University of Alberta

Excess-of-Loss reinsurance under taxes and fixed costs

by

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To my parents, HongYing Wang and JiShun Yang

Abstract

A problem of risk control and dividend optimization for a financial corporation is considered in this thesis. More specifically we investigate the case of excess-of-loss reinsurance for an insurance company. Under this scheme the insurance company diverts part of its premium stream to another company in exchange of an obligation to pick up that amount of each claim which exceeds a certain level a. This reduces the risks but it also reduces the potential profits. The objective of the corporation is to maximize the expected total discounted dividend distribution prior to the bankruptcy time. We consider the cases when there is debt liability for the company and when there is no debt liability. In both cases, we solve the problems explicitly and construct value function together with optimal policy.

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1
Introduction

Actuarial mathematics originated toward the end of the 17th century, when E. Halley's famous mortality table permitted the mathematical treatment and calculation of annuity values for the first time. However rigorous models for insurance in actuarial sciences were born only in 1903, when Filip Lundberg defended his Ph.D thesis and showed the collective risk model for insurance claim data. At this time, mathematical finance was already established through the Ph.D thesis of Bachelier in 1900. However, the two areas began to fall apart during the first half of the last century. When the actuarial sciences persevered developing by the works of Cramer, Essher and many others, mathematical finance stepped slowly. Fortunately, mathematical finance had a marvelous progress through the works of Paul Samuelson, Robert Merton, Black and Scholes, ..., etcetera. during the second half of the last century.

The classical approach to the assessment of the value of the company is to consider the total dividend pay-outs. Recently there have been an upsurge in optimizing dividend in the context of optimization of the corporate policies (see Asmussen and Taksar 1997, Jeanblanc-Piqué and Shiryaev 1995). These models were extended to more complicated models in order to incorporate both control of profit/risk related activities and the dividend pay-outs. This is important when one is modeling a large insurance portfolio of a firm. Usually besides the distribution of the part of the reserve as dividends, the management of the insurance company faces a problem of how much risk to avoid. This phenomena is called reinsurance.

1.1 Reinsurance

Reinsurance means controlling revenues by diverting part of the premium to another insurance company, thus reducing its own risk as well as profit. In other words, reinsurance is the one of the major risk-management tools that permits insurance companies to be protected. Some of the reasons to employ reinsurance are summarized in the following.

- Against adverse fluctuations.
- The appearance of excessively large claims, or an unusually large number of claims: The most dangerous risk comes from the large claims and also a large number of claims can lead to a disastrous situation.
- Reinsurance can be considered to increase the capacity of the company by offering more services to its clients.

In general, one of the most common reasons why an insurance companies would employ reinsurance is to diminish the impact of large claims. There also exists a variety of reinsurance forms. One most commonly used type of reinsurance is proportional (or quota-share) reinsurance where a certain proportion of the total portfolio is reinsured. Proportional reinsurance means that it is possible for the cedent to divert a proportion $1-a$ of all premiums to another company with the obligation from the latter to pay $1-a$ fraction of each claim, where a denotes the fraction of the claim covered by the cedent. It was introduced by Buhlmann (1970), Gerber (1979) and Sundt (1993). The type of reinsurance considered in this thesis is the most common example of a nonproportional reinsurance, well known in industry as excess-of-loss reinsurance. In this type of reinsurance the reinsurer takes on a share of each loss in excess of a previously agreed threshold, or attachment point (retention level a, where a is the dollar amount in excess of which the reinsurer picks up the full claim for excess-of-loss reinsurance), however only up to a maximum amount (which can be infinite–meaning an unlimited cover). The motivation for our focus on the excess-of-loss reinsurance is that not only it is more profitable than the proportional reinsurance for an insurance company, but also the actual assessment of a given excess coverage is similar to the valuation of an option spread on a common stock.

1.2 Cramer-Lunderberg models

The first model for the reserve (risk) process of an insurance company is called the Cramer-Lundberg model and is described by

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i, \tag{1.1}$$

where x is the initial capital, p is the premium rate, U_i is the size of the i^{th} claim. Random variables U_i are independently identically distributed and are independent of the Poisson process N_t with finite first and second moments. This model shows the case of the reserve or risk process of the insurance company that takes the full risk. If a risk U_i is too dangerous (for instance if U_i has an infinite third moment i.e $E(U_i^3) = \infty$), the insurer may want to transfer part of the risk U_i to another insurer. This risk transfer from a first insurer that transfers (part of) his risk is called a cedent. Often the reinsurance company does the same, i.e. it passes part of its own risk to a third company, and so on. By passing on parts of risks, large risks are split into number of smaller portions taken up by different risk carriers. This procedure of risk exchange makes large claims less dangerous to the individual insurers, while the total risk remains the same. Therefore, when the insurance company considers reinsurance, say that the company assumes risks with sizes $U_i^{(a)}, (= f(U_i, a))$ where a is the retention level, and diverts $U_i - U_i^{(a)}$ to another insurance company. Therefore, the reserve process of the cedent takes the following form

$$X_t^{(a,\eta)} = x + p^{(a,\eta)}t - \sum_{i=1}^{N_t} U_i^{(a)}, \tag{1.2}$$

where $p^{(a,\eta)} = (1+\eta)E(U_i^{(a)})$, η denotes the safety loading. Generally, reinsurance can be classified into two main groups. The first group is called proportional reinsurance. In this case, the cedent assumes aU_i (here $0 \leq a \leq 1$) as risk for a claim with size U_i, and its reserve process follows

$$X_t^a := x + apt - \sum_{i=1}^{N_t} aU_i.$$

The second main group of reinsurance is called the non-proportional reinsurance. Here in my thesis, i will focus on an important example of this type of reinsurance, called excess-of-loss reinsurance. For this example of reinsurance, the cedent assumes $a \wedge U_i$ for a claim with size U_i, where a is the retention level ($a \geq 0$).

1.3 Diffusion models

One of the main reasons why diffusion models are needed to model the business activities of an insurance company, lies in the fact that when we have big portfolios, the model define in (1.2) is not suitable, and a limiting of it is more adequate. In the probability theory literature, it is proved that as η goes to zero, $\left(\eta X_{t/\eta^2}^{(a,\eta)}\right)_{t \geq 0}$ converges to $BM(\mu(a), \sigma^2(a))$ in the space of $D[0, \infty)$ (the space of right continuous functions with left limits endowed with the Skorohod topology). The resulting limit process is a Brownian motion (a particular example of diffusion process). We refer the reader to Iglehart (1969), Grandell (1977), (1978), (1990), Emanuel et al. (1975), Harrison (1977), Asmussen (1984), Schmidli (1992), (1993) and Møller (1994) for more examples of diffusion approximations in risk theory. Furthermore, during the recent decade, there has been an upsurge in applying diffusion models to insurance mathematics and in particular in (re)-insurance modeling setting. Historically, Iglehart (1969) attempted to treat the insurance surplus as a diffusion process and the diffusion model was brought in by Whittle (1983) and Dayananda (1970), see also Harrison,J.M (1977), Asmussen,S. and Taksar,M.(1997), B.Höjgaard and M.Taksar (1998a,1998b), and the references therein. In these models the liquid assets processes of the corporation are driven by a Brownian motion with constant drift and diffusion coefficients.

1.4 The role of stochastic control

In this subsection, we will present two main arguments in order to justify the usefulness of stochastic control in insurance and/or in finance.

The first argument is a historical one and it is incorporated in the speech of K. Borch to the Royal Statistical Society of London in 1967. K. Borch pointed out the value of the control theory for actuarial science:"*The theory of control processes seems to be "tailor-made" for the problems which actuaries have struggled to formulate for more than a century. It may be interesting and useful to meditate a little how the theory would have developed, if actuaries and engineers had realized that they were studying the same problems and joined forces over 50 years ago. A little reflection should teach us that a "highly specialized" problem may, when given the proper mathematical formulation, be identical to a series of other, seemingly unrelated problems.*"

The second argument is explaining how naturally the stochastic control intervenes in our analysis. This needs brief description of our model. This model applies to a financial corporation whose liquid assets are modeled by a Brownian motion with constant drift and diffusion coefficients. The drift is the profit per unit time, while the diffusion term is the risk that the company faces. The larger the diffusion coefficient the greater the business risk the company takes on. If the company wants to decrease the risk from its business activities, it also faces a decrease in its potential profit. This sets a scene for an optimal stochastic control where the controls affect not only the drift but also the diffusion part of the dynamic of the system.

1.5 Summary of the thesis

In this thesis we study a model of a financial corporation, such as a large insurance company, in which both taxes and fixed costs are present. The mathematical setup results in a mixed classical-impulse stochastic control problem, in which one maximizes the expected total discounted dividend distribution prior to the bankruptcy time. This thesis is organized as follows. In the second chapter, we will provide the economy as well as its rigorous mathematical formulation, and we will state the objective. In other word, we will set the optimal control problem that we tackle in the remaining part of this thesis. In chapter 3, we transform the stochastic control problem into a quasi-variational inequality (QVI hereafter) and provide some properties of the value function. In Chapter 4 and 5, the constructions of the smooth solution to the QVI together with the constructions of the optimal policy in the case with no debt liability rate and with debt liability rate are presented, respectively. Some numerical examples to chapter 4 and 5 are illustrated in Chapter 6.

2

The mathematical and economical model

This chapter describes the economical model that we undertake in this thesis in the first stage. Then we state rigorously the mathematical formulation of the model as well as some of the mathematical/ statistical tools necessary for the definition and the analysis of the model.

2.1 The economy

Our economical model consists of a firm which has control of the dividend payment stream, the timing at which these payments are made, and its risk as well as potential profit by choosing different business activities among those available to it. Our model is one of the extension of the classical Miller-Modigliani theory of firm valuation to the situation of controllable business activities in a stochastic environment. The value of the firm is associated with the expected present value of the net dividend distributions (under the optimal policy). More precisely our model applies to a large corporation, such as an insurance company, whose liquid assets in the absence of control fluctuate as a Brownian motion with a constant positive drift and a constant diffusion coefficient. The diffusion coefficient can be interpreted as risk exposure, while the drift represents potential profit.

The firm is bankrupt at the first time, τ, at which the cash reserve falls to zero (τ may be infinite), and the firm's objective is to maximize the expected total discounted dividends from the beginning to the bankruptcy time τ, given an initial reserve x. If we denote this maximum by $v(x)$, we want to calculate $v(x)$ as explicit as we can, as a function of the exogenous parameters of the model.

2.2 The mathematical formulation

To rigorously formalize the economy, we start with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ such that

$$\mathcal{F}_\infty := \sigma(\bigcup_{t\geq 0} \mathcal{F}_t) \subset \mathcal{F}, \ \ \underset{t+}{\mathcal{F}} := \sigma(\bigcap_{\varepsilon\geq 0} \mathcal{F}_{t+\varepsilon}) = \mathcal{F}_t, \ \ t \geq 0,$$

and all $\mathcal{F}$-negligible sets belong to $\mathcal{F}_0$. Suppose that on this filtered space, we can define a Brownian motion $W = (W_t)_{t\geq 0}$, the unique source of randomness that drives our economic model. Here below, we recall the definition of the Brownian motion.

Definition 2.1 *The process $W = (W_t)_{0\leq t\leq s}$ is called an $(\mathcal{F}_t)_{t\geq 0}$-Brownian Motion, (also called a Wiener process), if W satisfies the following,*

1. *W_t is $\mathcal{F}_t$-measurable, for all $t \geq 0$,*

2. *For all $0 \leq s < t, W_t - W_s$ is independent of $\mathcal{F}_s$ and it follows a normal distribution with mean zero and standard deviation $\sqrt{t-s}$.*

3. *For almost all $w \in \Omega$,*

$$t \to W_t(w)$$

   is a continuous map.

The claims for the insurance company have a random size that is described mathematically by nonnegative independently identically distributed random variables $(U_i)_{i\geq 0}$ with the cumulative distribution function, F satisfying

$$\int_0^\infty x^2 F(dx) = E(U_i^2) < +\infty.$$

Let a be a nonnegative number that represents the retention level. That is a level, up to which our company assume the risk and any risk beyond that level is assumed by another company called the reinsurer. In return this reinsurer takes a part of the premiums. In other words, for any claim with size U_i, our company assumes the risk of size

$$U_i^{(a)} := \min(a, U_i), \quad a > 0.$$

Now consider the following functions of the retention level $a \geq 0$

$$\begin{aligned} \mu(a) &:= E(U_i^{(a)}) = \int_0^a \overline{F}(x)dx, \\ \sigma^2(a) &:= E((U_i^{(a)})^2) = \int_0^a 2x\overline{F}(x)dx, \end{aligned} \tag{2.1}$$

where $\overline{F}(x) = P(U_i > x) \equiv 1 - F(x)$, for all $x \geq 0$.

Definition 2.2 *1. Any non-negative random variable $\tau : \Omega \to [0, \infty)$, $\mathcal{F}$-measurable is called a random time.*

2. Let τ be a random time. Then τ is called a stopping time (or a Markovian time) if for any $t \geq 0$,

$$(\tau \leq t) \in \mathcal{F}_t.$$

These notations allow us to describe the dynamic of the reserve process of our company as follows.

$$X_t := x + \int_0^t (\mu(a(s)) - \delta)ds + \int_0^t \sigma(a(s))dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}}\xi_n, \quad t \geq 0. \tag{2.2}$$

Define the upper bound of the support of F by

$$N := \inf\{x \geq 0 : \overline{F}(x) = 0\}. \tag{2.3}$$

Then, both functions $\mu(\cdot)$ and $\sigma^2(\cdot)$ are increasing on $[0, N]$, while on $[N, \infty]$ they are constants equal to $\mu_\infty = \mu(N)$ and $\sigma_\infty^2 = \sigma^2(N)$ respectively.

Here, X_t is the cash reserve of the insurance company at time t, while x is the initial reserve. δ represents the debt liability rate, which is a constant payment of the firm's debt, such as the bond liability, mortgage, or loan amortization. The profit rate, μ, is the difference between the premium rate that the insurance company receives and the expected payments on claims per unit of time. The volatility rate (σ) is always greater than zero. The risk process, $a = (a_s)_{s \geq 0}$, is a predictable process such that

$$a_s(w) \geq 0, \ \ \forall s \geq 0, \ \ \forall w \in \Omega.$$

τ_n is the time of the n^{th} intervention, and ξ_n is the amount of the n^{th} dividend payment. Note that $\tau_0 = 0, \xi_0 = 0$, and

$$\mathcal{T} = \{\tau_0 = 0 < \tau_1 < \tau_2 < \tau_3 < ... < \tau_n < ...\} \tag{2.4}$$

is an increasing sequence of stopping times; $\xi = (\xi_i)_{i \geq 1}$ is a sequence of nonnegative random variables such that each ξ_n is $\mathcal{F}_{\tau_n}$-measurable for all $n \geq 1$. Let

$$\pi := (a, \mathcal{T}, \xi) = (a_s; \tau_1, \tau_2, ..., \tau_n, ...; \xi_1, \xi_2, ..., \xi_n, ...) \tag{2.5}$$

denotes the control for this model.

Definition 2.3 *A triple*

$$\pi := (a, \mathcal{T}, \xi) = (a_s; \tau_1, \tau_2, ..., \tau_n, ...; \xi_1, \xi_2, ..., \xi_n, ...)$$

is called an admissible control or an admissible policy if $a_s : \sigma \times [0, \infty) \mapsto [0, 1]$ is an $\{\mathcal{F}_t\}_{t \geq 0}$-adapted process, τ_i, $i = 1, 2, ...$ is a stopping time with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, $0 \leq \tau_1 < \tau_2 < \tau_3 < ... < \tau_n < ...$a.s., and the random variable ξ_i, $i = 1, 2, ...$ is $\mathcal{F}_{\tau_i}$ measurable with $0 \leq \xi_i \leq X(\tau_i -)$. The class of all admissible controls is denoted by $\mathcal{A}(x)$.

Then the time of bankruptcy of the company can be defined as

$$\tau := \tau^{\pi} := \inf\{t \geq 0 : X(t) = X^{\pi}(t) = 0\}. \tag{2.6}$$

And for $\forall t \geq \tau$, we assume that $X^{\pi}(t) = X(t) = 0$. This leads to put

$$X(t) = \begin{cases} x + \int_0^t [\mu(a(s)) - \delta] ds + \int_0^t \sigma(a(s)) dW_s - \sum_{n=1}^{\infty} I_{\{\tau_n < t\}} \xi_n, & \text{if } 0 \leq t < \tau \\ 0, & \text{if } t \geq \tau. \end{cases} \tag{2.7}$$

Consider the following function

$$g(\eta) := \begin{cases} -K + k\eta, & \eta > 0, \\ 0, & \eta = 0. \end{cases} \tag{2.8}$$

Here $K > 0$ represents the cost for each dividend payout, and $1 - k$ denotes the tax rate that the shareholders pay ($0 < k \leq 1$). Therefore, for any amount of dividend, η, paid by the company, the shareholder will receive $g(\eta)$ only.

The objective is to maximize the value of the firm. This value coincides with the expected total discounted dividends received by the shareholders at the optimal policy. Hence, our main concern focus on solving the following stochastic control problem.

Problem 2.1 *1. Find $\pi^* = (u^*, \mathcal{T}^*, \xi^*)$ an admissible policy that maximizes the index function i.e.*

$$J(x;\pi^*) = \sup_{\pi \in \mathcal{A}(x)} J(x,\pi) \tag{2.9}$$

$$J(x;\pi) := E_x\left[\sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{\{\tau_n < \tau\}}\right], \tag{2.10}$$

where λ is a positive number that represents the ***discount factor****.*

2. Compute the value function v defined by

$$v(x) := \sup_{\pi \in \mathcal{A}(x)} J(x,\pi). \tag{2.11}$$

Proposition 2.2.1 *For any admissible $\pi \in \mathcal{A}(x)$. Let $X = X^{\pi}$, then we have*

$$E\Big[e^{-\lambda(t\wedge T)} X_{t\wedge T}\Big] \leq xe^{-\lambda t} + |\mu_\infty - \delta| te^{-\lambda t}, \tag{2.12}$$

where T is any stopping time.

Proof. If we put

$$Y_t := \sum_{i=1}^{\infty} \xi_i I_{\{\tau_i \leq t\}},$$

then Y is an increasing process (since $\xi_i \geq 0$), and

$$X_t = x + \int_0^t [\mu(a_s) - \delta] ds + \int_0^t \sigma(a_s) dW_s - Y_t. \tag{2.13}$$

Consider $(T_n)_{n\geq 1}$ a sequence of stopping time such that $\int_0^{t\wedge T_n} \sigma(a_s)dW_s$ is a true martingale. Then, due to $Y \geq 0$, we have

$$X_{t\wedge T_n} \leq x + \int_0^{t\wedge T_n} [\mu(a_s) - \delta]ds + \int_0^{t\wedge T_n} \sigma(a_s)dW_s.$$

Therefore, by taking the expectation, we obtain

$$E(X_{t\wedge T_n \wedge T}) \leq x + |\mu_\infty - \delta| E(t \wedge T_n \wedge T).$$

Letting n goes to infinite, and using Fatou's lemma, we get

$$E(X_{t\wedge T}) \leq x + |\mu_\infty - \delta| t.$$

Finally, the following inequalities follow

$$\begin{aligned} E(e^{-\lambda(t\wedge T)} X_{t\wedge T}) &\leq e^{-\lambda t} E(X_{t\wedge T}) \\ &\leq xe^{-\lambda t} + te^{-\lambda t}|\mu_\infty - \delta|. \end{aligned}$$

This completes the proof of the Proposition. ■

3

The quasi-variational inequalities

In this chapter, we will illustrate the first main step for solving Problem 2.1. This step focus on transforming the stochastic control problem (Problem 2.1) into quasi-variational inequalities, or Hamilton-Jacobi-Bellman equations when the optimal return function is smooth enough. This step is mainly based on the argument of the dynamic principle. Due to the incorporation of costs in our financial model, the resulting Problem 2.1 lies within the family of stochastic impulse control problems. This sort of control problems was already addressed in the literature, in some sense, due to its importance in Engineering. We can cite among the existing literature, the paper of Tang and Yong (1993). Therefore, we will present the Bellman's dynamic principle and the quasi-variational inequalities related to Problem 2.1 without proof and refer the readers to this paper. Furthermore, we will state and prove the growth property for the value function $v(x)$.

Theorem 3.0.1 Bellman's Dynamic Principle

The value function v satisfies the following optimality principle: for all $x \geq 0$, and for all stopping time T,

$$v(x) = \sup_{\pi \in \mathcal{A}(x)} E\Big[e^{-\lambda T} I_{\{T < \tau^\pi\}} v(X^{x,\pi}(T)) + \sum_{\tau_i \leq T} I_{\{T < \tau^\pi\}} e^{-\lambda \tau_i} g(\xi_i)\Big]. \tag{3.1}$$

Next, we define the following operator for any function ϕ and any $x \geq 0$,

$$M\phi(x) := \sup_{0<\eta\leq x}\Big[\phi(x-\eta)+g(\eta)\Big], \tag{3.2}$$

where g is given by (2.8). This operator is also called the impulse obstacles. As a result of Theorem (3.0.1), we state

Theorem 3.0.2 *The value function v has the following properties:*

1. *For any $x \geq 0$,*

$$v(x) \geq Mv(x), \tag{3.3}$$

2. *If for $x \geq 0$, (3.3) holds with strict inequality, then*

$$v(x) = \sup_{\pi\in\mathcal{A}(x)} E[e^{-\lambda T}v(X_T^\pi)I_{\{T<\tau^\pi\}}].$$

Theorem 3.0.3 *Suppose that the value function v is smooth. Then it satisfies the following Hamilton-Jacobi-Bellman equation:*

$$\max\left(Mv(x)-v(x), \max_{a\geq 0}\mathcal{L}^a v(x)\right) = 0,$$

where $Mv(x)$ is defined in (3.2), and

$$\mathcal{L}^a v(x) := \frac{\sigma^2(a)}{2}v''(x) + (\mu(a)-\delta)v'(x) - \lambda v(x), \;\; a \geq 0. \tag{3.4}$$

The proofs of these Theorems follow similar arguments as in Tang and Yong (1993).

Theorem 3.0.4 *The value function v is continuous and satisfies*

$$v(x) \leq k\frac{|\mu_\infty - \delta|}{\lambda} + kx, \;\; x \geq 0. \tag{3.5}$$

Proof. The proof of the continuity of the value function v follows the same arguments as in Tang and Yong (1993) (Theorem 3.1). Now, we will prove the growth condition (3.5). Consider an admissible control $\pi \in \mathcal{A}(x)$. Then, recall that

$$g(\xi_i) \leq k\xi_i,$$
$$\triangle\, Y_{\tau_i} = Y_{\tau_i} - Y_{\tau_i-} = \xi_i \geq 0,$$
$$\triangle\, X_{\tau_i} = X_{\tau_i} - X_{\tau_i-} = -\triangle\, Y_{\tau_i} \leq 0.$$

Then, due to the above (in)equalities, we write

$$\begin{aligned} J(x;\pi) &= E_x\Big[\sum_{i=1}^{\infty} e^{-\lambda\tau_i} g(\xi_i) I_{\{\tau_i<\tau^\pi\}}\Big] \\ &\leq kE_x\Big[\sum_{i=1}^{\infty} e^{-\lambda\tau_i}(\triangle\, Y_{\tau_i}) I_{\{\tau_i<\tau^\pi\}}\Big] \\ &= kE_x\left[\int_0^{\tau^\pi} e^{-\lambda s} dY_s\right]. \end{aligned} \tag{3.6}$$

Thanks to (2.13), we derive

$$\begin{aligned} E_x\Big[\int_0^{\tau^\pi} e^{-\lambda s} dY_s\Big] &= E\Big[\int_0^{\tau^\pi} e^{-\lambda s}(\mu(a_s)-\delta)ds + \int_0^{\tau^\pi} e^{-\lambda s}\sigma(a_s)dW_s - \int_0^{\tau^\pi} e^{-\lambda s} dX_s\Big] \\ &= E\Big[\int_0^{\tau^\pi} e^{-\lambda s}\Big(\mu(a_s)-\delta-\lambda X_t\Big)ds + \int_0^{\tau^\pi} e^{-\lambda s}\sigma(a_s)dW_s - e^{-\lambda\tau}X_{\tau^\pi} + x\Big]. \end{aligned}$$

Since $X_\tau = 0, \;\; X \geq 0$, we deduce that

$$E_x\Big[\int_0^{\tau} e^{-\lambda s} dY_s\Big] \leq \frac{|\mu_\infty - \delta|}{\lambda} + x. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$J(x;\pi) \leq k\frac{|\mu_\infty - \delta|}{\lambda} + kx,$$

and (3.5) follows immediately. ■

4

The case of no debt liability

This chapter focuses on the case when there is no debt liability rate (i.e. δ=0). Therefore, from Chapter 3, we deduce the Hamilton-Jacobi-Bellman equation (hereafter HJB) whose solution is a candidate for the control Problem 2.1 as follows.

$$0 = \max\left(\max_{a\geq 0}\left[\frac{1}{2}\sigma^2(a)V''(x) + \mu(a)V'(x) - \lambda V(x)\right], MV(x) - V(x)\right), \tag{4.1}$$

where the operator M is defined in (3.2), and

$$V(0) = 0. \tag{4.2}$$

Our primarily concern here is to construct a smooth solution to (4.1)-(4.2). To this end, we consider the following threshold

$$x_D := \inf\{x \geq 0 : MV(x) = V(x)\}. \tag{4.3}$$

Then for all $x < x_D, MV(x) \neq V(x)$, and the equation (4.1) becomes

$$0 = \max_{a\geq 0}\left(\frac{1}{2}\sigma^2(a)V''(x) + \mu(a)V'(x) - \lambda V(x)\right). \tag{4.4}$$

Thus, now we will concentrate on finding a smooth solution to (4.4).

The maximizer $a(x)$ of (4.4) satisfies

$$\sigma(a)\sigma'(a)V''(x) + \mu'(a)V'(x) = 0. \tag{4.5}$$

Thanks to

$$\sigma^2(a) = \int_0^a 2x\overline{F}(x)dx \text{ and } \mu(a) = \int_0^a \overline{F}(x)dx,$$

we have

$$\sigma'(a)\sigma(a) = a\overline{F}(a) \text{ and } \mu'(a) = \overline{F}(a),$$

and therefore (4.5) can be written as

$$a\overline{F}(a)V''(x) + \overline{F}(a)V'(x) = 0. \tag{4.6}$$

It is obvious that the description of the final solution to (4.6) requires to distinguish cases depending on whether the function $\overline{F}$ vanishes on $[0, \infty)$ or not. This is equivalent to whether the upper bound of the claim's sizes, N defined in (2.3), is finite or not respectively.

4.1 The case of unbounded claim's size.

This Subsection deals with the case of $N = \infty$. In this case, we have $\overline{F}(a) > 0$, for every $a < +\infty$. Therefore the solution to (4.6), denoted by $a(x)$, is given by

$$a(x) = \begin{cases} -\frac{V'(x)}{V''(x)}, & V''(x) \neq 0, \\ \infty, & V''(x) = 0. \end{cases} \tag{4.7}$$

On the set $\{x : x < x_D, a(x) < \infty\}$, (4.4) becomes

$$0 = \frac{1}{2}\sigma^2(a(x))V''(x) + \mu(a(x))V'(x) - \lambda V(x). \tag{4.8}$$

According to (4.7), we have $V''(x) = -\frac{V'(x)}{a(x)}$, and by substituting this into (4.8), we get

$$-\frac{1}{2}\sigma^2(a(x))\left(\frac{V'(x)}{a(x)}\right) + \mu(a(x))V'(x) - \lambda V(x) = 0.$$

This can be written as

$$h_0(a(x)) = \frac{\lambda V(x)}{V'(x)}, \tag{4.9}$$

where

$$h_0(a) = \begin{cases} \frac{-\sigma^2(a)}{2a} + \mu(a), & a > 0, \\ 0, & a = 0. \end{cases} \tag{4.10}$$

Notice that h_0 is a continuous, differentiable, and strictly increasing function on $(0,\infty)$. Indeed,

$$h_0'(a) := \frac{\sigma^2(a)}{2a^2} > 0, \quad a > 0.$$

Hence h_0^{-1} exists and

$$h_0^{-1} : [0, h_0(\infty)) \to [0,\infty),$$

where $h_0(\infty) = \mu(\infty) = E(U) < +\infty$. Then (4.9) implies,

$$a(x) = h_0^{-1}\left(\frac{\lambda V(x)}{V'(x)}\right), \quad 0 < x < x_D. \tag{4.11}$$

Since $h_0(0) = 0$ and $V(0) = 0$, then $a(0) = h_0^{-1}(0) = 0$ and $a(x)$ is a differentiable function. Therefore, by differentiating $h_0(a(x))V'(x) = \lambda V(x)$, we get

$$h_0'(a(x))a'(x)V'(x) + h_0(a(x))V''(x) = \lambda V'(x).$$

Again, by substituting $V''(x) = \frac{-V'(x)}{a(x)}$ into this equation, we derive

$$h_0'(a(x))a'(x)V'(x) + h_0(a(x))\frac{-V'(x)}{a(x)} = \lambda V'(x).$$

This leads to

$$h_0'(a(x))a'(x) = \frac{h_0(a(x))}{a(x)} + \lambda,$$

or equivalently

$$a'(x) = \frac{2a(x)h_0(a(x)) + 2\lambda a^2(x)}{\sigma^2(a(x))} > 0.$$

Thus, $a(x)$ is an increasing function and satisfies

$$\frac{\sigma^2(a(x))a'(x)}{2a(x)h_0(a(x)) + 2\lambda a^2(x)} = 1.$$

By integrating both sides in the equation above, we get

$$\int_0^x \frac{\sigma^2(a(t))a'(t)}{2a(t)h_0(a(t)) + 2\lambda a^2(t)}dt = x, \ \ x \geq 0.$$

By changing the variable (i.e. using $s = a(t)$), we get

$$g_0(a(x)) := x, \ \ x \geq 0, \tag{4.12}$$

where

$$g_0(a) := \int_0^a \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2}ds, \quad a \geq 0. \tag{4.13}$$

Lemma 4.1.1 *The function $a(x)$ defined by (4.7) satisfies*

$$a(x) = \begin{cases} g_0^{-1}(x), & 0 \le x < x_\infty, \\ \infty, & x_\infty \le x < x_D, \end{cases} \tag{4.14}$$

where

$$x_\infty := g_0(\infty) < +\infty. \tag{4.15}$$

Proof. The proof of the lemma is reduced to show that g_0^{-1} exists and $g_0(\infty) < +\infty$. g_0 is a continuously differentiable and strictly increasing function on $(0, \infty)$. Therefore, g_0^{-1} exists.

$$\begin{aligned} g_0(\infty) &= \int_0^\infty \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} ds \\ &= \int_0^1 \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} ds + \int_1^\infty \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} ds. \end{aligned}$$

In one hand, it is easy to prove that

$$\lim_{s \to 0} \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} = \frac{\overline{F}(0)}{\overline{F}(0) + 2\lambda} < +\infty,$$

and then

$$\int_0^1 \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} ds < +\infty.$$

On the other hand, we have

$$\begin{aligned} \int_1^\infty \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} ds &\le \frac{\sigma^2(\infty)}{2\lambda} \int_1^\infty \frac{1}{s^2} ds \\ &= \frac{\sigma^2(\infty)}{2\lambda} < +\infty. \end{aligned}$$

Thus, $g_0(\infty) < +\infty$. Then for $0 \le x < x_\infty = g_0(\infty)$, the equation (4.12) implies $a(x) = g_0^{-1}(x)$. Since $a(x)$ is continuous and increasing, then equation (4.12) allows us to conclude that $a(x) = \infty$, for $x \ge x_\infty$. This completes the proof of the lemma. ■

Next, on $(0, x_\infty)$ we have

$$\frac{-V''(x)}{V'(x)} = \frac{1}{g_0^{-1}(x)}.$$

By integrating both sides of this equation, we get

$$\int_x^{x_\infty} \frac{V''(t)}{V'(t)} dt = \int_x^{x_\infty} \frac{-dt}{g_0^{-1}(t)}, \quad 0 < x \le x_\infty,$$

which is equivalent to

$$\ln V'(x_\infty) - \ln V'(x) = \int_x^{x_\infty} \frac{-dt}{g_0^{-1}(t)}, \quad 0 < x \le x_\infty.$$

This implies

$$V'(x) = V'(x_\infty) \exp\left(\int_x^{x_\infty} \frac{dt}{g_0^{-1}(t)}\right), \quad 0 < x < x_\infty. \tag{4.16}$$

Corollary 4.1.1 *Using the above notation, we have*

$$V'(0+) = \infty.$$

Proof. Recall that

$$V'(0+) = V'(x_\infty) \exp\left(\int_0^{x_\infty} \frac{dt}{g_0^{-1}(t)}\right).$$

For any $0 < x \le x_\infty$, we have $g_0^{-1}(x) > 0$, and

$$\begin{aligned}\int_x^{x_\infty} \frac{dt}{g_0^{-1}(t)} &= \int_{g_0^{-1}(x)}^{\infty} \frac{\sigma^2(s)}{2s^2 h_0(s) + 2\lambda s^3} ds \\ &\le \frac{\sigma^2(\infty)}{2\lambda} \int_{g_0^{-1}(x)}^{\infty} \frac{1}{s^3} ds = \frac{\sigma^2(\infty)}{4\lambda (g_0^{-1}(x))^2} < +\infty.\end{aligned}$$

Then due to

$$\lim_{s \to 0} \frac{\sigma^2(s)}{2sh_0(s) + 2\lambda s^2} = \frac{\overline{F}(0)}{\overline{F}(0) + 2\lambda} > 0,$$

we have for $x > 0$ (very close to zero),

$$\int_0^x \frac{dt}{g_0^{-1}(t)} \approx \frac{\overline{F}(0)}{2\lambda + \overline{F}(0)} \int_0^x \frac{dt}{t} = +\infty.$$

Therefore,

$$V'(0+) = +\infty.$$

■

As a result, using $V(0) = 0$, and integrating (4.16), we get

$$V(x) = V'(x_\infty) \int_0^x \exp\left(\int_y^{x_\infty} \frac{dt}{g_0^{-1}(t)}\right) dy, \quad 0 \le x < x_\infty. \tag{4.17}$$

Due to (4.14), the equation (4.4) for $x_\infty \le x < x_D$ becomes

$$0 = \frac{1}{2}\sigma_\infty^2 V''(x) + \mu_\infty V'(x) - \lambda V(x). \tag{4.18}$$

Therefore the value function takes the following form:

$$V(x) = C_1 e^{r_+(x-x_\infty)} + C_2 e^{r_-(x-x_\infty)}, \quad x_\infty \le x < x_D, \tag{4.19}$$

where

$$r_\pm = \frac{-\mu_\infty \pm \sqrt{\mu_\infty^2 + 2\lambda\sigma_\infty^2}}{\sigma_\infty^2}. \tag{4.20}$$

Lemma 4.1.2 *For $x \ge x_D$, we have*

$$V(x) = MV(x) = V(\widetilde{x}) + k(x - \widetilde{x}) - K, \tag{4.21}$$

where $\widetilde{x} < x_D$ is a root of

$$V'(x) = k. \tag{4.22}$$

Proof. First of all, we assume the following

$$MV(x) = V(x), \quad x \ge x_D. \tag{4.23}$$

Recall that $MV(x) = \sup_{0<\eta\le x} (V(x-\eta) + k\eta - K)$. Then we can claim that for any $x \ge 0$, there exists $0 < \eta(x) \le x$, such that

$$MV(x) = V(x - \eta(x)) + k\eta(x) - K = V(x). \tag{4.24}$$

To prove this claim, we proceed as follows. For all $n \ge 1$, there exists $0 < \eta_n \le x$, such that

$$\lim_{n\to\infty} [V(x - \eta_n) + k\eta_n - K] = MV(x).$$

Since the sequence $(\eta_n)_{n\ge1}$ is bounded, then there exists a subsequence of it, $(\eta_{\varphi(n)})_{n\ge0}$, that converges to a number $\eta(x)$ satisfying $0 \le \eta(x) \le x$ and

$$\begin{aligned} MV(x) &= \lim_{n\to\infty} \Big(V(x - \eta_{\varphi(n)}) + k\eta_{\varphi(n)} - K \Big) \\ &= V(x - \eta(x)) + k\eta(x) - K = V(x). \end{aligned}$$

Then $\eta(x)$ can not be zero due to $V(x) > V(x) - K$. Furthermore, $\eta(x)$ satisfies

$$V'(x - \eta(x)) = k. \tag{4.25}$$

Then by differentiating (4.24) and using the above equation, we deduce that

$$V'(x) = k, \quad x \geq x_D. \tag{4.26}$$

Next consider

$$\widetilde{x} := \inf\{x \geq 0 : V'(x) = k\}. \tag{4.27}$$

Thus, it is obvious that

$$x_D > x_D - \eta(x_D) \geq \widetilde{x}.$$

Now, we will prove that $V'(x) - k$ vanishes at most one time on $(0, x_D)$.
Denote $f(x) = V'(x) - k$ on $(0, x_D)$, and suppose that there exist $\widehat{x}_1$ and $\widehat{x}_2$ such that $\widetilde{x} \leq \widehat{x}_1 < \widehat{x}_2 < x_D$, and

$$f(\widehat{x}_1) = f(\widehat{x}_2) = f(x_D) = 0.$$

Then, there exist $\widehat{x}_1 < \widetilde{x}_1 < \widehat{x}_2 < \widetilde{x}_2 < x_D$ such that

$$f'(\widetilde{x}_1) = f'(\widetilde{x}_2) = 0.$$

This leads to the existence of $\widetilde{x}_1 < \overline{x} < \widetilde{x}_2$ such that

$$f''(\overline{x}) = 0.$$

On the other hand,

$$f''(x) = V'''(x) = \begin{cases} V'(x_\infty)\left(\left(\frac{-1}{g_0^{-1}(t)}\right)^2 + \left(-\frac{1}{g_0^{-1}(x)}\right)'\right)\exp\left(\int_x^{x_\infty}\frac{dt}{g_0^{-1}(t)}\right) > 0, & \text{if } 0 < x < x_\infty, \\ C_1 r_+^3 e^{r_+(x - x_\infty)} + C_2 r_-^3 e^{r_-(x - x_\infty)} > 0, & \text{if } x_\infty < x < x_D. \end{cases}$$

This is a contradiction. Hence, we cannot have another point on $(0, x_D)$ at which f vanishes other than $\widetilde{x}$. As a consequence we see that solution to $V'(x) = k$ exists and is unique on $(0, x_D)$, and is given by that $x_D - \eta(x_D) = \widetilde{x}$. Then, due to

$$V(x_D) = V(\widetilde{x}) + k(x_D - \widetilde{x}) - K,$$

and (4.26), we get

$$V(x) = V(\widetilde{x}) + k(x - \widetilde{x}) - K, \quad \forall x \geq x_D. \tag{4.28}$$

To achieve the proof of this lemma, we need to prove the assumption (4.23). In fact, suppose that there exists $x_1 > x_D$ such that $V(x_1) \neq MV(x_1)$, and consider

$$\underline{x} = \inf\{x \geq x_D, V(x) \neq MV(x)\},$$

$$\overline{x} = \sup\{x \geq x_1, V(x) \neq MV(x)\}.$$

Then on the set $(\underline{x}, \overline{x})$, $V(x)$ takes the form of

$$V(x) = \alpha_1 e^{r_+(x-\underline{x})} + \alpha_2 e^{r_-(x-\underline{x})}, \quad \underline{x} \leq x \leq \overline{x}.$$

Due to $V'(\underline{x}) = k, \quad V''(\underline{x}-) = 0$ (since on $(x_D, \underline{x}), V(x) = MV(x)$ that leads to $V'(x) = k$), and V is twice continuously differentiable on (x_D, ∞), we get

$$\alpha_1 = \frac{-r_-k}{r_+(r_+ - r_-)}, \quad \alpha_2 = \frac{r_+k}{r_-(r_+ - r_-)}.$$

This leads to conclude that $V'' > 0$ on $(\underline{x}, \overline{x})$ in one hand. On the other hand, $V'(\underline{x}) = k = V'(\overline{x})$ implies that V'' vanishes on $(\underline{x}, \overline{x})$. This is a contradiction. Therefore, the assumption (4.23) holds. This completes the proof of the lemma. ■

By combining (4.17), (4.19) and (4.21) together, we write

$$V(x) = \begin{cases} C\int_0^x \exp\left(\int_x^{x_\infty} \frac{dt}{g_0^{-1}(t)}\right) dy, & 0 \leq x < x_\infty, \\ C_1 e^{r_+(x-x_\infty)} + C_2 e^{r_-(x-x_\infty)}, & x_\infty \leq x < x_D, \\ V(\widetilde{x}) + k(x - \widetilde{x}) - K, & x \geq x_D. \end{cases}$$

where $C, C_1, C_2, \widetilde{x}$ and x_D are parameters to be calculated. The smooth fit of V' and V'' at the point x_∞ implies

$$V'(x_\infty-) = V'(x_\infty) \Rightarrow C = C_1 r_+ + C_2 r_-,$$

$$V''(x_\infty-) = V''(x_\infty) \Rightarrow 0 = C_1 r_+^2 + C_2 r_-^2.$$

By solving these two equations, we get

$$C_1 r_+ = \frac{-Cr_-}{r_+ - r_-},$$
$$C_2 r_- = \frac{Cr_+}{r_+ - r_-}.$$

Therefore,

$$V(x)=\begin{cases} C\int_0^x \exp\left(\int_x^{x_\infty}\frac{dt}{g_0^{-1}(t)}\right)dy, & 0\le x<x_\infty,\\ C\left(\frac{-r_-}{r_+(r_+-r_-)}e^{r_+(x-x_\infty)}+\frac{r_+}{r_-(r_+-r_-)}e^{r_-(x-x_\infty)}\right), & x_\infty\le x<x_D,\\ V(\widetilde{x})+k(x-\widetilde{x})-K, & x\ge x_D.\end{cases} \tag{4.29}$$

The points $\widetilde{x}$ and x_D are roots of

$$V'(x)=k. \tag{4.30}$$

To solve this equation, we write

$$V'(x)=\begin{cases} CH(x), & x<x_D,\\ k, & x\ge x_D,\end{cases} \tag{4.31}$$

where

$$H(x):=\begin{cases} \exp\left(\int_x^{x_\infty}\frac{dt}{g_0^{-1}(t)}\right), & 0<x<x_\infty,\\ \frac{-r_-}{r_+-r_-}e^{r_+(x-x_\infty)}+\frac{r_+}{r_+-r_-}e^{r_-(x-x_\infty)}, & x\ge x_\infty.\end{cases} \tag{4.32}$$

For $0<x<x_\infty$, we have

$$H'(x)=-\frac{1}{g_0^{-1}(x)}\exp\left(\int_x^{x_\infty}\frac{dt}{g_0^{-1}(t)}\right)<0,$$

$$\begin{aligned} H''(x)&=\left(-\frac{1}{g_0^{-1}(x)}\right)'H(x)+\left(-\frac{1}{g_0^{-1}(x)}\right)H'(x),\\ &=\left(-\frac{1}{g_0^{-1}(x)}\right)'H(x)+\left(-\frac{1}{g_0^{-1}(x)}\right)^2H(x)>0.\end{aligned}$$

For $x=x_\infty$,

$$H'(x_\infty)=\frac{-r_-r_+}{r_+-r_-}+\frac{r_+r_-}{r_+-r_-}=0.$$

For $x>x_\infty$,

$$\begin{aligned} H'(x)&=\frac{-r_-r_+}{r_+-r_-}e^{r_+(x-x_\infty)}+\frac{r_+r_-}{r_+-r_-}e^{r_-(x-x_\infty)}>0,\\ H''(x)&=\frac{-r_+^2r_-}{r_+-r_-}e^{r_+(x-x_\infty)}+\frac{r_+r_-^2}{r_+-r_-}e^{r_-(x-x_\infty)},\\ &=\frac{-r_+r_-}{r_+-r_-}\left(r_+e^{r_+(x-x_\infty)}-r_-e^{r_-(x-x_\infty)}\right)>0.\end{aligned}$$

Thus, it is easy to see that H is a continuously differentiable convex function, and the unique root of

$$H'(x)=0$$

is $x = x_\infty$ with $H(x_\infty) = 1$. Then H is strictly decreasing on $(0, x_\infty]$ and strictly increasing on $[x_\infty, \infty)$. Next, put

$$\widehat{H} := H\,|_{(0,x_\infty]} \text{ and } \overline{H} := H\,|_{[x_\infty,\infty)}.$$

Then the two functions $\widehat{H}^{-1}$ and $\overline{H}^{-1}$ exist and both are defined on $[1, \infty)$. Notice that the equation (4.30) is equivalent to

$$H(x) = \frac{k}{C}.$$

Then the existence of $\widetilde{x}^C$ and x_D^C satisfying $C\widehat{H}(\widetilde{x}^C) = C\overline{H}(x_D^C) = k$ is guaranteed by the condition that $\frac{k}{C}$ should belong to the range of $\widehat{H}$ and $\overline{H}$, which is equivalent to $\frac{k}{C} \geq 1$. Hence for $0 < C \leq k$, we derive

$$\widetilde{x}^C = \widehat{H}^{-1}\left(\frac{k}{C}\right), \quad x_D^C = \overline{H}^{-1}\left(\frac{k}{C}\right).$$

Clearly, $\widetilde{x}^C$ is an increasing function of C, while x_D^C is a decreasing function of C. Therefore, to completely describe V, we need to calculate the only remaining parameter, C. To this end, we use the smooth fit of V at the point x_D, that is the following equation.

$$V(x_D) = V(\widetilde{x}) + k(x_D - \widetilde{x}) - K. \tag{4.33}$$

Thanks to (4.31), this is equivalent to

$$I(C) = K,$$

where

$$I(C) := \int_{\widetilde{x}^C}^{x_D^C} (k - CH(x))dx, \;\; 0 < C \leq k.$$

$I(C)$ is a continuous and decreasing function of C because both the integrand and the interval are continuous and decreasing with respect to C. As C approaches to 0, $I(C)$ is maximized and becomes an infinite number.

Obviously, if $C = k$ then $\widetilde{x}^C = x_D^C = x_\infty$. This corresponds to the case when $K = 0$.

Since $\lim_{C\to 0} \widetilde{x}^C = \widehat{H}^{-1}(\infty) = 0$ and $\lim_{C\to 0} x_D^C = \overline{H}^{-1}(\infty) = \infty$, we get

$$\lim_{C\to 0} I(C) = \int_0^\infty k dx = \infty.$$

Therefore, there exists $\widetilde{C}$ such that $0 < \widetilde{C} \leq k$ and

$$I(\widetilde{C}) = \int_{\widetilde{x}^{\widetilde{C}}}^{x_D^{\widetilde{C}}} (k - \widetilde{C}H(x))dx = K, \tag{4.34}$$

where $K \geq 0$.

Therefore, this analysis leads to state the following.

Theorem 4.1.1 *The function*

$$\widehat{V}(x) = \begin{cases} C \int_0^x \exp\left(\int_y^{x_\infty} \frac{dt}{g_0^{-1}(t)}\right) dy, & 0 \leq x < x_\infty, \\ C\left(\frac{-r_-}{r_+(r_+-r_-)} e^{r_+(x-x_\infty)} + \frac{r_+}{r_-(r_+-r_-)} e^{r_-(x-x_\infty)}\right), & x_\infty \leq x < x_D, \\ \widehat{V}(\widetilde{x}) + k(x-\widetilde{x}) - K, & x \geq x_D, \end{cases} \tag{4.35}$$

with $C = \widetilde{C}, \widetilde{x} = \widetilde{x}^{\widetilde{C}} = \widehat{H}^{-1}\left(\frac{k}{\widetilde{C}}\right), x_D = x_D^{\widetilde{C}} = \overline{H}^{-1}\left(\frac{k}{\widetilde{C}}\right)$,
is a smooth solution (continuously differentiable on $(0, \infty)$ *and twice differentiable on* $(0, x_D) \cup (x_D, x_\infty)$ *to (4.1)-(4.2), satisfying a (3.5) type growth condition.*

proof. In the following subsection, We will explain how this theorem can be seen as a particular case of Theorem 4.2.1. Therefore, the proof of the current theorem will follow immediately from that of Theorem 4.2.1. ∎

4.2 The case of bounded claim's size.

In this subsection, we consider the case where N, defined in (2.3), is finite. Therefore, the maximizer $a_N(x)$ of (4.4), which is the root of (4.6), is given by a version of Lemma 4.1.1 as follows.

Lemma 4.2.1 *The function* $a_N(x)$ *is given by*

$$a_N(x) = \begin{cases} g_0^{-1}(x), & 0 \leq x < x_N, \\ N, & x \geq x_N, \end{cases} \tag{4.36}$$

where g_0 *is defined in (4.13) and* x_N *is given by*

$$x_N := g_0(N) = \int_0^N \frac{\sigma^2(s)}{2\lambda s^2 + 2sh_0(s)} ds. \tag{4.37}$$

Similarly as in Subsection 4.1, we will start describing the function V solution to (4.1)-(4.2) that we will denote in this subsection by V_N (to show the dependence in N).

Thanks to (4.36) and (4.7), we write

$$\frac{-V_N''(x)}{V_N'(x)} = \frac{1}{g_0^{-1}(x)}, \quad 0 < x < x_N,$$

By integrating both sides of this equation, we get

$$\int_x^{x_N} \frac{V_N''(t)}{V_N'(t)} dt = \int_x^{x_N} \frac{-dt}{g_0^{-1}(t)}, \quad 0 < x < x_N,$$

which is equivalent to

$$\ln V_N'(x_N) - \ln V_N'(x) = \int_x^{x_N} \frac{-dt}{g_0^{-1}(t)}, \quad 0 < x < x_N.$$

Thus, due to $V_N(0) = 0$, we obtain

$$V_N(x) = V_N'(x_N) \int_0^x \exp\left(\int_y^{x_N} \frac{dt}{g_0^{-1}(t)} \right) dy, \quad 0 \leq x < x_N. \tag{4.38}$$

On the set $[x_N, x_D)$, (4.4) is equivalent to

$$0 = \frac{1}{2}\sigma^2(N) V_N''(x) + \mu(N) V_N'(x) - \lambda V_N(x), \tag{4.39}$$

whose solution takes the following form

$$V_N(x) = C_1 e^{r_+(x-x_N)} + C_2 e^{r_-(x-x_N)}, \quad x_N \leq x < x_D, \tag{4.40}$$

where $r_\pm$ are given in (4.20). Next, due to Lemma 4.1.2, we have

$$V_N(x) = V_N(\widetilde{x}) + k(x - \widetilde{x}) - K, \quad x \geq x_D. \tag{4.41}$$

Then, according to the equations (4.38), (4.40) and (4.41), we conclude that

$$V_N(x) = \begin{cases} C \int_0^x \exp\left(\int_y^{x_N} \frac{dt}{g_0^{-1}(t)} \right) dy, & 0 \leq x < x_N, \\ C_1 e^{r_+(x-x_N)} + C_2 e^{r_-(x-x_N)}, & x_N \leq x < x_D, \\ V_N(\widetilde{x}) + k(x - \widetilde{x}) - K, & x \geq x_D, \end{cases} \tag{4.42}$$

where $C = V_N'(x_N)$ and C_1 and C_2, will be determined in term of C using the smooth fit of the functions V_N' and V_N'' at the point x_N. In fact, we have

$$\begin{aligned} &V_N'(x_N-) = V_N'(x_N) \Rightarrow C = C_1 r_+ + C_2 r_-, \\ &V_N''(x_N-) = V_N''(x_N) \Rightarrow C\frac{-1}{g_0^{-1}(x_N)} = C_1 r_+^2 + C_2 r_-^2 \Rightarrow \frac{-C}{N} = C_1 r_+^2 + C_2 r_-^2. \end{aligned} \tag{4.43}$$

By solving the above two obtained equations, we get

$$C_1 r_+ = \frac{-C(1+Nr_-)}{N(r_+ - r_-)},$$
$$C_2 r_- = \frac{C(1+Nr_+)}{N(r_+ - r_-)}.$$

Thus,

$$V_N'(x) = \begin{cases} C\exp\left(\int_x^{x_N} \frac{dt}{g_0^{-1}(t)}\right), & 0 < x < x_N, \\ C\left(\frac{-(1+Nr_-)}{N(r_+-r_-)}e^{r_+(x-x_N)} + \frac{1+Nr_+}{N(r_+-r_-)}e^{r_-(x-x_N)}\right), & x_N \le x < x_D, \\ k, & x \ge x_D, \end{cases} \tag{4.44}$$

or equivalently

$$V_N'(x) = \begin{cases} CH_N(x), & x < x_D, \\ k, & x \ge x_D. \end{cases} \tag{4.45}$$

Here

$$H_N(x) := \begin{cases} \exp\left(\int_x^{x_N} \frac{dt}{g_0^{-1}(t)}\right), & 0 < x < x_N, \\ \frac{-(1+Nr_-)}{N(r_+-r_-)}e^{r_+(x-x_N)} + \frac{(1+Nr_+)}{N(r_+-r_-)}e^{r_-(x-x_N)}, & x \ge x_N. \end{cases} \tag{4.46}$$

We know that $H_N'(x_N) = -\frac{1}{N} < 0$, which means that H_N keeps decreasing from 0 to x_N. Next, for $x > x_N$, we have

$$H_N'(x) = \frac{-r_+(1+Nr_-)}{N(r_+ - r_-)}e^{r_+(x-x_N)} + \frac{r_-(1+Nr_+)}{N(r_+ - r_-)}e^{r_-(x-x_N)},$$

and

$$\lim_{x\to\infty} H_N'(x) = \begin{cases} +\infty, & \text{if } 1 + Nr_- < 0, \\ -\infty, & \text{if } 1 + Nr_- > 0. \end{cases} \tag{4.47}$$

Notice that if $1 + Nr_- \ge 0$ $(\frac{1}{N} \ge -r_-)$, then there is no solution for $H_N'(x) = 0$. Since $H_N'(x_N) = -\frac{1}{N} < 0$ and $\lim_{x\to\infty} H_N'(x) = +\infty$ for

$$\lambda > (\frac{\sigma^2}{2N^2} - \frac{\mu}{N})^+, \tag{4.48}$$

then $H_N'(x) = 0$ has a unique root $\widehat{x}$ which is given by

$$\widehat{x} := x_N + \frac{1}{r_+ - r_-}\ln\Big[\frac{r_-(1+Nr_+)}{r_+(1+Nr_-)}\Big] > x_N. \tag{4.49}$$

Then, H_N is strictly decreasing on $(0, \widehat{x}]$ and strictly increasing on $[\widehat{x}, \infty)$. Put

$$\widehat{H}_N := H_N|_{(0,\widehat{x}]}, \quad \overline{H}_N := H_N|_{[\widehat{x},\infty)}.$$

Recall that

$$H_N(\infty) = \infty = H_N(0), \quad H_N(\widehat{x}) = \min_{x \geq 0} H_N(x), \quad H_N(x_N) = 1.$$

Therefore, it is clear that $\widehat{H}_N^{-1}$ and $\overline{H}_N^{-1}$ exist and both are defined on $[H_N(\widehat{x}), \infty)$, and

$$\widetilde{x}^C = \widehat{H}_N^{-1}\left(\frac{k}{C}\right), \quad x_D^C = \overline{H}_N^{-1}\left(\frac{k}{C}\right). \tag{4.50}$$

These two levels exist if and only if $\frac{k}{C}$ belongs to the range of both $\widehat{H}_N$ and $\overline{H}_N$, which is equivalent to the condition

$$\frac{k}{C} \geq H_N(\widehat{x}). \tag{4.51}$$

Then for any $0 < C \leq \frac{k}{H_N(\widehat{x})}$, $\widetilde{x}^C$ and x_D^C exist and we can define

$$I_N(C) := \int_{\widetilde{x}^C}^{x_D^C} (k - CH_N(x))dx.$$

The function $I_N(C)$ is continuous and decreasing ($\widetilde{x}^C$ is an increasing function of C, and x_D^C is a decreasing function of C). As C approaches to 0, $I_N(C)$ is maximized and takes an infinite number. Obviously, if $C = \frac{k}{H_N(\widehat{x})}$, then $\widetilde{x}^C = x_D^C = \widehat{x}$. This corresponds to the case when $K = 0$. Since $\lim_{C \to 0} \widetilde{x}^C = \widehat{H}_N^{-1}(\infty) = 0$ and $\lim_{C \to 0} x_D^C = \overline{H}_N^{-1}(\infty) = \infty$, we get

$$\lim_{C \to 0} I_N(C) = \int_0^\infty k dx = \infty.$$

Therefore, there exists $\widetilde{C}(N)$ such that $0 < \widetilde{C}(N) \leq \frac{k}{H_N(\widehat{x})}$ and

$$I_N(\widetilde{C}) = K, \tag{4.52}$$

where $K \geq 0$. Finally, the possible smooth candidate for the HJB equation (4.1)-(4.2) in this subsection is given by

$$\widehat{V}_N(x) = \begin{cases} C \int_0^x \exp\left(\int_y^{x_N} \frac{dt}{g^{-1}(t)}\right) dy, & 0 \leq x < x_N, \\ C\left(\frac{-(1+Nr_-)}{Nr_+(r_+ - r_-)} e^{r_+(x-x_N)} + \frac{1+Nr_+}{Nr_-(r_+ - r_-)} e^{r_-(x-x_N)}\right), & x_N \leq x < x_D, \\ \widehat{V}_N(\widetilde{x}) + k(x - \widetilde{x}) - K, & x \geq x_D, \end{cases} \tag{4.53}$$

with $C = \widetilde{C}(N), \widetilde{x} = \widetilde{x}^{\widetilde{C}(N)} = \widehat{H}_N^{-1}\left(\frac{k}{\widetilde{C}(N)}\right), x_D = x_D^{\widetilde{C}(N)} = \overline{H}_N^{-1}\left(\frac{k}{\widetilde{C}(N)}\right)$.

Notice that this case of bounded claim's sizes cover the previous case of $N = \infty$. Indeed, for N defined in (2.3), we have $0 < N \leq \infty$, and by adapting the notations of this current section, we can prove that

- x_N defined in (4.37) converges to x_∞ defined in (4.15), as N becomes infinite. Also notice that $\widehat{x}$ defined in (4.49) converges to /coincides with x_∞ when N becomes infinite. Indeed, from the expression (4.49), we can easily prove that the positive quantity $\widehat{x} - x_N$ goes to zero as N becomes infinite.

- $\widetilde{C}(N)$ in (4.52) converges to /coincides with $\widetilde{C}$ defined in (4.34) as N becomes infinite.

- The function $\widehat{V}_N(x)$ defined in (4.53) converges to /coincides with $\widehat{V}(x)$ defined in (4.35), where N becomes infinite.

- The condition (4.48) becomes redundant when N is finite.

Now we are in stage to state the first main result of this chapter.

Theorem 4.2.1 *Suppose that the condition (4.48) holds. Then the function $\widehat{V}_N$ defined in (4.53), is continuously differentiable on $(0, \infty)$, twice continuously differentiable on $(0, x_D) \cup (x_D, \infty)$, and is a smooth solution to the HJB equation (4.1) - (4.2). Furthermore, $\widehat{V}_N$ satisfies the growth condition in the following sense.*

$$\widehat{V}_N(x) \leq \widehat{V}_N(\widetilde{x}) + kx, \quad \forall x \geq 0. \tag{4.54}$$

Proof. It is clear that the function $\widehat{V}_N$, defined in (4.53), is continuously differentiable on $(0, \infty)$ and twice continuously differentiable on $(0, x_D) \cup (x_D, \infty)$ by construction. Next, notice that for $0 < x < \widehat{x}$, $\widehat{V}'_N(x) > k$, since $\widehat{V}'_N$ is decreasing on $(0, \widehat{x})$ $(\widehat{x} > x_N > \widetilde{x})$, and $\widehat{V}'_N(\widetilde{x}) = k$. Then for any $0 < x < \widetilde{x}$, the function $\eta \in (0, x] \to \widehat{V}_N(x - \eta) + k\eta - K$ is decreasing (since $k - \widehat{V}'_N(x - \eta) < 0$). Therefore for all $0 < x < \widetilde{x}$,

$$\begin{aligned} M\widehat{V}_N(x) &= \sup_{0<\eta\leq x} [\widehat{V}_N(x - \eta) + k\eta - K] \\ &= \lim_{\eta \to 0} [\widehat{V}_N(x - \eta) + k\eta - K] \\ &= \widehat{V}_N(x) - K < \widehat{V}_N(x). \end{aligned}$$

Due to

$$\widehat{V}_N'(x) = \begin{cases} > k, & x < \widetilde{x}, \\ \leq k, & \widetilde{x} \leq x, \end{cases}$$

we deduce the following

- $\sup_{x-\widetilde{x}<\eta\leq x} (\widehat{V}_N(x-\eta) + k\eta - K) = \widehat{V}_N(\widetilde{x}) + k(x-\widetilde{x}) - K, \quad x > \widetilde{x}.$

- $\sup_{0<\eta\leq x-\widetilde{x}} (\widehat{V}_N(x-\eta) + k\eta - K) = \widehat{V}_N(\widetilde{x}) + k(x-\widetilde{x}) - K.$

Thus, for all $x > \widetilde{x}$,

$$\begin{aligned} M\widehat{V}_N(x) &= \sup_{0<\eta\leq x} (\widehat{V}_N(x-\eta) + k\eta - K) \\ &= \widehat{V}_N(\widetilde{x}) + k(x-\widetilde{x}) - K, \ \ \forall x > \widetilde{x}. \end{aligned}$$

In particular for $x \geq x_D$,

$$M\widehat{V}_N(x) = \widehat{V}_N(\widetilde{x}) + k(x-\widetilde{x}) - K = \widehat{V}_N(x).$$

For $\widetilde{x} < x < x_D$, we have

$$\begin{aligned} M\widehat{V}_N(x) &= \widehat{V}_N(\widetilde{x}) + k(x-\widetilde{x}) - K \\ &= \widehat{V}_N(\widetilde{x}) + k(x_D-\widetilde{x}) - K - k(x_D - x) \\ &= \widehat{V}_N(x_D) - k(x_D - x) < \widehat{V}_N(x). \end{aligned}$$

The last inequality comes from the fact that $\widehat{V}_N'(x) < k$ for $\widetilde{x} < x < x_D$, and by integration over $[x, x_D)$, the inequality under consideration follows. Therefore, we get

- For $0 < x < x_D$, $\ M\widehat{V}_N(x) < \widehat{V}_N(x)$.

- For $x \geq x_D$, $\ M\widehat{V}_N(x) = \widehat{V}_N(x)$.

Furthermore, by construction, $\widehat{V}_N$ is a solution to (4.8) on $(0, x_N)$, and solution to (4.39) on (x_N, x_D). Therefore, V is a solution to (4.4) on $(0, x_D)$, and $V(0) = 0$. Next, notice that for all $a \geq 0$, and for all $x \geq x_D$,

$$\begin{aligned} \mathcal{L}^a\widehat{V}_N(x) &= \frac{\sigma^2(a)}{2}\widehat{V}_N''(x) + (\mu(a)-\delta)\widehat{V}_N'(x) - \lambda\widehat{V}_N(x) \\ &= k(\mu(a)-\delta) - \lambda\widehat{V}_N(x) \\ &\leq k(\mu(a)-\delta) - \lambda\widehat{V}_N(x_D). \end{aligned}$$

(Since $\widehat{V}_N' \geq 0$, then $\widehat{V}_N$ is increasing. Therefore, $\widehat{V}_N(x) \geq \widehat{V}_N(x_D)$.) Notice that for any $x < x_D$, $\mathcal{L}^a\widehat{V}_N(x) = 0$. Then,

$$0 = \mathcal{L}^a\widehat{V}_N(x_D-) = \frac{\sigma^2(a)}{2}\widehat{V}_N''(x_D-) + (\mu(a) - \delta)k - \lambda\widehat{V}_N(x_D)$$
$$\geq (\mu(a) - \delta)k - \lambda\widehat{V}_N(x_D).$$

Since $\widehat{V}_N'$ and $\widehat{V}_N$ are continuous, the last inequality follows from the fact that $\widehat{V}_N''(x_D-) \geq 0$. To prove this, we recall that there exists $\widehat{x} < x_D$ such that

$$\widehat{V}_N''(\widehat{x}) = 0,$$

and $\widehat{V}_N''$ is increasing on $(0, x_D)$. Therefore, $\widehat{V}_N''(x_D-) \geq 0$. In conclusion, for all $a \geq 0$, and for all $x \geq x_D$, we drive

$$\mathcal{L}^a\widehat{V}_N(x) \leq 0.$$

This completes the proof of the theorem. ■

Next in the following subsection, we will describe the optimal policies and prove that the function $\widehat{V}_N$ defined in (4.53) coincides with the the value function v defined in (2.11).

4.3 Optimal Policies

To state the main result of this subsection, we recall that the feedback function $a_N(x)$ is given by (4.36), and the function $\widehat{V}_N$ is described by (4.53).

Theorem 4.3.1 *Suppose that C is a root of (4.52) and $\widetilde{x}(C)$ and $x_D(C)$ are given by (4.50). Let $a(x)$ be given by (4.36). Then the following control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, ..., \tau_n^*, ...; \xi_1^*, \xi_2^*, ..., \xi_n^*, ...)$$

defined by

$$u_t^* = a_N(X_t^*), \;\; t \geq 0, \tag{4.55}$$

$$\tau_1^* := \inf\{t \geq 0 : X^*(t) = x_D(C)\}, \tag{4.56}$$

$$\xi_1^* := x_D(C) - \widetilde{x}(C), \tag{4.57}$$

and for every $n \geq 2$:

$$\tau_n^* := \inf\{t \geq \tau_{n-1} : X^*(t) = x_D(C)\}, \tag{4.58}$$

$$\xi_n^* := x_D(C) - \widetilde{x}(C), \tag{4.59}$$

where X^* is the solution to the stochastic differential equation

$$X_t^* = x_0^* + \int_0^t \Big[\mu(a_N(X_s^*)) - \delta\Big] ds + \int_0^t \sigma(a_N(X_s^*)) dW_s - (x_D(C) - \widetilde{x}(C)) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}} \tag{4.60}$$

is optimal and the function $\widehat{V}_N$, defined by (4.53), coincides with the value function. That is,

$$\widehat{V}_N(x) = v(x) = J(x; \pi^*), \ \ \forall x \geq 0. \tag{4.61}$$

Proof. For any admissible policy $\pi \in \mathcal{A}(x)$, we can write

$$e^{-\lambda(t\wedge\tau)} \widehat{V}_N(X_{t\wedge\tau}) \leq e^{-\lambda t} \widehat{V}_N(\widetilde{x}) + k e^{-\lambda(t\wedge\tau)} X_{t\wedge\tau}.$$

Combining this inequality together with (2.12), we conclude that $E\Big[e^{-\lambda(t\wedge\tau)} \widehat{V}_N(X_{t\wedge\tau})\Big]$ goes to zero when t becomes infinite. Next, applying Itô formula to $e^{-\lambda(t\wedge\tau)} \widehat{V}_N(X_{t\wedge\tau})$, we derive

$$\begin{aligned}
e^{-\lambda(t\wedge\tau)} \widehat{V}_N(X_{t\wedge\tau}) =& \widehat{V}_N(x) + \int_0^{t\wedge\tau} e^{-\lambda s} \Big[\frac{1}{2}\sigma(a_s) \widehat{V}_N''(X_s) + (\mu(a_s) - \delta) \widehat{V}_N'(X_{s-}) - \lambda \widehat{V}_N(X_s)\Big] ds \\
&+ \int_0^{t\wedge\tau} e^{-\lambda s} \sigma(a_s) \widehat{V}_N'(X_s) dW_s - \int_0^{t\wedge\tau} e^{-\lambda s} \widehat{V}_N'(X_{s-}) dY_s \\
&+ \sum_{0<s\leq t\wedge\tau} e^{-\lambda s} \Big[\widehat{V}_N(X_s) - \widehat{V}_N(X_{s-}) - \widehat{V}_N'(X_{s-})(\triangle\, X_s)\Big] \\
=& \widehat{V}_N(x) + \int_0^{t\wedge\tau} e^{-\lambda s} \mathcal{L}^{a(s)} \widehat{V}_N(X_s) ds + \int_0^{t\wedge\tau} e^{-\lambda s} \sigma(a_s) \widehat{V}_N'(X_s) dW_s \\
&+ \sum_{0<s\leq t\wedge\tau} e^{-\lambda s} \Big[\widehat{V}_N(X_s) - \widehat{V}_N(X_{s-})\Big] \\
\leq& \widehat{V}_N(x) + \int_0^{t\wedge\tau} e^{-\lambda s} \sigma(a_s) \widehat{V}_N'(X_s) dW_s + \sum_{0<s\leq t\wedge\tau} e^{-\lambda s} \Big[\widehat{V}_N(X_s) - \widehat{V}_N(X_{s-})\Big].
\end{aligned} \tag{4.62}$$

Since $\int_0^{t\wedge\tau} \sigma(a_s) \widehat{V}_N'(X_s) dW_s$ is a local martingale, then its expected value does not exceed zero, and we deduce that

$$E\Big[e^{-\lambda(t\wedge\tau)} \widehat{V}_N(X_{t\wedge\tau})\Big] \leq \widehat{V}_N(x) + E\Big[\sum_{0<s\leq t\wedge\tau} e^{-\lambda s} \Big(\widehat{V}_N(X_s) - \widehat{V}_N(X_{s-})\Big)\Big]. \tag{4.63}$$

Remark that

$$\sum_{0<s\leq t\wedge\tau} e^{-\lambda s}\Big[\widehat{V}_N(X_s)-\widehat{V}_N(X_{s-})\Big]=\sum_{i=1}^{\infty}e^{-\lambda\tau_i}\Big[\widehat{V}_N(X_{\tau_i})-\widehat{V}_N(X_{\tau_i-})\Big]I_{\{\tau_i\leq t\wedge\tau\}}.$$

Also, notice that $\triangle\ X_{\tau_i}=-\ \triangle\ Y_{\tau_i}\leq 0$, and due to Theorem 4.2.1, $\widehat{V}_N$ satisfies

$$\widehat{V}_N(y)\geq\widehat{V}_N(y-\eta)+k\eta-K,\ \ \forall y\geq 0,\ \ \forall 0\geq\eta\leq y.$$

Then by taking $y=X_{\tau_i-}$ and $0\leq\eta=-\ \triangle\ X_{\tau_i}=X_{\tau_i-}-X_{\tau_i}\leq y$, we get

$$\widehat{V}_N(X_{\tau_i-})\geq\widehat{V}_N(X_{\tau_i})+k(\triangle\ Y_{\tau_i})-K,$$

or equivalently

$$\widehat{V}_N(X_{\tau_i})-\widehat{V}_N(X_{\tau_i-})\leq -g(\xi_i).$$

Therefore, we conclude that

$$E\Big[e^{-\lambda(t\wedge\tau)}\widehat{V}_N(X_{t\wedge\tau})\Big]\leq\widehat{V}_N(x)-E\Big[\sum_{i=1}^{\infty}e^{-\lambda\tau_i}g(\xi_i)I_{\{\tau_i\leq t\wedge\tau\}}\Big].$$

Thanks to (4.54) and (2.12), the right-hand side term in the above inequality converges to zero when t goes to infinite. Then, due to Fatou's lemma, we get

$$0\leq\widehat{V}_N(x)-J(x;\pi).$$

Hence, we obtain

$$\widehat{V}(x)\leq\widehat{V}_N(x).$$

In order to complete the proof of the theorem, it's enough to prove that

$$\widehat{V}_N(x)=J(x;\pi^*). \tag{4.64}$$

Next, using similar arguments as in (4.62), we derive for any stopping time T,

$$\begin{aligned} e^{-\lambda(\tau^*\wedge T)}\widehat{V}_N(X^*_{\tau^*\wedge T}) =&\widehat{V}_N(x)+\int_0^{\tau^*\wedge T}e^{-\lambda s}\mathcal{L}^{a_N(X^*_s)}\widehat{V}_N(X^*_s)ds\\ &+\int_0^{\tau^*\wedge T}e^{-\lambda s}\sigma(a_N(X^*_s))\widehat{V}'_N(X^*_s)dW_s\\ &+\sum_{i=0}^{\infty}e^{-\lambda\tau_i}\Big[\widehat{V}_N(X^*_{\tau_i})-\widehat{V}_N(X^*_{\tau_i-})\Big]I_{\{\tau_i\leq\tau^*\wedge T\}}. \end{aligned} \tag{4.65}$$

Due to the construction of X^*, π^*, and thanks to the previous analysis, we claim that

$$\begin{aligned}
&\mathcal{L}^{a_N(x^*)}\widehat{V}_N(X^*) = 0 \\
&\triangle\, X^*_{\tau_i^*} = -(x_D - \widetilde{x}), \;\; X^*_{\tau_i^*} = \widetilde{x}, \;\; X^*_{\tau_i^*-} = x_D \\
&\widehat{V}_N(X^*_{\tau_i^*-}) - \widehat{V}_N(X^*_{\tau_i^*}) = k(x_D - \widetilde{x}) - K = g(\xi_i^*) \\
&0 \le X^* \le x_D, \;\; X^*_{\tau^*} = 0.
\end{aligned}$$

Consider $(T_n)_n$ a sequence of stopping times such that $\int_0^{T_n\wedge t} e^{-\lambda s}\sigma\Big(a_N(X_s^*)\Big)\widehat{V}_N'(X_s^*)dW_s$ is a true martingale whose expectation vanishes. Then by putting $T = T_n$ in (4.65), and taking expectation , we derive

$$E\Big(e^{-\lambda(\tau^*\wedge T_n)}\widehat{V}_N(X^*_{\tau^*\wedge T_n})\Big) = \widehat{V}_N(x) - E\Big(\sum_{i=1}^{\infty} e^{-\lambda\tau_i} g(\xi_i^*) I_{\{\tau_i^* \le \tau^*\wedge T_n\}}\Big).$$

The right-hand side term in the above equality converges to zero when n goes to infinity, and then (4.64) follows. This completes the proof of the theorem. ∎

5

Effect of nonzero debt liability rate

This chapter is concerned with the impact of non-zero debt liability rate (i.e. $\delta > 0$) on the optimal policies and the value function. First, we will start constructing a candidate for the value function. Hence, here, we will consider

$$0 = \max\left(\max_{a\geq 0}\Big[\frac{1}{2}\sigma^2(a)V''(x) + (\mu(a) - \delta)V'(x) - \lambda V(x)\Big], MV(x) - V(x)\right), \quad (5.1)$$

where $MV(x)$ is defined in (3.2) and

$$V(0) = 0. \quad (5.2)$$

Then for all $x < x_D$, where x_D is defined in (4.3), we have

$$MV(x) \neq V(x),$$

and the equation (5.1) becomes

$$0 = \max_{a\geq 0}\left(\frac{1}{2}\sigma^2(a)V''(x) + (\mu(a) - \delta)V'(x) - \lambda V(x)\right). \quad (5.3)$$

Thus, now we will concentrate on finding a smooth solution to the resulting equation (5.3). The maximizer $a_\delta(x)$ of (5.3) is a root of (4.5) which is equivalent to (4.6). Similarly as in Chapter 4, in order to completely solve (4.6), we need to distinguish whether N is finite or not.

5.1 The case of unbounded claim's size.

In this case, $N = \infty$, and for any $a < +\infty, \overline{F}(a) > 0$. Therefore, the solution to (4.5) in this subsection is denoted by $a_\delta(x)$ and is given by the expression (4.7).

On the set $\{x : x < x_D,\ a(x) < \infty\}$, (5.3) can be written as follows.

$$0 = \frac{1}{2}\sigma^2(a_\delta(x))V''(x) + (\mu(a_\delta(x)) - \delta)V'(x) - \lambda V(x). \tag{5.4}$$

By substituting (4.7) into (5.4), we get

$$\frac{1}{2}\sigma^2(a_\delta(x))\left(-\frac{V'(x)}{a_\delta(x)}\right) + (\mu(a_\delta(x)) - \delta)V'(x) - \lambda V(x) = 0,$$

or equivalently

$$h_\delta(a_\delta(x)) = \frac{\lambda V(x)}{V'(x)}, \tag{5.5}$$

where

$$h_\delta(a) := \begin{cases} \frac{-\sigma^2(a)}{2a} + \mu(a) - \delta, & \text{for } a > 0 \\ -\delta, & \text{for } a = 0. \end{cases} \tag{5.6}$$

Notice h_δ converges pointwise to h_0, defined in (4.10), when δ goes to zero. Furthermore, h_δ is a continuously differentiable function, and strictly increasing on $(0, \infty)$. Indeed,

$$h'_\delta(a) := \frac{\sigma^2(a)}{2a^2} > 0, \quad a > 0.$$

Hence h_δ^{-1} exists and is defined by

$$h_\delta^{-1} : [-\delta, \mu(\infty) - \delta) \to [0, \infty),$$

and $h_\delta(\infty) = \mu(\infty) - \delta < +\infty$. Thus,

$$a_\delta(x) = h_\delta^{-1}\left(\frac{\lambda V(x)}{V'(x)}\right), \quad 0 < x < x_D. \tag{5.7}$$

Since $h_\delta(0) = -\delta < 0$, then $a_\delta(0) = h_\delta^{-1}(0) > 0$. Furthermore, due to (5.7), $a_\delta(x)$ is a continuously differentiable function. Hence, by differentiating $h_\delta(a_\delta(x))V'(x) = \lambda V(x)$, we derive

$$h'_\delta(a_\delta(x))a'_\delta(x)V'(x) + h_\delta(a_\delta(x))V''(x) = \lambda V'(x).$$

Again, by substituting (4.7) into this equation, we derive

$$h'_\delta(a_\delta(x))a'_\delta(x)V'(x) + h_\delta(a_\delta(x))\frac{-V'(x)}{a_\delta(x)} = \lambda V'(x).$$

This leads to

$$h'_\delta(a_\delta(x))a'_\delta(x) = \frac{h_\delta(a_\delta(x))}{a_\delta(x)} + \lambda,$$

or equivalently,

$$a'_\delta(x) = \frac{2a_\delta(x)h_\delta(a_\delta(x)) + 2\lambda a_\delta^2(x)}{\sigma^2(a_\delta(x))}.$$

Thus, $a_\delta(x)$ satisfies

$$\frac{\sigma^2(a_\delta(x))a'_\delta(x)}{2a_\delta(x)h_\delta(a_\delta(x)) + 2\lambda a_\delta^2(x)} = 1,$$

and due to (5.7) and the increase of h_δ^{-1}, we deduce that $a_\delta(x) \geq a_\delta(0) = h_\delta^{-1}(0) > 0$. This leads to deduce that $2a_\delta(x)h_\delta(a_\delta(x)) + 2\lambda a_\delta^2(x) > 0$, and hence $a_\delta(x),\ \ x \geq 0$ is an increasing function. By integrating both sides in the above equation, we get

$$\int_0^x \frac{\sigma^2(a_\delta(t))a'_\delta(t)}{2a_\delta(t)h_\delta(a_\delta(t)) + 2\lambda a_\delta^2(t)} dt = x, \ \ 0 < x < x_D.$$

By changing the variables (precisely using $s = a_\delta(t)$), we get

$$g_\delta(a_\delta(x)) = x, \ \ 0 < x < x_D, \tag{5.8}$$

where

$$g_\delta(a) := \int_{h_\delta^{-1}(0)}^{a} \frac{\sigma^2(s)}{2sh_\delta(s) + 2\lambda s^2} ds, \ \ a \geq h_\delta^{-1}(0), \tag{5.9}$$

and then we state the following.

Lemma 5.1.1 *The function $a_\delta(x)$ defined in (4.7) takes the following form.*

$$a_\delta(x) = \begin{cases} g_\delta^{-1}(x), & 0 \leq x < x_\infty(\delta), \\ \infty, & x_\infty(\delta) \leq x < x_D, \end{cases} \tag{5.10}$$

where

$$x_\infty(\delta) := g_\delta(\infty) < +\infty. \tag{5.11}$$

Proof. The proof of the lemma is reduced to show that g_δ^{-1} exists and $g_\delta(\infty) < +\infty$. g_δ is a continuously differentiable and strictly increasing function on $(0, \infty)$.

Therefore, g_δ^{-1} exists, and

$$g_\delta(\infty) = \int_{h_\delta^{-1}(0)}^{\infty} \frac{\sigma^2(s)ds}{2sh_\delta(s) + 2\lambda s^2} \leq \frac{\sigma_\infty^2}{2\lambda} \int_{h_\delta^{-1}(0)}^{\infty} \frac{ds}{s^2} = \frac{\sigma_\infty^2}{2\lambda h_\delta^{-1}(0)} < +\infty.$$

Then for $0 \leq x < x_\infty(\delta) = g_\delta(\infty)$, the equation (5.8) implies $a_\delta(x) = g_\delta^{-1}(x)$. Again, equation (5.8) allows us to conclude that $a_\delta(x) = \infty$, for all $x \geq x_\infty(\delta)$. This completes the proof of the lemma. ■

Now, we will focus on describing the function V. Due to the above lemma and (4.7), we write

$$\frac{-V''(x)}{V'(x)} = \frac{1}{g_\delta^{-1}(x)}, \quad 0 \leq x < x_\infty(\delta).$$

Then by integrating both sides, we get

$$\ln V'(x_\infty(\delta)) - \ln V'(x) = \int_x^{x_\infty(\delta)} \frac{-dt}{g_\delta^{-1}(t)}, \quad 0 \leq x < x_\infty(\delta).$$

As a result, due to $V(0) = 0$ and integration of both sides, we obtain

$$V(x) = V'(x_\infty(\delta)) \int_0^x \exp\left(-\int_y^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right) dy, \quad 0 \leq x < x_\infty(\delta). \tag{5.12}$$

Again, due to the previous lemma, we have $a_\delta(x) = \infty$ for $x_\infty(\delta) \leq x < x_D$. Thus (5.3) becomes

$$0 = \frac{1}{2}\sigma_\infty^2 V''(x) + (\mu_\infty - \delta)V'(x) - \lambda V(x). \tag{5.13}$$

The solution to this equation is given by

$$V(x) = C_1 e^{r_+(\delta)(x - x_\infty(\delta))} + C_2 e^{r_-(\delta)(x - x_\infty(\delta))}, \quad x_\infty(\delta) \leq x < x_D, \tag{5.14}$$

where

$$r_\pm(\delta) := \frac{-(\mu_\infty - \delta) \pm \sqrt{(\mu_\infty - \delta)^2 + 2\lambda\sigma_\infty^2}}{\sigma_\infty^2}. \tag{5.15}$$

Again, similarly as in Chapter 4, we can prove that for $x \geq x_D$,

$$MV(x) = V(x),$$

and the solution to (5.3) is given by (4.21).

Next, we can state a complete description of V' as follows.

$$V'(x) = \begin{cases} C \exp\{\int_x^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\}, & 0 \leq x < x_\infty(\delta), \\ C_1 r_+ e^{r_+(x - x_\infty(\delta))} + C_2 r_- e^{r_-(x - x_\infty(\delta))}, & x_\infty(\delta) \leq x < x_D \\ k, & x \geq x_D. \end{cases}$$

where $C = V'(x_\infty(\delta))$, and C_1 and C_2 will be calculated in term of C using the smooth fit of functions V' and V'' at the point $x_\infty(\delta)$.

$$V'(x_\infty(\delta)-) = V'(x_\infty(\delta)) \Rightarrow C = C_1 r_+ + C_2 r_-,$$

$$V''(x_\infty(\delta)-) = V''(x_\infty(\delta)) \Rightarrow 0 = C_1 r_+^2 + C_2 r_-^2.$$

By solving these two equations, we get

$$C_1 r_+ = \frac{-Cr_-}{r_+ - r_-},$$

$$C_2 r_- = \frac{Cr_+}{r_+ - r_-}.$$

Then, we derive

$$V'(x) = \begin{cases} C \exp\left(\int_x^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right), & 0 \leq x < x_\infty(\delta), \\ C\left(\frac{-r_-}{r_+ - r_-} e^{r_+(x - x_\infty(\delta))} + \frac{r_+}{r_+ - r_-} e^{r_-(x - x_\infty(\delta))}\right), & x_\infty(\delta) \leq x < x_D, \\ k, & x \geq x_D. \end{cases} \tag{5.16}$$

There also exists $\widetilde{x} < x_D$ such that

$$V'(\widetilde{x}) = k, \tag{5.17}$$

and

$$V(x_D) = V(\widetilde{x}) + k(x_D - \widetilde{x}) - K, \tag{5.18}$$

or equivalently

$$\int_{\widetilde{x}}^{x_D} (k - V'(x))dx = K. \tag{5.19}$$

Now we rewrite (5.16) as follows.

$$V'(x) = \begin{cases} CH_\delta(x), & x < x_D, \\ k, & x \geq x_D, \end{cases} \tag{5.20}$$

where

$$H_\delta(x) = \begin{cases} \exp\left(\int_x^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right), & 0 \leq x < x_\infty(\delta), \\ \frac{-r_-}{r_+ - r_-} e^{r_+(x - x_\infty(\delta))} + \frac{r_+}{r_+ - r_-} e^{r_-(x - x_\infty(\delta))}, & x \geq x_\infty(\delta). \end{cases} \tag{5.21}$$

For $x < x_\infty(\delta)$, we have

$$H'_\delta(x) = -\frac{1}{g_\delta^{-1}(x)} \exp\left(\int_x^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right) < 0,$$

$$H''_\delta(x) = \left(-\frac{1}{g_\delta^{-1}(x)}\right)' H_\delta(x) + \left(-\frac{1}{g_\delta^{-1}(x)}\right) H'_\delta(x)$$
$$= \left(-\frac{1}{g_\delta^{-1}(x)}\right)' H_\delta(x) + \left(-\frac{1}{g_\delta^{-1}(x)}\right)^2 H_\delta(x) > 0.$$

For $x = x_\infty(\delta)$,

$$H'_\delta(x_\infty(\delta)) = \frac{-r_-r_+}{r_+ - r_-} + \frac{r_+r_-}{r_+ - r_-} = 0.$$

For $x > x_\infty(\delta)$,

$$H'_\delta(x) = \frac{-r_-r_+}{r_+ - r_-} e^{r_+(x - x_\infty(\delta))} + \frac{r_+r_-}{r_+ - r_-} e^{r_-(x - x_\infty(\delta))} > 0,$$

$$H''_\delta(x) = \frac{-r_+^2 r_-}{r_+ - r_-} e^{r_+(x - x_\infty(\delta))} + \frac{r_+ r_-^2}{r_+ - r_-} e^{r_-(x - x_\infty(\delta))}$$
$$= \frac{-r_+r_-}{r_+ - r_-} \left(r_+ e^{r_+(x - x_\infty(\delta))} - r_- e^{r_-(x - x_\infty(\delta))}\right) > 0.$$

Thus, it is easy to see that H_δ is a continuously differentiable convex function and the unique root of

$$H'_\delta(x) = 0$$

is $x = x_\infty(\delta)$ with $H_\delta(x_\infty(\delta)) = 1$. Then H_δ is strictly decreasing on $[0, x_\infty(\delta)]$ and increasing on $[x_\infty(\delta), \infty)$.

Let

$$\widehat{H}_\delta = H_\delta \mid_{[0, x_\infty(\delta)]} \text{ and } \overline{H}_\delta = H_\delta \mid_{[x_\infty(\delta), \infty)} .$$

Then $\widehat{H}_\delta^{-1}$ exists and $\widehat{H}_\delta^{-1} : [1, H_\delta(0)] \to [0, x_\infty(\delta)]$, and also $\overline{H}_\delta^{-1}$ exists and $\overline{H}_\delta^{-1} : [1, \infty) \to [x_\infty(\delta), \infty)$. Remark that

$$H_\delta(0) < +\infty, \tag{5.22}$$

and this is one of the properties of this case of $\delta > 0$. In fact,

$$\ln(H_\delta(0)) = \int_0^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)} = \int_{g_\delta^{-1}(x)}^{\infty} \frac{\sigma^2(s)}{2s^2 h_\delta(s) + 2\lambda s^3} ds$$
$$\leq \frac{\sigma^2(\infty)}{2\lambda} \int_{a_\delta(0) = h_\delta^{-1}(0)}^{\infty} \frac{1}{s^3} ds = \frac{\sigma^2(\infty)}{4\lambda (h_\delta^{-1}(0))^2} < +\infty.$$

Due to (5.17) - (5.20), $\widetilde{x}^C$ and x_D^C satisfies $\widetilde{x}^C < x_\infty(\delta) < x_D^C$, and are two roots of

$$H_\delta(x) = \frac{k}{C}.$$

For these two cash reserve levels to exist, $\frac{k}{C}$ is required to belong to the range of $\widehat{H}_\delta$ and $\overline{H}_\delta$, which we will describe. H_δ is strictly convex (since $H''_\delta(x) > 0$, for $\forall x \geq 0$). We know that $H'_\delta(x) < 0$, for $x < x_\infty(\delta)$, $H'_\delta(x_\infty(\delta)) = 0$, and $H'_\delta(x) > 0$, for $x > x_\infty(\delta)$. Then

$$\min_{x\geq 0} H_\delta(x) = H_\delta(x_\infty(\delta)) = 1, \quad H_\delta(\infty) = \infty.$$

If $\frac{k}{C} < 1 = H_\delta(x_\infty(\delta)) \leq H_\delta(x), \forall x \geq 0$ (or equivalently $CH_\delta(x) > k$), then neither $\widetilde{x}^C$ nor x_D^C exist. This leads to one of the necessary conditions for the existence of $\widetilde{x}^C$ and x_D^C given by

$$\frac{k}{C} \geq 1.$$

Suppose that $\frac{k}{C} > H_\delta(0)$ (or equivalently $k > CH_\delta(0)$), $\widetilde{x}^C$ does not exist. Thus, for $\widetilde{x}^C$ and x_D^C to exist, we need

$$\frac{k}{C} \leq H_\delta(0).$$

As a result, we get

$$1 \leq \frac{k}{C} \leq H_\delta(0). \tag{5.23}$$

Therefore, when (5.23) holds, $\widetilde{x}^C$ and x_D^C exist and are given by

$$\widetilde{x}^C = \widehat{H}_\delta^{-1}\left(\frac{k}{C}\right), \quad x_D^C = \overline{H}_\delta^{-1}\left(\frac{k}{C}\right).$$

Clearly, $\widetilde{x}^C$ is an increasing function of C, while x_D^C is a decreasing function of C. Consider the following function

$$I_\delta(C) := \int_{\widetilde{x}^C}^{x_D^C} (k - CH_\delta(x))dx, \quad \frac{k}{H_\delta(0)} \leq C \leq k.$$

$I_\delta(C)$ is a continuous and decreasing function of C because both the integrand and the interval are continuous and decreasing with respect to C. The only parameter, in the expression of V' given in (5.16), to be calculated is C. This will be done using the equation (5.18), or equivalently

$$I_\delta(C) = K, \tag{5.24}$$

where $K \geq 0$. Obviously, if $C = k$ then $\widetilde{x}^C = x_D^C = x_\infty(\delta)$. This corresponds to the case when $K = 0$.

If $C = \frac{k}{H_\delta(0)}$, then we define

$$K_{max}(\delta) := \int_0^{\overline{H}_\delta^{-1}(H_\delta(0))} \left(1 - \frac{H_\delta(x)}{H_\delta(0)}\right) dx. \tag{5.25}$$

Then, we can write

$$\max_{\frac{k}{H_\delta(0)} \leq C \leq k} I_\delta(C) = I_\delta\Big(\frac{k}{H_\delta(0)}\Big) = kK_{max}(\delta). \tag{5.26}$$

Therefore (5.24) admits a solution if and only if

$$K \leq kK_{max}(\delta). \tag{5.27}$$

Then, under the condition (5.27), there exists $\widetilde{C}$ such that $\frac{k}{H_\delta(0)} \leq \widetilde{C} \leq k$, and

$$I_\delta(\widetilde{C}) = \int_{\widetilde{x}^{\widetilde{C}}}^{x_D^{\widetilde{C}}} (k - \widetilde{C}H_\delta(x))dx = K. \tag{5.28}$$

This analysis proves the following.

Proposition 5.1.1 *Suppose that $N = \infty$, and consider the previous notations. Then the following assertions hold.*

1. *If $K > K_{max}(\delta)$, then the equation (5.1)-(5.2) has no smooth solution.*

2. *If $K \leq K_{max}(\delta)$, then there exists a range for tax rates, $1 - k$, precisely. k should belong to $\left[\frac{K}{K_{max}(\delta)}, 1\right]$, for which the equations (5.1)-(5.2) admit a smooth solution given by*

$$\widehat{V}_\delta(x) = \begin{cases} C\int_0^x \exp\left(\int_y^{x_\infty(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right) dy, & 0 \leq x < x_\infty(\delta), \\ C\left(\frac{-r_-}{r_+(r_+-r_-)}e^{r_+(x-x_\infty(\delta))} + \frac{r_+}{r_-(r_+-r_-)}e^{r_-(x-x_\infty(\delta))}\right), & x_\infty(\delta) \leq x < x_D, \\ \widehat{V}(\widetilde{x}) + k(x-\widetilde{x}) - K, & x \geq x_D, \end{cases} \tag{5.29}$$

where $\widetilde{x} = \widehat{H}_\delta^{-1}\left(\frac{k}{\widetilde{C}}\right)$, $x_D = \overline{H}_\delta^{-1}\left(\frac{k}{\widetilde{C}}\right)$, and $C = \widetilde{C}$ is a root of (5.24).

In particular, when $K = K_{max}(\delta)$, the equations (5.1)-(5.2) admit a smooth solution if there are no taxes on the dividend pay-outs.

Remark 5.1.1 *1. These scenarios (assertions 1 and 2 of this proposition) illustrate one of the impacts of a non-zero debt liability rate on the model.*

2. *The second assertion of the above proposition explains the interplay between the costs (K) and the taxes ($1-k$) as well as with other exogenous parameters of the model, mainly μ, σ and F.*

5.2 The case of bounded claim's size.

This subsection is an extension of Subsection (4.2) to the case where δ (the debt liability rate) is positive. Therefore, it can be seen as a result of combining Subsection 4.1 and Subsection 5.1.

Lemma 5.2.1 *$a(x)$ takes the following form*

$$a(x) = \begin{cases} g_\delta^{-1}(x), & 0 \leq x < x_N(\delta), \\ N, & x \geq x_N(\delta), \end{cases} \tag{5.30}$$

where

$$x_N(\delta) := g_\delta(N).$$

Then, using this lemma, we derive

$$\frac{-V_N''(x)}{V_N'(x)} = \frac{1}{g_\delta^{-1}(x)}, \quad 0 \leq x < x_N(\delta).$$

Integration of both sides leads to

$$\int_x^{x_N(\delta)} \frac{V_N''(t)}{V_N'(t)} dt = \int_x^{x_N(\delta)} \frac{-dt}{g_\delta^{-1}(t)}.$$

As a result, we get

$$V_N(x) = V_N'(x_N(\delta)) \int_0^x \exp\left(\int_y^{x_N(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right) dy, \quad 0 \leq x < x_N(\delta). \tag{5.31}$$

Again the lemma above implies that (5.3) is equivalent to

$$0 = \frac{1}{2}\sigma^2(N)V_N''(x) + (\mu(N) - \delta)V_N'(x) - \lambda V_N(x).$$

The solution of this equation is

$$V_N(x) = C_1 e^{r_+(N)(x - x_N(\delta))} + C_2 e^{r_-(N)(x - x_N(\delta))}, \quad x_N(\delta) \leq x < x_D, \tag{5.32}$$

where $r_\pm(N) = r_\pm$ are given by (5.15). Notice that lemma 4.1.2 will apply in this context, and

$$V_N(x) = V_N(\tilde{x}) + k(x - \tilde{x}) - K. \tag{5.33}$$

Hence, the piecewise constructed V_N takes the following form

$$V_N(x) = \begin{cases} C\int_0^x \exp\left(\int_y^{x_N(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right) dy, & 0 \le x < x_N(\delta), \\ C_1 e^{r_+(x - x_N(\delta))} + C_2 e^{r_-(x - x_N(\delta))}, & x_N(\delta) \le x < x_D, \\ V_N(\tilde{x}) + k(x - \tilde{x}) - K, & x \ge x_D, \end{cases} \tag{5.34}$$

where C, C_1 and C_2 will be determined. Using the smooth fit of the functions V' and V'' at the point $x_N(\delta)$, we derive

$$\begin{aligned} &V_N'(x_N(\delta)-) = V_N'(x_N(\delta)) \Rightarrow C = C_1 r_+ + C_2 r_-, \\ &V_N''(x_N(\delta)-) = V_N''(x_N(\delta)) \Rightarrow C\frac{-1}{g^{-1}(x_N(\delta))} = C_1 r_+^2 + C_2 r_-^2 \Rightarrow \frac{-C}{N} = C_1 r_+^2 + C_2 r_-^2. \end{aligned} \tag{5.35}$$

These leads to

$$\begin{aligned} C_1 r_+ &= \frac{-C(1 + N r_-)}{N(r_+ - r_-)}, \\ C_2 r_- &= \frac{C(1 + N r_+)}{N(r_+ - r_-)}, \end{aligned}$$

and we can write

$$V_N'(x) = \begin{cases} C\exp\left(\int_x^{x_N(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right), & 0 \le x < x_N(\delta), \\ C\left(\frac{-(1+Nr_-)}{N(r_+ - r_-)} e^{r_+(x - x_N(\delta))} + \frac{1+Nr_+}{N(r_+ - r_-)} e^{r_-(x - x_N(\delta))}\right), & x_N(\delta) \le x < x_D, \\ k, & x \ge x_D. \end{cases} \tag{5.36}$$

This also can be given by

$$V_N'(x) = \begin{cases} CH_\delta(x), & x < x_D, \\ k, & x \ge x_D. \end{cases} \tag{5.37}$$

where

$$H_\delta(x) := \begin{cases} \exp\left(\int_x^{x_N(\delta)} \frac{dt}{g_\delta^{-1}(t)}\right), & 0 \le x < x_N(\delta), \\ \frac{-(1+Nr_-)}{N(r_+ - r_-)} e^{r_+(x - x_N(\delta))} + \frac{(1+Nr_+)}{N(r_+ - r_-)} e^{r_-(x - x_N(\delta))}, & x \ge x_N(\delta). \end{cases} \tag{5.38}$$

We know that $H_\delta'(x_N(\delta)) = -\frac{1}{N} < 0$, which means H_δ keeps decreasing from 0 to $x_N(\delta)$. Let $x > x_N(\delta)$, then

$$H_\delta'(x) = \frac{-r_+(\delta)(1 + Nr_-(\delta))}{N(r_+(\delta) - r_-(\delta))} e^{r_+(\delta)(x - x_N(\delta))} + \frac{r_-(\delta)(1 + Nr_+(\delta))}{N(r_+(\delta) - r_-(\delta))} e^{r_-(\delta)(x - x_N(\delta))}.$$

Thus,

$$\lim_{x\to\infty} H_\delta'(x) = \begin{cases} +\infty, & \text{if } 1+Nr_-(\delta)<0, \\ -\infty, & \text{if } 1+Nr_-(\delta)>0. \end{cases} \tag{5.39}$$

Notice that

- $1+Nr_-(\delta)<0$ if and only if

$$\lambda > \left(\frac{\sigma^2}{2N^2} - \frac{(\mu-\delta)}{N}\right)^+. \tag{5.40}$$

- If $1+Nr_-(\delta)\geq 0$, then $H_\delta'(x)<0$, for all $x\geq 0$, and $H_\delta'(x)=0$ has no solution.

Then $H_\delta'(x)=0$ admits a solution if (5.40) holds. In this the solution is unique, denoted by $\widehat{x}(\delta,N)>x_N(\delta)$, and given by

$$\widehat{x}(\delta,N) := x_N(\delta) + \frac{1}{r_+(\delta)-r_-(\delta)}\ln\Big[\frac{r_-(\delta)(1+Nr_+(\delta))}{r_+(\delta)(1+Nr_-(\delta))}\Big] > x_N(\delta). \tag{5.41}$$

Then, H_δ is strictly decreasing for $[0,\widehat{x}]$ and strictly increasing for $[\widehat{x},\infty)$.

Let

$$\widehat{H}_\delta = H_\delta|_{[0,\widehat{x}]},\quad \overline{H}_\delta = H_\delta|_{[\widehat{x},\infty)}.$$

Remark that

$$H_\delta(\infty)=\infty,\quad H_\delta(0)<+\infty,\quad H_\delta(\widehat{x})=\min_{x\geq 0} H_\delta(x),\quad H_\delta(x_N(\delta))=1.$$

Then, $\widehat{H}_\delta^{-1}$ and $\overline{H}_\delta^{-1}$ exist with $\widehat{H}_\delta^{-1}:[H_\delta(\widehat{x}),H_\delta(0)]\to[0,\widehat{x}]$ and $\overline{H}_\delta^{-1}:[H_\delta(\widehat{x}),\infty)\to[\widehat{x},\infty)$. The only remaining undescribed parameters of V_N are $\widetilde{x}^C, x_D^C$ and C. Recall that $\widetilde{x}^C$ and x_D^C satisfy $\widetilde{x}^C<\widehat{x}<x_D^C$ and are roots of

$$H_\delta(x)=\frac{k}{C}. \tag{5.42}$$

This equation allows us to determine $\widetilde{x}^C$ and x_D^C in term of C. Therefore, $\widetilde{x}^C$ and x_D^C exist if and only if $\frac{k}{C}$ belongs to the range of $\widehat{H}_\delta$ and $\overline{H}_\delta$. As in the previously subsection, this is equivalent to

$$\frac{k}{H_\delta(0)} \leq C \leq \frac{k}{H_\delta(\widehat{x})}. \tag{5.43}$$

This condition on C is a kind of combination of (4.51) and (5.23). Then under the condition (5.43), we calculate

$$\widetilde{x}^C = \widehat{H}_\delta^{-1}\left(\frac{k}{C}\right),\quad x_D^C = \overline{H}_\delta^{-1}\left(\frac{k}{C}\right). \tag{5.44}$$

Now, we will focus on calculating $C > 0$. To this end, we use the equation

$$I(C) := \int_{\widetilde{x}^C}^{x_D^C} (k - CH_\delta(x))dx = K. \tag{5.45}$$

Due to (5.44), $\widetilde{x}^C$ is an increasing function of C, while x_D^C is a decreasing function of C. Thus, $I(C)$ is a continuous and decreasing function of C because both the integrand and the interval are continuous and decreasing with respect to C. Obviously, when $C = \frac{k}{H_\delta(\widehat{x})}$, we get $\widetilde{x}^C = x_D^C = \widehat{x}$. This corresponds to the case when $K = 0$.
When $C = \frac{k}{H_\delta(0)}$, we obtain

$$K_{\max}(\delta, N) := \max_{\frac{k}{H_\delta(0)} \le C \le \frac{k}{H_\delta(\widehat{x})}} I(C) = \int_0^{\overline{H}_\delta^{-1}(H_\delta(0))} \left(1 - \frac{H_\delta(x)}{H_\delta(0)}\right) dx. \tag{5.46}$$

Then the equation (5.1)-(5.2) admits a solution if and only if

$$K \le kK_{max}(\delta, N). \tag{5.47}$$

Thus, there exists $\widetilde{C}(N)$ such that $\frac{k}{H_\delta(0)} \le \widetilde{C}(N) \le \frac{k}{H_\delta(\widehat{x})}$ and root of

$$I(\widetilde{C}) = K, \tag{5.48}$$

where $K \le kK_{max}(\delta, N)$.

Therefore, we completely describe the candidate for the solution to (5.1) as follows.

$$\widehat{V}_{N,\delta}(x) := \begin{cases} C \int_0^x \exp\left(\int_y^{x_N(\delta)} \frac{dt}{g_\delta^{-1}(t)} dy\right), & 0 \le x < x_N(\delta), \\ C\left(\frac{-(1+Nr_-)}{Nr_+(r_+-r_-)} e^{r_+(x-x_N(\delta))} + \frac{1+Nr_+}{Nr_-(r_+-r_-)} e^{r_-(x-x_N(\delta))}\right), & x_N(\delta) \le x < x_D, \\ \widehat{V}_{N,\delta}(\widetilde{x}) + k(x - \widetilde{x}) - K, & x \ge x_D, \end{cases} \tag{5.49}$$

with $C = \widetilde{C}(N), \widetilde{x} = \widetilde{x}^{\widetilde{C}(N)} = \widehat{H}_\delta^{-1}\left(\frac{k}{\widetilde{C}(N)}\right), x_D = x_D^{\widetilde{C}(N)} = \overline{H}_\delta^{-1}\left(\frac{k}{\widetilde{C}(N)}\right)$.

Now we are in stage to state the main result of this chapter.

Theorem 5.2.1 *The function $\widehat{V}_{N,\delta}$ defined in (5.49), is continuously differentiable on $(0, \infty)$, twice continuously differentiable on $(0, x_D) \cup (x_D, \infty)$, and is a smooth solution to the HJB equation (5.1) - (5.2).*

Proof. The proof of this theorem is similar to the proof of the Theorem 4.2.1. ■

5.3 Optimal policies

Theorem 5.3.1 *Suppose that C is a root of (5.48) and $\widetilde{x}(C)$ and $x_D(C)$ are given by (5.44). Let $a(x)$ be given by (5.30). Then the control*

$$\pi^* = (u^*, \mathcal{T}^*, \xi^*) = (u^*; \tau_1^*, \tau_2^*, ..., \tau_n^*, ...; \xi_1^*, \xi_2^*, ..., \xi_n^*, ...)$$

defined by

$$u_t^* := a_N(X_t^*), \quad t \geq 0, \tag{5.50}$$

$$\tau_1^* := \inf\{t \geq 0 : X^*(t) = x_D(C)\}, \tag{5.51}$$

$$\xi_1^* := x_D(C) - \widetilde{x}(C), \tag{5.52}$$

and for every $n \geq 2$:

$$\tau_n^* := \inf\{t \geq \tau_{n-1} : X^*(t) = x_D(C)\}, \tag{5.53}$$

$$\xi_n^* := x_D(C) - \widetilde{x}(C), \tag{5.54}$$

where X^* is the solution to the stochastic differential equation

$$X_t^* = X_0^* + \int_0^t \Big[\mu(a_N(X_s^*)) - \delta\Big] ds + \int_0^t \sigma(a_N(X_s^*)) dW_s - (x_D(C) - \widetilde{x}(C)) \sum_{n=1}^{\infty} I_{\{\tau_n^* < t\}} \tag{5.55}$$

is optimal and the function $\widehat{V}_{N,\delta}$, defined in (5.49), coincides with the value function. That is,

$$\widehat{V}_{N,\delta}(x) = J(x; \pi^*) = J(x; u^*, \mathcal{T}^*, \xi^*). \tag{5.56}$$

Proof. The proof of this theorem can be obtained by mimicking the proof of the Theorem 4.3.1. ∎

6

Numerical Examples

Here are some numerical examples for the functions $\widehat{V}'(x)$ and $\widehat{V}'_N(x)$ when $\delta = 0$ and when $\delta > 0$.

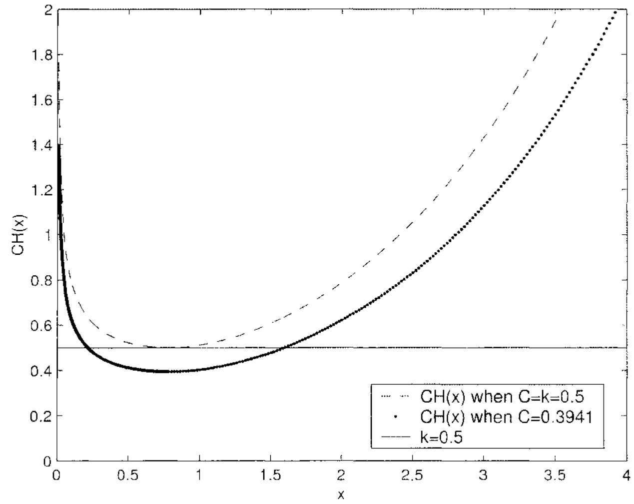

Figure 6.1: The relationship between x and $C * H(x)$. This is the example for the case of $N = \infty, \delta = 0$. In this case, we have $\overline{F}(x) = 1 - F(x) = e^{-x}$, where F is the claim size distribution. $\mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1, x_\infty = 0.7811, K = 0.05, \widetilde{C} = 0.3941, \widetilde{x}^{\widetilde{C}} = 0.2129, x_D^{\widetilde{C}} = 1.6011$.

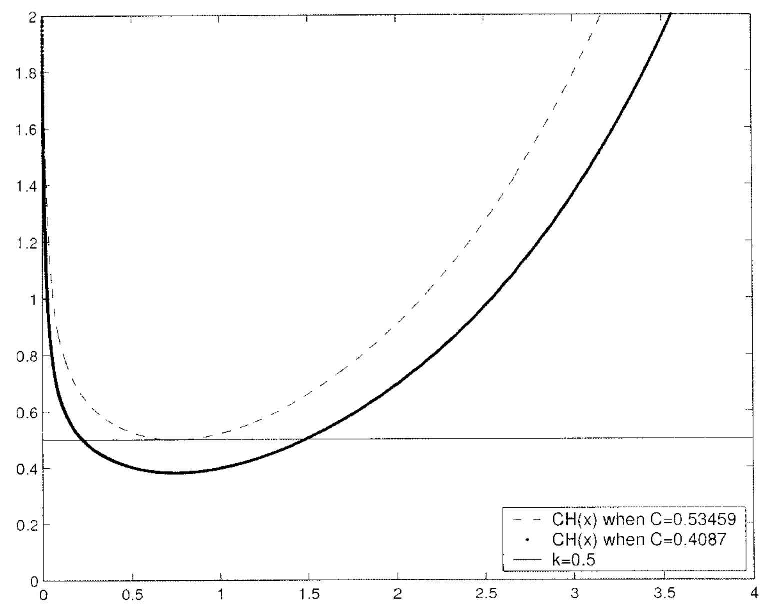

Figure 6.2: The relationship between x and $C * H(x)$. This is the example for the case of $N < \infty, \delta = 0$. In this case, we have $\overline{F}(x) = 1 - F(x) = \begin{cases} 0, & x > N, \\ 1 - \frac{x}{N}, & 0 \leq x \leq N, \end{cases}$ where F is the claim size distribution. $N = 2, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1, \widehat{x} = 0.7475, K = 0.05, \widetilde{C} = 0.4087, \widetilde{x}^{\widetilde{C}} = 0.2232, x_D^{\widetilde{C}} = 1.4865$.

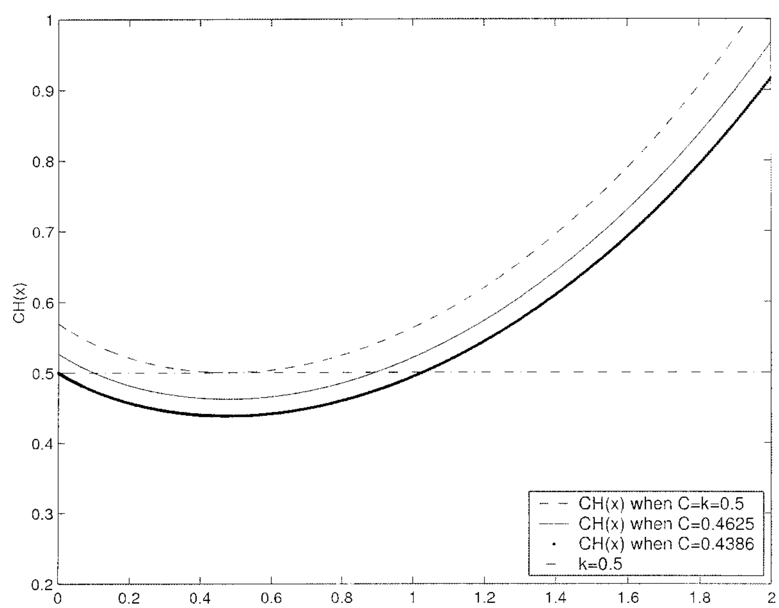

Figure 6.3: The relationship between x and $C * H(x)$. This is the example for the case of $N = \infty, \delta > 0$. In this case, we have $\overline{F}(x) = 1 - F(x) = e^{-x}$, where F is the claim size distribution. $\mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1, \delta = \frac{1}{2}, x_\infty = 0.4792, K_{max} = 0.0213, \widetilde{C} = 0.4386, \widetilde{x}^{\widetilde{C}} = 0, x_D^{\widetilde{C}} = 1.0292$. When $K = 0.01, \widetilde{C} = 0.4625, \widetilde{x}^{\widetilde{C}} = 0.1023, x_D^{\widetilde{C}} = 0.8942$.

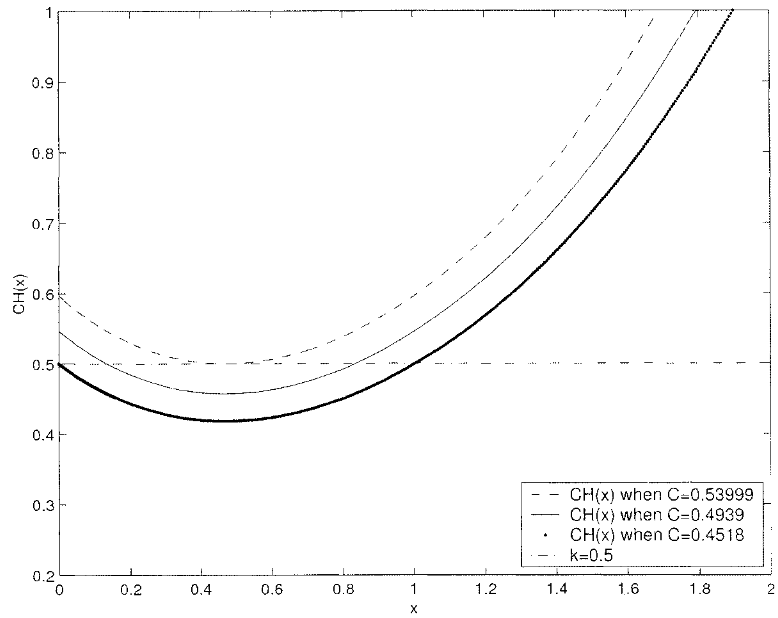

Figure 6.4: The relationship between x and $C * H(x)$. This is the example for the case of $N < \infty, \delta > 0$. In this case, we have $\overline{F}(x) = 1 - F(x) = \begin{cases} 0, & x > N, \\ 1 - \frac{x}{N}, & 0 \leq x \leq N, \end{cases}$ where F is the claim size distribution. $N = 2, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1, \delta = \frac{1}{2}, \widehat{x} = 0.47088, K_{max} = 0.0276, \widetilde{C} = 0.4518, \widetilde{x}^{\widetilde{C}} = 0, x_D^{\widetilde{C}} = 1.009$. When $K = 0.01, \widetilde{C} = 0.4939, \widetilde{x}^{\widetilde{C}} = 0.1354, x_D^{\widetilde{C}} = 0.839$.

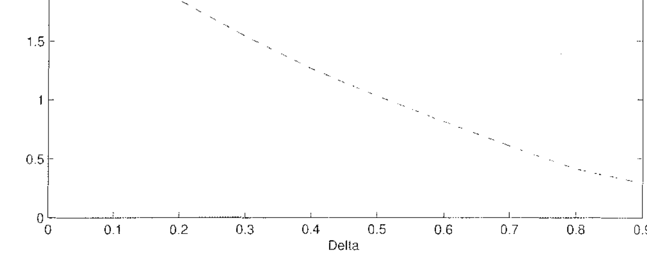

Figure 6.5: The relationship between δ and $x_D(\delta)$. This is the example for the case of $N = \infty, \delta > 0, \mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1$.

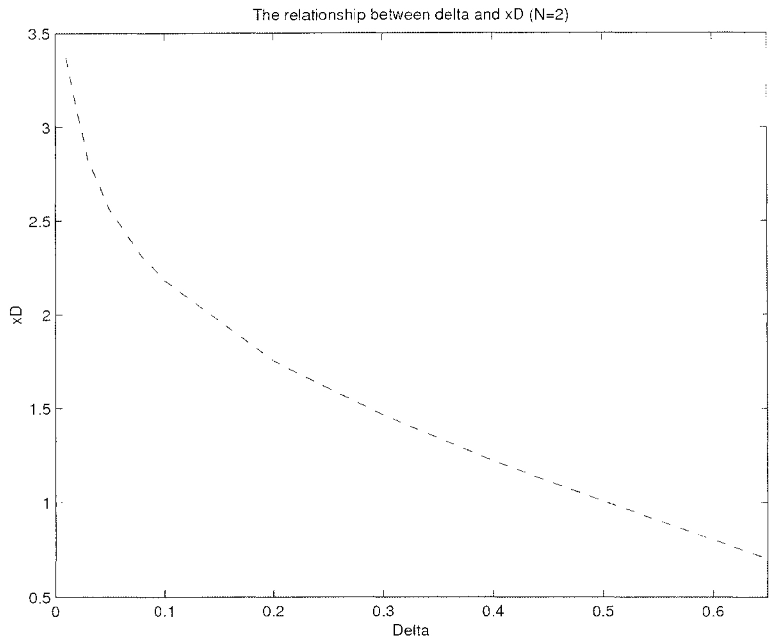

Figure 6.6: The relationship between δ and $x_D(\delta)$. This is the example for the case of $N = 2, \delta > 0, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1$.

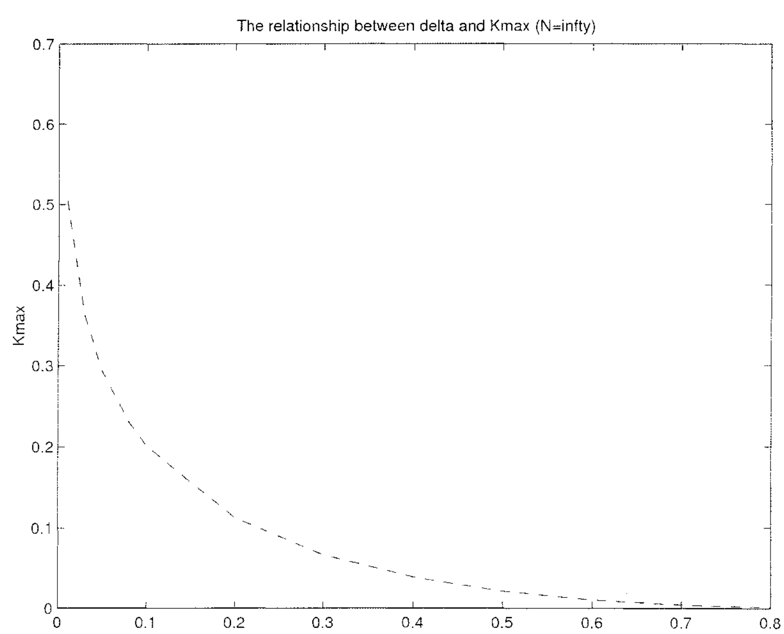

Figure 6.7: The relationship between δ and $K_{max}(\delta)$. $K_{max}(\delta)$ is the maximum cost permitted. This is the example for the case of $N = \infty, \delta > 0, \mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1$.

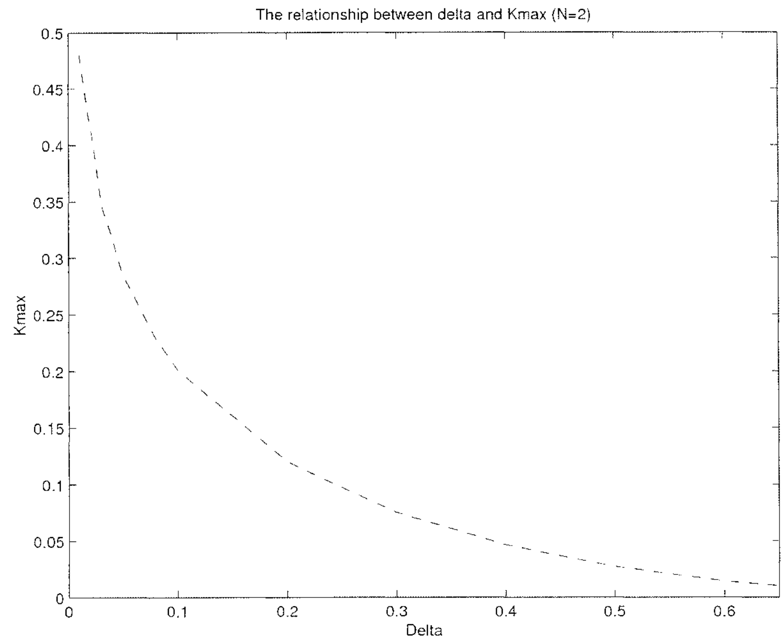

Figure 6.8: The relationship between δ and $K_{max}(\delta)$. This is the example for the case of $N = 2, \delta > 0, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1$.

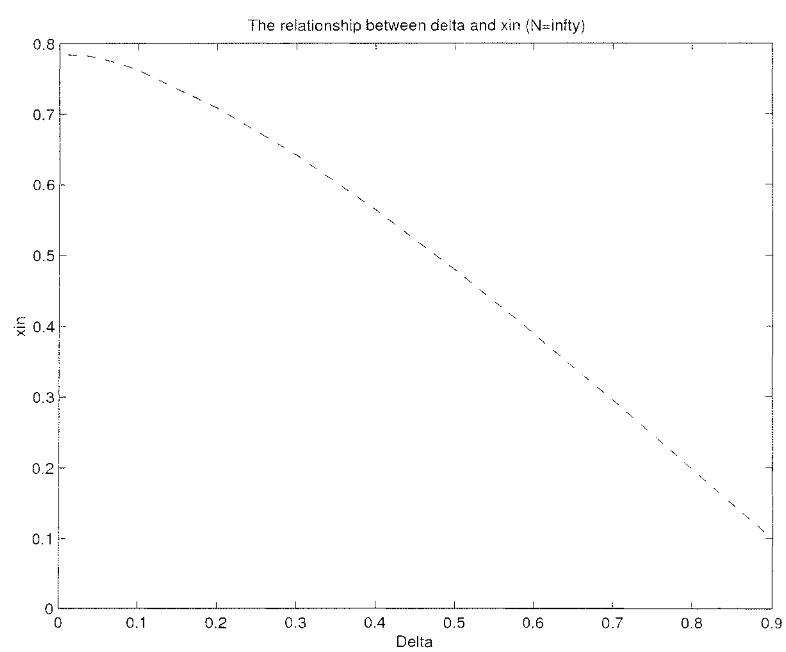

Figure 6.9: The relationship between δ and $x_\infty(\delta)$. This is the example for the case of $N = \infty, \delta > 0, \mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1$.

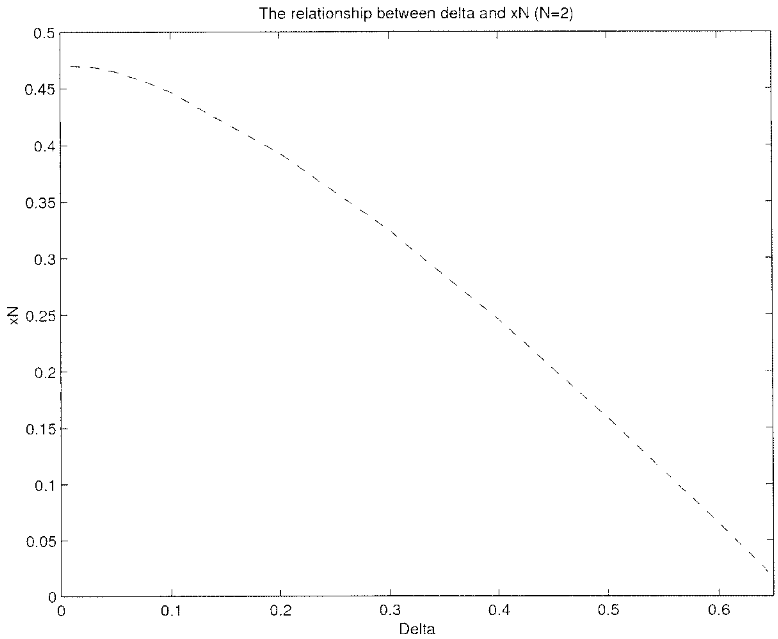

Figure 6.10: The relationship between δ and $x_N(\delta)$. This is the example for the case of $N = 2, \delta > 0, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1$.

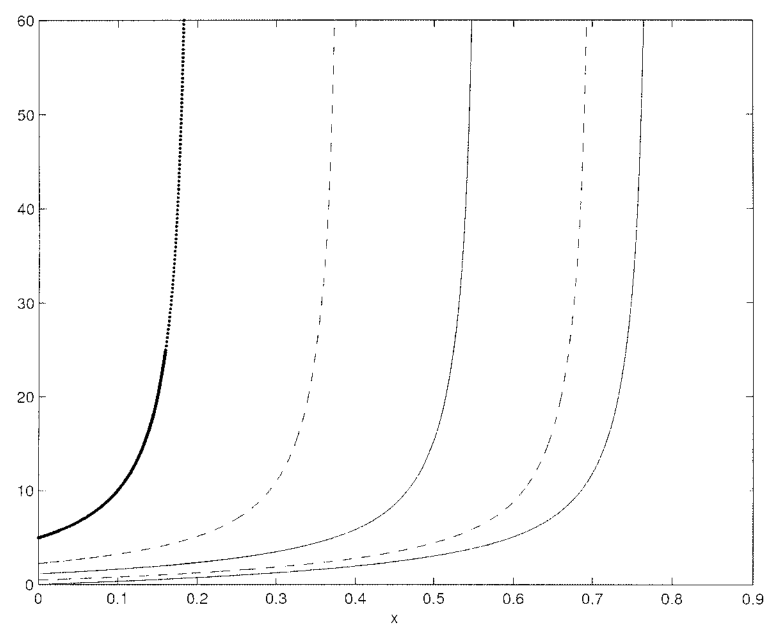

Figure 6.11: Graph for $a_0^*(x), a_{0.2}^*(x), a_{0.4}^*(x), a_{0.6}^*(x), a_{0.8}^*(x)$ with $N = \infty, \delta > 0, \mu_\infty = 1, \sigma_\infty^2 = 2, \lambda = 1$.

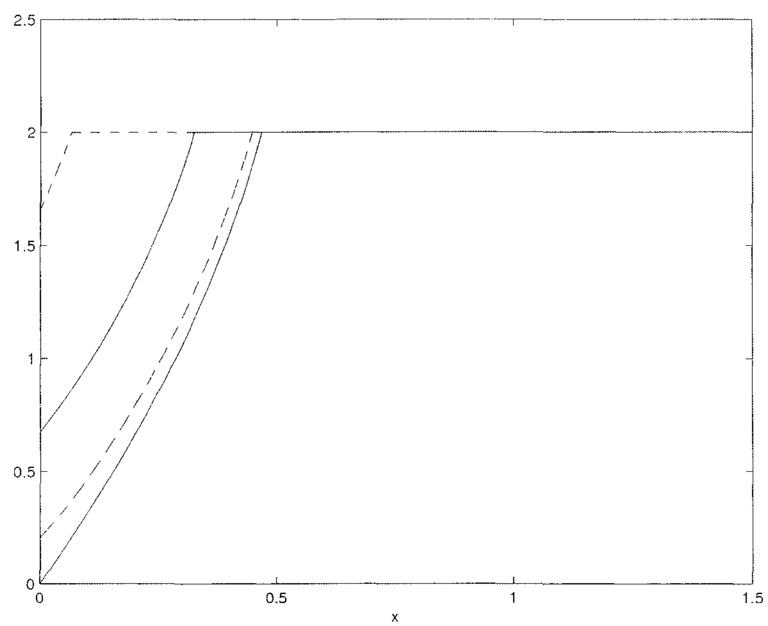

Figure 6.12: Graph for $a_0^*(x), a_{0.1}^*(x), a_{0.3}^*(x), a_{0.6}^*(x)$ with $N = 2, \delta > 0, \mu_N = 1, \sigma_N^2 = \frac{4}{3}, \lambda = 1$.

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