University of Alberta

Opera Prima: Nonobtuse Triangulation

by

Miss Winslowe Lacesso

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NAME OF AUTHOR: Miss Winslowe Lacesso
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(Signed) W. Lacesso . . . .

Permanent Address:
Apt. 3, 10757 - 83 Avenue
Edmonton, Alberta
CANADA T6E 2E5

Date: 92-04-24

"For me there is only traveling on paths that have a heart,
on any path that may have a heart;
there I travel, and the only worthwhile challenge for me
is to traverse its full length.
And there I travel - looking, looking, breathlessly."
– *don Juan Matus*

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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled **Opera Prima: Nonobtuse Triangulation** submitted by **Miss Winslowe Lacesso** in partial fulfillment of the requirements for the degree of Master of Science.

B. Joe
(Supervisor)

P. Rudnicki

A. Liu

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Dedicated to the most respectful and inspirational
Signores *T.A.*, *A.C.*, Herren *G.P.T.*, *F.J.H.*
and especially Signore *A.L.V.*
all of whose instruction enlightened my working hours,
tempering me with lively rhythms and sweet harmonies.

Abstract

This research work attempts to outline algorithms for guaranteeing a nonobtuse triangulation, under various geometric constraints, of convex polygonal regions. To begin with, the polygonal region is simplified to consist merely of an arbitrary quadrilateral. Five cases can be identified, and at least one guaranteed nonobtuse algorithm is presented for each simplified case. In the process of doing so, some classification characteristics of arbitrary quadrilaterals are identified. Due to time constraints, the algorithms developed were not modified to have the geometric constraints reintroduced. This work only serves as a small theoretical basis for much further work that this melding of Computational Geometry and Finite Element Analysis (Engineering) requires.

Acknowledgements

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Contents

List of Figures

Chapter 1

Introduction

The focus of this research is to attempt to devise an optimal method and/or establish results about triangulating, or tiling with nonoverlapping triangles, an arbitrary convex polygon such that no angles in the triangulation exceed 90°, that is, that all angles are *nonobtuse*. Right angles are valid nonobtuse angles, but for reasons outlined later, a distinction is made between the desirability of *right* versus *acute* angles.

Ideally, as well, a number of constraints in addition to nonobtuseness are to be satisfied by the triangulation method(s) developed, or at least as many of them are to be satisfied as possible. During development of solution approaches, most of the constraints are dropped in the process of simplifying the problem. The idea was to reintroduce the constraints and refine the solution method(s) to take them into account, so that the nonobtuse triangulation satisfies the necessary constraints; however, due to time constraints, that was not accomplished.

The main problem addressed in this work is the simplified problem of developing an optimal guaranteed method for a nonobtuse tiling of an arbitrary convex quadrilateral; the criterion of "optimality" is interpreted as that of using as few added vertices and edges as possible in the guaranteed tiling method. An additional criterion which is strongly desired, but not demanded, is that the placement of added vertices be as flexible as possible while still guaranteeing a nonobtuse tiling of a given quadrilateral. Quadrilaterals are considered to fall into cases of having zero, one, two opposite, two adjacent, or three obtuse angles.

This research focusing on arbitrary convex quadrilaterals has shown even the highly simplified instance to be much more complex than expected. In brief, the results of this thesis are a few classification characteristics of the quadrilateral cases mentioned above, and a few simple results, or tools, that may prove useful in later research, plus at least one guaranteed tiling method for each of the cases of quadrilaterals.

This thesis is organized in the following manner:

Chapter 2 contains necessary preliminaries in the way of notation and definition standards, and a few basic lemmas, that are used throughout this work.

Chapter 3 explains the motivation behind triangulation in general, why nonobtuseness is relevant in triangulation, and how and why the issue of quadrilateral triangulation relates to Professor Joe's work on automatic triangulation programs. The statement of the simplified constrained triangulation problem as the focus of this research is restated as a number of subproblems.

Chapters 4 to 7 develop solutions to the subproblems, where an attempt is made to present an algorithm to fully solve each subproblem. At the very least, a fairly positive statement can be made about the worst case theoretical solution to the subproblems.

Chapter 8 contains one attempt to apply the theoretical solutions developed here to actual larger (non-quadrilateral) triangulation problem (taken from [J86]), where the solution methods (as developed in this research) had to be "stretched" in application, to accommodate problem areas for which the solution methods were not designed. Chapter 8 concludes with a brief summary of the main results, some proposals for further research, and some final comments and observations.

Chapter 2

Preliminaries

In this chapter some notation and definitions are introduced, used in descriptions throughout this work, and some basic lemmas are stated that can be used in devising solutions and proving them correct.

Definitions do not change in any part of this work; however, some notational changes are made later on, for convenience. They are found in the section to which they pertain.

2.1 Definitions

Let θ denote the measure of an angle, in degrees.

Define:

- *acute* angle: $0° < \theta < 90°$.
- *right* angle: $\theta = 90°$.
- *nonobtuse*: $0° < \theta \leq 90°$.
- *obtuse* angle: $90° < \theta < 180°$.
- *straight* angle: $\theta = 180°$.
- In an *obtuse* triangle, the maximum of the three angle measures is an obtuse angle.
- In a *right* triangle, the maximum of the three angle measures is a right angle.
- In a *nonobtuse* triangle, no angle exceeds $90°$.
- In an *acute* triangle, no angle exceeds or equals $90°$.
- The (x, y) coordinates of a point E are denoted by (E_x, E_y).

- A *simple* polygon has no internal interfaces ("holes") or edges that cross each other.

2.2 Notation

- *Vertices* are denoted by capital letters, such as A, B, H, etc., regardless of whether they are *permanently* added to a polygon, or used only as *temporary markers*. An edge or line segment joining vertices A, B is denoted as AB.
- *Lengths* of edges are denoted by lower case letters, e.g., $len(MP) = |MP| = r$.
- *Angle measures* are denoted both by naming three vertices that define the angle, prefixed with the symbol '$\angle$', and by lower case Greek letters, e.g., angle $\angle ABC = \beta$, angle $\angle CAB = \alpha$, angle $\angle BCA = \gamma$. All angle measures are in degrees.
- A partitioned angle has subangles denoted by the angle measure with a subscript, i.e. corner C of a polygon Q with angle measure γ: adding edges to corner C (thus partitioning the angle) creates angles γ_1, γ_2, etc.
- *Regions* are denoted by calligraphic letters, e.g., $\mathcal{F}, \mathcal{H}$. Regions can consist only of edges or lines, or part of edges or lines; or of an area defined by curves and lines or edges, either including the boundary curves and lines/edges, or not.
- *Edges* actually added to the interior of a polygon are solid; edges or lines used within a polygon to delimit areas, or as temporary edges (i.e., altitudes) are of various kinds of dashes.
- *Semicircles* or *semidisks* are, in the context of this work, always defined by a line segment or edge that is taken to be the diameter of the curve. A semicircle is denoted by $\ominus AB$, where an edge or line segment with endpoints A and B is taken as the diameter of the semicircle. A semidisk is denoted by $\odot AB$, where A and B are similarly taken as the diameter endpoints. A semicircle $\ominus AB$ consists only of the curve itself, whereas a semidisk $\odot AB$ consists of the curve *plus* the area enclosed between the curve and the diameter AB.
- The boundary edges of a polygon P are collectively called ∂P.

2.3 Lemmas for General Use

The following lemmas will be useful for later chapters on the nonobtuse triangulation of quadrilaterals.

Lemma 1: Let $\triangle ABC$ be an obtuse triangle with obtuse angle $\angle ACB$. Adding altitude DC, where $D \in AB$, partitions obtuse angle ACB into two acute angles, and partitions $\triangle ABC$ into two right triangles, $\triangle ACD$ and $\triangle BCD$.

Proof: Obtuse angle $\angle ACB$ is partitioned into angles $\angle ACD$ and $\angle BCD$, which are both acute, since $\angle DAC + \angle ACD = 90°$ in triangle $\triangle ACD$, and $\angle DBC + \angle BCD = 90°$ in triangle $\triangle BCD$. □

Lemma 2 (the Semicircle Rule): Angle $\angle ACB$ of $\triangle ABC$ is obtuse, right, or acute, if C is inside, on, or outside, respectively, the circle with diameter AB. (See Figure 2.3.1.)

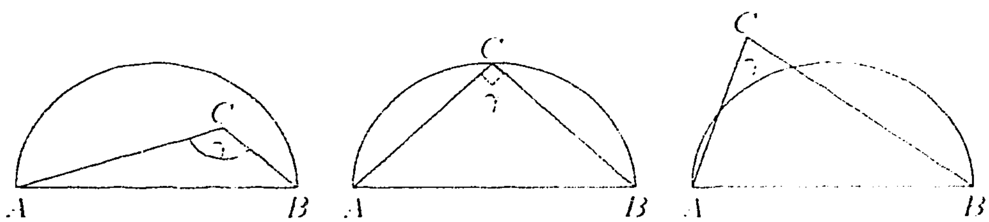


Figure 2.3.1: γ is obtuse, right or acute, if C is inside, on, or outside the semicircle, respectively.

Proof: Let $\angle ACB$ be an inscribed triangle in a circle, as in Figure 2.3.2, left illustration. A well known theorem of plane geometry states that the measure γ of the inscribed angle is $\gamma = \frac{s}{2}$ where s = the measure, (*not* length) of arc AB. Thus when AB = a diameter of a circle and vertex C of $\triangle ABC$ lies *on* the circle, $s = 180°$, so $\gamma = 90°$. (See Figure 2.3.2, right illustration.)

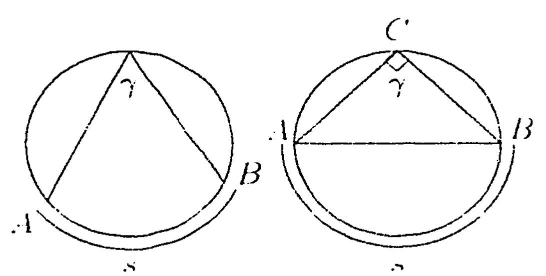


Figure 2.3.2: C on semicircle $\widehat{AB}$, so $\gamma = s/2 = 90°$

For the case where C is *inside* the semicircle, no matter where inside the semicircle C lies, we can extend the line AC so that it intersects the semicircle at point C''. See Figure 2.3.3, left illustration. Since C'' is *on* the semicircle, angle $\angle AC''B = 90°$.

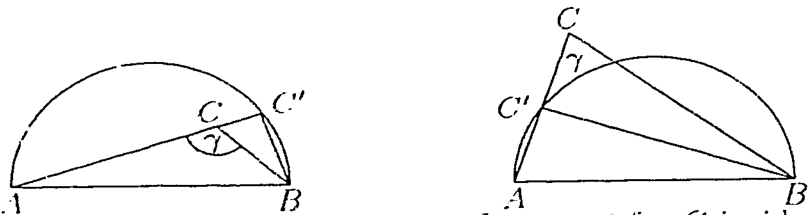


Figure 2.3.3: Illustrations to prove Lemma 2 for C inside or outside semicircle

Then since angles $\angle CAB$ and $\angle C''AB$ are equal, and angle $\angle CBA < \angle C''BA$, it follows that $\gamma = \angle BCA > \angle BC''A = 90°$, so γ is obtuse.

Similarly, when C lies *outside* the semicircle, let C'' be the point where AC intersects the semicircle. See Figure 2.3.3. right illustration. Again, $\angle CAB = \angle C''AB$ but now $\angle CBA > \angle C''BA$; then $\angle AC''B = 90° > \angle ACB = \gamma$; so γ is acute. □

Lemma 3 (The Right Angle Bound Rule): Suppose AD and DC are edges of a convex polygon, and the angle δ at D is obtuse or straight. Extend two lines from D inside the polygon that form right angles with AD and DC. These lines will be referred to as *Right Angle Bound* lines. A second vertex E existing or placed anywhere such that it is *bracketed* by the Right Angle Bound lines, and joined to D, is guaranteed to be positioned such that the two subangles formed by partitioning obtuse angle δ are both *acute*. If E lies *on* one of the Right Angle Bound lines, one of the subangles is a right angle, and the other is acute if $\delta < 180°$, or a right angle if $\delta = 180°$. If E lies *outside* one of the Right Angle Bound lines, one of the subangles is obtuse, and the other is acute. If δ is a *straight* angle, the Right Angle Bound lines coincide to form a perpendicular line $DV = DU$ to AC at D; in this case, if E lies *on* $DV (= DU)$ then both subangles are right angles; in any other location, joining E to D creates one obtuse and one acute angle.

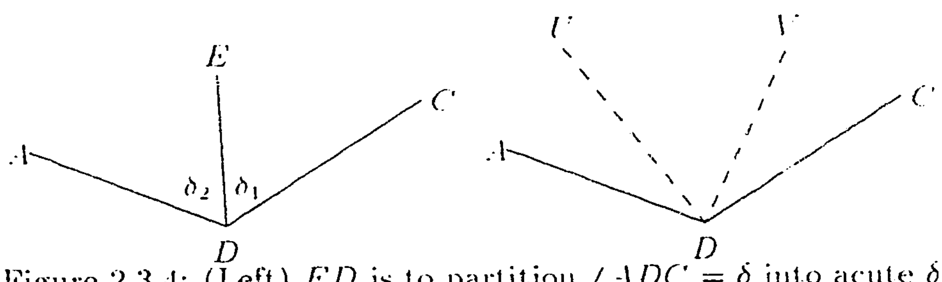


Figure 2.3.4: (Left) ED is to partition $\angle ADC = \delta$ into acute δ_1 and δ_2; (Right) Right angle bounds DU, DV such that $\angle ADV = \angle CDU = 90°$

Proof: Joining E to D will partition angle $\angle ADC = \delta$ into two angles; let the right one be angle $\angle CDE = \delta_1$, and the left one be angle $\angle ADE = \delta_2$. See Figure 2.3.4, left illustration.

Form Right Angle Bounds DV, such that angle $\angle ADV = 90°$, and DU, such that angle $\angle CDU = 90°$. See Figure 2.3.4, right illustration.

If E is placed between DA and DU, outside the Right Angle Bound line DU - for example at position E_1 in Figure 2.3.5, left illustration - then $\delta_1 > \angle CDU = 90°$, so δ_1 is obtuse. Then $\delta_2 = \delta - \delta_1$, and since $\delta \leq 180°$, δ_2 is acute. Similarly, placing E between DC and DV, outside the Right Angle Bound line ADV - for example, at position E_2 in Figure 2.3.5, right illustration - makes $\delta_2 > \angle ADV = 90°$, so δ_2 is obtuse. Then $\delta_1 = \delta - \delta_2$, and since $\delta \leq 180°$, δ_1 is acute.

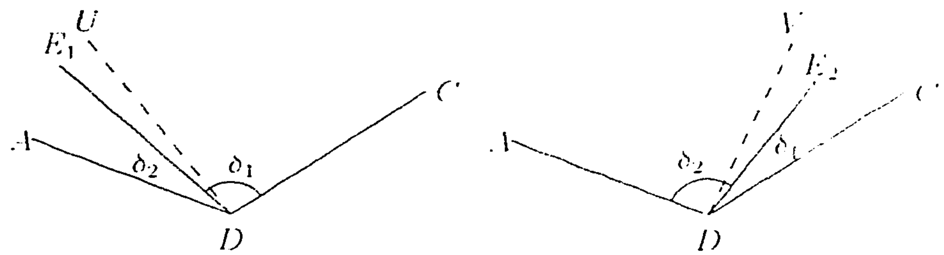


Figure 2.3.5: E_1 to left of $DU \Rightarrow \delta_1$ obtuse; E_2 to right of $DV \Rightarrow \delta_2$ obtuse

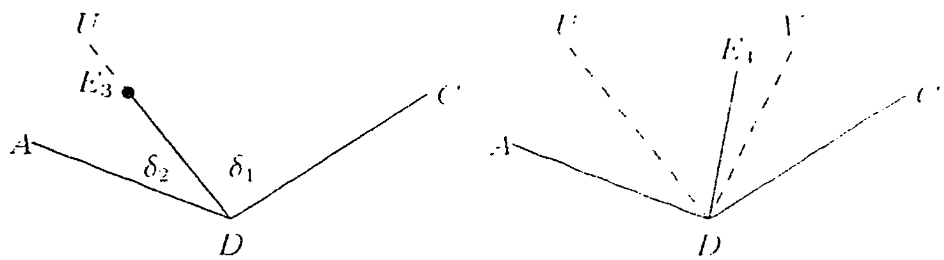


Figure 2.3.6: Left: E_3 on Right Angle Bound line $DU \Rightarrow \delta_1 = 90°$.
Right: E_4 bracketed by $[U, V] \Rightarrow$ both δ_1, δ_2 acute.

Placing E on line DU - for example, at E_3 in Figure 2.3.6, left illustration - makes $\delta_1 = \angle CDU = 90°$. Then, if δ is a straight angle, it must be that $\delta_2 = 90°$ too; otherwise, $\delta_2 = \delta - \delta_1 < 90°$, that is, acute. The case for placing E on the DV Right Angle Bound line is similar: E on $DV \Rightarrow \delta_2 = \angle ADV = 90°$ and δ_1 is acute.

Finally, placing E such that it is bracketed by the Right Angle Bound lines - for example, at E_4 in Figure 2.3.6, right illustration - guarantees that $\delta_1 < \angle CDU$ and $\delta_2 < \angle ADV$; so both are acute. □

In using Right Angle Bounds, the property of "bracketing" may be viewed in two ways. Firstly, that the Right Angle Bounds intersect the edges ∂P of a polygon, as in Figure 2.3.7: in this case, *regardless of the orientation of the polygon*, the part of the boundary ∂P considered to be enclosed by the Right Angle Bounds is found by intersecting ∂P with the Right Angle Bound lines. For example, in Figure 2.3.7, left illustration, the Right Angle Bounds are $\angle ADV$ and $\angle CDU$, equivalent in meaning to Right Angle Bound *lines* DU, DV; then $\partial ABCD \cap$ (sector VDU) $= [V, U] = [U, V]$ on edge BC. Similarly in the right illustration in Figure 2.3.7, intersecting ∂P with the Right Angle Bounds DU, DV is understood to be $[U, V] = [V, U]$ on edge AB, NOT $[V, U] =$ segments $VB + BC + CD + DA + AU$ of ∂P.

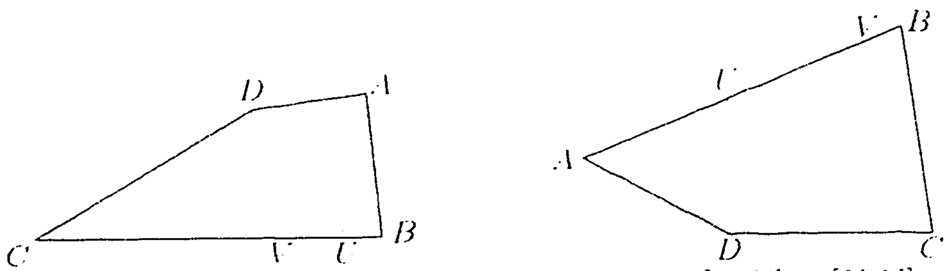


Figure 2.3.7: Left: $\angle ADV \cap \angle CDU \cap ABCD = [V, U] = [U, V]$ on edge BC. Right: Similar interaction/intersection: $[U, V] = [V, U]$ on edge AB, despite the different orientation of the polygon.

Another way to "view" or use the Right Angle Bounds is to consider P to be oriented with one edge taken as a horizontal "base", and consider the Right Angle Bound "mark" at its intersection with the base. See Figure 2.3.8. This is a more useful method in algorithms, where detecting if a vertex (like B) is bracketed by Right Angle Bounds (say, DU and DV) becomes a matter of comparing x- or $y-$coordinates. In Figure 2.3.8, right illustration, the comparison $V_x, U_x < B_x$, showing that B is not "bracketed" by Right Angle Bounds DU, DV is more easily detected (computationally) than discerning that DU, DV both intersect side BC of P, and so do not bracket B. In Figure 2.3.8, left illustration, having oriented P so edge AB is a base, the comparisons $B_x < Y_x, X_x < A_x$ easily show that Right Angle Bounds CY, CX enclose an interval $[Y, X]$ on edge AB; likewise, the comparisons $V_y < B_y < U_y$ show that DU, DV "bracket" corner B.

Lemma 4: In a convex, simple polygon of four or more sides, any vertex followed by two vertices both with nonobtuse angles, has an altitude existing from the first vertex to the edge between the two following nonobtuse-angle vertices.

Proof: See Figure 2.3.9. Let vertices D, C and B exist in that order in a polygon, with angle δ followed by nonobtuse angles γ and β, respectively. If γ is

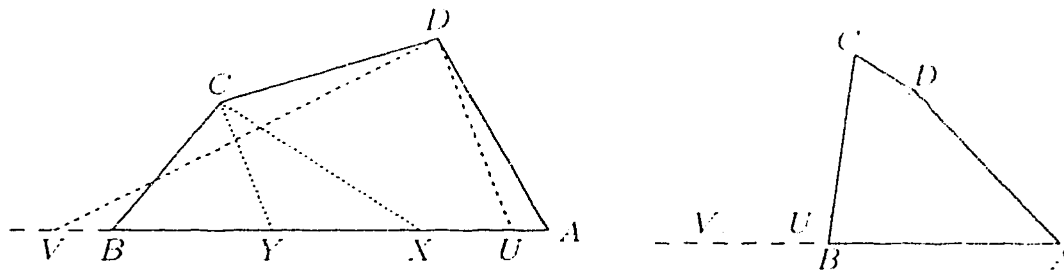


Figure 2.3.8: Left: Right Angle Bounds $[Y, X]$ from C and $[V, U]$ from D.
Right: $V_x, U_x < B_x \Rightarrow B$ not "bracketed" by Right Angle Bounds DU, DV.

a right angle, the altitude is trivially DC. If γ is acute, it should be possible to project CD onto CB, and this projection is the basepoint of the altitude from D to CB. If CD *cannot* be projected onto CB, it must be because edge CB is too short. But then in following the rest of the polygon edges from B back to D, either β is obtuse, or the polygon is not convex, or edges cross other edges. But none of those is the case. Therefore, the projection of CD onto edge CB, and hence the altitude from D to edge CB, does exist. □

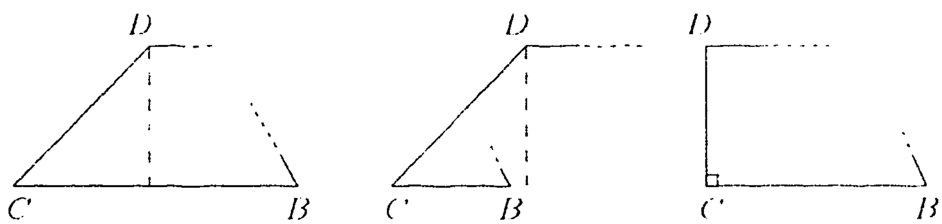


Figure 2.3.9: (left) Project CD onto CB; (centre) If the projection doesn't exist, then since β is acute, tracing the path from B back to D must mean the polygon is nonconvex, OR that angle β must be obtuse.
(Right) The altitude basepoint is trivially C when $\gamma = 90°$.

Chapter 3

The Research Problem

In this chapter an overview is given of motivation for the triangulation problem in general, and some current theory and approaches to solutions: a brief explanation follows of the larger problem of triangulation (or *tiling*) of convex regions. from which this research problem is derived: the refinement to the problem of tiling convex regions, of insisting that all angles be nonobtuse, is then presented, followed by a simplification and restatement of the research problem.

3.1 Triangulation Problem Overview

The finite element method for solving partial differential equations seeks approximations to the solution of the PDE at the vertices of a mesh covering a polygonal area in which the solution function is defined. (A "polygonal area" may have internal interfaces ("holes"), whereas "polygon" refers to a region without internal interfaces.) Quadrilaterals and triangles are two common mesh elements used to tile polygonal areas. Compared to quadrilaterals, triangles are more flexible in filling areas that have highly irregular boundaries (with respect to edge length and change of direction), and likewise for openings within the polygonal region ([L89]). Triangles also are more suitable because they are simplices in two dimensions: that is, they are the simplest space-filling configurations into which their spaces can be partitioned ([F87]).

It is well known that every polygon can be dissected into nonobtuse triangles ([M60]). However, the triangulation. to be useful in engineering problems, must satisfy a number of constraints that greatly complicate the triangulation problem. If vertices are allowed on the sides of triangles (as in Figure 3.1.1). then the existence of a nonobtuse triangulation is obvious and a nearly equilateral triangulation is possible ([G84]). Unfortunately, one of the constraints is that vertices cannot be allowed on the sides of triangles.

A wide range in angle sizes within the triangulation can affect the computations for which the triangulation is intended ([T80]). Error analyses indicate that when

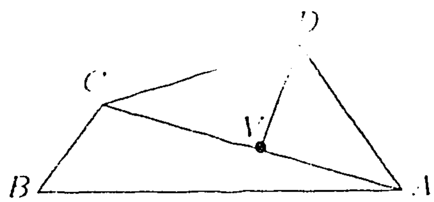


Figure 3.1.1: Illegal triangulation: V on side CA of $\triangle CAB$.

using triangular elements to tile a region, the greatest accuracy (of the solution of the finite element problem) is obtained when all angles are nearly equal – that is, all triangles are nearly equilateral triangles. Not only angle sizes are to be as nearly equilateral as possible (in a really good triangulation), but within a given subregion of the domain to be tiled, the triangles should be all of approximately the same size. This is because the boundary edges are viewed "as defining a series of length scales for the triangulation, in the vicinity of the boundary" ([JS86]). Very small boundary edge lengths imply very small triangular elements to be used in that region; where the boundary edges are long, a more "comfortable" triangular element size can be chosen to help the triangulation meet other criteria (such as the number of triangular elements requested to be present in the triangulation, or a maximum triangle size).

It used to be thought that avoidance of small angles was necessary for convergence of the finite element method, but this was found not to be so ([BA76]). Nevertheless, small angles may lead to ill-conditioned matrices (encountered in the finite element method), so it is best to avoid them ([BGS88]). What has been shown to be essential is that no angle in the triangulation be too close to 180°. Since significant changes in triangle size across subregion boundaries can cause small angles in the triangulation ([JS86]), such size changes along boundaries of adjacent subregions must be limited. A gradual change in triangular element size throughout the triangulated region is thus necessary if the polygonal region has highly irregular boundaries; this type of mesh, in which not all triangles are of the same size, is known as a *graded* mesh.

A great deal of work has been done in the areas of controlling triangle size and shape, both in an initial phase of mesh generation, and in later phases of mesh *smoothing* where both interior and boundary edge mesh vertices may be subtly adjusted in position to improve the triangulation overall (that is, make it better satisfy certain criteria). Very little work has been addressed to the problem of controlling *angle* size within the mesh, and almost no research has been dedicated to *guaranteeing* angle sizes within a mesh. The present research work concentrates *only* on the phase of mesh generation, concentrates primarily on guaranteeing nonobtuse angles, and only secondarily on controlling triangle shape and size.

Within each partition of a polygonal area P (which may include part of the boundary of P), the tiling element size is a function of the node spacing (distance between neighbouring nodes) or node density (nodes per unit length or area) ([F87]). The node spacing may be determined by an explicit function supplied by a user, or may be implicitly defined by the variation in edge lengths ([F87], [JS86]). Where the spacing function is implicitly defined, "the given boundary data control the spacings on the boundary, and this in turn controls the element size in the interior of the domain" ([F87]). Thus the number of vertices added, both on the boundary of P and in the interior of P, is determined by the spacing function, whether explicitly or implicitly defined. The spacing and number of added vertices on the boundary of P is highly dependent on the complexity characteristics of the edges comprising the boundary of P, and in the vicinity of the boundary, the spacing and number of added interior vertices is likewise controlled by the boundary characteristics. Further within the interior of P, however, the shapes and sizes of the final triangles can be more independent of the boundary characteristics, and a freer hand can be used to control their shape and size, although it must be kept in mind that "the shapes of the final triangles, and the resulting 'quality' of the mesh, depend on the method used to generate prospective nodes, and the various tests to which they are subjected" ([F87]). In most previous work, the objectives of "good" interior node placement strategies have been triangle size and shape, with no or little regard for resulting angle sizes.

Discussion in other papers (i.e., [F87]) has looked at the sensitive interaction between the spacing function and the resulting mesh quality, as "the minimum value of the spacing function ... controls the node density in the most refined portion of the mesh" ([F87]).

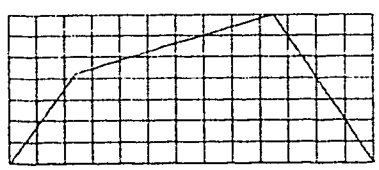


Figure 3.1.2: Overlay of rectangular grid: add diagonal to each rectangle, then deal with edge cases.

A number of schemes have been developed that use the idea of overlaying a rectangular grid onto P (or onto a partition of P, where different sizes of grids are used to vary the triangle size in different partitions of P), and taking the interior grid points as the mesh points of the triangulation (see Figure 3.1.2). Each rectangle entirely within P is easily tiled nonobtusely by simply adding a diagonal; the real difficulties arise along the boundary, where rectangular elements are only partly within P; more difficulties arise where the boundary is very complex, with a wide variety of edge lengths. Nonetheless, this method *has* been "massaged" into a guaranteed all-nonobtuse triangulation method ([BGS88]), where edge cases (along the boundary) are laboriously dealt with, although successfully. However, it proves to be nonrobust with regard to internal interfaces in P. Also, small angles are created near the boundary of P with the approach taken by [BGS88]. Although *large* angles have proven to be the more pertinent concern, the problems (matrix ill-conditioning) caused by small angles cannot be ignored.

3.2 Convex Polygon Triangulation Problem

As mentioned before, [BGS88] have presented a guaranteed method for nonobtuse tiling of simply connected polygons, by using a rectangular grid overlay and then dealing with edge cases. Also, [C89] has presented a method of guaranteeing a tiling with angles in $[30°, 120°]$ that incorporates the notion of a regular *or* irregular grid overlay. However, [C89]'s solution requires conditions on the input, making it less general and less robust than could be wished. Specifically, a mathematical relation must exist amongst the boundary edges of the polygonal region P for [C89]'s method to work, whereas more robust methods make no such requirement of the input polygon P.

In addition to the lack of stringent requirements on P like the above, a really general and robust triangulation scheme should be "able to maintain the greatest possibility of mesh patterns, i.e. the number of elements around a node is not fixed and the relative positions of the nodes are not predetermined by mathematical formulae" ([L85]). According to the same author, any regular grid scheme cannot efficiently meet the above requirements *and* handle complex and/or irregular boundaries (including interior interfaces), and so "the concept of superimposing a rectangular grid has to be abandoned completely" ([L85]).

This is not to say that placing vertices (either on the boundary or within P) in any sort of uniform pattern is futile or degrades the quality of the generated mesh. In [J86] a *quasi-uniform* mesh is used in the interior of a convex polygon to guide the placement of interior vertices, allowing fair flexibility of placement while still guaranteeing triangle shape and size *and* no obtuse angles (strictly within an interior region of P). In fact for well-shaped nearly-equilateral triangles, at least a quasi-uniform grid of interior vertices is no less than expected.

The area to be triangulated may be convex or nonconvex; it may be *complex*, where [JS86] identifies several properties of the boundary that contribute to 'complexity'. One is a sizable variation in the lengths of the boundary edges of the area, as the range of lengths in the input boundary edges affect the uniformity of triangle size in the resulting triangulation ([J86]). If the polygon to be triangulated is very complex, or even just nonconvex, a simplifying approach is to partition the polygon in several convex subpolygons and then triangulate each subpolygon ([J86]).

Not just any tiling of the polygonal region will do. Some triangulation algorithms can require time-expensive checking for overlap of a (potential) triangle with an already-existing (inserted) triangle, or with the polygonal boundary. "By choosing (an appropriate) suitability criterion, the overlapping check could be complete avoided, and the boundary check limited to the most suitable of the potential elements. The 'max-min' angle criterion is the best of such criteria, and the resulting triangulation is the famous Delaunay triangulation" ([L89]). (For a definition of a Delaunay triangulation, see [PS85].)

Thus the *Delaunay triangulation* is found to be a very useful triangulation, so many efforts in automatic triangulation programs have been directed to that end ([PS85], [L89]). However, the Delaunay triangulation is defined for the convex hull of a domain. Therefore, nonconvex domains have to be first decomposed into simpler convex subregions ([L89]). Even if the region is already convex, if it is complex (and hence difficult to triangulate), decomposing the region into several simpler convex subdomains can greatly facilitate both triangulation itself and the grading process ([F87], [J86], [JS86]).

To be useful in engineering problems, the resulting triangulation must satisfy a number of constraints that greatly complicate the process of triangulation.

- A *minimum spacing* of added edge and interior vertices is imposed. The minimum spacing is either a function defined implicitly by the lengths of the input edges, or user-supplied.

- (related to above) A minimum spacing is also required between any added interior vertex and any input or added edge of P. This is to prevent a "narrowing" within P, as a small distance between an interior vertex of P and any edge of P effectively "defines" an edge of (small) length to take into account with respect to the spacing function ([J86]).

- Nearly equal or (optimally) equal triangle size, where the size is based on the spacing function.

- Nearly equal angles in all angles created in the tiling of P (with allowance made for the fact that the angles defined by ∂P cannot be other than left alone or optimally partitioned).

- No vertices existing on the side of a triangle in the tiling. Any two triangles in the tiling should intersect along one whole side, in a single point, or not at all ([BGS88]).

The *minimum spacing* and *triangle size* constraints together set a rough ceiling on the number of triangles that can exist in a tiling. However, the time complexity of a tiling algorithm is minimally equal to the number of edges in the tiling that the algorithm must report; therefore, a strong interest lies in tiling with the *minimum* number of triangles.

This also implies minimizing the number of added vertices on any edge and in the interior, since any vertex added onto ∂P must have an added edge (within P) connected to it, and any vertex added to the interior of P must have at least four edges connected to it to ensure nonobtuse angles at that vertex.

Therefore, "derived" constraints are

- Minimize the number of triangles used to tile P.
- Minimize the number of added vertices on any edge or within the interior. (Note: henceforth, "edge vertex" means "a vertex added on an edge", and "interior vertex" means "a vertex added to an interior area".)

There are a great many algorithms and variants of methods on tiling of convex regions. Within the larger (or "real") problem of satisfactorily tiling a convex region, is a refinement of the problems in insisting that all angles in the resulting tiling be nonobtuse. The use of nonobtuse tiling and the adjustments to the above tiling constraints that are required to incorporate nonobtuseness are outlined in the next section.

3.3 The Simplified Problem

If there are no obtuse angles in the triangulation, then the matrices encountered in the finite element method possess properties that are important with respect to the analysis of iterative methods for solving the linear system of equations in the finite element method ([BGS88]). Also, some techniques within the finite element method require the centre of the circumcircle of each triangle to satisfy a certain property which is true if and only if no angle is greater than 90° ([BEG90]). As well, nonobtuse triangulations are important in unstructured finite difference approximations to partial differential equations ([FF91]).

Thus nonobtuse triangulation is essential to certain techniques available within the finite element method, and this alone makes it worthwhile to incorporate guarantees of nonobtuseness into automatic mesh generation programs. Given that a polygonal region can be partitioned into convex subregions, a guaranteed method of nonobtusely tiling a convex subregion can then be made a part of the solution of nonobtusely tiling nonconvex regions.

[BE91] have recently announced the development of a guaranteed nonobtuse polygon triangulation method, which relies fundamentally on a procedure to nonobtusely tile a triangle that has any number of vertices situated on its edges. Briefly, for each triangle edge with n vertices situated on it, an interior edge is added to the triangle that "slices" off the edge containing the n vertices; the "slice" of the triangle is then tiled nonobtusely such that $n-1$ of the n vertices are projected onto the "new" interior edge of the triangle; simultaneously, the method guarantees that a *minimum* number of vertices are added as necessary to other edges of the triangle, to accommodate the nonobtuse tiling of the triangle "slice". Thus each iteration of the procedure reduces by one the number of vertices on a triangle edge, and once $n = 0$ the triangle is easily nonobtusely tiled.

This is an attractive idea; however, no constraint other than the number of added edge vertices is imposed upon the nonobtuse tiling of a polygon. As already explained, minimum spacing between edge vertices is a critical factor in ensuring smooth gradations of triangle size, and hence the reduced risk of small angles due to large changes in triangle size. In a given tiling by the method of [BE91] (by hand), even of a polygon that is merely a "nicely-shaped" triangle with one vertex on an edge, a huge variation in triangle sizes within the tiling occurs, as well as extremely small spacings, both between vertices on the polygon edges and between interior vertices and the polygon edges ("narrowing"). As a theoretical method it is quite interesting, but more work would be required to have it meet the previously mentioned triangulation constraints.

The present research took a different approach than [BE91], as the problem arose from [J86]'s work on tiling convex polygons. [J86] *shrinks* a convex polygon uniformly to obtain an interior area, $int(P)$: each boundary edge of $int(P)$ is at least a distance r from ∂P. Then $int(P)$ is triangulated nonobtusely on a quasi-uniform mesh. The strip between the boundary and $int(P)$ remains to be triangulated, and it is in this boundary strip that, unfortunately, difficulties arise in guaranteeing a nonobtuse triangulation. This research is intended to shed some light on or to solve that problem.

It may be possible to divide the boundary strip into quadrilaterals; if this can be done, and an algorithm can be developed guaranteeing that the quadrilaterals in the boundary strip can be tiled nonobtusely, then the entire convex region has been tiled nonobtusely. This would be the desired extension of [J86]'s work to guarantee a nonobtuse tiling of a convex region, which may be a partition of a larger (convex or nonconvex) region. Or, a polygonal region (convex or not) might be partitioned into a set of convex quadrilaterals, to obtain a nonobtuse tiling. Within the convex quadrilaterals to be tiled, all constraints as inherited from the parent problem must be satisfied, as well as any constraints arising from requiring all nonobtuse angles.

Initially, most constraints are disregarded to simplify the problem. However, it must eventually be remembered that to maintain approximately even triangle size within the larger convex region P, edge vertices must be added to the boundary of each quadrilateral Q according to the spacing function defined for P. In addition, where vertices must be added (in the process of nonobtuse tiling) to the edges of quadrilaterals that share edges with other quadrilaterals, those vertices must be "required" or "allowed" in both quadrilaterals, to satisfy the constraint that vertices do not exist on the sides of triangles in the final tiling. This "matching" problem was not addressed in this research due to time constraints.

Therefore, the current research problem is that of developing methods to guarantee a nonobtuse tiling of P where $P = Q$, a convex quadrilateral. At first it is assumed that Q has only the four corner vertices; solutions for this situation might then be extended to situations where Q has any number of added edge vertices, which would be considered as straight angle "corners" of $Q = P$ (no longer a "quadrilateral"). With no "shrunken" interior $int(Q)$, Q in this case would be an instance of the situation mentioned in [J86] where the shrunken area inside P, $int(P)$, is degenerate, in particular, where $int(P) = \emptyset$.

All the constraints from the problem of convex polygon tiling are inherited, with the change that the constraint:

- Nearly equal angles in all angles created in the tiling of Q.

must be amended to:

- All angles should be as nearly equilateral as possible, and no obtuse (or straight) angles should exist in the tiling; also, no angle should be smaller than $min\{$smallest angle $\in Q$, user-defined minimum angle$\}$.

As well, the nonobtuse angle requirement gives rise to a couple of other constraints. This is due to the fact that an interior degree-four vertex is limited in its usefulness with respect to flexibility of positioning. See Figure 3.3.1.

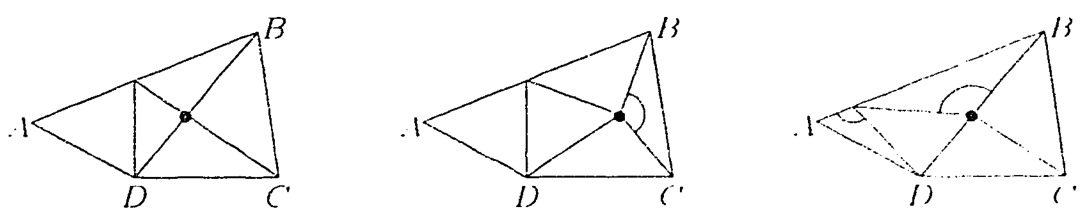


Figure 3.3.1: Degree-4 added vertex: (left) "nailed" with right angles to interior edges; (centre) if the added vertex moves with its attached edges, one or more obtuse angles result; (right) if an edge shifts, but the position of the added vertex remains fixed, obtuse angle(s) can result.

If the vertex must be moved, the right angles at the added interior degree-four vertex are lost, unless the edges connected to it can move *exactly correspondingly*. If the edges are attached to other vertices that are not moved, or even *cannot* be moved (for instance, corner vertices of Q), one or more obtuse angles result from moving the added interior degree-four vertex (Figure 3.3.1, centre illustration). Similarly, moving one of the other vertices to which the added interior degree-four vertex is connected, results in a loss of right angles at the added interior degree-four vertex, and so creates one or more obtuse angles at the added interior vertex (Figure 3.3.1, right illustration).

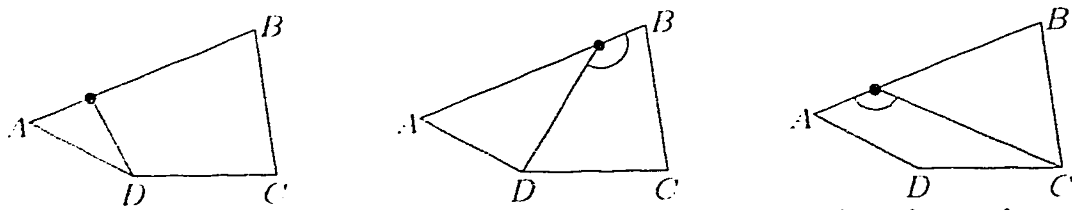


Figure 3.3.2: Degree-3 added vertex: (left) "nailed" with right angles to an edge; (centre) if the added vertex moves with its attached edge, an obtuse angle is created; also, (right) if the edge moves, but the position of the added vertex remains fixed, an obtuse angle is created.

A similar case applies to added edge vertices that are of degree three (see Figure 3.3.2), where two edges are the partitioned boundary edge (or "parent" edge, as it does not necessarily have to be a *boundary* edge of the original input polygon), and the third edge is an added interior edge connecting the added vertex to some other vertex in the polygon. To be part of an nonobtuse tiling, the angles at the degree-three added edge vertex *must* be *maintained at right angles*; any shifting of either the added vertex or the added interior edge will create an obtuse angle (Figure 3.3.2, centre and right illustrations).

Thus degree-three and degree-four vertices limit options as to placement of added edge and interior vertices. Both degree-three edge and degree-four interior vertices are, of course, perfectly valid elements of a nonobtuse tiling. However, it is best to have a *flexible* placement range, as this will probably facilitate re-introducing the dropped constraints (especially those referring to minimum vertex spacing, or average triangle size, or angle size). So, to maximize the flexibility of positioning the added edge and interior vertices in solutions, it is also highly desirable to:

- Minimize the use of degree-four interior and degree-three edge vertices in tiling Q.

3.4 The Refined Problem Statement; Case 0 Solution

The basic problem is that of tiling a quadrilateral Q with nonobtuse triangles. Q as given is convex, with arbitrary edge lengths. Q has no vertices other than its four corner vertices, and no interior edges connecting opposite corner vertices. This tiling is, optimally, to satisfy the constraints "inherited" (so to speak) from the parent problem in [J86].

To simplify the problem, the following constraints are *dropped* from consideration:

- Minimum spacing of added edge and interior vertices.
- Minimum spacing between input or added edges of Q, and any added interior vertex ("narrowing" allowed, if necessary, for now).
- Nearly equal triangle size.
- Nearly equal angles: allow arbitrarily small angles, bu *no angles greater than* 90°..

The following constraints are *retained*:

- No vertices on sides of triangles.
- Aim to *minimize* the use of degree-four interior vertices and degree-three edge vertices.
- Aim to tile Q with the *minimum* numbe. of nonobtuse triangles.
- Strong effort must be made to *minimize* the number of added edge and interior vertices.

The last two constraints retained effectively minimize the number of edges added to the interior of Q, and hence affect (by minimizing) the time complexity of any resulting algorithms.

Thus the simplified problem statement is:

Given a nondegenerate convex quadrilateral Q, consisting of four corner vertices and four non-crossing edges joining those vertices, determine the minimum number of nonobtuse triangles needed, in the worst case, to tile Q, subject to the previously mentioned four constraints. If Q has:

CASE 0 no obtuse or straight angles.

CASE 1 one obtuse or straight angle.

CASE 2o two opposite obtuse angles, one of which may be a straight angle.

CASE 2a two adjacent obtuse angles, one of which may be a straight angle.

CASE 3 three obtuse angles (none of which can be straight angles).

Also, in the worst case, what is the minimum number of extra vertices needed both on the boundary and in the interior of Q?

Case 0 is trivial, so it is solved in the next subsection. Cases 1, 2o, 2a, and 3 are discussed in Chapters 4, 5, 6 and 7, respectively.

3.4.1 Case 0: Q has No Obtuse Angles

Q is then necessarily a rectangle or square.

Solution: Add one diagonal to obtain two right triangles and no added (edge or interior) vertices.

Chapter 4

Case 1: Q has One Obtuse or Straight Angle

Let the corners of Q be denoted as A, B, C and D, in clockwise order, having angle measures α, β, γ and δ, respectively, with δ being the sole obtuse or straight angle. Thus B is opposite D. Q is convex, so $90° < \delta \leq 180°$. Each of α, β and γ is $\in (0, 90]$. Sections that deal with δ strictly $< 180°$ are separate from sections that consider the special case $\delta = 180°$.

4.1 The Problem With $Q \in$ Case 1

Sometimes, simply inserting a diagonal joining D and B can resolve the obtuse angle δ into nonobtuse angles and satisfactorily tile Q. (Hopefully this also does not partition already-nonobtuse β into β_1 and β_2 that are *too* small.) This is not impossible even if δ is a straight angle: in this case, D very conveniently lies at the basepoint of the altitude from B to triangle side AC. See Figure 4.1.1.

Define:

- $\angle BDC = \delta_1$.
- $\angle BDA = \delta_2$.

Thus the first step for a Case 1 quadrilateral will always be to try to resolve obtuse δ by joining D to its opposite corner B since, if this works, the utter minimum of added edges (one) and added vertices (none) are used.

It's easy to construct cases where joining D to B leaves one of δ_1 or δ_2 obtuse. See Figure 4.1.2.

In that situation, a vertex *must* be added to Q. The question of where the added vertex *can* and *should* be best placed, within the constraints retained for tiling Q, occupies the rest of this chapter.

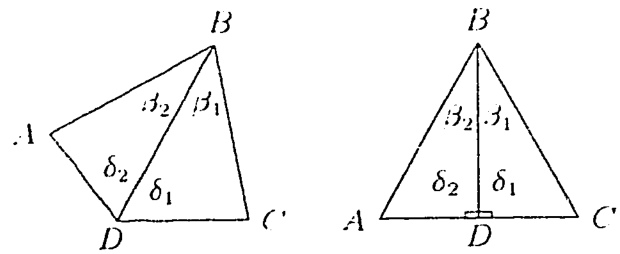


Figure 4.1.1: Edge BD alone can resolve δ into nonobtuse angles.

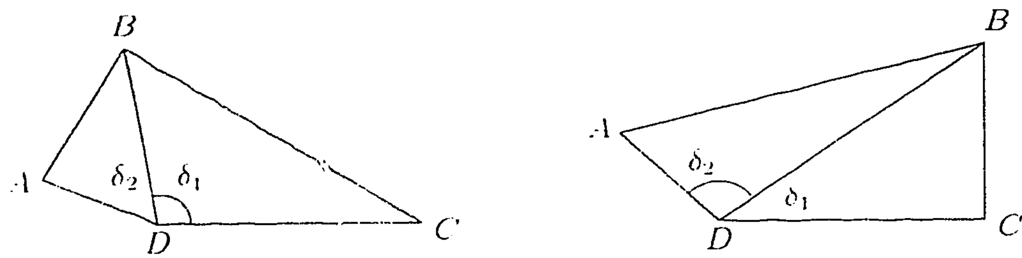


Figure 4.1.2: Joining D to B can leave an obtuse angle (shown as arc)

Note that one situation is simply a mirror reverse of the other. Acute corner B is always opposite obtuse corner D, and a mirror reverse of the situation where δ_1 remains obtuse is the case of δ_2 remaining obtuse, where the δ-angle subscripts 1 and 2 are switched, and vertex labels A and C are simply exchanged. Henceforth not a lot of attention will be paid to distinguishing very carefully between the situations; a solution devised for the situation where δ_1 remains obtuse, is perfectly valid for the δ_2 situation, with some adjustment of angle subscripts, and relabeling corners of Q (exchanging vertex labels $A \Leftrightarrow C$, and their angle labels $\alpha \Leftrightarrow \gamma$).

For the case δ = a straight angle, this is a case where D on edge AC (in $\triangle ABC$) is *not* conveniently the basepoint of the altitude from B to triangle side AC. Since the case $\delta = 180°$ is a limiting case, discussion of it is deferred to the end of this chapter.

4.2 The Nailed Vertex Solution

In this case, use Lemma 1, and drop an altitude from D through the remaining obtuse angle to firmly resolve it into two nonobtuse angles. This solution adds a "nailed" edge vertex (of degree three) to ∂Q.

- Let situation 1 = δ_1 is obtuse. To resolve δ_1, drop an altitude from D to edge BC, adding vertex H to edge BC. See Figure 4.2.1, left. Angle δ_2 is left as

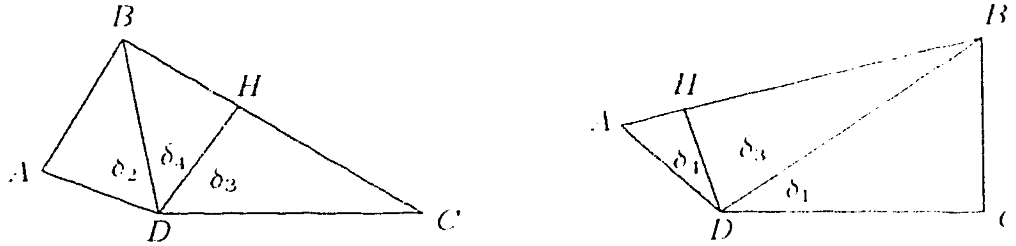


Figure 4.2.1: By Lemma 1, resolve obtuse angle by altitude DH

originally labeled; the partitioned angle δ_1 is replaced by δ_4 and δ_3.

- Let situation 2 = δ_2 is obtuse. To resolve δ_2, drop an altitude from D to edge BA, adding vertex H to edge BA. See Figure 4.2.1, right.
- For completeness' sake, let situation 3 = joining corners D and B leaves neither δ_1 nor δ_2 obtuse. This requires no further work. (See Figure 4.1.1.)

Since situation 3 requires no proof and Lemma 1 guarantees the solutions for situations 1 and 2, this "nailed vertex" method is guaranteed to produce a nonobtuse tiling of Q. **Therefore:**

Minimum (best case): two nonobtuse triangles, no added vertices (interior or boundary).

Maximum (worst case): three nonobtuse triangles (at least two of which are right triangles), one added nailed edge vertex.

4.2.1 Using the Semicircle Rule

Is there a simple way to detect situations 1 or 2 without actually calculating angle values? Yes, using Lemma 2 (the Semicircle Rule).

Basically, if corner $D \in \ominus BC$, then δ_1 is obtuse; if corner $D \in \ominus AB$, then δ_2 is obtuse. If neither of the previous is the case, then joining D to B resolves δ into two nonobtuse angles. In Figure 4.2.2, M is the midpoint of edge BC, so $|BM| = |MC| = r$, where r is the radius of the semicircle. The algorithm employing an altitude to resolve obtuse angles is called a Nailed Vertex algorithm.

```
Add edge DB to Q.
/* see if δ1 is obtuse */
r ← |BC|/2.
M ← (B + C)/2.
```

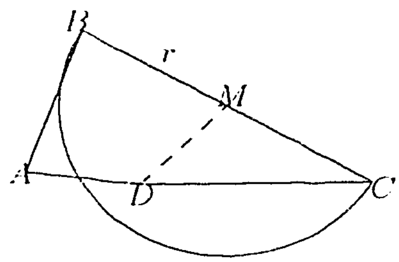


Figure 4.2.2: $len(MD) < r \Rightarrow D \in \frown BC$

```
IF (|DM| < r) THEN
      Add altitude DH, where H is on BC, to Q.
ELSE /* see if δ2 is obtuse */
      r ← |BA|/2.
      M ← (B + A)/2.
      IF (|DM| < r) THEN
            Add altitude DH, where H is on BA, to Q.
      ENDIF
ENDIF
```

By Lemmas 1 and 2, this algorithm is guaranteed. However, it is unattractive due to the possibility of adding a nailed edge vertex H.

Note that adding edge DB first doesn't permit skipping the tests for δ_1 and/or δ_2.

With the above algorithm for the nailed method, the best and worst case results are the same as before: optimally, joining D to B tiles Q with two nonobtuse triangles; otherwise an altitude is used to resolve the remaining obtuse angle, adding a nailed edge vertex and producing three nonobtuse triangles (two of which are right triangles) tiling Q.

4.2.2 Using the Right Angle Bound Rule

Using Lemma 3, establish right angle bound lines DU such that $\angle CDU = 90°$, and DV such that $\angle ADV = 90°$, and see if they bracket B: if so, D can be joined to B with assurance that neither δ_1 nor δ_2 will be obtuse. If B does not lie within DU and DV, we can still use the previous (inelegant) solution, i.e., drop an altitude through the remaining obtuse angle. The difference here is in the test used to detect which situation is occurring.

The Nailed Vertex algorithm using Lemma 3 to detect obtuse angles is:

Add edge DB to Q.
Orient Q so that the longer of CB or AB is a horizontal "base".
Extend DU and DV so that they intersect the horizontal "base".
(Call the intersection points U and V also. See Figure 4.2.3.)
/* U_x is x-coordinate of U. */
IF ($U_x < B_x$) THEN
        /* δ_1 is obtuse; use Lemma 1 to resolve it */
        Add altitude DH, where H is on CB, to Q.
ELSE
        IF ($B_x < V_x$) THEN
            /* δ_2 is obtuse; use Lemma 1 to resolve it */
            Add altitude DH, where H is on AB, to Q.
        ENDIF
ENDIF

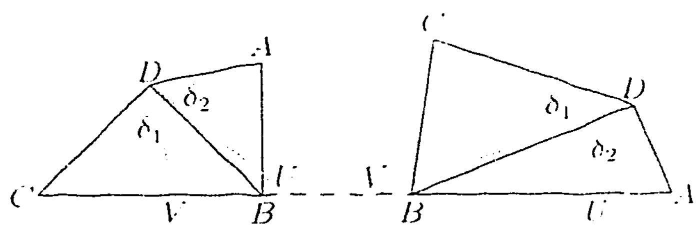


Figure 4.2.3: Orient Q so $max\{|AB|, |CB|\}$ is horizontal base; extend DU and DV to intersect the horizontal "base". For both cases (CB base or AB base). $B_x \in [V_x, U_x] \Rightarrow \delta_1, \delta_2$ both nonobtuse.

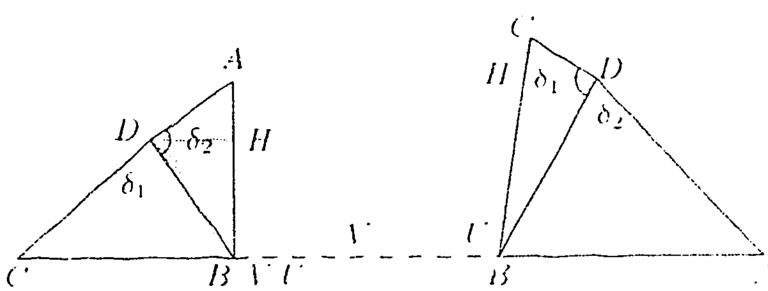


Figure 4.2.4: Left, CB base: $B_x < V_x, U_x \Rightarrow \delta_2$ obtuse;
Right, AB base: $V_x, U_x < B_x \Rightarrow \delta_1$ obtuse.

Given the ordering of vertices as $C - D - A$ as in Figures 4.2.3, 4.2.4, and 4.2.5, with Q oriented so that the longer of CB and AB is a horizontal "base", the

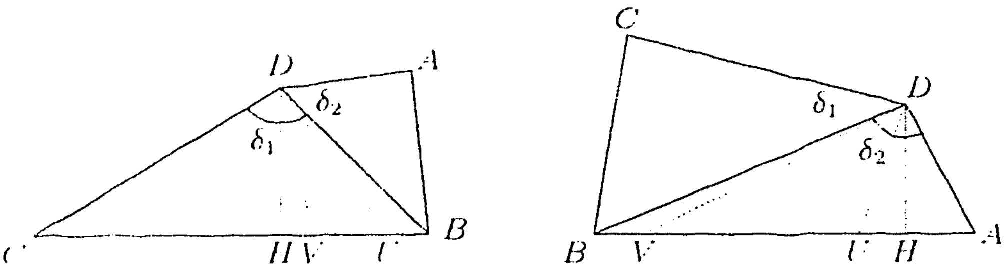


Figure 4.2.5: Left, CB base: $V_x, U_x < B_x \Rightarrow \delta_1$ obtuse;
Right, AB base: $B_x < V_x, U_x \Rightarrow \delta_2$ obtuse.

ordering of U and V is *always* with V to the left of U. Many following illustrations of example quadrilaterals are "flipped" so that the edge intersected by the Right Angle Bound lines DU and DV is now at the top of the figure, and so we will always see U to the left of V.

$[U, V]$ *never* brackets H: that is, $H \notin (U, V)$. More specifically, since we have used H both on edge CB and AB, as required, to partition whichever of δ_1 or δ_2, respectively, remains obtuse after joining D to B, we can say

Lemma 5: In a Case 1 quadrilateral, when one of δ_1 or δ_2 remains obtuse after joining D to B, the Right Angle Bounds from D never bracket the altitude basepoint H of DH.

Proof: Consider the case when δ_2 is obtuse and H lies on AB (the other case is just the mirror reverse), and assume $H \in (U, V)$. Consider $Q' = HBCD$, and let the angles of this quadrilateral be η, β, γ, and δ' at H, B, C, and D, respectively. Since $H \in (U, V)$, $\angle CDH < \angle CDU$, so δ' is acute. But then $\eta = 90°$ (since DH is an altitude to AB). δ' is acute, and, as given in Q, both γ, β are nonobtuse. But then $\beta + \gamma + \delta' + \eta < 360°$.

Therefore $H \notin (U, V)$. □

It may certainly be the case that $U = H$ when δ_2 is obtuse, or that $V = H$ when δ_1 is obtuse. This simply means that sides DC and AB, in the first case, or sides DA and CB, in the second case, are parallel, and that Q is a trapezoid. This may be taken as something of a limiting case, for given that δ is obtuse and $DC \| AB$ or $DA \| CB$, it *must* be that $\beta = 90°$, one of α or γ also $= 90°$, and the other must be acute. See Figure 4.2.6. In this very simple special case, nonobtuse tiling consists of adding altitude DH to Q, thus effectively cutting off the right-triangle "ear" ADH of the trapezoid, and adding a diagonal - either one - to the remaining rectangle. Thus finding $H = U$ or V is a special (and probably unusual, in practice) case. In those cases where a Right Angle Bound vertex label overlays an altitude basepoint label, the Right Angle Bound label will generally take precedence in the figure.

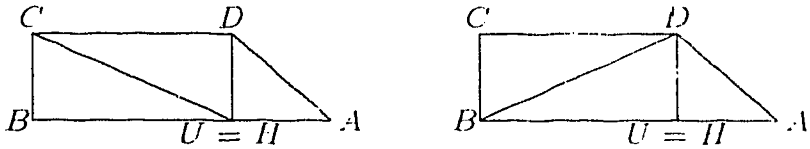


Figure 4.2.6: $U = H \Rightarrow Q$ is a trapezoid, easily tiled in one of two ways.

Using the above modification of the Nailed Vertex algorithm, the best and worst case results are the same as before: optimally, joining D to B tiles Q nonobtusely with two nonobtuse triangles; otherwise an altitude is used to resolve the remaining obtuse angle, adding a nailed edge vertex and producing three nonobtuse triangles (two of which are right triangles) tiling Q.

4.3 Floating Vertex Method for $Q \in$ Case 1

We now attempt to employ both Lemmas 2 and 3 in tiling Q. Assume that simply adding diagonal DB to Q *does not* resolve δ into nonobtuse δ_1 and δ_2, so that we need to add at least one vertex to Q.

If we considered adding an *interior* vertex W to Q, W would have to be of degree five, as it is highly unlikely that W of degree four, connected to the corners of Q, would have four right angles. Thus an added *interior* vertex implies an added *edge* vertex. Since we want to minimize the number of added vertices of either variety, this does not seem too attractive an idea.

Given that it is sometimes impossible to resolve Case 1 by adding only the DB edge to Q, is the Nailed Vertex method the best possible solution? That is, given that an edge vertex *must* be added to resolve an obtuse angle that remains after adding diagonal DB to Q, can the added edge vertex be changed from being "nailed" in position as an altitude basepoint, to having some range within which its positioning may vary, but will still enable Q to be tiled nonobtusely and satisfactorily? If so, the added edge vertex would be called a "floating" vertex, and the range within which it can be positioned, while still tiling Q nonobtusely and satisfactorily, a "viable area".

See Figure 4.3.1. Starting with diagonal DB and altitude DH, swap diagonal DB for the other diagonal of quadrilateral $DCBH$ - in the example in Figure 4.3.1, this is the CH diagonal; then move H along the top edge (in the example, this is edge AB) to a "better" position W. (The vertex is relabeled as W because the label H is still kept as a marker of the altitude from D to the top edge of Q.)

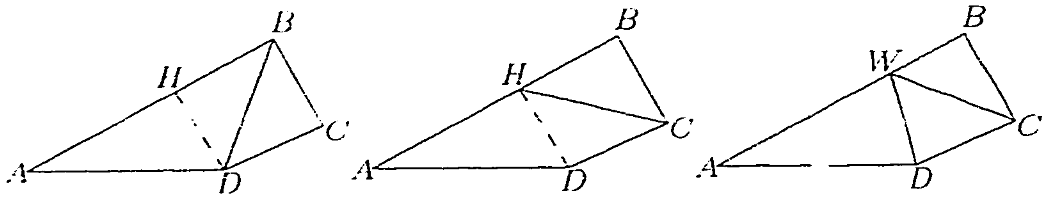


Figure 4.3.1: (Left) Start with altitude DH and diagonal DB; (Center) Switch diagonal DB for CH; (Right) Move H along AB to W where the tiling has nicely shaped nonobtuse angles.

In the example shown (Figure 4.3.1, right), all the angles are nonobtuse in the final tiling, and in shifting the position of W *slightly* along edge AB we find that the angles *do remain* nonobtuse.

This *floating vertex* method looks like a much more attractive solution than the previous nailed vertex method. The added edge vertex is of degree four rather than three; this alone "frees" it from being nailed. Also, the fact that the added edge vertex is floating, rather than nailed at an altitude basepoint, result in some leeway in positioning that will directly affect vertex spacing, triangle sizes, and angle sizes in the finished tiling. There is still the "problem" (if it can be called so) of partitioning an already nonobtuse angle, but that is unavoidable when Q has only one obtuse angle, and a degree-four edge vertex is being added to Q.

Since a guaranteed method (the Nailed Vertex method) has been proved, why would another solution using a floating vertex be desirable? This was discussed in Section 3.3, where the larger problem was explained as triangulating the boundary strip between ∂P and $int(P)$ from [J86]'s work. This could even arise in trying to nonobtusely tile a convex or nonconvex region, in which the first step is to partition the region into convex quadrilaterals. In either case, each quadrilateral is to be tiled nonobtusely, and there will be instances where quadrilaterals will share edges.

Therefore, where a vertex must be added (in the process of nonobtuse tiling) to an edge of a quadrilateral, where that edge is shared with another quadrilateral, that vertex must be "required" or "allowed" in both quadrilaterals, to satisfy the constraint that vertices do not exist on the sides of triangles in the final tiling. Allowing the edge vertex position to float may well be critical in getting the vertex position along the shared edge to coincide, while still guaranteeing that all angles are nonobtuse in the triangulation of the two quadrilaterals that share an edge.

4.3.1 Approaching the Floating Vertex Solution

The current problem is: change the added edge vertex from being nailed at an altitude basepoint H, to being able to "float" within some viable area; this includes being able to tell when such a viable area would exist or not, and what delimits the viable area when it does exist.

As in the example in Figure 4.3.1, one added floating edge vertex, of degree four, seems to be an ideal solution, if it can be discovered where the floating vertex can and can't be placed; when it can and can't, or shouldn't, be used; is it *always* better (if possible) to use the floating vertex method than the nailed vertex method with respect to triangle size, angle size, edge vertex spacing; etc.

Where **can't** W go, and why? If placement of W can be restricted by constraints that will guarantee nonobtuse angles in the final tiling of Q, whatever is left over after the restrictions must be a viable area. (Or so it is hoped!)

From now on in illustrations, Q will be oriented so that the edge intersected by both Right Angle Bound lines DU and DV is at the *top* of the figure. (If DU and DV intersect two different edges of Q, they must bracket a vertex of Q between them, which can then be joined to D with the guarantee that both subangles obtained by partitioning δ will be nonobtuse. We have assumed that this is not the case, so both DU and DV intersect the same edge of Q.) This edge will be referred to as the "top" edge, regardless of whether it is CB or AB. The opposite edge, having D as one endpoint, is the "bottom" edge. Thus it is expected to see U always to be to the left of V. The top edge will not be oriented horizontally; rather, one of the other edges of Q, containing D as an endpoint, will be oriented horizontally. Also, to simplify the discussion, figures will often (but not always) be restricted to one or the other case of AB being the top edge, or CB being the top edge. One is just a mirror-reverse of the other anyway, and can be "converted" by simply exchanging vertex labels $A \Leftrightarrow C$ (and similarly angle labels $\alpha \Leftrightarrow \gamma$).

Let the angle designations in the finished tiling be as shown in Figure 4.3.2.

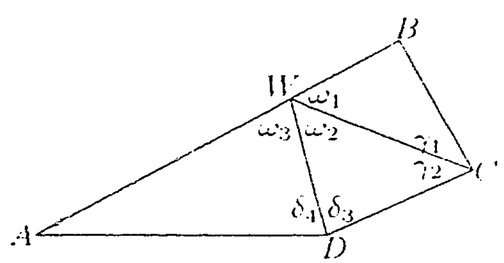


Figure 4.3.2: Angle designations in final tiling by floating vertex method.

There does not need to be any constraint to guarantee that angles γ_1 and γ_2 are guaranteed acute as their parent angle γ is nonobtuse to start with, except that the subangles of γ should not be too small if it can be helped. (Similarly with α_1, α_2, and their parent angle α in the mirror-reverse case.)

Therefore there are five angles that need guarantees:

1. $\angle CDW = \delta_3$.

2. $\angle ADW = \delta_4$.

3. $\angle BWC = \omega_1$.

4. $\angle CWD = \omega_2$.

5. $\angle AWD = \omega_3$.

(Recall that $\angle BDC = \delta_1$ and $\angle BDA = \delta_2$; so using a floating vertex W to partition δ creates two "new" angles, $\angle CDW = \delta_3$ and $\angle ADW = \delta_4$.)

If a viable area can be found for each angle, where placing W in that viable area guarantees that the particular angle will be nonobtuse, then the intersection of all viable areas must jointly guarantee all angles to be nonobtuse.

4.3.2 Guaranteeing Nonobtuse δ_3 and δ_4

To guarantee that both δ_3 and δ_4 will be nonobtuse, use Lemma 3. See Figure 4.3.3, where Right Angle Bounds DU and DV have been sketched in. Assume that DU and DV do not bracket B. From Lemma 3, $W \in [U, V]$ on the top edge of $Q \Rightarrow$ neither δ_3 nor δ_4 will be obtuse. Thus to guarantee δ_3 and δ_4, the viable area is $[U, V]$.

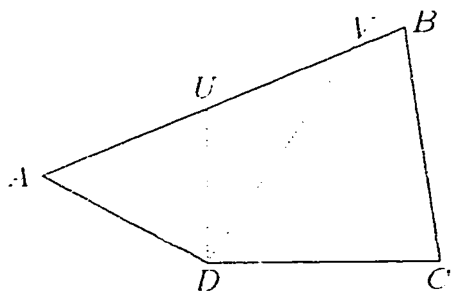


Figure 4.3.3: Right Angle Bounds DU, DV.

4.3.3 Guaranteeing Nonobtuse ω_1 and ω_3

To guarantee that angles ω_1 and ω_3 will be nonobtuse, Lemma 2, the Semicircle Rule, is used.

See Figure 4.3.1. Both semicircles from the sides of Q intersect the top edge of Q, since α and β are nonobtuse. Let the points of intersection be H and F as shown. That is, H will always be the point of intersection of a semicircle that has D as one diameter endpoint, and the top edge; and F will always be the intersection point of the other semicircle, with the top edge. $F = B$ if $\beta = 90°$, and $H = A$ if $\alpha = 90°$ (or $H = C$, if $\gamma = 90°$, in the mirror-reverse case).

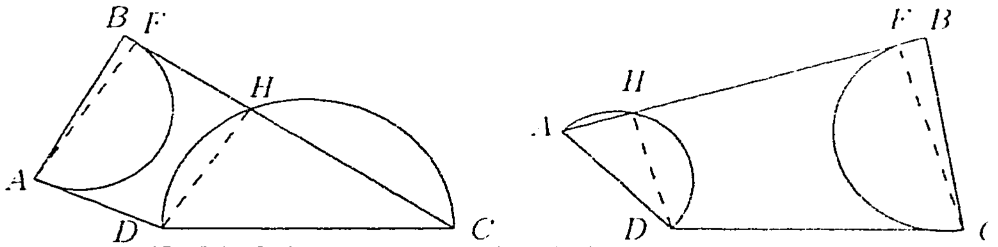


Figure 4.3.4: (Left) CB as top: $\ominus AB \cap BC = F$; $\ominus DC \cap BC = H$.
(Right) AB as top: $\ominus AD \cap AB = H$; $\ominus CB \cap AB = F$.

Figure 4.3.4 shows both situations (not merely mirror reverses of each other). In the right illustration, H is *on* semicircle $\ominus AD$, so that angle $\angle AHD = 90^\circ$, and thus DH is an altitude to top edge AB. Similarly, $F = \ominus BC \cap AB$ marks the altitude basepoint from C to AB. This is not surprising: by Lemma 4, both altitudes must exist, since both obtuse corner D and nonobtuse corner C are followed by two nonobtuse corners, A and B.

Likewise, as in Figure 4.3.4, left illustration, semicircle $\ominus CD$ intersects with top edge BC at altitude basepoint H, and F is similarly found to be the altitude basepoint of the other semicircle with the top edge.

Define regions $\mathcal{F} = [B, F)$ and $\mathcal{H} = [A, H)$ or $[C, H)$, as in Figure 4.3.5.

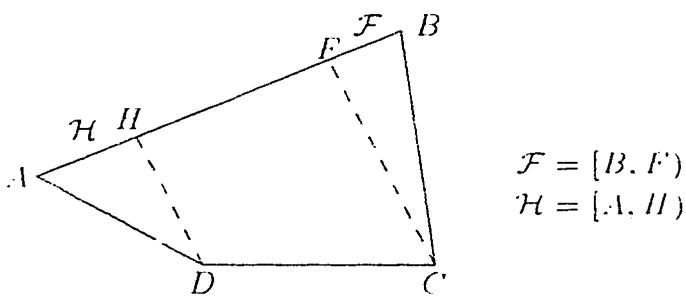


Figure 4.3.5: Altitudes from corners to top edge define regions $\mathcal{F}$ and $\mathcal{H}$

1. Since $\mathcal{H}$ = portion of top edge inside semicircle AD, $W \in$ region $\mathcal{H} \Rightarrow$ $\angle AWD = \omega_3$ is obtuse.

2. Since $\mathcal{F}$ = portion of top edge inside semicircle BC, $W \in$ region $\mathcal{F} \Rightarrow$ $\angle BWC = \omega_1$ is obtuse.

Therefore, to guarantee that angles ω_1 and ω_3 are both nonobtuse, the viable area must (so far) be $[F, H]$ on the top edge of Q.

Is it possible that this area is empty, that is, that $F = H$? No, since DH is the altitude to the top edge, and CF (or AF) is also an altitude to the top edge; therefore $F = H \Rightarrow D = C$ (or $D = A$). Also, we would never see the altitudes DH and CF (or AF) crossing so that F and H "overlap", since HD and FC (or FA) are parallel lines, both perpendicular to the top edge of Q.

4.3.4 Guaranteeing Nonobtuse $\delta_3, \delta_4, \omega_1$ and ω_3

At this point we will consider the intersection of viable areas for $\omega_1, \omega_3, \delta_3$ and δ_4. See Figure 4.3.6.

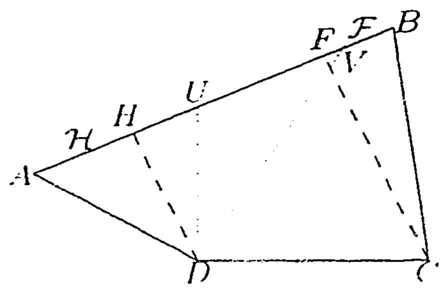


Figure 4.3.6: Regions $\mathcal{F} = [B, F)$, $\mathcal{H} = [A, H)$, and $[V, U]$.

By Lemma 5, $H \notin (U, V)$, where H is the altitude basepoint from D to the top edge of Q. Since $H \notin (U, V)$, $U \notin [A, H) = \mathcal{H}$. (However, notice that $U = H$ is a perfectly valid occurrence, only probably rare, as it implies that Q is a trapezoid.) Also, since U is *always* to the left of V (assuming $\delta < 180°$), $V \notin [A, H]$. However, $V \in (F, B]$ can easily happen, as it does in Figure 4.3.6. $U \in (F, B]$ can also happen; see Figure 4.3.7. In this case there is no viable area in which to place W guaranteeing that all of $\delta_3, \delta_4, \omega_1$ and ω_3 are nonobtuse. So the Floating Vertex method fails in this case, and the Nailed Vertex method must be used. (Since mirror-reverse drawings and discussion are not given, note that the roles of H and F (and $\mathcal{H}, \mathcal{F}$) are reversed in the above discussion if the bottom edge of Q is AD instead of CD.)

To finish developing the Floating Vertex method, assume that $[U, V] \cap [H, F] \neq \emptyset$; that is, that U is to the left of F, or $U = F$.

When the viable area is nonempty, $[H, F] \cap [U, V] = [U, L]$, where U either overlays H (but the label U takes precedence), or is strictly to the right of H; and L is the leftmost of F or V. This viable area $[U, L]$ guarantees angles $\omega_1, \omega_3, \delta_3$, and δ_4 to be nonobtuse. Restricting the viable area to (U, L) guarantees *all* of them to be acute (which may be very desirable); otherwise one or more of them are right angles.

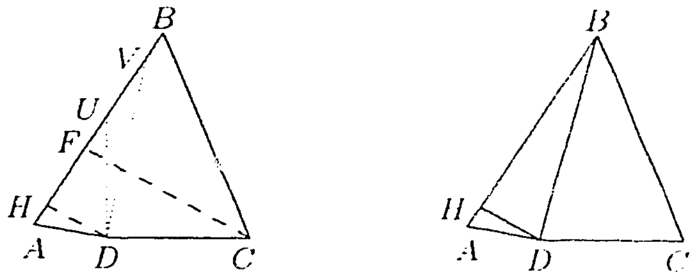


Figure 4.3.7: Left: When $[U, V] \in \mathcal{F}$, $W \in [U, V]$ would leave $\angle BWC = \omega_1$ obtuse. Right: Easiest solution is to use the Nailed Vertex method.

4.3.5 Guaranteeing Nonobtuse ω_2

To continue with the Floating Vertex idea, assume $[H, F] \cap [U, V] \neq \emptyset$. To guarantee the final angle, ω_2, use Lemma 2, the Semicircle Rule. Let the semidisk that has the bottom edge of Q as diameter be labeled, for convenience, as semidisk $\mathcal{Z} = \odot CD$ or $\odot AD$.

This case proved the most difficult to guarantee. Whether or not $\mathcal{Z}$ intersects the top edge of Q is highly dependent upon the interaction of edge lengths of Q, and not always obvious to predict. Certainly the "height" of the top edge of Q above the bottom edge has much to do with it, but so do the angles at the bottom, and lengths of the sides, of Q. However, a finite number of classes of interaction emerge from experimentation and drawings:

1. $\mathcal{Z} \cap$ top edge $= \emptyset$. Then W placed anywhere on the top edge of Q is outside $\mathcal{Z}$, so that the entire top edge is a viable area for ω_2. See Figure 4.3.8.

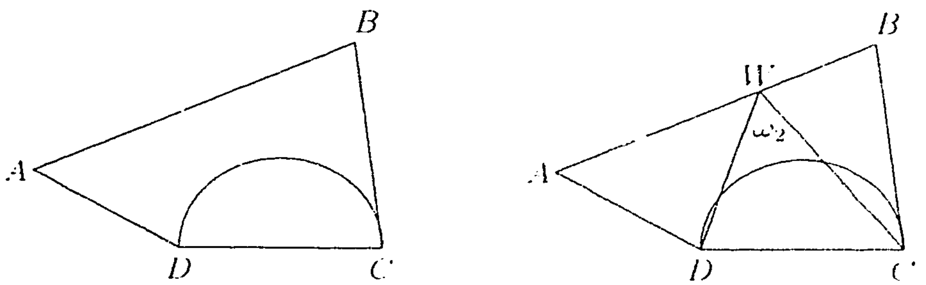


Figure 4.3.8: $\mathcal{Z} \cap AB = \emptyset \Rightarrow W$ anywhere on $AB \Rightarrow \omega_2$ will be acute.

2. $\mathcal{Z} \cap$ top edge $=$ one point Z_0. Then the top edge of Q is tangent to $\mathcal{Z}$ at Z_0, and W placed *at* $Z_0 \Rightarrow \omega_2 = 90°$; W placed anywhere else along the top edge $\Rightarrow \omega_2$ will be acute. Thus the viable area for ω_2 is still the entire top edge of Q.

3. $\mathcal{Z} \cap$ top edge $= [Z_1, Z_2]$. See Figure 4.3.9. $\mathcal{Z}$ "intrudes" into the top edge of Q. Neither "top" corner of Q could possibly be in $\mathcal{Z}$, as Q has only one obtuse angle δ. The viable area in which W can be placed to guarantee that ω_2 will be nonobtuse, becomes split into two subareas, $[A, Z_2] \cup [Z_1, B]$. Depending on the size of $\mathcal{Z}$ (clearly dependent on the length of the "bottom" edge of Q) and the amount $\mathcal{Z}$ "intrudes" into the top edge of Q (clearly dependent on the "height" of Q), the viable subareas can be quite small; what is of more concern, however, is that W may be forced into a position near a corner vertex of Q, and triangle sizes in the tiling can be very unequal.

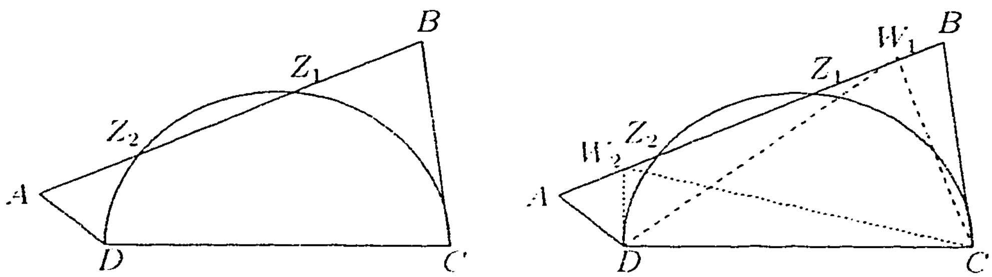


Figure 4.3.9: Left: When $AB - \mathcal{Z} = [A, Z_1] \cup [Z_2, B]$,
(Right) placing W in either viable area $\Rightarrow$ very unequal triangle sizes.

4.3.6 Guaranteeing All Nonobtuse Angles

Assume $[H, F] \cap [U, V] \neq \emptyset$. Again, to jointly guarantee ω_2 with the other angles, consider the intersection

$$[U, L] - (\mathcal{Z} \cap [U, L]) \cup S$$

of the viable areas for $\omega_1, \omega_2, \omega_3, \delta_3$, and δ_4, where

$$S = \begin{cases} \{Z_1, Z_2\} \text{ if both } Z_1, Z_2 \in [U, L]; \\ \{Z_1\} \text{ or } \{Z_2\}, \text{ whichever is the only one} \\ \text{in } [U, L] \text{ (as } Z_2 \text{ in Fig. 4.3.10, below);} \\ \{Z_0\} \text{ if } Z_0 \in [U, L]; \\ \emptyset \quad \text{otherwise.} \end{cases}$$

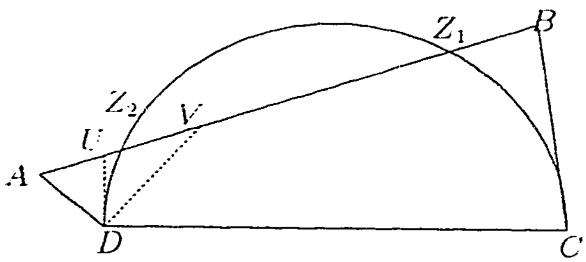
Again, because of the unpredictability of the interaction of $\mathcal{Z}$, a number of disjoint classes emerge:

1. $\mathcal{Z} \cap [U, L] = \emptyset$. Then viable area $= [U, L]$.

2. $\mathcal{Z} \cap [U, L] = Z_0$, one point. Then viable area $= [U, L]$.

3. $\Xi \cap [U, L] = [Z_2, Z_1]$. Then viable area $= [U, Z_2] \cup [Z_1, L]$.

4. $\Xi \cap [U, L] = [Z_2, L]$, as in Figure 4.3.10. Then the viable area is $[U, Z_2]$.



viable area $= [U, Z_2]$

Figure 4.3.10: Ξ can partially, but will never entirely, obscure $[U, V]$.

Thus it is seen that for each "class" of interaction, a viable area exists.

Is it possible that the viable area shrinks to nothing? No: see Figure 4.3.11. As the height of Q decreases, the viable area could get quite small. However, since $DU \perp DC$ and DC is the diameter of semicircle Ξ, there will *always* be room between U and Z_2 to place W.

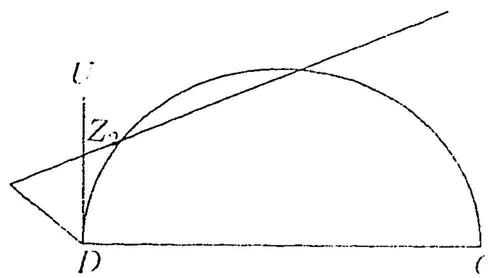


Figure 4.3.11: $DU \perp DC$, and DC being the diameter of Ξ $\Rightarrow |U, Z_2| = 0$ only when $U = Z_2 = D$.

So if $[H, F] \cap [U, V] \neq \emptyset$, a viable area *always* does exist in which to place W and use the Floating Vertex method. As Q "flattens", Ξ will obscure more of the top edge, and the distance between U and Z_2 will shrink. As Q gets taller, Z_2 and Z_1 will approach one another, coincide and meld into one point Z_0, then $\Xi \cap AB = \emptyset$, so that the viable area $= [U, L]$ (if not $\emptyset$).

Figures 4.3.12 and 4.3.13 show a few completed tilings using the Floating Vertex method.

$[U, L]$ (when not $\emptyset$) shrinks both as the "height" of Q decreases (as in Figure 4.3.10) and as $\delta \rightarrow 180°$. In both cases the width of the area shrinks to a locality to the right of and including U: in the former case, the right endpoint of the viable area is likely to be Z_2, in the latter case it is more likely to be V.

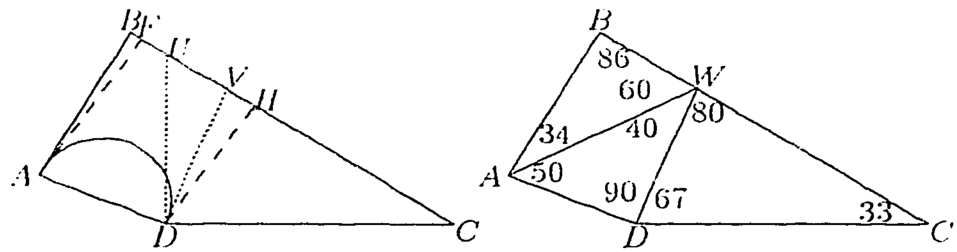


Figure 4.3.12: Viable area for $W = [U, V]$; W placed to tile Q acutely.

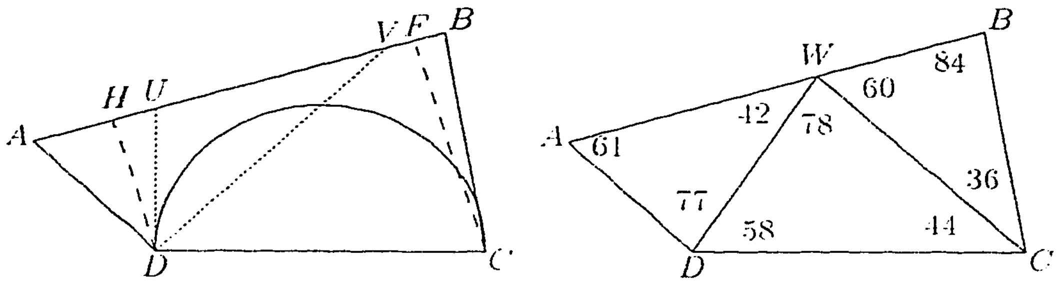


Figure 4.3.13: Viable area for $W = [U, V]$; W placed to tile Q acutely.

The algorithm for the floating vertex method is ordered such that the viable area for δ_1 and δ_2 is found first, as establishing Right Angle Bounds from D indicates which edge is the top edge. Having found the top edge, the altitude basepoints F and H can be found; otherwise it is not known whether to take altitude CF (to top edge AB) or AF (to top edge CB). The intersection with semicircle $\mathcal{Z}$ is taken last.

> Establish Right Angle Bounds DU', DV' such that $\angle CDU' = \angle ADV' = 90°$.
> Let top edge = edge on which $Q \cap DU'$ exists.
> IF (edge on which $Q \cap DV'$ exists $\neq$ top edge) THEN
>   /* $[U, V]$ bracket B */
>   Add edge DB to Q
> ELSE
>   Let I, J be the endpoints of the top edge, with I adjacent to D.
>   Orient Q so that the bottom edge is horizontal.
>   Let K = other endpoint of bottom edge.
> $F = \perp KJ \cap IJ$.
> $H = \perp DI \cap IJ$.
>   Viable $= [F, H] \cap [U, V]$.

```
IF (Viable = ∅) THEN
  /* Can't use a Floating vertex, use a Nailed vertex */
ELSE    /* Can use Floating Vertex */
  Let Ƶ = ⊙DK.
  Viable = [U, L] − (Ƶ ∩ [U, L]).
  Place W in viable area; add edges WD and WK to Q.
  /* Where W is placed depends on what is desired regarding
  minimum angle size/edge vertex spacing, triangle size, etc. */
ENDIF
ENDIF
```

When the floating vertex algorithm works (i.e., when $[U, L] \neq \emptyset$), the best and worst case results are a little improved: joining D to B tiles Q with two nonobtuse triangles; otherwise, a degree-four floating edge vertex is added, to tile Q into three nonobtuse triangles. If W is placed in the interior of the viable area then all new angles in the finished tiling are acute (except possibly ω_2, if $W = Z_0$, and/or α, β, γ if they are input as right angles). Otherwise one or more new angles will be right angles.

4.3.7 Final Issues

W placed anywhere in the viable area guarantees nonobtuse angles in the tiling of Q, by definition of the viable area. Adjusting the position of W changes triangle sizes and angle sizes. Experimentation seems to suggest that *usually* placing W at the position where the bisector of δ intersects the top edge is optimal: that is, of *all* the positions within the viable area, placing W at the δ-bisector maximizes the minimum angle in tiling Q by the floating vertex method. This is not strictly always the case: it seems to depend highly on the geometry of Q. If the location where the δ-bisector intersects the top edge is not within the viable area (i.e., it lies within $\mathcal{Z}$), then placing W as close to that location as possible has the same effect. This often means placing W at L, the rightmost end of the viable area, which would guarantee at least one right angle in the finished tiling, but still, the minimum angle of all possible tilings by the floating vertex method is often maximized with W at that position.

In Figure 4.3.14, the Right Angle Bounds, altitudes F and H, and semicircle $\mathcal{Z}$ are shown; the viable area that remains after the intersection of all these areas is $[U, Z_2]$. In Figure 4.3.15, U is chosen as the location of W; in Figure 4.3.16, Z_2, which is the closest point in the viable area to the intersection of the δ-bisector and the top edge, is chosen as the location of W. Note that $W = Z_2$ maximizes the minimum angle. Although in both cases, due to the geometry of Q, the triangle sizes are quite uneven, $W = Z_2$ is better than $W = U$. This can be especially noted in that with $W = Z_2$, it is more nearly the case that $|AW| = |DW| = |AD|$;

since AD is the edge of minimum length in Q, the position for W should adhere to a minimum vertex spacing as defined by the input edges. This is much more closely attained by using $W = Z_2$ than with $W = U$.

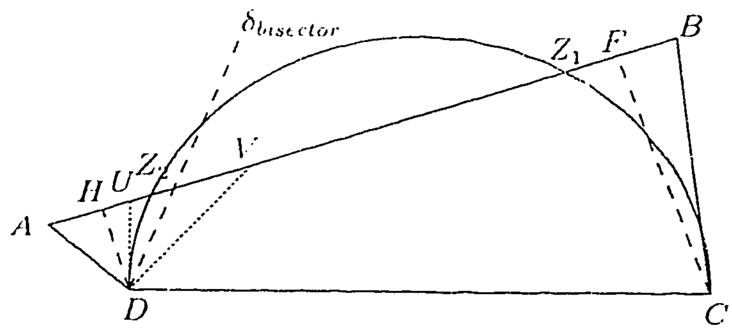


Figure 4.3.14: Viable area can become quite small as Q "flattens" towards its obtuse corner.

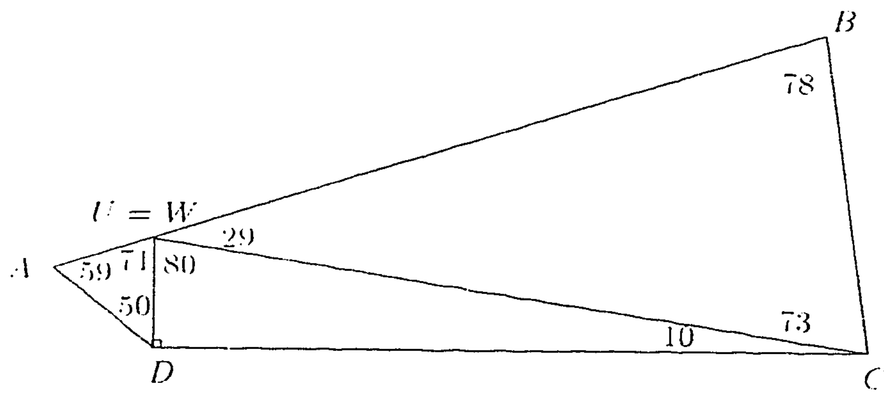


Figure 4.3.15: Position W at U: finished tiling (angle measures shown)

4.4 Tiling $Q \in$ Case 1 When $\delta = 180°$

When $\delta = 180°$, D becomes a vertex on edge AC of triangle $\triangle ABC$. Let E be the altitude basepoint on AC from vertex B. If $D = E$, then adding DB to Q resolves D into two right angles, by Lemma 1. (Again, it is hoped that β_1 and β_2 are not *too* small.) Otherwise adding DB to Q leaves either δ_1 or δ_2 obtuse. See Figure 4.4.1.

This can be easily resolved (by Lemma 1) with an altitude through the remaining obtuse angle. To avoid actually having to calculate angle measures, Lemma

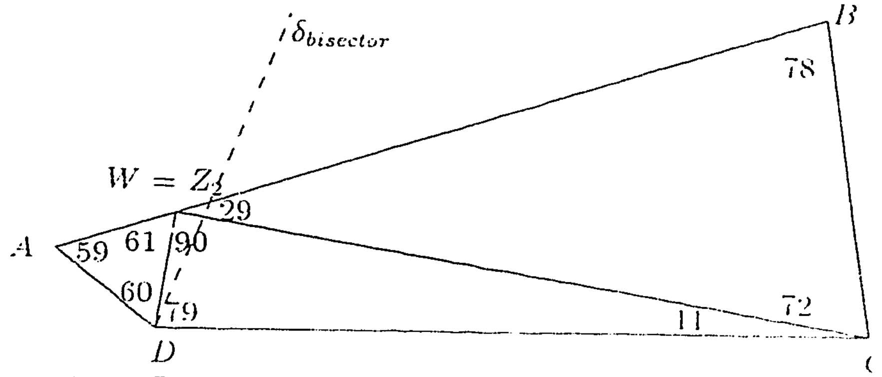


Figure 4.3.16: Position W at Z_2: finished tiling (angle measures shown)

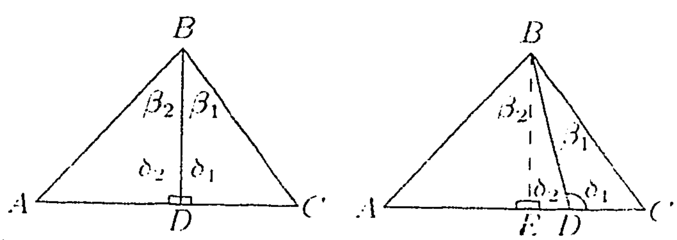


Figure 4.4.1: Adding DB to Q tiles Q if and only if DB is an altitude.

2 is used to detect which situation is occurring, by calculating distances between points. The details of doing so are not repeated here.

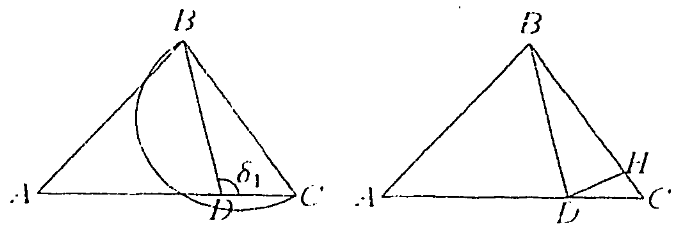


Figure 4.4.2: Using Lemma 2 to detect, and Lemma 1 to resolve, obtuse angle.

The algorithm in this case is (see Figure 4.4.2):

```
Add edge DB to Q.
IF (D ∈ ⊖BC) THEN
        /* δ1 is obtuse */
        Add altitude DH to Q, where H ∈ CB and DH ⊥ BC.
ELSE IF (D ∈ ⊖AB) THEN
        /* δ2 is obtuse */
        Add altitude DH to Q, where H ∈ AB and DH ⊥ AB.
ENDIF
```

The best and worst case results are the same as before: optimally, joining D to B tiles Q nonobtusely with two nonobtuse triangles; otherwise an altitude is used to resolve the remaining obtuse angle, adding a nailed edge vertex and producing three nonobtuse triangles (at least two of which are right triangles) tiling Q.

4.4.1 Using Lemma 3 when $\delta = 180°$

When $\delta = 180°$, the Right Angle Bounds from D behave like an altitude from D, and $U = V$. Again, if $D = E$, then $U = V = B$ and simply adding DB tiles Q into two right triangles. Otherwise $V = U$ (call it V) lies on AB or BC. Note that V is essentially nailed due to $\delta = 180°$.

See Figure 4.4.3. Adding DV to Q cuts off a right triangle "ear" and leaves an inner Case 1 $Q' = ABVD$ where $\nu_2 = \nu$ is the obtuse corner, α and β are acute, and $\delta' = \delta_2 = 90°$. Thus the Floating Vertex method cannot *really* be applied to Q in this case, as $[U, V] = V$ is a single point. However, either the Floating Vertex or Nailed Vertex method can be used on the inner Case 1 Q', as previously outlined, although attempting to use the Floating Vertex method on quadrilateral $Q' = ABVD$ can lead to four triangles in $Q = ABCD$ (three from the Floating Vertex method in Q', plus the one right triangle ear).

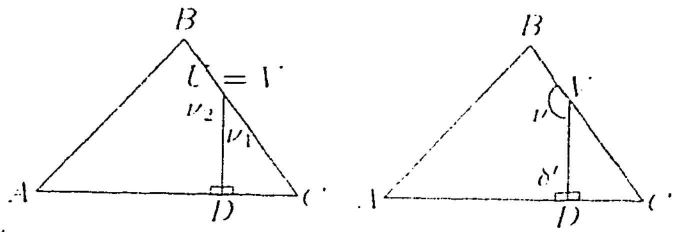


Figure 4.4.3: Right Angle Bounds $U = V$ "nailed" on BC: $Q' = ABVD$.

Chapter 5

Case 2o: Q has Two Opposite Obtuse Angles

Let the obtuse corners of Q be D and B, which are opposite one another. Assume for now that both $\delta, \beta < 180°$. (The case where one of them is a straight angle is deferred to the end of this chapter.)

If adding edge DB to Q leaves none of $\delta_1, \delta_2, \beta_1$ or β_2 obtuse, the tiling is finished. In fact, as D and B move *closer* to being overtop one another, the angles $\delta_1, \delta_2, \beta_1$ and β_2 all become more uniform in size - the smaller pair increase, the larger pair decrease, as in Figure 5.0.1.

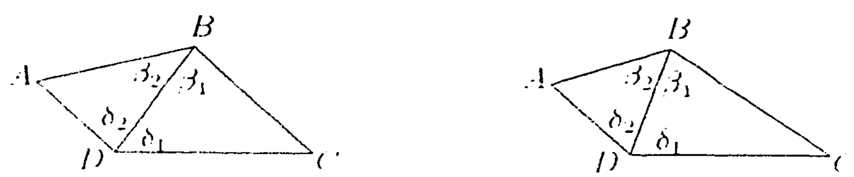


Figure 5.0.1: DB alone can tile Q nonobtusely, with the subangles becoming more uniform as B "moves over" D (so that $B_i \Rightarrow D_i$).

However, as D and B move *apart*, or as the nonobtuse angles α and γ decrease, it is more likely that one or more of $\delta_1, \delta_2, \beta_1$ and β_2 may become obtuse, as in Figure 5.0.2.

5.1 Using the Semicircle Rule

Obtuse angles remaining after edge DB has been added to Q are easily detected using Lemma 2, the Semicircle Rule. However, given that an obtuse corner of Q is followed by an nonobtuse-angled corner, and then by an obtuse-angled corner, there is *no guarantee* that altitudes from obtuse corners exist on both possible

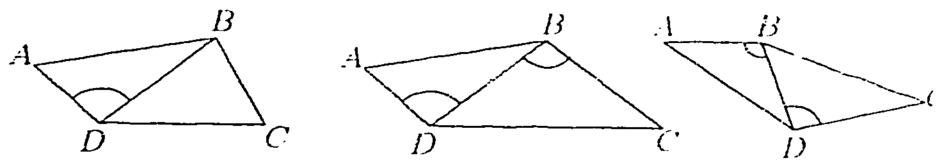


Figure 5.0.2: Adding BD to Q can easily leave obtuse angle or angles

edges of Q. Figure 5.1.1 illustrates that altitudes H_D and F_D from D exist, and altitude F_B from B exists, but altitude H_B from B to edge DC does not exist (within Q, that is!).

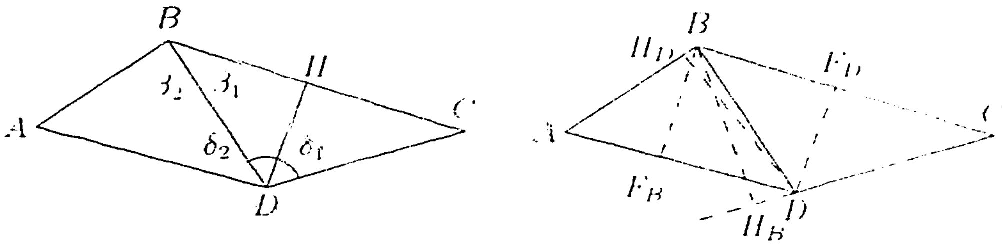


Figure 5.1.1: Left: Obtuse angles can be resolved with altitudes as needed. Right: Not all altitudes will exist, but those necessary will exist.

Still, using Lemma 2 to detect obtuse angles remaining after adding DB to Q, and then Lemma 1 to resolve them by dropping an altitude through them, will work. If δ_1 is obtuse (as in Figure 5.1.1), then β_1 cannot be; so an altitude cannot be needed from B to edge DC. Similarly with δ_2 being obtuse preventing β_2 from being obtuse, and vice versa. It is quite possible that two alternate interior angles will be obtuse - that is, δ_1 and β_2, or β_1 and δ_2. In that case two nailed added edge vertices are used, as in Figure 5.1.2.

Since it cannot be foretold whether 0, 1 or 2 obtuse angles will remain after adding edge DB to Q, no tests are skipped in the following basic algorithm. It should be possible to make it a bit more sophisticated, in that, given that both δ and β are $< 180°$, it is certain that if δ_1 is obtuse, neither its other "half", δ_2, nor the obtuse partition in its triangle, β_1, will be; and similarly for the rest of δ_2, β_1 and β_2.

In general, then, a simple Nailed Vertex algorithm is:

Add edge DB to Q.
IF (corner $B \in \angle DC$) THEN
/* angle β_1 is obtuse */
Drop altitude from D to V on edge CD.

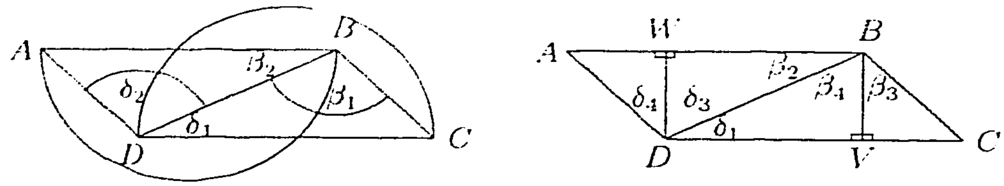


Figure 5.1.2: A corner of Q within a semicircle $\Rightarrow$ an obtuse angle; resolve with altitude. Similarly for δ_1 and/or β_2 obtuse.

```
ENDIF
IF (corner D ∈ ⊙AB) THEN
    /* angle δ2 is obtuse */
    Drop altitude from D to W on edge BA.
ENDIF
IF (corner B ∈ ⊙DA) THEN
    /* angle β2 is obtuse */
    Drop altitude from B to X on edge DA.
ENDIF
IF (corner D ∈ ⊙BC) THEN
    /* angle δ1 is obtuse */
    Drop altitude from B to Y on edge BC.
ENDIF
```

Right Angle Bounds from B and D can also be used as the test to detect which situation is occurring; but it is better to use the Semicircle Rule, as simpler computation is required (and the resulting tiling would be no different). However, Right Angle Bounds more easily show that in the worst case, four triangles (rather than three, as in Case 1) are necessary.

Consider Figure 5.1.2: since adding DB to Q leaves δ_2 obtuse, $B \notin$ Right Angle Bound ADV (where V would fall on edge AB). Likewise, since adding DB to Q leaves β_1 obtuse, $D \notin$ Right Angle Bound CBX (where X would fall on edge CD). Thus, form Right Angle Bounds CDU, ADV where $V, U \in AB$; likewise, form Right Angle Bounds CBX, ABY where $X, Y \in CD$. If $ADW, W \in [V, U]$, and $CBZ, Z \in [Y, X]$ can be formed to guarantee nonobtuse angles at D and B, then Q is partitioned into two nonobtuse triangles ADW and CBZ, and a quadrilateral $WBZD$ is left in the middle, which must be tiled by at least two triangles. Thus three triangles are not sufficient to tile a worst-case instance of $Q \in$Case 2o. Also, adding DB to Q will at worst leave two obtuse triangles ADB and BCD (with obtuse angles δ_2 and β_1, respectively, as in Figure 5.1.2), both of which by Lemma 1 can be tiled nonobtusely via an altitude from the obtuse angle. Thus four triangles are also the maximum necessary in the worst case.

So, *the above algorithm is guaranteed.*

The above discussion also shows how the Floating Vertex method can be applied to $Q \in$ Case 2o. In a case where adding DB to Q would leave two obtuse triangles (as in Figure 5.1.2), adding instead the two altitudes through the obtuse angles that would remain (for example, in Figure 5.1.2, these are edges DW and BV (and not DB), or in Figure 5.1.1, adding edges BF_B and DF_D) leaves an interior quadrilateral - for example, F_BBF_DD - with two opposite 90° angles; hence of the two remaining angles, either both are also 90° angles, or one is obtuse and one is acute (thus an inner Case 1). Thus at least one of the altitudes is guaranteed to partition an obtuse angle into nonobtuse angles.

For the inner Case 1, it is possible that a viable area may exist for the Floating Vertex to be applied. This is basically viewed as adding DB and both altitudes as indicated above, then, if it will leave no obtuse angles, switching the diagonal and moving one or both altitude basepoints into a viable area. For instance, in Figure 5.1.2, replace DB by VW; then W can be moved both to the right (closer towards B) up to the point where angle $\angle ADW$ would be a right angle; and to the left, no farther than its original position (the altitude basepoint). Right Angle Bounds from D intersected with the altitude basepoints from D and its adjacent vertex would delimit the viable area, if it exists (as in Case 1). Similarly it is possible that V can be moved to the left along CD, to tile Q in more of a "zigzag" pattern. This would still result (when it works) in tiling Q into four nonobtuse triangles, as in the worst case, but more flexibility of vertex positioning would be gained.

Therefore:

Minimum (best case): two nonobtuse triangles, no added vertices (interior or boundary).

Maximum (worst case): four right triangles, using two added nailed edge vertices.

5.2 Allowing Straight Angles

Assume, without loss of generality, that $\delta = 180°$ and $\beta \in (90, 180)$. The other case is simply a matter of switching vertex labels and angle labels.

As in Case 1, $\delta = 180°$ makes D a vertex on side AC (and Q can be oriented so that this is a 'base') of obtuse triangle $\triangle ABC$, where angle β is obtuse. If $D = E$, where E is the altitude basepoint from B to AC, then (by Lemma 1) adding edge DB to Q tiles Q satisfactorily. This is a convenient but not too likely situation.

More likely D will not lie at the altitude basepoint of B. Then joining D to B will leave one or more of $\delta_1, \delta_2, \beta_1$ and β_2 obtuse. It is impossible to have one of β_1 or β_2 obtuse without having either δ_2 or δ_1, respectively, obtuse as well; for if neither of δ_1 nor δ_2 is obtuse, they must both be right angles, and so by Lemma 1 DB is an altitude resolving β into nonobtuse angles.

Thus, the possibilities are that adding DB to Q leaves

- no obtuse angles;
- only angle δ_1 or δ_2 obtuse;
- a pair of angles ($[\beta_1, \delta_2]$ or $[\beta_2, \delta_1]$) obtuse.

5.2.1 Using the Semicircle Rule

Obtuse angles remaining after adding edge DB to Q can be resolved (by Lemma 1) with an altitude through them. Again Lemma 2 can be used to detect obtuse angles without calculating angle measures. See Figure 5.2.1.

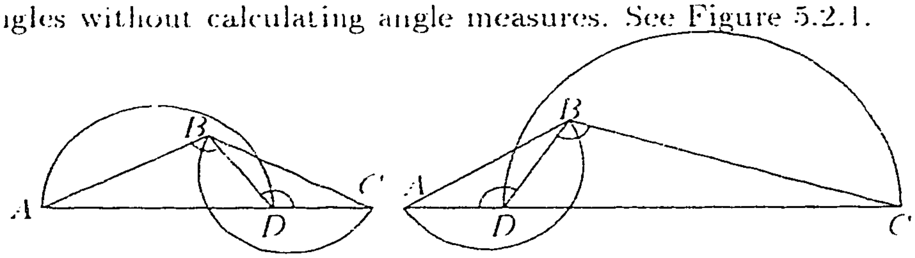


Figure 5.2.1: Lemma 2 detects obtuse angles; again, resolve via altitudes.

The algorithm in this case is basically an extension of that for a Case 1 quadrilateral, with tests added to check β_1, β_2 after checking δ_1, δ_2.

```
Add edge DB to Q.
IF (D ∈ ⌒BC) THEN
   /* δ1 is obtuse */
   Drop altitude from D to H on edge CB.
   /* It's possible that β2 is obtuse. */
   IF (B ∈ ⌒AD) THEN
      /* β2 is obtuse */
      Drop altitude from B to F on edge AD.
   ENDIF
ENDIF
IF (D ∈ ⌒AB) THEN
   /* δ2 is obtuse */
   Drop altitude from D to H on edge AB.
   /* It's possible that β1 is obtuse. */
   IF (B ∈ ⌒CD) THEN
      /* β1 is obtuse */
      Drop altitude from B to F on edge DC.
  ENDIF
ENDIF
```

Like the previous situation (where neither δ nor β were straight angles), Right Angle Bounds can be used to detect obtuse angles, but it is simpler to use the Semicircle Rule.

The best and worst case results are the same as before: at minimum, two right triangles tile Q, with one edge and no vertices added to Q; at worst, four right triangles tile Q, with three edges and two nailed vertices added to Q.

Chapter 6

Case 2a: Q has Two Adjacent Obtuse Angles

Suppose γ, δ are the angles larger than $90°$. As in the other cases, the situation where one of γ or δ is a straight angle is dealt with last. Therefore, until explicitly stated, in discussing Case 2a quadrilaterals it is assumed that neither γ nor δ is a straight angle.

6.1 Introduction

Let $Q = ABCD$ with angles $\delta, \gamma \in (90, 180)$. Q is considered to be oriented so that edge BA acts as a horizontal "base" (though the orientation should not really matter).

Simply drawing diagonals leaves obtuse angles - if not at $\delta_1, \delta_2, \gamma_1$ or γ_2, then elsewhere in Q. (See Figure 6.1.1.)

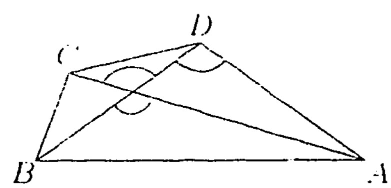


Figure 6.1.1: Diagonals can resolve some obtuse angles but create others.

Trying to apply the Nailed Vertex method to Case 2a fails; to attempt it, retain a diagonal, resolve the other obtuse corner by an altitude, then make the altitude basepoint (situated on the diagonal) into a degree-five interior vertex by connecting it to the other vertices of Q and to a nailed altitude basepoint. See Figure 6.1.2, left illustration. But obtuse angles can remain (i.e. δ_1 in the illustration), so essentially the Nailed Vertex method is unsuitable for Case 2a quadrilaterals

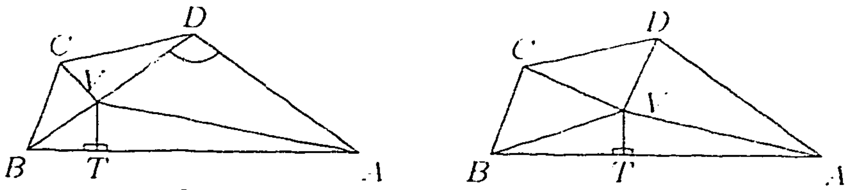


Figure 6.1.2: (Left) Altitude from C marks location of V; (Right) "Bend" the BD diagonal and V can somewhat "float", still tiling Q acutely.

After a bit of experimenting, it was realized that the interior vertex did not have to be on a diagonal. Consider "bending" the diagonal; then the interior vertex becomes more properly a degree-five interior vertex, where four edges connect to Q's corners and the last edge must become a nailed added edge vertex. See Figure 6.1.2, right illustration.

The interior vertex would have to be positioned so as to satisfactorily resolve both adjacent obtuse corners of Q, and none of the five angles of V could be obtuse. Even $90°$ angles would not be really desirable at V, as they would be sensitive to becoming obtuse if the vertex was shifted at all (for whatever reason). Hopefully the already-acute corners α and β of Q would not be partitioned into *very* small angles, but that could be only a hope, not a constraint.

However, since the aim is to use as few added vertices as possible, it seems to be even more useful to let the interior vertex "sink" to the base, and try to find a viable area for W as a Floating Vertex on the base of Q in Case 2a quadrilaterals. This uses only one added vertex and adds (at best) two edges to Q. In the final tiling of Q by the Floating Vertex method - if it works! - let the angle designations be as follows (see Figure 6.1.3):

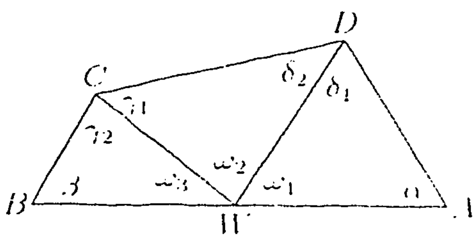


Figure 6.1.3: Case 2a angle designations for Floating Vertex method.

It is certainly not necessarily the case that edge AB is the longest edge, as it happens to be in most (or all) of the Case 2a figures herein. However, even if edge BC or AD is the longest edge (it cannot very well be the top edge CD), Lemma 4 guarantees that altitudes to edge AB from obtuse corners C and D exist; so edge

AB will always be the "base" edge, and edge AB is where a viable area is looked for, in which to place W, for the Floating Vertex Method.

6.2 Using the Semicircle Rule

Again, the attempt is to try to delimit a viable area in which to place W by using Lemma 2. This is tried first because Lemma 2 relies on comparing distances, not on calculating lines and intersection points (as an implementation using Lemma 3 would); so it is much simpler. If it does not work entirely, whatever does work can possibly be retained for further use; those aspects for which Lemma 2 is not suitable can be attempted to be guaranteed (if possible) by some other approach.

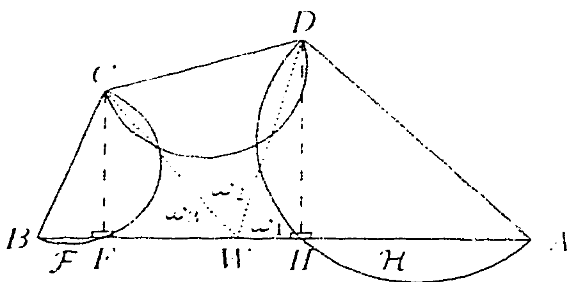


Figure 6.2.1: ω-angle guarantees and regions $\mathcal{F}, \mathcal{H}$ defined by Semicircle Rule.

See Figure 6.2.1. Similar to Case 1, the intersection of $\frown BC$ with base AB indicates the altitude basepoint F from corner C; likewise $\frown DA \cap AB = H$, the altitude basepoint from D. By Lemma 1, both F and H are guaranteed to exist on AB, since β and α are both nonobtuse. Again, altitude basepoints F and H delimit regions $\mathcal{F} = [B, F)$ and $\mathcal{H} = (H, A]$. It is guaranteed that interval $[F, H] \neq \emptyset$, as $F = H \Leftrightarrow C = D$ (and then Q is a degenerate quadrilateral).

As in Case 1, semidisk $\frown CD = \mathcal{Z}$ may or may not touch or intersect base AB, depending on the geometry of Q. (Recall that by the Semicircle Rule, $W \in \mathcal{Z} \Rightarrow \angle CWD = \omega_2$ will be obtuse.) However, it *is* always guaranteed that $[F, H] \cap (AB - \mathcal{Z}) \neq \emptyset$. If $\mathcal{Z}$ does not intersect AB, or does so at one point Z_0, this is obvious; it is perhaps less obvious but still true when $\mathcal{Z}$ "intrudes" into AB, so that $AB - \mathcal{Z} = [A, Z_1] \cup [Z_2, B]$. If $\mathcal{Z}$ could "enclose" $[F, H]$, then none of ω_1, ω_2 or ω_3 would be guaranteeable (to be nonobtuse).

To see that $\mathcal{Z}$ never "encloses" both F and H, assume first that $CD \| AB$. Then $CF, DH \perp CD$ and $\frown CD = \mathcal{Z}$ having CD as diameter, means there is always space between CF, DH and $\mathcal{Z}$ until $F = C$ and $H = D$ (and then Q is degenerate). This is very similar to Section 4.3.6 (see Figure 4.3.11). Next, assume CD is not parallel to AB; however, CF and DH are always parallel, so that if one of them "shifts into" the $\mathcal{Z}$ region, the other will correspondingly shift away from $\mathcal{Z}$. Thus one of F or H may be $\in \mathcal{Z}$, but not both.

So there is *always* a viable area $[F, H] \cap (AB - \mathcal{Z})$ in which to place W that guarantees nonobtuse angles ω_1, ω_2 and ω_3.

By Lemma 2,

- $W \notin \mathcal{F} \Rightarrow \omega_3$ nonobtuse.
- $W \notin \mathcal{H} \Rightarrow \omega_1$ nonobtuse.
- $W \notin \mathcal{Z} \Rightarrow \omega_2$ nonobtuse.

Observations:

It is not possible for both α and β to be right angles because both γ and δ are obtuse. Thus, it may be that $B = F$, or $A = H$; but both cannot simultaneously be true in any instance of a Case 2a quadrilateral.

Also, the higher obtuse corner vertex cannot be adjacent to a right angle base corner, i.e., if C is higher than D, it cannot be that $B = F$, or conversely if $D_y > C_y$, it cannot be that $A = H$. See Figure 6.2.2. For example, if C is higher than D, for angle C to be obtuse, edge CD must lie above the dashed $90°$ line. So if $\beta = 90°$, $D_y > C_y$, and if $\alpha = 90°$, $C_y > D_y$.

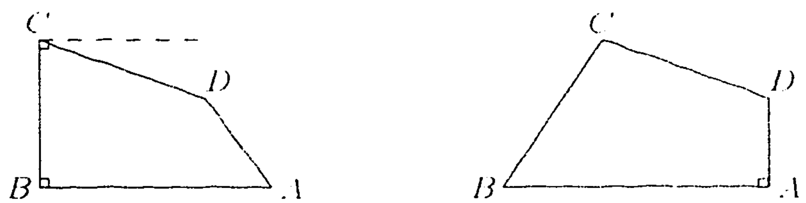


Figure 6.2.2: The higher corner cannot be obtuse if it is adjacent to a right angle lower corner; only the lower obtuse corner can.

To continue trying to use Lemma 2 to outline/implement the Floating Vertex method for Case 2a quadrilaterals, the idea is to continue subtracting, from Q, areas in which it is known that one of $\omega_1, \omega_2, \omega_3, \delta_1, \delta_2, \gamma_1$, or γ_2 will be obtuse. Whatever is left of Q after this, is a viable area in which to place W such that each of the angles mentioned above is nonobtuse.

Therefore, as in Figure 6.2.3,

- $D \notin \odot AW$ verifies δ_1 acute.
- $C \notin \odot BW$ verifies γ_2 acute.
- $D \notin \odot CW$ verifies δ_2 acute.

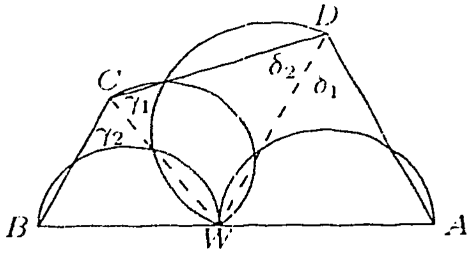


Figure 6.2.3: Four semicircle checks can verify W is placed to guarantee acute angles $\delta_1, \delta_2, \gamma_1$ and γ_2 - but not where to place W on AB.

- $C \notin \ominus DW$ verifies γ_1 acute.

Angles β and α are given as nonobtuse. These checks can *verify* if W is placed to nonobtusely tile Q, but they do not indicate where to place W, that is, they do not delimit a viable area for W, *assuming one exists!*

6.3 Intersecting with Right Angle Bounds

Since Lemma 2 appears inadequate to establish a viable area, let us see what use can be made of Lemma 3. Whatever is useful from applying Lemma 2 can be retained; to wit, guaranteeing angles ω_1, ω_2 and ω_3 by restricting $W \notin \mathcal{F}, \mathcal{H}$ or $\mathcal{Z}$. To use Lemma 3, orient Q so that AB acts as a horizontal "base" and establish Right Angle Bounds $[Y, X]$ from C and $[V, U]$ from D, extending them (and the baseline) as necessary. See Figure 6.3.1.

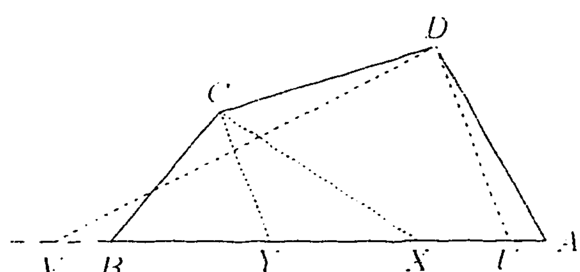


Figure 6.3.1: Right Angle Bounds $[Y, X]$ from C and $[V, U]$ from D.

Any of Y, X, U or V may pass through the sides of Q, as V does in Figure 6.3.1, rather than directly intersecting the base AB. Of course if the two Right Angle Bound lines from an obtuse corner intersect different edges of Q, this means they bracket some vertex of Q, as in Figure 6.3.1 $[V, U]$ brackets B. When this happens, for purposes of comparison in using the Right Angle Bound rule, the Right Angle

Bound line (i.e., DV in Figure 6.3.1) is extended to intersect the horizontal base line.

The Right Angle Bounds can guarantee angles as follows:

- $W \in [Y, X] \cap [B, A] \Rightarrow \gamma_1, \gamma_2$ nonobtuse.
- $W \in [V, U] \cap [B, A] \Rightarrow \delta_1, \delta_2$ nonobtuse.

Therefore to guarantee all of $\gamma_1, \gamma_2, \delta_1$ and δ_2, the viable area is $[Y, X] \cap [V, U] \cap [B, A]$. Combining this with the information gained from using Lemma 2, the viable area is

$$[Y, X] \cap [V, U] \cap [F, H] \cap (AB - \mathcal{Z})$$

where $AB - \mathcal{Z}$ is the area left over after the interaction with semidisk $\odot CD = \mathcal{Z}$; as in Case 1, it may be that $AB - \mathcal{Z} =$ entire edge AB, or $AB - \mathcal{Z} = [A, Z_1] \cup [Z_2, B]$.

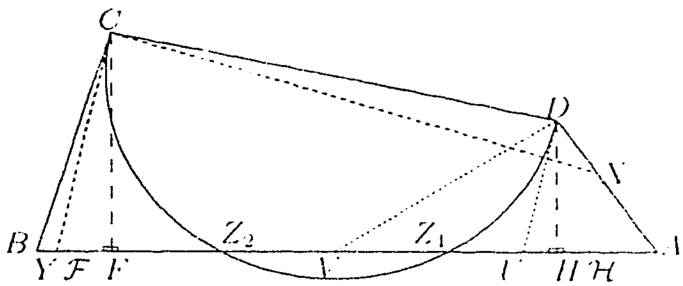


Figure 6.3.2: A fairly complex Case 2a example with viable area $[Z_1, U]$.

Figure 6.3.2 shows a fairly complex situation where the viable area for W is found as follows:

$$[Y, X] \cap [V, U] = [V, U] \quad \Rightarrow \quad [V, U] \cap [F, H] = [V, U] \quad \Rightarrow \quad [V, U] \cap \mathcal{Z} = [Z_1, U]$$

The larger the area in which W can be placed while still guaranteeing all angles, or at least *guaranteeing as many of them as possible, if they cannot all be guaranteed*, the more flexibility is gained.

In all Case 2a quadrilaterals, lines CY and DU are parallel, as they are both defined as 90° with the top edge CD of Q. As γ becomes larger, X draws near to Y, and $X = Y$ when $\gamma = 180°$. Likewise as δ becomes larger, point V approaches U, and $V = U$ when $\delta = 180°$.

Also, $H = U \Leftrightarrow CD \| AB$; then also $Y = F$. See Figure 6.3.3. When D is closer to BA, both Y and U shift to the left, away from overlaying points F and H, respectively. Thus, D lower than $C \Rightarrow [Y, U] \cap [F, H] = [F, U]$. Similarly, C lower than $D \Rightarrow [Y, U] \cap [F, H] = [Y, H]$.

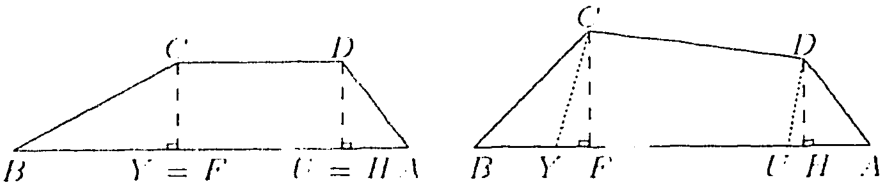


Figure 6.3.3: Left: $CD \| BA \Rightarrow F = Y, H = U$; If $D_y < C_y$, Y and U shift left of F, H; they shift right if $C_y < D_y$.

6.4 Solutions for $Q \in$ Case 2a

At first the interaction between elements of Case 2a quadrilaterals seems unpredictable. There is a fairly complex and/or subtle interaction between angle sizes and edge lengths. Angle sizes can remain exactly the same, and only edge lengths (hence vertex position) differ, creating entirely different situations. It is difficult to grasp the dynamics of this interaction, which is unfortunate, as some understanding of it would aid in developing theories and algorithms for arbitrary quadrilaterals of Case 2a. "Experience alone can teach most people the immense complexity of interactions between many factors, and the mathematical solution of such problems seems to be the only means of clearly conceiving the nature of such phenomena" ([H4]).

Despite this, a number of solutions for $Q \in$ Case 2a can be shown to guarantee a nonobtuse tiling, using at best one added edge vertex, tiling Q into three nonobtuse triangles, and at worst two added edge vertices, tiling Q into four nonobtuse triangles.

In the following discussion, the notation is that $\odot CD = \mathcal{Z}$, where it must be understood that the *boundary* of $\mathcal{Z}$ is an acceptable location for a vertex, creating a right angle, but that the interior is *not*. If $\mathcal{Z}$ intrudes into edge AB (partitioning it), the notation $AB - \mathcal{Z}$ must be understood to include the *boundary* of the semicircle $\odot CD$, but not the interior, so that $AB - \mathcal{Z} = [A, Z_1] \cup [Z_2, B]$.

6.4.1 Solutions With One Added Floating Edge Vertex

First of all, assume that a viable area in which to place W on AB exists; that is, that

$$[Y, X] \cap [V, U] \cap [F, H] \cap (AB - \mathcal{Z}) \neq \emptyset.$$

Then W can be placed as optimally as possible within the viable area, tiling Q into three nonobtuse triangles with one floating added vertex W on edge BA. By definition of the viable area, all angles in Figure 6.1.3 are guaranteed nonobtuse, and Q is tiled satisfactorily. A few examples are shown in Figures 6.4.1 to 6.4.4. As can be seen, placement of W was made with some particular aspect of the resulting tiling in mind, though of course any location within the viable area results in a (slightly) different but valid nonobtuse tiling.

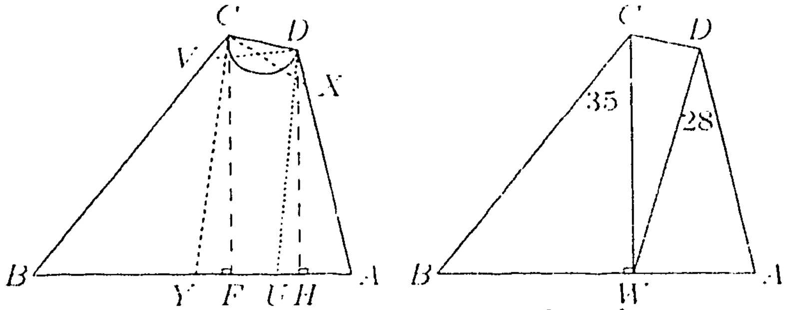


Figure 6.4.1: Example: viable area $= [F, U]$; to maximize δ_1, choose $W = F$.

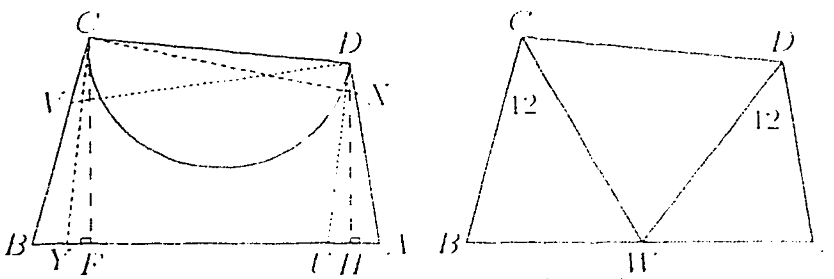


Figure 6.4.2: Example: viable area $= [F, U]$; $\delta_1 = \gamma_2$ when W = midpoint $[F, U]$.

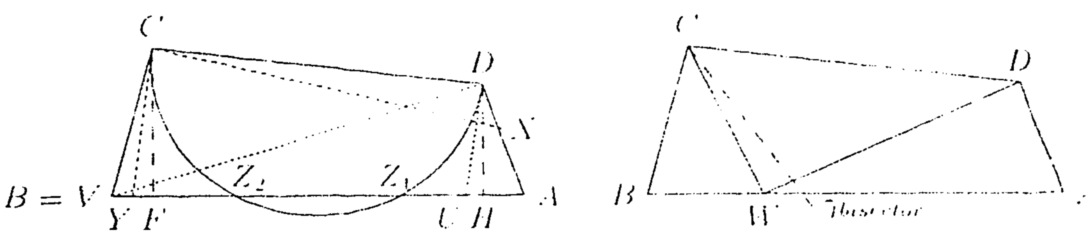


Figure 6.4.3: Example: viable area $= [F, Z_2] \cup [Z_1, U]$; Right: place W in left viable area: choose $W = Z_2$, closest to the γ bisector, to maximize γ_2.

6.5 When No Viable Area Exists

Next assume that the viable area is empty, that is, that

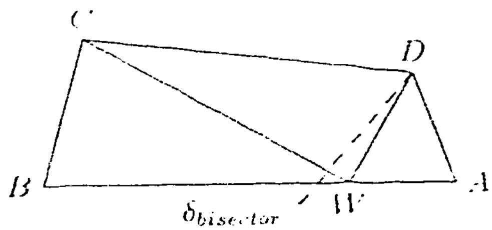


Figure 6.4.4: Same example (viable area = $[F, Z_2] \cup [Z_1, U]$). Now place W in right viable area. Choose $W = Z_1$, closest to the δ bisector, to maximize δ_1.

$$[Y, X] \cap [V, U] \cap [F, H] \cap (AB - \mathcal{Z}) = \emptyset.$$

The solution in this situation requires retaining the altitude from the *higher* of C or D, or, if $C_y = D_y$, either altitude. This is useful as it is guaranteed, by the following Lemma, to partition Q into a right triangle "ear" and an instance of an "inner" Case 1 quadrilateral and Case 1 is known to be solvable.

Lemma 6: Dropping altitudes CF and DH from obtuse corners C and D, respectively, of $Q \in$ Case 2a, resolves at least one of the obtuse corners into two acute angles. If only one corner is resolved into two acute angles, it is the "higher" of C and D, where "higher" means the corner with the greater length altitude from base BA. If both obtuse corners of Q are resolved by dropping altitudes, then $C_y = D_y$, and the interior section of Q is a rectangle.

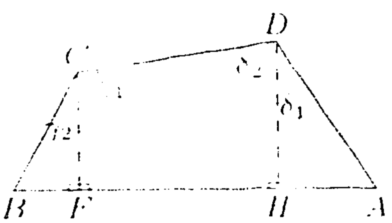


Figure 6.5.1: One of γ or δ (or both) is resolved by altitude to BA.

Proof: See Figure 6.5.1. Drop altitude CF from obtuse corner C of Q, and likewise drop altitude DH from obtuse corner D. We then have two right triangles as "ears" on either side, and a quadrilateral in the centre, of which the bottom two angles are both right. Thus the sum of the top two angles is 180°. Then either both the top angles in the interior section are right angles (and so the interior section of Q is a rectangle), or one is obtuse and one is acute. □

For the moment, and without loss of generality, assume $D_y > C_y$ as in the accompanying diagrams. The other situation merely involves switching vertex and angle labels. Therefore assume edge DH is added to Q, cutting off right triangle ear $\triangle AHD$ and leaving inner Case 1 instance $Q_1 = HBCD$ where the angles will be designated $\eta(= 90°)$, β, obtuse γ, and acute $\delta(= \delta_2)$. See Figure 6.5.2.

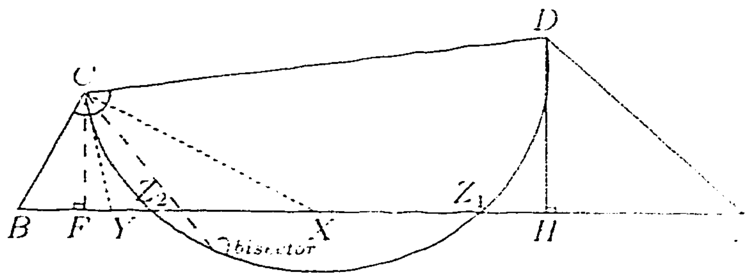


Figure 6.5.2: Keep altitude $DH \Rightarrow$ Case 1 = $HBCD$ remains, with γ obtuse.

As recalled from the Case 1 discussion, adding an edge from C to its opposite corner H either (i) resolves γ into acute γ_1, γ_2; (ii) leaves γ_2 obtuse; or, (iii) leaves γ_1 obtuse.

6.5.1 Proof That γ Cannot Be Resolved by CH

Given that in the "original" $Q = ABCD$, $[Y, X]$ [illegible] $= \emptyset$, it is impossible for situation (i) to arise. The next Lemma aids in showing this.

Lemma 7: In a Case 2a quadrilateral, the Right Angle Bounds from the higher obtuse corner strictly bracket the basepoint of the altitude from the higher obtuse corner to the base BA; whereas the Right Angle Bounds from the lower obtuse corner do not bracket the basepoint of the altitude from the lower obtuse corner to the base BA. If $C_y = D_y$, then $Y = F$ and $U = H$, and so the Right Angle Bounds may be said to bracket the altitude basepoint.

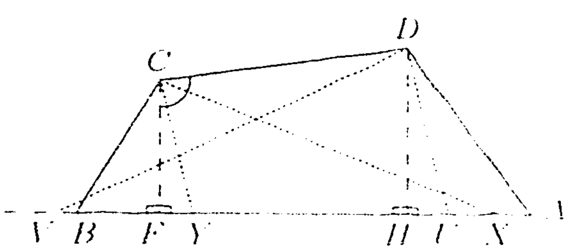


Figure 6.5.3: Lemma 7: Right Angle Bounds from the higher corner bracket altitude basepoint from that corner; not so for the lower corner of $Q \in$ Case 2a.

Proof: See Figure 6.5.3. Without loss of generality, let D be higher than C. Then by Lemma 6, $\angle DCF > 90°$, and since $\angle DCY = 90°$, Y is to the right of F. (Recall that $Y = F \Leftrightarrow CD \| AB$; then neither C nor D is higher than the other, and that is not the case here.) Since $CF \| DH$ (both are altitudes to BA) and $CY \| DU$, U is to the right of H. From Lemma 6, D being the higher corner of Q means that altitude DH partitions obtuse corner δ into acute $\delta_1 = \angle ADH$ (since in triangle $\triangle AHD$, angle $\angle AHD$ is a right angle), and $\delta_2 = \angle HDC$. Since by definition $\angle ADV = 90°$ and δ_1 is acute, V is always to the left of H. Therefore, corner D higher than corner $C \Rightarrow [U, V]$ strictly brackets H.

As for the lower obtuse corner C, as just mentioned, Y is to the right of F. When $\gamma = 180°, X = Y$. But since, in $Q' = HFCD$, $\gamma_1 + \delta_2 = 180°$, and $\delta_2 \neq 0°$, then $\gamma_1 < 180°$. So $X \neq Y$, but rather X is to the right of Y, and so $[X, Y]$ does not bracket F. □

This says nothing about $[U, V]$ bracketing F, which may or may not happen (in Figure 6.5.3 it does happen); likewise it says nothing about $[X, Y]$ bracketing H or not (which again in Figure 6.5.3 does happen).

Since in the "original" $Q = ABCD$, $[Y, X] \cap [V, U] \cap [F, H] \cap (AB - \Xi) = O$, and $C_y \leq D_y$, the Right Angle Bounds $[V, U]$ from D bracket H (even if $C_y = D_y$ and then $U = H$, this can be said to be true). Then it is impossible for this Case 1 to be resolved by joining the obtuse corner C to its opposite corner H; for, were this possible, the Right Angle Bounds $[Y, X]$ from C would have to bracket H. But since in this solution, altitude DH is kept *because* it resolves δ into acute δ_1 and δ_2, then by Lemma 6 it must be that $D_y \geq C_y$, and by Lemma 7 the Right Angle Bounds $[V, U]$ from D bracket the altitude basepoint H. Then if Right Angle Bounds $[Y, X]$ from C bracketed H also, it *must be* that $[Y, X] \cap [V, U] \cap [F, H] \neq O$. This area could never be within Ξ, for if it were, η would be obtuse, and it is known that η is a right angle. So simply adding edge CH to Q - the optimal Case 1 solution - in addition to edge DH (assuming $D_y \geq C_y$), will never solve Q_1.

Therefore, *only* situations (ii) and (iii) need be "solved". Situation (ii) is that the remaining obtuse angle (γ_2) is towards the "base" AB; let $Q \in$ Case 2a in which this situation develops be said be Q of Class BASE. Situation (iii) is that the remaining obtuse angle (γ_1) is towards the retained altitude; similarly, let $Q \in$ Case 2a in which this situation develops be said be Q of Class ALT.

6.6 Guaranteed solutions for Class BASE

Again, assume $C_y < D_y$, and altitude DH is retained to partition Q into a right triangle ear $\triangle AHD$ and an inner instance of Case 1 $Q_1 = HBCD$, with γ obtuse. (The mirror reverse case occurs when $D_y > C_y$ and altitude CF is retained to partition Q into right triangle ear $\triangle BFC$ and Case 1 $Q_1 = AFCD$ with δ obtuse.)

Since $\gamma < 180°$, $Y \neq X$; then $[Y, X]$ is on AB; this information, plus knowing that H is strictly to the right of X (by definition of Class BASE), and likewise Y is strictly to the right of $F \Rightarrow$ a viable area exists on AB in which to place W and tile $Q' = HBCD$ by the Floating Vertex method. Even though a nailed vertex (an altitude basepoint) has to be added, the other added edge vertex W can float to some degree (depending on the extent of the viable area). This tiles Q into four nonobtuse triangles, at least two of which are right triangles.

It is guaranteed that a Floating Vertex solution exists for the inner Case 1 instance, that is, that a nonempty viable area exists on BH (or FA, in the mirror reverse case). To see this, recall that $X \neq Y$ unless γ is a straight angle (which it has been assumed is not the case; that situation is dealt with later); so if the viable area for Q_1 is $[Y, X]$, it is therefore nonempty. When the viable area is $[Y, Z_2]$, this also is guaranteed nonempty; see Figure 6.6.1. As base BA rises towards the top of Q, both $|Y, Z_2|$ and $|Z_1, U|$ shrink. However, since $CY \perp CD$, $DU \perp CD$, and Ξ has CD as diameter, it is impossible that semicircle Ξ could intersect CY or DU. At worst, $Y = Z_2$ only when $Y = C = Z_2$, and then Q is degenerate; similarly with respect to D, U and Z_1. Thus if the viable area is $[Y, Z_2]$, it is also nonempty. So solving the inner Case 1 by the Floating Vertex method is *always* possible.

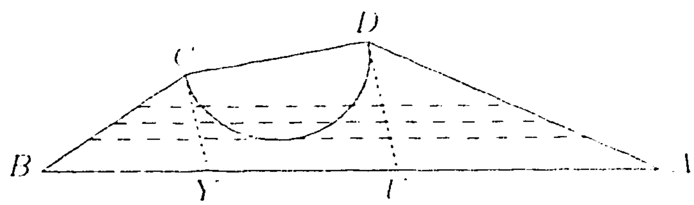


Figure 6.6.1: As BA edge rises, $|\Xi, CY|$, $|\Xi, DU|$ shrink, but do not disappear.

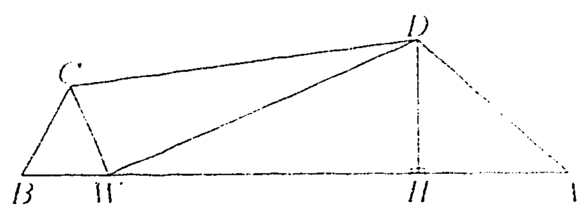


Figure 6.6.2: Place W in viable area $[Y, Z_2] \Rightarrow$ 2 right + 2 acute triangles.

Figure 6.6.2 shows a completed tiling for the example in Figure 6.5.2. W is placed at Z_2, as it is the closest point in the viable area to the γ bisector, so angle $\omega_2 = \angle CWD = 90°$.

Therefore, a guaranteed solution is to add vertices F or H and W to AB, where the altitude basepoint F or H is nailed, and W can float (within a viable area as defined); and three edges (CF or DH, and CW, DW) resolving Q into four triangles (at least two of which are right triangles).

6.7 Guaranteed solutions for Class ALT

Assume $C_y = D_y$ and, arbitrarily, altitude CF is retained to create the inner Case 1 $Q_1 = AFCD$. But $C_y = D_y \Rightarrow CD \| AB$: then Q_1 is a trapezoid with right angles $\angle AFC$ and $\angle CDA$. Then it is impossible that adding DF to Q_1 would leave angle $\angle FDC = \delta_2$ (towards retained altitude CF) obtuse, since triangle $\triangle FCD$ is a right triangle (with right angle $\gamma = \gamma_1$). So for $Q \in$ Class ALT of Case 2a, $C_y \neq D_y$.

Therefore assume without loss of generality that $C_y > D_y$, but do not *retain* altitude CF to form the inner Case 1 as before: just "sketch in" the altitude (for reference). Since $Q \in$ Class ALT, Right Angle Bounds from the (lower) obtuse corner D do not enclose the altitude basepoint F. Thus Right Angle Bound CDU *must be* such that U is to the left of F. Also, $CY \| DU \Rightarrow V$ to left of F also. Then, $\delta \leq 180° \Rightarrow U_x \geq V_x$; that is, V either overlays U (if δ is a straight angle) or is strictly to the left of U. See Figure 6.7.1.

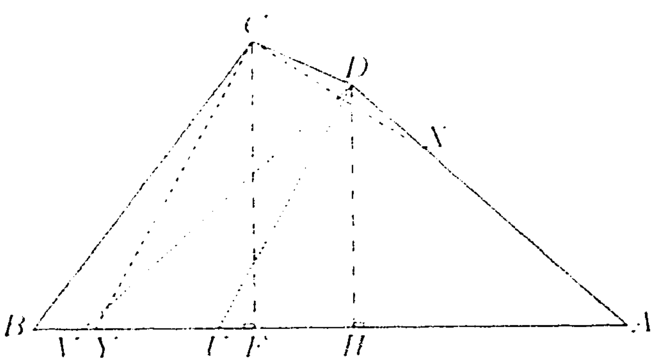


Figure 6.7.1: Class ALT: Edges $CF + DF$ leave δ_2, towards altitude CF, obtuse.

Thus in Class ALT (and this may be taken as an alternate definition), Right Angle Bounds from the obtuse corner that is *not* resolved by its altitude (this is the "lower" obtuse angle), lie on AB strictly *within* the region from the basepoint of the *other* altitude to the corner of the cut off right triangle "ear" of Q; that is, when $C_y < D_y$, Right Angle Bounds $[Y, X]$ from C lie within $\mathcal{H} \cup [A, D]$ (not shown); when $D_y < C_y$, Right Angle Bounds $[V, U]$ from D lie within $\mathcal{F} \cup [B, C]$ (as in Figure 6.7.1).

Without loss of generality, assume $C_y > D_y$. It is known that $W \in [V, U] \cap [Y, X]$ guarantees angles $\delta_1, \delta_2, \gamma_1$, and γ_2 to be nonobtuse. Y is to the left of U on AB since

$CY \| DU$ and X is to the right of F since $\angle BCF < 90°$, so $\emptyset \neq [V,U] \cap [Y,X] \subset \mathcal{F}$.

By definition of F as semicircle $\frown BC$ intersected with edge AB, $W \in (B,F) \Rightarrow$ angle $\angle BWC = \omega_3$ is obtuse. Then, since the sum $\angle BWC + \angle CWD + \angle DWA = \omega_3 + \omega_2 + \omega_1 = 180°$, and ω_3 is known to be obtuse, $\omega_2 + \omega_1 < 90°$; so both ω_2 and ω_1 are acute.

Therefore, ω_1, ω_2 are guaranteed acute by placing $W \in [Y,X] \cap [V,U] \subseteq \mathcal{F}$.

Thus placing W in the viable overlap $[Y,X] \cap [V,U] \subseteq \mathcal{F}$ (for the mirror case, $\mathcal{H}$) guarantees angles $\delta_1, \delta_2, \gamma_1, \gamma_2, \omega_2$, and ω_1 to be nonobtuse. (The mirror reverse situation is not discussed, but it should be obviously the same with exchanging of vertex and angle labels, notably $\omega_3 \Leftrightarrow \omega_1$.) This tiles Q into three triangles (see Figure 6.7.2, left illustration): nonobtuse triangles $\triangle AWD$ and $\triangle WCD$, and obtuse $\triangle BWC$ with $\angle BWC = \omega_3$ obtuse. By Lemma 1 ω_3 is easily resolved by an altitude from W to T on BC, as in Figure 6.7.2, right illustration. (The mirror reverse situation would place T on AD.)

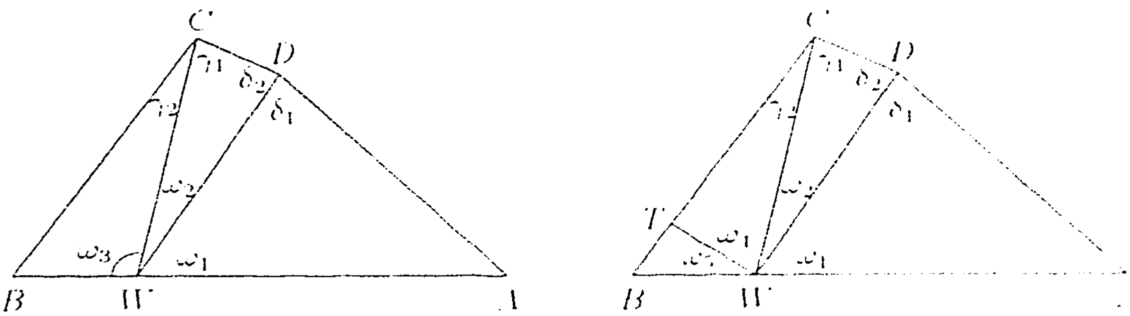


Figure 6.7.2: Left: Place W in $[V,U] \in \mathcal{F} \Rightarrow \omega_3$ still obtuse.
Right: The solution for Class ALT: use Floating W and semi-nailed T.

Thus Q in Class ALT is guaranteed to be nonobtusely tiled into four triangles, at least two of which are right triangles, using added edge vertices W on AB and T on BC (or AD). Since the position of W can float within $[Y,X] \cap [V,U]$ (assuming $\delta, \gamma < 180°$), and the position of T is solely dependent on the position of W, T is only "semi-nailed"; that is, the positions of both vertices are flexible to some degree, with T's position dependent on W's position.

6.8 Other Two-Vertex Solutions

A couple of other solutions are presented here; they are as "optimal" as the previous ones with respect to number of triangles used to tile Q, and some of them may be simpler to implement, since they rely on altitude basepoints (easier to calculate?) than Right Angle Bounds. Also, when flexibility of position for the added edge vertices is an issue (during the process of "matching" added edge vertices in adjacent quadrilaterals), one of them may be "more suitable" than the previously discussed methods. At any rate, the "extra" methods show that there *is* a "choice" of more than one solution.

6.8.1 Using Altitudes Exclusively in Class BASE

Assume that $C_y < D_y$, and so altitude DH has been retained. Adding altitude CF to the inner Case 1 Q_1 cuts off a right triangle ear $\triangle FBC$, leaving *yet another* inner Case 1 $Q_1' = HFCD$, still with $\gamma' = \angle FCD$ obtuse. However, then Right Angle Bound FCX is such that $CX \| AB$, and thus $H \in [Y, X]$; so adding edge CH to Q assuredly resolves the remaining obtuse angle $\angle FCD$. This is illustrated in Figures 6.8.1 and 6.8.2. This tiles Q into four triangles, three of which are right triangles, and adds two nailed vertices F and H to edge AB.

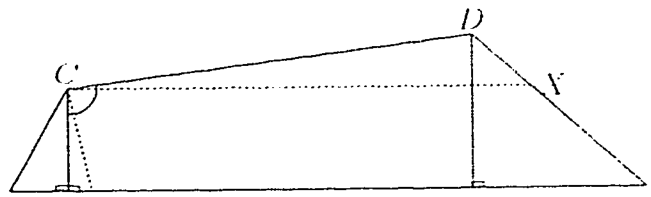


Figure 6.8.1: Keep both altitudes CF and DH $\Rightarrow$ two triangle "ears" + inner Case 1 = $HFCD$ with γ obtuse; but now $H \in [X, Y]$.

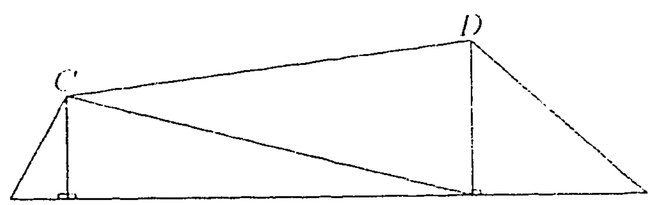


Figure 6.8.2: Since $H \in [X, Y]$, resolve γ with a diagonal.

Or, this can be seen equivalently as adding edge CH to $Q_1 = HBCD$, and resolving obtuse $\gamma_2 = \angle HCB$ by altitude CF; this is then using the Nailed Vertex method.

This choice of solution (looking at it either way) is quite unappealing; essentially, it is keeping both altitudes and joining either the remaining obtuse corner to its opposite corner in the inner Case 1 (the altitude basepoint of the other obtuse corner); or, if $C_y = D_y$ and $HFCD$ is a rectangle, either diagonal can be used.

Therefore, this guaranteed solution consists of adding both altitude basepoints F and H to AB, and three edges (DH, CF, and CH or DF), resolving Q into four triangles (at least three of which are right triangles). No calculation of Right Angle Bounds is necessary.

6.8.2 Placing T on Top of Q

Assume that $AB - \Xi = [A, Z_1] \cup [Z_2, B]$, and that $[Y, X] \cap [V, U] \cap [F, H] \in (Z_1, Z_2)$. Then placing W in viable area $[Y, X] \cap [V, U] \cap [F, H]$ would leave ω_2 obtuse, but all other angles would be guaranteed nonobtuse. Another solution in this situation is to place W within the viable area $\in \Xi$, and the remaining obtuse angle ω_2 can be resolved by dropping an altitude through it to a nailed vertex T on edge CD. This tiles Q into two right triangles (created by T) in the middle, and two acute "ears" on either side. This is illustrated in Figure 6.8.3.

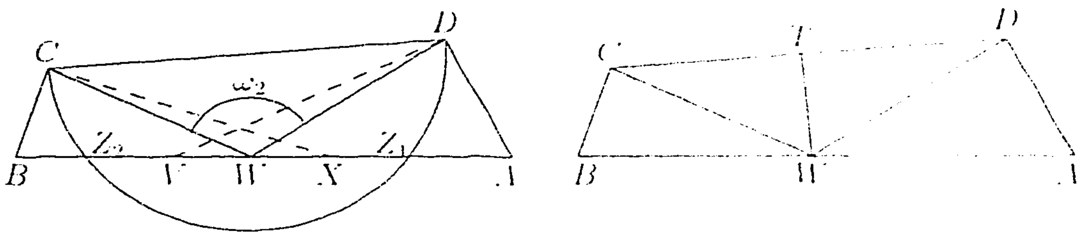


Figure 6.8.3: Left: Place W in viable area (here, $[Y, X] \in \Xi$), leaving ω_2 obtuse. Right: Resolve ω_2 with altitude to T on CD (much more equal triangle sizes).

Using added edge vertices W on AB and T on CD, is again a "semi nailed" method, in that the position of W can float within the viable area, and the position of T is dependent on the position of W. This tiles Q in four triangles (at least two of which will be right triangles).

6.8.3 Allowing Straight Angles

Assume, without loss of generality, that $\gamma = 180^\circ$ and $\delta \in (90, 180)$. The other case is simply a matter of switching vertex labels and angle labels.

This situation is exactly that of Lemma 2 in [BE91]. In the worst case, [BE91]'s solution results in five right triangles, using one nailed interior vertex and one nailed edge vertex; see Figure 6.8.4. In the best case, [BE91]'s solution results in three right triangles, when the perpendicular to BC from C *and* the perpendicular to AD from D intersect very conveniently at a point on AB (which we can call W); see Figure 6.8.5 (the intersection point becomes the nailed added edge vertex). This is a convenient, but not too [illegible] circumstance. Otherwise, [BE91]'s solution results in four right triangles, using two nailed edge vertices: see Figure 6.8.6. In each situation, the added nailed edge vertices are *always* guaranteed to be on the "base" of the obtuse triangle, where the "base" is defined as the edge opposite the obtuse angle.

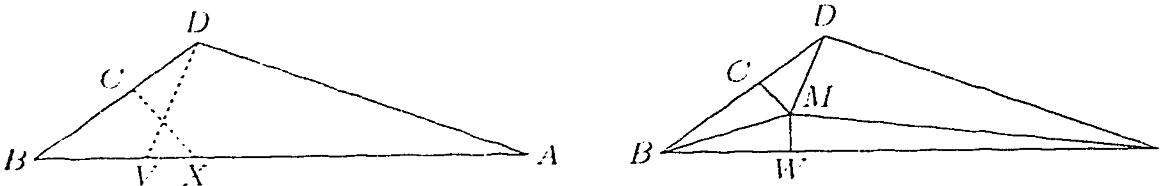


Figure 6.8.4: [BE91] worst case. Left: Perpendiculars from C and Right Angle Bound ADV meet at M, inside triangle; add edges BM, CM, DM and MA; then resolve obtuse $\angle AMB$ with altitude. Right: Result: 5 right triangles.

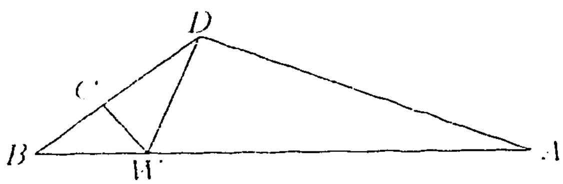


Figure 6.8.5: [BE91] best case. Perpendicular from C and ADV intersect on base edge AB; add that as a nailed vertex W. Result: three right triangles.

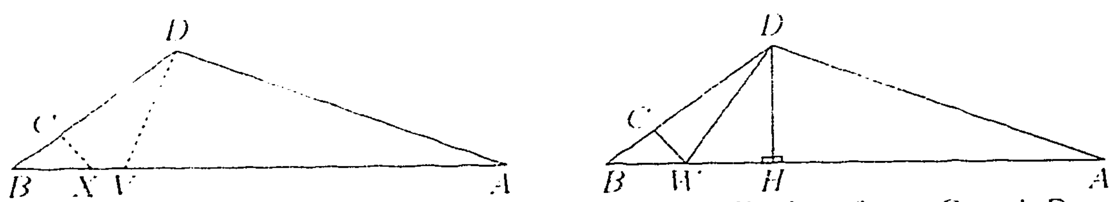


Figure 6.8.6: [BE91] "average" case. Left: Perpendiculars from C and D don't meet inside triangle: Right: add $CX \cap AB = W$ and altitude DH. Right: Result: four right triangles, using two nailed vertices.

The solutions presented in this chapter are all easily extensible to the case of a straight angle. Figure 6.8.7 shows the situation where a viable area - in this case, since $\gamma = 180°$, then $X = Y$ and a viable "spot" is all that remains - exists as $X = Y = [V, U] \cap [F, H] \cap [Y, X]$. Placing W at $X = Y$ tiles Q into three triangles, of which two are right triangles, using one nailed vertex on the "base" edge. The difference between this solution and that of [BE91] is that in [BE91], the best case solution *only* "happens" precisely when $X = Y = V$, since [BE91] only consider one right angle bound line. In the solution presented here, the best case solution is possible whenever $X = Y \in [V, U] \cap [F, H]$, which is much more likely to occur. Also, the [BE91] solution of three right triangles alters subtly to become two right triangles and one triangle that is a right triangle if $X = Y = V$, but an acute triangle otherwise.

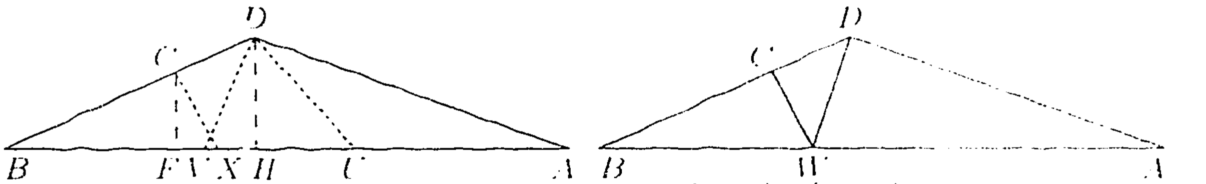


Figure 6.8.7: Left: Viable "spot" $= X = Y \in [V, U] \cap [F, H] \Rightarrow$ use $W = X = Y$. Right: Best case: 1 nonobtuse, 2 right triangles. W still "nailed" at 90° to C.

Figure 6.8.8 illustrates the Class BASE solution, when $X = Y$ lies outside of $[V, U] \cap [F, H]$; altitude DH is retained since $C_y < D_y$, creating an inner Case 1 where adding edge CH would leave γ_2 obtuse. The solution again is to choose $W = X = Y$, a solution exactly the same (in this situation) as that of [BE91].

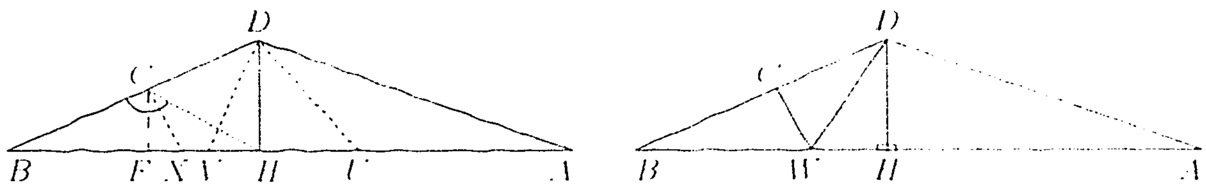


Figure 6.8.8: Left: Class BASE: $[V, U] \cap X(= Y) = \emptyset$: add $DH \Rightarrow$ inner Case 1: adding CH would leave γ_2 obtuse. Right: Class BASE solution: W at viable "spot" $\Rightarrow$ four right triangles using two nailed vertices on AB.

Figure 6.8.9 illustrates the Class ALT solution, when $X = Y$ lies within $\mathcal{H}$. Since the Right Angle Bound lines $CX = CY$ and DV intersect within Q, it is here that [BE91] would add that intersection point M to Q and end up at its worst case solution (five triangles, one nailed edge vertex and one nailed interior vertex). Using the ALT solution, we add $W = X = Y \in \mathcal{H}$, and edges WC and WD. Then the remaining obtuse angle $AWD = \omega_1$ is resolved via an altitude to T on AD. This results in four right triangles, using two nailed edge vertices, which are added to *different* edges of Q.

Observe that in all solutions of Case 2a where one of the obtuse angles is a straight angle, *none* of the solution methods partition the already-acute angles α

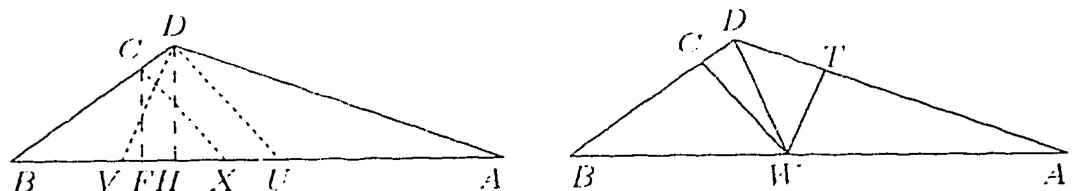


Figure 6.8.9: Left: Class ALT: C's Right Angle Bounds $X = Y$ (= viable spot) lie in $\mathcal{H}$; Right: Class ALT solution: W at viable "spot" $\in \mathcal{H}$ + nailed altitude basepoint T on $AD \Rightarrow$ 4 right triangles, 2 nailed vertices, one each on base, side.

and β - not even the worst case (ALT) solution. [BE91]'s worst case solution does partition both α and β, but perhaps this is considered a reasonable price to ensure that only one nailed edge vertex is added, and to guarantee to which edge it will be added.

6.8.4 Conclusion

Guaranteeing a solution for whatever situation of $Q \in$ Case 2a has been accomplished. All the solution methods are relatively simple. *None* of the solution methods partition the already-acute angles α and β of Q. The established results are:

Minimum (best case): Three triangles tile Q, using one added floating vertex on edge AB.

Maximum (worst case): Four triangles, at least two of which are right triangles, tile Q, using two added edge vertices: either

- floating vertex W on AB and nailed altitude basepoint F or H (on AB).
- floating vertex W on AB and semi-nailed altitude basepoint T on a "side" AD or BC of Q.
- floating vertex W on AB and semi-nailed altitude basepoint T on the "top" edge CD of Q.

Chapter 7

Case 3: Q has Three Obtuse Angles

Let the obtuse angles be β, γ and δ in clockwise order. Note that in Case 3, there cannot be a situation where any one of the obtuse angles is a straight angle; otherwise there could be at most only one other obtuse angle. Therefore, in Case 3, $\alpha \in (0, 90^\circ)$, and each of $\beta, \gamma, \delta \in (90^\circ, 180^\circ)$.

A Few Notation Updates

The given quadrilateral is Q, and its boundary edges (collectively) are denoted as ∂Q.

In Case 3 quadrilaterals, using Lemma 2 (the Semicircle Rule) will create a semicircle with a given edge as a diameter; any such semicircle will be designated as $\ominus IJ$, where IJ is the edge (or line segment) that is the diameter of the semicircle. Using Lemma 3, with three obtuse angles, gives six Right Angle Bounds vertices on ∂Q. These will be $[Y, X]$ from C and $[V, U]$ from D, as before, and $[S, R]$ from B.

Since none of the corners in a Case 3 quadrilateral is followed by two acute angles, Lemma 4 cannot guarantee that any altitudes will exist. It is most likely that some will, but that will be by chance. Partitioned angles are not always subscripted when concentrating on a subpolygon that includes the partitioned angle.

7.1 Using the Semicircle Rule

We can first see if any use can be drawn from applying the Semicircle Rule plus the Right Angle Bound Rule. First, consider how the idea of one interior degree-five vertex plus one added edge vertex can be transferred, if at all, to Case 3 from Case 2a where it was originally tried.

To do this, a viable area within Q must be established in which to place a central vertex E, such that joining E to each of the corners leaves no obtuse angles at those corners. Since this leaves E of degree four, either there are four right angles at E, which is acceptable but not preferred: or there remains at least one obtuse angle at E, which, as before, can be resolved by dropping an altitude through it.

Designate the angles at vertex E by naming $\angle BEC$ as ϵ_1, and the rest $\epsilon_2, ..., \epsilon_n$ in a clockwise fashion. The size of n depends on how many edges must be added to E (and joined to vertices added elsewhere within or on Q) to adequately tile Q.

If E can exist at the intersection point of four semicircles that have edges of Q as diameters, then E can be of degree four and Q is tiled into four right triangles. See Figure 7.1.1. There cannot be an area within Q that is outside all four semicircles, otherwise the sum of the four angles at E would sum to $< 360°$. Therefore E can be outside of at most three semicircles (and so it will definitely be inside one of them).

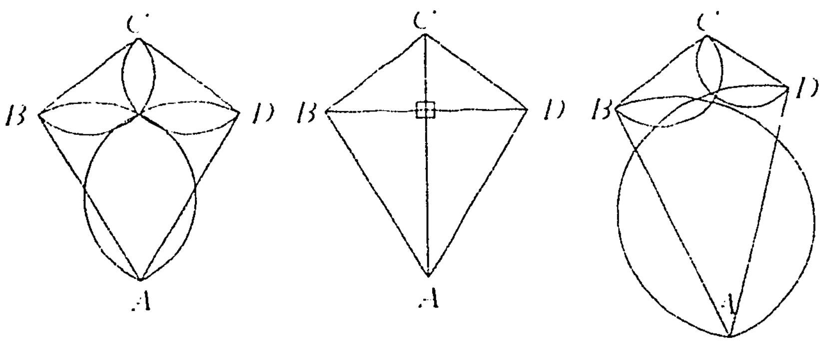


Figure 7.1.1: Case 3 where degree 4 vertex tiles Q: but usually this won't be so.

A more workable idea would be that E should be outside as many semicircles as possible. Placing E within the fourth semicircle creates an obtuse angle at E, in the triangle consisting of the edge of Q (which is the diameter of the fourth semicircle) joined to E. The most obvious method to resolve this obtuse angle is via dropping an altitude to the edge of Q to partition the obtuse triangle into two right triangles (by Lemma 1), adding a nailed edge vertex W.

Basically, the above (inelegant!) method consists of choosing one edge of Q to be the recipient of an added nailed vertex W, since E will necessarily be of degree five (at least). Semicircles from the other three edges of Q are calculated and a spot outside all of them found: E is placed there, connected to each corner of Q, and an altitude dropped from E to the chosen edge of Q to resolve the obtuse angle at E. Thus, four "choices" of solution are immediately available. Each situation is illustrated, in Figure 7.1.2 to 7.1.5.

If nothing else (although inelegant) it *does* provide for some choice in the placement of the added edge vertex, should such flexibility be needed.

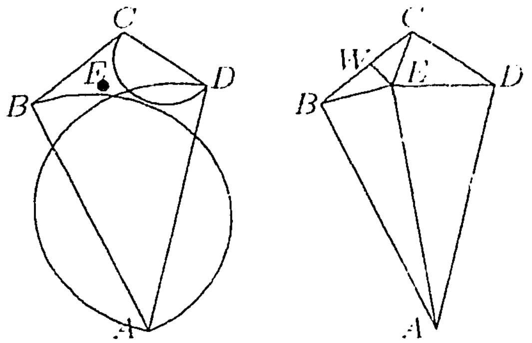


Figure 7.1.2: Choose edge BC for W; place $E \in (Q - (\circ AB \cup \circ CD \cup \circ DA))$.

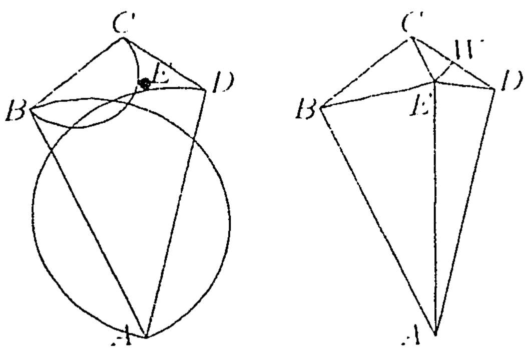


Figure 7.1.3: Choose edge CD for W; place $E \in (Q - (\circ AB \cup \circ BC \cup \circ DA))$.

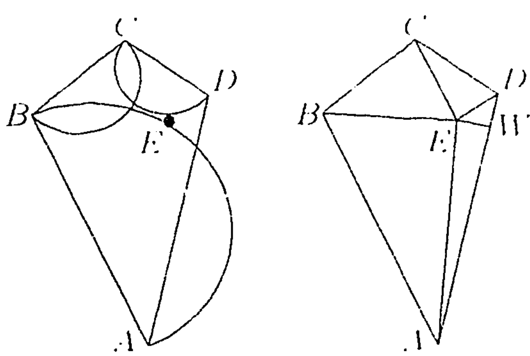


Figure 7.1.4: Choose edge AD for W; place $E \in (Q - (\circ AB \cup \circ BC \cup \circ CD))$.

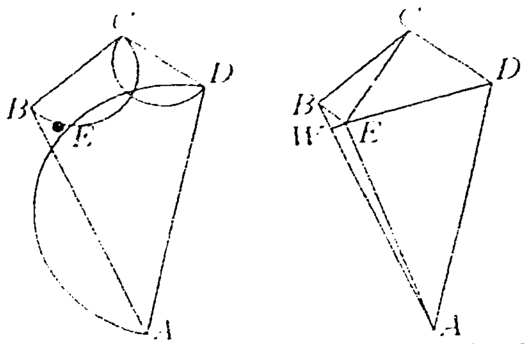


Figure 7.1.5: Choose edge AB for W: place $E \in (Q - (\ominus BC \cup \ominus CD \cup \ominus DA))$.

7.2 Adding the Right Angle Bounds

However, it is not entirely suitable as is: the flaw lies in the fact that, in Cases 1 and 2a, the Right Angle Bounds from an obtuse corner of Q *always* encompass part of the boundary ∂Q of Q. In particular, for Case 2a, the intersection of Right Angle Bounds from the two obtuse angles always only "mattered" (in devising the Floating Vertex method) along ∂Q. In Case 3, however, the Right Angle Bounds from three obtuse angles will intersect so as **not** to include any part of ∂Q, but rather intersect in a region entirely *within* Q. Since β is obtuse, the Right Angle Bounds from B do not include edges AB and BC except at vertex B. Likewise, since γ is obtuse, the Right Angle bounds from C do not include edges BC and CD except at C. Finally, the Right Angle Bounds from D do not include edges CD and DA except at D. Therefore, the intersection of Right Angle Bounds from B, C and D must be in the interior of Q.

The result of this, as seen in the example of Figure 7.2.1, is a strip, or area, along ∂Q that does not lie within the intersection of the Right Angle Bounds from all obtuse angles. Yet all or part of it may well lie *outside* three semicircles drawn from three edges of Q, where, again, the fourth edge of Q has been "chosen" to be the recipient of an added nailed edge vertex W. The area within the semicircle that has the "chosen" edge as diameter, is not considered to be "invalid" placement area for the added interior floating vertex E.

7.2.1 An Algorithm (Guarantee Not Included)

Figure 7.2.2 shows that placing E outside of three semicircles may still place it outside of the intersection of the Right Angle Bounds of the three obtuse angles. So the above method based solely on use of the Semicircle Rule is not guaranteed.

Recombining this knowledge with the previous "inelegant" idea.

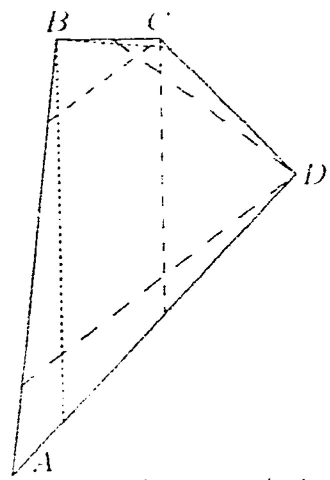


Figure 7.2.1: Right Angle Bounds intersect *within* Q, and include no part of ∂Q.

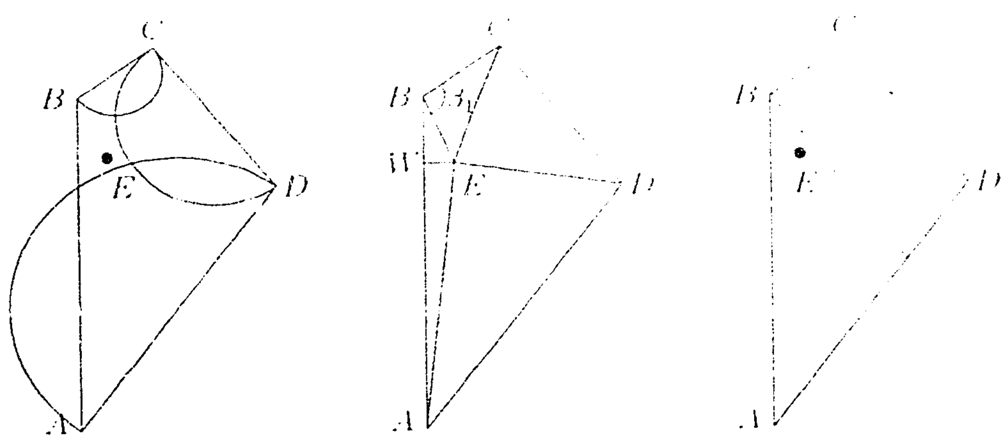


Figure 7.2.2: E outside three semicircles, but also outside Right Angle Bounds of B. Therefore, viable = take intersection of *all* Bounds *and* (Q − semidisks).

First, "choose" an edge to be the recipient of the nailed added edge vertex W;
(In Figure 7.2.3, this is edge AB.)
Establish the largest possible area outside the three semicircles from the other edges of Q;

For example, in both Figures 7.2.2 and 7.2.3, this area is $Q - (\frown BC \cup \frown CD \cup \frown DA)$, where it must be remembered that the semicircle *boundary* is a viable location in which to place a vertex (it creates a right angle), but the *inside* of the semicircle (or disk) is *not* part of the viable area (that location would create an obtuse angle).

Establish the Right Angle Bounds for each obtuse corner of Q and take their intersection. (This will be an area *within* Q.)
Take the intersection of the area within all Right Angle Bound lines, *and* the area *outside* the three semidisks. Call this area $\mathcal{A}$. (Assume, for now, that $\mathcal{A}$ is nonempty.)
Place E anywhere within $\mathcal{A}$ (possibly to satisfy some constraints regarding minimum angle size, or triangle size, or minimum distance of E from an edge of Q).
Drop an altitude from E to the chosen edge of Q.

This tiles Q into two right and three nonobtuse (right or acute) triangles, using one interior floating vertex E and one nailed edge vertex W, as in the example of Figure 7.2.3.

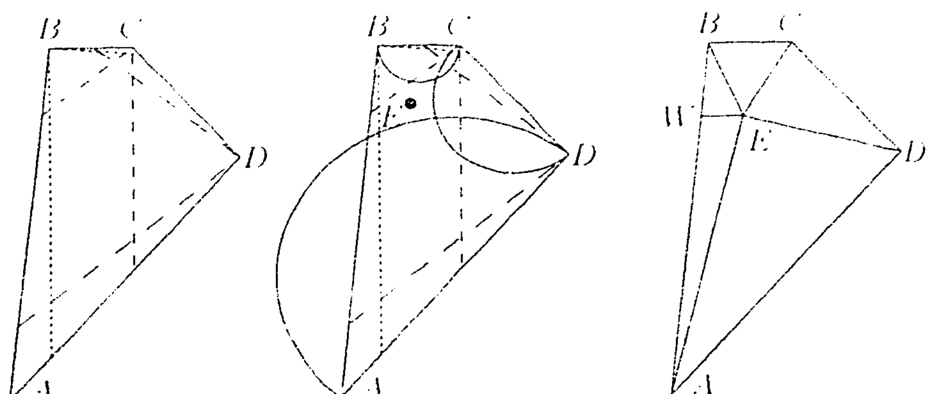


Figure 7.2.3: Choose edge AB for W; place $E \in$ (Bounds $\cap$ ($Q-$ semicircles)).

It is by no means guaranteed that $\mathcal{A}$ is nonempty. If it is, another edge must be "chosen" as the recipient edge for W and the attempt repeated of taking the area of intersection within all Right Angle Bound lines, and the area outside one new and two of the previous semidisks.

Using an interior degree-five vertex, and an altitude to resolve any remaining obtuse angle is simply the most obvious thing to do, although inelegant.

Questions:

1. Is it guaranteed that a viable area *always* exists? No; [L92] has suggested a counterexample where a viable area does not exist; see Figure 7.2.4. Whether a viable area exists or not appears heavily dependent on the geometry of Q, the relationship between edge lengths, vertex position, and angles of Q; until this elusive relationship can be identified, it cannot be used, only observed in action, so to speak. It would be useful to have some criteria (other than the result of an actual attempt to find a viable area) based on the geometry of a quadrilateral instance, to "pre-screen" or further classify quadrilaterals.

2. Why partition corner A? It's already acute! It can or should be left alone.

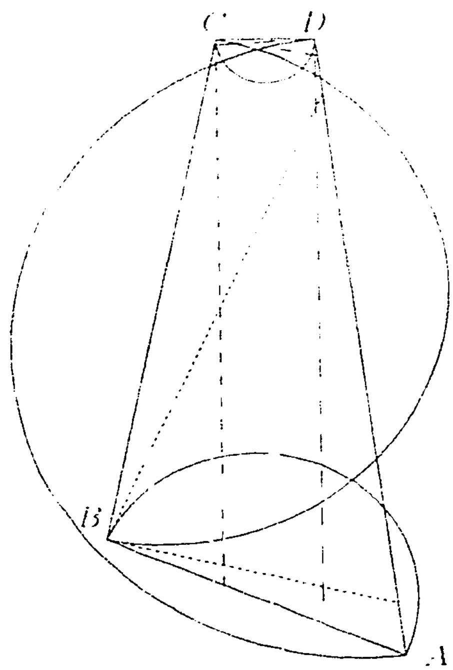


Figure 7.2.4: No viable area exists *inside* the Right Angle Bounds intersection (the inner trapezoid) and also *outside* 3 semicircles.

7.2.2 A Guaranteed Algorithm for Class 1 of $Q \in$ Case 3

Rather than answering these questions at the moment, another tack was found via which to arrive at a guaranteed nonobtuse triangulation for a certain class of $Q \in$ Case 3.

Lemma 8: Joining the two opposite obtuse angles of an instance of a Case 3 quadrilateral resolves AT LEAST ONE of the opposite obtuse angles into two acute angles.

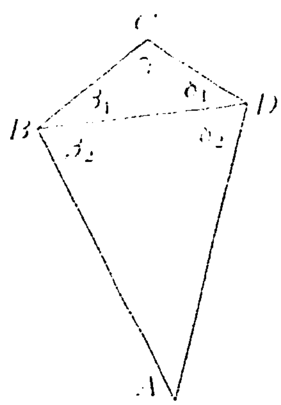


Figure 7.2.5: β_1, δ_1, and at least one of β_2 and δ_2, are guaranteed acute.

Proof: See Figure 7.2.5. Add edge DB to Q, partitioning Q into two triangles. Label the two angles nearest C as δ_1 and β_1, the others as δ_2, β_2. Since γ is obtuse, $\beta_1 + \delta_1 < 90°$, so both $\beta_1, \delta_1 < 90°$.

Angle α is acute and $> 0°$; therefore $\beta_2 + \delta_2 < 180°$. Thus *both* of β_2 and δ_2 cannot be obtuse; either they are both acute or one is acute and the other is obtuse.

Therefore, adding edge DB to Q resolves at least one, and possibly both, of δ and β into two acute angles. □

Figure 7.2.6 illustrates the three possibilities of Lemma 8. In the center and right drawing, the angle sizes are exactly the same, and there is only a shift in the vertex positions (hence in edge length) that creates the entirely different situations.

Given this, let Class 1 be $Q \in$ Case 3 such that adding edge DB to Q resolves both δ and β into acute angles; let Class 2 be $Q \in$ Case 3 such that one of β_2 or δ_2 remains obtuse after adding DB to Q.

Then a guaranteed method to tile $Q \in$ Class 1 with nonobtuse triangles is easily found, using one added interior vertex and one added edge vertex. It is not

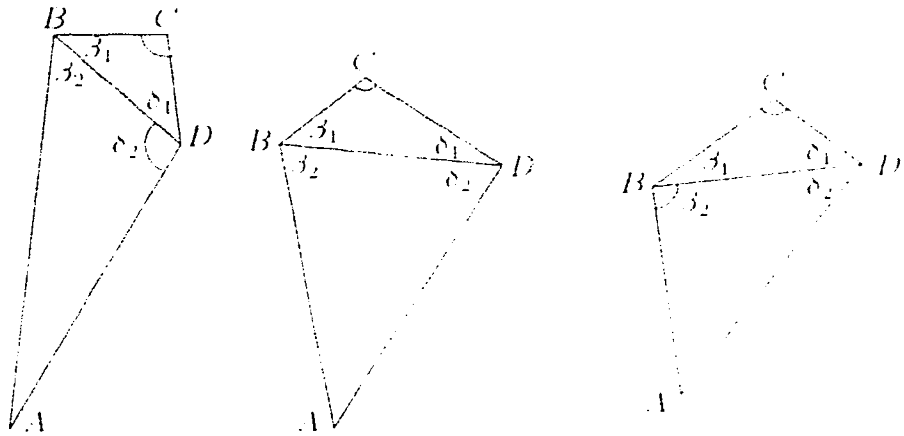


Figure 7.2.6: Add DB to Q: Left, β resolved into acute angles, δ not; Center: both obtuse angles resolved; Right: δ resolved, β not.

attractive, since both vertices are nailed, but it is guaranteed, and so is a (better) last resort than before.

Algorithm for $Q \in$ Class 1 of Case 3

Add edge BD to Q. (Resolves both β and δ.)
Drop an altitude from C to E on BD. (Resolves γ.)
Add edge EA to Q.

Adding edge EA creates two angles at E (angles $\angle DEA$ and $\angle AEB$); since $\angle BED$ is a straight angle, either they are both right angles, or one is acute and the other is obtuse.

If an obtuse angle remains, drop an altitude from E through it to W on either BA or DA.

Result: Four right triangles, with one interior nailed vertex (as in Figure 7.1.1 centre); or, one nonobtuse and four right triangles, with one nailed interior and one nailed edge vertex. Figure 7.2.7 is an example of the latter situation.

Even for $Q \in$ Class 1 of Case 3, this solution has some highly unattractive features:

1. An interior degree four vertex, as explained in Section 3.3, is not really desirable, being very sensitive to any change in vertex or edge position.

2. If not the above, *two* nailed vertices? Especially a nailed interior vertex. Having a nailed edge vertex is bad enough!

   Idea: Bend the BD diagonal downwards, so let E float. But again there would be the problem of finding a viable area within which to place E so

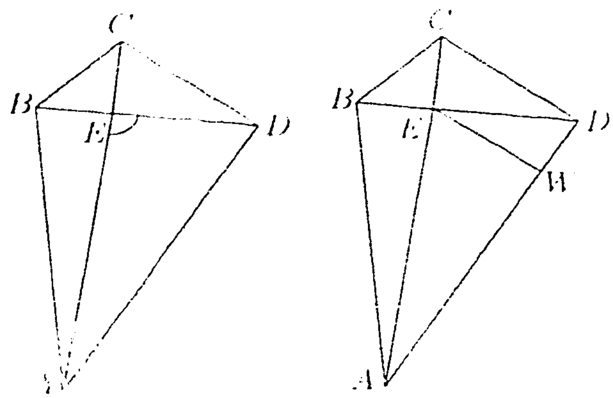


Figure 7.2.7: If DB resolves both δ and β, there is easily a guaranteed nonobtuse triangulation algorithm.

angles are guaranteed; this turns out workably (when such a viable area exists) but inelegantly.

3. This can very easily result in a *wide* range in triangle sizes: compare triangles $\triangle WED$ and $\triangle AEW$ in Figure 7.2.7.

4. Some *extremely* small angles can easily result, as angles $\angle BAE, \angle WED$ in Figure 7.2.7. Again, why partition corner A? It's already acute! It should be left alone!

7.3 A Guaranteed Algorithm for Class 2 of $Q \in$ Case 3

At this point it seems one must be resigned to adding more than one interior and one edge vertex for $Q \in$ Class 2. The idea of adding more interior vertices is somewhat more appealing than adding more edge vertices, if the interior vertices may less directly affect the number of edge vertices that must be added (and more importantly, perhaps affect less the completed triangulations in adjoining quadrilaterals).

Assume edge BD is added to $Q \in$ Class 2; then an altitude is dropped through γ from C to E on BD. This makes triangle ABD into a Case 2a "quadrilateral" $Q_{2a} = ABED$ where one of β_2 or δ_2 is obtuse, and angle $\angle BED = \epsilon$ is a straight angle. See Figure 7.3.1. The solutions in this case are found in Section 6.8.

The solutions for Q_{2a} are detailed in Section 6.8.3. It may be possible to tile Q_{2a} into three nonobtuse triangles, with W in a viable area, as in Figure 7.3.2, right illustration. Q_{2a} in Class ALT or BASE are tiled into four right triangles,

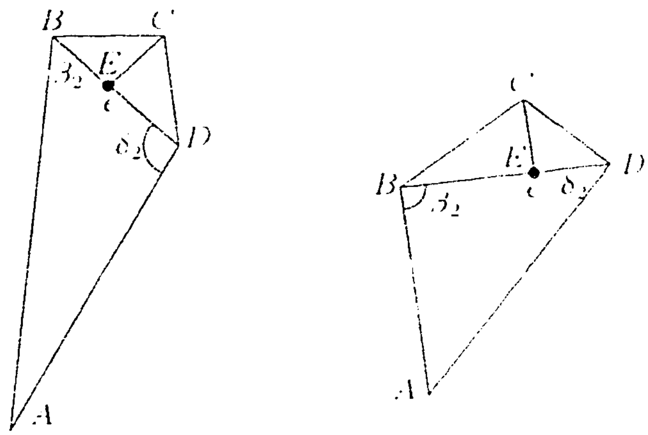


Figure 7.3.1: $BD \perp CE \Rightarrow$: Left: Case 2a with $\delta_{2,\epsilon} > 90°$. Right: Case 2a with $\beta_{2,\epsilon} > 90°$.

using either two nailed vertices on the same edge, or on adjacent edges (see Figure 6.8.8 and 6.8.9). An example of Q_{2a} in Class ALT is shown in Figure 7.3.2, left illustration. With the top two right triangles $\triangle BEC$ and $\triangle CED$, this tiles Q altogether into, at best, five triangles, and at worst, six right triangles.

7.4 More Elegance, Less Certainty

A less cumbersome approach might be to cut off the "ear" containing A (so that already-acute α can be left unpartitioned), adding two vertices to the sides of Q; then use one degree-five vertex inside the upper pentagon. See Figure 7.4.1, which also shows the angle designations in the finished tiling (if such can be accomplished).

The advantages for which to aim in developing this solution are:

- Let interior vertex E float within a viable area inside pentagon $BCDWT$ (if a viable area can be found!). (Note: [92] has shown that this is not always possible; see the example of Figure 7.2.1. However, the idea to develop, should this approach sound worthwhile, is to be able to pre-screen or apply some criteria to Case 3 quadrilaterals, to identify likely candidates for this approach. To date this has not been researched.)

- The side vertices W and T should be "as floating as possible" as well.

This idea seems very attractive, but is only partially developed. As before, it is not guaranteed that a viable area for E can be found; if it can, a sequence of Semicircle Rule and Right Angle Bound Rule checks can verify that W and T are

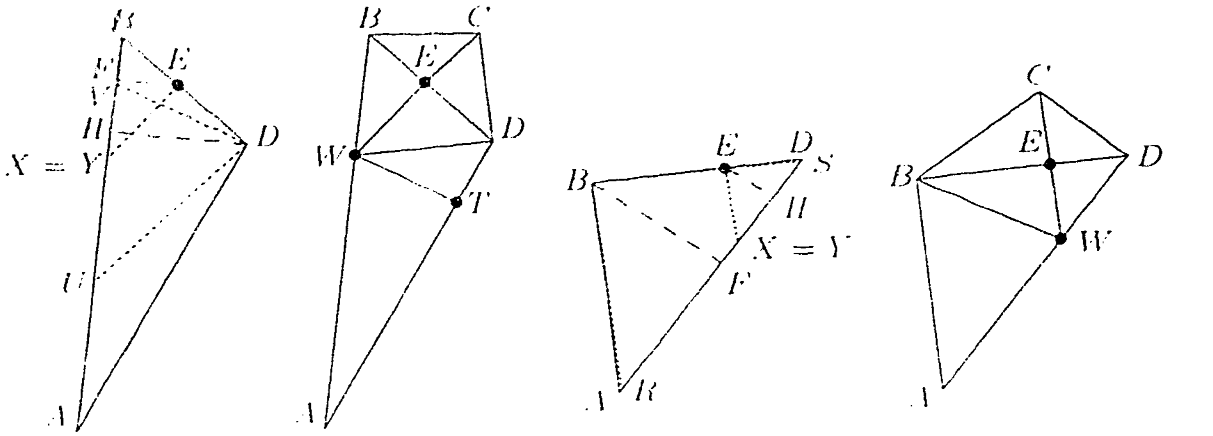


Figure 7.3.2: Case 2a solutions for $Q_{2a} \in$ Class 2 of Case 3. Left: $Q_{2a} \in$ Class ALT: solve with $W = X = Y$ $(\in \mathcal{H})$ and T on AD; Right: Q_{2a} has a viable "spot" $X = Y \in [R, S] \cap [F, H] = [F, H]$: place W there to tile Q_{2a} optimally.

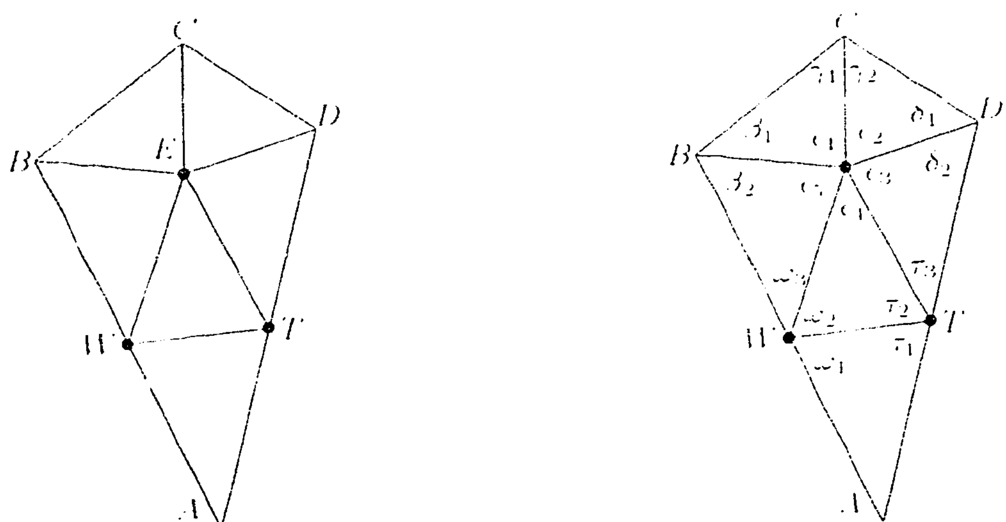


Figure 7.4.1: Left: A "nice idea" (if it works!); Right: Angle designations in finished tiling (if possible).

placed adequately to tile Q nonobtusely, but do not indicate where to place W and T. An iterative or intuitive approach seems, so far, to work best! All the information to hand is not understood/"massaged" well enough into one or more "rules" to either guaranteed if a viable area exists for E, or to state (unambiguously) exactly where the side vertices should be placed. Due to space constraints, the partial results are not presented here.

7.5 Conclusion

Lemma 8 partitions $Q \in$ Case 3 into Class 1 quadrilaterals, where adding BD leaves a Case 1 "quadrilateral" $Q_1 = ABED$ where the only obtuse angle is $\epsilon = 180°$; and into Class 2 quadrilaterals, where adding BD leaves a Case 2a "quadrilateral" $Q_{2a} = ABED$, where one of β_2 or δ_2 is obtuse, and $\epsilon = 180°$.

Section 7.2.2 details the solution for Q_1: at best Q_1 is tiled into two right triangles, at worst into three nonobtuse triangles, at least two of which are right triangles. This tiles the "parent" quadrilateral Q into at best four right triangles, and at worst five triangles, of which four are right triangles.

Section 6.8 details the solution for Q_{2a}: at best Q_{2a} is tiled into three triangles, of which two are right triangles; at worst Q_{2a} is tiled into four right triangles. This tiles the "parent" quadrilateral Q into at best five triangles, of which four are right triangles; and at worst, six right triangles. It is an open problem to determine whether five nonobtuse triangles suffice in the worst case.

From even the few examples in this chapter, it is easily observed that very small angles and very small triangles can frequently occur in "successful" tilings. An attempt to find a "nicer" method ends up (so far) relying on intuition in placing vertices (difficult to incorporate into a computer program!) or iterative, "position test-reposition as necessary" methods. At least one stumbling block appears to be the lack of criteria upon which to base a classification of $Q \in$ Case 3: the only criteria found so far is that from Lemma 8.

Some exploratory work was done on finding subcases 1, 2o or 2a within Case 3 quadrilaterals - that is, adding an edge or edges to Q to partition Q into instances of Case 1, 2o, or 2a, for which a solution is known; then the subtilings would (ideally) constitute a nonobtuse tiling of the larger Q. The variety of ways in which other Case instances could be found within an instance of $Q \in$ Case 3 was something of a surprise. This was an opportunity to see how well the solution methods for Cases 1, 2o and 2a lent themselves to the "matching" problem, where vertices added to edges shared by more than one quadrilateral must be "required" or "allowed" in both quadrilaterals, to maintain an existing nonobtuse tiling in both adjoining quadrilaterals. Again, due to time and space constraints, those results are not reported here; the little research done in this area served mainly to indicate that the problem can be quite tricky!

Chapter 8

Applications and Concluding Remarks

This chapter consists of a "real" example drawn from [J86], a list of topics for further research, and some concluding remarks.

8.1 "Real-Life" Application: An Example

This section presents a "real" example drawn from [J86], to which results in this thesis are applied, or attempted to be applied: in partitioning a nonconvex polygon into convex subregions, a pentagon, rather than a quadrilateral, is encountered, and the solutions in Chapters 1 to 7 were not really designed for pentagons!

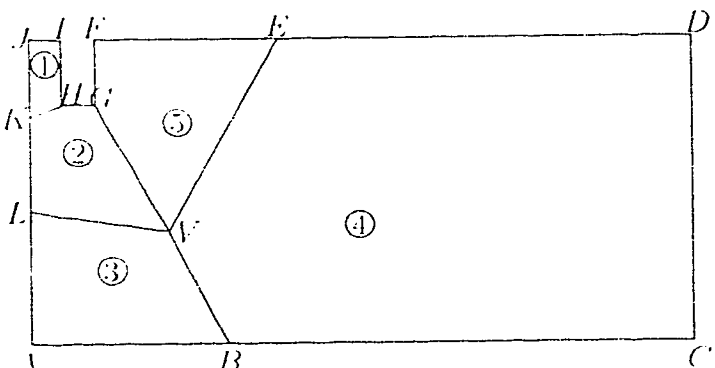


Figure 8.1.1: Example 1 from [J86].

Figure 8.1.1 is taken from [J86]; a nonconvex domain has been partitioned into five convex subdomains, and each subdomain is to be tiled. Of course each subdomain is to be tiled with triangles of a certain size, varying across subdomain boundaries by a set limit, so that the final mesh over the area is smoothly graded;

but for now that constraint has been dropped, and the only aim is to see how it goes applying the techniques learned in research to this input to obtain a guaranteed nonobtuse tiling of the entire nonconvex domain.

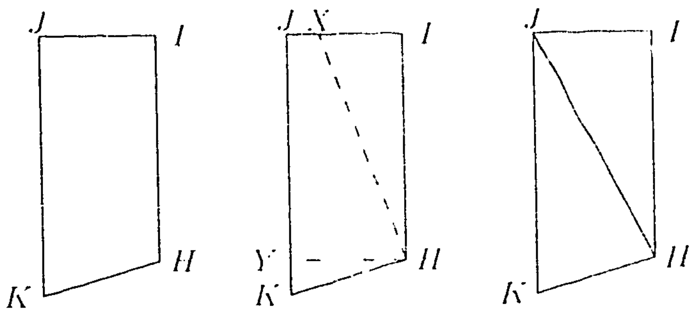


Figure 8.1.2: Polygon P1 = quadrilateral $HIJK$; $J \in [X, Y] \Rightarrow HJ$ tiles P1.

Convex partition P1 is quadrilateral $HIJK$, with angle $\angle KHI$ obtuse. See Figure 8.1.2. Note that $J \in [Y, X]$ = Right Angle Bounds from H, so joining H to J resolves $\angle KHI$ into acute angles, requiring one added edge and no added vertices.

Convex partition P2 is pentagon $VGHKL$, with *four* adjacent obtuse angles, corresponding to the last four vertices; therefore the obtuse angles are designated $\gamma, \eta, \kappa, \lambda$, in that order, counterclockwise. See Figure 8.1.3.

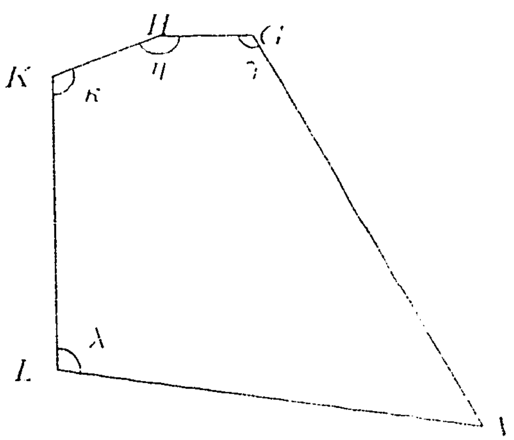


Figure 8.1.3: Polygon P2 = pentagon $VJHKL$ with four adjacent obtuse angles.

To "solve" this polygon we attempt to find a viable area for a central degree-five vertex by taking the intersection of Right Angle Bounds from the obtuse angles, which will be a region inside the pentagon; this region is then intersected with what is left of the polygon after the area of as many semidisks (based on polygon edges) as possible is subtracted from it. It turns out in this case that even with a semicircle drawn from *each* edge (that is, with each edge as diameter), a part of the polygon is still "viable", and that the intersection with all Right Angle Bounds from obtuse angles, is *still* not empty! See Figure 8.1.4. So, vertex W is placed anywhere in the viable area, and joined to each corner of the pentagon, easily resulting in a nonobtuse tiling, requiring one added floating interior vertex W, and five added edges.

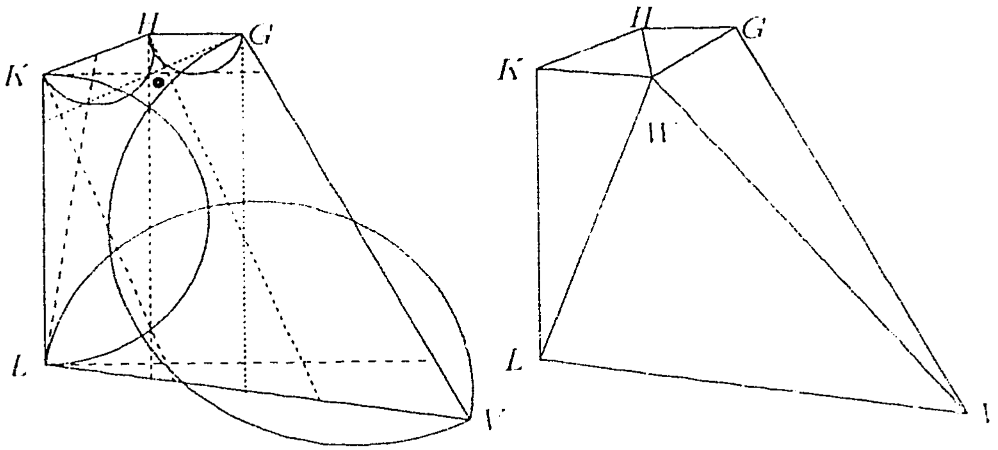


Figure 8.1.4: A small viable area remains! Place W there, join to all corners.

Convex partition P3 is quadrilateral $ABVL$, with angle $\angle BVL$ obtuse (and angle $\angle LAB$ a right angle). See Figure 8.1.5. Again, $A \in [Y, X]$ = Right Angle Bounds from V, so joining A to V resolves $\angle BVL$ into acute angles, requiring one added edge and no added vertices.

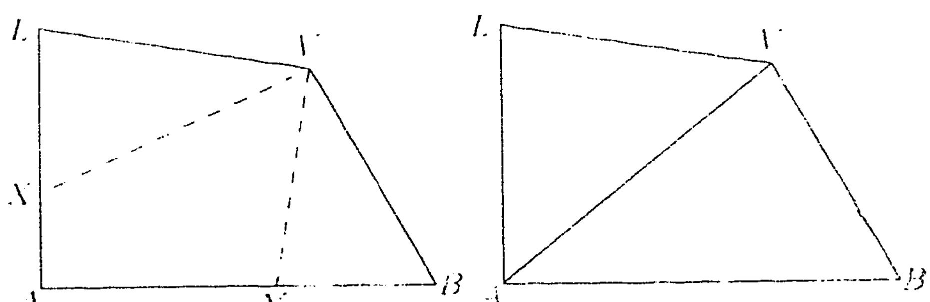


Figure 8.1.5: Polygon P3 = quadrilateral $ABVL$; $A \in [X, Y] \Rightarrow AV$ tiles P3.

Convex partition P4 is again a pentagon, $BCDEV$, with this time only three adjacent obtuse angles, and two adjacent right angles. The obtuse angles are designated ϵ, ν, and β, counterclockwise. See Figure 8.1.6.

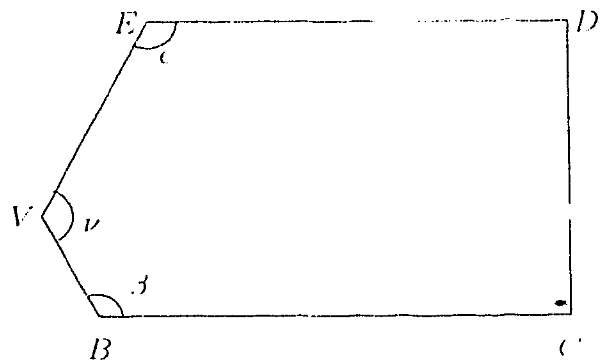


Figure 8.1.6: Polygon P4 = pentagon with three adjacent obtuse angles.

To "solve" this polygon we again attempted to find a viable area for a central degree-five vertex by the same method. A fair size area existed as the intersection of Right Angle Bounds from corners B, E and V; and a small spot *did* exist as (P4 - all semicircles); however, the intersection of this small spot, and the Right Angle Bounds area, did not overlap. See Figure 8.1.7.

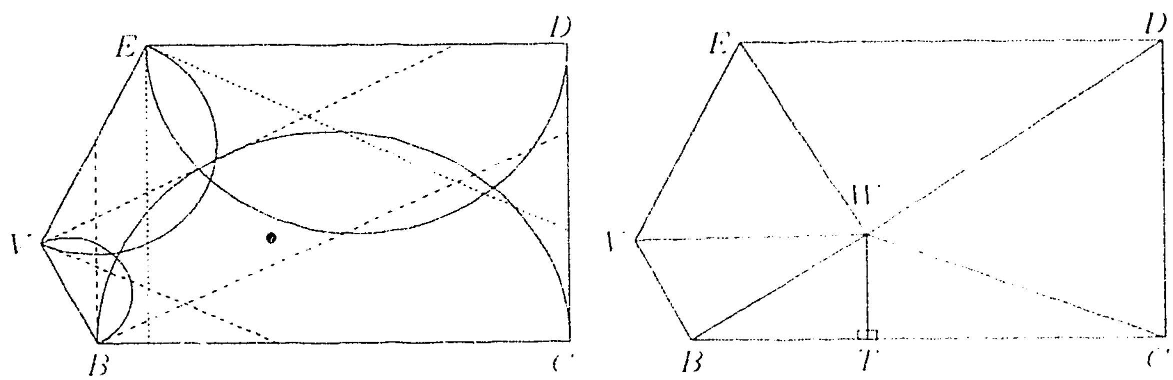


Figure 8.1.7: P4: No viable area exists! Choose recipient edge for T, place W in semicircle from that edge (in RAB intersection), add edges to corners and T.

So, an edge (arbitrarily, the longest edge in the polygon) was chosen to be the recipient of an added edge vertex T - in Figure 8.1.7, the edge chosen was the bottom BC edge; vertex W was placed inside the semicircle extending from BC, but outside all other semicircles, and within the Right Angle Bounds overlap. Thus only one angle at W will be obtuse - angle $\angle BWC$. W is joined to each corner of P4, and an altitude is dropped through the last remaining obtuse angle $\angle BWC$ to nailed edge vertex T. This results in a nonobtuse tiling requiring one added floating interior vertex W, one nailed edge vertex T, and six added edges.

Convex partition P5 is quadrilateral $EFGV$, with angle $\angle FGV$ obtuse (and angle $\angle EFG$ a right angle). See Figure 8.1.8. $E \in [Y, X]$ = Right Angle Bounds from G, so joining E to G resolves $\angle FGV$ into acute angles, requiring one added edge and no added vertices.

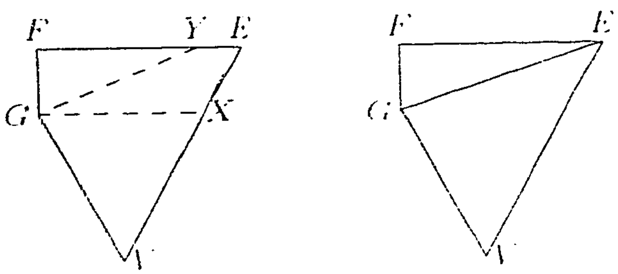


Figure 8.1.8: Polygon P5 = quadrilateral $EFGV$: $E \in [X, Y] \Rightarrow EG$ tiles P5.

The completed tiling (see Figure 8.1.9) has, in addition to the original interior vertex V, two interior vertices added, which are relabeled as W_2 and W_4, as they are inserted into convex partitions P2 and P4; and one nailed edge vertex added, T.

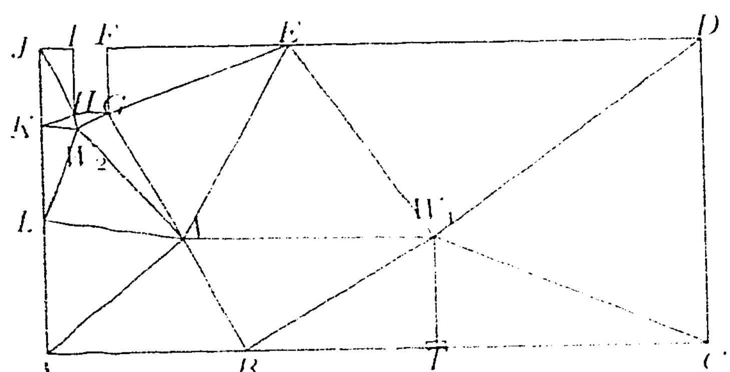


Figure 8.1.9: Example 1 from [JS6] - completed tiling.

8.2 Summary of Results

Eight lemmas regarding resolution of obtuse angles and some characteristics of convex quadrilaterals have been established and proved useful. At least one guaranteed method for nonobtuse tiling of quadrilaterals has been shown and proved for each case of quadrilateral. The best and worst case results of each guaranteed method are as follows:

Case 0: **Minimum (best case) = Maximum (worst case):** one added edge and no added vertices tile $Q \in$ Case 0 into two right triangles.

Case 1: **Minimum (best case):** one added edge and no added vertices tile $Q \in$ Case 1 into two nonobtuse triangles.

Maximum (worst case): two added edges and one added vertex, whose position, under certain conditions, may be flexible within a limited area, tile $Q \in$ Case 1 into three nonobtuse triangles.

Case 2o: **Minimum (best case):** one added edge and no added vertices tile $Q \in$ Case 2o into two nonobtuse triangles.

Maximum (worst case): three added edges and two added vertices tile $Q \in$ Case 2o into four nonobtuse triangles.

Case 2a: **Minimum (best case):** Two added edges and one added floating vertex tile $Q \in$ Case 2a into three nonobtuse triangles.

Maximum (worst case): Three added edges and two added vertices tile $Q \in$ Case 2a into four nonobtuse triangles.

Case 3: **Minimum (best case):** A special case exists, which would probably be quite rarely encountered in practice, in which four added edges and one added nailed vertex tile $Q \in$ Case 3 into four right triangles. *When a viable area exists* in which to place a floating vertex, five added edges, and one each added interior and edge vertex, can tile $Q \in$ Case 3 into five triangles.

Maximum (worst case): A guaranteed method has been presented such that six added edges and three added vertices tile $Q \in$ Case 3 into six right triangles.

8.3 Possible Extensions of Research

Further research: Topics that sound tantalizing with respect to further research in this area include:

- Firmly reintroduce the dropped constraints, requiring further changes in the algorithms, and see if the algorithms survive this step!

- Start out with a quadrilateral Q that has evenly spaced edge vertices on some or all of its edges. These would be considered "corner" vertices where the angles at these vertices are straight angles; so, strictly speaking, Q would not be a quadrilateral. However, Q would have only four non-straight angles.

- Start out with a quadrilateral Q that, having undergone the shrinking process described in [J86], has a degenerate evenly shrunk interior $int(Q)$ = a point or a line inside Q, not connected to its edges ("floating" within Q). Try using a line with or without vertices added, at even spacing, and then not at even spacing.

- Investigate the possibility that the area of intersection, for $P = Q \in$ Case 3, of Right Angle Bounds from three obtuse angles (as in Figure 7.2.1) could be used as $int(P)$ in an adaptation of [J86].

- Extend the results herein to convex polygon P sides $n = 5, 6, 7...$ This has already been done somewhat in the examples in Section 8.1, though the results haven't been "formally" extended or proved or anything.

- Given that the algorithms can survive the reintroduction of some of the dropped constraints (to make them practical), what would a complexity analysis of even these few simple algorithms be? What is the complexity of finding the intersection of Right Angle bounds from n obtuse (not straight) angles? Does it linearly increase with n or not?

- It might be interesting to try incorporating some of the ideas developed here as elements of a refinement algorithm to be used on an existing mesh. If a given mesh can be smoothed to almost entirely reduce or eliminate obtuse angles, that would probably be just as acceptable as an originally generated nonobtuse mesh.

- Could the principles found in this research be extended to make sense in three dimensions?

8.4 Conclusion

This research attempted to find and prove an optimal guaranteed method for nonobtuse tiling of arbitrary convex quadrilaterals in each case of the quadrilateral having zero, one, two opposite, two adjacent, or three obtuse angles. To that extent the research has been successful, as at least one guaranteed method has been found for each case of quadrilateral. The criterion of "optimality" can be questioned with regard to quadrilaterals with three obtuse angles, as whether five or six triangles is required in the worst case, has not been fully answered. Imposing only the condition of convexity on the input quadrilateral leaves a very wide margin for both "nice" and "pathological" input quadrilaterals: to a fair extent this was one of the most challenging aspects of the problem. However, the methods for nonobtuse tiling herein have been proved for even the most pathological cases.

In addition, a number of simple lemmas have been proved that serve both to guide in resolving obtuse angles, and to classify arbitrary convex quadrilaterals. A brief look at how well the results developed are cross-transferable to situations for which they were not really designed, showed that the essence of the ideas may have some promise in being expanded to become more generally useful, if only to serve as springboards for more sophisticated ideas about guaranteeing nonobtuse tilings.

An intangible but crucial result, that is equally as important as the proved methods and lemmas herein, is the fundamental realization that although the tiling problem at first looks to be quite simple in the case of quadrilaterals, it can easily become considerably more complex than expected.

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