UNIVERSITY OF ALBERTA

SIMILARITY SOLUTIONS FOR A CLASS OF NON-NEWTONIAN FLUID FLOWS

by

© Jean-Paul Pascal

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ABSTRACT

In this thesis we investigate the similarity solutions of a nonlinear parabolic equation with fixed and moving boundary conditions. This problem governs the flow of a class of non-Newtonian fluids in a porous medium, and thus has applications in oil reservoir engineering.

We first determine general similarity transformations which reduce the governing equation to an ordinary differential equation. From these transformations we select those which satisfy the boundary conditions. At the same time we determine the forms of the boundary conditions and the source term that will allow similar solutions.

The ordinary differential equations obtained by the similarity transformation have been solved analytically for a class of particular cases. For other cases, numerical solutions have been obtained. Solutions were also obtained by a perturbation method and they were found to be in good agreement with those obtained numerically.

The location and velocity of the moving pressure disturbance front have also been investigated. The conditions under which the disturbance front propagates with finite velocity are found.

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Introduction

In this thesis we shall consider the unsteady flow of a class of non-Newtonian fluids through a porous medium. We consider power low fluids for which the shear stress is related to the shear rate by the following rheological equation of state [10]:

$$\tau = H|\dot{\gamma}|^{n-1}\dot{\gamma} \tag{I.1}$$

where τ is the shear stress, $\dot{\gamma}$ is the shear rate and H and n are rheological parameters obtained through measurements. For a shear thinning or viscoplastic fluid $0 < n < 1$, for a dilatant fluid $n > 1$ and for a Newtonian fluid $n = 1$. These fluids are discussed in [1] and [8].

The one dimensional Darcy's Law for power law fluids is given by [8]:

$$v = -\operatorname{sgn}(\frac{\partial p}{\partial x})[\frac{k}{\mu_{ef}}|\frac{\partial p}{\partial x}|]^{\frac{1}{n}} \tag{I.2}$$

where v is the fluid velocity, p is the fluid pressure, μ_{ef} is the effective viscosity coefficient and k is the permeability coefficient of the porous medium.

The material balance equation for a one-dimensional unsteady fluid flow in the presence of a source is given by:

$$\frac{\partial(\rho v)}{\partial x} = -\phi\frac{\partial \rho}{\partial t} - g(x,t,p) \tag{I.3}$$

where ρ is the density of the fluid, $g(x,t,p)$ is the source term and ϕ is the porosity of the porous medium.

For slightly compressible fluids the variation of density with pressure is given by [8] :

$$\rho = \rho_0[1 + \beta(p - p_0)]$$

where β is an experimentally determined quantity, known as the compressibility coefficient of the fluid, and ρ_0 and p_0 are the reference density and reference pressure respectively.

Substituting the expression for ρ into (I.3) we obtain:

$$\rho_0[1 + \beta(p - p_0)]\frac{\partial v}{\partial x} + \rho_0\beta v\frac{\partial p}{\partial x} = -\phi\rho_0\beta\frac{\partial p}{\partial t} - g(x,t,p). \qquad (I.4)$$

Taking into account that for slightly compressible fluids we have $\beta(p-p_0) \ll 1$ and $\beta v\frac{\partial p}{\partial x} \ll \frac{\partial v}{\partial x}$ we can approximate (I.4) as:

$$\frac{\partial v}{\partial x} = -\beta\phi\frac{\partial p}{\partial t} - \frac{1}{\rho_o}g(x,t,p). \qquad (I.5)$$

Substituting (I.2) into (I.5) results in:

$$\beta\phi\frac{\partial p}{\partial t} + \frac{1}{\rho_0}g(x,t,p) = [\frac{k}{\mu_{ef}}]^{\frac{1}{n}}\frac{\partial}{\partial x}[\frac{\partial p}{\partial x}|\frac{\partial p}{\partial x}|^{\frac{1}{n}-1}].$$

Thus, the equation governing the unsteady one-dimensional flow of a slightly compressible non-Newtonian fluid, of the above type, through a porous medium is:

$$a^2\frac{\partial p}{\partial t}+\psi(x,t,p)=\frac{\partial}{\partial x}[\frac{\partial p}{\partial x}|\frac{\partial p}{\partial x}|^{\frac{1}{n}-1}] \qquad (I.6)$$

where

$$a^2=\beta\phi(\frac{\mu_{ef}}{k})^{\frac{1}{n}}$$

and

$$\psi(x,t,p)=\frac{1}{\rho_0}(\frac{\mu_{ef}}{k})^{\frac{1}{n}}g(x,t,p).$$

In the case of axisymmetric plane radial flow of non-Newtonian fluids through a porous medium, Darcy's Law is given by [8]:

$$v(r,t)=-\operatorname{sgn}(\frac{\partial p}{\partial r})[\frac{k}{\mu_{ef}}|\frac{\partial p}{\partial r}|]^{\frac{1}{n}}. \qquad (I.7)$$

The material balance equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v)=-\phi\frac{\partial \rho}{\partial t}-g(r,t,p). \qquad (I.8)$$

Substituting (I.4) into (I.8) we obtain

$$\frac{v}{r}\rho_0[1+\beta(p-p_0)]+v\rho_0\beta\frac{\partial p}{\partial r}+\rho_0[1+\beta(p-p_0)]\frac{\partial v}{\partial r}=-\phi\rho_0\beta\frac{\partial p}{\partial t}-g(r,t,p). \qquad (I.9)$$

But for slightly compressible fluids we have $\beta(p-p_\circ)\ll 1$ and $\beta v\frac{\partial p}{\partial r}\ll\frac{\partial v}{\partial r}$, thus (I.9) can be approximated by:

$$\frac{v}{r}+\frac{\partial v}{\partial r}=-\beta\phi\frac{\partial p}{\partial t}-\frac{1}{\rho_0}g(r,t,p). \qquad (I.10)$$

Substituting (I.7) into (I.10) we are led to the governing equation of fluid flow for the axisymmetric plane radial flow:

$$a^2 \frac{\partial p}{\partial t} + \psi(r,t,p) = \frac{1}{n} \left|\frac{\partial p}{\partial r}\right|^{\frac{1-n}{n}} \left[\frac{\partial^2 p}{\partial r^2} + \frac{n}{r}\frac{\partial p}{\partial r}\right] \tag{I.11}$$

where

$$a^2 = \beta\phi\left[\frac{\mu_{ef}}{k}\right]^{\frac{1}{n}}$$

and

$$\psi(r,t,p) = \frac{1}{\rho_0}\left(\frac{\mu_{ef}}{k}\right)^{\frac{1}{n}} g(r,t,p).$$

In order to obtain physically feasible initial and boundary conditions, let us consider the problem of injection and production of a non-Newtonian fluid in a one-dimensional reservoir of semi-infinite extent with prescribed pressure at the initial moment and on the boundary. For the case of axisymmetric plane radial flow, we consider a cylindrical reservoir of infinite extent with a centrally located well of known radius $R_w > 0$, which is very small compared to the dimensions of the reservoir.

At the initial moment, ie. $t = 0$, let us take the fluid pressure in the reservoir to be constant, i.e. $p(x, 0) = p_0$ for $0 \leq x < \infty$ and $p(r, 0) = p_0$, for $R_w \leq r < \infty$. The boundary condition at $x = 0, t > 0$ and $r = R_w, t > 0$ will be of the form $p(0,t) = p_0 + p_w(t)$ and $p(R_w, t) = p_0 + p_w(t)$ respectively. The function $p_w(t)$ will be positive in the case of injection and negative in the case of production.

It should be noted that fluid pressure is a non-negative quantity. Thus in the case of production we must ensure that $p_0 + p_w(t)$ is non-negative. If $p_w(t) = W =$ const. then we simply require that $-W \le p_0$. But if $p_w(t)$ is not constant and $p_0 + p_w(t)$ becomes negative after a finite time t_f, then we only determine the pressure distribution in the reservoir for $0 \le t \le t_f$.

The pressure disturbance produced at $x = 0$ and $r = R_w$ propagates in the reservoir. The speed of propagation may be finite or infinite. For the case when this speed is finite, let $\ell(t)$ denote the location of the interface separating the region of disturbed and undisturbed pressure, which from now on we will refer to as the moving pressure disturbance front. Since at $t = 0$ the pressure is undisturbed we have $\ell(0) = 0$ for the one-dimensional case and $\ell(0) = R_w$ for the plane radical case. For the case when the speed of propagation is infinite, $\ell(t) = \infty$.

Therefore the initial and boundary conditions pertaining to equation (I.6) are:

$$\begin{aligned} p(x,0) &= p_0 \\ p(0,t) &= p_0 + p_w(t), \quad t > 0 \\ p(\ell(t),t) &= p_0 \\ \frac{\partial p}{\partial x}\Big|_{x=\ell(t)} &= 0 \end{aligned} \tag{I.12}$$

and the conditions pertaining to equation (I.11) are:

$$\begin{aligned}
&p(r,0) = p_0 \\
&p(R_w,t) = p_0 + p_w(t), \quad t>0 \\
&p(\ell(t),t) = p_0 \\
&\frac{\partial p}{\partial r}|_{r=\ell(t)} = 0.
\end{aligned} \tag{I.13}$$

Our goal is primarily to determine admissible similarity transformations of equations (I.6) and (I.11) and the forms of the functions $\psi(x,t,p)$ and $\psi(r,t,p)$ which allow us to reduce the two partial differential equations to ordinary differential equations. We will also show that the problems (I.6) (I.12) and (I.11) (I.13) admit similarity solutions only if the function $p_w(t)$ are of specific forms.

These two problems, with $\psi \equiv 0$ and $p_w(t) =$ const., have already been solved in [9]. However besides considering this case we determine similarity transformations for cases in which $\psi(x,t,p) \not\equiv 0$ $(\psi(r,t,p) \not\equiv 0)$ and $p_w(t) \not\equiv$ const. Although we have not established a general existence theorem we have obtained closed form solutions for classes of particular cases. We also present a numerical scheme which has provided solutions to these ordinary differential equations for a large number of parameter values.

Equation (I.6), with $\psi \equiv 0$ and $-\infty < x < \infty$, has also been previously studied but only connected with an initial value problem. For example, in [5] and [7] the authors investigate the existence and uniqueness of solutions and the existence of pressure disturbances moving with finite speed for the Cauchy problem associated with equation (I.6) with $\psi \equiv 0$ and $-\infty < x < \infty$.

Examples of physical phenomenon governed by a non-linear parabolic equation with a source term are given in [6], where asymptotic solutions of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2(u^m)}{\partial x^2} - u^p$$

are studied.

In determining suitable similarity transformation for our partial differential equations we employ the Clarkson-Kruskal (C.K) technique introduced in [2], [3]. In these articles the authors use this particular technique to obtain general similarity transformations for several other partial differential equations. However, there, these differential equations are not connected with any initial and boundary conditions. Initial and boundary conditions introduce additional constraints on the similarity transformations for specific problems.

CHAPTER 1

General Constraints for Similarity Transformations

In this chapter we consider similarity solutions for the problems outlined in the introduction. An attempt has been made to keep the solutions as general as possible. We begin with a change of dependent variable by setting

$$p(x,t) = p_0 + u(x,t)$$

for the problem (I.6), (I.12), and

$$p(r,t) = p_0 + u(r,t)$$

for the problem (I.11), (I.13).

The problem (I.6), (I.12) reduces to solving the equation

$$a^2 \frac{\partial u}{\partial t} + \varphi(x,t,u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} \left|\frac{\partial u}{\partial x}\right|^{\frac{1}{n}-1}\right) \tag{1.1}$$

in the domain $(0,\infty) \times (0,\infty)$ with initial and boundary conditions:

$$\begin{aligned} u(x,0) &= 0, \qquad x \geq 0 \\ u(0,t) &= p_w(t), \qquad t > 0 \\ u(\ell(t),t) &= 0 \\ \frac{\partial u}{\partial x}\Big|_{x=\ell(t)} &= 0 \end{aligned} \tag{1.2}$$

where

$$\varphi(x,t,u) = \psi(x,t,p_0 + u).$$

The problem (I.11), (I.13) reduces to finding the function $u(r,t)$ satisfying the equation

$$a^2 \frac{\partial u}{\partial t} + \varphi(r,t,u) = \frac{1}{n} \left|\frac{\partial u}{\partial r}\right|^{\frac{1}{n}-1} \left[\frac{\partial^2 u}{\partial r^2} + \frac{n}{r}\frac{\partial u}{\partial r}\right] \tag{1.3}$$

in the domain $(R_w, \infty) \times (0, \infty)$ where $\varphi(r,t,u) = \psi(r,t,p_0 + u)$ and the initial and boundary conditions:

$$\begin{aligned} u(r,0) &= 0, \qquad r \geq R_w \\ u(R_w,t) &= p_w(t), \qquad t > 0 \\ u(\ell(t),t) &= 0 \\ \frac{\partial u}{\partial r}\Big|_{r=\ell(t)} &= 0 \end{aligned} \tag{1.4}$$

To begin with we concentrate our efforts on obtaining transformations that will reduce equations (1.1) and (1.3) to ordinary differential equations. According to the CK-technique [3] we will seek solutions to the equations (1.1) and (1.3) in the form.

$$u(y,t) = U[y,t,f(\xi(y,t))] \tag{1.5}$$

where $U[y,t,f]$ and $\xi(y,t)$ are functions to be determined.

Note that equation (1.1) can be written as:

$$a^2 \frac{\partial u}{\partial t} - \frac{1}{n}\frac{\partial^2 u}{\partial x^2}\left|\frac{\partial u}{\partial x}\right|^{\frac{1}{n}-1} + \varphi(x,t,u) = 0. \tag{1.6}$$

Upon substituting (1.5) into (1.6) and (1.3), for $y = x$ we have:

$$a^2[U_t + U_f f'\xi_t] - \frac{1}{n}\{U_{xx} + (2U_{xf}\xi_x + U_f\xi_{xx})f' + U_{ff}\xi_x^2 f'^2 + U_f\xi_x^2 f''\}\{|U_x + U_f\xi_x f'|^{\frac{1}{n}-1}\} + \varphi(x,t,U) = 0 \tag{1.7}$$

and for $y = r$ we have:

$$a^2[U_t + U_f\xi_t f'] - \frac{1}{n}\{U_{rr} + \frac{n}{r}U_r + (2U_{rf} + U_f\xi_{rr} + \frac{n}{r}U_f\xi_r)f' + U_{ff}\xi_r^2 f'^2 + U_f\xi_r^2 f''\}\{|U_r + U_f\xi_r f'|^{\frac{1}{n}-1}\} + \varphi(r,t,U) = 0. \tag{1.8}$$

In order for equations (1.7) and (1.8) to be ordinary differential equations in $f(\xi)$, the ratio of the coefficients of different derivatives of $f(\xi)$ must be functions of f and ξ only. For example, using the coefficient of f'' as a normalizing coefficient, considering the coefficient of $(f')^2$ we must have:

$$U_{ff} = F(f,\xi)U_f$$

where F is a function to be determined. So,

$$\frac{U_{ff}}{U_f} = F(f,\xi).$$

Integrating twice we obtain

$$U = g(y,t) + T(y,t)\Gamma(f,\xi) \tag{1.9}$$

where $\Gamma(f,\xi)$ is the result of integrating $F(f,\xi)$ twice, and $g(y,t)$ and $T(y,t)$ are integration functions. Note that y stands for x in the case of equation (1.1) and for r in the case of equation (1.3).

This result shows us that it is sufficient to consider, for equations (1.1) and (1.3), similarity transformations of the form [3]:

$$u(x,t) = g(x,t) + T(x,t)f\big(\xi(x,t)\big) \tag{1.10}$$

and

$$u(r,t) = g(r,t) + T(r,t)f\big(\xi(r,t)\big) \tag{1.11}$$

respectively.

Now, substituting (1.10) into (1.6) and (1.11) into (1.3) yields:

$$\begin{aligned} a^2[g_t + T_t f + T\xi_t f'] - \frac{1}{n}[g_{xx} + T_{xx} f + (2T_x\xi_x + T\xi_{xx})f' \\ + T\xi_x^2 f''][|g_x + T_x f + T\xi_x f'|^{\frac{1}{n}-1}] + \varphi(x,t,u) = 0 \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} a^2[g_t + T_t f + T\xi_t f'] - \frac{1}{n}[g_{rr} + \frac{n}{r}\, g_r + (T_{rr} + \frac{n}{r}\, T_r)f \\ + (\frac{n}{r}\, T\xi_r + 2T_r\xi_r + T\xi_{rr})f' + T\xi_r^2 f''] \\ \times [|g_r + T_r f + T\xi_r f'|^{\frac{1}{n}-1}] + \varphi(r,t,u) = 0. \end{aligned} \tag{1.13}$$

In order for equations (1.12) and (1.13) to be ordinary differential equations in $f(\xi)$ certain constraints must be imposed on the functions g, T and ξ. These restrictions will be obtained by requiring that the ratio of coefficients of different derivatives and powers of $f(\xi)$ be functions of ξ only. The appropriate ratios of coefficients are determined by performing the following operation on both equations: In the second square parenthesis we normalize by the coefficient of f'', in the third square brackets by the coefficient of f', and in the first square brackets by $T\xi_x^2|T\xi_x|^{\frac{1}{n}-1}$ and $T\xi_r^2|T\xi_r|^{\frac{1}{n}-1}$ respectively.

We thus obtain the following sets of constraints. For the one-dimensional case we have:

$$g_x = F_1(\xi)T\xi_x \tag{1.14}$$

$$T_x = F_2(\xi)T\xi_x \tag{1.15}$$

$$g_{xx} = F_3(\xi)T\xi_x^2 \tag{1.16}$$

$$T_{xx} = F_4(\xi)T\xi_x^2 \tag{1.17}$$

$$2T_x\xi_x + T\xi_{xx} = F_5(\xi)T\xi_x^2 \tag{1.18}$$

$$g_t = F_6(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.19}$$

$$T_t = F_7(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.20}$$

$$T\xi_t = F_8(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.21}$$

and $\varphi(x,t,u)$ must be of the form:

$$\varphi(x,t,u) = F_9(\xi, f)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.22}$$

Note that if $\varphi(x,t,u)$ is of the form:

$$\varphi(x,t,u) = b(x,t)u + h(x,t) \tag{1.23}$$

then by (1.10) we have:

$$\varphi(x,t,u) = b(x,t)T(x,t)f(\xi) + b(x,t)g(x,t) + h(x,t).$$

Substituting this expression into (1.12) and collecting coefficients of f^0, f and f' then the constraints (1.19) - (1.21) are replaced by:

$$a^2 g_t + bg + h = F_6(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.19a}$$

$$a^2 T_t + bT = F_7(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.20a}$$

$$a^2 T\xi_t = F_8(\xi)T\xi_x^2|T\xi_x|^{\frac{1}{n}-1} \tag{1.21a}$$

and obviously constraint (1.22) no longer applies.

It should be noted that in the particular case when $n = \frac{1}{2}$ the exponent $\frac{1}{n} - 1$ appearing in equation (1.12) is equal to one. In this case certain algebraic manipulations can be performed on this equation which lead to different constraints for the functions g, T and ξ. However, in the next section we discover that if we take $g(x,t) \equiv 0$ then we obtain a similarity transformation suitable for our boundary value problem. In this case the constraints for $n = \frac{1}{2}$ are identical to (1.14) - (1.22).

If the constraints (1.14) - (1.22) are satisfied then equation (1.12) becomes:

$$\begin{aligned} A^2(F_6(\xi) + F_7(\xi)f + F_8(\xi)f') - \frac{1}{n}[F_3(\xi) + F_4(\xi)f + F_5(\xi)f' + f''] \\ * |F_1(\xi) + F_2(\xi)f + f'|^{\frac{1}{n}-1} + F_9(\xi, f) = 0 \end{aligned} \tag{1.24}$$

where $A^2 = a^2$ if $\varphi(x,t,u) \neq b(x,t)u + h(x,t)$ and $A^2 = 1$ and $F_9(\xi, f) \equiv 0$ if $\varphi(x,t,u) = b(x,t)u + h(x,t)$.

For the plane radial case, the constraints are:

$$g_r = F_1(\xi)T\xi_r \tag{1.25}$$

$$T_r = F_2(\xi)T\xi_r \tag{1.26}$$

$$g_{rr} + \frac{n}{r}g_r = F_3(\xi)T\xi_r^2 \tag{1.27}$$

$$T_{rr} + \frac{nT_r}{r} = F_4(\xi)T\xi_r^2 \tag{1.28}$$

$$(\frac{nT}{r} + 2T_r)\xi_r + T\xi_{rr} = F_5(\xi)T\xi_r^2 \tag{1.29}$$

$$g_t = F_6(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.30}$$

$$T_t = F_7(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.31}$$

$$T\xi_t = F_8(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.32}$$

and $\varphi(r,t,u)$ must be of the form:

$$\varphi(r,t,u) = F_9(\xi,f)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1}. \tag{1.33}$$

Note that if

$$\varphi(r,t,u) = b(r,t)u + h(r,t) = b(r,t)T(r,t)f(\xi) + b(r,t)g(r,t) + h(r,t) \tag{1.34}$$

then as in the one-dimensional case the constraints (1.30) - (1.32) are replaced by

$$a^2 g_t + bg + h = F_6(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.30a}$$

$$a^2 T_t + bT = F_7(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.31a}$$

$$a^2 T\xi_t = F_8(\xi)T\xi_r^2|T\xi_r|^{\frac{1}{n}-1} \tag{1.32a}$$

and the constraint (1.33) no longer applies.

For the particular case when $n = \frac{1}{2}$, the observation made for the one-dimensional case apllies here as well.

If the constraints (1.25) - (1.33) are satisfied the equation (1.13) becomes:

$$A^2(F_6(\xi) + F_7(\xi)f + F_8(\xi)f') - \frac{1}{n}(F_3(\xi) + F_4(\xi)f + F_5(\xi)f' + f'') \\ * |F_1(\xi) + F_2(\xi)f + f'|^{\frac{1}{n}-1} + F_9(\xi, f) = 0 \tag{1.35}$$

where $A^2 = a^2$ if $\varphi(r,t,u) \neq b(r,t)u + h(r,t)$ and $A^2 = 1$ and $F_9(\xi, f) \equiv 0$ if $\varphi(r,t,u) = b(r,t)u + h(r,t)$.

The differential equations (1.24) and (1.35) obtained for the one-dimensional and plane radial cases respectively are identical. But the functions g, T and ξ which appear in the transformations (1.10) and (1.11) can be different as functions of their variables since the third, fourth and fifth equations from the list of constraints are different for the two cases.

Any set of functions g, T and ξ which together with the functions $F_1, F_2, \ldots, F_8$ satisfy the constraints (1.14) - (1.21) or (1.25) - (1.32) determine a similarity transformation for the partial differential equation (1.1) or (1.3) when the function φ satisfies the constraint (1.22) (or (1.23)) or (1.33) (or (1.34)) respectively.

Note that in both cases the one-dimensional case and the plane radial case we have eleven unknown functions: $g, T, \xi, F_1, F_2, \ldots, F_8$ which must sat- eight equations (1.14)- (1.21) or (1.25) - (1.32). Thus there are three degrees of freedom which can be used to determine a particular transformation compatible with the given boundary conditions.

CHAPTER 2

Constraints Arising from the Boundary Conditions

Let us now determine the constraints that the boundary conditions of our problem impose on the similarity transformation.

For example the boundary condition $u(\ell(t),t) = 0, \quad t > 0$ gives us

$$g(\ell(t),t) + T(\ell(t),t)f[\xi(\ell(t),t)] = 0. \tag{2.1}$$

Note that this relationship between g, T and f is the same for both problems.

On the other hand in order to obtain a fixed domain for the ordinary differential equation in $f(\xi)$ we impose the following restrictions on the functions $\xi(x,t)$ and $\xi(r,t)$:

$$\xi[\ell(t),t] = \xi_1 = \quad \text{const.} \tag{2.2}$$

and,

$$\xi(0,t) = \xi_0 = \quad \text{const.,} \tag{2.3}$$

$$\xi(R_w,t) = \xi_0. \tag{2.4}$$

However, upon examining the general constraints we realize that we are not able to satisfy all the constraints with $R_w \neq 0$. Yet, the well radius is extremely small compared to the dimension of the porous medium. Thus we can approximate the condition (2.4) with (2.3).

The condition (2.2) together with (2.1) imply that:

$$g(\ell(t),t) = -f(\xi_1)T(\ell(t),t)$$

which is possible if we take

$$g(x,t) = \quad \text{const.} \quad T(x,t)$$

for the one-dimensional case and

$$g(r,t) = \quad \text{const.} \quad T(r,t)$$

for the plane radial case. Substituting these into the transformations (1.10) and (1.11) respectively and denoting const. $+ f(\xi)$ by $f(\xi)$ we obtain

$$u(x,t) = T(x,t)f(\xi) \tag{2.5}$$

and

$$u(r,t) = T(r,t)f(\xi). \tag{2.6}$$

So in the constraints (1.14), (1.16), (1.19), (1.25), (1.27) and (1.30) we will take $g \equiv 0$ which implies that $F_1(\xi) \equiv 0, F_3(\xi) \equiv 0$ and $F_6(\xi) \equiv 0$ for both problems.

Let us now focus our attention on the constraint (1.15) and its counterpart in the plane radial case, (1.26). Denoting the spatial variable by y and integrating with respect to it we have:

$$T(y,t) = H(t)\Gamma(\xi)$$

where $\Gamma(\xi) = e^{\int F_2(\xi)d\xi}$ and $H(t)$ is an integration function.

Substituting this result into (2.5) and (2.6) and again denoting the spatial variable by y, we obtain

$$u(y,t) = H(t)\Gamma(\xi)f(\xi).$$

Reassigning f and T we are lead to the following similarity transformations:

$$u(x,t) = T(t)f\big(\xi(x,t)\big) \tag{2.7}$$

$$u(r,t) = T(t)f\big(\xi(r,t)\big) \tag{2.8}$$

Now, since T is only a function of t, the constraints (1.15), (1.17) and (1.26), (1.28) imply that $F_2(\xi) \equiv 0$ and $F_4(\xi) \equiv 0$.

In order to determine $\xi(x,t)$ we use the constraint (1.18), which now can be written as

$$\frac{\xi_{xx}}{\xi_x} = F_5(\xi)\xi_x.$$

Integrating with respect to x we obtain

$$\xi_x = \tau(t)e^{\int F_5(\xi)d\xi}$$

where $\tau(t)$ is an integration function. For convenience we will take $\tau(t) > 0$ for $t \geq 0$.

Integrating again we have:

$$\int e^{-\int F_5(\xi)d\xi} d\xi = \tau(t)x + \eta(t)$$

where $\eta(t)$ is an integration function.

Assuming that the function on the left hand side of this equation has an inverse, solving for ξ we obtain:

$$\xi = \Omega\big(x\tau(t) + \eta(t)\big). \tag{2.9}$$

Note that if $F_5(\xi) \equiv 0$ then $\xi(x,t) = x\tau(t) + \eta(t)$.

Now we will show that without loss of generality we can take $F_5(\xi) \equiv 0$. Indeed, substituting (2.9) into the transformations (2.7) we obtain:

$$u(x,t) = T(t)f[\Omega\big(x\tau(t) + \eta(t)\big)].$$

Reassigning f, this equation can be rewritten as:

$$u(x,t) = T(t)f\big(x\tau(t) + \eta(t)\big)$$

which together with (2.7) implies that:

$$\xi(x,t) = x\tau(t) + \eta(t)$$

and from (1.18) it results that

$$F_5(\xi) \equiv 0.$$

On the other hand, the condition (2.3) implies that $\eta(t) = \xi_0$ which can conveniently be taken equal to zero. Then,

$$\xi(x,t) = x\tau(t). \tag{2.10}$$

Before determining $\tau(t)$ and $T(t)$ from the remaining constraints we will find the expression of $\xi(r,t)$ for the plane radial case. In light of the fact that T is a function of t only, the constraint (1.29) becomes

$$\frac{\xi_{rr}}{\xi_r} + \frac{n}{r} = F_5(\xi)\xi_r. \tag{2.11}$$

In the case when $\xi_{rr} = 0$,

$$\xi(r,t) = \tau(t)r + \eta(t).$$

Then by the condition (2.3) with $\xi_0 = 0$,

$$\eta(t) \equiv 0$$

thus,

$$\xi(r,t) = r\tau(t). \tag{2.12}$$

Substituting this result into (2.11) we obtain

$$F_5(\xi) = \frac{n}{\xi}\,. \tag{2.13}$$

If $\xi_{rr} \neq 0$ then integrating (2.11) with respect to r we obtain:

$$\xi_r = \frac{\tau(t)}{r^n}\, e^{\int F_5(\xi)d\xi}$$

where $\tau(t)$ is an integration function, which for convenience will taken to be positive for $t \geq 0$.

Integrating again we obtain:

$$\int e^{-\int F_5(\xi)d\xi} d\xi = \frac{1}{1-n}\,\tau(t)r^{1-n} + \eta(t)$$

where $\eta(t)$ is an integration function. Reasoning as in the one-dimensional case we came to the conclusion that

$$\xi(r,t) = \frac{1}{1-n}\,\tau(t)r^{1-n} + \eta(t)$$

and that $F_5(\xi) \equiv 0$.

In order for this expression for $\xi(r,t)$ to satisfy the condition (2.3) we must have $n < 1$ and $\eta(t) = \xi_0$. So, reassigning $\tau(t)$ and taking $\xi_0 = 0$ we may write:

$$\xi(r,t) = \tau(t)r^{1-n}. \tag{2.14}$$

It is easily verified that this form of the similarity variable can be reduced to the previous form (i.e. $\xi(r,t) = r\tau(t)$) by reassigning the functions $\tau(t)$ and $f(\xi)$. However we will consider the expression for $\xi(r,t)$ given by (2.14) since the resulting form for the source term may be more appealing.

In order to determine the remaining unknown functions associated with our transformation we must distinguish between different forms of the function φ.

CHAPTER 3

The Ordinary Differential Equations for the Case of

$\varphi \neq bu + h$

In this case we must take into consideration the constraints (1.20) - (1.22) and (1.31) - (1.33). From (1.20) and (1.21) or from (1.30) and (1.32) we obtain:

$$\frac{T'(t)}{T(t)} = \frac{F_7(\xi)}{F_8(\xi)}\,\xi_t.$$

Taking into account the expression for ξ given by (2.10), (2.12) or (2.14) we obtain:

$$\xi_t = \frac{\tau'(t)}{\tau(t)}\,\xi \tag{3.1}$$

for both problems.

Thus,

$$\frac{T'(t)}{T(t)} = \frac{F_7(\xi)}{F_8(\xi)}\,\xi\,\frac{\tau'(t)}{\tau(t)}.$$

So, we must have:

$$\frac{F_7(\xi)}{F_8(\xi)}\,\xi = \gamma = \text{ const.}$$

which implies that

$$\frac{T'(t)}{T(t)} = \gamma\,\frac{\tau'(t)}{\tau(t)}.$$

Integrating we obtain:

$$T(t) = K[\tau(t)]^{\gamma}$$

where K is an integration constant. But then we would have $u = K[\tau(t)]^{\gamma} f(\xi)$ and reassigning $f(\xi)$ as $Kf(\xi)$ implies that

$$T(t) = [\tau(t)]^{\gamma}. \tag{3.2}$$

Now, in order to determine $\tau(t)$ we substitute this expression for $T(t)$ and the appropriate expression for ξ into the constraints (1.21) and (1.32):

$$\xi\tau'(t) = F_8(\xi)[\tau(t)]^{2+\frac{1}{n}+\gamma(\frac{1}{n}-1)} \tag{3.3}$$

if ξ is given by (2.10) or (2.12), and

$$[\tau(t)]^{-\gamma(\frac{1}{n}-1)-\frac{2n-n^2+1}{(1-n)n}}\tau'(t) = (1-n)^{\frac{1}{n}+1} F_8(\xi)\xi^{\frac{-2}{1-n}} \tag{3.4}$$

if ξ is given by (2.14).

First we analyse the equation (3.3). From this equation it is evident that $F_8(\xi)$ should be of the form:

$$F_8(\xi) = \alpha\xi \tag{3.5}$$

where α is a constant. This implies that $F_7(\xi) = \gamma\alpha$. The equation (3.3) becomes:

$$[\tau(t)]^{-2-\frac{1}{n}-\gamma(\frac{1}{n}-1)}\tau'(t) = \alpha.$$

Integrating, we obtain for $\gamma \neq \frac{1+n}{n-1}$

$$\tau(t) = \{[-1 - \frac{1}{n} - \gamma(\frac{1}{n} - 1)]\alpha t + c_1\}^{\frac{-n}{n+1+\gamma(1-n)}} \tag{3.6}$$

where c_1 is an integration constant. For $\gamma = \frac{1+n}{n-1}$ we obtain

$$\tau(t) = c_1 e^{\alpha t} \tag{3.7}$$

where c_1 is an integration constant.

On the other hand, substituting the expression for ξ given by (2.10) and (2.12) into (2.2) we obtain an expression for the moving pressure disturbance front:

$$\ell(t) = \xi_1[\tau(t)]^{-1}. \tag{3.8}$$

If the front propagates with infinite velocity this implies that $\xi_1 = \infty$ and no restriction is imposed on $\tau(t)$. However if the velocity is finite then $\ell(t)$ must satisfy $\ell(0) = 0$. This implies that $[\tau(0)]^{-1} = 0$.

This fact suggests that the case $\gamma = \frac{n+1}{n-1}$ does not constitute a similarity transformation that is compatible with our problem. Whereas for the case $\gamma \neq \frac{n+1}{n-1}$ we have to have $c_1 = 0$, and

$$n + 1 + \gamma(1 - n) > 0. \tag{3.9}$$

If we choose $\alpha = \frac{-n}{n+1+\gamma(1-n)}$ then we may write:

$$\tau(t) = t^{\frac{-n}{n+1+\gamma(1-n)}} \tag{3.10}$$

$$T(t) = t^{\frac{-n\gamma}{n+1+\gamma(1-n)}} \tag{3.11}$$

$$\xi(y,t) = yt^{\frac{-n}{n+1+\gamma(1-n)}}, \quad y = x \quad \text{or} \quad y = r \tag{3.12}$$

$$\text{and} \quad \ell(t) = \xi_1 t^{\frac{n}{n+1+\gamma(1-n)}}. \tag{3.13}$$

Now we will determine the form of the function φ such that the conditions (1.22) and (1.33) are satisfied. If $\varphi \equiv 0$ then $F_9 \equiv 0$ and all constraints are satisfied. If not then since we already have expressions for $T(t)$ and $\xi(y,t)$, we may write:

$$T\xi_y^2|T\xi_y|^{\frac{1}{n}-1} = T^{\frac{1}{\gamma}(1+\frac{1}{n})+\frac{1}{n}} \quad \text{for} \quad \gamma \neq 0,$$

and

$$T\xi_y^2|T\xi_y|^{\frac{1}{n}-1} = [\tau(t)]^{1+\frac{1}{n}} = t^{-1} \quad \text{for} \quad \gamma = 0.$$

Substituting this result into (1.22) and (1.33) and taking into account that $u = Tf$ we obtain:

$$\varphi(y,t,Tf) = F_9(\xi,f)T^{\frac{1}{\gamma}(1+\frac{1}{n})+\frac{1}{n}} \quad \text{for} \quad \gamma \neq 0 \tag{3.14}$$

and

$$\varphi(y,t,f) = F_9(\xi,f)t^{-1} \quad \text{for} \quad \gamma = 0. \tag{3.15}$$

The equation (3.14) implies that we must have:

$$\varphi(y,t,Tf) = F(\xi)(T|f|)^q \tag{3.16}$$

where

$$q = \frac{1}{\gamma}(1+\frac{1}{n}) + \frac{1}{n} \quad \text{or} \quad \gamma = \frac{n+1}{nq-1}. \tag{3.17}$$

Thus in order to have similarity solutions to our problems, φ in the equations (1.1) and (1.13) should be of the form

$$\varphi(y,t,u) = F\big(yt^{\frac{-(nq-1)}{(n+1)(q-1)}}\big)|u|^q \tag{3.18}$$

where $y = x$ and $y = r$ respectively.

The equation (3.17) and the constraint (3.9) impose restrictions on q, namely:

$$\begin{aligned} &q < 1 \quad \text{or} \quad q > \tfrac{1}{n} \qquad \text{if} \quad n < 1 \\ \text{and,} \qquad &q > 1 \quad \text{or} \quad q < \tfrac{2}{n} - 1 \quad \text{if} \quad n > 1. \end{aligned} \tag{3.19}$$

From (3.16) we deduce that

$$F_9(\xi, f) = F(\xi)|f|^q \tag{3.20}$$

Now, for the case when $\gamma = 0$, the equation (3.15) and the fact that $u = f$ imply that:

$$\varphi(y,t,u) = F(yt^{\frac{-n}{n+1}}, u)t^{-1} \tag{3.21}$$

where

$$F(yt^{\frac{-n}{n+1}}, u) \neq b(yt^{\frac{-n}{n+1}})u + h(yt^{\frac{-n}{n+1}}).$$

In this case $F_9(\xi, f)$ from the constraints (1.22) and (1.33) is given by:

$$F_9(\xi, f) = F(\xi, f). \tag{3.22}$$

We may now write the differential equation in $f(\xi)$. For the one-dimensional case we have:

$$\frac{na^2}{1+n+\gamma(1-n)}(\gamma f + \xi f') + \frac{1}{n} f''|f'|^{\frac{1}{n}-1} - F(\xi)|f|^q = 0 \tag{3.23}$$

for $\gamma = \frac{n+1}{nq-1}$ and,

$$\frac{na^2}{n+1}\xi f' + \frac{1}{n} f''|f'|^{\frac{1}{n}-1} - F(\xi, f) = 0 \tag{3.24}$$

for $\gamma = 0$.

For the plane radial case with $\xi(r,t)$ given by (3.12) we have

$$\frac{na^2}{1+n+\gamma(1-n)}(\gamma f + \xi f') + \frac{1}{n}(\frac{n}{\xi} f' + f'')|f'|^{\frac{1}{n}-1} - F(\xi)|f|^q = 0 \tag{3.25}$$

for $\gamma = \frac{n+1}{nq-1}$ and

$$\frac{na^2}{n+1}\xi f' + \frac{1}{n}(\frac{n}{\xi} f' + f'')|f'|^{\frac{1}{n}-1} - F(\xi, f) = 0 \tag{3.26}$$

for $\gamma = 0$.

Let us now return to the case when $\xi(r,t)$ is given by (2.14). In this case, as noted before, $\tau(t)$ is determined by the equation (3.4). This equation indicates that $F_8(\xi)$ should be of the form:

$$F_8(\xi) = \alpha' \xi^{\frac{2}{1-n}}, \quad n < 1.$$

This implies that $F_7(\xi) = \gamma \alpha' \xi^{\frac{1+n}{1-n}}$, and the equation (3.4) becomes:

$$[\tau(t)]^{-2-\frac{1}{n}-\gamma(\frac{1}{n}-1)-\frac{1+n}{1-n}} \tau'(t) = \alpha$$

where $\alpha = (1-n)^{\frac{1}{n}+1} \alpha'$.

Integrating we get:

$$\tau(t) = [\frac{-[1+n+\gamma(1-n)^2]}{n(1-n)} \alpha t + c]^{\frac{-n(1-n)}{1+n+\gamma(1-n)^2}} \tag{3.27a}$$

for $\gamma \neq \frac{-(1+n)}{(1-n)^2}$ and

$$\tau(t) = ce^{\alpha t} \quad \text{for} \quad \gamma = \frac{-(1+n)}{(1-n)^2}$$

where c is an integration constant.

The location of the moving pressure front is now expressed by:

$$\ell(t) = \xi_1^{\frac{1}{1-n}} [\tau(t)]^{\frac{-1}{1-n}}. \tag{3.28}$$

Reasoning as for the equation (3.8) we conclude that we must have $[\tau(0)]^{-1} = 0$ and therefore $\tau(t)$ is given by (3.27a) with $c = 0$ and $\gamma > \frac{-(1+n)}{(1-n)^2}$. Thus if we choose $\alpha = \frac{-n(1-n)}{1+n+\gamma(1-n)^2}$ then we have:

$$\tau(t) = t^{\frac{-n(1-n)}{1+n+\gamma(1-n)^2}}, \tag{3.27}$$

$$\xi(r,t) = r^{1-n} t^{\frac{-n(1-n)}{n+1+\gamma(1-n)^2}}$$

and

$$\ell(t) = \xi_1^{\frac{1}{1-n}} t^{\frac{n}{n+1+\gamma(1-n)^2}}.$$

With these expressions for $\xi(r,t)$ and $T(t)$ we find that:

$$T\xi_r^2 |T\xi_r|^{\frac{1}{n}-1} = (1-n)^{\frac{1}{n}+1} r^{-(1+n)} T^{\frac{1}{\gamma}(1+\frac{1}{n})+\frac{1}{n}} \quad \text{for} \quad \gamma \neq 0$$

and,

$$T\xi_r^2 |T\xi_r|^{\frac{1}{n}-1} = (1-n)^{\frac{1}{n}+1} r^{-(1+n)} t^{-1} \quad \text{for} \quad \gamma = 0.$$

Substituting this result into (1.33) we obtain

$$\varphi(r,t,Tf) = F_9(\xi,f)(1-n)^{\frac{1}{n}+1} r^{-(1+n)} T^{\frac{1}{\gamma}(1+\frac{1}{n})+\frac{1}{n}} \quad \text{for} \quad \gamma \neq 0 \tag{3.29}$$

$$\varphi(r,t,f) = F_9(\xi,f)(1-n)^{\frac{1}{n}+1} r^{-(1+n)} t^{-1} \quad \text{for} \quad \gamma = 0. \tag{3.30}$$

The equation (3.29) implies that we must have:

$$\varphi(r,t,Tf) = F(\xi) r^{-(1+n)} (T|f|)^{q} \tag{3.31}$$

where q is given by (3.17).

Thus in orde- to have similarity solutions to our problem, $\varphi(r,t,u)$ in equation (1.3) should be of the form:

$$\varphi(r,t,u) = F\left(r^{1-n}t^{\frac{-(nq-1)(1-n)}{(n+1)(q+n-2)}}\right) r^{-(1+n)}|u|^q \tag{3.32}$$

where q is subjected to the conditions $q > \frac{1}{n}$ or $q < 2-n$. From (3.31) we deduce that $F_9(\xi,f)$ is given by: $F_9(\xi,f) = \frac{1}{(1-n)^{\frac{1}{n}+1}} F(\xi)|f|^q$.

Now for the case when $\gamma = 0$, the equation (3.30) implies that we must have:

$$\varphi(r,t,u) = F(r^{1-n}t^{\frac{-n(1-n)}{n+1}}, u)t^{-1}r^{-(1+n)} \tag{3.33}$$

with $F(r^{1-n}t^{\frac{-n}{n+1}}, u) \neq b(r^{1+n}t^{\frac{-n}{n+1}})u + h(r^{1+n}t^{\frac{-n}{n+1}})$ In this case $F_9(\xi,f) = \frac{1}{(1-n)^{\frac{1}{n}+1}} F(\xi,f)$.

We may write the differential equation in $f(\xi)$ for the plane radial case with $\xi(r,t)$ given by (2.14):

$$\frac{-na^2}{(1-n)^{\frac{1}{n}}[1+n+\gamma(1-n)^2]}\, \xi^{\frac{1+n}{1-n}}(\gamma f+\xi f') - \frac{1}{n}\, f''|f'|^{\frac{1}{n}-1} + \frac{1}{(1-n)^{\frac{1}{n}+1}}\, F(\xi)|f|^q = 0 \tag{3.34}$$

for $\gamma = \frac{n+1}{nq-1}$ and

$$\frac{na^2}{(n+1)(1-n)^{\frac{1}{n}}}\, \xi^{\frac{2}{1-n}} f' + \frac{1}{n}\, f''|f'|^{\frac{1}{n}-1} - \frac{1}{(1-n)^{\frac{1}{n}+1}}\, F(\xi,f) = 0 \tag{3.35}$$

for $\gamma = 0$.

It remains now to determine the forms of the function $p_w(t)$ and the boundary conditions for $f(\xi)$ required by the similarity transformation. The boundary conditions for the one-dimensional problem at $x = 0$ and for the plane radial problem at $r = R_w$ are $u(0,t) \underset{t>0}{=} p_w(t)$ and $u(R_w, t) \underset{t>0}{=} p_w(t)$ respectively.

These conditions indicate that the similarity transformations for the function u impose specific forms on the function $p_w(t)$. In Chapter 2 (equations (2.7) and (2.8)) we established that $u(y,t) = T(t)f(\xi)$, where y represents the spatial variable. This implies that $T(t)f(0) \underset{t>0}{=} p_w(t)$. Thus $p_w(t)$ must be of the form:

$$p_w(t) = Wt^{\alpha\gamma}, \quad t > 0$$

where W is a constant, $\alpha = \frac{-n}{1+n+\gamma(1-n)}$ and $1 + n + \gamma(1-n) > 0$ if the similarity variable ξ is given by (2.10) or (2.12) and $\alpha = \frac{-n(1-n)}{1+n+\gamma(1-n)^2}$ and $1 + n + \gamma(1-n)^2 > 0$ if ξ is given by (2.14).

In order to determine the boundary conditions for $f(\xi)$ we need to refer to the boundary conditions for u : (1.2) and (1.4). From $T(t)f(0) \underset{t>0}{=} p_w(t)$ and the expressions for $p_w(t)$ we deduce that for all cases we must have:

$$f(0) = W. \tag{3.36}$$

From $u(\ell(t), t) = 0$ and $u_y(\ell(t), t) = 0, \quad t > 0$ we deduce that:

$$f(\xi_1) = 0 \tag{3.37}$$

and

$$f'(\xi_1) = 0. \tag{3.38}$$

It remains now to solve the ordinary differential equations in $f(\xi)$ with the boundary conditions (3.36)–(3.38). We will deal with this task in a later section.

CHAPTER 4

The Ordinary Differential Equations for

The Case of $\varphi = bu + h, b \neq 0$

In this case we must take into consideration the constraints (1.19a) - (1.21a) and (1.30a) - (1.32a). Using y to denote the spatial variable, from (1.20a) and (1.21a) or from (1.31a) and (1.32a) we deduce that:

$$\frac{T_t}{T} + \frac{b(y,t)}{a^2} = \frac{F_7(\xi)}{F_8(\xi)}\,\xi_t$$

or taking into account (3.1),

$$\frac{T'(t)}{T(t)} + \frac{b(y,t)}{a^2} = \frac{F_7(\xi)}{F_8(\xi)}\,\xi\,\frac{\tau'(t)}{\tau(t)}\,. \tag{4.1}$$

Here we will consider two cases: $b = b(y,t)$ and $b = b(t)$. If $b = b(y,t)$ then it must be of the form:

$$b(y,t) = \Gamma[\xi(y,t)]\Omega(t) + \Lambda(t).$$

Then,

$$\frac{T'(t)}{T(t)} = \frac{-1}{a^2}\,\Lambda(t),$$

$$\frac{\tau'(t)}{\tau(t)} = \frac{1}{a^2}\,\Omega(t), \tag{4.2}$$

and

$$\frac{F_7(\xi)}{F_8(\xi)} = \frac{1}{\xi}\,\Gamma(\xi).$$

Integrating the equation (4.2) we obtain

$$\tau(t) = k e^{\frac{1}{a^2}\int_0^t \Omega(\eta)d\eta} \tag{4.3}$$

where k is an integration constant.

The expressions for $\ell(t)$ given by (3.8) and (3.28) and the condition $\ell(0) = 0$ imply that $[\tau(0)]^{-1} = 0$. But this condition cannot be satisfied by the expression for $\tau(t)$ given by (4.3). Therefore b must be a function of t only: $b = b(t)$.

In this case we must have

$$\frac{F_7(\xi)}{F_8(\xi)} = \frac{\gamma}{\xi}$$

and then the equation (4.1) becomes:

$$\frac{T'(t)}{T(t)} + \frac{b(t)}{a^2} = \gamma\,\frac{\tau'(t)}{\tau(t)}.$$

Integration of this equation results in:

$$T(t) = [\tau(t)]^{\gamma} e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}. \tag{4.4}$$

Now, having obtained expressions for $T(t)$ and $\xi(y,t)$, we will determine $\tau(t)$ from the equations (1.21a) and (1.32a).

If we consider the expressions for ξ given by (2.10) and (2.12) then (1.21a) and (1.32a) become:

$$a^2\xi\,\frac{\tau'(t)}{\tau(t)} = F_8(\xi)[\tau(t)]^{1+\frac{1}{n}+\gamma(\frac{1}{n}-1)}e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^t b(\eta)d\eta}. \tag{4.5}$$

If ξ is given by (2.14) then (1.32a) becomes

$$a^2\xi\,\frac{\tau'(t)}{\tau(t)} = F_8(\xi)(1-n)^{\frac{1}{n}+1}r^{-(1+n)}[\tau(t)]^{1+\frac{1}{n}+\gamma(\frac{1}{n}-1)}e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^t b(\eta)d\eta}.$$

Expressing r in terms of ξ we may write the above equation as:

$$a^2\xi\,\frac{\tau'(t)}{\tau(t)} = F_8(\xi)(1-n)^{\frac{1}{n}+1}\xi^{-\frac{1+n}{1-n}}[\tau(t)]^{1+\frac{1}{n}+\gamma(\frac{1}{n}-1)+\frac{1+n}{1-n}}e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^t b(\eta)d\eta}. \tag{4.6}$$

4.a First we analyse the equation (4.5). From this equation it is evident that $F_8(\xi)$ should be of the form:

$$F_8(\xi) = \alpha\xi$$

where α is a constant. This implies that $F_7(\xi) = \gamma\alpha$. The equation (4.5) becomes

$$[\tau(t)]^{-2-\frac{1}{n}-\gamma(\frac{1}{n}-1)}\tau'(t) = \frac{\alpha}{a^2}\,e^{-(\frac{1}{n}-1)\frac{1}{a^2}\int_0^t b(\eta)d\eta}.$$

Integrating we get:

$$\tau(t) = \{\frac{-\alpha}{a^2}\,[1+\frac{1}{n}+\gamma(\frac{1}{n}-1)]\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu + c\}^{\frac{-n}{n+1+\gamma(1-n)}} \tag{4.7}$$

for $\gamma \neq \frac{1+n}{n-1}$

$$\text{and} \qquad \tau(t) = c\exp\left[\frac{\alpha}{a^2}\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu\right] \quad \text{for} \quad \gamma = \frac{1+n}{n-1}$$

where c is an integration constant.

As stated above, the condition $\ell(0) = 0$ implies that $[\tau(0)]^{-1} = 0$. From this requirement we conclude that γ should be choosen such that $n+1+\gamma(1-n) > 0$ and thus $\tau(t)$ is given by (4.7) with $c = 0$.

If we choose $\alpha = \frac{-na^2}{n+1+\gamma(1-n)}$ then:

$$\tau(t) = \left[\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu\right]^{\frac{-n}{n+1+\gamma(1-n)}}, \tag{4.8}$$

$$T(t) = \left[\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu\right]^{\frac{-n\gamma}{n+1+\gamma(1-n)}} e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}, \tag{4.9}$$

$$\xi(y,t) = y\left[\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu\right]^{\frac{-n}{n+1+\gamma(1-n)}} \tag{4.10}$$

and

$$\ell(t) = \xi_1\left[\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu\right]^{\frac{n}{n+1+\gamma(1-n)}}. \tag{4.11}$$

Turning our attention to the constraints (1.19a) and (1.30a), we determine the form of $h(y,t)$ for which we may obtain similarity solutions to our problems. If $h \equiv 0$ then $F_6 \equiv 0$ and all the constraints are satisfied. If not then from the expressions for $\xi(y,t), T(t)$ and $\tau(t)$ given above we obtain that:

$$T\xi_y^2|T\xi_y|^{\frac{1}{n}-1} = e^{\frac{-1}{na^2}\int_0^t b(\eta)d\eta}[\tau(t)]^{\frac{\gamma}{n}+1+\frac{1}{n}}.$$

This substituted into the constraints (1.19a) and (1.30a) tells us that $h(y,t)$ should be of the form:

$$h(y,t) = F\big(y\tau(t)\big)e^{\frac{-1}{na^2}\int_0^t b(\eta)d\eta}[\tau(t)]^{\frac{1}{n}+1+\frac{\gamma}{n}}. \tag{4.12}$$

If $h(y,t)$ is of this form then

$$F_6(\xi) = F(\xi).$$

We may now write the differential equation for the function $f(\xi)$:

$$\frac{na^2}{n+1+\gamma(1-n)}\,(\gamma f + \xi f') + \frac{1}{n}\, f''|f'|^{\frac{1}{n}-1} - F(\xi) = 0 \tag{4.13}$$

for the one-dimensional case, and

$$\frac{na^2}{n+1+\gamma(1-n)}\,(\gamma f + \xi f') + \frac{1}{n}\,(\frac{n}{\xi}\, f' + f'')|f'|^{\frac{1}{n}-1} - F(\xi) = 0 \tag{4.14}$$

for the plane radial case.

4.b We now determine $\tau(t)$ from the equation (4.6). In this equation $F_8(\xi)$ should be of the form:

$$F_8(\xi) = \alpha'\xi^{\frac{2}{1-n}}.$$

This implies that $F_7(\xi) = \gamma\alpha'\xi^{\frac{1+n}{1-n}}$, and the equation (4.6) becomes:

$$[\tau(t)]^{-2-\frac{1}{n}-\gamma(\frac{1}{n}-1)-\frac{1+n}{1-n}}\tau'(t) = \frac{\alpha}{a^2}e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^t b(\eta)d\eta}$$

where $\alpha = (1-n)^{\frac{1}{n}+1}\alpha'$.

Integrating we get:

$$\tau(t) = [\frac{-[1+n+\gamma(1-n)^2]}{n(1-n)}\ \frac{\alpha}{a^2}\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu + c]^{\frac{-n(1-n)}{1+n+\gamma(1-n)^2}} \quad (4.15)$$

for $\gamma \neq \frac{-(1+n)}{(1-n)^2}$ and

$$\tau(t) = c\ \exp[\frac{\alpha}{a^2}\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu]$$

for $\gamma = \frac{-(1+n)}{(1-n)^2}$ where c is an integration constant.

The requirement that $[\tau(0)]^{-1} = 0$ connected with the condition $\ell(0) = 0$ implies that $\tau(t)$ is given by (4.15) with $c = 0$ where γ is such that $\gamma > \frac{-(1+n)}{(1-n)^2}$.

If we choose $\alpha = \frac{-n(1-n)a^2}{1+n+\gamma(1-n)^2}$ then

$$\tau(t) = [\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu]^{\frac{-n(1-n)}{1+n+\gamma(1-n)^2}}, \quad (4.16)$$

$$T(t) = [\tau(t)]^\gamma e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}, \quad (4.17)$$

$$\xi(r,t) = r^{1-n}\tau(t),$$

and

$$\ell(t) = \xi_1^{\frac{1}{1-n}} [\tau(t)]^{\frac{-1}{1-n}}. \tag{4.18}$$

In this case,

$$T\xi^2 r |T\xi_r|^{\frac{1}{n}-1} = (1-n)^{\frac{1}{n}+1} r^{-(1+n)} e^{\frac{-1}{na^2}\int_0^t b(\eta)d\eta} [\tau(t)]^{\frac{\gamma}{n}+1+\frac{1}{n}}$$

which can also be written as:

$$T\xi_r^2 |T\xi_r|^{\frac{1}{n}-1} = (1-n)^{\frac{1}{n}+1} \xi^{-\frac{1+n}{1-n}} e^{-\frac{1}{na^2}\int_0^t b(\eta)d\eta} [\tau(t)]^{\frac{\gamma}{n}+1+\frac{1}{n}+\frac{1+n}{1-n}}.$$

Substituting this into the constraint (1.30a) tells us that $h(r,t)$ should be of the form:

$$h(r,t) = F\big(r^{1-n}\tau(t)\big) e^{-\frac{1}{na^2}\int_0^t b(\eta)d\eta} [\tau(t)]^{\frac{\gamma}{n}+1+\frac{1}{n}+\frac{1+n}{1-n}}.$$

If $h(r,t)$ is of the form then

$$F_6(\xi) = (1-n)^{\frac{-1}{n}-1} \xi^{\frac{1+n}{1-n}} F(\xi).$$

We may now write the differential equation for the function $f(\xi)$:

$$\frac{na^2}{(1-n)^{\frac{1}{n}}[1+n+\gamma(1-n)^2]} \xi^{\frac{1+n}{1-n}} (\gamma f + \xi f') + \frac{1}{n} f'' |f'|^{\frac{1}{n}-1} - (1-n)^{-\frac{1}{n}-1} \xi^{\frac{1+n}{1-n}} F(\xi) = 0. \tag{4.19}$$

Let us now determine the forms of the function $p_w(t)$ required by the similarity transformation. The boundary condition at $x = 0$ or $r = R_w$

implies that $p_w(t)_{t>0} = WT(t)$, $W =$ const.. Thus $p_w(t)$ must be of the form:

$$p_w(t) = W\Big[\int_0^t e^{-\frac{1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta} d\nu\Big]^{\alpha\gamma} e^{-\frac{1}{a^2}\int_0^t b(\eta)d\eta}, \quad t > 0$$

where $\alpha = \dfrac{-n}{1+n+\gamma(1-n)}$ and $1+n+\gamma(1-n) > 0$ if the similarity variable ξ is given by (2.10) or (2.12), and $\alpha = \dfrac{-n(1-n)}{1+n+\gamma(1-n)^2}$ and $1+n+\gamma(1-n)^2 > 0$ if ξ given by (2.14).

The boundary conditions for $f(\xi)$ are the same as those obtained for the case of $\varphi \neq bu + h$.

CHAPTER 5

The Moving Pressure Disturbance Front

In the previous sections we have deriv' the expressions for the location of the moving pressure disturbance front $\ell(t)$. For both the one dimensional and plane radial cases, if $\xi(y,t) = y\tau(t)$, where y is the spatial variable, we have determined that $\ell(t)$ is given by:

$$\ell(t) = \xi_1[\tau(t)]^{-1}.$$

But the expression for $\tau(t)$ depends on the form of the source term $\varphi(y,t,u)$. Thus the corresponding expressions for $\ell(t)$ are:

$$\ell(t) = \xi_1 t^{\lambda} \tag{5.1}$$

if

$$\varphi(y,t,u) \neq b(t)u + h(y,t)$$

and

$$\ell(t) = \xi_1\Big[\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^{\nu} b(\eta)d\eta}\, d\nu\Big]^{\lambda} \tag{5.2}$$

if $\varphi(y,t,u) = b(t)u + h(y,t)$ with $h(y,t)$ of the form (4.12). In (5.1) and (5.2)

$$\lambda = \frac{n}{1 + n + \gamma(1-n)} > 0$$

according to (3.8).

Let us now investigate the behavior of the moving pressure disturbance front as t tends to infinity. But to do this we must restrict ourselves to those cases for which $p_w(t) \geq -p_0$ for $t > 0$.

If $\ell(t)$ is given by (5.1) then the pressure disturbance propagates to infinity as $t \to \infty$. The behavior of $\ell(t)$ given by (5.2) as $t \to \infty$ depends on the sign of $(\frac{1}{n} - 1)\int_0^t b(\eta)d\eta$.

If $(\frac{1}{n} - 1)\int_0^t b(\eta)d\eta < 0$ as $t \to \infty$ then $\lim_{t\to\infty} \ell(t) = \infty$.

If $(\frac{1}{n} - 1)\int_0^t b(\eta)d\eta > 0$ as $t \to \infty$ then $\int_0^\infty e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu$ is convergent and thus if we let $A = \lim_{t\to\infty} \int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^\nu b(\eta)d\eta}d\nu$ then $\lim_{t\to\infty} \ell(t) = \xi_1[A]^\lambda$. Therefore, in this case, we can conclude that the pressure disturbance propagates only to a finite location.

In order to obtain a physical interpretation of these results we will consider the particular case when $n < 1$, $b(t) = b =$ const and $\varphi(y,t,u) = bu$. In this case the location of the pressure disturbance is given by

$$\ell(t) = \xi_1[\frac{a^2 n}{1-n}]^\lambda[\frac{1}{b}(1 - e^{\frac{-1}{a^2}(\frac{1}{n}-1)bt})]^\lambda. \tag{5.3}$$

Taking the limit of (5.3) as $t \to \infty$ we obtain:

$$\lim_{t\to\infty} \ell(t) = \begin{cases} \infty & \text{if } b < 0 \\ \frac{\eta}{b^\lambda} & \text{if } b > 0 \end{cases} \tag{5.4}$$

where $\eta = \xi_1[\frac{a^2 n}{1-n}]^\lambda$.

Note that this result is consisted with the physical process. Indeed, in the case of injection, i.e. $u = p - p_0 > 0$, the presence of sources, i.e. $\varphi < 0$, would indicate that $b < 0$. Thus, for the case of injection in the presence of sources (5.4) indicates that the front propagates to infinity.

If we have injection in the presence of sinks, i.e. $\varphi > 0$, then this would indicate that $b > 0$. Therefore, for the case of injection in the presence of sinks (5.4) indicates that the front reaches a maximum location: $\ell^* = \frac{\eta}{b^\lambda}$ from which the pressure disturbance cannot propagate.

If we have production, i.e. $p - p_0 < 0$, in the presence of sources, then $b > 0$. So, for the case of production in the presence of sources (5.4) indicates that the pressure disturbance propagates only to $\ell^* = \frac{\eta}{b^\lambda}$.

Finally consider the case of production in the presence of sinks $b < 0$. Then, (5.4) indicates that the front propagates to infinity as $t \to \infty$.

Like the behavior of the pressure disturbance front as $t \to \infty$, the rate at which it propagates through the fluid is relevant to the understanding and prediction of some physical processes of practical interest. The velocity of the disturbance front is obtained by differentiating the expressions for $\ell(t)$ given by (5.1) and (5.2) with respect to time. Thus, if $\varphi(y,t,u) \neq b(t)u + h(y,t)$ then the velocity of the front is given by:

$$\frac{d\ell}{dt} = \lambda \xi_1 t^{\lambda - 1} \tag{5.5}$$

while for the cases with $\varphi = bu$ the velocity of the front is given by:

$$\frac{d\ell}{dt} = \lambda\eta \, \frac{1}{a^2}\left(\frac{1}{n} - 1\right)e^{\frac{-b}{a^2}(\frac{1}{n}-1)t}\left\{\frac{-1}{b}\left[e^{\frac{-b}{a^2}(\frac{1}{n}-1)t} - 1\right]\right\}^{\lambda-1}. \tag{5.6}$$

We note that in the equation (5.5) $\frac{d\ell}{dt}$ decreases in time if $\lambda < 1$. So in this case the motion of the disturbance front is deaccelerated. If $\lambda > 1$ then $\frac{d\ell}{dt}$ increases in time implying that the motion of the disturbance front is accelerated.

Studying $\frac{d\ell}{dt}$ given by (5.6) by evaluating $\frac{d^2\ell}{dt^2}$ we conclude the following: If we first consider the case of $\lambda \leq 1$ then we may say that if $b > 0$ then the motion of the disturbance front is deaccelerated while if $b < 0$ then the motion is deaccelerated for $t < \frac{a^2 n}{1-n} \frac{\ell n \, \lambda}{b}$ and accelerated for $t > \frac{a^2 n}{1-n} \frac{\ell n \, \lambda}{b}$.

Now if $\lambda > 1$ and $b < 0$ then the motion of the disturbance front is accelerated while if $b > 0$ then the motion is accelerated for $t < \frac{a^2 n}{1-n} \frac{\ell n \, \lambda}{b}$ and deaccelerated for $t > \frac{a^2 n}{1-n} \frac{\ell n \, \lambda}{b}$.

For the radial case, besides the similarity variable $\xi(r,t) = r\tau(t)$ we determined a transformation with the similarity variable $\xi(r,t) = r^{1-n}\tau(t)$ in which $\tau(t)$ is given by (3.27) or (4.16).

For this transformation $\ell(t)$ is given by:

$$\ell(t) = \xi_1^{\frac{1}{1-n}} t^{\frac{n}{n+1+\gamma(1-n)^2}}$$

or

$$\ell(t) = \xi_1^{\frac{1}{1-n}} [\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1) \int_0^\nu b(\eta)d\eta} d\nu]^{\frac{n}{1+n+\gamma(1-n)^2}} .$$

We note that these expressions for $\ell(t)$ are similar to those given by (5.1), (5.2). Thus the conclusions concerning the behavior of the moving pressure disturbance front made above apply to this case as well.

CHAPTER 6

Particular Cases

In the previous chapters we have determined the similarity transformation and the forms of the functions p_w and ψ which allow the existence of similarity solutions. For both problems the similarity transformation is:

$$p(y,t) = p_0 + T(t)f(\xi) \tag{6.1}$$

where

$$\xi(y,t) = y\tau(t) \tag{6.2}$$

$$T(t) = [\tau(t)]^{\gamma} \quad \text{and} \quad \tau(t) = t^{\frac{-n}{1+n+\gamma(1-n)}} \quad \text{if} \quad \psi \neq b(t)(p-p_0) + h(y,t),$$

$$T(t) = [\tau(t)]^{\gamma} e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}$$

and

$$\tau(t) = [\int_0^t e^{\frac{-1}{a^2}(\frac{1}{n}-1)\int_0^{v} b(\eta)d\eta} dv]^{\frac{-n}{1+n+\gamma(1-n)}} \qquad if \quad \psi = b(t)(p-p_0) + h(y,t)$$

$$\text{with} \quad 1+n+\gamma(1-n) > 0, \tag{6.3}$$

and y is the spatial variable x or r.

$f(\xi)$ is the solution of an ordinary differential equation which depends on $p_w(t)$ and $\psi(y,t,p)$. For the one dimensional case $f(\xi)$ is the solution to the equation:

$$\alpha a^2(\gamma f + \xi f') - \frac{1}{n} f'' |f'|^{\frac{1}{n}-1} + F(\xi)|f|^q = 0 \tag{6.4}$$

and for the plane radial case $f(\xi)$ is the solution to the equation:

$$\alpha a^2(\gamma f + \xi f') - \frac{1}{n}(\frac{n}{\xi} f' + f'')|f'|^{\frac{1}{n}-1} + F(\xi)|f|^q = 0 \tag{6.5}$$

where

$$\alpha = \frac{-n}{1+n+\gamma(1-n)} \quad \text{for both cases.} \tag{6.6}$$

If ψ is of the form $\psi(y,t,p) = F(\xi)|p-p_0|^q$ then $\gamma = \frac{n+1}{nq-1}$ and q must satisfy the conditions

$$\begin{gathered} q < 1 \quad \text{or} \quad q > \frac{1}{n} \quad \text{if} \quad n < 1 \\ \text{and} \\ q > 1 \quad \text{or} \quad q < \frac{2}{n} - 1 \quad \text{if} \quad n > 1. \end{gathered} \tag{6.7}$$

In this case we must have $p_w(t) = Wt^{\frac{-n\gamma}{1+n+\gamma(1-n)}}, \quad t > 0.$

If $\psi(y,t,p) = F(\xi)|p-p_0|^q t^{-1}, q \neq 1,$ then $\gamma = 0$ and we must have $p_w(t) = W,\ t > 0.$

Finally, if $\psi(y,t,p) = b(t)(p-p_0) + F(\xi)e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}[\tau(t)]^{\frac{1}{n}+1+\frac{\gamma}{n}}$ and $p_w(t) = W[\tau(t)]^\gamma e^{\frac{-1}{a^2}\int_0^t b(\eta)d\eta}$ then $f(\xi)$ is a solution of (6.4) or (6.5) with $q = 0$ and no restriction on γ other than (6.3).

As it was shown previously the boundary conditions for $f(\xi)$ are:

$$f(0) = W \tag{6.8}$$

$$f(\xi_1) = 0 \tag{6.9}$$

$$f'(\xi_1) = 0. \tag{6.10}$$

The additional boundary condition allows us to find the constant ξ_1 which in turn determines the location of the pressure disturbance front, denoted by $\ell(t)$ and given by the formula:

$$\ell(t) = \xi_1[\tau(t)]^{-1}.$$

In this chapter we deal with solving the ordinary differential equations mentioned above. We first consider the cases in which a closed form solution can be found. In the cases when this is not possible we employ numerical and approximation methods.

6.1 Closed Form Solutions For One-Dimensional Flow

It is easily seen that the differential equation (6.4) has solutions in closed form for $F(\xi) = 0$ and $\gamma = 0$ or $\gamma = 1$. Therefore for these cases we are able to write the solution of the problem (I.6), (I.12) in closed form.

6.1a $F(\xi) \equiv 0, \quad \gamma = 0$

In this case the equation (6.4) becomes:

$$\frac{na^2}{n+1}\,\xi f' + \frac{1}{n}\, f''|f'|^{\frac{1}{n}-1} = 0$$

which yields the following first integral:

$$|f'(\xi)| = [c - \frac{a^2 n(1-n)}{2(1+n)}\,\xi^2]^{\frac{n}{1-n}} \qquad (6.1.1)$$

where c is an integration constant satisfying $c \geq 0$.

Considering the equation (6.10), we realize that we must distinguish between the two cases $n < 1$ and $n > 1$. For the case when $n > 1$ the condition (6.10) implies that $\xi_1 = \infty$, whereas for the case when $n < 1$ we have:

$$c = \frac{a^2 n(1-n)}{2(1+n)}\,\xi_1^2, \quad n < 1,$$

and thus,

$$|f'(\xi)| = [\frac{a^2 n(1-n)}{2(1+n)}]^{\frac{n}{1-n}}(\xi_1^2 - \xi^2)^{\frac{n}{1-n}}, \; n < 1 \quad \text{and} \quad 0 \le \xi \le \xi_1. \qquad (6.1.2)$$

The physical considerations of the problem we are dealing with imply that $f'(\xi)$ has the same sign on the interval $[0, \xi_1)$: $f'(\xi) < 0$ in the case of injection and $f'(\xi) > 0$ in the case of production. Note that $p_w(t) > 0,\ t > 0$ indicates injection while $p_w(t) < 0,\ t > 0$ indicates production. So, we conclude that if $W > 0$ we have injection and if $W < 0$ we have production. Therefore $\operatorname{sgn}(f'(\xi)) = -\operatorname{sgn}(W)$ for $0 \le \xi \le \xi_1$.

Integrating the first integral for $n > 1$ we obtain:

$$f(\xi) = W - \operatorname{sgn}(W)\int_0^{\xi} \frac{d\eta}{[c + \frac{a^2 n(n-1)}{2(n+1)}\,\eta^2]^{\frac{n}{n-1}}}.$$

The condition (6.9) which states that $f(\infty) = 0$ implies that:

$$\int_0^{\infty} \frac{d\eta}{[c + \frac{a^2 n(n-1)}{2(n+1)}\,\eta^2]^{\frac{n}{n-1}}} = |W|. \qquad (6.1.3)$$

Clearly equation (6.1.3) uniquely determines $c > 0$.

Now since $\xi_1 = \infty$ we have $\ell(t) = \infty$ for any $t > 0$. This implies that for $n > 1$ the pressure disturbance propogates with infinite velocity since it is felt instantaneously at all points in the domain.

Now we can write the solution to the problem (I.6), (I.12) for $n > 1$ in the following two cases:

In the case when $\psi(x,t,p) \equiv 0$ and $p_w(t) = W$

$$p(x,t) = p_0 + W - \operatorname{sgn}(W) \int_0^{\xi} \frac{d\eta}{[c + \frac{a^2 n(n-1)}{2(n+1)} \eta^2]^{\frac{n}{n-1}}}, \quad 0 < \xi$$

where $\xi(x,t) = x t^{\frac{-n}{n+1}}$.

In the case when $\psi(x,t,p) = b(p - p_0)$ and $p_w(t) = W e^{\frac{-bt}{a^2}}$

$$p(x,t) = p_0 + \{W - \operatorname{sgn}(W) \int_0^{\xi} \frac{d\eta}{[c + \frac{a^2 n(n-1)}{2(n+1)} \eta^2]^{\frac{n}{n-1}}}\} e^{\frac{-bt}{a^2}}$$

where

$$\xi(x,t) = x[\frac{na^2}{1-n}]^{\frac{-n}{1+n}} \{\frac{1}{b} [1 - \exp(\frac{-b(1-n)}{na^2} t)]\}^{\frac{-n}{1+n}}.$$

In both cases c is determined from (6.1.3).

For the case $n < 1$, integrating the first integral (6.1.1) leads to:

$$f(\xi) = W - \operatorname{sgn}(W)[\frac{n(1-n)a^2}{2(n+1)}]^{\frac{n}{1-n}} \int_0^{\xi} (\xi_1^2 - \eta^2)^{\frac{n}{1-n}} d\eta.$$

Making a change of variable, this equation can be written as:

$$f(\xi) = W - \operatorname{sgn}(W)[\frac{n(1-n)a^2}{2(1+n)}]^{\frac{n}{1-n}} \xi_1^{\frac{1+n}{1-n}} \int_0^{\frac{\xi}{\xi_1}} [1 - \mu^2]^{\frac{n}{1-n}} d\mu.$$

This expression for $f(\xi)$ together with the condition (6.9) gives us the value of ξ_1 :

$$\xi_1 = |W|^{\frac{1-n}{1+n}}[\frac{n(1-n)a^2}{2(1+n)}]^{\frac{-n}{1+n}}[B(\frac{1}{2},\frac{1}{1-n})]^{\frac{-(1-n)}{1+n}}$$

where $B(\alpha,\beta)$ is the beta function. So, for $n < 1$ the solution of the problem (I.6), (I.12) with $\psi(x,t,p) \equiv 0$ and $p_w(t) = W$ is given by

$$p(x,t) = p_0 + W - \operatorname{sgn}(W)[\frac{n(1-n)a^2}{2(1+n)}]^{\frac{n}{1-n}}\xi_1^{\frac{1+n}{1-n}}\int_0^{\frac{\xi}{\xi_1}}[1-\mu^2]^{\frac{n}{1-n}}d\mu$$

for $0 < \xi < \xi_1$ where $\xi = xt^{\frac{-n}{n+1}}$, thus $0 < x < \ell(t)$ and $p(x,t) = p_0$ for $x \geq \ell(t)$ where

$$\ell(t) = |W|^{\frac{1-n}{1+n}}[\frac{n(1-n)a^2}{2(1+n)}]^{\frac{-n}{1+n}}[B(\frac{1}{2},\frac{1}{1-n})]^{\frac{-(1-n)}{1+n}}t^{\frac{n}{1+n}}.$$

For this particular case we have found the same result as the one given in [9]. If in (I.6) $\psi(x,t,p) = b(p-p_0)$ and in (I.12) $p_w(t) = We^{\frac{-bt}{a^2}}$ then

$$p(x,t) = p_0 + \{W - \operatorname{sgn}(W)[\frac{n(1-n)a^2}{2(1+n)}]^{\frac{n}{1-n}}\xi_1^{\frac{1+n}{1-n}}\int_0^{\frac{\xi}{\xi_1}}[1-\mu^2]^{\frac{n}{1-n}}d\mu\}e^{\frac{-bt}{a^2}}$$

for $0 < \xi < \xi_1$ or $0 < x < \ell(t)$, and $p(x,t) = p_0$ for $x \geqq \ell(t)$, where

$$\xi = x[\frac{1-n}{na^2}]^{\frac{n}{1+n}}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}t)]\}^{\frac{-n}{n+1}}$$

and

$$\ell(t) = \xi_1[\frac{na^2}{1+n}]^{\frac{n}{1+n}}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]\}^{\frac{n}{n+1}}.$$

6.1b $F(\xi) \equiv 0$, $\gamma = 1$

In this case, the equation (6.4) becomes:

$$\frac{na^2}{2}\frac{d}{d\xi}(\xi f) + \frac{d}{d\xi}[|f'|^{\frac{1}{n}-1}f'] = 0.$$

Integrating this equation we obtain:

$$-\frac{na^2}{2}\xi f = |f'|^{\frac{1}{n}-1}f' + c$$

where c is an integration constant. But the conditions (6.9) and (6.10) imply that $c = 0$, thus yielding

$$\mathrm{sgn}(W)\frac{na^2}{2}\xi f = |f'|^{\frac{1}{n}}$$

and integrating again we obtain:

$$f(\xi) = \mathrm{sgn}(W)[c - (\frac{na^2}{2})^n \frac{1-n}{1+n}\xi^{n+1}]^{\frac{1}{1-n}}$$

where the integration constant $c \geq 0$ will be determined by the condition (6.9). Again, here we must distinguish between the two cases $n < 1$ and $n > 1$.

For $n < 1$ we deduce that

$$f(\xi) = \mathrm{sgn}(W)(\frac{na^2}{2})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}(\xi_1^{n+1} - \xi^{n+1})^{\frac{1}{1-n}}.$$

To find the value of ξ_1 we appeal to the condition (6.8) obtaining:

$$\xi_1 = |W|^{\frac{1-n}{1+n}}(\frac{2}{na^2})^{\frac{n}{n+1}}(\frac{1+n}{1-n})^{\frac{1}{n+1}}.$$

With this expression for $f(\xi)$ and ξ_1, we can write the solution of the problem (I.6), (I.12) with $n < 1$, $\psi(x,t,p) \equiv 0$ and $p_w(t) = Wt^{\frac{-n}{2}}$, $t > 0$ as

$$p(x,t) = p_0 + \operatorname{sgn}(W)(\frac{na^2}{2})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}t^{\frac{-n}{2}}(\xi_1^{n+1} - \xi^{n+1})^{\frac{1}{1-n}},$$

for $0 < \xi < \xi_1$ therefore $0 < x < \ell(t)$, and $p(x,t) = p_0$ for $x \geqq \ell(t)$ where $\xi = xt^{\frac{-n}{2}}$ and the location of the moving pressure disturbance front, $\ell(t)$ is given by:

$$\ell(t) = |W|^{\frac{1-n}{1+n}}(\frac{2}{na^2})^{\frac{n}{n+1}}(\frac{1+n}{1-n})^{\frac{1}{n+1}}t^{\frac{n}{2}}.$$

Also if in (I.6), $n < 1$, $\psi(x,t,p) = b(p - p_0)$ and in (I.12) $p_w(t) = W[\frac{na^2}{1-n}]^{-n/2}e^{-bt/a^2}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\,t)]\}^{-n/2}$ then the solution of this problem is given by

$$p(x,t) = p_0 + \operatorname{sgn}(W)(\frac{na^2}{2})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}(\xi_1^{1+n} - \xi^{1+n})^{\frac{1}{1-n}}(\frac{na^2}{1-n})^{\frac{-n}{2}}$$
$$\times\{\frac{1}{b}\,[1 - \exp(\frac{-b(1-n)}{na^2}\,t)]\}^{-\frac{n}{2}}e^{-\frac{bt}{a^2}}$$

for $0 < \xi < \xi_1$ and therefore $0 < x < \ell(t)$ and

$$p(x,t) = p_0 \quad \text{for} \quad x \geq \ell(t).$$

Here $\xi(x,t) = x(\frac{na^2}{1-n})^{-\frac{n}{2}}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\ t]\}^{\frac{-n}{2}}$ and the location of the moving pressure disturbance front, $\ell(t)$ is given by:

$$\ell(t) = \xi_1(\frac{na^2}{1-n})^{\frac{n}{2}}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\ t)]\}^{\frac{n}{2}}.$$

It should be pointed out that the two cases for which we have just provided the solution are of interest only from a mathematical point of view. This is because in these two situations the prescribed pressure at the boundary $x = 0$ tends to infinity as t approaches zero, which is not physically feasible.

Thus far in this section we have obtained closed form solutions for the differential equations in $f(\xi)$ for some particular values for γ and some particular forms for the source term ψ. On the basis of these results let us now attempt to find closed form solutions for other cases.

6.1c $f'(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta$

In the Section 6.1a we found that for $n < 1$, $\gamma = 0$, and $F(\xi) \equiv 0$, $f'(\xi)$ is given by (6.1.2). Now we will investigate other cases with $n < 1$ for which $f'(\xi)$ is of the form

$$f'(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta \tag{6.1.4}$$

where $\lambda > 0$, $\beta > 0$, and

$$\operatorname{sgn}(A) = -\operatorname{sgn}(W). \tag{6.1.5}$$

Upon substituting (6.1.4) into (6.4) we obtain:

$$\frac{-na^2}{1+n+\gamma(1-n)}\,[\gamma f + A\xi(\xi_1^\lambda - \xi^\lambda)^\beta] + \frac{A}{n}\,|A|^{\frac{1}{n}-1}\beta\lambda\xi^{\lambda-1}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-1} + F(\xi)|f|^q = 0. \qquad (6.1.6)$$

This equation must be studied separately for $\lambda = 1$ and $\lambda \neq 1$.

If $\lambda = 1$ then we can integrate (6.1.4) and thus obtain:

$$f(\xi) = \frac{-A}{\beta+1}\,(\xi_1 - \xi)^{\beta+1}. \qquad (6.1.7)$$

In this case (6.1.6) becomes

$$\alpha a^2[A\xi_1(\xi_1-\xi)^\beta - A(1+\frac{\gamma}{\beta+1})(\xi_1-\xi)^{\beta+1}] + \frac{A}{n}\,|A|^{\frac{1}{n}-1}\beta(\xi_1-\xi)^{\frac{\beta}{n}-1} + F(\xi)\,|\frac{A}{\beta+1}|^q(\xi_1-\xi)^{q(\beta+1)} = 0. \qquad (6.1.8)$$

Therefore $F(\xi)$ must be of the form:

$$F(\xi) = D_1(\xi_1-\xi)^{\beta-q(\beta+1)} + D_2(\xi_1-\xi)^{(\beta+1)(1-q)} + D_3(\xi_1-\xi)^{\frac{\beta}{n}-1-q(\beta+1)}.$$

The undetermined constants in the expression for $f(\xi)$ can be determined in terms of the given constants in the expression for $F(\xi)$.

As an example we will take $F(\xi) = D =$ const., which corresponds to a source term $\psi(x,t,p) = D(p-p_0)^q$. In the case (6.1.8) is satisfied only if:

$$1 + \frac{\gamma}{\beta+1} = 0,$$
$$\beta = \frac{\beta}{n} - 1 = q(\beta+1)$$

and

$$\alpha a^2 A\xi_1 + \frac{A}{n}\,|A|^{\frac{1}{n}-1}\beta + D|\frac{A}{\beta+1}|^q = 0. \tag{6.1.9}$$

From the first three equations we obtain:

$$\beta = \frac{n}{1-n}\,,$$
$$\gamma = \frac{-1}{1-n}\,,$$
$$\text{and} \quad q = n.$$

Substituting these results into the equation (6.1.9) gives us:

$$-a^2 A\xi_1 + \frac{A|A|^{\frac{1}{n}-1}}{1-n} + D(1-n)^n|A|^n = 0. \tag{6.1.10}$$

On the other hand the condition (6.8) together with (6.1.7) imply that

$$W = -A(1-n)\xi_1^{\frac{1}{1-n}}. \tag{6.1.11}$$

From (6.1.10) and (6.1.11) we obtain the value for A and ξ_1.

For example if $D = 0$ and $A > 0\,(W < 0)$ then:

$$A = (-W)^{\frac{n}{1+n}}(1-n)^{\frac{n^2}{1-n^2}}$$

and

$$\xi_1 = \frac{1}{a^2}\,(-W)^{\frac{1-n}{1+n}}(1-n)^{\frac{-1}{1+n}}. \tag{6.1.12}$$

Therefore, in this case $f(\xi)$ is given by:

$$f(\xi) = -(-W)^{\frac{n}{1+n}}(1-n)^{\frac{1}{1-n^2}}(\xi_1 - \xi)^{\frac{1}{1-n}}. \qquad (6.1.13)$$

Now we may write the solution of the problem (I.6), (I.12) with $\psi(x,t,p) \equiv 0$ and $p_w(t) = Wt^{\frac{1}{1-n}}$, $W < 0$:

$$p(x,t) = p_0 - (-W)^{\frac{n}{1+n}}(1-n)^{\frac{1}{1-n^2}}(\xi_1 - \xi)^{\frac{1}{1-n}} t^{\frac{1}{1-n}}, \quad \text{for} \quad 0 < \xi < \xi_1$$

where $\xi(x,t) = xt^{-1}$ and the location of the moving pressure disturbance front is given by:

$$\ell(t) = \xi_1 t$$

with ξ_1 given by (6.1.12).

However if $p_w(t)$ is of this form the prescribed pressure at the boundary $x = 0$ is negative for $t > t_f$ where $t_f = (\frac{p_0}{-W})^{1-n}$. Thus the solution given above is only valid for $0 < t \leq t_f$.

The function defined by (6.1.13) also provides us with the solution to the problem (I.6), (I.12) with $\psi(x,t,p) = b(p - p_0)$ and

$$p_w(t) = W[\frac{na^2}{1-n}]^{\frac{1}{1-n}}\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]\}^{\frac{1}{1-n}} e^{\frac{-bt}{a^2}}, \quad W < 0.$$

The solution is

$$p(x,t) = p_0 - (-W)^{\frac{n}{1+n}}(1-n)^{\frac{-n}{1-n^2}}(na^2)^{\frac{1}{1-n}}(\xi_1 - \xi)^{\frac{1}{1-n}}$$
$$\times \{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]\}^{\frac{1}{1-n}} e^{\frac{-bt}{a^2}}, \quad 0 < \xi \leq \xi_1$$

where $\xi(x,t) = x(\frac{1-n}{na^2})\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]\}^{-1}$ and the location of the moving pressure disturbance front is given by:

$$\ell(t) = \xi_1(\frac{na^2}{1-n})\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]$$

with ξ_1 given by (6.1.12).

Clearly, if $b < 0$ the function $p_w(t)$ considered here decreases in time and tends to minus infinity as t approaches infinity. Therefore there exists a value for t, t_f such that $p_0 + p_w(t) < 0$ for $t > t_f$ which implies that the above solution is only valid for $0 < t \leq t_f$.

If $b > 0$ then it can be shown that $p_w(t)$ has a minimum at

$$t_{\min} = \frac{na^2}{b(1-n)}\, \ell n(\frac{n+1}{n})$$

so, if p_0 is such that $p_0 \geq p_w(t_{\min})$ then the above solution is valid for $t > 0$.

Note that in both of the above problems $p(x,t) = p_0$ for $x \geq \ell(t)$.

We will now study the equation (6.1.6) for $\lambda \neq 1$. In this case $f(\xi)$ is not of the form $\text{const.}\,(\xi_1^\lambda - \xi^\lambda)^\mu$ and since $q \neq 1$, we must have $\gamma = 0$ and $q = 0$. Then (6.1.6) becomes:

$$\alpha a^2 A\xi(\xi_1^\lambda - \xi^\lambda)^\beta + \frac{A}{n}\,|A|^{\frac{1}{n}-1}\beta\lambda\xi^{\lambda-1}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-1} + F(\xi) = 0 \qquad (6.1.14)$$

where $\alpha = \frac{-n}{1+n}$.

For the case when $F(\xi) \equiv 0$ the above equation holds only if $\lambda = 2$, $\beta = \frac{n}{1-n}$ and $|A| = [\frac{a^2 n(1-n)}{2(1+n)}]^{\frac{n}{1-n}}$. Therefore

$$|f'(\xi)| = [\frac{a^2 n(1-n)}{2(1+n)}]^{\frac{n}{1-n}} (\xi_1^2 - \xi^2)^{\frac{n}{1-n}}.$$

Note that this same result was obtained in Section 6.1a.

Now if $F(\xi) \not\equiv 0$ then it must be of the form:

$$F(\xi) = D_1 \xi (\xi_1^\lambda - \xi^\lambda)^\beta + D_2 \xi^{\lambda-1} (\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-1}. \qquad (6.1.15)$$

With this expression for $F(\xi)$ (6.1.14) holds only if:

$$\beta\lambda = \frac{-D_2}{D_1} (\frac{a^2 n}{1+n})^{\frac{1}{n}} \frac{n}{|D_1|^{\frac{1}{n}-1}} \quad \text{and} \quad A = \frac{D_1(1+n)}{na^2}. \qquad (6.1.16)$$

Note that this implies that there must exist a relationship between the constants appearing in the source term.

Once β, λ and A are determined the expression for $f(\xi)$ can be obtained by integrating (6.1.4):

$$f(\xi) = \int_0^\xi A(\xi_1^\lambda - \eta^\lambda)^\beta d\eta + W.$$

From this expression for $f(\xi)$ and the condition $f(\xi_1) = 0$ we deduce that

$$\xi_1 = (\frac{-W\lambda}{A})^{\frac{1}{\lambda\beta+1}} [B(\frac{1}{\lambda}, \beta+1)]^{\frac{-1}{\lambda\beta+1}}. \qquad (6.1.17)$$

But ξ_1 appears in the expression for $F(\xi)$, thus it is given by the source term. Therefore W, which is a function of ξ_1 by the above relation, is imposed by the source term.

As an example let us consider the problem (I.6), (I.12) with

$$\psi(x,t,p) = F(\xi)t^{-1} = [\frac{na^2}{1+n}\,\xi(\xi_1^3-\xi^3)^{2n} - 6\xi^2(\xi_1^3-\xi^3)]t^{-1}$$

where $\xi(x,t)$ is the similarity transformation variable and it is given by:

$$\xi(x,t) = xt^{\frac{-n}{1+n}}.$$

This expression for $F(\xi)$ is of the form of (6.1.15) with:

$$D_1 = \frac{na^2}{1+n}, \quad D_2 = -6, \quad \beta = 2n \quad \text{and} \quad \lambda = 3.$$

From (6.1.16) we obtain $A = 1$ and from (6.1.17) $W = -\frac{1}{3}\,\xi_1^4 B(\frac{1}{3}\,, 2n+1)$. So $f(\xi)$ is given by:

$$f(\xi) = -\frac{1}{3}\,\xi_1^4 B(\frac{1}{3}\,, 2n+1) + \int_0^\xi (\xi_1^3-\eta^3)^{2n} d\eta.$$

So, if in the boundary condition at $x = 0$ $p_w(t)$ is given by:

$$p_w(t) = -\frac{1}{3}\,\xi_1^4 B(\frac{1}{3}\,, 2n+1), \quad t > 0$$

then the solution to our problem is:

$$p(x,t) = p_0 - \frac{1}{3}\,\xi_1^4 B(\frac{1}{3}\,, 2n+1) + \int_0^\xi (\xi_1^3-\eta^3)^{2n} d\eta, \quad \text{for} \quad 0 < x < \ell(t)$$

and $p(x,t) = p_0$ for $x \geq \ell(t)$ where $\ell(t) = \xi_1 t^{\frac{n}{1+n}}$.

6.1d $f(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta$

In the subsection 6.1b, we found that for $n < 1, \gamma = 1,$ and $\psi \equiv 0$, $f(\xi)$ is of the form:

$$f(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta \tag{6.1.18}$$

where $\lambda = 1+n$, $\beta = \frac{1}{1-n}$ and $A = \text{Sgn}(W)(\frac{na^2}{2})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}$. Let us look for a solution to (6.4) of this form with $\lambda > 0, \beta > 0$ and $\text{sgn}(A) = \text{sgn}(W)$ for other cases with $n < 1$. In the previous section we have considered a solution of this form with $\lambda = 1$. So in this section we will only consider $\lambda \neq 1$.

Upon substituting (6.1.18) into (6.4) we obtain:

$$\alpha a^2 A[\gamma(\xi_1^\lambda - \xi^\lambda)^\beta - \lambda\beta\xi^\lambda(\xi_1^\lambda - \xi^\lambda)^{\beta-1}] + \frac{1}{n}(\lambda\beta)^{\frac{1}{n}} A|A|^{\frac{1}{n}-1}\xi^{\frac{\lambda}{n}-1-\frac{1}{n}}$$
$$(6.1.19)$$
$$\times(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-1-\frac{1}{n}}[(\lambda-1)(\xi_1^\lambda - \xi^\lambda) - \lambda(\beta-1)\xi^\lambda] + F(\xi)|A|^q(\xi_1^\lambda - \xi^\lambda)^{\beta q} = 0.$$

If $F(\xi) \equiv 0$ then (6.1.19) holds only if

$$\lambda = 1+n, \quad \beta = \frac{1}{1-n}, \quad \gamma = 1, \quad \text{and} \quad |A| = (\frac{na^2}{2})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}.$$

But this is the result obtained in Section 6.1b by directly integrating the equation (6.4) with $\gamma = 1$ and $F(\xi) \equiv 0$.

If in (6.1.19) $F(\xi) \not\equiv 0$ then it must be of the form:

$$F(\xi) = D_1(\xi_1^\lambda - \xi^\lambda)^{\beta-\beta q} + D_2\xi^\lambda(\xi_1^\lambda - \xi^\lambda)^{\beta-1-\beta q}$$

$$+ D_3\xi^{\frac{\lambda}{n}-\frac{1}{n}-1}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-\frac{1}{n}-\beta q} + D_4\xi^{\frac{\lambda}{n}-\frac{1}{n}-1+\lambda}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-\frac{1}{n}-1-\beta q}. \tag{6.1.20}$$

With this expression for $F(\xi)$ (6.1.19) is valid if:

$$\alpha a^2 A\gamma = -|A|^q D_1 \tag{6.1.21}$$

$$-\alpha a^2 A\lambda\beta = -|A|^q D_2 \tag{6.1.22}$$

$$\frac{1}{n}(\lambda - 1)(\lambda\beta)^{\frac{1}{n}} A|A|^{\frac{1}{n}-1} = -|A|^q D_3 \tag{6.1.23}$$

$$-\frac{1}{n}\lambda(\beta - 1)(\lambda\beta)^{\frac{1}{n}} A|A|^{\frac{1}{n}-1} = -|A|^q D_4. \tag{6.1.24}$$

These equations imply that:

$$\lambda\beta = -\frac{D_2}{D_1}\gamma, \quad \frac{\lambda - 1}{-\lambda(\beta - 1)} = \frac{D_3}{D_4}$$

and $\quad A = \operatorname{sgn}(W)[\dfrac{n\gamma a^2}{D_1(1 + n + \gamma(1 - n))}]^{\frac{1}{q-1}}$

where $\quad \gamma = \dfrac{1 + n}{nq - 1}\,.$

As an example, let us take $q = n$, $D_1 = -(1-n)$, $D_2 = -1$, $D_3 = 1-n$ and $D_4 = -1$. In this case $\gamma = \frac{-1}{1-n}$ and $\lambda = \beta = \frac{1}{1-n}$. Then:

$$F(\xi) = (n-1)(\xi_1^{\frac{1}{1-n}} - \xi^{\frac{1}{1-n}}) - \xi^{\frac{1}{1-n}} + (1-n)\xi^{\frac{n}{1-n}}(\xi_1^{\frac{1}{1-n}} - \xi^{\frac{1}{1-n}}) - \xi^{\frac{1+n}{1-n}},$$

$$A = \operatorname{sgn}(W)[\frac{a}{1-n}]^{\frac{2}{n-1}},$$

$$f(\xi) = \operatorname{sgn}(W)[\frac{a}{1-n}]^{\frac{2}{n-1}}(\xi_1^{\frac{1}{1-n}} - \xi^{\frac{1}{1-n}})^{\frac{1}{1-n}}$$

$$\text{and} \qquad \xi = \frac{x}{t}.$$

From the condition $f(0) = W$ it follows that

$$|W| = [\frac{a}{1-n}]^{\frac{2}{1-n}}\xi_1^{\frac{1}{(1-n)^2}}.$$

With these results we can write the solution to the problem (I.6), (I.12) with

$$\psi(x,t,p) = |p-p_0|^n\{(n-1)\xi_1^{\frac{1}{1-n}} - n(\frac{x}{t})^{\frac{1}{1-n}} + (n-2)(\frac{x}{t})^{\frac{1+n}{1-n}} + (1-n)\xi_1^{\frac{1}{1-n}}(\frac{x}{t})^{\frac{n}{1-n}}\}$$

and

$$p_w(t) = \pm[\frac{a}{1-n}]^{\frac{2}{1-n}}\xi_1^{\frac{1}{(1-n)^2}}t^{\frac{1}{1-n}}, \quad t>0, \quad n<1$$

as

$$p(x,t) = p_0 \pm [\frac{a}{1-n}]^{\frac{2}{1-n}}[\xi_1^{\frac{1}{1-n}} - (\frac{x}{t})^{\frac{1}{1-n}}]^{\frac{1}{1-n}} t^{\frac{1}{1-n}} \quad \text{for} \quad 0 < x < \ell(t)$$

and

$$p(x,t) = p_0 \quad \text{for} \quad x \geq \ell(t)$$

where $\ell(t) = \xi_1 t$.

It is interesting to note that in this case the location of the pressure disturbance front is a linear function of time, and therefore the front propagates with constant velocity.

We may also write the solution to the problem (I.6), (I.12) with

$$\psi(x,t,p) = b(p-p_0) + [(n-1)\xi_1^{\frac{1}{1-n}} - n\xi^{\frac{1}{1-n}} + (1-n)\xi_1^{\frac{1}{1-n}}\xi^{\frac{n}{1-n}}$$
$$+ (n-2)\xi^{\frac{1+n}{1-n}}][\tau(t)]^{\frac{-n}{1-n}} e^{\frac{-bt}{a^2}}$$

where

$$\xi(x,t) = x\tau(t) \quad \text{and} \quad \tau(t) = (\frac{1-n}{na^2})\{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}\, t)]\}^{-1}$$

and with $p_w(t) = \pm(\frac{a}{1-n})^{\frac{2}{1-n}}\xi_1^{\frac{1}{(1-n)^2}}[\tau(t)]^{\frac{-1}{1-n}} e^{\frac{-bt}{a^2}}$, $t > 0$ and $n < 1$. In this case the solution is:

$$p(x,t) = p_0 \pm (\frac{a}{1-n})^{\frac{2}{1-n}}[\xi_1^{\frac{1}{1-n}} - \xi^{\frac{1}{1-n}}]^{\frac{1}{1-n}}[\tau(t)]^{\frac{-1}{1-n}} e^{\frac{-bt}{a^2}} \quad \text{for} \quad 0 < x < \ell(t)$$

and $p(x,t) = p_0$ for $x \geq \ell(t)$ where $\ell(t) = \xi_1[\tau(t)]^{-1}$.

6.2 Closed Form Solutions for the Plane Radial Flow

The differential equation (6.5) corresponding to this case can be solved analytically for $F(\xi) \equiv 0$ and $\gamma = 0$.

6.2a $F(\xi) \equiv 0, \quad \gamma = 0$

In this case (6.5) becomes:

$$\frac{na^2}{n+1}\,\xi f' + \frac{1}{n}\,(\frac{n}{\xi}\,f' + f'')|f'|^{\frac{1}{n}-1} = 0.$$

If we let $g = |f'|$ then this reduces to Bernoulli's equation:

$$\frac{na^2}{n+1}\,\xi + \frac{1}{\xi}\,g^{\frac{1}{n}-1} + \frac{1}{n}\,g' g^{\frac{1}{n}-2} = 0.$$

Solving this equation yields:

$$f'(\xi) = -\operatorname{sgn}(W)\xi^{-n}[c + \frac{n(n-1)a^2}{(n+1)(3-n)}\,\xi^{3-n}]^{\frac{n}{1-n}} \quad \text{if} \quad n \neq 3 \qquad (6.2.1)$$

and

$$f'(\xi) = -\operatorname{sgn}(W)\xi^{-3}[c + \frac{3}{2}\,a^2 \ell n\,\xi]^{\frac{-3}{2}} \quad \text{if} \quad n = 3 \qquad (6.2.2)$$

where c is an integration constant.

In order to determine the values of c and ξ_1 we refer to the boundary conditions for $f(\xi)$ outlined above. However, we must distinguish between different values for n :

CASE 1: $n < 1$

The condition (6.10) implies that in this case we must have $c = \frac{-n(n-1)a^2}{(n+1)(3-n)} \xi_1^{3-n}$, which leads to:

$$f'(\xi) = -\operatorname{sgn}(W)[\frac{n(n-1)a^2}{(n+1)(3-n)}]^{\frac{n}{1-n}} \xi^{-n} [\xi_1^{3-n} - \xi^{3-n}]^{\frac{n}{1-n}}, \quad 0 \le \xi \le \xi_1.$$

Integrating this equation we obtain:

$$f(\xi) = W - \operatorname{sgn}(W)[\frac{n(n-1)a^2}{(n+1)(3-n)}]^{\frac{n}{1-n}} \xi_1^{\frac{1+n}{1-n}} \int_0^{\xi/\xi_1} \mu^{-n}(1-\mu^{3-n})^{\frac{n}{1-n}} d\mu.$$

Now, substituting this result into the equation (6.9) we have:

$$|W| - [\frac{n(n-1)a^2}{(n+1)(3-n)}]^{\frac{n}{1-n}} \xi_1^{\frac{1+n}{1-n}} \int_0^1 \mu^{-n}(1-\mu^{3-n})^{\frac{n}{1-n}} d\mu = 0.$$

Making the appropriate change of variable in the integral we may write the following expression for ξ_1 :

$$\xi_1 = |W|^{\frac{1-n}{1+n}} (3-n)^{\frac{1}{1+n}} [\frac{n+1}{n(1-n)a^2}]^{\frac{n}{1+n}} [B(\frac{n-1}{n-3}, \frac{1}{1-n})]^{-\frac{1-n}{1+n}}.$$

Having determined the function $f(\xi)$ and ξ_1, we may write the solution to the problem (I.11), (I.13) with $\psi(r,t,p) \equiv 0$ and $p_w(t) = W$, for $n < 1$:

$$p(r,t) = p_0 + W - \operatorname{sgn}(W)[\frac{n(1-n)a^2}{(n+1)(3-n)}]^{\frac{n}{1-n}} \xi_1^{\frac{1-n}{1-n}} \int_0^{\xi/\xi_1} \mu^{-n}(1-\mu^{3-n})^{\frac{n}{1-n}} d\mu,$$

for $0 < \xi \le \xi_1$ therefore $R_w \le r \le \ell(t)$, where $\xi(r,t) = rt^{-\frac{n}{1+n}}$ and $\ell(t) = \xi_1 t^{\frac{n}{1+n}}$, and $p(r,t) = p_0$ for $r \geqq \ell(t)$.

In this particular case we have obtained the same result as the one presented in [9].

The same function $f(\xi)$ enables us to write the solution of the problem (I.11), (I.13) with $\psi(r,t,p) = b(p - p_0)$ and $p_w(t) = We^{-\frac{bt}{a^2}}$, $n < 1$ as:

$$p(r,t) = p_0 + \{W - \text{ sgn } (W)[\frac{n(1-n)a^2}{(1+n)(3-n)}]^{\frac{n}{1-n}} \xi_1^{\frac{1+n}{1-n}} \int_0^{\frac{\xi}{\xi_1}} \mu^{-n}(1-\mu^{3-n})d\mu\}^{\frac{-bt}{a^2}}$$

for $0 < \xi \leq \xi_1$ i.e. $R_w \leq r < \ell(t)$ where

$$\xi(t) = r(\frac{na^2}{1-n})^{\frac{-n}{1+n}} \{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}t)]\}^{\frac{-n}{1+n}} \quad \text{and}$$

$$\ell(t) = \xi_1(\frac{na^2}{1-n})^{\frac{n}{1+n}} \{\frac{1}{b}[1 - \exp(\frac{-b(1-n)}{na^2}t)]\}^{\frac{n}{1+n}} \quad \text{and}$$

$$p(r,t) = p_0 \qquad \text{for} \quad r \geqq \ell(t).$$

Case 2 $1 < n < 3$

Integrating the equations (6.2.1) and taking into consideration the condition (6.8) we obtain

$$f(\xi) = W - \text{ sgn}(W) \int_0^{\xi} \eta^{-n}[c + \frac{n(n-1)a^2}{(n+1)(3-n)}\eta^{3-n}]^{\frac{n}{1-n}} d\eta.$$

However, analysing the integral in this expression we realize that it is divergent for all $\xi > 0$ with any value for the constant c.

CASE 3 $n \geq 3$

First we rewrite (6.2.1) as:

$$f'(\xi) = -\operatorname{sgn}(W)\xi^{\frac{-2n}{n-1}}[c\xi^{n-3} - \frac{n(n-1)}{(n+1)(n-3)}]^{\frac{-n}{n-1}} \quad \text{for} \quad n > 3.$$

It is evident that with any c this function, as well as the one given by (6.2.2) is undefined for sufficiently small ξ.

From the arguments presented above we conclude that the problem we are currently concerned with does not have a similarity solution if $n > 1$.

6.2b $f(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta$

As in the one-dimensional case, let us seek a solution to (6.5) with $n < 1$ of the form

$$f(\xi) = A(\xi_1^\lambda - \xi^\lambda)^\beta$$

with $\lambda > 0$, $\beta > 0$ and $\operatorname{sgn}(A) = \operatorname{sgn}(W)$.

Substituting this expression for $f(\xi)$ into (6.5) yields:

$$\alpha a^2 A[\gamma(\xi_1^\lambda - \xi^\lambda)^\beta - \lambda\beta\xi^\lambda(\xi_1^\lambda - \xi^\lambda)^{\beta-1}] + \frac{A(\lambda\beta)^{\frac{1}{n}}|A|^{\frac{1}{n}-1}}{n}$$
$$[(n+\lambda-1)\xi^{\frac{\lambda}{n}-\frac{1}{n}-1}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-\frac{1}{n}} - \lambda(\beta-1)\xi^{\lambda+\frac{\lambda}{n}-\frac{1}{n}-1}(\xi_1^\lambda - \xi^\lambda)^{\frac{\beta}{n}-\frac{1}{n}-1}]$$
$$+ F(\xi)|A|^q(\xi_1^\lambda - \xi^\lambda)^{\beta q} = 0 \tag{6.2.3}$$

If $F(\xi) \equiv 0$ then (6.2.3) holds only if

$$\lambda = 1+n,\ \beta = \frac{1}{1-n},\ \gamma = 2 \quad \text{and} \quad A = \operatorname{sgn}(W)(\frac{na^2}{3-n})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}.$$

In this case $f(\xi)$ is given by:

$$f(\xi) = \operatorname{sgn}(W)(\frac{na^2}{3-n})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}(\xi_1^{1+n} - \xi^{1+n})^{\frac{1}{1-n}}.$$

From the condition $f(0) = W$ we obtain:

$$\xi_1 = |W|^{\frac{1-n}{1+n}}(\frac{3-n}{na^2})^{\frac{n}{1+n}}(\frac{1+n}{1-n})^{\frac{1}{1+n}}. \tag{6.2.4}$$

Now we can write the solution to the problem (I.11), (I.13) with $\psi(r,t,p) \equiv 0$ and $p_w(t) = Wt^{\frac{-2n}{3-n}}$, $t > 0$ and $n < 1$ as

$$p(r,t) = p_0 + \operatorname{sgn}(W)(\frac{na^2}{3-n})^{\frac{n}{1-n}}(\frac{1-n}{1+n})^{\frac{1}{1-n}}(\xi_1^{1+n} - \xi^{1+n})^{\frac{1}{1-n}}t^{\frac{-2n}{3-n}}$$

for $t > 0$ and $R_w < r \leq \ell(t)$ and $p(r,t) = p_0$ for $r \geq \ell(t)$, where $\xi(r,t) = rt^{\frac{-n}{3-n}}$, $\ell(t) = \xi_1 t^{\frac{n}{3-n}}$ and ξ_1 is given by (6.2.4).

We can also write the solution to the problem (I.11), (I.13) with $\psi(r,t,p) = b(p-p_0)$, $n < 1$ and $p_w(t) = W[\tau(t)]^2 e^{\frac{-bt}{a^2}}$, where

$$\tau(t) = (\frac{na^2}{1-n})^{\frac{-n}{3-n}}\{\frac{1}{b}[1 - \quad exp(\frac{-b(1-n)}{na^2}t)]\}^{\frac{-n}{3-n}}$$

namely:

$$p(r,t) = p_0 + \operatorname{sgn}(W)(\frac{na^2}{3-n})^{\frac{n}{1-n}}(\xi_1^{1+n} - \xi^{1+n})^{\frac{1}{1-n}}[\tau(t)]^2 e^{\frac{-bt}{a^2}},\ t > 0$$

for $t > 0$ and $R_w < r < \ell(t)$, and

$$p(r,t) = p_0 \quad \text{for} \quad r \geqq \ell(t),$$

where $\xi(r,t) = r\tau(t)$, $\ell(t) = \xi_1[\tau(t)]^{-1}$ and ξ_1 is given by (6.2.4).

Now we turn our attention to the equation (6.5) with $F(\xi)$ of the form (6.1.20). In order for (6.2.3) to be valid the parameters appearing in $F(\xi)$ must satisfy the relations (6.1.21), (6.1.22), (6.1.24) and

$$\frac{1}{n}(n+\lambda-1)(\lambda\beta)^{\frac{1}{n}}A|A|^{\frac{1}{n}-1} = -|A|^q D_3.$$

From these relations we deduce that

$$\lambda\beta = -\frac{D_2}{D_1}\gamma, \quad \frac{n+\lambda-1}{-\lambda(\beta-1)} = \frac{D_3}{D_4}.$$

As an example let us take $q = n$, $D_1 = -(1-n)$, $D_2 = -1$, $D_3 = (1-n)(2-n)$, $D_4 = -1$. In this case we have

$$\gamma = \frac{-1}{1-n}, \quad \lambda = \beta = \frac{1}{1-n}.$$

So the solution to the problem (I.11), (I.13) with

$$\psi(r,t,p) = |p-p_0|^n\{(n-1)\xi_1^{\frac{1}{1-n}} - n(\frac{r}{t})^{\frac{1}{1-n}} + (1-n)(2-n)\xi_1^{\frac{1}{1-n}}(\frac{r}{t})^{\frac{n}{1-n}}$$
$$-(3-3n+n^2)(\frac{r}{t})^{\frac{1+n}{1-n}}\}$$

and

$$p_w(t) = \pm(\frac{a}{1-n})^{\frac{2}{1-n}}\xi_1^{\frac{1}{(1-n)^2}}t^{\frac{1}{1-n}}, \quad t>0,\ n<1$$

is

$$p(r,t) = p_0 \pm (\frac{a}{1-n})^{\frac{2}{1-n}}[\xi_1^{\frac{1}{1-n}} - (\frac{r}{t})^{\frac{1}{1-n}}]^{\frac{1}{1-n}} t^{\frac{1}{1-n}} \quad \text{for} \quad R_w < r < \ell(t)$$

and

$$p(r,t) = p_0 \quad \text{for} \quad r \geq \ell(t)$$

where $\ell(t) = \xi_1 t$.

Here again we note that the front propagates with constant velocity.

6.3 Numerical Solutions

For the cases in which a closed form solution for the equations (6.4) and (6.5) cannot be found we resort to numerical methods.

In solving the problems (6.4), (6.8) - (6.10) and (6.5), (6.8) - (6.10) we must also determine the right end point of the interval $[0, \xi_1]$. We deal with this task by using the shooting method combined with the fifth order Runge-Kutta method.

We solve the equation (6.4) or (6.5) with initial conditions $f(0) = W$ and $f'(0) = \Omega$ with the Runge-Kutta method. Then by varying the shooting parameter Ω we determine the first point at which both f and f' equal zero. This is then the required right end point ξ_1.

To illustrate this technique we present the following example. In the equation (I.6) we take $n = 0.75$, $a^2 = 1$ and $\psi(x,t,p) \equiv 0$ and in the boundary condition at $x = 0$ we take $p_w(t) = -t^3$. This implies that in (6.4)

$\gamma = -3.5$, $\alpha = -6/7$ and $F(\xi) \equiv 0$. In this case (6.4) becomes:

$$9(-3.5f + \xi f') + 14f''(f')^{\frac{1}{3}} = 0.$$

So we are looking for the solution to this equation and the constant ξ_1 such that $f(\xi) < 0$ and $f'(\xi) > 0$ for $\xi \in (0, \xi_1)$ and

$$f(0) = -1$$

$$f(\xi_1) = 0$$

$$f'(\xi_1) = 0.$$

In fig. 1 we have plotted the function $f(\xi)$ for three different shooting parameters. For $\Omega = 1.376$ on the interval (2.28, 2.30) both f and f' change sign. Thus we may say that 2.29 is a good approximation for the right end point ξ_1.

Thus for this example the location of the pressure disturbances front is given by:

$$\ell(t) = 2.29t^{\frac{6}{7}}.$$

Let us now consider situations in which sources or sinks are present. With the same parameters as above we have solved the equation (6.4) with $F(\xi) = 1$ and $q = \frac{2}{3}$. This corresponds to $\psi(x,t,p) = |p - p_0|^{\frac{2}{3}}$. In this case we have found that $\xi_1 = 2.90$ implying that

$$\ell(t) = 2.90t^{\frac{6}{7}}.$$

Now if in (6.4) we take $F(\xi) = -1$ which corresponds to $\psi(x,t,p) = -|p-p_0|^{\frac{2}{3}}$ indicating the presence of sources, we find that $\xi_1 = 1.95$ which implies that

$$\ell(t) = 1.95t^{\frac{6}{7}}.$$

We have plotted these results in fig.2 and fig.3. Fig.2 contains the graph of $f(\xi)$ in the presence of sources, in the presence of sinks and in the absence of both in the case of production with $p_w(t) = -t^3$. In fig.3 we have plotted the location of the pressure disturbance front versus time for these three situations.

Examining fig.3 we observe that our numerical results are consistent with the physical expectations which indicate that in the case of production the moving pressure disturbance front propagates faster in the presence of sinks and slower in the presence of sources.

We have also investigated the plane radial problem with the parameters considered above. We found that in the presence of sinks $\xi_1 = 4.05$, in the presence of sources of equal magnitude $\xi_1 = 1.55$, and if neither sinks nor sources are present then $\xi_1 = 1.90$. Fig. 4 contains the graph of $f(\xi)$ in each of these three cases.

To show the accuracy of our numerical scheme, in table 1 we present a comparison between the exact solution and the solution obtained numerically for the equation

$$-4f + \xi f' + \frac{4}{3}f''|f'|^{\frac{1}{3}} = 0.$$

The exact solution of this equation is:

$$f(\xi) = -4^{\frac{-16}{5}}(4^{\frac{4}{7}} - \xi)^4. \tag{6.4}$$

6.4 Perturbation Method

In sections 6.1 and 6.2 we have obtained closed form solutions for the equations (6.4) and (6.5) with various values for γ. Now with values for γ near the ones mentioned above we may obtain approximate solutions by employing the perturbation method.

To illustrate the application of this method we consider the problem (6.4) (6.8) - (6.10) with $n = 0.5$, $W < 0$, and $F(\xi) \equiv 0$. In this case the equation (6.4) becomes:

$$\frac{a^2}{3+\gamma}(\gamma f + \xi f') + 2f'f'' = 0. \tag{6.4.1}$$

In section 6.1a we have obtained the solution to this problem with $\gamma = 0$. So we will now seek an approximate solution for $|\gamma| << 1$ of the form:

$$f = f_0 + \gamma f_1. \tag{6.4.2}$$

Since f must satisfy the conditions (6.9) - (6.10) we will impose on f_0 and f_1 the following conditions:

$$f_0(\xi_1) = f_0'(\xi_1) = 0 \tag{6.4.3}$$

and

$$f_1(\xi_1) = f_1'(\xi_1) = 0 \tag{6.4.4}$$

where ξ_1 is to be determined from the additional condition (6.8).

Substituting (6.4.2) into (6.4.1) and equating the coefficients of powers of γ to zero we obtain:

$$2f_0'' + \frac{a^2}{3}\,\xi = 0 \tag{6.4.5}$$

and

$$f_1'' + \frac{a^2}{6}\,\frac{f_0}{f_0'} - \frac{a^2}{18}\,\xi = 0. \tag{6.4.6}$$

Solving the problem (6.4.5) (6.4.3) we have:

$$f_0(\xi) = \frac{-a^2}{36}\,(\xi_1 - \xi)^2(\xi + 2\xi_1). \tag{6.4.7}$$

Substituting (6.4.7) into (6.4.6) and integrating the resulting equations with the conditions (6.4.4) yields:

$$f_1(\xi) = \frac{a^2\xi_1^2}{9}\,[\xi_1 - \xi + (\xi_1 + \xi)\ell n(\frac{\xi_1 + \xi}{2\xi_1})].$$

Now the condition (6.8) indicates that

$$f_0(0) + \gamma f_1(0) = W$$

which yields

$$\xi_1 = \{\frac{-18W}{a^2[1 - 2\gamma(1 - \ell n2)]}\}^{\frac{1}{3}}.$$

In table 2 we present the values of the function $f(\xi)$ for $\gamma = -0.5$ and $\gamma = 0.1$ obtained by the perturbation method and by the numerical method described in the previous section. Note the excellent agreement between the two methods even for the relatively large value for $|\gamma|$.

ıble 1 - Comparison between exact and numerical solutions

$n=0.75$, $W=-1$, $a=1$, $\gamma=-4$

ξ	exact soln. f	num. soln. f
0.000	-1.00000	-1.00000
0.050	-0.91246	-0.91246
0.100	-0.83079	-0.83079
0.150	-0.75473	-0.75474
0.200	-0.68402	-0.68403
0.250	-0.61840	-0.61841
0.300	-0.55762	-0.55762
0.350	-0.50143	-0.50144
0.400	-0.44960	-0.44960
0.450	-0.40190	-0.40190
0.500	-0.35809	-0.35810
0.550	-0.31797	-0.31798
0.600	-0.28132	-0.28132
0.650	-0.24793	-0.24794
0.700	-0.21761	-0.21761
0.750	-0.19015	-0.19016
0.800	-0.16538	-0.16539
0.850	-0.14312	-0.14312
0.900	-0.12318	-0.12319
0.950	-0.10540	-0.10541
1.000	-0.08962	-0.08963
1.050	-0.07568	-0.07569
1.100	-0.06343	-0.06344
1.150	-0.05274	-0.05275
1.200	-0.04345	-0.04347
1.250	-0.03545	-0.03547
1.300	-0.02861	-0.02863
1.350	-0.02281	-0.02283
1.400	-0.01794	-0.01797
1.450	-0.01390	-0.01392
1.500	-0.01058	-0.01061
1.550	-0.00789	-0.00792
1.600	-0.00575	-0.00579
1.650	-0.00408	-0.00412
1.700	-0.00281	-0.00285
1.750	-0.00185	-0.00190
1.800	-0.00117	-0.00122
1.850	-0.00069	-0.00075
1.900	-0.00038	-0.00045
1.950	-0.00019	-0.00026
2.000	-0.00008	-0.00016
2.050	-0.00003	-0.00012
2.100	-0.00001	-0.00010

Table 2 - Comparison between perturbation and numerical method for n=0.5 , W=-1 ,a=1

	ξ	f_o	f_1	pert.meth. f	num.meth. f
γ=0.1	0.00	-1.06538	0.65383	-1.00000	-1.00000
	0.10	-1.00571	0.60012	-0.94569	-0.94564
	0.20	-0.94620	0.54928	-0.89127	-0.89116
	0.30	-0.88702	0.50121	-0.83690	-0.83674
	0.40	-0.82834	0.45581	-0.78276	-0.78255
	0.50	-0.77033	0.41300	-0.72904	-0.72878
	0.60	-0.71316	0.37269	-0.67589	-0.67559
	0.70	-0.65698	0.33482	-0.62350	-0.62316
	0.80	-0.60197	0.29930	-0.57204	-0.57167
	0.90	-0.54830	0.26607	-0.52169	-0.52129
	1.00	-0.49612	0.23507	-0.47261	-0.47219
	1.10	-0.44561	0.20623	-0.42499	-0.42455
	1.20	-0.39693	0.17951	-0.37898	-0.37853
	1.30	-0.35026	0.15483	-0.33477	-0.33432
	1.40	-0.30575	0.13216	-0.29253	-0.29207
	1.50	-0.26357	0.11144	-0.25243	-0.25197
	1.60	-0.22389	0.09263	-0.21463	-0.21419
	1.70	-0.18689	0.07568	-0.17932	-0.17889
	1.80	-0.15271	0.06054	-0.14665	-0.14625
	1.90	-0.12153	0.04719	-0.11681	-0.11644
	2.00	-0.09352	0.03558	-0.08997	-0.08962
	2.10	-0.06885	0.02566	-0.06628	-0.06598
	2.20	-0.04767	0.01742	-0.04593	-0.04567
	2.30	-0.03016	0.01080	-0.02908	-0.02887
	2.40	-0.01648	0.00579	-0.01591	-0.01575
	2.50	-0.00681	0.00235	-0.00657	-0.00648
	2.60	-0.00130	0.00044	-0.00125	-0.00123
	2.70	-0.00012	0.00004	-0.00012	0.00207
γ =-0.5	0.00	-0.76520	0.46961	-1.00000	-1.00000
	0.10	-0.71734	0.42667	-0.93067	-0.92929
	0.20	-0.66965	0.38628	-0.86279	-0.86011
	0.30	-0.62230	0.34836	-0.79648	-0.79259
	0.40	-0.57544	0.31280	-0.73184	-0.72686
	0.50	-0.52925	0.27953	-0.66902	-0.66305
	0.60	-0.48390	0.24846	-0.60813	-0.60130
	0.70	-0.43954	0.21952	-0.54931	-0.54174
	0.80	-0.39635	0.19265	-0.49268	-0.48450
	0.90	-0.35450	0.16777	-0.43838	-0.42973
	1.00	-0.31414	0.14483	-0.38656	-0.37757
	1.10	-0.27545	0.12377	-0.33734	-0.32814
	1.20	-0.23860	0.10453	-0.29086	-0.28160
	1.30	-0.20374	0.08707	-0.24728	-0.23809
	1.40	-0.17106	0.07133	-0.20672	-0.19774
	1.50	-0.14070	0.05728	-0.16934	-0.16070
	1.60	-0.11284	0.04487	-0.13528	-0.12712
	1.70	-0.08766	0.03405	-0.10468	-0.09714
	1.80	-0.06530	0.02479	-0.07770	-0.07091
	1.90	-0.04595	0.01705	-0.05447	-0.04856
	2.00	-0.02976	0.01080	-0.03516	-0.03025
	2.10	-0.01690	0.00600	-0.01990	-0.01613
	2.20	-0.00755	0.00262	-0.00886	-0.00635
	2.30	-0.00186	0.00063	-0.00217	-0.00107
	2.40	0.00000	0.00000	0.00000	0.00208

Illustration of the shooting method

n=.75 $\gamma=-3.5$ $F(\xi)=0$

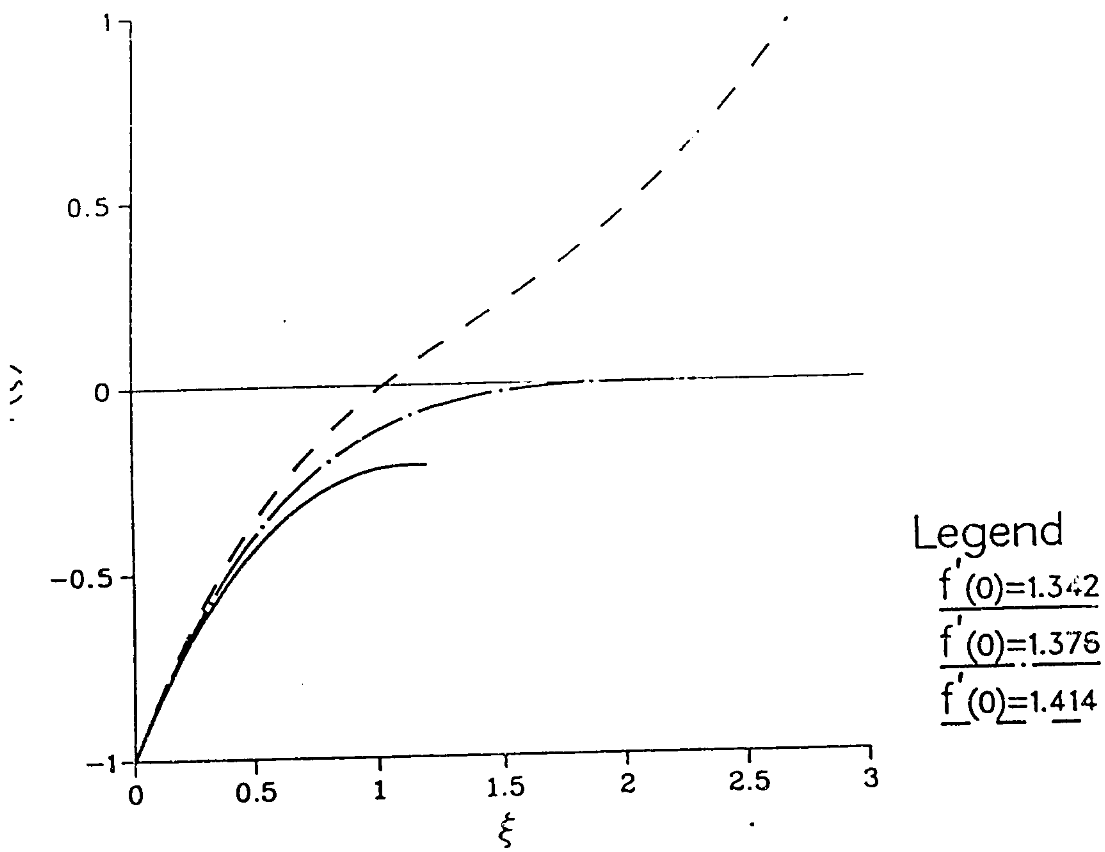
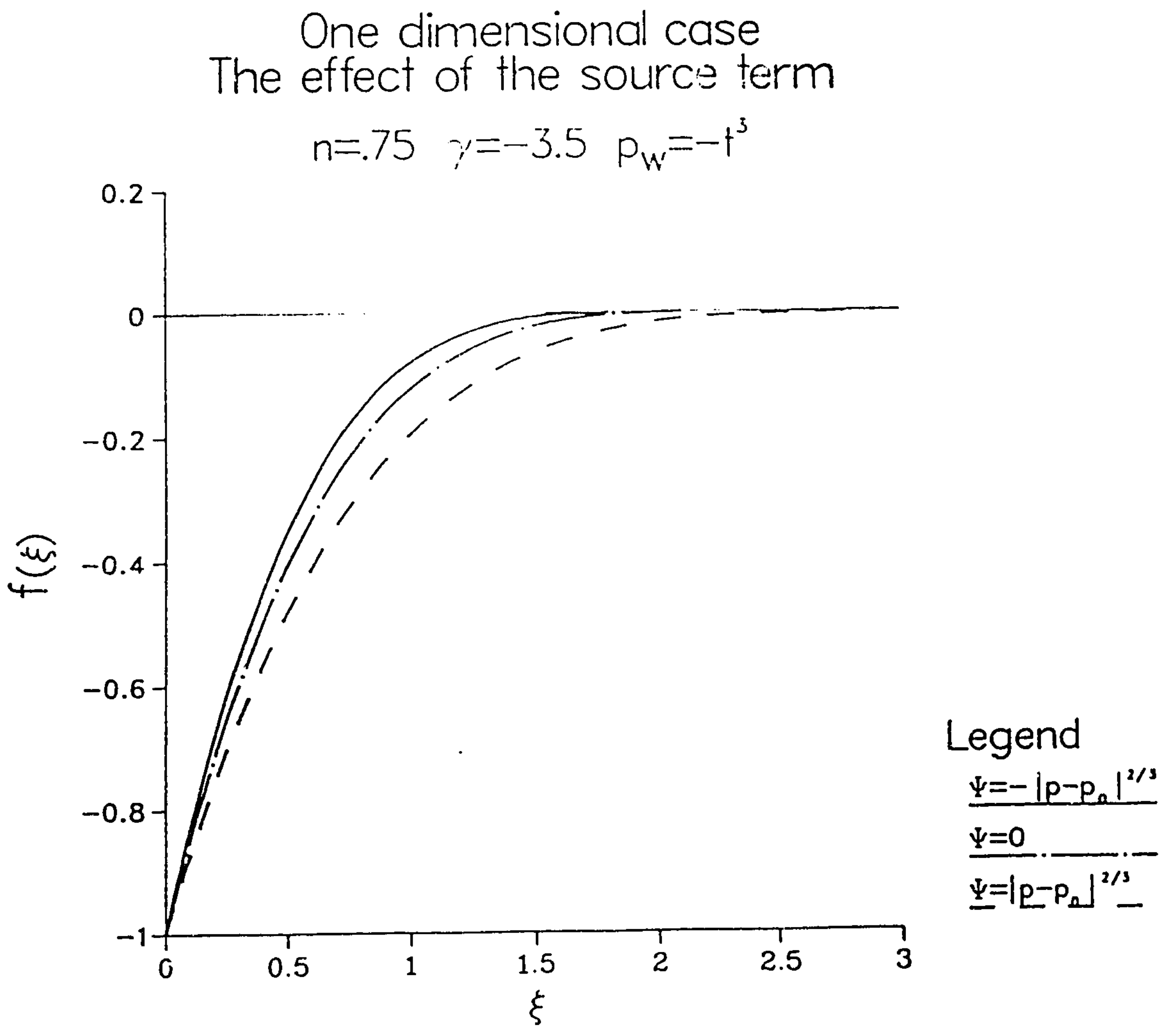


fig.1



One dimensional case
The effect of the source term
n=.75 γ=−3.5 pW=−t^3
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
f(ξ)
0
0.5
1
1.5
2
2.5
3
ξ
Legend
Ψ=− |p−p0|^2/3
Ψ=0
Ψ=|p−p0|^2/3

fig. 2

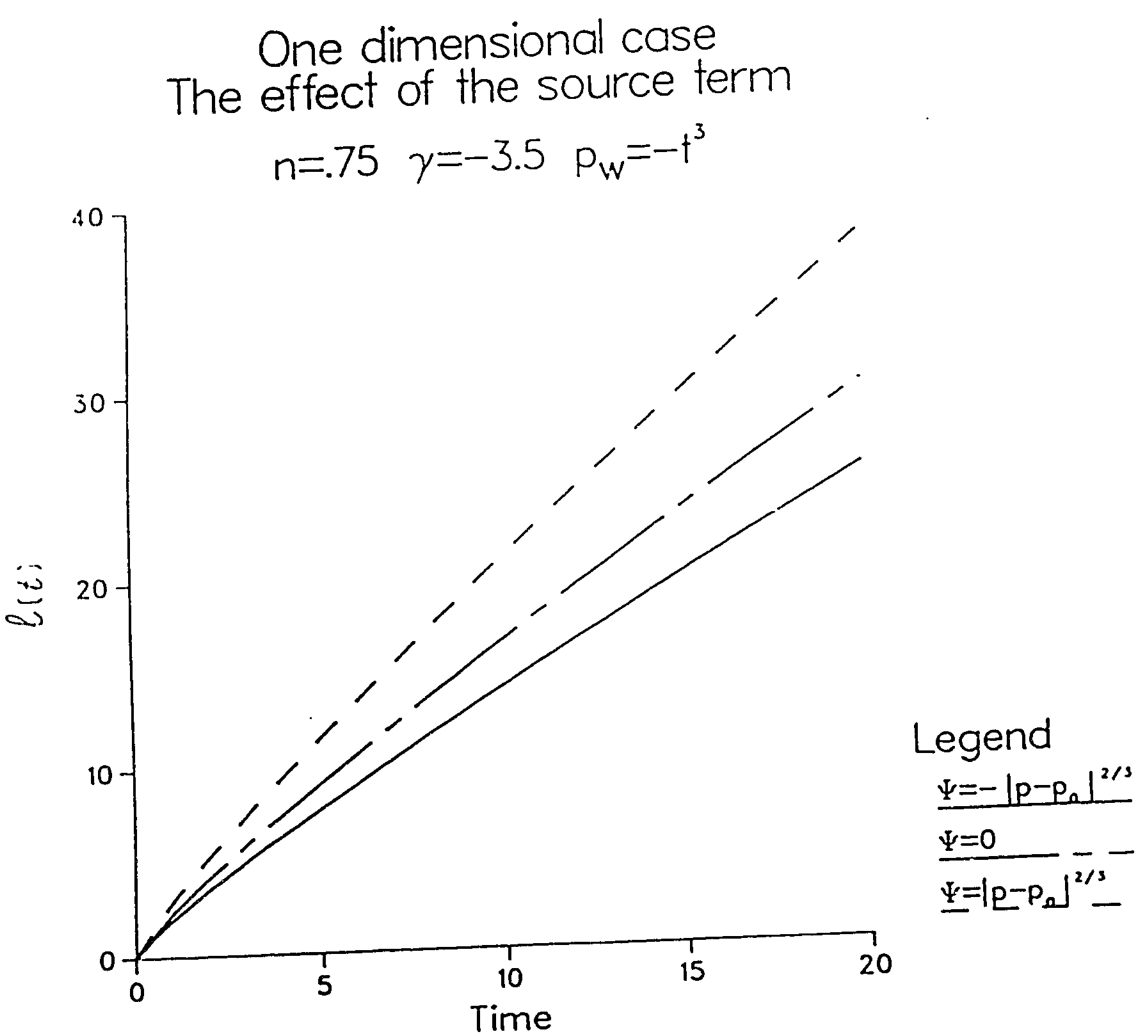
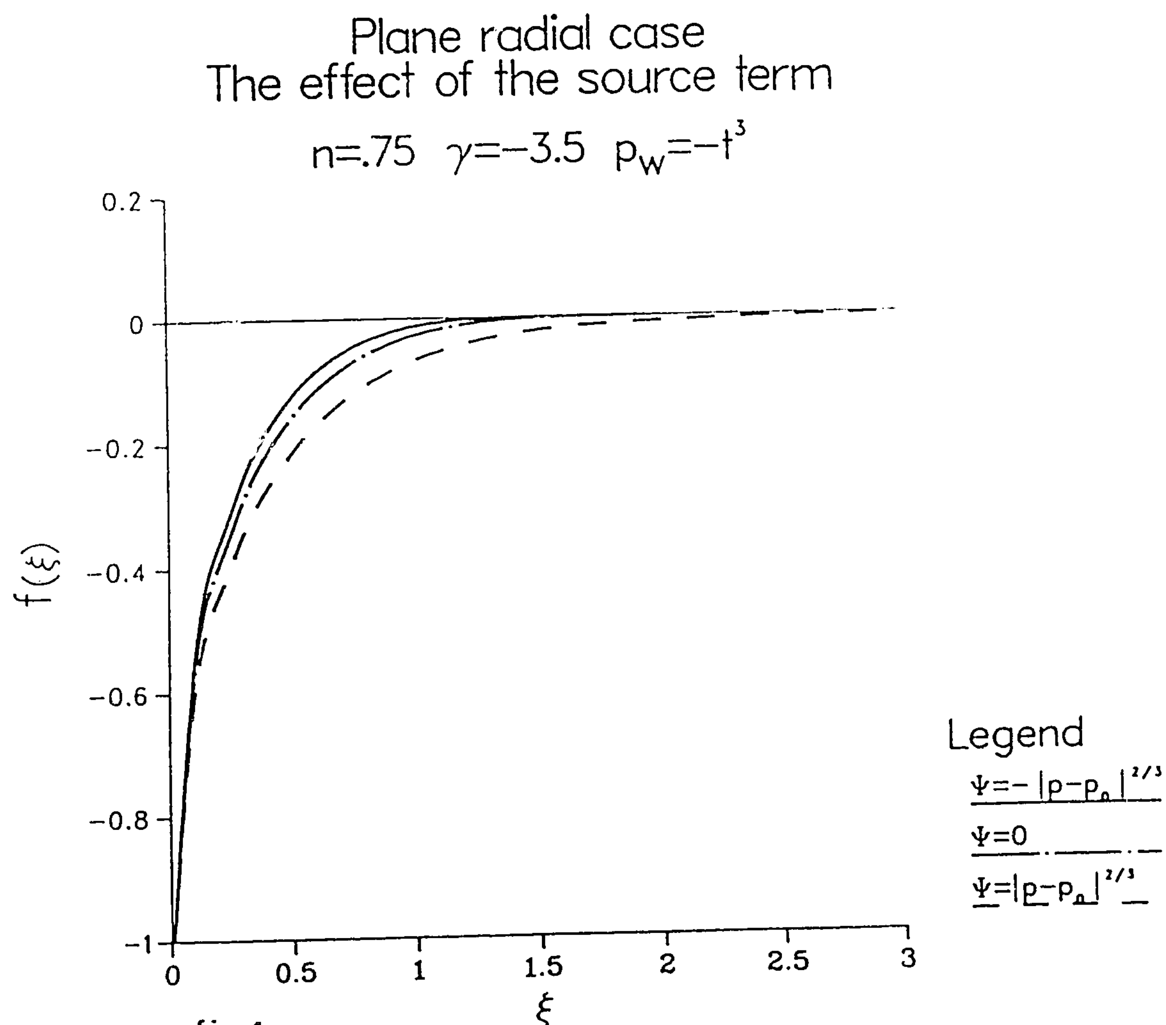

One dimensional case
The effect of the source term
n=.75 γ=-3.5 p_w=-t^3
40
30
20
10
0
ℓ(t)
0
5
10
15
20
Time
Legend
Ψ=-|p-p_0|^{2/3}
Ψ=0
Ψ=|p-p_0|^{2/3}

fig. 3



Plane radial case
The effect of the source term
n=.75 γ=−3.5 p_W=−t^3
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
f(ξ)
0
0.5
1
1.5
2
2.5
3
ξ
Legend
Ψ=− |p−p_o|$^{2/3}$
Ψ=0
Ψ=|p−p_o|$^{2/3}$

fig 4

CONCLUSION

In this thesis we have investigated initial and boundary value problems related to two nonlinear parabolic equations which govern the flow of power law fluids through porous media with the purpose of finding similarity solutions. We have determined the forms of the source term ψ and the corresponding forms of the prescribed pressure at the fixed boundary which allow similarity solutions. Only the case with $\psi \equiv 0$ and $p_w(t) =$ const. have previously been treated [9].

We have not studied the existence and uniqueness of the solution to the boundary value problem corresponding to the resulting ordinary differential equations. However we were able to obtain closed form solutions for classes of particular cases. For other cases numerical solutions have been found which were in good agreement with the exact solutions when these were available.

The solutions obtained indicate that for shear thinning fluids $(0 < n < 1)$ the pressure disturbance front propagates with finite velocity, while for dilatant fluids $(n > 1)$ it propagates with infinite velocity a property characteristic of Newtonian fluids.

For the cases when the pressure disturbance front propagates with finite velocity we have investigated its acceleration and its behavior as t tends to infinity. We found that the front propagates to infinity as t tends to infinity for all the allowable source terms except for the case when $\psi = b(t)(p - p_0) +$

$h(y,t)$ with $(\frac{1}{n}-1)\int_0^t d(\eta)b\eta > 0$ as $t \to \infty$, in which case it propagates to a finite location.

Although not all physical situations can be modeled by equations with a source term that allows similarity solutions one might be able to approximate them by one of these forms, thus obtaining some practical insight into the problem.

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