UNIVERSITY OF ALBERTA

CARDINAL INTERPOLATION BY SPLINES AND REFINABLE FUNCTIONS

by

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To *University of Alberta*
the place my dream comes true

ABSTRACT

This thesis investigates three problems: cardinal interpolation by fundamental cardinal splines, cardinal interpolation by certain refinable functions, and Boolean methods.

Cardinal interpolation is important for smooth approximation of empirical tables, and has wide applications. We will investigate *boundness properties* of two cardinal interpolants. A function f is called *fundamental* if it satisfies the interpolation condition $f(j) = \delta(j),\ j \in \mathbb{Z}$. We shall prove in Chapter 2 that the fundamental cardinal spline ψ_m satisfies

$$(-1)^j\ \psi_m(x) \ <\ (-1)^j\ \frac{\sin(\pi x)}{\pi x} \quad \text{for} \quad x \in (j,\ j+1),\ j = 0, 1, 2, ...$$

This results improved the existing analogous results in the literature. In Chapter 3, we shall prove that the DD function φ_N, which is fundamental and is supported on $[-2N+1,\ 2N-1]$, $N \in \mathbb{N}$, satisfies

$$0 < \varphi_N(x) < \frac{\sin(\pi x)}{\pi x} \ \text{ for } x \in (0,\ 1) \quad \text{and} \quad |\varphi_N(x)| \le \left|\frac{\sin(\pi x)}{\pi x}\right| \ \text{ for } x \in \mathbb{R}.$$

Boolean methods are effective in computer aided geometric design to produce surfaces. In Chapter 4, we will demonstrate that the Boolean sum constructed from φ_N provides desired polynomial reproducibility and approximation order.

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Table of Contents

Chapter 1

Introduction

In this thesis, we shall study cardinal interpolation by cardinal splines and refinable functions. As outlined in the following sections, we will investigate cardinal interpolation by the fundamental cardinal splines in Chapter 2, and by the DD-functions in Chapter 3 . As an application, we discuss the Boolean sum constructed from the DD-functions in Chapter 4.

1.1 Cardinal interpolation and cardinal splines

We are interested in cardinal interpolation, that is, interpolation of given data at the integer points on the real line.

Let $\{y_j\}$, $j = 0,\ \pm 1,\ \pm 2,\ \cdots$, be a doubly-infinite sequence of data. Whittaker in [36] introduced the so-called *cardinal series*

$$f(x) = \sum_{j \in \mathbb{Z}} y_j \, \frac{\sin\big(\pi(x-j)\big)}{\pi(x-j)}. \tag{1.1.1}$$

Obviously, $f(j) = y_j$ for all $j \in \mathbb{Z}$. The Whittaker cardinal series appears in many fields, from the theory of entire functions to sampling theory and communication theory. The modern extensions of the cardinal series have been nicely surveyed by Jetter [22], Riemenschneider [30] and Higgins [20].

In his paper [31], I. J. Schoenberg introduced the so-called *spline functions* with knots at the integers which have come to be known as the birth of (univariate) *cardinal splines*. In [32] and [33], Schoenberg initiated a beautiful theory of cardinal interpolation by splines. His initial motivation can be described as follows. Let $M_2(x)$ be the so-called "roof-function" given by $M_2(x) = x + 1$ in $[-1,\ 0]$, $M_2(x) = -x + 1$ in $[0,\ 1]$, and $M_2(x) = 0$ elsewhere, then

$$\phi_2(x) = \sum_{j \in \mathbb{Z}} y_j M_2(x-j) \tag{1.1.2}$$

is clearly the piecewise linear interpolant.

Cardinal spline functions are simply defined in the following way: Let $n \geq 0$ be an integer. We denote by $\mathcal{S}_n$ the class of functions S satisfying the following two conditions

(i) $S \in C^{n-1}(\mathbb{R})$,

(ii) $S \in \Pi_n$ in each interval $(j,\ j+1)$, $j \in \mathbb{Z}$

where Π_n represents the class of polynomials of degree at most n, over the field $\mathbb{C}$ of complex numbers.

The elements of $\mathbb{S}_n$ are called *cardinal spline functions* of degree n.

During the development of cardinal splines, two topics have drawn many mathematicians' attention. The first one is a limit property. Assume that the sequence y_j in (1.1.1) is fixed, in particular, let $y_j = \delta(j)$ be the *fundamental sequence* defined by

$$\delta(j) = \begin{cases} 1 & j = 0, \\ 0 & j \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{1.1.3}$$

How does the interpolant S behave as its degree goes to infinity? In his paper [33] (also see [32], [34]), Schoenberg proved that the *sinc*-function, $\frac{\sin(\pi x)}{\pi x}$, is the limit function for those cardinal splines which interpolate on δ_j. This beautiful result was extended to a general class of (multivariate) cardinal series by de Boor, Höllig and Riemenschneider in 1985 (see [8–10]).

Moreover, observe that the sinc-function has *sign-regularity* property, i.e.

$$\operatorname{sign}\left(\frac{\sin(\pi x)}{\pi x}\right) = (-1)^j, \qquad j < x < j+1,\ j = 0, 1, 2,\ \ldots \tag{1.1.4}$$

The second topic, which is derived naturally from the limit property, is to investigate if these cardinal splines which interpolate to δ_j have this property. de Boor and Schoenberg answered this question positively (see [11] and also [26] for a simpler proof).

The objective of this thesis is to further develop cardinal interpolation with respect to the fundamental sequence δ_j. For $f \in C(\mathbb{R})$, f is called a *fundamental function* if

$$f(j) = \delta(j), \qquad \text{for all} \quad j \in \mathbb{Z}.$$

where $\delta(j)$ is given in (1.1.3).

Chapter 2 is devoted to further investigation of the behavior of the *fundamental cardinal splines.* We prove that the fundamental cardinal splines, denoted by ψ_m, $m \in \mathbb{N}$, satisfy a restraint relation with the sinc-function. With the aid of (1.1.4), we shall prove that for $m \in \mathbb{N}$, $m \geq 11$,

$$0 < (-1)^j \, \psi_m(x) < (-1)^j \, \frac{\sin(\pi x)}{\pi x}, \qquad x \in (j, \ j+1), \ \ j = 0, 1, 2, \ ... \tag{1.1.5}$$

1.2 The DD-functions

Chapter 3 is devoted to another construction of the fundamental functions. Our initial motivation comes from the observation of the fundamental cardinal splines ψ_m, $m \in \mathbb{N}$.

A difficulty with ψ_m, if we can say so, is that they are supported on the whole real line. In 1986, Dubuc [15] introduced a fundamental function which is compactly supported. Deslauries and Dubuc [12] extended the idea used in [15] and introduced a family of compactly supported fundamental functions, which we call them the *DD-*

functions and denote by φ_N, $N \in \mathbb{N}$.

Recently, more and more properties of φ_N are discovered. In 1988, suggested by Meyer, Daubechies [4] recognized that φ_N also plays an analogous role to the *Daubechies wavelets* [5]. Micchelli [28] further observed a tight connection between φ_N and the Daubechies wavelets.

We found that

$$\lim_{N\to\infty} \varphi_N(x) = \frac{\sin(\pi x)}{\pi x} \qquad \text{uniformly.}$$

But surprisingly, $\varphi_N(x)$ does *not* have sign-regularity property anymore as $x > 2$ ([15], Theorem 14). This fact could be the biggest difference between the fundamental cardinal splines and the DD-functions. Due to the failure of sign-regularity property, an analogous result to (1.1.5) is not valid for φ_N. But we shall demonstrate that

$$0 < \varphi_N(x) < \frac{\sin(\pi x)}{\pi x} \qquad \text{for} \quad x \in (-1,\ 0) \cup (0,\ 1) \tag{1.2.1}$$

and

$$\left|\varphi_N(x)\right| \le \left|\frac{\sin(\pi x)}{\pi x}\right| \qquad \text{for all} \quad x \in \mathbb{R}. \tag{1.2.2}$$

The reader might find that the techniques used in Chapter 2 and 3 are quite similar. In fact, observe that for either (1.1.5) or (1.2.2), we are essentially required to measure how large the difference between ψ_m (or φ_N) and the sinc-function would be. Therefore, we are led to consider their Fourier transforms

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x}\, d\xi.$$

With the help of the Poisson summation formula, the Fourier transform of ψ_m (or of φ_N), combining with the Fourier transform of the sinc-function together, can be expressed as a doubly-infinite series. We then pick up a *main term* from the series such that its (absolute) value dominates the absolute value of the remainder terms of the series, and consequently we can estimate the difference between ψ_m (or φ_N) and the sinc-function.

1.3 The Boolean sum and the DD-functions

A practical application of the DD-functions is considered in Chapter 4.

In 1968, Gordon [17] developed the so-called *Boolean method* (see also [13–14]) for interpolation and smoothing of data given on a *mesh* $\triangle$:

$$\triangle = \left[\frac{j}{n}, \frac{k}{m}\right] \times \left[\frac{j+1}{n}, \frac{k+1}{m}\right] \qquad j,\ k \in \mathbb{Z},\ \ n,\ m \in \mathbb{N}. \tag{1.3.1}$$

This method is more effective than those using tensor product of interpolants in producing surfaces. But this method requires that all interpolants involved must *commute* with each other. Obviously, this assumption has restricted its potential application. In 1992, R. Q. Jia [23] paid attention to this question. He initiated a new theory and completely overcame the commutativity restraint.

The results presented in Chapter 4 have heavily benefited from Jia's work. We first give a simple expression of Boolean sums in the bivariate case. Then we use the

DD-functions as the interpolants to create bivariate interpolants $\mathcal{P}$. We prove that those interpolants $\mathcal{P}$ reproduce certain (bivariate) polynomials. We then show that these Boolean interpolants $\mathcal{P}$ constructed from the DD-functions φ_N possess a good approximation property. The reader might find that this approximation property is much better than those derived by using the tensor product. Moreover, to interpolate and smooth down data given on (1.3.1), our result (see §4.2, Theorem 4.5) is more effective than the Biermann interpolation given in [14].

Finally, as a practical applications, we demonstrate that Boolean interpolants $\mathcal{P}$ constructed from the DD-functions can be used to produce surfaces.

Chapter 2

Cardinal Interpolation by Fundamental Cardinal Splines

2.1 Some basic properties of the central cardinal B-splines

We are interested in the following **cardinal interpolation problem**: Given a continuous function ϕ on $\mathbb{R}$ with compact support and a sequence $b \in \ell_\infty(\mathbb{Z})$, find a sequence $a \in \ell_\infty(\mathbb{Z})$ such that the function

$$f := \sum_{k \in \mathbb{Z}} a(k)\phi(\cdot - k) \tag{2.1.1}$$

interpolates $b(j)$ at j, $j \in \mathbb{Z}$, that is,

$$f(j) = b(j) \qquad \forall\, j \in \mathbb{Z}. \tag{2.1.2}$$

The basic theory of cardinal interpolation was developed by Schoenberg [32]. It was proved in [33, Lemma 11] that the cardinal interpolation with ϕ is uniquely solvable if and only if

$$\sum_{j \in \mathbb{Z}} \phi(j) e^{ij\xi} \neq 0 \qquad \forall\, \xi \in \mathbb{R}. \tag{2.1.3}$$

Throughout this chapter, ϕ is assumed to be a central cardinal B-spline which is defined as follows. Let M_1 be the function defined as:

$$M_1(x) = \begin{cases} 1/2 & x = 0, \\ 1 & 0 < x < 1, \\ 1/2 & x = 1, \\ 0 & x \in \mathbb{R} \setminus [0,\ 1]. \end{cases} \tag{2.1.4}$$

For $m \geq 2$, let M_m be defined by

$$M_m(x) := M_{m-1} * M_1(x) = \int_0^1 M_{m-1}(x - t)\,dt. \tag{2.1.5}$$

It is well known that M_m is a piecewise polynomial function supported on $[0,\ m]$. Moreover, M_m is continuous for $m \geq 2$. The **central cardinal B-spline** ϕ_m of order m is a shift of M_m:

$$\phi_m := M_m\left(\cdot + \frac{m}{2}\right). \tag{2.1.6}$$

It was proved in [32] that the cardinal interpolation problem with ϕ_m ($m \geq 2$) is uniquely solvable.

Let

$$a_m(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ij\xi}}{\tilde{S}_{\phi_m}} \, d\xi, \qquad j \in \mathbb{Z}, \tag{2.1.7}$$

where $\tilde{S}_{\phi_m}$ is the **symbol** of ϕ_m given by

$$\tilde{S}_{\phi_m}(\xi) = \sum_{j \in \mathbb{Z}} \phi_m(j) e^{ij\xi}.$$

Then the function ψ_m given by

$$\psi_m(x) = \sum_{j \in \mathbb{Z}} a_m(j) \phi_m(x - j), \qquad x \in \mathbb{R} \tag{2.1.8}$$

has the property $\psi_m(j) = \delta(j)$ for all $j \in \mathbb{Z}$ where $\delta(j)$ is the fundamental sequence given in (1.1.3). The function ψ_m is called **the fundamental cardinal spline** of the cardinal interpolation associated with ϕ_m. Given $b \in \ell_\infty(\mathbb{Z})$, the unique solution f of the cardinal interpolation problem can be written in the Lagrange form

$$f = \sum_{j \in \mathbb{Z}} b(j) \psi_m(\cdot - j).$$

Schoenberg [34] found a striking property of the fundamental cardinal splines ψ_m as m goes to infinity. His result can be stated as follows:

$$\lim_{m \to \infty} \psi_m(x) = \frac{\sin(\pi x)}{\pi x} \qquad \forall\, x \in \mathbb{R}.$$

In [11], de Boor and Schoenberg showed that

$$\operatorname{sign}(\psi_m(x)) = \operatorname{sign}\left(\frac{\sin(\pi x)}{\pi x}\right) \qquad \forall\, x \in \mathbb{R}.$$

In other words,

$$(-1)^j\, \psi_m(x) > 0 \qquad \forall\, x \in (j,\ j+1),\ \ j \in \mathbb{Z}_+. \tag{2.1.9}$$

where Z_+ is the set of all nonnegative integers. de Boor, Höllig and Riemenschneider [8–10] investigated convergence of cardinal series and established a characterization of the limits of cardinal series.

As usual, we use **sinc** to denote the function $x \mapsto \frac{\sin(\pi x)}{\pi x}$, $x \in \mathbb{R}$. The behavior of this convergence has attracted interest of many mathematicians. In particular, de Boor conjectured that ψ_m converges to the sinc-function monotonically as m goes to infinity by the parity of m, i.e., $\psi_{2m-1}(x) < \psi_{2m+1}(x)$ and $\psi_{2m}(x) < \psi_{2m+2}(x)$, $m \in \mathbb{N}$.

In this chapter we take the first step toward solving the conjecture of de Boor. Our main result is the following: Let $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$

Theorem 2.1 *For* $m \geq 11$,

$$0 < (-1)^j\, \psi_m(x) < (-1)^j\, \frac{\sin(\pi x)}{\pi x} \qquad \textit{for} \quad x \in (j,\ j+1), \quad j \in \mathbb{Z}_+. \tag{2.1.10}$$

Let us compare our theorem with the existent results in the literature. In [29], Reimer proved that $||\psi_m||_\infty = 1$ for even m. In his Ph.D thesis [26], using the results from Schoenberg [35] and de Boor [7], Lee demonstrated that for $m \in \mathbb{N}$,

$$-1 < \psi_m(x) \leq 1 \qquad \forall\, x \in \mathbb{R}.$$

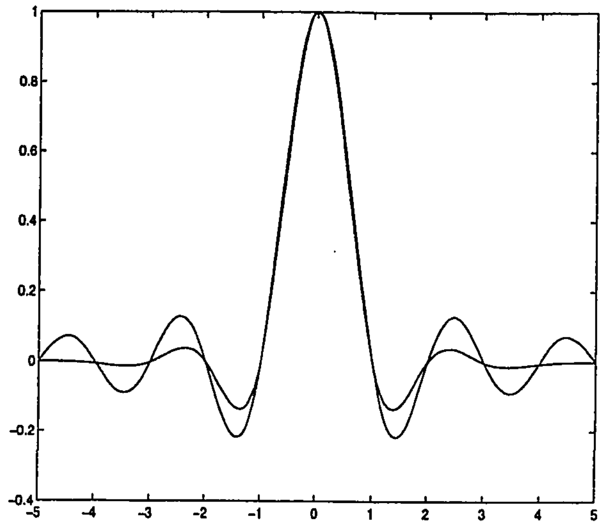

Figure 2.1: Graphs of ψ_4 and $\frac{\sin(\pi x)}{\pi x}$ over $[-5,\ 5]$

Clearly, our theorem presented here has more precise description for the behavior of ψ_m (see Figure 2.1).

2.2 Motivation and the main technique

As we mentioned before, the first thing that should be investigated is the relationship between $\frac{\sin(\pi x)}{\pi x}$ and $\psi_m(x)$. But it is seen that the inequality (2.1.10) describes the difference between the fundamental function $\psi_m(x)$ and the sinc-function $\frac{\sin(\pi x)}{\pi x}$. To measure how great the difference will be we first modify the expression (2.1.8) so that we have better observation to the structure of $\psi_m(x)$. Recall that $a_m(j)$ has the integral expression (2.1.7) where $\tilde{S}_{\phi_m}$ is the symbol of ϕ_m . Then

$$\psi_m(x) = \sum_{j\in\mathbb{Z}} a_m(j)\phi_m(x-j)$$

can be rewritten as

$$\psi_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{j\in\mathbb{Z}} \phi_m(x-j)e^{-ij\xi}\right)\left(\sum_{k\in\mathbb{Z}} \phi_m(k)e^{ik\xi}\right)^{-1} d\xi.$$

With the help of the Poisson summation formula, the first integrand above can be written as

$$\sum_{j\in\mathbb{Z}} \phi_m(x-j)e^{-ij\xi} = \sum_{k\in\mathbb{Z}} \hat{\phi}_m\big(-(\xi+2k\pi)\big)e^{-ix(\xi+2k\pi)}$$

and by choosing $x=0$, it becomes

$$\sum_{j\in\mathbb{Z}} \phi_m(j)e^{ij\xi} = \sum_{k\in\mathbb{Z}} \hat{\phi}_m\big(-(\xi+2k\pi)\big).$$

Since $\hat{\phi}_m(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^m$, $\xi\in\mathbb{R}$,

$$\hat{\phi}_m\big(-(\xi+2k\pi)\big) = (-1)^{km}\, 2^m \left(\frac{\sin(\xi/2)}{\xi+2k\pi}\right)^m.$$

For convenience, let us introduce

$$u(\xi) := \sum_{j\in\mathbb{Z}} \frac{(-1)^{jm}}{(\xi+2j\pi)^m}. \tag{2.2.1}$$

After a simple calculation one has

$$\psi_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k\in\mathbb{Z}} \frac{(-1)^{km}e^{-ix(\xi+2k\pi)}}{(\xi+2k\pi)^m u(\xi)}\right) d\xi, \qquad x\in\mathbb{R}.$$

Noting that

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^{km}\cos\big(x(\xi+2k\pi)\big)}{(\xi+2k\pi)^m u(\xi)} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \frac{(-1)^{km}\sin\big(x(\xi+2k\pi)\big)}{(\xi+2k\pi)^m u(\xi)}$$

are even and odd functions of ξ respectively, we then obtain an expression of the fundamental function $\psi_m(x)$:

$$\psi_m(x) = \frac{1}{\pi}\int_0^\pi \left(\sum_{k\in\mathbb{Z}} \frac{(-1)^{km}\cos\big(x(\xi+2k\pi)\big)}{(\xi+2k\pi)^m u(\xi)}\right) d\xi, \qquad x \in \mathbb{R}. \tag{2.2.2}$$

Finally, rewriting $\frac{\sin(\pi x)}{\pi x} = \frac{1}{\pi}\int_0^\pi \cos(\xi x) d\xi$ we obtain, for $x \in \mathbb{R}$,

$$\begin{aligned} I &:= \frac{\sin(\pi x)}{\pi x} - \psi_m(x) \\ &= \frac{2}{\pi}\int_0^\pi \left(\sum_{k\in\mathbb{Z}} \frac{(-1)^{km}\sin(k\pi x)\sin\big(x(\xi+k\pi)\big)}{(\xi+2k\pi)^m u(\xi)}\right) d\xi \\ &= \frac{2}{\pi}\sum_{k\in\mathbb{Z}}(-1)^{km}\sin(k\pi x)\left(\int_0^\pi \frac{\sin\big(x(\xi+k\pi)\big)}{(\xi+2k\pi)^m u(\xi)}\right) d\xi. \end{aligned} \tag{2.2.3}$$

Starting from §3 we shall prove (2.1.10) by estimating the integral (2.2.3). A main technique we will use repeatedly is to pick up a term from the bi-infinite series in (2.2.3) such that it dominates all others. The term corresponding to $k = -1$ is certainly a good candidate for our purpose. We will show in each following section that (the absolute value of) this main term is always large enough to control the absolute value of other terms. To do this, we will frequently use properties of $u(\xi)$. Hence we prove the following technical lemma:

Lemma 2.2 *The function u given by*

$$u(\xi) = \sum_{j\in\mathbb{Z}} \frac{(-1)^{jm}}{(2j\pi+\xi)^m}$$

is decreasing on $(0,\ \pi]$. *Consequently,*

$$g(\xi) = (2\pi - \xi)^m u(\xi) \tag{2.2.4}$$

is decreasing and nonnegative on $(0,\ \pi]$.

Proof: If m is an even positive integer, then

$$u(\xi) = \sum_{j\in\mathbb{Z}} \frac{1}{(2j\pi+\xi)^m} = \sum_{j=0}^{\infty} \left(\frac{1}{(2j\pi+\xi)^m} + \frac{1}{(2(j+1)\pi-\xi)^m} \right).$$

It follows that

$$u'(\xi) = -m \sum_{j=0}^{\infty} \left(\frac{1}{(2j\pi+\xi)^{m+1}} - \frac{1}{(2(j+1)\pi-\xi)^{m+1}} \right).$$

Clearly, $u'(\xi) < 0$ if $0 < \xi < \pi$ and $u'(\xi) = 0$ if $\xi = \pi$. Consequently, $u(\xi)$ is decreasing on $(0, \pi]$.

Assume that m is an odd positive integer. Note that the series is absolutely convergent for $\xi \in (0, \pi]$. Hence

$$u(\xi) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{(2j\pi+\xi)^m} + \frac{1}{(2(j+1)\pi-\xi)^m} \right),$$

and

$$u'(\xi) = -m \left(\frac{1}{\xi^{m+1}} + \sum_{j=1}^{\infty} (-1)^j \left(\frac{1}{(2j\pi-\xi)^{m+1}} + \frac{1}{(2j\pi+\xi)^{m+1}} \right) \right).$$

Therefore, for $0 < \xi < \pi$

$$\begin{aligned} u''(\xi) &= m(m+1) \left(\sum_{j=0}^{\infty} (-1)^{j+1} \left(\frac{1}{(2j\pi+\xi)^m} + \frac{1}{(2(j+1)\pi-\xi)^m} \right) \right) \\ &= m(m+1) \sum_{j=0}^{\infty} \left(\frac{1}{(2j\pi+\xi)^m} - \frac{1}{(2(j+1)\pi+\xi)^m} \right) \\ &\quad + m(m+1) \sum_{j=1}^{\infty} \left(\frac{1}{(2j\pi-\xi)^m} - \frac{1}{(2(j+1)\pi-\xi)^m} \right) > 0. \end{aligned}$$

Moreover, it is easily seen that $u'(\pi) = 0$. It follows immediately that $u'(\xi) < 0$ for $0 < \xi < \pi$ and consequently, $u(\xi)$ is decreasing. ■

2.3 The case $0 < x < 1$

Following the notation in (2.2.3), we shall prove that $I > 0$ for $0 < x < 1$ in this section. For this purpose, write

$$I = \frac{2\sin(\pi x)}{\pi x}(I_1 + I_2) \tag{2.3.1}$$

where

$$I_1 := \int_0^\pi \frac{\sin\big(x(\pi - \xi)\big)}{(2\pi - \xi)^m u(\xi)}\, d\xi = \int_0^\pi \frac{\sin\big(x(\pi - \xi)\big)}{g(\xi)}\, d\xi \tag{2.3.2}$$

with $g(\xi)$ in (2.2.4) and

$$I_2 := \sum_{k \in \mathbf{Z}\setminus\{0,-1\}} \frac{(-1)^{km}\sin(k\pi x)}{\sin(\pi x)} \int_0^\pi \frac{\sin\big(x(k\pi + \xi)\big)}{(2k\pi + \xi)^m u(\xi)}\, d\xi. \tag{2.3.3}$$

Note that if $0 < x < 1$, then $\sin\big(x(\pi - \xi)\big) \geq 0$ for $0 \leq \xi \leq \pi$. Moreover, by Lemma 2.2, $g(\xi) \geq 0$. Then I_1 is positive. We shall prove $I_1 > |I_2|$, under the restriction $0 < x < 1$, and thereby, combining (2.1.9), the theorem follows immediately.

We first investigate I_1. To estimate a lower bound for I_1, an upper bound for $g(\xi)$, $0 \leq \xi \leq \pi$, should be considered. From Lemma 2.2 this is the case that ξ approaches the lower boundary $t\pi$ of the interval of the integration

$$\int_{t\pi}^\pi \frac{\sin\big(x(\pi - \xi)\big)}{g(\xi)}\, d\xi$$

where $0 < t < 1$. It should be pointed out that $g(\xi)$ approaches to infinity as $\xi \to 0$. Therefore t should be chosen carefully so that an appropriate lower bound for I_1 is obtained.

Let t be a real number such that $\frac{1}{2} < t < 1$. Then

$$I_1 \geq \int_{t\pi}^{\pi} \frac{\sin\bigl(x(\pi-\xi)\bigr)}{g(\xi)}\, d\xi.$$

For $t\pi \leq \xi \leq \pi$ and $m \geq 3$,

$$\begin{aligned} g(\xi) &\leq 1 + \left(\frac{2\pi-\xi}{\xi}\right)^m + \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} \left|\frac{2\pi-\xi}{\xi+2k\pi}\right|^m \\ &\leq \left(\frac{2-t}{t}\right)^m + 1 + \sum_{k=1}^{\infty}\left(\frac{2-t}{2k+t}\right)^m + \sum_{k=2}^{\infty}\left(\frac{2-t}{2k-t}\right)^m \\ &< 2 + \left(\frac{2-t}{t}\right)^m. \end{aligned}$$

It follows that

$$I_1 \geq \frac{1}{2+\left(\frac{2-t}{t}\right)^m} \int_{t\pi}^{\pi} \sin\bigl(x(\pi-\xi)\bigr)\, d\xi.$$

But

$$\int_{t\pi}^{\pi} \sin\bigl(x(\pi-\xi)\bigr)\, d\xi = \frac{2}{x}\left(\sin\left(\frac{\pi x(1-t)}{2}\right)\right)^2 \geq 2(1-t)^2 x.$$

Hence

$$I_1 \geq \frac{2(1-t)^2 x}{2+\left(\frac{2-t}{t}\right)^m}.$$

Choosing $t = 7/8$ in the above estimation, we obtain a lower bound for I_1:

$$I_1 \geq \frac{x}{32\bigl(2+(9/7)^m\bigr)}. \tag{2.3.4}$$

Next, let us find an upper bound for $|I_2|$. Since

$$\bigl|\sin(k\pi x)\bigr| \leq |k| \sin(\pi x), \qquad \text{for} \quad 0 < x < 1,$$

we have

$$|I_2| \leq \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} |k| \int_0^{\pi} \frac{\bigl|\sin\bigl(x(\xi+k\pi)\bigr)\bigr|}{|\xi+2k\pi|^m u(\xi)}\, d\xi.$$

Thus an upper bound for $|I_2|$ will be obtained once a lower bound for $|\xi+2k\pi|^m u(\xi)$, $0<\xi<\pi$, is provided. But either m being odd or even, we always have

$$\begin{aligned} u(\xi) &= \sum_{j\in\mathbb{Z}} \frac{(-1)^{jm}}{(\xi+2j\pi)^m} \\ &\geq -\frac{1}{(4\pi-\xi)^m} + \frac{1}{(2\pi-\xi)^m} + \frac{1}{\xi^m} - \frac{1}{(\xi+2\pi)^m} \\ &\geq \frac{1}{\xi^m}\left(1-\left(\frac{\xi}{4\pi-\xi}\right)^m\right) \\ &\geq \frac{1}{\xi^m}\left(1-\frac{1}{3^m}\right) \qquad \text{for} \quad 0<\xi<\pi. \end{aligned}$$

Then it follows that, for $0<\xi<\pi$,

$$\begin{aligned} |\xi+2k\pi|^m u(\xi) &\geq \left|\frac{\xi+2k\pi}{\xi}\right|^m \left(1-\frac{1}{3^m}\right) \\ &\geq |2k+1|^m\left(1-\frac{1}{3^m}\right) \qquad \text{for} \quad k\in\mathbb{Z}. \end{aligned} \tag{2.3.5}$$

Moreover, for $0<\xi\leq\pi$,

$$\left|\sin\big(x(k\pi+\xi)\big)\right| \leq x|k\pi+\xi| \leq x(|k|+1)\pi.$$

Hence

$$|I_2| \leq \frac{3^m\pi^2 x}{3^m-1}\left(\sum_{k=1}^{\infty}\frac{k(k+1)}{(2k+1)^m} + \sum_{k=2}^{\infty}\frac{k(k+1)}{(2k-1)^m}\right).$$

But note that $\sum_{k=3}^{\infty}\frac{2k^2}{(2k-1)^m} < \sum_{k=3}^{\infty}\frac{1}{(2k-1)^{m-2}} < \frac{1}{3^m}$ for $m\geq 10$. Hence

$$\sum_{k=1}^{\infty}\frac{k(k+1)}{(2k+1)^m} + \sum_{k=2}^{\infty}\frac{k(k+1)}{(2k-1)^m} = \frac{8}{3^m} + \sum_{k=3}^{\infty}\frac{2k^2}{(2k-1)^m} < \frac{1}{3^{m-2}}.$$

Therefore we obtained an upper bound for $|I_2|$

$$|I_2| \leq \frac{9\pi^2 x}{3^m-1}. \tag{2.3.6}$$

Combining (2.3.4) and (2.3.6), one can easily verify that for $m \geq 10$,

$$\frac{1}{32\left(2+(9/7)^m\right)} \geq \frac{9\pi^2}{3^m - 1}$$

and we conclude that

$$\frac{\sin(\pi x)}{\pi x} - \psi_m(x) > 0$$

for $0 < x < 1$ and $m \geq 10$. ■

2.4 The case $1 < x < 8$

In this section we will extend our theorem to the case $1 < x < 8$.

First of all, Lemma 2.3 below suggests that new techniques should be created to cope with an issue arisen from the alternation of the sign of $\sin\big(x(\pi - \xi)\big)$ because of $x > 1$. On the other hand, $1 < x < 8$ is investigated individually because most of techniques established in the previous section can be used in this case. We still take, for instance, the interval $[7\pi/8,\ \pi]$ to estimate a lower bound for I_1. But more importantly, one might find that the condition $x > 8$ will help us to obtain better estimates in the next section (see (2.5.9) and (2.5.10)). Briefly, this section plays a role like a bumper so that our estimates either for I_1 or I_2 wouldn't jump sharply as x across the turning point 1.

Lemma 2.3 *For $x > 1$,*

$$I_1 = \int_0^{\pi} \frac{\sin\left(x(\pi-\xi)\right)}{g(\xi)}\, d\xi \geq \int_{\pi-\frac{\pi}{x}}^{\pi} \sin\left(x(\pi-\xi)\right)\left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)}\right) d\xi.$$

Consequently, for $1 < x < 8$,

$$I_1 \geq \int_{\frac{7\pi}{8}}^{\pi} \sin\left(x(\pi - \xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi.$$

Proof: For $1 < x < 2$, we have $\frac{\pi}{x} > \pi - \frac{\pi}{x}$. Then

$$\begin{aligned} I_1 &= \left(\int_{\frac{\pi}{x}}^{\pi} + \int_{\pi-\frac{\pi}{x}}^{\frac{\pi}{x}} + \int_0^{\pi-\frac{\pi}{x}} \right) \frac{\sin\left(x(\pi-\xi)\right)}{g(\xi)}\, d\xi \\ &= \int_{\frac{\pi}{x}}^{\pi} \sin\left(x(\pi-\xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi \\ &\quad + \int_{\pi-\frac{\pi}{x}}^{\frac{\pi}{x}} \frac{\sin\left(x(\pi-\xi)\right)}{g(\xi)}\, d\xi \\ &\geq \int_{\pi-\frac{\pi}{x}}^{\pi} \sin(x(\pi-\xi)) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi. \end{aligned}$$

For $2 < x < 8$, (i.e., $2j < x < 2(j+1)$, $j = 1, 2, 3$)

$$\begin{aligned} I_1 &= \left(\int_{\pi-\frac{2\pi}{x}}^{\pi} + \cdots + \int_0^{\pi-\frac{2j\pi}{x}} \right) \frac{\sin\left(x(\pi-\xi)\right)}{g(\xi)}\, d\xi \\ &\geq \int_{\pi-\frac{2\pi}{x}}^{\pi} \frac{\sin\left(x(\pi-\xi)\right)}{g(\xi)}\, d\xi \\ &= \int_{\pi-\frac{\pi}{x}}^{\pi} \sin\left(x(\pi-\xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi. \quad \blacksquare \end{aligned}$$

Remark: This lemma will be also used in Chapter 3 and we will restate it for the reader's convenience (see §3.4, Lemma 3.14).

To estimate a lower bound for I_1, Lemma 2.3 motivates us to investigate the difference between $g(\xi)$ and $g(\xi - \pi/x)$. To this end, we have

Lemma 2.4 *For* $\pi/2 \leq \xi \leq \pi$ *and* $m \geq 3$,

$$\left(\frac{2\pi - \xi}{\xi} \right)^m \leq g(\xi) \leq 2 + \left(\frac{2\pi - \xi}{\xi} \right)^m.$$

Proof: Write

$$g(\xi) = 1 + \left(\frac{2\pi - \xi}{\xi}\right)^m + \sum_{j \in \mathbf{Z}\setminus\{0,-1\}} (-1)^{jm} \left(\frac{2\pi - \xi}{2j\pi + \xi}\right)^m.$$

Observe that

$$\begin{aligned}
\left|\sum_{j \in \mathbf{Z}\setminus\{0,-1\}} (-1)^{jm} \left(\frac{2\pi - \xi}{2j\pi + \xi}\right)^m\right| &\le \sum_{j \in \mathbf{Z}\setminus\{0,-1\}} \left|\frac{2\pi - \xi}{2j\pi + \xi}\right|^m \\
&= \sum_{j=1}^{\infty} \left(\frac{2\pi - \xi}{2j\pi + \xi}\right)^m + \sum_{j=2}^{\infty} \left(\frac{2\pi - \xi}{2j\pi - \xi}\right)^m \\
&\le \sum_{j=1}^{\infty} \left(\frac{2\pi - \frac{\pi}{2}}{2j\pi + \frac{\pi}{2}}\right)^m + \sum_{j=2}^{\infty} \left(\frac{2\pi - \frac{\pi}{2}}{2j\pi - \frac{\pi}{2}}\right)^m \\
&= 3^m \left(\frac{1}{5^m} + \frac{1}{7^m} + \cdots\right) \\
&< 1 \quad \text{for} \quad \frac{\pi}{2} \le \xi \le \pi \quad \text{and} \quad m \ge 3. \qquad \blacksquare
\end{aligned}$$

Lemma 2.5 *For* $7\pi/8 \le \xi \le \pi$ *and* $m \ge 10$

$$g\left(\xi - \frac{\pi}{8}\right) \ge 3g(\xi).$$

Proof: By Lemma 2.4, for $7\pi/8 \le \xi \le \pi$ we have

$$g\left(\xi - \frac{\pi}{8}\right) \ge \left(\frac{2\pi - \xi + \pi/8}{\xi - \pi/8}\right)^m \qquad \text{and} \qquad g(\xi) \le 2 + \left(\frac{2\pi - \xi}{\xi}\right)^m.$$

Thus, to prove $g\left(\xi - \frac{\pi}{8}\right) \ge 3g(\xi)$, it suffices to show that

$$\left(\frac{2\pi - \xi + \pi/8}{\xi - \pi/8}\right)^m \ge 6 + 3\left(\frac{2\pi - \xi}{\xi}\right)^m,$$

in other words,

$$\left(\frac{\xi}{2\pi - \xi}\right)^m \left(\frac{2\pi - \xi + \pi/8}{\xi - \pi/8}\right)^m \ge 6\left(\frac{\xi}{2\pi - \xi}\right)^m + 3.$$

But

$$\frac{\xi}{\xi-\pi/8} \geq \frac{\pi}{\pi-\pi/8} = \frac{8}{7} \quad \text{and} \quad \frac{2\pi-\xi+\pi/8}{2\pi-\xi} \geq \frac{2\pi-7\pi/8+\pi/8}{2\pi-7\pi/8} = \frac{10}{9}.$$

Note that $(80/63)^m > 9$ for $m \geq 10$, we obtain the desired result. ■

We now turn to the estimations of I_1. We have, for $1 < x < 8$,

$$I_1 \geq \int_{\frac{7\pi}{8}}^{\pi} \sin\big(x(\pi-\xi)\big)\left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi-\frac{\pi}{x}\right)}\right) d\xi.$$

Note that $g(\xi) \leq 2 + \left(\frac{9}{7}\right)^m$ for $\frac{7\pi}{8} \leq \xi \leq \pi$. Then it follows from Lemma 2.5 that

$$\begin{aligned} I_1 &\geq \int_{\frac{7\pi}{8}}^{\pi} \sin\big(x(\pi-\xi)\big)\left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi-\frac{\pi}{x}\right)}\right) d\xi \\ &\geq \frac{2}{3}\int_{\frac{7\pi}{8}}^{\pi} \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi \\ &\geq \frac{2}{3(2+(9/7)^m)}\int_{\frac{7\pi}{8}}^{\pi} \sin\big(x(\pi-\xi)\big)\, d\xi. \end{aligned}$$

But for $1 < x < 8$,

$$\int_{\frac{7\pi}{8}}^{\pi} \sin\big(x(\pi-\xi)\big)\, d\xi = \frac{2\sin^2(\pi x/16)}{x} \geq \frac{2\sin(\pi x/16)\frac{2}{\pi}(\pi x/16)}{x} \geq \frac{\sin(\pi/16)}{4}$$

Hence a lower bound for I_1 is given as

$$I_1 = \int_0^{\pi} \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi \geq \frac{\sin(\pi/16)}{6(2+(9/7)^m)}. \tag{2.4.1}$$

To estimate I_2, we observe that

$$\left|\sin(k\pi x)\right| \leq |k|\left|\sin(\pi x)\right| \qquad \text{and} \qquad \left|\sin(k\pi x + x\xi)\right| \leq 1.$$

Moreover, from (2.3.5),

$$|\xi + 2k\pi|^m u(\xi) \geq |2k+1|^m\left(1-\frac{1}{3^m}\right) \qquad \text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad 0 < \xi \leq \pi.$$

Hence

$$|I_2| = \left| \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} \frac{(-1)^{km}\sin(k\pi x)}{\sin(\pi x)} \int_0^{\pi} \frac{\sin(x(k\pi+\xi))}{(\xi+2k\pi)^m u(\xi)}\, d\xi \right|$$

$$\leq \frac{\pi}{1-(1/3)^m} \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} \frac{|k|}{|2k+1|^m}.$$

But

$$\sum_{k\in\mathbf{Z}\setminus\{0,-1\}} \frac{|k|}{|2k+1|^m} = \sum_{k=1}^{\infty} \frac{k}{(2k+1)^m} + \sum_{k=2}^{\infty} \frac{k}{(2k-1)^m}$$

$$= \sum_{k=2}^{\infty} \frac{2k-1}{(2k-1)^m} = \frac{1}{3^{m-1}} + \frac{1}{5^{m-1}} + \cdots$$

$$< \frac{1}{3^{m-1}} + \frac{1}{3^m} = \frac{4}{3^m} \qquad \text{for} \quad m \geq 3.$$

This shows that

$$|I_2| \leq \frac{4\pi}{3^m - 1}. \tag{2.4.2}$$

Combining (2.4.1) and (2.4.2), we have, for $m \geq 9$,

$$\frac{\sin(\pi/16)}{6\big(2+(9/7)^m\big)} > \frac{4\pi}{3^m-1}.$$

Therefore, we conclude that

$$0 < (-1)^j\, \psi_m(x) < (-1)^j\, \frac{\sin(\pi x)}{\pi x}$$

for $x \in (j,\ j+1), \quad j = 1, \cdots, 7$ and $m \geq 10$. ■

2.5 The case $x > 8$

The idea used in the previous sections will be applied again in this section. In other words, we will show that I_1 dominates the remainders.

One might note that in §2.3 and §2.4 we only chose the interval $[7\pi/8,\ \pi]$, instead of $[0,\ \pi]$, to be the interval of integration of I_1 to estimate the lower bounds for I_1 because this interval contributes the most to the value of I_1. This technique, modified slightly, will be still applied in the case $x > 8$. More precisely, for a fixed x, let $n = \lfloor \frac{x}{4} \rfloor$ and $1 \le j \le n$. Define

$$\begin{aligned} I_1^j &:= \left(\int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} + \int_{\pi-2j\pi/x}^{\pi-(2j-1)\pi/x} \right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi \\ &= \int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\big(x(\pi-\xi)\big) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi. \end{aligned} \tag{2.5.1}$$

Lemma 2.6 *I_1^j is positive for $1 \le j \le n$.*

Proof: In fact, since

$$\pi - \frac{(2j-1)\pi}{x} < \xi < \pi - \frac{(2j-2)\pi}{x},$$

then $(2j-2)\pi < x(\pi-\xi) < (2j-1)\pi$, and hence $\sin\big(x(\pi-\xi)\big) > 0$. The desired result follows immediately by applying Lemma 2.2 in §2.2. ■

Lemma 2.7 *I_1 is positive for $x > 1$.*

Proof: There are two posibilities.

Case 1. If $x = 2N + t$, $N \in \mathbb{N}$ and $0 < t < 1$, then

$$I_1 = \left(\int_{\pi-\frac{\pi}{2N+t}}^{\pi} + \cdots + \int_{\pi-\frac{2N\pi}{2N+t}}^{\pi-\frac{(2N-1)\pi}{2N+t}} \right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi + \int_0^{\pi-\frac{2N\pi}{2N+t}} \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi.$$

The first N pairs of integrals must be positive by Lemma 2.6. Moreover, if

$$0 < \xi < \pi - \frac{2N\pi}{2N+t},$$

then $2N\pi < (2N+t)(\pi-\xi) < (2N+t)\pi$ which implies that the last integral is also positive.

Case 2. If $x = 2N+1+t$, then

$$I_1 = \left(\int_{\pi-\frac{\pi}{2N+1+t}}^{\pi} + \cdots + \int_{\pi-\frac{2N\pi}{2N+1+t}}^{\pi-\frac{(2N-1)\pi}{2N+1+t}}\right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi$$
$$+ \left(\int_{\pi-\frac{(2N+1)\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} + \int_0^{\pi-\frac{(2N+1)\pi}{2N+1+t}}\right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi.$$

Again, the first N pairs of integrals are positive. To see that the sum of the last two integrals is positive, noting that

$$\pi - \frac{(2N+1)\pi}{2N+1+t} < \frac{\pi}{2N+1+t} < \pi - \frac{2N\pi}{2N+1+t},$$

and if

$$\pi - \frac{(2N+1)\pi}{2N+1+t} < \xi < \pi - \frac{2N\pi}{2N+1+t},$$

we have $\sin\big(x(\pi-\xi)\big) > 0$, then

$$\int_{\pi-\frac{(2N+1)\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi \ge \int_{\frac{\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi.$$

Therefore

$$\left(\int_{\pi-\frac{(2N+1)\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} + \int_0^{\pi-\frac{(2N+1)\pi}{2N+1+t}}\right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi$$
$$\ge \left(\int_{\frac{\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} + \int_0^{\pi-\frac{(2N+1)\pi}{2N+1+t}}\right) \frac{\sin\big(x(\pi-\xi)\big)}{g(\xi)}\, d\xi$$
$$\ge \int_{\frac{\pi}{2N+1+t}}^{\pi-\frac{2N\pi}{2N+1+t}} \sin\big(x(\pi-\xi)\big) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi-\frac{\pi}{x}\right)}\right) d\xi$$
$$> 0. \quad \blacksquare$$

Combining Lemma 2.6 and Lemma 2.7, we have actually proved,

Corollary 2.8 *For* $1 \le j \le n,\ n = \lfloor \frac{x}{4} \rfloor$,

$$I_1 \ge \sum_{j=1}^{n} \int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\left(x(\pi-\xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi.$$

Consequently, $I_1 \ge \sum_{j=1}^{n} I_1^j$. ■

Corollary 2.9 *For* $x > 1$. *Write*

$$I_1 = I_{11} + I_{12}$$

where

$$I_{11} := \int_{\pi-\frac{\pi}{x}}^{\pi} \sin\left(x(\pi-\xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi$$

and

$$I_{12} := \int_{0}^{\pi-\frac{\pi}{x}} \sin\left(x(\pi-\xi)\right) \left(\frac{1}{g(\xi)} - \frac{1}{g\left(\xi - \frac{\pi}{x}\right)} \right) d\xi.$$

Then $I_{11} > 0$ *and* $I_{12} < 0$. ■

Remark: Corollary 2.9 will be employed in Chapter 3 only.

We now begin to prove our main theorem for $8 < x < \infty$ in four steps.

Step 1. A lower bound for I_1^j where I_1^j is given in (2.5.1).

With the help of the mean-value theorem we now estimate a lower bound for I_1^j for $j = 1, 2, \cdots, n$. One can easily calculate

$$\left(\frac{1}{g(\xi)} \right)' = \frac{2m\pi}{(2\pi-\xi)^{m+1}\left(u(\xi)^2\right)} \sum_{j\in\mathbf{Z}} (-1)^{jm} \frac{j+1}{(2j\pi+\xi)^{m+1}}.$$

Then a lower bound for $\left(\frac{1}{g}\right)'$ will be obtained once we estimate an upper bound for $u(\xi)$ (recall that $u(\xi) > 0$ for $0 < \xi < \pi$) and a lower bound for $\sum_{k\in\mathbf{Z}}(-1)^{jm} \frac{j+1}{(2j\pi+\xi)^{m+1}}$ respectively.

At first, we claim that for $m \geq 11$ and $\xi \in (0, \pi]$,

$$u(\xi) \leq \frac{1}{\xi^m}\left(2 + \frac{1}{3^{10}}\right).$$

In fact, if m is even

$$\begin{aligned}
u(\xi) &= \sum_{j\in\mathbf{Z}} \frac{1}{(2j\pi+\xi)^m} \\
&= \frac{1}{\xi^m}\left(1 + \left(\frac{\xi}{2\pi-\xi}\right)^m + \sum_{j\in\mathbf{Z}\setminus\{0,-1\}} \left(\frac{\xi}{2j\pi+\xi}\right)^m\right) \\
&\leq \frac{1}{\xi^m}\left(1 + 1 + \frac{1}{3^m} + \frac{1}{3^m} + \frac{1}{5^m} + \frac{1}{5^m} + \cdots\right) \\
&\leq \frac{1}{\xi^m}\left(2 + \frac{1}{3^{10}}\right).
\end{aligned}$$

The case of m odd is much easier and we omit it here. On the other hand, we observe that for $\xi \in (0, \pi]$,

$$\sum_{j\in\mathbf{Z}}(-1)^{jm} \frac{j+1}{(2j\pi+\xi)^{m+1}} \geq \frac{1}{\xi^{m+1}}\left(1 - \frac{1}{3^m}\right).$$

Indeed, if m is either even or odd, we always have

$$\begin{aligned}
\sum_{j\in\mathbf{Z}}(-1)^{jm} \frac{j+1}{(2j\pi+\xi)^{m+1}} &\geq \frac{1}{\xi^{m+1}} - \frac{2}{(2\pi+\xi)^{m+1}} - \frac{1}{(4\pi-\xi)^{m+1}} \\
&= \frac{1}{\xi^{m+1}}\left(1 - 2\left(\frac{\xi}{2\pi+\xi}\right)^{m+1} - \left(\frac{\xi}{4\pi-\xi}\right)^{m+1}\right) \\
&\geq \frac{1}{\xi^{m+1}}\left(1 - \frac{1}{3^m}\right).
\end{aligned}$$

Then for $m \geq 11$ and $\xi \in (0, \pi]$, a lower bound for $\left(\frac{1}{g}\right)'$ is given by

$$\left(\frac{1}{g(\xi)}\right)' \geq \frac{2m\pi}{(2+1/3^{10})^2}\left(1-\frac{1}{3^m}\right)\frac{\xi^{m-1}}{(2\pi-\xi)^{m+1}}.$$

Consequently, (note that $\xi - \frac{\pi}{x} < \eta < \xi$):

$$\begin{aligned}
I_1^j &= \frac{\pi}{x}\int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\big(x(\pi-\xi)\big)\left(\frac{1}{g}\right)'(\eta)\, d\xi \\
&\geq \frac{2m\pi^2}{x(2+1/3^{10})^2}\left(1-\frac{1}{3^m}\right) \\
&\quad \times \int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\big(x(\pi-\xi)\big)\frac{\eta^{m-1}}{(2\pi-\eta)^{m+1}}\, d\xi \\
&\geq \frac{2m\pi^2}{x(2+1/3^{10})^2}\left(1-\frac{1}{3^m}\right)\frac{\left(\pi-\frac{(2j-1)\pi}{x}-\frac{\pi}{x}\right)^{m-1}}{\left(2\pi-\pi+\frac{(2j-1)\pi}{x}+\frac{\pi}{x}\right)^{m+1}} \\
&\quad \times \int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\big(x(\pi-\xi)\big)\, d\xi.
\end{aligned}$$

But

$$\int_{\pi-(2j-1)\pi/x}^{\pi-(2j-2)\pi/x} \sin\big(x(\pi-\xi)\big)\, d\xi = \frac{2}{x}.$$

Hence

$$\begin{aligned}
I_1^j &\geq \frac{2m\pi^2}{x(2+1/3^{10})^2}\left(1-\frac{1}{3^m}\right)\frac{x^2}{\pi^2}\frac{1}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1}\frac{2}{x} \\
&= \frac{4m}{(2+1/3^{10})^2}\left(1-\frac{1}{3^m}\right)\frac{1}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1}.
\end{aligned}$$

Then for $m \geq 10$, a lower bound for I_1^j is given as follows

$$\begin{aligned}
I_1^j &\geq \frac{4m}{(2+1/3^{10})^2}\left(1-\frac{1}{3^{10}}\right)\frac{1}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1} \\
&> \frac{0.9m}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1}.
\end{aligned} \qquad (2.5.2)$$

Step 2. An upper bound for $|I_2^{2j}| + |I_2^{2j-1}|$ where I_2^j is defined in (2.5.4) below.

Let us now estimate $|I_2|$. For $k \in \mathbb{Z} \setminus \{0, -1\}$, denote

$$g_k(\xi) = (2k\pi + \xi)^m \, u(\xi).$$

Then I_2 can be rewritten as

$$I_2 = \sum_{k \in \mathbb{Z} \setminus \{0,-1\}} \frac{(-1)^{km} \sin(k\pi x)}{\sin(\pi x)} \int_0^\pi \frac{\sin\big(x(k\pi + \xi)\big)}{g_k(\xi)} \, d\xi.$$

We will examine I_2 on the intervals $[0,\ \pi - \frac{4n\pi}{x}]$ and $[\pi - \frac{4n\pi}{x},\ \pi]$ respectively. Before proceeding further, a strategy should be stated. To cope with the new issue arisen from the alternation of sign of $\sin\big(x(\pi - \xi)\big)$, the main term I_1 has been split up I_1^j, $1 \le j \le n$ and (2.5.2) describes a lower bound for I_1^j's. Therefore, I_2 should be divided to many pieces as well so that each such piece is still controlled by the (corresponding) I_1^j. Consequently, $|I_2|$ would be dominated by I_1.

We first estimate $|I_2|$ on $[\pi - \frac{4n\pi}{x},\ \pi]$. Let $k \in \mathbb{Z} \setminus \{0, -1\}$ be fixed for the time being and let $1 \le l \le 2n$. Set

$$\begin{aligned} I_2^{k,l} &:= \left(\int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} + \int_{\pi-2l\pi/x}^{\pi-(2l-1)\pi/x} \right) \frac{\sin\big(x(k\pi + \xi)\big)}{g_k(\xi)} \, d\xi \\ &= \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \sin\big(x(k\pi + \xi)\big) \left(\frac{1}{g_k(\xi)} - \frac{1}{g_k\left(\xi - \frac{\pi}{x}\right)} \right) d\xi. \end{aligned} \tag{2.5.3}$$

Noting that $\left|\sin\big(x(\pi - \xi)\big)\right| \le 1$, we have

$$|I_2^{k,l}| \le \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \left| \left(\frac{1}{g_k(\xi)} - \frac{1}{g_k\left(\xi - \frac{\pi}{x}\right)} \right) \right| d\xi.$$

Following the same idea we estimated $|I_1^j|$, we will give an upper bound for $\left|I_2^{k,l}\right|$ with the aid of the mean-value theorem again. For this purpose, let us consider

$$\left(\frac{1}{g_k(\xi)} \right)' = -\frac{2m\pi}{(2k\pi + \xi)^{m+1} \big(u(\xi)\big)^2} \sum_{j \in \mathbb{Z}} (-1)^{jm} \, \frac{j - k}{(2j\pi + \xi)^{m+1}}.$$

Then an upper bound for $\left| \left(\frac{1}{g_k}\right)' \right|$ will be obtained as long as we estimate an upper bound for

$$\left|\sum_{j\in\mathbb{Z}}(-1)^{jm}\,\frac{j-k}{(2j\pi+\xi)^{m+1}}\right|.$$

As a matter of fact, we claim that, for $m \geq 11$ and $0 < \xi \leq \pi$,

$$\left|\sum_{j\in\mathbb{Z}}(-1)^{jm}\,\frac{j-k}{(2j\pi+\xi)^{m+1}}\right| \leq \frac{1}{\xi^{m+1}}\left(2|k|\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right).$$

Indeed,

$$\begin{aligned}
\left|\sum_{j\in\mathbb{Z}}(-1)^{jm}\,\frac{j-k}{(2j\pi+\xi)^{m+1}}\right| &\leq \sum_{j\in\mathbb{Z}}\frac{|j|}{|2j\pi+\xi|^{m+1}}+\sum_{j\in\mathbb{Z}}\frac{|k|}{|2j\pi+\xi|^{m+1}} \\
&= \sum_{j=0}^{\infty}\frac{|j|+|k|}{|2j\pi+\xi|^{m+1}}+\sum_{j=1}^{\infty}\frac{|j|+|k|}{|2j\pi-\xi|^{m+1}} \\
&\leq \frac{1}{\xi^{m+1}}\left(\sum_{j=0}^{\infty}(j+|k|)\left(\frac{1}{2j+1}\right)^{m+1}\right) \\
&\quad + \frac{1}{\xi^{m+1}}\left(\sum_{j=1}^{\infty}(j+|k|)\left(\frac{1}{2j-1}\right)^{m+1}\right) \\
&\leq \frac{1}{\xi^{m+1}}\left(2|k|\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right).
\end{aligned}$$

Moreover, we proved in §2.3 that $u(\xi) \geq \frac{1}{\xi^m}\left(1-\frac{1}{3^m}\right)$. Therefore we obtain for $m \geq 11$ and $0 < \xi \leq \pi$,

$$\left|\left(\frac{1}{g_k(\xi)}\right)\right|' \leq \frac{2m\pi}{(1-1/3^m)^2}\left(2|k|\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right)\frac{\xi^{m-1}}{|2k\pi+\xi|^{m+1}}.$$

Hence for $k \in \mathbb{Z}\setminus\{0,-1\}$ and $1 \leq l \leq 2n$, set

$$I_2^l = \sum_{k\in\mathbb{Z}\setminus\{0,-1\}} I_2^{k,l}, \tag{2.5.4}$$

and noting that $\left|\sin(k\pi x)\right| \le |k|\left|\sin(\pi x)\right|$. Then

$$
\begin{aligned}
|I_2^l| &\le \sum_{k\in\mathbb{Z}\setminus\{0,-1\}} |I_2^{k,l}| \\
&\le \frac{\pi}{x} \sum_{k\in\mathbb{Z}\setminus\{0,-1\}} |k| \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \left| \left(\frac{1}{g_k}\right)'(\eta) \right| d\xi \\
&\le \frac{2m\pi^2}{x(1-1/3^m)^2} \sum_{k\in\mathbb{Z}\setminus\{0,-1\}} |k| \left(2|k|\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \\
&\quad \times \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \frac{\eta^{m-1}}{|2k\pi+\eta|^{m+1}}\, d\xi .
\end{aligned}
$$

Writing

$$
\sum_{k\in\mathbb{Z}\setminus\{0,-1\}} = \sum_{k=1}^{\infty} + \sum_{k=2}^{\infty},
$$

we have

$$
\begin{aligned}
|I_2^l| \le{}& \frac{2m\pi^2}{x(1-1/3^m)^2} \sum_{k=1}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \frac{\eta^{m-1}}{(2k\pi+\eta)^{m+1}}\, d\xi \\
&+ \frac{2m\pi^2}{x(1-1/3^m)^2} \sum_{k=2}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \frac{\eta^{m-1}}{(2k\pi-\eta)^{m+1}}\, d\xi
\end{aligned}
$$

Note that both $\frac{\eta^{m-1}}{(2k\pi+\eta)^{m+1}}$ and $\frac{\eta^{m-1}}{(2k\pi-\eta)^{m+1}}$ are increasing functions of η. Moreover, since $\xi - \frac{\pi}{x} \le \eta \le \xi$, then

$$
\begin{aligned}
|I_2^l| \le{}& \frac{2m\pi^2}{x\left(1-\frac{1}{3^m}\right)^2} \sum_{k=1}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \frac{\xi^{m-1}}{(2k\pi+\xi)^{m+1}}\, d\xi \\
&+ \frac{2m\pi^2}{x\left(1-\frac{1}{3^m}\right)^2} \sum_{k=2}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \int_{\pi-(2l-1)\pi/x}^{\pi-(2l-2)\pi/x} \frac{\xi^{m-1}}{(2k\pi-\xi)^{m+1}}\, d\xi \\
\le{}& \frac{2m\pi}{\left(1-\frac{1}{3^m}\right)^2} \sum_{k=1}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \frac{(x-2l+2)^{m-1}}{((2k+1)x-2l+2)^{m+1}} \\
&+ \frac{2m\pi}{\left(1-\frac{1}{3^m}\right)^2} \sum_{k=2}^{\infty} k \left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) \frac{(x-2l+2)^{m-1}}{((2k-1)x+2l-2)^{m+1}} .
\end{aligned}
$$

The above estimate for $|I_2^l|$ should be simplified before it is employed to the next estimate. Obviously, only their first terms contribute the most to the serieses expressed above and one can verify that for $m \geq 11$, the remainder of the first series

$$\sum_{k=2}^{\infty} k\left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right)\frac{(x-2l+2)^{m-1}}{((2k+1)x-2l+2)^{m+1}} \leq \frac{(x-2l+2)^{m-1}}{(3x-2l+2)^{m+1}}$$

and the the remainder of the second series

$$\sum_{k=3}^{\infty} k\left(2k\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right)\frac{(x-2l+2)^{m-1}}{((2k-1)x+2l-2)^{m+1}} \leq \frac{(x-2l+2)^{m-1}}{(3x+2l-2)^{m+1}}.$$

Moreover, note that the coefficient of the first term (i.e. $k=1$) of the first series is

$$2\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}} < 3.00002$$

and the coefficient of the first term (i.e. $k=2$) of the second series is

$$2\left(4\left(1+\frac{2}{3^{12}}\right)+1+\frac{5}{3^{12}}\right) < 10.00005.$$

Then multiplied by the common factor $\frac{2m\pi}{(1-1/3^m)^2}$, we obtain an upper bound for $|I_2^l|$

$$\begin{aligned}
|I_2^l| &\leq \frac{2(3.00002+1)m\pi}{\left(1-\frac{1}{3^m}\right)^2}\frac{(x-2l+2)^{m-1}}{(3x-2l+2)^{m+1}} + \frac{2(10.00005+1)m\pi}{\left(1-\frac{1}{3^m}\right)^2}\frac{(x-2l+2)^{m-1}}{(3x+2l-2)^{m+1}} \\
&\leq \frac{2(3.00002+1)m\pi}{\left(1-\frac{1}{3^m}\right)^2}\frac{(x-2l+2)^{m-1}}{(3x-2l+2)^{m+1}} + \frac{2(10.00005+1)m\pi}{\left(1-\frac{1}{3^m}\right)^2}\frac{(x-2l+2)^{m-1}}{(3x-2l+2)^{m+1}} \\
&< \frac{94.25m}{\left(1-\frac{1}{3^m}\right)^2}\frac{(x-2l+2)^{m-1}}{(3x-2l+2)^{m+1}}.
\end{aligned}$$

For each j, $j = 1, 2, \cdots, n$, our goal is to use I_1^j to control two terms of $|I_2|$, namely, $|I_2^{2j-1}|$ and $|I_2^{2j}|$. For this purpose, replacing l by $2j-1$ and $2j$ in the above inequality, we obtain

$$|I_2^{2j}| + |I_2^{2j-1}| \leq \frac{94.25m}{\left(1-\frac{1}{3^m}\right)^2}\left(\frac{(x-4j+4)^{m-1}}{(3x-4j+4)^{m+1}} + \frac{(x-4j+2)^{m-1}}{(3x-4j+2)^{m+1}}\right).$$

But

$$\frac{(x-4j+4)^{m-1}}{(3x-4j+4)^{m+1}} \geq \frac{(x-4j+2)^{m-1}}{(3x-4j+2)^{m+1}}.$$

Then an upper bound for the sum of $|I_2^{2j}|$ and $|I_2^{2j-1}|$ is given by

$$|I_2^{2j}| + |I_2^{2j-1}| < \frac{188.5m}{\left(1-\frac{1}{3^m}\right)^2(3x-4j+2)^2}\left(\frac{x-4j+4}{3x-4j+4}\right)^{m-1} \tag{2.5.5}$$

Step 3. An upper bound for $|I_3^j|$ where $|I_3^j|$ is given in (2.5.6) below.

We now estimate I_2 on $[0,\ \pi - \frac{4n\pi}{x}]$. For $1 \leq j \leq n$, denote

$$I_3^{k,j} := \int_{\pi-\frac{4n\pi}{x}-jr}^{\pi-\frac{4n\pi}{x}-(j-1)r} \frac{\sin\big(x(k\pi+\xi)\big)}{g_k(\xi)}\, d\xi$$

where $r := \frac{\pi-4n\pi/x}{n} = \frac{\pi}{n} - \frac{4\pi}{x}$. Then

$$|I_3^{k,j}| \leq \int_{\pi-\frac{4n\pi}{x}-jr}^{\pi-\frac{4n\pi}{x}-(j-1)r} \frac{1}{|g_k(\xi)|}\, d\xi.$$

For each $1 \leq j \leq n$, set

$$I_3^j = \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} I_3^{k,j}. \tag{2.5.6}$$

Then

$$\begin{aligned}
|I_3^j| &\leq \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} |I_3^{k,j}| \\
&\leq \frac{1}{\left(1-\frac{1}{3^m}\right)} \sum_{k\in\mathbf{Z}\setminus\{0,-1\}} |k| \int_{\pi-\frac{4n\pi}{x}-jr}^{\pi-\frac{4n\pi}{x}-(j-1)r} \frac{\xi^m}{|2k\pi+\xi|^m}\, d\xi \\
&\leq \frac{1}{\left(1-\frac{1}{3^m}\right)} \sum_{k=1}^{\infty} k \int_{\pi-\frac{4n\pi}{x}-jr}^{\pi-\frac{4n\pi}{x}-(j-1)r} \left(\frac{\xi}{2k\pi+\xi}\right)^m d\xi \\
&\quad + \frac{1}{\left(1-\frac{1}{3^m}\right)} \sum_{k=2}^{\infty} k \int_{\pi-\frac{4n\pi}{x}-jr}^{\pi-\frac{4n\pi}{x}-(j-1)r} \left(\frac{\xi}{2k\pi-\xi}\right)^m d\xi.
\end{aligned}$$

Recall that $r = \frac{\pi}{n} - \frac{4n\pi}{x}$. Moreover for $k \geq 1$, $\frac{\xi}{2k\pi+\xi}$ and $\frac{\xi}{2k\pi-\xi}$ attain their maximal values at $\pi - \frac{4n\pi}{x} - (j-1)\left(\frac{\pi}{n} - \frac{4n\pi}{x}\right)$ in the above interval of integration. Then we have

$$\begin{aligned}
|I_3^j| &\leq \frac{r}{\left(1-\frac{1}{3^m}\right)} \sum_{k=1}^{\infty} k \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k+1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m \\
&\quad + \frac{r}{\left(1-\frac{1}{3^m}\right)} \sum_{k=2}^{\infty} k \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k-1+\frac{4n}{x}+(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m \\
&\leq \frac{r}{\left(1-\frac{1}{3^m}\right)} \sum_{k=1}^{\infty} k \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k+1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m \\
&\quad + \frac{r}{\left(1-\frac{1}{3^m}\right)} \sum_{k=2}^{\infty} k \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k-1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m \\
&= \frac{r}{\left(1-\frac{1}{3^m}\right)} \sum_{k=1}^{\infty} (2k+1) \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k+1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m .
\end{aligned}$$

Once again, one can verify that for $m \geq 11$,

$$\sum_{k=2}^{\infty} (2k+1) \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{2k+1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m \leq \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{3-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m .$$

Hence we obtain an upper bound for $|I_3^j|$

$$|I_3^j| \leq \frac{4r}{\left(1-\frac{1}{3^m}\right)} \left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{3-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)} \right)^m . \tag{2.5.7}$$

Step 4. $I_1 > |I_2|$.

We are now in a position to prove that I_1 dominates $|I_2|$. In fact it suffices to show $I_1^j > |I_2^{2j-1}| + |I_2^{2j}| + |I_3^j|$ for $1 \leq j \leq n$, $n = \lfloor \frac{x}{4} \rfloor$.

Combining (2.5.2) and (2.5.5), we first show that, for $m \geq 11$,

$$\frac{9}{10} I_1^j \geq |I_2^{2j}| + |I_2^{2j-1}| , \tag{2.5.8}$$

i.e.,

$$\frac{9}{10}\left(\frac{0.9m}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1}\right) \geq \frac{188.5m}{(1-\frac{1}{3^m})^2} \times \frac{1}{(3x-4j+2)^2}\left(\frac{x-4j+4}{3x-4j+4}\right)^{m-1}.$$

Observe that for $x > 8$ and $j = 1, 2, \cdots, n$,

$$\frac{0.9}{(x+2j)^2} \geq \frac{3}{2}\frac{1}{(1-\frac{1}{3^{11}})^2}\frac{1}{(3x+4j+2)^2}. \tag{2.5.9}$$

Indeed, let $c_1 = \frac{3}{2\times 0.9}\frac{1}{(1-1/3^{11})^2} \approx 1.6667...$ and

$$f(x) := (3x-4j+2)^2 - c_1(x+2j)^2.$$

Treating the right hand side of the above expression as a (quadratic) polynomial of j, one can easily calculate that the root of j is not in the domain of j, $1 \leq j \leq \lfloor\frac{x}{4}\rfloor$. Hence the extreme values of $f(x)$ are attained at the two endpoints, $j = 1$ and $j = n$. Substituting these two numbers into $f(x)$, we obtain, for $j = 1$,

$$f(x) = (9-c_1)x^2 - 4(3-c_1)x + 4(1-c_1)$$

which is positive as $x > 1.2$. For $j = x/4$,

$$f(x) = \left(4 - \frac{9c_1}{4}\right)x^2 + 8x + 4$$

which is positive for any positive x. Consequently, (2.5.9) is valid.

Next we claim that for $m \geq 11$,

$$\frac{9}{10}\left(\frac{x-2j}{x+2j}\right)^{m-1} \geq 188.5\left(\frac{2}{3}\left(\frac{x-4j+4}{3x-4j+4}\right)^{m-1}\right). \tag{2.5.10}$$

In fact, let $c_2 = \left(\frac{10\times 188.5\times 2}{3\times 9}\right)^{\frac{1}{10}} \approx 1.63869...$ and let

$$f_1(x) := (3x - 4j + 4)(x - 2j) - c_2(x + 2j)(x - 4j + 4),$$

then the root of j, $\frac{10x - 2c_2x + 8 + 8c_2}{16 + 16c_2}$, is in $[1,\ n]$. Substituting it into $f_1(x)$ we obtain

$$f_1(x) = 0.82608x^2 - 5.91608x - 5.27738$$

and this quadratic polynomial is positive if $x > 8$. Therefore (2.5.10) is valid. In light of (2.5.9) and (2.5.10), (2.5.8) follows immediately.

Finally, by (2.5.2) and (2.5.7), we show

$$\frac{1}{10} I_1^j \geq |I_3^j|, \tag{2.5.11}$$

i.e.,

$$\frac{1}{10}\left(\frac{0.9m}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1}\right) \geq \left(\frac{4r}{1-\frac{1}{3^m}}\right)\left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{3-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}\right)^m.$$

Since $n = \lfloor \frac{x}{4} \rfloor$, we write $\frac{x}{4} = n + t$ and assume $0 < t < \frac{1}{2}$ for the time being. Then

$$1 - \frac{4n}{x} - (j-1)\left(\frac{1}{n} - \frac{4}{x}\right) = \frac{t}{n+t}\left(1 - \frac{j-1}{n}\right).$$

Hence

$$\left(\frac{1-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}{3-\frac{4n}{x}-(j-1)\left(\frac{1}{n}-\frac{4}{x}\right)}\right)^m = \left(\frac{\frac{t}{n+t}\left(1-\frac{j-1}{n}\right)}{2+\frac{t}{n+t}\left(1-\frac{j-1}{n}\right)}\right)^m.$$

Note that $0 < t < \frac{1}{2}$ and $x > 8$. Then

$$\frac{\frac{t}{n+t}\left(1-\frac{j-1}{n}\right)}{2+\frac{t}{n+t}\left(1-\frac{j-1}{n}\right)} < \frac{1}{x} < \frac{1}{8}. \tag{2.5.12}$$

On the other hand, write

$$\frac{0.9}{(x+2j)^2}\left(\frac{x-2j}{x+2j}\right)^{m-1} = \frac{0.9}{(x-2j)(x+2j)}\left(\frac{x-2j}{x+2j}\right)^{m}$$

and note that $r = \frac{\pi}{n} - \frac{4\pi}{x} = \frac{\pi t}{n(n+t)}$. We observe that

$$(x-2j)(x+2j)r = (x^2-4j^2)\,\frac{16\pi t}{(x-4t)x} < \frac{16\pi(x^2-4)}{(x-4)x}.$$

Since the maximal value of $\frac{x^2-4}{(x-4)x}$ is $15/8$ as $x > 8$, hence

$$(x-2j)(x+2j)r < 30\pi. \tag{2.5.13}$$

Moreover, for $m \geq 11$, $\frac{1}{1-1/3^m} \leq \frac{1}{1-1/3^{11}}$. Therefore, to show (2.5.11), in light of (2.5.12) and (2.5.13), it suffices to show

$$\left(\frac{x-2j}{x+2j}\right)^m > \left(\frac{30\pi \times 10 \times 4}{0.9 \times (1-\frac{1}{3^{11}})^2 \times m}\right)\left(\frac{1}{8}\right)^m. \tag{2.5.14}$$

But for $m \geq 11$,

$$\left(\frac{30\pi \times 10 \times 4}{0.9 \times (1-\frac{1}{3^{11}})^2 \times m}\right)^{\frac{1}{m}} < 1.8\,.$$

Our desired result then follows from

$$\frac{x-2j}{x+2j} > \frac{1}{3} > 1.8\left(\frac{1}{8}\right).$$

Consequently, (2.5.11) is valid. Therefore, combining (2.5.8) and (2.5.11), we have proved that for $m \geq 11$, $x > 8$ and $1 \leq j \leq n = \lfloor \frac{x}{4} \rfloor$,

$$I_1^j > |I_2^{2j}| + I_2^{2j-1}| + |I_3^j|.$$

Remark: One might observe that our proof above is based on the assumption of $0 < t < \frac{1}{2}$. In fact the same proof works for $\frac{1}{2} < t < 1$. To see this, if this is the case, we choose $n = \lceil \frac{x}{4} \rceil$ such that $\frac{x}{4} = n - s$ where $0 < s < \frac{1}{2}$. We then reset the domain of l in (2.5.3) to be $1 \leq l \leq 2n - 1$ and hence we have the exact same upper bound of $|I_2^l|$ as in (2.5.5) except that the interval $\left[\pi - \frac{4n\pi}{x},\ \pi\right]$ is replaced by $\left[\pi - \frac{(4n-2)\pi}{x},\ \pi\right]$. Subsequently, we should choose $r = \frac{\pi - (4n-2)\pi/x}{n}$. Therefore, (2.5.7) (the estimation of $|I_3^j|$ on $\left[0,\ \pi - \frac{(4n-2)\pi}{x}\right]$, $1 \leq j \leq n$) is replaced by

$$|I_3^j| < \left(\frac{4r}{1 - \frac{1}{3^m}}\right)\left(\frac{1 - \frac{4n-2}{x} - (j-1)\left(\frac{1}{n} - \frac{4n-2}{nx}\right)}{3 - \frac{4n-2}{x} - (j-1)\left(\frac{1}{n} - \frac{4n-2}{nx}\right)}\right)^m. \tag{2.5.15}$$

Note that $\frac{x}{4} = n - s$. We then obtain

$$1 - \frac{(4n-2)}{x} - (j-1)\left(\frac{1}{n} - \frac{(4n-2)}{nx}\right) = \frac{2-4s}{4(n-s)}\left(1 - \frac{j-1}{n}\right).$$

Hence we still have

$$\frac{\frac{2-4s}{4(n-s)}\left(1 - \frac{j-1}{n}\right)}{2 + \frac{2-4s}{4(n-s)}\left(1 - \frac{j-1}{n}\right)} \leq \frac{1-2s}{4(n-s)} < \frac{1}{x}$$

which is the key step in proving (2.5.12). Therefore we have showed that I_1 dominates $|I_2|$ and the proof of the main theorem is complete. ■

Chapter 3

Cardinal Interpolation by the DD-functions

3.1 Some basic properties of the DD-functions

In this chapter we shall investigate properties of fundamental and refinable functions on $\mathbb{R}$ with compact support. Recall that a function φ is said to be **fundamental** if it is continuous and satisfies

$$\varphi(j) = \delta(j), \quad \text{for } j \in \mathbb{Z},$$

where $\delta(0) = 1$ and $\delta(j) = 0$ for all $j \in \mathbb{Z} \setminus \{0\}$. With the help of the Poisson summation formula, we see that a compactly supported continuous function φ is fundamental if and only if

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2k\pi) = 1, \qquad \forall\, \xi \in \mathbb{R}. \tag{3.1.1}$$

A function φ is said to be **refinable** if it satisfies the following **refinement equation**

$$\varphi = \sum_{j\in\mathbb{Z}} a(j)\varphi(2\cdot -j), \tag{3.1.2}$$

where a is a finitely supported sequence on $\mathbb{Z}$, called the **refinement mask**. If $\sum_{j\in\mathbb{Z}} a(j) = 2$, it was proved by Cavaretta, Dahmen and Micchelli (see [2]) that there exists a unique compactly supported distributional solution φ to the refinement equation (3.1.2) such that $\hat{\varphi}(0) = 1$. Throughout this chapter, we always assume that $\sum_{j\in\mathbb{Z}} a(j) = 2$. We call such a solution the **normalized solution** to (3.1.2) associated with a and we denote it by φ_a. In fact, φ_a is given by

$$\hat{\varphi}_a(\xi) := \prod_{j=1}^{\infty} \left(\frac{\tilde{a}(e^{-i\xi/2^j})}{2} \right), \qquad \xi \in \mathbb{R},$$

where $\tilde{a}$ is the **symbol** of a defined as

$$\tilde{a}(z) := \sum_{j\in\mathbb{Z}} a(j) z^j, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Let $H(\xi) := \tilde{a}(e^{-i\xi})/2,\ \xi \in \mathbb{R}$. Note that H is 2π-periodic. Then

$$\hat{\varphi}_a(\xi) = H\left(\frac{\xi}{2}\right) \hat{\varphi}_a\left(\frac{\xi}{2}\right), \qquad \forall\, \xi \in \mathbb{R}.$$

If, in addition, φ_a is fundamental, then for $\xi \in \mathbb{R}$ we have

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}} \hat{\varphi}_a(2\xi + 2k\pi) = \sum_{k\in\mathbb{Z}} H(\xi + k\pi)\hat{\varphi}_a(\xi + k\pi) \\ &= H(\xi) \sum_{j\in\mathbb{Z}} \hat{\varphi}_a(\xi + 2j\pi) + H(\xi + \pi) \sum_{j\in\mathbb{Z}} \hat{\varphi}_a(\xi + \pi + 2j\pi) = H(\xi) + H(\xi + \pi). \end{aligned}$$

Thus, a necessary condition for φ_a to be a refinable fundamental function is

$$H(\xi) + H(\xi + \pi) = 1 \qquad \forall\, \xi \in \mathbb{R}, \tag{3.1.3}$$

which is equivalent to saying that

$$a(j) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \text{ is an } \textit{even} \text{ integer}. \end{cases}$$

If a mask a satisfies the above condition and $\sum_{j\in\mathbf{Z}} a(j) = 2$, then we say that a is an **interpolatory refinement mask.**

It should be pointed out that the (forward) cardinal B-spline M_m (see §2.1) is refinable for $\forall\, m \in \mathbb{N}$. In fact, it is evident that

$$M_1(x) = M_1(2x) + M_1(2x-1) \qquad \forall\, x \in \mathbb{R},$$

i.e., M_1 is refinable. Moreover, if f and g are refinable with mask a and b respectively, then so is $f * g$ with mask $a * b/2$. Indeed,

$$f(x) = \sum_{j\in\mathbf{Z}} a(j) f(2x - j), \qquad g(x) = \sum_{k\in\mathbf{Z}} b(k) g(2x-k).$$

Then

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} \sum_{j\in\mathbf{Z}} a(j) f(2x - 2y - j) \sum_{k\in\mathbf{Z}} b(k) g(2y-k)dy \\ &= \sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} a(j)b(k) \int_{\mathbb{R}} f\left(2x - 2\left(y - \frac{j}{2}\right)\right) g\left(2\left(y-\frac{k}{2}\right)\right) dy \\ &= \frac{1}{2} \sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} a(j)b(k) f * g(2x - j - k) = \sum_{\ell\in\mathbf{Z}} c(\ell) f * g(2x - \ell) \end{aligned}$$

where $c(\ell) = \sum_{j\in\mathbb{Z}} a(j)b(\ell-j)/2 = a*b/2$. It follows immediately that M_m is refinable for all $m \in \mathbb{N}$ with a mask a given by its symbol $2^{1-m}(1+z)^m$. Similarly, when m is an even positive integer, the central cardinal B-spline ϕ_m satisfies (3.1.2) with a mask given by its symbol $2^{1-m}(z^{-1}+2+z)^{m/2}$. Finally, recall that the fundamental cardinal spline ψ_m is given by

$$\hat{\psi}_m(\xi) = \hat{\phi}_m(\xi)/\tilde{S}_{\phi_m}(\xi)$$

where $\tilde{S}_{\phi_m}(\xi) \neq 0$, given in Chapter 2, is a 2π-periodic function. Therefore, when m is even, ψ_m also satisfies (3.1.2) with its mask a_m given by

$$a_m(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\tilde{S}_{\phi_m}(\xi)}{\tilde{S}_{\phi_m}(2\xi)}\, 2^{1-m}(1+e^{-i\xi})^2 e^{-ij\xi}\, d\xi, \qquad j \in \mathbb{Z}.$$

Obviously a_m is not finitely supported and ψ_m is not compactly supported.

In [12], Deslauriers and Dubuc constructed a family of refinable and fundamental functions with *compact support*. In what follows we shall take the approach adopted in [19] to the construction of refinable and fundamental function. For this purpose, let us recall the definition of sum rules from [6]. For a positive integer k, a mask a is said to satisfy the **sum rules of order k** if

$$\sum_{j\in\mathbb{Z}} a(2j)(2j)^\ell = \sum_{j\in\mathbb{Z}} a(1+2j)(1+2j)^\ell \qquad \forall\, 0 \le \ell < k.$$

See [24–25] for an extension of sum rules to multivariate masks. Note that in the univariate case a mask a satisfies the sum rules of order k if and only if

$$\tilde{a}(z) = \left(\frac{1+z}{2}\right)^k \tilde{b}(z) \tag{3.1.4}$$

for some finitely supported sequence b. The following theorem is taken from [19]:

Theorem 3.1 *For each positive integer N, there exists a unique interpolatory refinement mask b_N supported on $[1-2N,\ 2N-1]$ such that b_N satisfies the sum rules of order $2N$.*

Proof: For each positive integer N, if b_N is an interpolatory refinement mask, then $b_N(0) = 1$ and $b_N(2j) = 0$ for all $j \in \mathbb{Z} \setminus \{0\}$. Then b_N satisfies the sum rules of order $2N$ if and only if

$$\sum_{j=-N+1}^{N} b_N(2j-1) = 1 \quad \text{and} \quad \sum_{j=-N+1}^{N} b_N(2j-1)(2j-1)^{\ell} = 0, \qquad 0 < \ell \leq N-1.$$

The above equations regarding $b_N(2j-1)$ can be rewritten in the following matrix form:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ (-2N+1) & (-2N+3) & \cdots & (2N-1) \\ \vdots & \vdots & \ddots & \vdots \\ (-2N+1)^{2N-1} & (-2N+3)^{2N-1} & \cdots & (2N-1)^{2N-1} \end{bmatrix} \begin{bmatrix} b_N(-2N+1) \\ b_N(-2N+3) \\ \vdots \\ b_N(2N-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.1.5}$$

Since $\left((2j-1)^{\ell}\right)_{-N+1 \leq j \leq N, 0 \leq \ell \leq N+1}$ is a Vandermonde matrix, the equation (3.1.5) has a unique solution for $[b_N(-2N+1), b_N(-2N+3), \ldots, b_N(2N-1)]$, which can be easily found as follows:

$$b_N(2j-1) = (-1)^{j+N-1} \frac{\prod_{\ell=1-N}^{N}(2\ell-1)}{2^{2N-1}(2j-1)(j+N-1)!(N-j)!}, \quad 1-N \leq j \leq N. \quad \blacksquare$$

The normalized solution to the refinement equation (3.1.2) associated with the mask b_N, denoted by φ_N, is exactly the fundamental function introduced by Deslauriers and Dubuc (see [15] and [12]). Throughout this chapter we call φ_N the DD-family of fundamental refinable functions, or simply, the **DD-functions**. For the reader's convenience, we list b_N for $N = 2, 3$ in the following:

$$\tilde{b}_2(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3,$$

$$\tilde{b}_3(z) = \frac{3}{256}z^{-5} - \frac{25}{256}z^{-3} + \frac{75}{128}z^{-1} + 1 + \frac{75}{128}z - \frac{25}{256}z^3 + \frac{3}{256}z^5.$$

In this chapter, we focus our attention to the Fourier transform $\hat{\varphi}_N$ of φ_N given by

$$\hat{\varphi}_N(\xi) = \prod_{k=1}^{\infty} H_N\left(\frac{\xi}{2^k}\right), \qquad \xi \in \mathbb{R}, \tag{3.1.6}$$

where $H_N(\xi) := \tilde{b}_N(e^{-i\xi})/2$. Note that the infinite product (3.1.6) is uniformly convergent on each compact subset of $\mathbb{R}$. Indeed, since $H_N(0) = 1$ and H_N is continuous, $|H_N(\xi) - H_N(0)| \leq c|\xi|$ for some constant c. It follows that $\sum_{j=1}^{\infty} |1 - H_N(\xi/2^j)|$, hence $\prod_{j=1}^{\infty} H_N(\xi/2^j)$ is convergent. The following expression of H_N is given by Meyer [27] and Micchelli [28].

Proposition 3.2

$$H_N(\xi) = 1 - \frac{1}{\rho_N}\int_0^{\xi} (\sin t)^{2N-1} dt \tag{3.1.7}$$

where

$$\rho_N = \int_0^{\pi} (\sin t)^{2N-1} dt. \tag{3.1.8}$$

Consequently, $0 \le H_N(\xi) \le 1$ *for all* $\xi \in \mathbb{R}$ *and* $H_N(\pi) = 0$.

Proof: Since $\tilde{b}_N$ satisfies the sum rules of order $2N$, $\tilde{b}_N(z)$ has a factor $(1+z)^{2N}$. Hence,

$$\tilde{b}_N^{(j)}(-1) = \tilde{b}_N^{(j)}(e^{i\pi}) = 0, \qquad j = 1, \ldots, 2N-1.$$

Moreover, by (3.1.3),

$$H_N(\xi) = 1 - H_N(\xi + \pi), \quad \text{i.e.,} \quad \tilde{b}_N(z) = 1 - \tilde{b}_N(-z).$$

Therefore,

$$\tilde{b}_N^{(j)}(1) = 0, \qquad j = 1, \ldots, 2N-1.$$

But $\tilde{b}'_N(z) = z^{-2N}p(z)$ for some polynomial of degree at most $4N-2$. Hence, there is a constant c such that

$$\tilde{b}'_N(z) = c(1+z)^{2N-1}(1-z)^{2N-1}z^{-2N} = cz^{-1}(z^{-1}-z)^{2N-1}.$$

Letting $z = e^{i\xi}$ in the above equation, we obtain

$$i\tilde{b}'_N(e^{i\xi})e^{i\xi} = c(\sin\xi)^{2N-1}(-1)^{N-1}2^{2N-1}.$$

This in connection with $\tilde{b}_N(e^{i\pi}) = 0$ and $\tilde{b}_N(e^{i2\pi}) = 2$ yields

$$\tilde{b}_N(e^{i\xi}) = 2\frac{\int_\pi^\xi (\sin t)^{2N-1}dt}{\int_\pi^{2\pi}(\sin t)^{2N-1}dt}.$$

It follows that

$$H_N(\xi) = \frac{1}{2}\tilde{b}_N(e^{i\xi}) = \frac{\int_\xi^\pi (\sin t)^{2N-1}dt}{\int_0^\pi (\sin t)^{2N-1}dt} = \frac{\int_0^\pi (\sin t)^{2N-1}dt - \int_0^\xi (\sin t)^{2N-1}dt}{\int_0^\pi (\sin t)^{2N-1}dt}$$
$$= 1 - \frac{1}{\rho_N}\int_0^\xi (\sin t)^{2N-1}dt. \quad \blacksquare$$

The following technical lemmas, combining with the propositions given in this section, will be used to estimate $\hat{\varphi}_N(\xi)$ in the rest of the chapter.

Lemma 3.3 *For $N \in \mathbb{N}$,*

$$\sqrt{\frac{N-1/2}{\pi}} < \frac{1}{\rho_N} < \sqrt{\frac{N}{\pi}}.$$

Proof: In fact, for $0 < t < \frac{\pi}{2}$,

$$(\sin t)^{2N+1} < (\sin t)^{2N} < (\sin t)^{2N-1}.$$

Integrating the above inequalities from 0 to $\frac{\pi}{2}$, we have

$$\int_0^{\pi/2} (\sin t)^{2N+1} dt < \int_0^{\pi/2} (\sin t)^{2N} dt < \int_0^{\pi/2} (\sin t)^{2N-1} dt. \tag{3.1.9}$$

Set

$$J_m := \int_0^{\pi/2} \sin^m x dx.$$

Clearly, $\rho_N = 2J_{2N-1}$. From elementary calculus we have

$$J_{2n} = \int_0^{\pi/2} \sin^{2n} x dx = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n\cdot(2n-2)\cdots 4\cdot 2}\frac{\pi}{2} := \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

and

$$J_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x dx = \frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 3\cdot 1} := \frac{(2n)!!}{(2n+1)!!}.$$

Therefore, (3.1.9) is rewritten as

$$\frac{(2N)!!}{(2N+1)!!} < \frac{(2N-1)!!}{(2N)!!}\frac{\pi}{2} < \frac{(2N-2)!!}{(2N-1)!!}. \tag{3.1.10}$$

On the other hand,

$$\frac{1}{\rho_N} = \frac{1}{2J_{2N-1}} = \frac{(2N-1)!!}{2(2N-2)!!}.$$

But from the second inequality of (3.1.10),

$$\frac{1}{\rho_N}\,\frac{\pi}{2N} < \frac{1}{2}\,\rho_N$$

and hence $\frac{1}{\rho_N} < \sqrt{\frac{N}{\pi}}$. Similarly, from the first inequality of (3.1.10),

$$\frac{1}{\pi}\left(\frac{2N^2}{2N+1}\right) < \frac{1}{\rho_N^2}.$$

Note that $\frac{2N^2}{2N+1} > N - \frac{1}{2}$. We have $\sqrt{\frac{N-1/2}{\pi}} < \frac{1}{\rho_N}$. ■

Lemma 3.4 *For* $0 < \xi < \frac{\pi}{2}$, *define*

$$g_N(\xi) = \frac{1}{\rho_N}\int_0^{\xi}(\sin t)^{2N-1}dt.$$

Then

$$g_N(\xi) \leq \frac{1}{2}\sqrt{N\pi}\,(\sin\xi)^{2N-1}. \tag{3.1.11}$$

Proof: Note that for $0 < \xi < \frac{\pi}{2}$

$$\int_0^{\xi}(\sin t)^{2N-1}dt \leq \frac{\pi}{2}\,(\sin\xi)^{2N-1}$$

with $0 < \sin\xi < 1$. By Lemma 3.3, $\frac{1}{\rho_N} < \sqrt{\frac{N}{\pi}}$. Therefore

$$g_N(\xi) \leq \frac{\pi}{2}\sqrt{\frac{N}{\pi}}\,(\sin\xi)^{2N-1} = \frac{1}{2}\sqrt{N\pi}\,(\sin\xi)^{2N-1}. \quad ■$$

Lemma 3.5 *For $N \geq 10$ and $\pi \leq \eta \leq \pi + \pi/4$,*

$$\prod_{j=2}^{\infty} H_N\left(\frac{\eta}{2^j}\right) \geq \frac{3}{4}.$$

Proof: By (3.1.11), for $0 \leq \xi < \frac{\pi}{2}$,

$$g_N(\xi) \leq \frac{1}{2}\sqrt{N\pi}\,(\sin\xi)^{2N-1}.$$

But for $0 \leq \xi \leq 5\pi/16$ and $N \geq 10$, we have

$$g_N(\xi) \leq \frac{1}{2}\sqrt{N\pi}\,(\sin\xi)^{2N-1} \leq \frac{1}{2}\sqrt{N\pi}\left(\sin\frac{5\pi}{16}\right)^{2N-1} < \frac{1}{4}.$$

On the other hand recall that

$$1 - t \geq e^{-2t} \qquad \text{for } 0 \leq t \leq \frac{1}{4}.$$

Hence,

$$H_N(\xi) = 1 - g_N(\xi) \geq \exp(-2g_N(\xi)) \geq \exp\left(-\sqrt{N\pi}\,(\sin\xi)^{2N-1}\right).$$

One can calculate that, for $N \geq 10$ and $\pi \leq \eta \leq \pi + \pi/4$,

$$\prod_{j=2}^{\infty} H_N\left(\frac{\eta}{2^j}\right) \geq \exp\left(-\sqrt{N\pi}\sum_{j=2}^{\infty}\left(\sin\frac{5\pi}{2^{j+2}}\right)^{2N-1}\right) > \frac{3}{4}. \quad \blacksquare$$

Proposition 3.6

$$H_N(\xi) = \left(\cos^2\left(\frac{\xi}{2}\right)\right)^N P_N\left(\sin^2\left(\frac{\xi}{2}\right)\right)$$

where

$$P_N(y) = \sum_{k=0}^{N-1}\binom{N+k-1}{k} y^k. \tag{3.1.12}$$

Consequently, $\hat{\varphi}_N$ is an even function of ξ.

Proof: Since b_N satisfies the sum rules of order $2N$, then it follows from (3.1.4) that

$$H_N(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^{2N} \tilde{p}(e^{i\xi})$$

for some finitely supported sequence p. Moreover, by (3.1.7), $H_N(\xi)$ is nonnegative. Hence

$$\begin{aligned} H_N(\xi) &= \left(\frac{1+e^{i\xi}}{2}\right)^{2N} \tilde{p}(e^{i\xi}) = \left|\frac{1+e^{i\xi}}{2}\right|^{2N} |\tilde{p}(e^{i\xi})| \\ &= \left(\frac{1+e^{i\xi}}{2}\right)^{N} \left(\frac{1+e^{-i\xi}}{2}\right)^{N} |\tilde{p}(e^{i\xi})| = \left(\frac{e^{i\xi}+e^{-i\xi}+2}{4}\right)^{N} |\tilde{p}(e^{i\xi})| \\ &= \left(\frac{2\cos\xi+2}{4}\right)^{N} |\tilde{p}(e^{i\xi})| = \left(\cos^2\left(\frac{\xi}{2}\right)\right)^{N} |\tilde{p}(e^{i\xi})|. \end{aligned}$$

On the other hand

$$\sum_j c_j e^{ij\xi} = |\tilde{p}(e^{i\xi})| = |\tilde{p}(e^{-i\xi})| = \sum_j c_{-j} e^{ij\xi}.$$

Thus, $c_j = c_{-j}$. It follows that $|\tilde{p}(e^{i\xi})|$ is a polynomial in $\cos\xi$, or, equivalently, in $\sin^2(\xi/2)$. Then there exists a polynomial P_N such that $P_N(\sin^2(\xi/2)) = |\tilde{p}(e^{i\xi})|$. Therefore,

$$H_N(\xi) = \left(\cos^2\left(\frac{\xi}{2}\right)\right)^{N} P_N\left(\sin^2\left(\frac{\xi}{2}\right)\right). \tag{3.1.13}$$

Moreover, since b_N is interpolatory, which is equivalent to (3.1.3)

$$H_N(\xi) + H_N(\xi+\pi) = 1.$$

With $y = \cos^2(\xi/2)$ and (3.1.13), the above equation becomes

$$y^N P_N(1-y) + (1-y)^N P_N(y) = 1 \qquad \forall\, y \in [0,\ 1]. \tag{3.1.14}$$

We now solve the equation (3.1.14) for P_N. Note that (see p168, [5]) there exist two unique polynomials q_1 and q_2 of degree less than N such that

$$(1-y)^N q_1(y) + y^N q_2(y) = 1. \tag{3.1.15}$$

To find q_1 and q_2, substituting $1-y$ for y in (3.1.15) lead to

$$(1-y)^N q_2(1-y) + y^N q_1(1-y) = 1.$$

The uniqueness of q_1 and q_2 implies $q_2(y) = q_1(1-y)$. It follows that $P_N(y) = q_1(y)$ is a solution of (3.1.14). From (3.1.15) we obtain

$$q_1(y) = (1-y)^{-N}\left[1 - y^N q_2(1-y)\right].$$

Note that

$$(1-y)^{-N} = 1 + (-N)(-y) + \frac{(-N)(-N-1)}{2}(-y)^2 + \cdots.$$

It follows that

$$q_1(y) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} y^k + O(y^N).$$

But q_1 is a polynomial of degree less than N. Hence $O(y^N) = 0$. Therefore we obtain

$$P_N(y) = q_1(y) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} y^k. \qquad \blacksquare$$

Theorem 3.7 *For each positive integer N, $\hat{\varphi}_N$ is integrable and φ_N is a fundamental function.*

Proof: For every positive integer $n \in \mathbb{Z}_+$, define

$$\hat{f}_n(\xi) := \prod_{j=1}^{n} H_N\left(\frac{\xi}{2^j}\right) \chi_{[-2^n\pi,\, 2^n\pi]}(\xi).$$

Since $0 \le H_N(\xi) \le 1$ for all $\xi \in \mathbb{R}$, we observe that $\hat{f}_n(\xi) \ge 0$ for all $\xi \in \mathbb{R}$. We demonstrate that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_n(\xi) e^{ij\xi} d\xi = \delta(j) \qquad \forall\, j \in \mathbb{Z},\ n \in \mathbb{Z}_+ \tag{3.1.16}$$

or equivalent,

$$\sum_{k\in\mathbb{Z}} \hat{f}_n(\xi + 2k\pi) = 1, \qquad \text{a.e. } \xi \in \mathbb{R},\ n \in \mathbb{Z}_+. \tag{3.1.17}$$

It is evident that (3.1.17) holds true for $n = 0$ since $\hat{f}_0(\xi) = \chi_{[-\pi,\,\pi]}$. Assume (3.1.17) holds true for n. We observe that

$$\hat{f}_{n+1}(\xi) = H_N\left(\frac{\xi}{2}\right) \hat{f}_n\left(\frac{\xi}{2}\right).$$

Therefore, by induction hypothesis,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \hat{f}_{n+1}(\xi + 2k\pi) &= H_N\left(\frac{\xi}{2}\right) \sum_{k\in\mathbb{Z}} \hat{f}_n\left(\frac{\xi}{2} + 2k\pi\right) \\
&\quad + H_N\left(\frac{\xi + 2\pi}{2}\right) \sum_{k\in\mathbb{Z}} \hat{f}_n\left(\frac{\xi + 2\pi}{2} + 2k\pi\right) \\
&= H_N\left(\frac{\xi}{2}\right) + H_N\left(\frac{\xi + 2\pi}{2}\right) = 1
\end{aligned}$$

where we have used (3.1.3) in the last equality. Since $\hat{\varphi}_N(\xi) \le \hat{f}_n(\xi)$ for all $|\xi| \le 2^n\pi$, we have

$$\int_{-2^n\pi}^{2^n\pi} \hat{\varphi}_N(\xi) d\xi \le \int_{-2^n\pi}^{2^n\pi} \hat{f}_n(\xi) d\xi = \int_{\mathbb{R}} \hat{f}_n(\xi) d\xi = 1 \qquad \forall\, n \in \mathbb{N}.$$

Note that $\hat{\varphi}_N(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Therefore, by Fatou's Lemma,

$$\int_{\mathbb{R}} \hat{\varphi}_N(\xi)d\xi \leq \lim_{n\to\infty} \int_{-2^n\pi}^{2^n\pi} \hat{\varphi}_N(\xi)d\xi \leq 1$$

which implies that $\hat{\varphi}_N$ is integrable. On the other hand, in the light of (3.1.13), we observe that $H_N(\xi) > 0$ for all $\xi \in (-\pi,\ \pi)$. Since $H_N(\xi)$ is a trigonometric polynomial and $H_N(0) = 1$ and $\hat{\varphi}_N(\xi) = \prod_{j=1}^{\infty} H_N(\xi/2^j)$, $\hat{\varphi}_N$ must be continuous and $\hat{\varphi}_N(\xi) > 0$ for all $\xi \in [-\pi,\ \pi]$. The reason lies in that if $\hat{\varphi}_N(\xi_0) = 0$ for some ξ_0, then $H_N(\xi_0/2^j) = 0$ for some $j \in \mathbb{N}$. Contradict $H_N(\xi) > 0$ for all $\xi \in (-\pi,\ \pi)$. Since $\hat{\varphi}_N(0) = 1$, there exists a positive constant c such that $\hat{\varphi}_N(\xi) \geq c$, for all $\xi \in [-\pi,\ \pi]$. Observe that

$$\hat{f}_n(\xi) = \begin{cases} \hat{\varphi}_N(\xi)/\hat{\varphi}_N(\xi/2^n) & \text{if } \xi \in [-2^n\pi,\ 2^n\pi], \\ 0 & \text{if } \xi \in \mathbb{R} \setminus [-2^n\pi,\ 2^n\pi]. \end{cases}$$

Therefore, we have

$$|\hat{f}_n(\xi)| \leq c^{-1}\hat{\varphi}_N(\xi) \qquad \forall\, \xi \in \mathbb{R}.$$

It then follows from the Lebesgue dominated convergence theorem that

$$\varphi_N(j) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_N(\xi)e^{ij\xi}d\xi = \frac{1}{2\pi}\lim_{n\to\infty}\int_{\mathbb{R}} \hat{f}_n(\xi)e^{ij\xi}d\xi = \delta(j), \quad j \in \mathbb{Z}.$$

Hence, φ_N is a fundamental function. ■

Finally, it is easily seen that

$$||\varphi_N||_\infty \leq 1.$$

Indeed, noting that $\hat{\varphi}_N$ is nonnegative, we have

$$|\varphi_N(x)| = \frac{1}{2\pi}\left|\int_{\mathbb{R}} \hat{\varphi}_N(\xi)e^{ix\xi}d\xi\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{\varphi}_N(\xi)|d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_N(\xi)d\xi = \varphi_N(0) = 1.$$

More basic properties about $\hat{\varphi}_N$, H_N and ρ_N will be given individually when they are needed in the rest of the chapter.

The behaviour of the DD-functions φ_N has drawn attention of several mathematicians. Following Meyer's suggestion, Daubechies noticed that there is a similarity between the techniques used in [4], and those in [12]. Micchelli [28] observed a connection between the DD-functions and the Daubechies wavelets [5]. In [4], Daubechies constructed a family of orthogonal refinable functions g_N, $N = 2, 3, \ldots$ from which orthogonal wavelets were derived. Each g_N is the normalized solution to the refinement equation

$$g_N = \sum_{j\in\mathbb{Z}} a_N(j) g_N(2\cdot -j)$$

where the mask a_N is supported on $[0,\ 2N-1]$, $\sum_{j\in\mathbb{Z}} a_N(j) = 2$, and a_N satisfies the sum rules of order N. Moreover, $\{g_N(\cdot - j) : j \in \mathbb{Z}\}$ forms an orthonormal sequence.

Let f_N be the **autocorrelation** of g_N, that is,

$$f_N(x) = \int_{\mathbb{R}} g_N(x+y) g_N(y) dy, \qquad x \in \mathbb{R}.$$

Then f_N is the normalized solution to the refinement equation

$$f_N = \sum_{j\in\mathbb{Z}} c_N(j) f_N(2\cdot -j)$$

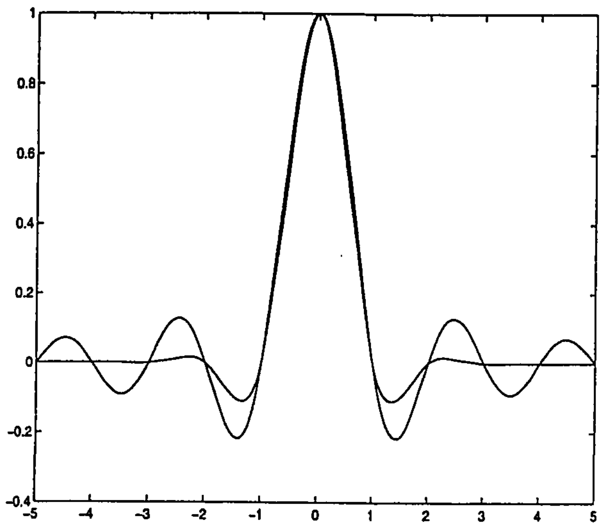

Figure 3.1: Graphs of φ_2 and $\frac{\sin(\pi x)}{\pi x}$ over $[-3,\ 3]$

where

$$c_N(j) = \sum_{k\in\mathbb{Z}} a_N(j+k)a_N(k)/2, \qquad j \in \mathbb{Z}.$$

Hence, c_N is supported on $[1-2N,\ 2N-1]$. Note that a_N satisfies the sum rules of order N. Thus, c_N satisfies the sum rules of order $2N$. Furthermore, since $\{g_N(\cdot - j) : j \in \mathbb{Z}\}$ forms an orthonormal sequence, f_N is fundamental:

$$f_N(j) = \int_{\mathbb{R}} g_N(j+y)g_N(y)dy = \delta(j), \qquad j \in \mathbb{Z}.$$

By Theorem 3.1, c_N and b_N must be the same. This shows that the DD-function φ_N is the autocorrelation of the Daubechies orthogonal refinable function g_N. This fact was observed by Micchelli [28].

For us, the DD-functions are also good candidates (see Figure 3.1) to be seen whether the techniques and results established in Chapter 2 can be extended to a new field or not. We believe that our study could partially answer these questions. Similar to what we concerned in Chapter 2, we concentrate ourselves to the following three subjects. First, does the DD-function converges (uniformly) to the sinc-function as $N \to \infty$? Second, does φ_N possesses the same sign-regularity property as sinc-function does? Finally, is φ_N (or its absolute value) bounded above by the sinc-function?

The answer to the second question is a surprisingly *no.*

Example 1: Consider φ_2 associated with the mask $\{ -\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\}$. Let us compute the values of $\varphi_2(\cdot/2^j)$, $j = 1, 2, 3$. By (3.1.2) we have

$$\varphi_2(x) = -\frac{1}{16}\varphi_2(2x-3) + \frac{9}{16}\varphi_2(2x-1) + \varphi_2(2x) + \frac{9}{16}\varphi_2(2x+1) - \frac{1}{16}\varphi_2(2x+3).$$

Note that the DD-functions φ_N are fundamental. Hence we may apply the above refinement equation iterately to find those $\varphi(n/2^j)$, $n \in \mathbb{Z}$, $j = 1, 2, 3$: First

$$\varphi_2\left(\frac{n}{2}\right) = -\frac{1}{16}\varphi_2(n-3) + \frac{9}{16}\varphi_2(n-1) + \varphi_2(n) + \frac{9}{16}\varphi_2(n+1) - \frac{1}{16}\varphi_2(n+3).$$

Thus $\varphi_2(\pm 1/2) = 9/16$ and $\varphi_2(\pm 3/2) = -1/16$. Similarly

$$\begin{aligned}\varphi_2\left(\frac{n}{4}\right) = -\frac{1}{16}\,\varphi_2\left(\frac{n-6}{2}\right) + \frac{9}{16}\,\varphi_2\left(\frac{n-2}{2}\right) + \varphi_2\left(\frac{n}{2}\right) + \\ \frac{9}{16}\,\varphi_2\left(\frac{n+2}{2}\right) - \frac{1}{16}\,\varphi_2\left(\frac{n+6}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\varphi_2\left(\frac{n}{8}\right) = -\frac{1}{16}\,\varphi_2\left(\frac{n-12}{4}\right) + \frac{9}{16}\,\varphi_2\left(\frac{n-4}{4}\right) + \varphi_2\left(\frac{n}{4}\right) + \\ \frac{9}{16}\,\varphi_2\left(\frac{n+4}{4}\right) - \frac{1}{16}\,\varphi_2\left(\frac{n+12}{4}\right).\end{aligned}$$

x	$\varphi_2(x)$	$\varphi_2(x+1)$	$\varphi_2(x+2)$
0	1	0	0
$\frac{1}{8}$	$\frac{243}{256}$	$-\frac{53}{1024}$	$\frac{9}{2048}$
$\frac{2}{8}$	$\frac{27}{32}$	$-\frac{9}{128}$	$\frac{1}{256}$
$\frac{3}{8}$	$\frac{1459}{2048}$	$-\frac{207}{4096}$	$\frac{9}{4096}$
$\frac{4}{8}$	$\frac{9}{16}$	$-\frac{1}{16}$	0
$\frac{5}{8}$	$\frac{837}{2048}$	$-\frac{207}{4096}$	$-\frac{1}{4096}$
$\frac{6}{8}$	$\frac{33}{128}$	$-\frac{9}{256}$	0
$\frac{7}{8}$	$\frac{117}{1024}$	$-\frac{33}{2048}$	0
$\frac{8}{8}$	0	0	0

Table 3.1: The values of φ_2 at $\frac{n}{8}$

One may observe that $\varphi_2(21/8)$ is negative at which the value of the sinc-function is positive, and also φ_2 has roots at non-integer points (see Table 3.1).

Example 2: Consider φ_3 with mask $\left\{\frac{3}{256}, 0, -\frac{25}{256}, 0, \frac{75}{128}, 0, 1, 0, \frac{75}{128}, 0, -\frac{25}{256}, 0, \frac{3}{256}\right\}$.

$$\begin{aligned}\varphi_3(x) = {} & \frac{3}{256}\,\varphi_3(2x-5) - \frac{25}{256}\,\varphi_3(2x-3) + \frac{75}{128}\,\varphi_3(2x-1) + \varphi_3(2x) + \\ & \frac{75}{128}\,\varphi_3(2x+1) - \frac{25}{256}\,\varphi_3(2x+3) + \frac{3}{256}\,\varphi_3(2x+5).\end{aligned}$$

Let us calculate the values of $\varphi_3(\cdot/2^j)$, $j = 1, 2$.

$$\begin{aligned}\varphi_3\left(\frac{n}{2}\right) = {} & \frac{3}{256}\,\varphi_3(n-5) - \frac{25}{256}\,\varphi_3(n-3) + \frac{75}{128}\,\varphi_3(n-1) + \varphi_3(n) + \\ & \frac{75}{128}\,\varphi_3(n+1) - \frac{25}{256}\,\varphi_3(n+3) + \frac{3}{256}\,\varphi_3(n+5)\end{aligned}$$

x	$\varphi_3(x)$	$\varphi_3(x+1)$	$\varphi_3(x+2)$	$\varphi_3(x+3)$
0	1	0	0	0
$\frac{1}{4}$	$\frac{57075}{65536}$	$-\frac{7491}{65536}$	$\frac{1075}{65536}$	$-\frac{75}{65536}$
$\frac{2}{4}$	$\frac{75}{128}$	$-\frac{25}{256}$	$\frac{3}{256}$	0
$\frac{3}{4}$	$\frac{17175}{65536}$	$-\frac{3375}{65536}$	$\frac{375}{65536}$	$\frac{9}{65536}$
$\frac{4}{4}$	0	0	0	0

Table 3.2: The values of φ_3 at $\frac{n}{4}$

and

$$\varphi_3\left(\frac{n}{4}\right) = \frac{3}{256}\varphi_3\left(\frac{n-10}{2}\right) - \frac{25}{256}\varphi_3\left(\frac{n-6}{2}\right) + \frac{75}{128}\varphi_3\left(\frac{n-2}{2}\right) + \varphi_3\left(\frac{n}{2}\right) +$$
$$\frac{75}{128}\varphi_3\left(\frac{n+2}{2}\right) - \frac{25}{256}\varphi_3\left(\frac{n+6}{2}\right) + \frac{3}{256}\varphi_3\left(\frac{n+10}{2}\right).$$

Thus $\varphi_3(1/2) = 75/128$, $\varphi_3(3/2) = -25/256$ and $\varphi_3(5/2) = 3/256$. All the (nonzero) values of $\varphi_3(\cdot/4)$ are listed on Table 3.2. Observe that $\varphi_3(15/4) > 0$ at which the value of the sinc-function is negative, and also φ_3 has roots at non-integer points (see Table 3.2).

Remark. In fact, it was proved that φ_N, $N \in \mathbb{N}$, have infinitely many roots at non-integer points ([15], Theorem 14). Therefore, the DD-functions do not possess sign-regularity property

The structure of this chapter is as follows. In §2, we will show that φ_N, $N \in \mathbb{N}$, converges uniformly to the sinc-function as N goes to infinity. We then show, in §3, that the curves of φ_N are contained inside the curve of the sinc-function for

$-1 < x < 1$ which is the property that we have known for ψ_m in Chapter 2, that is,

$$0 < \varphi_N(x) < \frac{\sin(\pi x)}{\pi x} \qquad \forall\, x \in (-1,\ 0) \cup (0,\ 1).$$

Finally, in §4, we prove

$$|\varphi_N(x)| \le \left|\frac{\sin(\pi x)}{\pi x}\right|, \qquad \forall\, x \in \mathbb{R}.$$

3.2 A limit property of the DD-functions

We prove that $\varphi_N(x)$, $N \in \mathbb{N}$, converges uniformly to the sinc-function as $N \to \infty$ in this section.

Recall that in the case of the fundamental cardinal splines ψ_m there are two approaches to show this uniformly convergent property. The first one is given in [32], in which the so-called *eigenspline* derived from the central cardinal B-splines ϕ_m plays an essential role. The second approach is to consider the Fourier transform of ψ_m. By showing

$$\lim_{m\to\infty} \hat{\psi}_m(\xi) = \begin{cases} 1 & \text{if } |\xi| < \pi, \\ \frac{1}{2} & \text{if } |\xi| = \pi, \\ 0 & \text{if } |\xi| > \pi. \end{cases}$$

The desired result then follows from the dominated convergence theorem. For the DD-functions φ_N we adopt the second approach because less information is obtained from φ_N themselves. We now show

Theorem 3.8 *For $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \varphi_N(x) = \frac{\sin(\pi x)}{\pi x} \qquad \textit{uniformly.}$$

For this purpose, we need the following lemma:

Lemma 3.9 *For $0 < \xi < \pi$*

$$\lim_{N\to\infty} \hat{\varphi}_N(\xi) = 1.$$

Proof: We first prove the theorem by assumption of Lemma 3.7. Note that

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\xi}d\xi,$$

and

$$\varphi_N(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_N(\xi)e^{-ix\xi}d\xi.$$

Hence

$$\left|\frac{\sin(\pi x)}{\pi x} - \varphi_N(x)\right| \le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\left|(1-\hat{\varphi}_N(\xi))\right|d\xi + \int_{|\xi|>\pi}|\hat{\varphi}_N(\xi)|d\xi\right).$$

Moreover, since $0 \le \hat{\varphi}_N(\xi) \le 1$ and $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_N(\xi)d\xi = \varphi_N(0) = 1$, we have

$$\begin{aligned}\left|\frac{\sin(\pi x)}{\pi x} - \varphi_N(x)\right| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\hat{\varphi}_N(\xi))d\xi + 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\varphi}_N(\xi)d\xi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(1-\hat{\varphi}_N(\xi))d\xi \\ &= \frac{2}{\pi}\int_{0}^{\pi}(1-\hat{\varphi}_N(\xi))d\xi.\end{aligned}$$

The fact that $\hat{\varphi}_N(\xi)$ is an even function of ξ is used in the last equality above. Since $1-\hat{\varphi}_N(\xi) \le 1$ and $\hat{\varphi}_N(\xi) \to 1$ for $\xi \in (0,\pi)$ as $N \to \infty$ by the lemma , therefore by the dominated convergence theorem, $\varphi_N(x) \to \frac{\sin(\pi x)}{\pi x}$ uniformly for $x \in \mathbb{R}$. ■

Proof of lemma 3.9 By (3.1.6) and (3.1.7)

$$\hat{\varphi}_N(\xi) = \prod_{j=1}^{\infty} H_N\left(\frac{\xi}{2^j}\right), \qquad \xi \in \mathbb{R},$$

with

$$H_N(\xi) = 1 - g_N(\xi).$$

In light of (3.1.11), for sufficiently large N,

$$H_N(\xi) = 1 - g_N(\xi) \geq \exp(-2g_N(\xi)) \geq \exp\left(-\sqrt{N\pi}\,(\sin\xi)^{2N-1}\right).$$

It follows that

$$1 \geq \hat{\varphi}_N(\xi) \geq \exp\left(-\sqrt{N\pi}\sum_{j=1}^{\infty}\left(\sin\frac{\xi}{2^j}\right)^{2N-1}\right).$$

Note that for $0 < \xi < \pi$ and $N \geq 2$,

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\sin\frac{\xi}{2^j}\right)^{2N-1} &\leq \left(\sin\frac{\xi}{2}\right)^{2N-1} + \sum_{j=2}^{\infty}\left(\frac{\xi}{2^j}\right)^{2N-1} \\ &\leq \left(\sin\frac{\xi}{2}\right)^{2N-1} + 2\left(\frac{\xi}{4}\right)^{2N-1}.\end{aligned}$$

Therefore

$$1 \geq \hat{\varphi}_N(\xi) \geq \exp\left(-\sqrt{N\pi}\left(\left(\sin\frac{\xi}{2}\right)^{2N-1} + 2\left(\frac{\xi}{4}\right)^{2N-1}\right)\right) \to 1$$

as $N \to \infty$. ■

3.3 The case $0 < x < 1$

We will show in this section that the curves of φ_N are contained in the curve of the sinc-function for $0 < x < 1$ and $N \geq 42$. More precisely, we will show:

Theorem 3.10 *For $x \in (0,\ 1)$ and $N \geq 42$, the following inequalities hold:*

$$0 < \frac{\sin(\pi x)}{\pi x} - \varphi_N(x) < \frac{\sin(\pi x)}{\pi x}.$$

The proof of Theorem 3.10 is divided in the following five steps:

Step 1: An expression of $I = \frac{\sin(\pi x)}{\pi x} - \varphi_N(x)$.

Motivated by what we did in Chapter 2, we first derive a series-expression of the Fourier transform of I. Since $\varphi_N(j) = \delta_j,\ j \in \mathbb{Z}$, then by (3.1.1)

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}_N(\xi + 2k\pi) = 1.$$

Consequently,

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\xi} d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi} \hat{\varphi}_N(\xi + 2k\pi)e^{-ix\xi} d\xi. \tag{3.3.1}$$

Moreover, since φ_N is real and symmetric, we have

$$\varphi_N(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_N(\xi)e^{-i\xi x} d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi} \hat{\varphi}_N(\xi + 2k\pi)e^{-ix(\xi+2k\pi)} d\xi. \tag{3.3.2}$$

It follows from (3.3.1) and (3.3.2) that

$$\begin{aligned} I &= \frac{\sin(\pi x)}{\pi x} - \varphi_N(x) \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\left(e^{-ix\xi} - e^{-ix(\xi+2k\pi)}\right) d\xi \\ &= \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\int_{0}^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\big[\cos(x\xi) - \cos\big(x(\xi+2k\pi)\big)\big] d\xi \\ &= \frac{2}{\pi}\sum_{k\in\mathbb{Z}}\int_{0}^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\sin(k\pi x)\sin\big(x(\xi+k\pi)\big) d\xi. \end{aligned} \tag{3.3.3}$$

One might note that (3.3.3) is quite similar to what we obtained for the fundamental cardinal splines ψ_m (see (2.2.3) of §2.2). For convenience we write

$$I = \frac{\sin(\pi x)}{\pi x} - \varphi_N(x) = \frac{2}{\pi}\sin(\pi x)(I_1 + I_2^+ + I_2^- + I_3^+ + I_3^-),$$

where

$$I_1 := \int_0^{\pi} \hat{\varphi}_N(2\pi - \xi)\sin\big(x(\pi - \xi)\big)\,d\xi, \tag{3.3.4}$$

corresponding to $k = -1$, is the term that plays the same role as I_1 in Chapter 2 and

$$I_2^+ := \sum_{k=1}^{16} \frac{\sin(k\pi x)}{\sin(\pi x)} \int_0^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\sin\big(x(\xi + k\pi)\big)\,d\xi, \tag{3.3.5}$$

$$I_2^- := \sum_{k=-2}^{-16} \frac{\sin(k\pi x)}{\sin(\pi x)} \int_0^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\sin\big(x(\xi + k\pi)\big)\,d\xi,$$

$$I_3^+ := \sum_{k=17}^{\infty} \frac{\sin(k\pi x)}{\sin(\pi x)} \int_0^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\sin\big(x(\xi + k\pi)\big)\,d\xi, \tag{3.3.6}$$

and

$$I_3^- := \sum_{k=-17}^{-\infty} \frac{\sin(k\pi x)}{\sin(\pi x)} \int_0^{\pi} \hat{\varphi}_N(\xi + 2k\pi)\sin\big(x(\xi + k\pi)\big)\,d\xi.$$

Step 2: A lower bound for I_1.

Similarly to what we did in Chapter 2, we intend to demonstrate that I_1 is what contributes the most to the bi-infinite series (3.3.3). For this purpose, we need a lower bound for I_1 and upper bounds for $|I_\ell^+|$ and $|I_\ell^-|$ such that the lower bound for I_1 is greater than the sum of the upper bounds for $|I_\ell^+|$ and $|I_\ell^-|$, $\ell = 2, 3$.

We first estimate a lower bound for I_1 where

$$I_1 = \int_0^{\pi} \hat{\varphi}_N(2\pi - \xi)\sin\big(x(\pi - \xi)\big)d\xi = \int_0^{\pi} \hat{\varphi}_N(\pi + \xi)\sin(x\xi)d\xi.$$

Since $0 < x < 1$ and $0 \le \xi \le \pi$, $\sin(x\xi) \ge 0$. Moreover,

$$I_1 = \int_0^{\pi} \hat{\varphi}_N(\pi + \xi)\sin(x\xi)d\xi \ge \int_0^{\pi/8} \hat{\varphi}_N(\pi + \xi)\sin(x\xi)d\xi.$$

But it follows from (3.1.6) and (3.1.7) that

$$\begin{aligned}\hat{\varphi}'_N(\eta) &= \sum_{j=1}^{\infty} \frac{1}{2^j} H'_N\left(\frac{\eta}{2^j}\right) \prod_{k \ne j} H_N\left(\frac{\eta}{2^k}\right) \\ &= \sum_{j=1}^{\infty} \left(-\frac{1}{2^j \rho_N}\right) \sin^{2N-1}\left(\frac{\eta}{2^j}\right) \prod_{k=1}^{j-1} H_N\left(\frac{\eta}{2^k}\right) \prod_{k=j+1}^{\infty} H_N\left(\frac{\eta}{2^k}\right).\end{aligned}$$

Consequently, $\hat{\varphi}'_N(\eta) < 0$ for $0 < \eta < 2\pi$ and $\hat{\varphi}_N$ is decreasing on the interval $[0,\ 2\pi]$. Hence for $0 < \xi < \pi/8$, $\hat{\varphi}_N(\pi + \xi) > \hat{\varphi}_N(9\pi/8)$, so

$$I_1 \ge \hat{\varphi}_N\left(\frac{9\pi}{8}\right) \int_0^{\frac{\pi}{8}} \sin(x\xi)d\xi. \tag{3.3.7}$$

We now estimate a lower bound for $\hat{\varphi}_N(9\pi/8)$. By (3.1.6) and (3.1.7) again,

$$\hat{\varphi}_N\left(\frac{9\pi}{8}\right) = H_N\left(\frac{9\pi}{16}\right) \prod_{j=2}^{\infty} H_N\left(\frac{9\pi}{2^{j+3}}\right)$$

where

$$H_N\left(\frac{9\pi}{16}\right) = 1 - \frac{1}{\rho_N} \int_0^{9\pi/16} (\sin t)^{2N-1} dt.$$

We next examine $H_N(9\pi/16)$ and $\prod_{j=2}^{\infty} H_N(9\pi/2^{j+3})$ respectively. Observe that $H_N(\pi) = 0$. Thus

$$H_N\left(\frac{9\pi}{16}\right) = 1 - \frac{1}{\rho_N} \int_0^{9\pi/16} (\sin t)^{2N-1} dt = \frac{1}{\rho_N} \int_0^{7\pi/16} (\sin t)^{2N-1} dt.$$

Therefore, a lower bound for $H_N(9\pi/16)$ is given by

$$H_N\left(\frac{9\pi}{16}\right) \geq \frac{1}{\rho_N}\int_{13\pi/32}^{7\pi/16}(\sin t)^{2N-1}dt \geq \frac{\pi}{32\rho_N}\left(\sin\left(\frac{13\pi}{32}\right)\right)^{2N-1}. \quad (3.3.8)$$

A lower bound for I_1 will be obtained once we give a lower bound for $\prod_{j=2}^{\infty} H_N(9\pi/2^{j+3})$ and this is where Lemma 3.5 comes to play. It follows from Lemma 3.5 and (3.3.8) that a lower bound for $\hat{\varphi}_N(9\pi/8)$ is given by

$$\hat{\varphi}_N\left(\frac{9\pi}{8}\right) = H_N\left(\frac{9\pi}{16}\right)\prod_{j=2}^{\infty} H_N\left(\frac{9\pi}{2^{j+3}}\right) \geq \frac{3\pi}{128\rho_N}\left(\sin\left(\frac{13\pi}{32}\right)\right)^{2N-1}.$$

Finally, for $0 < x < 1$,

$$\int_0^{\frac{\pi}{8}} \sin(x\xi)d\xi = -\frac{1}{x}\cos(x\xi)\Big|_0^{\pi/8} = \frac{2}{x}\left(\sin\left(\frac{\pi x}{16}\right)\right)^2 \geq \frac{2}{x}\left(\frac{2}{\pi}\frac{\pi x}{16}\right)^2 = \frac{x}{32}.$$

Therefore, in light of (3.3.7), a lower bound for I_1 is given as

$$I_1 \geq \frac{3\pi x}{4096\rho_N}\left(\sin\left(\frac{13\pi}{32}\right)\right)^{2N-1}. \quad (3.3.9)$$

Step 3: An upper bound for $|I_3^+|$ and $|I_3^-|$.

We now investigate $|I_\ell^+|$ and $|I_\ell^-|$, $\ell = 2, 3$. The property of exponential decay of $\hat{\varphi}_N$ has been discussed in many papers. See [5], [3], and [16], for example. With the aid of Proposition 3.4 in §3.1, the following lemma describes this property more precisely and it plays a crucial role in the rest of our estimates.

Lemma 3.11 *For* $2^J\pi \leq \eta \leq 2^{J+1}\pi$, $J \in \mathbb{N}$,

$$\hat{\varphi}_N(\eta) \leq \frac{1}{2^{2J+1}}\left(2\left(\frac{3}{4}\right)^J\right)^{N-1}.$$

Proof: Since $H_N(\xi) \le 1$ for all $\xi \in \mathbb{R}$

$$\hat{\varphi}_N(\eta) = \prod_{j=1}^{\infty} H_N\left(\frac{\eta}{2^j}\right) = \prod_{j=1}^{J+1} H_N\left(\frac{\eta}{2^j}\right) \prod_{j=J+2}^{\infty} H_N\left(\frac{\eta}{2^j}\right) \le \prod_{j=1}^{J+1} H_N\left(\frac{\eta}{2^j}\right).$$

In light of (3.1.13), we obtain

$$\hat{\varphi}_N(\eta) \le \left(\prod_{j=2}^{J+2} \cos\left(\frac{\eta}{2^j}\right)\right)^{2N} \prod_{j=2}^{J+2} P_N\left(\sin^2\left(\frac{\eta}{2^j}\right)\right). \tag{3.3.10}$$

We now estimate $\left(\prod_{j=2}^{J+2} \cos(\eta/2^j)\right)^{2N}$ and $\prod_{j=2}^{J+2} P_N\left(\sin^2(\eta/2^j)\right)$ respectively.

First observe that

$$\prod_{j=2}^{J+2} \cos\left(\frac{\eta}{2^j}\right) = \frac{\sin(\eta/2)}{2^{J+1}\sin(\eta/2^{J+2})}$$

and if $2^J\pi \le \eta \le 2^{J+1}\pi$, $\pi/4 \le \eta/2^{J+2} \le \pi/2$, then $\sin(\eta/2^{J+2}) \ge 1/\sqrt{2}$. Hence

$$\prod_{j=2}^{J+2} \left|\cos\left(\frac{\eta}{2^j}\right)\right| \le \frac{1}{2^{J+1}\left|\sin(\eta/2^{J+2})\right|} \le \frac{\sqrt{2}}{2^{J+1}}$$

and

$$\left(\prod_{j=2}^{J+2} \cos\left(\frac{\eta}{2^j}\right)\right)^{2N} \le \left(\frac{1}{2^{2J+1}}\right)^{N}. \tag{3.3.11}$$

We now estimate $\prod_{j=2}^{J+2} P_N\left(\sin^2(\eta/2^j)\right)$. We first show (see [3], [5]) that for $y = \sin^2(\eta/2)$,

$$P_N(y) \le 3^{N-1} \qquad \text{if} \quad 0 \le y \le \frac{3}{4}, \tag{3.3.12}$$

and

$$P_N(4y(1-y))P_N(y) \le 3^{2(N-1)} \qquad \text{if} \quad \frac{3}{4} \le y \le 1. \tag{3.3.13}$$

In fact, it follows from (3.1.12) that $P_N(y)$ is an increasing function of y. Then for $0 \le y \le 1/2$, by (3.1.14)

$$y^N P_N(1-y) + (1-y)^N P_N(y) = 1,$$

we have

$$P_N(y) \le P_N\left(\frac{1}{2}\right) = 2^{N-1}.$$

For $1/2 \le y \le 1$, we claim that

$$P_N(y) \le (4y)^{N-1}.$$

Indeed, if $1/2 \le y \le 1$, then $1 \le 2y$. Hence

$$\begin{aligned} P_N(y) &= \sum_{k=0}^{N-1} \binom{N+k-1}{k} y^k = \sum_{k=0}^{N-1} \binom{N+k-1}{k} 2^{-k}(2y)^k \\ &\le (2y)^{N-1} \sum_{k=0}^{N-1} \binom{N+k-1}{k} 2^{-k} = (2y)^{N-1} P_N\left(\frac{1}{2}\right) = (4y)^{N-1}. \end{aligned}$$

In particular, if $0 \le y \le 3/4$, then $P_N(y) \le 3^{N-1}$. So (3.3.12) is valid.

For $3/4 \le y \le 1$, let us consider $P_N\big(4y(1-y)\big)P_N(y)$. If $0 \le 4y(1-y) \le 1/2$,

$$P_N\big(4y(1-y)\big)P_N(y) \le 2^{N-1}P_N(y) \le 2^{N-1}(4y)^{N-1} = (8y)^{N-1} \le 3^{2(N-1)}.$$

If $1/2 \le 4y(1-y) \le 1$, since $P_N(y) \le (4y)^{N-1}$,

$$\begin{aligned} P_N\big(4y(1-y)\big)P_N(y) &\le \big[4\big(4y(1-y)\big)\big]^{N-1} P_N(y) \\ &\le \big[4\big(4y(1-y)\big)\big]^{N-1} (4y)^{N-1} \\ &= \big[64y^2(1-y)\big]^{N-1}. \end{aligned}$$

It is evident that the function $y^2(1-y) = y^2 - y^3$ attains its maximum at $y = 3/4$ on the interval $[3/4,\ 1]$. Therefore $\left[64y^2(1-y)\right]^{N-1} \leq 3^{2(N-1)}$ and (3.3.13) is valid.

We now estimate $\prod_{j=2}^{J+2} P_N\left(\sin^2\left(\eta/2^j\right)\right)$. First observe that for $y = \sin^2(\eta/2^j)$,

$$\sin^2(\eta/2^{j-1}) = 4y(1-y).$$

It follows from (3.3.12) and (3.3.13) that if $\sin^2(\eta/2^j) \leq 3/4$, then

$$P_N\left(\sin^2(\eta/2^j)\right) \leq 3^{N-1}.$$

If $\sin^2(\eta/2^j) > 3/4$, then

$$P_N\left(\sin^2(\eta/2^{j-1})\right) P_N\left(\sin^2(\eta/2^j)\right) \leq 3^{2(N-1)}.$$

Consequently

$$\prod_{j=2}^{J+2} P_N\left(\sin^2\left(\frac{\eta}{2^j}\right)\right) \leq (3^{N-1})^{\ell} \left(\prod_{j=2}^{J+2-\ell} P_N\left(\sin^2\left(\frac{\eta}{2^j}\right)\right)\right) \quad \text{for} \quad \ell \in \{1,2\}.$$

We now play the same trick for $\eta/2^{J-\ell}$ and keep doing so until we can't go on and at that moment no term left if J is odd and only the first term $P_N\left(\sin^2(\eta/2^2)\right)$ left if J is even. Therefore we obtain

$$\prod_{j=2}^{J+2} P_N\left(\sin^2\left(\frac{\eta}{2^j}\right)\right) \leq \left(3^{N-1}\right)^J \sup_{0\leq y\leq 1} P_N(y).$$

But $P_N(y) \leq 4^{N-1}$ for $0 \leq y \leq 1$. Therefore

$$\prod_{j=2}^{J+2} P_N\left(\sin^2\left(\frac{\eta}{2^j}\right)\right) \leq \left(4 \cdot 3^J\right)^{N-1} \tag{3.3.14}$$

Substituting (3.3.11) and (3.3.14) to (3.3.10), we obtain

$$\hat{\varphi}_N(\eta) \leq \frac{1}{2^{2J+1}} \left(2\left(\frac{3}{4}\right)^J \right)^{N-1} \qquad \text{for } 2^J\pi \leq \eta \leq 2^{J+1}\pi. \quad \blacksquare$$

Remark: It is evident that, in order to apply this lemma, we require $2(3/4)^J < 1$, which is the case for $J \geq 3$. But recall that (3.3.9) provides a lower bound for I_1 with exponential decay $(\sin(13\pi/32))^{2N-1}$. Observe that $\sin(13\pi/32) \approx 0.957$ and on the other hand $\sqrt{2}(3/4)^{3/2} \approx 0.919$ which is very close to $\sin(13\pi/32)$. In other words, the value of J has to be enlarged for better exponential decay of φ_N. Our computations suggest that J should start from 5 and this is the reason that I_ℓ^+ and I_ℓ^-, $\ell = 2, 3$, are introduced. This lemma serves for estimate of upper bounds for I_3^+ and I_3^-.

We now estimate upper bounds for I_3^+ and I_3^-. To apply the lemma, we require $2^{J-1} \leq k \leq 2^J - 1$, $J \geq 5$. Consequently, by (3.3.6)

$$\begin{aligned} |I_3^+| &\leq \sum_{J=5}^{\infty} \sum_{k=2^{J-1}}^{2^J-1} \left| \frac{\sin(k\pi x)}{\sin(\pi x)} \right| \int_0^\pi \frac{1}{2^{2J+1}} \left(2\left(\frac{3}{4}\right)^J \right)^{N-1} \left|\sin\big(x(\xi + k\pi)\big)\right| d\xi \\ &= 2^{N-1} \sum_{J=5}^{\infty} \frac{1}{2^{2J+1}} \left(\left(\frac{3}{4}\right)^{N-1} \right)^J \sum_{k=2^{J-1}}^{2^J-1} \left| \frac{\sin(k\pi x)}{\sin(\pi x)} \right| \int_0^\pi \left|\sin\big(x(\xi + k\pi)\big)\right| d\xi. \end{aligned}$$

Note that $|\sin(k\pi x)| \leq k|\sin(\pi x)|$. Moreover, if

$$2^{J-1} \leq k \leq 2^J - 1 \quad \text{and} \quad 0 \leq \xi \leq \pi,$$

then $\xi + k\pi \leq 2^J\pi$. Hence

$$\left|\sin\big(x(\xi + k\pi)\big)\right| \leq x(\xi + k\pi) \leq x2^J\pi,$$

so

$$\sum_{k=2^{J-1}}^{2^J-1}\left|\frac{\sin(k\pi x)}{\sin(\pi x)}\right|\int_0^\pi\left|\sin\big(x(\xi+k\pi)\big)\right|d\xi\le x2^J\pi^2\sum_{k=2^{J-1}}^{2^J-1}k.$$

But

$$\sum_{k=2^{J-1}}^{2^J-1}k=\frac{(2^{J-1}+2^J-1)(2^J-1-2^{J-1}+1)}{2}\le\frac{2^{J+1}2^{J-1}}{2}=2^{2J-1}.$$

Therefore,

$$\begin{aligned}|I_3^+|&\le 2^{N-1}\sum_{J=5}^{\infty}\frac{1}{2^{2J+1}}\left(\left(\frac{3}{4}\right)^{N-1}\right)^J x2^J\pi^2 2^{2J-1}\\&=\frac{x\pi^2}{4}2^{N-1}\sum_{J=5}^{\infty}\left(2\left(\frac{3}{4}\right)^{N-1}\right)^J.\end{aligned}$$

Observe that the first term of the series $\sum_{J=5}^{\infty}\left(2\,(3/4)^{N-1}\right)^J$ is $32\,(3/4)^{5(N-1)}$, and its common ratio $2\,(3/4)^{N-1}<1/5$ for $N\ge 10$. Hence

$$\sum_{J=5}^{\infty}\left(2\left(\frac{3}{4}\right)^{N-1}\right)^J\le 40\left(\frac{3}{4}\right)^{5(N-1)}\qquad\text{for } N\ge 10.$$

Therefore, an upper bound for $|I_3^+|$ is given by

$$\begin{aligned}|I_3^+|&\le\left(\frac{x\pi^2}{4}2^{N-1}\right)\left(40\left(\frac{3}{4}\right)^{5(N-1)}\right)\\&=10x\pi^2\left(2\left(\frac{3}{4}\right)^5\right)^{N-1}\\&=\frac{10x\pi^2}{\sqrt{2}(3/4)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.\end{aligned}$$

Similarly, replacing k by $-\ell$, we have

$$\begin{aligned}|I_3^-|&=\left|\sum_{k=-2}^{-\infty}\frac{\sin(k\pi x)}{\sin(\pi x)}\int_0^\pi\hat{\varphi}_N(\xi+2k\pi)\sin\big(x(\xi+k\pi)\big)d\xi\right|\\&\le\sum_{\ell=2}^{\infty}\left|\frac{\sin(k\pi x)}{\sin(\pi x)}\right|\int_0^\pi\hat{\varphi}_N(2\ell\pi-\xi)\left|\sin\big(x(\ell\pi-\xi)\big)\right|d\xi\\&\le\frac{10x\pi^2}{\sqrt{2}(3/4)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.\end{aligned}$$

Therefore

$$|I_3^+| + |I_3^-| \le \frac{20x\pi^2}{\sqrt{2}(3/4)^{5/2}} \left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}. \tag{3.3.15}$$

Step 4: An upper bound for $|I_2^+|$ and $|I_2^-|$.

We next estimate $|I_2^+|$ and $|I_2^-|$.

Lemma 3.12 *For $J = 1, 2, 3, 4$, i.e., $2\pi < \eta < 32\pi$,*

$$\hat{\varphi}_N(\eta) \le \frac{N\pi}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}.$$

Proof: We first consider the case $J = 1$, i.e., $2\pi < \eta < 4\pi$. Note that

$$\hat{\varphi}_N(\eta) = \prod_{k=1}^{\infty} H_N\left(\frac{\eta}{2^k}\right) \le H_N\left(\frac{\eta}{2}\right) H_N\left(\frac{\eta}{4}\right).$$

Moreover, recalling that $\frac{1}{\rho_N}\int_0^\pi (\sin t)^{2N-1} dt = 1$, we have

$$\begin{aligned} H_N\left(\frac{\eta}{4}\right) &= 1 - \frac{1}{\rho_N}\int_0^{\eta/4} (\sin t)^{2N-1} dt \\ &= \frac{1}{\rho_N}\int_0^{\pi} (\sin t)^{2N-1} dt - \frac{1}{\rho_N}\int_0^{\eta/4} (\sin t)^{2N-1} dt \\ &= \frac{1}{\rho_N}\int_{\eta/4}^{\pi} (\sin t)^{2N-1} dt \\ &= \frac{1}{\rho_N}\int_0^{\pi-\eta/4} \left(\sin\left(t + \frac{\eta}{4}\right)\right)^{2N-1} dt. \end{aligned}$$

Since $0 \le t \le \pi - \eta/4$ and $2\pi < \eta < 4\pi$, $\pi/2 < \eta/4 < t + \eta/4 < \pi$. Therefore $\sin(t + \eta/4) < \sin(\eta/4)$. Then

$$H_N\left(\frac{\eta}{4}\right) \le \frac{1}{\rho_N}\int_0^{\pi-\eta/4} \left(\sin\left(\frac{\eta}{4}\right)\right)^{2N-1} dt = \frac{1}{\rho_N}\int_0^{\zeta} (\sin(\pi - \zeta))^{2N-1} dt$$

where $\eta = 4(\pi - \zeta)$. Note that for $2\pi < \eta < 4\pi$, $0 < \zeta < \pi/2$. Therefore

$$H_N\left(\frac{\eta}{4}\right) \le \frac{\zeta}{\rho_N}(\sin\zeta)^{2N-1} \qquad \forall\, 0 < \zeta < \frac{\pi}{2}.$$

Similarly

$$\begin{aligned} H_N\left(\frac{\eta}{2}\right) &= 1 - \frac{1}{\rho_N}\int_0^{\eta/2}(\sin t)^{2N-1}dt \\ &= \frac{1}{\rho_N}\int_0^{\pi}(\sin t)^{2N-1}dt - \frac{1}{\rho_N}\int_0^{2(\pi-\zeta)}(\sin t)^{2N-1}dt \\ &= -\frac{1}{\rho_N}\int_{\pi}^{2(\pi-\zeta)}(\sin t)^{2N-1}dt \\ &= \frac{1}{\rho_N}\int_0^{\pi-2\zeta}(\sin t)^{2N-1}dt. \end{aligned}$$

If $0 < \zeta < \pi/4$, we have

$$\hat{\varphi}_N(\eta) \le H_N\left(\frac{\eta}{2}\right)H_N\left(\frac{\eta}{4}\right) \le H_N\left(\frac{\eta}{4}\right) \le \frac{\pi}{4\rho_N}\left(\sin\left(\frac{\pi}{4}\right)\right)^{2N-1} < \frac{N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}.$$

If $\pi/4 \le \zeta < \pi/2$, then $0 < \pi - 2\zeta \le \pi/2$. Hence

$$\begin{aligned} \hat{\varphi}_N(\eta) &\le H_N\left(\frac{\eta}{2}\right)H_N\left(\frac{\eta}{4}\right) \le \left(\frac{1}{\rho_N}\int_0^{\pi-2\zeta}(\sin t)^{2N-1}dt\right)\left(\frac{1}{\rho_N}\zeta\left(\sin\zeta\right)^{2N-1}\right) \\ &\le \frac{1}{\rho_N^2}(\pi - 2\zeta)\,\zeta\,\left(\sin 2\zeta\,\sin\zeta\right)^{2N-1}. \end{aligned}$$

Observe that

$$\max_{\pi/4 \le \zeta < \pi/2}(\pi - 2\zeta)\,\zeta = \frac{\pi^2}{8} \quad \text{and} \quad \max_{\pi/4 \le \zeta < \pi/2}\sin 2\zeta\,\sin\zeta = \frac{4\sqrt{3}}{9}.$$

Moreover, by Lemma 3.3 in §3.1, $1/\rho_N < \sqrt{N/\pi}$. Thus

$$\hat{\varphi}_N(\eta) \le \frac{N\pi}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}.$$

Similarly, for $J = 2, 3, 4$, by choosing

$$\hat{\varphi}_N(\eta) \le H_N\left(\frac{\eta}{4}\right)H_N\left(\frac{\eta}{8}\right),$$

$$\hat{\varphi}_N(\eta) \leq H_N\left(\frac{\eta}{8}\right) H_N\left(\frac{\eta}{16}\right)$$

and

$$\hat{\varphi}_N(\eta) \leq H_N\left(\frac{\eta}{16}\right) H_N\left(\frac{\eta}{32}\right)$$

respectively, one can easily verify that

$$\hat{\varphi}_N(\eta) \leq \frac{N\pi}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}$$

using the same procedure we did for $J = 1$. ■

We now estimate $|I_2^+|$ and $|I_2^-|$. It follows from (3.3.5) and the lemma that

$$\begin{aligned} |I_2^+| &\leq \sum_{k=1}^{16} \left|\frac{\sin(k\pi x)}{\sin(\pi x)}\right| \int_0^\pi \hat{\varphi}_N(\xi + 2k\pi)\left|\sin\big(x(\xi + k\pi)\big)\right| d\xi \\ &\leq \frac{N\pi}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1} \sum_{k=1}^{16} kx(\pi + k\pi)\pi \\ &= \frac{1632x\pi^3 N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}. \end{aligned}$$

Similarly, replacing k by $-\ell$, we have

$$\begin{aligned} |I_2^-| &\leq \sum_{\ell=2}^{16} \left|\frac{\sin(\ell\pi x)}{\sin(\pi x)}\right| \int_0^\pi \hat{\varphi}_N(2\ell\pi - \xi)\left|\sin\big(x(\ell\pi - \xi)\big)\right| d\xi \\ &\leq \frac{N\pi}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1} \sum_{\ell=2}^{16} \ell x(\ell\pi)\pi \\ &= \frac{1495x\pi^3 N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}. \end{aligned}$$

Therefore an upper bound for $|I_2^+| + |I_2^-|$ is given by

$$|I_2^+| + |I_2^-| \leq \frac{3127x\pi^3 N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}. \tag{3.3.16}$$

Step 5: Proof of theorem 3.10

We first show

$$I := \frac{\sin(\pi x)}{\pi x} - \varphi_N(x) > 0.$$

We are now in a position to demonstrate that I_1 is the main term in the bi-infinite series (3.3.3). Comparing with (3.3.9), (3.3.15) and (3.3.16), one can easily verify that for $N \geq 42$ and $0 < x < 1$,

$$\frac{9}{10} I_1 > |I_2^+| + |I_2^-| \quad \text{and} \quad \frac{1}{10} I_1 > |I_3^+| + |I_3^-|.$$

Consequently, we obtain that $I > 0$ for $N \geq 42$ and $0 < x < 1$, i.e.,

$$\varphi_N(x) < \sin(\pi x)/\pi x.$$

We next show, for $N \geq 30$ and $0 < x < 1$, $\varphi_N(x) > 0$, or, equivalently,

$$I < \frac{\sin(\pi x)}{\pi x}.$$

To this end, it suffices to show

$$I_1 + |I_2^+| + |I_2^-| + |I_3^+| + |I_3^-| < \frac{1}{2x}.$$

For this purpose, instead of considering a lower bound for I_1, we need an upper bound for I_1. By cutting the interval $[0, \ \pi]$ of the integration of (3.3.4) to two pieces, we obtain

$$I_1 = \int_0^\pi \hat{\varphi}_N(\pi + \xi)\sin(x\xi)d\xi = I_{11} + I_{12}$$

where

$$I_{11} := \int_0^{\frac{\pi}{3}} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi$$

and

$$I_{12} := \int_{\frac{\pi}{3}}^{\pi} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi$$

We first estimate I_{11}. Note that

$$\hat{\varphi}_N(\pi) \leq H_N\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

and $\hat{\varphi}_N(\eta)$ is a decreasing function of η for $0 \leq \eta \leq 2\pi$. Then an upper bound for I_{11} is given by

$$I_{11} \leq \hat{\varphi}_N(\pi)\int_0^{\frac{\pi}{3}} \sin(x\xi)d\xi \leq -\frac{1}{2x}\cos(x\xi)\Big|_0^{\pi/3} = \frac{1}{x}\left(\sin\left(\frac{\pi x}{6}\right)\right)^2 \leq \frac{1}{x}\left(\sin\left(\frac{\pi}{6}\right)\right)^2,$$

i.e.

$$I_{11} \leq \frac{1}{4x}. \tag{3.3.17}$$

We now investigate

$$I_{12} = \int_{\frac{\pi}{3}}^{\pi} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi.$$

Observe that

$$I_{12} \leq \hat{\varphi}_N\left(\frac{4\pi}{3}\right)\int_{\frac{\pi}{3}}^{\pi} \sin(x\xi)d\xi \leq H_N\left(\frac{2\pi}{3}\right)\int_{\frac{\pi}{3}}^{\pi} \sin(x\xi)d\xi.$$

Moreover

$$\begin{aligned} H_N\left(\frac{2\pi}{3}\right) &= 1 - \frac{1}{\rho_N}\int_0^{\frac{2\pi}{3}} (\sin t)^{2N-1}dt \\ &= \frac{1}{\rho_N}\int_0^{\frac{\pi}{3}} (\sin t)^{2N-1}dt \\ &\leq \frac{1}{\rho_N}\left(\sin\left(\frac{\pi}{3}\right)\right)^{2N-1}\frac{\pi}{3}. \end{aligned}$$

But $\frac{1}{\rho_N} < \sqrt{\frac{N}{\pi}}$ and

$$\int_{\frac{\pi}{3}}^{\pi} \sin(x\xi)d\xi = -\frac{1}{x}\cos(x\xi)\Big|_{\pi/3}^{\pi} = \frac{2}{x}\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{2\pi x}{3}\right) \le \frac{\sqrt{3}}{x},$$

therefore

$$I_{12} \le \frac{1}{x}\left(\frac{\pi}{3}\sqrt{\frac{3N}{\pi}}\left(\sin\left(\frac{\pi}{3}\right)\right)^{2N-1}\right) = \frac{1}{x}\left(\sqrt{\frac{N\pi}{3}}\left(\sin\left(\frac{\pi}{3}\right)\right)^{2N-1}\right). \qquad (3.3.18)$$

From (3.3.17) and (3.3.18), an upper bound for I_1 is given by, for $N \ge 22$,

$$I_1 = I_{11} + I_{12} < \frac{1}{x}\left(\frac{1}{4} + \sqrt{\frac{N\pi}{3}}\left(\sin\left(\frac{\pi}{3}\right)\right)^{2N-1}\right) \le 0.26\,\frac{1}{x}. \qquad (3.3.19)$$

On the other hand by (3.3.15)

$$|I_3^+| + |I_3^-| \le \frac{20x\pi^2}{\sqrt{2}(3/4)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.$$

But $0 < x < 1$, then

$$|I_3^+| + |I_3^-| \le \frac{20\pi^2}{x\sqrt{2}(3/4)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.$$

Similarly, from (3.3.16)

$$|I_2^+| + |I_2^-| \le \frac{3127\pi^3 N}{8x}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1}.$$

One can calculate that for $N \ge 30$,

$$\frac{20\pi^2}{\sqrt{2}(3/4)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1} + \frac{3127\pi^3 N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1} < 0.24. \qquad (3.3.20)$$

Therefore, it follows from (3.3.19) and (3.3.20) that

$$I_1 + |I_2^+| + |I_2^-| + |I_3^+| + |I_3^-| < \frac{1}{2x}.$$

Consequently, we proved $I < \frac{\sin(\pi x)}{\pi x}$. Therefore we conclude that for $N \geq 42$ and $0 < x < 1$,

$$0 < \frac{\sin(\pi x)}{\pi x} - \varphi_N(x) < \frac{\sin(\pi x)}{\pi x}$$

and the proof of Theorem 3.10 is complete. ■

3.4 The case $1 < x < 2N - 1$

As we pointed out before, the DD-functions φ_N do not possess sign-regularity property. In other words, the graphs of φ_N usually are not contained inside the graph of the sinc-function. Therefore, instead of carrying on this mission, we prove

Theorem 3.13 *For $N \geq 40$ and $x \in \mathbb{R}$*

$$|\varphi_N(x)| \leq \left|\frac{\sin(\pi x)}{\pi x}\right|.$$

To this end, it suffices to show that for $N \geq 40$ and $x \in (j,\ j+1)$, $j \in \mathbb{N}$,

$$(-1)^j I > 0 \tag{3.4.1}$$

and

$$(-1)^j I < 2(-1)^j \left(\frac{\sin(\pi x)}{\pi x}\right). \tag{3.4.2}$$

with

$$I = \frac{\sin \pi x}{\pi x} - \varphi_N(x) = \frac{2}{\pi} \sin(\pi x)(I_1 + I_2^+ + I_2^- + I_3^+ + I_3^-)$$

where I_1, I_ℓ^+ and I_ℓ^-, $\ell = 2, 3$, are given in §3.3.

First we recall a result which we proved for the fundamental cardinal spline $\psi_m(x)$ (see §2.4, Lemma 2.3) and it is easily seen that this result is valid for the DD-functions $\varphi_N(x)$ as well.

Lemma 3.14 *For $x > 1$,*

$$I_1 = \int_0^\pi \hat{\varphi}_N(2\pi - \xi)\sin\big(x(\pi - \xi)\big)d\xi$$
$$\geq \int_{\pi-\frac{\pi}{x}}^{\pi} \left(\hat{\varphi}_N(2\pi - \xi) - \hat{\varphi}_N\left(2\pi - \xi + \frac{\pi}{x}\right)\right)\sin\big(x(\pi - \xi)\big)d\xi > 0. \quad \blacksquare$$

Proof of theorem 3.13: We first show (3.4.1). For the case $1 < x < 8$, from the lemma above

$$I_1 \geq \int_{\frac{7\pi}{8}}^{\pi} \left(\hat{\varphi}_N(2\pi - \xi) - \hat{\varphi}_N\left(2\pi - \xi + \frac{\pi}{8}\right)\right)\sin\big(x(\pi - \xi)\big)d\xi > 0.$$

On the other hand, by the mean value theorem, for $7\pi/8 \leq \xi \leq \pi$,

$$\hat{\varphi}_N(2\pi - \xi) - \hat{\varphi}_N\left(2\pi - \xi + \frac{\pi}{8}\right) = -\hat{\varphi}_N'(\eta)\frac{\pi}{8}$$

where η depends on ξ and satisfies $2\pi - \xi < \eta < 2\pi - \xi + \pi/8$. Hence

$$\begin{aligned} I_1 &\geq \int_{7\pi/8}^{\pi} \left(\hat{\varphi}_N(2\pi - \xi) - \hat{\varphi}_N\left(2\pi - \xi + \frac{\pi}{8}\right)\right)\sin\big(x(\pi - \xi)\big)d\xi \\ &= \left(\frac{\pi}{8}\right)\int_{7\pi/8}^{\pi} (-\hat{\varphi}_N'(\eta))\sin\big(x(\pi - \xi)\big)d\xi. \end{aligned} \tag{3.4.3}$$

But recall that

$$\hat{\varphi}_N'(\eta) = \sum_{j=1}^{\infty}\left(-\frac{1}{2^j\rho_N}\right)\sin^{2N-1}\left(\frac{\eta}{2^j}\right)\prod_{k=1}^{j-1} H_N\left(\frac{\eta}{2^k}\right)\prod_{k=j+1}^{\infty} H_N\left(\frac{\eta}{2^k}\right).$$

Thus

$$
\begin{aligned}
-\hat{\varphi}_N'(\eta) &= \sum_{j=1}^{\infty}\left(\frac{1}{2^j\rho_N}\right)\sin^{2N-1}\left(\frac{\eta}{2^j}\right)\prod_{k=1}^{j-1}H_N\left(\frac{\eta}{2^k}\right)\prod_{k=j+1}^{\infty}H_N\left(\frac{\eta}{2^k}\right) \\
&\geq \left(\frac{1}{2\rho_N}\right)\sin^{2N-1}\left(\frac{\eta}{2}\right)\prod_{k=2}^{\infty}H_N\left(\frac{\eta}{2^k}\right).
\end{aligned}
\tag{3.4.4}
$$

Since

$$
2\pi-\xi<\eta<2\pi-\xi+\frac{\pi}{8} \quad \text{and} \quad \frac{7\pi}{8}\leq\xi\leq\pi,
$$

then $\pi \leq \eta \leq \pi+\pi/4$. It follows from Lemma 3.5 in § 3.1 that, for $N \geq 10$,

$$
\prod_{k=2}^{\infty}H_N\left(\frac{\eta}{2^k}\right)\geq\frac{3}{4}.
$$

Hence

$$
-\hat{\varphi}_N'(\eta)\geq\frac{3}{8\rho_N}\left(\sin\left(\frac{5\pi}{8}\right)\right)^{2N-1}, \qquad \pi<\eta<\frac{5\pi}{4}. \tag{3.4.5}
$$

Moreover, for $1 < x < 8$

$$
\int_{\frac{7\pi}{8}}^{\pi}\sin\bigl(x(\pi-\xi)\bigr)d\xi=\frac{2\sin^2\left(\frac{\pi x}{16}\right)}{x}\geq\frac{2\sin\left(\frac{\pi x}{16}\right)\frac{2}{\pi}\left(\frac{\pi x}{16}\right)}{x}\geq\frac{\sin\left(\frac{\pi}{16}\right)}{4}. \tag{3.4.6}
$$

Substituting (3.4.5) and (3.4.6) to (3.4.3), we obtain a lower bound for I_1 in the case $1 < x < 8$

$$
I_1\geq\frac{3\pi\sin\left(\frac{\pi}{16}\right)}{256\rho_N}\left(\sin\left(\frac{5\pi}{8}\right)\right)^{2N-1}. \tag{3.4.7}
$$

The estimate for I_ℓ^+, $\ell = 2, 3$, is much easier than for the case $0 < x < 1$. Instead of using $\bigl|\sin\bigl(x(\xi+k\pi)\bigr)\bigr| \leq x(\xi+k\pi) \leq 2^{J+1}\pi$, we use $\bigl|\sin\bigl(x(\xi+k\pi)\bigr)\bigr| \leq 1$.

Consequently, by (3.3.15) and Lemma 3.11 in §3.3

$$|I_3^+| \le \sum_{J=5}^{\infty} \sum_{k=2^{J-1}}^{2^J-1} \left|\frac{\sin(k\pi x)}{\sin(\pi x)}\right| \int_0^{\pi} \frac{1}{2^{2J+1}} \left(2\left(\frac{3}{4}\right)^J\right)^{N-1} \left|\sin\left(x(\xi+k\pi)\right)\right| d\xi$$
$$\le 2^{N-1}\pi \sum_{J=5}^{\infty} \frac{1}{2^{2J+1}} \left(\left(\frac{3}{4}\right)^{N-1}\right)^J \sum_{k=2^{J-1}}^{2^J-1} k.$$

Recall that $\sum_{k=2^{J-1}}^{2^J-1} k \le 2^{2J-1}$. Thus,

$$|I_3^+| \le 2^{N-1}\pi \sum_{J=5}^{\infty} \frac{1}{2^{2J+1}} \left(\left(\frac{3}{4}\right)^{N-1}\right)^J 2^{2J-1} = \frac{2^{N-1}\pi}{4} \sum_{J=5}^{\infty} \left(\left(\frac{3}{4}\right)^{N-1}\right)^J.$$

It is easy to verify that for $N \ge 4$, $\sum_{J=5}^{\infty} \left((3/4)^{N-1}\right)^J \le 2\,(3/4)^{5(N-1)}$, therefore an upper bound for $|I_3^+|$ is given by

$$|I_3^+| \le \frac{\pi}{2} \left(2\left(\frac{3}{4}\right)^5\right)^{N-1} = \frac{\pi}{2^{3/2}(3/4)^{5/2}} \left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}$$

and the same result holds for $|I_3^-|$. Therefore

$$|I_3^+| + |I_3^-| \le \frac{\pi}{\sqrt{2}(3/4)^{5/2}} \left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}. \tag{3.4.8}$$

Similarly, by (3.3.5) and Lemma 3.12 in §3.3

$$|I_2^+| \le \sum_{k=1}^{16} \left|\frac{\sin(k\pi x)}{\sin(\pi x)}\right| \int_0^{\pi} \hat{\varphi}_N(\xi+2k\pi) \left|\sin\left(x(\xi+k\pi)\right)\right| d\xi$$
$$\le \frac{\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1} \sum_{k=1}^{16} k = \frac{136\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1},$$

and

$$|I_2^-| \le \frac{\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1} \sum_{\ell=2}^{16} \ell = \frac{135\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1}.$$

Therefore

$$|I_2^+| + |I_2^-| \le \frac{271\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1}. \tag{3.4.9}$$

It follows from (3.4.7), (3.4.8) and (3.4.9) that, for $N \geq 35$ and $1 < x < 8$,

$$I_1 \geq |I_2^+| + |I_2^-| + |I_3^+| + |I_3^-|.$$

We next consider the case $8 < x < 2N-1$. Again, by Lemma 3.14 and the mean value theorem,

$$\begin{aligned} I_1 &= \int_0^{\pi} \hat{\varphi}_N(2\pi - \xi)\sin\big(x(\pi-\xi)\big)d\xi \\ &\geq \int_{\pi-\frac{\pi}{x}}^{\pi} \left(\hat{\varphi}_N(2\pi-\xi) - \hat{\varphi}_N\left(2\pi - \xi + \frac{\pi}{x}\right)\right)\sin\big(x(\pi-\xi)\big)d\xi > 0 \\ &= -\frac{\pi}{x}\int_{\pi-\frac{\pi}{x}}^{\pi} \hat{\varphi}_N'(\eta)\sin\big(x(\pi-\xi)\big)d\xi. \end{aligned}$$

By (3.4.4)

$$-\hat{\varphi}_N'(\eta) \geq \left(\frac{1}{2\rho_N}\right)\sin^{2N-1}\left(\frac{\eta}{2}\right)\prod_{k=2}^{\infty} H_N\left(\frac{\eta}{2^k}\right)$$

for $2\pi - \xi < \eta < 2\pi - \xi + \frac{\pi}{x}$. Observe that in this case, since $2\pi - \xi < \eta < 2\pi - \xi + \frac{\pi}{x}$ and $\pi - \frac{\pi}{x} < \xi < \pi$, we have $\pi < \eta < \pi + \frac{2\pi}{x}$. Hence

$$\frac{\pi}{2} < \frac{\eta}{2} < \frac{\pi}{2} + \frac{\pi}{x}.$$

It follows that, for $x > 8$

$$\sin\left(\frac{\eta}{2}\right) > \cos\left(\frac{\pi}{x}\right) > \cos\left(\frac{\pi}{8}\right).$$

On the other hand, note that for $\pi < \eta < \pi + \frac{2\pi}{x}$ and $x > 8$, $\pi < \eta < \pi + \frac{\pi}{4}$. Therefore for $N \geq 10$, by Lemma 3.5 in §3.1 again,

$$\prod_{k=2}^{\infty} H_N\left(\frac{\eta}{2^k}\right) > \frac{3}{4}.$$

Hence we have

$$-\hat{\varphi}_N'(\eta) \geq \left(\frac{1}{2\rho_N}\right)\sin^{2N-1}\left(\frac{\eta}{2}\right)\prod_{k=2}^{\infty} H_N\left(\frac{\eta}{2^k}\right) \geq \frac{3}{8\rho_N}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1}. \tag{3.4.10}$$

Finally $\int_{\pi-\frac{\pi}{x}}^{\pi} \sin(x(\pi-\xi))d\xi = \frac{2}{x}$. In the light of (3.4.10), we obtain a lower bound for I_1 in the case $8 < x < 2N-1$:

$$I_1 \geq \frac{3\pi}{4x^2\rho_N}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1} > \frac{3\pi}{4(2N-1)^2\rho_N}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1}. \tag{3.4.11}$$

Comparing (3.4.9), (3.4.8) and (3.4.11), one can calculate that for $8 < x < 2N-1$ and $N \geq 40$

$$\frac{3\pi}{4(2N-1)^2\rho_N}\left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1} > \frac{271\pi^2 N}{8}\left(\frac{4\sqrt{3}}{9}\right)^{2N-1} + \frac{\pi}{\sqrt{2}\left(\frac{3}{4}\right)^{5/2}}\left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.$$

Consequently, we obtain (3.4.1) for $1 < x < 2N-1$ and $N \geq 40$.

We now prove (3.4.2). For

$$I = \frac{2}{\pi}\sin(\pi x)(I_1 + I_2^+ + I_2^- + I_3^+ + I_3^-),$$

it suffices to show (remember that I_1 is positive)

$$I_1 + |I_2^+| + |I_2^-| + |I_3^+| + |I_3^-| \leq \frac{1}{x}. \tag{3.4.12}$$

As was explained before, we only need to give an upper bound for I_1. Write

$$\begin{aligned}
I_1 &= \int_0^{\pi} \hat{\varphi}_N(2\pi - \xi)\sin(x(\pi-\xi))d\xi \\
&= \int_0^{\pi} \hat{\varphi}_N(\pi + \xi)\sin(x\xi)d\xi \\
&= \left(\int_0^{\frac{\pi}{x}} + \int_{\frac{\pi}{x}}^{\pi}\right)\hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi \\
&= I_{11} + I_{12}
\end{aligned}$$

where

$$I_{11} := \int_0^{\frac{\pi}{x}} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi$$

and

$$I_{12} := \int_{\frac{\pi}{x}}^{\pi} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi.$$

It is easily seen that Corollary 2.9 in §2.5 is valid for φ_N, i.e., $I_1 > 0$ and $I_{12} < 0$. Therefore we have

$$0 < I_1 < I_{11} = \int_0^{\frac{\pi}{x}} \hat{\varphi}_N(\pi+\xi)\sin(x\xi)d\xi.$$

Rewrite

$$I_{11} = \int_0^{\frac{\pi}{x}} \frac{1}{2}\sin(x\xi)d\xi \; - \; \int_0^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi+\xi)\right)\sin(x\xi)d\xi.$$

Hence

$$I_1 < \int_0^{\frac{\pi}{x}} \frac{1}{2}\sin(x\xi)d\xi \; - \; \int_0^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi+\xi)\right)\sin(x\xi)d\xi. \tag{3.4.13}$$

Observe that the first integral in (3.4.13) is $\frac{1}{x}$. Moreover, since $\hat{\varphi}_N(\pi) \le \frac{1}{2}$ and $\hat{\varphi}_N(\eta)$ is a decreasing function on $[0,\ 2\pi]$,

$$\int_0^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi+\xi)\right)\sin(x\xi)d\xi > \int_{\frac{\pi}{2x}}^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi+\xi)\right)\sin(x\xi)d\xi > 0.$$

Consequently, in order to obtain (3.4.12), we prove

$$J := \int_{\frac{\pi}{2x}}^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi+\xi)\right)\sin(x\xi)d\xi > |I_2^+| + |I_2^-| + |I_3^+| + |I_3^-|. \tag{3.4.14}$$

We now estimate J. Since

$$\hat{\varphi}_N(\pi+\xi) \le H_N\left(\frac{\pi+\xi}{2}\right) = 1 - \frac{1}{\rho_N}\int_0^{\frac{\pi+\xi}{2}} (\sin t)^{2N-1}dt = \frac{1}{\rho_N}\int_0^{\frac{\pi-\xi}{2}} (\sin t)^{2N-1}dt$$

and

$$\frac{1}{2} = \frac{1}{\rho_N} \int_0^{\frac{\pi}{2}} (\sin t)^{2N-1} dt,$$

we have

$$\begin{aligned}
\frac{1}{2} - \hat{\varphi}_N(\pi + \xi) &\geq \frac{1}{\rho_N} \int_{\frac{\pi-\xi}{2}}^{\frac{\pi}{2}} (\sin t)^{2N-1} dt \\
&= \frac{1}{\rho_N} \int_0^{\frac{\xi}{2}} (\cos t)^{2N-1} dt \\
&\geq \frac{1}{\rho_N} \int_0^{\min(\xi/2,\, \pi/8)} (\cos t)^{2N-1} dt \\
&\geq \frac{1}{\rho_N} \left(\cos \frac{\pi}{8}\right)^{2N-1} \min\left(\frac{\xi}{2}, \frac{\pi}{8}\right).
\end{aligned}$$

Since $\frac{\pi}{2x} < \xi < \frac{\pi}{x}$ and $1 < x < 2N-1$, then $\frac{\pi}{2(2N-1)} < \xi < \frac{\pi}{2}$. Therefore for $N \geq 3$

$$\frac{1}{2} - \hat{\varphi}_N(\pi + \xi) \geq \frac{1}{\rho_N} \left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1} \frac{\pi}{4(2N-1)}. \tag{3.4.15}$$

Following (3.4.15) and noting that $\int_{\pi/2x}^{\pi/x} \sin(x\xi) d\xi = \frac{1}{x}$, we have

$$J = \int_{\frac{\pi}{2x}}^{\frac{\pi}{x}} \left(\frac{1}{2} - \hat{\varphi}_N(\pi + \xi)\right) \sin(x\xi) d\xi \geq \frac{1}{\rho_N} \left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1} \frac{\pi}{4(2N-1)} \left(\frac{1}{x}\right).$$

But since $x < 2N-1$, we obtain a lower bound for J as follows:

$$J \geq \frac{\pi}{4(2N-1)^2 \rho_N} \left(\cos\left(\frac{\pi}{8}\right)\right)^{2N-1}.$$

Now it is easily seen that for $N \geq 40$

$$\frac{\pi}{4(2N-1)^2 \rho_N} \left(\cos \frac{\pi}{8}\right)^{2N-1} \geq \frac{271\pi^2 N}{8} \left(\frac{4\sqrt{3}}{9}\right)^{2N-1} + \frac{\pi}{\sqrt{2}(3/4)^{5/2}} \left(\sqrt{2}\left(\frac{3}{4}\right)^{5/2}\right)^{2N-1}.$$

Thus (3.4.14) is valid. Consequently (3.4.2) is valid. In the light of (3.4.1), we obtain

$$|\varphi_N(x)| \leq \left|\frac{\sin(\pi x)}{\pi x}\right|, \qquad \text{for } N \geq 40 \text{ and } x \in \mathbb{R}$$

and the proof of Theorem 3.13 is complete. ■

Chapter 4

The Boolean Sum and Approximation by the DD-functions

4.1 The Boolean sum

In this chapter we shall investigate a scheme of approximation by the DD-functions. First, let us review some basic definitions involved in this chapter.

A **Boolean algebra** (see [1]) is a set B of elements $a, b, c, \cdots$ with the following properties:

(i) B has two binary operations, $\wedge$ (wedge) and $\vee$ (vee), which satisfy the idempotent laws $a \wedge a = a \vee a = a$, the commutative laws $a \wedge b = b \wedge a$, $a \vee b = b \vee a$, and the associative laws $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$.

(ii) These operations satisfy the absorption laws: $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.

(iii) These operations are mutually distributive: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(iv) B contains universal bounds O, I which satisfy $O \wedge a = O$, $O \vee a = a$, $I \wedge a = a$, $I \vee a = I$.

(v) B has a unary operation $a \to a'$ of complementation, which obeys the laws $a \wedge a' = O$, $a \vee a' = I$.

A **lattice** is a set L of elements, with two binary operations $\wedge$ and $\vee$ which satisfy (i) and (ii). If in addition the distributive laws (iii) hold, L is called a **distributive lattice**.

The theory of distributive lattices was applied by Gordon [17] to multivariate interpolation. Following [14], we call such a method the **Boolean method**, since Boolean algebras play a key role in its development. In its early application, the Boolean method requires that the operators involved (see (4.1.1) below) must commute with each other [17]. This property is heavily used in [14] as well (see also [13]). In several important situations, however, some basic approximation operators fail to meet the requirement (see [18] and [23]). Therefore, new Boolean methods which are valid for both commutative and noncommutative operators are needed. Recently, a general theory of Boolean methods is developed by Jia in such a way that it can be applied to noncommutative operators (see [23]).

In what follows, we adopt the method used in [23] and apply the new Boolean method to the bivariate approximation. It should be emphasized that *all* theorems presented in this chapter are taken, or, are specified from those in [23] and the author rewrites them here just for the reader's convenience.

To begin with, let us introduce some basic notation employed throughout the chapter. We denote by $\mathbb{N}$ and by $\mathbb{Z}_+$, as usual, the sets of all positive and nonnegative integers respectively. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. The **length** of α is defined to be

$$|\alpha| := \alpha_1 + \alpha_2.$$

Given $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, $\alpha \leq \beta$ means

$$\alpha_i \leq \beta_i \qquad \text{for all} \quad i = 1, 2.$$

We define

$$\alpha \wedge \beta := \big(\min\{\alpha_1,\ \beta_1\},\ \min\{\alpha_2,\ \beta_2\}\big),$$

and

$$\alpha \vee \beta := \big(\max\{\alpha_1,\ \beta_1\},\ \max\{\alpha_2,\ \beta_2\}\big).$$

Then it is easily seen that $(\mathbb{N}^2, \wedge, \vee)$ is a distributive lattice.

Now let G be a free abelian group with a basis $\{\Omega_\alpha : \alpha \in \mathbb{N}^2\}$ (see [21]). Then any element f of G has a unique representation of the form

$$f = \sum_{\alpha \in \mathbb{N}^2} a_\alpha \Omega_\alpha,$$

where a_α are integers and $a_\alpha = 0$ except for finitely many α. Given two elements $f, g \in G$, $f = \sum_{\alpha \in \mathbb{N}^2} a_\alpha \Omega_\alpha$ and $g = \sum_{\beta \in \mathbb{N}^2} b_\beta \Omega_\beta$, the product of f and g, denoted by $f \circ g$, is defined by

$$f \circ g := \sum_{\alpha, \beta \in \mathbb{N}^2} a_\alpha b_\beta \Omega_{\alpha \wedge \beta}.$$

It is easily seen that

$$f \circ g = g \circ f, \quad (f \circ g) \circ h = f \circ (g \circ h), \quad f \circ (g + h) = f \circ g + f \circ h,$$

where $+$ is the addition in G. Now the **Boolean sum** of f and g in G is defined by the rule

$$f \oplus g := f + g - f \circ g. \tag{4.1.1}$$

In contrast to what were discussed in [17] and [14], here $(G, \circ, \oplus)$ does not constitute a (distributive) lattice. But the Boolean addition given in (4.1.1) is commutative and associative. The commutative follows immediately from the definition of $\wedge$. To see the associativity, we let $f, g, h \in G$ and observe that

$$(f \oplus g) \oplus h = f + g + h - f \circ g - g \circ h - h \circ f + f \circ g \circ h = f \oplus (g \oplus h).$$

Thus, for any finitely many elements $f_1, f_2, \cdots, f_n$ of G, the Boolean sum

$$\bigoplus_{i=1}^{n} f_i := f_1 \oplus f_2 \oplus \cdots \oplus f_n$$

is well defined.

Let A be a nonempty finite subset of $\mathbb{N}^2$ and G be the free abelian group above. Consider the Boolean sum

$$\bigoplus_{\alpha \in A} \Omega_\alpha = \sum_{\beta \in \mathbb{N}^2} b_\beta \Omega_\beta. \tag{4.1.2}$$

We want to determine the coefficients b_β, $\beta \in \mathbb{N}^2$. For this purpose, we put $R_\alpha := \{\beta \in \mathbb{N}^2 : \beta \leq \alpha\}$, and $R_A := \bigcup_{\alpha \in A} R_\alpha$. In particular

$$E_2 := \{(\varepsilon_1, \varepsilon_2) : \varepsilon_j = 0 \text{ or } 1 \text{ for } j = 1, 2\} = R_e$$

where $e = (1,1)$. It is evident that

$$\sum_{\varepsilon \in R_\alpha} (-1)^{|\varepsilon|} = \left(\sum_{0 \leq \varepsilon_1 \leq \alpha_1} (-1)^{\varepsilon_1} \right) \left(\sum_{0 \leq \varepsilon_2 \leq \alpha_2} (-1)^{\varepsilon_2} \right) = \delta_{\alpha 0} \qquad \text{for all} \quad \alpha \leq e. \tag{4.1.3}$$

The set R_A is a lower set in the sense that

$$\beta \in R_A \quad \text{and} \quad \gamma \leq \beta \quad \Rightarrow \quad \gamma \in R_A.$$

Let χ_{R_A} denote the characteristic function of the set R_A. The following theorem provides a practical method to calculate coefficients b_β in two-dimensional case, which is an immediate consequence from Jia's work ([23], Theorem 3.1).

Theorem 4.1 *The coefficients* b_β, $\beta \in \mathbb{N}^2$, *in* (4.1.2) *are given by*

$$\begin{aligned} b_\beta &= \sum_{\varepsilon \in E_2} (-1)^{|\varepsilon|} \chi_{R_A}(\beta + \varepsilon) \\ &= \chi_{R_A}(\beta) - \chi_{R_A}(\beta + (0,1)) - \chi_{R_A}(\beta + (1,0)) + \chi_{R_A}(\beta + (1,1)). \end{aligned} \tag{4.1.4}$$

In particular, $b_\beta = 0$ *if* $\beta \notin R_A$ *or* $\beta + e \in R_A$.

Of particular interest for us is the case when

$$A = \{\alpha \in \mathbb{Z}_+^2 : |\alpha| = k\} \tag{4.1.5}$$

for some $k \in \mathbb{N}$. In this case, the Boolean sum has the following simple expression.

Corollary 4.2 *Let A be a nonempty subset of $\mathbb{Z}_+^2$ expressed in* (4.1.5). *Then*

$$\bigoplus_{|\alpha|=k} \Omega_\alpha = \sum_{|\beta|=k} \Omega_\beta - \sum_{|\beta|=k-1} \Omega_\beta. \tag{4.1.6}$$

Proof: By Theorem 4.1 we have $b_\beta = 0$ if $|\beta| > k$ or $|\beta| \le k-2$. The latter is true because $|\beta| \le k-2$ implies

$$|\beta + e| = |\beta| + |e| \le k - 2 + 2 = k.$$

Then $\beta + e \in R_A$ and hence $b_\beta = 0$ by Theorem 4.1. Therefore only those coefficients b_β whose indices β satisfy the condition $k-1 \le |\beta| \le k$ are nonzero. On the other hand, observe that

$$\chi_{R_A}(\beta + \varepsilon) = 1 \Longleftrightarrow |\beta + \varepsilon| \le k \Longleftrightarrow |\varepsilon| \le k - |\beta|.$$

Now, if $|\beta| = k$, then $\chi_{R_A}(\beta + \varepsilon) = 1$ only if $\varepsilon = 0$; hence in this case $b_\beta = \chi_{R_A}(\beta) = 1$. If $|\beta| = k-1$, then $\chi_{R_A}(\beta + \varepsilon) = 1$ if and only if $\varepsilon = (0,1),\ (1,0),$ or $(1,1)$; hence in this case

$$b_\beta = -\chi_{R_A}(\beta + (0,1)) - \chi_{R_A}(\beta + (1,0)) + \chi_{R_A}(\beta + (1,1)) = -1. \quad \blacksquare$$

Remark: In the multivariate case, a formula for calculating b_β, $\beta \in \mathbb{Z}_+^d$, where $d \ge 2$, was given by Delvos in [13] in which the commutativity of Ω_α is required. By using a different approach, Jia obtained the same result which is valid for both commutative and noncommutative operators ([23] p.124).

4.2 Approximation by the DD-functions

In this section, we shall apply the Boolean sum established in (4.1.6) to construct a bivariate operator $\mathcal{P}$ which gives rise to a desirable approximation scheme. Then $\mathcal{P}$ will be used to produce surfaces when initial datas are given. Before proceeding further, let us introduce some notation.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and let q_α be the monomial $x^{\alpha_1}y^{\alpha_2}$ where $(x, y) \in \mathbb{R}^2$. We denote by Π_k the linear span of all bivariate polynomials of total degree at most k. Under some circumstances, Π_k also stands for the linear space of all *univariate* polynomials of degree $\leq k$.

Next let us recall some facts regarding the (univariate) **polynomial reproducibility**. Assume that φ is a compactly supported distribution on $\mathbb{R}$ and b is a sequence in $\ell(\mathbb{Z})$. The **semiconvolution** of φ with b is defined by

$$\varphi *' b := \sum_{j \in \mathbb{Z}} \varphi(\cdot - j) b(j).$$

Let $\mathbb{S}(\varphi)$ denote the linear space $\{\varphi *' b : \ b \in \ell(\mathbb{Z})\}$. We call $\mathbb{S}(\varphi)$ the **shift-invariant** space generated by φ. We say that φ has **accuracy** k, if $\mathbb{S}(\varphi) \supset \Pi_{k-1}$.

Now suppose φ_N, $N \in \mathbb{N}$, is the DD-functions introduced in Chapter 3. It was proved by Deslauriers and Dubuc ([12] Lemma 4.1) that φ_N has accuracy $2N$ for each $N \in \mathbb{N}$.

Remark. The fact that φ_N has accuracy $2N$ can be also derived from a more general discussion regarding the sum rules (see [24–25] for the details).

Let $N \in \mathbb{N}$. We define the cardinal interpolation operator $\mathcal{P}_N$ associated with the DD-functions φ_N, $N \in \mathbb{N}$, by

$$\mathcal{P}_N(f) := \sum_{j \in \mathbb{Z}} f(j)\varphi_N(\cdot - j). \tag{4.2.1}$$

Since φ_N is a fundamental function, $\mathcal{P}_N(f)(j) = f(j)$ for all $j \in \mathbb{Z}$ and $\mathcal{P}_N$ is a projector on $\mathbb{S}(\varphi_N)$ (i.e. $\mathcal{P}_N^2 = \mathcal{P}_N$). Note that $\mathbb{S}(\varphi_N) \supset \Pi_{2N-1}$. Thus

$$\mathcal{P}_N(p) = p \qquad \forall\, p \in \Pi_{2N-1}. \tag{4.2.2}$$

We caution the reader that

$$\mathcal{P}_{N_1}\mathcal{P}_{N_2} \neq \mathcal{P}_{N_2}\mathcal{P}_{N_1} \qquad \text{if} \quad N_1 \neq N_2.$$

In other words, $\mathcal{P}_N$ is one of such examples that the commutativity does *not* hold.

For any $n \in \mathbb{N}$, define

$$\mathcal{P}_{N,n}(f)(x) := \sum_{j \in \mathbb{Z}} f(j/n)\varphi_N(nx - j), \qquad x \in \mathbb{R}. \tag{4.2.3}$$

It is evident that

$$\mathcal{P}_{N,n}(p) = p \qquad \forall\, p \in \Pi_{2N-1}. \tag{4.2.4}$$

In the following, we shall discuss polynomial reproducibility. To distinguish between univariate and bivariate cases, we always assume that $\alpha,\ \beta,\ \gamma \in \mathbb{Z}_+^2$ and $\alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{Z}_+,\ i = 1, 2$.

Let $\mathcal{P}^x_{\alpha_1,n}$ and $\mathcal{P}^y_{\alpha_2,m}$ be the **parametrically extended cardinal interpolation operators** on $C(\mathbb{R}^2)$ defined by

$$\mathcal{P}^x_{\alpha_1,n}(F)(x,y) = \sum_{j\in\mathbb{Z}} F(j/n, y)\varphi_{\alpha_1}(nx-j), \qquad n,\ \alpha_1 \in \mathbb{N},$$

and

$$\mathcal{P}^y_{\alpha_2,m}(F)(x,y) = \sum_{k\in\mathbb{Z}} F(x, k/m)\varphi_{\alpha_2}(my-k), \qquad m,\ \alpha_2 \in \mathbb{N}.$$

For $x,\ y \in \mathbb{R}$, let $\mathcal{P}_{(\alpha_1,\,\alpha_2)}$ be the tensor product of $\mathcal{P}^x_{\alpha_1,n}$ and $\mathcal{P}^y_{\alpha_2,m}$ (we omit the indices $n,\ m$ if the scalings are clear from the context) given by

$$\begin{aligned}\mathcal{P}_{(\alpha_1,\,\alpha_2)}(F)(x,y) &:= \mathcal{P}^x_{\alpha_1,n} \otimes \mathcal{P}^y_{\alpha_2,m}(F)(x,y)\\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} F\left(\frac{j}{n}, \frac{k}{m}\right)\varphi_{\alpha_1,n}(nx-j)\varphi_{\alpha_2,m}(my-k).\end{aligned} \tag{4.2.5}$$

The following lemma demonstrates that the univariate polynomial reproducibility (4.2.2) and (4.2.4) can be transferred to the multivariate case (for us, it is bivariate polynomial reproducibility). Let $\mathcal{P}_{(\alpha_1,\,\alpha_2)}$ be given in (4.2.5).

Lemma 4.3

(i) $\mathcal{P}_{(\alpha_1,\,\alpha_2)}(q_\gamma) = q_\gamma$ *for all* $\gamma \le 2\alpha - e$.

(ii) Given $\alpha = (\alpha_1,\ \alpha_2)$, $\beta = (\beta_1\ \beta_2)$ *and* $\gamma = (\gamma_1,\ \gamma_2) \in \mathbb{N}^2$. *Assume that* α, β *and* γ *satisfy the following conditions:* $\alpha_1 = \beta_1$ *and* $\gamma_2 \le \min\{2\alpha_2 - 1,\ 2\beta_2 - 1\}$, *or,* $\alpha_2 = \beta_2$

and $\gamma_1 \le \min\{2\alpha_1 - 1,\ 2\beta_1 - 1\}$. *Then*

$$\mathcal{P}_{(\alpha_1,\,\alpha_2)}(q_\gamma) = \mathcal{P}_{(\beta_1,\,\beta_2)}(q_\gamma).$$

Proof: First, observe that if $\gamma \le 2\alpha - e$, then $\gamma_i \le 2\alpha_i - 1$ for $i = 1, 2$. Hence a repeated use of (4.2.4) derives (i). Next, let α and β be two elements of $\mathbb{N}^2$ satisfying $\alpha_1 = \beta_1$ or $\alpha_2 = \beta_2$. Without loss of any generality, we may assume that the former holds. Then for fixed x, $q_\gamma(x, \cdot)$ is a monomial of degree γ_2. Moreover, since $\gamma_2 \le \min\{2\alpha_2 - 1,\ 2\beta_2 - 1\}$, by (4.2.4) again we have

$$\left(\mathcal{P}^y_{\alpha_2,\,m} - \mathcal{P}^y_{\beta_2,\,m}\right)\left(q_\gamma(x, \cdot)\right) = 0.$$

On the other hand, since $\alpha_1 = \beta_1$

$$\left(\mathcal{P}_{(\alpha_1,\,\alpha_2)} - \mathcal{P}_{(\beta_1,\,\beta_2)}\right)\left(q_\gamma(x, y)\right) = \mathcal{P}^x_{\alpha_1,n}\left(\mathcal{P}^y_{\alpha_2,m} - \mathcal{P}^y_{\beta_2,m}\right)\left(q_\gamma(x, y)\right).$$

Combining the above two equalities gives (ii), as desired. ■

For practical applications, we usually require that the DD-functions φ_N maintain some smoothness. Hence, in what follows we assume that $N \ge 2$. Set

$$A := \{\alpha = (\alpha_1,\ \alpha_2) \in \mathbb{N}^2 :\ \alpha_1 + \alpha_2 = k \ \text{ and } \ (\alpha_1,\ \alpha_2) \ge (N,\ N)\} \qquad (4.2.6)$$

One of main reasons that the Boolean methods are investigated and are employed successfully in many fields is that the Boolean sum of the elements $\mathcal{P}_{(\alpha_1,\,\alpha_2)}$ combines certain properties of every individual member. The precise meaning of this idea is formulated as follows. In light of Corollary 4.2, we have

Theorem 4.4 *Let A be a nonempty subset of $\mathbb{N}^2$ expressed in (4.2.6). If $k > 2N$, then the operator*

$$\mathcal{P} := \bigoplus_{\substack{\alpha_1+\alpha_2=k \\ \alpha_1,\,\alpha_2 \geq N}} \mathcal{P}_{(\alpha_1,\,\alpha_2)} = \sum_{\substack{\beta_1+\beta_2=k \\ \beta_1,\,\beta_2 \geq N}} \mathcal{P}_{(\beta_1,\,\beta_2)} - \sum_{\substack{\beta_1+\beta_2=k-1 \\ \beta_1,\,\beta_2 \geq N}} \mathcal{P}_{(\beta_1,\,\beta_2)} \tag{4.2.7}$$

satisfies

$$\mathcal{P}(q) = q \qquad \text{for all} \quad q \in \Pi_{2(k-N)-1}. \tag{4.2.8}$$

Proof: Let k and N be fixed for the time being. First it is easily seen that to show (4.2.8), it suffices to prove that (4.2.8) holds for all q_γ where $|\gamma| = \gamma_1 + \gamma_2 = 2(k-N) - 1$. Rewrite (4.2.7) to be

$$\mathcal{P} = \sum_{i=0}^{k-2N} \mathcal{P}_{(k-N-i,\, N+i)} - \sum_{j=0}^{k-2N-1} \mathcal{P}_{(k-N-1-j,\, N+j)}. \tag{4.2.9}$$

Suppose $\gamma_1 = 2(k-N) - 1 - \ell'$ and $\gamma_2 = \ell'$, $0 \leq \ell' \leq 2(k-N) - 1$. Choose $\mathcal{P}_{(k-N-\ell,\, N+\ell)}$ from the first sum in (4.2.9), $0 \leq \ell \leq k - 2N$, such that

$$2(k-N) - 1 - \ell' \leq 2(k - N - \ell) - 1 \quad \text{and} \quad \ell' \leq 2(N+\ell) - 1,$$

i.e.,

$$2\ell \leq \ell' \leq 2(N+\ell) - 1. \tag{4.2.10}$$

We now regroup (4.2.9) as

$$
\begin{aligned}
\mathcal{P} = {} & \mathcal{P}_{(k-N,\, N)} + \mathcal{P}_{(k-N-1,\, N+1)} + \cdots \\
& + \boldsymbol{\mathcal{P}_{(k-N-\ell,\, N+\ell)}} + \mathcal{P}_{(k-N-\ell-1,\, N+\ell+1)} + \cdots + \mathcal{P}_{(N,\, k-N)} \\
& - \mathcal{P}_{(k-N-1,\, N)} - \mathcal{P}_{(k-N-2,\, N+2)} - \cdots \\
& - \mathcal{P}_{(k-N-\ell-1,\, N+\ell)} - \cdots \quad - \mathcal{P}_{(N,\, k-N-1)} \\
= {} & \Big(\mathcal{P}_{(k-N,\, N)} - \mathcal{P}_{(k-N-1,\, N)}\Big) + \Big(\mathcal{P}_{(k-N-1,\, N+1)} - \mathcal{P}_{(k-N-2,\, N+1)}\Big) + \cdots \\
& \qquad + \boldsymbol{\mathcal{P}_{(k-N-\ell,\, N+\ell)}} \\
& + \Big(\mathcal{P}_{(k-N-\ell-1,\, N+\ell+1)} - \mathcal{P}_{(k-N-\ell-1,\, N+\ell)}\Big) + \cdots + \Big(\mathcal{P}_{(N,\, k-N)} - \mathcal{P}_{(N,\, k-N-1)}\Big).
\end{aligned}
$$

Observe that the indices of all operators involved in the first ℓ brackets have the same *second* components, N, $N+1$, $\cdots$ and $N+\ell-1$ respectively and the indices of all operators in the last $k-2N-\ell-1$ brackets have the same *first* components, $k-N-\ell-1$, $\cdots$, N respectively. We now claim that $\mathcal{P}$ reproduce q_γ, i.e.,

$$\mathcal{P}(q_\gamma) = q_\gamma.$$

It follows from the choice of $\mathcal{P}_{(k-N-\ell,\, N+\ell)}$ (see (4.2.10)) and Lemma 4.3 (i) that

$$\mathcal{P}_{(k-N-\ell,\, N+\ell)}(q_\gamma) = q_\gamma.$$

Moreover, for the first ℓ brackets in the above, we want to apply Lemma 4.3 (ii). For this purpose, we need to verify $\gamma_1 \le 2(k-N-i)-1$, $0 \le i \le \ell$. Since $i \le \ell$, then

$$2(k-N)-1-2\ell \le 2(k-N-i)-1.$$

But by (4.2.10) $\ell' \ge 2\ell$. Hence

$$\gamma_1 = 2(k-N)-1-\ell' \le 2(k-N)-1-2\ell \le 2(k-N-i)-1.$$

Therefore, it follows from Lemma 4.3 (ii) that

$$\left(\mathcal{P}_{(k-N-i,\,N)} - \mathcal{P}_{(k-N-1-i,\,N)}\right)(q_\gamma) = 0 \qquad \text{for} \quad 0 \le i \le \ell.$$

Similarly, for the last $k-2N-\ell-1$ brackets, we need to check

$$\gamma_2 = \ell' \le 2(N+\ell+j)-1 \qquad \text{for} \quad 1 \le j \le k-2N-\ell-1,$$

which is obviously true by (4.2.10). Hence by Lemma 4.3 (ii) again

$$\left(\mathcal{P}_{(k-N-\ell-j,\,N+\ell+j)} - \mathcal{P}_{(k-N-\ell-j,\,N+\ell-1+j)}\right)(q_\gamma) = 0 \qquad \text{for} \quad 1 \le j \le k-2N-\ell-1.$$

Consequently, we proved that

$$\mathcal{P}(q) = q \qquad \text{for all} \quad q \in \Pi_{2(k-N)-1}. \quad \blacksquare$$

We now apply Theorem 4.4 to an approximation scheme. To this end, let $\triangle_{n,m}$ be a **mesh** of $\mathbb{R}^2$ defined by

$$\triangle_{n,m} := \left[\frac{j}{n},\, \frac{j+1}{n}\right] \times \left[\frac{k}{m},\, \frac{k+1}{m}\right] \qquad j,\ k \in \mathbb{Z},\ \ n,\ m \in \mathbb{N}.$$

The **length** $|\triangle_{n,m}|$ of mesh $\triangle_{n,m}$ is defied by

$$|\triangle_{n,m}| = \max\left\{\frac{1}{n},\, \frac{1}{m}\right\}. \tag{4.2.11}$$

Denote by D_j the partial derivative operator with respect to the jth coordinate, and by D^α the partial differential operator $D_1^{\alpha_1}D_2^{\alpha_2}$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$.

Define

$$|f|_{s,\infty} := \sum_{|\alpha|=s} ||D^\alpha f||_\infty.$$

Theorem 4.5 *If $k > 2N$, then for any $s \le 2(k-N)-1$, any function $f \in C^s(\mathbb{R}^2)$, the operator given in* (4.2.7) *satisfies*

$$||f - \mathcal{P}(f)||_\infty \le C_k(|\triangle_{n,m}|^s)|f|_{s,\infty}. \tag{4.2.12}$$

where C_k is a constant depends on k only and $|\triangle_{n,m}|$ is the length of mesh $\triangle_{n,m}$ defined in (4.2.11).

Proof: Note that φ_N is continuous and compactly supported on $[1-2N,\ 2N-1]$. Hence

$$\max_{x\in[0,\,1]} \sum_{j\in\mathbb{Z}} |\varphi_N(x-j)| = \text{const}_N. \tag{4.2.13}$$

Let $(x,\ y) \in \mathbb{R}^2$ be fixed for the time being. First let us consider the parametrically extended operator $\mathcal{P}^{1,1}_{(\alpha_1,\,\alpha_2)}$ defined in (4.2.5) for $n = m = 1$:

$$\mathcal{P}^{1,1}_{(\alpha_1,\,\alpha_2)} g(x,\ y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} g\,(j,k)\,\varphi_{\alpha_1,1}(x-j)\varphi_{\alpha_2,1}(y-k),$$

and the associated Boolean sum operator given in (4.2.7)

$$\mathcal{P}^{1,1} = \sum_{\substack{\beta_1+\beta_2=k \\ \beta_1,\,\beta_2 \ge N}} \mathcal{P}^{1,1}_{(\beta_1,\,\beta_2)} - \sum_{\substack{\beta_1+\beta_2=k-1 \\ \beta_1,\,\beta_2 \ge N}} \mathcal{P}^{1,1}_{(\beta_1,\,\beta_2)}.$$

Note that in this case the largest support of the DD-functions in the expression of $\mathcal{P}^{1,1}_{(\alpha_1,\,\alpha_2)}$ is $[1-2(k-N),\ 2(k-N)-1]$. Then it is easily seen that

$$\left|\mathcal{P}^{1,1} g(x,\ y)\right| \le C_k \max_{(x',y')} \left\{ \left|g(x',y')\right| : |x-x'|,\ |y-y'| \le 2(k-N)-1 \right\}. \tag{4.2.14}$$

Let q be the Taylor polynomial of f of degree $s-1$ about (x,y). By Theorem 4.4, $\mathcal{P}^{1,1}(q)=q$. Consequently

$$(f-\mathcal{P}^{1,1}f)(x,y)=q(x,y)-\mathcal{P}^{1,1}f(x,y)=\mathcal{P}^{1,1}(q-f)(x,y).$$

Then it follows from (4.2.14) that

$$\begin{aligned}\left|(f-\mathcal{P}^{1,1}f)(x,y)\right| &\le C_k \max_{(x',y')}\left\{\left|(f-q)(x',y')\right| : |x-x'|,\ |y-y'|\le 2(k-N)-1\right\}\\ &\le C_k|f|_{s,\infty}.\end{aligned}$$

The above estimate is valid for every $(x,\ y)\in\mathbb{R}^2$. Hence

$$\left|\left|f-\mathcal{P}^{1,1}f\right|\right|_\infty \le C_k|f|_{s,\infty}. \tag{4.2.15}$$

Generally, for $n,\ m\in\mathbb{N}$, we define

$$f_{n,m}(x,\ y):=f(x/n,\ y/m).$$

Then it is easy to see that

$$\left(\mathcal{P}^{1,1}_{(\alpha_1,\alpha_2)}f_{n,m}\right)(x,\ y)=\left(\mathcal{P}^{n,m}_{(\alpha_1,\alpha_2)}f\right)(nx,\ my)$$

where $\mathcal{P}^{n,m}_{(\alpha_1,\alpha_2)}$ stands for $\mathcal{P}_{(\alpha_1,\alpha_2)}$ in (4.2.5). Consequently,

$$\left(\mathcal{P}^{1,1}f_{n,m}\right)(x,\ y)=\left(\mathcal{P}^{n,m}f\right)(nx,\ my)$$

where $\mathcal{P}^{n,m}$ is indeed the $\mathcal{P}$ given in (4.2.7). Therefore, it follows from (4.2.15) that

$$\left|\left|f-\mathcal{P}^{n,m}f\right|\right|_\infty=\left|\left|f_{n,m}-\mathcal{P}^{1,1}f_{n,m}\right|\right|_\infty\le C_k\left|f_{n,m}\right|_{s,\infty}\le C_k\left(|\triangle_{n,m}|^s\right)|f|_{s,\infty}$$

The last inequality holds since $\left|f_{n,m}\right|_{s,\infty}\le\left(|\triangle_{n,m}|^s\right)|f|_{s,\infty}$ by the (bivariate) Taylor formula. ■

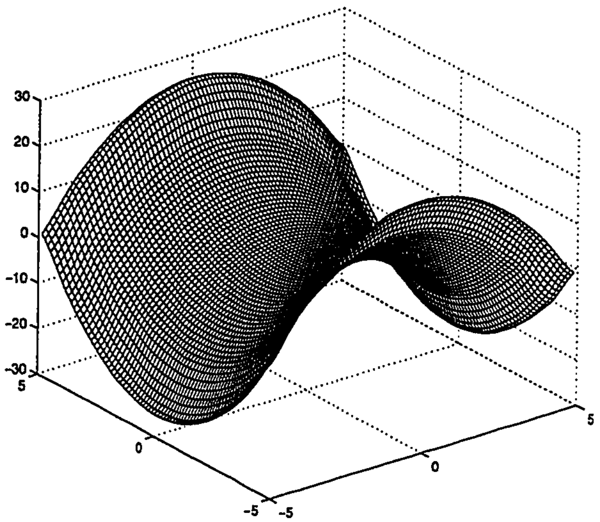

Figure 4.1: The graph of $z = x^2 - y^2$

Figure 4.1 and Figure 4.2 demonstrate that $\mathcal{P}$ in (4.2.7) can be used to produce not only smooth surfaces (see Fig 4.1 which shows the quadratic surface $z = x^2 - y^2$) but also complicate geographic surfaces (see Fig 4.2 which shows a piece of Loon Lake, Alberta, Canada). To produce these surfaces, we choose $N = 2$, α_1, α_2, β_1, β_2 are either 2 or 3, and $k = 5$ in Theorem 4.4 to construct those parametrically extended cardinal interpolation operators $\mathcal{P}_{(\alpha_1,\,\alpha_2)}$, $\mathcal{P}_{(\beta_1,\,\beta_2)}$ and their Boolean sum $\mathcal{P}$. Hence, by Theorem 4.4, $\mathcal{P}$ can reproduce polynomials with degree up to 5 and thereby has the accuracy 6.

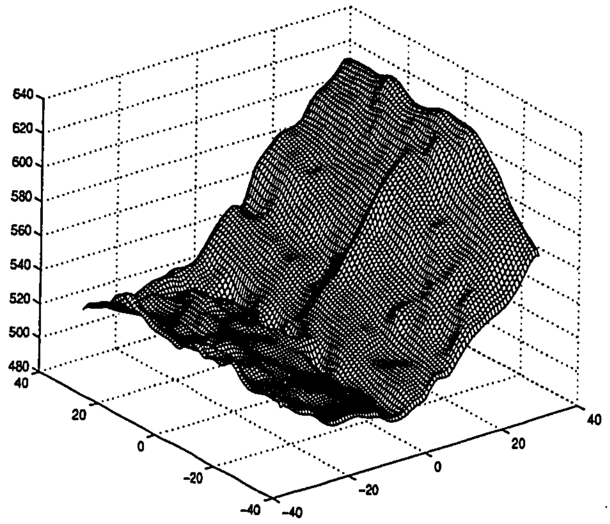

Figure 4.2: The graph of Loon Lake area, Alberta, Canada

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