation of longitudinal plasma oscillations by transforming the Landau (Fokker-Planck) equation in time, space and velocity to obtain an integral equation for the transformation of the distribution function. Ogaswara (1963) used the Boltzman collision integral and expanded the distribution function in a complete set of velocity dependent polynomials. Buti and Jain (1965) have used Comisar's method to study the damping of longitudinal as well as transverse plasma oscillations. Thourson and Lewis (1965) calculated the high frequency conductivity of a fully ionized plasma by using the BBGKY hierarchy. They assumed that the terms representing Coulomb potential can be replaced by a cutoff Coulomb potential with a range equal to the Debye length.

More recently this problem has been treated by Shkarofsky (1968), Matsuda (1969), and McBride (1969) through the use of the Fokker-Planck equation or the BBGKY hierarchy. Shkarofsky studied the problem of longitudinal as well as transverse plasma waves by using the Fokker-Planck equation and also used the Boltzman equation to study the electron-neutral collision. Matsuda obtained the dispersion relation including collisional effects by applying the BBGKY equation and the pair correlation function given by Rostoker's supperposition principle, (Rostoker's test particle method). McBride solved the linearized kinetic equation for the perturbed electron distribution function for the case of longitudinal plasma oscillations along an external magnetic field, where the collision term is specified to be the Landau (Fokker-Planck)collision integral.

The problem of collisional damping has also been studied using quantum mechanical methods. DuBois, Gilinisky and Kivelson (1963) calculated the collisional damping for longitudinal and transverse waves from the conductivity which is derived by using the diagrammatic techniques of field theory. Similar work has been done for transverse waves by Perel and Eliashberg (1962), Ron and Tzoar (1963) and DuBois and Gilinisky (1964). For a basic understanding of this technique one may refer to Chappel and Britten (1966), Rauscher (1968), Harris (1969) as well as chapter II of this thesis.

In the present chapter, the quantum mechanical approach is adopted and the collisional damping as arising out of the scattering of two particles through effective Coulomb field and emission (absorption) of one plasmon is calculated directly without reference to the dielectric function or the conductivity. In addition, the scattering of a particle pair with the emission (absorption) of two plasmons will be examined. This process may be called a nonlinear collisional process similar to that of nonlinear Landau damping which involves two plasmons (fig. 4). In section 3.2, the theory of chapter II is extended to calculate the scattering matrix elements for the one and two wave processes. Section 3.3 deals with the linear collisional damping of longitudinal oscillations and section 3.4 with the two waves collisional process.

3.2 Scattering Matrix for Linear and Nonlinear Collision

The Feynman graphs for processes involving scattering of two particles through an effective Coulomb field and emission of one plasmon are obtained from the following S-matrix

$$S^{(2)} = \frac{(-i)^2}{2!\hbar^2}\iint d^4x_1 d^4x_2 T\left\{H_{w-p}(x_1)H_{p-p}(x_2)\right\} \quad (1)$$

where H_{w-p} and H_{p-p} are the interaction Hamiltonian densities for wave-particle and particle-particle scattering respectively. These are given by

$$H_{w-p} = \Psi^+(x_1)e_j\phi(x_1)\Psi(x_1) \quad (2.a)$$

$$H_{p-p} = \int \Psi_2^+\Psi_3^+V(\vec{x}_2-\vec{x}_3)\Psi_3\Psi_2 d^4x_3 \quad (2.b)$$

Substituting (2.a) and (2.b) into (1) yields

$$S^{(2)} = \frac{(-i)^2}{2!\hbar^2}\int d^4x_1 d^4x_2 d^4x_3 T\left\{\Psi_1^+e_j\phi_1\Psi_1\Psi_2^+\Psi_3^+ \times V(|\vec{x}_2-\vec{x}_3|)\Psi_3\Psi_2\right\} \quad (3)$$

The T-product can be written as follows

$$T\left\{\Psi_1^+\phi_1\Psi_1\Psi_2^+\Psi_3^+V(|\vec{x}_2-\vec{x}_3|)\Psi_3\Psi_2\right\}$$

$$= N\{\, . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \,\}$$

$$+ \; 2\Big[N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}$$

$$+ \; N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}$$

$$+ \; N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}$$

$$+ \; N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}$$

$$+ \; N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}$$

$$+ \; N\{\Psi^+_1 \phi_1 \Psi_1 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_3 \Psi_2\}\Big] \qquad (4)$$

where the factor 2 is due to the equal contraction terms. Substituting (4) into (3), one obtains

$$S^{(2)} = \frac{(-i)^2}{\hbar^2} \int d^4x_1 d^4x_2 d^4x_3 \Big[\Big[\Psi^+_1 \Psi_3 \Psi^+_2 \Psi^+_3 V(2,3) \Psi_2 \Psi_1$$

$$+ \; \Psi^+_3 \Psi_1 \Psi^+_1 \Psi^+_2 V(2,3) \Psi_3 \Psi_2 \; + \; (-1) \Psi^+_1 \Psi_3 \Psi^+_2 \Psi_1 V(2,3) \Psi^+_2 \Psi_2$$

$$+ \text{ Similar terms } 2 \leftrightarrow 3 \Big] e_j \phi_1 \Big] \qquad (5)$$

where the first term can be represented by Feynman graphs according to the theory of chapter II as follows: at the point x_1 a fermion is annihilated and another is created due to the emission (absorption) of a boson. The created fermion propagates to the point x_3, and is then scattered into its final state due to the Coulomb collision field V(2,3). It

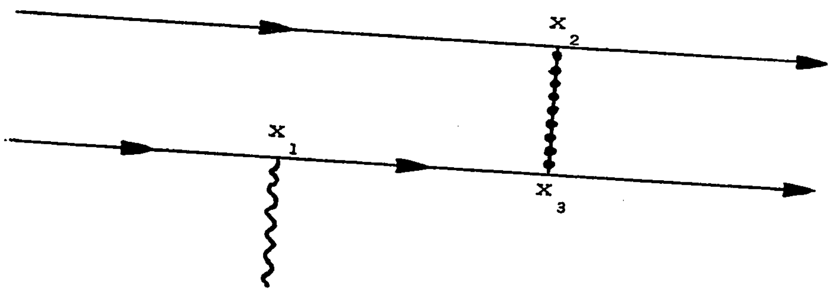


Fig. 12(a) Linear collisional process

can also be seen that one fermion is annihilated and another is created at the point x_2. These processes are shown in fig. 12(a).

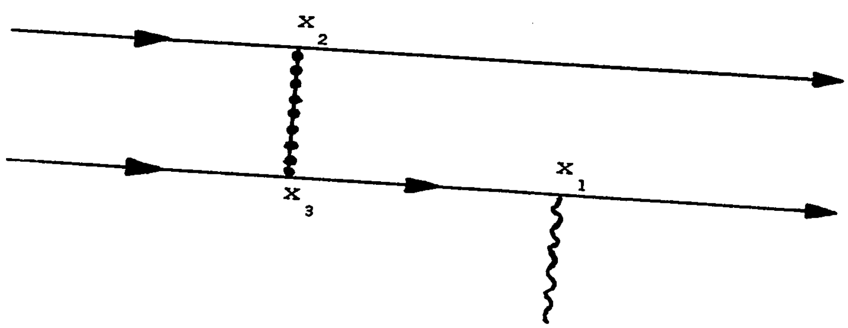


Fig. 12(b) Linear collisional process

Similarly, the second term is represented by fig. 12(b) where the fermion propagator in this case is given by $\overline{\psi_3^+\psi_1}$. Since figs.12(a) and (b) are nonequivalent, their sum will yield the correct S-matrix for the linear process. These diagrams will be constructed in the momentum space and the calculation of the S-matrix will be carried on in sec. 3.3.

A diagram similar to that of Landau damping (fig. 1) is obtained from the third term of (5). This is shown in fig. 13. However this diagram is nonconsistent, because the

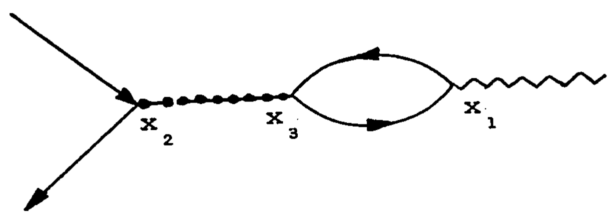


Fig. 13 Indirect linear Landau damping

Coulomb potential becomes singular at the plasmon frequency. The similar terms in (5) yield diagrams similar to that of figs. 12 and 13 where the plasmon is emitted (absorbed) by a different species of fermions.

A process of pair particle scattering via the Coulomb field, with the emission of two plasmons, can be obtained from the third order term of the S-matrix which contains two wave-particle interaction Hamiltonians and only one particle-particle interaction Hamiltonian. This is given by

$$S^{(3)} = \frac{(-i)^3}{3!\hbar^3} \int \ldots \int d^4x_1 \ldots d^4x_4 \, T\left\{ H_{\omega-p}(x_1) H_{\omega-p}(x_2) H_{p-p}(x_3,x_4) \right\} \quad (6)$$

As has been discussed in the previous chapter, the

T-product contains connected as well as non significant disconnected diagrams. Each of the connected diagrams has 3! = 6 equivalence. In addition there will be several nonequivalent diagrams that must be added. Keeping only the connected diagrams, (6) yields

$$
\begin{aligned}
S^{(3)} = \frac{(-i)^3}{\hbar^3} \int \cdots \int d^4x_1 \cdots d^4x_4 \Big[& \overline{\psi^+_1 \psi_2}\, \overline{\psi^+_2 \psi_3}\, \psi^+_3 \psi^+_4 V(3,4) \psi_1 \psi_4 \\
& + \overline{\psi^+_1 \psi_3}\, \overline{\psi^+_3 \psi_2}\, \psi^+_2 \psi^+_4 V(3,4) \psi_1 \psi_4 + \overline{\psi^+_2 \psi_1}\, \overline{\psi^+_1 \psi_3}\, \psi^+_3 \psi^+_4 V(3,4) \psi_2 \psi_4 \\
& + \overline{\psi^+_2 \psi_3}\, \overline{\psi^+_3 \psi_1}\, \psi^+_1 \psi^+_4 V(3,4) \psi_2 \psi_4 + \overline{\psi^+_3 \psi_1}\, \overline{\psi^+_1 \psi_2}\, \psi^+_2 \psi^+_4 V(3,4) \psi_3 \psi_4 \\
& + \overline{\psi^+_3 \psi_2}\, \overline{\psi^+_2 \psi_1}\, \psi^+_1 \psi^+_4 V(3,4) \psi_3 \psi_4 + \overline{\psi^+_1 \psi_2}\, \overline{\psi^+_2 \psi_3}\, \overline{\psi^+_3 \psi_1}\, V(3,4) \psi^+_4 \psi_4 \\
& + \overline{\psi^+_1 \psi_3}\, \overline{\psi^+_3 \psi_2}\, \overline{\psi^+_2 \psi_1}\, V(3,4) \psi^+_4 \psi_4 + \text{similar terms } 3 \leftrightarrow 4 \Big] e_j^2 \phi_1 \phi_2
\end{aligned} \tag{7}
$$

The first six terms describe the nonlinear collisional processes. These can be represented diagrammatically by six nonequivalent diagrams. Only the process describing the first

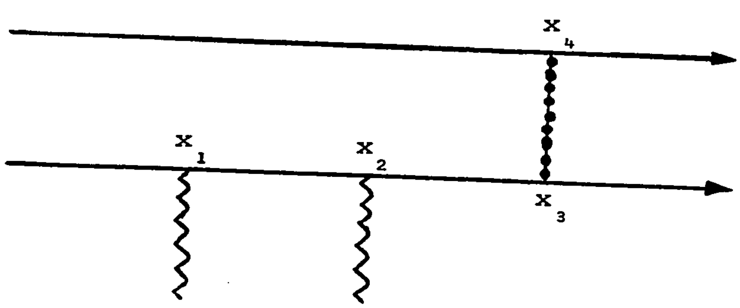


Fig. 14 Nonlinear collisional process

term is shown in fig. 14. The six diagrams will be constructed later in section 3.4 The indirect nonlinear scattering is described by the last two terms and is shown diagrammatically in fig. 15. The similar terms will yield

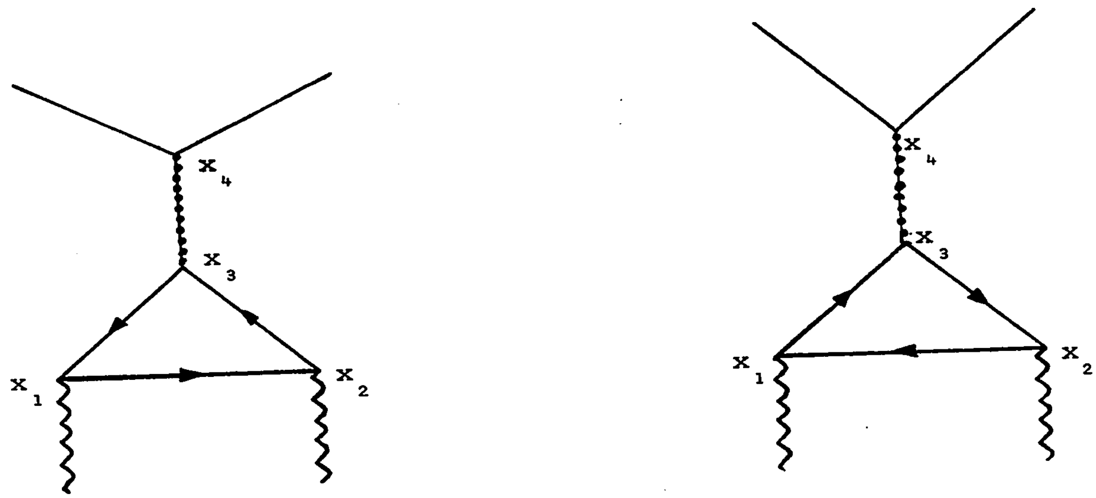


Fig. 15 Indirect nonlinear scattering

diagrams similar to that of figs. 14 and 15 where the plasmons are emmitted by a different species of particles.

3.3 Linear Collisional Damping of Plasma Oscillations

The scattering matrix for the linear process is easily calculated by transforming diagrams 11(a) and (b) to the momentum space. In doing so, one has to bear in mind the conservation of momentum and energy of the initial and final states as well as the conservation of momentum and energy parameters at each vertex. This is shown in fig. 16.

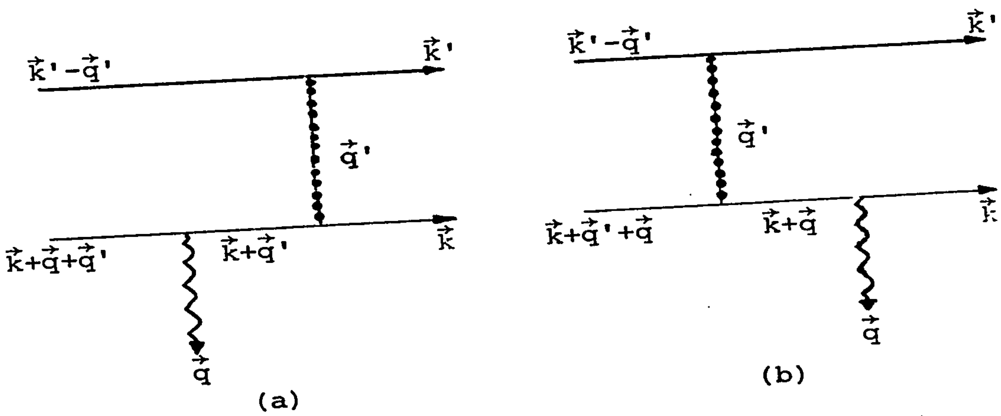


Fig. 16 Linear collisional process: The upper line refers to a heavy ion and the lower one to an electron.

There are four diagrams like the one shown in fig. 16(a). Of these, half will be those in which the plasmon will be emitted by the heavy ion. Such diagrams give a negligible contribution (of the order m/m_i); thus only contributions due to diagrams which emit plasmons at the electron line will be considered. Following the S-matrix rules of chapter II, one can immediately write the S-matrix element for the processes shown in fig. 16 as follows

$$S_{fi}^{(2)} = 2\pi \frac{(-i)^2}{\hbar^2} M_0(\vec{q}) \frac{4\pi e^2}{Vq'^2 \varepsilon(\vec{q}',\omega')} \left[\frac{i}{E-E_{\vec{k}+\vec{q}'}} + \frac{i}{E'-E_{\vec{k}+\vec{q}}}\right] \tag{8}$$

$$\times\ \delta(E_i - E_f)$$

where the energy parameters E and E' are known and there is no internal variable over which to sum or integrate. The factor (2π) is because the number of vertices exceed the number of internal lines by unity. From the conservation of energy parameters at each vertex, one obtains

$$E = E_{\vec{k}+\vec{q}+\vec{q}\,'} - \omega_{\vec{q}} \tag{8.a}$$

$$E' = E_{\vec{k}} + \omega_{\vec{q}} \tag{8.b}$$

Substituting (8.a) and (8.b) into (8) yields

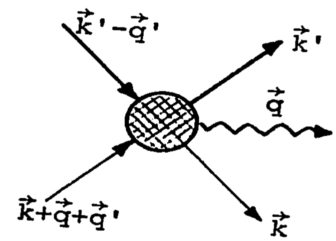
$$S_{fi}^{(2)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{M_0(\vec{q})}{\hbar} \frac{4\pi e^2}{Vq'^2 \varepsilon(\vec{q}\,',\omega')} \left[\frac{1}{(E_{\vec{k}+\vec{q}+\vec{q}\,'} - \omega_{\vec{q}} - E_{\vec{k}+\vec{q}\,'})} + \frac{1}{(E_k + \omega_q - E_{k+q})}\right] \delta(E_i - E_f) \tag{9}$$

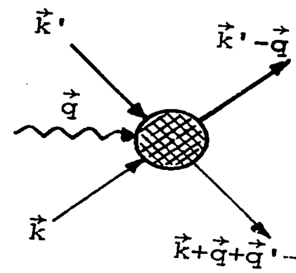
Putting $\hbar\vec{k} = m\vec{v}$ and expanding (9) in power of $\hbar$, one obtains by virtue of the S_{fi}, M_{fi} relationship ((86) of Chapter II)

$$M_{fi} = M_0(\vec{q}) \frac{4\pi e^2}{mVq'^2 \varepsilon(\vec{q}\,',\omega')} \cdot \frac{\vec{q}\cdot\vec{q}\,'}{(\omega_{\vec{q}} - \vec{q}\cdot\vec{v})^2} \tag{10}$$

where higher powers of $\hbar$ have been neglected, since these will not contribute in the classical limit.

An equation for the rate of change of $N_{\vec{q}}$, the number of plasmons of momentum $\hbar\vec{q}$ can be written schematically as

$$\frac{\partial N_{\vec{q}}}{\partial t} = \sum_{\vec{k},\vec{k}\,',\vec{q}\,'} \Bigg[$$ 

$$-$$ 

$$\Bigg] \qquad (11)$$

This means that one adds all the processes in which a plasmon of momentum $\hbar\vec{q}$ is emitted and subtracts all those in which one is absorbed. The difference gives the net increase in $N_{\vec{q}}$. Taking $f(\vec{k})$, the number of particles of momentum $\hbar\vec{k}$; the schematic equation can be converted into a mathematical equation by replacing each diagram by the transition proba-bility per unit time for the process: Thus, (11) yields

$$\frac{\partial N_{\vec{q}}}{\partial t} = \frac{2\pi}{\hbar^2} \sum_{\vec{k},\vec{k}\,',\vec{q}\,'} |M_{fi}|^2 \Big[f(\vec{k}+\vec{q}+\vec{q}\,')\, f(\vec{k}\,'-\vec{q}\,') \big(1-f(\vec{k})\big)\big(1-f(\vec{x}\,')\big) \times (N_{\vec{q}}+1) - f(\vec{k}) f(\vec{k}\,') \big(1-f(\vec{k}+\vec{q}+\vec{q}\,')\big)\big(1-f(\vec{k}\,'-\vec{q}\,')\big) N_q \Big] \delta\big(E_{\vec{k}+\vec{q}+\vec{q}\,'} + E_{\vec{k}\,'-\vec{q}\,'} - \omega_{\vec{q}} - E_{\vec{k}} - E_{\vec{k}\,'}\big) \qquad (12)$$

where M_{fi} is given by (10) and $E_{\vec{k}}$ is the kinetic energy of the particle expressed in units of $\hbar$. Expanding the f's and E's to the lowest order of $\hbar$, (12) becomes;

$$\frac{\partial N_{\vec{q}}}{\partial t} = \frac{2\pi}{\hbar^2} \sum_{\vec{k},\vec{k}\,',\vec{q}\,'} |M_{fi}|^2 \Big[(\vec{q}+\vec{q}\,') \cdot \frac{\partial f(\vec{k})}{\partial \vec{k}}\, f(\vec{k}\,')\, (N_q+1)$$

$$- \vec{q}\,'\cdot\frac{\partial f(\vec{k}\,')}{\partial \vec{k}\,'}\; f(\vec{k})N_q\Big]\,\delta\left(\vec{q}+\vec{q}\,'\cdot\frac{\partial E_{\vec{k}}}{\partial \vec{k}} - \vec{q}\,'\cdot\frac{\partial E_{\vec{k}\,'}}{\partial \vec{k}\,'} - \omega_{\vec{q}}\right) \qquad (13)$$

Taking $N_q >> 1$ and substituting

$$E_k = \frac{\hbar k^2}{2m}\,, \qquad (14.a)$$

$$\hbar\vec{k} = m\vec{v} \qquad (14.b)$$

$$\frac{1}{N_q}\frac{\partial N_q}{\partial t} = 2\gamma \qquad (14.c)$$

$$\sum_{\vec{q}\,'} \rightarrow V\int\frac{d^3q'}{(2\pi)^3} \qquad (14.d)$$

$$\sum_{\vec{k},\vec{k}\,'} f(\vec{k})f(\vec{k}\,') \rightarrow V^2\int d^3v d^3v' f(\vec{v})f(\vec{v}\,')\,, \qquad (14.e)$$

into (13), one obtains in the classical limit

$$2\gamma_{LC} = \frac{1}{N_q}\frac{dN_q}{dt} = \frac{\omega_{pe}^6}{(2\pi)^2}\,\frac{m\omega_q}{n_0^3F_q}\int\frac{\cos^2\theta\, d^3q'}{q'^2|\varepsilon(\vec{q}\,',\omega')|^2}\int\frac{d^3v d^3v'}{(\omega_{\vec{q}}-\vec{q}\cdot\vec{v})^4}$$

$$\cdot\left[\frac{(\vec{q}+\vec{q}\,')}{m}\cdot\frac{\partial f(\vec{v})}{\partial \vec{v}}\, f(\vec{v}\,') - \frac{\vec{q}\,'}{m_i}\cdot\frac{\partial f(\vec{v}\,')}{\partial \vec{v}\,'}\, f(\vec{v})\right]$$

$$\cdot\,\delta\left(\vec{q}\,'\cdot\vec{v}\,' - (\vec{q}+\vec{q}\,')\cdot\vec{v} + \omega_{\vec{q}}\right) \qquad (15)$$

where γ_{LC}, is the rate at which the wave amplitude decays, $\vec{q}\cdot\vec{q}\,' = qq'\cos\theta$, $\vec{v}$ and $\vec{v}\,'$ are the electron and ion velocities respectively. The volume of the box in which the system is quantized tends to infinity so that the sums go over into integrals, hence (14.d) and (14.e). Finally, $f(\vec{v})$ and $f(\vec{v}')$ are the classical particle distribution functions. If electrons and ions are in thermal equilibrium with Maxwellian distribution, (15) becomes

$$2\gamma_{LC} = -\frac{\omega_{pe}^6}{(2\pi)^2 v_e^2}\frac{\omega_q^2}{n_0^3 F_q}\int\frac{\cos^2\theta d^3q'}{q'^2|\varepsilon(q',\omega')|^2}\int\frac{d^3v d^3v' f(\vec{v}) f(\vec{v}')}{(\omega_q-\vec{q}\cdot\vec{v})^4}$$

$$\cdot\delta\left(\vec{q}'\cdot\vec{v}'-(\vec{q}+\vec{q}')\cdot\vec{v}+\omega_{\vec{q}}\right) \tag{16}$$

Performing first the $\vec{v}'$-integration, then using the asymptotic expansion method (Fried and Conte, 1961) to perform the $\vec{v}$-integration, one obtains

$$2\gamma_{LC} = -\frac{\omega_{pe}^6}{(2\pi)^{3/2} n_0 v_e^3}\cdot\frac{1}{\omega^2 F_q}\iint\frac{dq'}{|q'|}\cos^2\theta\sin\theta d\theta\left(1+\frac{10q^2v_e^2}{\omega_q^2}\sin^2\theta\right) \tag{17}$$

which yields after performing the q' and θ integrations

$$\gamma_{LC} = -\frac{\omega_{pe}^6}{4\pi^{3/2} n_0 v_e^3}\frac{1}{\omega_q^2 F_q}\left[\frac{\sqrt{2}}{3}+\frac{4\sqrt{2}}{3}\frac{q^2v_e^2}{\omega_q^2}\right]\ell n\left(\frac{q'_{max}v_e}{\omega_{pe}}\right) \tag{18}$$

Substituting $\omega_q^2 = \omega_{pe}^2\left[1+\frac{3q^2v_e^2}{\omega_{pe}^2}\right]$ and

$$F_q = \left|\frac{\partial}{\partial\omega}\left(\omega\varepsilon(\vec{q},\omega)\right)\right|_{\omega=\omega_q}$$

$$= \left| \frac{\partial}{\partial\omega} \left\{ \omega \left[1 - \frac{\omega^2_{pe}}{\omega^2} \left(1 + \frac{3q^2 v_e^2}{\omega^2} \right) \right] \right\} \right|_{\omega=\omega_q}$$

$$= 1 + \frac{\omega_{pe}^2}{\omega^2} + 9 \frac{q^2 v_e^2 \omega_{pe}^2}{\omega_q^4}$$

$$= 2\left[1 + \frac{3q^2 v_e^2}{\omega_{pe}^2}\right]$$ into (18) yields

$$\gamma_{LC} = - \frac{\omega_{pe}^4}{8\pi^{3/2} n_0 v_e^3} \left[\frac{\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} \frac{q^2 v_e^2}{\omega_{pe}^2}\right] \ln\left(\frac{q'_{max} v_e}{\omega_{pe}}\right) \quad (19)$$

This result agrees with previous calculations (DuBois, Gilinisky and Kivelson (1963), Shkarofsky (1968), Matsuda (1969,, and McBride (1969)).

3.4 Nonlinear Collisional Process

In the previous section, the Coulomb scattering of a particle pair via emission of one plasmon is investigated. It has been shown that for waves of long wave-length ($q \to 0$), the collisional damping dominates over Landau damping. However, it appears that the nonlinear collisional process is also more important than the nonlinear Landau damping in the long wave length limit.

In studying the nonlinear collisional process, it can be shown that similar to the linear collisional process described in section 3.3, diagrams involving emission of plasmons by the heavy ions will make a negligible contribution, and also there are six nonequivalent diagrams which will give a significant contribution. These diagrams are described by the first six terms of (7) which in the momentum space yields the diagrams shown in fig. 17.

Following the rules of the scattering matrix explained in chapter II, and the previous section, one can easily write

$$S_{fi}^{(3)} = \frac{(-i)^3}{\hbar^3} M_0(q_1) M_0(q_2) \frac{4\pi e^2}{Vq'^2 \varepsilon(q',\omega')} \cdot$$

$$\Bigg[\frac{i^2}{(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{\vec q_1} - E_{\vec k+\vec q_2+\vec q'})(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{\vec q_1} - \omega_{\vec q_2} - E_{\vec k+\vec q'})}$$

$$+ \frac{i^2}{(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{\vec q_2} - E_{\vec k+\vec q_1+\vec q'})(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{\vec q_1} - \omega_{\vec q_2} E_{k+q'})}$$

$$+ \frac{i^2}{(E_{\vec k} + \omega_{q_2} - E_{\vec k+\vec q_2})(E_{\vec k} + \omega_{q_2} + \omega_{q_1} - E_{\vec k+\vec q_1+\vec q_2})}$$

$$+ \frac{i^2}{(E_{\vec k} + \omega_{q_1} - E_{\vec k+\vec q_1})(E_{\vec k} + \omega_{q_1} + \omega_{q_2} - E_{\vec k+\vec q_1+\vec q_2})}$$

$$+ \frac{i^2}{(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{q_1} - E_{\vec k+\vec q_2+\vec q'})(E_{\vec k} + \omega_{q_2} - E_{\vec k+\vec q_2})}$$

$$+ \frac{i^2}{(E_{\vec k+\vec q_1+\vec q_2+\vec q'} - \omega_{q_2} - E_{\vec k+\vec q_1+\vec q'})(E_{\vec k} + \omega_{q_1} - E_{\vec k+\vec q_1})} \Bigg] \cdot \qquad (20)$$

$$\delta(E_i - E_f)$$

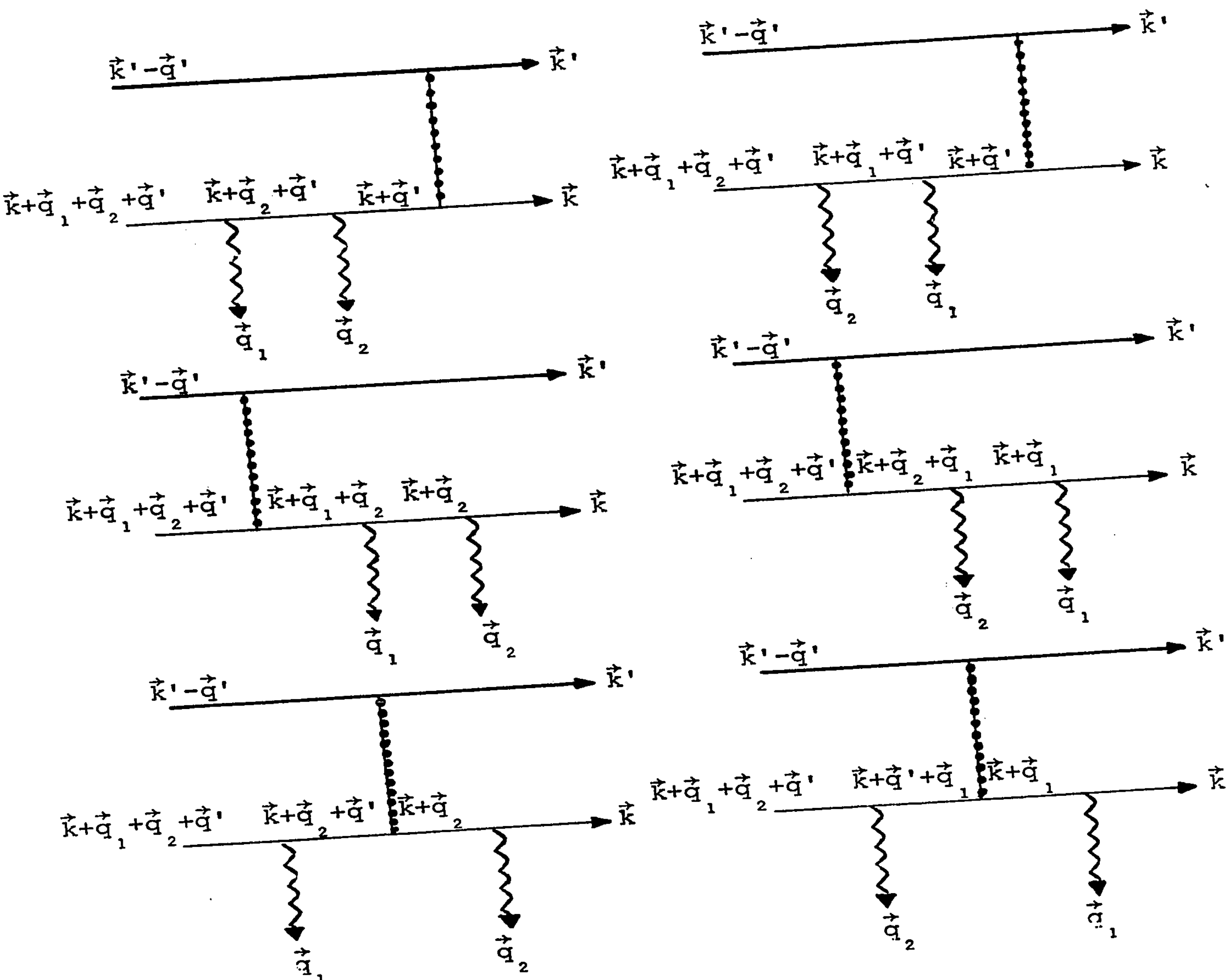


Fig. 17 Nonlinear collisional process: The upper line refers to a heavy ion and the lower one to an electron.

Substituting $E_k = \frac{\hbar k^2}{2m}$, $\hbar\vec{k} = m\vec{v}$ into (20), expanding the denominators up to the order of $\hbar^2$, and using relation (86) of chapter II, one obtains

$$M_{fi} = \frac{\omega_{pe}^4}{4n_0^2V^2q'^2\varepsilon(\vec{q}',\omega')}\left[\frac{\hbar\omega_{q_1}\,\hbar\omega_{q_2}}{q_1^2F_{q_1}\,q_2^2F_{q_2}}\right]^{1/2}\Lambda(\vec{q}_1,\vec{q}_2,\vec{q}',\vec{v}) \qquad (21)$$

where

$$\Lambda(\vec{q}_1,\vec{q}_2,\vec{q}',\vec{v}) = -\frac{\left\{q_1^4+[q_1^2+2\vec{q}_1\cdot(\vec{q}_2+\vec{q}')]\right\}}{\lambda_1^2\lambda_2(\lambda_1+\lambda_2)}$$

$$-\frac{\left\{q_2^4+[q_2^2+2\vec{q}_2\cdot(\vec{q}_1+\vec{q}')]\right\}}{\lambda_2^2\lambda_1(\lambda_2+\lambda_1)}$$

$$+\frac{\left\{(\vec{q}_1+\vec{q}_2)^4+[(\vec{q}_1+\vec{q}_2)^2+2\vec{q}'\cdot(\vec{q}_1+\vec{q}_2)]^2\right\}}{\lambda_1\lambda_2(\lambda_1+\lambda_2)^2}$$

$$-\frac{\left\{q_1^2(\vec{q}_1+\vec{q}_2)^2+[q_1^2+2\vec{q}_1\cdot(q_2+\vec{q}')][(\vec{q}_1+\vec{q}_2)^2+2\vec{q}'\cdot(\vec{q}_1+\vec{q}_2)]\right\}}{\lambda_1^2(\lambda_1+\lambda_2)^2}$$

$$+\frac{\left\{q_2^2(\vec{q}_1+\vec{q}_2)^2+[q_2^2+2\vec{q}_2\cdot(\vec{q}_1+\vec{q}')][(\vec{q}_2+\vec{q}_1)^2+2\vec{q}'\cdot(\vec{q}_2+\vec{q}_1)]\right\}}{\lambda_2^2(\lambda_1+\lambda_2)}$$

$$-\frac{\left\{q_1^2[q_2^2+2\vec{q}_2\cdot(\vec{q}_1+\vec{q}')]+q_2^2[q_1^2+2\vec{q}_1\cdot(\vec{q}_2+\vec{q}')]\right\}}{\lambda_1^2\lambda_2^2} \qquad (21.a)$$

and

$$\lambda_i = \omega_{q_i} - \vec{q}_i\cdot\vec{v} \qquad (21.b)$$

In obtaining (21), terms of order $\hbar^0$, and $\hbar$ vanished identically and that of order $\hbar^2$ yielded the correct classical limit of the matrix element. Inserting the square of (21) into the rate equation for N_{q_1} yields

$$\frac{\partial N_{\vec{q}_1}}{\partial t} = \frac{2\pi}{\hbar^2} \sum_{\vec{q}_2,\vec{q}',\vec{k},\vec{k}'} |M_{fi}|^2 \Big\{ \big(N_{\vec{q}_1} + N_{\vec{q}_2}\big) f(\vec{k}) f(\vec{k}')$$

$$+ N_{\vec{q}_1} N_{\vec{q}_2} \left[(\vec{q}_1 + \vec{q}_2 + \vec{q}') \cdot \frac{\partial f(\vec{k})}{\partial \vec{k}} f(\vec{k}') - \vec{q}' \cdot \frac{\partial f(\vec{k}')}{\partial \vec{k}'} f(\vec{k}) \right] \Big\} \cdot$$

$$\delta\big(E_{\vec{k}} + E_{\vec{k}'} + \omega_{q_1} + \omega_{q_2} - E_{\vec{k}+\vec{q}_1+\vec{q}_2+\vec{q}'} - E_{\vec{k}'-\vec{q}'}\big) \cdot \tag{22}$$

which can be rewritten after converting summations into integrations and taking the classical limit

$$\frac{\partial I(\vec{q}_1)}{\partial t} = \frac{(2\pi)^{-5} \omega_{pe}^8}{16 n_0^4} \frac{\omega_{\vec{q}_1}}{q_1^2 F_{q_1}} \int d^3q_2 d^3q' d^3v d^3v' \, x$$

$$\frac{\Lambda^2(\vec{q}_1, \vec{q}_2, \vec{q}', \vec{v}) \omega_{\vec{q}_2}}{q_2^2 F_{q_2} q'^4 |\varepsilon(\vec{q}', \omega')|^2} \cdot \left[\Big(I(\vec{q}_1) + \frac{\omega_{q_1}}{\omega_{q_2}} I(\vec{q}_2) \Big) f(\vec{v}) f(\vec{v}') \right.$$

$$\left. + I(\vec{q}_1) \frac{I(\vec{q}_2)}{\omega_{\vec{q}_2}} \left\{ \frac{\vec{q}_1 + \vec{q}_2 + \vec{q}'}{m} \cdot \frac{\partial f(\vec{v})}{\partial \vec{v}} f(\vec{v}') - \frac{\vec{q}'}{m_i} \cdot \frac{\partial f(\vec{v}')}{\partial \vec{v}'} f(\vec{v}) \right\} \right]$$

$$\delta\big(\vec{q}' \cdot \vec{v}' + \omega_{\vec{q}_1} + \omega_{\vec{q}_2} - (\vec{q}_1 + \vec{q}_2 + \vec{q}') \cdot \vec{v}\big), \tag{23}$$

where $I(\vec{q}_i) = \hbar\omega_{\vec{q}_i} N_{\vec{q}_i}$ is the plasmon energy in the mode $\vec{q}_i$. Considering the special case of longitudinal waves where $\omega_{\vec{q}_1} \simeq \omega_{\vec{q}_2} \simeq \omega_{pe}$, and assuming that the electrons and ions obey a Maxwellian distributions as before, (23) yields after performing the velocity integrations

$$2\gamma_{NLC} = \frac{1}{I(\vec{q})}\frac{\partial I(\vec{q})}{\partial t} = \frac{(2\pi)^{-5}\omega_{pe}^2}{64\sqrt{2\pi}v_e n_0^2}\int \frac{d^3q_2\, d^3q'\, \Lambda^2(\vec{q},\vec{q}_2,\vec{q}\,')}{q^2 q_2^2 q'^4 |\vec{q}+\vec{q}_2+\vec{q}\,'|\, |\varepsilon(q',\omega')|^2}$$

$$\cdot\left[\left(1+\frac{I(\vec{q}_2)}{I(\vec{q})}\right) - 2\,\frac{I(\vec{q}_2)}{mv_e^2}\right] \qquad (24)$$

where γ_{NlC} is the nonlinear collisional damping (growth),

$$\Lambda^2(\vec{q},\vec{q}_2,\vec{q}\,') = (\vec{q}\cdot\vec{q}_2)^2|\vec{q}+\vec{q}_2|^2 q'^2\cos^2\theta$$

$$+\ 8(\vec{q}\cdot\vec{q}_2)(\vec{q}\cdot\vec{q}\,')(\vec{q}_2\cdot\vec{q}\,')|\vec{q}+\vec{q}_2|q'\cos\theta$$

$$+\ 16(\vec{q}\cdot\vec{q}\,')^2(\vec{q}_2\cdot\vec{q}\,')^2 \qquad (25)$$

and θ is the angle between $\vec{q}+\vec{q}_2$ and $\vec{q}\,'$ as shown in fig. 18.

Substituting (25) into (24) and performing the $\vec{q}\,'$-integration one obtains

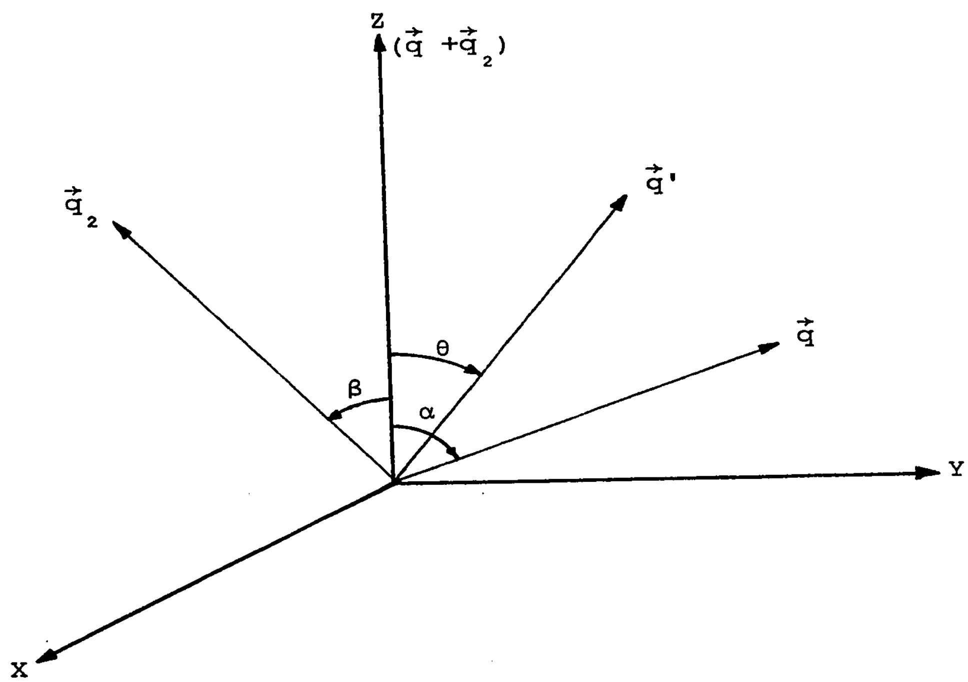


Fig. 18

$$2\gamma_{NLC} = \frac{1}{I(\vec{q})}\frac{\partial I(\vec{q})}{\partial t} = \frac{(2\pi)^{-5}\omega_{pe}^2}{64\sqrt{2\pi}v_e n_0^2}\,\ell n\left(\frac{q'_{max}v_e}{\omega_{pe}}\right)\int\frac{d^3q_2\,\Lambda(\vec{q},\vec{q}_2)}{q^2q_2^2}\,\cdot$$

$$\left[\left(1+\frac{I(\vec{q}_2)}{I(\vec{q})}\right) - 2\,\frac{I(\vec{q}_2)}{mv_e^2}\right], \qquad (26)$$

where

$$\begin{aligned}\Lambda(\vec{q},\vec{q}_2) = {} & \frac{4\pi}{3}(\vec{q}\cdot\vec{q}_2)^2|\vec{q}+\vec{q}_2|^2 \\ & -8\pi q q_2(\vec{q}\cdot\vec{q}_2)|\vec{q}+\vec{q}_2|^2\left(\frac{4}{5}\cos\alpha\cos\beta + \cos(\phi_2-\phi)\right) \\ & + 8\pi q^2q_2^2|\vec{q}+\vec{q}_2|^2\left\{\frac{32}{35}\cos^2\alpha\cos^2\beta\right. \\ & + \frac{8}{105}(\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\alpha) \\ & + \frac{8}{105}\sin2\alpha\sin2\beta\cos(\phi_2-\phi) \\ & \left. - \frac{16}{105}\sin^2\alpha\sin^2\beta\left[1+2\cos^2(\phi_2-\phi)\right]\right\} \end{aligned} \qquad (27)$$

The angles α and β are shown in fig. 18 and ϕ, ϕ_2 are the azimuthal angles of $\vec{q}$ and $\vec{q}_2$ respectively. In order to perform the $\vec{q}_2$-integration, the plasmon energy $I(\vec{q}_2)$ will be considered independent of $\vec{q}_2$, and is replaced by $\bar{I}(t)$, the average value of $I(\vec{q}_2)$. This approximation is good enough for plasmons which are not far removed from equilibrium. Furthermore, the integration over $\vec{q}_2$ will be cut off at q_D, the Debye wave vector. Thus, for an average energy spectrum, (26) gives

$$2\gamma_{NLC} = \frac{1}{\bar{I}}\frac{d\bar{I}(t)}{dt} = \frac{37\sqrt{2}\ \times 10^{-2}}{21}\ \frac{\pi^{3/2}\omega_{pe}}{(n_0\lambda_D^3)^2}\ \ell n\left(\frac{q'_{max}v_e}{\omega_{pe}}\right)\left[1-\frac{\bar{I}}{\chi T}\right] \tag{28}$$

where terms of order q^2 have been neglected.

The nonlinear contribution in (28) is of a higher order in the plasma parameter $(n_0\lambda_D^3)^{-1}$. In equilibrium the plasmons obey Bose distribution giving $\bar{I} = \chi T$ and rendering $\gamma_{NLC} = 0$. When the plasmons are slightly below or above the equilibrium level of energy, the above equation shows that there is growth or decay respectively. In either case, the nonlinear collisional process tries to restore equilibrium. This is quite visible from the solution of the differential equation (28) which yields

$$\frac{\bar{I}}{\bar{I}_0} = \frac{\exp\left\{(74\times 10^{-2}/21\sqrt{2})\left[\pi^{3/2}\omega_{pe}\big/(n_0\lambda_D^3)^2\right]\ell n(q'_{max}v_e/\omega_{pe})t\right\}}{1-(\bar{I}_0/\chi T)+(\bar{I}_0/\chi T)\exp\left\{\frac{74\times 10^{-2}}{21\sqrt{2}}\ \frac{\pi^{3/2}\omega_{pe}}{(n_0\lambda_D^3)^2}\ \ell n\left(\frac{q'_{max}v_e}{\omega_{pe}}\right)t\right\}} \tag{29}$$

where $\bar{I}_0$ is the equilibrium value of $\bar{I}(t)$ at $t = 0$.

CHAPTER IV

A STABILITY MECHANISM FOR ION ACOUSTIC WAVES

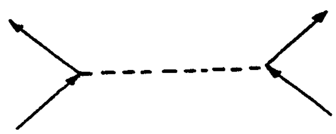
4.1 Introduction

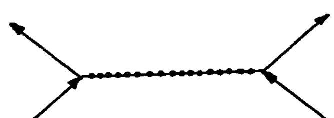
The problem of linearly unstable ion acoustic modes has been studied by Kadomtsev (1965) and more recently by Sloan and Drummond (1970). The ion acoustic waves become unstable when the component of electron beam velocity in the direction of wave propagation exceeds the phase velocity $c_s = \sqrt{\frac{T_e}{m_i}}$ (Jackson, 1960), where T_e is the electron temperature and m_i is the ion mass. In this case, direct electron-ion Coulomb interactions are negligible as has been discussed by Kadomtsev and Pogutse (1967). Kadomtsev argued that nonlinear Landau damping of ion acoustic waves by the ions resembles the damping of Langmuir waves by the electrons, and therefore, he suggested ion nonlinear damping as the mechanism for terminating the instability of the ion acoustic modes. Thus, the electron nonlinear Landau growth was thought to be of little significance for ion acoustic instability. This result was later used by Sagdeev (1967) to calculate the well-known Sagdeev formula for anomalous resistivity. Sloan and Drummond (1970), questioned the validity of Kadomtsev's latter assumption and therefore, the anomalous resistivity results of Sagdeev. They calculated the electron nonlinear Landau growth and compared it with the ion nonlinear Landau damping and concluded that the dominant nonlinear mechanism is electron nonlinear Landau growth which would be further destabilizing. It therefore, became necessary to consider other processes such as mode coupling (Drummond and Pines, 1962; Vedenov et al., 1962; Aamodt and Drummond, 1964). In mode coupling, however, transfer of energy from one mode to another may arrest the growth of one unstable mode at the cost of exciting others and is not necessarily a stabilization mechanism.

Since the ion-acoustic mode plays an important role in the heating of charged particles, for example, in the magnetosphere causing auroral precipitation (Buneman, 1959; Swift, 1965; Kennel and Petschek, 1969), as well as in laboratory plasmas (Jones and Alexeff, 1965; Alexeff et al., 1968; Okamoto, Tamagawa, 1971), this phenomenon will be further investigated in this chapter. As a stabilization mechanism for the ion-acoustic modes, the amplitude dependent frequency shift (Fukai, et al. 1969, 1970) which is of the same order of magnitude as the nonlinear Landau damping (growth) will be considered and its consequences discussed. It will also be demonstrated that the nonlinear dielectric properties of the medium modifies the wave-particle coupling constant and that it has stabilizing influence on the wave (Selim and Krishan to be published in the Phys. of Fluids, 1971-1972). In section 2, the frequency and growth rate of the waves for the linear approximation is derived and section 3 contains the calculations of the nonlinear dielectric function and the amplitude dependent frequency shift. In section 4 a comparison between the result obtained in this chapter and that estimated by Sloan and Drummond (1970) is made.

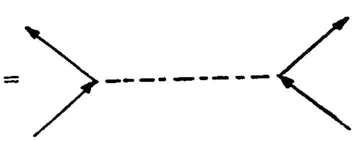
4.2 Linear Dielectric Function and Growth Rate

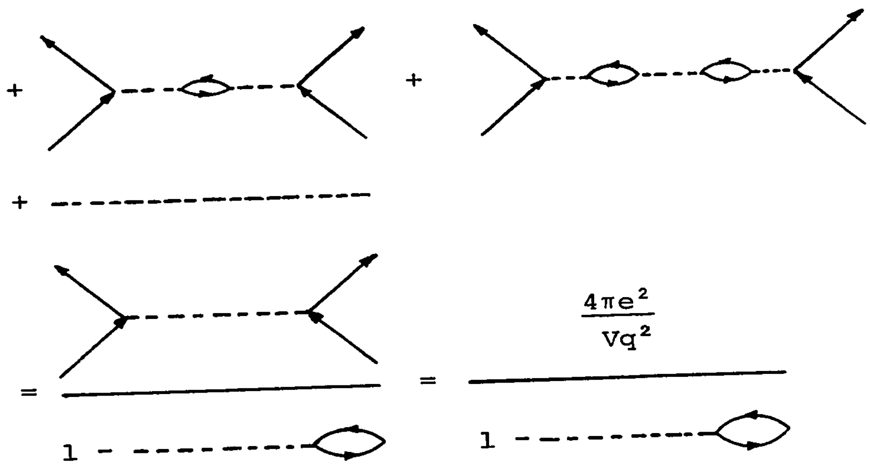
In the Feynman diagram representation as shown in chapter II, the bare Coulomb potential as well as the effective potential (which takes into account the screening of the Coulomb field) are given by

$V_C =$  $= \dfrac{4\pi e^2}{Vq^2}$ (1)

$V_{eff} =$  $= \dfrac{4\pi e^2}{Vq^2 \varepsilon(\vec{q},\omega)}$ (2)

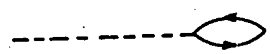
where e is the electronic charge, V is the volume of the system, $\vec{q}$ is the wave vector for the qth Fourier component and $\varepsilon(\vec{q},\omega)$ is the linear dielectric function of the plasma. In finding $\varepsilon(\vec{q},\omega)$ in the random phase approximation, one adds an infinite series such as

$V_{eff} =$  $=$ 



(3)

by using the Dyson equation. From equations (2) and (3) it is obvious that the linear dielectric function can be written as

$$\varepsilon(\vec{q},\omega) = 1 - \sum_j$$  (4)

where j is the summation over the plasma species with reference to electron and ion loops. The 'bubble' in the series which is the second term in (4) will be calculated by using the S-matrix theory which has been developed in chapter II. Accordingly, use will be made of the following rules:

(1) in the expression of the second order matrix element a factor $\frac{4\pi e^2}{Vq^2}$ appears in the product for each Coulomb line,

(2) a factor $iG(E,E_{\vec{k}}) = \frac{i}{E-E_{\vec{k}}+i\delta}$ appears for each internal particle line, where $E_{\vec{k}}$ and E are energy and energy parameters of the particle in units of $\hbar$.

(3) sum over the internal variable $\vec{k}$ and integrate over the internal energy parameters E,

(4) multiply by (-1) for each fermion loop and introduce $\delta(E_i - E_f)$ which appears in the theory and serves to conserve energy between initial and final states,

(5) use the relationship ((86) of chapter II) between the S and M matrices to find the matrix elements of the second, third, fourth and other terms in (3).

Applying these rules to the calculation of $\varepsilon(\vec{q},\omega)$, one obtains

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi e^2}{Vq^2} \sum_{j,\vec{k}} \frac{1}{(2\pi i\hbar)} \int \frac{dE}{(E-E_{\vec{k}}+i\delta)(E+\omega-E_{\vec{k}+\vec{q}}+i\delta)} \qquad (5)$$

Performing the E-integration yields

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi e^2}{Vq^2\hbar} \sum_{j,\vec{k}} \left\{ \frac{f_j(\vec{k})}{(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})} + \frac{f_j(\vec{k}+\vec{q})}{(E_{\vec{k}+\vec{q}}-\omega-E_{\vec{k}})} \right\} \qquad (6)$$

A considerable simplification can be made by the linear transformation $\vec{k}+\vec{q}\rightarrow\vec{k}$ in the second term since, in any event, the summation is over all values of $\vec{k}$. Thus, there is a common factor $f_j(\vec{k})$ which is the number of particles in the state $\vec{k}$. Therefore, (6) yields

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi e^2}{\hbar q^2} \sum_j \int d^3v f_j(\vec{v}) \left[\frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)} - \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)} \right] \qquad (7)$$

by substituting $E_{\vec{k}} = \frac{\hbar k^2}{2m_j}$ and $\hbar\vec{k} = m_j\vec{v}$ after changing $\vec{k}$-

summation into $\vec{v}$ integration, (7) thus simplifies into

$$\varepsilon(\vec{q},\omega) = 1 - \sum_j \frac{4\pi e^2}{m_j} \int d^3v \, \frac{f_j(\vec{v})}{(\omega-\vec{q}\cdot\vec{v})^2} \tag{8}$$

The distribution functions for electrons streaming with velocity $\vec{u}$ and ions with zero mean velocity are

$$f_e(\vec{v}) = \frac{n_0}{(2\pi v_e^2)^{3/2}} \, e^{-1/2 \frac{(\vec{v}-\vec{u})^2}{v_e^2}} \tag{9}$$

$$f_i(\vec{v}) = \frac{n_0}{(2\pi v_i^2)^{3/2}} \, e^{-1/2 \frac{v^2}{v_i^2}} \tag{10}$$

where v_e, v_i are the electron and ion thermal velocities respectively. Substituting (9) and (10) into (8) and performing the velocity integrations by asymptotic expansion (Fried and Conte, 1961), the linear dielectric function for a wave of frequency ω and wave vector $\vec{q}$ is given by

$$\varepsilon(\vec{q},\omega) = 1 + \frac{1}{q^2\lambda_{De}^2}\left(1 - i\sqrt{\frac{\pi}{2}}\,\frac{\vec{q}\cdot\vec{u}-\omega}{qv_e}\right) - \frac{\omega_{pi}^2}{\omega^2} \tag{11}$$

where $\lambda_{De} = (T_e/4\pi e^2 n_0)^{1/2}$, the Debye wave-length and ω_{pi}, the ion plasma frequency. The dielectric function given by (11) is identical to the results obtained by standard techniques. The frequency ω can be written as $\omega = \omega_{\vec{q}} + i\gamma_{\vec{q}}$, which when substituted in the equation

$$\varepsilon(\vec{q},\omega) = 0 \tag{12}$$

will determine the frequency of oscillations and the growth rate for the ion acoustic wave. Solving (12) for $\omega_{\vec{q}}$ and γ_q yields

$$\omega_{\vec{q}} = q\sqrt{\frac{T_e}{m_i}} = qc_s \tag{13}$$

$$\gamma_q = \sqrt{\frac{\pi}{8}}\,\frac{c_s}{v_e}\,(\vec{q}\cdot\vec{u} - \omega_{\vec{q}}) \tag{14}$$

where $\gamma_q > 0$ for $\vec{q}\cdot\vec{u} > \omega_{\vec{q}}$ corresponds to growth of the wave. This means that the ion acoustic mode is unstable in a cone of $\vec{q}$- space centered about the direction of the drift velocity u with an apex angle 2α given by $\cos\alpha = c_s/u$.

Another approach through which one can implement derivation of the growth rate γ_q, is the use of the rate equation. If $N(\vec{q})$ is the number of phonons in state $\vec{q}$ and $f_e(\vec{k})$ is the number of electrons in state $\vec{k}$, then, the rate equation for the sound waves can be written as

$$\frac{dN(\vec{q})}{dt} = \frac{2\pi}{\hbar^2}\sum_{\vec{k}} |M_0|^2 N(\vec{q})\vec{q}\cdot\frac{\partial f_e(\vec{k})}{\partial\vec{k}}\,\delta(E_{\vec{k}+\vec{q}}-\omega_{\vec{q}}-E_{\vec{k}})$$
$$= 2\pi\left(\frac{V}{\hbar}\right)\int d^3v|M_0|^2 N(\vec{q})\,\frac{\vec{q}}{m}\cdot\frac{\partial f_e(\vec{v})}{\partial\vec{v}}\,\delta(\vec{q}\cdot\vec{v}-\omega_{\vec{q}}) \tag{15}$$

where the direct matrix element M_0 (derived in chapter II) is of the order $\left(\frac{\hbar}{V}\right)^{1/2}$. This yields the correct classical limit. Substituting the electron distribution function (9) into (15) and performing the velocity integration, one obtains

$$\frac{1}{N(\vec{q})}\frac{dN(\vec{q})}{dt} = 2\gamma_{\vec{q}} = \frac{\sqrt{2\pi}}{mv_e^2}\left(\frac{n_0 V}{\hbar}\right)|M_0|^2 \frac{(\vec{q}\cdot\vec{u}-\omega_{\vec{q}})}{qv_e} \qquad (16)$$

which is the same as equation (14) if one substitutes

$$M_0 = \left[\frac{4\pi e^2\hbar\omega_{\vec{q}}}{Vq^2F_{\vec{q}}}\right]^{1/2}, \quad F_{\vec{q}} = \left|\frac{\partial}{\partial\omega}\left(\omega\varepsilon(\vec{q},\omega)\right)\right|_{\omega=\omega_{\vec{q}}} \qquad (16.a)$$

In calculating $F_{\vec{q}}$, the linear dielectric function and the linear frequency of the waves have often been used. However, for the case of unstable oscillations, one should use the nonlinear frequency of the waves and the nonlinear dielectric function in calculating $F_{\vec{q}}$. These modifications reduce the coupling constant and consequently the growth rate.

4.3 Nonlinear Dielectric Function and Frequency Shift

In order to calculate the nonlinear dielectric function $\varepsilon_{NL}(\omega,\vec{q})$, one has to know the nonlinear processes (J. Coste, 1969). The basic processes responsible for the nonlinearities in the medium are those where the bare Coulomb potentials represented by dotted lines in (3), is replaced by the effective potentials which are represented by beaded lines and the simple loops in the same equation are replaced by the square and triangular loops shown in figs. 19 and 20. Then, following the method of the previous section, one obtains a formula for $\varepsilon_{NL}(\omega,\vec{q})$.

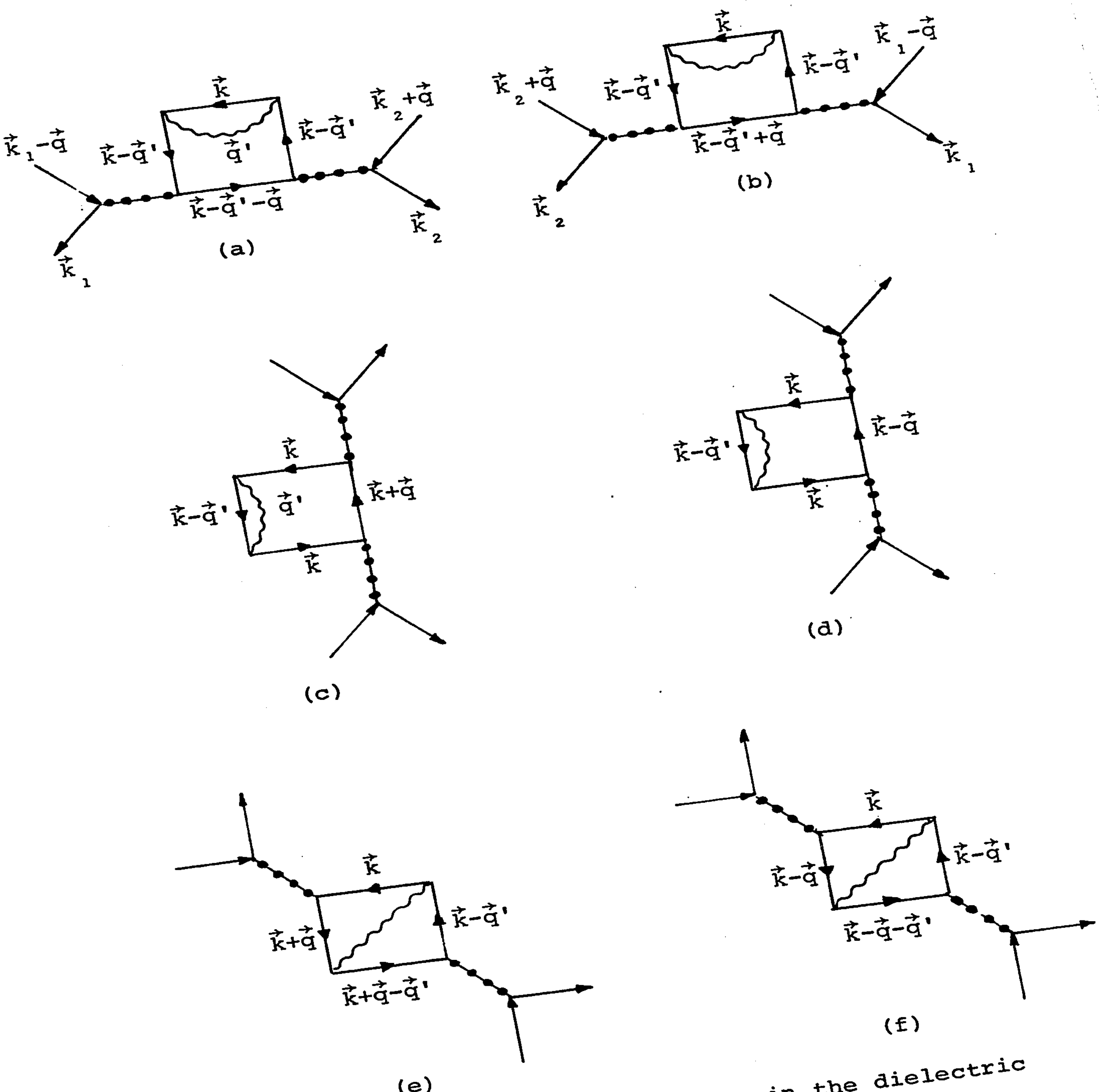


Fig. 19 Nonlinear effects in the dielectric function (square loops)

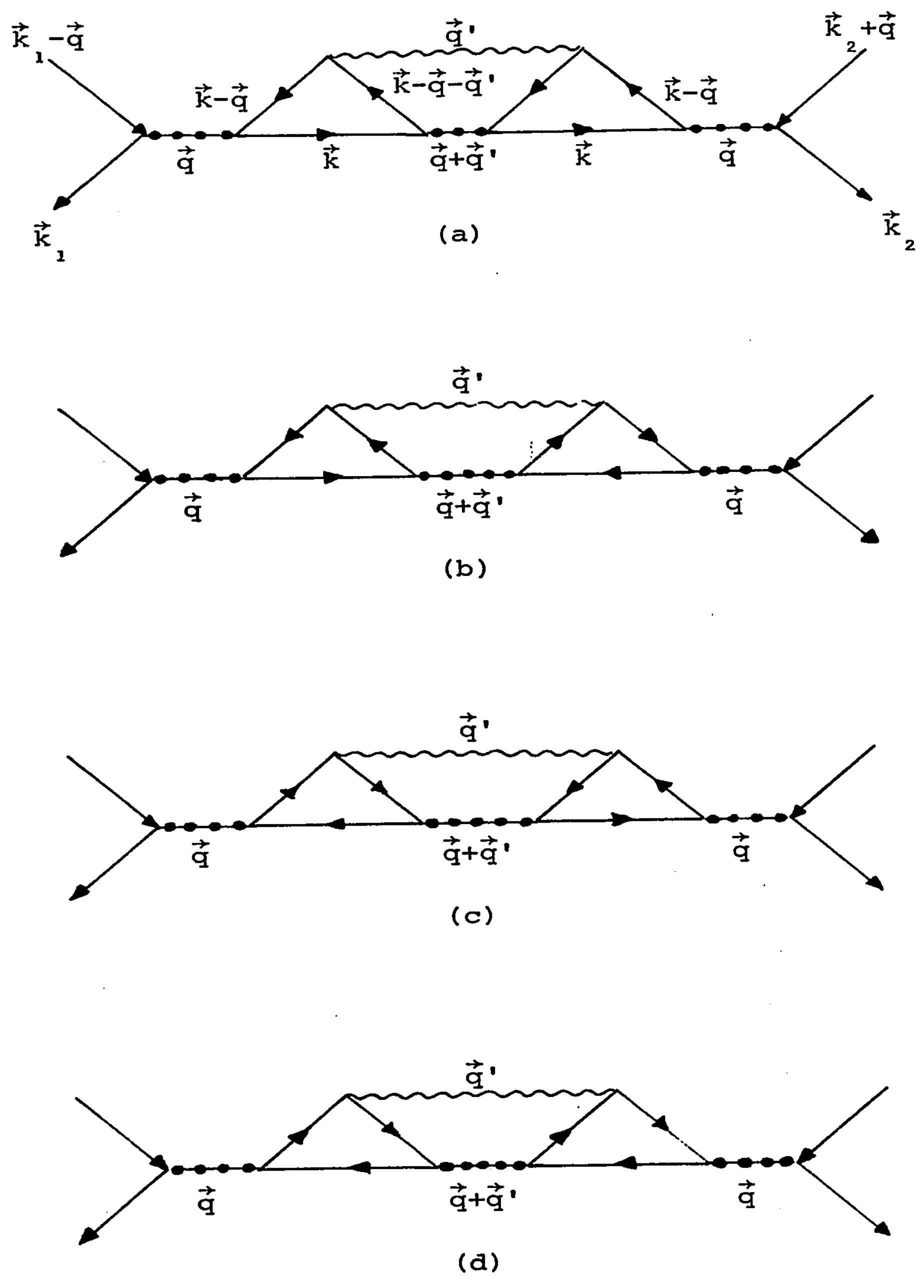
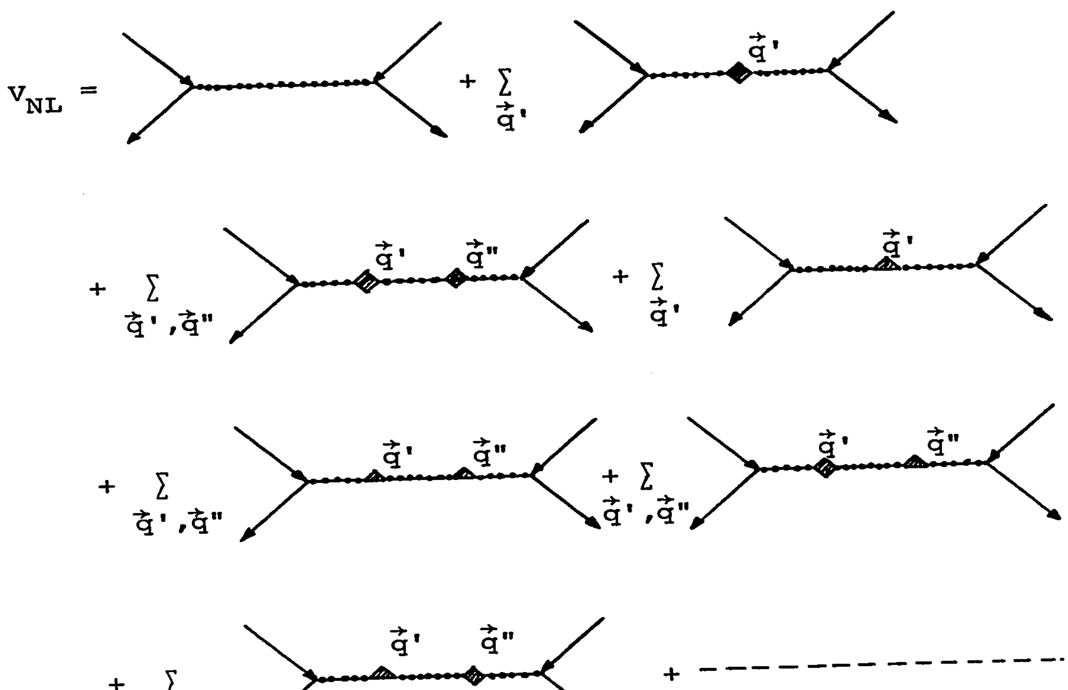


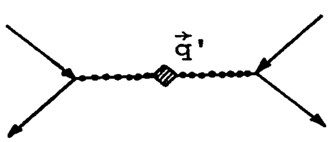
Fig. 20 Nonlinear effects in the dielectric function (triangle loops)

In the nonlinear limit, the effective potential V_{NL} is given by the sum of an infinite series as shown diagrammatically in the following equation



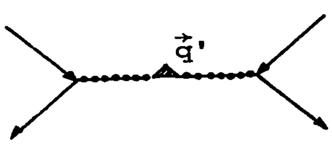
(17)

where

$\sum_{\vec{q}',j}$ 

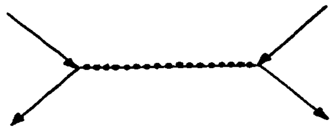
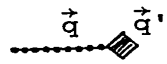
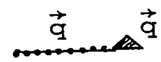
= Sum of the diagrams in fig. 19, (18)

and

$\sum_{\vec{q}',j}$ 

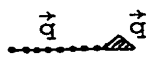
= Sum of the diagrams in fig. 20. (19)

Summing the infinite series (17), one obtains

$V_{NL} =$  $\left(1 - \sum_{\vec{q}',j}\right.$  $- \sum_{\vec{q}',j}$  $\left.\right)^{-1}$

$$= \frac{4\pi e^2}{Vq^2 \varepsilon_{NL}} \qquad (20)$$

where $\varepsilon_{NL}(\vec{q},\omega)$, the nonlinear dielectric function is given by

$\varepsilon_{NL} = \varepsilon_L \left[1 - \sum_{\vec{q}',j}\right.$  $- \sum_{\vec{q}',j}$  $\left.\right]$, (21)

In the above equation ε_L is the linear dielectric function given by (11),

$\sum_{\vec{q}',j}$  $= \alpha_1$ = Sum of the diagrams in fig. 19/V_{eff}, (22)

$\sum_{\vec{q}'j}$  $= \alpha_2$ = Sum of the diagrams in fig. 20/V_{eff}, (23)

and the numerators of (22) and (23) are the transition matrix elements of the interactions shown in figs. 19 and 20. These matrix elements can be calculated with the help of the rules developed in chapter II. Let $S^{(4)}_{r=a\ldots f}$ be the S-matrix element corresponding to the diagrams in fig. 19, then one can write

$$S_a^{(4)} = \frac{(-i)^4}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L(\vec{q},\omega)}\right]^2 \sum_{j,\vec{q}\,'}\left[\frac{4\pi e^2\hbar\omega_{\vec{q}\,'}}{Vq'^2F_{\vec{q}\,'}}\right]N(\vec{q}\,')(-1)\times$$

$$\left[\sum_{\vec{k}}\int dE\,\frac{i^4}{(E-E_{\vec{k}}+i\delta)(E-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'}+i\delta)^2(E-\omega-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}-\vec{q}\,'}+i\delta)}\right.$$

$$\left.+\ \text{Similar term with } \omega_{\vec{q}\,'} \rightarrow -\omega_{\vec{q}\,'}\right]\delta(E_i-E_f) \qquad (24)$$

where $(-i/\hbar)^4$, due to the four vertices in the diagram; $(4\pi e^2/Vq^2\varepsilon_L)$, is the effective Coulomb field; $(4\pi e^2\hbar\omega_{\vec{q}\,'}/Vq'^2F_{\vec{q}\,'})^{1/2}$, is the coupling constant due to wave-particle interaction; (-1), is because of the single fermion loop, the four energy denominators are due to the four virtual particle lines and $N(\vec{q}\,')$, is due to integration over the energy parameter of the virtual plasmons because, there are $N(\vec{q}\,')$ plasmons in the state $\vec{q}\,'$. Thus, the summation over the plasmons states $\vec{q}\,'$ simply becomes the summation over the wave vector $\vec{q}\,'$. Finally, the summation j is over the electron and ion loops, and the last term is due to the existence of plasmons propagating in the opposite direction, i.e. a plasmon can be emitted where it was absorbed before and vice versa without violating the conservation of momentum and energy parameters.

Similarly, one can write the scattering matrix elements for diagrams b and c.

$$S_b^{(4)} = \frac{(-i)^4}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'}\left[\frac{4\pi e^2\hbar\omega_{\vec{q}\,'}}{Vq'^2F_{\vec{q}\,'}}\right]N(\vec{q}\,')(-1)\ \times$$

$$\left[\sum_{\vec{k}}\int dE\ \frac{i^4}{(E-E_{\vec{k}}+i\delta)^2(E-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'}+i\delta)(E+\omega-E_{\vec{k}+\vec{q}}+i\delta)}\right.$$

$$\left. + \text{ similar terms with } \omega_{\vec{q}\,'} \to -\omega_{\vec{q}\,'}\right]\delta(E_i-E_f) \qquad (25)$$

$$S_c^{(4)} = \frac{(-i)^4}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'}\left[\frac{4\pi e^2\hbar\omega_{\vec{q}\,'}}{Vq'^2F_{\vec{q}\,'}}\right]N(\vec{q}\,')(-1)$$

$$\left[\sum_{\vec{k}}\int dE\ \frac{i^4}{(E-E_{\vec{k}}+i\delta)(E-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'}+i\delta)(E-\omega_{\vec{q}\,'}+\omega-E_{\vec{k}-\vec{q}\,'+\vec{q}}+i\delta)}\right.$$

$$\times\ \frac{1}{(E+\omega-E_{\vec{k}+\vec{q}}+i\delta)}$$

$$\left. + \ .\ .\ .\ .\ .\ .\right]\delta(E_i-E_f) \qquad (26)$$

Performing the E-integration in (24) one obtains

$$S_a^{(4)} = \frac{4\pi e^2}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'}\frac{\hbar\omega_{\vec{q}\,'}N(\vec{q}\,')}{Vq'^2F_{\vec{q}\,'}}\ (-2\pi i)\,\delta(E_i-E_f)$$

$$\left[\sum_{\vec{k}}\left[\frac{f_j(\vec{k})}{(E_{\vec{k}}-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'})^2(E_{\vec{k}}-\omega-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}-\vec{q}\,'})}\right.\right.$$

$$+\ \frac{f_j(\vec{k}-\vec{q}\,'-\vec{q})}{(E_{\vec{k}-\vec{q}\,'-\vec{q}}+\omega-E_{\vec{k}-\vec{q}\,'})^2(E_{\vec{k}-\vec{q}\,'-\vec{q}}+\omega_{\vec{q}\,'}+\omega-E_{\vec{k}})}$$

$$- \frac{f_j(\vec{k}-\vec{q}\,')}{(E_{\vec{k}-\vec{q}\,'}-\omega-E_{\vec{k}-\vec{q}\,'-\vec{q}})^2(E_{\vec{k}-\vec{q}\,'}+\omega_{\vec{q}\,'}-E_{\vec{k}})}$$

$$\left. - \frac{f_j(\vec{k}-\vec{q}\,')}{(E_{\vec{k}-\vec{q}\,'}+\omega_{\vec{q}\,'}-E_{\vec{k}})^2(E_{\vec{k}-\vec{q}\,'}-\omega-E_{\vec{k}-\vec{q}\,'-\vec{q}})}\right]$$

$$\left. + \cdot\cdot\cdot\cdot\cdot\cdot \right] \delta(E_i-E_f) \qquad (27)$$

In equation (27), transforming $\vec{k}-\vec{q}\,'-\vec{q} \rightarrow \vec{k}$ in the second term and $\vec{k}-\vec{q}\,' \rightarrow \vec{k}$ in the third and fourth terms, permits rewriting equation (27) with a common factor $f_j(\vec{k})$

$$S_a^{(4)} = \frac{4\pi e^2}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'} \frac{\hbar\omega_{\vec{q}\,'}N(\vec{q}\,')}{Vq'^2F_{\vec{q}\,'}} \left[\sum_{\vec{k}} (-2\pi i) f_j(\vec{k}) \times\right.$$

$$\left[\frac{1}{(E_{\vec{k}}-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'})^2(E_{\vec{k}}-\omega-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'-\vec{q}})}\right.$$

$$+ \frac{1}{(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})^2(E_{\vec{k}}+\omega_{\vec{q}\,'}+\omega-E_{\vec{k}+\vec{q}\,'+\vec{q}})}$$

$$- \frac{1}{(E_{\vec{k}}-\omega-E_{\vec{k}-\vec{q}})^2(E_{\vec{k}}+\omega_{\vec{q}\,'}-E_{\vec{k}+\vec{q}\,'})}$$

$$\left. - \frac{1}{(E_{\vec{k}}+\omega_{\vec{q}\,'}-E_{\vec{k}+\vec{q}\,'})^2(E_{\vec{k}}-\omega-E_{\vec{k}-\vec{q}})}\right]$$

$$\left. + \cdot\cdot\cdot\cdot\cdot\cdot \right] \delta(E_i-E_f) \qquad (28)$$

Similarly, performing the E-integration and making the necessary transformation in $\vec{k}$, equations (25) and (26) yield

$$S_b^{(4)} = \frac{4\pi e^2}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}'} \frac{\hbar\omega_{\vec{q}'}N(\vec{q}')}{Vq'^2F_{\vec{q}'}} \left[\sum_{\vec{k}} (-2\pi i) f_j(\vec{k}) \times \right.$$

$$\left[-\frac{1}{(E_{\vec{k}}-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'})^2(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})}\right.$$

$$-\frac{1}{(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})^2(E_{\vec{k}}-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'})}$$

$$+\frac{1}{(E_{\vec{k}}+\omega_{\vec{q}'}-E_{\vec{k}+\vec{q}'})^2(E_{\vec{k}}+\omega_{\vec{q}'}+\omega-E_{\vec{k}+\vec{q}'+\vec{q}})}$$

$$\left.+\frac{1}{(E_{\vec{k}}-\omega-E_{\vec{k}-\vec{q}})^2(E_{\vec{k}}-\omega-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'-\vec{q}})}\right]$$

$$\left. + \; .\;.\;.\;.\;.\;.\; \right]\delta(E_i-E_f) \tag{29}$$

$$S_c^{(4)} = \frac{4\pi e^2}{\hbar^4}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{\vec{q}'} \frac{\hbar\omega_{\vec{q}'}N(\vec{q}')}{Vq'^2F_{\vec{q}'}} \left[\sum_{\vec{k}} (-2\pi i) f_j(\vec{k})\right.$$

$$\left[\frac{1}{(E_{\vec{k}}-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'})(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})(E_{\vec{k}}-\omega_{\vec{q}'}+\omega-E_{\vec{k}-\vec{q}'+\vec{q}})}\right.$$

$$+\frac{1}{(E_{\vec{k}}+\omega_{\vec{q}'}-E_{\vec{k}+\vec{q}'})(E_{\vec{k}}+\omega-E_{\vec{k}+\vec{q}})(E_{\vec{k}}+\omega_{\vec{q}'}+\omega-E_{\vec{k}+\vec{q}'+\vec{q}})}$$

$$+ \frac{1}{(E_{\vec{k}}+\omega_{\vec{q}'}-E_{\vec{k}+\vec{q}'})(E_{\vec{k}}-\omega-E_{\vec{k}-\vec{q}})(E_{\vec{k}}+\omega_{\vec{q}'}-\omega-E_{\vec{k}+\vec{q}'-\vec{q}})}$$

$$+ \frac{1}{(E_{\vec{k}}-\omega_{\vec{q}'}-E_{\vec{k}+\vec{q}'})(E_{\vec{k}}-\omega-E_{\vec{k}-\vec{q}})(E_{\vec{k}}-\omega_{\vec{q}'}-\omega-E_{\vec{k}-\vec{q}'-\vec{q}})}$$

$$+ \;.\;.\;.\;.\;.\;.\;.\Bigg]\delta(E_i-E_f) \tag{30}$$

In the classical limit, one substitutes $E_k = \frac{\hbar k^2}{2m_j}$, $\hbar\vec{k} = m_j\vec{v}$ and changes the k-summation into integration to obtain

$$S_a^{(4)} = \left(\frac{-2\pi i}{\hbar}\right)\frac{4\pi e^2}{\hbar^3}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}'} \frac{\hbar\omega_{\vec{q}'}N(\vec{q}')}{q'^2F_{\vec{q}'}} \Bigg[\int d^3v f_j(\vec{v})$$

$$\Bigg[- \frac{1}{\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}+\vec{q}')^2\right)}$$

$$+ \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)^2\left(\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}+\vec{q}')^2\right)}$$

$$- \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)^2\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)}$$

$$+ \frac{1}{\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)}\Bigg]$$

$$+ \;.\;.\;.\;.\;.\;.\;.\Bigg]\delta(E_i-E_f)\;, \tag{31}$$

$$S_b^{(4)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{4\pi e^2}{\hbar^3} \left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'} \frac{\hbar\omega_{\vec{q}\,'} N(\vec{q}\,')}{q'^2 F_{\vec{q}\,'}} \left[\int d^3v f_j(\vec{v})\right.$$

$$\left[-\frac{1}{\left(\omega_{\vec{q}\,'}-\vec{q}\,'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)}\right.$$

$$+\frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)^2\left(\omega_{\vec{q}\,'}-\vec{q}\,'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)}$$

$$+\frac{1}{\left(\omega_{\vec{q}\,'}-\vec{q}\,'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega+\omega_{\vec{q}\,'}-(\vec{q}+\vec{q}\,')\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}+\vec{q}\,')^2\right)}$$

$$\left.-\frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)^2\left(\omega+\omega_{\vec{q}\,'}-(\vec{q}+\vec{q}\,')\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}+\vec{q}\,')^2\right)}\right]$$

$$\left.+\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\right]\delta(E_i-E_f)\;, \tag{32}$$

$$S_c^{(4)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{4\pi e^2}{\hbar^3} \left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}\,'} \frac{\hbar\omega_{\vec{q}\,'} N(\vec{q}\,')}{q'^2 F_{\vec{q}\,'}} \left[\int d^3v f_j(\vec{v})\right.$$

$$\left[-\frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}\,'}-\vec{q}\,'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)\left(\omega-\omega_{\vec{q}\,'}-(\vec{q}-\vec{q}\,')\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}-\vec{q}\,')^2\right)}\right.$$

$$+\frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}\,'}-\vec{q}\,'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)\left(\omega+\omega_{\vec{q}\,'}-(\vec{q}+\vec{q}\,')\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}+\vec{q}\,')^2\right)}$$

$$+ \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q}{2m_j}\right)\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)\left(\omega-\omega_{\vec{q}'}-(\vec{q}-\vec{q}')\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}-\vec{q}')^2\right)}$$

$$- \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)\left(\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}+\vec{q}')^2\right)}\Bigg]$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \Bigg]\delta(E_i-E_f) \qquad (33)$$

The S-matrix for diagrams d, e and f can be obtained from a, b and c respectively by making the transform $\vec{q} \rightleftharpoons -\vec{q}$ and $\omega \rightleftharpoons -\omega$

$$S_d^{(4)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{4\pi e^2}{\hbar^3} \left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}'} \frac{\hbar\omega_{\vec{q}'}N(\vec{q}')}{q'^2F_{\vec{q}'}} \Bigg[\int d^3v f_j(\vec{v})$$

$$\Bigg[- \frac{1}{\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega_{\vec{q}'}-\omega-(\vec{q}'-\vec{q})\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}'-\vec{q})^2\right)}$$

$$+ \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)^2\left(\omega_{\vec{q}'}-\omega-(\vec{q}'-\vec{q})\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}'-\vec{q})^2\right)}$$

$$- \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)^2\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)}$$

$$- \frac{1}{\left(\omega_{\vec{q}'}-q'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)^2\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)}\Bigg]$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \Bigg] \delta(E_i - E_f), \tag{34}$$

$$S_e^{(4)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{4\pi e^2}{\hbar^3} \left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{k}'} \frac{\hbar\omega_{\vec{q}'} N(\vec{q}')}{q'^2 F_{\vec{q}'}} \Bigg[\int d^3v f_j(\vec{v})$$

$$\Bigg[+ \frac{1}{\left(\omega_{\vec{q}'} - \vec{q}'\cdot\vec{v} + \frac{\hbar q'^2}{2m_j}\right)^2 \left(\omega - \vec{q}\cdot\vec{v} + \frac{\hbar q^2}{2m_j}\right)}$$

$$+ \frac{1}{\left(\omega - \vec{q}\cdot\vec{v} + \frac{\hbar q^2}{2m_j}\right)^2 \left(\omega_{\vec{q}'} - \vec{q}'\cdot\vec{v} + \frac{\hbar q'^2}{2m_j}\right)}$$

$$+ \frac{1}{\left(\omega_{\vec{q}'} - \vec{q}'\cdot\vec{v} - \frac{\hbar q'^2}{2m_j}\right)^2 \left(\omega_{\vec{q}'} - \omega - (\vec{q}' - \vec{q})\cdot\vec{v} - \frac{\hbar}{2m_j}(\vec{q}' - \vec{q})^2\right)}$$

$$- \frac{1}{\left(\omega - \vec{q}\cdot\vec{v} - \frac{\hbar q^2}{2m_j}\right)^2 \left(\omega_{\vec{q}'} - \omega - (\vec{q}' - \vec{q})\cdot\vec{v} + \frac{\hbar}{2m_j}(\vec{q}' - \vec{q})^2\right)}\Bigg]$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \Bigg] \delta(E_i - E_f), \tag{35}$$

$$S_f^{(4)} = \left(\frac{-2\pi i}{\hbar}\right) \frac{4\pi e^2}{\hbar^3} \left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{j,\vec{q}'} \frac{\hbar\omega_{\vec{q}'} N(\vec{q}')}{q'^2 F_{\vec{q}'}} \Bigg[\int d^3v f_j(\vec{v})$$

$$\Bigg[- \frac{1}{\left(\omega - \vec{q}'\cdot\vec{v} + \frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}'} - \vec{q}'\cdot\vec{v} + \frac{\hbar q'^2}{2m_j}\right)\left(\omega_{\vec{q}'} + \omega - (\vec{q} + \vec{q}')\cdot\vec{v} + \frac{\hbar}{2m_j}(\vec{q} + \vec{q}')^2\right)}$$

$$- \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}+\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\omega-(\vec{q}'-\vec{q})\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}'-\vec{q})^2\right)}$$

$$+ \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}-\frac{\hbar q'^2}{2m_j}\right)\left(\omega_{\vec{q}'}+\ -(\vec{q}'+\vec{q})\cdot\vec{v}-\frac{\hbar}{2m_j}(\vec{q}'+\vec{q})^2\right)}$$

$$+ \frac{1}{\left(\omega-\vec{q}\cdot\vec{v}-\frac{\hbar q^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v}+\frac{\hbar q'^2}{2m_j}\right)\left(\omega_{\vec{q}'}-\omega-(\vec{q}'-\vec{q})\cdot\vec{v}+\frac{\hbar}{2m_j}(\vec{q}'-\vec{q})^2\right)}\Bigg]$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \Bigg]\delta(E_i-E_f) \qquad (36)$$

By adding the S-matrix elements as obtained from the six diagrams in fig. 19 one obtains the transition matrix element M_1. In the above relations, the factor $(-2\pi i/\hbar)$ which appears in the S-matrix will be cancelled. Since $\hbar\omega_{\vec{q}'}N(\vec{q}')$ is a classical quantity defining the wave energy $I(\vec{q}')$ and the classical limit of the M-matrix has to be independent of $\hbar$, one has to expand the denominators in equations (31) - (35) up to the order of $\hbar^3$. In doing so, terms of order of $\hbar^0$, $\hbar$ and $\hbar^2$ must vanish identically. After some tedious algebra, this trivial condition has been verified and therefore, the matrix element M_1 is given by

$$M_1 = \frac{V_{eff}}{n_0 V q^2 \varepsilon_L} \sum_j \frac{\omega_{pj}^4}{m_j} \sum_{\vec{q}'\neq\vec{q}} \frac{I(\vec{q}')}{q'^2 F_{\vec{q}'}} \int d^3v f_j(\vec{v}) \Lambda_j(\vec{q}_1 \vec{q}', \vec{v}) \qquad (37)$$

where

$$\Lambda_j(\vec{q},\vec{q}',\vec{v}) = \frac{8q^3q'^3}{\lambda^3\lambda'^3}\cos\theta + 4\Bigg\{q^4q'^2\left[\frac{2}{\lambda'^2(\lambda'^2-\lambda^2)^2} - \frac{1}{\lambda^2\lambda'^2(\lambda'^2-\lambda^2)}\right]$$

$$+ q^2q'^4\left[\frac{2}{\lambda^2(\lambda'^2-\lambda^2)^2} + \frac{1}{\lambda^2\lambda'^2(\lambda'^2-\lambda^2)}\right]\Bigg\}\cos^2\theta$$

$$- \frac{16q^3q'^3}{\lambda\lambda'(\lambda'^2-\lambda^2)^2}\cos^3\theta\ , \qquad (38)$$

$$\lambda = \omega - \vec{q}\cdot\vec{v},\quad \lambda' = \omega_{\vec{q}'} - \vec{q}'\cdot\vec{v}, \qquad (38.a)$$

and θ is the angle between $\vec{q}$ and $\vec{q}'$.

Following the same rules, one can write the S-matrix elements for processes shown in fig. 20 and consequently the transition matrix element M_2.

$$S_a^{(5)} = \frac{(-i)^5}{\hbar^5}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{q'\neq q} \frac{4\pi e^2\hbar\omega_{\vec{q}'}N(\vec{q}')}{Vq'^2F_{\vec{q}'}}\left[\frac{4\pi e^2}{V|\vec{q}+\vec{q}'|^2\varepsilon(\vec{q}+\vec{q}',\omega+\omega_{\vec{q}'})}\right.$$

$$\Bigg\{\sum_{j,\vec{k}}\int dE\,\frac{i^3}{(E-E_{\vec{k}})(E-\omega-E_{\vec{k}-\vec{q}})(E-\omega-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'-\vec{q}})}$$

$$\sum_{s,\vec{k}'}\int dE'\,\frac{i^3}{(E'-E_{\vec{k}})(E'-\omega-E_{\vec{k}-\vec{q}})(E'-\omega-\omega_{\vec{q}'}-E_{\vec{k}-\vec{q}'-\vec{q}})}\Bigg\}$$

$$\left. + \text{Similar terms with } \omega_{\vec{q}'} \to -\omega_{\vec{q}'}\right]\delta(E_i-E_f) \qquad (39)$$

where the factors, $(4\pi e^2/Vq^2\varepsilon_L)$, is due to the effective Coulomb field; $[4\pi e^2\hbar\omega q'/Vq'^2F_{\vec{q}'}]^{1/2}$, is the wave-particle coupling constant; $4\pi e^2/V|\vec{q}+\vec{q}\,'|^2\varepsilon(\vec{q}+\vec{q}\,',\omega+\omega_{q'})$, is the effective Coulomb field line between the two triangles and $(-i)^5/\hbar^5$, due to the five vertices. The six energy denominators are due to the six virtual fermion lines, and the 'similar terms' have the same meaning as in the square process.

Also, the S-matrix element due to diagram (b) of fig. 20 is

$$S_b^{(5)} = \frac{(-i)^5}{\hbar^5}\left[\frac{4\pi e^2}{Vq^2\varepsilon_L}\right]^2 \sum_{\vec{q}\,'\neq q} \frac{4\pi e^2\hbar\omega_{\vec{q}\,'}N(\vec{q}\,')}{Vq'^2F_{\vec{q}\,'}} \left[\frac{4\pi e^2}{V|\vec{q}+\vec{q}\,'|^2\varepsilon(\vec{q}+\vec{q}\,',\omega+\omega_{\vec{q}\,'})}\right.$$

$$\left\{\sum_{j,\vec{k}}\int dE\,\frac{i^3}{(E-E_{\vec{k}})(E-\omega-E_{\vec{k}-\vec{q}})(E-\omega-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'-\vec{q}})}\right. \times$$

$$\left.\sum_{s,\vec{k}}\int dE'\,\frac{i^3}{(E'-E_{\vec{k}})(E'+\omega-E_{\vec{k}+\vec{q}})(E'+\omega+\omega_{\vec{q}\,'}-E_{\vec{k}+\vec{q}\,'+\vec{q}})}\right\} \times$$

$$\left. + \;.\;.\;.\;.\;.\;.\;\right]\delta(E_i-E_f) \qquad (40)$$

Performing the E'-integration in (39) and (40), and making the necessary transformation of $\vec{k}-\vec{q}\,'-\vec{q} \to \vec{k}$, etc., and substituting $E_k = \frac{\hbar k^2}{2m_j}$, $\hbar\vec{k} = m_j\vec{v}$ and changing the k-summation into integration, one obtains by adding

$$S_a^{(5)}+S_b^{(5)} = \left(\frac{-2\pi i}{\hbar}\right)\left[\frac{(4\pi e^2)^2}{Vq^2\varepsilon_L}\right]^2 \sum_{\vec{q}\,'\neq\vec{q}} \frac{\hbar\omega_{\vec{q}\,'}N(\vec{q}\,')}{V^2q'^2F_{\vec{q}\,'}|\vec{q}+\vec{q}\,'|^2}\left[\frac{1}{\varepsilon(\vec{q}+\vec{q}\,',\omega+\omega_{\vec{q}\,'})}\right.$$

$$\left\{\frac{-i}{2\pi\hbar^2}\sum_{j,\vec{k}}\int\frac{dE}{(E-E_{\vec{k}})(E-\omega-E_{\vec{k}-\vec{q}})(E-\omega-\omega_{\vec{q}\,'}-E_{\vec{k}-\vec{q}\,'-\vec{q}})}\right. \times$$

$$\sum_s \int d^3v f_s(\vec{v}) \left(\frac{\hbar}{m_s}\right)^2 \frac{1}{(\omega-\vec{q}\cdot\vec{v})(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v})(\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v})}$$

$$\left[\frac{[\vec{q}'\cdot(\vec{q}+\vec{q}')]q^2}{(\omega-\vec{q}\cdot\vec{v})} + \frac{[\vec{q}\cdot(\vec{q}+\vec{q}')]q'^2}{(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v})} + \frac{\vec{q}\cdot\vec{q}'(\vec{q}+\vec{q}')^2}{\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v}}\right]\Bigg\}$$

$$+ \cdot\cdot\cdot\cdot\cdot\cdot\cdot \Bigg]\delta(E_i-E_f) \qquad (41)$$

Similarly, the sum of diagrams (c) and (d) yield

$$S_c^{(5)}+S_d^{(5)} = \left(\frac{-2\pi i}{\hbar}\right)\left[\frac{(4\pi e^2)^2}{Vq^2\varepsilon_L}\right]^2 \sum_{\vec{q}'\neq\vec{q}} \frac{\hbar\omega_{\vec{q}'} N(\vec{q}')}{V^2 q'^2 F_{\vec{q}'} |\vec{q}+\vec{q}'|}\left[\frac{1}{\varepsilon(\vec{q}+\vec{q}',\omega+\omega_{\vec{q}'})}\right.$$

$$\left\{\frac{-i}{2\pi\hbar^2}\sum_{j,\vec{k}} \int \frac{dE}{(E-E_{\vec{k}})(E+\omega-E_{\vec{k}+\vec{q}})(E+\omega+\omega_{\vec{q}'}-E_{\vec{k}+\vec{q}'+\vec{q}})} \times\right.$$

$$\sum_s \int d^3v f_s(\vec{v}) \left(\frac{\hbar}{m_s}\right)^2 \frac{1}{(\omega-\vec{q}\cdot\vec{v})(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v})(\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v})}$$

$$\left[\frac{\vec{q}'\cdot(\vec{q}+\vec{q}')q^2}{(\omega-\vec{q}\cdot\vec{v})} + \frac{\vec{q}\cdot(\vec{q}+\vec{q}')q'^2}{(\omega_{\vec{q}'}-\vec{q}'\cdot\vec{v})} + \frac{(\vec{q}\cdot\vec{q}')(\vec{q}+\vec{q}')^2}{\omega+\omega_{\vec{q}'}-(\vec{q}+\vec{q}')\cdot\vec{v}}\right]\Bigg\}$$

$$+ \cdot\cdot\cdot\cdot\cdot\cdot\cdot \Bigg]\delta(E_i-E_f) \qquad (42)$$

By adding (41) and (42) and carrying out the same procedure as for the previous case, one obtains

$$M_2 = \left[\frac{(4\pi e^2)^2}{Vq^2\varepsilon_L(\vec{q},\omega)}\right]^2 \sum_{\vec{q}'\neq\vec{q}} \frac{\hbar\omega_{\vec{q}'} N(\vec{q}')}{q'^2 F_{\vec{q}'}} \times$$

$$\left[\frac{1}{|\vec{q}+\vec{q}\,'|^2\varepsilon(\vec{q}+\vec{q}\,',\omega+\omega_{\vec{q}\,'})}\left\{\sum_j \int d^3v\,\frac{f_j(\vec{v})}{m_j{}^2}\,Y_1\right\}^2\right.$$

$$\left.+\frac{1}{|\vec{q}-\vec{q}\,'|^2\varepsilon(\vec{q}-\vec{q}\,',\omega-\omega_{\vec{q}\,'})}\left\{\sum_j \int d^3v\,\frac{f_j(\vec{v})}{m_j^2}\,Y_2\right\}^2\right] \qquad (43)$$

where

$$Y_1 = \frac{1}{\lambda\lambda'(\lambda+\lambda')}\left[\frac{\vec{q}\,'\cdot(\vec{q}+\vec{q}\,')q^2}{\lambda}+\frac{\vec{q}\cdot(\vec{q}+\vec{q}\,')q'^2}{\lambda'}+\frac{\vec{q}\cdot\vec{q}\,'(q+\vec{q}\,')^2}{\lambda+\lambda'}\right],$$

$$Y_2 = \frac{1}{\lambda\lambda'(\lambda-\lambda')}\left[\frac{\vec{q}\,'\cdot(\vec{q}-\vec{q}\,')q^2}{\lambda}+\frac{\vec{q}\cdot(\vec{q}-\vec{q}\,')q'^2}{\lambda'}+\frac{\vec{q}\cdot\vec{q}\,'(\vec{q}-\vec{q}\,')^2}{\lambda-\lambda'}\right] \qquad (44)$$

and λ, λ' are defined by (38).

The matrix element M_2 can be obtained directly from the calculations given in the last section of chapter II if one recognizes that each diagram in fig. 20 is the square of three-wave interaction (apart from some coupling constant factors). Substituting (36) and (43) into (22) and (23) respectively yield

$$\alpha_1 = \frac{1}{n_0 V q^2 \varepsilon_L}\sum_j \frac{\omega_{pj}^4}{m_j}\sum_{\vec{q}\,'\neq\vec{q}}\frac{I(\vec{q}\,')}{q'^2 F_{\vec{q}\,'}}\int d^3v f_j(\vec{v})\Lambda_j(\vec{q},\vec{q}\,',\vec{v}), \qquad (45)$$

and

$$\alpha_2 = \frac{(4\pi e^2)^3}{Vq^2\varepsilon_L}\sum_{\vec{q}\,'\neq\vec{q}}\frac{I(\vec{q}\,')}{q'^2F_{q'}}\left[\frac{1}{|\vec{q}+\vec{q}\,'|^2\varepsilon(\vec{q}+\vec{q}\,',\omega\ \ \omega_{\vec{q}\,'})}\times\right. \qquad (46)$$

$$\left.\left\{\sum_j \int \frac{d^3vf_j(\vec{v})}{m_j^2}\,Y_1\right\}^2+\frac{1}{|\vec{q}-\vec{q}\,'|^2\varepsilon(\vec{q}-\vec{q}\,',\omega-\omega_{\vec{q}\,'})}\left\{\sum_j\int\frac{d^3vf_j(\vec{v})}{m_j^2}Y_2\right\}^2\right]$$

These equations occur as equations (15) and (18) in the work of Selim and Krishan (1971-1972).

The velocity integrals in (45) and (46) are calculated by the asymptotic expansion method (Fried and Conte, 1961) where the ion integrals converge asymptotically in the limit $\frac{v_i}{c_s} << 1$. On the other hand the electron integrals converge asymptotically in the limit $\frac{c_s}{v_e} << 1$. In order to perform the velocity integrals, one assumes for simplicity that $\vec{q}'$ lies within a cone of angle $\theta_0 << 1$ about the $\vec{q}$ direction which is nearly parallel to the direction of the electron beam velocity. Furthermore, the calculations will be restricted to a spectrum which is nearly constant within an interval about q_D and zero outside, where q_D is the ion Debye wave number. Also it will be assumed that the energy is of the form

$$I(\vec{q}',t) = I(t)\delta(\theta-\theta_0) \tag{47}$$

This choice of spectrum is more general than that of Sagdeev and Galeev (1969), where they have considered only two spectrum lines in $\vec{q}'$-space.

Substituting equations (9), (10) and the spectrum distribution (47) into (45) and (46) and integrating over the velocities and wave vector $\vec{q}'$, one obtains

$$\alpha_1 = \frac{1}{q^2\lambda_{De}^2\varepsilon_L}\frac{224\pi^2}{27N_D}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2}\theta_0, \tag{48}$$

and

$$\alpha_2 = \frac{1}{q^2\lambda_{De}^2\varepsilon_L}\frac{256\pi^2}{9N_D}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2}\frac{1}{\theta_0} \tag{49}$$

where $N_D = \frac{4\pi}{3} n_0 \lambda^3_{De}$ is the number of electrons in Debye sphere. In obtaining the result for α_1 and α_2, terms of order higher than (q/q_D) have been neglected and to keep (48) and (49) accurate up to order I(t) only, the linear wave frequency is substituted. Since $\theta_0 << 1$, one can easily see that, the contribution from the triangle diagrams (49) exceeds the square diagrams (48) by a large factor $(1/\theta_0)$ and, therefore, the former will be retained. Substituting (49) into (21) yields

$$\varepsilon_{NL} = \varepsilon_L - \frac{1}{q^2\lambda_{De}^2}\frac{256\pi^2}{9N_D\theta_0}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2} \tag{50}$$

where $\varepsilon_L(\vec{q},\omega)$ is the real part of the linear dielectric function given by equation (11). Solving equation (50) for $\varepsilon_{NL} = 0$ as has been done in the previous section, and retaining only terms of order I(t), one obtains

$$\omega_{\vec{q}} = qc_s + \Delta\omega_{\vec{q}} \tag{51}$$

where the frequency shift $\Delta\omega_{\vec{q}}$ is given by

$$\Delta\omega_{\vec{q}} = \frac{128\pi^2}{9N_D\theta_0} \frac{I(t)}{mv_e^2} \left(\frac{T_e}{T_i}\right)^{3/2} qc_s \tag{52}$$

Next, to obtain the function $F_{\vec{q}}$ in the nonlinear approximation, one has to differentiate (21) first with respect to ω, and then, substituting $\omega = qc_s$ and performing the $\vec{q}\,'$-integration. Equations (21), (45), (46) and

$$F_{\vec{q}} = \left|\frac{\partial}{\partial\omega}\left(\omega\varepsilon_{NL}(\vec{q},\omega)\right)\right|_{\omega=\omega_{\vec{q}}} \tag{53}$$

yield (appendix A)

$$F_{\vec{q}} = \omega_{\vec{q}}\left[\frac{(2/qc_s)}{q^2\lambda_{De}^2}\left\{\left(1- \frac{256}{9}\alpha\right)^{3/2} + \frac{128}{3\theta_0^2}\alpha\right\}\right], \tag{54}$$

where

$$\alpha = \frac{\pi^2}{\theta_0 N_D} \frac{I(t)}{mv_e^2} \left(\frac{T_e}{T_i}\right)^{3/2}, \tag{55}$$

Substituting (54) into (16.a) yields the modified matrix element

$$|\tilde{M}_0|^2 = \frac{4\pi e^2\hbar\omega_{\vec{q}}}{Vq^2F_{\vec{q}}} = \frac{4\pi e^2\hbar}{2V}\lambda^2_{De}qc_s\left[(1- \frac{256}{9}\alpha)^{3/2} + \frac{128}{3\theta_0^2}\alpha\right]^{-1} \tag{56}$$

then, inserting (56) and (51) into (16) to get

$$\gamma_{\vec{q}} = \sqrt{\pi/8}\; qc_s\left(\frac{u}{v_e}\right)\left[(1- \frac{256}{9}\alpha)^{3/2} + \frac{128}{3\theta_0^2}\alpha\right]^{-1} \times\left\{\left(1- \frac{c_s}{u}\right) - \frac{128c_s}{9u}\alpha\right\} \tag{57}$$

Thus, since I(t) increases with time, one notices that the factor within the square bracket (which is obtained from the nonlinearities associated with the matrix element) increases and consequently the growth rate decreases. Also, the expression in the curly bracket (where the last term is due to the frequency shift) decreases as I(t) increases until the wave energy assymtotically approaches its maximum value,

$$\frac{I_{max}}{mv_e^2} \simeq \frac{9\theta_0 N_D}{128\pi^2} \left(\frac{u}{c_s} - 1\right)\left(\frac{T_i}{T_e}\right)^{3/2} \tag{58}$$

making the growth vanish and thereby stabilizing the wave. This is so, because the phase velocity of an initially un-

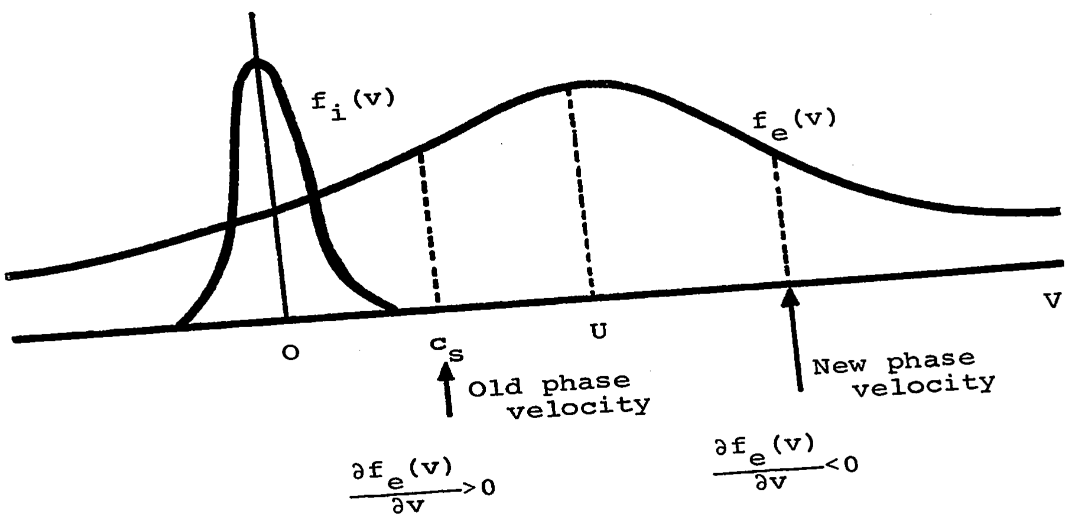


Fig. 21

stable wave has shifted from the unstable part of the electron distribution to the damping part. This is shown schematically in fig. 21.

As a result of the instability there will be some heating of the plasma particles over any finite time. To discuss this heating mechanism, one writes the rate equations for electrons, as well as ions, from which the rise in temperatures can be estimated in the stationary state. The rate equations for particles can be written as

$$\frac{\partial f_j(\vec{v})}{\partial t} = \frac{2\pi}{\hbar^2} \sum_{\vec{q}} |\tilde{M}_0|^2 \Big\{ \Big[\big(N(\vec{q})+1\big) f_j(\vec{k}+\vec{q}) \big(1-f_j(\vec{k})\big) - N(\vec{q}) f_j(\vec{k}) \big(1-f_j(\vec{k}+\vec{q})\big) \Big] \delta\big(E_{\vec{k}+\vec{q}}-\omega_q-E_{\vec{k}}\big) + \Big[N(\vec{q}) f_j(\vec{k}-\vec{q}) \big(1-f_j(\vec{k})\big) - \big(N(\vec{q})+1\big) f_j(\vec{k}) \big(1-f_j(\vec{k}-\vec{q})\big) \Big] \delta\big(E_{\vec{k}}-\omega_q-E_{\vec{k}-\vec{q}}\big) \Big\} \quad (59)$$

where $\tilde{M}_0$ is the modified matrix element (55) and $E_{\vec{k}}$ is expressed in unit of $\hbar$. By expanding $f(\vec{k}\pm\vec{q})$, $E_{\vec{k}\pm\vec{q}}$, and the δ-function in powers of $\hbar$, and substituting $E_{\vec{k}} = \frac{\hbar k^2}{2m_j}$, $\hbar\vec{k} = m_j\vec{v}$, (59) yields to the lowest order in $\hbar$

$$\frac{\partial f_j(\vec{v})}{\partial t} = \frac{2\pi}{m_j^2} \sum_{\vec{q}} |\tilde{M}_0|^2 N(\vec{q}) \Big[\Big(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\Big)^2 f(\vec{v})\, \delta(\vec{q}\cdot\vec{v}-\omega_q) + q^2 \Big(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\Big) f(\vec{v})\, \delta'(\vec{q}\cdot\vec{v}-\omega_q) \Big] \quad (60)$$

Substituting the value of $\tilde{M}_0$ and $\hbar\omega_q N(\vec{q}) = I(\vec{q},t)$ into (60), and changing the $\vec{q}$-summation into integration, one obtains the rate equations for the plasma particles

$$\frac{\partial f_e(\vec{v})}{\partial t} = \frac{\pi v_e^2}{m}\left(1+\frac{128}{3\theta_0^2}\alpha\right)^{-1}\int\frac{d^3q}{(2\pi)^3}\, I(\vec{q})\left[\left(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\right)^2 f_e(\vec{v})\,\delta(\vec{q}\cdot\vec{v}-\omega_q) + q^2\left(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\right) f_e(\vec{v})\,\delta'(\vec{q}\cdot\vec{v}-\omega_q)\right], \tag{61}$$

$$\frac{\partial f_i(\vec{v})}{\partial t} = \frac{\pi v_e^2}{m}\left(\frac{m}{m_i}\right)^2\left(1+\frac{128}{3\theta_0^2}\alpha\right)^{-1}\int\frac{d^3q}{(2\pi)^3}\, I(\vec{q},t)\times\left[\left(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\right)^2 f_i(\vec{v})\,\delta(\vec{q}\cdot\vec{v}-\omega_q)+q^2\left(\vec{q}\cdot\frac{\partial}{\partial\vec{v}}\right) f_i(\vec{v})\,\delta'(\vec{q}\cdot\vec{v}-\omega_q)\right] \tag{62}$$

Multiplying both sides of equations (61) and (62) by mv^2 and $m_i v^2$ respectively and integrating the result over velocity space, one obtains the rates of change of temperature

$$\frac{\partial T_e}{\partial t} = -\frac{m}{3}\frac{\partial u^2(t)}{\partial t} - \frac{\sqrt{2}\pi}{3n}\frac{c_s}{v_e}\left(1+\frac{128\alpha}{3\theta_0^2}\right)^{-1}\int\frac{d^3q}{(2\pi)^3}\, I(\vec{q})\,(\vec{q}\cdot\vec{u}-\omega_{\vec{q}}) \tag{63}$$

$$\frac{\partial T_i}{\partial t} = \frac{\sqrt{2}\pi}{3n_0}\left(\frac{T_e}{T_i}\right)\left(\frac{T_i}{m_i}\right)^{1/2}\left(1+\frac{128\alpha}{3\theta_0^2}\right)^{-1}\int\frac{d^3q}{(2\pi)^3}\, qI(\vec{q}) \tag{64}$$

To eliminate the first term in (63), multiply (61) by $\vec{v}$ and integrate over the velocity to obtain

$$\frac{\partial}{\partial t}(mu^2) = -\frac{\sqrt{2}\pi}{n_0}\left(1+\frac{128\alpha}{3\theta_0^2}\right)^{-1}\int\frac{d^3q}{(2\pi)^3}\,\frac{u}{v_e}\,(\vec{q}\cdot\vec{u}-\omega_{\vec{q}})\,qI(\vec{q}), \tag{65}$$

then, substituting (65) into (63) yields

$$\frac{\partial T_e}{\partial t} = \frac{\sqrt{2n}}{3n_0}\left[1+\frac{128\alpha}{3\theta_0^2}\right]^{-1}\int\frac{d^3q}{(2\pi)^3}\frac{(\vec{q}\cdot\vec{u}-\omega_{\vec{q}})^2}{qv_e}I(\vec{q}) \tag{66}$$

Solving (64) and (66) for $\frac{u}{c_s} >> 1$, one obtains

$$\frac{T_i}{T_e} \simeq \left(\frac{m_i}{m}\right)^{1/5}\left(\frac{c_s}{u}\right)^{4/5}\left[1 - \left(\frac{T_0}{T_e}\right)^{5/2}\right]^{2/5} \tag{67}$$

where T_0 is the initial electron temperature.

Since the maximum value of the last factor is unity, one can write

$$\frac{T_i}{T_e} \leqslant \left(\frac{m_i}{m}\right)^{1/5}\left(\frac{c_s}{u}\right)^{4/5} \tag{68}$$

This result is different from the one obtained by Sagdeev (1965) where it was shown that

$$\frac{T_i}{T_e} \leqslant \left(\frac{c_s}{u}\right) \tag{69}$$

However it has been shown by Sloan and Drummond (1970) that (69) is incorrect.

4.4 Comparison

The stabilization theory discussed in the previous sections can be strengthend by proving that the amplitude dependent term in the growth rate obtained in the previous section is larger than the electron nonlinear Landau growth obtained by Sloan and Drummond (1970). From equation (5) of the above reference, one obtains

$$\frac{dN(q)}{dt} = \iint d^3q'' d^3v N(\vec{q}) N(\vec{q}'') A\delta(\omega' - \vec{q}' \cdot \vec{v}) \times \left[\omega_{pe}^2 B\vec{q}' \cdot \frac{\partial f_e(\vec{v})}{\partial \vec{v}}\right] \tag{70}$$

where $\omega' = \omega + \omega''$, $\vec{q}' = \vec{q} + \vec{q}''$, $A = \frac{2\pi}{q'^2} \frac{\partial \varepsilon(q', \omega')}{\partial \omega'}$, and B is given by equation (6) of the above reference. Substituting the value of $\varepsilon(q', \omega')$ into A, and calculating B, one gets

$$A = \frac{4\pi}{\omega' q'^4 \lambda_{De}^2} , \tag{71}$$

$$B = \frac{\pi e^2}{m^2} \frac{q'^4 \lambda_{De}^4}{v_e^4} \omega\omega'\omega'' , \tag{72}$$

then, inserting (71) and (72) into (70) yields

$$\frac{dN(\vec{q})}{dt} = \frac{\pi}{n_0 m} \iint d^3q'' d^3v \omega N(\vec{q}) \omega'' N(\vec{q}'') \times \vec{q}' \cdot \frac{\partial f_e(\vec{v})}{\partial \vec{v}} \delta(\omega' - \vec{q}' \cdot \vec{v}) \tag{73}$$

The number of plasmons in the state $\vec{q}''$ is given by

$$N(\vec{q}'') = \frac{\partial \varepsilon(q'', \omega'')}{\partial \omega''} \frac{|E_{\vec{q}''}|^2}{8\pi} = \frac{2}{\omega'' q''^2 \lambda_{De}^2} I(\vec{q}'') \tag{74}$$

Substituting (74) into (73) and performing the velocity integration, (73) becomes

$$\frac{1}{N(\vec{q})}\frac{dN(\vec{q})}{dt} = 2\gamma_{NL} = \frac{\sqrt{2\pi}}{n_0\lambda_{De}^2}\left(\frac{qc_s}{v_e}\right)\int\frac{d^3q''}{q'q''^2}\frac{I(\vec{q}'')}{mv_e^2}(\vec{q}'\cdot\vec{u}-\omega') \qquad (75)$$

Performing the $\vec{q}''$-integration by taking, as before, the spectrum form $I(\vec{q}'') = I(t)\delta(\theta-\theta_0)$, (72) yields

$$\gamma_{NL} = \gamma_L \frac{32\pi^3}{3N_D}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{1/2}\theta_0 \qquad (76)$$

where

$$\gamma_L = \sqrt{\frac{\pi}{8}}\frac{qc_s}{v_e}(u-c_s) \qquad (77)$$

From (57), the dominant nonlinear term in the growth rate is given by

$$\gamma_C = -\gamma_L\frac{128\pi^2}{3\theta_0^3 N_D}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2} \qquad (78)$$

Comparing (76) with (78), one obtains

$$\left|\frac{\gamma_{NL}}{\gamma_C}\right| = \frac{\pi}{4}\left(\frac{T_i}{T_e}\right)\theta_0^4 << 1 \qquad (79)$$

Thus, the nonlinear term due to the electron nonlinear Landau growth is negligible.

CHAPTER V

FREQUENCY SHIFT IN ELECTROMAGNETIC WAVES

5.1 Introduction

The Krylov-Bogoliubov-Mitropolskii (1947, 1961) techniques familiar in nonlinear mechanics have been used by many authors to study the behaviour of some types of plasma waves. In particular, Montgomery and Tidman (1964) considered the nonlinear propagation of an electromagnetic wave in a cold plasma and found an expression for the second order frequency shift for both travelling and standing waves. Tidman and Stainer (1965) have calculated frequency and wave number shifts for nonlinear waves in a finite temperature plasma considering both cyclotron waves and electron plasma oscillations. Boyd (1967) obtained an amplitude frequency shift for extraordinary waves, i.e. with a propagation vector perpendicular to the direction of the external magnetic field. Sluijter and Montgomery (1965) and Das (1968) used the same techniques to obtain the amplitude dependent frequency shift for both light waves and extraordinary modes for which the relativistic corrections are included. All the above authors based their calculations on the dynamical equations for a cold plasma together with Maxwell's equations. This method will be first briefly described. Then, the quantum mechanical method will be employed to calculate the frequency shift in light, whistlers and Alfvén waves. The non-relativistic equation for momentum conservation is given by

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\frac{\partial\vec{v}}{\partial\vec{x}} + \frac{e}{m}(\vec{E} + \frac{1}{c}\,\vec{v}\Lambda\vec{B}) = 0 \tag{1}$$

and the corresponding Maxwell's equations are

$$\nabla\cdot\vec{E} = 4\pi e(n_0-N), \tag{2}$$

$$\nabla\Lambda\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0, \tag{3}$$

$$\nabla\Lambda\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \frac{4\pi}{c}\,eN\vec{v}, \tag{4}$$

where N is the electron number density; $\vec{v}$, the electron velocity; $\vec{E},\vec{B}$, the electric and magnetic fields and n_0 is the uniform background positive ion density. To the above set of equations, the Krylov-Bogoliubov-Mitropolskii (KBM) perturbation expansion is applied

$$N = n_0+\varepsilon n^{(0)}(a,\phi)+\varepsilon^2 n^{(1)}(a,\phi)+\ldots. \tag{5}$$

$$\vec{v} = 0+\varepsilon\vec{v}^{(0)}(a,\phi)+\varepsilon^2\vec{v}^{(1)}(a,\phi)+\ldots. \tag{6}$$

$$\vec{E} = 0+\varepsilon\vec{E}^{0}(a,\phi)+\varepsilon^2\vec{E}^{(1)}(a,\phi)+\ldots. \tag{7}$$

$$\vec{B} = \vec{B}_0+\varepsilon\vec{B}^{(0)}(a,\phi)+\varepsilon^2\vec{B}^{(1)}(a,\phi)+\ldots \tag{8}$$

in which the perturbation correction terms are functions of the wave amplitude a and the phase ϕ, and $\varepsilon << 1$, is the expansion parameter. In the KBM theory one uses

$$\frac{\partial\phi}{\partial t} = \omega+\varepsilon^2 A(a)+0(\varepsilon^3), \tag{9}$$

$$\frac{\partial a}{\partial t} = \varepsilon^2 D(a)+0(\varepsilon^3), \tag{10}$$

in which A (the frequency shift of that mode under investigation) and D are determined by the requirement that there be no secular terms (terms $\propto\phi$) in the perturbation expansion.

In (1969), Tam developed a perturbation scheme by which a small amplitude dispersion effect can be accounted for. This scheme, based on the method of Krylov, Bogoliubov and Mitropolskii (1961), takes advantage of the fact that the frequency of a nonlinear oscillator depends on the amplitude of oscillation. Accordingly, for small amplitude plasma waves, the frequency, wave number and amplitude can be represented by a power series in the energy density of the waves as follows:

$$\omega = \omega_0+\omega_2(\vec{q})a^2+\omega_4(\vec{q})a^4+\ldots \tag{11}$$

where the first term is the linearized result and the additional terms represent the effect of nonlinearity. Tam applied the above technique to the investigation of whistler modes. He concluded that, if whistlers are propagating parallel to the external magnetic field, the frequency shift is zero, and that the linearized solution is exact for the full nonlinear two-component cold plasma. However, it has been demonstrated by Selim and Krishan (1971) that these observations are incorrect; and later Sudan (1971) proved that Tam's result is restrictive.

In this chapter, a general theory has been developed for the studying of whistlers, Alfvén and transverse waves. The various physical processes responsible for the amplitude dependent frequency shift are represented by Feynman graphs using the theory developed in chapter II. The wave-particle and two-wave-particle interaction Hamiltonians are obtained in sections 2 and 3 respectively. Section 4 contains the calculation of the scattering matrix element and a general formula for the amplitude dependent frequency shift. These shifts for light, whistlers and Alfvén waves are given in section 5.

5.2 Wave-Particle Interaction

The first two terms given by (26.b) of chapter II represent the electromagnetic wave-particle interaction. The corresponding interaction Hamiltonian can be written in the second quantisation formalism

$$H_{int} = \sum_j \frac{e_j}{2cm_j} \int \Psi^+(x) \Big[\big(\vec{P}_j + \frac{e_j}{c} \vec{A}_0(\vec{x})\big) \cdot \vec{A}_1(x)$$

$$+ \vec{A}_1(x) \cdot \big(\vec{P}_j + \frac{e_j}{c} \vec{A}_0(\vec{x})\big) \Big] \Psi(x) d^3x \qquad (12)$$

Here, $\vec{A}_0(\vec{x})$ is the vector potential due to uniform external magnetic field directed along the z-axis such that $\vec{A}_0(\vec{x}) = (-B_0 y, 0, 0)$, and Ψ's are the field operators

$$\Psi(x) = \sum_{n,\vec{k}} C_{n,\vec{k}}\, \xi_{n,\vec{k}}(\vec{x}) \qquad (13)$$

$$\Psi^+(x) = \sum_{n,\vec{k}} C^+_{n,\vec{k}}\, \xi^+_{n,\vec{k}}(\vec{x}) \qquad (14)$$

$C_{\vec{k},n}$ and $C^+_{\vec{k},n}$ are the annihilation and creation operators of particle in state $\vec{k}$ with quantum number n, and $\xi_{n,\vec{k}}$ is the solution of the Schrõdinger equation

$$\frac{1}{2m_j} \left[\vec{P}_j + \frac{e_j}{c} \vec{A}_0\right]^2 \xi_{n,\vec{k}} = E_{n,\vec{k}} \xi_{n,\vec{k}} \qquad (15)$$

and is given by (Landau and Lifshitz, 1958)

$$\xi_{n,k_x,k_z}(\vec{x}) = \frac{1}{L} G_n(y-y_c) e^{i(k_x x + k_z Z)} \qquad (16)$$

The Harmonic oscillator function $G_n(y-y_c)$ has the form

$$G_n(y-y_c) = \frac{\sqrt{m_j \omega_{cj}/\hbar}}{[n!2^{n/2}\pi^{1/4}]^{1/2}} H_n\left(\sqrt{m_j \omega_{cj}/\hbar}\,(y-y_c)\right) \cdot e^{-[m_j \omega_{cj}/2\hbar (y-y_c)^2]} \qquad (17)$$

where $y_c = \frac{\hbar k_x}{m_j \omega_{cj}}$, $\omega_{cj} = \left|\frac{e_j B_0}{m_j c}\right|$, and H_n is the Hermite polynomial. Inserting (16) into (15), one obtains the corresponding eigenvalues

$$E_{n,k_x,k_z} = \hbar\omega_{cj}(n+1/2) + \frac{\hbar^2 k_z^2}{2m_j} \tag{18}$$

This means that the particle behaves like an oscillator perpendicular to the magnetic field and like a free particle parallel to it.

Substituting (13) and (14) into (12) yields for electrons

$$H_{int} = -\frac{e}{2mc} \sum_{\vec{k},\vec{k}',n,n'} C^{+}_{\vec{k},n}(t)\, C_{\vec{k}',n'} \int \xi^{+}_{k,n} \Big[(\vec{P} + \frac{eB_0}{c} y\vec{e}_x)\cdot\vec{A}_1 + \vec{A}_1\cdot(\vec{P} + \frac{eB_0}{c} y\vec{e}_x)\Big] \xi_{\vec{k}',n'} d^3x \tag{19}$$

Since the problem under investigation concerns transverse circularly polarized electromagnetic waves propagating parallel to the external magnetic field $\vec{B}_0$, the vector potential $\vec{A}_1$ given by equation (28) of chapter II will be

$$\vec{A}_1(x) = \sum_{\vec{q}} \left[\frac{2\pi\hbar c^2}{V\omega_T(q) F_T(q)}\right]^{1/2} \frac{(\vec{e}_x \pm i\vec{e}_y)}{\sqrt{2}} \left[A_q(t)e^{iqz} + A^{+}_q(t)e^{-iqz}\right] \tag{20}$$

where, in $(e_x \pm ie_y)$ the plus sign refers to the clockwise polarization and the minus to the counterclockwise. The whistler mode have a clockwise polarization, while Alfvén and light waves can be polarized in the clockwise or counter-clockwise direction. Substituting (16) and (20) into (19) and performing the x and z integrations, one obtains

$$H_{int} = -\frac{e}{2mc}\sum_{q}\left[\frac{4\pi\hbar c^2}{V\omega_T(q)F_T(q)}\right]^{1/2}\sum_{k_x,k_z,n,n'} C^{+}_{k_x,k_z+q,n}C_{k,k_z,n'}A_q$$
$$\int dy\left[(m\omega_{ce})G_n(y-y_c)(y-y_c)G_{n'}(y-y_c)\right.$$
$$\left.\pm\hbar G_n(y-y_c)\frac{\partial}{\partial y}G_{n'}(y-y_c)\right]+H\cdot C. \qquad (21)$$

To perform the y-integration, one can make use of the following relations:

$$\left(\frac{m\omega_{ce}}{\hbar}\right)^{1/2}(y-y_c)G_{n'}(y-y_c) = \sqrt{\frac{n'+1}{2}}\,G_{n'+1}+\sqrt{\frac{n'}{2}}\,G_{n'-1}, \qquad (22)$$

$$\frac{\partial}{\partial y}G_n(y-y_c) = \sqrt{\frac{m\omega_{ce}}{\hbar}}\left[\sqrt{\frac{n}{2}}\,G_{n-1}-\sqrt{\frac{n+1}{2}}\,G_{n+1}\right], \qquad (23)$$

where $G_n(y-y_c) = G_n[\alpha(y-y_c)]$, $\alpha^2 = \frac{m\omega_{ce}}{\hbar}$. Inserting (22) and (23) into (21) yields

$$H_{int} = -\frac{e\hbar}{2mc}\sum_{q,k_x,k_z,n,n'}\left[\frac{8\pi c^2 m\omega_{ce}}{V\omega_T(q)F_T(q)}\right]^{1/2} C^{+}_{k_x,k_z+q,n}C_{k_x,k_z,n'}A_q$$
$$\left\{\int dy\sqrt{n'}\,G_n(y-y_c)G_{n'-1}+\int dy\sqrt{n'+1}\,G_n(y-y_c)G_{n'+1}\right\}+H\cdot C. \qquad (24)$$

Using the integral

$$\int dy G_n(y-y_c)G_{n'}(y-y_c) = \delta_{n,n'}, \qquad (25)$$

one obtains

$$H_{int} = -\frac{e\hbar}{2mc}\sum_{q}\sum_{k_x,k_z,n}\left[\sqrt{\frac{8\pi c^2 m\omega_{ce} n}{V\omega_T(q)F_T(q)}}\, C^+_{k_x,k_z+q,n-1} C_{k_x,k_z,n} A_q \right.$$

$$\left. + \sqrt{\frac{8\pi c^2 m\omega_{ce}(n+1)}{V\omega_T(q)F_T(q)}}\, C^+_{k_x,k_z+q,n+1} C_{k_x,k_z,n} A_q \right] + H.C. \quad (26)$$

In (26) the first term is due to the clockwise polarization, and the second term to counterclockwise. These two terms are represented diagramatically in fig. 22(a) and (b) respectively.

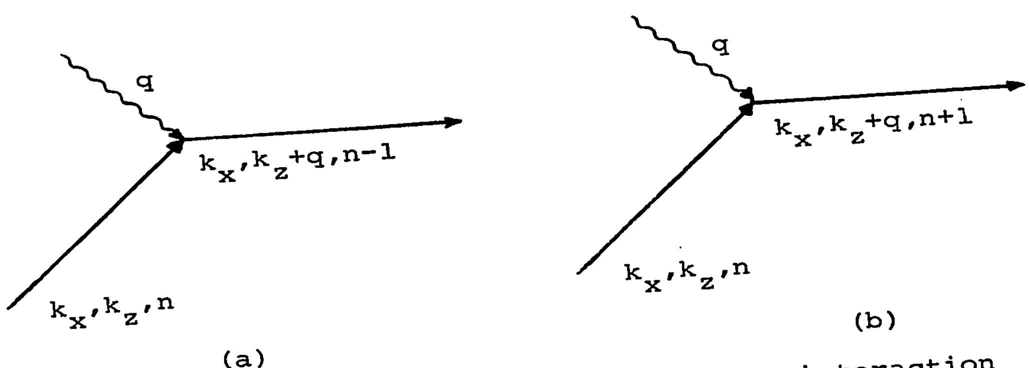


Fig. 22 Electromagnetic wave-particle interaction

5.3 Two-Wave-Particle Interaction

The interaction Hamiltonian given by equation (26.3) of chapter II is responsible for wave-wave-particle interactions. In the second quantisation formalism, this interaction Hamiltonian can be written

$$H_{int} = \int \Psi^{+}(\vec{x}) \frac{e^2}{2mc^2} (\vec{A}_1 \cdot \vec{A}_1^{*}) \Psi(\vec{x}) d^3x \tag{27}$$

where the Ψ's and $\vec{A}_1(x)$ are given by (13), (14) and (20) respectively. Substituting these equations into (27) and performing the integration one obtains

$$H_{int} = \frac{\pi e^2 \hbar}{mV} \sum_{q,q',k_x,k_z,n} \left[\omega_T(q)\,\omega_T(q')\,F_T(q)\,F_T(q')\right]^{-1/2}$$

$$\left[C^{+}_{n,k_x,k_z+q'-q} C_{n,k_x,k_z} A^{+}_{q} A_{q'} + C^{+}_{n,k_x,k_z-q-q'} C_{n,k_x,k_z} A^{+}_{q} A^{+}_{q'} \right. \tag{28}$$

$$\left. + \text{H.C.}\right]$$

(28) can be represented by Feynman diagrams as follow: The term containing $C^{+}_{n,k_x,k_z+q'-q} C_{n,k_x,k_z} A^{+}_{q} A_{q}$ describes the process in which one particle in the state n,k_x,k_z is annihilated,

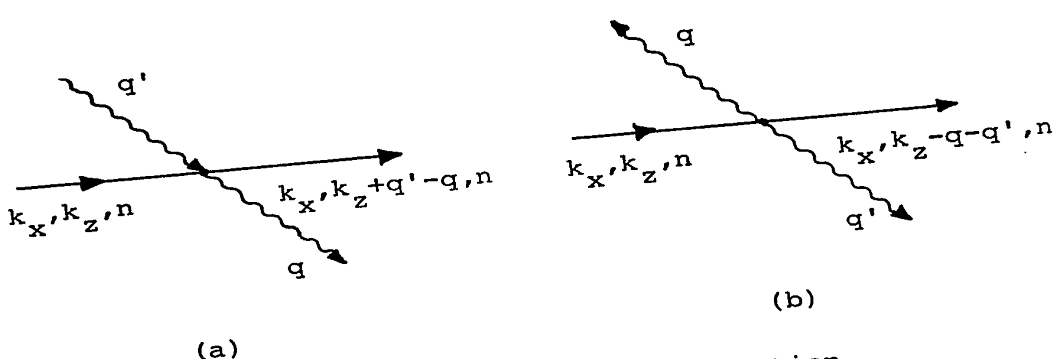


Fig. 23 Two-wave-particle interaction

and one is created in the state $n, k_x, k_z+q'-q$; an electromagnetic wave with momentum $\hbar\vec{q}\,'$ is destroyed and one with momentum $\hbar\vec{q}$ is created. Similarly the second term describes the process of destroying one particle and creating another with the creation of two electromagnetic waves. These processes are shown in fig. 23(a) and (b).

5.4 Scattering Amplitude and Frequency Shift

In order to obtain a second order term in the frequency shift, a four wave interaction must be considered. The total Hamiltonian can be written (Harris, 1969)

$$H = H_0 + H' \tag{29}$$

$$\begin{aligned} H_0 = & \sum \tfrac{1}{2}\hbar\omega_{q_1} A^+_{q_1} A_{q_1} + \sum \tfrac{1}{2}\hbar\omega_{q_2} A^+_{q_2} A_{q_2} \\ & + \sum \tfrac{1}{2}\hbar\omega_{q_3} A^+_{q_3} A_{q_3} + \sum \tfrac{1}{2}\hbar\omega_{q_4} A^+_{q_4} A_{q_4} \\ & + \text{H.C.} \end{aligned} \tag{30}$$

$$\begin{aligned} H' = & \sum M_1 A^+_1 A^+_2 A^+_3 A^+_4 + \sum M_2 A^+_1 A^+_2 A^+_3 A_4 \\ & + \sum M_3 A^+_1 A^+_2 A_3 A^+_4 + \sum M_4 A^+_1 A_2 A^+_3 A^+_4 \\ & + \sum M_5 A_1 A^+_2 A^+_3 A^+_4 + \sum M_6 A^+_1 A^+_2 A_3 A_4 \\ & + \sum M_7 A^+_1 A_2 A^+_3 A_4 + \sum M_8 A^+_1 A_2 A_3 A^+_4 \\ & + \text{H.C.} \end{aligned} \tag{31}$$

where $A_i \equiv A_{q_i}$, ω_{q_1}, ω_{q_2}, ω_{q_3} and ω_{q_4} are the frequencies of the four waves in the absence of interactions. The perturbed frequencies are contained implicitly in H' which includes all the possible four-wave interactions.

Since H' contains expressions involving four plasmons, the appropriate Feynman representation must include diagrams with four external plasmons. These are given by three different types. First, diagrams similar to that of fig. 10 in which there should be always two plasmons annihilated and two created. The conservation of momentum must be satisfied and the quantum numbers obey selection rules given by the interaction Hamiltonian (26) at each vertex. Second, since there are creation and annihilation of two plasmons at the same vertex as shown in fig. 23 due to the term $(\vec{A}\cdot\vec{A}^*)$, it is possible to construct diagrams similar to that of fig. 9 with two plasmons at one vertex and one at each of the other two vertices. Third, loops, corresponding to the second order matrix elements, with two plasmons at each vertex are also possible. Retaining only terms containing the annihilation of two plasmons and creation of another in (29), one obtains

$$H = \sum_{q_1} \frac{1}{2}\hbar\omega_{q_1} A^+_{q_1} A_{q_1} + \sum_{q_2} \frac{1}{2}\hbar\omega_{q_2} A^+_{q_2} A_{q_2}$$

$$+ \sum_{q_4} \frac{1}{2}\hbar\omega_{q_3} A^+_{q_3} A_{q_3} + \sum_{q_4} \frac{1}{2}\hbar\omega_{q_4} A^+_{q_4} A_{q_4}$$

$$+ \sum M_1 A^+_{q_1} A^+_{q_2} A_{q_3} A_{q_4} + \sum M_2 A^+_{q_1} A_{q_2} A^+_{q_3} A_{q_4}$$

$$+ \sum M_3 A^+_{q_1} A_{q_2} A_{q_3} A^+_{q_4} + H\cdot C. \qquad (32)$$

If the waves are taken to be monochromatic i.e. $q_1=q_2=q_3=q_4$ (Montgomery and Tidman, 1964), (32) becomes

$$H = \frac{1}{2}\hbar\omega_q A^+_q A_q + M A^+_q A_q A^+_q A_q + H\cdot C$$

$$= \frac{1}{2}\hbar A^+_q A_q (\omega_q + \frac{2M}{\hbar} A^+_q A_q) + H\cdot C$$

$$= \frac{1}{2}\hbar A^+_q A_q (\omega_q + \Delta\omega_q) + H\cdot C. \qquad (33)$$

where $\Delta\omega_q$ is the frequency shift, given by

$$\Delta\omega_q = \frac{2M}{\hbar} A^+_q A_q = \frac{2M}{\hbar^2\omega_q} \hbar\omega_q N(q) = \frac{2MI(q)}{\hbar^2\omega_q} \qquad (34)$$

and M is the sum of the matrix elements for the three possible processes discussed earlier and it has to be at least of order of $\hbar^2$.

The matrix element for the first process shown in fig. 24 for the clockwise polarization can be written as

$$M_I = -\left(\frac{\hbar}{2\pi i}\right) \frac{(-i)^4}{\hbar^4} \sum_j \left(\frac{e\hbar}{2m_j}\right)^4 \left(\frac{8\pi m_j \omega_{cj}}{V\omega_q F_q}\right)^2 (-1)$$

$$\sum_{k_x,k_z,n} \int dE \left[\frac{n(n+1)\, i^4}{(E-E_{k_z,n})^2 (E+\omega_q - E_{k_z+q,n-1})(E-\omega_q - E_{k_z-q,n+1})}\right.$$

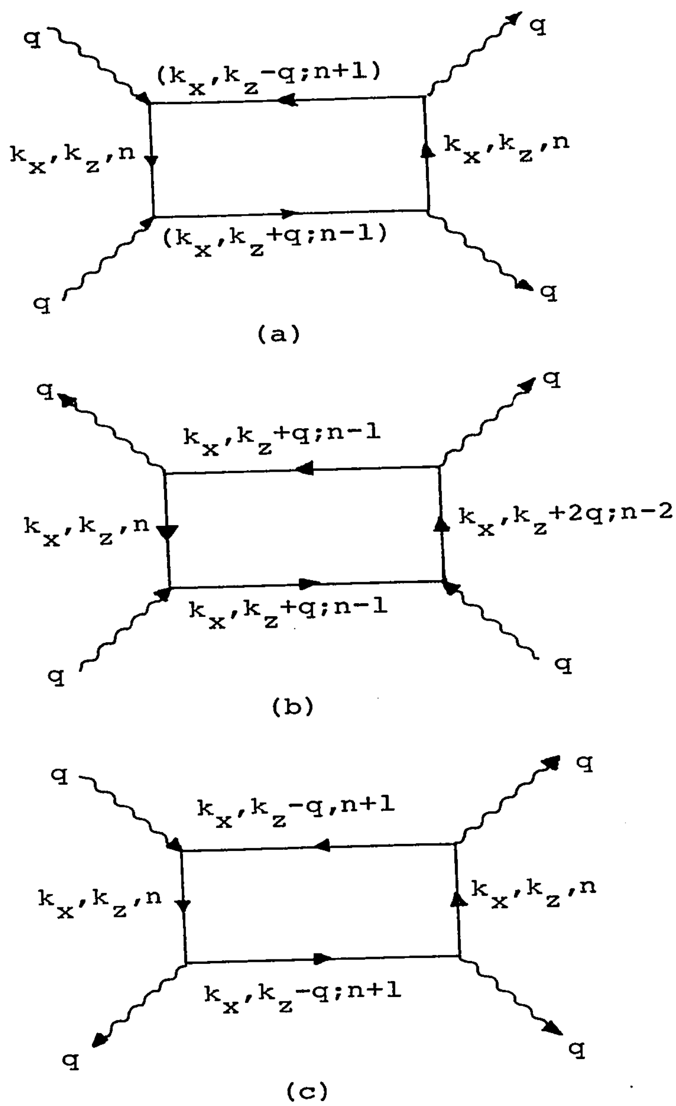


Fig. 24 The contribution to the scattering amplitude due to four-plasmon-particle vertices. This comes from $\vec{p}\cdot\vec{A}$ term.

$$+\frac{n(n-1)i^4}{(E-E_{k_z,n})(E+\omega_q-E_{k_z+q,n-1})^2(E+2\omega_q-E_{k_z+2q,n-2})}$$

$$\left.+\frac{(n+1)^2 i^4}{(E-E_{k_z,n})^2(E-\omega_q-E_{k_z-q,n+1})^2}\right] . \qquad (35)$$

where the factor $(-\hbar/2\pi i)$ comes from the relation between the S and M matrix. The source of various factors, as have been explained in chapter II. The three different terms in the bracket are due to the three different diagrams in fig. 24. Performing the E-integration, (35) yields

$$M_I = \sum_j \left(\frac{e}{2m_j}\right)^4 \left[\frac{8\pi m_j\omega_{cj}}{V\omega_q F_q}\right]^2 \sum_{k_x,k_z,n} \hbar\left[n(n+1)f(k_x,k_z,n)\right.$$

$$\left\{-\frac{1}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2(E_{k_z},\omega_q-E_{k_z-q,n+1})}\right.$$

$$\left.-\frac{1}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})(E_{k_z,n}-\omega_q-E_{k_z-q,n+1})^2}\right\}$$

$$+\frac{n(n+1)f(k_x,k_z+q,n-1)}{(E_{k_z+q,n-1}-\omega_q-E_{k_z,n})^2(E_{k_z+q,n-1}-2\omega_q-E_{k_z-q,n+1})}$$

$$+\frac{n(n+1)f(k_x,k_z-q,n+1)}{(E_{k_z-q,n+1}+\omega_q-E_{k_z,n})^2(E_{k_z-q,n+1}+2\omega_q-E_{k_z+q,n-1})}$$

$$+\frac{n(n-1)f(k_x,k_z,n)}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2(E_{k_z,n}+2\omega_q-E_{k_z+2q,n-2})}$$

$$- \frac{n(n-1)f(k_x,k_z+q,n-1)}{\left(E_{k_z+q,n-1}-\omega_q-E_{k_z,n}\right)^2\left(E_{k_z+q,n-1}+\omega_q-E_{k_z+2q,n-2}\right)}$$

$$- \frac{n(n-1)f(k_x,k_z+q,n-1)}{\left(E_{k_z+q,n-1}-\omega_q-E_{k_z,n}\right)\left(E_{k_z+q,n-1}+\omega_q-E_{k_z+2q,n-2}\right)^2}$$

$$+ \frac{n(n-1)f(k_x,k_z+2q,n-2)}{\left(E_{k_z+2q,n-2}-2\omega_q-E_{k_z,n}\right)\left(E_{k_z+2q,n-2}-\omega_q-E_{k_z+q,n-1}\right)^2}$$

$$\left. - \frac{2(n+1)\ f(k_x,k_z,n)}{\left(E_{k_z,n}-\omega_q-E_{k_z-q,n+1}\right)^3} - \frac{2(n+1)^2 f(k_x,k_z-q,n+1)}{\left(E_{k_z-q,n+1}+\omega_q-E_{k_z,n}\right)^3}\right] \qquad (36)$$

After making the necessary transformation in k_z and n (for example $k_z+q \rightarrow k_z$ and $n-1 \rightarrow n$ in the third term, etc.), (36) can be rewritten

$$M_I = \sum_j \left(\frac{e}{2m_j}\right)^4 \left[\frac{8\pi m_j \omega_{cj}}{V\omega_q F_q}\right]^2 \sum_{k_x,k_z,n} 2\hbar f(k_x,k_z,n) \times$$

$$\left[- \frac{n(n+1)}{\left(E_{k_z,n}+\omega_q-E_{k_z+q,n-1}\right)^2\left(E_{k_z,n}-\omega_q-E_{k_z-q,n+1}\right)}\right.$$

$$- \frac{n(n+1)}{\left(E_{k_z,n}+\omega_q-E_{k_z+q,n-1}\right)\left(E_{k_z,n}-\omega_q-E_{k_z-q,n+1}\right)^2}$$

$$+ \frac{(n+1)(n+2)}{\left(E_{k_z,n}-\omega_q-E_{k_z-q,n+1}\right)^2\left(E_{k_z,n}-2\omega_q-E_{k_z-2q,n+2}\right)}$$

$$+ \frac{n(n-1)}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2(E_{k_z,n}+2\omega_q-E_{k_z+2q,n-2})}$$

$$\left. - \frac{(n+1)^2}{(E_{k_z,n}-\omega_q-E_{k_z-q,n+1})^3} - \frac{n^2}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^3}\right] \tag{37}$$

Substituting $E_{k_z,n} = \omega_{cj}(n+1/2) + \frac{\hbar k_z^2}{2m_j}$, $\frac{\hbar k_z}{m_j} = v_z$ and changing the summation into integration, (37) becomes

$$M_I = 2\hbar \sum_j \left(\frac{e}{2m_j}\right)^4 \left[\frac{8\pi m_j \omega_{cj}}{\sqrt{V}\ \omega_q F_q}\right]^2 \int d^3v f_j(\vec{v})$$

$$\left[\frac{n(n+1)}{\left(\lambda_j - \frac{\hbar q^2}{2m_j}\right)^2\left(\lambda_j + \frac{\hbar q^2}{2m_j}\right)} - \frac{n(n+1)}{\left(\lambda_j - \frac{\hbar q^2}{2m_j}\right)\left(\lambda_j + \frac{\hbar q^2}{2m_j}\right)^2}\right.$$

$$- \frac{(n+1)(n+2)}{2\left(\lambda_j + \frac{\hbar q^2}{2m_j}\right)^2\left(\lambda_j + \frac{2\hbar q^2}{2m_j}\right)} + \frac{n(n-1)}{2\left(\lambda_j - \frac{\hbar q^2}{2m_j}\right)^2\left(\lambda_j - \frac{2\hbar q^2}{2m_j}\right)}$$

$$\left. + \frac{(n+1)^2}{\left(\lambda_j + \frac{\hbar q^2}{2m_j}\right)^3} - \frac{n^2}{\left(\lambda_j - \frac{\hbar q^2}{2m_j}\right)^3}\right] \tag{38}$$

where $$\lambda_j = \omega_q + \omega_{cj} - qv_z \tag{38.a}$$

Since the matrix element must be of order of $\hbar^2$, one has to expand the denominators of (38) up to the order of $\hbar^3$ to obtain

$$M_I = \hbar^2 q^2 \sum_j \left(\frac{e}{2m_j}\right)^4 \left[\frac{8\pi m_j \omega_{cj}}{\sqrt{V}\ \omega_q F_q}\right]^2 \int \frac{d^3 v f_j(\vec{v})}{m_j \lambda_j^4} \times$$

$$\left[1 - \frac{5q^2\hbar\omega_{cj} n}{m_j \omega_{cj}\lambda_j} - \frac{5}{2}\frac{q^4\hbar^2\omega_{cj}^2 n^2}{m_j^2\omega_{cj}^2\lambda_j^2} + 10\ \left(\frac{\hbar q^2}{2m_j\lambda_j}\right)n^2\right.$$

$$\left. - 5\ \frac{\hbar q^2}{2m_j\lambda_j} + 16\left(\frac{\hbar q^2}{2m_j\lambda_j}\right)^2\right] \tag{39}$$

Substituting $\hbar\omega_{cj}n = \frac{1}{2} m_j V_\perp^2$ and taking the classical limit, one obtains

$$\left(\frac{M_I}{\hbar^2}\right)_{\hbar\to 0} = q^2 \sum_j \left(\frac{e}{2m_j}\right)^4 \left[\frac{8\pi m_j \omega_{cj}}{\sqrt{V}\ \omega_q F_q}\right]^2 \times \int \frac{d^3 v f_j(\vec{v})}{m_j \lambda_j^4} \left[1 - \frac{5}{2}\frac{q^2 V_\perp^2}{\omega_{cj}\lambda_j} - \frac{5}{8}\frac{q^4 V_\perp^4}{\omega_{cj}^2\lambda_j^2}\right] \tag{40}$$

Following the same procedures, one can write the matrix element for the second process shown in fig. 25.

$$M_{II} = \frac{-\hbar}{2\pi i}\left(\frac{-i}{\hbar}\right)^3 \sum_j \frac{\pi e^2\hbar}{m_j V}\left(\frac{e\hbar}{2m_j}\right)^2 \frac{8\pi m_j\omega_{cj}}{V\omega_q F_q}\ (-1)$$

$$\sum_{k_x,k_z,n}\int dE\left[\frac{n i^3}{(E-E_{k_z,n})^2(E+\omega_q-E_{k_z+q,n-1})}\right.$$

$$\left. + \frac{(n+1)i^3}{(E-E_{k_z,n})^2(E-\omega_q-E_{k_z-q,n+1})}\right] \tag{41}$$

Performing the E-integration yields

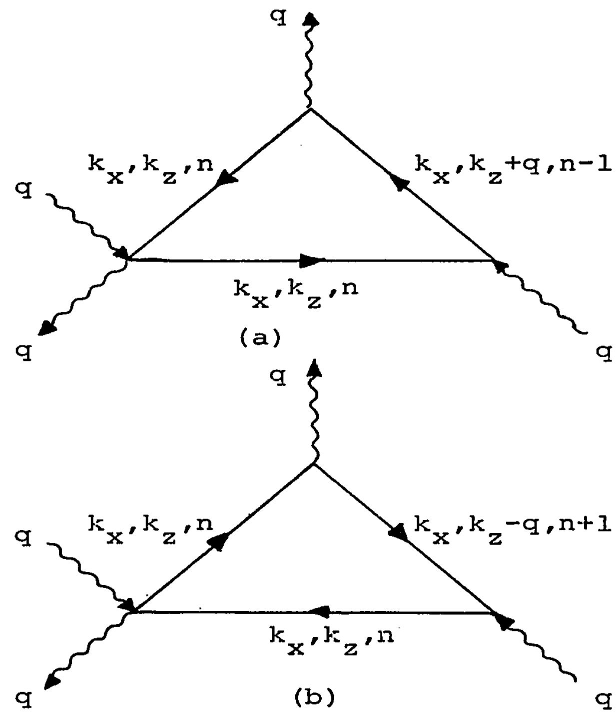


Fig. 25 The contribution to four-wave scattering from two plasmon-particle vertices and one two-plasmon-particle vertix. This comes from A^2 and $\vec{p}\cdot\vec{A}$ terms.

$$M_{II} = \hbar \sum_j \frac{\pi e^2}{V} \left(\frac{e}{2m_j}\right)^2 \frac{8\pi\omega_{cj}}{V\omega_q F_q} \times \sum_{k_x,k_z,n} \left[- \frac{nf(k_x,k_z,n)}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2} \right.$$

$$+ \frac{nf(k_x,k_z+q,n-1)}{(E_{k_z+q,n-1}-\omega_q-E_{k_z,n})^2} - \frac{(n+1)f(k_x,k_z,n)}{(E_{k_z,n}-\omega_q-E_{k_z-q,n+1})^2}$$

$$\left. + \frac{(n+1)f(k_x,k_z-q,n+1)}{(E_{k_z-q,n+1}+\omega_q-E_{k_z,n})^2} \right] \tag{42}$$

Making the necessary transformation to obtain a common factor $f(k_x,k_z,n)$, one obtains

$$M_{II} = \hbar \sum_j \frac{\pi e^2}{V} \left(\frac{e}{2m_j}\right)^2 \frac{8\pi\omega_{cj}}{V\omega_q F_q} \sum_{k_x,k_z,n} f(k_x,k_z,n)$$

$$\times \left[- \frac{n}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2} + \frac{n+1}{(E_{k_z,n}-\omega_q-E_{k_z-q,n+1})^2} \right.$$

$$\left. - \frac{n+1}{(E_{k_z,n}-\omega_q-E_{k_z-q,n+1})^2} + \frac{n}{(E_{k_z,n}+\omega_q-E_{k_z+q,n-1})^2} \right] = 0 \tag{43}$$

This means that the contribution due to fig. 25(b) exactly cancels that of fig. 25(a).

The matrix element due to the third possible process shown in fig. 26 is given by

$$M_{III} = - \left(\frac{\hbar}{2\pi i}\right) \frac{(-i)^2}{\hbar^2} \sum \left[\frac{\pi e^2 \hbar}{m_j V\omega_q F_q}\right]^2 (-1) \times$$

$$\sum_{k,n}\int dE\left[\frac{i^2}{(E-E_{k_z,n})^2}+\frac{i^2}{(E-E_{k_z,n})(E+2\omega_q-E_{k_z+2q,n})}\right] \qquad (44)$$

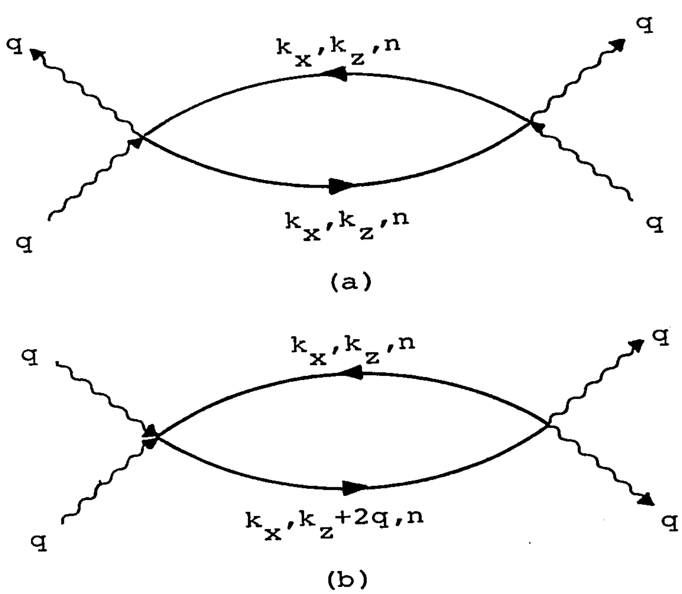


Fig. 26 The contribution to four-wave scattering from two-plasmon-particle vertices. This comes from the A^2 term in the Hamiltonian.

Performing the E-integration in (56), it is obvious that the first term is equal to zero. This means that the process shown in fig. 26(a) gives zero contribution. Thus, only process 26(b) contributes to M_{III}, which becomes

$$M_{III} = \hbar \sum_j \left[\frac{\pi e^2}{m_j V\omega_q F_q}\right]^2 \sum_{k_x,k_z,n}\left[\frac{f(k_x,k_z,n)}{(E_{k_z,n}+2\omega_q-E_{k_z+2q,n})}\right.$$

$$+ \frac{f(k_x,k_z+2q,n)}{\left(E_{k_z+2q,n}-2\omega_q-E_{k_z,n}\right)}\Bigg] \quad (45)$$

Making the transformation $k_z + 2q \rightarrow k$ in the second term and substituting $E_{k_z,n} = \omega_{cj}(n+1/2) + \frac{\hbar k_z^2}{2m_j}$, $\hbar k_z = m_j\vec{v}$, into (45) and changing summation into integration, one obtains

$$\left(\frac{M_{III}}{\hbar^2}\right)_{\hbar\to 0} = \sum_j \left[\frac{\pi e^2}{m_j\sqrt{V}\omega_q F_q}\right]^2 \int d^3v f_j(\vec{v})\ \frac{q^2}{m_j(\omega_q-qv_z)^2} \quad (46)$$

Adding (40) and (46) yields the total matrix element which is given by

$$\left(\frac{M}{\hbar^2}\right)_{\hbar\to 0} = \sum_j \frac{\omega_{pj}^4}{\omega_q^2}\ \frac{q^2}{16n_0^2Vm_jF_q^2}\int d^3v f_j(\vec{v})\ \times$$

$$\left[\frac{4\omega_{cj}^2}{\left(\omega_q+\omega_{cj}-qv_z\right)^4}\left(1-\frac{5}{2}\ \frac{q^2v_\perp^2}{\omega_{cj}\lambda_j} - \frac{5}{8}\ \frac{q\ v_\perp}{\omega_{cj}^2\lambda_j^2}\right) + \frac{1}{(\omega_q-qv_z)^2}\right], \quad (47)$$

and, then substituting (47) into (34), one obtains an expression for the amplitude dependent frequency shift.

$$\Delta\omega_q = \frac{q^2 I(\vec{q})}{8n_0^2VF_q^2}\sum \frac{\omega_{pj}^4}{\omega_q^3}\int \frac{d^3vF_j(\vec{v})}{m_j} \times Y(q,v_z,v_\perp,\omega_{cj}) \quad (48)$$

where

$$Y(q,v_z,v_\perp,\omega_{cj}) = \frac{4\omega_{cj}^2}{(\omega_q+\omega_{cj}-qv_z)^4}\left(1-\frac{5}{2}\ \frac{q^2v_\perp^2}{\omega_{cj}\lambda_j} - \frac{5}{8}\ \frac{q^4v_\perp^4}{\omega_{cj}^2\lambda_j^2}\right)$$

$$+ \frac{1}{(\omega_q - qv_z)^2} \qquad (49)$$

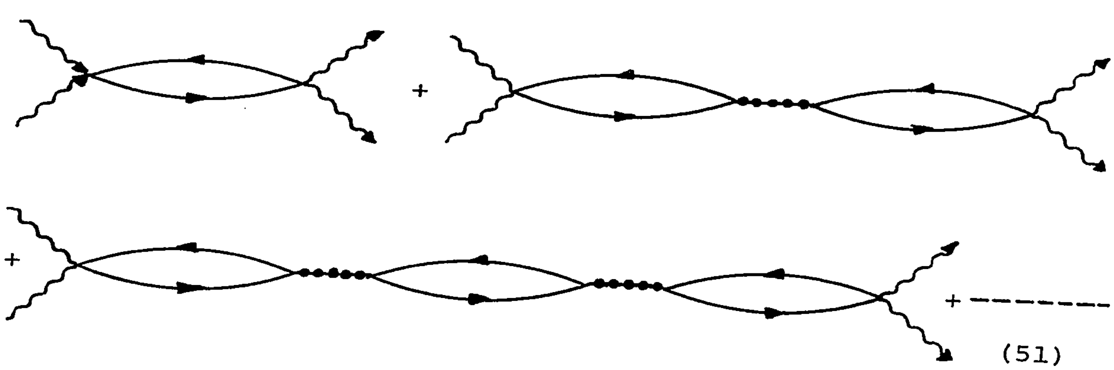
5.5 Amplitude Dependent Frequency Shift

a. Light Waves

For the circularly polarized high frequency electromagnetic wave ($\omega_q^2 = \omega_{pe}^2 + q^2c^2$, $\omega_q >> \omega_{cj}$), propagating parallel to the external magnetic field, only the last term in (49) is significant (this corresponds to the contribution from the process shown in fig. 26(b)) as compared to the contribution from fig. 24. Therefore, expression (48) becomes

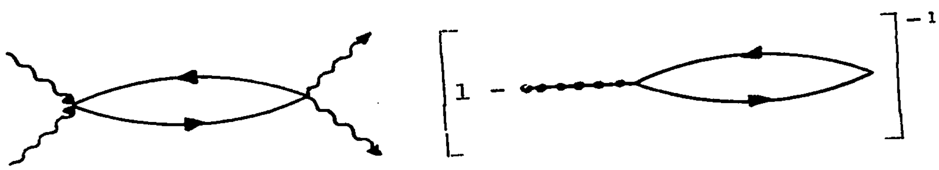
$$\Delta\omega_q = \frac{q^2 I(q)}{8n_0^2 V F_q^2} \sum_j \frac{\omega_{pi}^4}{\omega_q^3} \int \frac{d^3v f_j(\vec{v})}{m_j(\omega_q - qv_z)^2} \qquad (50)$$

If the contribution to the frequency shift from higher order terms has to be considered, one will have to sum the series



(51)

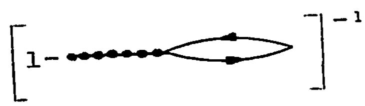
where the first term is given by (50), and the beaded line refers to the screened Coulomb potential. Therefore, summing the series, one obtains



(52)

Hence, the modified expression for the frequency shift in the light wave will be

$$\Delta\omega_T = \frac{q^2 I(q)}{8n_0^2 VF_q^2} \sum_j \frac{\omega_{pj}^4}{\omega_q^3} \int \frac{d^3 v f_j(\vec{v})}{m_j(\omega_q - qv_z)}$$



(53)

Since, for a cold plasma $f(\vec{v}) = n_0 \delta(\vec{v})$, (53) yields (see appendix B)

$$\Delta\omega_T = \frac{q^2 I(q)}{16 n_0 mVF_q^2} \frac{\omega_{pe}^4}{\omega_q^5} \frac{3\omega_q^2 + q^2 c^2}{\omega_q^2 + q^2 c^2}$$

$$= \frac{3}{32} \frac{q^2 e^2 \omega_{pe}^2}{m^2 \omega_q^5} |\vec{E}|^2 \left\{1 + \frac{q^2 c^2}{3\omega_q^2}\right\} \qquad (54)$$

where $\vec{E}$ is the electric field associated with the light wave. The same result can be obtained for a wave polarized in the counterclockwise direction.

Comparing (54) with expression (50) of Montgomery and Tidman (1964) one obtains

$$\frac{\Delta\omega_T}{\Delta\omega} = \frac{9}{8}\left(1+\frac{q^2c^2}{3\omega_q^2}\right)^2 \qquad (55)$$

where $\Delta\omega$ is the frequency shift in the above reference which is due to the linearly polarized transverse wave. Thus, it is obvious that the result obtained in this subsection is nearly the same as that of Montgomery and Tidman except for a small factor of $\frac{9}{8}\left(1+\frac{q^2c^2}{3\omega_q^2}\right)^2$.

b. Whistlers and Alfvén Modes

It has been seen that processes shown in figs. 25 and 26(a) do not give rise to a frequency shift either for light waves, whistler or Alfvén modes. However, the processes shown in figs. 24 and 26(b) do yield nonzero contribution as shown by (48) for a cold as well as hot plasmas. Performing the $\vec{v}$-integration in (48) for a cold plasma, one obtains

$$\Delta\omega_q = \frac{q^2 I(q)}{2n_0 V\omega_q^3 F_q^2}\left[\frac{\omega_{pe}^4\omega_{ce}^2}{m(\omega_q+\omega_{ce})^4} + \frac{\omega_{pi}^4\omega_{ci}^2}{m_i(\omega_q-\omega_{ci})^4} + \frac{\omega_{pe}^4}{4m\omega_q^2}\right] \qquad (56)$$

where the second term is due to ion contribution and the last term is due to the process shown in fig. 26(b).

For whistlers, where $\omega_{ci} << \omega_q << \omega_{ce}$, (56) yields

$$\Delta\omega_q \simeq \frac{q^2 I(q)}{8n_0 Vm\omega_q F_q^2}\frac{\omega_{pe}^4}{\omega_q^4} \qquad (57)$$

Substituting $\frac{I(q)}{VF_q^2} = \frac{|E|^2}{4\pi}$ into (57), one obtains the frequency shift for whistlers

$$\Delta\omega_w = \frac{q^2 e^2 \omega_{pe}^2}{8m^2 \omega_q^5} |\vec{E}|^2 \tag{58}$$

where E is the electric field associated with the whistler mode and $\omega_q = \frac{q^2 c^2}{\omega_{pe}^2} |\omega_{ce}|$, is the linear frequency.

Similarly, one can obtain from (56) the frequency shift for Alvén modes where $\omega << \omega_{cj}$

$$\Delta\omega_A = \frac{q^2 e^2 \omega_{pe}^2}{16m^2 \omega_q^5} |\vec{E}|^2 \tag{59}$$

again, $\vec{E}$ is the electric field in Alfvén mode, $\omega_q = qV_A$ and $V_A = \sqrt{B^2/4\pi n_0 m}$, the Alfvén velocity.

The frequency shift for electromagnetic waves propagating at an angle θ with respect to the external magnetic field can be obtained similarly.

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APPENDIX A

The nonlinear dielectric function ε_{NL} given by equation (21) of chapter IV can be written

$$\varepsilon_{NL} = \varepsilon_L - \varepsilon_L(\alpha_1 + \alpha_2) \tag{A.1}$$

where α_1 and α_2 are given by equations (45) and (46) of the same chapter

$$\alpha_1 = \frac{1}{n_0 V q^2 \varepsilon_L} \sum_{\vec{q}' \neq \vec{q}} \frac{I(\vec{q}')}{q'^2 F_{\vec{q}'}} \sum_j \frac{\omega_{pj}^4}{m_j} \int d^3v f_j(\vec{v}) \Lambda_j(\vec{q},\vec{q}',\vec{v}), \tag{A.2}$$

$$\alpha_2 = \frac{(4\pi e^2)^3}{Vq^2 \varepsilon_L(\vec{q},\omega)} \sum_{\vec{q}' \neq \vec{q}} \frac{I(\vec{q}')}{q'^2 F_{\vec{q}'}} \times \left[\frac{1}{|\vec{q}+\vec{q}'|^2 \varepsilon(\vec{q}+\vec{q}', \omega+\omega_{\vec{q}'})} \left\{ \sum_j \int \frac{d^3v f_j(\vec{v})}{m_j^2} Y_1 \right\}^2 + \frac{1}{|\vec{q}-\vec{q}'|^2 \varepsilon(\vec{q}-\vec{q}', \omega-\omega_{\vec{q}'})} \left\{ \sum \int \frac{d^3v f_j(\vec{v})}{m_j^2} Y_2 \right\}^2 \right] \tag{A.3}$$

and Λ_j and Y_1, Y_2 are given by (37) and (44) respectively.

Since α_1 gives negligible contribution as it has been seen from (48) and (49), there is no necessity for finding $\frac{\partial}{\partial\omega}(\varepsilon_L \alpha_1)$. Equation (A.3) yields after performing the velocity integrations

$$\alpha_2 = \frac{(4\pi e^2)^3}{Vq^2 \varepsilon_L} \frac{n_0^2}{m^4 v_e^8} \sum_{\vec{q}' \neq \vec{q}} \frac{I(q')}{q'^2 F_{\vec{q}'}} \left[\frac{1}{|\vec{q}+\vec{q}'|^2 \varepsilon(\vec{q}+\vec{q}', \omega+\omega_{\vec{q}'})} \times \right.$$

$$\left\{1+ \frac{qq'(q+q')}{\omega\omega_{\vec{q}'}(\omega+\omega_{\vec{q}'})}\left(\frac{q}{\omega} + \frac{q'}{\omega_{\vec{q}'}} + \frac{q+q'}{\omega+\omega_{\vec{q}'}}\right)\right\}^2$$

$$+ \frac{1}{|\vec{q}-\vec{q}'|^2\varepsilon(\vec{q}-\vec{q}',\omega-\omega_{\vec{q}'})}\left\{1+ \frac{qq'(q-q')}{\omega\omega_{\vec{q}'}(\omega-\omega_{\vec{q}'})} \times \right.$$

$$\left.\left.\left(\frac{q}{\omega} + \frac{q'}{\omega_{\vec{q}'}} + \frac{q-q'}{\omega-\omega_{\vec{q}'}}\right)\right\}^2\right] \qquad \text{(A.4)}$$

Substituting $q'^2 F_{\vec{q}'} = 2/\lambda_{De}^2$ and

$$\varepsilon(\vec{q}\pm\vec{q}',\omega\pm\omega_{\vec{q}'}) = 1+ \frac{1}{|\vec{q}\pm\vec{q}'|^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{(\omega\pm\omega_{\vec{q}'})^2} \qquad \text{(A.5)}$$

into (A.4), one obtains

$$\alpha_2 = \frac{1}{q^2\lambda_{De}^2\varepsilon_L} \cdot \frac{1}{n_0 V} \sum_{\vec{q}'\neq\vec{q}} \frac{I(\vec{q}')}{2mv_e^2}\left[\frac{(\omega+\omega_{\vec{q}'})^2}{(\omega+\omega_{\vec{q}'})^2-|\vec{q}+\vec{q}'|^2c_s^2}\right.$$

$$\times \left\{1+ \frac{qq'(q+q')}{\omega\omega_{\vec{q}'}(\omega+\omega_{\vec{q}'})}\left(\frac{q}{\omega} + \frac{q'}{\omega_{\vec{q}'}} + \frac{(q+q')}{\omega+\omega_{\vec{q}'}}\right)\right\}^2$$

$$+ \frac{(\omega-\omega_{\vec{q}'})^2}{(\omega-\omega_{\vec{q}'})^2-|\vec{q}-\vec{q}'|^2c_s^2}\left\{1+ \frac{qq'(q-q')}{\omega\omega_{\vec{q}'}(\omega-\omega_{\vec{q}'})}\right.$$

$$\left.\left.\left(\frac{q}{\omega} + \frac{q'}{\omega_{\vec{q}'}} + \frac{q-q'}{\omega-\omega_{\vec{q}'}}\right)\right\}^2\right] \qquad \text{(A.6)}$$

Multiply (A.6) by ε_L and differentiate with respect to ω, then substitute $\omega = qc_s$ and $\omega_{\vec{q}'} = q'c_s$ to get

$$\frac{\partial}{\partial\omega}(\varepsilon_L\alpha_2) = -\frac{16}{q^2\lambda_{De}^2 n_0 V}\sum_{\vec{q}'\neq\vec{q}}\frac{I(\vec{q}')}{mv_e^2}\times\left[\frac{(q+q')^3}{q^2q'^2c_s\theta_0^4}+\frac{(q-q')^3}{q^2q'^2c_s\theta_0^4}\right] \tag{A.7}$$

Performing the $\vec{q}'$-integration, using the spectrum distribution (47), one obtains

$$\frac{\partial}{\partial\omega}(\varepsilon_L\alpha_2) = -\frac{1}{qc_s}\frac{1}{q^2\lambda_{De}^2}\frac{256\pi^2}{3N_D\theta_0^3}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2} \tag{A.8}$$

Differentiate (A.1) with respect to ω and retaining only terms of order $I(t)$, to get

$$\begin{aligned} F_{\vec{q}} &= \left|\omega\frac{\partial}{\partial\omega}(\varepsilon_{NL})\right|_{\omega=\omega_{\vec{q}}} \\ &= \left|\omega\frac{\partial\varepsilon_L}{\partial\omega}\right|_{\omega=\omega_{\vec{L}}} - \omega_{\vec{q}}\frac{\partial}{\partial\omega}(\varepsilon_L\alpha_2) \\ &= \omega_{\vec{q}}\left[\frac{2\omega_{pi}^2}{\omega_{\vec{q}}^3}+\frac{1}{(qc_s)q^2\lambda_{De}^2}\frac{256\pi^2}{3N_D\theta_0^3}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2}\right] \end{aligned} \tag{A.9}$$

Equation (50) yields for $\varepsilon_{NL} = 0$.

$$\frac{\omega_{pi}^2}{\omega_{\vec{q}}^2} = \frac{1}{q^2\lambda_{De}^2}\left[1-\frac{256\pi^2}{9N_D\theta_0}\frac{I(t)}{mv_e^2}\left(\frac{T_e}{T_i}\right)^{3/2}\right] \tag{A.10}$$

Substituting (A.10) into (A.9), one obtains

$$F_{\vec{q}} = \omega_{\vec{q}}\left[\frac{2}{q^2\lambda_{De}^2(qc_s)}\left\{\left(1-\frac{256}{9}\alpha\right)^{3/2}+\frac{128}{3\theta_0^2}\alpha\right\}\right], \tag{A.11}$$

and $\alpha = (\pi^2/\theta_0 N_D)(I(t)/mv_e^2)(T_e/T_i)^{3/2}$ (A.12)

APPENDIX B

The common ratio of the infinite series given by equation (51) of chapter V can be written as

 $$= \frac{\text{Second term in the series}}{\text{first term}} \qquad \text{(B.1)}$$

where the first term is given by

$$M = \frac{\omega_{pe}^4 q^2 h^2}{16 n_0^2 V F_q^2 \omega_q^2 m} \int \frac{f(\vec{v})\, d^3v}{(\omega_q - q v_z)^2} \qquad \text{(B.2)}$$

and

$$\text{Second term} = \left(\frac{-h}{2\pi i}\right)\left(\frac{\pi e^2 h}{mV\omega_q F_q}\right)^2 \frac{(-i)^3}{h^3} \frac{4\pi e^2}{4q^2 V \varepsilon(29,2\omega_q)}$$

$$\left[\sum_{k_x,k_z,n} \int dE \frac{i^2}{(E-E_{k,n})(E+2\ q-E_{k+2q,n})}\right]^2$$

$$= \frac{\omega_{pe}^6 h^2 q^2}{64 m n_0^3 V \omega_q^2 F_q^2} \frac{1}{\varepsilon(2q,2\omega_q)} \left[\int \frac{d^3v f(\vec{v})}{(\omega_q - q v_z)^2}\right]^2 \qquad \text{(B.3)}$$

Substituting (B.2) and (B.3) into (B.1), one obtains

 $$= \frac{\omega_{pe}^2}{4 n_0 \varepsilon(2q,2\omega_q)} \cdot \int \frac{d^3v f(v)}{(\omega_q - q v_z)^2} \qquad \text{(B.4)}$$

This yields for a cold plasma

 $$= \frac{\omega_{pe}^2}{4\omega_q^2} \cdot \frac{1}{\varepsilon(2q,2\omega_q)} \qquad \text{(B.5)}$$

and

$\left[1-\right.$  $\left.\right] = 1 - \frac{\omega_{pe}^2}{4\omega_q^2}\frac{1}{\varepsilon(2q,2\omega_q)}$ (B.6)

where the dielectric function for light waves is

$$\varepsilon(2q,2\omega_q) = 1 - \frac{\omega_{pe}^2}{4\omega_q^2} \qquad \text{(B.7)}$$

Substituting (B.7) into (B.6) yields

$\left[1-\right.$  $\left.\right]^{-1} = \frac{3}{2}\frac{1+q^2c^2/3\omega_q^2}{1+q^2c^2/\omega_q^2}$ (B.8)