Some SPDEs/SDEs driven by fractional Gaussian noises and their related properties

by

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Abstract

In this dissertation, various problems related to stochastic (partial) differential equations are investigated. These problems include well-posedness, Hölder continuity of the solution, moments of the solution and their asymptotics. This thesis is divided into three parts. The first part studies the existence and uniqueness problems of nonlinear stochastic differential equations including stochastic heat equation and stochastic wave equation driven by multiplicative Gaussian noises. The main feature of this part is that the Gaussian noise has the covariance of a fractional Brownian motion with Hurst parameter $H \in (1/4, 1/2)$ in the spatial variable. Our contributions are to remove an artificial assumption on diffusion coefficient in the nonlinear stochastic heat equation and to surmount the barrier caused by the absence of semi-group property of wave kernel. The second part of the dissertation explores intermittency properties for various stochastic PDEs with varieties of space-time Gaussian noises via matching upper and lower moment bounds. This part introduces the Feynman diagram formula for the moments of the solution and the small ball nondegeneracy for the Green's function to obtain the sharp lower bounds for all moments for various interesting equations, including stochastic heat equations, stochastic wave equations, stochastic heat equations with fractional Laplacians, and stochastic diffusions which are both fractional in time and in space. The third part of this thesis considers stability problems in the mean square sense for stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Both the mean square stability of the solution and its stochastic theta scheme for linear and nonlinear equations are investigated by introducing a set of analytic and probabilistic tools. Numerical examples are carried out to illustrate our theoretical results.

Preface

This thesis is based on one published paper, one accepted paper and two complete preprints. In particular

- Chapter 2 of this thesis is a joint work with Prof. Yaozhong Hu which has been published as "Stochastic Heat Equation with general noise" in *Annales de l'institut Henri Poincaré (B) Probability and Statistics.*

- Chapter 3 of this thesis is a joint work with Prof. Yaozhong Hu and Shuhui Liu with the title "Nonlinear stochastic wave Equation driven by rough noise." It has been accepted by *Journal of Differential Equations.*

- Chapter 4 of this thesis is based on a complete work with Prof. Yaozhong Hu. This preprint is entitled "Intermittency properties for a large class of stochastic PDEs driven by fractional space-time noises."

- Chapter 5 of this thesis is a joint work with Prof. Yaozhong Hu, Prof. Chengming Huang and Dr. Min Li. This preprint is entitled "Mean square stability of stochastic theta method for stochastic differential equations driven by fractional Brownian motion."

Xiong Wang (PhD Candidate)

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Chapter 1

Introduction and Summary

1.1 Introduction

In natural sciences and engineering sciences, partial differential equations (PDEs) are essential tools for modeling physical phenomena. PDEs are foundational in the modern scientific understanding of sound, heat, diffusion, fluid dynamics and so forth. Joseph Fourier ([FD+22]) developed the heat equation via the law of heat conduction (also called Fourier's law). That is, given an open subset $U \subset \mathbb{R}^d$ and and a subinterval $[0,T] \subset \mathbb{R}^+$, we say $u(t,x) : [0,T] \times U \to \mathbb{R}$ is a solution of the heat equation with source if

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + f(t,x)\,, \tag{1.1.1}$$

with $f(t,x)$ denoting a heat source. Another example is the wave equation, which describes water waves, and sound waves that arise in the fields of electromagnetism and fluid dynamics. Historically, the vibrating string problem was studied by many famous mathematicians such as Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli and others. We refer [CD81] to the readers for history of vibration theory. Classical physics tells us that the displacement of a string (one dimension) or a water wave (two dimension) $v(t,x) : [0,T] \times U \to \mathbb{R}$ solves the following PDE:

$$\frac{\partial^2 v(t,x)}{\partial t^2} = \Delta v(t,x) + g(t,x)\,, \tag{1.1.2}$$

where $g(t,x)$ is a wave source.

We are interested in the following question:

"What if the sources $f(t,x)$ and $g(t,x)$ in (1.1.1) and (1.1.2)

depend on some random noises?"

In Walsh's note [Wal86], there are many interesting interpretations of this question by introducing the theory of stochastic partial differential equations (SPDEs).

Before SPDEs, the mathematical theory of ordinary differential equations (ODEs) perturbed by terms dependent on some random noises was developed in the 1940s. The most popular noise is the white noise, which can be formally viewed as the derivative of Brownian motion B_t (or Wiener process W_t). Kiyosi Itô ([Itô42, Itô44]) extended the classical calculus to Brownian motion and developed the the theory of stochastic differential equations (SDEs). Alternatively, physicist Ruslan Stratonovich ([Str57]) proposed another stochastic calculus which is also frequently used. A typical SDE is in the form of

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t\,. \tag{1.1.3}$$

Some interesting examples and their applications to other fields, such as mathematical finance, can be found in [Øks03].

However, in stochastic partial differential equations, people are more interested in noises relying on both time and space variables, for example, the space-time white noise. In this thesis, we primarily focus on what is so-called Gaussian noises in the form of $\dot{W}(t,x) = \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x)$. See section 1.2 for a brief introduction to Gaussian noises. There is a large amount of research on SPDEs driven by the Gaussian noises $\dot{W}(t,x)$. Let us close the introduction section with some common examples.

If $f(t,x)$ and $g(t,x)$ in (1.1.1) and (1.1.2) are replaced by $\dot{W}(t,x)$. The equations can be rewritten as

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \dot{W}(t,x)\,, \\ u(0,x) = u_0(x)\,. \end{cases} \qquad \begin{cases} \frac{\partial^2 v(t,x)}{\partial t^2} = \Delta v(t,x) + \dot{W}(t,x)\,, \\ v(0,x) = u_0(x)\,, \frac{\partial}{\partial t} u(0,x) = v_0(x)\,. \end{cases} \tag{1.1.4}$$

They are referred to as stochastic heat equation (SHE) with additive noise and stochastic wave equation (SWE) with additive noise. If we set all the initial conditions to 0, then the solutions to (1.1.4) as random fields are still Gaussian processes. Some critical properties such as upper and lower bounds, strong local nondeterminism and exact modulus of continuity have been investigated. For more related results, we refer the interested readers to [Adl90, Hu17, Tal14, Xia06] and references therein.

Let $f(t,x) = u(t,x)\dot{W}(t,x)$ and $g(t,x) = v(t,x)\dot{W}(t,x)$ in (1.1.1) and (1.1.2), respectively. Then we get the parabolic Anderson model (PAM) and hyperbolic Anderson model (HAM):

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x)\,, \\ u(0,x) = u_0(x)\,. \end{cases} \qquad \begin{cases} \frac{\partial^2 v(t,x)}{\partial t^2} = \Delta v(t,x) + v(t,x)\dot{W}(t,x)\,, \\ v(0,x) = u_0(x)\,, \frac{\partial}{\partial t}u(0,x) = v_0(x)\,. \end{cases} \tag{1.1.5}$$

There are many interesting properties of the solutions to Anderson models (1.1.5). We refer the readers to the seminal work [CM94] by Carmona and Molchanov. Among these properties, intermittency is the most attractive one to us. Roughly speaking, it is characterized by the structures of sharp peaks. Let us mention an interesting example in [BC14], the solar magnetic field is intermittent since more than 99% of the magnetic energy concentrates on less than 1% of the surface area. A more comprehensive discussion of this topic can be found in [Kho14] and references therein. Other than intermittency, the Kardar-Parisi-Zhang (KPZ) equation and the KPZ universality class have drawn increasing attention in recent years. When $d = 1$, $\dot{W}(t,x)$ is the space-time white nose, there is a fundamental connection between PAM and KPZ equation. This is, the Hopf-Cole solution $h(t,x) = \log u(t,x)$ formally solves the KPZ equation

$$\frac{\partial h(t,x)}{\partial t} = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}[\nabla h(t,x)]^2 + \dot{W}(t,x)\,. \tag{1.1.6}$$

The solvability of (1.1.6) has been rigorously justified by Martin Hairer in [Hai13] by regularity structures.

In addition, when $f(t,x) = \sigma(u(t,x))\dot{W}(t,x)$ and $g(t,x) = \sigma(v(t,x))\dot{W}(t,x)$ for some nonlinear functions, we refer them as nonlinear SPDEs. If $\sigma(\cdot)$ is Lipschitz and the noise

$W(t,x)$ can be viewed as a Brownian motion in an infinite dimensional Hilbert space, the existence and uniqueness problems of nonlinear SPDEs have been well developed. The readers can find the classical results in [Dal99, DKM+09, DPZ14]. Moreover, if $\sigma(\cdot)$ is non-Lipschitz, the qualitative properties of solutions to these nonlinear SPDEs, such as the support property ([Mue91, Shi94]) have been studied in last decades. In particular, the stochastic heat equation with $\sigma(u) = \sqrt{u}$ is related to super-Brownian motion. There are many researches concentrating on this direction, especially on the pathwise uniqueness problem ([MPS06, MP11]).

1.2 Preliminaries

In most cases, the equations such as (1.1.1) and (1.1.2) do not have a 'classical solution' with $f(t,x)$ and $g(t,x)$ replaced by some random forces. This section briefly reviews the fractional Gaussian noises we mainly focus on throughout this thesis and the precise definitions of solutions to SPDEs.

The fractional Gaussian noise can be formally written as $\dot{W}(t,x) = \frac{\partial^2}{\partial t \partial x} W(t,x)$ where $W(t,x)$ is a centered Gaussian process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with covariance given by (see e.g. [Hu19, CH21] for more details)

$$\begin{aligned} &\mathbb{E}[W(t,x)W(s,y)] = \mathcal{C}_{H_0}(t,s) \prod_{i=1}^{d} \mathcal{C}_{H_i}(x,y)\,, \quad s,t \geq 0\,, x,y \in \mathbb{R}^d\,, \qquad (1.2.1)\\ &\text{where} \qquad \mathcal{C}_H(a,b) = \frac{1}{2}(|a|^{2H} + |b|^{2H} - |a-b|^{2H})\,, \quad \forall\, a,b \in \mathbb{R}\,. \end{aligned}$$

In this dissertation, we always assume $\frac{1}{2} \leq H_0 < 1$ and $0 < H_i < 1$. Denote $\mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ the space of real-valued functions with compact support that are infinitely differentiable. Since $W(t,x)$ is not differentiable in general, we need to identify the noise to a mean zero Gaussian family $\{W(\phi) : \phi \in \mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^d)\}$, with the covariance structure defined by

$$\mathbb{E}[W(\phi)W(\psi)] = C_{H_0,H} \int_{(\mathbb{R}^+\times\mathbb{R}^d)^2} \widehat{\phi}(s,\xi)\overline{\widehat{\psi}(r,\xi)}\gamma(r-s) \prod_{i=1}^{d} |\xi_i|^{1-2H_i} d\xi ds dr\,, \qquad (1.2.2)$$

where $\gamma(r-s) = C_{H_0}|s-r|^{-\gamma} = H_0(2H_0-1)|s-r|^{2H_0-2}$ and $\widehat{\phi}$ means the Fourier transform on space variables. Moreover, when $H_0 = 1/2$, we always replace $H_0(2H_0-1)|s-r|^{2H_0-2}$ by $\delta(s-r)$. On the other hand, if $\frac{1}{2} \le H_0 < 1$ and $\frac{1}{2} \le H_i < 1$, one can rewrite (1.2.2) as

$$\mathbb{E}[W(\phi)W(\psi)] = \int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} \phi(r,x)\psi(s,y)\gamma(r-s)\Lambda(x-y)drdxdsdy \tag{1.2.3}$$

where $\Lambda(x) = C\prod_{j=1}^d |x_j|^{-\lambda_j}$ with $\lambda_j \in (0,1)$, $j = 1,\cdots,d$. Specially, when $d = 1$ and $H_0 = H = 1/2$, we call $\dot{W}(t,x)$ the space-time white noise. In addition, people also pay attention to Riesz potential $\Lambda(x) = |x|^{-\lambda}$ with $0 < \lambda < d$ in (1.2.3) that can be compared with fractional Gaussian noise. See [CJK13, CJKS13, Che16], for example. The case of Riesz potential has connections to many classical laws in physics.

Next, we shall briefly introduce the form of SPDEs we are dealing with. We consider the following stochastic partial differential equation in the Eulcidean space $\mathbb{R}^d$:

$$\mathscr{L}u(t,x) = \sigma(t,x,u(t,x))\dot{W}(t,x)\,, \qquad t>0,\ x\in\mathbb{R}^d \tag{1.2.4}$$

with some given initial condition(s). Here $\mathscr{L}$ denotes a general (including fractional order) partial differential operator. The examples include $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$ (heat operator), $\mathscr{L} = \partial_t^2 - \Delta$ (wave operator), $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$ (α-heat operator), and $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$ (fractional diffusion operator) and so on. The Green's function associated with $\mathscr{L}$ is a (possibly generalized) function $G_{t-s}(x,y), 0 \le s < t < \infty, x, y \in \mathbb{R}^d$ or a measure $G_{t-s}(x,y)dy := G_{t-s}(x,dy)$ (we omit the explicit dependence of G on $\mathscr{L}$). Then by Duhamel's principle, the solution to (1.2.4) is given by the mild solution form

$$u(t,x) = I_0(t,x) + \int_0^t\int_{\mathbb{R}^d} G_{t-s}(x,y)\sigma(s,y,u(s,y))W(ds,dy)\,, \tag{1.2.5}$$

where the term $I_0(t,x)$ depends on the initial data and the Green's function. For instance, when $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$ (heat operator), the Green's function (heat kernel) and its Fourier transform in spatial variable are respectively:

$$G_t^{\mathrm{h}}(x) = \frac{1}{(2\pi t)^{d/2}}\exp\left(-\frac{|x|^2}{2t}\right) \quad \text{and} \quad \mathcal{F}[G_t^{\mathrm{h}}(\cdot)](\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right). \tag{1.2.6}$$

When $\mathscr{L} = \partial_t^2 - \Delta$ (wave operator), the associated Green's function (wave kernel) has different forms for different dimensions. More precisely, it is given by

$$\begin{cases} G_t^{\mathrm{w}}(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}\,, & d=1\,, \\ G_t^{\mathrm{w}}(x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}\,, & d=2\,, \\ G_t^{\mathrm{w}}(dx) = \frac{1}{4\pi}\frac{\sigma_t(dx)}{t}\,, & d=3\,. \end{cases} \tag{1.2.7}$$

The Fourier transform of $G_t^{\mathrm{w}}(\cdot)$ has the same form given by

$$\mathcal{F}[G_t^{\mathrm{w}}(\cdot)](\xi) = \frac{\sin(t|\xi|)}{|\xi|}\,, \quad \xi \in \mathbb{R}^d\,.$$

To make things precise we give here the definitions of strong and weak solutions.

Definition 1.2.1. *Let $\{u(t,x)\,, t\geq 0\,, x\in\mathbb{R}\}$ be a real-valued adapted stochastic process such that for all $t\in[0,T]$ and $x\in\mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(s,y,u(s,y))1_{[0,t]}(s)\}$ is integrable with respect to W.*

(i) We say that $u(t,x)$ is a strong (mild) solution *to (1.2.4) if for all $t\in[0,T]$ and $x\in\mathbb{R}$ we have (1.2.5) holds almost surely.*

(ii) We say (1.2.4) has a weak solution *if there exists a probability space with a filtration $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$, a Gaussian random field $\widetilde{W}$ identical to W in law, and an adapted stochastic process $\{u(t,x)\,, t\geq 0\,, x\in\mathbb{R}\}$ on this probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ such that $u(t,x)$ is a mild solution with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ and $\widetilde{W}$.*

1.3 Summary of the work

This dissertation is a collection of joint works with my advisor and other collaborators. It consists of four research articles, which are listed as follows.

1. Stochastic Heat Equation with general noise, *with Yaozhong Hu, Ann. Inst. Henri Poincaré Probab. Stat. 58 (2022), no. 1, 379-423*;

2. Nonlinear stochastic wave Equation driven by rough noise, *with Yaozhong Hu and Shuhui Liu, accepted by Journal of Differential Equations*;

3. Intermittency properties for a large class of stochastic PDEs driven by fractional space-time noises, *with Yaozhong Hu, arXiv preprint*;

4. Mean square stability of stochastic theta method for stochastic differential equations driven by fractional Brownian motion, *with Yaozhong Hu, Chengming Huang and Min Li, arXiv preprint.*

In the Chapter 2 and Chapter 3, the existence and uniqueness problems of nonlinear stochastic heat equation and stochastic wave equation driven by Gaussian noise are studied based on paper 1 and preprint 2. We set $d = 1$ and the Hurst parameters of Gaussian noise to be $H_0 = 1/2$ and $H \in (1/4, 1/2)$ in these two chapters. Considering the nonlinear stochastic heat equation (namely, (1.2.4) with $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$), and assuming the diffusion coefficient $\sigma(t, x, u)$ in a reduced form $\sigma(u)$ satisfying $\sigma(0) = 0$, the authors of [HHL+17] successfully proven the strong existence and uniqueness of the SPDE by introducing some new function spaces with some Hölder norms. We keep using the reduced form $\sigma(u)$ to talk about the results in Chapter 2 briefly. The main effort of Chapter 2 is to remove this artificial condition on $\sigma(\cdot)$. The idea is to work on a weighted space $\mathcal{Z}^p_{\lambda,T}$ (see(2.4.4) for details) for the spatial power decay weight $\lambda(x)$. Our key tasks are to establish some contractive type inequalities of heat kernel with regard to the weight $\lambda(x)$, which are done in Section 2.2. Taking $\lambda(x) = c_H(1 + x^2)^{H-1}$ and without assuming $\sigma(0) = 0$, we show that the SPDE has a weak solution in $\mathcal{Z}^p_\lambda(T)$ for $p > 3/H$ under the uniform linear growth condition and uniform Lipschitz condition (see **(H1)** in Chapter 2). Moreover, it has a unique strong solution in $\mathcal{Z}^p_\lambda(T)$ for $p > \frac{6}{4H-1}$ under **(H2)** in Chapter 2. This assumption is stronger than before but is satisfied for some crucial cases such as affine functions $\sigma(u) = au + b$. In addition, for any $\gamma < H - \frac{3}{p}$, the process $u(t, x)$ is almost surely Hölder continuous on any compact sets in $[0, T] \times \mathbb{R}$ of Hölder exponent $\gamma/2$ and γ with respect to the temporal variable t and the spatial variable x, respectively. Furthermore, in the additive case, i.e. $\sigma(u) = 1$, we obtain some exact asymptotics related to the solution $u_{\text{add}}(t, x)$ as t and x go to infinity. These results depend on Talagrand's majorizing measure theorem ([Tal14]).

In Chapter 3, the existence and uniqueness of the strong solution to one spatial dimension nonlinear stochastic wave equation ((1.2.4) with $\mathscr{L} = \partial_{tt} - \Delta$ and $\sigma(t, x, u) =$

$\sigma(u)$ for the simple introduction of Chapter 3) are obtained under the constraint $\sigma(0) = 0$. Further research remains to get rid of this condition. In this chapter, some techniques are developed to overcome the difficulties because of missing semi-group property of the wave kernel. These can be found in Section 3.6. Thus, assuming that $\sigma(u)$ satisfies the uniform bounded condition and uniform Lipschitz condition on its derivative with respect to u (see hypothesis **(H2)** in Chapter 3) and that $I_0(t,x)$ is in $\mathcal{Z}^p(T) := \mathcal{Z}^p_\lambda(T)$ (with $\lambda(x) = 1$) for some $p > \frac{2}{4H-1}$, we prove that it has a unique strong solution in $\mathcal{Z}^p(T)$ for $p > \frac{2}{4H-1}$ whose sample paths are in $\mathcal{C}([0,T] \times \mathbb{R})$ almost surely. Moreover, it is proven that both the temporal and the spatial Hölder exponents of the random field $u(t,x)$ are $\gamma < H - \frac{1}{p}$ on any compact subsets of $[0,T] \times \mathbb{R}$. We remark that by selecting suitable large p, the exponents of Hölder continuous in time and space of the solutions to SHE (Chapter 2) and SWE (Chapter 3) close the optimal ones as possible as we can.

Intermittent random fields as functions of space variable x consist of 'high peaks' which give the most contribution to the processes. Taking from is from preprint 3, we mainly investigate this property for the following four type Anderson models: $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$ (SHE), $\mathscr{L} = \partial_t^2 - \Delta$ (SWE), $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$ (α-SHE), and $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$ (SFD) in Chapter 4. See equations (1.1.5) for examples. We consider $H_0 \in [1/2, 1)$, $H_i \in [1/2, 1)$ $\forall i = 1, \cdots, d$ and $\sigma(t,x,u) = u$ in this chapter. It is well known the lower p-th moments and upper p-th moments of SHE match with each other (see [Hu19] and references therein). But the approach of showing this sharp result highly depends on Feynman-Kac formula which is no longer applicable in general, e.g. the stochastic wave equation. However, people conjectured this should hold for hyperbolic Anderson models and other cases. In Chapter 4, this has been answered in general. We apply Feynman diagram formula (see section 4.5) for the moments of the solution and then select the diagrams in the principle of small ball non-degeneracy **(G2)**. In this way, we successfully obtain the sharp lower bounds. More preciesely, we find that (see also (4.3.10))

$$
\begin{aligned}
&c_1 \exp\left(c_2 \cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} \cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right) \\
&\qquad\qquad \leq \mathbb{E}\left[|u(t,x)|^p\right] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} \cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right)
\end{aligned}
$$

for all $t \geq 0$, $x \in \mathbb{R}^d$, $p \geq 2$, where c_1, c_2, C_1, C_2 are some positive constants, independent of t, x, p. The meaning of all parameters is omitted and can be found in Chapter 4.

In the Chapter 5, considering $d = 1$, we study the nonlinear SDEs driven by fractional Brownian motion (fBm) $B^H(t)$ with $H \in [1/2, 1)$ (i.e. replace B_t by B_t^H in (1.1.3)). More precisely, we investigate the nonlinear SDE ((5.1.9))

$$dX(t) = f(t, X(t))dt + g(t, X(t)) \circ dB^H(t)\,,$$

with $f(t, X)$ satisfying monotone condition and linear growth condition and $g(t, X)$ satisfying uniform linear growth condition (see Assumption 1 in Chapter 5). The noise $B^H(t)$ is also a Gaussian process that can be defined similarly to $W(t, x)$ in (1.2.1) without spatial parameter. The stochastic integral with respect to $B^H(t)$ is understood in Stratonovich sense in this chapter (see [HØ03, Mis08, Hu13]). Throughout this chapter, we focus on the problem of mean square stability; namely, the second moment of the original solution or numerical solution vanishes as t or t_n goes to infinity. This reflects a numerical algorithm is stable or not. Euler θ-method is a popular numerical scheme to simulate the solution of differential equations and stochastic differential equations. See stochastic theta method (STM) (5.1.13) for the full description of this numerical scheme in stochastic setting. However, the mean square stability of STM for SDEs driven by fBm remains open after the paper [Hig00b] which deals with the Brownian motion case. Moreover, even the problem of stability in the mean square sense of the original solution has not been well studied due to the presence of long memory. We answer part of these questions by developing an entirely new set of techniques to address the dependence generated by the long memory of the fBm. Our method relies on the asymptotic property of confluent hypergeometric functions, Gaussian correlation inequality, and law of large numbers for correlated random variables. In conclusion, for the linear case, the STM reproduces the mean square stability when θ is larger than $\theta_0 \approx 0.77$ and is not mean square stable when $\theta < 0.5$. For the nonlinear case, the original solution to a special form (including linear case) is proven to be mean square stable and the STM for the numerical solution to the general form is proven to be stable in the mean square scene when θ is larger than $\theta_1 \approx 0.87$.

Chapter 2

Stochastic Heat Equation with general rough noise

2.1 Introduction and main results

In this chapter, we consider the following one dimensional (in space variable) nonlinear stochastic heat equation driven by the Gaussian noise which is white in time and fractional in space:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \sigma(t,x,u(t,x))\dot{W}(t,x), \quad t \geqslant 0, \quad x \in \mathbb{R}\,, \tag{2.1.1}$$

where $W(t,x)$ is a centered Gaussian process with covariance given by

$$\mathbb{E}[W(t,x)W(s,y)] = \frac{1}{2}(s \wedge t)\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big) \tag{2.1.2}$$

and where $\frac{1}{4} < H < \frac{1}{2}$ and $\dot{W}(t,x) = \frac{\partial^2}{\partial t \partial x} W(t,x)$.

There has been a lot of work on stochastic heat equations driven by general Gaussian noises. We refer to [Hu19] for a short survey and for more references. The main feature of this work is that the noise is rough (e.g. $\frac{1}{4} < H < \frac{1}{2}$) in the space variable. We mention three works that are directly related to this specific Gaussian noise structure. The first two are [BJQS15] and [HHL$^+$18], where the authors study the existence, uniqueness and some properties such as moment bounds of the mild solution when the diffusion coefficient

σ is affine (i.e. $\sigma(t,x,u) = \sigma(u) = au + b$) in [BJQS15] or linear (i.e. $\sigma(u) = au$) in [HHL⁺18]. After these works researchers tried to study (2.1.1) for general nonlinear σ. However, the method effective for affine (and linear) equations cannot no longer work. One difficulty for general nonlinear diffusion coefficient σ is that we cannot no longer bound $|\sigma(u_1) - \sigma(u_2) - \sigma(v_1) + \sigma(v_2)|$ by a multiple of $|u_1 - u_2 - v_1 + v_2|$ (which is possible only in the affine case). A breakthrough was made in [HHL⁺17]. However, to solve equation (2.1.1) the authors in [HHL⁺17] have to assume that $\sigma(0) = 0$, which does not even cover the affine case studied in [BJQS15]! The main motivation of this chapter is to remove the condition $\sigma(0) = 0$ assumed in [HHL⁺17]. To this end we need to understand why this condition is so crucial there. We first find out that this condition $\sigma(0) = 0$ can ensure the solution lives in the space $\mathcal{Z}_T^p$ (see [HHL⁺17] or (2.4.4) in Section 2.4 of this chapter with $\lambda(x) = 1$). As we shall see that even in the simplest case $\sigma(u) \equiv 1$ (of the case $\sigma(0) \neq 0$), the solution is no longer in $\mathcal{Z}_T^p$ (see e.g. Theorem 2.1.1 and Proposition 2.3.11). Moreover, the initial condition in [HHL⁺17] must be integrable to guarantee the solution belongs to $\mathcal{Z}_T^p$, which means $u_0(x) = 1$ is excluded. Thus, to remove the restriction $\sigma(0) = 0$, we must find another appropriate solution space. Our idea is to introduce a decay weight (as the spatial variable x goes to infinity) to enlarge the solution space $\mathcal{Z}_T^p$ to a weighted space $\mathcal{Z}_{\lambda,T}^p$ for some suitable power decay function $\lambda(x)$. This weight function will have to be chosen appropriately (not too fast and not too slow. See Section 2 for details).

The introduction of the weight makes all the tools used in [HHL⁺17] collapse. As we can see we shall need a whole set of new understandings of the heat kernel to complete our program. People may wonder whether one can still just use $\mathcal{Z}_T^\infty$ for our solution space. This question is natural since we work on the whole real line $\mathbb{R}$ for the space variable. A constant function is in $L^\infty(\mathbb{R})$ but not in $L^p(\mathbb{R})$ for any finite p. If it happens to be possible to use $\mathcal{Z}_T^\infty$ (without weight), then many computations in [HHL⁺17] will still be valid and the problem becomes greatly simplified.

To see if this is possible or not we consider the solution $u_{\text{add}}(t,x)$ to the equation with additive noise, which is the solution to (2.1.1) with $\sigma(u) = 1$ and with initial condition $u_0(x) = 0$. This is the simplest case that $\sigma(0) \neq 0$. To find out if $u_{\text{add}}(t,x)$ is in $\mathcal{Z}_T^\infty$

or not (or to see if the introduction of decay weight λ is necessary or not), we shall find the sharp bound of the solution $u_{\text{add}}(t,x)$ as x goes to infinity. In other words, we shall find the exact explosion rate of $\sup_{|x|\le L}|u_{\text{add}}(t,x)|$ as L goes to infinity. This problem has a great value of its own. To study the supremum of a family of random variables, there are two powerful tools: one is to use the independence and the other one is to use the martingale inequalities. However, $u_{\text{add}}(t,x)$ is not a martingale with respect to the spatial variable x (nor it is a martingale with respect to the time variable t) and since the noise $\dot W$ is not independent in the spatial variable either, the application of independence may be much more involved (We refer, however, to [Che16, CHNT17, CJK13, CJKS13] for some successful applications of the independence in the stochastic heat equation (2.1.1)). In this work, we shall use instead the idea of majorizing measure to obtain sharp growth of $\sup_{|x|\le L}|u_{\text{add}}(t,x)|$ and $\sup_{0\le t\le T,|x|\le L}|u_{\text{add}}(t,x)|$, as L and T go to infinity, both in terms of expectation and almost surely. More precisely, we have

Theorem 2.1.1. *Let the Gaussian field $u_{\text{add}}(t,x)$ be the solution to (2.1.1) with $\sigma(t,x,u)$ $=1$ and $u_0(x)=0$. Then, we have the following statements.*

(1) There are two positive constants c_H and C_H, independent of T and L, such that

$$c_H\,\Psi(T,L)\le\mathbb{E}\left[\sup_{\substack{0\le t\le T\\ -L\le x\le L}}u_{\text{add}}(t,x)\right]\le\mathbb{E}\left[\sup_{\substack{0\le t\le T\\ -L\le x\le L}}|u_{\text{add}}(t,x)|\right]\le C_H\,\Psi(T,L)\,,\tag{2.1.3}$$

where $\Psi_0(T,L):=1+\sqrt{\log_2\left(L/\sqrt{T}\right)}$, $L\ge\sqrt{T}$ and

$$\Psi(T,L):=\begin{cases}T^{\frac{H}{2}}\Psi_0(T,L) & \text{if } L\ge\sqrt{T},\\ T^{\frac{H}{2}} & \text{if } L<\sqrt{T}\,.\end{cases}\tag{2.1.4}$$

(2) There are two strictly positive random constants c_H and C_H, independent of T and L, such that almost surely

$$\begin{aligned}c_H\,T^{\frac{H}{2}}\Psi_0(T,L)&\le\sup_{(t,x)\in\Upsilon(T,L)}u_{\text{add}}(t,x)\\&\le\sup_{(t,x)\in\Upsilon(T,L)}|u_{\text{add}}(t,x)|\le C_H\,T^{\frac{H}{2}}\Psi_0(T,L)\,,\end{aligned}\tag{2.1.5}$$

where $\Upsilon(T,L) = \{(t,x) \in [0,T] \times [-L,L] \ : \ L \geq \sqrt{T}\}$.

Let us point out that Theorem 2.1.1 is an extension of Theorem 1.2 of [CJK13] and Theorem 2.3 of [CJKS13] to spatial rough noise.

It is well-known that the solution to equation (2.1.1), if exists, is usually Hölder continuous on any bounded domain. But usually it is not Hölder continuous on the whole space. An interesting question to ask is how the Hölder coefficient depends on the size of the domain. Since the additive solution $u_{\text{add}}(t,x)$ is a Gaussian random field we will be able to obtain sharp dependence on the size of the domain of the Hölder coefficient. In the following theorem we state our result on the Hölder continuity in spatial variable over unbounded domain.

Theorem 2.1.2. *Let* $u_{\text{add}}(t,x)$ *be the solution to* (2.1.1) *with* $\sigma(t,x,u)=1$ *and* $u_0(x)=0$ *and denote*

$$\Delta_h u_{\text{add}}(t,x) := u_{\text{add}}(t,x+h) - u_{\text{add}}(t,x)\,.$$

Let $0<\theta<H$ *be given. Then, there are positive constants* c, c_H *and* $C_{H,\theta}$ *such that the following inequalities hold true:*

$$\begin{aligned} c_H\,|h|^H \Psi_0(t,L) &\leq \mathbb{E}\left[\sup_{-L\leq x\leq L} \Delta_h u_{\text{add}}(t,x)\right] \\ &\leq \mathbb{E}\left[\sup_{-L\leq x\leq L} |\Delta_h u_{\text{add}}(t,x)|\right] \leq C_{H,\theta}\, t^{\frac{H-\theta}{2}} |h|^\theta \Psi_0(t,L) \end{aligned} \tag{2.1.6}$$

for all $L \geq \sqrt{t} > 0$ *and* $0 < |h| \leq c(\sqrt{t} \wedge 1)$. *Moreover, there are two (strictly) positive random constants* c_H *and* $C_{H,\theta}$

$$\begin{aligned} c_H\,|h|^H \Psi_0(t,L) &\leq \sup_{-L\leq x\leq L} \Delta_h u_{\text{add}}(t,x) \\ &\leq \sup_{-L\leq x\leq L} |\Delta_h u_{\text{add}}(t,x)| \leq C_{H,\theta}\, t^{\frac{H-\theta}{2}} |h|^\theta \Psi_0(t,L) \end{aligned} \tag{2.1.7}$$

for all $L \geq \sqrt{t} > 0$ *and* $0 < |h| \leq c(\sqrt{t} \wedge 1)$.

Next, we study the Hölder continuity in time over the unbounded domain. We state the following.

Theorem 2.1.3. *Let $u_{\text{add}}(t,x)$ be the solution to (2.1.1) with $\sigma(t,x,u)=1$ and $u_0(x)=0$ and denote*

$$\Delta_\tau u_{\text{add}}(t,x) := u_{\text{add}}(t+\tau,x) - u_{\text{add}}(t,x)\,.$$

Let $0<\theta<H/2$. Then, there are positive constants c, c_H and $C_{H,\theta}$ such that

$$\begin{aligned} c_H\,\tau^{\frac{H}{2}}\Psi_0(t,L) &\le \mathbb{E}\left[\sup_{-L\le x\le L}\Delta_\tau u_{\text{add}}(t,x)\right] \\ &\le \mathbb{E}\left[\sup_{-L\le x\le L}|\Delta_\tau u_{\text{add}}(t,x)|\right] \le C_{H,\theta}\,t^{\frac{H}{2}-\theta}\tau^{\theta}\Psi_0(t,L) \end{aligned} \tag{2.1.8}$$

for all $L\ge\sqrt{t}>0$ and $0<\tau\le c(t\wedge 1)$. We also have the almost sure version of the above result.

$$\begin{aligned} c_H\,\tau^{\frac{H}{2}}\Psi_0(t,L) &\le \sup_{-L\le x\le L}\Delta_\tau u_{\text{add}}(t,x) \\ &\le \sup_{-L\le x\le L}|\Delta_\tau u_{\text{add}}(t,x)| \le C_{H,\theta}\,t^{\frac{H}{2}-\theta}\tau^{\theta}\Psi_0(t,L) \end{aligned} \tag{2.1.9}$$

for all $L\ge\sqrt{t}>0$ and $0<\tau\le c(t\wedge 1)$. Now, c_H and $C_{H,\theta}$ are random.

The above Theorems 2.1.1-2.1.3 are proved in Section 2.3. Now let us return to the equation (2.1.1). To make things precise we give here the definitions of strong and weak solutions.

Definition 2.1.4. *Let $\{u(t,x)\,,t\ge 0\,,x\in\mathbb{R}\}$ be a real-valued adapted stochastic process such that for all $t\in[0,T]$ and $x\in\mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(s,y,u(s,y))\mathbf{1}_{[0,t]}(s)\}$ is integrable with respect to W (see Definition 2.2.4), where $G_t(x):=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$ is the heat kernel on $\mathbb{R}$ associated with the Laplacian operator Δ.*

(i) We say that $u(t,x)$ is a strong (mild) solution *to (2.1.1) if for all $t\in[0,T]$ and $x\in\mathbb{R}$ we have*

$$u(t,x) = G_t * u_0(x) + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(ds,dy) \tag{2.1.10}$$

almost surely, where the stochastic integral is understood in the sense of Definition 2.2.4.

(ii) *We say* (2.1.1) *has a* weak solution *if there exists a probability space with a filtration* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$, *a Gaussian random field* $\widetilde{W}$ *identical to* W *in law, and an adapted stochastic process* $\{u(t,x)\,, t \geq 0\,, x \in \mathbb{R}\}$ *on this probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ *such that* $u(t,x)$ *is a mild solution to* (2.1.1) *with respect to* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ *and* $\widetilde{W}$.

Before stating our theorem on the existence of a weak solution, we make the following assumption.

(H1) $\sigma(t,x,u)$ is jointly continuous over $[0,T] \times \mathbb{R}^2$ and is at most of linear growth in u uniformly in t and x. This means

$$\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u)| \leq C(|u|+1) \tag{2.1.11}$$

for some positive constant C. We also assume that it is uniformly Lipschitzian in u, namely, $\forall\, u, v \in \mathbb{R}$

$$\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u) - \sigma(t,x,v)| \leq C|u-v|\,, \tag{2.1.12}$$

for some constant $C > 0$.

We can now state our main theorem of the Chapter.

Theorem 2.1.5. *Let* $\lambda(x) = c_H(1+|x|^2)^{H-1}$ *satisfy* $\int_{\mathbb{R}} \lambda(x)dx = 1$. *Assume that* $\sigma(t,x,u)$ *satisfies hypothesis* ***(H1)*** *and that the initial data* u_0 *belongs to* $\mathcal{Z}^p_{\lambda,0}$ *for some* $p > \frac{6}{4H-1}$ *(see Section 4.1 for the definition of* $\mathcal{Z}^p_{\lambda,T}$*). Then, there exists a weak solution to* (2.1.1) *with sample paths in* $\mathcal{C}([0,T] \times \mathbb{R})$ *almost surely. In addition, for any* $\gamma < H - \frac{3}{p}$, *the process* $u(\cdot,\cdot)$ *is almost surely Hölder continuous on any compact sets in* $[0,T] \times \mathbb{R}$ *of Hölder exponent* $\gamma/2$ *with respect to the time variable* t *and of Hölder exponent* γ *with respect to the spatial variable* x.

From Theorem 2.1.1 we see that when $\sigma(0) \neq 0$ we expect that the solution will not be in the space $\mathcal{Z}^p_T$. We enlarge it to the weighted space $\mathcal{Z}^p_{\lambda,T}$ in the above theorem. As we said earlier that the introduction of the weight λ makes the computations in [HHL+17] no longer applicable. For example, now we need to control, roughly speaking, a certain norm of $\lambda(\cdot)\int_0^*\int_{\mathbb{R}} G_{*-s}(\cdot-y)u(s,y)dW(s,y)$ and its fractional derivative (with respect to spatial variable) by the similar norm of $\lambda(\cdot)u(*,\cdot)$ and its fractional derivative. This would require us to study the delicate properties of $\lambda(x)G_t(x-y)\lambda^{-1}(y)$. Thus, we need some very subtle and very sharp bounds on the heat kernel $G_t(x-y)$ with respect to the weight function $\lambda(x)$, which are of interest in their own. This is done in Section 2.2. After these preparations, we shall show the above theorem in Section 2.4. Although the techniques of [HHL+17] are no longer effective in our new situation we still follow the same spirit there.

It is always interesting to have existence and uniqueness of the strong solution. As we said earlier, due to the roughness of the noise we need to handle, as in [HHL+17], the square increment $|\sigma(u_1)-\sigma(u_2)-\sigma(v_1)+\sigma(v_2)|$. It seems too complicated for the weighted space. So, to show the existence and uniqueness of strong solution we assume that the derivative of the diffusion coefficient in (2.1.1) possesses a decay itself as $x\to\infty$. More precisely, we make the following assumptions.

(H2) Assume that $\sigma(t,x,u)\in\mathcal{C}^{0,1,1}([0,T]\times\mathbb{R}^2)$ satisfies the following conditions: $|\sigma'_u(t,x,u)|$ and $|\sigma''_{xu}(t,x,u)|$ are uniformly bounded, i.e. there is a constant $C>0$ such that

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma'_u(t,x,u)|\le C\,; \tag{2.1.13}$$

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma''_{xu}(t,x,u)|\le C\,. \tag{2.1.14}$$

Moreover, we assume that for some $p>\frac{6}{4H-1}$,

$$\sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{-\frac{1}{p}}(x)\,|\sigma'_u(t,x,u_1)-\sigma'_u(t,x,u_2)|\le C|u_1-u_2|\,. \tag{2.1.15}$$

Theorem 2.1.6. *Let σ satisfy the above hypothesis **(H2)** and assume that for some $p>\frac{6}{4H-1}$, $u_0\in\mathcal{Z}^p_{\lambda,0}$. Then (2.1.1) has a unique strong solution with sample paths in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely. Moreover, the process $u(\cdot,\cdot)$ is uniformly Hölder continuous almost surely on any compact subset of $[0,T]\times\mathbb{R}$ with the same temporal and spatial Hölder exponents as those in Theorem 2.1.5.*

This theorem will be proved in Section 2.5. Let us point out that if $\sigma(u)$ is affine, then it satisfies the assumption **(H2)**.

2.2 Auxiliary Lemmas

In this section, we shall obtain some estimates about the heat kernel $G_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}$ associated with the Laplacian Δ combined with the decay weight $\lambda(x)$. These estimates are the key ingredients to establish our results.

2.2.1 Covariance structure

We start by recalling some notations used in [HHL+17]. Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ with compact support, and by $\mathcal{D}'$ the dual of $\mathcal{D}$ with respect to the $L^2(\mathbb{R}, dx)$. The Fourier transform of a function $f \in \mathcal{D}$ is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx,$$

and the inverse Fourier transform is then given by $\mathcal{F}^{-1}g(x) = \frac{1}{2\pi}\mathcal{F}g(-x)$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and let $H \in (\frac{1}{4}, \frac{1}{2})$ be given and fixed. Our noise $\dot{W}$ is a zero-mean Gaussian family $\{W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ with covariance structure given by

$$\mathbb{E}\left[W(\phi)W(\psi)\right] = c_{1,H} \int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H} d\xi ds, \tag{2.2.1}$$

where $c_{1,H}$ is given below by (2.2.7) and $\mathcal{F}\phi(s,\xi)$ is the Fourier transform with respect to the spatial variable x of the function $\phi(s,x)$. Let $\mathcal{F}_t$ be the filtration generated by W. This means

$$\mathcal{F}_t = \sigma\{W(\phi(x)\mathbf{1}_{[0,r]}(s)) : r \in [0,t], \phi(x) \in \mathcal{D}(\mathbb{R})\}.$$

Equation (2.2.1) defines a Hilbert scalar product on $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$. To express this product without the use of Fourier transform, we recall the Marchaud fractional derivative D_-^β of order $\beta \in (0,1)$. For a function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, the Marchaud fractional derivative D_-^β

is defined as:

$$D_-^\beta \phi(t,x) = \lim_{\varepsilon\downarrow 0} D_{-,\varepsilon}^\beta \phi(t,x) = \lim_{\varepsilon\downarrow 0} \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\phi(t,x)-\phi(t,x+y)}{y^{1+\beta}} dy. \tag{2.2.2}$$

We also define the Riemann-Liouville fractional integral of order β of a function ϕ by

$$I_-^\beta \phi(t,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t,y)(y-x)^{\beta-1} dy.$$

Set

$$\mathfrak{H} = \{\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \ ; \ \ \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \ s.t. \ \phi(t,x) = I_-^{\frac{1}{2}-H} \psi(t,x)\}. \tag{2.2.3}$$

With these notations we can express the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to the scalar product given by (2.2.1) in the following proposition (see e.g. [PT00] for a proof).

Proposition 2.2.1. *The function space $\mathfrak{H}$ is a Hilbert space equipped with the scalar product*

$$\begin{aligned} \langle \phi, \psi \rangle_{\mathfrak{H}} =& c_{1,H} \int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)} |\xi|^{1-2H} d\xi ds & (2.2.4)\\ =& c_{2,H} \int_{\mathbb{R}_+\times\mathbb{R}} D_-^{\frac{1}{2}-H}\phi(t,x) D_-^{\frac{1}{2}-H}\psi(t,x) dx dt & (2.2.5)\\ =& c_{3,H} \int_{\mathbb{R}_+} \int_{\mathbb{R}^2} [\phi(t,x+y)-\phi(t,x)][\psi(t,x+y)-\psi(t,x)] |y|^{2H-2} dx dy dt \,, & (2.2.6) \end{aligned}$$

where

$$\begin{aligned} c_{1,H} =& \frac{1}{2\pi} \Gamma(2H+1) \sin(\pi H) \,; & (2.2.7)\\ c_{2,H} =& \left[\Gamma\Big(H+\frac{1}{2}\Big)\right]^2 \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}}\right]^2 dt + \frac{1}{2H} \right)^{-1} \,; & (2.2.8)\\ c_{3,H} =& \sqrt{H(\frac{1}{2}-H)}\, c_{2,H}^{-1/2} \,. & (2.2.9) \end{aligned}$$

The space $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathfrak{H}$.

The Gaussian space $\mathfrak{H}$ is the same as the homogeneous Sobolev space $\dot{\mathcal{H}}^\beta$ for $\beta =$

$\frac{1}{2} - H \in (0, \frac{1}{2})$ in harmonic analysis ([BCD11]). The Gaussian family $W = \{W(\phi),\ \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ can be extended to an isonormal Gaussian process $W = \{W(\phi),\ \phi \in \mathfrak{H}\}$ indexed by the Hilbert space $\mathfrak{H}$. This means that W is a centered Gaussian family such that $\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathfrak{H}}$. It is easy to see that $\phi(t,x) = \chi_{\{[0,t]\times[0,x]\}}$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, is in $\mathfrak{H}$ (we set $\chi_{\{[0,t]\times[0,x]\}} = -\chi_{\{[0,t]\times[x,0]\}}$ if x is negative). We denote $W(t,x) = W(\chi_{\{[0,t]\times[0,x]\}})$.

2.2.2 Stochastic integration

We first define stochastic integral for elementary integrands and then extend it to general ones.

Definition 2.2.2. *An elementary process g is a process of the following form*

$$g(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(t) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The stochastic integral of such an elementary process with respect to W is defined as

$$\begin{aligned}\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x) W(dt,dx) &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \big[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j)\big].\end{aligned} \tag{2.2.10}$$

Proposition 2.2.3. *Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbb{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then, the space of elementary processes defined in Definition 2.2.2 is dense in Λ_H.*

Definition 2.2.4. *For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+\times\mathbb{R}} g(t,x)W(dt,dx)$ is defined as the $L^2(\Omega)$ limit of stochastic integrals of the elementary processes approximating $g(t,x)$ in Λ_H, and we have the following isometry equality*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} g(t,x)W(dt,dx)\right)^2\right] = \mathbb{E}\left[\|g\|_{\mathfrak{H}}^2\right]. \tag{2.2.11}$$

2.2.3 Auxiliary Lemmas

We shall find a solution to equation (2.1.1) in the space $\mathcal{Z}^p_{\lambda,T}$. To deal with the weight λ we need a few technical results concerning the interaction between the weight $\lambda(x)$ and the Green's function $G_t(x-y)$.

Lemma 2.2.5. *For any $\lambda \in \mathbb{R}$, $\lambda(x) = \frac{1}{(1+|x|^2)^\lambda}$ and $T > 0$, we have*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \frac{1}{\lambda(x)}\int_{\mathbb{R}} G_t(x-y)\lambda(y)dy < \infty. \tag{2.2.12}$$

Remark 2.2.6. *To avoid using too many notations we use the symbol λ for a real number and the function induced. Apparently, there will be no confusion.*

Proof. Let us rewrite (2.2.12) as

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} G_t(y)\frac{\lambda(y+x)}{\lambda(x)}dy \le \sup_{0\le t\le T}\int_{\mathbb{R}} G_t(y)\sup_{x\in\mathbb{R}}\frac{\lambda(y+x)}{\lambda(x)}dy.$$

We discuss the cases $\lambda \ge 0$ and $\lambda < 0$ separately. When $\lambda \ge 0$, we have

$$\sup_{x\in\mathbb{R}}\frac{\lambda(y+x)}{\lambda(x)} \le C_\lambda \sup_{x\in\mathbb{R}}\left(\frac{1+|x|}{1+|x+y|}\right)^{2\lambda} \le C_\lambda(1+|y|)^{2\lambda}.$$

On the other hand when $\lambda < 0$ we have

$$\sup_{x\in\mathbb{R}}\left(\frac{1+|x+y|^2}{1+|x|^2}\right)^{-\lambda} \le C_\lambda \sup_{x\in\mathbb{R}}\left(\frac{1+|x+y|}{1+|x|}\right)^{-2\lambda} \le C_\lambda(1+|y|)^{-2\lambda}.$$

In both cases we see

$$\sup_{0\le t\le T}\int_{\mathbb{R}} G_t(y)\sup_{x\in\mathbb{R}}\frac{\lambda(y+x)}{\lambda(x)}dy \le C_\lambda \sup_{t\in[0,T]}\int_{\mathbb{R}} G_t(y)(1+|y|)^{2|\lambda|}dy < \infty.$$

This finishes the proof. ∎

Lemma 2.2.7. *Denote $J(x) := \int_0^\infty e^{-\eta^2}\eta^\beta cos(x\eta)d\eta$, where $\beta > -1$. We have*

$$|J(x)| \le C_\beta\left(1\wedge\frac{1}{|x|^{\beta+1}}\right). \tag{2.2.13}$$

Proof. Notice that this is to estimate the decay rate of the Fourier transform of $e^{-\eta^2}\eta^\beta$ when $|x|$ is large. Since $J(-x) = J(x)$ and since we are only concerned with the large x behaviour we may assume $x \geq 1$. We split the integral $J(x)$ into two parts:

$$J(x) = \int_0^{s(x)} e^{-\eta^2}\eta^\beta \cos(x\eta)d\eta + \int_{s(x)}^{\infty} e^{-\eta^2}\eta^\beta \cos(x\eta)d\eta := J_1(x) + J_2(x),$$

where $s(x) > 0$ is a function to be determined shortly.

First, it is easy to see

$$|J_1(x)| \leq \int_0^{s(x)} \eta^\beta d\eta \leq C_\beta [s(x)]^{\beta+1}.$$

For $J_2(x)$, an integration by parts implies

$$\begin{aligned}
|J_2(x)| &= \Big| \int_{s(x)}^{\infty} e^{-\eta^2}\eta^\beta \cos(x\eta)d\eta \Big| \\
&= \Big| \frac{1}{x} \int_{s(x)}^{\infty} e^{-\eta^2}\eta^\beta d\sin(x\eta) \Big| \\
&\leq C_\beta \frac{[s(x)]^\beta}{x} + \frac{C_\beta}{x} \Big| \int_{s(x)}^{\infty} \eta^{\beta-1} e^{-\eta^2} \sin(x\eta)d\eta \Big| \\
&\quad + \frac{C_\beta}{x} \Big| \int_{s(x)}^{\infty} \eta^{\beta+1} e^{-\eta^2} \sin(x\eta)d\eta \Big|.
\end{aligned}$$

Let $k = \lceil \beta \rceil$ denote the least integer greater than or equal to β. Continuing the above application of integration by parts another k times yields

$$|J_2(x)| \leq \frac{C_\beta}{x^{k+1}} + C_\beta \sum_{j=0}^{k} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^{j+1}}.$$

Combining the estimates of $J_1(x)$ and $J_2(x)$ we have

$$|J(x)| \leq C_\beta [s(x)]^{\beta+1} + \frac{C_\beta}{x^{k+1}} + C_\beta \sum_{j=0}^{k} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^{j+1}}.$$

The lemmas follows with the choice of $s(x) = \frac{1}{x}$. □

Let us associate two increments related to the Green function $G_t(x)$, given as follows.

The first one is a first order difference:

$$D_t(x,h) := G_t(x+h) - G_t(x)\,. \tag{2.2.14}$$

Denote $D(x,h) = \sqrt{\pi}D_{1/4}(x,h) = e^{-(x+h)^2} - e^{-x^2}$. The second one is a second order difference:

$$\Box_t(x,y,h) := G_t(x+y+h) - G_t(x+y) - G_t(x+h) + G_t(x)\,. \tag{2.2.15}$$

As above, we denote $\Box(x,y,h) = \sqrt{\pi}\Box_{1/4}(x,y,h)$:

$$\Box(x,y,h) = e^{-(x+y+h)^2} - e^{-(x+h)^2} - e^{-(x+y)^2} + e^{-x^2}\,. \tag{2.2.16}$$

For these two increments, we have the following identities which are needed later.

Lemma 2.2.8. *For any $\alpha, \beta \in (0,1)$, we have*

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{-1-2\beta} dh dx = \frac{C_\beta}{t^{\frac{1}{2}+\beta}} \tag{2.2.17}$$

and

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dy dh dx = \frac{C_{\alpha,\beta}}{t^{\frac{1}{2}+\alpha+\beta}}. \tag{2.2.18}$$

Proof. With change of variables, it suffices to show

$$\begin{aligned} &\int_{\mathbb{R}^2} |D(x,h)|^2 |h|^{-1-2\beta} dh dx < \infty\,; \\ &\int_{\mathbb{R}^3} |\Box(x,y,h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dy dh dx < \infty. \end{aligned} \tag{2.2.19}$$

The above two inequalities will be derived from Plancherel's identity. The Fourier transforms with respect to the variable x of $D(x,h)$ and $\Box(x,y,h)$ are, respectively,

$$\hat{D}(\xi,h) = \mathcal{F}[D(\cdot,h)](\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}[e^{ih\xi} - 1]$$

and

$$\hat{\Box}(\xi,y,h) = \mathcal{F}[\Box(\cdot,y,h)](\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}[e^{iy\xi} - 1][e^{ih\xi} - 1]\,.$$

Thus, we have

$$\int_{\mathbb{R}} |D(x,h)|^2 dx = \int_{\mathbb{R}} |\hat{D}(\xi,h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}} [1-\cos(h\xi)] d\xi$$

and

$$\int_{\mathbb{R}} |\Box(x,y,h)|^2 dx = \int_{\mathbb{R}} |\hat{\Box}(\xi,y,h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}} [1-\cos(h\xi)][1-\cos(y\xi)] d\xi.$$

By Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^2} |D(x,h)|^2 |h|^{-1-2\beta} dh dx &= C \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}} d\xi \int_{\mathbb{R}} [1-\cos(h\xi)] |h|^{-1-2\beta} dh \\ &= C \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}} |\xi|^{2\beta} d\xi \int_{\mathbb{R}} [1-\cos(h)] |h|^{-1-2\beta} dh < \infty \end{aligned} \tag{2.2.20}$$

since $\int_0^\infty \frac{1-\cos(t)}{t^\theta} dt$ is finite for all $\theta \in (1,3)$ which requires $\alpha, \beta \in (0,1)$. This proves the first inequality in (2.2.19). Same argument shows the second inequality in (2.2.19) under the condition of the lemma. □

Remark 2.2.9. *In the rest of our chapter, we shall use the lemma for* $\alpha = \beta = \frac{1}{2} - H \in (0, 1/4)$.

Lemma 2.2.10. *For* $D(x,h)$ *and* $D_t(x,h)$ *defined in* (2.2.14), *we have*

$$F(x) := \int_{\mathbb{R}} |D(x,h)|^2 |h|^{2H-2} dh \le C_H \left(1 \wedge |x|^{2H-2}\right), \tag{2.2.21}$$

and when $t > 0$

$$F_t(x) := \int_{\mathbb{R}} |D_t(x,h)|^2 |h|^{2H-2} dh \le C_H \left(t^{H-\frac{3}{2}} \wedge \frac{|x|^{2H-2}}{\sqrt{t}}\right), \tag{2.2.22}$$

where $0 < H < \frac{1}{2}$.

Proof. The assertion (2.2.22) is an easy consequence of (2.2.21) by change of variables so we only need to provide a proof for (2.2.21).

Recall that the Fourier transform of $D(x,h)$ (as a function of x) is

$$\hat{D}(\eta,h)=\mathcal{F}[D(\cdot,h)](\eta)=\sqrt{\pi}e^{-\frac{\eta^2}{4}}[e^{ih\eta}-1]\,.$$

By the inverse Fourier transformation $D(x,h)$ can also be written as

$$D(x,h)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{D}(\eta,h)e^{ix\eta}d\eta=\frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}}e^{-\frac{\eta^2}{4}}[e^{ih\eta}-1]e^{ix\eta}d\eta\,.$$

Therefore, we can write

$$\begin{aligned}F(x)&=C_H\pi^2\int_{\mathbb{R}^2}e^{-\frac{\eta_1^2+\eta_2^2}{4}}\int_{\mathbb{R}}[e^{ih\eta_1}-1]\overline{[e^{ih\eta_2}-1]}|h|^{2H-2}dh\ e^{ix(\eta_1-\eta_2)}d\eta_1d\eta_2\\&=C_H\int_{\mathbb{R}^2}e^{-\frac{\eta_1^2+\eta_2^2}{4}}H(\eta_1,\eta_2)e^{ix(\eta_1-\eta_2)}d\eta_1d\eta_2\,,\end{aligned}$$

where similar to (2.2.20), we have

$$\begin{aligned}H(\eta_1,\eta_2)&=C_H\int_{\mathbb{R}}[e^{ih\eta_1}-1]\overline{[e^{ih\eta_2}-1]}|h|^{2H-2}dh\\&=C_H\left(|\eta_1|^{1-2H}+|\eta_2|^{1-2H}-|\eta_1-\eta_2|^{1-2H}\right)\,.\end{aligned}\tag{2.2.23}$$

It is easy to see that $\sup_{x\in\mathbb{R}}|F(x)|\le C<\infty$. Now, we want to get the desired decay estimate when x goes to infinity. We have

$$\begin{aligned}F(x)\le\ &C_H\left|\int_{\mathbb{R}^2}e^{-\frac{\eta_1^2+\eta_2^2}{4}}|\eta_2|^{1-2H}e^{ix(\eta_1-\eta_2)}d\eta_1d\eta_2\right|\\&+C_H\left|\int_{\mathbb{R}^2}e^{-\frac{\eta_1^2+\eta_2^2}{4}}|\eta_1-\eta_2|^{1-2H}e^{ix(\eta_1-\eta_2)}d\eta_1d\eta_2\right|\\\le\ &C_He^{-x^2}\left|\int_{\mathbb{R}}e^{-\frac{\eta_2^2}{4}}|\eta_2|^{1-2H}e^{-ix\eta_2}d\eta_2\right|\\&+C_H\left|\int_{\mathbb{R}}\left[|\eta|^{1-2H}e^{-ix\eta}\int_{\mathbb{R}}e^{-\frac{|\eta_2|^2+|\eta_2+\eta|^2}{4}}d\eta_2\right]d\eta\right|\\\le\ &C_He^{-x^2}\left|\int_{\mathbb{R}_+}e^{-\frac{\eta^2}{4}}|\eta|^{1-2H}cos(x\eta)d\eta\right|+C_H\left|\int_{\mathbb{R}_+}e^{-\frac{\eta^2}{8}}|\eta|^{1-2H}cos(x\eta)d\eta\right|\end{aligned}$$

since

$$\int_{\mathbb{R}}e^{-\frac{|\eta_2|^2+|\eta_2+\eta|^2}{4}}d\eta_2=Ce^{-\frac{|\eta|^2}{8}}\,.$$

Now the inequality (2.2.21) follows from Lemma 2.2.7. $\square$

Lemma 2.2.11. *Recall that* $\Box_t(x,y,h)$ *and* $\Box(x,y,h)$ *are defined by* (2.2.15) *and* (2.2.16). *We have*

$$F(x) := \int_{\mathbb{R}^2} |\Box(x,y,h)|^2 |h|^{2H-2}|y|^{2H-2} dydh \le C_H \left(1 \wedge |x|^{2H-2}\right) . \tag{2.2.24}$$

Moreover, for any $t>0$ *we have*

$$F_t(x) := \int_{\mathbb{R}^2} |\Box_t(x,y,h)|^2 |h|^{2H-2}|y|^{2H-2} dydh \le C_H \left(t^{2H-2} \wedge \frac{|x|^{2H-2}}{t^{1-H}}\right) . \tag{2.2.25}$$

Proof. As in the proof of Lemma 2.2.10 we only need to prove (2.2.24) and last inequality can be derived from (2.2.24) by a change of variable.

The proof of (2.2.24) is similar to that of Lemma 2.2.10. Recall the Fourier transform of $\Box(x,y,h)$ as a function of x:

$$\hat{\Box}(\eta,y,h) = \mathcal{F}[\Box(\cdot,y,h)](\eta) = \sqrt{\pi} e^{-\frac{\eta^2}{4}} [e^{iy\eta}-1][e^{ih\eta}-1] .$$

This means

$$\Box(x,y,h) = \sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{\eta^2}{4}} [e^{iy\eta}-1][e^{ih\eta}-1] e^{ix\eta} d\eta .$$

Thus, we have

$$\begin{aligned} F(x) &= \int_{\mathbb{R}^4} e^{-\frac{\eta_1^2+\eta_2^2}{4}} [e^{iy\eta_1}-1][e^{ih\eta_1}-1] \cdot \overline{[e^{iy\eta_2}-1]} \\ &\qquad \overline{[e^{ih\eta_2}-1]} |h|^{2H-2}|y|^{2H-2} e^{ix(\eta_1-\eta_2)} dydhd\eta_1 d\eta_2 \\ &= 2\pi^2 \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2+\eta_2^2}{4}} H^2(\eta_1,\eta_2) e^{ix(\eta_1-\eta_2)} d\eta_1 d\eta_2 , \end{aligned} \tag{2.2.26}$$

where $H(\eta_1,\eta_2)$ is defined in (2.2.23) or

$$\begin{aligned} H^2(\eta_1,\eta_2) = C_H^2 \Big(& |\eta_1|^{2-4H} + |\eta_2|^{2-4H} + |\eta_1|^{1-2H}|\eta_2|^{1-2H} + |\eta_1-\eta_2|^{2-4H} \\ & - |\eta_1|^{1-2H}|\eta_1-\eta_2|^{1-2H} - |\eta_2|^{1-2H}|\eta_1-\eta_2|^{1-2H} \Big) . \end{aligned}$$

It is easy to see that $\sup_{x\in\mathbb{R}} |F(x)| \le C < \infty$. Now we want to show the desired decay rate as $x \to \infty$. By the symmetry $F(-x) = F(x)$, we can and will assume $x \ge 1$. The argument in the proof of Lemma 2.2.10 can be used to obtain the desired bound for each of the above terms except the terms $|\eta_1 - \eta_2|^{2-4H}$ and $|\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H}$ (and $|\eta_2|^{1-2H}|\eta_1 - \eta_2|^{1-2H}$, which can be handled analogously).

For term $|\eta_1 - \eta_2|^{2-4H}$, letting $\xi_1 = \eta_1 - \eta_2$ and $\xi_2 = \eta_1 + \eta_2$ implies

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{-\frac{\eta_1^2+\eta_2^2}{4}} |\eta_1 - \eta_2|^{2-4H} e^{ix(\eta_1-\eta_2)} d\eta_1 d\eta_2 \\ =&C \int_{\mathbb{R}^2} e^{-\frac{\xi_1^2+\xi_2^2}{8}} |\xi_1|^{2-4H} e^{ix\xi_1} d\xi_1 d\xi_2 = C \int_{\mathbb{R}_+} e^{-\frac{\xi^2}{8}} |\xi|^{2-4H} \cos(x\xi) d\xi.\end{aligned}$$

Then using Lemma 2.2.7, we see that this term is bounded by $1 \wedge |x|^{4H-3} \lesssim 1 \wedge |x|^{2H-2}$ for $\frac{1}{4} < H < \frac{1}{2}$.

In order to deal with the second term $|\eta_1|^{1-2H}|\eta_1 - \eta_2|^{1-2H}$, we make the substitution $\xi = \eta_1$ and $\eta = \frac{1}{2}(\eta_1 - \eta_2)$ to obtain

$$\begin{aligned}J(x) :=& \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2+\eta_2^2}{4}} |\eta_1|^{1-2H} |\eta_1 - \eta_2|^{1-2H} e^{ix(\eta_1-\eta_2)} d\eta_1 d\eta_2 \\ =&C \int_{\mathbb{R}^2} \exp\left(-\frac{(\xi-\eta)^2}{2}\right) \exp\left(-\frac{\eta^2}{2}\right) |\xi|^{1-2H} |\eta|^{1-2H} e^{i2x\eta} d\xi d\eta\,.\end{aligned}$$

Denote

$$E(\eta) := \int_{\mathbb{R}} \exp\left(-\frac{(\xi-\eta)^2}{2}\right) |\xi|^{1-2H} d\xi\,.$$

We need to show a similar inequality to that in Lemma 2.2.7:

$$|J(x)| = \left| \int_0^\infty e^{-\frac{\eta^2}{2}} \eta^{1-2H} E(\eta) \cos(2x\eta) d\eta \right| \le C_H \left(1 \wedge |x|^{2H-2}\right).$$

First, we observe that $|E(\eta)| \le C_H(1 + |\eta|^{1-2H})$ and both $|E'(\eta)|$ and $|E''(\eta)|$ can be bounded by a multiple of

$$\int_{\mathbb{R}} \exp\left(-\frac{(\xi-\eta)^2}{4}\right) |\xi|^{1-2H} d\xi \le C_H \left(1 + |\eta|^{1-2H}\right).$$

We only need to care the case when x is large. Let us split $J(x)$ into two parts of which one integrates from 0 to $s(x)$, denoted by $J_1(x)$, and the other integrates from $s(x)$ to infinity, denoted by $J_2(x)$, such that $s(x) \to 0$ as x goes to infinity and whose precise form will be given later. For the first part

$$|J_1(x)| \le [s(x)]^{1-2H} \int_0^{s(x)} |E(\eta)| d\eta \le C_H \left([s(x)]^{2-2H} + [s(x)]^{3-4H} \right).$$

For $J_2(x)$, an integration by parts yields

$$\begin{aligned}
&|J_2(x)| = \Big| \int_{s(x)}^{\infty} e^{-\frac{\eta^2}{2}} \eta^{1-2H} E(\eta) \cos(2x\eta) d\eta \Big| \\
=&C \Big| \frac{1}{x} \int_{s(x)}^{\infty} e^{-\frac{\eta^2}{2}} \eta^{1-2H} E(\eta) d \sin(2x\eta) \Big| \\
\le& C_H \frac{[s(x)]^{1-2H}}{x} e^{-\frac{[s(x)]^2}{2}} |E(s(x))| + \frac{C_H}{x} \Big| \int_{s(x)}^{\infty} \eta^{-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \Big| \\
+& \frac{C_H}{x} \Big| \int_{s(x)}^{\infty} \eta^{2-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \Big| + \frac{C_H}{x} \Big| \int_{s(x)}^{\infty} \eta^{1-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E'(\eta) d\eta \Big| \\
=:& J_{21} + J_{22} + J_{23} + J_{24}.
\end{aligned}$$

The first term is bounded by

$$J_{21}(x) \le C_H \frac{1}{x} [s(x)]^{1-2H}.$$

As for $J_{22}(x)$ an integration by parts yields

$$\begin{aligned}
J_{22}(x) :=& \frac{1}{x} \Big| \int_{s(x)}^{\infty} \eta^{-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \Big| \\
\le& C \frac{E(s(x))}{x^2} [s(x)]^{-2H} + \frac{C}{x^2} \int_{s(x)}^{\infty} \left| \frac{d}{d\eta} \left[\eta^{-2H} E(\eta) e^{-\frac{\eta^2}{2}} \right] \right| d\eta \\
\le& \frac{C_H}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} [s(x)]^{1-4H} + \frac{C_H}{x^2}.
\end{aligned}$$

In the same way we can bound $J_{23}(x)$ as follows.

$$\begin{aligned}
J_{23}(x) :=& \frac{1}{x} \Big| \int_{s(x)}^{\infty} \eta^{2-2H} e^{-\frac{\eta^2}{2}} \sin(2x\eta) E(\eta) d\eta \Big| \\
\le& C \frac{E(s(x))}{x^2} [s(x)]^{2-2H} + \frac{C}{x^2} \int_{s(x)}^{\infty} \left| \frac{d}{d\eta} \left[\eta^{2-2H} E(\eta) e^{-\frac{\eta^2}{2}} \right] \right| d\eta
\end{aligned}$$

$$\leq \frac{C_H}{x^2}[s(x)]^{2-2H} + \frac{C_H}{x^2}[s(x)]^{3-4H} + \frac{C_H}{x^2}.$$

The term $J_{24}(x)$ satisfies

$$\begin{aligned}
J_{24}(x) :=& \frac{1}{x}\Big|\int_{s(x)}^{\infty} \eta^{1-2H} e^{-\frac{\eta^2}{2}} \sin(x\eta) E'(\eta) d\eta\Big| \\
\leq & C\frac{E'(s(x))}{x^2}[s(x)]^{1-2H} + \frac{C}{x^2}\int_{s(x)}^{\infty}\left|\frac{d}{d\eta}\left[\eta^{1-2H}E'(\eta)e^{-\frac{\eta^2}{2}}\right]\right| d\eta \\
\leq & \frac{C_H}{x^2}[s(x)]^{1-2H} + \frac{C_H}{x^2}[s(x)]^{2-4H} + \frac{C_H}{x^2}.
\end{aligned}$$

Noticing that $\frac{1}{4} < H < \frac{1}{2}$, and taking $s(x) = \frac{1}{x}$ imply our result. □

Lemma 2.2.12. *Denote $\lambda(x) = \frac{1}{(1+|x|^2)^{1-H}}$ and recall $D_t(x,h)$ defined by (2.2.14) and $\Box_t(x,y,h)$ defined by (2.2.15). We have*

$$\begin{aligned}
&\int_{\mathbb{R}^2} |D_t(x,h)|^2|h|^{2H-2}\lambda(z-x)dxdh \leq C_{T,H}t^{H-1}\lambda(z), \\
&\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2|h|^{2H-2}|y|^{2H-2}\lambda(z-x)dxdydh \leq C_{T,H}t^{2H-\frac{3}{2}}\lambda(z).
\end{aligned} \tag{2.2.27}$$

Proof. Set

$$R(x,z) = \frac{\lambda(z-x)}{\lambda(z)} \simeq \left(\frac{1+|z|}{1+|x-z|}\right)^{2-2H},$$

where and throughout the chapter for two functions f and g, notation $f \simeq g$ means that there exist two positive constants c_H and C_H such that $c_H g \leq f \leq C_H g$. By Lemma 2.2.8, we have by change of variables $x \to x\sqrt{t}$, $h \to h\sqrt{t}$ and $z \to z\sqrt{t}$

$$\begin{aligned}
&\int_{\mathbb{R}^2} |D_t(x,h)|^2|h|^{2H-2}R(x,z)dxdh \\
\leq & C_H t^{H-1}\int_{\mathbb{R}^2} |D(x,h)|^2|h|^{2H-2}R(\sqrt{t}x,\sqrt{t}z)dxdh \\
\leq & C_H t^{H-1}\int_{\mathbb{R}} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x,\sqrt{t}z)dx.
\end{aligned} \tag{2.2.28}$$

Similarly, making substitutions $x \to x\sqrt{t}$, $y \to y\sqrt{t}$, $h \to h\sqrt{t}$ and $z \to z\sqrt{t}$ we can get rid of the t in $\Box_t$:

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2|h|^{2H-2}|y|^{2H-2}R(x,z)dxdydh$$

$$
\begin{aligned}
&= C_H t^{2H-\frac{3}{2}} \int_{\mathbb{R}^3} |\Box(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} R(\sqrt{t}x, \sqrt{t}z) dx dy dh \\
&\le C_H t^{2H-\frac{3}{2}} \int_{\mathbb{R}} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x, \sqrt{t}z) dx. \qquad (2.2.29)
\end{aligned}
$$

Notice that the above change of variable with respect to z is not essential because we will take its supremum over $\mathbb{R}$. But it will be convenient for us to split the intervals. From (2.2.28) and (2.2.29) to show our lemma it is sufficient to show

$$
\sup_{t\in[0,T]} \sup_{z\in\mathbb{R}} \int_{\mathbb{R}} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x, \sqrt{t}z) dx < \infty. \qquad (2.2.30)
$$

Notice that we assume that $t \in [0,T]$ is bounded. If z is bounded, then $R(\sqrt{t}x, \sqrt{t}z)$ is also bounded. Then, we have

$$
\sup_{t\in[0,T]} \sup_{|z|\le 2} \int_{\mathbb{R}} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x, \sqrt{t}z) dx \ \le \ C_{T,H} \int_{\mathbb{R}} 1 \wedge |x|^{2H-2} dx < \infty. \qquad (2.2.31)
$$

This means that we only need to consider the case $|z| \ge 2$. Due to the symmetry $R(-\sqrt{t}x, -\sqrt{t}z) = R(\sqrt{t}x, \sqrt{t}z)$, we can assume $z \ge 2$.

Next we split the domain of the integral into two parts.

(i) The domain $x \le z/2$ or $x \ge 2z$. On this domain $R(\sqrt{t}x, \sqrt{t}z)$ is bounded. Thus

$$
\sup_{t\in[0,T]} \sup_{|z|\ge 1} \int_{\left\{ \substack{x\le z/2 \\ x\ge 2z} \right\}} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x, \sqrt{t}z) dx \ \le \ C_T \int_{\mathbb{R}} 1 \wedge |x|^{2H-2} dx < \infty. \quad (2.2.32)
$$

(ii) The domain $z/2 \le x \le 2z$. On this domain we have $x \ge z/2 \ge (z+1)/3 \ge 1$ and then

$$
1 \wedge |x|^{2H-2} \le |x|^{2H-2} \le \frac{3^{2-2H}}{(1+z)^{2-2H}} \,.
$$

Thus,

$$
\begin{aligned}
I := & \int_{\frac{z}{2}<x<2z} (1 \wedge |x|^{2H-2}) R(\sqrt{t}x, \sqrt{t}z) dx \\
\le & C_H \left(\frac{1+\sqrt{t}z}{1+z} \right)^{2-2H} \int_0^{2z} \frac{1}{(1+\sqrt{t}|x-z|)^{2-2H}} dx \,.
\end{aligned}
$$

By the symmetry of the above integrand we know that the integrals $\int_0^z$ and $\int_z^{2z}$ are the same. Hence, we have

$$\begin{aligned}
I \leq & C_H \frac{1+(\sqrt{t}z)^{2-2H}}{1+z^{2-2H}} \int_z^{2z} \frac{1}{\left(1+\sqrt{t}|x-z|\right)^{2-2H}} dx \\
= & C_H \frac{1+(\sqrt{t}z)^{2-2H}}{\sqrt{t}\,(1+z^{2-2H})} \left[1-(1+\sqrt{t}z)^{2H-1}\right] \\
\leq & C_H T^{\frac{1}{2}-H} \frac{1+(\sqrt{t}z)^{2-2H}}{\sqrt{t}z\left(1+(\sqrt{t}z)^{1-2H}\right)} \left[1-(1+\sqrt{t}z)^{2H-1}\right] .
\end{aligned}$$

Consider now the function

$$f(u) = \frac{1+u^{2-2H}}{u(1+u^{1-2H})} \left[1-(1+u)^{2H-1}\right] , \quad u > 0 .$$

This is a continuous function on $(0, \infty)$. When $u \to 0$ and when $u \to \infty$ we have

$$\lim_{u \to 0+} f(u) = 1 - 2H , \quad \lim_{u \to \infty} f(u) = 1 .$$

Thus, $f(u)$ is bounded on $(0, \infty)$ and this in turn proves

$$\sup_{t \in [0,T]} \sup_{z \geq 1} \int_{z/2 \leq x \leq 2z} \left(1 \wedge |x|^{2H-2}\right) R(\sqrt{t}x, \sqrt{t}z) dx \ < \infty. \tag{2.2.33}$$

Combining (2.2.31)-(2.2.32) together with our above symmetry argument we prove (2.2.30) and hence we complete the proof of the lemma. □

Remark 2.2.13. *From this lemma, we see why we choose the above decay rate for our weight function. If we consider* $\lambda(x) = (1+|x|^2)^{-\lambda}$ *with* $\lambda > 1-H$*, then for* $|z|$ *sufficiently large one has*

$$\int_{\mathbb{R}} \left(1 \wedge |x|^{2H-2}\right) R(x,z) dx \gtrsim \int_{|x-z|<1} |x|^{2H-2} R(x,z) dx \gtrsim \frac{(1+|z|)^{\lambda}}{|z|^{2-2H}},$$

which diverges as $|z| \to \infty$*. This elementary fact suggests us that* λ *must be in* $(\frac{1}{2}, 1-H]$*, and it is obvious* $L^p_\lambda(\mathbb{R})$ *is the largest space when* $\lambda = 1 - H$*.*

2.3 Additive noise

When the diffusion coefficient $\sigma(t,x,u)=1$ (or a general constant), the noise is additive and the solution to (2.1.1) can be written explicitly as

$$u(t,x)=\int_{\mathbb{R}} G_t(x-y)u_0(y)dy+\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy)\,, \tag{2.3.1}$$

where $G_t(x)=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$ is the heat kernel. To focus on the stochastic part we assume $u_0=0$. Thus, the resulting solution is written as

$$u_{\rm add}(t,x)=\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)W(ds,dy)\,. \tag{2.3.2}$$

This solution $u_{\rm add}(t,x)$ defines a (symmetric) centered Gaussian process. We shall study how it grows as the parameters t and x go to infinity. It is expected that $u_{\rm add}(t,x)$ is Hölder continuous in t and x. More precisely, for any positive constants $\gamma<H$, $T,L\in(0,\infty)$, there is a constant $C_{T,L,\gamma}$, depending only on T, L and γ, such that

$$\sup_{0\le s,t\le T\,,|x|,|y|\le L}|u_{\rm add}(s,x)-u_{\rm add}(t,y)|\le C_{T,L,\gamma}\left(|t-s|^{\gamma/2}+|x-y|^{\gamma}\right)\,.$$

We want to consider the Hölder continuity of $u_{\rm add}(t,x)$ on the whole space $\mathbb{R}$. Namely, we want to know how the sharp constant $C_{T,L,\gamma}$ grows as T and L go to infinity (for any fixed γ).

2.3.1 Majorizing measure theorem

To find the sharp bound for $C_{T,L,\gamma}$ we shall utilize Talagrand's majorizing measure theorem which we recall below.

Theorem 2.3.1. (Majorizing Measure Theorem, see e.g. [Tal14, Theorem 2.4.2]). *Let T be a given set and let $\{X_t,t\in T\}$ be a centered Gaussian process indexed by T. Denote $d(t,s)=(\mathbb{E}|X_t-X_s|^2)^{\frac12}$, the associated natural metric on T. Then*

$$\mathbb{E}\Big[\sup_{t\in T}X_t\Big]\asymp\gamma_2(T,d):=\inf_{\mathcal{A}}\sup_{t\in T}\sum_{n\ge0}2^{n/2}{\rm diam}\left(A_n(t)\right), \tag{2.3.3}$$

where the infimum is taken over all increasing sequence $\mathcal{A} := \{\mathcal{A}_n, n = 1, 2, \cdots\}$ *of partitions of* T *such that* $\#\mathcal{A}_n \leq 2^{2^n}$ *(*$\#A$ *denotes the number of elements in the set* A*),* $A_n(t)$ *denotes the unique element of* $\mathcal{A}_n$ *that contains* t*, and* $\mathrm{diam}\,(A_n(t))$ *is the diameter (with respect to the natural distance* d*) of* $A_n(t)$*.*

This theorem provides a powerful general principle for the study of the supremum of Gaussian process.

Remark 2.3.2. *The natural metric* $d(t,s)$ *is actually only a pseudo-metric because* $d(t,s) = 0$ *does not necessarily imply* $t = s$ *(e.g.* $X_t \equiv 1$*). It is also called the canonical metric.*

It is more convenient for us to use the following theorem to obtain the lower bound.

Theorem 2.3.3. (Sudakov minoration theorem, see e.g. [Tal14, Lemma 2.4.2]). *Let* $\{X_{t_i}, i = 1, \cdots, L\}$ *be a centered Gaussian family with natural distance* d *and assume*

$$\forall p, q \leq L,\ p \neq q \Rightarrow d(t_p, t_q) \geq \delta.$$

Then, we have

$$\mathbb{E}\Big(\sup_{1\leq i\leq L} X_{t_i}\Big) \geq \frac{\delta}{C}\sqrt{\log_2(L)}, \tag{2.3.4}$$

where C *is a universal constant.*

The following "concentration of measure" type theorem allows us to obtain deviation inequalities for the supremum of a Gaussian family.

Theorem 2.3.4. (Borell, see e.g. [Adl90, Theorem 2.1]). *Let* $\{X_t, t \in T\}$ *be a centered separable Gaussian process on some topological index set* T *with almost surely bounded sample paths. Then* $\mathbb{E}\Big(\sup_{t\in T} X_t\Big) < \infty$*, and for all* $\lambda > 0$

$$\mathbf{P}\left\{\left|\sup_{t\in T} X_t - \mathbb{E}\Big(\sup_{t\in T} X_t\Big)\right| > \lambda\right\} \leq 2\exp\left(-\frac{\lambda^2}{2\sigma_T^2}\right), \tag{2.3.5}$$

where $\sigma_T^2 := \sup_{t\in T} \mathbb{E}(X_t^2)$*.*

We have the following observation which can be deduced immediately from [Tal14, Lemma 2.2.1]. This simple fact tells us $\mathbb{E}[\sup_{t\in T}|X_t|] \simeq \mathbb{E}[\sup_{t\in T} X_t]$. So, we only need to consider $\mathbb{E}[\sup_{t\in T} X_t]$.

Lemma 2.3.5. *If the process $\{X_t, t \in T\}$ is symmetric, then we have*

$$\mathbb{E}\big[\sup_{t\in T}|X_t|\big] \leqslant 2\mathbb{E}\big[\sup_{t\in T} X_t\big] + \inf_{t_0\in T}\mathbb{E}\big[|X_{t_0}|\big]\,. \tag{2.3.6}$$

2.3.2 Asymptotics of the Gaussian solution

For the mild solution $u_{\mathrm{add}}(t,x)$ to (2.1.1) with additive noise (e.g. $\sigma(t,x,u) = 1$), defined by (2.3.2), we shall first obtain the sharp upper and lower bounds for its associated natural metric:

$$d_1((t,x),(s,y)) = \sqrt{\mathbb{E}|u_{\mathrm{add}}(t,x) - u_{\mathrm{add}}(s,y)|^2}\,, \tag{2.3.7}$$

The following lemma gives a sharp bounds for this induced natural metric for the Gaussian solution $u_{\mathrm{add}}(t,x)$.

Lemma 2.3.6. *Let $d_1((t,x),(s,y))$ be the natural metric defined by (2.3.7). Then, there are positive constants c_H, C_H such that*

$$\begin{aligned} c_H(|x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}) &\leq d_1((t,x),(s,y)) \\ &\leq C_H(|x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}) \end{aligned} \tag{2.3.8}$$

for any $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}$.

Remark 2.3.7. *The above property of the natural metric can also be written as*

$$d_1((t,x),(s,y)) \asymp d_{1,H}((t,x),(s,y)) := |x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}. \tag{2.3.9}$$

$d_{1,H}((t,x),(s,y))$ is no longer a distance but it is very convenient for us to obtain the desired results.

Proof. Without loss of generality, let us assume $t > s$. Plancherel's identity and the independence of the stochastic integrals over the time intervals $[0,s]$ and $[s,t]$ give

$$
\begin{aligned}
d_1^2((t,x),(s,y)) &= \mathbb{E}|u_{\text{add}}(t,x) - u_{\text{add}}(s,y)|^2 \\
=&\mathbb{E}\left|\int_0^s \int_{\mathbb{R}} [G_{t-r}(x-z) - G_{s-r}(y-z)]W(dr,dz)\right|^2 \\
&+ \mathbb{E}\left|\int_s^t \int_{\mathbb{R}} G_{t-r}(x-z)W(dr,dz)\right|^2 \\
=& \int_{\mathbb{R}_+} [1-\exp(-2s\xi^2)][1+\exp(-2(t-s)\xi^2) \\
&\qquad - 2\exp(-(t-s)\xi^2)\cos(|x-y|\xi)]\cdot \xi^{-1-2H} d\xi + 2^{H-1}\kappa_H (t-s)^H ,
\end{aligned} \tag{2.3.10}
$$

where $\kappa_H = H^{-1}\Gamma(1-H)$ is a positive constant. We start to obtain the upper bound of (2.3.8). The triangle inequality gives

$$
d_1((t,x),(s,y)) \leq d_1((t,x),(s,x)) + d_1((s,x),(s,y)) . \tag{2.3.11}
$$

Let us deal with the two terms on the right hand side of the above inequality separately. For the first term, Plancherel's identity (2.3.10) implies

$$
\begin{aligned}
d_1^2((t,x),(s,x)) &= \kappa_H \left[2^{H-1}t^H + 2^{H-1}s^H - (t+s)^H\right] + (2^{H-1}+1)\kappa_H (t-s)^H \\
&\leq C_H (t-s)^H ,
\end{aligned}
$$

because $2^{H-1}t^H + 2^{H-1}s^H - (t+s)^H \leq 0$ when $t \geq s$. Again from (2.3.10), the second term on the right hand side of (2.3.11) is given by

$$
\begin{aligned}
d_1^2((s,x),(s,y)) &= \int_0^s \int_{\mathbb{R}} \exp[-2(s-r)\xi^2]\cdot|\xi|^{1-2H}|1-\cos(\xi|x-y|)|d\xi dr \\
&= C_H|x-y|^{2H} \int_{\mathbb{R}_+} \left[1-\exp\left(-\frac{2s\xi^2}{|x-y|^2}\right)\right]\cdot \xi^{-1-2H}[1-\cos(\xi)]d\xi ,
\end{aligned}
$$

which can be controlled by $C_H|x-y|^{2H}$. On the other hand, we have

$$
\begin{aligned}
d_1^2((s,x),(s,y)) =&\mathbb{E}[|u_{\text{add}}(s,x) - u_{\text{add}}(s,y)|^2] \\
\leq&2\big(\mathbb{E}[|u_{\text{add}}(s,x)|^2] + \mathbb{E}[|u_{\text{add}}(s,y)|^2]\big) \leq C_H s^H .
\end{aligned}
$$

Thus, the quantity of $d_1^2((s,x),(s,y))$ is bounded by the minimum of $C_H|x-y|^{2H}$ and $C_H s^H$. We can summarize the above argument as

$$d_1((t,x),(s,y)) \le C_H(|x-y|^H \wedge s^{\frac{H}{2}} + (t-s)^{\frac{H}{2}})\,, \tag{2.3.12}$$

which is the upper bound part of (2.3.8).

Now we turn to the lower bound part of (2.3.8). From Plancherel's identity it is sufficient to bound the first summand in (2.3.10) from below by $c_H(|x-y|^H \wedge s^{\frac{H}{2}})$ for some constant $c_H > 0$. We denote this first summand by I:

$$\begin{aligned} I :=& \int_{\mathbb{R}} [1-\exp(-2s\xi^2)][1+\exp(-2(t-s)\xi^2) \\ &- 2\exp(-(t-s)\xi^2)\cos(|x-y|\xi)]|\xi|^{1-2H} d\xi \\ =& c|x-y|^{2H} \int_{\mathbb{R}_+} \left[1-\exp\left(-\frac{2s\xi^2}{|x-y|^2}\right)\right] \cdot \xi^{-1-2H} \\ & \qquad \cdot \left[1-\exp\left(-\frac{(t-s)\xi^2}{|x-y|^2}\right)\cos(\xi)\right]^2 d\xi\,. \end{aligned} \tag{2.3.13}$$

To bound it from below, we divide our argument into two cases:

$$|x-y| > \sqrt{s} \qquad \text{and} \qquad |x-y| \le \sqrt{s}\,.$$

When $|x-y| \le \sqrt{s}$, we can bound (2.3.13) from below by

$$\begin{aligned} I \ge& c_H|x-y|^{2H} \sum_{n=1}^{\infty} \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}} \left[1-\exp\left(-2\xi^2\right)\right] \cdot \xi^{-1-2H} d\xi \\ \ge& c_H|x-y|^{2H}\,, \end{aligned} \tag{2.3.14}$$

since $1-\exp(-2s\xi^2/|x-y|^2) \ge 1-\exp(-2\xi^2)$ by the assumption and $\cos(\xi)$ is negative on the intervals $\bigcup_{n=1}^{\infty}[2n\pi+\frac{\pi}{2}, 2n\pi+\frac{3\pi}{2}]$.

The case $|x-y| > \sqrt{s}$ is a little bit more involved. Denote

$$n_0 := \inf\left\{n \in \mathbb{N}_0 : 2n\pi + \frac{\pi}{2} \ge \sqrt{\frac{-\ln(1-c^*)}{2s}}|x-y|\right\}$$

with the choice $c^* = 1 - \exp(-\pi^2/2)$ such that

$$\sqrt{\frac{-\ln(1-c^*)}{2s}}|x-y| \geq \frac{\pi}{2}\,.$$

It is then easy to see that n_0 is a well defined finite positive integer. This way, we have the lower bound for (2.3.13):

$$\begin{aligned} I \geq & \sum_{n=n_0} |x-y|^{2H} \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}} \left[1 - \exp\left(-\frac{2s\xi^2}{|x-y|^2}\right)\right] \cdot \xi^{-1-2H} d\xi \\ \geq & c^*|x-y|^{2H} \sum_{n\geq n_0} \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}} \xi^{-1-2H} d\xi \geq \frac{c^*}{2}|x-y|^{2H} \int_{2n_0\pi+\frac{\pi}{2}}^{\infty} \xi^{-1-2H} d\xi, \end{aligned}$$

where the last inequality follows from the fact that ξ^{-1-2H} is a decreasing function on $(0,\infty)$. From the definition of n_0, it follows

$$I \geq c_H|x-y|^{2H}\left(\sqrt{\frac{-\ln(1-c^*)}{2s}}|x-y| + 2\pi\right)^{-2H} \geq c_H s^H \tag{2.3.15}$$

since $|x-y| > \sqrt{s}$ and consequently

$$\begin{aligned} & |x-y|^{2H}\left(\sqrt{\frac{-\ln(1-c^*)}{2s}}|x-y| + 2\pi\right)^{-2H} \\ & = \left(\sqrt{\frac{-\ln(1-c^*)}{2s}} + \frac{2\pi}{|x-y|}\right)^{-2H} \geq \left(\sqrt{\frac{-\ln(1-c^*)}{2s}} + \frac{2\pi}{\sqrt{s}}\right)^{-2H} = c_H s^H\,. \end{aligned}$$

Thus, (2.3.14) together with (2.3.15) imply

$$d_1((t,x),(s,y)) \geq c_H(|x-y|^H \wedge s^{\frac{H}{2}} + (t-s)^{\frac{H}{2}}). \tag{2.3.16}$$

Combining (2.3.12) and (2.3.16), we complete the proof of this lemma. □

Now we are ready to prove Theorem 2.1.1, which gives a sharp bound for

$$\mathbb{E}\left[\sup_{\substack{0\leq t\leq T\\ -L\leq x\leq L}} |u_{\text{add}}(t,x)|\right]\,.$$

Proof of the first part of Theorem 2.1.1. To simplify notation we denote

$$\mathbb{T} = [0, T] \quad \text{and} \quad \mathbb{L} = [-L, L] \,.$$

Since $u_{\text{add}}(t, x)$ is a symmetric and centred Gaussian process Lemma 2.3.5 states that

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} |u_{\text{add}}(t, x)|\right] \simeq \mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u_{\text{add}}(t, x)\right] . \tag{2.3.17}$$

Hence, to show (2.1.3) it is equivalent to show

$$c_H \, \Psi(T, L) \leq \mathbb{E}\left[\sup_{t\in\mathbb{T}, x\in\mathbb{L}} u_{\text{add}}(t, x)\right] \leq C_H \, \Psi(T, L) \,, \tag{2.3.18}$$

where $\Psi(T, L)$ is defined by (2.1.4). We shall prove the upper and lower bound parts of (2.3.18) separately. Let us first consider the upper bound part in (2.3.18). We shall use the majorizing measure method (Theorem 2.3.1) and our bound for the natural distance (Lemma 2.3.6). Let us separate the proof into the cases $L > \sqrt{T}$ and $L \leq \sqrt{T}$. First, we assume $L > \sqrt{T}$. We choose the admissible sequences $(\mathcal{A}_n)$ as uniform partition of $\mathbb{T} \times \mathbb{L} = [0, T] \times [-L, L]$ such that $\text{card}(\mathcal{A}_n) \leq 2^{2^n}$. More precisely, we partition $[0, T] \times [-L, L]$ as

$$\begin{cases} [0, T] = \displaystyle\bigcup_{j=0}^{2^{2^{n-1}}-1} \left[j \cdot 2^{-2^{n-1}} T, (j + 1) \cdot 2^{-2^{n-1}} T\right) \,, \\ \\ [-L, L] = \displaystyle\bigcup_{k=-2^{2^{n-2}}}^{2^{2^{n-2}}-1} \left[k \cdot 2^{-2^{n-2}} L, (k + 1) \cdot 2^{-2^{n-2}} L\right) \,. \end{cases}$$

Theorem 2.3.1 states

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u_{\text{add}}(t, x)\right] \leq C\gamma_2(T, d) \leq C \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \sum_{n\geq 0} 2^{n/2} \text{diam}\left(A_n(t, x)\right) . \tag{2.3.19}$$

Here $A_n(t, x)$ is the element of uniform partition $\mathcal{A}_n$ that contains (t, x), i.e.

$$A_n(t, x) = \left[j \cdot 2^{-2^{n-1}} T, (j + 1) \cdot 2^{-2^{n-1}} T\right) \times \left[k \cdot 2^{-2^{n-2}} L, (k + 1) \cdot 2^{-2^{n-2}} L\right)$$

such that $j \cdot 2^{-2^{n-1}} T \leq t < (j+1) \cdot 2^{-2^{n-1}} T$ and $k \cdot 2^{-2^{n-2}} L \leq x < (k+1) \cdot 2^{-2^{n-2}} L$. We only need to estimate diameter of each $A_n(t,x)$. Since $(\mathcal{A}_n)$ is an uniform partition, the diameter of $A_n(t,x)$ with respect to $d_{1,H}((t,x),(s,y))$ defined in (2.3.9) can be estimated as

$$\operatorname{diam}\left(A_n(t,x)\right) \leq C_H \left(T^{\frac{H}{2}} \wedge (2^{-H2^{n-2}} L^H)\right) + C_H 2^{-H2^{n-2}} T^{\frac{H}{2}}.$$

For $L \geq \sqrt{T}$, we can split it into two cases: $\sqrt{T} \leq L < 2\sqrt{T}$ and $L \geq 2\sqrt{T}$. It is clear that the case $L \geq 2\sqrt{T}$ is more complicated. We consider it first. Let N_0 be the smallest integer such that $2^{-2^{n-2}} L \leq \sqrt{T}$, i.e. $\log_2(\log_2(L/\sqrt{T}))+2 \leq N_0 < \log_2(\log_2(L/\sqrt{T}))+3$. By (2.3.19) we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right] \\
\leq & C_H \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \left[\sum_{n=0}^{N_0} 2^{n/2} \operatorname{diam}\left(A_n(t,x)\right) + \sum_{n=N_0+1}^{\infty} 2^{n/2} \operatorname{diam}\left(A_n(t,x)\right)\right] \\
\leq & C_H T^{\frac{H}{2}} \left[\sum_{n=0}^{N_0} 2^{n/2} + \sum_{n=N_0+1}^{\infty} 2^{n/2} \left(\frac{2^{2^{N_0-2}}}{2^{2^{n-2}}}\right)^H\right] + C_H T^{\frac{H}{2}} \\
\leq & C_H T^{\frac{H}{2}} \cdot 2^{N_0/2} + C_H T^{\frac{H}{2}} \leq C_H T^{\frac{H}{2}} \Psi_0(T,L)\,, \qquad (2.3.20)
\end{aligned}$$

where $\Psi_0(T,L) = 1 + \sqrt{\log_2(L/\sqrt{T})}$ and $L \geq 2\sqrt{T}$. The case $\sqrt{T} \leq L < 2\sqrt{T}$ is easy because $\Psi_0(T,L)$ is bounded now. One can prove directly that $\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right] \leq C_H T^{\frac{H}{2}}$ same as (2.3.21). This concludes proof of the upper bound in (2.3.18) when $L \geq \sqrt{T}$.

Now, we prove the upper bound part in (2.3.18) when $L < \sqrt{T}$. The same uniform partition discussed above is still applicable. We have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} |u(t,x)|\right] \\
\leq & C_H \left[\sum_{n=0}^{\infty} 2^{n/2} \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \operatorname{diam}\left(A_n(t,x)\right)\right] \\
\leq & C_H T^{\frac{H}{2}} \sum_{n=0}^{\infty} 2^{n/2} \cdot 2^{-H2^{n-1}} + C_H T^{\frac{H}{2}} \leq C_H T^{\frac{H}{2}}\,, \qquad (2.3.21)
\end{aligned}$$

because

$$\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \operatorname{diam}(A_n(t,x)) \le C_H\left[\left(2^{-2^{n-2}}L\right)^H + \left(2^{-2^{n-1}}T\right)^{\frac{H}{2}}\right] \le C_H 2^{-H2^{n-2}}T^{\frac{H}{2}}\,.$$

This completes the upper bounds part of (2.3.18).

We will utilize Theorem 2.3.3 (Sudakov minoration Theorem) to prove the lower bound in (2.3.18). We also divide the proof into two cases: $L \ge \sqrt{T}$ and $L < \sqrt{T}$.

First, we consider the case $L \ge \sqrt{T}$. Select δ in Theorem 2.3.3 as $c_H T^{\frac{H}{2}}$ with certain relatively small $c_H > 0$. For the sequence $\{u(T,x_i), i = 0, 1, \cdots, \pm N\}$, where $N = \lfloor L/\sqrt{T}\rfloor$ (≥ 1 by the assumption) and

$$x_0 = 0, x_{\pm 1} = \pm\sqrt{T}, \cdots, x_{\pm N} = \pm N\sqrt{T}\,,$$

we have

$$d_{1,H}((T,x_i),(T,x_j)) \ge c_H T^{\frac{H}{2}} = \delta \quad \text{if } i \ne j\,.$$

Sudakov's minoration theorem implies

$$\begin{aligned}\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} |u(t,x)|\right] &\ge \mathbb{E}\left[\sup_i u(T,x_i)\right] \\ \ge & c_H\delta\sqrt{\log_2(2N+1)} \ge c_H T^{\frac{H}{2}}\Psi_0(T,L)\,.\end{aligned} \tag{2.3.22}$$

The lower bound in (2.3.18) is established when $L \ge \sqrt{T}$.

Now we prove the lower bound part in (2.3.18) when $L < \sqrt{T}$. We choose $\delta = c_H T^{\frac{H}{2}}$ as above and we choose $u(T/2,0), u(T,0)$ as our comparison set. We have $d_{1,H}((T/2,0),(T,0)) \ge c_H(T/2)^{\frac{H}{2}} \ge \delta$. Theorem 2.3.3 gives

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right] \ge \mathbb{E}[u(T/2,0)\vee u(T,0)] \ge c_H T^{\frac{H}{2}}\,. \tag{2.3.23}$$

Thus, the proof of the lower bound part in (2.3.18) is completed. ☐

Notice that from (2.3.9), it follows that for any fixed $t \in \mathbb{R}_+$

$$d_1((t,x),(t,y)) \asymp d_{t,H}(x,y) := t^{\frac{H}{2}} \wedge |x-y|^H \,, \tag{2.3.24}$$

and for fixed $x \in \mathbb{R}$

$$d_1((t,x),(s,x)) \asymp d_{1,H}(t,s) := |t-s|^{\frac{H}{2}} \,. \tag{2.3.25}$$

Using a similar argument to that in the proof of inequality (2.1.3) we have the following corollary.

Corollary 2.3.8. *Let the Gaussian field $u_{\mathrm{add}}(t,x)$ be defined by (2.3.2). There are positive universal constants c_H and C_H such that*

$$\begin{cases} c_H t^{\frac{H}{2}} \sqrt{\log_2(L)} \le \mathbb{E}\left[\sup_{-L\le x\le L} |u_{\mathrm{add}}(t,x)|\right] \\ \qquad\qquad \le \mathbb{E}\left[\sup_{-L\le x\le L} u_{\mathrm{add}}(t,x)\right] \le C_H t^{\frac{H}{2}} \sqrt{\log_2(L)}\,; \\ \\ c_H T^{\frac{H}{2}} \le \mathbb{E}\left[\sup_{0\le t\le T} u_{\mathrm{add}}(t,x)\right] \le \mathbb{E}\left[\sup_{0\le t\le T} |u_{\mathrm{add}}(t,x)|\right] \le C_H T^{\frac{H}{2}}\,. \end{cases} \tag{2.3.26}$$

Next, we shall explain that the almost sure version of Theorem 2.1.1 is a consequence of (2.1.3) with the aid of Borell's inequality (Theorem 2.3.4).

Proof of the second part of Theorem 2.1.1. First, we shall prove (2.1.5) for $T = n^\alpha$ for some α and for all sufficiently large integer n. Denote $\mathbb{L} := [-L, L]$, $\mathbb{T}^\alpha = [0, n^\alpha]$. Let $\varepsilon > 0$ and let $L \ge n^{\frac{(1+\varepsilon)\alpha}{2}}$ be sufficiently large. We start with the lower bound. Theorem 2.1.1 gives

$$\mathbb{E}\left[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\right] \ge c_H\left(n^{\frac{\alpha H}{2}} + n^{\frac{\alpha H}{2}}\sqrt{\log_2\left(\frac{L}{n^{\alpha/2}}\right)}\right)$$

for some positive number c_H. Denote

$$\lambda_H := \lambda_H(\mathbb{T}^\alpha\times\mathbb{L}) = \frac{1}{2}\mathbb{E}\left[\sup_{x\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\right],$$

and

$$\sigma_H^2 := \sigma_H^2(\mathbb{T}^\alpha \times \mathbb{L}) = \sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} \mathbb{E}[|u_{\mathrm{add}}(t,x)|^2] = C_H n^{\frac{\alpha H}{2}} .$$

Then, Borell's inequality implies

$$\begin{aligned}&\mathbf{P}\left\{\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x) < \frac{1}{2}\mathbb{E}\Big[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\Big]\right\} \le 2\exp\left(-\frac{\lambda_H^2}{2\sigma_H^2}\right)\\ &\le 2\exp\left(-c_H\left[1+\log_2\left(\frac{L}{n^{\alpha/2}}\right)\right]\right) \le C_H\left[\frac{n^\alpha}{n^{\alpha(1+\varepsilon)}}\right]^{\frac{c_H}{2}} \le C_H n^{-\alpha\varepsilon\cdot\frac{c_H}{2}} ,\end{aligned} \tag{2.3.27}$$

where $c_H, C_H > 0$ are some constants independent of n. Select real number α sufficiently large such that $\alpha\varepsilon \cdot \frac{c_H}{2} > 1$ and define the events F_n

$$F_n := \left\{\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x) < \frac{1}{2}\mathbb{E}\Big[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\Big]\right\}.$$

The bound (2.3.27) means $\sum_{n=1}^\infty \mathbf{P}(F_n) < \infty$. An application of Borel-Cantelli's lemma yields that $\mathbf{P}(\limsup_n F_n) = 0$. This means that

$$\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x) \ge \frac{1}{2}\mathbb{E}\Big[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\Big] \ge c_H\, T^{\frac{H}{2}}\,\Psi_0(T,L)\,, \tag{2.3.28}$$

almost surely for sufficiently large values of $T = n^\alpha$. Then letting $\varepsilon \to 0$ proves lower bound part of (2.1.5).

The proof of the upper bound in (2.1.5) can be done in exactly the same manner as in the proof of the lower bound except now we replace (2.3.27) by

$$\begin{aligned}&\mathbf{P}\left\{\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x) > \frac{3}{2}\mathbb{E}\Big[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\Big]\right\} \le 2\exp\left(-\frac{\lambda_H^2}{2\sigma_H^2}\right)\\ &\le 2\exp\left(-c_H\left[1+\log_2\left(\frac{L}{n^{\alpha/2}}\right)\right]\right) \le C_H\left[\frac{n^\alpha}{n^{\alpha(1+\varepsilon)}}\right]^{\frac{c_H}{2}} \le C_H n^{-\alpha\cdot\frac{c_H\varepsilon}{2}} ,\end{aligned} \tag{2.3.29}$$

with some positive constant c_H, C_H independent of n. Similar to (2.3.28) we have

$$\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x) \le \frac{3}{2}\mathbb{E}\Big[\sup_{(t,x)\in\mathbb{T}^\alpha\times\mathbb{L}} u_{\mathrm{add}}(t,x)\Big] \le C_H\, T^{\frac{H}{2}}\,\Psi_0(T,L) \tag{2.3.30}$$

almost surely for sufficiently large $T = n^\alpha$. And then $\varepsilon \to 0$ implies the upper bound in

(2.1.5).

Finally, we conclude the proof of (2.1.5) for $\sup_{(t,x)} u_{\mathrm{add}}(t,x)$ by combining (2.3.28), (2.3.30) and the property that $\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u_{\mathrm{add}}(t,x)$ is an increasing function of L and T almost surely. On the other hand, it is easy to see

$$\sup_x |f(x)| \le \sup_x [f(x)] + \sup_x [-f(x)]$$

since $|f(x)| \le \sup_x[f(x)] + \sup_x[-f(x)]$ for any function $f(x)$. Since $u_{\mathrm{add}}(t,x)$ is symmetric, we see that $\sup_{t,x}[-u_{\mathrm{add}}(t,x)]$ and $\sup_{t,x}[u_{\mathrm{add}}(t,x)]$ have the same law. Then, we have

$$\sup_{t,x} |u_{\mathrm{add}}(t,x)| \le 2 \sup_{t,x}[u_{\mathrm{add}}(t,x)]\,. \tag{2.3.31}$$

This completes the proof of (2.1.5). □

One can show the following asymptotic (2.3.32) by combining (2.3.26) and Borell's inequality and we omit the details.

Corollary 2.3.9. *Let $u_{\mathrm{add}}(t,x)$ be defined by (2.3.2) and let T satisfy $T \le L^2$. Then, there are two positive random constants c_H and C_H such that for any fixed $t \in [0,T]$ we have*

$$\begin{aligned} c_H\, t^{\frac{H}{2}} \sqrt{\log_2(L)} &\le \sup_{-L\le x\le L} u_{\mathrm{add}}(t,x) \\ &\le \sup_{-L\le x\le L} |u_{\mathrm{add}}(t,x)| \le C_H\, t^{\frac{H}{2}} \sqrt{\log_2(L)} \quad \textit{almost surely}\,. \end{aligned} \tag{2.3.32}$$

Remark 2.3.10. *As in [CJK13, CJKS13], the inequality (2.3.32) implies that there exist some constants $c, C > 0$ such that*

$$ct^{\frac{H}{2}} \le \liminf_{|x|\to\infty} \frac{u_{\mathrm{add}}(t,x)}{\sqrt{\log_2(|x|)}} \le \limsup_{|x|\to\infty} \frac{u_{\mathrm{add}}(t,x)}{\sqrt{\log_2(|x|)}} \le Ct^{\frac{H}{2}}\,, \tag{2.3.33}$$

for any $t \in \mathbb{R}_+$ almost surely.

We now turn to show Theorem 2.1.2.

Proof of Theorem 2.1.2. $\Delta_h u_{\rm add}(t,x)$ is centered symmetric and stationary Gaussian process. As before, we only need to find appropriate bounds for $\Delta_h u_{\rm add}(t,x)$. The conclusion with respect to $|\Delta_h u_{\rm add}(t,x)|$ follows from (2.3.31). Our strategy to prove Theorem 2.1.2 for $\Delta_h u_{\rm add}(t,x)$ is also to apply Talagrand's majorizing measure theorem and Sudakov's minoration theorem to the following Gaussian process

$$\begin{aligned}\Delta_h u_{\rm add}(t,x) :=& u_{\rm add}(t,x+h) - u_{\rm add}(t,x)\\ &= \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h-z) - G_{t-s}(x-z)] W(ds,dz),\end{aligned} \tag{2.3.34}$$

with fixed $t > 0$ and fixed $h \neq 0$. Without loss of generality, we assume $h > 0$. The natural metric is given by

$$d_{2,t,h}(x,y) := \left(\mathbb{E}|\Delta_h u_{\rm add}(t,x) - \Delta_h u_{\rm add}(t,y))|^2\right)^{\frac{1}{2}} .$$

We need to obtain good upper and lower bounds of $d_{2,t,h}(x,y)$. Let us first focus on the upper bound. Similar to (2.3.10) Plancherel's identity yields

$$d_{2,t,h}^2(x,y) = C_H \int_{\mathbb{R}_+} [1-\exp(-2t\xi^2)][1-\cos(|x-y|\xi)][1-\cos(h\xi)] \cdot \xi^{-1-2H} d\xi \,.$$

By the same argument as in the proof of the upper bound of $d_1((s,x),(s,y))$ in Lemma 2.3.6 it is easy to see that for any $0 \le \theta \le 1$

$$\begin{aligned}d_{2,t,h}^2(x,y) &\le C_H \int_{\mathbb{R}_+} [1-\exp(-2t\xi^2)][1-\cos(h\xi)] \cdot \xi^{-1-2H} d\xi\\ &\le C_H t^H \wedge h^{2H} \le C_H t^{H-\theta} h^{2\theta} \,.\end{aligned}$$

On the other hand, an application of the elementary inequality $1-\cos(x) \le C_\theta x^{2\theta}$, where $\theta \in (0,H)$ is as above, and a substitution $\xi \to \xi/|x-y|$ yield

$$\begin{aligned}d_{2,t,h}^2(x,y) &\le C_{\theta,H} h^{2\theta} |x-y|^{2H-2\theta} \int_{\mathbb{R}_+} [1-\cos(\xi)]\xi^{2\theta-1-2H} d\xi\\ &\le C_{\theta,H} h^{2\theta}|x-y|^{2H-2\theta} \,.\end{aligned}$$

In conclusion, we have the following bound analogous to upper bound part of (2.3.9):

$$d_{2,t,h}(x,y) \leq C_{H,\theta} h^{\theta} (|x-y|^{H-\theta} \wedge t^{\frac{H-\theta}{2}}) \,, \tag{2.3.35}$$

for any $\theta \in (0, H)$.

Now we can follow the same argument ((2.3.20) in particular) as in the proof of Theorem 2.1.1 by invoking Talagrand's majorizing measure theorem (Theorem 2.3.1) to prove the upper bound part of (2.1.6):

$$\mathbb{E}\left[\sup_{x\in\mathbb{L}} \Delta_h u_{\text{add}}(t,x)\right] \leq C_{H,\theta} |h|^{\theta} t^{\frac{H-\theta}{2}} \Psi_0(t,L) \,,$$

if $L \geq \sqrt{t}$. Now we turn to prove the lower bound part of (2.1.6). To this end, we need the inverse part of (2.3.35) and we shall use again the Sudakov minoration theorem. Observe that we only need to consider the case when $|x-y| \geq \sqrt{t}$. We claim

$$d_{2,t,h}^2(x,y) \geq c_H h^{2H} \quad \text{when } |x-y| \geq \sqrt{t} \text{ and } h \leq \sqrt{\tfrac{t\pi^2}{8\ln 2}} \wedge 1 \,.$$

In fact, notice that

$$1 - \exp\left(-\frac{2t\xi^2}{|x-y|^2}\right) \geq \frac{1}{2} \qquad \forall \;\; \xi \geq \tfrac{|x-y|\pi}{4h} \text{ and } h \leq \sqrt{\tfrac{t\pi^2}{8\ln 2}} \wedge 1 \,.$$

The simple inequality

$$1 - \cos(x) \geq x^2/4 \quad \text{if } |x| \leq \pi/2$$

implies

$$1 - \cos\left(\frac{h\xi}{|x-y|}\right) \geq \frac{h^2\xi^2}{4|x-y|^2} \qquad \text{if } \xi \leq \tfrac{|x-y|\pi}{2h} \,.$$

Therefore, a substitution $\xi \to \xi/|x-y|$ yields

$$\begin{aligned}
&d^2_{2,t,h}(x,y)\\
=&c_H|x-y|^{2H}\int_{\mathbb{R}_+}\left[1-\exp\left(-\frac{2t\xi^2}{|x-y|^2}\right)\right]\left[1-\cos\left(\frac{h\xi}{|x-y|}\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot[1-\cos(\xi)]\xi^{-1-2H}d\xi\\
\geq&c_H|x-y|^{2H}\int_{\frac{|x-y|\pi}{4h}}^{\frac{|x-y|\pi}{2h}}\left[1-\cos\left(\frac{h\xi}{|x-y|}\right)\right][1-\cos(\xi)]\cdot\xi^{-1-2H}d\xi\\
\geq&c_Hh^2|x-y|^{2H-2}\int_{\frac{|x-y|\pi}{4h}}^{\frac{|x-y|\pi}{2h}}[1-\cos(\xi)]\cdot\xi^{1-2H}d\xi\,.
\end{aligned}$$

Set

$$k_0=\inf\left\{k\in\mathbb{N}_0:\frac{(2k+1)\pi}{2}\geq\frac{|x-y|\pi}{4h}\right\};$$

and

$$k_1=\sup\left\{k\in\mathbb{N}_0:\frac{(2k+3)\pi}{2}\leq\frac{|x-y|\pi}{2h}\right\}.$$

If h is sufficiently small, then

$$\begin{aligned}
&\int_{\frac{|x-y|\pi}{4h}}^{\frac{|x-y|\pi}{2h}}[1-\cos(\xi)]\cdot\xi^{1-2H}d\xi=\sum_{k\geq0}\int_{I_k\cap[\frac{|x-y|\pi}{4h},\frac{|x-y|\pi}{2h}]}[1-\cos(\xi)]\cdot\xi^{1-2H}d\xi\\
\geq&\sum_{k=k_0}^{k_1}\int_{I_k}[1-\cos(\xi)]\cdot\xi^{1-2H}d\xi\geq\frac{1}{2}\int_{\frac{(2k_0+1)\pi}{2}}^{\frac{(2k_1+3)\pi}{2}}\xi^{1-2H}d\xi\\
=&c_H\left[\left(\frac{(2k_1+3)\pi}{2}\right)^{2-2H}-\left(\frac{(2k_0+1)\pi}{2}\right)^{2-2H}\right]\geq c_H\left(\frac{|x-y|}{h}\right)^{2-2H},
\end{aligned}$$

due to the fact that ξ^{1-2H} is an increasing function. Thus, we have for $|x-y|\geq\sqrt{t}$

$$d_{2,t,h}(x,y)\geq c_Hh^H \tag{2.3.36}$$

if $h\leq C(\sqrt{t}\wedge1)$ for some small positive quantity C. On the interval $\mathbb{L}=[-L,L]$ for L large enough, let us select $x_j=jL/\sqrt{t}$ for $j=0,\pm1,\cdots,\pm\lfloor L/\sqrt{t}\rfloor$. Similar to (2.3.22), applying the Sudakov minoration theorem (Theorem 2.3.3) with $\delta=c_H|h|^H$ yields

$$\mathbb{E}\left[\sup_{x\in\mathbb{L}}\Delta_hu_{\text{add}}(t,x)\right]\geq\mathbb{E}\left[\sup_{x_i}\Delta_hu_{\text{add}}(t,x)\right]\geq c_H|h|^H\Psi_0(t,L)\,.$$

The proof of (2.1.7) follows from exactly the same argument as in the proof of (2.1.5) by Borel-Cantelli's lemma. The only difference is that now we have

$$\begin{cases} \sigma_t^2(h) = \sup_{x\in\mathbb{L}^\alpha} \mathbb{E}[|\Delta_h u_{\text{add}}(t,x)|^2] \le C_{H,\theta} t^{H-\theta}|h|^{2\theta}\,; \\ \\ \lambda_L := \frac{1}{2}\mathbb{E}\left[\sup_{x\in\mathbb{L}^\alpha} \Delta_h u_{\text{add}}(t,x)\right]\,; \\ \\ \exp\left(-\frac{\lambda_L^2}{2\sigma_t^2(h)}\right) \le C_{H,\theta} \exp\left(-\left[\frac{h^2}{t}\right]^{H-\theta} \log_2\left[\frac{n^\alpha}{\sqrt{t}}\right]\right)\,, \end{cases}$$

where $\mathbb{L}^\alpha := [-n^\alpha, n^\alpha]$. We can then complete the proof of the theorem by choosing α appropriately. We omit the details here. □

Proof of Theorem 2.1.3. We will use the same method as in the proof of Theorem 2.1.2. The natural metric associated with the time increment of the solution is

$$d_{3,t,\tau}(x,y) = (\mathbb{E}|\Delta_\tau u_{\text{add}}(t,x) - \Delta_\tau u_{\text{add}}(t,y)|^2)^{\frac{1}{2}}\,.$$

Using

$$\Delta_\tau u_{\text{add}}(t,x) = \int_0^{t+\tau}\int_{\mathbb{R}} G_{t+\tau-s}(x-z)W(ds,dz) - \int_0^t \int_{\mathbb{R}} G_{t-s}(x-z)W(ds,dz)\,,$$

and using the isometric property of stochastic integral and Plancherel's identity one derives

$$d_{3,t,\tau}^2(x,y) = 2\int_{\mathbb{R}_+} f(t,\tau,\xi)[1-\cos(|x-y|\xi)]\cdot \xi^{-1-2H} d\xi\,, \qquad (2.3.37)$$

where

$$\begin{aligned} & f(t,\tau,\xi) \\ =& [1-\exp(-2(t+\tau)\xi^2)] + [1-\exp(-2t\xi^2)] - 2\exp(-\tau\xi^2)[1-\exp(-2t\xi^2)] \\ =& [1-\exp(-2\tau\xi^2)] + [1-\exp(-2t\xi^2)][1+\exp(-2\tau\xi^2) - 2\exp(-\tau\xi^2)]. \end{aligned}$$

Notice that when $x \geq 0$, $1 - e^{-x} \leq C_\theta x^\theta$ and $1 + e^{-2x} - 2e^{-x} = (1 - e^{-x})^2 \leq C_\theta^2 x^{2\theta}$ for any $\theta \in (0, 1)$. Then, we have

$$f(t, \tau, \xi) \leq C_\theta (\tau \xi^2)^\theta \,, \quad \forall \, \theta \in (0, 1) \,.$$

Inserting this bound into (2.3.37) yields

$$\begin{aligned} d_{3,t,\tau}^2(x, y) \leq & C_\theta \tau^\theta \int_{\mathbb{R}_+} [1 - \cos(|x - y|\xi)] \cdot \xi^{-1-2H+2\theta} d\xi \\ \leq & C_{H,\theta} \tau^\theta |x - y|^{2H-2\theta} \qquad \text{for any } 0 < \theta < H. \end{aligned}$$

On the other hand, a substitution $\xi \to \xi/\sqrt{\tau}$ yields

$$\begin{aligned} d_{3,t,\tau}^2(x, y) \leq & C \int_{\mathbb{R}_+} [1 - \exp(-2\tau\xi^2)] \xi^{-1-2H} d\xi \\ & + \int_{\mathbb{R}_+} [1 - \exp(-2t\xi^2)][1 - \exp(-\tau\xi^2)]^2 \xi^{-1-2H} d\xi \\ \leq & C_H \tau^H + C_{H,\theta} \tau^\theta t^{H-\theta} \leq C_{H,\theta} \tau^\theta t^{H-\theta} \end{aligned}$$

when $\tau \leq Ct$. Thus, we have

$$d_{3,t,\tau}(x, y) \leq C_{H,t,\theta} \tau^{\theta/2} (|x - y|^{H-\theta} \wedge t^{\frac{H-\theta}{2}}) \,, \tag{2.3.38}$$

where $0 < \theta < H$, which is the bound needed for us to prove the upper bound part of (2.1.8).

The Sudakov minoration Theorem 2.3.3 will still be used to prove the lower bound. We need to obtain an appropriate lower bound of $d_{3,t,\tau}(x, y)$ for $|x - y| \geq \sqrt{t}$. It is easy to see

$$\begin{aligned} d_{3,t,\tau}^2(x, y) & \geq c \int_{\mathbb{R}_+} [1 - \exp(-2\tau\xi^2)][1 - \cos(|x - y|\xi)] \cdot \xi^{-1-2H} d\xi \\ & \geq c\tau |x - y|^{2H-2} \int_{\frac{|x-y|}{2\sqrt{\tau}}}^{\frac{|x-y|}{\sqrt{\tau}}} [1 - \cos(\xi)] \xi^{1-2H} d\xi. \end{aligned} \tag{2.3.39}$$

Analogous to the obtention of (2.3.36) we can conclude that the integral in (2.3.39) is bounded below by a multiple of $\left(\frac{|x-y|}{\sqrt{\tau}}\right)^{2-2H}$. Thus, we obtain

$$d_{3,t,\tau}(x,y) \geq c_H \tau^{H/2} \tag{2.3.40}$$

if $\tau \leq C(t \wedge 1)$ for some constant C. This is the bound needed to use Theorem 2.3.3 to show the lower bound part of (2.1.8).

Once again, Borell's inequality (Theorem 2.3.4) can be combined with Borel-Cantelli's lemma to show the almost sure asymptotics (2.1.7), and the proof Theorem 2.1.3 is completed. □

In [HHL+17] (see also next section) to show the existence and uniqueness of the solution to (2.1.1) (for Hurst parameter $H \in (1/4, 1/2)$) it is extensively used the following quantity

$$\mathcal{N}_{\frac{1}{2}-H} u(t,x) = \left(\int_{\mathbb{R}} |u(t,x+h) - u(t,x)|^2 \cdot |h|^{2H-2} dh\right)^{\frac{1}{2}}, \tag{2.3.41}$$

which plays the role of fractional derivative of u. It is because of the difficulty to appropriately bound this quantity (see [HHL+17] or the next section) it is assumed that $\sigma(0) = 0$ in [HHL+17]. After our work on the bound of the solution $u_{\text{add}}(t,x)$ we want to argue that

$$\mathbb{E}\left[\sup_{x\in\mathbb{L}} \mathcal{N}^2_{\frac{1}{2}-H} u_{\text{add}}(t,x)\right] \geq c_{t,H} \log_2(L) \quad \text{if } L \text{ is sufficently large.} \tag{2.3.42}$$

This fact illustrates that the argument in [HHL+17] for the pathwise uniqueness (see Lemma 4.9 in [HHL+17] for this argument) is not applicable in the general setting when $\sigma(0) \neq 0$. Here is the precise statement of our result, which is also interesting for its own sake.

Proposition 2.3.11. *Let $u_{\text{add}}(t,x)$ be defined by (2.3.2) and let $\mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t,x)$ be defined by (2.3.41).*

(i) For any fixed $t > 0$ and $L \geq \sqrt{t}$ we have

$$\mathbb{E}\left[\sup_{-L\leq x\leq L} \mathcal{N}^2_{\frac{1}{2}-H} u_{\text{add}}(t,x)\right] \geq c_{t,H} \log_2(L), \tag{2.3.43}$$

where $c_{t,H}$ is a positive constant.

(ii) Moreover, we have almost surely if $L \ge \sqrt{t}$

$$\sup_{-L\le x\le L} \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t,x) \le C_H\, t^{2H-\frac{1}{2}}[1-\log(\sqrt{t}\wedge 1)]\Psi_0(t,L)\,, \tag{2.3.44}$$

where C_H is a positive random constant.

Proof. First, we consider the upper bound (2.3.44). Let $0 < \theta < \frac{1-2H}{2}$. Applying Theorem 2.1.2 when $|h| \le \sqrt{t}\wedge 1$ and Theorem 2.1.1 when $|h| > \sqrt{t}\wedge 1$, respectively, and using the notation $\Delta_h u_{\text{add}}(t,x) := u_{\text{add}}(t,x+h) - u_{\text{add}}(t,x)$ we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{L}} &\mathcal{N}^2_{\frac{1}{2}-H} u_{\text{add}}(t,x) = \sup_{x\in\mathbb{L}} \int_{\mathbb{R}} |\Delta_h u_{\text{add}}(t,x)|^2 \cdot |h|^{2H-2} dh \\
&\le \int_{\mathbb{R}} \left(\sup_{x\in\mathbb{L}} |\Delta_h u_{\text{add}}(t,x)|\right)^2 \cdot |h|^{2H-2} dh \\
&\le \int_{\{|h|\le\sqrt{t}\wedge 1\}} \left(\sup_{x\in\mathbb{L}} |\Delta_h u_{\text{add}}(t,x)|\right)^2 \cdot |h|^{2H-2} dh \\
&\quad + \int_{\{|h|>\sqrt{t}\wedge 1\}} \left(\sup_{x\in\mathbb{L}} |\Delta_h u_{\text{add}}(t,x)|\right)^2 \cdot |h|^{2H-2} dh \\
&\le C_{H,\theta} t^{H-\theta}\Psi_0(t,L) \int_{\{|h|\le\sqrt{t}\wedge 1\}} |h|^{2H-2+2\theta} dh + C_H t^H \Psi_0(t,L) \\
&\quad \cdot \left[\int_{\{|h|>\sqrt{t}\wedge 1\}} |h|^{2H-2} dh + \int_{\{|h|>\sqrt{t}\wedge 1\}} \log_2(|h|/\sqrt{t})|h|^{2H-2} dh\right],
\end{aligned}$$

where we applied an elementary inequality

$$|\log_2 |L+h|| \le \begin{cases} \log_2(L) + 1 & \text{when } |h| \le 1\,; \\ \log_2(L) + \log_2(|h|) + 1 & \text{when } |h| \ge 1\,. \end{cases}$$

Most of terms of above integrals can be evaluated easily except the one involving with $\log_2(|h|/\sqrt{t})$, which equals to

$$\begin{aligned}
&\int_{\{|h|>\sqrt{t}\wedge 1\}} \left[\log_2(|h|) - \log_2(\sqrt{t})\right]|h|^{2H-2} dh \\
&\le \int_{\{|h|>\sqrt{t}\wedge 1\}} \left[\log_2(|h|) - \log_2(\sqrt{t}\wedge 1)\right]|h|^{2H-2} dh \lesssim (\sqrt{t}\wedge 1)^{2H-1}[1-\log(\sqrt{t}\wedge 1)]\,.
\end{aligned}$$

This yields (2.3.44).

Now we turn to the lower bound (2.3.43). A simple observation and an application of Jensen's inequality give

$$\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\mathbb{L}}\mathcal{N}^2_{\frac{1}{2}-H}u_{\text{add}}(t,x)\right]\\
\geq & c_H\mathbb{E}\left[\left(\sup_{x\in\mathbb{L}}\int_{\mathbb{R}}\Delta_h u_{\text{add}}(t,x)\varrho(h)dh\right)^2\right]\geq c_H\left(\mathbb{E}\left[\sup_{x\in\mathbb{L}}\int_{\mathbb{R}}\Delta_h u_{\text{add}}(t,x)\varrho(h)dh\right]\right)^2,
\end{aligned} \tag{2.3.45}$$

where $\varrho(h) = C_H\left[|h|^{2H-\frac{3}{2}}\mathbf{1}_{\{|h|\leq 1\}} + |h|^{2H-2}\mathbf{1}_{\{|h|>1\}}\right]$ such that it is probability density. Denote

$$u_\varrho(t,x) = \int_{\mathbb{R}}\Delta_h u_{\text{add}}(t,x)\varrho(h)dh = \int_0^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}}D_{t-s}(h,x-z)\varrho(h)dh\right)W(ds,dz)\,,$$

where $D_t(h,x)$ is defined in (2.2.14). The above $u_\varrho(t,x)$ is a well-defined Gaussian random field since $\varrho(h)$ is integrable for $\frac{1}{4}<H<\frac{1}{2}$. Introduce the induced natural metric

$$d_{4,t}(x,y) := \left(\mathbb{E}|u_\varrho(t,x)-u_\varrho(t,y)|^2\right)^{\frac{1}{2}}.$$

We need to bound this distance for $|x-y|\geq 1$. Applying Plancherel's identity we can find

$$\begin{aligned}
d^2_{4,t}(x,y) = c_H\int_{\mathbb{R}_+}&[1-\exp(-2t\xi^2)][1-\cos(|x-y|\xi)]\\
&\cdot\left(\int_{\mathbb{R}_+}[1-\cos(h\xi)]\varrho(h)dh\right)^2\cdot\xi^{-1-2H}d\xi\,.
\end{aligned}$$

When $\xi\geq 1$, we have

$$\int_{\mathbb{R}_+}[1-\cos(h\xi)]\varrho(h)dh \geq \xi^{\frac{1}{2}-2H}\int_0^\xi[1-\cos(h)]\cdot h^{2H-\frac{3}{2}}dh\geq c\xi^{\frac{1}{2}-2H}.$$

Thus, we conclude that if $|x-y|\geq 1$, then

$$d^2_{4,t}(x,y)\geq c_H[1-\exp(-2t)]\int_1^\infty[1-\cos(|x-y|\xi)]\cdot\xi^{-6H}d\xi$$

$$\geq c_H[1-\exp(-2t)] \tag{2.3.46}$$

by the same argument as in proof of lower bound of $\mathbb{E}[\sup_{x\in\mathbb{L}} \Delta_h u_{\text{add}}(t,x)]$ in Theorem 2.1.2. An application of the Sudakov minoration Theorem 2.3.3 implies the lower bound (2.3.43). □

2.4 Weak Existence and Regularity of Solutions

2.4.1 Basic settings

This section is devoted to prove the existence of a weak solution to (2.1.1). Let us briefly recall some notations and facts in [HHL+17]. Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$. Let $\beta\in(0,1)$ be a fixed number. For any function $f:\mathbb{R}\to B$ denote

$$\mathcal{N}_\beta^B f(x) = \left(\int_{\mathbb{R}} \|f(x+h)-f(x)\|_B^2 |h|^{-1-2\beta} dh\right)^{\frac{1}{2}}, \tag{2.4.1}$$

if the above quantity is finite. When $B=\mathbb{R}$, we abbreviate the notation $\mathcal{N}_\beta^{\mathbb{R}} f$ as $\mathcal{N}_\beta f$ (see also (2.3.41)). As in [HHL+17] throughout this chapter we are particularly interested in the case $B=L^p(\Omega)$, and in this case we denote $\mathcal{N}_\beta^B$ by $\mathcal{N}_{\beta,p}$:

$$\mathcal{N}_{\beta,p} f(x) = \left(\int_{\mathbb{R}} \|f(x+h)-f(x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh\right)^{\frac{1}{2}}. \tag{2.4.2}$$

The following Burkholder-Davis-Gundy inequality is well-known (see e.g. [HHL+17]).

Proposition 2.4.1. *Let W be the Gaussian noise defined by the covariance (2.2.1), and let $f\in\Lambda_H$ be a predictable random field. Then for any $p\geq 2$ we have*

$$\left\|\int_0^t\int_{\mathbb{R}} f(s,y)W(ds,dy)\right\|_{L^p(\Omega)} \leq \sqrt{4p}\, c_H \left(\int_0^t\int_{\mathbb{R}} \left[\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)\right]^2 dyds\right)^{\frac{1}{2}}, \tag{2.4.3}$$

where c_H is a constant depending only on H and $\mathcal{N}_{\frac{1}{2}-H,p}f(s,y)$ denotes the application of $\mathcal{N}_{\frac{1}{2}-H,p}$ with respect to the space variable y.

In the work [HHL+17], the authors have already proved the existence and uniqueness result in a solution space $\mathcal{Z}_T^p$ (see [HHL+17] or formula (2.4.4) in next paragraph for the definition of $\mathcal{Z}_T^p$) under the condition $\sigma(t,x,0)=0$. When $\sigma(t,x,0)\neq 0$ or even in the simplest case $\sigma(t,x,u)=1$ (as we see from (2.3.43)) we cannot expect that the solution is still in $\mathcal{Z}_T^p$. So, the method powerful in [HHL+17] is no longer valid to solve (2.1.1) for general $\sigma(t,x,u)$. Our idea is to add an appropriate weight $\lambda(x)$ to the space $\mathcal{Z}_T^p$ to obtain a weighted space $\mathcal{Z}_{\lambda,T}^p$.

Let $\lambda(x)\geq 0$ be a Lebesgue integrable positive function with $\int_{\mathbb{R}}\lambda(x)dx=1$. Introduce a norm $\|\cdot\|_{\mathcal{Z}_{\lambda,T}^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_{\lambda,T}^p} := \sup_{t\in[0,T]}\|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}v(t), \tag{2.4.4}$$

where $p\geq 2$, $\frac{1}{4}<H<\frac{1}{2}$,

$$\|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} = \left(\int_{\mathbb{R}}\mathbb{E}\left[|v(t,x)|^p\right]\lambda(x)dx\right)^{\frac{1}{p}},$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,p}v(t) = \left(\int_{\mathbb{R}}\|v(t,\cdot)-v(t,\cdot+h)\|^2_{L_\lambda^p(\Omega\times\mathbb{R})}|h|^{2H-2}dh\right)^{\frac{1}{2}}. \tag{2.4.5}$$

Then $\mathcal{Z}_{\lambda,T}^p$ is the function space consisting of all the random fields $v=v(t,x)$ such that $\|v\|_{\mathcal{Z}_{\lambda,T}^p}$ is finite. When the function is independent of t, the corresponding space is denoted by $\mathcal{Z}_{\lambda,0}^p$.

2.4.2 Some bounds for stochastic convolutions

To prove the existence of weak solution, we need some delicate estimates of stochastic integral with respect to the weight.

Proposition 2.4.2. *Denote the weight function*

$$\lambda(x)=\lambda_H(x)=c_H(1+|x|^2)^{H-1}, \tag{2.4.6}$$

where c_H is a constant such that $\int \lambda(x)dx = 1$, and denote

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy). \tag{2.4.7}$$

We have the following estimates. (In the following $C_{T,p,H,\gamma}$ denotes a constant, depending only on T, p, H and γ).

(i) *If $p > \frac{3}{H}$, then*

$$\Big\| \sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\Phi(t,x) \Big\|_{L^p(\Omega)} \le C_{T,p,H} \|v\|_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.8}$$

(ii) *If $p > \frac{6}{4H-1}$, then*

$$\Big\| \sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x) \Big\|_{L^p(\Omega)} \le C_{T,p,H} \|v\|_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.9}$$

(iii) *If $p > \frac{3}{H}$, and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then*

$$\Big\| \sup_{\substack{t,t+h\in[0,T]\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x)\big[\Phi(t+h,x) - \Phi(t,x)\big] \Big\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^{\gamma} \|v\|_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.10}$$

(iv) *If $p > \frac{3}{H}$, and $0 < \gamma < H - \frac{3}{p}$, then*

$$\left\| \sup_{\substack{t\in[0,T]\\ x,y\in\mathbb{R}}} \frac{\Phi(t,x) - \Phi(t,y)}{\lambda^{-\frac{1}{p}}(x) + \lambda^{-\frac{1}{p}}(y)} \right\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |x-y|^{\gamma} \|v\|_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.11}$$

Remark 2.4.3. *The method provided in the following proof depends on the semigroup property of the heat kernel because we need to use the factorization method (e.g. [?GJ2014]. see also (2.4.13) below). This means that we can not apply our approach directly to the stochastic wave equation since the wave kernel (the fundamental solution of the wave equation in [BJQS15]) lacks the semigroup property.*

Proof. For any $\alpha \in (0,1)$ we set

$$J_\alpha(r,z) := \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} G_{r-s}(z-y)v(s,y)W(ds,dy). \tag{2.4.12}$$

A stochastic version of Fubini's theorem implies

$$\Phi(t,x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) J_\alpha(r,z) dz dr. \tag{2.4.13}$$

We are going to show the four different parts of the proposition separately. We divide our proof into six steps. Let us recall $D_t(x,h) := G_t(x+h) - G_t(x)$, and $\Box_t(x,y,h) = D_t(x+y,h) - D_t(x,h)$ defined in (2.2.14) and (2.2.15).

Step 1. The first two steps are to prove part **(i)**. In this step we will obtain the desired growth estimate of $\Phi(t,x)$ in term of $J_\alpha(r,z)$. Applying the bounds of (2.2.22) and (2.2.12) to (2.4.13) we have

$$\begin{aligned}
&\sup_{t,x} \lambda^\theta(x) \left|\Phi(t,x)\right| \\
&\simeq \sup_{t,x} \lambda^\theta(x) \left| \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) J_\alpha(r,z) dz dr \right| \\
&\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}} |G_{t-r}(x-z)\lambda^{-\frac{1}{p}}(z)|^q dz \right)^{\frac{1}{q}} \|J_\alpha(r,\cdot)\|_{L^p_\lambda(\mathbb{R})} dr \\
&\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}} (t-r)^{\frac{1-q}{2}} G_{(t-r)/q}(x-z) \lambda^{-\frac{q}{p}}(z) dz \right)^{\frac{1}{q}} \|J_\alpha(r,\cdot)\|_{L^p_\lambda(\mathbb{R})} dr \\
&\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t-r)^{\alpha-1} \cdot (t-r)^{\frac{1-q}{2q}} \lambda^{-\frac{1}{p}}(x) \cdot \|J_\alpha(r,\cdot)\|_{L^p_\lambda(\mathbb{R})} dr \,.
\end{aligned}$$

Setting $\theta = \frac{1}{p}$ and then applying the Hölder inequality we obtain

$$\begin{aligned}
\sup_{t,x} \lambda^\theta(x)|\Phi(t,x)| &\lesssim \sup_{t\in[0,T]} \int_0^t (t-r)^{\alpha-\frac{3}{2}+\frac{1}{2q}} \cdot \|J_\alpha(r,\cdot)\|_{L^p_\lambda(\mathbb{R})} dr \\
&\lesssim \sup_{t\in[0,T]} \left[\int_0^t (t-r)^{q(\alpha-\frac{3}{2}+\frac{1}{2q})} dr \right]^{\frac{1}{q}} \cdot \left[\int_0^T \|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} dr \right]^{\frac{1}{p}} \\
&\lesssim \left[\int_0^T \|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} dr \right]^{\frac{1}{p}}
\end{aligned} \tag{2.4.14}$$

if $q(\alpha - \frac{3}{2} + \frac{1}{2q}) > -1$, i.e. if

$$\alpha > \frac{3}{2p}\,. \tag{2.4.15}$$

This is possible if $p > 3/2$. Thus, to prove part **(i)**, we only need to show that there exists a constant C, independent of $r \in [0,T]$, such that

$$\mathbb{E}\|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} \le C\|v\|^p_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.16}$$

Step 2. We shall prove the above bound (2.4.16) in this step and to do this let us introduce the following two notations

$$\mathcal{D}_1(r,z) := \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\big|D_{r-s}(y,h)\big|^2 \|v(s,y+z)\|^2_{L^p(\Omega)}|h|^{2H-2}dhdyds\right)^{\frac{p}{2}},$$

and

$$\mathcal{D}_2(r,z) := \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}|G_{r-s}(y)|^2 \|\Delta_h v(s,y+z)\|^2_{L^p(\Omega)}|h|^{2H-2}dhdyds\right)^{\frac{p}{2}},$$

where $\Delta_h v(t,x) := v(t,x+h) - v(t,x)$. From the definition (2.4.12) of J and by Burkholder-Davis-Bundy's inequality (3.2.5) stated in Lemma 2.4.1, we have

$$\begin{aligned}\mathbb{E}\|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} &\lesssim \int_{\mathbb{R}} \Bigg\{ \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\Big[\mathbb{E}\big|G_{r-s}(y+h-z)v(s,y+h) \\ &\qquad - G_{r-s}(y-z)v(s,y)\big|^p\Big]^{2/p} h^{2H-2}dhdyds\Bigg\}^{p/2}\lambda(z)dz \\ &\lesssim \int_{\mathbb{R}} \big[\mathcal{D}_1(r,z)+\mathcal{D}_2(r,z)\big]\lambda(z)dz =: D_1 + D_2\,.\end{aligned}$$

For the first term I_1, thanks to Minkowski's inequality, we have

$$\begin{aligned} D_1 &\lesssim \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\big|D_{r-s}(y,h)\big|^2 \cdot \|\Delta_y v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dhdyds\right)^{\frac{p}{2}} \\ &\quad + \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha}\big|D_{r-s}(y,h)\big|^2 \cdot \|v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dhdyds\right)^{\frac{p}{2}} \\ &\lesssim \left(\int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha-\frac{1}{2}}\|\Delta_y v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|y|^{2H-2}dyds\right)^{\frac{p}{2}} \\ &\qquad + \left(\int_0^r (r-s)^{-2\alpha+H-1}\|v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}ds\right)^{\frac{p}{2}}, \end{aligned} \tag{2.4.17}$$

where the last inequality follows from inequalities (2.2.22) and (2.2.17).

For the second term I_2, we can again use Minkowski's inequality, Jensen's inequality with respect to $(r-s)^{1/2}G^2_{r-s}(y)dy \simeq G_{\frac{r-s}{2}}(y)dy$ (since when $p>2$, the function $\phi(x)=x^{2/p}$, $x>0$, is concave), and then we use (2.2.12) to obtain

$$\begin{aligned}
D_2 &\lesssim \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} G^2_{r-s}(y) \left(\int_{\mathbb{R}} \|\Delta_h v(s,y+z)\|^p_{L^p(\Omega)} \lambda(z) dz \right)^{2/p} |h|^{2H-2} dy dh ds \\
&\lesssim \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha-\frac{1}{2}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} G_{\frac{r-s}{2}}(y) \|\Delta_h v(s,z)\|^p_{L^p(\Omega)} dz \cdot \lambda(z-y) dy \right)^{\frac{2}{p}} |h|^{2H-2} dh ds \\
&\lesssim \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha-\frac{1}{2}} \|\Delta_h v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |h|^{2H-2} dh ds\,. \qquad (2.4.18)
\end{aligned}$$

Recall that

$$\|v\|_{\mathcal{Z}^p_{\lambda,T}} := \sup_{s\in[0,T]} \|v(s,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})} + \sup_{s\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} v(s)\,,$$

where $\mathcal{N}^*_{\frac{1}{2}-H,p}v(t)$ is defined in (2.4.5). The estimates obtained in (2.4.17) and (2.4.18) imply

$$\mathbb{E}\|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} \lesssim \|v\|^p_{\mathcal{Z}^p_{\lambda,T}} \left(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} + (r-s)^{-2\alpha+H-1} dr \right)^{\frac{p}{2}}. \qquad (2.4.19)$$

If we have $-2\alpha+H-1>-1$ and $-2\alpha-\frac{1}{2}>-1$, i.e. $\alpha<\frac{H}{2}$, then (2.4.16) follows.

However, the condition $\alpha<H/2$ should be combined with (2.4.15). This gives $\frac{3}{2p}<\alpha<\frac{H}{2}$ which implies $p>\frac{3}{H}$. Thus, under the condition of the proposition, the inequality (2.4.16) holds true. This finishes the proof of **(i)**.

Step 3. In this and next steps we prove **(ii)**. The spirit of the proof is similar to that of the proof of **(i)** but is more involved. In order to obtain the desired decay rate of $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$, we still use the equation (2.4.13) to express $\Phi(t,x)$ by J.

$$\begin{aligned}
&\Phi(t,x+h)-\Phi(t,x) \\
=&\frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} D_{t-r}(x-z,h) J_\alpha(r,z) dz dr \\
=&\frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) \Delta_h J_\alpha(r,z) dz dr\,,
\end{aligned}$$

where $\Delta_h J_\alpha(t,x) := J_\alpha(t,x+h) - J_\alpha(t,x)$.

Invoking Minkowski's inequality and then Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ we get

$$
\begin{aligned}
&\int_{\mathbb{R}} |\Phi(t,x+h)-\Phi(t,x)|^2 |h|^{2H-2} dh \\
\simeq &\int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) \Delta_h J_\alpha(r,z) dz dr \right|^2 \cdot |h|^{2H-2} dh \\
\lesssim &\Big(\int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{1}{2}} dz dr \Big)^2 \\
\lesssim &\left(\int_0^t \int_{\mathbb{R}} (t-r)^{q(\alpha-1)} G^q_{t-r}(x-z) \lambda^{-\frac{q}{p}}(z) dz dr \right)^{\frac{2}{p}} \\
&\qquad \times \left(\int_0^T \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{p}{2}} \lambda(z) dz dr \right)^{\frac{2}{p}} \\
\lesssim &\lambda(x)^{-\frac{2}{p}} \left[\int_0^t (t-r)^{q(\alpha-\frac{3}{2}+\frac{1}{2q})} dr \right]^{\frac{2}{q}} \times \left(\int_0^T \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{p}{2}} \lambda(z) dz dr \right)^{\frac{2}{p}},
\end{aligned}
$$

where in the above last inequality we used $G^q_{t-r}(x-z) = (t-r)^{\frac{1-q}{2}} G_{\frac{t-r}{q}}(x-z)$ and inequality (2.2.12). If we take $\theta = \frac{1}{p}$, and $q(\alpha - \frac{3}{2} + \frac{1}{2q}) > -1$, i.e.

$$
\alpha > \frac{3}{2p}, \tag{2.4.20}
$$

then

$$
\begin{aligned}
&\sup_{t,x} \lambda(x)^\theta \left(\int_{\mathbb{R}} |\Phi(t,x+h)-\Phi(t,x)|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}} \\
\lesssim &\left(\int_0^T \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{p}{2}} \lambda(z) dz dr \right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, to prove part **(ii)** we only need to prove that there exists some constant C_1, independent of $r \in [0,T]$, such that

$$
\mathcal{I} := \mathbb{E} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{p}{2}} \lambda(z) dz \le C_1 \|v\|^p_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.21}
$$

Step 4. In this step we show the above inequality (2.4.21). By the definition (2.4.12) of J and by an application of Minkowski's inequality we have

$$\begin{aligned}\mathcal{I} &\lesssim \Big(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathbb{E}|\Delta_h J_\alpha(r,z)|^p\lambda(z)dz\Big]^{\frac{2}{p}}|h|^{2H-2}dh\Big)^{\frac{p}{2}}\\ &\lesssim \Big(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathbb{E}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\big|D_{r-s}(z-y-l,h)v(s,y+l)\\ &\qquad - D_{r-s}(z-y,h)v(s,y)\big|^2|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}\lambda(z)dz\Big]^{\frac{2}{p}}|h|^{2H-2}dh\Big)^{\frac{p}{2}}.\end{aligned}$$

We introduce two notations:

$$\mathcal{I}_1(r,z,h) := \mathbb{E}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\big|D_{r-s}(z-y,h)\big|^2\times\big|\Delta_l v(s,y)\big|^2|l|^{2H-2}dldyds\Big)^{\frac{p}{2}},$$

and

$$\mathcal{I}_2(r,z,h) := \mathbb{E}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\Big|\Box_{r-s}(z-y,l,h)\Big|^2\times\big|v(s,y)\big|^2|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}.$$

Then, we have

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}|\Delta_h J_\alpha(r,z)|^2|h|^{2H-2}dh\Big]^{\frac{p}{2}}\lambda(z)dz\\ &\lesssim \Big(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathcal{I}_1(r,z,h)\lambda(z)dz\Big]^{\frac{2}{p}}|h|^{2H-2}dh\Big)^{\frac{p}{2}}\\ &\qquad + \Big(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathcal{I}_2(r,z,h)\lambda(z)dz\Big]^{\frac{2}{p}}|h|^{2H-2}dh\Big)^{\frac{p}{2}} =: I_1^{p/2}+I_2^{p/2}.\end{aligned}$$

We shall bound I_1 and I_2 one by one. For the first term, a change of variable $y\to z-y$ and an application of Minkowski's inequality yield

$$\begin{aligned}I_1 &\lesssim \int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\mathbb{E}\Big(\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\alpha}\big|D_{r-s}(y,h)\big|^2\\ &\qquad\times\big|\Delta_l v(s,y+z)\big|^2|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}\lambda(z)dz\Big|^{\frac{2}{p}}|h|^{2H-2}dh\\ &\lesssim \int_0^r\int_{\mathbb{R}^3}(r-s)^{-2\alpha}\big|D_{r-s}(y,h)\big|^2|l|^{2H-2}|h|^{2H-2}\\ &\qquad\times\Big(\int_{\mathbb{R}}\mathbb{E}\big|\Delta_l v(s,z)\big|^p\lambda(z-y)dz\Big)^{\frac{2}{p}}dydhdlds. \qquad (2.4.22)\end{aligned}$$

By (2.2.17) with $\beta = \frac{1}{2} - H$ we see that

$$\int_{\mathbb{R}^2} \big|D_{r-s}(y,h)\big|^2 |h|^{2H-2} dhdy \lesssim (r-s)^{H-1}\,,$$

which is finite. Since $x^{2/p}, x > 0$ is a concave function for $p \geq 2$ we can apply Jensen's inequality with respect to the probability measure $(r-s)^{1-H}\big[G_{r-s}(y) - G_{r-s}(y+h)\big]^2 |h|^{2H-2} dydh$. Thus, we have for $p \geq 2$:

$$\begin{aligned}
I_1 \lesssim & \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha+H-1} \bigg(\int_{\mathbb{R}^3} (r-s)^{1-H} \big|D_{r-s}(y,h)\big|^2 \\
& \qquad |h|^{2H-2} \int_{\mathbb{R}} \mathbb{E}\big|\Delta_l v(s,z)\big|^p \lambda(z-y) dzdydh \bigg)^{\frac{2}{p}} \times |l|^{2H-2} dlds \\
\lesssim & \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha+H-1} \|\Delta_l v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |l|^{2H-2} dldr \qquad (2.4.23)
\end{aligned}$$

by the first inequality in Lemma 2.2.12.

In order to bound $\mathcal{I}_2(t,x,h)$, we make a change of variable $y \to z - y$ and then split it to two terms. More precisely, we have

$$\begin{aligned}
&\mathcal{I}_2(r,z,h) \lesssim \mathcal{I}_{21}(r,z,h) + \mathcal{I}_{22}(r,z,h) \qquad (2.4.24) \\
:=& \mathbb{E}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |\Box_{r-s}(y,l,h)|^2 |v(s,z)|^2 |l|^{2H-2} dldyds \right)^{\frac{p}{2}} \\
&+ \mathbb{E}\left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |\Box_{r-s}(y,l,h)|^2 |\Delta_y v(s,z)|^2 |l|^{2H-2} dldyds \right)^{\frac{p}{2}} .
\end{aligned}$$

Using Minkowski's inequality, Lemma 2.2.8, and Lemma 2.2.11, one can check that

$$\begin{aligned}
I_{21} :=& \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \mathcal{I}_{21}(r,z,h)\lambda(z)dz \Big]^{\frac{2}{p}} |h|^{2H-2} dh \\
\lesssim & \int_0^r \int_{\mathbb{R}^3} (r-s)^{-2\alpha} |\Box_{r-s}(y,l,h)|^2 \|v(s)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |l|^{2H-2} |h|^{2H-2} dldhdyds \qquad (2.4.25) \\
\lesssim & \int_0^r (r-s)^{-2\alpha+2H-\frac{3}{2}} \|v(s)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} ds,
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &:= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \mathcal{I}_{22}(r,z,h)\lambda(z)dz \Big]^{\frac{2}{p}} |h|^{2H-2}dh \\
&\lesssim \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^2} |\Box_{r-s}(y,l,h)|^2 |l|^{2H-2}|h|^{2H-2} dldh \Big) \cdot \|\Delta_y v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} dyds \\
&\lesssim \int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha+H-1} \|\Delta_y v(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |y|^{2H-2} dyds.
\end{aligned} \tag{2.4.26}$$

Recalling the definition of $\|\cdot\|^p_{\mathcal{Z}^p_{\lambda,T}}$, and combining (2.4.23), (2.4.25) and (2.4.26), we obtain

$$\begin{aligned}
&\mathbb{E} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 |h|^{2H-2} dh \Big]^{\frac{p}{2}} \lambda(z)dz \\
\le& C_2 \|v\|^p_{\mathcal{Z}^p_{\lambda,T}} \Big(\int_0^r (r-s)^{-2\alpha+2H-\frac{3}{2}} + (r-s)^{-2\alpha+H-1} dr \Big)^{\frac{p}{2}}.
\end{aligned} \tag{2.4.27}$$

Once we have $-2\alpha + 2H - \frac{3}{2} > -1$ and $-2\alpha + H - 1 > -1$, i.e. $\alpha < H - \frac{1}{4}$, we see that (2.4.21) follows from (2.4.27). This condition on α is combined with (2.4.20) to become $\frac{3}{2p} < \alpha < H - \frac{1}{4}$. Therefore, we have proved that if $p > \frac{6}{4H-1}$, then (2.4.21) holds, finishing the proof of **(ii)**.

Step 5. We are going to prove part **(iii)**. We continue to use (2.4.13). Without loss of generality, we can assume $h > 0$ and $t \in [0,T]$ such that $t + h \le T$. We have

$$\begin{aligned}
&\Phi(t+h,x) - \Phi(t,x) \\
=& \frac{\sin(\pi\alpha)}{\pi} \Big[\int_0^{t+h} \int_{\mathbb{R}} (t+h-r)^{\alpha-1} G_{t+h-r}(x-z) J_\alpha(r,z) drdz \\
&\qquad - \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) \times J_\alpha(r,z) drdz \Big] \\
\lesssim& \sum_{i=1}^3 \mathcal{J}_i(t,h,x),
\end{aligned}$$

where

$$\mathcal{J}_1(t,h,x) := \int_0^t \int_{\mathbb{R}} \big[(t+h-r)^{\alpha-1} - (t-r)^{\alpha-1} \big] G_{t-r}(x-z) J_\alpha(r,z) drdz,$$

$$\mathcal{J}_2(t,h,x) := \int_0^t \int_{\mathbb{R}} (t+h-r)^{\alpha-1} \big[G_{t+h-r}(x-z) - G_{t-r}(x-z)\big] J_\alpha(r,z) drdz,$$

and

$$\mathcal{J}_3(t,h,x) := \int_t^{t+h} \int_{\mathbb{R}} (t+h-r)^{\alpha-1} G_{t+h-r}(x-z) J_\alpha(r,z) drdz.$$

As in the proof of **(i)** and **(ii)**, we insert additional factors of $\lambda^{-\frac{1}{p}}(z) \cdot \lambda^{\frac{1}{p}}(z)$ and apply Hölder's inequality in the expression for $\mathcal{J}_1$. Then, $\mathcal{J}_1$ is estimated as follows.

$$\begin{aligned} \mathcal{J}_1(t,h,x) \leq & \lambda^{-\frac{1}{p}}(x) \int_0^t \big|(t+h-r)^{\alpha-1} - (t-r)^{\alpha-1}\big| (t-r)^{\frac{1-q}{2q}} \|J_\alpha(r,\cdot)\|_{L^p_\lambda(\mathbb{R})} dr \\ \leq & \lambda^{-\frac{1}{p}}(x) \left(\int_0^t \big|(t+h-r)^{\alpha-1} - (t-r)^{\alpha-1}\big|^q (t-r)^{\frac{1-q}{2}} dr \right)^{\frac{1}{q}} \\ & \times \left(\int_0^T \|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} dr \right)^{\frac{1}{p}}. \end{aligned} \tag{2.4.28}$$

Fix $\gamma \in (0,1)$. It is easy to see

$$\big|(t+h-r)^{\alpha-1} - (t-r)^{\alpha-1}\big| \lesssim \big|t-r\big|^{\alpha-1-\gamma} h^\gamma. \tag{2.4.29}$$

Thus, we have

$$\begin{aligned} \sup_{t,x} \lambda^{1/p}(x) |\mathcal{J}_1(t,h,x)| \lesssim & h^\gamma \sup_{t\in[0,T]} \left(\int_0^t (t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}} dr \right)^{\frac{1}{q}} \\ & \times \left(\int_0^T \|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} dr \right)^{\frac{1}{p}}. \end{aligned}$$

In other word, if $\gamma + \frac{3}{2p} < \alpha < \frac{H}{2}$ or equivalently, if $\gamma < \frac{H}{2} - \frac{3}{2p}$, then we have

$$\mathbb{E} \left| \sup_{t,x} \lambda^\theta(x) |\mathcal{J}_1(t,h,x)| \right|^p \lesssim |h|^{p\gamma} \|v\|^p_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.30}$$

Let us proceed to bound $\mathcal{J}_2(t,h,x)$. One finds easily

$$\begin{aligned} \mathcal{J}_2(t,h,x) \leq & \left(\int_0^t \int_{\mathbb{R}} (t+h-r)^{q(\alpha-1)} \big| G_{t+h-r}(x-z) \right. \\ & \left. - G_{t-r}(x-z) \big|^q \lambda^{-\frac{q}{p}}(z) dz dr \right)^{\frac{1}{q}} \left(\int_0^T \|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})} dr \right)^{\frac{1}{p}}. \end{aligned} \tag{2.4.31}$$

To bound the above first factor we use the following inequality

$$\left|\exp\left(-\frac{x^2}{t+h}\right)-\exp\left(-\frac{x^2}{t}\right)\right| \le C_\gamma h^\gamma t^{-\gamma}\exp\left(-\frac{\gamma x^2}{2(t+h)}\right) \quad \forall\, \gamma\in(0,1)\,.$$

Combining the above inequality with (2.4.29) (with $\alpha = 1/2$), we have

$$\begin{aligned}&|G_{t+h-r}(x-z)-G_{t-r}(x-z)|\\ \le& C_\gamma h^\gamma (t-r)^{-\gamma}\left[G_{\frac{2}{\gamma}(t+h-r)}(x-z)+G_{\frac{2}{\gamma}(t-r)}(x-z)\right]\,.\end{aligned} \tag{2.4.32}$$

Thus, the first factor in (2.4.31) is bounded by

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(t+h-r)^{q(\alpha-1)}\big|G_{t+h-r}(x-z)-G_{t-r}(x-z)\big|^q\lambda^{-\frac{q}{p}}(z)dzdr\\ \lesssim& h^{q\gamma}\int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}G_{\frac{2(t+h-r)}{\gamma q}}(x-z)\lambda^{-\frac{q}{p}}(z)dzdr\\ +&h^{q\gamma}\int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}G_{\frac{2(t-r)}{\gamma q}}(x-z)\lambda^{-\frac{q}{p}}(z)dzdr\\ \lesssim& h^{q\gamma}\lambda^{-\frac{q}{p}}(x)\int_0^t(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}dr\,,\end{aligned}$$

where the last inequality follows from Lemma 2.2.5. Hence, if $\gamma+\frac{3}{2p}<\alpha<\frac{H}{2}$, namely, if $\gamma<\frac{H}{2}-\frac{3}{2p}$, then we have the following estimation:

$$\mathbb{E}\left|\sup_{t,x}\lambda^\theta(x)|\mathcal{J}_2(t,h,x)|\right|^p \lesssim |h|^{p\gamma}\|v\|^p_{\mathcal{Z}^p_{\lambda,T}}. \tag{2.4.33}$$

Now we are going to bound $\mathcal{J}_3(t,x,h)$. Exactly in the same way as for (2.4.28), we have

$$\begin{aligned}\mathcal{J}_3(t,x,h)\le&\lambda^{-\frac{1}{p}}(x)\left(\int_t^{t+h}(t+h-r)^{q(\alpha-1)}(t+h-r)^{\frac{1-q}{2}}dr\right)^{\frac{1}{q}}\\ &\left(\int_0^T\|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})}dr\right)^{\frac{1}{p}}\\ =&C_p\lambda^{-\frac{1}{p}}(x)h^{\alpha-\frac{3}{2p}}\left(\int_0^T\|J_\alpha(r,\cdot)\|^p_{L^p_\lambda(\mathbb{R})}dr\right)^{\frac{1}{p}}\,.\end{aligned}$$

If $\frac{3}{2p} < \alpha < \frac{H}{2}$, which is possible if $\gamma < \alpha - \frac{3}{2p} < \frac{H}{2} - \frac{3}{2p}$, then

$$\mathbb{E}\left|\sup_{t,x} \lambda^{\theta}(x)|\mathcal{J}_3(t,h,x)|\right|^p \leq C_3|h|^{p\alpha-\frac{3}{2}}\|v\|_{\mathcal{Z}^p_{\lambda,T}} = C_3\,|h|^{p\gamma}\|v\|^p_{\mathcal{Z}^p_{\lambda,T}}\,. \tag{2.4.34}$$

Combining (2.4.30), (2.4.33) and (2.4.34) we prove (2.4.10).

Step 6. We prove part **(iv)** of the proposition. As before, we shall again use the representation formula (2.4.13) and then we apply the Hölder inequality to find

$$\begin{aligned}
&\Phi(t,x) - \Phi(t,y)\\
=&\frac{\sin(\pi\alpha)}{\pi}\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\big[G_{t-r}(x-z) - G_{t-r}(y-z)\big]J_\alpha(r,z)dzdr\\
\lesssim&\left(\int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1)}|G_{t-r}(x-z) - G_{t-r}(y-z)|^q\lambda^{-\frac{q}{p}}(z)dzdr\right)^{\frac{1}{q}}\\
&\times\left(\int_0^T\int_{\mathbb{R}}|J_\alpha(r,z)|^p\lambda(z)dzdr\right)^{\frac{1}{p}}.
\end{aligned}$$

Denote the above first factor by

$$\mathcal{K}(t,x,y) := \int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1)}|G_{t-r}(x-z) - G_{t-r}(y-z)|^q\lambda^{-\frac{q}{p}}(z)dzdr.$$

Fix $\gamma \in (0,1)$. Using Hölder's inequality we have

$$\begin{aligned}
&\mathcal{K}(t,x,y)\\
\lesssim&\int_0^t(t-r)^{q(\alpha-1)}\left(\int_{\mathbb{R}}|G_{t-r}(x-z) - G_{t-r}(y-z)|^{pq(1-\gamma)}\lambda^{-q}(z)dz\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_{\mathbb{R}}|G_{t-r}(x-z) - G_{t-r}(y-z)|^{q^2\gamma}dz\right)^{\frac{1}{q}}dr\,.
\end{aligned} \tag{2.4.35}$$

To bound the integral inside the above second bracket, we make the substitutions $\tilde{x} = \frac{x}{\sqrt{t-r}}$, $\tilde{y} = \frac{y}{\sqrt{t-r}}$ and $\tilde{z} = \frac{z}{\sqrt{t-r}}$ to obtain for any $\rho > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}}|G_{t-r}(x-z) - G_{t-r}(y-z)|^\rho dz\\
\simeq&(t-r)^{\frac{1-\rho}{2}}\int_{\mathbb{R}}|\exp(-|\tilde{x}-\tilde{z}|^2) - \exp(-|\tilde{y}-\tilde{z}|^2)|^\rho d\tilde{z}\\
\lesssim&(t-r)^{\frac{1-\rho}{2}}|\tilde{x}-\tilde{y}|^\rho = (t-r)^{\frac{1-2\rho}{2}}|x-y|^\rho\,.
\end{aligned}$$

Substituting this bound into (2.4.35) with $\rho = q^2 r$ we have

$$
\begin{aligned}
&\mathcal{K}(t,x,y)\\
&\lesssim |x-y|^{q\gamma} \cdot \int_0^t (t-r)^{q(\alpha-1)+\frac{1-pq(1-\gamma)}{2p}+\frac{1-2q^2\gamma}{2q}}\\
&\qquad\qquad \times \left(\int_{\mathbb{R}} \left[G_{\frac{t-r}{pq(1-\gamma)}}(x-z) + G_{\frac{t-r}{pq(1-\gamma)}}(y-z) \right] \lambda^{-q}(z) dz \right)^{\frac{1}{p}} dr\\
&\lesssim |x-y|^{q\gamma} \cdot \left[\lambda^{-\frac{q}{p}}(x) + \lambda^{-\frac{q}{p}}(y) \right] \cdot \int_0^t (t-r)^{q(\alpha-\frac{3}{2}+\frac{1}{2q})-\frac{q\gamma}{2}} dr\,,
\end{aligned}
$$

where the last inequality follows from Lemma 2.2.5.

If $q(\alpha - \frac{3}{2} + \frac{1}{2q}) - \frac{q\gamma}{2} > -1$ and $\alpha < \frac{H}{2}$, namely, if $\frac{3}{2p} + \frac{\gamma}{2} < \alpha < \frac{H}{2}$, then with $\theta = \frac{1}{p}$ we have

$$
\begin{aligned}
&\mathbb{E} \left| \sup_{\substack{t\in[0,T]\\ x,y\in\mathbb{R}}} \left(\lambda^{-\theta}(x) + \lambda^{-\theta}(y)\right)^{-1} |\mathcal{K}(t,x,y)|^{\frac{1}{q}} \times \left(\int_0^T \int_{\mathbb{R}} |J_\alpha(r,z)|^p \lambda(z) dz dr \right)^{\frac{1}{p}} \right|^p\\
&\lesssim |x-y|^{p\gamma} \cdot \int_0^T \int_{\mathbb{R}} \mathbb{E} |J_\alpha(r,z)|^p \lambda(z) dz dr \le C_4 |x-y|^{p\gamma} \|v\|^p_{\mathcal{Z}^p_{\lambda,T}}\,. \qquad (2.4.36)
\end{aligned}
$$

This proves (2.4.11). The proof of the proposition is then completed. □

2.4.3 Weak existence of the solution

In this subsection we show the weak existence of a solution with paths in $\mathcal{C}([0,T]\times\mathbb{R})$, the space of all continuous real valued functions on $[0,T]\times\mathbb{R}$, equipped with a metric

$$
d_{\mathcal{C}}(u,v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\le t\le T, |x|\le n} (|u(t,x) - v(t,x)| \wedge 1). \qquad (2.4.37)
$$

We state a tightness criterion of probability measures on $(\mathcal{C}([0,T]\times\mathbb{R}), \mathscr{B}(\mathcal{C}([0,T]\times\mathbb{R})))$ that we are going to use (see Section 2.4 in [KS88] for the case where $[0,T]\times\mathbb{R}$ is replaced by $[0,\infty)$. It is also true for our case as indicated there).

Theorem 2.4.4. *A sequence $\{\mathbf{P}_n\}_{n=1}^{\infty}$ of probability measures on $(\mathcal{C}([0,T]\times\mathbb{R}), \mathscr{B}(\mathcal{C}([0,T]\times\mathbb{R})))$ is tight if and only if*

(1) $\lim_{\lambda\uparrow\infty} \sup_{n\ge1} \mathbf{P}_n \left(\{\omega \in \mathcal{C}([0,T]\times\mathbb{R}) : \ |\omega(0,0)| > \lambda\}\right) = 0,$

(2) for any $T>0$, $R>0$ and $\varepsilon>0$

$$\lim_{\delta\downarrow 0}\sup_{n\geq 1}\mathbf{P}_n\left(\left\{\omega\in\mathcal{C}([0,T]\times\mathbb{R}):\ m^{T,R}(\omega,\delta)>\varepsilon\right\}\right)=0$$

where

$$m^{T,R}(\omega,\delta):=\max_{\substack{|t-s|+|x-y|\leq\delta\\ 0\leq t,s\leq T;0\leq|x|,|y|\leq R}}|\omega(t,x)-\omega(s,y)|$$

is the modulus of continuity on $[0,T]\times[-R,R]$.

We approximate the noise W with respect to the space variable by the following smoothing of the noise. That is, for $\varepsilon>0$ we define

$$\frac{\partial}{\partial x}W_\varepsilon(t,x)=\int_{\mathbb{R}}G_\varepsilon(x-y)W(t,dy)\,. \tag{2.4.38}$$

The noise W_ε induces an approximation to mild solution

$$u_\varepsilon(t,x)=G_t*u_0(x)+\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)\sigma(s,y,u_\varepsilon(s,y))W_\varepsilon(ds,dy), \tag{2.4.39}$$

where the stochastic integral is understood in the Itô sense. As in [HHL+17] due to the regularity in space, the existence and uniqueness of the solution $u_\varepsilon(t,x)$ to above equation is well-known.

The lemma below asserts that the approximate solution $u_\varepsilon(t,x)$ is uniformly bounded in the space $\mathcal{Z}^p_{\lambda,T}$. More precisely, we have

Lemma 2.4.5. *Let $H\in(\frac{1}{4},\frac{1}{2})$ and let $\lambda(x)$ be defined by (2.4.6). Assume $\sigma(t,x,u)$ satisfies hypothesis **(H1)**. Assume also that the initial value $u_0(x)\in\mathcal{Z}^p_{\lambda,0}$. Then the approximate solutions u_ε satisfy*

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}:=\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})}+\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)<\infty. \tag{2.4.40}$$

Proof. For notational simplicity we assume $\sigma(t,x,u)=\sigma(u)$ without loss of generality because of hypothesis **(H1)**. We shall use some similar thoughts to that in [HHL+17] but now with special attention to the weight $\lambda(x)$. To this end, we define the Picard iteration as follows:

$$u_\varepsilon^0(t,x) = G_t * u_0(x)\,,$$

and recursively for $n = 0, 1, 2, \cdots$,

$$u_\varepsilon^{n+1}(t,x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y))W_\varepsilon(ds,dy)\,. \tag{2.4.41}$$

From [HHL$^+$18, Lemma 4.12] it follows that for any fixed $\varepsilon > 0$ when n goes to infinity, the sequence $u_\varepsilon^n(t,x)$ converges to $u_\varepsilon(t,x)$ a.s. In the following steps 1 and 2, we shall first bound $\|u_\varepsilon^n\|_{\mathcal{Z}^p_{\lambda,T}}$ uniformly in n, and ε. Then, in step 3 we use Fatou's lemma to show (3.3.24).

Step 1. In this step, we derive a Gronwall-type inequality to bound the $L^p_\lambda(\Omega\times\mathbb{R})$ norm of $u_\varepsilon^{n+1}(t,x)$ by the $\mathcal{Z}^p_{\lambda,T}$ norm of $u_\varepsilon^n(t,x)$. Rewrite (2.4.41) as

$$u_\varepsilon^{n+1}(t,x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} \Big[\Big(G_{t-s}(x-\cdot)\sigma(u_\varepsilon^n(s,\cdot))\Big) * G_\varepsilon\Big](y)W(ds,dy)\,.$$

In the following, we will continue to use the notations $D_t(x,h)$ and $\Box_{t-s}(x,y,h)$ defined in (2.2.14) and (2.2.15) previously. Applying the Burkholder-Davis-Gundy inequality (Proposition 2.4.1) and the isometry equalities (2.2.4)-(2.2.6) and then noting $|\sigma(u)| \lesssim |u|+1$, we have

$$\begin{aligned}
&\mathbb{E}\big[|u_\varepsilon^{n+1}(t,x)|^p\big] \\
\le& C_p|G_t * u_0^n(x)|^p + C_p\mathbb{E}\left(\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}\big[G_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))\big](\xi)\Big|^2 e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi ds\right)^{\frac{p}{2}} \\
\le& C_p|G_t * u_0(x)|^p + C_p\mathbb{E}\Bigg(\int_0^t\int_{\mathbb{R}^2}\Big|G_{t-s}(x-y-h)\sigma(u_\varepsilon^n(s,y+h)) \\
&\qquad\qquad - G_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y))\Big|^2|h|^{2H-2}dhdyds\Bigg)^{\frac{p}{2}} \\
\le& C_p\left(|G_t * u_0(x)|^p + \mathcal{D}_1^{\varepsilon,n}(t,x) + \mathcal{D}_2^{\varepsilon,n}(t,x)\right)\,,
\end{aligned} \tag{2.4.42}$$

where the constant C_p is independent of ε because $e^{-\varepsilon|\xi|^2} \le 1$, and where we denote

$$\mathcal{D}_1^{\varepsilon,n}(t,x) := \left(\int_0^t\int_{\mathbb{R}^2}\big|D_{t-s}(y,h)\big|^2\left(1+\|u_\varepsilon^n(s,x+y)\|^2_{L^p(\Omega)}\right)|h|^{2H-2}dhdyds\right)^{\frac{p}{2}},$$

and $\Delta_h u_\varepsilon^n(t,x) := u_\varepsilon^n(t,x+h) - u_\varepsilon^n(t,x)$,

$$\mathcal{D}_2^{\varepsilon,n}(t,x) := \left(\int_0^t \int_{\mathbb{R}^2} |G_{t-s}(y)|^2 \|\Delta_h u_\varepsilon^n(t,x+y)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds \right)^{\frac{p}{2}} .$$

This means

$$\begin{aligned} \|u_\varepsilon^{n+1}(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 = &\left(\int_{\mathbb{R}} \mathbb{E}\big[|u_\varepsilon^n(t,x)|^p\big] \lambda(x) dx \right)^{\frac{2}{p}} \\ \leq & C_p \left(\|u_0(x)\|_{L_\lambda^p(\mathbb{R})} + I_1^{\varepsilon,n} + I_2^{\varepsilon,n} \right) , \end{aligned} \tag{2.4.43}$$

where $I_1^{\varepsilon,n}$ and $I_2^{\varepsilon,n}$ are defined and bounded as follows.

$$I_1^{\varepsilon,n} := \left(\int_{\mathbb{R}} \mathcal{D}_1^{\varepsilon,n}(t,x) \lambda(x) dx \right)^{\frac{2}{p}} \leq C_{p,H} \int_0^t (t-s)^{H-1} \Big(1 + \|u_\varepsilon^n(s,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 \Big) ds , \tag{2.4.44}$$

and

$$I_2^{\varepsilon,n} := \left(\int_{\mathbb{R}} \mathcal{D}_2^{\varepsilon,n}(t,x) \lambda(x) dx \right)^{\frac{2}{p}} \leq C_{p,H} \int_0^t \frac{\left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s) \right]^2}{\sqrt{t-s}} ds . \tag{2.4.45}$$

The above bounds on $I_1^{\varepsilon,n}, I_2^{\varepsilon,n}$ together with (2.4.43) yield

$$\begin{aligned} \|u_\varepsilon^{n+1}(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 \leq C_{p,H} \Bigg(& \|u_0\|_{L_\lambda^p(\omega\times\mathbb{R})}^2 + \int_0^t (t-s)^{H-1} \ \|u_\varepsilon^n(s,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 ds \\ & + \int_0^t (t-s)^{-1/2} \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s) \right]^2 ds \Bigg) . \end{aligned} \tag{2.4.46}$$

Step 2. Next, we obtain a bound for $\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^{n+1}(t)$ analogous to (2.4.46). Similar to (3.3.26) we have

$$\begin{aligned} & \mathbb{E}\big[|u_\varepsilon^{n+1}(t,x) - u_\varepsilon^{n+1}(t,x+h)|^p\big] \\ \leq & C_p \big| G_t * u_0(x) - G_t * u_0(x+h) \big|^p \\ & + C_p \mathbb{E} \Bigg(\int_0^t \int_{\mathbb{R}^2} \Big| D_{t-s}(x-y-z,h) \sigma(u_\varepsilon^n(s,y+z)) \\ & \qquad - D_{t-s}(x-z,h) \sigma(u_\varepsilon^n(s,z)) \Big|^2 |y|^{2H-2} dz dy ds \Bigg)^{\frac{p}{2}} \\ \leq & C_p \left(\mathcal{I}_0(t,x,h) + \mathcal{I}_1^{\varepsilon,n}(t,x,h) + \mathcal{I}_2^{\varepsilon,n}(t,x,h) \right) , \end{aligned}$$

where

$$\mathcal{I}_0(t,x,h) := \left|G_t * u_0(x) - G_t * u_0(x+h)\right|^p,$$

$$\mathcal{I}_1^{\varepsilon,n}(t,x,h) := \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \Big|D_{t-s}(x-y-z,h)\Big|^2 \big|\sigma(u_\varepsilon(s,y+z)) - \sigma(u_\varepsilon(s,z))\big|^2 |y|^{2H-2} dzdyds\bigg)^{\frac{p}{2}},$$

$$\mathcal{I}_2^{\varepsilon,n}(t,x,h) := \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \Big|\Box_{t-s}(x-z,y,h)\Big|^2 \times \big|\sigma(u_\varepsilon(s,z))\big|^2 |y|^{2H-2} dzdyds\bigg)^{\frac{p}{2}}.$$

By Minkowski's inequality we have

$$\begin{aligned}\Big[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^{n+1}(t)\Big]^2 &\le C_p \sum_{j=0}^{2} \int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} \mathcal{I}_j^{\varepsilon,n}(t,x,h)\lambda(x)dx\bigg)^{\frac{2}{p}} |h|^{2H-2} dh \\ &=: J_0 + J_1 + J_2. \end{aligned} \tag{2.4.47}$$

Our strategy is to control the above three quantities by using the ideas similar to those when we deal with the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ in the step 4 of the proof of Proposition 2.4.2 **(ii)**. First, from Lemma 2.2.5 it follows.

$$\begin{aligned} J_0 \le & C_p \int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} G_t(x-y)\lambda(x)dx\Big] \left|\Delta_h u_0(y)\right|^p dy\bigg)^{\frac{2}{p}} |h|^{2H-2} dh \\ \le & C_p \int_{\mathbb{R}} \bigg(\int_{\mathbb{R}} \left|\Delta_h u_0(y)\right|^p \lambda(y) dy\bigg)^{\frac{2}{p}} |h|^{2H-2} dh = C_p \Big[\mathcal{N}^*_{\frac{1}{2}-H,p} u_0\Big]^2. \end{aligned} \tag{2.4.48}$$

For the term J_1, we can use the method similar to that when we obtain (2.4.22) and (2.4.23). This is, a change of variable $y \to z - y$, and applications of Minkowski's inequality, Jensen's inequality and Lemma 2.2.12 give

$$\begin{aligned} J_1 \le & C_{p,H} \int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \bigg(\int_{\mathbb{R}^3} (t-s)^{1-H} \Big|D_{t-s}(z,h)\Big|^2 |h|^{2H-2} \\ & \qquad \times \mathbb{E}\Big[\big|\Delta_y u_\varepsilon^n(t,x)\big|^p\Big] \lambda(x-z) dxdzdh\bigg)^{\frac{2}{p}} |y|^{2H-2} dyds \\ \le & C_{p,H} \int_0^t (t-s)^{H-1} \Big[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\Big]^2 ds. \end{aligned} \tag{2.4.49}$$

Next, we obtain a bound for J_2. Similar to the obtention of (2.4.24), (2.4.25) and (2.4.26) we also make a change of variable $y \to z - y$, and then split it to two terms to obtain

$$
\begin{aligned}
&\mathcal{I}_2^{\varepsilon,n}(t,x,h) \leq C_p \left(\mathcal{I}_{21}^{\varepsilon,n}(t,x,h) + \mathcal{I}_{22}^{\varepsilon,n}(t,x,h)\right) \\
:=&C_p \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(y,z,h)|^2 |\sigma(u_\varepsilon^n(s,x))|^2 |y|^{2H-2} dy dz ds\right)^{\frac{p}{2}} \\
+&C_p \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(y,z,h)|^2 |\sigma(u_\varepsilon^n(s,x+z)) - \sigma(u_\varepsilon^n(s,x))|^2 |y|^{2H-2} dy dz ds\right)^{\frac{p}{2}} .
\end{aligned}
$$

Applying Minkowski's inequality, the condition $|\sigma(u)| \lesssim |u| + 1$, and Lemma 2.2.8 one has

$$
\begin{aligned}
J_{21} :=& \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{I}_{31}^{\varepsilon,n}(t,x,h)\lambda(x)dx \right|^{\frac{2}{p}} |h|^{2H-2} dh \\
\leq& C_{p,H} \int_0^t (t-s)^{2H-\frac{3}{2}} \left(1 + \|u_\varepsilon^n(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\right) ds .
\end{aligned}
\tag{2.4.50}
$$

Again by Minkowski's inequality, the Lipschitz condition (3.2.10) on σ, and Lemma 2.2.11 we obtain

$$
\begin{aligned}
J_{22} :=& \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{I}_{32}^{\varepsilon,n}(t,x,h)\lambda(x)dx \right|^{\frac{2}{p}} |h|^{2H-2} dh \\
\leq& C_{p,H} \int_0^t (t-s)^{H-1} \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\right]^2 ds .
\end{aligned}
\tag{2.4.51}
$$

Using that fact that $J_3 \leq J_{31} + J_{32}$ and using (2.4.47)-(2.4.51) we obtain

$$
\begin{aligned}
\left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^{n+1}(t)\right]^2 \leq& C_{p,H} \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_0\right]^2 + C_{p,H} \int_0^t (t-s)^{H-1} \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\right]^2 ds \\
&+ C_{p,H} \int_0^t (t-s)^{2H-\frac{3}{2}} \left(1 + \|u_\varepsilon^n(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\right) ds .
\end{aligned}
\tag{2.4.52}
$$

Step 3. Set

$$
\Psi_\varepsilon^n(t) := \|u_\varepsilon^n(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} + \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(t)\right]^2 .
$$

Thus, combining all the estimates (3.3.31), (3.3.32), (3.3.33) and (2.4.51) yields

$$
\Psi_\varepsilon^{n+1}(t) \leq C_{p,H,T} \left(\|u_0\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} + \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_0\right]^2 + \int_0^t (t-s)^{2H-\frac{3}{2}} \Psi_\varepsilon^n(s) ds \right) .
$$

Now it is relatively easy to see by fractional Gronwall lemma (e.g. [LHH21, Lemma 1])

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\Psi_\varepsilon^n(t)\leq C_{T,p,H}<\infty\,.$$

For any fixed $\varepsilon>0$ since u_ε^n converges to u_ε a.s. as $n\to\infty$, we have by Fatou's lemma

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})}&=\left(\int_{\mathbb{R}}\mathbb{E}\left[\lim_{n\to\infty}|u_\varepsilon^n(t,x)|^p\right]\lambda(x)dx\right)^{\frac{1}{p}}\\&\leq\varliminf_{n\to\infty}\left(\int_{\mathbb{R}}\mathbb{E}\left[|u_\varepsilon^n(t,x)|^p\right]\lambda(x)dx\right)^{\frac{1}{p}}\leq\sup_{n\geq 1}\sup_{t\in[0,T]}\Psi_\varepsilon^n(t)<\infty\,.\end{aligned}$$

Thus, we conclude that $\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})}$ is finite. On the other hand, for any t,x and h we have $|u_\varepsilon^n(t,x+h)-u_\varepsilon^n(t,x)|^2\to|u_\varepsilon(t,x+h)-u_\varepsilon(t,x)|^2$ a.s. So, on the domain $|h|\leq 1$

$$\begin{aligned}&\int_{|h|\leq 1}\|u_\varepsilon(t,\cdot+h)-u_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dh\\&\leq\varliminf_{n\to\infty}\int_{|h|\leq 1}\|u_\varepsilon^n(t,\cdot+h)-u_\varepsilon^n(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dh.\end{aligned}$$

For $|h|\geq 1$, we simply bound $\|u_\varepsilon(t,\cdot+h)-u_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}$ by $2\|u_\varepsilon^n(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}$, which is uniform bounded with respect to t,ε and n. When $H<\frac{1}{2}$, $\int_{|h|>1}|h|^{2H-2}<\infty$. Thus, we have that

$$\begin{aligned}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)=&\sup_{t\in[0,T]}\left(\int_{\mathbb{R}}\|u_\varepsilon(t,\cdot+h)-u_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dh\right)^{\frac{1}{2}}\\\leq&C_H\sup_{n\geq 1}\sup_{t\in[0,T]}\Psi_\varepsilon^n(t)<\infty\,.\end{aligned}\tag{2.4.53}$$

Therefore, $\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)$ is finite.

In conclusion, we have proved $\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}:=\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})}+\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)$ is finite. □

Recall that $(\mathcal{C}([0,T]\times\mathbb{R}),d_{\mathcal{C}})$ is the metric space with the metric $d_{\mathcal{C}}$ defined by (2.4.37).

Lemma 2.4.6. *Let $u_\varepsilon\in\mathcal{Z}^p_{\lambda,T}$. If $u_\varepsilon\to u$ almost surely in $(\mathcal{C}([0,T]\times\mathbb{R}),d_{\mathcal{C}})$ as $\varepsilon\to 0$, then u is also in $\mathcal{Z}^p_{\lambda,T}$.*

Proof. Since u_ε converges to u in $(\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}})$ almost surely, we have $u_\varepsilon(t,x)\to u(t,x)$ for each $(t,x)\in[0,T]\times\mathbb{R}$ almost surely. Thus

$$\|u(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})} \lesssim \varliminf_{\varepsilon\to 0}\left(\int_{\mathbb{R}}\mathbb{E}\left[|u_\varepsilon(t,x)|^p\right]\lambda(x)dx\right)^{\frac{1}{p}} < \infty. \tag{2.4.54}$$

This means that $\sup\limits_{t\in[0,T]}\|u(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})}$ is finite.

On the other hand, for any x,h we have $|u_\varepsilon(t,x+h)-u_\varepsilon(t,x)|^2\to|u(t,x+h)-u(t,x)|^2$ almost surely. So, on the domain $|h|\le 1$ and $|h|\ge 1$, we can simply repeat the same procedure as in the Step 3 of the proof of Lemma 2.4.5 but replacing $\varliminf\limits_{n\to\infty}$ by $\varliminf\limits_{\varepsilon\to 0}$, and bound $\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}$ by $2\|u(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}$, which is finite. Thus, similar to (2.4.53) we have

$$\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u(t) = \sup_{t\in[0,T]}\left(\int_{\mathbb{R}}\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}|h|^{2H-2}dh\right)^{\frac{1}{2}} < \infty.$$

Together with (2.4.54), this implies that $u\in\mathcal{Z}^p_{\lambda,T}$. ∎

Lemma 2.4.7. *Let u_ε be the approximate mild solution defined by (3.3.23) and assume that $u_0(x)$ belongs to $\mathcal{Z}^p_{\lambda,0}$. Then, we have the following statements.*

(i) *If $p>\frac{6}{4H-1}$, then*

$$\left\|\sup_{t\in[0,T],x\in\mathbb{R}}\lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}u_\varepsilon(t,x)\right\|_{L^p(\Omega)} \le C_{T,H}(\|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}+1). \tag{2.4.55}$$

(ii) *If $p>\frac{3}{H}$, then*

$$\left\|\sup_{\substack{t,t+h\in[0,T]\\x\in\mathbb{R}}}\lambda^{\frac{1}{p}}(x)\left[u_\varepsilon(t+h,x)-u_\varepsilon(t,x)\right]\right\|_{L^p(\Omega)} \le C_{T,H}|h|^\gamma(\|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}+1), \tag{2.4.56}$$

for all $0<\gamma<\frac{H}{2}-\frac{3}{2p}$.

(iii) *If $p>\frac{3}{H}$, then*

$$\left\|\sup_{\substack{t\in[0,T]\\x,y\in\mathbb{R}}}\frac{u_\varepsilon(t,x)-u_\varepsilon(t,y)}{\lambda^{-\frac{1}{p}}(x)+\lambda^{-\frac{1}{p}}(y)}\right\|_{L^p(\Omega)} \le C_{T,H}|x-y|^\gamma(\|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}+1), \tag{2.4.57}$$

for all $0 < \gamma < H - \frac{3}{p}$.

Proof. Denote for $\alpha \in [0,1]$

$$J_\alpha^\varepsilon(r,\xi) = \int_0^r \int_{\mathbb{R}} \int_{\mathbb{R}} (r-s)^{-\alpha} G_{r-s}(\xi - z)\sigma(u_\varepsilon(s,z))G_\varepsilon(z-y)dzW(ds,dy).$$

Then, Fubini's theorem implies

$$\begin{aligned} u_\varepsilon(t,x) =& G_t * u_0(x) + \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-\xi) J_\alpha^\varepsilon(r,\xi) d\xi dr \\ =& u_1(t,x) + u_{2,\varepsilon}(t,x)\,. \end{aligned}$$

Applying Proposition 2.4.2 **(ii)**, **(iii)**, **(iv)** to $u_{2,\varepsilon}(t,x)$ yields (2.4.55)-(2.4.57) without the constant term 1. However, from the assumption that $u_0(x)$ belongs to $\mathcal{Z}_{\lambda,0}^p$ we see that left hand sides of (2.4.55)-(2.4.57) are finite when $u_\varepsilon(t,x)$ is replaced by $u_1(t,x)$. Combining the bounds for $u_1(t,x)$ and $u_{2,\varepsilon}(t,x)$ proves the lemma. □

Proof of Theorem 2.1.5. We still assume $\sigma(t,x,u) = \sigma(u)$ to simplify the notations. From Lemma 2.4.5 and Lemma 3.4.2 **(ii)** and **(iii)** it follows that the two conditions of Theorem 2.4.4 are satisfied. Hence, the probability measures on the space $(\mathcal{C}([0,T]\times\mathbb{R}), \mathscr{B}(\mathcal{C}([0,T]\times\mathbb{R})), d_{\mathcal{C}})$ corresponding to the processes $\{u_\varepsilon\,, \varepsilon \in (0,1]\}$ are tight. Thus, there is a subsequence $\varepsilon_n \downarrow 0$ such that $u_n = u_{\varepsilon_n}$ convergence weakly. By Skorohod representation theorem, there is a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ carrying the subsequence $\widetilde{u}_{n_j}$ and noise $\widetilde{W}$ such that the finite dimensional distributions of $(\widetilde{u}_{n_j}, \widetilde{W})$ and (u_{n_j}, W) coincide. Moreover, we have

$$\widetilde{u}_{n_j}(t,x) \to \widetilde{u}(t,x) \text{ in } (\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}}) \quad \widetilde{\mathbf{P}}\text{-almost surely} \tag{2.4.58}$$

for a certain stochastic process $\widetilde{u}$ as $j \to \infty$. By Lemma 2.4.6 we see that $\widetilde{u}$ belongs to space $\widetilde{\mathcal{Z}}_{\lambda,T}^p$ with respect to the new probability $\widetilde{\mathbf{P}}$. We want to show that $\widetilde{u}$ is a weak solution to (2.1.1).

Define the filtration $\widetilde{\mathcal{F}}_t$ to be the filtration generated by $\widetilde{W}$. We claim that $\widetilde{u}_{n_j}$ satisfies

(2.1.1) with W replaced by $\widetilde{W}$, namely,

$$\widetilde{u}_{n_j}(t,x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(\widetilde{u}_{n_j}(s,\cdot)) * G_{\varepsilon_j}(y)\widetilde{W}(ds,dy)\,. \tag{2.4.59}$$

To show the above identity it is sufficient to prove that for any $Z \in L^2(\widetilde{\Omega}, \widetilde{\mathbf{P}})$ one has

$$\begin{aligned}\widetilde{\mathbb{E}}[\widetilde{u}_{n_j}(t,x)Z] = \widetilde{\mathbb{E}}\Big[& G_t * u_0(x)Z \\ & + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(\widetilde{u}_{n_j}(s,\cdot)) * G_{\varepsilon_j}(y)\widetilde{W}(ds,dy)Z\Big]\,,\end{aligned} \tag{2.4.60}$$

where $\widetilde{\mathbb{E}}$ means the expectation under $\widetilde{\mathbf{P}}$.

For any $\phi \in \mathcal{D}(\mathbb{R})$, denote

$$\widetilde{W}_t(\phi) = \int_{\mathbb{R}} \phi(x)\tilde{W}(t,dx)\,; \quad W_t(\phi) = \int_{\mathbb{R}} \phi(x)W(t,dx)\,.$$

It is routine to argue that the set

$$\mathcal{S} := \left\{ f(\widetilde{W}_{t_1}(\phi), \cdots, \widetilde{W}_{t_n}(\phi))\,,\ 0 \le t_1 < \cdots < t_n \le T\ \ f \in \mathcal{C}_0(\mathbb{R}^n) \right\}$$

are dense in $L^2(\widetilde{\Omega}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_T)$. This means that it is sufficient to choose $Z = f(\widetilde{W}_{t_1}(\phi), \cdots, \widetilde{W}_{t_n}(\phi))$ in (2.4.60), which is true because we have the following identities:

$$\widetilde{\mathbb{E}}[\widetilde{u}_{n_j}(t,x)f(\widetilde{W}_{t_1}(\phi), \cdots, \widetilde{W}_{t_n}(\phi))] = \mathbb{E}[u_{n_j}(t,x)f(W_{t_1}(\phi), \cdots, W_{t_n}(\phi))]\,;$$

$$\widetilde{\mathbb{E}}\left[G_t * u_0(x)f(\widetilde{W}_{t_1}(\phi), \cdots, \widetilde{W}_{t_n}(\phi))\right] = \mathbb{E}\left[G_t * u_0(x)f(W_{t_1}(\phi), \cdots, W_{t_n}(\phi))\right]\,;$$

and

$$\begin{aligned}&\widetilde{\mathbb{E}}\left[\int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(\widetilde{u}_{n_j}(s,\cdot)) * G_{\varepsilon_j}(y)\widetilde{W}(ds,dy)f(\widetilde{W}_{t_1}(\phi), \cdots, \widetilde{W}_{t_n}(\phi))\right] \\ =&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(u_{n_j}(s,\cdot)) * G_{\varepsilon_j}(y)W(ds,dy)f(W_{t_1}(\phi), \cdots, W_{t_n}(\phi))\right]\end{aligned}$$

due to the fact that the finite dimensional distributions of $(\widetilde{u}_{n_j}, \widetilde{W})$ coincide with that of (u_{n_j}, W). Therefore, $\tilde{u}_{n_j}(t,x)$ satisfies (2.4.60), and hence it satisfies (2.4.59).

From (2.4.58) and (2.4.59) it follows that $\widetilde{u}$ is a mild solution to (2.1.1) with W replaced by $\widetilde{W}$. Therefore, we have proved the existence of a weak solution to (2.1.1).

Moreover, for any $\gamma \in (0, H - \frac{3}{p})$ and for any compact set $\mathbf{T} \subseteq [0,T] \times \mathbb{R}$, Lemma 3.4.2 (parts **(ii)** and **(iii)**) implies that there exists constant C such that

$$\widetilde{\mathbb{E}}\left(\sup_{(t,x),(s,y)\in\mathbf{T}} \left| \frac{\tilde{u}(t,x) - \tilde{u}(s,y)}{|t-s|^{\frac{\gamma}{2}} + |x-y|^{\gamma}} \right|^p \right) \leq C\|\tilde{u}\|^p_{\mathcal{Z}^p_{\lambda,T}} . \tag{2.4.61}$$

This combined with the Kolmogorov lemma implies the desired Hölder continuity. □

2.5 Pathwise Uniqueness and Strong Existence of solutions

In this section we prove the pathwise uniqueness and the existence of strong solution for the equation (2.1.1). It is well known that once pathwise uniqueness is achieved, together with the existence of weak solution proved in previous section, we can conclude the existence of the unique strong solutions to (2.1.1) by, for example, the Yamada-Watanabe theorem ([IW89]). Therefore, we only need to focus on the proof of pathwise uniqueness.

Proof of Theorem 2.1.6. The proof follows the strategy in the proof of Theorem 4.3 of [HHL+17] combined with Proposition 2.4.2 (part **(ii)**).

Define the following stopping times

$$\begin{aligned} T_k := \inf\Big\{ t\in[0,T] : \sup_{0\le s\le t, x\in\mathbb{R}} \lambda^{\frac{2}{p}}(x)\mathcal{N}_{\frac12-H}u(s,x) \geq k, \\ \text{or} \sup_{0\le s\le t, x\in\mathbb{R}} \lambda^{\frac{2}{p}}(x)\mathcal{N}_{\frac12-H}v(s,x) \geq k \Big\}, \quad k=1,2,\cdots \end{aligned}$$

Proposition 2.4.2, part **(ii)** implies that $T_k \uparrow T$ almost surely as $k\to\infty$. We need to find appropriate bounds for the following two quantities:

$$I_1(t) = \sup_{x\in\mathbb{R}} \mathbb{E}\left[\mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x)|^2\right]$$

and

$$I_2(t) = \sup_{x\in\mathbb{R}} \mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x) - u(t,x+h) + v(t,x+h)|^2 |h|^{2H-2} dh\right].$$

First, it is easy to see

$$\begin{aligned}
&\mathbf{1}_{\{t<T_k\}}(u(t,x) - v(t,x)) \\
=&\mathbf{1}_{\{t<T_k\}} \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\mathbf{1}_{\{s<T_k\}}[\sigma(s,y,u(s,y)) - \sigma(s,y,v(s,y))]W(ds,dy).
\end{aligned}$$

Recall $D_t(x,h)$ defined in (2.2.14) and denote $\triangle(t,x,y) = \sigma(t,x,u(t,y)) - \sigma(t,x,v(t,y))$. We can decompose

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\{t<T_k\}}|u(t,x) - v(t,x)|^2\big] \\
\lesssim&\mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s<T_k\}} |D_{t-s}(x-y,h)|^2 [\triangle(s,y,y)]^2 |h|^{2H-2} dhdyds\bigg) \\
+&\mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s<T_k\}} G^2_{t-s}(x-y-h)[\triangle(s,y+h,y) - \triangle(s,y,y)]^2 |h|^{2H-2} dhdyds\bigg) \\
+&\mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s<T_k\}} G^2_{t-s}(x-y)[\triangle(s,y,y+h) - \triangle(s,y,y)]^2 |h|^{2H-2} dhdyds\bigg) \\
&=: J_1 + J_2 + J_3\,.
\end{aligned} \tag{2.5.1}$$

The assumption (3.2.12) of σ and the equality (2.2.17) can be used to dominate the above first term J_1. This is,

$$\begin{aligned}
J_1 \lesssim&\mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s<T_k\}} |D_{t-s}(x-y,h)|^2 |u(s,y) - v(s,y)|^2 |h|^{2H-2} dhdyds\bigg) \\
\lesssim& \int_0^t (t-s)^{H-1} \sup_{y\in\mathbb{R}} \mathbb{E}\left[\mathbf{1}_{\{s<T_k\}} |u(s,y) - v(s,y)|^2\right] ds = \int_0^t (t-s)^{H-1} I_1(s) ds\,.
\end{aligned}$$

Using the properties (3.2.12) of σ, we have if $|h| > 1$

$$\begin{aligned}
&[\triangle(s,y+h,y) - \triangle(s,y,y)]^2 \lesssim |u(s,y) - v(s,y)|^2 \\
=&\left|\int_u^v [\sigma'_\xi(s,y+h,\xi) - \sigma'_\xi(s,y,\xi)] d\xi\right|^2 \lesssim |u(s,y) - v(s,y)|^2\,,
\end{aligned}$$

and if $|h| \le 1$ (with the help of additional properties (3.2.13))

$$\begin{aligned}&[\triangle(s,y+h,y)-\triangle(s,y,y)]^2\\ =&\left|\int_u^v[\sigma'_\xi(s,y+h,\xi)-\sigma'_\xi(s,y,\xi)]d\xi\right|^2\lesssim|h|^2|u(s,y)-v(s,y)|^2\,.\end{aligned}$$

Thus, the second term J_2 in (4.5.1) is bounded by

$$\begin{aligned}J_2=&\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}}\int_{|h|>1}\mathbf{1}_{\{s<T_k\}}G^2_{t-s}(x-y-h)|u(s,y)-v(s,y)|^2|h|^{2H-2}dhdyds\bigg)\\&+\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}}\int_{|h|\le1}\mathbf{1}_{\{s<T_k\}}G^2_{t-s}(x-y-h)|u(s,y)-v(s,y)|^2|h|^{2H}dhdyds\bigg)\\ \lesssim&\int_0^t I_1(s)\left(\int_{\mathbb{R}}G^2_{t-s}(x-y)dy\right)ds\lesssim\int_0^t(t-s)^{-\frac12}I_1(s)ds\,.\end{aligned}$$

For the last term J_3 in (4.5.1) we have by (3.2.12), (3.2.14)

$$\begin{aligned}&\big|\triangle(s,y,y+h)-\triangle(s,y,y)\big|^2\\ =&\Big|\int_0^1[u(s,y+h)-v(s,y+h)]\sigma'_\xi(s,y,\theta u(s,y+h)+(1-\theta)v(s,y+h))d\theta\\&-\int_0^1[u(s,y)-v(s,y)]\sigma'_\xi(s,y,\theta u(s,y)+(1-\theta)v(s,y))d\theta\Big|^2\,.\end{aligned}$$

Noticing the additional uniform decay assumption (3.2.12), we have

$$\begin{aligned}&\big|\triangle(s,y,y+h)-\triangle(s,y,y)\big|^2\\ \lesssim&|u(s,y+h)-v(s,y+h)-u(s,y)+v(s,y)|^2\\ &+\lambda^{\frac2p}(y)|u(s,y)-v(s,y)|^2\cdot\big[|u(s,y+h)-u(s,y)|^2+|v(s,y+h)-v(s,y)|^2\big]\,.\end{aligned}$$

Thus, we can dominate the last term in (4.5.1) by

$$J_3\lesssim k\int_0^t(t-s)^{-\frac12}\big[I_1(s)+I_2(s)\big]ds\,.$$

Summarizing the above estimates we have

$$I_1(t)\lesssim k\int_0^t(t-s)^{H-1}\big[I_1(s)+I_2(s)\big]ds\,.$$

The similar procedure can be applied to estimate the term $I_2(t)$ to obtain

$$I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{3}{2}} \big[I_1(s) + I_2(s)\big] ds\,.$$

As a consequence,

$$I_1(t) + I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{3}{2}} \big[I_1(s) + I_2(s)\big] ds.$$

Now Gronwall's lemma implies $I_1(t) + I_2(t) = 0$ for all $t \in [0,T]$. In particular, we have

$$\mathbb{E}\big[\mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x)|^2\big] = 0\,.$$

Thus, we have $u(t,x) = v(t,x)$ almost surely on $\{t < T_k\}$ for all $k \geq 1$, and the fact $T_k \uparrow \infty$ a.s as k tends to infinity necessarily indicate $u(t,x) = v(t,x)$ a.s. for every $t \in [0,T]$ and $x \in \mathbb{R}$.

It is clear that the hypothesis **(H2)** implies the hypothesis **(H1)**. So the existence of a Hölder continuous modification version of the solution follows from Theorem 2.1.5. We have then completed the proof of Theorem 2.1.6. □

Chapter 3

Nolinearstochastic wave equation driven by rough noise

3.1 Introduction

In this chapter, we consider the following one (spatial) dimensional stochastic nonlinear wave equation (SWE for short) driven by rough spatial Gaussian noise which is white in time and fractional in space:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} + \sigma(t,x,u(t,x))\dot{W}(t,x), \quad t \in [0,T], \quad x \in \mathbb{R}\,, \\ u(0,x) = u_0(x)\,, \quad \frac{\partial}{\partial t}u(0,x) = v_0(x)\,. \end{cases} \tag{3.1.1}$$

Here $W(t,x)$ is a centered Gaussian process with covariance given by

$$\mathbb{E}[W(t,x)W(s,y)] = \frac{1}{2}(s \wedge t)\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big) \tag{3.1.2}$$

and $\dot{W}(t,x) = \frac{\partial^2}{\partial t \partial x}W(t,x)$. The main feature of this work is our assumption that the Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. Namely, the noise is rough and fractional in space variable. This make the study of this equation challenging. Before we continue let us briefly summarize some relevant works. 1). When the noise is less singular, more precisely, when the noise is general Gaussian which is white in time and satisfies the so-called Dalang's condition, there are some results about the well-posedness of the equation and the properties of the solutions (e.g. [DKM+09, DSS09, HHN14]). When we apply Dalang's condition to fractional Gaussian noise, then we need to assume the spatial Hurst parameter $H > 1/2$. 2). When $H < 1/2$, namely, when the noise is rough in space (in this case the spatial dimension must be one dimensional), there are very limited results. The only result we know, to the best of our knowledge, is the work [BJQS15], where the noise coefficient $\sigma(t,x,u) = au + b$ is affine. There has been no work to tackle the case when $\sigma(t,x,u)$ is nonlinear (or not affine) function of u. 3). On the other hand, when $\frac{\partial^2}{\partial t^2}$ on the left hand of (3.1.1) is replaced by $\frac{\partial}{\partial t}$, this is, in the case nonlinear stochastic heat equations (SHE for short) driven by spatial rough noise, the authors of [HHL+17] studied the equation in the case $\sigma(t,x,0) = 0$. They prove the strong existence and uniqueness of solution. This condition $\sigma(t,x,0) = 0$ is removed in [HW22], where the authors obtained the existence of weak solution.

The objective of this work is to obtain the strong existence and uniqueness of the SWE (3.1.1) while still assuming $\sigma(t,x,0) = 0$. For the moment we are are not sure if our approach can be applied to remove this condition. When we start to consider the well-posedness of the equation (3.1.1) we immediately encounter a similar problem as that in [HHL+17, HW22]: One cannot bound the L_p norm of $\int_0^t \int_{\mathbb{R}} h_t(s,y) W(ds,dy)$ by the L_p norm of $h_t(s,y)$ itself, where $h_t(s,y) = G_{t-s}(x,y)\sigma(s,y,u(s,y))$ and $G_t(x,y)$ is the heat or wave kernel. Instead, one has to use the L_p norm of $h_t(s,y)$ itself plus the L_p norm of its fractional derivative. This makes thing very much sophisticated. In particular, as indicated in [HHL+17, HW22], due to the existence of our rough noise $\dot{W}$ we need to bound $|\sigma(u_1) - \sigma(u_2) - \sigma(v_1) + \sigma(v_2)|$ by a multiple of $|u_1 - u_2 - v_1 + v_2|$ (which is possible only in the affine case). To get around this difficulty the authors in [HHL+17, HW22] use a priori bound of $L_p \times L_\infty$ norm $\mathbb{E} \sup_{0\le t\le T} |u(t,x)|^p_{L_p(\mathbb{R})}$ and the similar norm of the fractional derivative of $u(t,x)$ for the solution $u(t,x)$. We shall follow the same strategy. However, this immediately poses some new challenges.

1. The first one is that $\int_0^t \int_{\mathbb{R}} h_t(s,y) W(ds,dy)$ is not a martingale in t (nor it is a semimartingale), it is hard to bound the L_p norm of $\sup_{0\le t\le T} \int_0^t \int_{\mathbb{R}} h_t(s,y) W(ds,dy)$ since we can no longer use the powerful Burkholder-Davis-Gundy inequality. In the case of SHE, this is overcome by a clever exploitation of the semigroup property

of the heat kernel. Unfortunately, this idea is not reproducible in our case simply because the wave kernel $G_t(x,y)$ associated with our SWE (3.1.1) does not have the semigroup property. To surmount this barrier we have luckily found a way to decompose $G_t(x-y)$ to four complicated parts (see (3.3.2) in Section 3) so that we can bound the L_p norm of $\sup_{0\le t\le T}\int_0^t\int_{\mathbb{R}} h_t(s,y)W(ds,dy)$ by the L_p norm of $h_t(s,y)$ itself plus the L_p norm of its fractional derivative. Of course, one also needs to bound L_p norm of the $\sup_{0\le t\le T}$ norm of the fractional derivative of $\int_0^t\int_{\mathbb{R}} h_t(s,y)W(ds,dy)$.

2. Since the wave kernel and heat kernels are of completely different nature, all the estimates in [HHL+17, HW22] are no longer useful here and we need an entirely new set of analysis of our decomposed kernels toward our final purpose. Since the wave kernel can decomposed into these kernels, we hope our estimates may also be useful in future study of stochastic wave equations.

After achieving the necessary estimation of the decomposed kernels, the proof of the existence and uniqueness of the mild solution is routine and we omit them to save the space of the chapter.

In the study of fractional noise, the number 1/4 seems to be a magic number. It appears in a number of occurrences. Here we are interested in the problem if $H > 1/4$ is necessary for (3.1.1) to have a classical (L_2) solution. We shall provide an affirmative answer. To this end we consider the hyperbolic Anderson model, namely, $\sigma(t,x,u) = u$. In this case the equation (3.1.3) becomes

$$\begin{cases} \frac{\partial^2 v(t,x)}{\partial t^2} = \frac{\partial^2 v(t,x)}{\partial x^2} + v(t,x)\dot{W}(t,x), \quad t\in[0,T], \quad x\in\mathbb{R}\,, \\ v(0,x) = u_0(x)\,, \quad \frac{\partial}{\partial t}v(0,x) = v_0(x)\,. \end{cases} \tag{3.1.3}$$

Under some conditions on the initial data, we shall prove that $v(t,x)$ is square integrable only if $H > 1/4$. After the completion of this work, we discover that the necessity of $H > 1/4$ is implied in [BJQS17, Proposition 3.7] (see also [SSX20, Proposition 3.4]). To make the chapter more comprehensive, we keep our alternative proof of the necessity of $H > 1/4$. Our method may be useful to study the properties of (3.1.1) with additive noise ($\sigma \equiv 1$). Let us also mention a recent work [CH21] that for the parabolic Anderson model when the dimension $d = 1$ and when the noise is white in time and fractional in space with Hurst parameter H, then $H > 1/4$ is also the necessary and sufficient condition for the solution to be square integrable.

Here is the organization of this chapter. In Section 3.2 we briefly recall some necessary concept about stochastic integral and wave kernel and so on to fix the notations used in the chapter and we also state our main results obtained in this work. Sections 3.3 and 3.4 are the core of the chapter. In Section 3.3 we decompose the wave kernel into four parts and then we use this decomposition to obtain the necessary bound of the stochastic integral (stochastic convolution with the wave kernel). There are a lot of computations to obtain the bound for the stochastic convolution. We postpone some of these computations in the Appendix 3.6 and 3.7. Section 3.4 obtains the existence and uniqueness of the strong solution. Some of the computations are moved to Appendix 3.8 for the fluency of the proof. Section 3.5 is about the necessity of $H > 1/4$ for strong solution to exist.

Throughout the chapter, $A \lesssim B$ (and $A \gtrsim B$) means that there are universal constants $C_1, C_2 \in (0, \infty)$ such that $A \le C_1 B$ (and $A \ge C_2 B$). We also denote throughout the chapter

$$\Delta_\tau f(t,x) := f(t+\tau, x) - f(t,x)\,, \tag{3.1.4}$$

$$\mathfrak{D}_h f(t,x) := f(t, x+h) - f(t,x)\,, \tag{3.1.5}$$

and

$$\begin{aligned}\Box_{h,l} f(t,x) &:= \mathfrak{D}_l \mathfrak{D}_h f(t,x) = \mathfrak{D}_h f(t, x+l) - \mathfrak{D}_h f(t,x) \\ &= [f(t, x+h+l) - f(t, x+l)] - [f(t, x+h) - f(t,x)]\,. \end{aligned} \tag{3.1.6}$$

3.2 Preliminaries and Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $W = (W(t,x), t \ge 0\,, x \in \mathbb{R})$ be a mean zero Gaussian random field whose covariance is given by (3.1.2). For any $t \ge 0$, $\mathcal{F}_t = \sigma(W(s,x)\,, s \in [0,t], x \in \mathbb{R})$ be the σ-algebra generated by the Gaussian field W. We recall briefly some notations and facts in [HHL+17] and refer to that reference for more details.

Denote $\mathcal{S}$ the set of smooth functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support. For any $f, g \in \mathcal{S}$, define

$$\langle f, g \rangle_{\mathfrak{H}} = c_H^2 \int_{\mathbb{R}_+ \times \mathbb{R}^2} [f(t, x+y) - f(t,x)][g(t,x+y) - g(t,x)]|y|^{2H-2} dx dy dt \,, \qquad (3.2.1)$$

where

$$c_H^2 = H(\frac{1}{2} - H) \left[\Gamma\Big(H + \frac{1}{2}\Big) \right]^{-2} \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}} \right]^2 dt + \frac{1}{2H} \right) .$$

Let $\mathfrak{H}$ be the Hilbert space obtained by completing $\mathcal{S}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. Let us start with the stochastic integration of elementary process with respect to W, and then extend it to general process.

Definition 3.2.1. *A random field $f = (f(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ is called adapted to the filtration $\mathcal{F}_t$ if $f(t,x) \in \mathcal{F}_t$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. An elementary process g is $\mathcal{F}_t$-adapted random field of the following form:*

$$g(t,x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \mathbf{1}_{(a_i, b_i]}(t) \mathbf{1}_{(c_j, d_j]}(x) \,,$$

where n and m are positive integers, $0 \le a_1 < b_1 < \cdots < a_n < b_n < +\infty$, $c_j < d_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \cdots, n, j = 1, \cdots, m$. The stochastic integral of such an elementary process g with respect to W is defined as

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x) W(dt, dx) &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W\left(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(c_j,d_j]} \right) \\ &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \left[W(b_i, d_j) - W(a_i, d_j) - W(b_i, c_j) + W(a_i, c_j) \right] . \end{aligned} \qquad (3.2.2)$$

In fact, we have the following proposition (e.g. [HHL+17]).

Proposition 3.2.2. *Let Λ_H be the space of adapted random field g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that $g \in \mathfrak{H}$ a.s. and $\mathbb{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then we have the following statements.*

1. *The space of elementary process defined in Definition 3.2.1 is dense in Λ_H;*

2. *For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+\times\mathbb{R}} g(t,x)W(dt,dx)$ is defined as the $L^2(\Omega)$-limit of stochastic integrals of elementary processes approximating $g(t,x)$ in Λ_H, and for this stochastic integral we have the following isometry equality*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} g(t,x)W(dt,dx)\right)^2\right] = \mathbb{E}[\|g\|_{\mathfrak{H}}^2].$$

Now we introduce some norms and spaces used in this chapter. Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$. Let $\beta \in (0,1)$ be a fixed number. For any function $f : \mathbb{R} \to B$ denote

$$\mathcal{N}_\beta^B f(x) := \left(\int_{\mathbb{R}} \|\mathfrak{D}_h f(x)\|_B^2 |h|^{-1-2\beta} dh\right)^{\frac{1}{2}}, \tag{3.2.3}$$

if the above quantity is finite, where we recall $\mathfrak{D}_h f(x) = f(x+h) - f(x)$. When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}_\beta^{\mathbb{R}} f$ as $\mathcal{N}_\beta f$. With this notation, the norm of the homogeneous Sobolev space $\dot{H}^\beta$ can be given by using $\mathcal{N}_\beta f$: $\|f\|_{\dot{\mathcal{H}}_\beta} = \|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}$. As in [HHL+17] throughout this chapter we are particularly interested in the case $B = L^p(\Omega)$, and in this case we denote $\mathcal{N}_\beta^B$ by $\mathcal{N}_{\beta,p}$:

$$\mathcal{N}_{\beta,p} f(x) := \left(\int_{\mathbb{R}} \|\mathfrak{D}_h f(x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh\right)^{\frac{1}{2}}. \tag{3.2.4}$$

We shall set $\beta = \frac{1}{2} - H$. The following Burkholder-Davis-Gundy inequality is well-known (see e.g. [HHL+17, HW22]).

Proposition 3.2.3. *Let W be the Gaussian noise defined by the covariance (3.1.2), and let $f \in \Lambda_H$ be a predictable random field. Then for any $p \geq 2$ we have*

$$\begin{aligned} &\left\| \sup_{0\le r\le t} \int_0^r \int_{\mathbb{R}} f(s,y)W(ds,dy) \right\|_{L^p(\Omega)} \\ &\le C_H\sqrt{p}\left(\int_0^t \int_{\mathbb{R}} \left[\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)\right]^2 dyds\right)^{\frac{1}{2}} \\ &= C_H\sqrt{p}\left(\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \|\mathfrak{D}_h f(s,y)\|_{L^p(\Omega)}^2 |h|^{2H-2} dhdyds\right)^{\frac{1}{2}}, \end{aligned} \tag{3.2.5}$$

where C_H is a constant depending only on H, $\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)$ denotes the application of

$\mathcal{N}_{\frac{1}{2}-H,p}$ *to the space variable* y *and* $\mathfrak{D}_h$ *is defined by* (3.1.5).

We introduce the solution space $\mathcal{Z}^p(T)$. It consists of all continuous functions f from $[0,T]\times\mathbb{R}$ to $L^p(\Omega)$ such the following norm is finite:

$$\begin{aligned}\|f\|_{\mathcal{Z}^p(T)} &= \|f\|_{\mathcal{Z}_1^p(T)} + \|f\|_{\mathcal{Z}_2^p(T)} \\ &:= \sup_{t\in[0,T]} \big\|f(t,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} f(t)\,,\end{aligned} \tag{3.2.6}$$

where $\big\|f(t,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})} = \big(\int_{\mathbb{R}}\mathbb{E}[|f(t,x)|^p]dx\big)^{1/p}$ and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} f(t) := \left(\int_{\mathbb{R}} \big\|\mathfrak{D}_h f(t,\cdot)\big\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh\right)^{\frac{1}{2}}\,.$$

It is proved that $\mathcal{Z}^p(T)$ is a Banach space (e.g. [HHL+17, Section 4.1]).

After defining the stochastic integral, let us return to the stochastic wave equation. Since we are working in dimension $d=1$, the Green's function associated with (3.1.1) is

$$G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}\,, \quad t\in\mathbb{R}_+\,, x\in\mathbb{R}\,. \tag{3.2.7}$$

Notice that $G_t(x)$ does not satisfy semigroup property.

Now we give the definitions of strong and weak solutions to (3.1.1).

Definition 3.2.4. *Let* $\{u(t,x)\,, t\geq 0\,, x\in\mathbb{R}\}$ *be a real-valued adapted random field such that for all fixed* $t\in[0,T]$ *and* $x\in\mathbb{R}$, *the random field*

$$\{G_{t-s}(x-y)\sigma(u(s,y))\mathbf{1}_{[0,t]}(s)\,, (s,y)\in\mathbb{R}_+\times\mathbb{R}\}$$

is integrable with respect to W *(namely it is in* Λ_H*).*

(i) *We say that* $u(t,x)$ *is a* strong (mild, random field) solution *to* (3.1.1) *if for all* $t\in[0,T]$ *and* $x\in\mathbb{R}$ *we have almost surely*

$$\begin{aligned}u(t,x) &= \frac{\partial}{\partial t}G_t * u_0(x) + G_t * v_0(x) + G_t \circledast \sigma(\cdot,\cdot,u)(x) \\ &= I_0(t,x) + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(ds,dy)\,,\end{aligned} \tag{3.2.8}$$

where

$$
\begin{aligned}
I_0(t,x) :=& G_t * v_0(x) + \frac{\partial}{\partial t} G_t * u_0(x) \\
=& \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}[u_0(x+t) + u_0(x-t)] \,.
\end{aligned}
\tag{3.2.9}
$$

(ii) *We say* (3.1.1) *has a* weak solution *if there exists a probability space with a filtration* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$*, an* $\widetilde{\mathcal{F}}_t$*-adapted Gaussian random field* $\widetilde{W}$ *identical to* W *in law, and an* $\widetilde{\mathcal{F}}_t$*-adapted random field* $\{u(t,x)\,,(t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ *on this probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ *such that* $u(t,x)$ *is a mild solution with respect to* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ *and* $\widetilde{W}$*.*

To obtain the existence and uniqueness of strong (mild) solution to (3.1.1), we make the following assumptions on σ.

(H1) $\sigma(t,x,u)$ is jointly continuous over $[0,T]\times\mathbb{R}^2$, $\sigma(t,x,0)=0$, and it is Lipschitz in u (uniformly in t and x). This means $\forall\, u,v\in\mathbb{R}$

$$
\sup_{t\in[0,T],x\in\mathbb{R}} |\sigma(t,x,u)-\sigma(t,x,v)| \le C|u-v| \,,
\tag{3.2.10}
$$

for some constant $C>0$.

One easily observes that the hypothesis (3.2.10) and the condition $\sigma(t,x,0)=0$ imply that

$$
\sup_{t\in[0,T],x\in\mathbb{R}} |\sigma(t,x,u)| \le C|u| \,,
\tag{3.2.11}
$$

for some constant $C>0$.

(H2) Assume $|\frac{\partial}{\partial u}\sigma(t,x,u)|$ and $|\frac{\partial^2}{\partial x\partial u}\sigma(t,x,u)|$ exist and are uniformly bounded, i.e. there is some constant $C>0$ such that

$$
\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} \left|\frac{\partial}{\partial u}\sigma(t,x,u)\right| \le C\,;
\tag{3.2.12}
$$

$$
\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} \left|\frac{\partial^2}{\partial x\partial u}\sigma(t,x,u)\right| \le C\,.
\tag{3.2.13}
$$

Moreover, we assume

$$\sup_{t\in[0,T],x\in\mathbb{R}} \left|\frac{\partial}{\partial u}\sigma(t,x,u_1) - \frac{\partial}{\partial u}\sigma(t,x,u_2)\right| \le C|u_1 - u_2|\,. \tag{3.2.14}$$

Notice that (3.2.10) is a consequence of (3.2.12). But we keep the former one in the assumption **(H1)** since we shall use **(H1)** for the existence of the weak solution and **(H2)** for the existence and uniqueness of the strong solution.

Now we state the main results of this chapter.

Theorem 3.2.5. *Assume that $\sigma(t,x,u)$ satisfies the hypothesis **(H1)** and that $I_0(t,x)$ is in $\mathcal{Z}^p(T)$ for some $p > \frac{1}{H}$. Then, there exists a weak solution to (3.1.1) whose sample paths are in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely. Moreover, for any $\gamma < H - \frac{1}{p}$, the process $u(t,x)$ is almost surely Hölder continuous of exponent γ with respect to t and x on any compact subsets of $[0,T]\times\mathbb{R}$.*

Theorem 3.2.6. *Assume that $\sigma(t,x,u)$ satisfies the hypothesis **(H2)** and that $I_0(t,x)$ is in $\mathcal{Z}^p(T)$ for some $p > \frac{2}{4H-1}$. Then (3.1.1) has a unique strong solution whose sample paths are in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely. Moreover, the random field $u(t,x)$ is Hölder continuous a.s. on compact subsets of $[0,T]\times\mathbb{R}$ with the same exponent as in Theorem 3.2.5.*

Theorem 3.2.7. *If the hyperbolic Anderson model (3.1.3) has a solution in $\mathcal{Z}^p(T)$ for some $p \ge 2$ and for some $T > 0$, then the Hurst parameter H must satisfy $H > 1/4$.*

3.3 Uniform moment bounds

In this section, we obtain the uniform moment estimates of the stochastic convolution with the noise $\dot{W}$ which appears in the definition of the mild solution. These estimates are used later on to prove the existence and uniqueness of solution to SWE (3.1.1).

3.3.1 Uniform moment bounds of stochastic convolution

Define

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy)\,, \tag{3.3.1}$$

where $G_t(x)$ is the Green's function associated with the wave operator (3.1.1), given by (3.2.7).

As we mentioned before, the major difficulty here is that the wave Green's function $G_t(x)$ does not satisfy the semigroup property so that the stochastic Fubini technique used for stochastic heat equation is no longer applicable (see Remark 4.3 in [HW22]). To get around this obstacle, we decompose it into sum of convolutions of some 'nice' kernels. More precisely, we have the following simple and important lemma which is the key starting point of our approach and which plays the role of semigroup property of the heat kernel when the heat equation is investigated (e.g. [HHL+17, HW22]).

Lemma 3.3.1. *The wave kernel $G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$ can be expressed as*

$$\begin{aligned} G_{t-s}(x-y) = &\int_{\mathbb{R}} \mathcal{C}_\beta(t-r, x-z)\mathcal{S}_{1-\beta}(r-s, z-y)dz \\ &+ \int_{\mathbb{R}} \mathcal{S}_\alpha(t-r, x-z)\mathcal{C}_{1-\alpha}(r-s, z-y)dz \\ &+ \int_{\mathbb{R}} \mathcal{S}(t-r, x-z)\mathcal{E}(r-s, z-y)dz \\ &+ \int_{\mathbb{R}} \mathcal{E}(t-r, x-z)\mathcal{S}(r-s, z-y)dz\,, \end{aligned} \tag{3.3.2}$$

where $\alpha, \beta \in (0,1)$, $\mathcal{S}(t,x) = \mathcal{S}_1(t,x) = G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$ and

$$\begin{cases} \mathcal{E}(t,x) := \frac{1}{\pi}\frac{t}{t^2+x^2}\,, \\ \mathcal{S}_\alpha(t,x) := \frac{\Gamma(1-\alpha)}{2\pi}\cos\left(\frac{\alpha\pi}{2}\right)\left[(t+|x|)^{\alpha-1} + \operatorname{sgn}(t-|x|)\big|t-|x|\big|^{\alpha-1}\right]\,, \\ \mathcal{C}_{1-\alpha}(t,x) := \frac{\Gamma(\alpha)}{2\pi}\Bigg[\cos\left(\frac{\alpha\pi}{2}\right)\left[\big|t+|x|\big|^{-\alpha} + \big|t-|x|\big|^{-\alpha}\right] \\ \qquad\qquad -2\cos\left(\alpha\tan^{-1}\left(\frac{|x|}{t}\right)\right)[t^2+x^2]^{-\frac{\alpha}{2}}\Bigg]\,. \end{cases} \tag{3.3.3}$$

Proof. We prove (3.3.2) via Fourier transform

$$\hat{f}(\xi) = \mathcal{F}[f(\xi)] = \int_{\mathbb{R}} e^{-\iota x\xi} f(x)dx\,, \quad \text{where} \quad \iota = \sqrt{-1}\,.$$

The Fourier transform of $G_{t+s}(x)$ is

$$\hat{G}_{t+s}(\xi) = \frac{\sin((t+s)|\xi|)}{|\xi|}\,.$$

We can decompose $\hat{G}_{t+s}(\xi)$ into the summation of following four items:

$$\begin{aligned}
\hat{G}_{t+s}(\xi) &= \frac{\sin(t|\xi|)\cos(s|\xi|)}{|\xi|} + \frac{\sin(s|\xi|)\cos(t|\xi|)}{|\xi|} \\
&= \frac{\sin(t|\xi|)}{|\xi|^{\alpha}} \cdot \frac{\cos(s|\xi|) - e^{-s|\xi|}}{|\xi|^{1-\alpha}} + \frac{\sin(t|\xi|)}{|\xi|} \cdot e^{-s|\xi|} \\
&+ \frac{\sin(s|\xi|)}{|\xi|^{\beta}} \cdot \frac{\cos(t|\xi|) - e^{-t|\xi|}}{|\xi|^{1-\beta}} + \frac{\sin(s|\xi|)}{|\xi|} \cdot e^{-t|\xi|}\,.
\end{aligned}$$

On the other hand, the Fourier transforms of $\mathcal{E}(t,x)$, $\mathcal{S}_\alpha(t,x)$ and $\mathcal{C}_{1-\alpha}(t,x)$ are given as follows (see Lemma 3.6.1):

$$\hat{\mathcal{E}}(t,\xi) = e^{-t|\xi|}\,, \quad \hat{\mathcal{S}}_\alpha(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|^{\alpha}}\,, \quad \hat{\mathcal{C}}_{1-\alpha}(t,\xi) = \frac{\cos(t|\xi|) - e^{-t|\xi|}}{|\xi|^{1-\alpha}}\,. \tag{3.3.4}$$

We then conclude the proof of (3.3.2) by the fact the Fourier transformation transforms the convolution to product. □

Remark 3.3.2. *Readers may wonder why we don't use the following simpler decomposition as we originally attempted:*

$$\begin{aligned}
\hat{G}_{t+s}(\xi) &= \frac{\sin((t+s)|\xi|)}{|\xi|} \\
&= \frac{\sin(t|\xi|)\cos(s|\xi|)}{|\xi|} + \frac{\sin(s|\xi|)\cos(t|\xi|)}{|\xi|} \\
&= \frac{\sin(t|\xi|)}{|\xi|^{\alpha}} \cdot \frac{\cos(s|\xi|)}{|\xi|^{1-\alpha}} + \frac{\cos(t|\xi|)}{|\xi|^{\beta}} \cdot \frac{\sin(s|\xi|)}{|\xi|^{1-\beta}}\,.
\end{aligned}$$

The reason is that the following quantity

$$C_\beta(t,x) := \mathcal{F}^{-1}\left[\frac{\cos(t|\xi|)}{|\xi|^{\beta}}\right] = c_\beta\left[(t+|x|)^{\beta-1} + |t-|x||^{\beta-1}\right]$$

is not integrable. w.r.t. $x \in \mathbb{R}$ when $0 \le \beta \le 1$.

Analogously to idea used in [HHL+17], we shall seek the solution of (3.1.1) in the space

$\mathcal{Z}^p(T)$. To this end we need to bound the $\|\cdot\|_{\mathcal{Z}^p(T)}$ norm of the stochastic convolution $\Phi(t,x)$ defined by (3.3.1) and its variant $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$ as stated in the following theorem.

Proposition 3.3.3. *For the stochastic convolution $\Phi(t,x)$, we have the following estimates:*

(i) *If $p > \frac{1}{H}$, then*

$$\Big\| \sup_{t\in[0,T],x\in\mathbb{R}} |\Phi(t,x)| \Big\|_{L^p(\Omega)} \le C_{T,p,H}\|v\|_{\mathcal{Z}^p(T)}\,. \tag{3.3.5}$$

(ii) *If $p > \frac{2}{4H-1}$, then*

$$\Big\| \sup_{t\in[0,T],x\in\mathbb{R}} \Big|\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)\Big| \Big\|_{L^p(\Omega)} \le C_{T,p,H}\|v\|_{\mathcal{Z}^p(T)}\,. \tag{3.3.6}$$

Proof. We shall use Lemma 3.3.1 to prove this proposition. We divide the proof into two steps.

Step 1: In this step, we shall prove part **(i)** of the proposition. For any $\theta\in(0,1)$ and $i=1,2,3,4$, set

$$J_\theta^{\mathcal{K}_i}(r,z) := \int_0^r\int_{\mathbb{R}} (r-s)^{-\theta}\mathcal{K}_i(r-s,z-y)v(s,y)W(dy,ds)\,, \tag{3.3.7}$$

where

$$\mathcal{K}_1=\mathcal{C}_\alpha,\ \mathcal{K}_2=\mathcal{S}_\alpha,\ \mathcal{K}_3=\mathcal{S},\ \text{and } \mathcal{K}_4=\mathcal{E}. \tag{3.3.8}$$

And we define $\bar{\mathcal{K}}_i$ to be the complements of $\mathcal{K}_i$ according to (3.3.2), namely,

$$\bar{\mathcal{K}}_1=\mathcal{S}_{1-\alpha},\ \bar{\mathcal{K}}_2=\mathcal{C}_{1-\alpha},\ \bar{\mathcal{K}}_3=\mathcal{E},\ \text{and } \bar{\mathcal{K}}_4=\mathcal{S}. \tag{3.3.9}$$

Let us set

$$\Phi_i(t,x) := \frac{\sin(\theta\pi)}{\pi}\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z)J_\theta^{\mathcal{K}_i}(r,z)dzdr\,,\ i=1,2,3,4\,.$$

Then a stochastic version of Fubini's theorem and Lemma 3.3.1 yield

$$\Phi(t,x) = \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy)$$

$$
\begin{aligned}
&=\frac{\sin(\theta\pi)}{\pi}\int_0^t\int_{\mathbb{R}}\int_s^t(t-r)^{\theta-1}(r-s)^{-\theta}dr\times G_{t-s}(x-y)v(s,y)W(dy,ds)\\
&=\sum_{i=1}^4\frac{\sin(\theta\pi)}{\pi}\int_0^t\int_{\mathbb{R}}\int_s^t\int_{\mathbb{R}}(t-r)^{\theta-1}(r-s)^{-\theta}\\
&\qquad\qquad\times\bar{\mathcal{K}}_i(t-r,x-z)\mathcal{K}_i(r-s,z-y)dzdr\times v(s,y)W(dy,ds)\\
&=\sum_{i=1}^4\frac{\sin(\theta\pi)}{\pi}\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z)J_\theta^{\mathcal{K}_i}(r,z)dzdr\\
&=\sum_{i=1}^4\Phi_i(t,x)\,, \qquad (3.3.10)
\end{aligned}
$$

where we have applied the identity

$$
\int_s^t(t-r)^{\theta-1}(r-s)^{-\theta}dr=\frac{\pi}{\sin(\theta\pi)}\,,\quad \theta\in(0,1)\,,0\le s\le t\,.
$$

This expression is essential for us to derive the desired estimates. In the following, we will use $\sum_i$ to denote $\sum_{i=1}^4$ and $\sup\limits_{t,x}$ to denote $\sup\limits_{t\in[0,T],x\in\mathbb{R}}$.

It is clear by the Hölder inequality with $1/p+1/q=1$ that for $i=1,\cdots,4$

$$
\begin{aligned}
\sup_{t,x}|\Phi_i(t,x)|\lesssim&\sup_{t,x}\int_0^t(t-r)^{\theta-1}\left(\int_{\mathbb{R}}|\bar{\mathcal{K}}_i(t-r,x-z)|^qdz\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\|J_\theta^{\mathcal{K}_i}(r,z)\|_{L^p(\mathbb{R})}dr\\
\lesssim&\left(\sup_t\int_0^t\int_{\mathbb{R}}r^{q(\theta-1)}|\bar{\mathcal{K}}_i(r,z)|^qdzdr\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_0^T\|J_\theta^{\mathcal{K}_i}(r,z)\|_{L^p(\mathbb{R})}^pdr\right)^{\frac{1}{p}}\\
=&(I_i^{(1)})^{1/q}\times(I_i^{(2)})^{1/p}\,, \qquad (3.3.11)
\end{aligned}
$$

where we change the variables $r\to t-r$ and $z\to x-z$ in the second inequality and then it is clear that $\sup_{t,x}$ becomes $\sup_t$ thanks to the translation invariance in space variable of the function. This technique will be freely used in the sequel without mention. We shall deal with $I_i^{(1)}, I_i^{(2)}$, $i=1,\cdots,4$, term by term in the subsequent paragraphs.

First, let us deal with $I_i^{(1)}$ when $i=1,2$. The cases $i=3,4$ can be treated similarly. When $i=1$, $\mathcal{K}_1=\mathcal{C}_\alpha$ and $\bar{\mathcal{K}}_1=\mathcal{S}_{1-\alpha}$ defined as (3.3.3). By the change of variable

$z \to rz$, it is easy to see $I_1^{(1)}$ can be bounded as

$$\begin{aligned} I_1^{(1)} &= \sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{S}_{1-\alpha}(r,z)|^q dz dr \\ &\lesssim \left[\sup_t \int_0^t r^{q\left(\theta-1-\alpha+\frac{1}{q}\right)} dr \right] \times \int_0^\infty \left| (1+|z|)^{-\alpha} + \operatorname{sgn}(1-|z|) \big|1-|z|\big|^{-\alpha} \right|^q dz \,. \end{aligned}$$

In order to make sure the above integrals converge, we need

$$\alpha q < 1\,, \ (\alpha+1)q > 1 \quad \Leftrightarrow \quad 0 < \alpha < \frac{1}{q} = 1 - \frac{1}{p}\,, \tag{3.3.12}$$

and also

$$q\left(\theta - \alpha - 1 + \frac{1}{q}\right) > -1 \quad \Leftrightarrow \quad \theta > 1 - \frac{2}{q} + \alpha\,. \tag{3.3.13}$$

When $i = 2$, $\mathcal{K}_2 = \mathcal{S}_\alpha$ and $\bar{\mathcal{K}}_2 = \mathcal{C}_{1-\alpha}$ which are defined in (3.3.3), we have

$$I_2^{(1)} = \sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{C}_{1-\alpha}(r,z)|^q dz dr \tag{3.3.14}$$

$$\begin{aligned} &\lesssim \left[\sup_t \int_0^t r^{q\left(\theta-1-\alpha+\frac{1}{q}\right)} dr \right] \\ &\quad \times \int_0^\infty \Bigg[\cos\left(\frac{\alpha\pi}{2}\right) \left[\big|1+|z|\big|^{-\alpha} + \big|1-|z|\big|^{-\alpha} \right] \\ &\qquad - 2\cos\left(\alpha \tan^{-1}(z)\right) \left[1+z^2\right]^{-\frac{\alpha}{2}} \Bigg]^q dz \,. \end{aligned} \tag{3.3.15}$$

By Lemma 3.6.1 in the Appendix 3.6, $\mathcal{C}_{1-\alpha}(r,z)$ can be bounded by

$$|\mathcal{C}_{1-\alpha}(r,z)| \lesssim \begin{cases} \big|r+|z|\big|^{-\alpha} + \big|r-|z|\big|^{-\alpha} + \left[r^2+z^2\right]^{-\frac{\alpha}{2}} & \text{if } |z| \approx r\,, \\ r\left(|z|^2 - r^2\right)^{-\frac{\alpha}{2}-1} & \text{if } |z| \approx \infty\,. \end{cases}$$

Thus, in order to make sure (3.3.14) is bounded, we need

$$\alpha q < 1\,, \ (\alpha+2)q > 1 \Leftrightarrow 0 < \alpha < \frac{1}{q}\,, \tag{3.3.16}$$

and

$$q\left(\theta - \alpha - 1 + \frac{1}{q}\right) > -1 \Leftrightarrow \theta > 1 - \frac{2}{q} + \alpha\,. \tag{3.3.17}$$

Therefore, to prove part **(i)** of the proposition we only need to show

$$\mathbb{E}\|J_\theta^{\mathcal{K}_i}(r,\cdot)\|_{L^p(\mathbb{R})}^p \le C\|v\|_{\mathcal{Z}^p(T)}^p, \quad i=1,2,3,4\,.$$

This is objective of Lemma 3.7.1, proved in the Appendix 3.7 under the following condition:

$$p>\frac{1}{H},\ 1-\frac{2}{q}+\alpha<\theta<H+\alpha-\frac{1}{2},\quad 1-H<\alpha<1-\frac{1}{p}. \tag{3.3.18}$$

Therefore, when $p>\frac{1}{H}$, we can choose α such that $1-H<\alpha<1-\frac{1}{p}$, and then we see (3.3.12), (3.3.13), and (3.3.18) are satisfied since $\frac{1}{H}>\frac{4}{2H+1}$ if $H<\frac{1}{2}$. Thus we have proved **(i)** of the proposition for $\Phi_1(t,x)$, $\Phi_2(t,x)$. The cases for $\Phi_3(t,x)$ and $\Phi_4(t,x)$ can be proved similarly. Thus, we complete the proof of part **(i)** of the proposition.

Step 2: Let us now consider part **(ii)** of the proposition. In order to obtain the desired decay rate of $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$, we still use the equation (3.3.10) to express $\Phi(t,x)$ by $J_\theta^{\mathcal{K}_i}$. Recall our notation $\mathfrak{D}_h\Phi(t,x):=\ \Phi(t,x+h)-\Phi(t,x)$ and same notations for $\mathfrak{D}_h\bar{\mathcal{K}}_i(t-r,z)$, $\mathfrak{D}_hJ_\theta^{\mathcal{K}_i}(r,z)$. Then

$$\begin{aligned}\mathfrak{D}_h\Phi(t,x) &= \frac{\sin(\theta\pi)}{\pi}\sum_i\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}\mathfrak{D}_h\bar{\mathcal{K}}_i(t-r,x-z)J_\theta^{\mathcal{K}_i}(r,z)dzdr\\ &\simeq \sum_i\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z)\mathfrak{D}_hJ_\theta^{\mathcal{K}_i}(r,z)dzdr\,,\end{aligned} \tag{3.3.19}$$

with the choice of $\mathcal{K}$ and $\bar{\mathcal{K}}$ defined by (3.3.8) and (3.3.9). Invoking Minkowski's inequality and then Hölder's inequality we get

$$\begin{aligned}&\sup_{t,x}\left(\int_{\mathbb{R}}|\mathfrak{D}_h\Phi(t,x)|^2|h|^{2H-2}dh\right)^{\frac{1}{2}}\\ &\lesssim \sup_{t,x}\sum_i\Bigg(\int_{\mathbb{R}}\Bigg|\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}\bar{\mathcal{K}}_i(t-r,x-z)\\ &\qquad\qquad\times\mathfrak{D}_hJ_\theta^{\mathcal{K}_i}(r,z)dzdr\Bigg|^2\cdot|h|^{2H-2}dh\Bigg)^{\frac{1}{2}}\\ &\lesssim \sup_{t,x}\sum_i\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}|\bar{\mathcal{K}}_i(t-r,x-z)|\\ &\qquad\qquad\times\left(\int_{\mathbb{R}}\left|\mathfrak{D}_hJ_\theta^{\mathcal{K}_i}(r,z)\right|^2|h|^{2H-2}dh\right)^{\frac{1}{2}}dzdr\end{aligned}$$

$$\lesssim \sum_i \left(\sup_t \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \left| \bar{\mathcal{K}}_i(r,z) \right|^q dz dr \right)^{\frac{1}{q}}$$

$$\times \left(\int_0^T \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left| \mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z) \right|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} dz dr \right)^{\frac{1}{p}}$$

$$=: (J_i^{(1)})^{\frac{1}{q}} \times (J_i^{(2)})^{\frac{1}{p}} . \tag{3.3.20}$$

The first factor $(J_i^{(1)})^{\frac{1}{q}}$ in (3.3.20) is finite if we require that α, θ, p, q satisfy (3.3.12) and (3.3.13). Therefore we only need to focus on the second factor $(J_i^{(2)})^{\frac{1}{p}}$ in (3.3.20). By Lemma 3.7.2, we see

$$\mathbb{E} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left| \mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z) \right|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} dz \le C_{T,p,\alpha,\theta} \|v\|^p_{\mathcal{Z}^p(T)} ,$$

under the conditions

$$p > \frac{1}{H}, \ 1 - 2/q + \alpha < \theta < 2H + \alpha - 1, \ \frac{3}{2} - 2H < \alpha < 1 - \frac{1}{p}. \tag{3.3.21}$$

If $p > \frac{2}{4H-1}$, then we can choose α such that $\frac{3}{2} - 2H < \alpha < 1 - \frac{1}{p}$, and then we see (3.3.12), (3.3.13) and (3.3.21) are satisfied since $\frac{2}{4H-1} > \frac{1}{H}$ when $H < \frac{1}{2}$. Thus, we complete the proof of part **(ii)** of the proposition. □

3.3.2 Uniform moment bounds of the approximate solutions

We approximate the noise W by the following smoothing of the noise with respect to the space variable. That is, for $\varepsilon > 0$ we define

$$\frac{\partial}{\partial x} W_\varepsilon(t,x) = \int_{\mathbb{R}} \rho_\varepsilon(x-y) W(t,dy) , \tag{3.3.22}$$

where $\rho_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp(-\frac{x^2}{2\varepsilon})$. The regulated noise W_ε induces an approximation of mild solution

$$u_\varepsilon(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sigma(s,y,u_\varepsilon(s,y)) W_\varepsilon(ds,dy), \tag{3.3.23}$$

where the stochastic integral is understood in the Itô sense. Due to the regularity in space of the noise, the existence and uniqueness of the solution $u_\varepsilon(t,x)$ to above equation is standard (even the higher dimensional case were known (e.g. [HHN14, Pes02] and references therein).

The lemma below asserts that the approximate solution $\{u_\varepsilon(t,x)\,,\varepsilon>0\}$ is uniformly bounded in the space $\mathcal{Z}^p(T)$. More precisely, we have

Lemma 3.3.4. *Let $H\in(\frac14,\frac12)$. Assume that $\sigma(t,x,u)$ satisfies the hypothesis* ***(H1)*** *and assume that $I_0(t,x)$ is in $\mathcal{Z}^p(T)$. Then the approximate solutions u_ε satisfy*

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathcal{Z}^p(T)}:=\sup_{\varepsilon>0}\|u_\varepsilon(t,\cdot)\|_{\mathcal{Z}_1^p(T)}+\sup_{\varepsilon>0}\|u_\varepsilon(t,\cdot)\|_{\mathcal{Z}_2^p(T)}<\infty. \tag{3.3.24}$$

Proof. For notational simplicity we assume $\sigma(t,x,u)=\sigma(u)$ without loss of generality because of hypothesis **(H1)**. We shall use some thoughts similar to those in [HW22]. To this end, we define the Picard iteration as follows:

$$u_\varepsilon^0(t,x)=I_0(t,x)\,,$$

and recursively for $n=0,1,2,\cdots$,

$$u_\varepsilon^{n+1}(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}}G_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y))W_\varepsilon(ds,dy)\,. \tag{3.3.25}$$

From [HHL$^+$18, Lemma 4.12] it follows that for any fixed $\varepsilon>0$ when n goes to infinity, the sequence $u_\varepsilon^n(t,x)$ converges to $u_\varepsilon(t,x)$ a.s. In the following steps 1 and 2, we shall first bound $\|u_\varepsilon^n\|_{\mathcal{Z}^p(T)}$ uniformly in n, and ε. Then, in step 3 we use Fatou's lemma to show (3.3.24).

In the following, we will continue to use the notations $\mathfrak{D}_hf(t,x)$ and $\Box_{h,l}f(t,x)$ previously defined in (3.1.5) and (3.1.6).

Step 1. In this step, we bound the $L^p(\Omega\times\mathbb{R})$ norm of $u_\varepsilon^{n+1}(t,x)$ by the $\mathcal{Z}^p$ norm of $u_\varepsilon^n(t,x)$. Rewrite (3.3.25) as

$$u_\varepsilon^{n+1}(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}}\Big[\Big(G_{t-s}(x-\cdot)\sigma(u_\varepsilon^n(s,\cdot))\Big)*G_\varepsilon\Big](y)W(ds,dy)\,.$$

Using $e^{-\varepsilon|\xi|^2} \le 1$ and the condition (3.2.11) on σ, we have from the Burkholder-Davis-Gundy inequality (3.2.5)

$$
\begin{aligned}
&\mathbb{E}\big[|u_\varepsilon^{n+1}(t,x)|^p\big] \\
\le& C_p|I_0(t,x)|^p + C_p\mathbb{E}\left(\int_0^t\int_{\mathbb{R}}\Big|\mathcal{F}\big[G_{t-s}(x-\cdot)\sigma(u_\varepsilon(s,\cdot))\big](\xi)\Big|^2 e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi ds\right)^{\frac{p}{2}} \\
\le& C_p|I_0(t,x)|^p + C_p\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2}\Big|G_{t-s}(x-y-h)\sigma(u_\varepsilon^n(s,y+h)) \\
&\qquad\qquad - G_{t-s}(x-y)\sigma(u_\varepsilon^n(s,y))\Big|^2|h|^{2H-2}dhdyds\bigg)^{\frac{p}{2}} \\
\le& C_p\left[|I_0(t,x)|^p + |\mathcal{D}_1^{\varepsilon,n}(t,x)|^{\frac{p}{2}} + |\mathcal{D}_2^{\varepsilon,n}(t,x)|^{\frac{p}{2}}\right], \qquad (3.3.26)
\end{aligned}
$$

where we have used the notations $\mathcal{D}_1^{\varepsilon,n}(t,x)$ and $\mathcal{D}_2^{\varepsilon,n}(t,x)$ similar to (3.7.2) and (3.7.3), namely,

$$
\mathcal{D}_1^{\varepsilon,n}(t,x) := \int_0^t\int_{\mathbb{R}^2}\big|\mathfrak{D}_h G_{t-s}(y)\big|^2 \cdot \|u_\varepsilon^n(s,x+y)\|_{L^p(\Omega)}^2|h|^{2H-2}dhdyds\,,
$$

and

$$
\mathcal{D}_2^{\varepsilon,n}(t,x) := \int_0^t\int_{\mathbb{R}^2}|G_{t-s}(y)|^2\|\mathfrak{D}_h u_\varepsilon^n(t,x+y)\|_{L^p(\Omega)}^2|h|^{2H-2}dhdyds\,.
$$

This means

$$
\begin{aligned}
\|u_\varepsilon^{n+1}(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 &= \left(\int_{\mathbb{R}}\mathbb{E}\big[|u_\varepsilon^{n+1}(t,x)|^p\big]dx\right)^{\frac{2}{p}} \\
&\le C_p\left[\|I_0(t,x)\|_{L^p(\Omega\times\mathbb{R})}^2 + D_1^{\varepsilon,n}(t) + D_2^{\varepsilon,n}(t)\right], \qquad (3.3.27)
\end{aligned}
$$

where $D_1^{\varepsilon,n}(t)$ and $D_2^{\varepsilon,n}(t)$ are defined and can be bounded similar to the argument used in the proof of Lemma 3.7.1:

$$
D_1^{\varepsilon,n}(t) := \left(\int_{\mathbb{R}}|\mathcal{D}_1^{\varepsilon,n}(t,x)|^{\frac{p}{2}}dx\right)^{\frac{2}{p}} \le C_{p,H}\int_0^t(t-s)^{2H}\|u_\varepsilon^n(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 ds\,, \qquad (3.3.28)
$$

and

$$D_2^{\varepsilon,n}(t) := \left(\int_{\mathbb{R}} |\mathcal{D}_2^{\varepsilon,n}(t,x)|^{\frac{p}{2}}\, dx\right)^{\frac{2}{p}} \le C_{p,H} \int_0^t (t-s) \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\right]^2 ds\,. \tag{3.3.29}$$

The above bounds on $D_1^{\varepsilon,n}(t), D_2^{\varepsilon,n}(t)$ together with (3.3.27) yield

$$\begin{aligned}\|u_\varepsilon^{n+1}(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} \le C_{p,H}\Bigg(&\|I_0(t,x)\|^2_{L^p(\Omega\times\mathbb{R})} + \int_0^t (t-s) \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\right]^2 \\ &+ \int_0^t (t-s)^{2H}\ \|u_\varepsilon^n(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} ds ds\Bigg)\,. \end{aligned}\tag{3.3.30}$$

Step 2. Next, we bound $\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^{n+1}(t)$ by the $\mathcal{Z}^p$ norm of $u_\varepsilon^n(t,x)$. Similar to (3.3.26) we have

$$\begin{aligned}\mathbb{E}\left[|\mathfrak{D}_h u_\varepsilon^{n+1}(t,x)|^p\right] \le& C_p\big|I_0(t,x) - I_0(t,x+h)\big|^p \\ &+ C_p\mathbb{E}\Bigg(\int_0^t \int_{\mathbb{R}^2} \Big|\mathfrak{D}_h G_{t-s}(x-y-z)\sigma(u_\varepsilon^n(s,y+z)) \\ &\qquad - \mathfrak{D}_h G_{t-s}(x-z)\sigma(u_\varepsilon^n(s,z))\Big|^2 |y|^{2H-2} dz dy ds\Bigg)^{\frac{p}{2}} \\ \le& C_p\left[\mathcal{I}_0(t,x,h) + \mathcal{I}_1^{\varepsilon,n}(t,x,h) + \mathcal{I}_2^{\varepsilon,n}(t,x,h)\right]\,,\end{aligned}$$

where

$$\mathcal{I}_0(t,x,h) := \big|I_0(t,x) - I_0(t,x+h)\big|^p\,,$$

$$\mathcal{I}_1^{\varepsilon,n}(t,x,h) := \mathbb{E}\Bigg(\int_0^t \int_{\mathbb{R}^2} \Big|\mathfrak{D}_h G_{t-s}(x-y-z)\Big|^2 \Big|\mathfrak{D}_y \sigma(u_\varepsilon(s,z))\Big|^2 |y|^{2H-2} dz dy ds\Bigg)^{\frac{p}{2}}\,,$$

and

$$\mathcal{I}_2^{\varepsilon,n}(t,x,h) := \mathbb{E}\Bigg(\int_0^t \int_{\mathbb{R}^2} \Big|\Box_{y,h} G_{t-s}(x-z)\Big|^2 \cdot \Big|\sigma(u_\varepsilon(s,z))\Big|^2 |y|^{2H-2} dz dy ds\Bigg)^{\frac{p}{2}}\,.$$

Thus, by Minkowski's inequality we have

$$\left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^{n+1}(t)\right]^2 = \int_{\mathbb{R}} \|\mathfrak{D}_h u_\varepsilon^{n+1}(t,x)\|^2_{L^p(\mathbb{R}\times\Omega)} |h|^{2H-2} dh$$

$$
\begin{aligned}
&\le C_p \sum_{j=0}^{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{I}_j^{\varepsilon,n}(t,x,h)dx \right)^{\frac{2}{p}} |h|^{2H-2}dh \\
&=: J_0 + J_1 + J_2.
\end{aligned}
$$

For the term J_0, it is clear that

$$
J_0 = C_p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|\mathfrak{D}_h I_0(t,x)\right|^p dx \right)^{\frac{2}{p}} |h|^{2H-2}dh = \left[\mathcal{N}^*_{\frac{1}{2}-H,p} I_0(t)\right]^2 . \tag{3.3.31}
$$

We can deal with the term J_1 in the similar manner as when we deal with (3.7.15) in the proof of Lemma 3.7.2. An application of Minkowski's inequality and then an application of Parseval's formula yield

$$
\begin{aligned}
J_1 \le & C_{p,H} \int_0^t \int_{\mathbb{R}^2} \left|\mathfrak{D}_h G_{t-s}(z)\right|^2 |h|^{2H-2}dhdz \\
& \qquad \times \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{E}\Big[\left|\mathfrak{D}_y u_\varepsilon^n(t,x)\right|^p\Big] dx \right)^{\frac{2}{p}} |y|^{2H-2}dyds \\
\le & C_{p,H} \int_0^t (t-s)^{2H} \left[\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(s)\right]^2 ds\,.
\end{aligned} \tag{3.3.32}
$$

Next, we bound J_2. By the condition (3.2.11) ($|\sigma(u)| \lesssim |u|$) and by a change of variable $z \to x - z$, we obtain

$$
\mathcal{I}_2^{\varepsilon,n}(t,x,h) \le C_p \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(z,y,h)|^2 |u_\varepsilon^n(s,x-z)|^2 |y|^{2H-2} dydzds \right)^{\frac{p}{2}} .
$$

In a similar way to that when we deal with (3.7.16) in the proof of Lemma 3.7.2, we have

$$
\begin{aligned}
J_2 :=& \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{I}_2^{\varepsilon,n}(t,x,h)dx \right|^{\frac{2}{p}} |h|^{2H-2}dh \\
\le& \, C_{p,H} \int_0^t \int_{\mathbb{R}^3} |\Box_{y,h} G_{t-s}(z)|^2 |y|^{2H-2} dy |h|^{2H-2} dhdz \\
& \qquad \times \left(\int_{\mathbb{R}} \mathbb{E}|u_\varepsilon^n(s,x-z)|^p dx \right)^{\frac{2}{p}} ds \\
\le& \, C_{p,H} \int_0^t (t-s)^{4H-1} \|u_\varepsilon^n(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} ds\,.
\end{aligned} \tag{3.3.33}
$$

Thus, we obtain

$$\left[\mathcal{N}^*_{\frac{1}{2}-H,p}u^{n+1}_\varepsilon(t)\right]^2 \le C_{p,H}\left[\mathcal{N}^*_{\frac{1}{2}-H,p}I_0(t)\right]^2 + C_{p,H}\int_0^t (t-s)^{2H}\left[\mathcal{N}^*_{\frac{1}{2}-H,p}u^n_\varepsilon(s)\right]^2 ds$$
$$+ C_{p,H}\int_0^t (t-s)^{4H-1}\|u^n_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}ds\,. \tag{3.3.34}$$

Step 3. Set

$$\Psi^n_\varepsilon(t) := \|u^n_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} + \left[\mathcal{N}^*_{\frac{1}{2}-H,p}u^n_\varepsilon(t)\right]^2.$$

Then combining the estimates (3.3.30) and (3.3.34) yields

$$\Psi^{n+1}_\varepsilon(t) \le C_{p,H,T}\left(\|I_0\|^2_{\mathcal{Z}^p(T)} + \int_0^t (t-s)^{4H-1}\Psi^n_\varepsilon(s)ds\right).$$

Now it is relatively easy to see by fractional Gronwall lemma (similar to [CHN16, Lemma A.2])

$$\sup_{\varepsilon>0}\sup_{n\ge1}\sup_{t\in[0,T]}\Psi^n_\varepsilon(t) \le C_{T,p,H} < \infty\,.$$

Thus, by the same argument as in the proof of [HW22, Lemma 4.5], we have that $\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ and $\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)$ are finite.

In conclusion, we have proved

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathcal{Z}^p(T)} := \sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)$$

is finite. □

3.4 Hölder continuity and well-posedness

In this section, we obtain some estimations which imply the Hölder regularity of the stochastic convolution with respect to our noise $\dot{W}$. Then the similar estimations of the solution to SWE (3.1.1) follow in a routine way. These estimations are devoted to prove the tightness associated with the solution to (3.1.1).

3.4.1 Hölder continuity of stochastic convolution

We have the following regularity results for stochastic convolution $\Phi(t,x)$ defined by (3.3.1) and the approximated solution u_ε defined by (3.3.23).

Proposition 3.4.1. *Let $v(\cdot,\cdot) \in \mathcal{Z}^p(T)$ and let the stochastic convolution $\Phi(t,x)$ be defined by (3.3.1). We have the following Hölder regularity in the space and time variables for $\Phi(t,x)$:*

(i) *If $p > \frac{1}{H}$ and $0 < \gamma < H - \frac{1}{p}$, then*

$$\Big\| \sup_{t,t+h\in[0,T],x\in\mathbb{R}} |\Phi(t+h,x) - \Phi(t,x)| \Big\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^\gamma \|v\|_{\mathcal{Z}^p(T)} \,. \tag{3.4.1}$$

(ii) *If $p > \frac{1}{H}$ and $0 < \gamma < H - \frac{1}{p}$, then*

$$\Big\| \sup_{t\in[0,T],x,y\in\mathbb{R}} |\Phi(t,x) - \Phi(t,y)| \Big\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |x-y|^\gamma \|v\|_{\mathcal{Z}^p(T)} \,. \tag{3.4.2}$$

Proof. **Step 1:** In this step, we concentrate on the analysis of the following quantity (we denote $\sup_{t,t+h\in[0,T],x\in\mathbb{R}}$ by $\sup_{t,x}$)

$$\sup_{t,x} |\Delta_h \Phi(t,x)| := \sup_{t,x} |\Phi(t+h,x) - \Phi(t,x)| \,.$$

Assuming $h \in (0,1)$ and $t \in [0,T]$ such that $t+h \le T$, then by the representation formula (3.3.10) and the triangle inequality we have

$$\begin{aligned}
\Delta_h \Phi(t,x) = \sum_i \frac{\sin(\pi\theta)}{\pi} \Big[\int_0^{t+h} \int_{\mathbb{R}} (t+h-r)^{\theta-1} \overline{\mathcal{K}}_i(t+h-r, x-z) J_\theta^{\mathcal{K}_i}(r,z) drdz \\
- \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1} \overline{\mathcal{K}}_i(t-r, x-z) J_\theta^{\mathcal{K}_i}(r,z) drdz \Big] \\
\lesssim \sum_{j=1}^3 \mathcal{I}_j(t,h,x) \,,
\end{aligned}$$

where

$$\mathcal{I}_1(t,h,x) := \sum_i \mathcal{I}_1^{(i)}(t,h,x)$$

$$:= \sum_i \int_0^t \int_{\mathbb{R}} \Delta_h (t-r)^{\theta-1} \overline{\mathcal{K}}_i(t-r, x-z) J_\theta^{\mathcal{K}_i}(r,z) dr dz$$

with $\Delta_h (t-r)^{\theta-1} := (t+h-r)^{\theta-1} - (t-r)^{\theta-1}$;

$$\begin{aligned} \mathcal{I}_2(t,h,x) &:= \sum_i \mathcal{I}_2^{(i)}(t,h,x) \\ &:= \sum_i \int_0^t \int_{\mathbb{R}} (t+h-r)^{\theta-1} \Delta_h \overline{\mathcal{K}}_i(t-r, x-z) J_\theta^{\mathcal{K}_i}(r,z) dr dz, \end{aligned}$$

with $\Delta_h \overline{\mathcal{K}}_i(t-r, x-z) := \overline{\mathcal{K}}_i(t+h-r, x-z) - \overline{\mathcal{K}}_i(t-r, x-z)$; and

$$\begin{aligned} \mathcal{I}_3(t,h,x) &:= \sum_i \mathcal{I}_3^{(i)}(t,h,x) \\ &:= \sum_i \int_t^{t+h} \int_{\mathbb{R}} (t+h-r)^{\theta-1} \overline{\mathcal{K}}_i(t+h-r, x-z) J_\theta^{\mathcal{K}_i}(r,z) dr dz. \end{aligned}$$

Our goal is to show that

$$\big\| \sup_{t,x} \mathcal{I}_j(t,h,x) \big\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^\gamma \|v\|_{\mathcal{Z}^p(T)}, \qquad j = 1,2,3, \tag{3.4.3}$$

under the conditions

$$p > \frac{1}{H}, \quad 1 - H < \alpha < 1 - \frac{1}{p}, \quad \gamma < H - \frac{1}{p}. \tag{3.4.4}$$

We shall first treat $\mathcal{I}_1(t,h,x)$ and $\mathcal{I}_3(t,h,x)$. The term $\mathcal{I}_2(t,h,x)$ is more complicated and shall be handled lastly.

For the term $\mathcal{I}_1(t,h,x)$, it is easy to see that for any fixed $\gamma \in (0,1)$,

$$\Delta_h (t-r)^{\theta-1} = |(t+h-r)^{\theta-1} - (t-r)^{\theta-1}| \lesssim |t-r|^{\theta-1-\gamma} h^\gamma. \tag{3.4.5}$$

Then by Hölder's inequality with $1/p + 1/q = 1$ and Lemma 3.7.1, under conditions (3.3.18) we have for $i = 1, \cdots, 4$

$$\big\| \sup_{t,x} \mathcal{I}_1^{(i)}(t,h,x) \big\|_{L^p(\Omega)}$$

$$
\leq \left(\sup_{t,x} \int_0^t \int_{\mathbb{R}} \left|\Delta_h (t-r)^{\theta-1}\right|^q |\overline{\mathcal{K}}_i(t-r,x-z)|^q dzdr \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)}
$$

$$
\leq \left(\sup_{t} \int_0^t \int_{\mathbb{R}} |r|^{(\theta-1-\gamma)q} |\overline{\mathcal{K}}_i(r,z)|^q dzdr \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)} \cdot |h|^{\gamma} \,, \tag{3.4.6}
$$

where in the last inequality of (3.4.6) we have used the change of variables $r \to t-r$ and $z \to z+x$. Now we only need to show

$$
\sup_{t} \int_0^t \int_{\mathbb{R}} |r|^{(\theta-1-\gamma)q} |\overline{\mathcal{K}}_i(r,z)|^q dzdr < +\infty.
$$

We shall only discuss the situation $i=1$. Other cases $i=2,3,4$ can be treated similarly. For $i=1$, we have $\mathcal{K}_1 = \mathcal{C}_\alpha$, $\overline{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha}$ as defined in (3.3.3). Hence, by changing variable $r \to rz$ we have

$$
\begin{aligned}
&\sup_{t} \int_0^t \int_{\mathbb{R}} |r|^{(\theta-1-\gamma)q} |\mathcal{S}_{1-\alpha}(r,z)|^q dzdr \\
&\leq \sup_{t} \int_0^t |r|^{(\theta-1-\gamma)q+1-\alpha q} dr \cdot \int_0^\infty \left| (1+|z|)^{-\alpha} + \mathrm{sgn}(1-|z|)|1-|z||^{-\alpha} \right|^q dz \,.
\end{aligned}
$$

Then by the same argument as in the proof of part **(i)** of Proposition 3.3.3, we have

$$
\left\| \sup_{t,x} \mathcal{I}_1(t,h,x) \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^{\gamma} \|v\|_{\mathcal{Z}^p(T)}
$$

under the conditions (3.3.18) and $(\theta-1-\gamma)q + 1 - \alpha q > -1$, which can be summarized as the following conditions

$$
p > \frac{1}{H}\,, \quad 1-H < \alpha < 1-\frac{1}{p}\,, \quad 1+\alpha-\frac{2}{q}+\gamma < \theta < H+\alpha-\frac{1}{2}\,. \tag{3.4.7}
$$

Since $p > \frac{1}{H} > 2$ implies $\gamma < H - 1/p < H + 2/q - 3/2$ it is clear that one can choose α and θ satisfying (3.4.7) under conditions (3.4.4).

Now let us deal with the term $\mathcal{I}_3$. Using Hölder's inequality, Lemma 3.7.1 and the change of variables $z \to z+x$ and $r \to r-t-h$, we have

$$
\left\| \sup_{t,x} \mathcal{I}_3(t,h,x) \right\|_{L^p(\Omega)}
$$

$$
\begin{aligned}
&\lesssim \sum_i \left(\int_0^h \int_{\mathbb{R}} r^{q(\theta-1)} |\overline{\mathcal{K}}_i(r,z)|^q dz dr \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)} \\
&=: \sum_i \left(\mathcal{I}_3^{(i)}(h) \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)} \,. \qquad (3.4.8)
\end{aligned}
$$

We want to show that $\left(\mathcal{I}_3^{(i)}(h)\right)^{1/q} \lesssim h^\gamma$ for $i = 1, \cdots, 4$ with $p,\ \alpha,\ \gamma$ satisfying (3.4.4). As before, we only consider the case $i = 1$, i.e. $\mathcal{K}_1 = \mathcal{C}_\alpha,\ \overline{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha}$. The other cases can be handled in similar way. In this case we have

$$
\begin{aligned}
\left(\sup_{t,x} \mathcal{I}_3^{(1)}(h) \right)^{1/q} &= \left(\int_0^h \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{S}_{1-\alpha}(r,z)|^q dz dr \right)^{1/q} \\
&\le \Bigg(\int_0^h r^{q(\theta-1)+1-q\alpha} dr \\
&\qquad \times \int_0^{+\infty} \left[(1+|z|)^{-\alpha} + \operatorname{sgn}(1-|z|)|1-|z||^{-\alpha} \right]^q dz \Bigg)^{1/q} \\
&\lesssim \left[|h|^{q(\theta-1)+2-q\alpha} \right]^{1/q} \le h^\gamma \qquad (3.4.9)
\end{aligned}
$$

if (3.4.7) is satisfied (and hence so does (3.4.4)). We have similar bound for $\mathcal{I}_3^{(i)}(h)$ for $i = 2, 3, 4$. Combing these bounds with (3.4.8) we have

$$
\left\| \sup_{t,x}\ \mathcal{I}_3(t,h,x) \right\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^\gamma \|v\|_{\mathcal{Z}^p(T)} \,,
$$

if $p,\ \alpha,\ \gamma$ satisfy (3.4.4).

Lastly, we are going to deal with $\mathcal{I}_2$, which is much more complicated. By Hölder's inequality,

$$
\begin{aligned}
\mathcal{I}_2(t,h,x) \ \le\ & \sum_i \left(\int_0^t \int_{\mathbb{R}} (t+h-r)^{q(\theta-1)} |\Delta_h \overline{\mathcal{K}}_i(t-r, x-z)|^q dz dr \right)^{1/q} \\
& \times \left(\int_0^T \| \ J_\theta^{\mathcal{K}_i}(r,z) \ \|_{L^p(\mathbb{R})}^p \ dr \right)^{1/p} . \qquad (3.4.10)
\end{aligned}
$$

The second factor inside the summation in (3.4.10) can be bounded by a multiple of $\|v\|_{\mathcal{Z}^p(T)}$ via Lemma 3.7.1 under the condition (3.3.18). By the change of variables $r \to$

$t-r,\ z \to x-z$, we see

$$
\begin{aligned}
&\big\| \sup_{t,x} \mathcal{I}_2(t,h,x)\big\|_{L^p(\Omega)} \\
&\qquad \le \sum_i \left(\sup_t \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} |\Delta_h \overline{\mathcal{K}}_i(r,z)|^q dz dr \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)} \\
&\qquad =: \sum_i \left(\sup_t \mathcal{I}_2^{(i)}(t,h) \right)^{1/q} \times \|v\|_{\mathcal{Z}^p(T)} .
\end{aligned}
$$

Thus, we shall need to show that for $i=1,2,3,4$

$$
\sup_t \mathcal{I}_2^{(i)}(t,h) = \sup_t \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} |\Delta_h \overline{\mathcal{K}}_i(r,z)|^q dz dr \le C_{T,p,H,\gamma} |h|^{\gamma q}, \tag{3.4.11}
$$

to obtain

$$
\big\| \sup_{t,x} \mathcal{I}_2(t,h,x)\big\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^{\gamma} \|v\|_{\mathcal{Z}^p(T)} . \tag{3.4.12}
$$

Now, we shall deal with $\mathcal{I}_2^{(i)}(t,h)$ for $i=1,2,3,4$ term by term.

Case i=1. Recall that $\overline{\mathcal{K}}_1(r,z) = \mathcal{S}_{1-\alpha}(r,z)$ and $\mathcal{K}_1(r,z) = \mathcal{C}_\alpha(r,z)$ are defined by (3.3.3). We shall show

$$
\begin{cases}
\sup_t \mathcal{I}_2^{(1)}(t,h) \le C_{T,p,\theta,\alpha} |h|^{\gamma q}, \quad \text{where} \\
\mathcal{I}_2^{(1)}(t,h) = \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} |\Delta_h \mathcal{S}_{1-\alpha}(r,z)|^q dz dr
\end{cases} \tag{3.4.13}
$$

for $p,\ \gamma$ and α satisfying (3.4.4). Set $A_1 := [|z|<r]$, $A_2 := [|z|>r+2h]$ and $A_3 := [r<|z|<r+2h]$. For fixed $\eta \in (0,1)$, we see

$$
\begin{cases}
\Delta_h |r+|z||^{-\alpha} = |r+|z|+h|^{-\alpha} - |r+|z||^{-\alpha} \lesssim |r+|z||^{-\alpha-\eta} |h|^\eta, & \text{on } \mathbb{R}\,; \\
\Delta_h |r-|z||^{-\alpha} = |r-|z|+h|^{-\alpha} - |r-|z||^{-\alpha} \lesssim |r-|z||^{-\alpha-\eta} |h|^\eta, & \text{on } A_1\,; \\
\Delta_h |r-|z||^{-\alpha} = |r-|z|+h|^{-\alpha} - |r-|z||^{-\alpha} \lesssim ||z|-r-h|^{-\alpha-\eta} |h|^\eta, & \text{on } A_2\,.
\end{cases} \tag{3.4.14}
$$

Then we have

$$
|\Delta_h \mathcal{S}_{1-\alpha}(r,z)|^q \quad \le \quad \left|\Delta_h \,|r+|z||^{-\alpha}\right|^q + \left|\Delta_h |r-|z||^{-\alpha} \cdot [\mathbf{1}_{A_1} + \mathbf{1}_{A_2}]\right|^q
$$

$$
\begin{aligned}
&+\Big||r+h-|z||^{-\alpha}+|r-|z||^{-\alpha}\Big|^q \mathbf{1}_{A_3}\\
\le\ & |r+|z||^{(-\alpha-\eta_1)q}h^{\eta_1 q}+|r-|z||^{(-\alpha-\eta_2)q}h^{\eta_2 q}\cdot \mathbf{1}_{A_1}\\
&+||z|-r-h|^{(-\alpha-\eta_3)q}h^{\eta_3 q}\cdot \mathbf{1}_{A_2}\\
&+\left||r+h-|z||^{-\alpha}+|r-|z||^{-\alpha}\right|^q \mathbf{1}_{A_3}\,,
\end{aligned}
$$

for some $\eta_1,\eta_2,\eta_3\in(0,1)$. Therefore,

$$
\begin{aligned}
\mathcal{I}_2^{(1)}(t,h)\lesssim & \int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}|r+|z||^{(-\alpha-\eta_1)q}h^{\eta_1 q}dzdr\\
&+\int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}|r-|z||^{(-\alpha-\eta_2)q}h^{\eta_2 q}\cdot \mathbf{1}_{A_1}dzdr\\
&+\int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}||z|-r-h|^{(-\alpha-\eta_3)q}h^{\eta_3 q}\cdot \mathbf{1}_{A_2}dzdr\\
&+\int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}\left||r+h-|z||^{-\alpha}+|r-|z||^{-\alpha}\right|^q\cdot \mathbf{1}_{A_3}dzdr\\
=:&\sum_{k=1}^4 \mathcal{I}_{2,k}^{(1)}(t,h)\,. \qquad (3.4.15)
\end{aligned}
$$

The procedures of dealing terms $\mathcal{I}_{2,k}^{(1)}(t,h)$, $k=1,2,3,4$ require standard but careful computations which are included in Appendix 3.8. By Lemma 3.8.1, for any $p>\frac{1}{H}$, $\gamma<H-\frac{1}{p}$, $\mathcal{I}_{2,k}^{(1)}(t,h)$ $(k=1,2,3,4)$ can be bounded by $h^{\gamma q}$ if α, θ satisfy (Π.1) and η_k, $k=1,2,3$ satisfy (3.8.1).

Case i=2. In this case, we have $\overline{\mathcal{K}}_2(r,z)=\mathcal{C}_{1-\alpha}(r,z)$ defined by (3.3.3). We want to show when $i=2$, i.e.

$$
\begin{cases}
\sup_t \mathcal{I}_2^{(2)}(t,h)\le C_{T,p,\theta,\alpha}|h|^{\gamma q}\,, \quad \text{where}\\
\mathcal{I}_2^{(2)}(t,h)=\displaystyle\int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}|\Delta_h \mathcal{C}_{1-\alpha}(r,z)|^q dzdr
\end{cases}
\qquad (3.4.16)
$$

with parameters p, γ and α satisfying (3.4.4).

For fixed $\eta\in(0,1)$, it is not hard to verify

$$
\begin{aligned}
\left|\Delta_h\left(r^2+|z|^2\right)^{-\frac{\alpha}{2}}\right| &\lesssim \left(r^2+|z|^2\right)^{-\frac{\alpha}{2}(1-\eta)}\left|\frac{(r+\xi h)\cdot h}{[(r+\xi h)^2+|z|^2]^{\alpha/2+1}}\right|^{\eta}\\
&\lesssim \left(r^2+|z|^2\right)^{-\frac{\alpha}{2}(1-\eta)}\left|r^2+|z|^2\right|^{-(\frac{\alpha}{2}+1)\eta}(r+h)^{\eta}|h|^{\eta}
\end{aligned}
$$

$$\lesssim \left|r^2+|z|^2\right|^{-\frac{\alpha}{2}-\eta}\cdot(r+h)^{\eta}|h|^{\eta}\,, \tag{3.4.17}$$

and

$$\left|\Delta_h \cos\left(\alpha\tan^{-1}\left(\frac{|z|}{r}\right)\right)\right|\lesssim\frac{|z|^{\eta}|h|^{\eta}}{(r^2+z^2)^{\eta}}. \tag{3.4.18}$$

Then by the above two inequalities (3.4.17) and (3.4.18), and the inequalities in (3.4.14), we have

$$\begin{aligned}\left|\Delta_h\mathcal{C}_{1-\alpha}(r,z)\right|^q \lesssim & |r+|z||^{(-\alpha-\eta_1)q}h^{\eta_1 q}+|r-|z||^{(-\alpha-\eta_2)q}h^{\eta_2 q}\cdot\mathbf{1}_{A_1}\\ &+||z|-r-h|^{(-\alpha-\eta_3)q}h^{\eta_3 q}\cdot\mathbf{1}_{A_2}\\ &+\left||r+h-|z||^{-\alpha}+|r-|z||^{-\alpha}\right|^q\mathbf{1}_{A_3}\\ &+\left|r^2+|z|^2\right|^{(-\frac{\alpha}{2}-\eta_4)q}\cdot(r+h)^{\eta_4 q}|h|^{\eta_4 q}+\frac{|z|^{\eta_5 q}|h|^{\eta_5 q}}{(r^2+z^2)^{\eta_5 q}}\\ =:&\sum_{k=1}^{6}M_k^{(2)}(r,z)\,. \end{aligned} \tag{3.4.19}$$

Substituting this bound into (3.4.16), we see that,

$$\sup_t\mathcal{I}_2^{(2)}(t,h)\le\sup_t\sum_{k=1}^{6}\mathcal{I}_{2,k}^{(2)}(t,h)\,,$$

where

$$\mathcal{I}_{2,k}^{(2)}(t,h)=\int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}M_k^{(2)}(r,z)dzdr,\ \ k=1,\cdots,6. \tag{3.4.20}$$

The first four terms $\mathcal{I}_{2,k}^{(2)}(t,h)$, $k=1,\cdots,4$ are treated in the same way as **Case i=1** and require conditions (Π.1) and (3.8.1) to guarantee

$$\sup_t\mathcal{I}_{2,k}^{(2)}(t,h)\lesssim|h|^{\gamma q},\quad k=1,\cdots,4.$$

We shall deal with the $\mathcal{I}_{2,5}^{(2)}(t,h)$ and $\mathcal{I}_{2,6}^{(2)}(t,h)$ in Appendix 3.8. By Lemma 3.8.3, $\sup_t\mathcal{I}_{2,k}^{(2)}(t,h)\lesssim|h|^{\gamma q}$ for $k=5,6$ under conditions (Π.1) and (3.8.6).

As a result, for any $p>\frac{1}{H}$, $\gamma<H-\frac{1}{p}$, we have $\sup_t\mathcal{I}_{2,k}^{(2)}(t,h)\lesssim|h|^{\gamma q}$ for $k=1,\cdots,6$, if α, θ satisfy (Π.1) and η_k $(k=1,\cdots,6)$ satisfy (3.8.1) and (3.8.6).

Case i=3. In this case we have $\overline{\mathcal{K}}_3(r,z) = \mathcal{E}(r,z) = \frac{1}{\pi}\frac{r}{r^2+z^2}$ and

$$
\begin{aligned}
|\Delta_h \mathcal{E}(r,z)|^q &\simeq \left| \frac{r+h}{(r+h)^2+z^2} - \frac{r}{r^2+z^2} \right|^q \\
&= \left| \frac{h}{(r+h)^2+z^2} + r\left[\frac{1}{(r+h)^2+z^2} - \frac{1}{r^2+z^2} \right] \right|^q \\
&\leq \left| \frac{h}{(r+h)^2+z^2} \right|^q + \frac{r^q \cdot |h|^q \cdot |2r+h|^q}{|(r+h)^2+z^2|^q \cdot |r^2+z^2|^q}. \qquad (3.4.21)
\end{aligned}
$$

By Hölder's inequality with $\frac{1}{m}+\frac{1}{n}=1$ and $|2r+h|^q \leq 2^q|r+h|^q$, we obtain

$$
\begin{aligned}
\mathcal{I}_2^{(3)}(t,h) &= \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \left|\Delta_h \mathcal{E}(r,z)\right|^q dzdr \\
&\lesssim \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \left| \frac{h}{(r+h)^2+z^2} \right|^q dzdr \\
&\qquad + |h|^q \cdot \int_0^t \int_{\mathbb{R}} \frac{|r+h|^{q\theta}}{|(r+h)^2+z^2|^q} \cdot \frac{|r|^q}{|r^2+z^2|^q} dzdr \\
&\lesssim \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \frac{|h|^q}{|(r+h)^2+z^2|^q} dzdr \\
&\qquad + |h|^q \left[\int_0^t \int_{\mathbb{R}} \left(\frac{|r+h|^{q\theta}}{|(r+h)^2+z^2|^q} \right)^m dzdr \right]^{\frac{1}{m}} \left[\int_0^t \int_{\mathbb{R}} \left(\frac{|r|^q}{|r^2+z^2|^q} \right)^n dzdr \right]^{\frac{1}{n}} \\
&=: \mathcal{I}_{2,1}^{(3)}(t,h) + |h|^q \left[\mathcal{I}_{2,2}^{(3)}(t,h) \right]^{1/m} \left[\mathcal{I}_{2,3}^{(3)}(t,h) \right]^{1/n}. \qquad (3.4.22)
\end{aligned}
$$

By the change of variable $z \to (r+h)z$ in $\mathcal{I}_{2,1}^{(3)}(t,h)$ and $\mathcal{I}_{2,2}^{(3)}(t,h)$, and by the change of variable $z \to rz$ in $\mathcal{I}_{2,3}^{(3)}(t,h)$, we have

$$
\begin{aligned}
\mathcal{I}_2^{(3)} &\lesssim |h|^q \cdot \int_0^t \int_{\mathbb{R}} |r+h|^{q(\theta-1)+1-2q} \frac{1}{(1+z^2)^q} dzdr \\
&\qquad + |h|^q \left[\int_0^t \int_{\mathbb{R}} \left(\frac{|r+h|^{1+mq\theta-2qm}}{|1+z^2|^{mq}} \right) dzdr \right]^{\frac{1}{m}} \left[\int_0^t \int_{\mathbb{R}} \left(\frac{|r|^{1-nq}}{|1+z^2|^{nq}} \right) dzdr \right]^{\frac{1}{n}} \\
&\lesssim |h|^q \int_0^t |r+h|^{q\theta+1-3q} dr + |h|^q \left[\int_0^t |r+h|^{1+mq\theta-2qm} dr \right]^{\frac{1}{m}} \left[\int_0^t |r|^{1-nq} dr \right]^{\frac{1}{n}} \\
&\lesssim |h|^{q(\theta-1)+2-q} + |h|^{q(\theta-1)+2/m} = |h|^{q(\theta-1)+2-q} + |h|^{q(\theta-1)+2-2/n} \lesssim |h|^{q\gamma},
\end{aligned}
$$

under condition

$$
\frac{2}{n} > q, \;\; \theta - \gamma > 2 - \frac{2}{q}. \qquad (3.4.23)
$$

Then $\mathcal{I}_2^{(3)}(t,h) \leq C_{T,p,H,\gamma}|h|^{\gamma q}$ under (3.4.23).

Case i=4. In this case we use $\overline{\mathcal{K}}_4(r,z)=\mathcal{S}(r,z)=\frac{1}{2}\mathbf{1}_{\{|z|<r\}}$. Since $(r+h)^{q(\theta-1)}\le r^{q(\theta-1)}$, we see

$$
\begin{aligned}
\mathcal{I}_2^{(4)}(t,h) &\simeq \int_0^t\int_{\mathbb{R}}(r+h)^{q(\theta-1)}\left|\mathbf{1}_{\{z|<r+h\}}-\mathbf{1}_{\{|z|<r+h\}}\right|^q dzdr\\
&\lesssim \int_0^t\int_{-(r+h)}^{-r}(r+h)^{q(\theta-1)}dzdr+\int_0^t\int_r^{r+h}(r+h)^{q(\theta-1)}dzdr\\
&=\int_0^t 2h(r+h)^{q(\theta-1)}dr\le h\int_0^t 2r^{q(\theta-1)}dr\lesssim |h|^{\gamma q},
\end{aligned}
$$

where the last inequality requires

$$
q(\theta-1)>-1,\ \gamma<\frac{1}{q}. \tag{3.4.24}
$$

Then under (3.4.24), we have

$$
\sup_{t,x}\mathcal{I}_2^{(4)}(t,h)\lesssim|h|^{\gamma q}.
$$

To conclude, with the choice of $1-H<\alpha<1-\frac{1}{p}\,,\ p>\frac{1}{H}\,,\ 0<\gamma<H-\frac{1}{p}$, we see that the condition (3.3.18) to guarantee

$$
\left(\int_0^T \|\ J_\theta^{\mathcal{K}_i}(r,z)\ \|_{L^p(\mathbb{R})}^p\ dr\right)^{1/p}\lesssim\|v\|_{\mathcal{Z}^p(T)}\ (i=1,2,3,4)
$$

and the conditions listed in **Case i=1,2,3,4** to guarantee (3.4.11) are all satisfied, so we have

$$
\|\sup_{t,x}\mathcal{I}_2(t,h,x)\|_{L^p(\Omega)}\le C_{T,p,H,\gamma}|h|^\gamma\|v\|_{\mathcal{Z}^p(T)}\,.
$$

This finishes the proof of **(i)**.

Step 2: In this step, we deal with

$$
\sup_{t\in[0,T],x,y\in\mathbb{R}}|\Phi(t,x)-\Phi(t,y)|\ .
$$

By (3.3.19) in the proof of part **(ii)** of Proposition 3.3.3, we have

$$
|\Phi(t,x)-\Phi(t,y)|=\left|\sum_{i=1}^4\frac{\sin(\theta\pi)}{\pi}\int_0^t\int_{\mathbb{R}}(t-r)^{\theta-1}[\bar{\mathcal{K}}_i(t-r,x-z)\right.
$$

$$
\left. - \bar{\mathcal{K}}_i(t-r,y-z)] J_\theta^{\mathcal{K}_i}(r,z) dzdr \right|
$$

$$
\lesssim \sum_{i=1}^{4} \left(\int_0^t \int_{\mathbb{R}} (t-r)^{q(\theta-1)} \left| \mathfrak{D}_\hbar \bar{\mathcal{K}}_i(t-r,z) \right|^q dzdr \right)^{1/q}
$$

$$
\times \left(\int_0^T \| J_\theta^{\mathcal{K}_i}(r,\cdot) \|_{L^p(\mathbb{R})}^p dr \right)^{1/p}, \tag{3.4.25}
$$

where $\hbar := |x-y|$ and $\mathfrak{D}_\hbar \bar{\mathcal{K}}_i(t-r,z) := \bar{\mathcal{K}}_i(t-r,z+\hbar) - \bar{\mathcal{K}}_i(t-r,z)$. Without loss of generality, we can suppose that $x > y$ and $\hbar = |x-y| < 1$ is sufficiently small. The term $\left(\int_0^T \| J_\theta^{\mathcal{K}_i}(r,\cdot) \|_{L^p(\mathbb{R})}^p dr \right)^{1/p}$ in (3.4.25) can be estimated via Lemma 3.7.1 which requires (3.3.18). Thus, we need to show for $i = 1, \cdots, 4$

$$
\sup_{t,x,y} \mathcal{J}^{(i)}(t,x,y) \le C_{T,p,H,\gamma} |\hbar|^{\gamma q}, \tag{3.4.26}
$$

where $\sup_{t,x,y}$ is the is abbreviation for $\sup_{t\in[0,T],x,y\in\mathbb{R}}$ and

$$
\mathcal{J}^{(i)}(t,x,y) := \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \left| \mathfrak{D}_\hbar \bar{\mathcal{K}}_i(r,z) \right|^q dzdr. \tag{3.4.27}
$$

We are going to bound $\mathcal{J}^{(i)}$ for $i = 1,2,3,4$ separately.

Case i=1. In this case $\bar{\mathcal{K}}_1(r,z) = \mathcal{S}_{1-\alpha}(r,z)$ which is defined by (3.3.3). We shall show that

$$
\sup_{t,x,y} \mathcal{J}^{(1)}(t,x,y) = \sup_{t,x,y} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \left| \mathfrak{D}_\hbar \mathcal{S}_{1-\alpha}(r,z) \right|^q dzdr \le C_{T,p,H,\gamma} |\hbar|^{\gamma q}, \tag{3.4.28}
$$

with α, p and γ satisfying (3.4.4). We split $\mathcal{J}^{(1)}(t,x,y)$ into two parts:

$$
\begin{aligned}
\mathcal{J}^{(1)}(t,x,y) &= \int_\hbar^t \int_{\mathbb{R}} r^{q(\theta-1)} \left| \mathfrak{D}_\hbar \bar{\mathcal{K}}_1(r,z) \right|^q dzdr \\
&\quad + \int_0^\hbar \int_{\mathbb{R}} r^{q(\theta-1)} \left| \mathfrak{D}_\hbar \bar{\mathcal{K}}_1(r,z) \right|^q dzdr \\
&=: \mathcal{J}_1^{(1)}(t,x,y) + \mathcal{J}_2^{(1)}(t,x,y).
\end{aligned}
$$

Let us treat the term $\mathcal{J}_1^{(1)}(t,x,y)$ first. In this case, $-r+\hbar<r-\hbar$. Set

$$\begin{cases} B_1=[z<-\hbar-r]\,, \quad B_2=[z>r+\hbar]\,, \quad B_3=[-r+\hbar<z<r-\hbar]\,; \\ B_4=[-r-\hbar<z<-r+\hbar]\,, \quad B_5=[r-\hbar<z<r+\hbar]\,. \end{cases} \tag{3.4.29}$$

By the triangle inequality and the inequalities (3.4.14), we have

$$\begin{aligned} |\mathfrak{D}_\hbar \mathcal{S}_{1-\alpha}(r,z)|^q &\simeq \Big|(r+|z+\hbar|)^{-\alpha}+\operatorname{sgn}(r-|z+\hbar|)\big|r-|z+\hbar|\big|^{-\alpha} \\ &\qquad -(r+|z|)^{-\alpha}-\operatorname{sgn}(r-|z|)\big|r-|z|\big|^{-\alpha}\Big|^q \\ &\lesssim \big|\mathfrak{D}_\hbar(r+|z|)^{-\alpha}\big|^q+\big|\mathfrak{D}_\hbar(r-|z|)^{-\alpha}\big|^q\cdot(\mathbf{1}_{B_1}+\mathbf{1}_{B_2}+\mathbf{1}_{B_3}) \\ &\quad +\big|(r-|z+\hbar|)^{-\alpha}+(r-|z|)^{-\alpha}\big|^q\cdot(\mathbf{1}_{B_4}+\mathbf{1}_{B_5}) \\ &\lesssim \big|r+|z|\big|^{-(\alpha+\eta_1)q} \\ &\quad +\big|r-|z|\big|^{-(\alpha+\eta_2)q}\cdot(\mathbf{1}_{B_1}+\mathbf{1}_{B_2}) \\ &\quad +\big|r-\hbar-|z|\big|^{-(\alpha+\eta_3)q}\cdot\mathbf{1}_{B_3} \\ &\quad +\big|(r-|z+\hbar|)^{-\alpha}+(r-|z|)^{-\alpha}\big|^q\cdot(\mathbf{1}_{B_4}+\mathbf{1}_{B_5}) \\ &=: \sum_{k=1}^{3} N_{1,k}^{(1)}(t,x,y)\,, \end{aligned} \tag{3.4.30}$$

Then

$$\mathcal{J}_1^{(1)}(t,x,y)\lesssim\sum_{k=1}^{3}\mathcal{J}_{1,k}^{(1)}(t,x,y):=\sum_{k=1}^{3}\int_\hbar^t\int_{\mathbb{R}} r^{q(\theta-1)}N_{1,k}^{(1)}(t,x,y)dzdr. \tag{3.4.31}$$

By Lemma 3.8.4, $\sup_{t,x,y}\mathcal{J}_{1,k}^{(1)}(t,x,y)\lesssim|\hbar|^{\gamma q}$ for $k=1,2,3$ if we require (II.1) and (3.8.9).

Next, we shall deal with $\mathcal{J}_2^{(1)}(t,x,y)$. In this case, $-r+\hbar\geq r-\hbar$. Setting

$$C_1=[z<-r-\hbar],\ C_2=[z>r+\hbar],\ C_3=[-r-\hbar<z<r+\hbar]\,, \tag{3.4.32}$$

then by the inequalities (3.4.14),

$$|\mathfrak{D}_\hbar \mathcal{S}_{1-\alpha}(r,z)|^q \simeq \Big|(r+|z+\hbar|)^{-\alpha}+\operatorname{sgn}(r-|z+\hbar|)\big|r-|z+\hbar|\big|^{-\alpha}$$

$$
\begin{aligned}
&\qquad - (r+|z|)^{-\alpha} - \text{sgn}(r-|z|)\big|r-|z|\big|^{-\alpha}\Bigg|^q \\
&\lesssim \big|\mathfrak{D}_\hbar(r+|z|)^{-\alpha}\big|^q + \big|\mathfrak{D}_\hbar(r-|z|)^{-\alpha}\big|^q \cdot (\mathbf{1}_{C_1} + \mathbf{1}_{C_2}) \\
&\qquad + \big|(r-|z+\hbar|)^{-\alpha} + (r-|z|)^{-\alpha}\big|^q \cdot \mathbf{1}_{C_3} \\
&\lesssim |r+|z||^{-(\alpha-\eta_1)q} + |r-|z||^{-(\alpha-\eta_4)q} \cdot (\mathbf{1}_{C_1} + \mathbf{1}_{C_2}) \\
&\qquad + \big|(r-|z+\hbar|)^{-\alpha} + (r-|z|)^{-\alpha}\big|^q \cdot \mathbf{1}_{C_3} \\
&=: \sum_{k=1}^{3} N_{2,k}^{(1)}(t,x,y)\,. \qquad (3.4.33)
\end{aligned}
$$

Thus

$$
\mathcal{J}_2^{(1)}(t,x,y) \lesssim \sum_{k=1}^{3} \mathcal{J}_{2,k}^{(1)}(t,x,y) := \sum_{k=1}^{3} \int_0^\hbar \int_{\mathbb{R}} r^{q(\theta-1)} N_{2,k}^{(1)}(t,x,y)dzdr. \qquad (3.4.34)
$$

By Lemma 3.8.5, $\sup_{t,x,y} \mathcal{J}_{2,k}^{(1)}(t,x,y) \lesssim |\hbar|^{\gamma q}$ for $k = 1,2,3$ under conditions (Π.1) and (3.8.14).

As a result, for any $p > \frac{1}{H}$, $\gamma < H - \frac{1}{p}$, we know that (3.4.28) holds if α, θ satisfy (Π.1) and $\eta_k, k = 1, \cdots, 4$ satisfy (3.8.9) and (3.8.14).

Case i=2. We consider $\bar{\mathcal{K}}_2(r,z) = \mathcal{C}_{1-\alpha}(r,z)$ defined by (3.3.3). We want to obtain

$$
\sup_{t,x,y} \mathcal{J}^{(2)}(t,x,y) = \sup_{t,x,y} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \left|\mathfrak{D}_\hbar \mathcal{C}_{1-\alpha}(r,z)\right|^q dzdr \leq C_{T,p,H,\gamma} |\hbar|^{\gamma q}\,, \qquad (3.4.35)
$$

with parameters $p,\ \alpha,\ \gamma$ satisfy (3.4.4). By the triangle inequality,

$$
\begin{aligned}
|\mathfrak{D}_\hbar \mathcal{C}_{1-\alpha}(r,z)|^q \lesssim& |\mathfrak{D}_\hbar |r+|z||^{-\alpha}|^q + |\mathfrak{D}_\hbar |r-|z||^{-\alpha}|^q + \big|\mathfrak{D}_\hbar (r^2+z^2)^{-\frac{\alpha}{2}}\big|^q \\
&+ \left[2\ \mathfrak{D}_\hbar \cos\left(\alpha \tan^{-1}\left(\frac{|z|}{r}\right)\right)\right]^q \left(r^2+z^2\right)^{-\frac{\alpha}{2}q} \\
=:& \sum_{k=1}^{4} N_k^{(2)}(r,z)\,. \qquad (3.4.36)
\end{aligned}
$$

Substituting (3.4.36) into (3.4.35), we have

$$
\sup_{t,x,y} \mathcal{J}^{(2)}(t,x,y) \leq \sup_{t,x,y} \sum_{k=1}^{4} \mathcal{J}_k^{(2)}(t,x,y)\,,
$$

where

$$\mathcal{J}_k^{(2)}(t,x,y) = \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} N_k^{(2)}(r,z)dzdr, \;\; k=1,\cdots,4.$$

For the term $\mathcal{J}_1^{(2)}(t,x,y)$, since for fixed $\eta_1 \in (0,1)$,

$$|\mathfrak{D}_\hbar|r+|z||^{-\alpha}|^q \leq |r+|z||^{-(\alpha+\eta_1)q}|\hbar|^{\eta_1 q},$$

similar to the estimation of $\mathcal{I}_{2,1}^{(1)}(t,h)$ in (3.4.15), we have $\sup\limits_{t,x,y} \mathcal{J}_1^{(2)}(t,x,y) \lesssim |\hbar|^{\gamma q}$ under the condition (3.8.2).

It is more complicated to deal with the term $\mathcal{J}_2^{(2)}(t,x,y)$ since $|\mathfrak{D}_\hbar|r-|z||^{-\alpha}|^q$ has different upper bounds on different domains of $|z|$. Similar to **Case i=1**, we split $\mathcal{J}^{(1)}(t,x,y)$ into two parts

$$\begin{aligned} \mathcal{J}_2^{(2)}(t,x,y) =& \int_{\frac{\hbar}{2}}^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathfrak{D}_\hbar|r-|z||^{-\alpha}|^q dzdr \\ &+ \int_0^{\frac{\hbar}{2}} \int_{\mathbb{R}} r^{q(\theta-1)} |\mathfrak{D}_\hbar|r-|z||^{-\alpha}|^q dzdr \\ =:& \mathcal{J}_{2,1}^{(2)}(t,x,y) + \mathcal{J}_{2,2}^{(2)}(t,x,y) \,. \end{aligned} \tag{3.4.37}$$

We first deal with $\mathcal{J}_{2,1}^{(2)}(t,x,y)$ when $r > \frac{\hbar}{2}$, namely $-r < r-\hbar$. Let us set

$$\begin{cases} D_1 = [z < -r-\hbar], \quad D_2 = [-r-\hbar < z < -r], \; D_3 = [-r < z < r-\hbar] \,, \\ D_4 = [r-\hbar < z < r], \; D_5 = [r < z < r+\hbar], \; D_6 = [r > z+\hbar] \,. \end{cases} \tag{3.4.38}$$

The first integral of (3.4.37) can be bounded by

$$\mathcal{J}_{2,1}^{(2)}(t,x,y) \lesssim \sum_{j=1}^6 \int_0^t \int_{D_j} r^{q(\theta-1)} |\mathfrak{D}_\hbar|r-|z||^{-\alpha}|^q dzdr =: \sum_{j=1}^6 \mathcal{J}_{2,1,j}^{(2)}(t,x,y) \,. \tag{3.4.39}$$

It is not hard to derive that for some $\eta \in (0,1)$

$$\left|\mathfrak{D}_\hbar |r-|z||^{-\alpha}\right| \lesssim \begin{cases} |r-|z||^{-\alpha-\eta}\hbar^\eta\,, & \text{on } D_1\cup D_5\cup D_6\,; \\ |r-|z+\hbar||^{-\alpha-\eta}\hbar^\eta\,, & \text{on } D_3\,; \\ |r-|z+\hbar||^{-\alpha}+|r-|z||^{-\alpha}\,, & \text{on } D_2\cup D_4\,. \end{cases}$$

Substituting this into (3.4.39) we obtain

$$\begin{aligned} \sum_{j=1}^{6}\mathcal{J}^{(2)}_{2,1,j}(t,x,y) \le & \int_0^t\int_{D_1\cup D_5} r^{q(\theta-1)}|r-|z||^{-(\alpha+\eta_2)q}\hbar^{\eta_2 q}dzdr \\ & + \int_0^t\int_{D_6} r^{q(\theta-1)}|r-|z||^{-(\alpha+\eta_3)q}\hbar^{\eta_3 q}dzdr \\ & + \int_0^t\int_{D_3} r^{q(\theta-1)}|r-|z+\hbar||^{-(\alpha+\eta_4)q}\hbar^{\eta_4 q}dzdr \\ & + \int_0^t\int_{D_2\cup D_4} r^{q(\theta-1)}\left(|r-|z+\hbar||^{-\alpha q}+|r-|z||^{-\alpha q}\right)dzdr\,. \end{aligned}$$

By Lemma 3.8.6 in Appendix 3.8, we have

$$\sup_{t,x,y}\mathcal{J}^{(2)}_{2,1,j}(t,x,y)\lesssim|\hbar|^{\gamma q},\ \ j=1,\cdots,6,$$

under conditions (Π.1) and (3.8.17).

In similar way we can obtain the same bound for $\mathcal{J}^{(2)}_{2,2}(t,x,y)$ by dividing the domain of $|z|$ into subdomains and estimating each terms. We omit the details here.

Now we turn to the third and last terms $\mathcal{J}^{(2)}_3(t,x,y)$ and $\mathcal{J}^{(2)}_4(t,x,y)$. Analogously to the obtention of (3.4.17) and (3.4.18), it is not hard to obtain for fixed $\eta\in(0,1)$,

$$\left|\mathfrak{D}_\hbar(r^2+z^2)^{-\frac{\alpha}{2}}\right| \le (r^2+z^2)^{-(\frac{\alpha}{2}+\eta)}|z+\hbar|^\eta|\hbar|^\eta, \tag{3.4.40}$$

and

$$\left|\mathfrak{D}_\hbar\cos\left(\alpha\tan^{-1}\left(\frac{|z|}{r}\right)\right)\right| \le \frac{r^\eta|\hbar|^\eta}{(r^2+z^2)^\eta}. \tag{3.4.41}$$

Then we have

$$
\begin{aligned}
&\mathcal{J}_3^{(2)}(t,x,y)+\mathcal{J}_4^{(2)}(t,x,y)\\
&\qquad\lesssim |\hbar|^{\eta_4 q}\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}(r^2+z^2)^{-(\frac{\alpha}{2}+\eta_4)q}|z+\hbar|^{\eta_4 q}dzdr\\
&\qquad\quad+|\hbar|^{\eta_5 q}\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}\frac{r^{\eta_5 q}}{(r^2+z^2)^{\eta_5 q}}(r^2+z^2)^{-\frac{\alpha}{2}q}dzdr. \qquad (3.4.42)
\end{aligned}
$$

By Lemma 3.8.7, $\sup\limits_{t,x,y}\mathcal{J}_3^{(2)}(t,x,y)$ and $\sup\limits_{t,x,y}\mathcal{J}_4^{(2)}(t,x,y)$ can be bounded by a multiple of $|\hbar|^{\gamma q}$ under conditions (Π.1) and (3.8.24).

As a result, for any $p>\frac{1}{H}$, $\gamma<H-\frac{1}{p}$, $\sup\limits_{t,x,y}\mathcal{J}^{(2)}(t,x,y)\lesssim|\hbar|^{\gamma q}$ if α, θ satisfy (Π.1), η_k ($k=1,\cdots,5$) satisfy (3.8.17) and (3.8.24).

Case i=3. In this case $\bar{\mathcal{K}}_3(r,z)=\mathcal{E}(r,z)=\frac{1}{\pi}\frac{r}{r^2+z^2}$. Then

$$
\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}\left|\mathfrak{D}_\hbar\bar{\mathcal{K}}_3(r,z)\right|^q dzdr=\int_0^t\int_{\mathbb{R}} r^{q\theta}\left|\frac{1}{r^2+(z+\hbar)^2}-\frac{1}{r^2+z^2}\right|^q dzdr. \qquad (3.4.43)
$$

The $\hbar=|x-y|$ in (3.4.43) plays the same role as h in the second term of (3.4.21). So using the similar method as in dealing with $\left|\Delta_h\left(\frac{1}{r^2+z^2}\right)\right|^q$ in **Case i=3** of **Step 1**, we have

$$
\int_0^t\int_{\mathbb{R}} r^{q\theta}\left|\mathfrak{D}_\hbar\left(\frac{1}{r^2+z^2}\right)\right|^q dzdr\lesssim\hbar^{\gamma q},
$$

if $\theta-\gamma>2-\frac{2}{q}$. Thus, under (3.4.4) we have

$$
\left(\mathcal{J}^{(3)}(t,x,y)\right)^{1/q}\times\left(\int_0^T\|J_\theta^{\mathcal{K}_3}(r,\cdot)\|_{L^p(\mathbb{R})}dr\right)^{1/p}\lesssim C_{T,p,H,\gamma}|x-y|^\gamma\|v\|_{\mathcal{Z}^p(T)}.
$$

Case i=4. In this case $\bar{\mathcal{K}}_4(r,z)=\mathcal{S}(r,z)=\frac{1}{2}\mathbf{1}_{\{|z|<r\}}$. Then

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}\left|\mathfrak{D}_\hbar\bar{\mathcal{K}}_4(r,z)\right|^q dzdr\\
&\quad=\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}\left|\frac{1}{2}\mathbf{1}_{\{|z+x-y|<r\}}-\frac{1}{2}\mathbf{1}_{\{|z|<r\}}\right|^q dzdr\\
&\quad\simeq\int_0^t r^{q(\theta-1)}\left(\int_{y-x-r}^{-r}dz+\int_{y-x+r}^{r}dz\right)dr\simeq\hbar\cdot\int_0^t r^{q(\theta-1)}\lesssim\hbar^{\gamma q},
\end{aligned}
$$

under the conditions $q(\theta-1)>-1$ and $\gamma<\frac{1}{q}$. Therefore, under (3.4.4) we have

$$\big(\mathcal{J}^{(4)}(t,x,y)\big)^{1/q}\times\left(\int_0^T\|J_\theta^{\mathcal{K}_4}(r,\cdot)\|_{L^p(\mathbb{R})}dr\right)^{1/p}\lesssim C_{T,p,H,\gamma}|x-y|^\gamma\|v\|_{\mathcal{Z}^p(T)}.$$

In conclusion, with the choice of $p>\frac{1}{H}, 1-H<\alpha<1-\frac{1}{p},\ 0<\gamma<H-\frac{1}{p}$, the conditions listed in **Case i=1,2,3,4** to ensure

$$\sup_{t,x,y}\mathcal{J}^{(i)}(t,x,y)\lesssim|\hbar|^{\gamma q},$$

and the condition (3.3.18) to ensure

$$\left(\int_0^T\|J_\theta^{\mathcal{K}_i}(r,\cdot)\|_{L^p(\mathbb{R})}dr\right)^{1/p}\lesssim\|v\|_{\mathcal{Z}^p(T)},$$

are all satisfied. Thus, we have

$$\|\sup_{t\in[0,T],x,y\in\mathbb{R}}|\Phi(t,x)-\Phi(t,y)|\,\|_{L^p(\Omega)}\lesssim C_{T,p,H,\gamma}|x-y|^\gamma\|v\|_{\mathcal{Z}^p(T)}\,.$$

This completes the proof of **(ii)**. □

3.4.2 Hölder continuity of the approximate solutions and well-ponsedness

Analogous to Proposition 3.4.1 we have the following regularity results for the approximated solution u_ε defined in (3.3.23). The proof is similar and we omit it.

Lemma 3.4.2. *Let u_ε be the approximation mild solution defined by (3.3.23) and assume that $I_0(t,x)$ belongs to $\mathcal{Z}^p(T)$.*

(i) *If $p>\frac{2}{4H-1}$, then*

$$\left\|\sup_{t\in[0,T],x\in\mathbb{R}}|\mathcal{N}_{\frac{1}{2}-H}u_\varepsilon(t,x)|\right\|_{L^p(\Omega)}\leq C_{T,p,H}\|u_\varepsilon\|_{\mathcal{Z}^p(T)}\,. \tag{3.4.44}$$

(ii) *If* $p > \frac{1}{H}$ *and* $0 < \gamma < H - \frac{1}{p}$*, then*

$$\left\| \sup_{t,t+h\in[0,T],x\in\mathbb{R}} |u_\varepsilon(t+h,x) - u_\varepsilon(t,x)| \right\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |h|^\gamma \|u_\varepsilon\|_{\mathcal{Z}^p(T)} . \tag{3.4.45}$$

(iii) *If* $p > \frac{1}{H}$ *and* $0 < \gamma < H - \frac{1}{p}$*, then*

$$\left\| \sup_{t\in[0,T],x,y\in\mathbb{R}} |u_\varepsilon(t,x) - u_\varepsilon(t,y)| \right\|_{L^p(\Omega)} \le C_{T,p,H,\gamma} |x-y|^\gamma \|u_\varepsilon\|_{\mathcal{Z}^p(T)} . \tag{3.4.46}$$

Finally, we are in position to prove our main results.

Proof of Theorem 3.2.5 and Theorem 3.2.6. As we know the uniformly Hölder continuity of the type specified in Lemma 3.4.2 is the most important ingredient in the proof ([HW22, Theorem 1.5]) of the existence of weak solution to the nonlinear stochastic heat equation. It is also the most important one to show the existence of weak solution for nonlinear stochastic wave equation (3.1.1). Hence we omit the details of the proof of Theorem 3.2.5. Since the pathwise uniqueness implies the existence of strong solution by the Yamada-Watanabe theorem (in the SPDEs setting, e.g. [KS88, Kur07]), we only need to focus on the proof of pathwise uniqueness. We follow the same strategy in [HHL+17, HW22] together with the crucial estimate (3.3.6) in Proposition 3.3.3.

Suppose $u(t,x)$ and $v(t,x)$ are two solution to (3.1.1). Define the following stopping times:

$$\begin{aligned}\mathfrak{T}_k := \inf \Big\{ t \in [0,T] : &\sup_{0\le s\le t, x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u(s,x) \ge k, \\ &\text{or} \sup_{0\le s\le t, x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} v(s,x) \ge k \Big\}, \quad k = 1, 2, \cdots\end{aligned}$$

Recall that the inequality (3.3.6) in Proposition 3.3.3 implies that $\mathfrak{T}_k \uparrow T$ almost surely as $k \to \infty$. This is a key fact to our method. We need to find appropriate bounds for the following two quantities:

$$\mathfrak{J}_1(t) = \sup_{x\in\mathbb{R}} \mathbb{E}\left[1_{\{t<\mathfrak{T}_k\}} |u(t,x) - v(t,x)|^2 \right]$$

and

$$\mathfrak{J}_2(t) = \sup_{x\in\mathbb{R}} \mathbb{E}\left[\int_{\mathbb{R}} 1_{\{t<\mathfrak{T}_k\}}|u(t,x)-v(t,x)-u(t,x+h)+v(t,x+h)|^2|h|^{2H-2}dh\right].$$

By the elementary properties of Itô's integral, we have

$$\begin{aligned}
&1_{\{t<\mathfrak{T}_k\}}[u(t,x)-v(t,x)] \\
&\quad = 1_{\{t<\mathfrak{T}_k\}}\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)1_{\{s<\mathfrak{T}_k\}}[\sigma(s,y,u(s,y))-\sigma(s,y,v(s,y))]W(ds,dy)\,.
\end{aligned}$$

Therefore, denoting $\triangle(t,x,y) := \sigma(t,x,u(t,y)) - \sigma(t,x,v(t,y))$ we have

$$\begin{aligned}
&\mathbb{E}\big[1_{\{t<\mathfrak{T}_k\}}|u(t,x)-v(t,x)|^2\big] \\
&\lesssim \mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2} 1_{\{s<T_k\}}|\mathfrak{D}_h G_{t-s}(x-y)|^2[\triangle(s,y,y)]^2|h|^{2H-2}dhdyds\bigg) \\
&+ \mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2} 1_{\{s<T_k\}}G^2_{t-s}(x-y-h)[\triangle(s,y+h,y)-\triangle(s,y,y)]^2|h|^{2H-2}dhdyds\bigg) \\
&+ \mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2} 1_{\{s<T_k\}}G^2_{t-s}(x-y)[\triangle(s,y,y+h)-\triangle(s,y,y)]^2|h|^{2H-2}dhdyds\bigg) \\
&=: I_{1,1}+I_{1,2}+I_{1,3}\,.
\end{aligned} \tag{3.4.47}$$

The assumption (3.2.10) on σ can be used to estimate $I_{1,1}$. This is,

$$\begin{aligned}
I_{1,1} &\lesssim \mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2} 1_{\{s<T_k\}}|\mathfrak{D}_h G_{t-s}(x-y)|^2|u(s,y)-v(s,y)|^2|h|^{2H-2}dhdyds\bigg) \\
&\lesssim \int_0^t (t-s)^{2H}\sup_{y\in\mathbb{R}}\mathbb{E}\left[1_{\{s<T_k\}}|u(s,y)-v(s,y)|^2\right]ds \\
&= \int_0^t (t-s)^{2H}\mathfrak{J}_1(s)ds\,.
\end{aligned}$$

Using the property (3.2.12) of σ, we have if $|h|>1$

$$\begin{aligned}
[\triangle(s,y+h,y)-\triangle(s,y,y)]^2 &= \left|\int_u^v\left[\frac{\partial}{\partial\xi}\sigma(s,y+h,\xi)-\frac{\partial}{\partial\xi}\sigma(s,y,\xi)\right]d\xi\right|^2 \\
&\lesssim |u(s,y)-v(s,y)|^2\,.
\end{aligned}$$

If $|h| \le 1$, with the help of additional property (3.2.13) we get

$$
\begin{aligned}
[\triangle(s, y+h, y) - & \triangle(s, y, y)]^2 \\
&= \left| \int_u^v \left[\frac{\partial}{\partial \xi} \sigma(s, y+h, \xi) - \frac{\partial}{\partial \xi} \sigma(s, y, \xi) \right] d\xi \right|^2 \\
&= \left| \int_u^v \int_0^h \frac{\partial^2}{\partial \eta \partial \xi} \sigma(s, y+\eta, \xi) d\eta d\xi \right|^2 \\
&\lesssim |h|^2 |u(s,y) - v(s,y)|^2 .
\end{aligned}
$$

Thus, the term $I_{1,2}$ in (3.4.47) is bounded by

$$
I_{1,2} \lesssim \int_0^t \mathfrak{J}_1(s) \left(\int_{\mathbb{R}} G_{t-s}^2 (x-y) dy \right) ds \lesssim \int_0^t (t-s) \mathfrak{J}_1(s) ds .
$$

For the last term $I_{1,3}$ in (3.4.47), by (3.2.12) and (3.2.14) we have

$$
\begin{aligned}
& \big| \triangle(s, y, y+h) - \triangle(s, y, y) \big|^2 \\
=& \Big| \int_0^1 [u(s, y+h) - v(s, y+h)] \frac{\partial}{\partial \xi} \sigma(s, y, \theta u(s, y+h) + (1-\theta) v(s, y+h)) d\theta \\
& \quad - \int_0^1 [u(s,y) - v(s,y)] \frac{\partial}{\partial \xi} \sigma(s, y, \theta u(s,y) + (1-\theta) v(s,y)) d\theta \Big|^2 \\
\lesssim & |u(s, y+h) - v(s, y+h) - u(s,y) + v(s,y)|^2 \\
+ & |u(s,y) - v(s,y)|^2 \cdot \left[|u(s, y+h) - u(s,y)|^2 + |v(s, y+h) - v(s,y)|^2 \right] .
\end{aligned}
$$

Thus, we can get

$$
I_{1,3} \lesssim k \int_0^t \left[\mathfrak{J}_1(s) + \mathfrak{J}_2(s) \right] ds .
$$

Summarizing the above estimates we have

$$
\mathfrak{J}_1(t) \le k \cdot C_T \int_0^t \left[\mathfrak{J}_1(s) + \mathfrak{J}_2(s) \right] ds ,
$$

where $C_T > 0$ and the constant k depends on the stopping times $\mathfrak{T}_k$.

A similar procedure to the obtention of (3.3.34) can be applied to estimate term $\mathfrak{J}_2(t)$

to obtain

$$\mathfrak{J}_2(t) \lesssim k \int_0^t (t-s)^{4H-1} \big[\mathfrak{J}_1(s) + \mathfrak{J}_2(s)\big] ds \,.$$

As a consequence,

$$\mathfrak{J}_1(t) + \mathfrak{J}_2(t) \lesssim k \int_0^t (t-s)^{4H-1} \big[\mathfrak{J}_1(s) + \mathfrak{J}_2(s)\big] ds.$$

Now Gronwall's lemma implies $\mathfrak{J}_1(t) + \mathfrak{J}_2(t) = 0$ for all $t \in [0,T]$. This means we have

$$\mathbb{E}\big[1_{\{t<\mathfrak{T}_k\}} |u(t,x) - v(t,x)|^2\big] = 0 \,.$$

Thus, we have $u(t,x) = v(t,x)$ almost surely on the set $\{t < \mathfrak{T}_k\}$ for all $k \geq 1$, and the fact $\mathfrak{T}_k \uparrow T$ a.s as k tends to infinity necessarily indicate $u(t,x) = v(t,x)$ a.s. for every $(t,x) \in [0,T] \times \mathbb{R}$.

It is clear that hypothesis **(H2)** implies the hypothesis **(H1)**. So equation (3.1.1) has a weak solution by Theorem 3.2.5. This combined with the above pathwise uniqueness yields Theorem 3.2.6.

□

3.5 Necessity of $H > \frac{1}{4}$

In Theorem 3.2.5 and Theorem 3.2.6, we see that $H > \frac{1}{4}$ is a sufficient condition for the solvability of equation (3.1.1). In this section we shall prove that it is also necessary for some specific stochastic wave equations, namely, the hyperbolic Anderson equation (3.1.3). It is known that if $\|v(t,x)\|_{L^2(\Omega)} < \infty$ the solution admits the following unique Wiener chaos expansion (see [Hu17, Nua06]):

$$v(t,x) = I_0(t,x) + \sum_{n=1}^{\infty} I_n(g_n(t,x)) \,, \tag{3.5.1}$$

where I_n denotes the multiple Itô-Wiener integrals and $g_n(t,x)$ $(n \geq 1)$ are defined by

$$g_n(\vec{s},\vec{x};t,x) = \frac{1}{n!} G_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots G_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) I_0(s_{\sigma(1)}, x_{\sigma(1)}) \,, \tag{3.5.2}$$

where $\vec{x} = (x_1, \dots, x_n)$ and $\vec{s} = (s_1, \dots, s_n)$ such that $0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} < t$ for a permutation σ. Then to verify the existence and uniqueness of the mild solution $v(t, x)$ is equivalent to show that

$$\mathbb{E}[|v(t,x)|^2] = \sum_{n=0}^{\infty} n! \|g_n(\cdot; t, x)\|^2_{\mathfrak{H}^{\otimes n}} < \infty \,, \tag{3.5.3}$$

where $\mathfrak{H}$ is defined by (3.2.1). In terms of Fourier transformation, we have

$$\|g_n(\cdot; t, x)\|^2_{\mathfrak{H}^{\otimes n}} = \int_{[0,t]^n} \int_{\mathbb{R}^n} \left| \mathcal{F} g_n(\vec{s}, \cdot; t, x)(\vec{\xi}) \right|^2 \mu(d\vec{\xi}) d\vec{s} \,,$$

with $\mu(d\vec{\xi}) = \prod_{j=1}^n |\xi_j|^{1-2H} d\vec{\xi}$.

For national simplicity, we abbreviate $I_k(g_k(t,x))$ as $I_k(t,x)$ for $k = 1, 2$, i.e.

$$\begin{aligned} I_1(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) I_0(s,y) W(ds, dy) \,, \\ I_2(t,x) &= \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) I_1(s,y) W(ds, dy) \,. \end{aligned}$$

Let us select some special initial conditions $u_0(x) = e^{-x^2}$ and $v_0(x) \equiv 0$ to proceed our argument. Then

$$\begin{aligned} I_0(t,x) =& \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2}[u_0(x+t) + u_0(x-t)] \\ =& \frac{1}{2} \left[e^{-(x+t)^2} + e^{-(x-t)^2} \right] \,. \end{aligned} \tag{3.5.4}$$

We do not consider the simple case $u_0(x) = 1$ and $v_0(x) = 0$. Because in this case, $I_0(t,x)$ is not in the space $\mathcal{Z}^p(T)$ for any $p \geq 1$.

Lemma 3.5.1. *Suppose $I_0(t,x)$ are given in (3.5.4). Then for $H \in (0, 1/2)$, there exist positive constants $c_{T,H}$ and $C_{T,H}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}$ and h small enough satisfying $0 < h < 1 \wedge \frac{t}{2}$,*

$$c_{t,H} \cdot |h|^{2H} \leq \mathbb{E}[|\mathfrak{D}_h I_1(t,x)|^2] \leq C_{T,H} \cdot |h|^{2H} \,. \tag{3.5.5}$$

Proof. first, from (3.5.4) we see easily that

$$|I_0(t,x)| \le C_T\,, \quad |\mathfrak{D}_l I_0(t,x)| \le C_T \cdot |l| \wedge 1\,. \tag{3.5.6}$$

Moreover, on the set $(t,x) \in [0,T] \times [-T,T]$, we have a lower bound for $|I_0(t,x)|$:

$$I_0(t,x) = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] \ge c_T\,. \tag{3.5.7}$$

Now we are in a position to estimate $\mathbb{E}[|\mathfrak{D}_h I_1(t,x)|^2]$. Let us consider the lower bound first. Recall an elementary inequality: $(a+b)^2 \ge \frac{3}{4}a^2 - 3b^2$, then

$$\begin{aligned}
\mathbb{E}[|\mathfrak{D}_h I_1(t,x)|^2] &= \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \mathfrak{D}_h G_{t-s}(x-y) \cdot I_0(s,y) W(ds,dy)\right|^2\right] \\
&= \int_0^t \int_{\mathbb{R}^2} \Big|\mathfrak{D}_h G_{t-s}(x-(y+l)) \cdot I_0(s,y+l) \\
&\qquad - \mathfrak{D}_h G_{t-s}(x-y) \cdot I_0(s,y)\Big|^2 \cdot |l|^{2H-2} dl dy ds \\
&\ge \frac{3}{4} \int_0^t \int_{\mathbb{R}^2} |\Box_{l,h} G_s(y) \cdot I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds \\
&\qquad - 3 \int_0^t \int_{\mathbb{R}^2} |\mathfrak{D}_h G_s(x-y)|^2 \cdot |\mathfrak{D}_l I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds\,.
\end{aligned}$$

By Hölder's inequality and (3.5.6), we see that

$$\sup_{s\in[0,T], y\in\mathbb{R}} \int_{\mathbb{R}} |\mathfrak{D}_l I_0(s,y)|^2 \cdot |l|^{2H-2} dl \le C_{T,H} < \infty,$$

for $H \in (0, \frac{1}{2})$. Then

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} |\mathfrak{D}_h G_s(x-y)|^2 \cdot |\mathfrak{D}_l I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds \\
&\qquad\qquad \lesssim \int_0^t \int_{\mathbb{R}} |\mathfrak{D}_h G_s(y)|^2 \, dy ds \le C_T \cdot |h|\,.
\end{aligned}$$

Moreover, we have

$$\int_0^t \int_{\mathbb{R}^2} |\Box_{l,h} G_s(y) \cdot I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds$$

$$\geq \int_0^t \int_{y>0} \int_{l\geq h} |\Box_{l,h} G_s(y) I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds\,.$$

Notice that on the set $\{y>0\}\times\{l\geq h\}$

$$\begin{aligned}|\Box_{l,h} G_s(y)|^2 &\simeq |1_{\{|y+l+h|<s\}} - 1_{\{|y+l|<s\}} - 1_{\{|y+h|<s\}} + 1_{\{|y|<s\}}|^2\\ &\simeq |1_{\{y+l<|s|<y+l+h\}} - 1_{\{y<|s|<y+h\}}|^2\\ &= 1_{\{y+l<|s|<y+l+h\}} + 1_{\{y<|s|<y+h\}}\,.\end{aligned}$$

Letting $h < 1\wedge \frac{t}{2}$ be small enough and noticing the lower bound (3.5.7), we have

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^2} |\Box_{l,h} G_s(y) I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds\\ &\qquad\gtrsim \int_{\frac{t}{2}}^t \int_{s-h}^s \int_{l\geq h} |l|^{2H-2} |I_0(s,y)|^2 dl dy ds\\ &\qquad\gtrsim \int_{\frac{t}{2}}^t \int_{s-h}^s |h|^{2H-1} |I_0(s,y)|^2 dy ds \gtrsim c_{t,H}\cdot h^{2H}\,.\end{aligned}$$

Thus, we obtain when $H\in(0,1/2)$ and $|h|$ is relatively small

$$\mathbb{E}[|\mathfrak{D}_h I_1(t,x)|^2] \gtrsim c_{t,H}\cdot h^{2H} - C_T\cdot|h| \gtrsim c_{t,H}\cdot h^{2H}.$$

The upper bound can be derived by the Fourier transformation. By (3.5.6), we have

$$\begin{aligned}\mathbb{E}[|\mathfrak{D}_h I_1(t,x)|^2] \leq{}& 2\int_0^t \int_{\mathbb{R}^2} |\Box_{l,h} G_s(y) I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds\\ &+ 2\int_0^t \int_{\mathbb{R}^2} |\mathfrak{D}_h G_s(x-y)|^2 \cdot |\mathfrak{D}_l I_0(s,y)|^2 \cdot |l|^{2H-2} dl dy ds\\ \lesssim{}& \int_0^t \int_{\mathbb{R}^2} |\Box_{l,h} G_s(y)|^2 \cdot |l|^{2H-2} dl dy ds + \int_0^t \int_{\mathbb{R}} |\mathfrak{D}_h G_s(y)|^2\, dy ds\\ \lesssim{}& \int_0^t \int_{\mathbb{R}} |e^{\iota h\xi} - 1|^2 \left(\frac{\sin(s|\xi|)}{\xi}\right)^2 |\xi|^{1-2H} d\xi ds + |h|\\ \lesssim{}& \int_0^t \int_{\mathbb{R}} [1-\cos(h|\xi|)] \frac{s^2}{1+s^2|\xi|^2} |\xi|^{1-2H} d\xi ds + |h|\\ ={}& C_{T,H}\cdot(|h|^{2H} + |h|) \lesssim C_{T,H}\cdot|h|^{2H}\,,\end{aligned}$$

for $H\in(0,1/2)$ and $|h|$ is sufficiently small. Therefore, we finish the proof. □

Now we begin to prove Theorem 3.2.7.

Proof of Theorem 3.2.7. We only need to consider $\|I_2(t,x)\|^2_{L^2(\Omega)}$ with some special initial data (3.5.4). Let us denote

$$F_{t,x}(s,y) := G_{t-s}(x-y)I_1(s,y)\,.$$

Noting that

$$\begin{aligned}|\mathfrak{D}_h F_{t,x}(s,y)|^2 &= |G_{t-s}(x-y)\mathfrak{D}_h I_1(s,y) + \mathfrak{D}_h G_{t-s}(x-y)I_1(s,y)|^2 \\ &\geq \frac{3}{4}|G_{t-s}(x-y)\mathfrak{D}_h I_1(s,y)|^2 - 3|\mathfrak{D}_h G_{t-s}(x-y)I_1(s,y)|^2\,,\end{aligned}$$

so we have

$$\begin{aligned}\mathbb{E}\left[|I_2(t,x)|^2\right] &= \mathbb{E}\int_0^t\int_{\mathbb{R}^2}|\mathfrak{D}_h F_{t,x}(s,y)|^2|h|^{2H-2}dhdyds \\ &\geq \frac{3}{4}\int_0^t\int_{\mathbb{R}^2}|G_{t-s}(x-y)|^2\mathbb{E}|\mathfrak{D}_h I_1(s,y)|^2|h|^{2H-2}dhdyds \qquad (3.5.8)\\ &\quad -3\int_0^t\int_{\mathbb{R}^2}|\mathfrak{D}_h G_{t-s}(x-y)|^2\mathbb{E}|I_1(s,y)|^2|h|^{2H-2}dhdyds\,. \qquad (3.5.9)\end{aligned}$$

Without loss of generality, we assume $t=2$ and estimate term (3.5.8) first. By Lemma 3.5.1 with $h<1\wedge\frac{t}{2}=1$, it is clear that when $H\leq\frac{1}{4}$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}|G_{t-s}(x-y)|^2\mathbb{E}|\mathfrak{D}_h I_1(s,y)|^2|h|^{2H-2}dhdyds \\ &\quad\gtrsim\int_1^2\int_{\mathbb{R}}\int_{|h|<1}|G_{2-s}(x-y)|^2\mathbb{E}|\mathfrak{D}_h I_1(s,y)|^2|h|^{2H-2}dhdyds \\ &\quad\gtrsim\int_1^2\int_{\mathbb{R}}\int_{|h|<1}|G_{2-s}(x-y)|^2\cdot|h|^{4H-2}dhdyds=\infty. \qquad (3.5.10)\end{aligned}$$

For any $H\in(0,1/2)$ we can get that $\sup_{y\in\mathbb{R}}\mathbb{E}|I_1(s,y)|^2\lesssim s^{2H}+s^2$ in the term (3.5.9), thus

$$\begin{aligned}&\int_0^2\int_{\mathbb{R}^2}|\mathfrak{D}_h G_{2-s}(x-y)|^2\mathbb{E}|I_1(s,y)|^2|h|^{2H-2}dhdyds \\ &\quad\lesssim\int_0^2(s^{2H}+s^2)\int_{\mathbb{R}}\left(\frac{\sin((2-s)|\xi|)}{|\xi|}\right)^2|\xi|^{1-2H}d\xi ds\end{aligned}$$

$$= \int_0^2 (s^{2H} + s^2)(2-s)^{2H} ds < \infty. \tag{3.5.11}$$

Plugging (3.5.10) and (3.5.11) into (3.5.8) and(3.5.9), we obtain that for $t = 2$

$$\mathbb{E}\left[|I_2(t,x)|^2\right] = \infty$$

when $H \leq \frac{1}{4}$. The proof is complete. □

3.6 Appendix A: some technical lemmas for wave kernel

In this Appendix, we show some technical lemmas used several times in our work. Let us start by proving the Fourier transform of $\mathcal{E}(t,x)$, $\mathcal{S}_\alpha(t,x)$ and $\mathcal{C}_{1-\alpha}(t,x)$.

Lemma 3.6.1. *Let $\mathcal{E}(t,x)$, $\mathcal{S}_\alpha(t,x)$ and $\mathcal{C}_{1-\alpha}(t,x)$ be defined by (3.3.3). Then they are all in $L^1(\mathbb{R})$, and their Fourier transforms are given by (3.3.4). Consequently, the wave kernel $G_{t-s}(x-y)$ can be expressed as the representation (3.3.2).*

Proof. We treat $\mathcal{E}(t,x)$ first,

$$\mathcal{E}(t,x) = \mathcal{F}^{-1}[\hat{\mathcal{E}}(t,\cdot)](x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t|\xi|} e^{\iota x\xi} d\xi = \frac{1}{\pi}\frac{t}{t^2+x^2}, \tag{3.6.1}$$

which is obviously in $L^1(\mathbb{R})$. Similarly, for $\mathcal{S}_\alpha(t,x)$,

$$\begin{aligned}
\mathcal{S}_\alpha(t,x) = \mathcal{F}^{-1}[\hat{\mathcal{S}}_\alpha(t,\cdot)](x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(t|\xi|)}{|\xi|^\alpha} e^{\iota x\xi} d\xi = \frac{1}{\pi}\int_0^\infty \frac{\sin(t\xi)}{\xi^\alpha}\cos(|x|\xi) d\xi \\
&= \frac{1}{2\pi}\int_0^\infty \frac{\sin\left((t+|x|)\xi\right)}{\xi^\alpha} d\xi + \frac{1}{2\pi}\int_0^\infty \frac{\sin\left((t-|x|)\xi\right)}{\xi^\alpha} d\xi \\
&= \frac{\Gamma(1-\alpha)}{2\pi}\cos\left(\frac{\alpha\pi}{2}\right)\left[\left(t+|x|\right)^{\alpha-1} + \operatorname{sgn}(t-|x|)\left(t-|x|\right)^{\alpha-1}\right],
\end{aligned} \tag{3.6.2}$$

where the last equality can be found in 17.33(2) in [GR15]. For fixed $t > 0$, if $|x|$ is close to t, $|\mathcal{S}_\alpha(t,x)|$ can be bounded by

$$\left(t+|x|\right)^{\alpha-1} + \left|t-|x|\right|^{\alpha-1}.$$

And when $|x|$ is large enough, $|\mathcal{S}_\alpha(t,x)|$ behaves like

$$\big(|x|-t\big)^{\alpha-1} - \big(|x|+t\big)^{\alpha-1},$$

which can be bounded by $t\big(|x|-t\big)^{\alpha-2}$. Therefore, $\mathcal{S}_\alpha(t,x)$ is in $L^1(\mathbb{R})$ since $\alpha \in (0,1)$.

The last one $\mathcal{C}_{1-\alpha}(t,x)$ is more involved because of the term $\mathcal{F}^{-1}\left[\frac{e^{-t|\xi|}}{|\xi|^{1-\alpha}}\right]$. But we can apply the formula 17.34(14) in [GR15] to get

$$\begin{aligned}
\mathcal{C}_{1-\alpha}(t,x) &= \mathcal{F}^{-1}[\hat{\mathcal{C}}_{1-\alpha}(t,\cdot)](x) \\
&= \frac{1}{\pi}\int_0^\infty \frac{\cos(t\xi)}{\xi^{1-\alpha}}\cos(|x|\xi)d\xi - \frac{1}{\pi}\int_0^\infty \frac{e^{-t|\xi|}}{\xi^{1-\alpha}}\cos(|x|\xi)d\xi \\
&= \frac{\Gamma(\alpha)}{2\pi}\Bigg[\cos\left(\frac{\alpha\pi}{2}\right)\Big[\big|t+|x|\big|^{-\alpha} + \big|t-|x|\big|^{-\alpha}\Big] \\
&\qquad - 2\cos\left(\alpha\tan^{-1}\left(\frac{|x|}{t}\right)\right)\big[t^2+x^2\big]^{-\frac{\alpha}{2}}\Bigg].
\end{aligned} \tag{3.6.3}$$

Similarly, when $|x|$ is close to t, $|\mathcal{C}_{1-\alpha}(t,x)|$ can be bounded by

$$\big|t+|x|\big|^{-\alpha} + \big|t-|x|\big|^{-\alpha} + \big[t^2+x^2\big]^{-\frac{\alpha}{2}}.$$

It is more interesting to know the above asymptotics when $|x|$ is large. Since

$$\begin{aligned}
\mathcal{C}_{1-\alpha}(t,x) \simeq{}& \cos\left(\frac{\alpha\pi}{2}\right)\big[|t+|x||^{-\alpha} + |t-|x||^{-\alpha}\big] - 2\cos\left(\frac{\alpha\pi}{2}\right)\big[t^2+x^2\big]^{-\frac{\alpha}{2}} \\
&+ 2\left[\cos\left(\frac{\alpha\pi}{2}\right) - \cos\left(\alpha\tan^{-1}\left(\frac{|x|}{t}\right)\right)\right](t^2+x^2)^{-\frac{\alpha}{2}},
\end{aligned} \tag{3.6.4}$$

setting $y_0 = \frac{|x|}{t}$, then for $|x|$ large enough,

$$\begin{aligned}
\cos\left(\frac{\alpha\pi}{2}\right) - \cos\left(\alpha\tan^{-1}\left(\frac{|x|}{t}\right)\right) &= \int_{y_0}^{+\infty}\frac{d}{d\omega}\left[\cos\left(\alpha\tan^{-1}(\omega)\right)\right]d\omega \\
&\le \alpha\int_{y_0}^{+\infty}\frac{1}{\omega^2}d\omega \simeq C_\alpha\cdot t|x|^{-1}.
\end{aligned} \tag{3.6.5}$$

Therefore,

$$\left[2\cos\left(\frac{\alpha\pi}{2}\right) - 2\cos\left(\alpha\tan^{-1}\left(\frac{|x|}{t}\right)\right)\right](t^2+x^2)^{-\frac{\alpha}{2}} \simeq C_\alpha\cdot t|x|^{-1}(t^2+x^2)^{-\frac{\alpha}{2}}, \tag{3.6.6}$$

which is integrable with respect to x when $|x|$ is large enough since $\alpha \in (0,1)$. Moreover, since the following important asymptotic behavior holds, which will be explained in Remark 3.6.2,

$$\begin{aligned} &\left|t+|x|\right|^{-\alpha}+\left|t-|x|\right|^{-\alpha} \\ &\qquad = 2\big(|x|^2-t^2\big)^{-\frac{\alpha}{2}}\cos\left[\alpha\tan^{-1}\left(\frac{\iota t}{|x|}\right)\right] \sim 2\big(|x|^2-t^2\big)^{-\frac{\alpha}{2}}, \end{aligned} \tag{3.6.7}$$

it is clear that

$$t\big(|x|^2+t^2\big)^{-\frac{\alpha}{2}-1}\lesssim \big(|x|^2-t^2\big)^{-\frac{\alpha}{2}}-\big(|x|^2+t^2\big)^{-\frac{\alpha}{2}}\lesssim t\big(|x|^2-t^2\big)^{-\frac{\alpha}{2}-1}.$$

We see that for $\alpha \in (0,1)$, $\mathcal{C}_\alpha(t,x)$ is also integrable with respect to x when $|x|$ sufficiently large. As a result, $\mathcal{C}_\alpha(t,x)$ is in $L^1(\mathbb{R})$.

Combining (3.6.1), (3.6.2) and (3.6.3), we can conclude (3.3.2). □

Remark 3.6.2. *We provide details of the equation (3.6.7) we used in the above proof of Lemma 3.6.1. Noticing that*

$$\arctan(z) = -\frac{\iota}{2}\ln\left(\frac{\iota - z}{\iota + z}\right) = -\frac{\iota}{2}\ln\left(\frac{1+\iota z}{1-\iota z}\right),$$

we have

$$\begin{aligned} \cos[\alpha\tan^{-1}(z)] &= \frac{1}{2}\left\{\exp\left[\iota\alpha\tan^{-1}(z)\right]+\exp\left[-\iota\alpha\tan^{-1}(z)\right]\right\} \\ &= \frac{1}{2}\left\{\exp\left[\frac{\alpha}{2}\ln\left(\frac{1+\iota z}{1-\iota z}\right)\right]+\exp\left[-\frac{\alpha}{2}\ln\left(\frac{1+\iota z}{1-\iota z}\right)\right]\right\} \\ &= \frac{1}{2}\left\{\left(\frac{1+\iota z}{1-\iota z}\right)^{\frac{\alpha}{2}}+\left(\frac{1-\iota z}{1+\iota z}\right)^{\frac{\alpha}{2}}\right\} \\ &= \frac{1}{2}\left\{(1+z^2)^{\frac{\alpha}{2}}\left[(1-\iota z)^{-\alpha}+(1+\iota z)^{-\alpha}\right]\right\}. \end{aligned}$$

Letting $z = \frac{\iota t}{|x|}$, we see the equation (3.6.7) holds.

Lemma 3.6.3. *If $\frac{1}{2} < \alpha < 1$, then $\hat{\mathcal{C}}_\alpha(t,\xi) := \frac{\cos(t|\xi|)-e^{-t|\xi|}}{|\xi|^\alpha}$ and $\hat{\mathcal{S}}_\alpha(t,\xi) := \frac{\sin(t|\xi|)}{|\xi|^\alpha}$ are in $L^2(\mathbb{R})$ for any $t > 0$. Hence, $\mathcal{C}_\alpha(t,x)$ and $\mathcal{S}_\alpha(t,x)$ are also in $L^2(\mathbb{R})$.*

3.7 Appendix B: lemmas for Proposition 3.3.3

Lemma 3.7.1. *If $p > \frac{1}{H}$, $1 - H < \alpha < 1 - \frac{1}{p}$ and $1 - 2/q + \alpha < \theta < H + \alpha - 1/2$, then there exists a constant C independent of r such that*

$$\mathbb{E}\|J_\theta^{\mathcal{K}_i}(r,\cdot)\|_{L^p(\mathbb{R})}^p \le C\|v\|_{\mathcal{Z}^p(T)}^p\,, \quad i = 1,2,3,4\,, \tag{3.7.1}$$

where $J_\theta^{\mathcal{K}_i}$ (depending on α,θ) and $\mathcal{K}_i$ (depending on α) are defined by (3.3.7) and (3.3.8) respectively.

Proof. We will prove the above bound (3.7.1) for $i = 1,2,3,4$ separately. We deal with the term $i = 1$ first. In this case $\mathcal{K}_1 = \mathcal{C}_\alpha$ and $\bar{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha}$ as defined by (3.3.3). From the definition (3.3.7) of $J_\theta^{\mathcal{K}_1}$ and from Burkholder-Davis-Gundy's inequality and the triangle inequality it follows

$$\int_\mathbb{R} \mathbb{E}\big|J_\theta^{\mathcal{C}_\alpha}(r,z)\big|^p dz \lesssim \int_\mathbb{R} \big|\mathcal{D}_1(r,z)\big|^{\frac{p}{2}} + \big|\mathcal{D}_2(r,z)\big|^{\frac{p}{2}} dz\,,$$

where we have used two notations

$$\begin{aligned}\mathcal{D}_1(r,z) := \int_0^r \int_{\mathbb{R}^2} &(r-s)^{-2\theta}\, |\mathfrak{D}_h \mathcal{C}_\alpha(r-s,y)|^2 \\ &\cdot \|v(s,y+z)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds\,,\end{aligned} \tag{3.7.2}$$

and

$$\begin{aligned}\mathcal{D}_2(r,z) := \int_0^r \int_{\mathbb{R}^2} &(r-s)^{-2\theta} |\mathcal{C}_\alpha(r-s,y)|^2 \\ &\cdot \|\mathfrak{D}_h v(s,z+y)\|_{L^p(\Omega)}^2 |h|^{2H-2} dh dy ds\,.\end{aligned} \tag{3.7.3}$$

By the definition of $\mathcal{Z}_1^p(T)$ in (3.2.6), we can bound $D_1(r) := \int_\mathbb{R} |\mathcal{D}_1(r,z)|^{\frac{p}{2}}\, dz$ as follows.

$$\begin{aligned} D_1(r) &\lesssim \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta}\, |\mathfrak{D}_h \mathcal{C}_\alpha(r-s,y)|^2\, |h|^{2H-2} dh dy ds\right)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_1^p(T)}^p \\ &\simeq \left(\int_0^r \int_{\mathbb{R}^2} s^{-2\theta} \left|\mathfrak{D}_h \hat{\mathcal{C}}_\alpha(s,\xi)\right|^2 |h|^{2H-2} dh d\xi ds\right)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_1^p(T)}^p\,. \end{aligned} \tag{3.7.4}$$

In the last line of (3.7.4), we have applied Parseval's formula which is legitimate since $\mathcal{C}_\alpha(s,\cdot)$ is in $L^2(\mathbb{R})$ when

$$\frac{1}{2} < \alpha \le 1\,, \tag{3.7.5}$$

by Lemma 3.6.3. Through (3.3.4), we can write (3.7.4) as

$$\begin{aligned} D_1(r) \lesssim& \bigg(\int_0^r \int_{\mathbb{R}^2} s^{-2\theta} \left|\frac{\cos(s|\xi|)-e^{-s|\xi|}}{|\xi|^\alpha}\right|^2 \qquad\qquad (3.7.6)\\ &\qquad\qquad \times [1-\cos(h|\xi|)]|h|^{2H-2} dh d\xi ds\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_1^p(T)} \\ \simeq& \bigg(\int_0^r \int_{\mathbb{R}} s^{-2\theta} \left|\cos(s|\xi|)-e^{-s|\xi|}\right|^2 |\xi|^{1-2\alpha-2H} d\xi ds\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_1^p(T)} \\ \simeq& \bigg(\int_0^r s^{2(H+\alpha-\theta-1)} ds \cdot \int_0^\infty \xi^{1-2\alpha-2H} \left|\cos(\xi)-e^{-\xi}\right|^2 d\xi\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_1^p(T)}\,, \qquad (3.7.7) \end{aligned}$$

which is finite if

$$\begin{aligned} &1-2\alpha-2H < -1\,,\ 2(H+\alpha-\theta-1) > -1, \\ &\Leftrightarrow\ \alpha > 1-H\,,\ \theta < H+\alpha-\frac{1}{2}\,. \end{aligned} \tag{3.7.8}$$

Similarly, by the definition of $\mathcal{Z}_2^p(T)$ in (3.2.6), for $D_2(r) := \int_{\mathbb{R}} |\mathcal{D}_2(r,z)|^{\frac{p}{2}} dz$, Parseval's formula implies

$$\begin{aligned} D_2(r) \lesssim& \bigg(\int_0^r \int_{\mathbb{R}} (r-s)^{-2\theta} |\mathcal{C}_\alpha(r-s,y)|^2 dy ds\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_2^p(T)} \\ \simeq& \bigg(\int_0^r \int_{\mathbb{R}} s^{-2\theta} |\hat{\mathcal{C}}_\alpha(s,\xi)|^2 d\xi ds\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_2^p(T)}\,, \end{aligned} \tag{3.7.9}$$

if α satisfies (3.7.5). Then plugging (3.3.4), we have

$$\begin{aligned} D_2(r) \lesssim& \bigg(\int_0^r \int_{\mathbb{R}} s^{-2\theta} \left|\frac{\cos(s|\xi|)-e^{-s|\xi|}}{|\xi|^\alpha}\right|^2 d\xi ds\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_2^p(T)} \\ \simeq& \bigg(\int_0^r s^{2(\alpha-\theta)-1} ds \cdot \int_0^\infty \xi^{-2\alpha} \left|\cos(s\xi)-e^{-s\xi}\right|^2 d\xi\bigg)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_2^p(T)}\,, \end{aligned} \tag{3.7.10}$$

which is finite since $\frac{1}{2} < \alpha \le 1$ and $\alpha > \frac{1}{2} - H + \theta > \theta$ by (3.7.8).

Thus, with the choice of $\theta < H + \alpha - 1/2$ and $\alpha > 1 - H$, we have finished the proof

(3.7.1) for $i = 1$.

Now let us deal with the case when $i = 2$. Similar to the proof in the case $i = 1$, now we need to show

$$\|J_\theta^{\mathcal{S}_\alpha}(r,z)\|_{L^p(\Omega\times\mathbb{R})}^p \leq C\|v\|_{\mathcal{Z}^p(T)}^p .$$

From the definition (3.3.7) of J_θ and from Burkholder-Davis-Gundy's inequality it follows

$$\int_{\mathbb{R}} \mathbb{E}\big|J_\theta^{\mathcal{S}_\alpha}(r,z)\big|^p dz \lesssim \int_{\mathbb{R}} \big[\widetilde{\mathcal{D}}_1(r,z)\big]^{\frac{p}{2}} + \big[\widetilde{\mathcal{D}}_2(r,z)\big]^{\frac{p}{2}} dz ,$$

where $\widetilde{\mathcal{D}}_1(r,z)$ and $\widetilde{\mathcal{D}}_2(r,z)$ are defined by (3.7.2) and (3.7.3), respectively, with $\mathcal{C}_\alpha$ replaced by $\mathcal{S}_\alpha$.

By the definition of $\mathcal{Z}_1^p(T)$ in (3.2.6) and Minkowski's inequality, we have

$$\begin{aligned}
\widetilde{D}_1(r) &:= \int_{\mathbb{R}} \Big|\widetilde{\mathcal{D}}_1(r,z)\Big|^{\frac{p}{2}} dz \\
&\lesssim \bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta}\big|\mathfrak{D}_h \mathcal{S}_\alpha(r-s,y)\big|^2 |h|^{2H-2} dh dy ds\bigg)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_1^p(T)}^p \\
&\lesssim \bigg(\int_0^r s^{2(H+\alpha-\theta-1)} ds \cdot \int_0^\infty \xi^{1-2\alpha-2H} \left|\sin(\xi)\right|^2 d\xi\bigg)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_1^p(T)}^p ,
\end{aligned} \tag{3.7.11}$$

which is finite under the condition (3.7.8). In a similar way we can get

$$\begin{aligned}
\widetilde{D}_2(r) &:= \int_{\mathbb{R}} \Big|\widetilde{\mathcal{D}}_2(r,z)\Big|^{\frac{p}{2}} dz \\
&\lesssim \bigg(\int_0^r \int_{\mathbb{R}} (r-s)^{-2\theta} |\mathcal{S}_\alpha(r-s,y)|^2 dy ds\bigg)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_2^p(T)}^p \\
&\lesssim \bigg(\int_0^r s^{2(\alpha-\theta)-1} dr \cdot \int_0^\infty \xi^{-2\alpha} \left|\sin(\xi)\right|^2 d\xi\bigg)^{\frac{p}{2}} \times \|v\|_{\mathcal{Z}_2^p(T)}^p ,
\end{aligned} \tag{3.7.12}$$

which is clearly bounded by (3.7.8) since $\frac{1}{2} < \alpha < 1$ and $\alpha > \theta$.

Therefore, with the choice of $\theta \in (1 - 2/q + \alpha, H + \alpha - 1/2)$, we finish the proof of (3.7.1) when $i = 2$. The remaining parts of (3.7.1), i.e. the cases $\mathcal{K}_3 = \mathcal{S}$ and $\mathcal{K}_4 = \mathcal{E}$ can be completed in the same spirit and we omit the details since they are actually simpler. □

Lemma 3.7.2. *If $p > \frac{1}{H}$, $\frac{3}{2} - 2H < \alpha < 1 - \frac{1}{p}$ and $1 - 2/q + \alpha < \theta < 2H + \alpha - 1$, then* *there exists a constant C independent of $r \in [0,T]$ such that for $i = 1,2,3,4$*

$$\mathbb{E}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\big|J_\theta^{\mathcal{K}_i}(r,z+h) - J_\theta^{\mathcal{K}_i}(r,z)\big|^2 |h|^{2H-2}dh\Big]^{\frac{p}{2}}dz \le C\|v\|^p_{\mathcal{Z}^p(T)}\,, \tag{3.7.13}$$

where $J_\theta^{\mathcal{K}_i}$ (depending on α,θ) and $\mathcal{K}_i$ (depending on α) are defined by (3.3.7) and (3.3.8) respectively.

Proof. Recall that $\mathfrak{D}_h J_\theta^{\mathcal{K}_i}(r,z) := J_\theta^{\mathcal{K}_i}(r,z+h) - J_\theta^{\mathcal{K}_i}(r,z)$. We still first consider the case when $i = 1$, i.e. $\mathcal{K}_1 = \mathcal{C}_\alpha$ and $\bar{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha}$ defined by (3.3.3). We only need to prove that there exists some constant C, independent of $r \in [0,T]$, such that

$$\begin{aligned}&\int_{\mathbb{R}}\mathbb{E}\Big[\int_{\mathbb{R}}\big|\mathfrak{D}_h J_\theta^{\mathcal{C}_\alpha}(r,z)\big|^2 |h|^{2H-2}dh\Big]^{\frac{p}{2}}dz\\ &\qquad\le \Big(\int_{\mathbb{R}}\|\mathfrak{D}_h J_\theta^{\mathcal{C}_\alpha}(r,z)\|^2_{L^p(\mathbb{R}\times\Omega)}|h|^{2H-2}dh\Big)^{\frac{p}{2}} \le C\|v\|^p_{\mathcal{Z}^p(T)}\,,\end{aligned} \tag{3.7.14}$$

where we employed Minkowski's inequality in the above first inequality.

Thanks to Burkholder-Davis-Gundy's inequality, the triangle inequality and then a change of variable $y \to z - y$, we have

$$\begin{aligned}&\mathbb{E}\big[|\mathfrak{D}_h J_\theta^{\mathcal{C}_\alpha}(r,z)|^p\big]\\ \le& C_p\Big(\int_0^r (r-s)^{-2\theta}\int_{\mathbb{R}^2}\Big[\mathbb{E}\Big|\mathfrak{D}_h\mathcal{C}_\alpha(r-s,z-y-l)v(s,y+l)\\ &\qquad\qquad - \mathfrak{D}_h\mathcal{C}_\alpha(r-s,z-y)v(s,y)\Big|^p\Big]^{\frac{2}{p}}|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}\\ \le& C_p\Big(\int_0^r (r-s)^{-2\theta}\int_{\mathbb{R}^2}|\mathfrak{D}_h\mathcal{C}_\alpha(r-s,y)|^2\|v(s,y+z)\|^2_{L^p(\Omega)}|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}\\ &+ C_p\Big(\int_0^r (r-s)^{-2\theta}\int_{\mathbb{R}^2}\Big|\square_{h,l}\mathcal{C}_\alpha(r-s,y)\Big|^2\|\mathfrak{D}_l v(s,y+z)\|^2_{L^p(\Omega)}|l|^{2H-2}dldyds\Big)^{\frac{p}{2}}.\end{aligned}$$

Therefore, by Minkowski's inequality

$$\begin{aligned}&\int_{\mathbb{R}}\|\mathfrak{D}_h J_\theta^{\mathcal{C}_\alpha}(r,\cdot)\|^2_{L^p(\mathbb{R}\times\Omega)}|h|^{2H-2}dh\\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\mathbb{E}\big[|\mathfrak{D}_h J_\theta^{\mathcal{C}_\alpha}(r,z)|^p\big]dz\Big)^{\frac{2}{p}}|h|^{2H-2}dh\\ &\le \int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\theta}|\mathfrak{D}_h\mathcal{C}_\alpha(r-s,y)|^2|h|^{2H-2}dhdyds \times \|v\|^2_{\mathcal{Z}_2^p(T)}\end{aligned}$$

$$+\int_0^r\int_{\mathbb{R}^3}(r-s)^{-2\theta}\Big|\Box_{h,l}\mathcal{C}_\alpha(r-s,y)\Big|^2|l|^{2H-2}|h|^{2H-2}dldhdyds\times\|v\|^2_{\mathcal{Z}_1^p(T)}$$
$$=:\mathcal{J}_1(r,z)\times\|v\|^2_{\mathcal{Z}_2^p(T)}+\mathcal{J}_2(r,z)\times\|v\|^2_{\mathcal{Z}_1^p(T)}\,.$$

Applying (3.3.4) and Parseval's formula again, one can find

$$\begin{aligned}\mathcal{J}_1(r,z)\simeq&\int_0^r\int_{\mathbb{R}^2}(r-s)^{-2\theta}\Big|\mathfrak{D}_h\hat{\mathcal{C}}_\alpha(r-s,\xi)\Big|^2|h|^{2H-2}dhd\xi ds\\ \lesssim&\int_0^r s^{2(H+\alpha-\theta-1)}dr\cdot\int_0^\infty\xi^{1-2\alpha-2H}\left|\cos(\xi)-e^{-\xi}\right|^2d\xi\,,\end{aligned}\tag{3.7.15}$$

which is finite if (3.7.8) is satisfied. Similarly, we have

$$\begin{aligned}\mathcal{J}_2(r,z)\simeq&\int_0^r\int_{\mathbb{R}^3}(r-s)^{-2\theta}|\xi|^{-2\alpha}\left|\cos\big((r-s)|\xi|\big)-e^{-(r-s)|\xi|}\right|^2\\ &\times[1-\cos(|l\xi|)][1-\cos(|h\xi|)]\cdot|l|^{2H-2}|h|^{2H-2}dldhd\xi ds\\ \simeq&\int_0^r(r-s)^{2(\alpha+2H-\theta)-3}ds\cdot\int_0^\infty\xi^{2(1-\alpha-2H)}|\cos(\xi)-e^{-\xi}|^2d\xi\,.\end{aligned}\tag{3.7.16}$$

In order to guarantee the integrals in (3.7.16) converge, we must have

$$\begin{aligned}&2(\alpha+2H-\theta)-3>-1\,,\ 2(1-\alpha-2H)<-1\\ &\Leftrightarrow\ \theta<\alpha+2H-1\,,\ \alpha>\frac{3}{2}-2H\,.\end{aligned}\tag{3.7.17}$$

Therefore, with the choice of $\theta\in(1-2/q+\alpha,2H+\alpha-1)$ and $\alpha\in(\frac{3}{2}-2H,1-\frac{1}{p})$ which implies $p>\frac{1}{H}$, by noting that $\frac{2}{4H-1}>\frac{1}{H}$ when $H<\frac{1}{2}$, then the conditions (3.7.8) and (3.7.17) are satisfied. Thus, we complete the proof of (3.7.14).

Now we show (3.7.13) for $i=2$, i.e. $\mathcal{K}_2=\mathcal{S}_\alpha$ and $\bar{\mathcal{K}}_2=\mathcal{C}_{1-\alpha}$ only briefly since the idea will be similar as in the above case $i=1$. We only need to show that there exists some constant C independent of $r\in[0,T]$, such that

$$\mathbb{E}\Big[\int_{\mathbb{R}}\left|\mathfrak{D}_hJ_\theta^{\mathcal{S}_\alpha}(r,z)\right|^2|h|^{2H-2}dh\Big]^{\frac{p}{2}}\le C\|v\|^p_{\mathcal{Z}^p(T)}\,.\tag{3.7.18}$$

Using Burkholder-Davis-Gundy's inequality, Minkowski's inequality and then the triangle inequality, we have the left hand side of (3.7.18) is bounded by

$$\Big(\widetilde{\mathcal{J}}_1(r,z)\Big)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_2^p(T)} + \Big(\widetilde{\mathcal{J}}_2(r,z)\Big)^{\frac{p}{2}} \times \|v\|^p_{\mathcal{Z}_1^p(T)}\,.$$

Applying (3.3.4) and Parseval's formula again, one finds

$$\begin{aligned}\widetilde{\mathcal{J}}_1(r,z) &:= \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta} |\mathfrak{D}_h \mathcal{S}_\alpha(r-s,y)|^2 |h|^{2H-2} dh dy ds \\ &\lesssim \int_0^r s^{2(H+\alpha-\theta-1)} dr \cdot \int_0^\infty \xi^{1-2\alpha-2H} \left|\sin(\xi)\right|^2 d\xi\,, \end{aligned} \tag{3.7.19}$$

which is obviously bounded if (3.7.8) is satisfied. Similarly, we have

$$\begin{aligned}\widetilde{\mathcal{J}}_2(r,z) &:= \int_0^r \int_{\mathbb{R}^3} (r-s)^{-2\theta} \left|\mathfrak{D}_h \mathcal{S}_\alpha(r-s,y+l) - \mathfrak{D}_h \mathcal{S}_\alpha(r-s,y)\right|^2 \\ &\qquad\qquad \times |l|^{2H-2} |h|^{2H-2} dl dh dy ds \\ &\simeq \int_0^r (r-s)^{2(\alpha+2H-\theta)-3} ds \cdot \int_0^\infty \xi^{2(1-\alpha-2H)} |\sin(\xi)|^2 d\xi\,, \end{aligned} \tag{3.7.20}$$

which is finite under (3.7.17).

Therefore, with the choice $\theta \in (1-\frac{2}{q}+\alpha, 2H+\alpha-1)$ and $\alpha \in (\frac{3}{2}-2H, 1-\frac{1}{p})$ which implies $p > \frac{1}{H}$, we see the conditions (3.7.8) and (3.7.17) are satisfied. So we finish the proof of (3.7.18). The other cases of (3.7.13) when $i=3$ and $i=4$ can be done by using the same strategy and we omit them here. □

3.8 Appendix C: lemmas for Proposition 3.4.1

Our aim is to show for any $p > \frac{1}{H}$ and $\gamma < H - \frac{1}{p}$, the temporal-spatial Hölder continuity in Proposition 3.4.1 hold by selecting appropriate α, θ and η. Above all, we list some conditions which will be used frequently in our technical lemmas.

Π.1 $1-H < \alpha < \frac{1}{q}$, $\alpha+\gamma < \frac{1}{q}$, $\frac{1}{p} < \theta < H+\alpha-\frac{1}{2}$;

Π.2 $\theta > 1+\alpha-\frac{2}{q}+2\eta$, $\eta > \gamma$;

Π.3 $\alpha+\eta > \frac{1}{q}$, $\eta > \gamma$;

Π.4 $\alpha+\eta<\frac{1}{q},\ \eta>\gamma\,.$

Throughout Appendix 3.8, we always assume $p>\frac{1}{H}$ and $\gamma<H-\frac{1}{p}$.

Lemma 3.8.1. *Suppose $\alpha,\ \theta$ satisfy (Π.1) and*

$$\begin{cases} \eta_1 \ \textit{satisfies (Π.2) and (Π.3)}; \\ \eta_2 \ \textit{satisfies (Π.2) and (Π.4)}; \\ \eta_3 \ \textit{satisfies (Π.3)}\,. \end{cases} \tag{3.8.1}$$

Then $\mathcal{I}^{(1)}_{2,k}(t,h),\ k=1,2,3,4$ in (3.4.15) can be bounded by $|h|^{\gamma q}$.

Proof. For $\mathcal{I}^{(1)}_{2,1}(t,h)$, since $(r+h)^{q(\theta-1)}\leq r^{q(\theta-1)}$ it can be bounded by

$$\begin{aligned}\mathcal{I}^{(1)}_{2,1}(t,h) &\lesssim h^{\eta_1 q}\cdot\int_0^t\int_{\mathbb{R}} r^{q(\theta-1)}\cdot r^{1-(\alpha+\eta_1)q}|1+\tilde{z}|^{(-\alpha-\eta_1)q}d\tilde{z}dr\\ &\simeq h^{\eta_1 q}\cdot\int_0^t r^{q(\theta-1)}\cdot r^{1-(\alpha+\eta_1)q}dr\ \simeq\ h^{\eta_1 q}\ \leq\ h^{\gamma q}\end{aligned}$$

where we require η_1 satisfy

$$\eta_1>\gamma\,,\quad (\alpha+\eta_1)q>1\,,\quad q(\theta-1)+1-(\alpha+\eta_1)q>-1\,,$$

which is

$$\eta_1>\gamma\,,\ \alpha+\eta_1>\frac{1}{q}\,,\ \theta>1+\alpha-\frac{2}{q}+\eta_1\,. \tag{3.8.2}$$

Similarly, for $\mathcal{I}^{(1)}_{2,2}(t,h)$ we have

$$\begin{aligned}\mathcal{I}^{(1)}_{2,2}(t,h) &\simeq h^{\eta_2 q}\cdot\int_0^t\int_0^r r^{q(\theta-1)}(r-z)^{(-\alpha-\eta_2)q}dzdr\\ &\simeq h^{\eta_2 q}\cdot\int_0^t r^{q(\theta-1)}r^{1-(\alpha+\eta_2)q}dr\ \simeq\ h^{\eta_2 q}\ \leq\ h^{\gamma q}\,,\end{aligned}$$

if we require

$$\eta_2>\gamma,\ \frac{1}{q}>\alpha+\eta_2,\ \theta>1+\alpha-\frac{2}{q}+\eta_2. \tag{3.8.3}$$

For $\mathcal{I}_{2,3}^{(1)}(t,h)$ we have

$$\mathcal{I}_{2,3}^{(1)}(t,h) \simeq h^{\eta_3 q} \cdot \int_0^t \int_h^\infty r^{q(\theta-1)} z^{(-\alpha-\eta_3)q} dz dr$$
$$\simeq h^{\eta_3 q} \cdot \int_0^t r^{q(\theta-1)} h^{1-(\alpha+\eta_3)q} dr \simeq h^{1-\alpha q} \leq h^{\gamma q},$$

under conditions

$$\frac{1}{q} > \gamma + \alpha,\ \alpha + \eta_3 > \frac{1}{q},\ \theta > 1 - \frac{1}{q} = \frac{1}{p}. \tag{3.8.4}$$

For the last term $\mathcal{I}_{2,4}^{(1)}(t,h)$ we have

$$\mathcal{I}_{2,4}^{(1)}(t,h) \lesssim h \cdot \int_0^t r^{q(\theta-1)} \cdot \int_{\mathbb{R}} \left[||r+h-|z||^{-\alpha q} + ||z|-r|^{-\alpha q} \right] \cdot 1_{A_3} dz dr$$
$$\simeq h \cdot \int_0^t r^{q(\theta-1)} \cdot \left[2 \int_0^h z^{-\alpha q} dz + \int_0^{2h} z^{-\alpha q} dz \right] dr$$
$$\leq h^{2-\alpha q} \cdot \int_0^t r^{q(\theta-1)} dr \simeq h^{2-\alpha q} \leq h^{\gamma q}$$

if we set

$$\alpha < \frac{1}{q} = 1 - \frac{1}{p}, \quad \theta > 1 - \frac{1}{q} = \frac{1}{p}, \quad \alpha + \gamma < \frac{1}{q} < \frac{2}{q}. \tag{3.8.5}$$

Notice that once α, θ satisfy (Π.1) and η_1, η_2 and η_3 satisfy (3.8.1), then the conditions (3.8.2)-(3.8.5) hold automatically. The proof is complete. □

Remark 3.8.2. *We remark here that the conditions (3.8.1) for α, θ, η's are compatible with $p > \frac{1}{H}$ and $\gamma < H - \frac{1}{p}$. Let us summarize all the restrictions in Lemma 3.8.1:*

1. $p > \frac{1}{H}$, $\gamma < H - \frac{1}{p}$;
2. $1 - H < \alpha < \frac{1}{q}$, $\alpha + \gamma < \frac{1}{q}$, $\frac{1}{p} < \theta < H + \alpha - \frac{1}{2}$;
3. $\eta_1 > \gamma$, $\theta > 1 + \alpha - \frac{2}{q} + 2\eta_1$, $\alpha + \eta_1 > \frac{1}{q}$;
4. $\eta_2 > \gamma$, $\theta > 1 + \alpha - \frac{2}{q} + 2\eta_2$, $\alpha + \eta_2 < \frac{1}{q}$;
5. $\eta_3 > \gamma$, $\alpha + \eta_3 < \frac{1}{q}$.

For any (fixed) $p > \frac{1}{H}$, we can choose for (small enough) $\varepsilon_k > 0$ $k = 1, \ldots, 6$

$$\begin{aligned} \gamma = H - \frac{1}{p} - \varepsilon_1 \,,\ \ \alpha =& 1 - H + \varepsilon_2 \,,\ \ \theta = H - \varepsilon_3 \,, \\ \eta_1 = H - \frac{1}{p} - \varepsilon_4 \,,\ \ \eta_2 =& H - \frac{1}{p} - \varepsilon_5 \,,\ \ \eta_3 = H - \frac{1}{p} - \varepsilon_6 \,. \end{aligned}$$

For arbitrary (small enough) $\varepsilon > 0$ let

$$\varepsilon_1 = 7\varepsilon \,, \varepsilon_2 = 4\varepsilon \,, \varepsilon_3 = \varepsilon \,, \varepsilon_4 = 3\varepsilon \,, \varepsilon_5 = 6\varepsilon \,, \varepsilon_6 = 6\varepsilon \,.$$

Then all the restrictions (1)-(5) are satisfied with γ arbitrarily close to $H - \frac{1}{p}$. The following lemmas can be verified similarly. We omit the details.

Lemma 3.8.3. *Suppose α, θ satisfy (Π.1) and*

$$\eta_4, \eta_5 \text{ satisfy (Π.2) and (Π.3)} \,. \tag{3.8.6}$$

Then the terms $\mathcal{I}^{(2)}_{2,5}(t,h)$ and $\mathcal{I}^{(2)}_{2,6}(t,h)$ in equation (3.4.20) can be bounded by $|h|^{\gamma q}$.

Proof. For the term $\mathcal{I}^{(2)}_{2,5}(t,h)$, from inequality (3.4.17) and $(r+h)^{\eta q} \leq r^{\eta q} + h^{\eta q}$ it follows

$$\begin{aligned} \mathcal{I}^{(2)}_{2,5}(t,h) \lesssim & \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \left| \Delta_h \left(r^2 + |z|^2 \right)^{-\frac{\alpha}{2}} \right|^q dz dr \\ \lesssim & h^{\eta_4 q} \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \left(r^2 + z^2 \right)^{-(\frac{\alpha}{2}+\eta_4)q} (r+h)^{\eta_4 q} dz dr \\ \lesssim & \, h^{\eta_4 q} \cdot \int_0^t r^{q(\theta-1)+1-(\alpha+\eta_4)q} dr \cdot \int_{\mathbb{R}} \left(1+z^2\right)^{-(\frac{\alpha}{2}+\eta_4)q} dz \\ & + h^{2\eta_4 q} \cdot \int_0^t r^{q(\theta-1)+1-(\alpha+2\eta_4)q} dr \cdot \int_{\mathbb{R}} \left(1+z^2\right)^{-(\frac{\alpha}{2}+\eta_4)q} dz \lesssim h^{q\gamma} \end{aligned}$$

if η_4 satisfies the following conditions

$$\eta_4 > \gamma \,, \quad \theta - 2\eta_4 > 1 + \alpha - \frac{2}{q} \,, \quad \alpha + 2\eta_4 > \frac{1}{q} \,. \tag{3.8.7}$$

Now we deal with $\mathcal{I}^{(2)}_{2,6}(t,h)$. For fixed $\eta \in (0,1)$ by (3.4.18) and then by changing of variable $z \to rz$,

$$\begin{aligned}\mathcal{I}_{2,6}^{(2)}(t,h) \lesssim & \int_0^t \int_{\mathbb{R}} (r+h)^{q(\theta-1)} \frac{|z|^{\eta_5 q}|h|^{\eta_5 q}}{(r^2+z^2)^{\eta_5 q}} \left(r^2+z^2\right)^{-\frac{\alpha}{2}q} dz dr \\ \lesssim & h^{\eta_5 q} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \frac{|z|^{\eta_5 q}}{(r^2+z^2)^{\eta_5 q}} \left(r^2+z^2\right)^{-\frac{\alpha}{2}q} dz dr \\ = & h^{\eta_5 q} \int_0^t r^{q(\theta-1)-\eta_5 q-\alpha q+1} dr \cdot \int_{\mathbb{R}} \frac{|z|^{\eta_5 q}}{(1+z^2)^{\eta_5 q}} \left(1+z^2\right)^{-\frac{\alpha}{2}q} dz,\end{aligned}$$

which can be bounded by $h^{\gamma q}$ under conditions (3.8.2) with η_1 replaced with η_5, i.e.

$$\eta_5 > \gamma\,, \;\; \alpha + \eta_5 > \frac{1}{q}\,, \;\; \theta > 1 + \alpha - \frac{2}{q} + \eta_5\,. \tag{3.8.8}$$

Therefore, under conditions (3.8.7) and (3.8.8), we have for $k = 5, 6$,

$$\sup_t \mathcal{I}_{2,k}^{(2)}(t,h) \lesssim |h|^{\gamma q}\,.$$

Notice that once α, θ satisfy (Π.1) and η_4, η_5 satisfy (3.8.6), then the conditions (3.8.7)-(3.8.8) hold automatically. The proof is complete. □

Lemma 3.8.4. *Suppose α, θ satisfy* (Π.1) *and*

$$\begin{cases} \eta_1 \text{ satisfies (Π.2) and (Π.3);} \\ \eta_2 \text{ satisfies (Π.3);} \\ \eta_3 \text{ satisfies (Π.2) and (Π.4)}\,. \end{cases} \tag{3.8.9}$$

Then the term $\mathcal{J}_{1,k}^{(1)}(t,x,y)$ in (3.4.31) *can be bounded as*

$$\sup_{t,x,y} \mathcal{J}_{1,k}^{(1)}(t,x,y) \lesssim C_{T,p,H,\gamma} |\hbar|^{\gamma q} \quad \text{for} \;\; k = 1, 2, 3.$$

Proof. Similar to the proof of $\mathcal{I}_{2,1}^{(1)}$ in part **(i)** of Proposition 3.4.1, $\mathcal{J}_{1,1}^{(1)}(t,x,y)$ can be bounded by $|\hbar|^{\gamma q}$ under the same condition as (3.8.2) which is implied by conditions on η_1 in (3.8.9).

Now we deal with $\mathcal{J}_{1,2}^{(1)}(t,x,y)$. By triangle inequality

$$\begin{aligned}
\mathcal{J}_{1,2}^{(1)}(t,x,y) &= \int_\hbar^t \int_{\mathbb{R}} r^{q(\theta-1)} \left|\mathfrak{D}_\hbar (r-|z|)^{-\alpha}\right|^q \cdot (1_{B_1}+1_{B_2}+1_{B_3})\, dzdr \\
&\leq \int_\hbar^t \int_{z<-r-\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_2)q} \hbar^{\eta_2 q} dzdr \\
&\quad + \int_\hbar^t \int_{z>r+\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_2)q} \hbar^{\eta_2 q} dzdr \\
&\quad + \int_\hbar^t \int_{-r+\hbar}^{r-\hbar} r^{q(\theta-1)} |r-\hbar-|z||^{(-\alpha-\eta_3)q} \hbar^{\eta_3 q} dzdr \\
&=: \sum_{j=1}^3 \mathcal{J}_{1,2,j}^{(1)}(t,x,y)\,., \qquad (3.8.10)
\end{aligned}$$

where B_1, B_2 and B_3 are defined by (3.4.29).

For the term $\mathcal{J}_{1,2,1}^{(1)}(t,x,y)$ in (3.8.10), we have

$$\begin{aligned}
\mathcal{J}_{1,2,1}^{(1)}(t,x,y) &\simeq \hbar^{\eta_2 q} \int_0^t r^{q(\theta-1)} \int_{z>\hbar} z^{-(\alpha+\eta_2)q} dzdr \\
&\simeq \hbar^{1-(\alpha+\eta_2)q+\eta_2 q} \int_0^t r^{q(\theta-1)} dr \simeq \hbar^{1-\alpha q} \lesssim \hbar^{\gamma q}\,,
\end{aligned}$$

under the same conditions as (3.8.4):

$$\alpha+\gamma < \frac{1}{q}\,,\ \alpha+\eta_2 > \frac{1}{q}\,,\ \theta > \frac{1}{p}\,. \qquad (3.8.11)$$

Similar to $\mathcal{J}_{1,2,1}^{(1)}(t,x,y)$, if the conditions in (3.8.11) hold, then we have

$$\begin{aligned}
\mathcal{J}_{1,2,1}^{(1)}(t,x,y) &= \int_\hbar^t \int_{r+\hbar}^{+\infty} r^{q(\theta-1)} (z-r)^{(-\alpha-\eta_2)q} \hbar^{\eta_2 q} dzdr \\
&= \hbar^{\eta_2 q} \int_\hbar^t \int_\hbar^{+\infty} r^{q(\theta-1)} z^{-(\alpha+\eta_2)q} dzdr \\
&\simeq \hbar^{1-\alpha q} \int_0^t r^{q(\theta-1)} dr \lesssim \hbar^{\gamma q}\,.
\end{aligned}$$

To estimate $\mathcal{J}_{1,2,3}^{(1)}(t,x,y)$ in (3.8.10), letting η_3 satisfy the conditions (3.8.3) with η_2 replaced by η_3, namely,

$$\eta_3 > \gamma,\ \frac{1}{q} > \alpha+\eta_3,\ \theta > 1+\alpha-\frac{2}{q}+\eta_3\,, \qquad (3.8.12)$$

we have

$$\begin{aligned}
\mathcal{J}^{(1)}_{1,2,3}(t,x,y) &= \int_{\hbar}^{t}\int_{-r+\hbar}^{0} r^{q(\theta-1)}|r-\hbar+z|^{(-\alpha-\eta_3)q}\hbar^{\eta_3 q}dzdr \\
&\quad + \int_{\hbar}^{t}\int_{0}^{r-\hbar} r^{q(\theta-1)}|r-\hbar-z|^{(-\alpha-\eta_3)q}\hbar^{\eta_3 q}dzdr \\
&\simeq \hbar^{\eta_3 q}\cdot \int_{\hbar}^{t}\int_{0}^{r-\hbar} r^{q(\theta-1)} z^{(-\alpha-\eta_3)q}dzdr \\
&\lesssim \hbar^{\eta_3 q}\cdot \int_{0}^{t} r^{1-(\alpha+\eta_3)q+q(\theta-1)}dr \lesssim \hbar^{\eta_3 q} \lesssim \hbar^{\gamma q}\,.
\end{aligned}$$

Now we proceed to deal with $\mathcal{J}^{(1)}_{1,3}(t,x,y)$ in (3.4.30). By the similar way as dealing with $\mathcal{J}^{(1)}_{1,2}(t,x,y)$, we have with B_4 and B_5 defined by (3.4.29).

$$\begin{aligned}
&\int_{\hbar}^{t}\int_{\mathbb{R}} r^{q(\theta-1)}\left|(r-|z+\hbar|)^{-\alpha}+(r-|z|)^{-\alpha}\right|^{q}\cdot(1_{B_4}+1_{B_5})\,dzdr \\
&\quad = \int_{\hbar}^{t}\int_{-r-\hbar}^{-r+\hbar} r^{q(\theta-1)}\left(|r-|z+\hbar||^{-\alpha q}+|r-|z||^{-\alpha q}\right)dzdr \\
&\quad\quad + \int_{\hbar}^{t}\int_{r-\hbar}^{r+\hbar} r^{q(\theta-1)}\left(|r-|z+\hbar||^{-\alpha q}+|r-|z||^{-\alpha q}\right)dzdr \\
&\quad \simeq \int_{\hbar}^{t}\int_{-\hbar}^{\hbar} r^{q(\theta-1)}|z|^{-\alpha q}dzdr + \int_{\hbar}^{t}\int_{0}^{2\hbar} r^{q(\theta-1)}|z|^{-\alpha q}dzdr \\
&\quad \lesssim \hbar^{1-\alpha q}\int_{0}^{t} r^{q(\theta-1)}dr \lesssim \hbar^{1-\alpha q} \lesssim \hbar^{\gamma q}\,,
\end{aligned}$$

under the same conditions as (3.8.5):

$$\alpha < \frac{1}{q} = 1-\frac{1}{p}\,,\ \theta > \frac{1}{p}\,,\ \alpha+\gamma < \frac{1}{q}\,. \tag{3.8.13}$$

Therefore, if α, θ satisfy (Π.1) and η_1, η_2, η_3 satisfy (3.8.9), then we have our desire upper bounds for $\sup_{t,x,y}\mathcal{J}^{(1)}_{1,k}(t,x,y)$ $(k=1,2,3)$. □

Lemma 3.8.5. *Suppose α, θ satisfy* (Π.1)*, and moreover*

$$\eta_4 \textit{ satisfies } (\Pi.2) \textit{ and } (\Pi.3)\,. \tag{3.8.14}$$

Then the terms $\mathcal{J}_{2,k}^{(1)}(t,x,y)$ *in* (3.4.34) *can be bounded as follows*

$$\sup_{t,x,y} \mathcal{J}_{2,k}^{(1)}(t,x,y) \lesssim C_{T,p,H,\gamma}|\hbar|^{\gamma q} \quad \textit{for} \;\; k=1,2,3.$$

Proof. Similar to the way when we deal with $\mathcal{I}_2^{(1)}$ in the proof of part **(i)** of Proposition 3.4.1, $\mathcal{J}_{2,1}^{(1)}(t,x,y)$ can be bounded by $\hbar^{\gamma q}$ under the condition (3.8.2) which holds under condition (3.8.14). Let us recall the definitions of C_1, C_2 and C_3 in (3.4.32), then for $\mathcal{J}_{2,2}^{(1)}(t,x,y)$ we have

$$\begin{aligned}
\mathcal{J}_{2,2}^{(1)}(t,x,y) &= \int_0^\hbar \int_{\mathbb{R}} r^{q(\theta-1)} \left|\mathfrak{D}_\hbar (r-|z|)^{-\alpha}\right|^q \cdot (1_{C_1}+1_{C_2})\, dzdr \\
&\le \int_0^\hbar \int_{z<-r-\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_4)q} \hbar^{\eta_4 q} dzdr \\
&\quad + \int_0^\hbar \int_{z>r+\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_4)q} \hbar^{\eta_4 q} dzdr\,. \qquad (3.8.15)
\end{aligned}$$

For the first term of the summation in (3.8.15), we have

$$\begin{aligned}
&\int_0^\hbar \int_{z<-r-\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_4)q} \hbar^{\eta_4 q} dzdr \\
&\qquad = \hbar^{\eta_4 q} \int_0^\hbar \int_{z<-r-\hbar} r^{q(\theta-1)} (-z-r)^{(-\alpha-\eta_4)q} dzdr \\
&\qquad = \hbar^{\eta_4 q} \int_0^\hbar \int_{z>\hbar} r^{q(\theta-1)} z^{(-\alpha-\eta_4)q} dzdr \\
&\qquad \simeq \hbar^{\eta_4 q} \hbar^{1-(\alpha+\eta_4)q} \int_0^\hbar r^{q(\theta-1)} dr \\
&\qquad \simeq \hbar^{\eta_4 q + 1 - (\alpha+\eta_4)q + 1 + q(\theta-1)} \lesssim \hbar^{\gamma q},
\end{aligned}$$

under the same conditions as (3.8.2) with η_1 replaced by η_4.

Similarly, we have for the second term of the sum in (3.8.15)

$$\begin{aligned}
&\int_0^\hbar \int_{z>r+\hbar} r^{q(\theta-1)} |r-|z||^{(-\alpha-\eta_4)q} \hbar^{\eta_4 q} dzdr \\
&\quad = \int_0^\hbar \int_{z>r+\hbar} r^{q(\theta-1)} (z-r)^{(-\alpha-\eta_4)q} \hbar^{\eta_4 q} dzdr \\
&\quad = \hbar^{\eta_4 q} \int_0^\hbar \int_{z>\hbar} r^{q(\theta-1)} z^{-(\alpha+\eta_4)q} dzdr
\end{aligned}$$

$$\simeq \hbar^{\eta_4 q}\hbar^{1-(\alpha+\eta_4)q}\int_0^{\hbar} r^{q(\theta-1)}dr \lesssim \hbar^{\eta_4 q+1-(\alpha+\eta_4)q+1+q(\theta-1)},$$

which can be bounded by $\hbar^{\gamma q}$ if the condition (3.8.2) with η_1 replaced by η_4 holds.

For the last term $\mathcal{J}^{(1)}_{2,3}(t,x,y)$, if $\alpha,\ \theta$ satisfy (Π.1), then the conditions

$$\alpha < \frac{1}{q} = 1 - \frac{1}{p}\,,\ \theta > \frac{1}{p}\,,\ \theta > 1+\alpha-\frac{2}{q}+\gamma\,, \tag{3.8.16}$$

are satisfied. So we have

$$\begin{aligned}
\mathcal{J}^{(1)}_{2,3}(t,x,y) &= \int_0^{\hbar}\int_{\mathbb{R}} r^{q(\theta-1)}\left|(r-|z+\hbar|)^{-\alpha}+(r-|z|)^{-\alpha}\right|^q \cdot 1_{C_3}dzdr \\
&= \int_0^{\hbar}\int_{-r-\hbar}^{r+\hbar} r^{q(\theta-1)}\left(\left|r-|z+\hbar|\right|^{-\alpha q}+\left|r-|z|\right|^{-\alpha q}\right)dzdr \\
&\simeq \int_0^{\hbar}\int_0^{r} r^{q(\theta-1)}|z|^{-\alpha q}dzdr + \int_0^{\hbar}\int_0^{\hbar} r^{q(\theta-1)}|z|^{-\alpha q}dzdr \\
&\simeq \int_0^{\hbar} r^{q(\theta-1)+1-\alpha q}dr + \hbar^{1-\alpha q}\int_0^{\hbar} r^{q(\theta-1)}dr \\
&\lesssim \hbar^{2-\alpha q+q(\theta-1)} \lesssim \hbar^{\gamma q}\,.
\end{aligned}$$

Thus, the proof is complete. □

Lemma 3.8.6. *Suppose $\alpha,\ \theta$ satisfy (Π.1) and*

$$\begin{cases} \eta_2 \ \textit{satisfies (Π.3)}; \\ \eta_3 \ \textit{satisfies (Π.4)}; \\ \eta_4 \ \textit{satisfies (Π.2) and (Π.3)}\,. \end{cases} \tag{3.8.17}$$

Then the $\mathcal{J}^{(2)}_{2,1,j}(t,x,y),\ j=1,\cdots,6$ in (3.4.39) can be bounded as follows.

$$\sup_{t,x,y}\mathcal{J}^{(2)}_{2,1,j}(t,x,y)\lesssim|\hbar|^{\gamma q}.$$

Proof. Let us recall the definitions of $D_1,\cdots,D_6$ in (3.4.38). Firstly, we deal with $\mathcal{J}^{(2)}_{2,1,1}$

and $\mathcal{J}^{(2)}_{2,1,5}$ on D_1 and D_5 successively. We have

$$
\begin{aligned}
&\mathcal{J}^{(2)}_{2,1,1}(t,x,y)+\mathcal{J}^{(2)}_{2,1,5}(t,x,y)\\
&=|\hbar|^{\eta_2 q}\int_0^t\int_{z<-r-\hbar} r^{q(\theta-1)}(-z-r)^{-(\alpha+\eta_2)q}dzdr\\
&\quad+|\hbar|^{\eta_2 q}\int_0^t\int_r^{r+\hbar} r^{q(\theta-1)}(z-r)^{-(\alpha+\eta_2)q}dzdr\\
&\lesssim|\hbar|^{\eta_2 q}\int_0^t r^{q(\theta-1)}dr\cdot\int_{\widetilde{z}>\hbar}(\widetilde{z})^{-(\alpha+\eta_2)q}dz\\
&\quad+|\hbar|^{\eta_2 q}\int_0^t r^{q(\theta-1)}dr\cdot\int_0^{\hbar}(\widehat{z})^{-(\alpha+\eta_2)q}dz\,,
\end{aligned}
\tag{3.8.18}
$$

through changing of variables $\widetilde{z}=-z-r$ and $\widehat{z}=z-r$. Thus, it can be bounded by $|\hbar|^{\gamma q}$ if

$$
\alpha+\eta_2>\frac{1}{q},\ \ \eta_2>\gamma. \tag{3.8.19}
$$

In the same way, we can deal with $\mathcal{J}^{(2)}_{2,1,6}(t,x,y)$ by changing of variable $\widehat{z}=z-r$,

$$
\begin{aligned}
&|\hbar|^{\eta_3 q}\int_0^t\int_{z>r+\hbar} r^{q(\theta-1)}(z-r)^{-(\alpha+\eta_3)q}dzdr\\
&\qquad\lesssim|\hbar|^{\eta_3 q}\int_0^t r^{q(\theta-1)}dr\cdot\int_{\widehat{z}>\hbar}(\widehat{z})^{-(\alpha+\eta_3)q}dz\lesssim|\hbar|^{\gamma q},
\end{aligned}
$$

which requires η_3 satisfying the conditions (3.8.19) and

$$
\alpha+\eta_3<\frac{1}{q},\ \ \eta_3>\gamma. \tag{3.8.20}
$$

Similarly, by changing of variable $z\to z+\hbar$ and then $z\to rz$, we have on D_3,

$$
\begin{aligned}
\mathcal{J}^{(2)}_{2,1,3}(t,x,y)&\lesssim\hbar^{\eta_4 q}\int_0^t\int_{-r}^{r-\hbar} r^{q(\theta-1)}|r-|z+\hbar||^{-(\alpha+\eta_4)q}dzdr\\
&\lesssim\hbar^{\eta_4 q}\int_0^t\int_{-r}^{r} r^{q(\theta-1)}|r-|z||^{-(\alpha+\eta_4)q}dzdr\\
&=\hbar^{\eta_4 q}\int_0^t r^{q(\theta-1)-(\alpha+\eta_4)q+1}dr\cdot\int_0^1|1-|z||^{-(\alpha+\eta_4)q}dz\lesssim\hbar^{\gamma q}\,,
\end{aligned}
\tag{3.8.21}
$$

which requires the same condition as (3.8.2) with η_1 replaced by η_4 here.

As for $\mathcal{J}^{(2)}_{2,1,2}(t,x,y)$ and $\mathcal{J}^{(2)}_{2,1,4}(t,x,y)$, we have

$$\begin{aligned}
&\mathcal{J}^{(2)}_{2,1,2}(t,x,y)+\mathcal{J}^{(2)}_{2,1,4}(t,x,y)\\
&\quad=\int_0^t\left(\int_{-r-\hbar}^{-r}+\int_{r-\hbar}^{r}\right)r^{q(\theta-1)}\left|\mathfrak{D}_\hbar|r-|z||^{-\alpha}\right|^q dzdr\\
&\quad\lesssim\int_0^t\left(\int_{-r-\hbar}^{-r}+\int_{r-\hbar}^{r}\right)r^{q(\theta-1)}|r-|z+\hbar||^{-\alpha q}dzdr\\
&\qquad+\int_0^t\left(\int_{-r-\hbar}^{-r}+\int_{r-\hbar}^{r}\right)r^{q(\theta-1)}|r-|z||^{-\alpha q}dzdr\\
&\quad\lesssim\int_0^t r^{q(\theta-1)}dr\cdot\int_0^\hbar|z|^{-\alpha q}dz\lesssim|\hbar|^{1-\alpha q}\lesssim|\hbar|^{\gamma q},
\end{aligned}\tag{3.8.22}$$

if we require

$$\theta>\frac{1}{p},\ \alpha<\frac{1}{q},\ \alpha+\gamma<\frac{1}{q}.\tag{3.8.23}$$

Thus, if $\alpha,\ \theta$ satisfy (Π.1) and if (3.8.17) holds, then all the restrictions on η's are satisfied. The proof is then complete. □

Lemma 3.8.7. *Suppose $\alpha,\ \theta$ satisfy (Π.1) and moreover*

$$\eta_4,\eta_5\ \textit{satisfy}\ (\Pi.2)\ \textit{and}\ (\Pi.3)\,.\tag{3.8.24}$$

Then the terms $\sup_{t,x,y}\mathcal{J}^{(2)}_3(t,x,y)$ and $\sup_{t,x,y}\mathcal{J}^{(2)}_4(t,x,y)$ in (3.4.42) can be bounded by a constant multiple of $|\hbar|^{\gamma q}$.

Proof. For the term $\mathcal{J}^{(2)}_3(t,x,y)$, by (3.4.40) and the inequality $|z+\hbar|^{\eta_4 q}\lesssim|z|^{\eta_4 q}+|\hbar|^{\eta_4 q}$, we have

$$\begin{aligned}
\mathcal{J}^{(2)}_3(t,x,y)\lesssim&|\hbar|^{\eta_4 q}\int_0^t\int_{\mathbb{R}}r^{q(\theta-1)}(r^2+z^2)^{-(\frac{\alpha}{2}+\eta_4)q}|z|^{\eta_4 q}dzdr\\
&+|\hbar|^{2\eta_4 q}\int_0^t\int_{\mathbb{R}}r^{q(\theta-1)}(r^2+z^2)^{-(\frac{\alpha}{2}+\eta_4)q}dzdr\\
=&|\hbar|^{\eta_4 q}\int_0^t r^{q(\theta-1)-(\alpha+2\eta_4)q+\eta_4 q+1}dr\cdot\int_{\mathbb{R}}(1+z^2)^{-(\frac{\alpha}{2}+\eta_4)q}|z|^{\eta_4 q}dz\\
&+|\hbar|^{2\eta_4 q}\int_0^t r^{q(\theta-1)-(\alpha+2\eta_4)q+1}dr\cdot\int_{\mathbb{R}}(1+z^2)^{-(\frac{\alpha}{2}+\eta_4)q}dz\,,
\end{aligned}\tag{3.8.25}$$

which can be bounded by $|\hbar|^{\gamma q}$ under the following conditions

$$\eta_4 > \gamma\,,\;\; \theta - 2\eta_4 > 1 + \alpha - \frac{2}{q}\,,\;\; \alpha + \eta_4 > \frac{1}{q}. \tag{3.8.26}$$

As for the term $\mathcal{J}_4^{(2)}(t,x,y)$, by inequality (3.4.41) and by changing of variable $z \to rz$,

$$\begin{aligned}\mathcal{J}_4^{(2)}(t,x,y) &\lesssim |\hbar|^{\eta_5 q} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} \frac{r^{\eta_5 q}}{(r^2+z^2)^{\eta_5 q}} (r^2+z^2)^{-\frac{\alpha}{2}q} dz dr \\ &\lesssim |\hbar|^{\eta_5 q} \int_0^t r^{q(\theta-1)-\eta_5 q - \alpha q + 1} dr \cdot \int_{\mathbb{R}} (1+z^2)^{-\frac{\alpha}{2}q - \eta_5 q} dz\,, \end{aligned} \tag{3.8.27}$$

which can be bounded by $|\hbar|^{\gamma q}$ under conditions (3.8.2) with η_1 substituted by η_5. So we complete the proof by noticing that (3.8.26) and (3.8.2) are implied by (3.8.24). □

Chapter 4

Intermittency properties for a large class of stochastic PDEs driven by fractional space-time noises

4.1 Introduction

In this chapter, we consider the following stochastic partial differential equation in the d-dimensional Euclidean space $\mathbb{R}^d$:

$$\mathscr{L}u(t,x) = u(t,x)\dot{W}(t,x)\,, \qquad t>0,\ x\in\mathbb{R}^d \tag{4.1.1}$$

with certain given initial condition. Here $\mathscr{L}$ denotes a general (including fractional order) partial differential operator and $\dot{W}(t,x) = \frac{\partial^{d+1}}{\partial t\partial x_1\cdots\partial x_d}W(t,x)$ is mean zero Gaussian noise. We would like to enable our approach to be applied to a large class of operators $\mathscr{L}$. For this reason, instead of giving the concrete form of $\mathscr{L}$, we shall impose conditions satisfied by the Green's function associated with $\mathscr{L}$ (see e.g. [HHNS15, NQS07] for a similar spirit).

One of the most studied properties of the solution is the intermittency arose from physics. An intermittent random field is a random function of space variable x consisting of 'high peaks' which give the most contribution to the processes. This property is related the moment bounds of the solution (p-th moment Lyapunov exponent, in particular). In order to formulate the mathematical definition of intermittency for a random field $u = \{u(t,x) : t \geq 0, x \in \mathbb{R}^d\}$, researchers consider the *upper and lower (moment) Lyapunov exponents*

$$\overline{\lambda}(p) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p] , \tag{4.1.2}$$

$$\underline{\lambda}(p) := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p] . \tag{4.1.3}$$

If $\overline{\lambda}(p) = \underline{\lambda}(p)$, then we call this common value the *p-th moment Lyapunov exponent*, denoted by $\lambda(p)$. Traditionally, the random-field u is called *intermittent* if $[1,\infty) \ni p \to \lambda(p)/p$ is strictly increasing. We refer to [CM94, Kho14] and references therein for more discussion.

When (4.1.1) is parabolic Anderson model, namely, when $\mathscr{L} = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$ is a heat operator, there are many results about the sharp (both lower and upper) moment bounds, see e.g. [BC16, CHKN18, Che17, CHSS18, CHSX15, HHL+18, HHNT15, Lyu20], and we also refer to [Hu19] for a recent survey. Let us also mention that there are many works on the behaviour of 'high peaks' of the parabolic Anderson model and general nonlinear parabolic stochastic PDEs. For instance, the macroscopic multifractal analysis of such random sets was studied in [KKX17, KKX18] and the estimates of the length of the 'intermittency islands' corresponding to 'high peaks' was studied in [CJK12].

However, when $\mathscr{L}$ is a wave operator, namely the hyperbolic Anderson model, or when $\mathscr{L}$ is (temporal) fractional differential operators, the situation is different. We summarize the known results (to the best of our knowledge) as follows.

(i) Similarly to the stochastic heat equation (parabolic Anderson model) we can use the chaos expansion and the hypercontractivity inequality to obtain the upper bounds (which we believe to be sharp). It is also possible to obtain the lower bound for the second moment. There are many contributions on these aspects and among them we mention only a few [BC14, BC16, BJQS17, CHHH17, SSX20] and the references therein. However, it is hard to obtain the sharp lower bound for any p-th moment which matches the upper bounds in terms of the growth of p.

(ii) Until now the success to obtain a sharp lower moment bounds for the solution of stochastic heat equations largely relies on the clever application of the Feynman-Kac formula. However, there is no effective analogous formula for other equations such as wave equations. The only work that we know is [DM09, Theorem 4.1], where the authors use an Feynman-Kac like formula for stochastic wave equation obtained in [DMT08] to obtain a nice lower bound for all moments when the Gaussian noise is white in time and "smooth" in space. However, this formula is hard to use to obtain sharp lower moment bounds for other more general Gaussian noises. Let us mention that after the completion of this work, we learn the announcement of a work [Qia], where the Gaussian noise is what they called Dobrić-Ojeda one, namely, noise is still white in time but with a weight and the equation is one dimensional stochastic wave equation. The idea is still to make more careful use of the Feynman-Kac like formula obtained in [DMT08].

(iii) For the time-independent noise, the authors of [BCC20, CE21] obtain the exact asymptotic behaviour of the p-th moment of the solution to (4.1.1) when $\mathscr{L}$ is wave operators or (temporal) fractional differential operators. However, it seems unlikely that their method is applicable to the time dependent noise.

The objective of this chapter is to obtain the sharp lower bounds (which matched the upper bounds) for all moments when the operator $\mathscr{L}$ in (4.1.1) is a wave operator of dimension one, two, or three or an operator which is fractional both in time and in space. The Gaussian noise $\dot{W}$ can be very general and it does not need to be white or even fractional in time.

The approach that we use is a generalization of the Feynman diagram formula for the moments of the solution. This formula allows us to keep track the extremely sophisticated terms in the expectation of the product of several multiple Wiener-Itô integrals. It is in some sense a brutal force method. We fully explore the positivity of the Green's function. This property enables us to throw away some complicated terms and to keep the main terms so that the remaining ones contain the essential contribution on one hand and on the other hand are possible to manage although still very sophisticated. After the (fortunately successful) isolation of the leading terms there still remains an extremely challenging problem of how to bound them from below. At the first glance this seems an impossible task since it needs to perform very sophisticated multiple integral computation involving the singular Green's function and the covariance structure of the Gaussian noise. To make the estimate of these multiple integrals possible, we discover a new property which we call the *small ball nondegeneracy* of the Green's functions, which helps us to significantly reduce the difficulty so that we are able to handle these multiple integrals. Of course, the remaining task to bound the multiple integrals is still highly technical but somehow possible. We shall show that many popular Green's functions satisfy this small ball nondegeneracy property. Similarly, to obtain the upper moment bounds we discover another property of Green's function, which we call the bounded Hardy-Littlewood-Sobolev total mass which can guarantee the upper moment bounds.

The Feynman diagram type formula that we obtain may be essentially analogous to the Feynman-Kac type formula obtained in [DMT08]. However, the former one is straightforward and is more convenient for us to manage. Since we only use the positivity and the small ball nondegeneracy properties for the lower moment bounds, our approach is applicable to a very large class of equations and to a large class of Gaussian noises. The equations include stochastic heat equation, stochastic wave equation (SWE, $\mathscr{L} = \partial_t^2 - \Delta$), stochastic heat equation which is inhomogeneous and fractional in space ((α, A)-SHE, $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$), where A is a positive definite symmetric matrix, stochastic partial differential equations which is both fractional in time and in space (SFDE, $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$) (see e.g. [BC14, BC16, CHHH17, CHSS18, CHW18, DM09] and references therein for the study of this type of equations). As for the noise, we can allow the noise structure to be very general: we only need to assume that the covariance function is bounded (above or/and below) by some singular power functions as those in [HHNT15]. In particular, they need not necessarily to be white or fractional in time or in space.

Here is the organization of the chapter. In Section 4.2 we give the noise structure and introduce the stochastic integral, mild solution, and chaos expansion. Section 4.3 proposes the general conditions satisfied by the Green's function associated with $\mathscr{L}$ and states our main results. Sections 4.4 is devoted to prove the upper moment bounds for the solution. This is done by using the chaos expansion and the hypercontractivity inequality. Our main tool to prove the lower moment bounds is the generalization of the Feynman diagram formula for the expected value of product of several multiple Wiener-Itô integrals. This formula is presented in Section 4.5. After this preparation, in Sections 4.6 we prove the lower moment bounds of the solution. To encure that some famous operators $\mathscr{L}$ satisfies the positivity **(G1)** and small ball nondegeneracy conditions **(G2)** so that our results can be applied to cover a large class of interesting stochastic partial equations, we verify these conditions for various interesting operators $\mathscr{L}$ in Section 4.7.

Throughout the entire chapter, we shall use the notations $\lesssim$, $\gtrsim$, and $\simeq$ extensively. The meaning are conventional. This is, $A \lesssim B$ (or $A \gtrsim B$) means that there are constants $C \in (0, \infty)$ such that $A \le CB$ (or $B \le CA$, respectively). The notation $A \simeq B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold true.

4.2 Noise covariance structure, mild solution and chaos expansion

In this section, we give the conditions satisfied by the covariance of the noise $\dot{W}$ in (4.1.1). For this Gaussian noise we also define the (Skorohod type) integral, the mild solution, and the chaos expansion of the solution candidate. These concepts are known, so we recall them very quickly to fix the notation throughout the chapter. We refer to [CHHH17, Hu17, HHL+17, HHL+18, HHNT15] and the references therein for more details. The existence and uniqueness of the solution in our new situation will be a consequence of the upper moment bounds.

4.2.1 Noise covariance structure

We assume that the noise $\dot{W}(t,x) = \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x)$ is mean zero Gaussian with the following covariance structure:

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \gamma(t-s)\Lambda(x-y)\,. \tag{4.2.1}$$

The restriction that the covariance of the noise has this product form of a function of time variables and a function of space variables is convenient. The reason that the time function is of the form $\gamma(t-s)$ means that the noise is stationary (or the original process W has stationary increment). The space function is of the form $\Lambda(x-y)$ means that the noise is homogeneous.

In order to simplify our presentation and in order to cover the typical examples we make the following assumptions. For γ we assume

(H1) There is a $\gamma \in (0,1)$ such that

$$c|t|^{-\gamma} \le \gamma(t) \le C|t|^{-\gamma}\,, \quad \forall\, t \in \mathbb{R}_+$$

for some positive constants c, C. For convenience, when $\gamma = 1$ we mean $\gamma(t) = \delta(t)$.

For $\Lambda(\cdot)$ we assume that it satisfies one of the following three conditions:

(H2) There is $\lambda \in (0,d)$ such that

$$c|x|^{-\lambda} \le \Lambda(x) \le C|x|^{-\lambda}\,, \quad \forall\, x \in \mathbb{R}^d\,.$$

(H3) There are constants $\lambda_j \in (0,1), j = 1, \cdots, d$ such that

$$c\prod_{j=1}^d |x_j|^{-\lambda_j} \le \Lambda(x) \le C\prod_{j=1}^d |x_j|^{-\lambda_j}\,, \quad \forall\, x \in \mathbb{R}^d\,.$$

In this case we denote $\lambda = \sum_{i=1}^d \lambda_i$.

(H4) When $d = 1$ and $\gamma = 1$, we assume $\Lambda(x) = \delta(x)$.

4.2.2 Stochastic integral

We follow the approach of [BC16, Hu17, HHNT15, HN09, Nua06] to define stochastic integral. First, let us recall the Fourier transform with respect to the spatial variables. Denote by $\mathcal{D}(\mathbb{R}^d)$ the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}^d$ (We can also introduce $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ in a similar way). The Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}[f(\cdot)](\xi) = \int_{\mathbb{R}^d} e^{-\iota \xi \cdot x} f(x) dx \,,$$

and the the inverse Fourier transform is given by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-d} \mathcal{F}[f(\cdot)](-x) \,.$$

Let $\mathcal{H}$ is the Hilbert space defined as the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$ equipped with the inner product given by

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \phi(t,x)\psi(s,y)\gamma(t-s)\Lambda(x-y) dt dx ds dy \tag{4.2.2}$$

$$= \frac{1}{(2\pi)^d} \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \gamma(t-s)\hat{\phi}(t,\cdot)(\xi)\overline{\hat{\psi}(s,\cdot)(\xi)}\mu(d\xi) \,, \tag{4.2.3}$$

where $\gamma : \mathbb{R} \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ are non-negative definite functions and satisfy **(H1)** and one of the conditions **(H2)**-**(H4)** introduced at the beginning of this section. Note that the space $\mathcal{H}$ contains generalized functions.

The noise $\dot{W}$ can be described by an isonormal family of mean zero Gaussian random variables $\{W(\phi)\,; \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ with the covariance $\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathcal{H}}$ for all ϕ and ψ in $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$. This isometry can be extended to $\mathcal{H}$ and is denoted by

$$W(\phi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \phi(t,x) W(dt,dx) \,, \quad \text{for all } \phi \in \mathcal{H} \,.$$

Let $\mathcal{P}$ be the set of smooth and cylindrical random variables of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)) \,,$$

with $\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (i.e. f and all its partial derivatives have polynomial growth). For $F \in \mathcal{P}$ of the above form we define DF as the $\mathcal{H}$-valued random variable by the following expression

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j \,.$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of $\mathcal{P}$ under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|^2_{\mathcal{H}}]}\,.$$

Given any element $u \in L^2(\Omega;\mathcal{H})$ if there is a $v \in L^2(\Omega)$ such that

$$\mathbb{E}\left(vF\right) = \mathbb{E}\left(\langle DF, u\rangle_{\mathcal{H}}\right) \qquad \text{for any } F \in \mathbb{D}^{1,2} \tag{4.2.4}$$

then we say that u is in the domain of δ and we call it the *Skorohod integral* of u, denoted by

$$v = \delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(dt,dx)\,.$$

Obviously, when such v (satisfying (4.2.4)) exists, it is unique. We refer to [Hu17, HHNT15, Nua06] for more details. Now with the Skorohod integral introduced, we give the concept of mild solution as follows. But first, let us briefly recall the concept of Green's function. Suppose $f(t,x), t \geq 0, x \in \mathbb{R}^d$ is a nice (smooth with compact support) function and consider the corresponding deterministic equation

$$\mathscr{L}u(t,x) = f(t,x)\,, \qquad t > 0,\ x \in \mathbb{R}^d\,. \tag{4.2.5}$$

with the same initial condition(s) as in (4.1.1). The Green's function associated with $\mathscr{L}$ is a (possibly generalized) function $G_{t,s}(x,y), 0 \leq s < t < \infty, x, y \in \mathbb{R}^d$ or a measure $G_{t,s}(x,y)dy := G_{t-s}(x,dy)$ (we omit the explicit dependence of G on $\mathscr{L}$) such that the solution to (4.2.5) is given explicitly by

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x,y) f(s,y) dyds\,, \tag{4.2.6}$$

where the term $I_0(t,x)$ depends on the initial data and the Green's function.

If we formally replace $f(s,y)$ in (4.2.6) by $u(s,y)\dot{W}(s,y)$ and replace $\dot{W}(s,y)dsdy$ by the Skorohod type stochastic integral $W(ds,dy)$, then the solution to (4.1.1) satisfies

$$u(t,x)=I_0(t,x)+\int_0^t\int_{\mathbb{R}^d}G_{t-s}(x,y)u(s,y)W(ds,dy)\,, \tag{4.2.7}$$

where the stochastic integral is interpreted in the Skorohod sense and $I_0(t,x)$ is from the initial condition(s) and the Green's function. . However, Unlike the previous identity (4.2.6) the expression (4.2.7) is still an equation on u. It is impossible to make sense for each of the terms $\mathscr{L}u(t,x)$ and $u(t,x)\dot W(t,x)$ in a straightforward way so it is impossible to find a solution satisfies (4.1.1) literally. But it is possible to find $u(t,x)$ satisfies (4.2.7). A random field $u(t,x)$ satisfying (4.2.7) will be called a mild solution (or random field solution) to (4.1.1). Here is its definition:

Definition 4.2.1. *An adapted random field $\{u(t,x): t\ge 0\,, x\in\mathbb{R}^d\}$ so that $\mathbb{E}[|u(t,x)|^2]<\infty$ for all $t\ge 0$ and $x\in\mathbb{R}^d$ is called a mild solution to equation (4.1.1) if for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ the process*

$$\{G_{t-s}(x,y)u(s,y)1_{[0,t]}(s)\ :\ s\ge 0\,,y\in\mathbb{R}^d\}$$

is Skorohod integrable, and $u(t,x)$ satisfies (4.2.7).

If u is a mild solution to (4.1.1), namely if u satisfies (4.2.7), then $u(s,y)=I_0(s,y)+\int_0^t\int_{\mathbb{R}^d}G_{s-r}(y,z)u(r,z)W(dr,dz)$. Substituting this expression into (4.2.7) we obtain

$$\begin{aligned}u(t,x) &= I_0(t,x)+\int_0^t\int_{\mathbb{R}^d}G_{t-s}(x,y)I_0(s,y)W(ds,dy)\\ &\quad+\int_0^t\int_0^r\int_{\mathbb{R}^{2d}}G_{t-s}(x,y)G_{s-r}(y,z)I_0(r,z)W(dr,dz)W(ds,dy)\,.\end{aligned}$$

Repeating this procedure we obtain a solution candidate for the equation (4.2.7):

$$u(t,x)=I_0(t,x)+\sum_{n=1}^\infty I_n(f_n(\cdot,t,x))\,. \tag{4.2.8}$$

Here

$$f_n(\cdot,t,x):=f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \tag{4.2.9}$$

$$=\frac{1}{n!}\sum_{\sigma\in S_n} G_{t-t_{\sigma(n)}}(x,x_{\sigma(n)})G_{t_{\sigma(n)}-t_{\sigma(n-1)}}(x_{\sigma(n)},x_{\sigma(n-1)})\cdots$$

$$\times G_{t_{\sigma(2)}-t_{\sigma(1)}}(x_{\sigma(2)},x_{\sigma(1)})I_0(t_{\sigma(1)},x_{\sigma(1)})1_{\{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<t\}}$$

is the symmetrization of

$$G_{t-t_n}(x,x_n)G_{t_n-t_{n-1}}(x_n,x_{n-1})\cdots$$

$$\times G_{t_2-t_1}(x_2,x_1)I_0(t_1,x_1)1_{\{0<t_1<\cdots<t_n<t\}}\,, \tag{4.2.10}$$

where S_n denotes the set of all permutations of $\{1,\dots,n\}$; and $I_n(f_n(\cdot,t,x))$ is the multiple Wiener-Itô integral (e.g. [Hu17, Nua06]). The expression (4.2.8) is called the Wiener chaos expansion (or simply chaos expansion) of the solution. It is known that if (4.2.8) is convergent in $L^2(\Omega)$, then (4.1.1) has a unique mild solution.

4.3 Small ball nondegeneracy and main results

4.3.1 Small ball nondegeneracy for Green's function

Our main aim of this chapter is to study the lower and upper asymptotics of the moments of mild solution defined in (4.2.7), which match with each other. What we need are the following assumptions on the Green's function associated with the operator $\mathscr{L}$. The following assumptions are made in order to derive the sharp lower asymptotics:

(G1) [**Positivity**]: $G_t(\cdot,\cdot)$ is a positive function, measure, or generalized function.

(G2) [**Small ball nondegeneracy**]: $G_t(\cdot,\cdot)$ satisfies the *small ball nondegeneracy (*$B(\mathfrak{a},\mathfrak{b})$*)*. This is, there exist real numbers $\mathfrak{a}$ and $\mathfrak{b}$ (depending on the Green's function) satisfying

$$\mathfrak{a}>-1\,,\quad \mathfrak{b}>0\,,\quad \text{and}\quad \mathfrak{b}(2\mathfrak{a}+1)-\lambda>0\,, \tag{4.3.1}$$

and there is a constant $C>0$ such that

$$\inf_{y\in B_\varepsilon(x)}\int_{B_\varepsilon(x)} G_t(y,z)dz \ge C\cdot t^{\mathfrak{a}}\,, \tag{4.3.2}$$

for all $0 < t \le \varepsilon^{\mathfrak{b}}(\le 1)$ and $x \in \mathbb{R}^d$, where $B_\varepsilon(x)$ is the ball of center x with radius ε.

To obtain the upper bound for moments, we need is the following hypothesis for the Green's function.

(G3) [**HLS-type mass property**]: $G_t(\cdot,\cdot)$ satisfies what we shall call the *Hardy-Littelewood-Sobolev type mass property* $M(\hbar)$. That is, there exist a real number $\hbar$ and a constant $C > 0$ satisfying

$$\hbar > -1\,, \tag{4.3.3}$$

and

$$\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t(x,y)\Lambda(y-y')G_t(x',y')dydy' \le C\cdot t^{\hbar}\,. \tag{4.3.4}$$

Remark 4.3.1. *The tasks to verify the assumptions (G1)-(G3), in particular to find the sharp indices $\mathfrak{a},\mathfrak{b},\hbar$ in* (4.3.2)-(4.3.4) *are not trivial. We shall dedicate one section (Section 4.7) to verify these conditions for various partial differential operators $\mathscr{L}$ that are currently interested by researchers. For different operators $\mathscr{L}$, we shall obtain the best indices $\mathfrak{a},\mathfrak{b},\hbar$ in the sense that our upper and lower p-th moment will match each other as p or t tends to infinity.*

Remark 4.3.2. *The hypothesis (G3) is quite standard for the upper moment bounds. When G_t is a function (rather than a measure), then we can easily apply Hardy-Littelewood-Sobolev inequality (*[LL97, Theorem 4.3]*) to obtain*

$$\begin{aligned}
&\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t(x,y)\Lambda(y-y')G_t(x',y')dydy' \\
&\qquad\le \sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t(x,y)|y-y'|^{-\lambda}G_t(x',y')dydy' \\
&\qquad\le \sup_{x\in\mathbb{R}^d}\left[\int_{\mathbb{R}^d}|G_t(x,y)|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}}.
\end{aligned}$$

We shall use this inequality to verify (4.3.4) *for some operators $\mathscr{L}$ in Section 4.7.*

Remark 4.3.3. *In this remark, we give some intuitive connections between (G2) and (G3). If we assume $G_t(x,y) = G_t(x-y)$ satisfies what we shall call the* total weighted mass property $\bar{M}(\mu,\nu)$*: there exist real numbers* μ *and* ν *(depending on the Green's function) satisfying*

$$\mu > -1\,, \quad \nu \in \mathbb{R}\,, \quad \text{and} \quad \mu+\nu > -1\,, \tag{4.3.5}$$

and there are two positive constants C_1 *and* C_2 *such that*

$$\begin{cases} \int_{\mathbb{R}^d} G_t(y)dy \le C_1 \cdot t^{\mu}\,, \\ \sup\limits_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_t(x-y)\Lambda(y)dy \le C_2 \cdot t^{\nu}\,. \end{cases} \tag{4.3.6}$$

Then we can easily see (4.3.4) *holds with* $\hbar = \mu + \nu > -1$*. Furthermore, we notice that* $\mu = \mathfrak{a}$*, where* $\mathfrak{a}$ *is the same parameter in* ***(G2)****.*

Let us discuss the SHE and SWE in one dimension as examples. It is easy to see from Hardy-Littlewood rearrangement inequality (see [LL97, Theorem 3.4]*) that*

$$\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} G_t^{\mathrm{h}}(x-y)\Lambda(x)dx \le \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} |x|^{-\lambda} dx \le C \cdot t^{-\frac{\lambda}{2}}\,;$$
$$\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} G_t^{\mathrm{w}}(x-y)\Lambda(x)dx \le \int_{-t}^{t} |x|^{-\lambda} dx \le C \cdot t^{1-\lambda}\,,$$

where G_t^{h} *and* G_t^{w} *are heat kernel and wave kernel, respectively. In addition, we know that*

$$\int_{\mathbb{R}} G_t^{\mathrm{h}}(x)dx = 1\,, \quad \int_{\mathbb{R}} G_t^{\mathrm{w}}(x)dx = t\,.$$

Thus, (4.3.4) *holds for SHE and SWE. Moreover, it will be shown in Section 4.7 that SHE and SWE satisfy* small ball nondegeneracy *with* $\mathfrak{a} = 0$ *and* $\mathfrak{a} = 1$*, respectively.*

4.3.2 Main results

In this subsection we present our main results. This is, we give the upper moment estimates in Theorem 4.3.4 and lower moment in Theorem 4.3.6. In fact, with $\gamma(\cdot)$, $\Lambda(\cdot)$ and $G_t(\cdot)$ satisfying conditions stated before, we also give the relation among the indices $\mathfrak{a}$, $\mathfrak{b}$ and $\hbar$ so that the exponents in t and p in the lower and upper moments match with each others (see the Table 4.1).

First we state the result for the upper moment bounds.

Theorem 4.3.4. *Assume $\gamma(\cdot)$ satisfies (the upper inequality in)* ***(H1)*** *and $\Lambda(\cdot)$ satisfies (the upper inequality in) one of* ***(H2)-(H4)****. Let the Green function $G_t(\cdot)$ satisfy* ***(G3)****. Assume that the initial condition term $I_0(t,x)$ is bounded, namely, there is a positive constant C such that $\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}|I_0(t,x)|\le C$. Then there is a unique mild solution $u(t,x)$ satisfying (4.2.7). Moreover, there are some constants C_1 and C_2 do not depend on t, p and x such that*

$$\mathbb{E}\left[|u(t,x)|^p\right] \le C_1 \exp\left(C_2 \cdot t^{1+\frac{1-\gamma}{\hbar+1}} \cdot p^{1+\frac{1}{\hbar+1}}\right) . \tag{4.3.7}$$

The proof of this result will be given in next section (Section 4.4) by using the hypercontractivity inequality.

Remark 4.3.5. *This result is new in the sense that it holds now for general operator $\mathscr{L}$ satisfying* ***(G3)****. When $\mathscr{L}$ is the heat operator, wave operator, fractional α-diffusion operator, or partial differential operator both fractional in time and space but homogeneous in space, the result is known (e.g. [BC16, CHHH17, Hu19, SSX20], references therein and other references.*

Our main contribution of this work is the following lower moment bounds for a general partial differential operator $\mathscr{L}$.

Theorem 4.3.6. *Assume $\gamma(\cdot)$ satisfies (the lower inequality in)* ***(H1)*** *and $\Lambda(\cdot)$ satisfies (the lower inequality in) one of* ***(H2)-(H4)****. Let the Green function $G_t(\cdot)$ satisfy (the lower inequality in)* ***(G1)*** *and* ***(G2)****. If the initial condition satisfies $\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d} I_0(t,x)\ge c_0$ for some constant $c_0>0$, then there are some positive constants c_1 and c_2 independent of t, p and x such that we have*

$$\mathbb{E}\left[|u(t,x)|^p\right] \ge\; c_1 \exp\left(c_2 \cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} \cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right) . \tag{4.3.8}$$

The proof of this theorem replies on the Feynman diagram formula for the moments of a chaos expansion. This formula will be presented in Section 4.5 and will be used in Section 4.6 to prove the above theorem.

Consequently, combing Theorem 4.3.4 and 4.3.6 we obtain the following theorem about the matching upper and lower moment bounds.

Theorem 4.3.7. *Assume $\gamma(\cdot)$ satisfy **(H1)** and $\Lambda(\cdot)$ satisfy one of **(H2)**-**(H4)**. Assume the Green function $G_t(\cdot)$ satisfy **(G1)**-**(G3)** with*

$$\hbar := 2\mathfrak{a} - \frac{\lambda}{\mathfrak{b}} > -1 \tag{4.3.9}$$

If the initial condition satisfies $c_0 \leq I_0(t,x) \leq C_0$ for some positive constants $0 < c_0 < C_0 < \infty$, then the mild solution $u(t,x)$ to (4.1.1) satisfies

$$\begin{aligned} c_1 \exp\left(c_2 \cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} \cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right) \\ \leq \mathbb{E}\left[|u(t,x)|^p\right] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} \cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right) \end{aligned} \tag{4.3.10}$$

for all $t \geq 0$, $x \in \mathbb{R}^d$, $p \geq 2$, where c_1, c_2, C_1, C_2 are some positive constants, independent of t, x, p.

Proof. It is obvious that under the conditions of this theorem both the conditions of Theorems 4.3.4 and 4.3.6 hold. Thus both (4.3.8) and (4.3.7) hold true. Replacing $\hbar$ by (4.3.9) we see that (4.3.7) becomes the second inequality in (4.3.10). The theorem is then proved. □

We shall demonstrate that (4.3.9) holds true for the Green's function of various partial differential operators: SHE, α-SHE, SWE and SFD (see (4.7.1), (4.7.13), (4.7.22) and (4.7.33) respectively). We summarize the results of that section here in following table. Notice that Table 4.1 only includes the exponent parts of (4.3.10).

SPDEs	$(\mathfrak{a},\mathfrak{b})$	$\hbar$	Moment	When $\gamma = 2 - 2H$
SHE	*(0,2)*	$-\frac{\lambda}{2}$	$t^{1+\frac{2\cdot(1-\gamma)}{2-\lambda}} \cdot p^{\frac{4-\lambda}{2-\lambda}}$	$t^{\frac{4H-\lambda}{2-\lambda}} \cdot p^{\frac{4-\lambda}{2-\lambda}}$
α-SHE	*(0,α)*	$-\frac{\lambda}{\alpha}$	$t^{1+\frac{\alpha\cdot(1-\gamma)}{\alpha-\lambda}} \cdot p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}$	$t^{\frac{2H\alpha-\lambda}{\alpha-\lambda}} \cdot p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}$
SWE	*(1,1)*	$2-\lambda$	$t^{1+\frac{1-\gamma}{3-\lambda}} \cdot p^{\frac{4-\lambda}{3-\lambda}}$	$t^{\frac{2H+2-\lambda}{3-\lambda}} \cdot p^{\frac{4-\lambda}{3-\lambda}}$
SFD	$(\beta-1,\frac{\alpha}{\beta})$	$2(\beta-1)-\frac{\lambda\beta}{\alpha}$	$t^{1+\frac{\alpha(1-\gamma)}{2\alpha\beta-\alpha-\beta\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}$	$t^{\frac{\alpha(2\beta+2H-2)-\beta\lambda}{2\alpha\beta-\alpha-\beta\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}$

Table 4.1: Matching lower and upper moments

4.4 Upper moment bounds

Our goal of this section is to prove the upper moment bounds assuming that **(G1)**, **(G3)** hold for the Green's function G associated with the operator $\mathscr{L}$ and assuming that **(H1)** and one of **(H2)**-**(H4)** or one of **(H2′)**-**(H3′)** hold true for the noise covariance structure.

Sometimes it is convenient to use Fourier transformation to represent the covariance function in spatial variables. Assume $\Lambda(x) \geq 0$ for all $x \in \mathbb{R}^d$ and assume that there is a measure μ on $\mathbb{R}^d$ such that

$$\Lambda(x) = \int_{\mathbb{R}^d} e^{\iota x\xi} \mu(d\xi) \,. \tag{4.4.1}$$

We now assume some conditions on the Fourier mode that are similar to **(H2)**-**(H3)** and **(G3)**.

(H2′) There is a $\hat{\Lambda} : \mathbb{R}^d \to \mathbb{R}$ such that $\mu(d\xi) = \hat{\Lambda}(\xi)d\xi$ and there are constants $\lambda_j \in (0,1), j = 1, \cdots, d$ and $C > 0$ such that

$$|\hat{\Lambda}(\xi)| \leq C \prod_{j=1}^{d} |\xi|^{\lambda_j - 1} \,, \quad \forall\, \xi \in \mathbb{R}^d \,.$$

In this case we denote $\lambda = \lambda_1 + \cdots + \lambda_d$.

(H3′) There is a $\hat{\Lambda} : \mathbb{R}^d \to \mathbb{R}$ such that $\mu(d\xi) = \hat{\Lambda}(\xi)d\xi$ and there are constants $\lambda \in (0, d)$ and $C > 0$ such that

$$|\hat{\Lambda}(\xi)| \le C|\xi|^{\lambda - d}, \quad \forall\, \xi \in \mathbb{R}^d.$$

(G3′) [**Majorized property**] $G_t(\cdot)$ satisfies the *majorized property* $(M(\hbar))$. This is, there exists a positive function or measure Q_t such that $G_t(x, y) \le Q_t(x - y)$ for any $t > 0$ and $x, y \in \mathbb{R}$, and there exist a real number $\hbar > -1$ (the same ones in **(G3)**) and a constant $C > 0$ such that

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{Q}_t(\xi - \eta)|^2 |\mu|(d\xi) \le C \cdot t^{\hbar}, \tag{4.4.2}$$

where $|\mu|(\xi) = |\hat{\Lambda}(\xi)|d\xi$ with **(H2′)** or **(H3′)** holds.

Theorem 4.4.1. *Let the Green function $G_t(\cdot)$ satisfy **(G3′)**. Assume $\gamma(\cdot)$ satisfies (the upper inequality in) **(H1)** and $\Lambda(\cdot)$ satisfies one of **(H2′)**-**(H3′)**. Assume that the initial condition term $I_0(t, x)$ is bounded, namely, there is a positive constant C such that $\sup_{(t,x)\in\mathbb{R}_+\times\in\mathbb{R}^d} |I_0(t, x)| \le C$. Then there is a unique mild solution $u(t, x)$ is satisfying (4.2.7). Moreover, there are some constants C_1 and C_2 do not depend on t, p and x such that (4.3.7) holds.*

Proof of Theorem 4.3.4 and Theorem 4.4.1. As indicated in [Hu19], there are mainly three approaches to obtain the upper moments, effective in different situations. In our case, we choose to use the approach of combining chaos expansion and hypercontractivity inequality.

Step 1: We shall show the upper bound under assumptions **(G3)**, **(H1)** and one of **(H2)**-**(H4)**. In the following, we shall only prove the case **(H2)**. The cases **(H3)** and **(H4)** can be done similarly. Recall the Wiener-Itô chaos expansion (4.2.8) for the mild solution to (4.2.7)

$$u(t, x) = I_0(t, x) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)),$$

where $f_n(\cdot, t, x)$ is given by (4.2.9). Denote $u_n(t, x) = I_n(f_n(\cdot, t, x))$. Then it follows from the Itô isometry for the multiple Wiener-Itô integral (e.g. [Hu17]) that

$$\|u_n(t,x)\|_{L^2}^2 = \mathbb{E}|I_n(f_n(\cdot;t,x))|^2$$
$$= n!\|f_n(\cdot;t,x)\|^2_{\mathcal{H}^{\otimes n}}\,.$$

To compute the above norm let us denote $\vec{t} = (t_1, \cdots, t_n)$, $\vec{s} = (s_1, \cdots, s_n)$, $\vec{x} = (x_1, \cdots, x_n)$, $\vec{y} = (y_1, \cdots, y_n)$ and

$$\Psi_n(\vec{t}, \vec{s}) := \int_{\mathbb{R}^{2nd}} f_n(\vec{t}, \vec{x}; t, x) \prod_{j=1}^{n} \Lambda(x_j - y_j) f_n(\vec{s}, \vec{y}; t, x) d\vec{x} d\vec{y}\,.$$

Then, we have

$$\|u_n(t,x)\|_{L^2}^2 = n!\|f_n(\cdot;t,x)\|^2_{\mathcal{H}^{\otimes n}}$$
$$= \frac{1}{n!}\Phi_n(t) := \frac{c_H^n}{n!}\int_{[0,t]^{2n}} \prod_{j=1}^{n} \gamma(t_j - s_j)\Psi_n(\vec{t}, \vec{s}) d\vec{t} d\vec{s}\,. \tag{4.4.3}$$

By the Cauchy-Schwarz inequality $\Psi_n(\vec{t}, \vec{s}) \le \left[\Psi_n(\vec{t}, \vec{t})\Psi_n(\vec{s}, \vec{s})\right]^{1/2}$ and Hardy-Littlewood-Sobolev inequality [HN09, Inequality (2.4)], we obtain with $\gamma = 2 - 2H$ (or $H = 1 - \frac{\gamma}{2}$) from (4.4.3)

$$\Phi_n(t) \lesssim \int_{[0,t]^{2n}} \prod_{j=1}^{n} \gamma(t_j - s_j) \left[\Psi_n(\vec{t}, \vec{t})\Psi_n(\vec{s}, \vec{s})\right]^{1/2} d\vec{t} d\vec{s}$$
$$\lesssim \left(\int_{[0,t]^n} |\Psi_n(\vec{s}, \vec{s})|^{1/H} d\vec{s}\right)^{2H}\,.$$

Now we need to resort to the key assumption **(G3)**, i.e. *Hardy-Littelewood-Sobolev type mass property* $M(\hbar)$ to obtain the bound for Ψ_n. Repeatedly using **(G3)** (namely, (4.3.4)), we have

$$\Psi_n(\vec{s}, \vec{s}) \le \int_{\mathbb{R}^{2nd}} f_n(\vec{s}, \vec{x}; t, x) \prod_{j=1}^{n} \Lambda(x_j - y_j) f_n(\vec{s}, \vec{y}; t, x) d\vec{x} d\vec{y}$$
$$\lesssim \prod_{j=1}^{n} |s_{\sigma(j+1)} - s_{\sigma(j)}|^{\hbar} 1_{\{0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t\}}\,, \tag{4.4.4}$$

where $\hbar > -1$ and $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < s_{\sigma(n+1)} = t$. Denote the simplex $\prod_n(t) =$

$\{(s_1, \cdots, s_n); 0 < s_1 < \cdots < s_n < t\}$. Then, using the bound we just obtained for Ψ_n, we obtain the upper bound for $\Phi_n(t)$:

$$\begin{aligned}\Phi_n(t) \leq & C_H^n \left(n! \int_{\Pi_n(t)} \prod_{j=1}^n |s_{j+1} - s_j|^{\hbar/2H} d\vec{s} \right)^{2H} \\ \leq & C_H^n \left(n! \frac{t^{n\hbar/2H+n}}{\Gamma(n\hbar/2H + n + 1)} \right)^{2H} \simeq C_H^n \frac{t^{n(\hbar+2H)}}{(n!)^{\hbar}}\end{aligned}$$

by Stirling's formula for Gamma function.

As a result, the second moment can be estimated as

$$\|u_n(t,x)\|_{L^2}^2 = \frac{1}{n!}\Phi_n(t) \leq C_H^n \frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}} .$$

It is now easy to bound the p-th moment from the above second moment bound by using the hypercontractivity inequality (e.g. [Hu17, p.54, Theorem 3.20])

$$\begin{aligned}\|u_n(t,x)\|_{L^p} \leq & (p-1)^{n/2} \|u_n(t,x))\|_{L^2} \\ \leq & C_H^n (p-1)^{n/2} \left[\frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}} \right]^{1/2} .\end{aligned}$$

Thus

$$\begin{aligned}\|u(t,x)\|_p \leq & C + \sum_{n=1}^{\infty} \|u_n(t,x))\|_{L^p} \\ \leq & C + \sum_{n=1}^{\infty} C_H^n (p-1)^{n/2} \left[\frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}} \right]^{1/2} \\ \leq & C \exp\left(C \cdot t^{\frac{\hbar+2H}{\hbar+1}} (p-1)^{\frac{1}{\hbar+1}} \right) .\end{aligned}$$

This means $\mathbb{E}[|u(t,x)|^p] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{1-\gamma}{\hbar+1}} p^{1+\frac{1}{\hbar+1}} \right)$ for some positive constants C_1 and C_2 and hence we conclude the proof of Theorem 4.3.4.

Step 2: We shall show the upper bound under assumptions **(G3′)**, **(H1)** and one of **(H2′)**-**(H3′)**. Denote

$$f_n^Q(\cdot, t, x) := f_n(t_1, x_1, \cdots, t_n, x_n, t, x) \tag{4.4.5}$$

$$
\begin{aligned}
=&\frac{1}{n!}\sum_{\sigma\in S_n} Q_{t-t_{\sigma(n)}}(x-x_{\sigma(n)})Q_{t_{\sigma(n)}-t_{\sigma(n-1)}}(x_{\sigma(n)}-x_{\sigma(n-1)})\cdots \\
&\qquad\qquad \times Q_{t_{\sigma(2)}-t_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})I_0(t_{\sigma(1)},x_{\sigma(1)})\,. \qquad (4.4.6)
\end{aligned}
$$

Namely, we replace G in the expression of $f_n(\cdot,t,x)$ by Q. Then by the positivity of G, Λ, and the fact that $G\le Q$, we have

$$
\begin{aligned}
\Psi_n(\vec{s},\vec{s})\lesssim&\int_{\mathbb{R}^{2nd}} f_n(\vec{s},\vec{x};t,x)\prod_{j=1}^n \Lambda(x_j-y_j)f_n(\vec{s},\vec{y};t,x)d\vec{x}d\vec{y}\\
\lesssim&\int_{\mathbb{R}^{2nd}} f_n^Q(\vec{s},\vec{x};t,x)\prod_{j=1}^n \Lambda(x_j-y_j)f_n^Q(\vec{s},\vec{y};t,x)d\vec{x}d\vec{y}\\
\lesssim&\int_{\mathbb{R}^{nd}} \left|\mathcal{F}[f_n^Q(\vec{s},\cdot;t,x)](\xi)\right|^2|\mu|(d\vec{\xi})\lesssim\prod_{j=1}^n|s_{\sigma(j+1)}-s_{\sigma(j)}|^{\hbar}\,.
\end{aligned}
$$

As a result, we get $\mathbb{E}[|u(t,x)|^p]\le C_1\exp\left(C_2\cdot t^{1+\frac{1-\gamma}{\hbar+1}}p^{1+\frac{1}{\hbar+1}}\right)$ for some positive constants C_1 and C_2. □

4.5 Feynman diagram formula

Now we turn to the proof of Theorem 4.3.6, i.e., the lower bounds for the moments. The main difficulty is the lack of the Feynman-Kac formula for general partial differential operators. To get round of this difficulty our strategy is a brutal force one. We try to handle the p-th moment of $u(t,x)$ directly, where p is an arbitrary positive integer and u is the mild solution to (4.1.1), given by its chaos expansion (4.2.8). Since the solution is an infinite sum of multiple Wiener-Itô integrals so we need first to use the product formulas of the multiple Wiener-Itô integrals (with respect to Gaussian noise). This is called the Feynman diagram formulas and they can be found in Theorem 5.7 and Theorem 5.8 in [Hu17] (for general Gaussian noise case), Theorem 10.2 in [Maj13] or Theorem 5.3 in [Maj14] (for White noise cases). In this section we shall present this formula "graphically" so that we can keep track the terms.

Recall that the Gaussian space $\mathcal{H}$ in our situation is the Hilbert space obtained by the completion of $\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^d)$ with respect to the scalar product defined by (4.2.2). Since the work of [Maj13] or [Maj14] are for the "White noise" case, we will follow the product formula of [Hu17, Theorem 5.7]. Since we are only interested in the expectation of the product of multiple integrals and since $\mathbb{E}[I_k(f)] = 0$ for all $k \geq 1$ we only need to sum the terms with $|\gamma| = 0$ in [Hu17, Theorem 5.7] when we take the expectation of the left hand side of [Hu17, Equation 5.3.5] (The notation γ used in [Hu17] is different than the one used in this chapter).

To visualize these summation terms graphically, we recall the concept of diagram associated with only these terms. A Feynman diagram D is a set of some vertices and some edges connecting them so that the vertices are arranged into some finite rows and each row contains some finite many vertices. The set of vertices of the diagram D can then be represented by $\mathscr{V}(\mathcal{D}) = \{(k, r) : 1 \leq k \leq m, 1 \leq r \leq n_m\}$. We use $\mathscr{E}(\mathcal{D}) = \{[(\overline{k}, \overline{r}), (\underline{k}, \underline{r})] : \overline{k} < \underline{k}\}$ denote the set of all edges of a diagram $\mathcal{D}$, where $\overline{k} < \underline{k}$ means $(\overline{k}, \overline{r})$ is the upper (row) and $(\underline{k}, \underline{r})$ is the lower (row) end point of a edge. The strict inequality is important here since two vertices in the same row are not allowed to form an edge. For an edge $[(\overline{k}, \overline{r}), (\underline{k}, \underline{r})] \in \mathscr{E}(\mathcal{D})$, we call $(\overline{k}, \overline{r})$ the upper vertice and $(\underline{k}, \underline{r})$ the lower vertice of the edge and we call a vertice associates with an edge if it is either upper or lower vertice of the edge. We use $\overline{\mathscr{V}}(\mathcal{D})$ and $\underline{\mathscr{V}}(\mathcal{D})$ to the sets of all upper and lower vertices, respectively. We require that one vertice associates with at most one edge. Thus we have $\overline{\mathscr{V}}(\mathcal{D}) \cap \underline{\mathscr{V}}(\mathcal{D}) = \emptyset$. After taking the expectation of [Hu17, Equation 5.3.5], the terms will be significantly reduced. To account the remaining terms we only need to consider the following special diagrams.

Definition 4.5.1. *A diagram $\mathcal{D} = (\mathscr{V}(\mathcal{D}), \mathscr{E}(\mathcal{D}))$ is called* admissible *if every vertice is associated with one and only one edge. The set of all* admissible diagrams *associated with the vertices $\{(k, r), 1 \leq k \leq m, 1 \leq r \leq n_k\}$ is denoted by $\mathbb{D}(n_1, \cdots, n_m)$.*

It is clear that if a diagram $\mathcal{D}$ is admissible then $n_1 + \cdots + n_m = 2|\mathscr{E}(\mathcal{D})|$, in particular, $n_1 + \cdots + n_m$ is an even integer.

Let $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$, $k = 1, \cdots, m$ be some given measurable functions. Associated with these functions we have naturally the set of Feynman diagrams $\tilde{\mathbb{D}}(f_1, \cdots, f_m)$. The correspondence is described as follows. Each Feynman diagram $\mathcal{D} \in \tilde{\mathbb{D}}(f_1, \cdots, f_m)$ contains m rows, corresponding to $f_1, \cdots, f_m$, and the k-th row of $\mathcal{D}$ contains n_k vertices, which is the number of independent variables of the function f_k. We use $\mathbb{D}(f_1, \cdots, f_m)$ to denote the set of all admissible Feynman diagrams associated with $f_1, \cdots, f_m$.

For the sake of convenience we consider (t, x) as one vector independent variable, where $t \geq 0, x \in \mathbb{R}^d$. So we shall say that $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$ has n_k independent (vector) variables. From the functions $f_k : (\mathbb{R}_+ \times \mathbb{R}^d)^{n_k} \to \mathbb{R}$, $k = 1, \cdots, m$, we define their concatenation $f_1 \circ \cdots \circ f_m$ as a function of $n_1 + \cdots + n_m$ independent vector variables. We name the n_k independent variables of the function f_k by $(t_{(k,1)}, x_{(k,1)}), \cdots, (t_{(k,n_k)}, x_{(k,n_k)})$, associated with the k-th row vertices. Thus for an admissible Feynman diagram $\mathcal{D} \in \mathbb{D}(f_1, \cdots, f_m)$, the concatenation $f_1 \circ \cdots \circ f_m$ is a function of $n_1 + \cdots + n_m$ vector variables and we write it as

$$f_1 \circ \cdots \circ f_m((t_{\overline{\mathscr{V}}(\mathcal{D})}, x_{\overline{\mathscr{V}}(\mathcal{D})}), (t_{\underline{\mathscr{V}}(\mathcal{D})}, x_{\underline{\mathscr{V}}(\mathcal{D})})) .$$

The edges in $\mathcal{D} \in \mathbb{D}(f_1, \cdots, f_m)$ are used to form the (tensor) scalar product in the Gaussian space $\mathcal{H}$ of the above concatenation. Here is the detail of this construction. If $[(\overline{k}, \overline{r}), (\underline{k}, \underline{r})]$ is an edge of the diagram $\mathcal{D}$, then we form a factor

$$\gamma(t_{(\overline{k},\overline{r})} - t_{(\underline{k},\underline{r})})\Lambda(x_{(\overline{k},\overline{r})} - x_{(\underline{k},\underline{r})}) .$$

For the set of $\mathscr{E}(\mathcal{D})$, we denote the product of all above factors as

$$\begin{aligned} &\gamma(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})})\Lambda(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}) \\ &\qquad = \prod_{[(\overline{k},\overline{r}),(\underline{k},\underline{r})] \in \mathscr{E}(\mathcal{D})} \gamma(t_{(\overline{k},\overline{r})} - t_{(\underline{k},\underline{r})})\Lambda(x_{(\overline{k},\overline{r})} - x_{(\underline{k},\underline{r})}) . \end{aligned} \tag{4.5.1}$$

With these notations, we define finally a real number associated with $f_1, \cdots, f_m$ and associated with an admissible diagram $\mathcal{D}$ as follows:

$$\begin{aligned} &F_{\mathcal{D}}(f_1, \cdots, f_m) \\ &\quad = \int f_1 \circ \cdots \circ f_m((t_{\overline{\mathscr{V}}(\mathcal{D})}, x_{\overline{\mathscr{V}}(\mathcal{D})}), (t_{\underline{\mathscr{V}}(\mathcal{D})}, x_{\underline{\mathscr{V}}(\mathcal{D})})) \\ &\qquad\quad \gamma(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})})\Lambda(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}) dt_{\overline{\mathscr{V}}(\mathcal{D})} dt_{\underline{\mathscr{V}}(\mathcal{D})} dx_{\overline{\mathscr{V}}(\mathcal{D})} dx_{\underline{\mathscr{V}}(\mathcal{D})} . \end{aligned} \tag{4.5.2}$$

To illustrate the above notation, we give one example.

Example 4.5.2. Given three functions of four independent of variables. Let us take the following admissible diagram $\mathcal{D} \in \mathbb{D}(4,4,4)$ in Figure 4.1 as an example. The upper vertices are colored in red and the lower vertices are colored in blue. The upper and lower variables are gives as follows:

$$
\begin{aligned}
x_{\overline{\mathscr{V}}(\mathcal{D})} &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,4)\}\,,\\
x_{\underline{\mathscr{V}}(\mathcal{D})} &= \{(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}\,,
\end{aligned}
$$

The corresponding set of edges of this diagram is

$$
\begin{aligned}
\mathscr{E}(\mathcal{D}) =& \{[(1,1),(3,1)],[(1,2),(2,3)],[(1,3),((3,4)],\\
& [(1,4),(2,2)],[(2,1),(3,2)],[(2,4),(3,3)]\}\,.
\end{aligned}
$$

In this case, $n_1 = n_2 = n_3 = 4$. It is easy to see that $|\mathscr{E}(\mathcal{D})| = 6 = (n_1 + n_2 + n_3)/2$ and

$$
\begin{aligned}
F_{\mathcal{D}}(f_1, f_2, f_3) = \int & f_1((t_{(1,1)}, x_{(1,1)}), (t_{(1,2)}, x_{(1,2)}), (t_{(1,3)}, x_{(1,3)}), (t_{(1,4)}, x_{(1,4)}))\\
& \cdot f_2((t_{(2,1)}, x_{(2,1)}), (t_{(2,2)}, x_{(2,2)}), (t_{(2,3)}, x_{(2,3)}), (t_{(2,4)}, x_{(2,4)}))\\
& \cdot f_3((t_{(3,1)}, x_{(3,1)}), (t_{(3,2)}, x_{(3,2)}), (t_{(3,3)}, x_{(3,3)}), (t_{(3,4)}, x_{(3,4)}))\\
& \cdot \gamma(t_{(1,1)} - t_{(3,1)})\gamma(t_{(1,2)} - t_{(2,3)})\gamma(t_{(1,3)} - t_{(3,4)})\\
& \cdot \gamma(t_{(1,4)} - t_{(2,2)})\gamma(t_{(2,1)} - t_{(3,2)})\gamma(t_{(2,4)} - t_{(3,3)})\\
& \cdot \Lambda(x_{(1,1)} - x_{(3,1)})\Lambda(x_{(1,2)} - x_{(2,3)})\Lambda(x_{(1,3)} - x_{(3,4)})\\
& \cdot \Lambda(x_{(1,4)} - x_{(2,2)})\Lambda(x_{(2,1)} - x_{(3,2)})\Lambda(x_{(2,4)} - x_{(3,3)})dtdx\,,
\end{aligned}
$$

where $dt = dt_{(1,1)} \cdots dt_{(3,4)}$ and similar notation for dx. Notice that the Feynman diagrams are used to track the terms and to provide guidance for the variables inside γ and Λ.

Let us also notice that the operation $F_{\mathcal{D}}$ can also be defined for any $f_k \in \mathcal{H}^{\otimes n_k}$, $k = 1, \cdots, m$, which may contain measures or generalized functions.

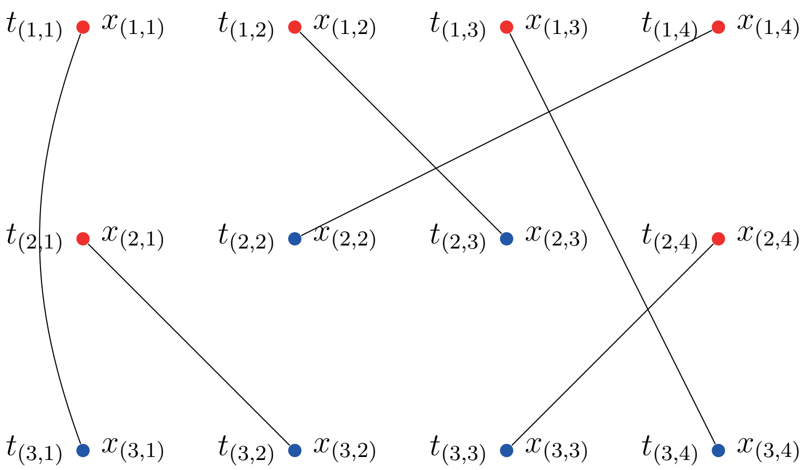

Figure 4.1: An example the admissible diagram

Theorem 4.5.3. *For* $f_1 \in \mathcal{H}^{\otimes n_1}, \dots, f_m \in \mathcal{H}^{\otimes n_m}$*, we have*

$$\mathbb{E}[I_{n_1}(f_1)\cdots I_{n_m}(f_m)] = \sum_{\mathcal{D}\in\mathbb{D}(f_1,\cdots,f_m)} F_{\mathcal{D}}(f_1,\cdots,f_m)\,. \tag{4.5.3}$$

where $F_{\mathcal{D}}(f_1,\cdots,f_m)$ *is given by* (4.5.2)*.*

Proof. This theorem is a consequence of [Hu17, Theorem 5.7] when we take the expectation of [Hu17, Equation 5.3.5] and notice that now the scalar product of $\mathcal{H}$ is defined by (4.2.2). □

If we apply the above formula to the f_n defined by (4.2.9), we have

Theorem 4.5.4. *Let* $f_n(\cdot,t,x)$ *be defined by* (4.2.9) *and let* $I_n(f_n(\cdot,t,x))$ *be the associated multiple Wiener-Itô integral. Then*

$$\begin{aligned}
&\mathbb{E}\Big[I_{n_1}(f_{n_1}(\cdot,t,x))\cdots I_{n_m}(f_{n_m}(\cdot,t,x))\Big]\\
&\quad= \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} F_{\mathcal{D}}\left(f_{n_1},\cdots,f_{n_m}\right)\\
&\quad= \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} \int \prod_{j=1}^{m}\prod_{r=1}^{n_j} G_{t_{(j,r+1)}-t_{(j,r)}}\left(x_{(j,r+1)}-x_{(j,r)}\right) 1_{\{0<t_{(j,1)}<\cdots<t_{(j,n_j)}\}}\\
&\qquad\qquad \times \gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D})}-t_{\underline{\mathcal{V}}(\mathcal{D})}\Big)\Lambda\Big(x_{\overline{\mathcal{V}}(\mathcal{D})}-x_{\underline{\mathcal{V}}(\mathcal{D})}\Big)dt_{\mathcal{D}}dx_{\mathcal{D}}\,,
\end{aligned} \tag{4.5.4}$$

where we use the notations $t_{(j,n_j+1)} = t$ *and* $x_{(j,n_j+1)} = x$ *for all* $1\le j\le m$ *and* $\gamma(\cdot)$ *and* $\Lambda(\cdot)$ *are defined as* (4.5.1)*.*

Proof. We only need to prove the second equality in (4.5.4). We may only consider the time variable without loss of generality (i.e. $d = 0$). Namely, we reduce the symmetric function $f_n(t_1, x_1, \cdots, t_n, x_n; t, x)$ to the symmetric function $f_n(t_1, \cdots, t_n; t)$. Then what we need to show is the following equality for any $n_1, \ldots, n_m$ and for any corresponding admissible diagrams $\mathbb{D}$, (4.5.4) holds true. We shall prove (4.5.4) recursively on n. Denote the function of (4.2.10) by $f_n(t_1, \cdots, t_n; t)$ and its symmetrization by $\tilde{f}_n(t_1, \cdots, t_n; t)$. Then

$$
\begin{aligned}
&\sum_{\mathcal{D}\in\mathbb{D}(\tilde{f}_{n_1},\cdots,\tilde{f}_{n_m})} F_{\mathcal{D}}\left(\tilde{f}_{n_1}(\cdot,t), \cdots, \tilde{f}_{n_m}(\cdot,t)\right) \\
&= \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} \int \prod_{j=1}^{m} \tilde{f}_{n_j}(t_{(j,1)}, \cdots, t_{(j,n_j)}; t) \times \gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D})} - t_{\underline{\mathcal{V}}(\mathcal{D})}\Big) dt_{\mathcal{D}} \\
&= \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} \frac{1}{n_1!} \sum_{\sigma\in S_{n_1}} \int f_{n_1}(t_{(1,\sigma(1))}, \cdots, t_{(1,\sigma(n_1))}; t) \\
&\qquad\qquad \cdot \prod_{j=2}^{m} \tilde{f}_{n_j}(t_{(j,1)}, \cdots, t_{(j,n_j)}; t) \times \gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D})} - t_{\underline{\mathcal{V}}(\mathcal{D})}\Big) dt_{\mathcal{D}} \\
&=: \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} \frac{1}{n_1!} \sum_{\sigma\in S_{n_1}} I_{\sigma(1),\cdots,\sigma(n_1),\mathcal{D}}\,,
\end{aligned}
\tag{4.5.5}
$$

where S_{n_1} denotes the set of all permutations of $\{1, 2, \cdots, n_1\}$ and $I_{\sigma(1),\cdots,\sigma(n_1),\mathcal{D}}$ denotes the above integral. Suppose that $\mathcal{D} \in \mathbb{D}(f_{n_1}, \cdots, f_{n_m})$ is a Feynman diagram. Then there there are $(j_1, r_1), \cdots, (j_{n_1}, r_{n_1})$ such that the following edges

$$[(1,1),(j_1,r_1)], \cdots, [(1,n_1),(j_{n_1},r_{n_1})]$$

are in $\mathscr{E}$. In this diagram $\mathcal{D}$, we replace the above edges by the following ones

$$[(1,\sigma(1)),(j_1,r_1)], \cdots, [(1,\sigma(n_1)),(j_{n_1},r_{n_1})]$$

and retain all other edges unchanged. Then we obtain another diagram $\mathcal{D}_\sigma$. See Figure 4.2 for an illustrating example.

This transformation $\mathcal{D} \to \mathcal{D}_\sigma$ has the following properties:

(i) If $\mathcal{D}$ is an admissible diagram, so is $\mathcal{D}_\sigma$.

Figure 4.2: The left is $\sigma = \{1,2,3,4\}$ and the right is $\sigma = \{2,1,4,3\}$

(ii) For any fixed permutation σ, the mapping $\mathcal{D} \to \mathcal{D}_\sigma$ is a bijection from $\mathbb{D}(f_{n_1}, \cdots, f_{n_m})$ to itself.

(iii) $\gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D}_\sigma)} - t_{\underline{\mathcal{V}}(\mathcal{D}_\sigma)}\Big)$ remains unchanged:

$$\gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D}_\sigma)} - t_{\underline{\mathcal{V}}(\mathcal{D}_\sigma)}\Big) = \gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D})} - t_{\underline{\mathcal{V}}(\mathcal{D})}\Big)\,.$$

These properties imply

$$\sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} I_{\sigma(1),\cdots,\sigma(n_1),\mathcal{D}} = \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} I_{1,\cdots,n_1,\mathcal{D}}\,.$$

Substituting it to (4.5.6) we have

$$\begin{aligned}
&\sum_{\mathcal{D}\in\mathbb{D}(\tilde f_{n_1},\cdots,\tilde f_{n_m})} F_{\mathcal{D}}\left(\tilde f_{n_1}(\cdot,t),\cdots,\tilde f_{n_m}(\cdot,t)\right)\\
&= \sum_{\mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_m})} \int f_{n_1}(t_{(1,1)},\cdots,t_{(1,n_1)};t) \qquad (4.5.6)\\
&\qquad \cdot \prod_{j=2}^m \tilde f_{n_j}(t_{(j,1)},\cdots,t_{(j,n_j)};t) \times \gamma\Big(t_{\overline{\mathcal{V}}(\mathcal{D})} - t_{\underline{\mathcal{V}}(\mathcal{D})}\Big) dt_{\mathcal{D}}\,.
\end{aligned}$$

This can be used to prove the theorem by induction. □

Example 4.5.5. The above formula (4.5.4) can be used to compute all moments of a chaos expansion. This will be done in the next section when we prove the lower moment bounds. As an example, it is interesting to consider the second moment. By orthogonality of multiple Wiener-Itô chaos expansion, we have

$$\mathbb{E}[|u(t,x)|^2] = 1 + \sum_{n=1}^\infty \mathbb{E}\left[|I_n(f_n)|^2\right]\,.$$

Then by (4.5.4) in Theorem 4.5.4, one finds

$$\begin{aligned}
&\mathbb{E}\Big[|I_n(f_n)|^2\Big] \\
&\quad= \sum_{\mathcal{D}\in\mathbb{D}(n,n)} \int \prod_{j=1}^{2}\prod_{r=1}^{n} G_{t^{(j)}_{r+1}-t^{(j)}_r}\left(x^{(j)}_{r+1}-x^{(j)}_r\right) 1_{\{0<t^{(j)}_1<\cdots<t^{(j)}_n<t\}} \qquad (4.5.7)\\
&\qquad\qquad \times \gamma\Big(t_{\overline{\mathscr{V}}(\mathcal{D})}-t_{\underline{\mathscr{V}}(\mathcal{D})}\Big)\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})}-x_{\underline{\mathscr{V}}(\mathcal{D})}\Big)dt_{\mathcal{D}}dx_{\mathcal{D}}\,.
\end{aligned}$$

An example of admissible diagram $\mathcal{D}\in\mathbb{D}(4,4)$ can be illustrated in the Figure 4.3. In this diagram, we have $T_{\overline{\mathscr{V}}(\mathcal{D})} := (t_{\overline{\mathscr{V}}(\mathcal{D})}, x_{\overline{\mathscr{V}}(\mathcal{D})}) = \{T^{(2)}_j : 1\le j\le 4\}$ colored in red, $T_{\underline{\mathscr{V}}(\mathcal{D})} := (t_{\underline{\mathscr{V}}(\mathcal{D})}, x_{\underline{\mathscr{V}}(\mathcal{D})}) = \{T^1_j : 1\le j\le 4\}$ colored in blue. Moreover,

$$\begin{aligned}
\gamma\Big(t_{\overline{\mathscr{V}}(\mathcal{D})}-t_{\underline{\mathscr{V}}(\mathcal{D})}\Big) &:= \gamma\Big(t^{(2)}_1-t^1_3\Big)\gamma\Big(t^{(2)}_2-t^1_1\Big)\gamma\Big(t^{(2)}_3-t^1_4\Big)\gamma\Big(t^{(2)}_4-t^1_2\Big)\,,\\
\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})}-x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) &:= \Lambda\Big(x^{(2)}_1-x^1_3\Big)\Lambda\Big(x^{(2)}_2-x^1_1\Big)\Lambda\Big(x^{(2)}_3-x^1_4\Big)\Lambda\Big(x^{(2)}_4-x^1_2\Big)\,.
\end{aligned}$$

and $\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})}-x_{\underline{\mathscr{V}}(\mathcal{D})}\Big)$ is also expressed in the same way. Obviously, there are 4! such diagram. If $\gamma(\cdot)=\delta(\cdot)$, then (4.5.7) reduced to

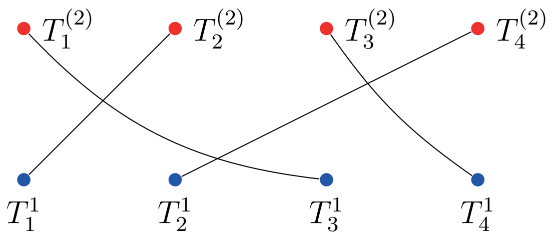

Figure 4.3: A admissible diagram $\mathcal{D}\in\mathbb{D}(4,4)$ with $T^{(j)}_l = (t^{(j)}_l, x^{(j)}_l)$

$$\begin{aligned}
\mathbb{E}\Big[|I^W_n(f_n)|^2\Big] = \int\prod_{r=1}^{n} G_{t_{r+1}-t_r}\left(x_{r+1}-x_r\right)\Lambda\Big(x_r-y_r\Big) \qquad (4.5.8)\\
\times G_{t_{r+1}-t_r}\left(y_{r+1}-y_r\right)\cdot 1_{\{0<t_1<\cdots<t_n<t\}}dt_{\mathcal{D}}dxdy\,.
\end{aligned}$$

This is because the only admissible admissible diagram is the 'trivial' one shown in Figure 4.4 in this case. Otherwise, in some non-trivial admissible diagrams (e.g. the one in Figure 4.3) the indicate function $1_{\{0<t^1_1<\cdots<t^1_n<t\}}$ is not compatible with $1_{\{0<t^{(2)}_1<\cdots<t^{(2)}_n<t\}}$.

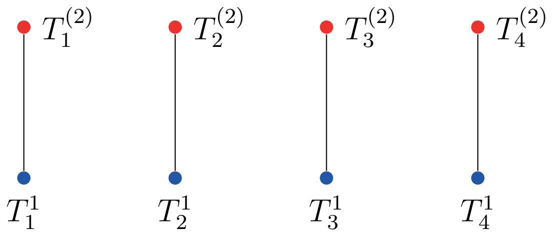

Figure 4.4: The 'trivial' admissible diagram $\mathcal{D} \in \mathbb{D}(4,4)$

4.6 Lower moment bounds

In this section we use the formula (4.5.4) to obtain the lower moment bounds for the mild solution of (4.1.1). In the remaining part of the chapter we shall use the index (t_j^l, x_j^l) to represent the independent variable $(t_{(l,j)}, x_{(l,j)})$ associated with the vertice (l,j): the superscript indicates the row that variable corresponds to and the subscript indicates the column that variable corresponds to. Again, in the following, we shall only prove the case **(H2)**. The cases **(H3)** and **(H4)** can be done similarly.

Proof of Theorem 4.3.6. Let $u(t,x)$ be the mild solution given by (4.2.8)-(4.2.9). Let p be an even positive integer. Applying Theorem 4.5.4, we have

$$\begin{aligned}\mathbb{E}\Big[\prod_{j=1}^p u(t,x_j)\Big] =& \mathbb{E}\Big[\prod_{j=1}^p \sum_{n_j=0}^\infty I_{n_j}(f_{n_j}(\cdot,t,x_j))\Big] \\ =& \sum_{n_1=0}^\infty \cdots \sum_{n_p=0}^\infty \mathbb{E}\Big[I_{n_1}(f_{n_1}(\cdot,t,x_j)) \cdots I_{n_p}(f_{n_p}(\cdot,t,x_j))\Big] \\ =& \sum_{m=0}^\infty \sum_{\substack{n_1+\cdots+n_p=2m \\ \mathcal{D}\in\mathbb{D}(f_{n_1},\cdots,f_{n_p})}} F_{\mathcal{D}}(f_{n_1},\cdots,f_{n_p})\,. \end{aligned} \tag{4.6.1}$$

Notice that the last equality follows from the fact that the number of all vertices of an admissible diagram $\mathcal{D}$ must be even.

Our next strategy is to find the suitable lower bounds for the term

$$\sum_{n_1+\cdots+n_p=2m} \sum_{\mathcal{D}\in\mathbb{D}} F_{\mathcal{D}}$$

in (4.6.1) when p and m are sufficiently large. We shall divide our proof into three steps.

Step 1: By the assumption **(G1)**, namely, all the kernels f_{n_k} are nonnegative, to obtain

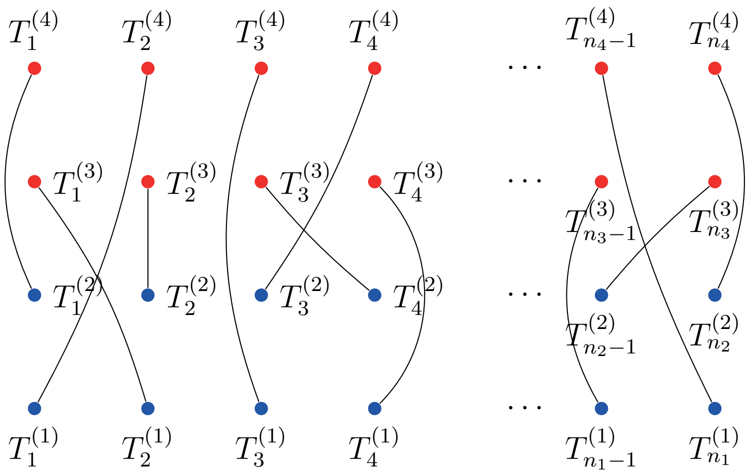

Figure 4.5: A particular scheme when $p = 4$

the lower bounds, we can discard any terms we wish. As in [DM09] we shall keep only those terms such that $n_1 = \cdots = n_p$ (see the Figure 4.5 for a graphical illustration). To be more precise, among all the admissible diagrams $\mathcal{D} \in \mathbb{D}(n_1, \cdots, n_p)$ such that $n_1 + \cdots + n_p = 2m$, we take into account only those diagrams satisfying the following conditions:

(D.1) We consider only the diagram so that the number of vertices in each row are the same. This is, we set

$$n_1 = \cdots = n_p = \frac{2m}{p} =: m_p \,. \tag{4.6.2}$$

(D.2) We set the first $\frac{p}{2}$ rows to be the upper vertices $T_{\overline{\mathscr{V}}(\mathcal{D})} := (t_{\overline{\mathscr{V}}(\mathcal{D})}, x_{\overline{\mathscr{V}}(\mathcal{D})})$ (which are colored in red in the Figure 4.5), and the remaining rows to be the lower vertices $T_{\underline{\mathscr{V}}(\mathcal{D})} := (t_{\underline{\mathscr{V}}(\mathcal{D})}, x_{\underline{\mathscr{V}}(\mathcal{D})})$ (which are colored in blue in the Figure 4.5).

Remark 4.6.1. *Fix the set of upper vertices. Any permutation of the lower vertices corresponds to an admissible diagram in one to one manner. Then there are* $m!$ *such* admissible diagrams *satisfying the conditions* ***(D.1)*** *and* ***(D.2)****.*

Step 2: Since $f_n(\cdot, t, x)$ is defined by (4.2.9) we can use (4.5.4) in Theorem 4.5.4 to bound $F_{\mathcal{D}}$ in (4.6.1).

We only consider particular scenario specified in **Step 1**. We denote the set of all admissible diagrams satisfying satisfying the conditions **(D.1)** and **(D.2)** by $\mathbb{D} :=$

$\mathbb{D}(f_{n_1}, \cdots, f_{n_m})$. When $\mathcal{D} \in \mathbb{D}$, we have

$$F_{\mathcal{D}}(f_{n_1}, \cdots, f_{n_p}) = \int \prod_{l=1}^{p} \prod_{j=1}^{m_p} G_{t^l_{j+1} - t^l_j}\left(x^l_{j+1}, x^l_j\right) 1_{\{0<t^l_1<\cdots<t^l_{m_p}<t\}} \times \gamma\Big(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})}\Big)\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) dt_{\mathcal{D}} dx_{\mathcal{D}}\,, \tag{4.6.3}$$

with the convention that $x^l_{m_p+1} = x$ and $t^l_{m_p+1} = t$ for all $1 \le l \le p$.

It seems very difficult to compute the multiple integral in (4.6.3). We need to find a suitable lower bounds of the integral that are the main parts and that are relatively easier to handle. Since $\Lambda(x) \to \infty$ when $x \to 0$, we shall first bound the above integral with respect to the spatial variables $dx_{\mathcal{D}}$ from below by the integration over small balls $B_\varepsilon(x)$ centered at $x = x_1 = \cdots = x_p$ with radius ε. This is, we concentrate on the domain

$$\Omega_\varepsilon := \cap_{l=1}^{p} \cap_{j=1}^{m_p} \{x^l_j \in B_\varepsilon(x)\}\,.$$

By the assumption **(H2)** or **(H3)**, it is easy to see $\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) \gtrsim \varepsilon^{-m\lambda}$ since $\#\{\overline{\mathscr{V}}(\mathcal{D})\} = \#\{\underline{\mathscr{V}}(\mathcal{D})\} = m$ and since we always have $|x_i - x_j| \le 2\varepsilon$ for any $i \in \overline{\mathscr{V}}(\mathcal{D})$ and $j \in \underline{\mathscr{V}}(\mathcal{D})$.

For the time variables. Let $t \in \mathbb{R}_+$. Denote $L = \frac{t}{2(m_p+1)}$, $t_j = \frac{j \cdot t}{2(m_p+1)}$ and $I_j = [a_j, b_j]$ for $j = 1, \dots, m_p$, where $a_j = t_j - L/4$ and $b_j = t_j + L/4$. We assure t^l_j is in I_j for $1 \le l \le p$ and $1 \le j \le m_p$. Moreover, we require (4.6.2) in **(D.1)** satisfying $m_p = \frac{2m}{p} \ge \frac{t}{\varepsilon^{\mathfrak{b}}}$, which is equivalent to the following conditon:

$$m \ge \frac{p \cdot t}{2\varepsilon^{\mathfrak{b}}}\,. \tag{4.6.4}$$

Then we have

$$\frac{t}{4m_p} \simeq \frac{t}{4(m_p+1)} \le t^l_{j+1} - t^l_j \le \frac{t}{m_p+1} \simeq \frac{t}{m_p} \le \varepsilon^{\mathfrak{b}}\,. \tag{4.6.5}$$

These restrictions are used to guarantee the *small ball nondegeneracy property* $B(\mathfrak{a}, \mathfrak{b})$

can be estblished. Overall, we shall consider the points $\{t_j^l\}$ on the following domain:

$$\tilde{\Omega}_\varepsilon := \cap_{l=1}^p \cap_{j=1}^{m_p} \{t_j^l \in I_j\},$$

with m (or m_p) satisfies condition (4.6.4). Similarly, it is obvious to control $\gamma\Big(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})}\Big) \gtrsim t^{-m\gamma}$ because $|t_i - t_j| \le t$ for any $i \in \overline{\mathscr{V}}(\mathcal{D})$ and $j \in \underline{\mathscr{V}}(\mathcal{D})$.

On each space-time line, we shall consider (t_j^l, x_j^l) on the set $\tilde{\Omega}_\varepsilon \times \Omega_\varepsilon$. Then the *small ball nondegeneracy property* $B(\mathfrak{a}, \mathfrak{b})$ implies

$$\int_{B_\varepsilon(x)} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) dx_j^l \ge C \cdot |t_{j+1}^l - t_j^l|^{\mathfrak{a}}$$

if x_j^l belong to $B_\varepsilon(x)$ for all l and j. Thus, on the domain Ω_ε we have from the simple fact $\Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) \gtrsim \varepsilon^{-m\lambda}$:

$$\begin{aligned}
\int \prod_{l=1}^p \prod_{j=1}^{m_p} & G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) \Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) dx_{\mathcal{D}} \\
&\gtrsim \int_{\Omega_\varepsilon} \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) \Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) dx_{\mathcal{D}} \\
&\gtrsim \varepsilon^{-m\lambda} \int_{B_\varepsilon(x)^{2m}} \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) dx_{\mathcal{D}} \\
&= \varepsilon^{-m\lambda} \int_{B_\varepsilon(x)^{2m-1}} \int_{B_\varepsilon(x)} G_{t_2^1 - t_1^1}\left(x_2^1, x_1^1\right) dx_1^1 \\
&\qquad\qquad \times \prod_{\substack{l=1,j=1 \\ l,j\neq 1}}^{p,m_p} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) dx_{\mathcal{D}} \backslash x_1^1 \\
&\gtrsim \varepsilon^{-m\lambda} \left|t_2^1 - t_1^1\right|^{\mathfrak{a}} \\
&\qquad \times \int_{B_\varepsilon(x)^{2m-1}} \prod_{\substack{l=1,j=1 \\ l,j\neq 1}}^{p,m_p} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) dx_{\mathcal{D}} \backslash x_1^1,
\end{aligned}$$

where we used (4.6.5) and $dx_{\mathcal{D}} \backslash x_1^1$ means that dx_1^1 is removed from $dx_{\mathcal{D}}$. We integrate the spatial variables iteratively to find

$$\int \prod_{l=1}^p \prod_{j=1}^{m_p} G_{t_{j+1}^l - t_j^l}\left(x_{j+1}^l, x_j^l\right) \Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) dx_{\mathcal{D}}$$

$$\gtrsim \varepsilon^{-m\lambda} \prod_{l=1}^{p} \prod_{j=1}^{m_p} \left[t_{j+1}^l - t_j^l\right]^{\mathfrak{a}} \tag{4.6.6}$$

for all $t_{\mathcal{D}} \in \tilde{\Omega}_\varepsilon$. From this inequality, Remark 4.6.1, and (4.6.5) we can bound (4.6.3) from below by

$$\begin{aligned}
&F_{\mathcal{D}}(f_{n_1}, \cdots, f_{n_p}) \\
&\quad \geq \int_{\tilde{\Omega}_\varepsilon \times \Omega_\varepsilon} \prod_{l=1}^{p} \prod_{j=1}^{m_p} G_{t_{j+1}^l - t_j^l} \left(x_{j+1}^l, x_j^l\right) 1_{\{0<t_1^l<\cdots<t_{m_p}^l<t\}} \\
&\qquad\qquad \times \gamma\Big(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})}\Big) \Lambda\Big(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\Big) dt_{\mathcal{D}} dx_{\mathcal{D}} \\
&\quad \geq \varepsilon^{-m\lambda} t^{-m\gamma} \cdot \int_{\tilde{\Omega}_\varepsilon} \prod_{l=1}^{p} \prod_{j=1}^{m_p} \left[t_{j+1}^l - t_j^l\right]^{\alpha} 1_{\{0<t_1^l<\cdots<t_{m_p}^l<t\}} dt_{\mathcal{D}} \\
&\quad \gtrsim \varepsilon^{-m\lambda} t^{-m\gamma} \cdot \left(\frac{t}{4m_p}\right)^{2m\mathfrak{a}} \int \prod_{l=1}^{p} \prod_{j=1}^{m_p} 1_{I_j}(t_j^l) 1_{\{0<t_1^l<\cdots<t_{m_p}^l<t\}} dt_{\mathcal{D}} \\
&\quad =: \varepsilon^{-m\lambda} t^{-m\gamma} \cdot \left(\frac{t}{4m_p}\right)^{2m\mathfrak{a}} I_{\varepsilon,p,m}\,,
\end{aligned} \tag{4.6.7}$$

where $I_{\varepsilon,p,m}$ denotes the above multiple integral with respect to $dt_{\mathcal{D}}$. Now let us deal with this integral $I_{\varepsilon,p,m}$. It is easy to see

$$I_{\varepsilon,p,m} = \left[\int \prod_{j=1}^{m_p} 1_{I_j}(t_j) dt_1 \cdots dt_{m_p}\right]^p = \left(\frac{L}{2}\right)^{m_p \times p} \simeq \left(\frac{t}{m_p}\right)^{2m}.$$

Let $\mathbb{D}(m_p)$ denote $\mathbb{D}(f_{m_p}, \cdots, f_{m_p})$. Substituting this bound into (4.6.7) we have for $\mathcal{D} \in \mathbb{D}(m_p)$,

$$\begin{aligned}
&F_{\mathcal{D}}(f_{m_p}, \cdots, f_{m_p}) \\
&\quad \gtrsim \varepsilon^{-m\lambda} t^{-m\gamma} \cdot \left(\frac{t}{4m_p}\right)^{2m\alpha} \int \prod_{l=1}^{p} \prod_{j=1}^{m_p} 1_{I_j}(t_j^l) 1_{\{0<t_1^l<\cdots<t_{m_p}^l<t\}} dt_{\mathcal{D}} \\
&\quad \gtrsim \varepsilon^{-m\lambda} t^{-m\gamma} \cdot \left(\frac{tp}{m}\right)^{2m(\mathfrak{a}+1)}.
\end{aligned}$$

Since there are $m!$ elements in $\mathbb{D}(m_p)$, we have

$$\sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p}) \gtrsim m!\varepsilon^{-m\lambda}t^{-m\gamma}\cdot\left(\frac{tp}{m}\right)^{2m(\mathfrak{a}+1)}. \tag{4.6.8}$$

Step 3: In this step, we obtain the asymptotic behaviors of the term appearing in (4.6.8) when m is sufficient large. According to Stirling's formula $m! \simeq \sqrt{2\pi m}\cdot(\frac{m}{e})^m$, we arrive at

$$\begin{aligned}&\sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p})\\ &\quad\gtrsim \varepsilon^{-m\lambda}t^{-m\gamma}\cdot\frac{(t\cdot p)^{2m(\mathfrak{a}+1)}}{m^{m(2\mathfrak{a}+1)}} \simeq \left(\varepsilon^{-\lambda}\times\frac{t^{2(\mathfrak{a}+1)-\gamma}\cdot p^{2(\mathfrak{a}+1)}}{m^{2\mathfrak{a}+1}}\right)^m.\end{aligned} \tag{4.6.9}$$

Let us recall that to obtain the above inequality we assumed that t, x are sufficiently large and $\mathfrak{b}$ is sufficiently small. Consequently, m is also large enough since it satisfies (4.6.4). Now in (4.6.9), we can take the value

$$m_0(\varepsilon) := \left[C\varepsilon^{-\lambda}t^{2(\mathfrak{a}+1)-\gamma}p^{2(\mathfrak{a}+1)}\right]^{\frac{1}{2\mathfrak{a}+1}} = C\cdot\varepsilon^{-\frac{\lambda}{2\mathfrak{a}+1}}t^{1+\frac{1-\gamma}{2\mathfrak{a}+1}}p^{1+\frac{1}{2\mathfrak{a}+1}}.$$

With this choice of $m = m_0(\varepsilon)$, the condition (4.6.4) i.e. $m \geq \frac{p\cdot t}{2\varepsilon^{\mathfrak{b}}}$ together with (4.3.1) (i.e. $\mathfrak{b}(2\mathfrak{a}+1)-\lambda > 0$) imply that

$$\begin{aligned}&\varepsilon^{\frac{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}{2\mathfrak{a}+1}} \gtrsim t^{-\frac{1-\gamma}{2\mathfrak{a}+1}}p^{-\frac{1}{2\mathfrak{a}+1}}\\ \Longleftrightarrow\ &\varepsilon \gtrsim t^{-\frac{1-\gamma}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}p^{-\frac{1}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}} =: \varepsilon_{t,p}.\end{aligned}$$

Thus, putting $\varepsilon = \varepsilon_{t,p}$ and $m = m_0(\varepsilon_{t,p})$ into (4.6.9) we obtain

$$\begin{aligned}\sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p}) &\gtrsim \exp\left(C\cdot\varepsilon_{t,p}^{-\frac{\lambda}{2\mathfrak{a}+1}}t^{1+\frac{1-\gamma}{2\mathfrak{a}+1}}p^{1+\frac{1}{2\mathfrak{a}+1}}\right)\\ &= \exp\left(C\cdot t^{\frac{1-\gamma}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}\cdot\frac{\lambda}{2\mathfrak{a}+1}}p^{\frac{1}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}\cdot\frac{\lambda}{2\mathfrak{a}+1}}\times t^{1+\frac{1-\gamma}{2\mathfrak{a}+1}}p^{1+\frac{1}{2\mathfrak{a}+1}}\right),\end{aligned}$$

where

$$1+\frac{1-\gamma}{2\mathfrak{a}+1}+\frac{1-\gamma}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}\cdot\frac{\lambda}{2\mathfrak{a}+1}=1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}$$

and

$$1+\frac{1}{2\mathfrak{a}+1}+\frac{1}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}\cdot\frac{\lambda}{2\mathfrak{a}+1}=1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}\,.$$

This is

$$\sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p}) \gtrsim \exp\left(C\cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right)\,. \tag{4.6.10}$$

As a result, from (4.6.1), (4.6.8) and (4.6.10) we obtain that

$$\begin{aligned}\mathbb{E}\Big[\prod_{j=1}^{p}u(t,x_j)\Big] &= \sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_p=2m}\sum_{\mathcal{D}\in\mathbb{D}(f_{m_1},\cdots,f_{m_p})} F_{\mathcal{D}}(f_{m_1},\cdots,f_{m_p})\\ &\gtrsim \sum_{p\cdot m_p=2m_0}\sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p})\\ &\gtrsim \exp\left(C\cdot t^{1+\frac{\mathfrak{b}\cdot(1-\gamma)}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\cdot p^{1+\frac{\mathfrak{b}}{\mathfrak{b}(2\mathfrak{a}+1)-\lambda}}\right)\,.\end{aligned}$$

We have completed the proof of Theorem 4.3.6. □

4.7 Some important SPDEs

In this section, we shall explain the *positivity property* **(G1)**, the *small ball nondegeneracy property* $(B(\mathfrak{a},\mathfrak{b}))$ **(G2)** and the *HLS total weighted mass property* **(G3)** for some important stochastic PDEs: SHE, α-SHE, SWE and SFD.

4.7.1 Stochastic heat equation (SHE)

Firstly, we consider the well known stochastic heat equation that has been extensively studied in literature, see [Hu19] and the references therein. The equation has the following form.

$$\text{(SHE)} \qquad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, \quad x \in \mathbb{R}^d\,, \\ u(0,x) = u_0(x)\,. \end{cases} \tag{4.7.1}$$

In this case the partial differential operator in the setting of equation (4.1.1) is

$$\mathscr{L}u(t,x) = \frac{\partial u(t,x)}{\partial t} - \frac{1}{2}\Delta u(t,x)\,.$$

There is only one initial condition $u(0,x) = u_0(x)$. The Green's function and its Fourier transform in spatial variable are respectively:

$$G_t^{\mathrm{h}}(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{and} \quad \mathcal{F}[G_t^{\mathrm{h}}(\cdot)](\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right)\,. \tag{4.7.2}$$

It is clear that $G_t^{\mathrm{h}}(x) \geq 0$ is a positive kernel. So, the assumption **(G1)** is obviously satisfied. We shall show the *small ball nondegeneracy property* ($B(\alpha,\beta)$) **(G2)** and the *HLS mass property* $M(\mu,\nu)$ **(G3)** in the following proposition 4.7.1 and proposition 4.7.2 respectively.

Proposition 4.7.1 (**Small Ball Nondegeneracy Property and Lower Moments for SHE**). *For the heat kernel $G_t^{\mathrm{h}}(x)$, the small ball nondegeneracy* B(0,2) *holds. In fact we have the following statements:*

(i) For all $d \in \mathbb{N}$, there exist some strict positive constants C_1 and C_2 independent of t, x and ε such that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{h}}(y-z)dz \geq C_1 \exp\left(-C_2 \frac{t}{\varepsilon^2}\right)\,. \tag{4.7.3}$$

(ii) Consequently, B(0,2) *holds for G_t^{h}, i.e. there exist a strict positive constant C independent of t, x and ε so that*

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{h}}(y-z)dz \geq C\,, \tag{4.7.4}$$

for $0 < t \le \varepsilon^2$.

As a result, assuming $\gamma(\cdot)$ *(with* $\gamma = 2-2H$*) and* $\Lambda(\cdot)$ *satisfy the same conditions of Theorem 4.3.6, there are some positive constants* c_1 *and* c_2 *independent of* t*,* p *and* x *such that*

$$\mathbb{E}[|u^{\mathrm{h}}(t,x)|^p] \ge c_1 \exp\left(c_2 \cdot t^{\frac{4H-\lambda}{2-\lambda}} p^{\frac{4-\lambda}{2-\lambda}}\right) .$$

Proof. We only need to prove (4.7.3), which is related to what is known as small ball property of Brownian motion. The readers can find the related result in immense literatures, for example (5.6.20) in [Hu17] for one dimension. We divide the proof into two steps.

Step 1: Clearly, we may assume $x = (0, \cdots, 0)$. It may be possible to work on the integral directly. However, we feel easier to use the spherical coordinate for the computation of the integral. We employ the following d-dimensional spherical coordinate $(z_1, \cdots, z_d) = \Phi(r, \theta, \phi_1, \cdots, \phi_{d-2})$:

$$\begin{cases} z_1 = r\cdot\cos(\phi_1) \\ z_2 = r\cdot\sin(\phi_1)\cos(\phi_2) \\ \qquad\qquad\qquad\qquad\cdots \\ z_{d-2} = r\cdot\sin(\phi_1)\cdots\sin(\phi_{d-3})\cos(\phi_{d-2}) \\ z_{d-1} = r\cdot\sin(\phi_1)\cdots\sin(\phi_{d-2})\cos(\theta) \\ z_{d-1} = r\cdot\sin(\phi_1)\cdots\sin(\phi_{d-2})\sin(\theta)\,, \end{cases} \tag{4.7.5}$$

where $0 \le \phi_n < \pi$, $n = 1, \cdots, d-2$, $0 \le \theta \le 2\pi$. The Jacobian determinant of this transformation is

$$J_d = r^{d-1} \prod_{k-1}^{d-2} \sin^{d-1-k}(\phi_k)\,.$$

Since $G_t^h(\cdot)$ is rotation invariant as a function in $\mathbb{R}^d$ we only need to consider $y = (r_0, 0, \cdots, 0)$ for some fixed $r_0 \in (0, \varepsilon)$. Set $B_\varepsilon(r_0) := B_\varepsilon(y)$, therefore

$$\int_{B_\varepsilon(0)} G_t^{\mathrm{h}}(y-z)dz \ge \int_{B_\varepsilon(r_0)\cap B_\varepsilon(0)} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|z|^2}{2t}\right) dz$$

$$\simeq \int_0^{\varepsilon} \int_{[0,\pi)^{d-2}} \int_0^{2\pi} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{r^2}{2t}\right) \times 1_{B_\varepsilon(r_0)}(\Psi(r,\theta,\phi)) \cdot |J_d| d\theta d\phi dr \,. \tag{4.7.6}$$

Notice that the identity

$$1_{B_\varepsilon(r_0)}(\Psi(r,\theta,\phi)) = 1_{B_\varepsilon(0)}((r_0,0,\ldots,0) - \Psi(r,\theta,\phi))$$

can be expressed as

$$\begin{aligned} &\{(r,\theta,\phi) \in [0,\varepsilon] \times [0,2\pi) \times [0,\pi)^{d-2} : r^2 \sin^2(\phi_1) + [r\cos(\phi_1) - r_0]^2 \le \varepsilon^2\} \\ &\quad = \{(r,\theta,\phi) \in [0,\varepsilon] \times [0,2\pi) \times [0,\pi)^{d-2} : r^2 + r_0^2 - 2r \cdot r_0 \cos(\phi_1) \le \varepsilon^2\} \,. \end{aligned} \tag{4.7.7}$$

In order to estimate the lower bound of (4.7.6), we need the following particular subset of $\{(r,\theta,\phi) \in [0,\varepsilon] \times [0,2\pi) \times [0,\pi)^{d-2} : \Psi(r,\theta,\phi) \in B_\varepsilon(r_0)\}$:

$$\begin{aligned} S_\varepsilon(r,\theta,\phi) := \{(r,\theta,\phi) \in [0,\varepsilon] \times [0,2\pi) \times [0,\pi/3) \times [0,\pi/2)^{d-3} : \\ r^2 + r_0^2 - 2r \cdot r_0 \cos(\phi_1) \le \varepsilon^2\} \,. \end{aligned} \tag{4.7.8}$$

Because for $\phi_1 \in [0,\pi/3)$, we always have

$$r^2 + r_0^2 - 2rr_0\cos(\phi_1) \le r^2 + r_0^2 - rr_0 \le \varepsilon^2 \,,$$

if $0 \le r, r_0 \le \varepsilon$. On the domain $S_\varepsilon(r,\theta,\phi)$, the indicate function $1_{S_\varepsilon}(r,\theta,\phi) := 1_{S_\varepsilon(r,\theta,\phi)}(r,\theta,\phi) = 1$. Then we have from (4.7.6)

$$\begin{aligned} &\int_{B_\varepsilon(0)} G_t^{\mathrm{h}}(y-z)dz \\ &\quad \gtrsim \int_0^{\varepsilon} \int_{[0,\pi)^{d-2}} \int_0^{2\pi} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{r^2}{2t}\right) \times 1_{S_\varepsilon}(r,\theta,\phi) \cdot |J_d| d\theta d\phi dr \\ &\quad \gtrsim \int_0^{\varepsilon} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{r^2}{2t}\right) \times r^{d-1} dr \simeq \int_0^{\left(\frac{\varepsilon}{\sqrt{t}}\right)^d} \exp\left(-\frac{\tilde{r}^{\frac{2}{d}}}{2}\right) d\tilde{r} \,, \end{aligned} \tag{4.7.9}$$

where we have used the change of varible $r \to \tilde{r} = r/\sqrt{t}$ in the last line.

Step 2: We shall prove (4.7.9) is greater than $C_1 \exp\left(-\frac{C_2 \cdot t}{\varepsilon^2}\right)$ by showing the following claim. For fixed $\nu > 0$, one can find a constant c_ν such that $c_\nu \cdot \int_0^\infty \exp\left(-\frac{1}{2}r^\nu\right) dr = 1$. We claim that there exists a constant $c > \frac{(\nu+1)^2}{4\nu}$ such that $\forall\ \delta := \left(\frac{\varepsilon}{\sqrt{t}}\right)^d > 0$,

$$\int_0^\delta e^{-\frac{r^\nu}{2}} dr \geq c_\nu^{-1} \cdot e^{-\frac{c}{\delta^\nu}}\,. \tag{4.7.10}$$

This is equivalent to prove

$$c_\nu \cdot \int_\delta^\infty \exp\left(-\frac{1}{2}r^\nu\right) dr + e^{-\frac{c}{\delta^\nu}} \leq 1\,.$$

Let

$$g(\delta) = c_\nu \cdot \int_\delta^\infty e^{-\frac{r^\nu}{2}} dr + e^{-\frac{c}{\delta^\nu}}\,.$$

It is easy to see that $g(\delta)$ is continuous and $g(0) = g(\infty) = 1$. So in order to prove $g(\delta) \leq 1$ for all $\delta > 0$, it suffices to show that if $c > \frac{(\nu+1)^2}{4\nu}$, then

$$g'(\delta) = \frac{\nu \cdot c}{\delta^{\nu+1}} e^{-\frac{c}{\delta^\nu}} - c_\nu \cdot e^{-\frac{\delta^\nu}{2}} = 0$$

has exactly one root. It is clear that this is equivalent to

$$\begin{aligned} \frac{\nu \cdot c}{c_\nu} \cdot e^{\frac{\delta^\nu}{2}} = \delta^{\nu+1} e^{\frac{c}{\delta^\nu}} &\Leftrightarrow \exp\left(\frac{c}{\delta^\nu} + (\nu+1)\ln(\delta) - \frac{\delta^\nu}{2} - \ln\left(\frac{\nu \cdot c}{c_\nu}\right)\right) = 1 \\ &\Leftrightarrow h(\delta) = \frac{c}{\delta^\nu} + (\nu+1)\ln(\delta) - \frac{\delta^\nu}{2} - \ln\left(\frac{\nu \cdot c}{c_\nu}\right) = 0 \end{aligned}$$

has exactly one root. One can notice that $h(0+) = +\infty$ and $h(+\infty) = -\infty$. Then $h(\varepsilon)$ has at least one root. Next, we shall show it has at most one root, which suffices to argue that

$$h'(\delta) = -\frac{1}{\delta^{\nu+1}}\left[\left(\delta^\nu - \frac{\nu+1}{2}\right)^2 + \nu \cdot c - \frac{(\nu+1)^2}{4}\right] = 0$$

has no root for $\delta > 0$. But this is verified when $c > \frac{(\nu+1)^2}{4\nu}$. Lastly, the fact $g'(\delta) = 0$ has only one root and the intermediate value theorem imply that the claim (4.7.10) holds.

Letting $\nu = 2/d$ and $\delta = (\frac{\varepsilon}{\sqrt{t}})^d$ in (4.7.10), we get (4.7.9) is greater than $C_1 \exp\left(-\frac{C_2 \cdot t}{\varepsilon^2}\right)$

for some constant C_1 and C_2. Thus, we have completed the proof of (4.7.3). □

Proposition 4.7.2 (HLS Mass Property and Upper Moments for SHE). *Assume $\gamma(\cdot)$ (with $\gamma = 2-2H$) and $\Lambda(\cdot)$ with $\lambda < 2$ satisfy the same conditions as in Theorem 4.3.4 or Theorem 4.4.1. Then for the heat kernel $G_t^{\mathrm{h}}(x-y)$, we have **(G3)** or **(G3')** with $M(-\frac{\lambda}{2})$ hold. In other words, for all $d \in \mathbb{N}$, there exist some strict positive constants C_1, C_2 and C_3 do not depend on t and x such that*

$$\sup_{x,x'\in\mathbb{R}^d} \int_{\mathbb{R}^{2d}} G_t^{\mathrm{h}}(x-y)\Lambda(y-y')G_t^{\mathrm{h}}(x'-y')dydy' \le C\cdot t^{-\frac{\lambda}{2}}\,, \tag{4.7.11}$$

or denoting $\mu(d\xi) = \hat{V}(\xi)d\xi$

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{G}_t^{\mathrm{h}}(\xi-\eta)|^2\mu(d\xi) \le C_3\cdot t^{-\frac{\lambda}{2}}\,. \tag{4.7.12}$$

As a result, we have the upper p-th ($p \ge 2$) moments for $u^{\mathrm{h}}(t,x)$ for any $d \ge 1$. More precisely, for some constants C_1 and C_2 that are independent of t, p and x we can get

$$\mathbb{E}[|u^{\mathrm{h}}(t,x)|^p] \le C_1\cdot \exp\left(C_2\cdot t^{\frac{4H-\lambda}{2-\lambda}} p^{\frac{4-\lambda}{2-\lambda}}\right)\,.$$

Proof. We only need to prove (4.3.6) with $\bar{M}(0,-\frac{\lambda}{2})$. This is,

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_t^{\mathrm{h}}(x-y)dy = \int_{\mathbb{R}^d} G_t^{\mathrm{h}}(y)dy = 1\,,$$
$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_t^{\mathrm{h}}(x-y)\Lambda(y)dy \lesssim \sup_{x\in\mathbb{R}^d}\mathbb{E}|\sqrt{t}X-x|^{-\lambda} \le C\cdot t^{-\frac{\lambda}{2}}\,,$$

where X is a standard normal random variable and the above last inequality follows from [HNS11, Lemma A.1].

For the (4.7.12), it is easy to

$$\begin{aligned}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{G}_t^{\mathrm{h}}(\xi-\eta)|^2\mu(d\xi) &= \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-t|\xi-\eta|^2}\mu(d\xi)\\ &\le t^{-\frac{\lambda}{2}}\cdot\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\xi|^{\lambda-d}}{1+|\xi-\eta|^2}d\xi \le C\cdot t^{\hbar}\,.\end{aligned}$$

So, we obtain the upper moment bound. □

4.7.2 Fractional spatial equations: Space nonhomogeneous case

The next model is the generalized d (≥ 1)-spatial dimensional fractional stochastic α-heat equation (α-SHE) that has been considered in [BC14, BC16, CHW18]:

$$(\alpha\text{-SHE}) \qquad \begin{cases} \frac{\partial u(t,x)}{\partial t} = -(-\nabla(a(x)\nabla))^{\alpha/2} u(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, \quad x\in\mathbb{R}^d\,, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{4.7.13}$$

where $0<\alpha<2$, $a(\cdot):\mathbb{R}^d\to\mathbb{R}^{d^2}$ is a matrix valued function whose entries are Hölder continuous, and there exists a constant $c\geq 1$ such that $c^{-1}\cdot Id \leq a(x) \leq c\cdot Id$. The operator $\mathscr{L}$ is

$$\mathscr{L}u(t,x) = \frac{\partial u(t,x)}{\partial t} + (-\nabla(a(x)\nabla))^{\alpha/2} u(t,x)$$

and the corresponding Green's function $G_t^{\mathrm{h},\alpha}(x)$ satisfies the following Nash's Hölder estimates (see e.g. [CHW18] for more details):

$$\frac{1}{C}\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq G_t^{\mathrm{h},\alpha}(x,y) \leq C\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \tag{4.7.14}$$

and $I_0(t,x) = G_t^{\mathrm{h},\alpha} * u_0(x)$. Clearly, (4.7.14) ensures the *positivity* of $G_t^{\mathrm{h},\alpha}(x)$ when $\alpha\in(0,2)$. We still need to take care of the *small ball nondegeneracy property* **(G2)** with *$B(\alpha,\beta)$* and the *HLS mass property* **(G3)** with $M(0,-\frac{\lambda}{\alpha})$.

Proposition 4.7.3 (**Small Ball Nondegeneracy Property and Lower Moments for α-SHE**). *For the heat kernel $G_t^{\mathrm{h},\alpha}(x)$, we have* B(0,$\alpha$) *holds:*

(i) For $\alpha\in(0,2)$ and $d\in\mathbb{N}$, there exist some strict positive constants C_1 and C_2 do not depend on t and ε such that

$$\inf_{y\in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{h},\alpha}(y,z)dz \geq C_1 \exp\left(-C_2\frac{t}{\varepsilon^\alpha}\right). \tag{4.7.15}$$

(ii) Consequently, B(0,α) *holds for $G_t^{\mathrm{h},\alpha}$, i.e. there exist a strict positive constant C*

independent of t and ε so that

$$\inf_{y\in B_\varepsilon(x)}\int_{B_\varepsilon(x)} G_t^{\mathrm{h},\alpha}(y,z)dz \geq C\,, \tag{4.7.16}$$

for $0 < t \leq \varepsilon^\alpha$.

As a result, assuming $\gamma(\cdot)$ (with $\gamma = 2-2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 4.3.6, we have the lower p-th ($p \geq 2$) moment bound: there are constants c_1 and c_2 independent of t, p and x such that

$$\mathbb{E}[|u^{\mathrm{h},\alpha}(t,x)|^p] \geq c_1 \exp\left(c_2 \cdot t^{\frac{2\alpha H-\lambda}{\alpha-\lambda}} p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}\right).$$

Proof. The proof is similar to the SHE case except now we have the Nash's Hölder estimates (4.7.14) instead of the the precise form of $G_t^{\mathrm{h},\alpha}(x)$.

By lower bound in the Nash's inequality (4.7.14), we have

$$G_t^{\mathrm{h},\alpha}(x,y) \gtrsim t^{-\frac{d}{\alpha}} \exp\left(-C_{\alpha,d}\cdot \frac{|x-y|^\alpha}{t}\right), \tag{4.7.17}$$

since $1\wedge|x|^{-1} \geq C_{1,\alpha}\cdot\exp\left(-C_{2,d}\cdot|x|^\alpha\right)$ for $\alpha > 0$. Thus (4.7.16) can be proved the same way as that of (4.7.15). □

Proposition 4.7.4 (HLS Mass Property and Upper Moments for α-SHE). *Assume $\gamma(\cdot)$ (with $\gamma = 2-2H$) and $\Lambda(\cdot)$ with $\lambda < \alpha$ satisfy the same conditions of Theorem 4.3.4 or Theorem 4.4.1. Then for the heat kernel $G_t^{\mathrm{h},\alpha}(x,y)$, we have **(G3)** or **(G3')** with $M(-\frac{\lambda}{\alpha})$ hold. In other words, for all $d\in\mathbb{N}$, there exist some strict positive constants C_1 and C_2 independent of t and x such that*

$$\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^{\mathrm{h},\alpha}(x,y)\Lambda(y-y')G_t^{\mathrm{h},\alpha}(x',y')dydy' \leq C\cdot t^{-\frac{\lambda}{\alpha}}. \tag{4.7.18}$$

Furthermore, there is a positive kernel $Q_t(x-y)$ such that $G_t^{\mathrm{h},\alpha}(x,y) \leq Q_t(x-y)$ and

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{Q}_t(\xi-\eta)|^2|\mu|(d\xi) \leq C_3\cdot t^{-\frac{\lambda}{\alpha}} \tag{4.7.19}$$

with $|\mu|(d\xi) = |\hat{V}(\xi)|d\xi$*.*

Consequently, we have the upper p-th ($p \geq 2$) *moment bounds. This is, for some constants* C_1 *and* C_2 *that are independent of* t, p *and* x *we have*

$$\mathbb{E}[|u^{\mathrm{h}}(t,x)|^p] \leq C_1 \cdot \exp\left(C_2 \cdot t^{\frac{2\alpha H-\lambda}{\alpha-\lambda}} p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}\right).$$

Proof. Presumably, we may use (4.7.14) to obtain the desired bounds. However, we will use Pollard's formula in [CHW18] to prove this proposition.

$$e^{-u^{\frac{\alpha}{2}}} = \int_0^\infty e^{-us} g(\alpha/2, s) ds\,, \;\; u \geq 0\,, \tag{4.7.20}$$

where $g(\alpha, s)$ is a probability density function of $s \geq 0$ and defined in (1.2) in [CHW18]. By Proposition 2.2 there, we have

$$\begin{aligned} G_t^{\mathrm{h},\alpha}(x,y) = &\int_0^\infty p(t^{\frac{2}{\alpha}} s, x, y) g(\alpha/2, s) ds \\ \leq &C \int_0^\infty t^{-\frac{d}{\alpha}} s^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct^{2/\alpha}s}\right) g(\alpha/2, s) ds =: Q_t(x-y)\,. \end{aligned} \tag{4.7.21}$$

Therefore, it is sufficient to show the assumption **(G3)** can be archived with $\bar{M}(0, -\frac{\lambda}{2})$ (i.e. the estimates (4.3.6)) for $Q_t(x-y)$. It is not hard to derive that

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} Q_t(x-y) dy \lesssim \int_0^\infty g(\alpha/2, s) ds < \infty\,,$$

and

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} Q_t(x-y)\Lambda(y) dy \\ &\quad \lesssim \int_0^\infty t^{-\frac{d}{\alpha}} s^{-\frac{d}{2}} \left[\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{Ct^{2/\alpha}s}\right)\Lambda(y) dy\right] g(\alpha/2, s) ds \\ &\quad \lesssim t^{-\frac{\lambda}{\alpha}} \cdot \int_0^\infty s^{-\frac{\lambda}{2}} g(\alpha/2, s) ds \leq C_2 \cdot t^{-\frac{\lambda}{\alpha}}\,, \end{aligned}$$

where we have applied rearrangement inequality and [CHW18, Proposition 2.1].

Moreover, for the Fourier transform of $Q_t(x)$ with respect to x, we have

$$\begin{aligned}\mathcal{F}[Q_t(\cdot)](\xi) &\simeq \int_0^\infty \exp\left(-Cs\cdot t^{2/\alpha}|\xi|^2\right) g(\alpha/2,s)ds \\ &\simeq \exp\left(-\left[Ct^{2/\alpha}|\xi|^2\right]^{\alpha/2}\right) = e^{-C_\alpha\cdot t|\xi|^\alpha}\,.\end{aligned}$$

Finally, it is relatively easy to see that the assumption (4.7.19) can be archived. Then the upper moment bound follows. □

4.7.3 Stochastic wave equations

The lower moment bounds for d-dimensional stochastic wave equation (SWE) is one of the SPDEs that motivated this study. This type of equations has been well-studied in literature. There are several works on the upper bounds for any moments. But the lower bounds are only known for the second moments except in a few cases. (see e.g. [BC16, DM09]). We give a more complete results for all moments. This equation has the following form (we consider only $d = 1, 2, 3$):

$$\text{(SWE)}\qquad \begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)\dot{W}(t,x), & t>0, \quad x\in\mathbb{R}^d\,, \\ u(0,x) = u_0(x)\,, \quad \frac{\partial}{\partial t}u(0,x) = v_0(x)\,. \end{cases} \tag{4.7.22}$$

The operator $\mathscr{L}$ has the form

$$\mathscr{L}u(t,x) = \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2}\,.$$

The associated Green's function has different forms for different dimensions. More precisely, it is given by

$$\begin{cases} G_t^{\mathrm{w}}(x) = \frac{1}{2}1_{\{|x|<t\}}\,, & d=1\,, \\ G_t^{\mathrm{w}}(x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}}\,, & d=2\,, \\ G_t^{\mathrm{w}}(dx) = \frac{1}{4\pi}\frac{\sigma_t(dx)}{t}\,, & d=3\,, \end{cases} \tag{4.7.23}$$

where $\sigma_t(dx)$ is a surface measure on the sphere $\partial B_t(0) \subseteq \mathbb{R}^3$ with center at 0 and radius t, with total mass $4\pi t^2$ and $G_t^{\mathrm{w}}(\mathbb{R}^3) = t$. It is well known that $G_t^{\mathrm{w}}(\cdot)$ may not be positive when $d \geq 4$. On the other hand for any dimension d, the Fourier transform of $G_t^{\mathrm{w}}(\cdot)$ has the same form given by

$$\mathcal{F}[G_t^{\mathrm{w}}(\cdot)](\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad \xi \in \mathbb{R}^d.$$

In this case we also have $I_0^{\mathrm{w}}(t,x) := \frac{\partial}{\partial t} G_t^{\mathrm{w}} * u_0(x) + G_t^{\mathrm{w}} * v_0(x)$. When $d = 1, 2$, $G_t^{\mathrm{w}}(x)$ are positive functions and when $d = 3$ it is a positive measure. Thus, the assumption **(G1)** is satisfied for wave kernel $G_t^W(dx)$. The next two propositions are devoted to **(G2)** and **(G3)**.

Proposition 4.7.5 (**Small Ball Nondegeneracy Property and Lower Moments for SWE**). *For the wave kernel $G_t^{\mathrm{w}}(x)$ defined by (4.7.23), we have* B(1,1) *holds:*

(i) When $d = 1$ and $d = 2$, there exist strict positive constants C_1 and C_2, independent of t, ε and y such that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{w}}(y - z) dz \geq C_1 \cdot t \exp\left(-C_2 \frac{t}{\varepsilon}\right). \tag{4.7.24}$$

Consequently, there exist a strict positive constant C independent of t, ε and y so that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{w}}(y - z) dz \geq C \cdot t, \tag{4.7.25}$$

for $0 < t \leq \varepsilon$.

(ii) When $d = 3$, there exists a strict positive constant C independent of t, ε and y such that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t^{\mathrm{w}}(y - dz) \geq C \cdot t, \tag{4.7.26}$$

for $0 < t \leq \varepsilon$.

As a consequence, assuming $\gamma(\cdot)$ (with $\gamma = 2 - 2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 4.3.6, we have the following lower moment bounds for the solution:

$$\mathbb{E}[|u^{\mathrm{w}}(t,x)|^p] \geq c_1 \exp\left(c_2 \cdot t^{\frac{2H+2-\lambda}{3-\lambda}} \cdot p^{\frac{4-\lambda}{3-\lambda}}\right)$$

for some constants c_1 and c_2 independent of t, p and x.

Remark 4.7.6. *The* small ball nondegeneracy property *of wave kernel G_t^{w} is motivated by the following fact when $d = 1$. Let us illustrate it with $x = y = 0$. Then the left hand of (4.7.24) can be evaluated exactly as*

$$\int_{-\varepsilon}^{\varepsilon} G_t^{\mathrm{w}}(z)dz = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2} 1_{\{|z|<t\}} dz = t \wedge \varepsilon \, .$$

And then it is not hard to see

$$t \wedge \varepsilon = \varepsilon \cdot \left(\frac{t}{\varepsilon} \wedge 1 \right) \geq \varepsilon \cdot \left(C_1 \cdot \frac{t}{\varepsilon} \exp\left(-C_2 \frac{t}{\varepsilon} \right) \right) = C_1 \cdot t \exp\left(-C_2 \frac{t}{\varepsilon} \right) \, ,$$

which is the right hand of (4.7.24).

Proof. We shall give the proof of Proposition 4.7.5 for $d = 1, 2, 3$ in three steps seperately.

Step 1 ($d = 1$): It is clear that we only need to show (4.7.24). Without loss of generality, we may assume $x = 0$. Let us consider $d = 1$ first. Because $G_t^{\mathrm{w}}(y - z) = \frac{1}{2} 1_{\{|y-z|<t\}}$, then (4.7.25) becomes

$$\begin{aligned} &\int_{\mathbb{R}} G_t^{\mathrm{w}}(y-z) G_\varepsilon^{\mathrm{w}}(z) dz \\ &\qquad \simeq \int_{\mathbb{R}} \mathcal{F}[G_t^{\mathrm{w}}(y - \cdot)](\xi) \mathcal{F}[G_\varepsilon^{\mathrm{w}}(\cdot)](\xi) d\xi \\ &\qquad \simeq \int_{\mathbb{R}} e^{-\imath y \xi} \frac{\sin(t|\xi|)}{|\xi|} \frac{\sin(\varepsilon|\xi|)}{|\xi|} d\xi \\ &\qquad \simeq \int_{\mathbb{R}} e^{-\imath y \xi} |\xi|^{-2} \left[\sin^2\left(\frac{1}{2} |t + \varepsilon||\xi| \right) - \sin^2\left(\frac{1}{2} |t - \varepsilon||\xi| \right) \right] d\xi \\ &\qquad \simeq (|t + \varepsilon| - y) 1_{\{|y|<|t+\varepsilon|\}} - (|t - \varepsilon| - y) 1_{\{|y|<|t-\varepsilon|\}} \, , \end{aligned} \tag{4.7.27}$$

where in the last line we have applied the Fourier transform (e.g. 17.34(21) in [GR15])

$$\mathcal{F}[x^{-2} \sin^2(ax)](\xi) = \mathcal{F}_c[x^{-2} \sin^2(ax)](\xi) = \frac{\pi}{2} (a - \xi/2) 1_{\{\xi<2a\}} \, .$$

The rest is routine. We split (4.7.27) into two cases: $t > \varepsilon$ and $t \leq \varepsilon$. Noticing $|y| \leq \varepsilon$, when $t > \varepsilon$ we can bound (4.7.27) below by

$$((t+\varepsilon)-y)-((t-\varepsilon)-y)1_{\{|y|<t-\varepsilon\}} \geq 2\varepsilon 1_{\{|y|<t-\varepsilon\}} + t1_{\{|y|\geq t-\varepsilon\}} \geq \varepsilon\,.$$

The case $t \leq \varepsilon$ can be done similarly, so we omit the details. Therefore, we obtain

$$\int_{B_\varepsilon(x)} G_t^{\mathrm{w}}(y-z)dz \geq t \wedge \varepsilon \geq C_1 \cdot t\exp\left(-C_2\frac{t}{\varepsilon}\right)\,.$$

We have completed the proof of (4.7.25) when $d=1$.

Step 2 ($d=2$): Recall that $G_t^{\mathrm{w}}(y-z) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|y-z|^2}}1_{\{|y-z|<t\}}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} G_t^{\mathrm{w}}(y-z)1_{B_\varepsilon}(z)dz \\
&\qquad\qquad \gtrsim \int_{\mathbb{R}^2} \frac{1}{t}1_{\{|y-z|<t\}}1_{\{|z|<\varepsilon\}}dz \\
&\qquad\qquad \simeq \frac{1}{t}\int_{\mathbb{R}^2} 1_{\{|y_1-z_1|<t\}}1_{\{|y_2-z_2|<t\}}1_{\{|z_1|<\varepsilon\}}1_{\{|z_2|<\varepsilon\}}dz \\
&\qquad\qquad \simeq \frac{1}{t}\left(\int_{\mathbb{R}} 1_{\{|y-z|<t\}}1_{\{|z|<\varepsilon\}}dz\right)^2 \\
&\qquad\qquad \gtrsim \frac{1}{t}\left(C_1\cdot t\exp\left(-C_2\frac{t}{\varepsilon}\right)\right)^2 = C_1\cdot t\exp\left(-C_2\frac{t}{\varepsilon}\right)\,,
\end{aligned} \tag{4.7.28}$$

where we have applied the result in $d=1$ to derive the inequality last line in (4.7.28). Thus, the proof of (4.7.25) when $d=2$ has been completed.

Step 3 ($d=3$): Let us recall that now $G_t^{\mathrm{w}}(dz) = \frac{1}{4\pi}\frac{\sigma_t(dz)}{t}$ where $\sigma_t(dz)$ is the surface measure on $\partial B_t(0)$. We may assume $x=0$ and simplify $B_\varepsilon(0)$ as B_ε. Then (4.7.26) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^3} 1_{B_\varepsilon}(z)G_t^{\mathrm{w}}(y-dz) \\
&\qquad\qquad = \frac{1}{4\pi t}\int_{\partial B_t} 1_{B_\varepsilon}(y-z)\sigma_t(dz) \\
&\qquad\qquad = \frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi} 1_{B_\varepsilon}(y-\Psi(\theta,\phi))\left\|\frac{\partial\Psi}{\partial\theta}\times\frac{\partial\Psi}{\partial\phi}\right\|d\phi d\theta \\
&\qquad\qquad = \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} 1_{B_\varepsilon}(y-\Psi(\theta,\phi))|\sin(\phi)|d\phi d\theta\,,
\end{aligned} \tag{4.7.29}$$

where the parametrization is the three dimensional spherical coordinate (i.e. $d=3$ in

(4.7.5)):

$$\Psi(\theta,\phi) = (z_1(\theta,\phi), z_2(\theta,\phi), z_3(\theta,\phi)) = (t\sin(\phi)\cos(\theta), t\sin(\phi)\sin(\theta), t\cos(\phi))\,.$$

Similarly, we can select the particular subset as in (4.7.8) so that we can bound (4.7.29) below as

$$\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} 1_{B_\varepsilon}(y-\Psi(\theta,\phi))|\sin(\phi)|d\phi d\theta \geq \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi/3}|\sin(\phi)|d\phi d\theta = t/4\,.$$

As a result, we have completed the proof of (4.7.26). □

Proposition 4.7.7 (HLS Mass Property and Upper Moments for SWE). *Assume $d=1,2,3$, $\gamma(\cdot)$ (with $\gamma = 2-2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 4.3.4 (under the condition $\lambda < d$) or Theorem 4.4.1 (under the condition $\lambda < 2\wedge d$). Then for the wave kernel $G_t^{\mathrm{w}}(x)$, we have **(G3)** with $\mathrm{M}(2-\lambda)$ or **(G3')** with $M(2-\lambda)$ hold. In other words, for $d=1,2,3$, there exists some strict positive constants C independent of t and x such that*

$$\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^{\mathrm{w}}(x-y)\Lambda(y-y')G_t^{\mathrm{w}}(x'-y')dydy' \leq C\cdot t^{2-\lambda}\,, \tag{4.7.30}$$

Denoting $\mu(d\xi) = \hat{V}(\xi)d\xi$

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{G}_t^{\mathrm{w}}(\xi-\eta)|^2|\mu|(d\xi) \leq C\cdot t^{2-\lambda}\,. \tag{4.7.31}$$

Consequently, we have the desired upper p-th ($p\geq 2$) moment bounds for the solution $u^{\mathrm{w}}(t,x)$ when $d=1,2,3$. This is, we can find constants C_1 and C_2 that are independent of t, p and x such that

$$\mathbb{E}[|u^{\mathrm{w}}(t,x)|^p] \leq C_1\cdot\exp\left(C_2\cdot t^{\frac{2H+2-\lambda}{3-\lambda}}\cdot p^{\frac{4-\lambda}{3-\lambda}}\right)\,.$$

Proof. It is clear we only need to show **(G3)** holds for $G_t^{\mathrm{w}}(x)$ with $M(2-\lambda)$, i.e. the estimates (4.7.30).

When $d=1,2$, we can easily apply Hardy-Littelewood-Sobolev inequality ([LL97, Theorem 4.3]) for $\lambda < d$ to bound

$$
\begin{aligned}
&\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^{\mathrm{w}}(x-y)\Lambda(y-y')G_t^{\mathrm{w}}(x'-y')dydy' \\
&\qquad\le \sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t(x-y)|y-y'|^{-\lambda}G_t(x'-y')dydy' \\
&\qquad\le \left[\int_{\mathbb{R}^d}|G_t^{\mathrm{w}}(y)|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}}.
\end{aligned}
$$

For $d=1$, we have

$$
\left[\int_{\mathbb{R}^d}|G_t^{\mathrm{w}}(y)|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}} \simeq \left[\int_{-t}^{t}|1/2|^{\frac{2}{2-\lambda}}dt\right]^{2-\lambda} \le C\cdot t^{2-\lambda}.
$$

For $d=2$, we have

$$
\begin{aligned}
\left[\int_{\mathbb{R}^d}|G_t^{\mathrm{w}}(y)|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}} &\simeq \left[\int_{\mathbb{R}^2}|t^2-x^2|^{-\frac{2}{4-\lambda}}1_{|x|<t}dx\right]^{\frac{4-\lambda}{2}} \\
&= t^{2-\lambda}\cdot\left[\int_{\mathbb{R}^2}|1-x^2|^{-\frac{2}{4-\lambda}}1_{|x|<1}dx\right]^{\frac{4-\lambda}{2}} \\
&= C\cdot t^{2-\lambda},
\end{aligned}
$$

where the integral is finite if $\lambda < 2$.

Now we shall apply the HLS inequality on sphere (see e.g. [LL97, Theorem 4.5]) to show (4.7.30) for $d=3$ and $\lambda<3$. Denote by $\mathcal{S}^3$ the unit sphere in $\mathbb{R}^3$. We have

$$
\begin{aligned}
&\sup_{x,x'\in\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathbb{R}^3} G_t^{\mathrm{w}}(x-y)\Lambda(y-y')G_t^{\mathrm{w}}(x'-y')dydy' \\
&\quad\le \sup_{x,x'\in\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathbb{R}^3}|y-y'|^{-\lambda}\frac{\sigma_t(x-dy)}{4\pi t}\frac{\sigma_t(x'-dy')}{4\pi t} \\
&\quad\simeq t^{2-\lambda}\cdot\sup_{x,x'\in\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathbb{R}^3}1_{x+\mathcal{S}^3}(y)|y-y'|^{-\lambda}1_{x'+\mathcal{S}^3}(y)\sigma_1(dy)\sigma_1(dy') \\
&\quad\lesssim t^{2-\lambda}\cdot\sup_{x\in\mathbb{R}^3}\left[\int_{\mathbb{R}^3}|1_{x+\mathcal{S}^3}(y)|^{\frac{6}{6-\lambda}}\sigma_1(dy)\right]^{\frac{6-\lambda}{3}} = C\cdot t^{2-\lambda},
\end{aligned}
$$

where we have made use of the scaling property of the surface measure $\sigma_t(dy) = t^2\sigma_1(d\tilde{y})$ with $y = t\tilde{y}$ in the third line and the HLS inequality [LL97, Theorem 4.5] on sphere in the last line. This proves (4.7.30).

In regard to the bound (4.7.31), it is easy to see that if $\lambda < 2 \wedge d$

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{G}_t^{\mathrm{w}}(\xi)|^2\mu(d\xi-\eta) = \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{\sin(t|\xi|)}{\xi}\right|^2\mu(d\xi-\eta)$$
$$\leq t^{2-\lambda}\cdot\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\xi|^{\lambda-d}}{1+|\xi+t\eta|^2}d\xi \leq C\cdot t^{2-\lambda}\,.$$

Thus, we complete the proof of Proposition 4.7.7. □

Remark 4.7.8. *The properties we obtained in Proposition 4.7.1 (i) and Proposition 4.7.5 (i) can be also rewritten as the following* small ball property *(*$\mathrm{B}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$*): if* $y \in B_\varepsilon(x)$*, then*

$$\int_{B_\varepsilon(x)} G_t(y-z)dz \geq C_1\cdot t^a\exp\left(-C_2\frac{t^b}{\varepsilon^c}\right)\,, \tag{4.7.32}$$

where $\mathfrak{a}$*,* $\mathfrak{b}$ *and* c *are parameters depending on the kernel. Obviously,* $\mathrm{B}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *is stronger than* $\mathrm{B}(\mathfrak{a},\mathfrak{b})$ *because* (4.7.32) *holds for all* $t > 0$ *other than* $0 < t \leq \varepsilon^\beta$*.*

For example, we have proved that $(\mathfrak{a},\mathfrak{b},\mathfrak{c})=(0,1,2)$ *for the heat kernel,* $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (0,1,\alpha)$ *for the* α*-heat kernel and* $(\mathfrak{a},\mathfrak{b},\mathfrak{c})=(1,1,1)$ *for the wave kernel.*

Our effort to take into account $\mathrm{B}(\mathfrak{a},\mathfrak{b})$ *rather than* B(a,b,c) *is mainly stimulated by Proposition 4.7.5 (ii). One should note that when* $d = 3$*, the wave kernel can not satisfy the* B(a,b,c)*. Because the three dimensional wave kernel is a surface measure on the sphere* $\partial B_t(0)$*, there might be no intersection between the surface measure* $G_t^{\mathrm{w}}(y-dz)$ *and the ball* $B_\varepsilon(x)$ *if* $t \gg \varepsilon$*. Then the lower bound in* (4.7.26) *might be* 0*.*

4.7.4 Fractional temporal and fractional spatial equations: space homogeneous case

In this section we consider the following d-spatial dimensional stochastic partial differential equation of fractional orders both in time and space variables, which will be called the stochastic fractional diffusion (SFD). The existence, uniqueness, upper moment bounds have been obtained earlier (e.g. [CHHH17] , [MN15] and references therein). But the sharp lower bounds for any moment has not been known. We shall apply Theorem 4.3.6 to obtain a sharp lower moment bounds for this equation.

This type of equations takes the following form:

$$\text{(SFD)} \quad \begin{cases} \partial_t^\beta u(t,x) = -\frac{1}{2}(-\Delta)^{\alpha/2} u(t,x) + u(t,x)\dot{W}(t,x)\,, & t>0, \quad x\in\mathbb{R}^d\,, \\ \partial_t^k u(t,x)|_{t=0} = u_k(x)\,, & 0\le k\le \lceil\beta\rceil - 1\,. \end{cases} \tag{4.7.33}$$

As in [CHHH17, MN15], we shall assume that $\beta \in (1/2, 2)$ and $\alpha \in (0, 2]$. We refer to [KST06] for the precise meaning of the fractional derivative in time and the fractional Laplacian. Notice that the SWE coincide with the case $(\alpha, \beta) = (2, 2)$ in (4.7.33) formally.

In this case, the operator $\mathscr{L}$ is given by

$$\mathscr{L} u(t,x) = \partial_t^\beta u(t,x) + \frac{1}{2}(-\Delta)^{\alpha/2} u(t,x)\,.$$

The associated Green's function can be represented by the Fox H-function.

$$G_t^Y(x) := G_t^{Y;\alpha,\beta,d}(x) = \frac{t^{\beta-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}t^\beta} \,\middle|\, \begin{matrix} (1,1),(\beta,\beta) \\ (\frac{d}{2},\frac{\alpha}{2}),(1,1),(1,\frac{\alpha}{2}) \end{matrix} \right)\,, \tag{4.7.34}$$

where H is a Fox H-function (e.g. [KS04]). When $\beta > 1$ we also need another Green function

$$G_t^Z(x) := G_t^{Z;\alpha,\beta,d}(x) = \frac{t^{\lceil\beta\rceil-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}t^\beta} \,\middle|\, \begin{matrix} (1,1),(\lceil\beta\rceil,\beta) \\ (\frac{d}{2},\frac{\alpha}{2}),(1,1),(1,\frac{\alpha}{2}) \end{matrix} \right) \tag{4.7.35}$$

to represent $I_0(t,x)$, namely,

$$I_0^{\mathrm{f}}(t,x) = \sum_{k=0}^{\lceil\beta\rceil-1} \int_{\mathbb{R}^d} u_{\lceil\beta\rceil-1-k}(y)\partial_t^k G_t^Z(x-y)dy\,. \tag{4.7.36}$$

The Fourier transforms of $G_t^Y(x)$ and $G_t^Z(x)$ are given by the following :

$$\begin{aligned} \mathcal{F}[G_t^Z(\cdot)](\xi) &= t^{\lceil\beta\rceil-1} E_{\beta,\lceil\beta\rceil}\left(-\frac{t^\beta|\xi|^\alpha}{2}\right)\,, \\ \mathcal{F}[G_t^Y(\cdot)](\xi) &= t^{\beta-1} E_{\beta,\beta}\left(-\frac{t^\beta|\xi|^\alpha}{2}\right)\,, \end{aligned} \tag{4.7.37}$$

where $E_{\beta,\beta'}$ is the Mittag-Leffler function (e.g. [KST06]).

As before, we may assume $u_0 = 1$ and $u_k = 0$ for $k \geq 1$ to simplify the form of moments without loss of generality (also see Remark 3.6 in [CHHH17]). We have $I_0^{\mathrm{f}}(t,x) = 1$ by our particular initial conditions. Whence, we can prove Theorem 4.3.4 with the notations introduced before.

Positivity of $G_t^Y(x)$ (as well as $G_t^Z(x)$) have been obtained in the following three cases in [CHHH17, Theorem 3.1]:

$$\begin{cases} d = 1,\ \beta \in (1,2) \text{ and } \alpha \in [\beta, 2]\,; \\ d = 2,3,\ \beta \in (1,2) \text{ and } \alpha = 2\,; \\ d \in \mathbb{N},\ \beta \in (0,1] \text{ and } \alpha \in (0,2]\,. \end{cases}$$

Notice that although β is allowed to be smaller than $\frac{1}{2}$, the existence and uniqueness of solutions to (4.7.33) can be proved only under the conditions $\beta \in (\frac{1}{2}, 2)$ and $\alpha \in (0,2]$. Therefore, we will replace last condition by

$$d \in \mathbb{N}\,, \beta \in (\frac{1}{2}, 1]\,, \text{ and } \alpha \in (0,2]\,.$$

This means that we will assume that (α, β, d) satisfies one of the following three conditions:

$$\begin{cases} (a)\ \beta \in (\frac{1}{2}, 1] \text{ and } \alpha \in (0,2], \quad d \in \mathbb{N}\,; \\ (b)\ \beta \in (1,2) \text{ and } \alpha \in (0,2], \quad d = 2,3\,; \\ (c)\ \beta \in (1,2) \text{ and } \alpha \in [\beta, 2], \quad d = 1\,. \end{cases} \tag{4.7.38}$$

As we indicated above the assumption **(G1)** is met under the above parameter range of (4.7.38). In the remaining part of this subsection, we shall prove **(G2)** and **(G3)** for the Green's function G_t^Y.

Proposition 4.7.9 (Small Ball Nondegeneracy Property and Lower Moments for SFD). *For the kernel $G_t^Y(x)$ defined in (4.7.34), the* small ball nondegeneracy property $\mathrm{B}(\beta - 1, \frac{\alpha}{\beta})$ *holds for the parameter ranges given in (4.7.38). More precisely, there* *exist a strictly positive constant C independent of t, ε and y such that*

$$\inf_{y\in B_\varepsilon(x)}\int_{B_\varepsilon(x)} G_t^Y(y-z)dz \ge C\cdot t^{\beta-1} \tag{4.7.39}$$

for any $0<t\le \varepsilon^{\frac{\alpha}{\beta}}$.

As a result, if $\gamma(\cdot)$ (with $\gamma=2-2H$) and $\Lambda(\cdot)$ satisfy the same conditions as in Theorem 4.3.6, the lower p-th ($p\ge 2$) moment bounds hold

$$\mathbb{E}[|u^{\mathrm{f}}(t,x)|^p] \ge c_1 \exp\left(c_2\cdot t^{\frac{\alpha(2\beta+2H-2)-\beta\lambda}{2\alpha\beta-\alpha-\beta\lambda}}\cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}\right)$$

for some constants c_1 and c_2 independent of t, p and x.

Proof. We divide the proof into three steps to deal with three cases in (4.7.38) seperately. **Step 1: case (a).** The special case $\beta=1$ was treated in (4.7.16), so we can assume $\beta\in(1/2,1)$. By the convolution property of [CHHH17], we get a subordination law for the Green's function:

$$\begin{aligned} G_t^Y(x) &= \frac{t^{\beta-1}}{\pi^{d/2}|x|^d} H_{2,3}^{2,1}\left(\frac{|x|^\alpha}{2^{\alpha-1}t^\beta}\middle|\begin{smallmatrix}(1,1),(\beta,\beta)\\ (\frac{d}{2},\frac{\alpha}{2}),(1,1),(1,\frac{\alpha}{2})\end{smallmatrix}\right)\\ &= \frac{\beta t^{\beta-1}}{\pi^{d/2}|x|^d}\int_0^\infty H_{1,2}^{1,1}\left(\frac{|x|^\alpha s^\beta}{2^{\alpha-1}}\middle|\begin{smallmatrix}(1,1)\\ (\frac{d}{2},\frac{\alpha}{2}),(1,\frac{\alpha}{2})\end{smallmatrix}\right) H_{1,1}^{1,0}\left((ts)^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)\frac{ds}{s}\,. \end{aligned} \tag{4.7.40}$$

When $y,z\in B_\varepsilon(x)$, $t\le\varepsilon^{\frac{\alpha}{\beta}}$ and when ε is small enough we have

$$\begin{aligned} &\int_{B_\varepsilon(x)} G_t^Y(y-z)dz\\ &\quad\simeq \int_{B_\varepsilon(x)}\frac{\beta t^{\beta-1}}{|y-z|^d}\int_0^\infty H_{1,2}^{1,1}\left(\frac{|y-z|^\alpha s^\beta}{2^{\alpha-1}}\middle|\begin{smallmatrix}(1,1)\\ (\frac{d}{2},\frac{\alpha}{2}),(1,\frac{\alpha}{2})\end{smallmatrix}\right)\\ &\qquad\qquad\times H_{1,1}^{1,0}\left((ts)^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)\frac{ds}{s}dz\\ &\quad\simeq \int_{B_\varepsilon(x)}\frac{\beta t^{\beta-1}}{|y-z|^d}\int_0^\infty H_{1,2}^{1,1}\left(\frac{|y-z|^\alpha s^\beta}{2^{\alpha-1}t^\beta}\middle|\begin{smallmatrix}(1,1)\\ (\frac{d}{2},\frac{\alpha}{2}),(1,\frac{\alpha}{2})\end{smallmatrix}\right)\\ &\qquad\qquad\times H_{1,1}^{1,0}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)\frac{ds}{s}dz\,. \end{aligned} \tag{4.7.41}$$

Notice that the second H-function is nonnegative by Lemma 4.5 in[CHHH17]. Moreover, recall that the characteristic function and the density of a centered, d-dimensional spherically symmetric α-stable random variable are given, respectively, by

$$f_{\alpha,d}(\xi) = \exp(-|\xi|^\alpha)\,, \qquad \xi \in \mathbb{R}^d\,, \tag{4.7.42}$$

and

$$\rho_{\alpha,d}(x) = \frac{1}{(\sqrt{\pi}|x|)^d} H^{1,1}_{1,2}\left(\frac{|x|^\alpha}{2^\alpha}\middle|\begin{smallmatrix}(1,1)\\(\frac{d}{2},\frac{\alpha}{2}),(1,\frac{\alpha}{2})\end{smallmatrix}\right)\,, \qquad x \in \mathbb{R}^d\,. \tag{4.7.43}$$

This means that the first Fox H-function is related to the spherically symmetric α-stable distribution (see also [CHHH17, Theorem 3.3] for more details). Therefore, one can apply the Pollard's formula in [CHW18] together with (4.7.42) and (4.7.43) to find

$$\begin{aligned}\frac{1}{|y-z|^d} H^{1,1}_{1,2}\left(\frac{|y-z|^\alpha s^\beta}{2^{\alpha-1}t^\beta}\middle|\begin{smallmatrix}(1,1)\\(\frac{d}{2},\frac{\alpha}{2}),(1,\frac{\alpha}{2})\end{smallmatrix}\right) &\simeq\; G^{\mathrm{h},\alpha}_{(t/s)^\beta}(y-z)\\ &\simeq \left(\frac{t}{s}\right)^{-\frac{\beta d}{\alpha}} \wedge \frac{(t/s)^\beta}{|y-z|^{d+\alpha}}\\ &\gtrsim \left(\frac{t}{s}\right)^{-\frac{\beta d}{\alpha}} \exp\left(-C_{\alpha,d}\cdot\frac{|y-z|^\alpha}{(t/s)^\beta}\right)\,,\end{aligned}$$

where $G^{\mathrm{h},\alpha}_t(x)$ is the α-heat kernel associated to (4.7.13). Whence, by Proposition 4.7.1 (i), and (4.7.41) we get

$$\begin{aligned}&\int_{B_\varepsilon(x)} G^Y_t(y-z)dz\\ &\qquad\gtrsim t^{\beta-1}\int_0^\infty \exp\left(-c\cdot\frac{(t/s)^\beta}{\varepsilon^\alpha}\right)\times H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)\frac{ds}{s}\\ &\qquad\gtrsim t^{\beta-1}\int_0^\infty \exp(-c\cdot s)\times H^{1,0}_{1,1}\left(s\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)\frac{ds}{s}\,,\end{aligned} \tag{4.7.44}$$

if $y, z \in B_\varepsilon(x)$ and $t < \varepsilon^{\alpha/\beta}$.

Next, we need to analyze $H^{1,0}_{1,1}\left(s\middle|\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)$. We only need to consider its asymptotics for s near to 0 and near ∞. We shall use the results in the Appendix of [CHHH17] replacing the notations there by $\Delta = \beta_1 - \alpha_1 = 1-\beta$, $a^* = \beta_1 - \alpha_1 = 1-\beta$, $\delta = \beta^{-\beta}$ and $\mu = 1-\beta$. Thus we can make use of the asymptotic expansion for the Fox H-function (e.g. [CHHH17, (A10)]):

$$H^{m,n}_{p,q}\left(s\,\middle|\,\begin{smallmatrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{smallmatrix}\right)=\sum_{j=1}^{m}\sum_{l=0}^{\infty}h^*_{jl}\cdot s^{\frac{b_j+l}{\beta_j}}\,. \tag{4.7.45}$$

Thus, when $s\to 0$ we have

$$H^{1,0}_{1,1}\left(s\,\middle|\,\begin{smallmatrix}(a_1,\alpha_1)\\(b_1,\beta_1)\end{smallmatrix}\right)=H^{1,0}_{1,1}\left(s\,\middle|\,\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)=\sum_{l=0}^{\infty}h^*_l\cdot s^{l+1}\,, \tag{4.7.46}$$

since $m=1$ and $(b_1,\beta_1)=(1,1)$, h^*_l is given by (e.g. [CHHH17, (A.12)])

$$h^*_l=\frac{(-1)^l}{l!\beta_1}\cdot\frac{1}{\Gamma\left(a_1-[b_1+l]\frac{\alpha_1}{\beta_1}\right)}=\frac{(-1)^l}{l!}\cdot\frac{1}{\Gamma\left(-\beta l\right)}\,.$$

Therefore, one can easily see that $h^*_0=0$, $h^*_1=-1/\Gamma(-\beta)>0$, and

$$H^{1,0}_{1,1}\left(s\,\middle|\,\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)=\sum_{l=0}^{\infty}h^*_l\cdot s^{l+1}\simeq h^*_1\cdot s\,,\qquad |s|\simeq 0\,.$$

When s goes to infinity, by [KS04, Corollary 1.10.2], we have the following asymptotic:

$$\begin{aligned}H^{1,0}_{1,1}\left(s\,\middle|\,\begin{smallmatrix}(\beta,\beta)\\(1,1)\end{smallmatrix}\right)&=O\left(s^{[3/2-\beta]/(1-\beta)}\exp\left[-C_\beta s^{1/(1-\beta)}\right]\right)\,,\qquad s\to\infty\,,\\&\gtrsim\exp(-C_\beta s^{1/(1-\beta)})\,,\end{aligned} \tag{4.7.47}$$

where $C_\beta=(1-\beta)\beta^{\beta/(1-\beta)}$. Whence we can observe that the integral in (4.7.44) is finite. So, we have for some constant $C_\beta>0$

$$\int_{B_\varepsilon(x)}G^Y_t(y-z)dz\gtrsim C_\beta\cdot t^{\beta-1}\,.$$

As a result, we have proved the small ball nondegeneracy property $B(\beta-1,\frac{\alpha}{\beta})$ for the **case (a)**.

Step 2: case (b). In this case $d=2$ or $d=3$, $\beta\in(1,2)$ and $\alpha=2$. By equations (43) and (85) in [Psk09], we have for $\beta\in[1,2)$

$$G^Y_t(x)=\Gamma_{\beta,d}(t,x)\,,$$

where

$$\Gamma_{\beta,2}(t,x) = \frac{C\cdot t^{-1}}{\Gamma(1/2)}\int_1^\infty \phi(-\beta/2,0,-|x|t^{-\frac{\beta}{2}}\tau)(\tau^2-1)^{-1/2}d\tau\,, \tag{4.7.48}$$

$$\Gamma_{\beta,3}(t,x) = Ct^{-\frac{\beta}{2}-1}\int_1^\infty \phi(-\beta/2,-\beta/2;-|x|t^{-\beta/2})d\tau\,. \tag{4.7.49}$$

Here $\phi(a,b,c)$ is the Wright function.

Let us check the *small ball nondegeneracy property* $B(\beta-1,\frac{2}{\beta})$ for $d=2$ first. If $y,z\in B_\varepsilon(x)$ and $t\le\varepsilon^{2/\beta}$, by the representation (4.7.48)

$$\begin{aligned}\int_{B_\varepsilon(x)} G_t^Y(y-z)dz &= \int_{B_\varepsilon(x)}\Gamma_{\beta,2}(t,y-z)dz\\ &\simeq t^{-1}\int_{B_\varepsilon(x)}\int_1^\infty \phi(-\beta/2,0,-|y-z|t^{-\frac{\beta}{2}}\tau)(\tau^2-1)^{-1/2}d\tau dz\\ &\simeq t^{-1}\int_{B_\varepsilon(x)}\int_0^\infty \phi(-\beta/2,0,-\tau)t^{\frac{\beta}{2}}\cdot\frac{t^{\frac{\beta}{2}}\cdot 1_{\{|y-z|\le\tau t^{\beta/2}\}}}{\sqrt{t^\beta\tau^2-|y-z|^2}}d\tau dz\\ &\gtrsim t^{-1}\int_0^\infty \phi(-\beta/2,0,-\tau)\cdot t^\beta\tau\cdot\exp\left(-\frac{t^{\beta/2}\tau}{\varepsilon}\right)d\tau\,,\end{aligned}$$

where the last inequality is derived analogously to the argument used in (4.7.28) for the wave kernel when $d=2$ and the fact that $\phi(-\beta/2,0,-\tau)$ is positive (see [Psk09, Section 2]). Then since $t^{\beta/2}\le\varepsilon$ and $\exp(-t^{\beta/2}\tau/\varepsilon)\ge\exp(-\tau)$, we obtain by the relation between the Wright function and the Fox H-function

$$\begin{aligned}\int_{B_\varepsilon(x)} G_t^Y(y-z)dz &= \int_{B_\varepsilon(x)}\Gamma_{\beta,2}(t,y-z)dz\\ &\gtrsim t^{\beta-1}\int_0^\infty \phi(-\beta/2,0,-\tau)\cdot\tau\exp(-\tau)\,d\tau\\ &\simeq t^{\beta-1}\int_0^\infty H_{1,1}^{1,0}\left(\tau\middle|\begin{matrix}(0,\beta/2)\\(0,1)\end{matrix}\right)\cdot\tau\exp(-\tau)\,d\tau\simeq C_\beta\cdot t^{\beta-1}\,,\end{aligned} \tag{4.7.50}$$

where the integral in the last equality of (4.7.50) is finite by the similar asymptotic analysis of $H_{1,1}^{1,0}$ as in **case (a)**. Thus, we proved $B(\beta-1,\frac{2}{\beta})$ for $d=2$.

Next, let us check the *small ball nondegeneracy property* $B(\beta-1,\frac{2}{\beta})$ for $d=3$. We have by the equation (4.7.49)

$$\begin{aligned}\int_{B_\varepsilon(x)} G_t^Y(y-z)dz &= \int_{B_\varepsilon(x)} \Gamma_{\beta,3}(t,y-z)dz \\ &\simeq \int_{B_\varepsilon(x)} t^{-\frac{\beta}{2}-1}\int_1^\infty \phi(-\beta/2,-\beta/2;-|y-z|t^{-\beta/2})d\tau dz \\ &\simeq t^{-\beta-1}\int_{B_\varepsilon(x)} \frac{1}{|y-z|}\int_0^\infty \phi(-\beta/2,-\beta/2;-\tau)1_{\{|y-z|\le t^{\beta/2}\tau\}}d\tau dz\,. \end{aligned} \tag{4.7.51}$$

Now we can apply the same three dimensional spherical coordinate transformation as in the proof of Proposition 4.7.5 (now for $d=3$). Assuming $x=0$, the integral with respect to z in (4.7.51) becomes

$$\begin{aligned}\int_{B_\varepsilon(0)} \frac{1}{|y-z|}1_{\{|y-z|\le t^{\beta/2}\tau\}}dz &\simeq \int_{B_{\tau t^{\beta/2}}(0)} \frac{1_{B_\varepsilon(0)}(y-z)}{|z|}dz \\ &\simeq \tau^2 t^\beta \int_0^{\tau t^{\beta/2}}\int_0^{2\pi}\int_0^\pi r\cdot 1_{B_\varepsilon}(y-\Psi(\theta,\phi))|\sin(\phi)|d\phi d\theta dr \\ &\gtrsim \tau^4 t^{2\beta}\cdot\int_0^{2\pi}\int_0^{\pi/3}|\sin(\phi)|d\phi d\theta \simeq \tau^4 t^{2\beta}\,.\end{aligned}$$

Thus, plugging it back to (4.7.51), we get

$$\int_{B_\varepsilon(x)} G_t^Y(y-z)dz \gtrsim t^{\beta-1}\int_0^\infty \phi(-\beta/2,-\beta/2;-\tau)\cdot\tau^4 d\tau \simeq t^{\beta-1}\,,$$

where the last equality follows from the asymptotic behavior of the Wright function. Hence we complete the proof of the proposition in **case (b)**.

Step 3: case (c). We have $d=1$, $\beta\in(1,2)$ and $\alpha\in[\beta,2]$. By Remark 3.2 (3) and convolution property Theorem 1.8 in [CHHH17], the Fox H-function admits an alternative representation:

$$\begin{aligned} G_t^Y(x) &= \frac{t^{\beta-1}}{|x|}H_{3,3}^{2,1}\left(\frac{|x|^\alpha}{t^\beta}\middle|\begin{smallmatrix}(1,1),(\beta,\beta),(1,\frac{\alpha}{2})\\(1,1),(1,\alpha),(1,\frac{\alpha}{2})\end{smallmatrix}\right) \\ &= \frac{\beta t^{\beta-1}}{|x|}\int_0^\infty H_{2,2}^{1,1}\left(|x|^\alpha s^\beta\middle|\begin{smallmatrix}(1,1),(1,\frac{\alpha}{2})\\(1,1),(1,\frac{\alpha}{2})\end{smallmatrix}\right)H_{1,1}^{1,0}\left((ts)^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s} \\ &= \frac{\beta t^{\beta-1}}{|x|}\int_0^\infty H_{2,2}^{1,1}\left(\frac{|x|^\alpha s^\beta}{t^\beta}\middle|\begin{smallmatrix}(1,1),(1,\frac{\alpha}{2})\\(1,1),(1,\frac{\alpha}{2})\end{smallmatrix}\right)H_{1,1}^{1,0}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}\,. \end{aligned} \tag{4.7.52}$$

(The representation is well defined since $\Delta_1=\sum_{j=1}^2\beta_j-\sum_{j=1}^2\alpha_j=0$, $a_1^*=\alpha_1-\alpha_2+$

$\beta_1 - \beta_2 = 2-\alpha$, $\delta_2 = \left(\frac{\alpha}{2}\right)^{\alpha/2}\left(\frac{\alpha}{2}\right)^{-\alpha/2} = 1$, $\mu_1 = 2-2 = 0$; $\Delta_2 = \beta_1 - \alpha_1 = \alpha - \beta$, $a_2^* = \beta_1 - \alpha_1 = \alpha-\beta$, $\delta_2 = \beta^{-\beta}$ and $\mu_2 = 1-\beta$.) Note that the second Fox H-function is nonnegative combining [KS04, Property 2.4] with [CHHH17, Lemma 4.5]. By [MLP01, (4.38)], the first Fox H-function can be identified as the Green function of neutral-fractional diffusion, namely,

$$\frac{1}{|x|}H^{1,1}_{2,2}\left(|x|^\alpha\middle|\begin{smallmatrix}(1,1),(1,\frac{\alpha}{2})\\(1,1),(1,\frac{\alpha}{2})\end{smallmatrix}\right) = N^0_\alpha(|x|) = K^0_{\alpha,\alpha}(|x|)$$
$$= \frac{1}{\pi}\frac{|x|^{\alpha-1}\sin[\alpha\pi/2]}{1+2|x|^\alpha\cos[\alpha\pi/2]+|x|^{2\alpha}}\,.$$

From (4.7.52) it then follows

$$G^Y_t(x) = \beta t^{\beta-1}\int_0^\infty\left(\frac{s}{t}\right)^{\beta/\alpha}N^0_\alpha\left(|x|(s/t)^{\beta/\alpha}\right)H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}\,.$$

Thus, we have (without loss of generality we can set $x=0$ in the following),

$$\begin{aligned}
&\int_{B_\varepsilon(x)} G^Y_t(y-z)dz\\
&= \int_{B_\varepsilon(0)}\beta t^{\beta-1}\int_0^\infty\left(\frac{s}{t}\right)^{\beta/\alpha}N^0_\alpha\left(|y-z|(s/t)^{\beta/\alpha}\right)H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}dz\\
&\gtrsim \sin\left[\frac{\alpha\pi}{2}\right]t^{\beta-1}\int_0^\infty\int_{B_\varepsilon(y)}\left(\frac{s}{t}\right)^{\beta/\alpha}\frac{[|z|(s/t)^{\beta/\alpha}]^{\alpha-1}}{[|z|(s/t)^{\beta/\alpha}]^{2\alpha}+1}dz\cdot H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}\\
&\gtrsim \sin\left[\frac{\alpha\pi}{2}\right]t^{\beta-1}\int_0^\infty\int_0^{(s/t)^{\beta/\alpha}\varepsilon}\frac{z^{\alpha-1}}{z^{2\alpha}+1}dz\cdot H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}\\
&\simeq \sin\left[\frac{\alpha\pi}{2}\right]t^{\beta-1}\int_0^\infty\arctan\left[\frac{s^\beta\varepsilon^\alpha}{t^\beta}\right]\cdot H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}\\
&\gtrsim \sin\left[\frac{\alpha\pi}{2}\right]t^{\beta-1}\int_0^\infty\arctan\left[s^\beta\right]\cdot H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)\frac{ds}{s}
\end{aligned}\tag{4.7.53}$$

for $y,z\in B_\varepsilon(x)$, and $t\le\varepsilon^{\frac{\alpha}{\beta}}$.

Next, we need to take care of the asymptotics of $H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right)$ (valid with the notations $\Delta = \beta_1-\alpha_1 = \alpha-\beta$, $a^* = \beta_1-\alpha_1 = \alpha-\beta$, $\delta = \beta^{-\beta}$ and $\mu = 1-\beta$) when s goes to infinity. Similar to (4.7.46) in **case (a)**, we find that

$$H^{1,0}_{1,1}\left(s^{-\beta}\middle|\begin{smallmatrix}(\beta,\beta)\\(1,\alpha)\end{smallmatrix}\right) = \sum_{l=0}^\infty h_l^*\cdot s^{-\beta(l+1)/\alpha}\simeq h_1^* s^{-2\beta/\alpha}\qquad\text{as } s\to\infty\,,$$

with $h_l^* = \frac{(-1)^l}{\alpha\cdot l!}\cdot\frac{1}{\Gamma(-\beta l)}$ and $h_1^*>0$. When $s\to 0$, similar to (4.7.47), we have the following asymptotic estimate

$$H_{1,1}^{1,0}(s^{-\beta}|\cdots) = O\left(s^{-\beta[3/2-\beta]/(\alpha-\beta)}\exp\left[-C_{\alpha,\beta}\cdot s^{-\beta/(\alpha-\beta)}\right]\right), \qquad s\to 0$$

for some constant $C_{\alpha,\beta}>0$.

Finally, we obtain from (4.7.53) and the asymptotics

$$\begin{aligned}&\int_{B_\varepsilon(x)} G_t^Y(y-z)dz\\ &\qquad \gtrsim \sin\left[\frac{\alpha\pi}{2}\right] t^{\beta-1}\int_0^\infty \arctan\left[s^\beta\right]\cdot H_{1,1}^{1,0}\left(s^{-\beta}\middle|\begin{matrix}(\beta,\beta)\\(1,\alpha)\end{matrix}\right)\frac{ds}{s}\gtrsim C_{\alpha,\beta}\cdot t^{\beta-1},\end{aligned}$$

for some constant $C_{\alpha,\beta}>0$. Thus, we complete the proof of the small ball nondegeneracy property $B(\beta-1,\frac{\alpha}{\beta})$ for **case (c)**. □

Proposition 4.7.10 (HLS Mass Property and Upper Moments for SFD). *Assume that $\gamma(\cdot)$ (with $\gamma-2-2H$) and $\Lambda(\cdot)$ satisfy the same conditions of Theorem 4.3.4 (under the condition $\lambda<\min(2\alpha-\alpha/\beta,d)$) or Theorem 4.4.1 (under the condition $\lambda<\min(\alpha,d)$). When the parameters are in the range given by (4.7.38) the Green's function $G_t^Y(x)$ satisfies the **(G3)** or **(G3')** with $\mathrm{M}(2(\beta-1)-\frac{\beta\lambda}{\alpha})$. In other words, there exist strict positive constants C_1 and C_2 independent of t and x such that*

$$\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^Y(x-y)\Lambda(y-y')G_t^Y(x'-y')dydy' \le C\cdot t^{2(\beta-1)-\frac{\beta\lambda}{\alpha}}, \tag{4.7.54}$$

and furthermore, denoting $\mu(d\xi)=\hat{V}(\xi)d\xi$

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\hat{G}_t^Y(\xi-\eta)|^2|\mu|(d\xi)\le C_3\cdot t^{2(\beta-1)-\frac{\beta\lambda}{\alpha}}. \tag{4.7.55}$$

Consequently, we have the upper p-th ($p\ge 2$) moment bounds for the solution $u^{\mathrm{f}}(t,x)$. Namely, there are positive constants C_1 and C_2 independent of t, p and x satisfying

$$\mathbb{E}[|u^{\mathrm{f}}(t,x)|^p]\le C_1\cdot\exp\left(C_2\cdot t^{\frac{\alpha(2\beta+2H-2)-\beta\lambda}{2\alpha\beta-\alpha-\beta\lambda}}\cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}\right).$$

Proof. We need to show $M(2(\beta-1)-\frac{\beta\lambda}{\alpha})$ under conditions (4.7.38) and $\lambda<\min(2\alpha-\alpha/\beta,d)$, i.e. the estimates (4.7.54). This gives the upper bound accordingly. This has been proved in [CHHH17, Theorem 3.14 and Lemma 7.3]. For the sake of completeness, we give some details here. Applying Hardy-Littelewood-Sobolev inequality ([LL97, Theorem 4.3]), we can find

$$\begin{aligned}
&\sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^Y(x-y)\Lambda(y-y')G_t^Y(x'-y')dydy' \\
&\quad\le \sup_{x,x'\in\mathbb{R}^d}\int_{\mathbb{R}^{2d}} G_t^Y(x-y)|y-y'|^{-\lambda}G_t^Y(x'-y')dydy' \\
&\quad\le \left[\int_{\mathbb{R}^d}|G_t^Y(y)|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}} \simeq \left[\int_{\mathbb{R}^d}\left|\frac{t^{\beta-1}}{|y|^d}H_{2,3}^{2,1}\left(\frac{|y|^\alpha}{2^{\alpha-1}t^\beta}\middle|\begin{matrix}--\\---\end{matrix}\right)\right|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}} \\
&\quad\simeq t^{2(\beta-1)-\frac{\beta\lambda}{\alpha}}\cdot\left[\int_{\mathbb{R}^d}\left|\frac{1}{|y|^d}H_{2,3}^{2,1}\left(|y|^\alpha\middle|\begin{matrix}--\\---\end{matrix}\right)\right|^{\frac{2d}{2d-\lambda}}dy\right]^{\frac{2d-\lambda}{d}} \le C\cdot t^{2(\beta-1)-\frac{\beta\lambda}{\alpha}},
\end{aligned}$$

where we have employed change of variable $y\to t^{\beta/\alpha}\cdot y$ and the estimate of H-function $H_{2,3}^{2,1}(y)$ obtained in [CHHH17, Lemma 7.1].

Next, we need to prove the inequality (4.7.55) under conditions (4.7.38) and $\lambda<\min(\alpha,d)$. Let us recall some useful estimates for the Mittag-Leffler function $E_{\beta,\beta}(-|z|^\beta)=\sum_{k=0}^\infty\frac{(-|z|)^k}{\Gamma(\beta(k+1))}$ (see [GLL02] or [WZO18] for example): when $z\to\infty$,

$$|E_{\beta,\beta}(-|z|)|\lesssim|z|^{-1}+|z|^{-2}.$$

On the other hand the Mittag-Leffler function $E_{\beta,\beta}(-|z|^\beta)$ is bounded when $|z|\simeq0$ for $\beta\in(0,2)$. Therefore, the following inequality holds

$$|E_{\beta,\beta}(-|z|^\alpha)|\lesssim 1\wedge|z|^{-\alpha}\lesssim\frac{1}{1+|z|^\alpha}.$$

Using the equation (4.7.37) and the assumptions on $\Lambda(\cdot)$, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}G_t^Y(x-y)\Lambda(y)dy &\lesssim \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}G_t^Y(x-y)|y|^{-\lambda}dy \\
&\simeq \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{G}_t^Y(\xi)\cdot e^{\iota x\cdot\xi}|\xi|^{\lambda-d}d\xi
\end{aligned}$$

$$\begin{aligned}&\lesssim t^{\beta-1} \cdot \int_{\mathbb{R}^d} \left| E_{\beta,\beta}\left(-\frac{t^\beta |\xi|^\alpha}{2}\right)\right| \cdot |\xi|^{\lambda-d} d\xi \\ &\lesssim t^{(\beta-1)-\frac{\beta\lambda}{\alpha}} \cdot \int_{\mathbb{R}^d} |E_{\beta,\beta}(-|\xi|^\alpha)| \cdot |\xi|^{\lambda-d} d\xi \,.\end{aligned}$$

And the integral is well defined since

$$\int_{\mathbb{R}^d} |E_{\beta,\beta}(-|\xi|^\alpha)| \cdot |\xi|^{\lambda-d} d\xi \lesssim \int_{\mathbb{R}^d} [1+|\xi|^\alpha]^{-1} \cdot |\xi|^{\lambda-d} d\xi < \infty \,,$$

under the assumption $\lambda < \min(\alpha, d)$. Thus, we complete the proof. □

Chapter 5

Mean square stability of stochastic theta method for stochastic differential equations driven by fractional Brownian motion

5.1 Introduction and main results

Numerical stability analysis of stochastic differential equations (SDEs) is an important topic in numerical analysis and scientific computing. In order to get insight into the stability behavior of numerical methods for SDEs, the authors in [Sch96, SM96] studied the mean square stability of several numerical schemes for the following stochastic test problem driven by standard Brownian motion (Bm)

$$dX(t) = \lambda X(t)dt + \mu X(t)dB(t), \quad \lambda, \mu \in \mathbb{C}\,, \tag{5.1.1}$$

with initial value $X(0) \neq 0$ with probability 1 and $\mathbb{E} \mid X(0) \mid^2 < \infty$, where $dB(t)$ is interpreted in Itô sense. The solution of (5.1.1) is said to be *mean square stable* if

$$\lim_{t\to\infty} \mathbb{E} \mid X(t) \mid^2 = 0\,. \tag{5.1.2}$$

As is well-known, the mean square stability of (5.1.1) is characterized by

$$Re(\lambda) + \frac{1}{2}|\mu|^2 < 0,$$

where $Re(\lambda)$ denotes the real part of λ. Higham [Hig00a, Hig00c] studied the mean square stability properties of stochastic theta method and stochastic theta Milstein method for the test equation (5.1.1). The A-stability (which means that the numerical method preserves the stability of the underlying test problem unconditionally) of stochastic theta method (STM) and the stochastic theta Milstein method are proved when $\theta \geq \frac{1}{2}$ and $\theta \geq \frac{3}{2}$, respectively.

Subsequently, the stability of the numerical method for nonlinear SDEs driven by Brownian motion

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t) \tag{5.1.3}$$

received much attention in the past decades. Assume that the drift coefficient f satisfies certain monotone condition, and the diffusion coefficient satisfies the linear growth condition. The authors in [HMS03, Sch01] proved that the backward Euler method and the split-step backward Euler method reproduce the exponential mean square stability of the underlying nonlinear problem. More recently, some scholars studied nonlinear stability under a coupled condition on the drift and diffusion coefficients. This condition allows that the diffusion coefficient grows super-linearly. For example, Szpruch and Mao [SM10] studied the asymptotic stability in this nonlinear setting for the STM. Huang [Hua14] proved that for all given step size $\Delta t > 0$, the STM with $\theta \in [1/2, 1]$ is mean square stable for stochastic delay differential equations under the following coupled condition:

$$u^T f(t, u, v) + \frac{1}{2} \mid g(t, u, v) \mid^2 \leq \widetilde{\alpha} \mid u \mid^2 + \widetilde{\beta} \mid v \mid^2, \qquad \forall t > 0, \quad u, v \in \mathbb{R}, \tag{5.1.4}$$

with $\widetilde{\alpha} + \widetilde{\beta} < 0$. If there exist positive constant K_1 and K_2 such that the drift coefficients f also satisfy

$$\mid f(t, u, v) \mid^2 \leq K_1 \mid u \mid^2 + K_2 \mid v \mid^2, \tag{5.1.5}$$

then the STM with $\theta \in [0, 1/2]$ is mean square stable under certain stepsize constraint.

For more details for nonlinear stability of numerical method for SDEs we refer to [HK21] and references therein.

In recent decades long memory processes have been widely studied and applied by mathematicians and statisticians. In particular, the theory of stochastic differential equations driven by fractional Brownian motions have been well-developed and have found applications in various fields (e.g. [BHØZ08, Mis08]). For example, the thermal dynamics characterized by a fractional Ornstein-Uhlenbeck process based on empirical observation in [BSZ02] is applied in the pricing of weather derivatives; The fractional Langevin model in [GZLC13], and the arbitrage in the financial market is eliminated in the case of geometric fBm in [HØ03, Gua06]. The readers can also find interesting applications of fBm in modeling anisotropic multidimensional data with self-similarity and long-range dependence in [MVN68, WD20] and references therein.

However, most of the SDEs driven by fBm do not have explicit solutions, whence numerical method are required in practice. So far, there have been many studies on the convergence of numerical methods for SDEs driven by fBm (cf. [NN07, DNT12, MS08, Dav08, HLN16, LT19, HLN21, KNP11, HHKW20, HHW21, CHL18]). However, the numerical stability studies of SDEs driven by fBm have rarely been addressed. In this works, we are concerned with the mean square stability analysis of *stochastic theta method* for some *stochastic test equations* driven by fractional Brownian motion (fBm) in $\mathbb{R}^d$. We focus our effort on the stability problem of the numerical scheme and try to avoid the complicate issues of existence and uniqueness for the solution when $H < 1/2$. For this reason we shall assume exclusively $H > 1/2$ throughout the chapter. We also assume $d = 1$. First, a natural choice of the test equation is the extension of (5.1.1), namely, we replace the Brownian motion in (5.1.1) by fractional Brownian motion. However, an easy computation (similar to the one shown below) immediately gives that for any parameters λ and μ, the solution to $dX(t) = -\lambda X(t)dt + \mu X(t)dB^H(t)$ with a nonzero initial condition $X(0) = x \in \mathbb{R}\backslash\{0\}$ will never be stable in the mean square sense (or any L_p sense for any finite p). So, the first thing we shall do is to modify (5.1.1) to the following new type of test equations:

$$dX(t) = -\lambda\kappa t^{\kappa-1}X(t)dt + \mu X(t)\circ dB^H(t)\,, \quad t\geq 0\,, \quad X(0)=X_0\,, \tag{5.1.6}$$

with $\lambda > 0, \mu \in \mathbb{R}$ and $\kappa \geq 2H$, and $\circ dB^H$ is for Stratonovich integration with respect to fBm with Hurst parameter $H > 1/2$. Here, for simplicity we assume that X_0 is a non-zero constant. The existence and uniqueness problems of (5.1.6) and the more general form (5.1.9) have been studied extensively in the last two decades. For precise results, we refer to [FZ21] and the references therein. Notice that (5.1.6) has an additional factor $t^{\kappa-1}$ than (5.1.1) in the drift term. By the chain rule formula (e.g. [Hu13, Proposition 2.7] or [Mis08, Lemma 2.7.1]), we have $X(t) = X_0\exp(-\lambda t^\kappa + \mu B^H(t))$ and hence

$$\mathbb{E}\mid X(t)\mid^2 = \mathbb{E}(X_0)^2\exp\left[2(-\lambda t^\kappa + \mu^2 t^{2H})\right]\,. \tag{5.1.7}$$

This formula implies the mean square stability of the solution to (5.1.6) if

$$\text{(i) } \kappa > 2H \text{ and } \lambda > 0 \quad \text{or} \quad \text{(ii) } \kappa = 2H \text{ and } -\lambda + \mu^2 < 0\,. \tag{5.1.8}$$

Otherwise, the solution of (5.1.6) diverges in mean square sense as t goes to infinity. So we only need to consider (5.1.6) for the above two parameter regions (5.1.8).

After we obtain the stability result for the above linear equations (5.1.6), we shall next focus our effort on the numerical stability of the STM for the following *nonlinear SDEs* which are long memory version of (5.1.3)

$$dX(t) = f(t,X(t))dt + g(t,X(t))\circ dB^H(t)\,, \tag{5.1.9}$$

where $B^H(t)$ is fBm with $H > 1/2$. Inspired by the conditions (5.1.4), (5.1.5) and (5.1.8), we shall assume the coefficients in the SDE (5.1.9) satisfy the following conditions (we assume $d = 1$):

Assumption 1. There exist constants $\kappa \geq 2H$, $\lambda > 0$, $\bar{\lambda} > 0$ and $\mu > 0$ such that for any $t > 0$ and $x \in \mathbb{R}$

$$\textbf{Monotone condition}: \quad xf(t,x) \le -\lambda\kappa t^{\kappa-1}x^2\,, \tag{5.1.10}$$

$$\textbf{Linear growth}: \quad |\, f(t,x) \,| \le \bar{\lambda}\kappa t^{\kappa-1} \,|\, x \,|\,, \tag{5.1.11}$$

$$\textbf{Uniform linear growth}: \quad |\, g(t,x) \,| \le \mu \,|\, x \,|\ . \tag{5.1.12}$$

Remark 5.1.1. *We mention that when $\kappa = 1$, conditions* (5.1.10) *and* (5.1.11) *reduce to the classical monotone condition and linear growth condition (which is discussed in the Brownian motion case, e.g.* [HMS03]*). After the completion of the first version of this chapter, the author found the mean square stability problem of* (5.1.6) *has been well studied in* [DHC19]*. See (2.4) with $A(t) = -\lambda\kappa t^{\kappa-1}$, $F(t,x) = 0$ and $C(t) = \mu$ therein. Moreover, our assumptions* (5.1.10)*-*(5.1.12) *can be compared with* [DHC19, (H1)-(H3)]*. However, the method (see also proof of Theorem 5.3.1) used in* [DHC19] *seems to be unapplicable to the general case* (5.1.9)*.*

Unlike the Brownian motion case, the stability problem of general SDE driven by fBm is still shrouded in mystery. To the best of our knowledge, the global almost surely exponential stability of SDE in the form of (5.1.6) with $\kappa = 1$ was considered in [GANS18]. In the same paper, the local almost surely exponential stability, i.e. initial condition must belong to a neighborhood of zero, was obtained for the general SDE (5.1.9) under some suitable conditions on f and g. It is well known that the second moment of solution to SDE under Bm exponentially decays to zero which implies the solution vanishes almost surely by Borel-Catelli lemma (the reverse does not hold in general). However, it seems there is no result on the long-time mean square stability analysis of the original solution of (5.1.9) under Assumption 1. We mention some papers about the related moment bounds of the solution $X(t)$ on finite time domain $[0,T]$. For example, the moment bounds is given in [HN07] when $f(t,X) = 0$ and $g(t,X) = \sigma(X)$. More recently, Fan and Zhang [FZ21] obtained the moment bounds with irregular drift term. We shall show that under the condition (5.1.10) and (5.1.11) and when $g(t,X(t)) = c(t)X(t)$, the solution $X(t)$ of (5.1.9) is mean square stable.

On the other hand, from the numerical viewpoint, we investigate the stability of the

STM for the general equation (5.1.9) rather than (5.3.1) based on Assumption 1. We hope the numerical results would also provide some insights for the theoretical stability analysis of the solution to (5.1.9).

The numerical scheme that we propose to study is the stochastic theta method (STM) to (5.1.9), which is some kind of implicit-explicit Euler-Maruyama scheme:

$$\text{(STM)}\begin{cases} \bar{X}_{n+1} = \bar{X}_n + \theta f(t_{n+1}, \bar{X}_{n+1})\Delta t + (1-\theta) f(t_n, \bar{X}_n)\Delta t + g(t_n, \bar{X}_n) V_n^H, \\ \qquad \text{where} \quad t_n = n \cdot \Delta t \quad \text{and} \quad V_n^H = B^H(t_{n+1}) - B^H(t_n), \\ \qquad\qquad n = 0, 1, 2, \cdots \quad \text{and} \quad \Delta t > 0 \text{ is a fixed stepsize.} \end{cases} \tag{5.1.13}$$

Remark 5.1.2. *Note that the function $F(x) = x - \theta f(t,x)\Delta t$ is one-to-one. By the Lipschitz condition on f, the function F is also surjective for Δt small enough. Hence, one sees that the STM (5.1.13) is well-defined.*

In particular, when $f(t,X) = -\lambda\kappa t^{\kappa-1} X$ and $g(t,X) = \mu X$, (5.1.13) becomes

$$\bar{X}_{n+1} = \bar{X}_n - \kappa\lambda\theta \cdot (t_{n+1})^{\kappa-1} \bar{X}_{n+1}\Delta t - \kappa\lambda(1-\theta) \cdot (t_n)^{\kappa-1} \bar{X}_n \Delta t + \mu \cdot \bar{X}_n V_n^H, \tag{5.1.14}$$

The main stability theorems we shall prove are displayed as follows:

Theorem 5.1.3. *Let $\Delta t > 0$ be fixed and let λ, μ satisfy (5.1.8). For the test equation (5.1.6) and the STM (5.1.14) we have the following statements.*

(i) If $\kappa \geq 2H$ and $\frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e + 1}} \leq \theta \leq 1$, then the STM (5.1.14) is mean square stable for the test equation (5.1.6), namely, $\lim_{n\to\infty} \mathbb{E} \mid \bar{X}_n \mid^2 = 0$.

(ii) If $\kappa > 3/2$ and $\frac{1}{2} < \theta \leq 1$, then the STM (5.1.14) is mean square stable for the test equation (5.1.6).

(iii) If $\kappa \geq 2H$ and $0 < \theta < \frac{1}{2}$, then the STM (5.1.14) is not unconditionally *mean square stable for the test equation (5.1.6).*

Remark 5.1.4. *It is not clear whether or not the STM (5.1.14) is mean square stable when $2H \leq \kappa \leq \frac{3}{2}$ and $\frac{1}{2} \leq \theta < \frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e + 1}}$, which will be a topic for future research.*

Remark 5.1.5. *The classical results of STM with $\theta < \frac{1}{2}$ for SDEs is mean square stable when the stepsize Δt is small enough. We do not expect such a result because the recurrence relation in the following equation (5.1.15) is related to both n and step size Δt if $\kappa \neq 1$.*

Some adaptive Euler-Maruyama scheme (e.g. [NSS19]) might be the topic of our future research.

Theorem 5.1.6. *Let $\Delta t > 0$ be fixed and let λ, μ in Assumption 1 satisfy (5.1.8). For the SDEs with fBm (5.1.9) and the STM (5.1.13) we have the following statement.*

(i) If (5.1.10) and (5.1.12) in Assumption 1 hold, then the STM (5.1.13) with $\theta = 1$ (i.e., the backward Euler method) is mean square stable for the equation (5.1.9).

(ii) If (5.1.10), (5.1.11) and (5.1.12) in Assumption 1 hold, then the STM (5.1.13) with $\frac{\sqrt{6}e\bar{\lambda}/\lambda}{\sqrt{6}e\bar{\lambda}/\lambda+1} \leq \theta < 1$ is mean square stable for the equation (5.1.9).

Remark 5.1.7. *For ODEs, the study of the asymptotic stability of numerical schemes for the linear test equation leads to a criterion for the asymptotic stability of numerical shcemes of nonlinear equation (see [Dah63, Dah78]). We still could not get similar results and shall leave it as a topic for future research.*

We shall prove Theorems 5.1.3 and 5.1.6 in Sections 2 and 3, respectively. Before we end this section we would point out the new difficulties we encounter compared with the classical Brownian motion (e.g. see subsection 2.5). We can write (5.1.14) as

$$\bar{X}_{n+1} = \left(\frac{1 - \kappa(1-\theta)\lambda(t_n)^{\kappa-1}\Delta t}{1 + \kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} + \frac{\mu V_n^H}{1 + \kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} \right) \bar{X}_n \, . \tag{5.1.15}$$

When $H = 1/2$ (i.e. the Brownian motion case), $\bar{X}_{n+1}$ is the product of independent variables and the corresponding computation is much easier. However, this is no longer true in our fBm setting. We encounter two major difficulties:

1. The increments $B^H(t_{n+1}) - B^H(t_n)$ of the fractional Brownian motion depend on the past history, which makes the stability analysis much more sophisticated.

2. The fractional Brownian motion lacks martingale property or Markov property so that some useful techniques such as conditional expectation seems impossible or at least over-sophisticated.

To get around these difficulties we shall employ some other analysis and computation techniques. In fact, in the proof of different parts of Theorem 5.1.3, we shall use different techniques. For example, in the proof of part (i) of Theorem 5.1.3 we use the technique of generalized polarization, raw moments formula of Gaussian distributions and the asymptotic properties of confluent hypergeometric function. On the other hand, the main tool to prove part (ii) is the celebrated Gaussian correlation inequality. Finally, the statement of part (iii) is proved through the strong law of large numbers of dependent random variables. All of these are done in Section 5.2. Let us mention that the test equation (5.1.6) has not been previously studied even when the fBm is replaced by the standard Brownian motion and it is interesting to carry out the stability analysis of the corresponding stochastic theta scheme for its own sake and for the comparison purpose. This is also done in Section 5.2. The proof of Theorem 5.1.6 is analogous to that of part (i) of Theorem 5.1.3 and is provided in Section 5.3. In Section Section 5.4, some numerical simulations are presented to validate our theoretical results. Finally, some concluding remarks are given in the last section.

Through the remaining part of this chapter, we use $a_n \asymp b_n$ to denote that there are two positive constants c_1 and c_2, independent of n, such that $c_1 \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \le c_2 \lim_{n\to\infty} a_n$ for all $n \ge 1$.

5.2 STM: Mean square linear stability analysis

In this section we shall prove our main result, i.e., Theorem 5.1.3. The parts (i), (ii) and (iii) are proved in subsection 5.2.2, 5.2.3 and 5.2.4, respectively.

Obviously, (5.1.14) is equivalent to the following recurrent equation

$$\bar{X}_{n+1} = \left(\alpha_n(\theta, \lambda, \Delta t) + \beta_n(\theta, \lambda, \mu, \Delta t) V_n^H\right) \bar{X}_n \,,$$

where $\kappa \geq 2H > 1$ and

$$\begin{cases} \alpha_n(\theta, \lambda, \Delta t) = \dfrac{1 - \kappa(1-\theta)\lambda(t_n)^{\kappa-1}\Delta t}{1 + \kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} = \dfrac{1 - \kappa(1-\theta)\lambda n^{\kappa-1}\Delta t^{\kappa}}{1 + \kappa\theta\lambda(n+1)^{\kappa-1}\Delta t^{\kappa}}, & (5.2.1) \\ \beta_n(\theta, \lambda, \mu, \Delta t) = \dfrac{\mu}{1 + \kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} = \dfrac{\mu}{1 + \kappa\theta\lambda(n+1)^{\kappa-1}\Delta t^{\kappa}}. & (5.2.2) \end{cases}$$

For notational simplicity, throughout the remaining part of the chapter we denote $\alpha_n(\theta, \lambda, \Delta t)$, $\beta_n(\theta, \lambda, \mu, \Delta t)$ by α_n and β_n, respectively. Note that (5.2.1) and (5.2.2) are well defined if we require the condition (5.1.8) or otherwise the denominators in the expressions of α_n and β_n could be 0.

5.2.1 Heuristic arguments

Before the proof, we would like to explain why Theorem 5.1.3 could hold true heuristically, namely, why the STM (5.1.14) is stable when $\theta > 1/2$ and is unstable when $\theta < 1/2$, formally. Denote

$$Z_n(\Delta t) = \alpha_n + \beta_n V_n^H. \tag{5.2.3}$$

Then we have

$$\bar{X}_{n+1} = X_0 \prod_{k=0}^{n} Z_k(\Delta t) = X_0 \prod_{k=0}^{n} \left(\alpha_k + \beta_k V_k^H\right). \tag{5.2.4}$$

Obviously, for fixed Δt, λ and μ,

$$\lim_{n\to\infty} \alpha_n = -\frac{1-\theta}{\theta}, \qquad \lim_{n\to\infty} \beta_n = 0.$$

Notice that this is quite different than the setting with $H = 1/2$ where α_n and β_n do not depend on n because of the absence of $(t_n)^{\kappa-1}$ for $\kappa = 2H = 1$ (see Section 3 for more details). Formally, if we could think of $\{V_k^H\}$ in (5.2.4) as a sequence of finite numbers, then by the limits of α_n and β_n, we would have

$$| \bar{X}_{n+1} |^2 = | X_0 |^2 \prod_{k=0}^{n} \left(\alpha_k + \beta_k V_k^H\right)^2 \asymp \left(\frac{1-\theta}{\theta}\right)^{2n} \to \begin{cases} 0\,, & \text{if } \frac{1}{2} < \theta \le 1\,; \\ \infty\,, & \text{if } 0 \le \theta < \frac{1}{2}\,, \end{cases}$$

However, the random variables $\{V_k^H\}$ in our setting are not uniformly bounded. Even worse, they are long range dependent. Therefore, the above heuristic argument cannot be applied directly to analyze (5.2.4), especially for the scenario of (mean square) stability. Presumably, there are two ways to break these barriers.

(1) Choose θ carefully so that the oscillation caused by $\{V_k^H\}$ can still be manageable.

(2) Take κ sufficiently large so that $\beta_k \cdot V_k^H$ converges to 0 fast enough so that influences of $\{V_k^H\}$ can be neglected.

Our proof will follow these spirits but with much more sophisticated tricks and computations. For example, we need to use the asymptotics of the confluent hypergeometric functions which comes from the moments of Gaussian variables.

5.2.2 The case of $\kappa \ge 2H$ and $\frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e}+1} \le \theta \le 1$

In this subsection we prove part (i) of the main theorem, namely, we consider the case when $\kappa \ge 2H$ and $\frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e+1}} \le \theta \le 1$. Firstly, we state a useful lemma, which is a generalization of polarization identity.

Lemma 5.2.1. [Kan08, Lemma 1] *Let $x_1, \ldots, x_n$ be real numbers, and let $s_1, \ldots, s_n$ be nonnegative integers and $s = \sum_{i=1}^n s_i$. Then, we have*

$$x_1^{s_1} \cdots x_n^{s_n} = \frac{1}{s!} \sum_{v_1=0}^{s_1} \cdots \sum_{v_n=0}^{s_n} (-1)^{\sum_{i=1}^n v_i} \binom{s_1}{v_1} \cdots \binom{s_n}{v_n} \cdot \left[\sum_{i=1}^n h_i x_i\right]^s ,$$

where $h_i = s_i/2 - v_i$.

Proof of part (i) of Theorem 5.1.3. Our goal is to show

$$\lim_{n\to\infty} \mathbb{E}[| \bar{X}_n |^2] = \lim_{n\to\infty} \mathbb{E}\left[X_0^2 \prod_{k=0}^{n-1} (Z_k(\Delta t))^2\right] = 0\,, \tag{5.2.5}$$

where $Z_k(\Delta t)$ is given by (5.2.3) and $\bar{X}_n$ is given by (5.2.4). We divide our proof into three steps.

Step 1: Bound $\mathbb{E} \mid \bar{X}_n \mid^2$ by a confluent hypergeometric function.

Applying Lemma 5.2.1 with $s_1 = \cdots = s_n = 2$ and $s = 2n$ we have

$$\prod_{k=0}^{n-1} Z_k^2(\Delta t) = \frac{1}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} (-1)^{\sum_{i=1}^n v_i} \binom{s_1}{v_1} \cdots \binom{s_n}{v_n} \cdot \left[\sum_{i=1}^{n} h_i Z_i(\Delta t) \right]^{2n} ,$$

with $h_i = 1 - v_i$. Note that $Z_i(\Delta t) = \alpha_i + \beta_i \cdot V_i^H \overset{d}{\sim} N(\mu_i, \sigma_i)$ with $\mu_i = \alpha_i$ and $\sigma_i^2 = \beta_i^2 \cdot (\Delta t)^{2H}$. Thus we have

$$\begin{aligned} \mathbb{E}\left[\prod_{k=0}^{n-1} Z_k^2(\Delta t) \right] &\leq \frac{2^n}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} \mathbb{E}\left[\sum_{i=1}^{n} (1 - v_i) \cdot Z_i(\Delta t) \right]^{2n} \\ &=: \frac{2^n}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} \mathbb{E}\left[Q_n \right]^{2n} , \end{aligned} \tag{5.2.6}$$

where $Q_n = Q_n(v_1, \cdots, v_n; Z_1(\Delta t), \cdots, Z_n(\Delta t)) = \sum_{i=1}^n (1 - v_i) \cdot Z_i(\Delta t)$. It is obvious that Q_n is still a normal random variable, with mean $\tilde{\mu}_n$ and variance $\tilde{\sigma}_n^2$ given by

$$\tilde{\mu}_n := \tilde{\mu}_n(v_1, \cdots, v_n) = \sum_{i=1}^{n} (1 - v_i) \cdot \mu_i = \sum_{i=1}^{n} (1 - v_i) \cdot \alpha_i ,$$

and

$$\begin{aligned} \tilde{\sigma}_n^2 := &\tilde{\sigma}_n^2(v_1, \cdots, v_n) = \mathbb{E}\left[\left[\sum_{i=1}^{n} (1 - v_i) \cdot \beta_i \cdot V_i^H \right]^2 \right] \\ &= \sum_{i,j=1}^{n} (1 - v_i)(1 - v_j) \cdot \beta_i \beta_j \cdot \mathbb{E}[V_i^H V_j^H] . \end{aligned}$$

From the raw moment formula ([Win12, Eq. (17)] or [AFVSF16, Appendix A]) it follows

$$\begin{aligned} \mathbb{E}\left[Q_n \right]^{2n} = &\frac{2^n}{\sqrt{\pi}} \tilde{\sigma}_n^{2n} \Gamma\left(\frac{2n+1}{2} \right) \cdot \Phi\left(-n, \frac{1}{2}, -\frac{\tilde{\mu}_n^2}{2\tilde{\sigma}_n^2} \right) \\ = &\frac{2^n}{\sqrt{\pi}} \tilde{\sigma}_n^{2n} \Gamma\left(\frac{2n+1}{2} \right) \cdot \exp\left(-\frac{\tilde{\mu}_n^2}{2\tilde{\sigma}_n^2} \right) \Phi\left(n + \frac{1}{2}, \frac{1}{2}, \frac{\tilde{\mu}_n^2}{2\tilde{\sigma}_n^2} \right) , \end{aligned} \tag{5.2.7}$$

where we used Kummer's transformation (see e.g. (5.7.5) of the appendix): $\Phi(\alpha, \gamma, z) =$

$\exp(z)\Phi(\gamma-\alpha,\gamma,-z)$. Here, $\Phi(\alpha,\gamma,z)$ is Kummer's confluent hypergeometric function (see (5.7.4) or Chapter 13 in [OLBC10] for more details).

By employing the differentiation formula (5.7.6) and then (5.7.8), we have with the substitution $\eta = \frac{\tilde{\mu}_n^2}{2\tilde{\sigma}_n^2}$

$$\begin{aligned}
&\frac{d}{d\eta}\left[e^{-\eta}\Phi\left(\frac{n+1}{2},\frac{1}{2},\eta\right)\right]\\
&\qquad = n\cdot e^{-\eta}\Phi\left(\frac{n+1}{2},\frac{3}{2},\eta\right)\\
&\qquad = n\cdot e^{-\eta}\cdot\frac{2^{\frac{n-3}{2}}\Gamma(\frac{n}{2})e^{\frac{\eta}{2}}}{\sqrt{2\eta\pi}}\times[U(n-1/2,-\sqrt{2\eta})-U(n-1/2,\sqrt{2\eta})]\\
&\qquad = n\cdot e^{-\frac{\tilde{\mu}_n^2}{4\tilde{\sigma}_n^2}}\cdot\frac{2^{\frac{n-3}{2}}\Gamma(\frac{n}{2})}{\sqrt{\pi\tilde{\mu}_n^2/\tilde{\sigma}_n^2}}\times\left[U\left(n-\frac{1}{2},-\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}\right)-U\left(n-\frac{1}{2},\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}\right)\right].
\end{aligned}$$

Using the identity (5.7.9), we have

$$\begin{aligned}
&U\left(n-\frac{1}{2},-\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}\right)-U\left(n-\frac{1}{2},\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}\right)\\
&=\frac{e^{-\frac{\tilde{\mu}_n^2}{4\tilde{\sigma}_n^2}}}{\Gamma(n)}\int_0^\infty w^{n-1}e^{-w^2/2}\cdot\left[e^{w\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}}-e^{-w\sqrt{\tilde{\mu}_n^2/\tilde{\sigma}_n^2}}\right]dw\geq 0\,.
\end{aligned}$$

This implies that $e^{-\eta}\Phi\left(\frac{n+1}{2},\frac{1}{2},\eta\right)$ is an increasing function with respect to the variable $\eta(=\frac{\tilde{\mu}_n^2}{2\tilde{\sigma}_n^2})$. Thus, $\mathbb{E}\left[Q_n\right]^{2n}$ can be bounded by the value at $\tilde{\mu}_n$ with $\tilde{\mathsf{m}}_n := \tilde{\mu}_n(0,\cdots,0) = \sum_{i=1}^n \alpha_i$ of this function, i.e.,

$$\mathbb{E}\left[Q_n\right]^{2n}\leq\frac{2^n}{\sqrt{\pi}}\tilde{\sigma}_n^{2n}\Gamma\left(\frac{2n+1}{2}\right)\cdot\exp\left(-\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right)\Phi\left(n+\frac{1}{2},\frac{1}{2},\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right). \tag{5.2.8}$$

Step 2: Analysis of the confluent hypergeometric function in (5.2.8).

A key ingredient of our proof is to analyze the asymptotic behavior as $n\to\infty$ of the right hand of (5.2.8) and this is the objective of this step. We claim that there exists a positive constant C which might change from line to line (we shall not point out the universal constants C unless necessary in this chapter) such that

$$\Phi\left(\frac{a}{2}+\frac{1}{4},\frac{1}{2},\frac{z^2}{2}\right)\leq C\cdot 2^{\frac{a}{2}-\frac{3}{4}}\Gamma\Big(\frac{a}{2}+\frac{3}{4}\Big)\times\frac{z^{a-\frac{1}{2}}\exp(\frac{z^2}{2})}{\Gamma(\frac{1}{2}+a)}\leq C\frac{z^{a-\frac{1}{2}}\exp(\frac{z^2}{2})}{2^{a/2}\Gamma(\frac{a}{2}+\frac{1}{4})}, \tag{5.2.9}$$

with $a = 2n + \frac{1}{2}$ and $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \geq C \cdot n^{2H}$ and n is sufficiently large. We shall show the key asymptotic behaviors of the confluent hypergeometric functions $\Phi(a, b, z)$. The idea is motivated by the Poincaré-type asymptotic forms (5.7.10) of confluent hypergeometric function. In our case, since we have $a = 2n + \frac{1}{2}$, the parameter a also goes to infinity. Fortunately, we have $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \geq C \cdot n^{2H}$ (see the proof in the Appendix A), and the parameter a is bounded from above by z since $H > 1/2$.

To prove the claim (5.2.9) we employ the integral representation of the parabolic cylinder functions (5.7.9). For $z > 0$, by the variable substitution, the parabolic cylinder functions are computed as follows:

$$\begin{aligned} U(a, z) &= \frac{z^{a+\frac{1}{2}} \exp(-\frac{z^2}{4})}{\Gamma(\frac{1}{2} + a)} \int_0^\infty t^{a-\frac{1}{2}} \exp(-z^2(\frac{1}{2}t^2 + t))dt \\ &= \frac{z^{a+\frac{1}{2}} \exp(\frac{z^2}{4})}{\Gamma(\frac{1}{2} + a)} \int_1^\infty (s-1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2})ds \end{aligned} \tag{5.2.10}$$

and

$$\begin{aligned} U(a, -z) &= \frac{z^{a+\frac{1}{2}} \exp(-\frac{z^2}{4})}{\Gamma(\frac{1}{2} + a)} \int_0^\infty t^{a-\frac{1}{2}} \exp(-z^2(\frac{1}{2}t^2 - t))dt \\ &= \frac{z^{a+\frac{1}{2}} \exp(\frac{z^2}{4})}{\Gamma(\frac{1}{2} + a)} \int_{-1}^\infty (s+1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2})ds\,. \end{aligned} \tag{5.2.11}$$

The sum of the integrals in (5.2.10)-(5.2.11) can be dominated as follow (with $a = 2n + \frac{1}{2}$ and $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2}$).

$$\begin{aligned} &\int_1^\infty (s-1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2})ds + \int_{-1}^\infty (s+1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2})ds \\ &\qquad \leq 2\int_{-1}^\infty (s+1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2})ds = 2\int_{-1}^\infty (s+1)^{2n} \exp(-\frac{\tilde{\mathsf{m}}_n^2 s^2}{2\tilde{\sigma}_n^2})ds\,. \end{aligned}$$

Basically, we know that $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \geq C \cdot n^{2H} \geq n$ for n large enough. So, for sufficient large n

$$(s+1)^{2(n+1)} \leq \exp\left(2(n+1)s\right) \leq \exp\left(\frac{\tilde{\mathsf{m}}_n^2 s^2}{4\tilde{\sigma}_n^2}\right)$$

for all $s \geq -1$. Therefore, we can easily obtain

$$\int_{-1}^{\infty} (s+1)^{2n} \exp\left(-\frac{\tilde{\mathsf{m}}_n^2 s^2}{2\tilde{\sigma}_n^2}\right) ds \leq \int_{-1}^{\infty} \exp\left(-\frac{\tilde{\mathsf{m}}_n^2 s^2}{4\tilde{\sigma}_n^2}\right) ds$$
$$\leq \int_{\mathbb{R}} \exp\left(-\frac{\tilde{\mathsf{m}}_n^2 s^2}{4\tilde{\sigma}_n^2}\right) ds = C \cdot \frac{\tilde{\sigma}_n}{\tilde{\mathsf{m}}_n} = C \cdot \frac{1}{z}\,.$$

Recall the relation between $\Phi(a, b, z)$ and the parabolic cylinder functions $U(a, z)$ given by (5.7.7). As a result, we get

$$\begin{aligned}\Phi\Big(\frac{a}{2}+\frac{1}{4}, \frac{1}{2}, \frac{z^2}{2}\Big) &= \frac{2^{\frac{a}{2}-\frac{3}{4}}}{\sqrt{\pi}}\Gamma\Big(\frac{a}{2}+\frac{3}{4}\Big)\exp(\frac{z^2}{4}) \times [U(a,z)+U(a,-z)] \\ &\leq C \cdot 2^{\frac{a}{2}-\frac{3}{4}}\Gamma\Big(\frac{a}{2}+\frac{3}{4}\Big)\exp(\frac{z^2}{4}) \times \frac{z^{a+\frac{1}{2}}\exp(\frac{z^2}{4})}{\Gamma(\frac{1}{2}+a)} \times \frac{1}{z} \\ &= C \cdot 2^{\frac{a}{2}-\frac{3}{4}}\Gamma\Big(\frac{a}{2}+\frac{3}{4}\Big) \times \frac{z^{a-\frac{1}{2}}\exp(\frac{z^2}{2})}{\Gamma(\frac{1}{2}+a)}\,. \end{aligned} \tag{5.2.12}$$

Thus we finish the proof of our claim (5.2.9).

Step 3: Completion of the proof of part (i) of Theorem 5.1.3. Applying (5.2.6), (5.2.8) and (5.2.12) with $a = 2n + \frac{1}{2}$, $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2} \geq C \cdot \frac{n^2}{n^{2-2H}} = n^{2H} \geq n$ we obtain

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^{n} Z_k^2(\Delta t)\right] &\leq \frac{2^n}{(2n)!}\sum_{v_1=0}^{2}\cdots\sum_{v_n=0}^{2}\mathbb{E}\left[Q_n\right]^{2n} \\ &\leq \frac{2^n \cdot 3^n}{(2n)!} \cdot \frac{2^n}{\sqrt{\pi}}\tilde{\sigma}_n^{2n}\Gamma\left(\frac{2n+1}{2}\right) \cdot \exp\left(-\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right)\Phi\left(n+\frac{1}{2}, \frac{1}{2}, \frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right) \\ &\leq \frac{2^n \cdot 3^n}{(2n)!} \cdot \frac{2^n}{\sqrt{\pi}}\tilde{\sigma}_n^{2n}\Gamma\left(\frac{2n+1}{2}\right) \cdot \exp\left(-\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right) \\ &\qquad \cdot \frac{C}{\Gamma(n+\frac{1}{2})}\left(\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right)^n \exp\left(\frac{\tilde{\mathsf{m}}_n^2}{2\tilde{\sigma}_n^2}\right) \\ &\leq \frac{6^n \cdot \tilde{\mathsf{m}}_n^{2n}}{(2n)!} \leq \frac{6^n \cdot n^{2n}}{\sqrt{4\pi n}\cdot(2n/e)^{2n}}\left(\frac{1-\theta}{\theta}\right)^{2n} \leq \frac{(\sqrt{3/2}e)^{2n}}{\sqrt{4\pi n}}\left(\frac{1-\theta}{\theta}\right)^{2n}, \end{aligned} \tag{5.2.13}$$

by Stirling's approximation, where we apply the claim (5.2.9) in the above forth inequality and the fact that $\frac{1}{2} < \theta < 1$ in the above last inequality. Now, it is obvious to see from

(5.2.13) that $\mathbb{E}\left[\prod_{k=1}^n Z_k^2(\Delta t)\right] \to 0$ as $n \to \infty$ if

$$\sqrt{3/2}e \cdot \frac{1-\theta}{\theta} \le 1 \Leftrightarrow \theta \ge \frac{\sqrt{3/2} \cdot e}{\sqrt{3/2} \cdot e + 1} \approx 0.77\,,$$

proving part (i) of Theorem 5.1.3. □

Remark 5.2.2. *We believe our method can also work under the condition that* $X_0 = 0$ *with probability* 0 *and* $\mathbb{E}[\mid X_0 \mid^2] < \infty$*. For example, one can apply Hölder inequality to* (5.2.5) *and then follow the same argument there. But this makes the computations much more involved. We are not pursuing the detail along this direction to simplify our presentation.*

Remark 5.2.3. *Following the same strategy as in our proof, we can prove more general results: For any integer* $p \ge 2$*, if* $\frac{1}{(1+M_p)} \le \theta < 1$*, where* $M_p = \frac{2}{e} \cdot \frac{1}{(p+1)\binom{p}{p/2}}$*, then* $\lim_{n\to\infty} \mathbb{E}(\bar{X}_n^p) \to 0$.

5.2.3 The case of $\kappa > 3/2$ and $\frac{1}{2} < \theta \le 1$

In this subsection we shall prove part (ii) of Theorem 5.1.3. To begin with, let us recall the celebrated Gaussian correlation inequality, and we are not giving the general statement here.

Lemma 5.2.4. [LaM17, Theorem 2] *Let* $n = n_1 + n_2$ *and* X *be an* n*-dimensional centered Gaussian vector. Then for any* $t_1, \cdots, t_n > 0$,

$$\begin{aligned} &\mathbb{P}\{ \mid X_1 \mid \le t_1, \ldots, \mid X_n \mid \le t_n\} \\ &\quad \ge \mathbb{P}\{\mid X_1 \mid \le t_1, \ldots, \mid X_{n_1} \mid \le t_{n_1}\} \cdot \mathbb{P}\{\mid X_{n_1+1} \mid \le t_{n_1+1}, \ldots, \mid X_n \mid \le t_n\}\,. \end{aligned}$$

Proof of part (ii) of Theorem 5.1.3. Let us consider

$$\begin{aligned} \bar{X}_{n+1}^2 &= \prod_{k=1}^{n} (Z_k(\Delta t))^2 = \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \prod_{k=p(n)+1}^{n} (Z_k(\Delta t))^2 \\ &= \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \prod_{k=p(n)+1}^{n} (Z_k(\Delta t))^2 \cdot \mathbf{1}_{\{Z_k(\Delta t) \le 0:\ p(n)+1 \le k \le n\}} \end{aligned} \tag{5.2.14}$$

$$+\left[\prod_{k=1}^{n}(Z_k(\Delta t))^2\right]\cdot\left[1-\prod_{k=p(n)+1}^{n}\mathbf{1}_{\{Z_k(\Delta t)\le 0\}}\right], \tag{5.2.15}$$

with $p(n) = n/p$, $q(n) = n/q$ and $1/p + 1/q = 1$. Formally, we know $Z_k(\Delta t)$ converges to $c_\theta = -\frac{1-\theta}{\theta} < 0$. Thus, the probability of the event $\{Z_k(\Delta t) \le 0 : \; p(n)+1 \le k \le n\}$ converges to one.

Firstly, applying the following bound on the geometric mean by the arithmetic one

$$\begin{cases} a_1^{p_1}\cdots a_n^{p_n} \le \left(\dfrac{p_1a_1+\cdots+p_na_n}{p_1+\cdots+p_n}\right)^{p_0+p_1+\cdots+p_n} \\ a_1,\cdots,a_n \ge 0,\;\; p_1,\cdots,p_n\in\mathbb{N}^+, \end{cases}$$

with $p_1 = p_2 = \cdots = p_{q(n)} = 2$ and $a_1 = -Z_{p(n)+1}(\Delta t),\cdots,a_{q(n)} = -Z_n(\Delta t)$ to the second factor of (5.2.14) yields

$$\begin{aligned}\bar{X}_{n+1}^2 &:= \prod_{k=1}^{p(n)}(Z_k(\Delta t))^2\cdot\prod_{k=p(n)+1}^{n}(-Z_k(\Delta t))^2\cdot\mathbf{1}_{\{Z_k(\Delta t)\le 0:\; p(n)+1\le k\le n\}} \\ &\le \prod_{k=1}^{p(n)}(Z_k(\Delta t))^2\cdot\left[\frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n}Z_k(\Delta t)\cdot\mathbf{1}_{\{Z_k(\Delta t)\le 0:\; p(n)+1\le k\le n\}}\right]^{2q(n)}.\end{aligned} \tag{5.2.16}$$

By the Hölder inequality, we then have

$$\begin{aligned}\mathbb{E}\bar{X}_{n+1}^2 &\le \left(\mathbb{E}\prod_{k=1}^{p(n)}(Z_k(\Delta t))^4\right)^{\frac12}\cdot\left(\mathbb{E}\left[\frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n}Z_k(\Delta t)\right]^{4q(n)}\right)^{\frac14} \\ &\qquad\cdot\Big(\mathbb{P}\{Z_k(\Delta t)\le 0:\; p(n)+1\le k\le n\}\Big)^{\frac14} \\ &\le \left(\mathbb{E}\prod_{k=1}^{p(n)}(Z_k(\Delta t))^4\right)^{\frac12}\cdot\left(\mathbb{E}\left[\frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n}Z_k(\Delta t)\right]^{4q(n)}\right)^{\frac14}.\end{aligned} \tag{5.2.17}$$

As we explained in Remark 5.2.3, by the same methods as in the proof of part (i), there exists an $M > 1$ such that the first factor of (5.2.17) can be bounded by

$$\left(\mathbb{E}\prod_{k=1}^{p(n)}(Z_k(\Delta t))^4\right)^{\frac{1}{2}} \le M^{p(n)} = M^{n/p}\,. \tag{5.2.18}$$

For the second term in (5.2.17), we have from raw moment formula and Kummer's transformation (5.7.5)

$$\begin{aligned}
&\mathbb{E}\left(\left[\frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n} Z_k(\Delta t)\right]^{4q(n)}\right) \\
&\quad= \frac{C}{\sqrt{\pi}}\bar{\sigma}_{p(n)}^{4q(n)}2^{2q(n)}\Gamma\left(\frac{4q(n)+1}{2}\right)\cdot\Phi\left(-2q(n),\frac{1}{2};-\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right) \\
&\quad= \frac{C}{\sqrt{\pi}}\bar{\sigma}_{p(n)}^{4q(n)}2^{2q(n)}\Gamma\left(\frac{4q(n)+1}{2}\right)\cdot\exp\left(-\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right)\Phi\left(2q(n)+\frac{1}{2},\frac{1}{2};-\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right),
\end{aligned} \tag{5.2.19}$$

where $\bar{\mu}_{p(n)} := \frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n}\alpha_k$ and $\bar{\sigma}_{p(n)}^2 := \mathbb{E}\left[|\frac{1}{n-p(n)}\sum_{k=p(n)+1}^{n}\beta_k\cdot V_k^H|^2\right]$. Then by (5.2.9) and the same procedure as in Step 2, we can bound (5.2.19), namely, the second factor of (5.2.17) by the following:

$$\begin{aligned}
&C\bar{\sigma}_{p(n)}^{4q(n)}2^{2q(n)}\Gamma\left(\frac{4q(n)+1}{2}\right)\exp\left(-\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right)\cdot\frac{\Gamma(1/2)}{\Gamma(2q(n)+1/2)}\left(\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right)^{2q(n)}\exp\left(\frac{\bar{\mu}_{p(n)}^2}{2\bar{\sigma}_{p(n)}^2}\right) \\
&\quad\le C\bar{\mu}_{p(n)}^{4q(n)}\le C\left(\frac{1-\theta}{\theta}\right)^{4q(n)}\to 0\,,\qquad (\text{as } n\to\infty)\,.
\end{aligned} \tag{5.2.20}$$

Combining (5.2.17)-(5.2.20), we conclude for the first summand (5.2.14)

$$\begin{aligned}
&\mathbb{E}\left[\bar{X}_n^2\right] \le M^{p(n)}\cdot|\,c_\theta\,|^{q(n)}\to 0\,, \\
&\quad\Leftrightarrow M^{1/p}\cdot|\,c_\theta\,|^{1/q}< 1\;\Leftrightarrow\; p > \frac{\ln(|\,c_\theta\,|)-\ln(M)}{\ln(|\,c_\theta\,|)} > 1\,.
\end{aligned}$$

Next, we treat (5.2.15). Denote $C_{\lambda,\mu,\Delta t}(k) := \frac{1-\kappa(1-\theta)\lambda k^{\kappa-1}\Delta t^{\kappa}}{\mu}$. It is easy to see that if $\beta_k > 0$ (i.e. $\lambda > 0,\ \mu > 0$), then

$$\mathbb{P}\{Z_k(\Delta t)\le 0\} = \mathbb{P}\{V_k^H \le -\alpha_k/\beta_k = -C_{\lambda,\mu,\Delta t}(k)\} = \mathbb{P}\{V_k^H \ge C_{\lambda,\mu,\Delta t}(k)\}\,,$$

and if $\beta_k < 0$ (i.e. $\lambda > 0,\ \mu < 0$), then

$$\mathbb{P}\{Z_k(\Delta t) \le 0\} = \mathbb{P}\{V_k^H \ge -\alpha_k/\beta_k = -C_{\lambda,\mu,\Delta t}(k)\} = \mathbb{P}\{V_k^H \le C_{\lambda,\mu,\Delta t}(k)\}\,.$$

Consequently, we have by the classical concentration inequality for normal variable V_k^H

$$\mathbb{P}\{Z_k(\Delta t) \le 0\} \ge \mathbb{P}\{\mid V_k^H \mid \le \mid C_{\lambda,\mu,\Delta t}(k) \mid\} \ge 1 - 2\exp\left[-\frac{\mid C_{\lambda,\mu,\Delta t}(k) \mid^2}{2(\Delta t)^{2H}}\right]. \tag{5.2.21}$$

Then, by the Gaussian correlation inequality (Lemma 5.2.4), we get

$$\begin{aligned}\mathbb{P}\{Z_k(\Delta t) \le 0 :\ p(n)+1 \le k \le n\} &\ge \mathbb{P}\{\mid V_k^H \mid \le \mid C_{\lambda,\mu,\Delta t}(k) \mid :\ p(n)+1 \le k \le n\}\\ &\ge \prod_{k=p(n)+1}^{n} \mathbb{P}\{\mid V_k^H \mid \le \mid C_{\lambda,\mu,\Delta t}(k) \mid\}\\ &\ge \prod_{k=p(n)+1}^{n} \left(1 - 2\exp\left[-\frac{\mid C_{\lambda,\mu,\Delta t}(k) \mid^2}{2(\Delta t)^{2H}}\right]\right). \end{aligned} \tag{5.2.22}$$

Denote

$$\bar{X}_n := 1 - \prod_{k=p(n)+1}^{n} \mathbf{1}_{\{Z_k(\Delta t) \le 0:\ p(n)+1 \le k \le n\}}\,.$$

By the Weierstrass product inequality:

$$\prod_{i=1}^{n}(1 - x_i) \ge 1 - \sum_{i=1}^{n} x_i\,, \quad \forall \quad x_1, \cdots, x_n \in (0,1)\,,$$

we have

$$\begin{aligned}\mathbb{E}\left[\bar{X}_n\right] &\le 1 - \prod_{k=p(n)+1}^{n} \left(1 - 2\exp\left[-\frac{\mid C_{\lambda,\mu,\Delta t}(k) \mid^2}{2(\Delta t)^{2H}}\right]\right)\\ &\le 2\sum_{k=p(n)+1}^{n} \exp\left[-\frac{\mid C_{\lambda,\mu,\Delta t}(k) \mid^2}{2(\Delta t)^{2H}}\right] \le C\exp\left[-\frac{\mid C_{\lambda,\mu,\Delta t}(p(n)) \mid^2}{2(\Delta t)^{2H}}\right], \end{aligned} \tag{5.2.23}$$

since when n is sufficiently large that $C\exp[- \mid C_{\lambda,\mu,\Delta t}(k) \mid^2 /2(\Delta t)^{2H}] < 1$. Because $\bar{X}_n$

is either 0 or 1, i.e. $\bar{X}_n^2 = \bar{X}_n$ we have for the second summand (5.2.15)

$$\mathbb{E}\left\{\left[\prod_{k=1}^{n}(Z_k(\Delta t))^2\right]\cdot\bar{X}_n\right\} \le \left(\mathbb{E}\Big[\prod_{k=1}^{n}(Z_k(\Delta t))^4\Big]\right)^{\frac{1}{2}}\cdot\Big(\mathbb{E}(\bar{X}_n)\Big)^{\frac{1}{2}}$$
$$\le CM^{2n}\cdot\exp\left[-\frac{|\,C_{\lambda,\mu,\Delta t}(p(n))\,|^2}{2(\Delta t)^{2H}}\right] \asymp M^{2n}\exp\left[-\frac{\lambda^2}{\mu^2}\cdot p(n)^{2(\kappa-1)}(\Delta t)^{2\kappa-2H}\right].$$

Here we applied $\mathbb{E}\Big[\prod_{k=1}^{n}(Z_k(\Delta t))^4\Big] \le M^{4n}$ for some constant $M>1$, which can be proved analogously as in the proof of part (i) of the theorem. Hence, it is easy to see if $\kappa > 3/2$, $p(n)^{2(\kappa-1)} \ge C_p\cdot n^{2(\kappa-1)} \gg n$, then the above term converges to 0. □

5.2.4 The case of $0<\theta<\frac{1}{2}$

In this subsection we prove part (iii) of Theorem 5.1.3. First, we state the following strong law of large numbers (SLLN).

Lemma 5.2.5. [HRV08, Theorem 1] *Let $\xi_1, \xi_2, \cdots, \xi_n$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants R_k such that*

$$\sup_{n\ge1}|\operatorname{Cov}(\xi_n,\xi_{n+k})|\le R_k\,,\ \ k\ge1\,,\qquad \sum_{k=1}^{\infty}\frac{R_k}{k^q}<\infty\quad\textit{for some } 0\le q<1\,,\tag{5.2.24}$$

and

$$\sum_{k=1}^{\infty}\frac{\mathbb{V}(\xi_n)\cdot[\log(n)]^2}{n^2}<\infty\,,\tag{5.2.25}$$

then the SLLN holds. More precisely, letting $S_n=\sum_{i=0}^{n}\xi_i$, one has

$$\lim_{n\to\infty}\frac{S_n-\mathbb{E}(S_n)}{n}=0\quad\textit{almost surely}.\tag{5.2.26}$$

With the help of this lemma we now give the proof of the last part of the theorem.

Proof of part (iii) of Theorem 5.1.3. Denote $Y_0=\ln X_0^2$, $Y_k=\ln\big(\alpha_k+\beta_k V_k^H\big)^2$ and $S_n=\sum_{k=0}^{n}Y_k$. In the above definition if $\alpha_k+\beta_kV_k^H=0$, then we put $Y_k:=0$. Notice that $(\alpha_k+\beta_kV_k^H)^2$ is positive almost surely, so Y_k are well defined for $k\ge0$. We shall apply Lemma 5.2.5 to $\xi_n=Y_n$. It is easier to verify that (5.2.25) holds. The main objective is to verify the conditions in (5.2.24). For $q\in(2H-1,1)$, the second condition of (5.2.24) holds if $\mathsf{R}_k \asymp | k |^{2H-2}$ for sufficiently large k. Thus, the proof of part (iii) in Theorem 5.1.3 is completed if we can show for some constant C

$$\sup_{n\geq 1} | \operatorname{Cov}(Y_n, Y_{n+k}) | \leq \mathsf{R}_k \leq C \cdot | k |^{2H-2} \,. \tag{5.2.27}$$

In fact, assume (5.2.27) and recall that if $0 < \theta < \frac{1}{2}$, then by noting $\lim_{n\to\infty} \alpha_n = -\frac{1-\theta}{\theta}$ and $\lim_{n\to\infty} \beta_n = 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(S_n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left[\ln(\alpha_k + \beta_k V_k^H)^2\right] = \ln\left(\frac{1-\theta}{\theta}\right)^2 =: C_\theta > 0\,.$$

Therefore, by Lemma 5.2.5 with $q \in (2H-1, 1)$, we get

$$\frac{S_n}{n} = \frac{\ln(\bar{X}_n)^2}{n} \xrightarrow{a.s.} C_\theta > 0\,.$$

This implies $(\bar{X}_n)^2 \to \infty$ almost surely. Consequently, by Fatou's Lemma, one has

$$\underline{\lim}_{n\to\infty} \mathbb{E} | \bar{X}_n |^2 \geq \mathbb{E}\left[\underline{\lim}_{n\to\infty} | \bar{X}_n |^2\right] = \infty\,,$$

which completes the proof of part (iii) of the theorem.

So, it suffices to show (5.2.27). We shall show that $| \operatorname{Cov}(Y_i, Y_j) | \lesssim | i-j |^{2H-2}$ as $| i-j | \to \infty$ which is obviously equivalent to (5.2.27). In fact,

$$\operatorname{Cov}(Y_i, Y_j) = \mathbb{E}\left[\ln(\alpha_i + \beta_i V_i^H)^2 \ln(\alpha_j + \beta_j V_j^H)^2\right] - \mathbb{E}\left[\ln(\alpha_i + \beta_i V_i^H)^2\right] \mathbb{E}\left[\ln(\alpha_j + \beta_j V_j^H)^2\right]\,.$$

Denote the probability densities of normal variables V_i^H and V_j^H by $f_i(x)$ and $f_j(y)$. The symmetric covariance matrix of V_i^H and V_j^H is given by

$$\Sigma = \begin{pmatrix} \sigma_i^2, & \rho_{ij}\sigma_i\sigma_j \\ \rho_{ij}\sigma_i\sigma_j, & \sigma_j^2 \end{pmatrix}\,,$$

where $\sigma_i := \sqrt{\mathbb{E}[(V_i^H)^2]} =\mid t_{i+1} - t_i \mid^H = \mid \Delta t \mid^H$ and $\sigma_j := \sqrt{\mathbb{E}[(V_j^H)^2]} =\mid t_{j+1} - t_j \mid^H = \mid \Delta t \mid^H$ are standard deviations of V_i^H and V_j^H, $\rho_{ij} := \frac{\mathbb{E}[V_i^H V_j^H]}{\sigma_i \sigma_j}$ is the correlation coefficient between V_i^H and V_j^H. Clearly, $\rho_{ij} \to 0$ as $\mid i-j \mid$ goes to infinity. Their joint distribution has the following form

$$\begin{aligned} f_{i,j}(x,y) &= \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma)}} \exp\left(-\frac{X^T \Sigma^{-1} X}{2}\right) \\ &= \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}} \exp\left(-\frac{1}{2(1-\rho_{ij}^2)}\left[\frac{x^2}{\sigma_i^2} - 2\rho_{ij}\frac{x \cdot y}{\sigma_i \cdot \sigma_j} + \frac{y^2}{\sigma_j^2}\right]\right), \end{aligned} \tag{5.2.28}$$

with $X = [x,y]^T$. Without loss of generality, we can assume that $i \geq j+1$. Then we have using the joint density (5.2.28):

$$\begin{aligned} \mathrm{Cov}(Y_i, Y_j) &= \int_{\mathbb{R}^2} \left[\ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2\right] \cdot \left[f_{i,j}(x,y) - f_i(x) f_j(y)\right] dxdy \\ &= \int_{\mathbb{R}^2} \left[\ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2\right] \cdot \exp\left(-\frac{\tilde{\rho}_{ij}}{2}\left[\frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2}\right]\right) \\ &\quad \times \left[-\exp\left(\frac{\tilde{\rho}_{ij}}{2}\left[\frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2}\right]\right) + \frac{1}{\sqrt{1-\rho_{ij}^2}} \exp\left(\bar{\rho}_{ij} \cdot \frac{xy}{\sigma_i \sigma_j}\right)\right] dF_i(x) dF_j(y), \end{aligned}$$

where $\tilde{\rho}_{ij} = \frac{\rho_{ij}^2}{1-\rho_{ij}^2}$, $\bar{\rho}_{ij} = \frac{\rho_{ij}}{1-\rho_{ij}^2}$. One should notice that $\tilde{\rho}_{ij}, \bar{\rho}_{ij} \to 0$ as $\mid i-j \mid$ goes to infinity. By the Hölder inequality, $\mathrm{Cov}(Y_i, Y_j)$ can be bounded by

$$\begin{aligned} &\left(\int_{\mathbb{R}^2} \left[\ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2\right]^2 dF_i(x) dF_j(y)\right)^{\frac{1}{2}} \\ &\times \Bigg(\int_{\mathbb{R}^2} \Bigg[\exp\left(\frac{\tilde{\rho}_{ij}}{2}\left[\frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2}\right]\right) \\ &\qquad - \frac{1}{\sqrt{1-\rho_{ij}^2}} \exp\left(\bar{\rho}_{ij} \cdot \frac{xy}{\sigma_i \sigma_j}\right)\Bigg]^2 dF_i(x) dF_j(y)\Bigg)^{\frac{1}{2}} =: A_{ij}^{\frac{1}{2}} \times B_{ij}^{\frac{1}{2}}. \end{aligned}$$

We proceed to estimate A_{ij} and B_{ij}. To estimate A_{ij} we only need to consider

$$\begin{aligned} &\int_{\mathbb{R}} \left[\ln(\alpha + \beta x)^2\right]^2 \times \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \asymp \int_{\mathbb{R}} \left[\ln(\alpha + \beta\sigma x)^2\right]^2 \times e^{-\frac{x^2}{2}} dx \\ &= \int_{|\alpha+\beta\sigma x| \leq 1} \left[\ln(\alpha + \beta\sigma x)^2\right]^2 \times e^{-\frac{x^2}{2}} dx + \int_{|\alpha+\beta\sigma x| \geq 1} \left[\ln(\alpha + \beta\sigma x)^2\right]^2 \times e^{-\frac{x^2}{2}} dx \end{aligned}$$

$$\leq \int_{|\alpha+\beta\sigma x|\leq 1} \left[\ln(\alpha+\beta\sigma x)^2\right]^2 dx + \int_{|\alpha+\beta\sigma x|\geq 1} [\alpha+\beta\sigma x]^4 \times e^{-\frac{x^2}{2}} dx\,.$$

Here, we neglect the subscripts of α_i and β_i to simplify the notations. Obviously, there exist constants c and C such that $A_{i,j} \leq C$, and α, β defined by (5.2.1)-(5.2.2) satisfying $0 < c < \alpha, \beta < C < \infty$. Next, for $\mid i-j \mid \to \infty$, we deal with B_{ij}:

$$B_{ij} = \int_{\mathbb{R}^2} \Bigg[\exp\left(\frac{\tilde{\rho}_{ij}}{2}\left[\frac{x^2}{\sigma_i^2}+\frac{y^2}{\sigma_j^2}\right]\right) - \frac{1}{\sqrt{1-\rho_{ij}^2}} \exp\left(\frac{\bar{\rho}_{ij}\cdot xy}{\sigma_i\sigma_j}\right)\Bigg]^2 \times \frac{1}{2\pi\sigma_i\sigma_j}\exp\left(-\frac{1}{2}\left[\frac{x^2}{\sigma_i^2}+\frac{y^2}{\sigma_j^2}\right]\right) dxdy.$$

By variable substitutions $x \to \sqrt{2}\sigma_i x, y \to \sqrt{2}\sigma_j y$, we have

$$\begin{aligned} B_{ij} &\leq C\int_{\mathbb{R}^2}\left[\exp\left(\tilde{\rho}_{ij}\left[x^2+y^2\right]\right) - \frac{1}{\sqrt{1-\rho_{ij}^2}}\exp\left(2\bar{\rho}_{ij}\cdot xy\right)\right]^2 \times \exp\left(-\left[x^2+y^2\right]\right) dxdy \\ &= C\int_{\mathbb{R}^2}\Bigg[\exp\left(2\tilde{\rho}_{ij}\left[x^2+y^2\right]\right) + \frac{1}{1-\rho_{ij}^2}\exp\left(4\bar{\rho}_{ij}\cdot xy\right) \\ &\qquad - \frac{2}{\sqrt{1-\rho_{ij}^2}}\exp\left(\tilde{\rho}_{ij}\left[x^2+y^2\right]+2\bar{\rho}_{ij}xy\right)\Bigg] \times \exp\left(-\left[x^2+y^2\right]\right) dxdy\,. \end{aligned}$$

The above three integrals can be explicitly evaluated as follows:

$$\int_{\mathbb{R}^2}\exp\left(2\tilde{\rho}_{ij}\left[x^2+y^2\right]\right)\times\exp\left(-\left[x^2+y^2\right]\right)dxdy = \frac{\pi}{1-2\tilde{\rho}_{ij}}\,,$$

$$\frac{1}{1-\rho_{ij}^2}\int_{\mathbb{R}^2}\exp\left(4\bar{\rho}_{ij}\cdot xy\right)\times\exp\left(-\left[x^2+y^2\right]\right)dxdy = \frac{1}{1-\rho_{ij}^2}\cdot\frac{\pi}{\sqrt{1-4\bar{\rho}_{ij}^2}}\,,$$

and

$$\begin{aligned} &\frac{2}{\sqrt{1-\rho_{ij}^2}}\int_{\mathbb{R}^2}\exp\left(\tilde{\rho}_{ij}\left[x^2+y^2\right]+2\bar{\rho}_{ij}xy\right)\times\exp\left(-\left[x^2+y^2\right]\right)dxdy \\ &\qquad = \frac{2}{\sqrt{1-\rho_{ij}^2}}\cdot\frac{\pi}{\sqrt{(1-\tilde{\rho}_{ij})^2-\bar{\rho}_{ij}^2}}\,. \end{aligned}$$

Thus to bound B_{ij} we need to know the asymptotics of ρ_{ij} and $\bar{\rho}_{ij}$. First, there exists a constant C_H such that

$$\begin{aligned}\mathbb{E}[V_i^H V_j^H] &= \mathbb{E}\left[(B^H(t_{i+1}) - B^H(t_i))(B^H(t_{j+1}) - B^H(t_j))\right] \\ &= \frac{1}{2}\left[(t_{i+1} - t_j)^{2H} - (t_{i+1} - t_{j+1})^{2H} - (t_i - t_j)^{2H} + (t_i - t_{j+1})^{2H}\right] \\ &= \frac{(\Delta t)^{2H}}{2}\left[(i - j + 1)^{2H} - 2(i - j)^{2H} + (i - j - 1)^{2H}\right] \qquad (5.2.29)\end{aligned}$$

Hence,

$$0 \le \rho_{ij} = \frac{\mathbb{E}[V_i^H V_j^H]}{\sigma_i \sigma_j} \le C\left[\mid i - j \mid^{2H-2}\right]. \qquad (5.2.30)$$

Consequently,

$$\bar{\rho}_{ij} \le C\left[\mid i - j \mid^{2H-2}\right], \quad \tilde{\rho}_{ij} \le C\left[\mid i - j \mid^{4H-4}\right]. \qquad (5.2.31)$$

Therefore, by the Taylor expansions as $x \to 0$, we have

$$\begin{aligned}B_{ij} \le& C\left[\frac{\pi}{1 - 2\tilde{\rho}_{ij}} + \frac{1}{1 - \rho_{ij}^2} \cdot \frac{\pi}{\sqrt{1 - 4\bar{\rho}_{ij}^2}} - \frac{2}{\sqrt{1 - \rho_{ij}^2}} \cdot \frac{\pi}{\sqrt{(1 - \tilde{\rho}_{ij})^2 - \bar{\rho}_{i,j}^2}}\right] \\ =& \frac{\pi}{\sqrt{1 - \rho_{ij}^2}}\left[\frac{\sqrt{1 - \rho_{ij}^2}}{1 - 2\tilde{\rho}_{ij}} + \frac{1}{\sqrt{1 - \rho_{ij}^2}} \cdot \frac{1}{\sqrt{1 - 4\bar{\rho}_{ij}^2}} - \frac{2}{\sqrt{(1 - \tilde{\rho}_{ij})^2 - \bar{\rho}_{ij}^2}}\right] \\ =& \frac{\pi}{\sqrt{1 - \rho_{ij}^2}}\left[\left(\frac{\sqrt{1 - \rho_{ij}^2}}{1 - 2\tilde{\rho}_{ij}} - 1\right) + \left(\frac{1}{\sqrt{1 - \rho_{ij}^2}} \cdot \frac{1}{\sqrt{1 - 4\bar{\rho}_{ij}^2}} - 1\right) + 2 - \frac{2}{\sqrt{(1 - \tilde{\rho}_{ij})^2 - \bar{\rho}_{ij}^2}}\right] \\ \le& C\left[\mid 2\tilde{\rho}_{ij} - \frac{1}{2}\rho_{ij}^2 \mid + \mid \frac{1}{2}(\rho_{ij}^2 + 4\bar{\rho}_{ij}^2 - 4\rho_{ij}^2\bar{\rho}_{ij}^2) \mid + \mid \frac{1}{2}(\tilde{\rho}_{ij}^2 - \bar{\rho}_{ij}^2 - 2\tilde{\rho}_{ij}) \mid\right] \\ \le& C \mid i - j \mid^{4H-4},\end{aligned}$$

when $\mid i - j \mid$ is sufficiently large. As a result, we have

$$\mathrm{Cov}(Y_i, Y_j) \le C\left[\mid i - j \mid^{2H-2} \wedge 1\right]. \qquad (5.2.32)$$

This completes the proof of (5.2.27) and hence we finish the proof of part (iii) of Theorem 5.1.3. □

5.2.5 Brownian motion case

In this section we consider the case when $H = 1/2$, namely, Brownian motion $B^H = B$. The equation (5.1.6) becomes

$$dX(t) = -\lambda\kappa t^{\kappa-1}X(t)dt + \mu X(t) \circ dB(t)\,, \quad X(0) = X_0\,, \tag{5.2.33}$$

where $\lambda, \mu \in \mathbb{R}$ and $\kappa \geq 2H = 1$. Here, we assume that $X_0 \neq 0$ with a positive probability and $\mid X_0 \mid$ is square integrable. We have $X(t) = X_0 \exp(\lambda t^\kappa + \mu B(t))$ and

$$\mathbb{E} \mid X(t) \mid^2 = \mathbb{E} \mid X_0 \mid^2 \exp\left(2(-\lambda t^\kappa + \mu^2 t)\right)\,. \tag{5.2.34}$$

So the solution is stable if (i) $\kappa > 1$ and $\lambda > 0$ or (ii) $\kappa = 1$ and $-\lambda + \mid \mu \mid^2 < 0$. Otherwise, the solution of (5.2.33) is unstable.

The STM for the SDEs (5.2.33) in the Stratonovich sense can write (5.1.14) as

$$\bar{X}_{n+1} = \left(\frac{1-\kappa(1-\theta)\lambda(t_n)^{\kappa-1}\Delta t}{1+\kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} + \frac{\mu V_n}{1+\kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t}\right)\bar{X}_n\,, \tag{5.2.35}$$

with $t_n = n \cdot \Delta t$ and $V_n = B(t_{n+1}) - B(t_n)$. Notice that V_n's are mutually independent. The equation (5.2.35) can also be rewritten as follows

$$\bar{X}_{n+1} = X_0 \prod_{k=1}^{n} Z_k(\Delta t) = X_0 \prod_{k=1}^{n} (\alpha_k + \beta_k V_k)\,. \tag{5.2.36}$$

where

$$\begin{cases} \alpha_n := \alpha_n(\theta, \lambda, \Delta t) = \dfrac{1-\kappa(1-\theta)\lambda(t_n)^{\kappa-1}\Delta t}{1+\kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} = \dfrac{1-\kappa(1-\theta)\lambda n^{\kappa-1}\Delta t^\kappa}{1+\kappa\theta\lambda(n+1)^{\kappa-1}\Delta t^\kappa}\,, \\ \beta_n := \beta_n(\theta, \lambda, \mu, \Delta t) = \dfrac{\mu}{1+\kappa\theta\lambda(t_{n+1})^{\kappa-1}\Delta t} = \dfrac{\mu}{1+\kappa\theta\lambda(n+1)^{\kappa-1}\Delta t^\kappa}\,. \end{cases}$$

Obviously, we have the following.

- If $\kappa > 1$, for every fixed $\Delta t > 0$, $\lambda > 0$ and μ

$$\lim_{n\to\infty} \alpha_n = -\frac{1-\theta}{\theta}, \qquad \lim_{n\to\infty} \beta_n = 0.$$

Therefore, we have

$$\begin{aligned}\mathbb{E}[\mid \bar{X}_{n+1} \mid^2] &= \mathbb{E}[\mid X_0 \mid^2] \prod_{k=1}^{n} \mathbb{E}\left[\mid \alpha_k + \beta_k V_k \mid^2\right] \\ &= \mathbb{E}[\mid X_0 \mid^2] \prod_{k=1}^{n} \left[\alpha_k^2 + \beta_k^2 \cdot \Delta t\right] \asymp \left(\frac{1-\theta}{\theta}\right)^{2n} \to \begin{cases} 0\,, & \text{if } \frac{1}{2} < \theta \leq 1\,; \\ \infty\,, & \text{if } 0 \leq \theta < \frac{1}{2}\,. \end{cases}\end{aligned}$$

- If $\kappa = 1$, (5.2.33) is reduced to the standard stochastic test equation (see also [KB12]). Then

$$\alpha_n = \bar{\alpha} = \frac{1-(1-\theta)\lambda\Delta t}{1+\theta\lambda\Delta t}\,, \qquad \beta_n = \bar{\beta} = \frac{\mu}{1+\theta\lambda\Delta t}\,.$$

Thus,

$$\mathbb{E} \mid \bar{X}_{n+1} \mid^2 = \mathbb{E}(\bar{\alpha} + \bar{\beta} \cdot V_n)^2 \mathbb{E} \mid \bar{X}_n \mid^2\ .$$

In this sense, the numerical stability (or non-stability) depends on the condition

$$\begin{aligned} & \bar{\alpha}^2 + \bar{\beta}^2 \cdot \Delta t < 1 \text{ (or } > 1)\,, \\ \Leftrightarrow\quad & (1-2\theta)\lambda^2\Delta t + (-2\lambda + \mid \mu \mid^2) < 0 \text{ (or } > 0)\,. \end{aligned}$$

Now, we can summarize the discussion above as the following proposition:

Proposition 5.2.6. *For the test equation (5.2.33) and the STM (5.2.35), we have*

(i) When $\kappa > 1$, for any fixed λ, μ, then the STM (5.2.35) is mean square stable for the test equation (5.2.33) if $\frac{1}{2} < \theta \leq 1$ and is not *mean square stable if $0 \leq \theta \leq \frac{1}{2}$;*

(ii) When $\kappa = 1$ and $-2\lambda + \mid \mu \mid^2 < 0$, then the STM (5.2.35) is mean square stable for the test equation (5.2.33) if either $\frac{1}{2} \leq \theta \leq 1$ for all $\Delta t > 0$ or $0 \leq \theta < \frac{1}{2}$ for Δt satisfying

$$0 < \Delta t < \frac{2\lambda - \mid \mu \mid^2}{(1-2\theta)\lambda^2}\,;$$

(iii) When $\kappa = 1$, $-2\lambda + \mid \mu \mid^2 > 0$ and $0 \leq \theta < 1/2$, then the STM (5.2.35) is not *mean square stable for the test equation (5.2.33) for all $\Delta t > 0$.*

5.3 STM: Mean square nonlinear stability analysis

In this section, we shall study the p-th moments stability and the numerical stability of the solution to the general SDEs driven by fBm.

5.3.1 Mean square stability

Beyond the well-posedness, as we mentioned in Section 1, it seems too complicated to find the long time asymptotic behavior of (5.1.9). To the best of our knowledge, there are few results on the convergence of $\mathbb{E}[\mid X(t) \mid^2]$ when t goes to infinity. Thus, we focus on the following simplified SDEs with $g(t, X(t)) = c(t)X(t)$ under the assumption (5.1.11) and (5.1.12)

$$dX(t) = f(t, X(t))dt + c(t)X(t) \circ dB^H(t)\,. \tag{5.3.1}$$

In this case, (5.1.12) means $\mid c(t) \mid \leq \mu$ for some $\mu > 0$.

Theorem 5.3.1. *Let $X(t)$ be the solution to SDE (5.3.1) with initial value X_0 is deterministic, and let p, κ and λ, μ satisfy*

$$\textit{(i) } \kappa > 2H \textit{ and } \lambda > 0 \quad \textit{or} \quad \textit{(ii) } \kappa = 2H \textit{ and } -\lambda + \frac{p}{2}\mu^2 < 0\,. \tag{5.3.2}$$

If $f(t, x)$ in (5.3.1) satisfies (5.1.10) in Assumption 1 and $c(t)$ satisfies $\mid c(t) \mid \leq \mu$ for $\mu > 0$, then $\mathbb{E}|X(t)|^p \to 0$ as $t \to \infty$.

Proof. We can assume that $p \geq 2$ is an even positive number (for p is an odd positive number, we can obtain the similar result from the Hölder inequality). Denote $F_t = \exp[-\int_0^t c(s)dB_s^H]$ and $Y_t = F_t \cdot X(t)$. Then by the chain rule formula (e.g. [Hu13, Proposition 2.7] or [Mis08, Lemma 2.7.1]) we have

$$\frac{d}{dt}Y_t = F_t \cdot f(t, X(t)) = F_t \cdot f(t, (F_t)^{-1}Y_t)\,.$$

Note that it is a deterministic ordinary differential equation for the function $t \to Y_t(\omega)$

for every $\omega \in \Omega$. Then by the condition (5.1.10) in Assumption 1, we get

$$\begin{aligned}\frac{d}{dt}Y_t^p &= pY_t^{p-1}\frac{dY_t}{dt}\\ &= pY_t^{p-1}(F_t)^{-1}F_t^2 f(t,(F_t)^{-1}Y_t) \le -p\lambda\kappa t^{\kappa-1}\cdot Y_t^p\,.\end{aligned}$$

Thanks to Gronwall's inequality, we have

$$Y_t^p \le Y_0^p \exp\left(-p\lambda\kappa\int_0^t s^{\kappa-1}ds\right) = X_0^p\exp(-p\lambda t^\kappa)\,,$$

and

$$X(t)^p \le X_0^p \exp\left(-p\lambda t^\kappa + p\int_0^t c(s)dB_s^H\right)\,.$$

Therefore, letting $C_H = H(2H-1)$, since $\int_0^t c(s)dB_s^H$ is a Gaussian process (in t) with $\mathbb{E}[|\int_0^t c(s)dB_s^H|^2] = C_H\int_0^t\int_0^t c(s)|s-r|^{2H}c(r)dsdr$, we have

$$\begin{aligned}\mathbb{E}[X(t)^p] &\le C\exp\left(-p\lambda t^\kappa + p^2 C_H\int_0^t\int_0^t c(s)\mid s-r\mid^{2H-2}c(r)drds\right)\\ &\le C\exp\left(-p\lambda t^\kappa + \frac{p^2}{2}\mu^2 t^{2H}\right)\,.\end{aligned}$$

So we have that $\mathbb{E}[X(t)^p]$ converges to 0 under the condition (5.3.2). □

5.3.2 Numerical stability

For the numerical stability of SDEs driven by fBm, we consider a more general diffusion coefficient. More precisely, instead of $g(t,X) = c(t)X$ we allow the diffusion term g to be generally nonlinear satisfying (5.1.12). We hope this will shed light to the stability of the original solution.

Now, we give the proof of Theorem 5.1.6.

Proof of Theorem 5.1.6. From STM (5.1.13), we have

$$\bar{X}_{n+1} - \theta f(t_{n+1},\bar{X}_{n+1})\Delta t = \bar{X}_n + (1-\theta)f(t_n,\bar{X}_n)\Delta t + g(t_n,\bar{X}_n)V_n^H\,. \tag{5.3.3}$$

By the condition (5.1.10), we have

$$| f(t,X) |^2 \geq (\lambda\kappa t^{\kappa-1})^2 | X |^2 \ .$$

We bound the square of left hand side of (5.3.3) as

$$\begin{aligned} &| \bar{X}_{n+1} - \theta f(t_{n+1}, \bar{X}_{n+1})\Delta t |^2 \\ &\quad = (\bar{X}_{n+1})^2 + \theta^2 | f(t_{n+1}, \bar{X}_{n+1}) |^2 (\Delta t)^2 - 2\theta \Delta t \bar{X}_{n+1} f(t_{n+1}, \bar{X}_{n+1}) \\ &\quad \geq (\bar{X}_{n+1})^2 + (\theta\ \lambda\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t)^2 (\bar{X}_{n+1})^2 + 2\ \theta\ \lambda\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t\ (\bar{X}_{n+1})^2 \\ &\quad = (\bar{X}_{n+1})^2 [1 + \theta\ \lambda\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t]^2 \,. \end{aligned} \tag{5.3.4}$$

Step 1 ($\theta = 1$): With the condition (5.1.12), it is clear that the square of right hand side of (5.3.3) can be bounded by

$$| \bar{X}_n + g(t_n, \bar{X}_n) V_n^H |^2 \leq 2 \left[(\bar{X}_n)^2 + \mu^2 \bar{X}_n^2 (V_n^H)^2 \right] \ . \tag{5.3.5}$$

Therefore, we have from (5.3.4) and (5.3.5)

$$| \bar{X}_n |^2 \leq 2 \left[\alpha_n^2 + \beta_n^2 \cdot (V_n^H)^2 \right] | \bar{X}_{n-1} |^2 \leq 2^n \prod_{j=1}^{n} \left[\alpha_j^2 + \beta_j^2 (V_j^H)^2 \right] X_0^2 \,, \tag{5.3.6}$$

where

$$\alpha_n = \frac{1}{1 + \lambda \cdot \kappa (t_n)^{\kappa-1} \Delta t} \,, \qquad \beta_n = \frac{\mu}{1 + \lambda \cdot \kappa (t_n)^{\kappa-1} \Delta t} \,.$$

Let us rewrite $\prod_{j=1}^{n} \left[\alpha_j^2 + \beta_j^2 (V_j^H)^2 \right]$ in (5.3.6) as

$$\prod_{j=1}^{2n} Z_j = \prod_{j=1}^{n} (\beta_j V_j^H + \iota \alpha_j)(\beta_j V_j^H - \iota \alpha_j) \,,$$

with ι being the imaginary number and

$$Z_{2j-1} = \beta_j V_j^H + \iota \alpha_j \,, \qquad Z_{2j} = \beta_j V_j^H - \iota \alpha_j \,.$$

Applying Lemma 5.2.1 with $s_1 = \cdots = s_{2n} = 1$, $s = \sum_{j=1}^{2n} s_j = 2n$, one has

$$\prod_{j=1}^{2n} Z_j = \frac{1}{(2n)!} \sum_{v_1=0}^{1} \cdots \sum_{v_{2n}=0}^{1} \binom{1}{v_1} \cdots \binom{1}{v_{2n}} (-1)^{\sum_{j=1}^{2n} v_j} \left[\sum_{j=1}^{2n} h_j Z_j \right]^{2n}, \tag{5.3.7}$$

where $h_j = \frac{1}{2} - v_j = \frac{(-1)^{v_j}}{2}$. Note that

$$\begin{aligned} \sum_{j=1}^{2n} h_j Z_j &= \sum_{j\ odd} h_j(\beta_j V_j^H + \iota\alpha_j) + \sum_{j\ even} h_j(\beta_j V_j^H - \iota\alpha_j) \\ &= \sum_{j=1}^{2n} h_j \beta_j \cdot V_j^H + \iota \left(\sum_{j\ odd} h_j\alpha_j - \sum_{j\ even} h_j\alpha_j \right). \end{aligned}$$

Thus, we get that

$$\begin{aligned} \mid \sum_{j=1}^{2n} h_j Z_j \mid^{2n} &= \left[\left(\sum_{j=1}^{2n} h_j\beta_j \cdot V_j^H \right)^2 + \left(\sum_{j\ odd} h_j\alpha_j - \sum_{j\ even} h_j\alpha_j \right)^2 \right]^n \\ &\le 2^n \left[\left(\sum_{j=1}^{2n} h_j\beta_j \cdot V_j^H \right)^{2n} + \left(\sum_{j\ odd} h_j\alpha_j - \sum_{j\ even} h_j\alpha_j \right)^{2n} \right]. \end{aligned}$$

Therefore, taking expectation on both sides of (5.3.6) and using Lemma 5.2.1, we obtain

$$\begin{aligned} \mathbb{E}[\mid \bar{X}_n \mid^2] &\le 2^n \cdot \mathbb{E}\left[\left| \prod_{j=1}^{2n} Z_j \right| \right] \\ &\le \frac{2^n}{(2n)!} \sum_{v_1=0}^{1} \cdots \sum_{v_{2n}=0}^{1} \binom{1}{v_0} \cdots \binom{1}{v_{2n}} \mathbb{E}\left[\left| \sum_{j=1}^{2n} h_j Z_j \right|^{2n} \right] \\ &\le \frac{2^n}{(2n)!} \sum_{v_1=0}^{1} \cdots \sum_{v_{2n}=0}^{1} \frac{1}{2^n} \left[\mathbb{E}\left(\sum_{j=1}^{2n} (-1)^{v_j} \beta_j \cdot V_j^H \right)^{2n} + \left(\sum_{j=1}^{2n} \alpha_j \right)^{2n} \right] \\ &=: \widetilde{I}_1 + \widetilde{I}_2 . \end{aligned}$$

Denote $R = \sum_{j=1}^{2n} (-1)^{v_j} \beta_j \cdot V_j^H$. Then R is a Gaussian random variable with mean zero and variance σ_R^2 given by

$$\sigma_R^2 = \mathbb{E}\left(\sum_{j=1}^{2n} (-1)^{v_j} \beta_j \cdot V_j^H \right)^2 \le C \cdot n^{2+2H-2\kappa}$$

(see the computation in the appendix A). Thus, we have

$$\mathbb{E}\left(\sum_{j=1}^{2n}(-1)^{v_j}\beta_j\cdot V_j^H\right)^{2n} = 2^n\Gamma(n+1/2)\cdot(\sigma_R)^{2n}$$
$$\leq C^n 2^n\Gamma(n+1/2)\cdot n^{(2+2H-2\kappa)n}\,.$$

By Stirling's formula, we further have

$$\begin{aligned}
\widetilde{I}_1 &\leq \frac{2^n}{(2n)!}\cdot\frac{2^{2n}}{2^n}\cdot\sup_{v_j}\mathbb{E}\left(\sum_{j=1}^{2n}(-1)^{v_j}\beta_j\cdot V_j^H\right)^{2n}\\
&\leq \frac{4^n C^n}{(2n)!}\cdot\Gamma(n+1/2)\cdot n^{n(2+2H-2\kappa)}\\
&\asymp \frac{1}{\left(\frac{2n}{e}\right)^{2n}}\cdot n^{n(2+2H-2\kappa)}\left(\frac{n-\frac{1}{2}}{e}\right)^{n-\frac{1}{2}}\\
&\asymp C^n\cdot n^{n(1+2H-2\kappa)}\to 0\,,
\end{aligned}$$

as $n\to\infty$ since $\kappa\geq 2H>1$. For the term $\widetilde{I}_2$, as $n\to\infty$, we also have

$$\widetilde{I}_2\leq\frac{2^n}{(2n)!}\cdot\frac{2^{2n}}{2^{n+1}}\left(\sum_{j=1}^{2n}\alpha_j\right)^{2n}\leq\frac{C^n\cdot n^{2(2-\kappa)n}}{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}\asymp C^n\cdot n^{2(1-\kappa)n}\to 0\,.$$

Step 2 ($\theta<1$): Similar to (5.3.5), it follows with additional condition (5.1.11) that the right hand side of (5.3.3) can be bounded by

$$\begin{aligned}
&|\ \bar{X}_n+(1-\theta)f(t_n,\bar{X}_n)\Delta t+g(t_n,\bar{X}_n)V_n^H\ |^2\\
&\quad\leq 3\left(\bar{X}_n^2+|\ (1-\theta)f(t_n,\bar{X}_n)\Delta t\ |^2+|\ g(t_n,\bar{X}_n)V_n^H\ |^2\right)\\
&\quad\leq 3\left(\bar{X}_n^2+(1-\theta)^2(\bar{\lambda}\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t)^2\bar{X}_n^2+\mu^2\bar{X}_n^2(V_n^H)^2\right)\,.
\end{aligned}\tag{5.3.8}$$

Combining (5.3.4) and (5.3.8), we have

$$[1+\theta\ \lambda\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t]^2(\bar{X}_{n+1})^2\leq 3\left(1+(1-\theta)^2(\bar{\lambda}\ \kappa\ t_{n+1}^{\kappa-1}\ \Delta t)^2+\mu^2(V_n^H)^2\right)(\bar{X}_n)^2\,.$$

We can the above inequality as

$$(\bar{X}_n)^2 \leq 3\left[\alpha_n^2 + \beta_n^2 \cdot (V_n^H)^2\right](\bar{X}_{n-1})^2\,,$$

where α_n and β_n are given by

$$\alpha_n^2 = \frac{1+(1-\theta)^2(\bar{\lambda}\ \kappa\ t_n^{\kappa-1}\ \Delta t)^2}{[1+\theta\ \lambda\ \kappa\ t_n^{\kappa-1}\ \Delta t]^2}\,, \qquad \beta_n^2 = \frac{\mu^2}{[1+\theta\ \lambda\ \kappa\ t_n^{\kappa-1}\ \Delta t]^2}\,. \tag{5.3.9}$$

Therefore,

$$\begin{aligned}\bar{X}_n^2 &\leq 3^n \prod_{j=1}^{n}\left(\alpha_j^2 + \beta_j^2 (V_j^H)^2\right) X_0^2 \\ &= 3^n \prod_{j=1}^{n} (\beta_j V_j^H + \iota\alpha_j)(\beta_j V_j^H - \iota\alpha_j)\,. \end{aligned}\tag{5.3.10}$$

Thus by the same procedure as in **Step 1 ($\theta = 1$)**, taking expectation on both sides of (5.3.10) gives

$$\begin{aligned}\mathbb{E}[|\ \bar{X}_n\ |^2] &\leq 3^n \cdot \mathbb{E}\left[\left|\prod_{j=1}^{2n} Z_j\right|\right] \\ &\leq \frac{3^n}{(2n)!}\sum_{v_1=0}^{1}\cdots\sum_{v_{2n}=0}^{1}\frac{1}{2^n}\left[\mathbb{E}\left(\sum_{j=1}^{2n}(-1)^{v_j}\beta_j \cdot V_j^H\right)^{2n} + \left(\sum_{j=1}^{2n}\alpha_j\right)^{2n}\right] \\ &=: \widetilde{I}_3 + \widetilde{I}_4\,.\end{aligned}$$

We can prove that the term $\widetilde{I}_3$ converges to 0 by the same technique used for the term $\widetilde{I}_1$ in **Step 1 ($\theta = 1$)**. For the term $\widetilde{I}_4$, we further have

$$\begin{aligned}\widetilde{I}_4 &\leq \frac{3^n}{(2n)!}\cdot\frac{2^{2n}}{2^{n+1}}\Big(\sum_{j=1}^{2n}\alpha_j\Big)^{2n} \\ &\asymp \frac{6^n}{\sqrt{2\pi\cdot 2n}\left(\frac{2n}{e}\right)^{2n}}\cdot(2n)^{2n}\left(\frac{1-\theta}{\theta}\cdot\frac{\bar{\lambda}}{\lambda}\right)^{2n}\,.\end{aligned}$$

Obviously, we should require that

$$\sqrt{6} \cdot e \cdot \frac{1-\theta}{\theta} \cdot \frac{\bar{\lambda}}{\lambda} < 1 \tag{5.3.11}$$

to ensure $\widetilde{I}_4 \to 0$ as $n \to \infty$. The inequality (5.3.11) is equivalent to $\theta > \frac{\sqrt{6} \cdot e \cdot \frac{\bar{\lambda}}{\lambda}}{\sqrt{6} \cdot e \cdot \frac{\bar{\lambda}}{\lambda} + 1} \geq \frac{\sqrt{6}e}{\sqrt{6}e+1} \approx 0.87$ since $\bar{\lambda} \geq \lambda$. This completes the proof of Theorem 5.1.6. □

5.4 Numerical Experiments

We shall carry simulations for the following three equations.

Example 5.4.1. Consider the following linear SDEs driven by fBm

$$dX(t) = -\lambda \cdot \kappa \cdot t^{\kappa-1} X(t)dt + \mu X(t)dB^H(t)\,, \tag{5.4.1}$$

with initial value $X(0) = 3$.

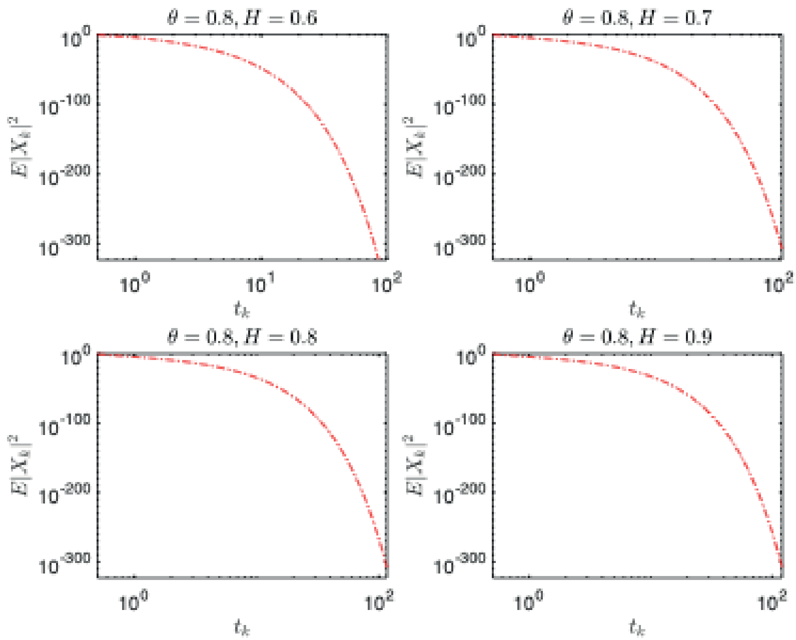

Figure 5.1: $\theta = 0.8$ for Example 5.4.1 with $\lambda = 9, \mu = 2$.

Example 5.4.2. Consider the following nonlinear SDEs driven by fBm

$$dX(t) = -\lambda \cdot \kappa \cdot t^{\kappa-1} X(t) - X^3(t)dt + \mu X(t)dB^H(t)\,, \tag{5.4.2}$$

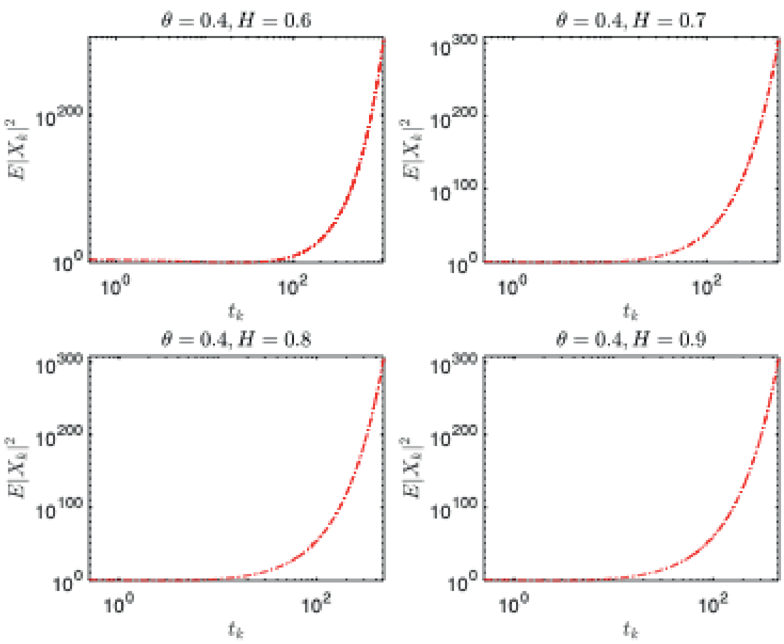

Figure 5.2: $\theta = 0.4$ for Example 5.4.1 with $\lambda = 9, \mu = 2$.

with initial value $X(0) = 3$.

Example 5.4.3. Consider the following nonlinear SDEs driven by fBm

$$dX(t) = -\lambda \cdot \kappa \cdot t^{\kappa-1} X(t) - X^3(t)dt + (X(t) + \sin(X(t)))dB^H(t)\,, \tag{5.4.3}$$

with initial value $X(0) = 3$.

For the first test, we first fix $\lambda = 9, \mu = 2, \kappa = 2H$ and apply the stochastic theta method with $\theta = 0.8$ and $\theta = 0.4$ for Example 5.4.1 with different Hurst parameters H, respectively. We take the stepsize $\Delta t = 0.5$, and the mean square of the numerical solutions over 5000 fBm samples are displayed in Figure 5.1 and Figure 5.2 on a log-log scale, respectively. Besides, we also let $\kappa = 2 > \frac{3}{2}$, and the corresponding results of numerical solution are shown in Figure 5.3. The expected stable and unstable behaviors verify our theoretical results.

For the test of Example 5.4.2, we choose $\lambda = 3$, $\kappa = 2$, $\mu = 4$ and $H = 0.6$. It is easy to verify that the coefficients of the equation satisfy (5.1.10) and (5.1.12) in Assumption 1. We take $\theta = 1$ and $\Delta t = 0.5$ and 1, respectively. The mean square of the numerical solutions are displayed in Figure 5.5. As expected, the stable behavior of the numerical

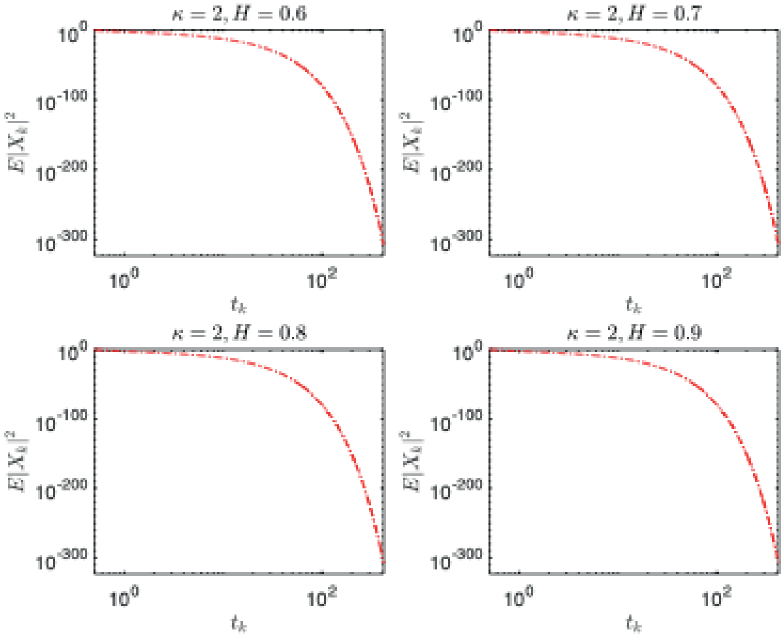

Figure 5.3: $\theta = 0.6$ for Example 5.4.1 with $\kappa = 2, \lambda = 9, \mu = 2$.

solution is in agreement with Theorem 5.1.6.

Moreover, we consider a more general diffusion term $g(t, X) = X(t) + \sin(X(t))$ instead of $g(t, X) = c(t)X(t)$ with $c(t) \leq \mu$, in Example 5.4.3. We take $\lambda = 3$, $\kappa = 2$, and $H = 0.8$. Figure 5.6 depicts the mean square of the numerical solutions with $\Delta t = 0.5$ and 1. The numerical results illustrate that the STM with $\theta = 1$ is also stable in this case. We hope the numerical results can shed some light on the asymptotic property of the solution of the nonlinear equations $dX(t) = f(t, X(t))dt + g(t, X(t))dB^H(t)$.

5.5 Concluding remarks

This work first focuses on the mean square stability of the stochastic theta method for the time non-homogeneous linear test equation driven by fBm,

$$dX(t) = -\lambda\kappa t^{\kappa-1}X(t)dt + \mu X(t)dB^H(t)\,, \quad X(0) = 3\,,$$

whose solution is stable in mean square sense. For $\kappa \geq 2H$, it is proved that the mean square A-stability of STM (5.1.14) is achieved for $\theta \geq \frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e + 1}}$, and the stochastic theta method cannot preserve the stability property of the test equation for $\theta < 0.5$ in the sense

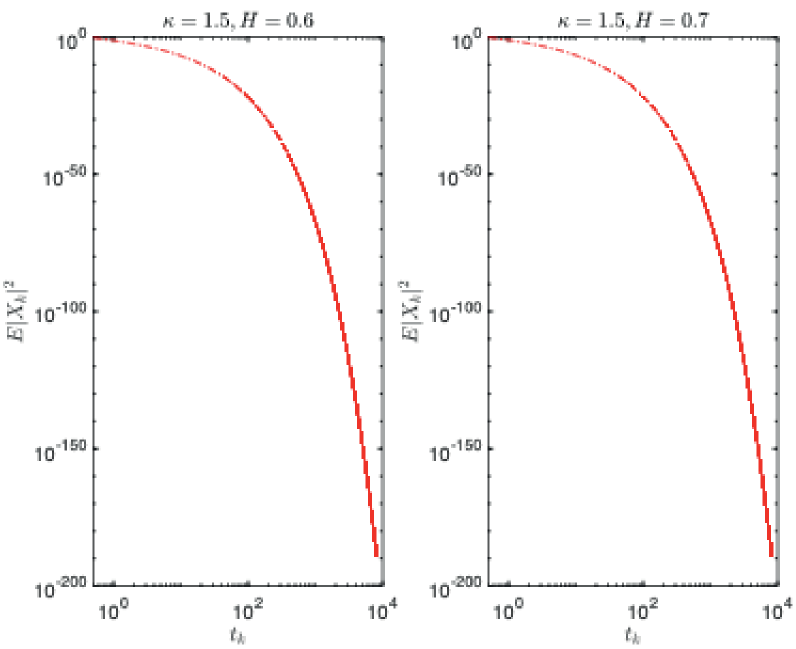

Figure 5.4: $\theta = 0.5$ for (5.4.1) with $\lambda = 9, \mu = 2$.

of almost surely and mean square. Moreover, if $\kappa > 3/2$ and $\frac{1}{2} < \theta \leq 1$, then the STM (5.1.14) is mean square stable for the above test equation. To illustrate our theoretical results, we give some simulation results for the equation (5.4.1) with $\lambda = 9, \mu = 2, \Delta t = 0.5$ with different H and θ. The simulation results agree well with our theoretical claims. On the other hand, we currently are not able to use our methods to deal with the case $\frac{1}{2} \leq \theta < \frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e + 1}}$ when $2H \leq \kappa \leq 3/2$. In this case, we simulate the equation (5.4.1) with $\lambda = 9, \mu = 2, \kappa = 1.4$ to test the stability by applying the stochastic theta method with $\theta = 0.5$, $\Delta t = 0.5$ over the time interval $0 \leq t \leq 2^{13}$, the numerical results in Figure 5.4 show that the method is still stable for $\theta = \frac{1}{2}$. Thus, we conjecture that when $2H \leq \kappa \leq 3/2$ and $\frac{1}{2} \leq \theta < \frac{\sqrt{3/2 \cdot e}}{\sqrt{3/2 \cdot e + 1}}$ the stochastic theta method is still mean square stable and this is our future research. Finally, we also study the stability of the STM for nonlinear non-autonomous case

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB^H(t)\,.$$

Under some conditions on the coefficients f and g, it is proved that the STM method is stable when $\theta = 1$. Moreover, under a stronger condition on the coefficient of drift term f, the STM method is stable when $\theta > \frac{\sqrt{6e\bar{\lambda}/\lambda}}{\sqrt{6e\bar{\lambda}/\lambda}+1}$ (where λ and $\bar{\lambda}$ are defined (5.1.10) and

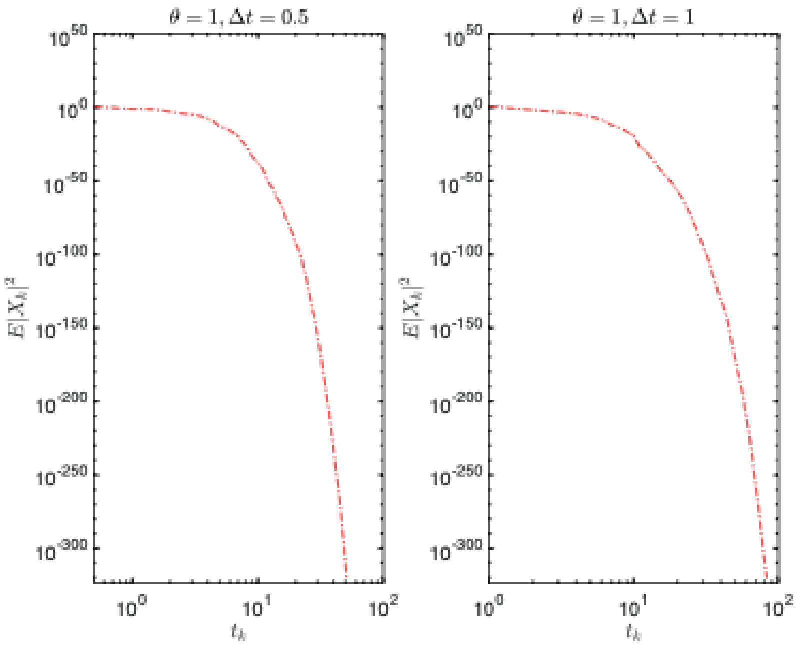

Figure 5.5: $\theta = 1$ for Example 5.4.2 (left:$\Delta t = 0.5$; right: $\Delta t = 1$).

(5.1.11) in Assumption 1, respectively).

5.6 Appendix A: Proof of $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \gg n$

In what follows, we show that $z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \geq C_H \cdot n^{2H} \gg n$ as $n \to \infty$.

By the property of fBm one can get with notation $\tilde{\beta}_j(\Delta t) := (1 - v_j)\beta_j(\Delta t)$

$$
\begin{aligned}
\tilde{\sigma}_n^2 &= \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t)\tilde{\beta}_j(\Delta t)\mathbb{E}(V_m^H V_j^H) \\
&= \frac{(\Delta t)^{2H}}{2} \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t)\tilde{\beta}_j(\Delta t) \left[\mid m-j+1 \mid^{2H} + \mid m-j-1 \mid^{2H} -2 \mid m-j \mid^{2H} \right].
\end{aligned}
\tag{5.6.1}
$$

When n and $\mid m-j \mid$ are large enough, we have

$$
\begin{aligned}
&\mid m-j+1 \mid^{2H} + \mid m-j-1 \mid^{2H} -2 \mid m-j \mid^{2H} \\
&\qquad = \mid m-j \mid^{2H} \cdot \left[\left|1 + \frac{1}{m-j}\right|^{2H} + \left|1 - \frac{1}{m-j}\right|^{2H} - 2 \right] \asymp \mid m-j \mid^{2H-2}.
\end{aligned}
$$

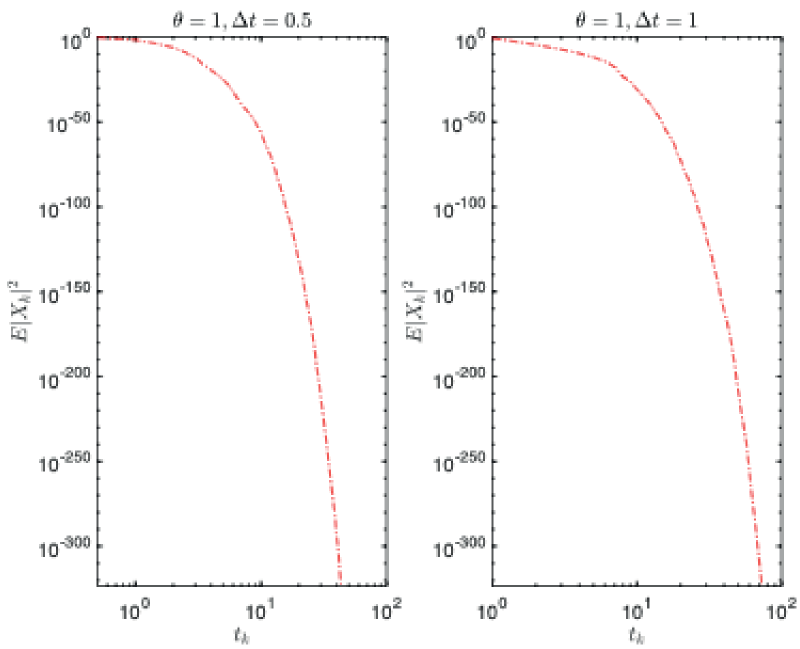

Figure 5.6: $\theta = 1$ for Example 5.4.3 (left:$\Delta t = 0.5$; right: $\Delta t = 1$).

Therefore, we can bound (5.6.1) by

$$\begin{aligned}\tilde{\sigma}_n^2 =& \frac{(\Delta t)^{2H}}{2} \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t)\tilde{\beta}_j(\Delta t) \left[| m-j+1 |^{2H} + | m-j-1 |^{2H} -2 | m-j |^{2H}\right] \\ \leq& C(\Delta t)^{2H} \sum_{m\neq j}^{n} \tilde{\beta}_m(\Delta t)\tilde{\beta}_j(\Delta t) | m-j |^{2H-2} \leq C(\Delta t)^{2H} \left(\sum_{m=1}^{n} | \tilde{\beta}_m(\Delta t) |^{\frac{1}{H}} \right)^{2H},\end{aligned} \tag{5.6.2}$$

where we have used the discrete type Hardy-Littlewood-Sobolev inequality (see e.g. Theorem 381 in [HLP88]) in the last step.

For any given $\Delta t > 0$ (and $-\lambda + | \mu |^2 < 0$), one observes that from (5.6.2)

$$\begin{aligned}\left(\sum_{m=0}^{n} |\tilde{\beta}_m(\Delta t)|^{\frac{1}{H}} \right)^{2H} &\leq \left(\sum_{m=0}^{n} \left| \frac{\mu}{1 - \kappa\theta\lambda(m+1)^{\kappa-1}\Delta t^{\kappa}} \right|^{\frac{1}{H}} \right)^{2H} \\ &= \left(\frac{\mu}{\kappa\theta\lambda(\Delta t)^{\kappa}} \right)^2 \left(\sum_{m=0}^{n} \left| \frac{1}{(m+1)^{\kappa-1} - \frac{1}{\kappa\theta\lambda(\Delta t)^{\kappa}}} \right|^{\frac{1}{H}} \right)^{2H} \\ &\leq C \left(\frac{\mu}{\kappa\theta\lambda(\Delta t)^{\kappa}} \right)^2 \cdot (n \vee 1)^{2(1-\kappa)+2H}.\end{aligned} \tag{5.6.3}$$

Thus, $\tilde{\sigma}_n^2 \leq C \cdot (n \vee 1)^{2+2H-2\kappa}$ Recall that $\tilde{\mathsf{m}}_n := \tilde{\mu}_n(0, \cdots, 0) = \sum_{i=1}^n \alpha_i$ and $\lim_{n\to\infty} \alpha_n =$

$-\frac{1-\theta}{\theta}$, we have if $\kappa \geq 2H$

$$z^2 = \frac{\tilde{\mathsf{m}}_n^2}{\tilde{\sigma}_n^2} \geq \frac{C \cdot n^2}{\tilde{\sigma}_n^2} \left(\frac{1-\theta}{\theta}\right)^2 \geq C_{\theta,H} n^{2H} \gg n \, .$$

5.7 Appendix B: Confluent Hypergeometric Functions

In this section, we gather some important properties of Kummer's confluent hypergeometric functions $\Phi(a, b, z)$ that are used in the main body of this work. The reader can find more details in Chapter 13 of [OLBC10]. Kummer's confluent hypergeometric functions $\Phi(a, b, z)$ is defined as

$$\Phi(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)2!} z^2 + \cdots \, , \tag{5.7.4}$$

where $z \in \mathbb{C}$. The following identity is called Kummer's transformation (see e.g. 13.2.29 in [OLBC10])

$$\Phi(a, b, z) = e^z \Phi(b-a, b, -z) \, . \tag{5.7.5}$$

A differentiation formula related to $\Phi(a, b, z)$ is helpful to us (13.3.20 in [OLBC10]):

$$\frac{d^n}{dz^n} [e^{-z} \Phi(a, b, z)] = (-1)^n \frac{(b-a)_n}{(b)_n} \Phi(a, b+n, z) \, . \tag{5.7.6}$$

Kummer's confluent hypergeometric functions $\Phi(a, b, z)$ can be represented by the so-called parabolic cylinder functions $U(a, z)$ (13.6.14 and 13.6.15 in [OLBC10]):

$$\Phi(a/2 + 1/4, 1/2, z^2/2) = \frac{2^{\frac{a}{2} - \frac{3}{4}} \Gamma(\frac{a}{2} + \frac{3}{4}) e^{\frac{z^2}{4}}}{\sqrt{\pi}} \times [U(a, z) + U(a, -z)] \, ; \tag{5.7.7}$$

$$\Phi(a/2 + 3/4, 3/2, z^2/2) = \frac{2^{\frac{a}{2} - \frac{5}{4}} \Gamma(\frac{a}{2} + \frac{1}{4}) e^{\frac{z^2}{4}}}{\sqrt{\pi} z} \times [U(a, -z) - U(a, z)] \, , \tag{5.7.8}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Recall the integral representation of the parabolic cylinder function $U(a,z)$ by 12.5.1 in [OLBC10]

$$U(a,z) = \frac{\exp(-\frac{z^2}{4})}{\Gamma(\frac{1}{2}+a)} \int_0^\infty w^{a-\frac{1}{2}} \exp(-w^2/2 - zw) dw \,, \qquad Re(a) > -\frac{1}{2} \,. \tag{5.7.9}$$

Lastly, the Poincaré-type asymptotic forms of confluent hypergeometric function hold (see 13.2.23 in [OLBC10]):

$$\mathbb{M}(a,b,z) = \frac{1}{\Gamma(b)} \Phi(a,b,z) \asymp \frac{z^{a-b}}{\Gamma(a)} \exp(z) \,, \qquad \text{as } z \to \infty \,, \tag{5.7.10}$$

where $\Gamma(\cdot)$ is the Gamma function.

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