University of Alberta

Young Tableaux and the Image of the Kashiwara Embedding for $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$

by

Alejandra Premat

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics

Department of Mathematical Sciences

Edmonton, Alberta

Spring 2000

0-612-60014-9

University of Alberta

Library Release Form

Name of Author: Alejandra Premat

Title of Thesis: Young Tableaux and the Image of the Kashiwara Embedding for $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$

Degree: Doctor of Philosophy

Year this degree granted: 2000

Alejandra Premat
1204-10101 Saskatchewan Drive
Edmonton, Alberta, T6E 4R6
Canada

April 12th, 2000

Para mi padre idealista quien, por luchar por un mundo libre de injusticias,
me hizo creer en un posible mundo ideal, y
para mi madre ideal quien, simplemente con su presencia,
siempre crea para mi tal mundo.

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled Young Tableaux and the Image of the Kashiwara Embedding for $\mathcal{U}_q(\hat{sl}(n))$ submitted by Alejandra Premat in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Dr. R. Moody

Dr. B. Allison

Dr. G. Cliff

Dr. T. Gannon

Dr. M. de Montigny

Dr. K. Misra
External Reader
North Carolina State Univ.
Raleigh, NC

Date: April 7, 2000

Abstract

In this thesis we describe the second component of the crystal base of an irreducible module of highest weight $N\Lambda_0$ of $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$, where $N \in \mathbb{N}$ and Λ_0 is a fundamental weight, as a set of N-tuples of Young Tableaux. This description differs from that given by Jimbo, Misra, Miwa and Okado, and it is related to certain Demazure modules. We use this description to obtain an explicit set of inequalities defining the cone whose lattice points give the image of one of the Kashiwara embeddings.

Acknowledgements

First and foremost, I would like to thank Dr. G. Cliff for suggesting this problem and for the - at times much needed - encouragement he provided.

I would also like to extend my thanks to Dr. B. Allison for his careful reading of this thesis, to Marion Benedict, Linda Drysdale, Christine Fischer, Leona Guthrie, Laura Heiland and Charlene Josey for always patiently doing more than their jobs require - it has been a pleasure dealing with them all, and to the Faculty of Graduate Studies and Research, Dr. G. Peschke and Dr. G. Cliff, for the much appreciated financial support.

Finalmente, a mi querido Lawrence, te agradezco por todo.

Contents

Introduction

Crystal bases, which are used as tools to study the representation theory of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, were introduced by Kashiwara in 1990, and since then have been the subject of extensive study. They can be thought of as "bases" of $\mathcal{U}_q(\mathfrak{g})$-modules at "q=0", and can be "lifted" to bases, called global bases, of these molules.

Let $(L(\lambda), B(\lambda))$ denote the crystal base of $V(\lambda)$ (the irreducible $\mathcal{U}_q(\mathfrak{g})$-module of highest weight λ), w an element of the Weyl group and $u_{w\lambda}$ an extremal vector of weight $w\lambda$ of $V(\lambda)$. Littelmann [**Lit95**] conjectured the existence of a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{\mathcal{U}_q^+(\mathfrak{g})u_{w\lambda} \cap L(\lambda)}{\mathcal{U}_q^+(\mathfrak{g})u_{w\lambda} \cap qL(\lambda)} = \sum_{b \in B_w(\lambda)} \mathbb{Q}b,$$

and proved this for $\mathfrak{g}$ of types A_n, B_n, C_n, D_n, E_6 and G_2.

In [**Kas93**], Kashiwara proved this conjecture in general by showing the existence of a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\mathcal{U}_q^+(\mathfrak{g})u_{w\lambda} = \sum_{b \in B_w(\lambda)} \mathbb{Q}(q)G_\lambda(b),$$

where $\{G_\lambda(b) : b \in B(\lambda)\}$ is the (lower) global base of $V(\lambda)$, and hence obtained the character formula

$$\mathrm{ch}(\mathcal{U}_q^+(\mathfrak{g})u_{w\lambda}) = \sum_{b \in B_w(\lambda)} e^{wt(b)}.$$

It is therefore of importance to describe the subset $B_w(\lambda)$ as explicitly as possible.

If $(L(\infty), B(\infty))$ denotes the crystal base of $\mathcal{U}_q^-(\mathfrak{g})$ and $T_\lambda = \{t_\lambda\}$ the one point crystal defined in 1.7, then it is known that there exists a full embedding of crystals $\tau_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda$ (see 1.3).

Kashiwara [**Kas93**] showed that for a sequence $\iota = (\dots, i_2, i_1)$ of elements of I - the index set of the simple roots - satisfying certain conditions (see 2.1(7)), the set of sequences of "coloured" integers

$$\mathbb{Z}_\iota^\infty := \{\dots a_1 a_0 : a_k \text{ is an } i_k - \text{coloured integer, and } a_k = 0 \text{ if } k >> 0\}$$

can be endowed with a crystal structure (see 1.17) such that the weight of $\dots a_1 a_0$ is equal to $-\sum_{k \geq 0} a_k \alpha_{i_k}$ and such that the crystal $B(\infty)$ is (isomorphic to) the connected component of $\mathbb{Z}_\iota^\infty$ containing $\dots 00$ (we identify $B(\infty)$ with this component). If $w = r_{i_l} \dots r_{i_1}$ is a reduced expression of an element of the Weyl group of $\mathfrak{g}$, then $B_w(\lambda) \overset{\tau_\lambda}{\hookrightarrow} B_w(\infty) \otimes T_\lambda$, where

$$B_w(\infty) := \{\dots a_1 a_0 \in B(\infty) : a_k = 0 \text{ if } k > l\}.$$

In [**NZ97**], $B(\infty)$ was shown to be the set of lattice points of a cone whose defining inequalities can be generated by applying certain operators to a given set of inequalities. The inequalities defining this cone were obtained for $\mathfrak{g}$ of rank 2 in [**Kas93**], and for the finite dimensional Lie algebras in [**Cli98**] and [**Lit98**] (for a particular sequence ι). In this thesis, we obtain an explicit formulation of these inequalities (see Theorem 4.9) for $\mathfrak{g}$ of type $A^{(1)}_{n-1}$ (also for a particular ι).

In Theorem 2.3, we show that if $\mathfrak{g}$ is affine (or of finite type) and ι is appropriately chosen,

$$B(\infty) = \{\dots a_1 a_0 \bar{c} : \ \bar{c} \in B(\infty, \mathfrak{g}') \text{ and } \dots a_1 a_0 \bar{0} \in B(\infty)\}$$

where $\mathfrak{g}'$ is a Lie algebra of lower rank than that of $\mathfrak{g}$.

Since the $B(\infty)$'s for the finite dimensional Lie algebras were described in [**Cli98**] and [**Lit98**], this Theorem reduces the problem of describing $B(\infty)$ for affine $\mathfrak{g}$ to that of describing its subset

$$\{\dots a_1 a_0 \bar{0} : \ \dots a_1 a_0 \bar{0} \in B(\infty)\}.$$

Since $\tau_{N\Lambda_i}(B(N\Lambda_i)) = \{\dots a_1 a_0 \bar{0} \otimes t_{N\Lambda_i} : \ \dots a_1 a_0 \bar{0} \in B(\infty) \text{ and } a_0 \le N\}$ (see Lemma 2.5), in order to describe $B(\infty)$, we would like to have an explicit description of $B(N\Lambda_i)$ and an explicit description of how the map $\tau_{N\Lambda_i}$ acts on it.

In [**JMMO91**], $B(N\Lambda_0)$ was explicitly described for $\mathfrak{g}$ of type $A^{(1)}_{n-1}$ as $N-$tuples of coloured Young diagrams such that an i-coloured box in an element Y of $B(N\Lambda_0)$ contributes $-\alpha_i$ to the weight of Y and hence contributes to an $i-$coloured integer in the image of $\tau_{N\Lambda_0}$.

Let $\iota = (\dots, 0, 1, \dots, n-1, 0, 1, \dots, n-1, 0, \iota')$, where ι' is appropriately chosen (see 2.1(7)) so that

$$B(N\Lambda_0) \overset{\tau_{N\Lambda_0}}{\hookrightarrow} \{\dots a_1\, a_0\, \bar{0} \otimes t_{N\Lambda_0}\}.$$

If we superimpose an element $Y \in B(\Lambda_0)$ on the following pattern:

$$\begin{array}{cccc} 0 & (n-1) & 2(n-1) & \cdots \\ 1 & (n-1)+1 & 2(n-1)+1 & \\ 2 & (n-1)+2 & 2(n-1)+2 & \\ 3 & (n-1)+3 & 2(n-1)+3 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

then $Y \overset{\tau_{\Lambda_0}}{\mapsto} \ldots a_1 a_0 \bar{0} \otimes t_{\Lambda_0}$, where a_k is the number of k's in the pattern which are enclosed by Y. (See Corollary 3.35).

For example, let n=2. Then, if the number in a box of $Y \in B(\Lambda_0)$ denotes the colour of that box,

$$\mathbf{Y} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \mapsto \ldots 0010212110\bar{0} \otimes t_{\Lambda_0},$$

since, when we superimpose $\mathbf{Y}$ on the pattern above, we obtain

$$\begin{array}{cccccc} 0 & 2 & 4 & 6 & 8 & \ldots \\ 1 & 3 & 5 & 7 & 9 & \\ 2 & 4 & 6 & 8 & 10 & \\ 3 & 5 & 7 & 9 & 11 & \ldots \\ \vdots & & & & \vdots & \end{array}$$

It can be shown (see Corollary 3.34) that if w is a subword of $\ldots r_{n-1} r_0 r_1 \ldots r_{n-1} r_0$, then the elements of $B_w(\Lambda_0)$ are those elements of $B(\Lambda_0)$ such that, when placed on the pattern, only enclose numbers less than or equal to the length of w.

For $N > 1$, however, the map $\tau_{N\Lambda_0}$ is not easily computed from the description of $B(N\Lambda_0)$ given in **[JMMO91]**.

In 3.9 and 3.10, following the ideas in **[JMMO91]**, we show that $\mathcal{Y}_N$, the set of N-tuples of coloured Young diagrams, can be endowed with a crystal structure for every total order of the set $\{1, \ldots, N\} \times \mathbb{N} \times (-\mathbb{N})$ such that if the order is as defined in 3.11, the crystal structure on $\mathcal{Y}_N$ coincides with that defined in **[JMMO91]**.

If the order is as defined in 3.14, we prove in Theorem 3.31 that the connected component of $\mathcal{Y}_N$ containing the N-tuple of empty Young diagrams is equal to the set $\mathcal{B}_N$ defined in 3.16.

In Theorem 3.23, we show that the map

$$\Phi : \mathcal{B}_N \hookrightarrow \mathbb{Z}_\iota^\infty \otimes T_\lambda$$

defined by

$$(Y_1, \dots, Y_N) \mapsto \dots a_1 a_0 \bar{0} \otimes t_{N\Lambda_0},$$

where $a_k = \sum_{j=1}^N$ (the number of k's in the pattern which are enclosed by Y_j), is a full embedding of crystals.

In Theorem 3.33, we prove that $\mathcal{B}_N$ is isomorphic to $B(N\Lambda_0)$, hence obtaining a description of $B(N\Lambda_0)$ as N-tuples of Young diagrams, which is different from that given in [**JMMO91**]. If we identify $\mathcal{B}_N$ with $B(N\Lambda_0)$, then $\Phi = \tau_{N\Lambda_0}$ (see Corollary 3.35.

One can show (see Corollary 3.34) that if w is a subword of $\dots r_{n-1} r_0 r_1 \dots r_{n-1} r_0$, then the elements of $B_w(N\Lambda_0)$ are those elements $(Y_1, \dots, Y_N) \in \mathcal{B}_N$ which are N-tuples of Young diagrams which when placed on the pattern, only enclose numbers less than or equal to the length of w.

In Chapter 4, we use our description of $B(N\Lambda_0)$ as $\mathcal{B}_N$ and the fact that with this description $\tau_{N\Lambda_0} = \Phi$ to explicitly find the inequalities defining the image of $\tau_{N\Lambda_0}$ (Theorem 4.7). This, together with our results from Chapter 2 and a result in [**Cli98**] and [**Lit98**] (see Appendix A), gives us an explicit description of $B(\infty)$ (Theorem 4.9).

CHAPTER 1

Preliminaries

In this chapter we set up the notation and state the definitions and known results which will be needed in the following chapters. Most of the definitions and results stated here are due to Kashiwara and can be found in [**Kas91**], [**Kas93**], and [**Kas94**].

1.1. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbb{Q}$, $\mathfrak{h}$ its Cartan subalgebra, $\{h_i : i \in I\}$ the set of simple coroots, $\{\alpha_i : i \in I\}$ the set of simple roots, P a lattice in $\mathfrak{h}^*$ such that $\alpha_i \in P$ for all $i \in I$, $P_+ = \{\lambda \in P : \langle h_i, \lambda\rangle \geq 0\}$, and $P^* = \{h \in \mathfrak{h} : \langle h, P\rangle \subseteq \mathbb{Z}\}$. The quantized universal enveloping algebra of $\mathfrak{g}$, $\mathcal{U}_q(\mathfrak{g})$, is a Hopf algebra over the field of rational functions of an indeterminate q generated by the set $\{e_i, f_i, q(h) : i \in I \text{ and } h \in P^*\}$ subject to some relations (see for example 1.1.14-1.1.18 in [**Kas91**].) This algebra contains a module L over the ring R of rational functions with no poles at 1 such that

(i) $\mathbb{Q}(q) \otimes_R L \simeq \mathcal{U}_q(\mathfrak{g})$ and

(ii) $L/(q-1)L \simeq \mathcal{U}(\mathfrak{g}) =$ the usual universal enveloping algebra of $\mathfrak{g}$.

A $\mathcal{U}_q(\mathfrak{g})$−module M is said to belong to the category $\mathcal{O}_{int}$ if

(i) $M = \bigoplus_{\lambda\in P} M_\lambda$ where $M_\lambda = \{u \in M : q(h)u = q^{\langle h,\lambda\rangle}u \;\; \forall h \in P^*\}$,

(ii) $\dim(M_\lambda) < \infty$ for all $\lambda \in P$,

(iii) for each $i \in I$, M is the union of finite dimensional $\mathcal{U}_q(\mathfrak{g}_i)$-modules where $\mathfrak{g}_i$ is the subalgebra of $\mathfrak{g}$ generated by $e_i, f_i, q(h_i)$ and $q(-h_i)$, and

(iv) $M = \bigoplus_{\lambda \in F+Q_-} M_\lambda$, where F is a finite subset of P and $Q_- = -\sum_{i\in I} \mathbb{N}\alpha_i$.

The category $\mathcal{O}_{int}$ is semisimple with irreducible objects $\{V(\lambda) : \lambda \in P_+\}$. $V(\lambda)$ is generated by a highest weight vector denoted by u_λ and if $\mathcal{U}_q^-(\mathfrak{g}) :=$ the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{f_i : i \in I\}$, then there exists a surjective $\mathcal{U}_q^-(\mathfrak{g})$-molule homomorphism $\pi_\lambda : \mathcal{U}_q^-(\mathfrak{g}) \to V(\lambda)$ such that $1 \to u_\lambda$. $V(\lambda)$ contains an $R-$module, which we denote by $(V(\lambda))_R$ such that

(i) $\mathbb{Q}(q) \otimes_R (V(\lambda))_R \simeq V(\lambda)$ and

(ii) $(V(\lambda))_R/(q-1)(V(\lambda))_R \simeq$ Verma module of highest weight λ.

1.2. If M is a $\mathbb{Q}(q)$ vector space, a **basis of M at $\mathbf{q = 0}$** is defined to be an ordered pair (L, B) where

(i) L is a free $A-$module such that $M \simeq \mathbb{Q}(q) \otimes_A L$ and

(ii) B is a basis of the $\mathbb{Q}$ vector space L/qL.

Here A is the ring of rational functions with no poles at 0.

1.3. If $M \in \mathcal{O}$ and $i \in I$,

$$M = \bigoplus_{\lambda \in P} \bigoplus_{0 \le n \le \langle h_i, \lambda\rangle} f_i^{(n)}(ker(e_i) \cap M_\lambda),$$

where $[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! := \prod_{k=1}^n [k]_i$, and $f_i^{(n)} := \frac{f_i^n}{[n]_i!}$, and q_i is as defined in [**Kas91**].

For $i \in I$, the operators $\tilde{e}_i$ and $\tilde{f}_i$ are defined as follows: for $u \in ker(e_i) \cap M_\lambda$ and $0 \le n \le \langle h_i, \lambda\rangle$, $\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}$ and $\tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}$. (Here $f_i^{(-1)} := 0$.)

A pair (L, B) is called a **(lower) crystal base** of M if

(i) (L, B) is a basis of M at $q = 0$,

(ii) $\tilde{e}_i(L) \subseteq L$ and $\tilde{f}_i(L) \subseteq L$ for all $i \in I$, and hence we can define $\tilde{e}_i$ and $\tilde{f}_i$ on B,

(iii) $\tilde{e}_i(B) \subseteq B \cup \{0\}$ and $\tilde{f}_i(B) \subseteq B \cup \{0\}$,

(iv) $L = \bigoplus_{\lambda \in P} L_\lambda$ and $B = \bigcup_{\lambda \in P} B_\lambda$ where $L_\lambda := L \cap M_\lambda$ and $B_\lambda := B \cap (L_\lambda / q L_\lambda)$, and

(v) for b_1 and $b_2 \in B$, $b_1 = \tilde{f}_i(b_2)$ if and only if $\tilde{e}_i(b_1) = b_2$.

Let $\lambda \in P_+$ and define $L(\lambda)$ to be the A–module generated by $\{\tilde{f}_{i_1} \dots \tilde{f}_{i_l} u_\lambda : i_1 \dots i_l \in I\}$ and let $B(\lambda) := \{\tilde{f}_{i_1} \dots \tilde{f}_{i_l} u_\lambda + qL(\lambda) : i_1 \dots i_l \in I\} \backslash \{0\}$. (We will also denote $u_\lambda + qL(\lambda)$ by u_λ). In [**Kas91**], Kashiwara shows that $(L(\lambda), B(\lambda))$ is the unique crystal base of $V(\lambda)$.

1.4. Let $\tilde{e}_i$ and $\tilde{f}_i$ be the operators on $\mathcal{U}_q^-(\mathfrak{g})$ defined in [**Kas91**].

A pair (L, B) is called a **crystal base** of $\mathcal{U}_q^-(\mathfrak{g})$ if

(i) (L, B) is a basis of $\mathcal{U}_q^-(\mathfrak{g})$ at $q = 0$,

(ii) $\tilde{e}_i(L) \subseteq L$ and $\tilde{f}_i(L) \subseteq L$ for all $i \in I$, and hence we can define $\tilde{e}_i$ and $\tilde{f}_i$ on B,

(iii) $\tilde{e}_i(B) \subseteq B \cup \{0\}$ and $\tilde{f}_i(B) \subseteq B$,

(iv) $L = \bigoplus_{\lambda \in P} L_\lambda$ and $B = \bigcup_{\lambda \in P} B_\lambda$ where $L_\lambda := L \cap M_\lambda$ and $B_\lambda := B \cap (L_\lambda / q L_\lambda)$, and

(v) for b_1 and $b_2 \in B$, $b_1 = \tilde{f}_i(b_2)$ if and only if $\tilde{e}_i(b_1) = b_2$.

Let $L(\infty)$ be the A–module generated by $\{\tilde{f}_{i_1} \dots \tilde{f}_{i_l} 1 : i_1 \dots i_l \in I\}$ and let $B(\infty) := \{\tilde{f}_{i_1} \dots \tilde{f}_{i_l} 1 + qL(\lambda) : i_1 \dots i_l \in I\}$. (We will denote $1 + qL(\lambda)$ by u_∞ and $B(\infty)$ by $B(\infty, \mathfrak{g})$ if we need to emphasize with which algebra we are dealing). In

[**Kas91**], Kashiwara shows that $(L(\infty), B(\infty))$ is the unique crystal base of $\mathcal{U}_q^-(\mathfrak{g})$, and that

$$\pi_\lambda(L(\infty)) = L(\lambda), \tag{1}$$

$$\bar{\pi}_\lambda : B(\infty)\backslash ker(\bar{\pi}_\lambda) \to B(\lambda) \text{ is a bijection,} \tag{2}$$

$$\text{where } \bar{\pi}_\lambda(u + L(\infty)) := \pi_\lambda(u) + qL(\lambda),$$

$$\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i \text{ for all } i \in I, \text{ and} \tag{3}$$

$$\text{if } b \in B(\infty) \text{with } \bar{\pi}_\lambda(b) \neq 0, \text{then } \tilde{e}_i(\bar{\pi}_\lambda(b)) = \bar{\pi}_\lambda(\tilde{e}_i(b)) \text{ for all } i \in I. \tag{4}$$

1.5. Define the antiautomorphism $* : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$ by, for $i \in I$ and $h \in P$, $e_i^* = e_i$, $f_i^* = f_i$ and $(q(h))^* = q(-h)$. It is shown in [**Kas91**] and [**Kas93**] that $(L(\infty))^* = L(\infty)$ and $(B(\infty))^* = B(\infty)$, respectively.

1.6. A **crystal** is defined to be a set B together with maps $wt : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ for $i \in I$, satisfying for $b \in B$ and $i \in I$:

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle$,

(ii) if $\tilde{e}_i(b) \neq 0$, $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$ and $wt(\tilde{e}_i(b)) = wt(b) + \alpha_i$,

(iii) if $\tilde{f}_i(b) \neq 0$, $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$ and $wt(\tilde{f}_i(b)) = wt(b) - \alpha_i$,

(iv) for b_1 and $b_2 \in B$, $b_2 = \tilde{f}_i(b_1)$ iff $\tilde{e}_i(b_2) = b_1$,

(v) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

(Note that $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$ and $\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$ are redundant in the definition as they follow from (i) and the rest of (ii) and (iii)).

The **crystal graph** of B is a graph whose set of vertices is B and whose edges are defined by: if b_1 and $b_2 \in B$ with $\tilde{f}_i(b_1) = b_2$ for some $i \in I$, there is a directed i—coloured (or i—labeled) edge from b_1 to b_2. i.e. $b_1 \xrightarrow{i} b_2$.

We will deal with the following examples of crystals in the coming chapters.

1.7. Examples. For $\lambda \in P_+$, and $b \in B(\lambda)$, if we define $wt(b) := \mu$ if $b \in B(\lambda)_\mu, \varepsilon_i(b) := \max\{n : \tilde{e}_i^n(b) \neq 0\}$, and $\varphi_i(b) := \max\{n : \tilde{f}_i^n(b) \neq 0\}$, then $B(\lambda)$ is a crystal.

For $b \in B(\infty)$, define $wt(b) := \mu$ if $b \in B(\infty)_\mu, \varepsilon_i(b) := \max\{n : \tilde{e}_i^n(b) \neq 0\}$, and $\varphi_i(b) := \varepsilon_i(b) + \langle h_i, wt(b)\rangle$, then $B(\infty)$ is a crystal.

For $i \in I$, define $B_i := \{b_i(n) : n \in \mathbb{Z}\}$, $wt(b_i(n)) := n\alpha_i$,

$$\varepsilon_j(b_i(n)) := \begin{cases} -n & \text{if } j = i, \\ -\infty & \text{if } j \in I\backslash\{i\}, \end{cases}$$

$$\varphi_j(b_i(n)) := \begin{cases} n & \text{if } j = i, \\ -\infty & \text{if } j \in I\backslash\{i\}, \end{cases}$$

$$\tilde{e}_j(b_i(n)) := \begin{cases} b_i(n+1) & \text{if } j = i, \\ 0 & \text{if } j \in I\backslash\{i\}, \end{cases}$$

$$\tilde{f}_j(b_i(n)) := \begin{cases} b_i(n-1) & \text{if } j = i, \\ 0 & \text{if } j \in I\backslash\{i\}, \end{cases}$$

Then B_i is a crystal.

For $\lambda \in P$, define $T_\lambda := \{t_\lambda\}$, $wt(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, and $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$ for $i \in I$. Then T_λ is a crystal.

1.8. If B_1 and B_2 are two crystals, a **morphism** from B_1 to B_2 is a map $\Psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that $\Psi(0) = 0$ and for $b \in B_1$ such that $\Psi(b) \in B_2$, and $i \in I$, the following are satisfied:

$$wt(\Psi(b)) = wt(b), \ \varepsilon_i(\Psi(b)) = \varepsilon_i(b), \text{ and } \varphi_i(\Psi(b)) = \varphi_i(b),$$

$$\text{if } \Psi(\tilde{e}_i(b)) \in B_2, \ \Psi(\tilde{e}_i(b)) = \tilde{e}_i(\Psi(b)),$$

$$\text{if } \Psi(\tilde{f}_i(b)) \in B_2, \ \Psi(\tilde{f}_i(b)) = \tilde{f}_i(\Psi(b)).$$

1.9. If B_1 and B_2 are two crystals, the **tensor product** of B_1 and B_2 is defined by

$B_1 \otimes B_2 := \{b_1 \otimes b_2 : b_1 \in B_1 \text{ and } b_2 \in B_2\}$, $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$, for $i \in I$, $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle\}$, $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle\}$,

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases} \tag{5}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases} \tag{6}$$

1.10. The category whose objects are crystals and morphisms are as defined above is shown in [**Kas93**] to be a tensor category.

1.11. If (L_i, B_i) is a crystal basis of M_i, where for $i = 1, 2$, $M_i \in \mathcal{O}$, then $(L_1 \oplus L_2, B_1 \oplus B_2)$, and $(L_1 \otimes L_2, B_1 \otimes B_2)$ are crystal bases of $M_1 \oplus M_2$ and $M_1 \otimes M_2$ respectively.

1.12. Let $\Psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ be a morphism of crystals. Ψ is called an **embedding** if Ψ is injective. In this case B_1 is called a **subcrystal** of B_2. Ψ is called an **isomorphism** if there exists a crystal morphism $\Phi : B_2 \cup \{0\} \to B_1 \cup \{0\}$ such that $\Phi \circ \Psi = \mathrm{id}_{B_1 \cup \{0\}}$ and $\Psi \circ \Phi = \mathrm{id}_{B_2 \cup \{0\}}$. (id := the identity map.) Ψ is called **strict** if $\Psi|_{B_1}$ commutes with all $\tilde{e}_i'$s and $\tilde{f}_i'$s, for $i \in I$. And if Ψ is an embedding, Ψ is called **full** if $\Psi|_{B_1}$ commutes with all $\tilde{e}_i'$s, for $i \in I$.

Note: the composition and the tensor product of two full (strict) embeddings of crystals is again a full (resp. strict) embedding. Also, Ψ is an isomorphism if and only if Ψ is injective and surjective, and $\Psi|_{B_1}$ commutes with all $\tilde{e}_i$'s and $\tilde{f}_i$'s for $i \in I$.

1.13. For $\lambda \in P_+$, define the map $\tau_\lambda : B(\lambda) \to B(\infty) \otimes T_\lambda$ as follows: if $b \in B(\infty)$ is such that $\bar{\pi}_\lambda(b) \in B(\lambda)$ (see 1.4(2)), $\tau_\lambda(\bar{\pi}_\lambda(b)) = b \otimes t_\lambda$. Using 1.4 (2) - (4), this map can be shown to be a full embedding of crystals. The image of this map is

$$\{b \otimes t_\lambda \in B(\infty) \otimes T_\lambda : \varepsilon_i(b^*) \leq \langle h_i, \lambda \rangle \forall i \in I\}$$

(see Proposition 8.2 in [**Kas94**]. G. Cliff pointed out that a proof of this result can be found in Proposition 2.8 of [**Nak99**].)

We will need the following lemmas.

1.14. Lemma. *Let $\Psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ be a full embedding of crystals, then for $i \in I$ and $b \in B_1$, $\tilde{f}_i(b) = 0$ iff $\tilde{f}_i(\Psi(b)) \notin Im\Psi \backslash \{0\}$.*

Proof. If $\tilde{f}_i(\Psi(b)) = \Psi(b')$, for some $b' \in B_1$, then

$$\Psi(b) = \tilde{e}_i(\Psi(b')) \text{ since } \Psi(b') \neq 0$$

$$= \Psi(\tilde{e}_i(b')) \text{ since } \Psi \text{ is a full embedding.}$$

So $b = \tilde{e}_i(b')$ since Ψ is injective. Hence $b' = \tilde{f}_i(b) \neq 0$.

If $\tilde{f}_i(b) \neq 0, \Psi(\tilde{f}_i(b)) \neq 0$ since Ψ is an embedding. Since Ψ is a morphism, $\tilde{f}_i(\Psi(b)) = \Psi(\tilde{f}_i(b)) \in \mathrm{Im}\Psi \backslash \{0\}$.

□

1.15. Lemma. *(Lemma 1.3.6 in* [**Kas93**]*). For $1 \leq k \leq m$, let C_k be a crystal and $b_k \in C_k$. For $i \in I$, define $a_k := \varepsilon_i(b_k) - \sum_{1 \leq \nu < k} \langle h_i, wt(b_\nu) \rangle$. Then if $a_k > a_\nu$ for $1 \leq \nu < k$ and $a_k \geq a_\nu$ for $k < \nu \leq m$,*

$$\tilde{e}_i(b_1 \otimes \ldots \otimes b_m) = b_1 \otimes \ldots \otimes \tilde{e}_i(b_k) \otimes \ldots \otimes b_m,$$

and if $a_k \geq a_\nu$ for $1 \leq \nu < k$ and $a_k > a_\nu$ for $k < \nu \leq m$,

$$\tilde{f}_i(b_1 \otimes \ldots \otimes b_m) = b_1 \otimes \ldots \otimes \tilde{f}_i(b_k) \otimes \ldots \otimes b_m.$$

(For a proof, see Proposition 2.1.1 in [**KN94**]).

1.16. For each positive integer j, let C_j be a crystal, and define $C_\infty := \{c_\infty\}$, $\mathrm{wt}\,(c_\infty) := 0$, $\varepsilon_i(c_\infty) := 0$ and $\tilde{e}_i(c_\infty) = 0$ for all $i \in I$.

Define the set

$$C := C_\infty \otimes \cdots \otimes C_2 \otimes C_1$$

$$:= \{c_\infty \otimes \cdots \otimes c_2 \otimes c_1 : \text{ for all } j \in \mathbb{N}_{\geq 1},\ c_j \in C_j; \text{ for all } i \in I, \mathrm{wt}\,(c_j) = 0$$
$$\text{and } \varepsilon_i(c_j) \leq 0 \text{ for all but finitely many } j\text{'s; and } \varepsilon_i(c_j) = 0 \text{ for}$$
$$\text{infinitely many } j\text{'s.}\}$$

For $j \in J = \{1,2,\ldots,\infty\}$, $i \in I$ and $c := c_\infty \otimes \cdots \otimes c_1 \in C$, define $a_{ji}(c) := \varepsilon_i(c_j) - \sum_{l>j}\langle h_i, \operatorname{wt}(c_l)\rangle$. Note: for $j >> 0$, $a_{ji}(c) \leq 0$.

We now define a crystal structure on C.

For $c = c_\infty \otimes \cdots \otimes c_2 \otimes c_1$ and $i \in I$, define $\operatorname{wt}(c) := \sum_{j\in J} \operatorname{wt}(c_j)$, $\varepsilon_i(c) := \max_{j\in J}\{a_{ji}(c)\}$, $\varphi_i(c) := \varepsilon_i(c) + \langle h_i, \operatorname{wt}(c)\rangle$,

$$\tilde{e}_i(c) := c_\infty \otimes \cdots \otimes \tilde{e}_i(c_k) \otimes \cdots \otimes c_1 \text{ if } a_{ki}(c) > a_{\nu i}(c) \text{ for all } \nu > k$$

$$\text{and } a_{ki}(c) \geq a_{\nu i}(c) \text{ for all } 1 \leq \nu \leq k,$$

and

$$\tilde{f}_i(c) := c_\infty \otimes \cdots \otimes \tilde{f}_i(c_k) \otimes \cdots \otimes c_1 \text{ if } a_{ki}(c) \geq a_{\nu i}(c) \text{ for all } \nu \geq k$$

$$\text{and } a_{ki}(c) > a_{\nu i}(c) \text{ for all } 1 \leq \nu < k.$$

Note 1: We don't need to define $\tilde{f}_i(c_\infty)$ since $a_{\infty i}(c) = 0$ and for some $j >> 0$, $a_{ji}(c) \leq 0$.

1.17. Lemma. *C with wt, ε_i, φ_i, $\tilde{e}_i$, and $\tilde{f}_i$, for $i \in I$, as defined above is a crystal.*

Proof. Let $c = c_\infty \otimes \cdots \otimes c_1 \in C$ and suppose $\tilde{e}_i(c) = c_\infty \otimes \cdots \otimes \tilde{e}_i(c_k) \otimes \cdots \otimes c_1 \neq 0$. Then $\operatorname{wt}(\tilde{e}_i(c)) = \operatorname{wt}(c) + \alpha_i$, and

$$a_{ji}(\tilde{e}_i(c)) = \begin{cases} a_{ji}(c) - 2 & \text{if } j < k \\ a_{ji}(c) - 1 & \text{if } j = k \\ a_{ji}(c) & \text{if } j > k. \end{cases}$$

Since $a_{ji}(c) < a_{ki}(c)$ if $j > k$,

$$\varepsilon_i(\tilde{e}_i(c)) := \max_{j\in J}\{a_{ji}(\tilde{e}_i(c))\} = a_{ki}(c) - 1 = \varepsilon_i(c) - 1.$$

Also, since $a_{ki}(\tilde{e}_i(c)) = a_{ki}(c) - 1 \geq a_{ji}(\tilde{e}_i(c))$ for all $j \geq k$, and $a_{ki}(\tilde{e}_i(c)) > a_{ji}(\tilde{e}_i(c))$ for all $j < k$,

$$\tilde{f}_i(\tilde{e}_i(c)) = c_\infty \otimes \cdots \otimes \tilde{f}_i \tilde{e}_i(c_k) \otimes \cdots \otimes c_1 = c \text{ since } \tilde{e}_i(c_k) \neq 0.$$

Now suppose that $\tilde{f}_i(c) = c_\infty \otimes \cdots \otimes \tilde{f}_i(c_k) \otimes \cdots \otimes c_1 \neq 0$. Then $\mathrm{wt}\,(\tilde{f}_i(c)) = \mathrm{wt}\,(c) - \alpha_i$ and

$$a_{ji}(\tilde{f}_i(c)) := \begin{cases} a_{ji}(c) + 2 & \text{if } j < k \\ a_{ji}(c) + 1 & \text{if } j = k \\ a_{ji}(c) & \text{if } j > k. \end{cases}$$

Since $a_{ji}(c) < a_{ki}(c)$ if $j < k$,

$$\varepsilon_i(\tilde{f}_i(c)) := \max_{j \in J}\{a_{ji}(\tilde{f}_i(c))\} = a_{ki}(c) + 1 = \varepsilon_i(c) + 1.$$

Also, since $a_{ki}(\tilde{f}_i(c)) = a_{ki}(c) + 1 \geq a_{ji}(\tilde{f}_i(c))$ if $j \leq k$ and $a_{ki}(\tilde{f}_i(c)) > a_{ji}(\tilde{f}_i(c))$ if $j > k$,

$$\tilde{e}_i(\tilde{f}_i(c)) = c_\infty \otimes \cdots \otimes \tilde{e}_i \tilde{f}_i(c_k) \otimes \cdots \otimes c_1 = c \text{ since } \tilde{f}_i(c_k) \neq 0.$$

1.6(i) follows from the definition, and 1.6(v) is true since $\varphi_i(C) \subseteq \mathbb{Z}$.

□

Note 2: If we define

$$C' := \cdots \otimes C_2 \otimes C_1$$

$$:= \{\cdots \otimes c_2 \otimes c_1 : \text{ for all } j \in \mathbb{N}_{\geq 1},\ c_j \in C_j;\ \text{for all } i \in I, \mathrm{wt}\,(c_j) = 0$$

$$\text{and } \varepsilon_i(c_j) \leq 0 \text{ for all but finitely many } j\text{'s; and } \varepsilon_i(c_j) = 0 \text{ for}$$

$$\text{infinitely many } j\text{'s}\}$$

then we can define wt, ε_i, φ_i and $\tilde{f}_i$ the same way we defined it for C since $a_{ji} \leq 0$ for all $j >> 0$, but the definition of $\tilde{e}_i$ only makes sense for those $c \in C'$ such that $\varepsilon_i(c) > 0$ otherwise there would not exist a maximal element $j_0 \in J$ such that $a_{j_0 i} = \varepsilon_i(c) = 0$. This is why we need the element c_∞. And although C' is not a crystal, for $c \in C'$, $\tilde{f}_i(c_\infty \otimes c) = c_\infty \otimes \tilde{f}_i(c)$ and if $\varepsilon_i(c) > 0$, $\tilde{e}_i(c_\infty \otimes c) = c_\infty \otimes \tilde{e}_i(c)$.

Note 3: For any positive integer j, $C_\infty \otimes \cdots \otimes C_{j+1}$ is a crystal and

$$C \simeq (C_\infty \otimes \cdots \otimes C_{j+1}) \otimes (C_j \otimes \cdots \otimes C_1).$$

CHAPTER 2

Kashiwara's map and its image

If $w = r_{i_l} \dots r_{i_1}$ is a reduced expression of an element w of the Weyl group of $\mathfrak{g}$, and if $u_{w\lambda}$ is the extremal vector of weight $w\lambda$ of $V(\lambda)$, then in [**Kas93**] Kashiwara shows the existence of a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\mathrm{ch}(\mathcal{U}_q^+(\mathfrak{g})u_{w\lambda}) = \sum_{b \in B_w(\lambda)} e^{wt(b)}$$

and the existence of a unique subset $B_w(\infty)$ of $B(\infty)$ satisfying $B_w(\lambda) = \bar{\pi}_\lambda(B_w(\infty)) \backslash \{0\}$, for all $\lambda \in P_+$.

He also shows the existence of a strict embedding of crystals

$$\Psi_\iota : B(\infty) \hookrightarrow \{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1},$$

for a sequence $\iota = (\dots, i_2, i_1)$ of elements of I satisfying 2.1(7) such that $\Psi_\iota(B_w(\infty)) = \Psi_\iota(B(\infty)) \cap \{u_\infty \otimes \dots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) : a_j = 0 \text{ for all } j > l\}$.

If $\mathfrak{g}$ is affine (or of finite type), $i \in I$, and $\mathfrak{g}_1$ is the finite dimensional Lie algebra whose Dynkin diagram is the Dynkin diagram of $\mathfrak{g}$ with the i^{th} node removed, and if ι and ι' are appropriately chosen (see 2.3), then we show that $\Psi_\iota(B(\infty, \mathfrak{g}))$ can be described by $\Psi_{\iota'}(B(\infty, \mathfrak{g}_1))$ and the images of $(\Psi_\iota \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$ (see 2.5).

We also show how in the simply-laced case, the images of Ψ_ι and $\Psi_{\iota'}$ are related if ι and ι' are two sequences of elements of I.

2.1. For $i \in I$, the map $\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes \boldsymbol{B}_i$ is defined (see [**Kas93**], Thm. 2.2.1) as follows: for $b \in B(\infty)$, we have that $b^* \in B(\infty)$ (see 1.5). Let m be such that $\tilde{e}_i^m(b^*) \neq 0$ and $\tilde{e}_i^{m+1}(b^*) = 0$. Now let $b_0 \in B(\infty)$ be such that $\tilde{e}_i^m(b^*) = b_0^*$, then $\Psi_i(b) := b_0 \otimes \tilde{f}_i^m b_i(0)$. In [**Kas93**], this map is shown to be a strict embedding of crystals.

For $i_1, \ldots, i_j \in I$, define

$$\Psi_{i_j \cdots i_1} := \left(\Psi_{i_j} \otimes \mathrm{id}_{B_{i_{j-1}} \otimes \cdots \otimes B_{i_1}}\right) \circ \cdots \circ \left(\Psi_{i_2} \otimes \mathrm{id}_{B_{i_1}}\right) \circ \Psi_{i_1}.$$

If $\iota = (\ldots, i_2, i_1)$ is a sequence of elements of I satisfying

$$\text{for each } i \in I,\ \{j : i_j = i\} \text{ is infinite} \tag{7}$$

(or if $\mathfrak{g}$ is finite dimensional and $w_0 = r_{i_1} \cdots r_{i_l}$ is a reduced expression of the longest word w_0 of the Weyl group of $\mathfrak{g}$, we may take $\iota = (i_1, i_2, \ldots, i_l)$ (see [**Jos95**], 6.1.15)), then for each $b \in B(\infty)$, there exists a j such that

$$\Psi_{i_j \cdots i_2 i_1}(b) \in \{u_\infty\} \otimes B_{i_j} \otimes \cdots \otimes B_{i_1}.$$

Using this, one obtains the **Kashiwara embedding** (see [**Kas93**])

$$\Psi_\iota : B(\infty) \hookrightarrow \{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}.$$

Here $\{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1} = \{u_\infty \otimes \cdots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) : a_s = 0 \text{ for } s >> 0\}$ is the crystal defined in 1.16. (Note that $\{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$ is isomorphic to a subset of $\varinjlim_k B(\infty) \otimes B_{i_k} \otimes \cdots \otimes B_{i_1}$.)

From the definition of the crystal structure on $\{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$, Lemma 1.15, and the fact that for all j, $\Psi_{i_j \cdots i_2 i_1}$ are strict embeddings of crystals it follows that Ψ_ι is a strict embedding of crystals.

We will denote $\{u_\infty\} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$ and $\cdots \otimes B_{i_2} \otimes B_{i_1} = \{\cdots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) : a_s = 0 \text{ for } s >> 0\}$ by $\{u_\infty\} \otimes B_\iota$ and B_ι respectively. Note that we did not define a crystal structure on B_ι although $\tilde{f}_i$ and $\tilde{e}_i$ act on $\{u_\infty\} \otimes B_\iota$ as they would if $\{u_\infty\} \otimes B_\iota$ were the tensor product of two crystals (see Note 2 at the end of 1.16).

In what follows, we write $(\ldots, a_2, a_1)$ $(u_\infty \otimes (\ldots, a_2, a_1))$ for the element $\ldots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) \in B_\iota$ (resp. $u_\infty \otimes \cdots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) \in \{u_\infty\} \otimes B_\iota$) if the sequence $(\ldots, i_2, i_1)$ is understood. Also we denote by $\overline{0}$ both the elements $(\ldots, 0, 0)$ and $(0, \ldots, 0)$.

2.2. Lemma. *Let $\iota = (\ldots, i_2, i_1)$ be a sequence of elements of I satisfying 2.1(7) and $S \subseteq \{u_\infty \otimes (\ldots, a_2, a_1) \in \{u_\infty\} \otimes B_\iota \cdot a_k \geq 0$ for all $k\}$. Then $S = \operatorname{Im}\Psi_\iota$ if and only if*

(a) $u_\infty \otimes (\ldots, 0, 0) \in S$;

(b) *for all $j \in I$, $\tilde{f}_j S \subseteq S$; and*

(c) *for all $j \in I$, $\tilde{e}_j S \subseteq S \cup \{0\}$.*

Proof. Suppose $S = \operatorname{Im}\Psi_\iota$. Then

(a) $u_\infty \otimes (\ldots, 0, 0) = \Psi_\iota(u_\infty) \in S$,

(b) and (c) are true since Ψ is a strict embedding of crystals.

Now suppose (a) - (c) are true. Then

$$\begin{aligned} \operatorname{Im}\Psi_\iota &= \{\Psi_\iota(\tilde{f}_{k_1} \cdots \tilde{f}_{k_s} u_\infty) : k_1, \ldots, k_s \in I\} \\ &= \{\tilde{f}_{k_1} \cdots \tilde{f}_{k_s}(u_\infty \otimes (\ldots, 0, 0)) : k_1, \ldots, k_s \in I\} \\ &\subseteq S \qquad \text{by (a) and (b).} \end{aligned}$$

To show that $S \subseteq \operatorname{Im} \Psi_\iota$, we use induction on the height of the weight of an element of S. Let $s \in S$. If $\mathrm{wt}(s) = 0$, then $s = u_\infty \otimes \overline{0} \in \operatorname{Im} \Psi_\iota$. If $\mathrm{wt}(s) \neq 0$, $s = u_\infty \otimes (\dots, 0, a_k, \dots, a_1)$ for some $a_1, \dots, a_k \in \mathbb{N}$, $a_k \neq 0$. Then $\varphi_{i_k}(u_\infty) = 0$ and $\varepsilon_{i_k}(s) \geq a_k > 0$.

Hence $\tilde{e}_{i_k} s = u_\infty \otimes \tilde{e}_{i_k}(\dots, 0, a_k, \dots, a_1) \in S$ by (c). By induction, $\tilde{e}_{i_k} s \in \operatorname{Im} \Psi_\iota$. So $s = \tilde{f}_{i_k} \tilde{e}_{i_k} s \in \operatorname{Im} \Psi_\iota$.

□

2.3. Let $\mathfrak{g}$ be affine (or of finite type) and I be as in Chapter 1. Let $i \in I$ and $\mathfrak{g}'$ be the Lie algebra whose Dynkin diagram is the Dynkin diagram of $\mathfrak{g}$ with the i^{th} node removed. Let $\iota' = (j_l, \dots, j_2, j_1)$ be a sequence of elements of $I \backslash \{i\}$ such that the image of the map $\Psi_{\iota'} : B(\infty, \mathfrak{g}') \hookrightarrow B(\infty, \mathfrak{g}') \otimes B_{j_l} \otimes \cdots \otimes B_{j_1}$ is inside $\{u_\infty\} \otimes B_{j_l} \otimes \cdots \otimes B_{j_1}$ (see 2.1(7)). Let $i_1, i_2, \dots \in I$ be such that, $\iota = (\dots, i_2, i_1, j_l, \dots, j_1)$ is as in 2.1(7) and let $\iota'' = (\dots, i_2, i_1)$. Then we have $\Psi_\iota : B(\infty, \mathfrak{g}) \hookrightarrow \{u_\infty\} \otimes B_\iota$. (We can assume $i_1 = i$.) We will also denote Ψ_ι by $\Psi_{\iota'', \iota'}$.

Theorem. *If ι, ι' and ι'' are as above, then*

$\operatorname{Im}(\Psi_\iota) = \{u_\infty \otimes b \otimes b' : b \in B_{\iota''},\ b' \in B_{\iota'},\ u_\infty \otimes b \otimes \bar{0} \in \operatorname{Im}(\Psi_\iota)$ *and* $u_\infty \otimes b' \in \operatorname{Im}(\Psi_{\iota'})\}$

Proof. Let S be the set on the right of the above equality. We show that (a), (b) and (c) of Lemma 2.2 are satisfied.

Clearly $u_\infty \otimes \bar{0} \in S$. So (a) is true.

Let $b \in B_{\iota''}$, and $b' \in B_{\iota'}$ be such that $u_\infty \otimes b \otimes \bar{0} \in \operatorname{Im}(\Psi_\iota)$ and $u_\infty \otimes b' \in \operatorname{Im}(\Psi_{\iota'})$. So $u_\infty \otimes b \otimes b' \in S$. We need to show that $\tilde{f}_j(u_\infty \otimes b \otimes b') \in S$.

$$\begin{aligned} \tilde{f}_j(u_\infty \otimes b \otimes b') &= u_\infty \otimes \tilde{f}_j(b \otimes b'), \qquad \text{see Notes 1 and 2 in 1.16} \\ &= \begin{cases} u_\infty \otimes \tilde{f}_j(b) \otimes b' & \text{if } \varphi_j(b) > \varepsilon_j(b') \\ u_\infty \otimes b \otimes \tilde{f}_j(b') & \text{if } \varphi_j(b) \leq \varepsilon_j(b'). \end{cases} \end{aligned}$$

(See Note 2 in 1.17 for the definitions of $\tilde{f}_j(b \otimes b')$, $\tilde{f}_j(b)$, and $\varphi_j(b)$.)

If $\varphi_j(b) > \varepsilon_j(b')$, then $\varphi_j(b) > \varepsilon_j(\bar{0})$ since

$$\varepsilon_j(b') \quad \begin{cases} = -\infty & = \varepsilon_j(\bar{0}) \quad \text{if } j = i \\ \geq 0 & = \varepsilon_j(\bar{0}) \quad \text{if } j \neq i \end{cases} \tag{8}$$

Hence $u_\infty \otimes \tilde{f}_j(b) \otimes \bar{0} = u_\infty \otimes \tilde{f}_j(b \otimes \bar{0}) = \tilde{f}_j(u_\infty \otimes b \otimes \bar{0}) \in \mathrm{Im}(\Psi_\iota)$ and $\tilde{f}_j(u_\infty \otimes b \otimes b') \in S$.

If $\varphi_j(b) \leq \varepsilon_j(b')$ and $j \neq i$, then $u_\infty \otimes \tilde{f}_j(b') = \tilde{f}_j(u_\infty \otimes b') \in \mathrm{Im}(\Psi_{\iota'})$. Hence $\tilde{f}_j(u_\infty \otimes b \otimes b') \in S$ and (b) is true.

We now show that $\tilde{e}_j(u_\infty \otimes b \otimes b') \in S \cup \{0\}$.

$$\tilde{e}_j(u_\infty \otimes b \otimes b') = \begin{cases} \tilde{e}_j(u_\infty \otimes b) \otimes b' & \text{if } \varphi_j(u_\infty \otimes b) \geq \varepsilon_j(b') \\ u_\infty \otimes b \otimes \tilde{e}_j(b') & \text{if } \varphi_j(u_\infty \otimes b) < \varepsilon_j(b') \end{cases}$$

(See Note 3 at the end of 1.17).

If $\varphi_j(u_\infty \otimes b) \geq \varepsilon_j(b')$, by (8) above $\varphi_j(u_\infty \otimes b) \geq \varepsilon_j(\bar{0})$. So $\tilde{e}_j(u_\infty \otimes b \otimes \bar{0}) = (\tilde{e}_j(u_\infty \otimes b)) \otimes \bar{0}$. If $0 \geq \varepsilon_j(b)$, $\tilde{e}_j(u_\infty \otimes b \otimes b') = 0$ and if $0 < \varepsilon_j(b)$, $\tilde{e}_j(u_\infty \otimes b \otimes b') = u_\infty \otimes \tilde{e}_j(b) \otimes b'$ and $u_\infty \otimes \tilde{e}_j(b) \otimes \bar{0} = \tilde{e}_j(u_\infty \otimes b \otimes \bar{0}) \in \mathrm{Im}\Psi_\iota$ (since $\tilde{e}_j(\mathrm{Im}\Psi_\iota) \subseteq \mathrm{Im}\Psi_\iota \cup \{0\}$). So $\tilde{e}_j(u_\infty \otimes b \otimes b') \in S \cup \{0\}$.

If $\varphi_j(u_\infty \otimes b) < \varepsilon_j(b')$, $j \neq i$ (since otherwise $\varepsilon_j(b') = -\infty$). Since $\tilde{e}_j(u_\infty \otimes b \otimes \bar{0}) \in \mathrm{Im}\Psi_\iota \cup \{0\}$, and since $u_\infty \otimes b \otimes \tilde{e}_j(\bar{0}) \notin \mathrm{Im}\Psi_\iota \cup \{0\}$, $\tilde{e}_j(u_\infty \otimes b \otimes \bar{0}) = (\tilde{e}_j(u_\infty \otimes b)) \otimes \bar{0}$. So $\varphi_j(u_\infty \otimes b) \geq \varepsilon_j(\bar{0}) = 0$. Thus since $0 \leq \varphi_j(u_\infty \otimes b) < \varepsilon_j(b')$, $u_\infty \otimes \tilde{e}_j(b') = \tilde{e}_j(u_\infty \otimes b') \in \mathrm{Im}\Psi_{\iota'}$. So $\tilde{e}_j(u_\infty \otimes b \otimes b') \in S$ and (c) is true. Hence $S = \mathrm{Im}\Psi_\iota$.

□

2.4. Note that for certain sequences ι's the image of the Kashiwara embedding is " known" for the finite dimensional Lie algebras (see [**Cli98**] and [**Lit98**]) or **Appendix A** for $\mathfrak{g}$ of type A_n); hence for affine $\mathfrak{g}$, we will "know" the image of

Ψ_ι if we can describe the elements of the set $\{u_\infty \otimes b \otimes \bar{0} \in \{u_\infty\} \otimes B_{\iota''} \otimes B_{\iota'} : u_\infty \otimes b \otimes \bar{0} \in \mathrm{Im}\Psi_\iota\}$.

2.5. Lemma. *For $N \in \mathbb{N}$, $i \in I$, and ι as in (7) of 2.1,*

$$\{u_\infty \otimes a \otimes \bar{0} : a \in B_{\iota''},\ u_\infty \otimes a \otimes \bar{0} \in \mathrm{Im}\Psi_\iota\}$$

$$= \{u_\infty \otimes a \otimes \bar{0} : u_\infty \otimes a \otimes \bar{0} \otimes t_{N\Lambda_i} \in \mathrm{Im}((\Psi_\iota \otimes \mathrm{id}_{N\Lambda_i} \circ \tau_{N\Lambda_i}) \textit{ for some } N \in \mathbb{N}\}.$$

Where $\mathrm{id}_{N\Lambda_i}$ *denotes the identity function on* $\tau_{N\Lambda_i}$.

Proof. $\mathrm{Im}\tau_{N\Lambda_i} = \{b \otimes t_{N\Lambda_i} \in B(\infty) \otimes T_{N\Lambda_i} : \varepsilon_j(b^*) \leq \langle h_j, N\Lambda_i\rangle = N\delta_{ij}$ for all $j \in I\}$. (See 1.13). For $b \in B(\infty)$, $\varepsilon_j(b^*) \leq N\delta_{ij}$ if and only if $\Psi_\iota(b) = u_\infty \otimes (\dots, k) \otimes \bar{0}$, for $k \leq N$. It follows that $\mathrm{Im}((\Psi_\iota \otimes id_{N\Lambda_i}) \circ \tau_{N\Lambda_i}) = \{u_\infty \otimes (\dots, k) \otimes \bar{0} \otimes t_{N\Lambda_i} : k \leq N$ and $u_\infty \otimes (\dots, k) \otimes \bar{0} \in Im\Psi_\iota\}$.

□

2.6. Lemma. *Let i, $j \in I$ be such that $\langle \alpha_i, h_j\rangle = -1$, and let C_1 and C_2 be crystals. Define the map $\beta := \beta_{iji} : C_1 \otimes B_i \otimes B_j \otimes B_i \otimes C_2 \to C_1 \otimes B_j \otimes B_i \otimes B_j \otimes C_2$ by $\beta(X \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y) =$*

$$X \otimes b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)) \otimes Y$$

for $X \in C_1$, $Y \in C_2$ and $a, b, c \in \mathbb{Z}$.

Then $\beta \circ \tilde{f}_k = \tilde{f}_k \circ \beta$.

Proof. Let $a, b, c \in \mathbb{Z}$. Then

$$\min(c, b-a) + \max(a, b-c) = \begin{cases} c + (b-c) & \text{if } c < b-a \\ (b-a) + a & \text{if } c \geq b-a \end{cases} \tag{9}$$

$$= b,$$

$$\begin{aligned}
\min(\max(a, b-c), a+c-\min(c, b-a)) &= \min \begin{cases} (b-c, a+c-c) & \text{if } c < b-a \\ (a, a+c-(b-a)) & \text{if } c \geq b-a \end{cases} \\
&= a, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\max(\min(c, b-a), a+c-\max(a, b-c)) &= \max \begin{cases} (c, a+c-(b-c)) & \text{if } c < b-a \\ (b-a, a+c-a) & \text{if } c \geq b-a \end{cases} \\
&= c.
\end{aligned}$$

Hence $\beta_{iji} \circ \beta_{jij}$ is the identity map.

Let $k \in I$. We first show that $\varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \varphi_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))$ and $\mathrm{wt}(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \mathrm{wt}(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))$. It will then follow that $\varepsilon_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \varepsilon_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))$.

$$\begin{aligned}
\mathrm{wt}(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) &= -a\alpha_i - b\alpha_j - c\alpha_i = -b\alpha_j - (a+c)\alpha_i \\
&= -\min(c, b-a)\alpha_j - (a+c)\alpha_i - \max(a, b-c)\alpha_j \text{ (by (9))} \\
&= \mathrm{wt}(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c))).
\end{aligned}$$

To show that $\varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \varphi_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))$, we consider three cases.

If $k \neq i$ and $k \neq j$, $\varphi_k(b_j(-\min(c, b-a) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))) = -\infty = \varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))$.

If $k = j$,

$$\begin{aligned}
&\varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \\
&\quad = \max(\varphi_k(b_i(-a)) + \langle \mathrm{wt}\,(b_j(-b) \otimes b_i(-c)), h_k\rangle, \varphi_k(b_j(-b) \otimes b_i(-c))) \\
&\quad = \varphi_k(b_j(-b) \otimes b_i(-c)), \qquad \text{since } \varphi_k(b_i(-a)) = -\infty \\
&\quad = \max(\varphi_k(b_j(-b)) + \langle \mathrm{wt}\,(b_i(-c)), h_k\rangle, \varphi_k(b_i(-c))) \\
&\quad = -b - \langle c\alpha_i, h_k\rangle, \qquad \text{since } \varphi_k(b_i(-c)) = -\infty \\
&\quad = -b + c,
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c))) \\
&= \max(\varphi_k(b_j(-\min(c, b-a)) + \langle \mathrm{wt}\,(b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))), h_k\rangle, \\
&\qquad\qquad \varphi_k(b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))) \\
&= \max(-\min(c, b-a) + \langle -(a+c)\alpha_i - (\max(a, b-c))\alpha_j, h_k\rangle, \\
&\qquad\qquad \max(-\infty, -\max(a, b-c))) \\
&= \max(-\min(c, b-a) + a + c - 2\max(a, b-c), -\max(a, b-c)) \\
&= \max(a + c - b - \max(a, b-c), -\max(a, b-c)), \qquad \text{(by (9))} \\
&= \begin{cases} a + c - b - \max(a, b-c) & \text{if } a + c - b \geq 0 \\ -\max(a, b-c) & \text{if } a + c - b < 0 \end{cases} \\
&= c - b
\end{aligned}$$

Finally, if $k = i$,

$$\varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \varphi_k(\gamma_{jij} \circ \gamma_{iji}(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)))$$

$$= \varphi_k(\gamma_{jij}(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c))))$$

$$= \varphi_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c))),$$

by the previous case applied to γ_{jij}. (Here $\gamma := \gamma_{iji}$ is defined by $\gamma_{iji}(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) := b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c))$, and γ_{jij} is its inverse.)

Now $\varepsilon_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y)$

$$= \max(\varepsilon_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)), \varepsilon_k(Y) - \langle \mathrm{wt}\,(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)), h_k \rangle)$$

$$= \max(\varepsilon_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))), \varepsilon_k(Y) -$$

$$\langle \mathrm{wt}\,(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))), h_k \rangle)$$

$$= \varepsilon_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y).$$

So

$$\tilde{f}_k(X \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y)$$

$$= \begin{cases} \tilde{f}_k(X) \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y \\ \qquad \text{if } \varphi_k(X) > \varepsilon_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \\ X \otimes \tilde{f}_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \\ \qquad \text{if } \varphi_k(X) \leq \varepsilon_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \end{cases}$$

$$\tilde{f}_k(X \otimes \gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y)$$
$$= \begin{cases} \tilde{f}_k(X) \otimes \gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y \\ \qquad \text{if } \varphi_k(X) > \varepsilon_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \\ X \otimes \tilde{f}_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \\ \qquad \text{if } \varphi_k(X) \leq \varepsilon_k((b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y) \end{cases}$$

$$\tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y)$$
$$= \begin{cases} \tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y \\ \qquad \text{if } \varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) > \varepsilon_k(Y) \\ (b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes \tilde{f}_k(Y) \\ \qquad \text{if } \varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \leq \varepsilon_k(Y) \end{cases}$$

$$\tilde{f}_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes Y)$$
$$= \begin{cases} \tilde{f}_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))) \otimes Y \\ \qquad \text{if } \varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) > \varepsilon_k(Y) \\ \gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \otimes \tilde{f}_k(Y) \\ \qquad \text{if } \varphi_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) \leq \varepsilon_k(Y) \end{cases}$$

If $k \neq i$ and $k \neq j$,

$$\tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = \tilde{f}_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))) = 0.$$

If $k = j$, $\tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)) = b_i(-a) \otimes b_j(-(b+1)) \otimes b_i(-c)$, $\gamma(\tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c))) = (b_j(-\min(c, b+1-a)), b_i(-(a+c)), b_j(-\max(a, b+1-c)))$, and

$$\tilde{f}_k(\gamma(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)))$$
$$= \tilde{f}_k(b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)))$$

$$= \begin{cases} b_j(-\min(c, b-a) - 1) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)) \\ \qquad \text{if } -\min(c, b-a) > \max(a, b-c) - (a+c) \\ b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c) - 1) \\ \qquad \text{if } -\min(c, b-a) \leq \max(a, b-c) - (a+c) \end{cases}$$

$$= \begin{cases} b_j(-\min(c, b-a) - 1) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)) \\ \qquad \text{if } b < a+c \\ b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c) - 1) \\ \qquad \text{if } b \geq a+c \end{cases}$$

$$= \begin{cases} b_j(-(b-a+1)) \otimes b_i(-(a+c)) \otimes b_j(-a) \\ \qquad \text{if } b < a+c \\ b_j(-c) \otimes b_i(-(a+c)) \otimes b_j(-(b-c+1)) \\ \qquad \text{if } b \geq a+c \end{cases}$$

$$= \gamma\, \tilde{f}_k(b_i(-a) \otimes b_j(-b) \otimes b_i(-c)).$$

So $\beta \tilde{f}_k(X \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y) = \tilde{f}_k \beta(X \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y)$. If $k = i$, using $\beta_{jij} \circ \beta_{iji} = \text{id}$ and the previous case, we get our result.

□

2.7. Corollary. *Let* $i, j, j_1, \ldots, j_l, i_1, i_2, \ldots \in I$. *If* $\langle \alpha_i, h_j \rangle = -1$, $\iota := (\ldots, i_1, i, j, i, j_l, \ldots, j_1)$ *and* $\iota' := (\ldots, i_1, j, i, j, j_l, \ldots, j_1)$, *then*

$$\mathrm{Im}\Psi_{\iota'} = \{u_\infty \otimes X \otimes b_j(-\min(c, b-a)) \otimes b_i(-(a+c)) \otimes b_j(-\max(a, b-c)) \otimes Y : X \in B_{(\ldots, i_2, i_1)}, Y \in B_{j_l} \otimes \cdots \otimes B_{j_1}, \text{ and } u_\infty \otimes X \otimes b_i(-a) \otimes b_j(-b) \otimes b_i(-c) \otimes Y \in \mathrm{Im}\Psi_\iota\}$$

Proof. With β as in Lemma 2.6,

$$\begin{aligned}
\mathrm{Im}\Psi_{\iota'} &= \{\tilde{f}_{k_1} \cdots \tilde{f}_{k_s}(u_\infty \otimes \overline{0}) : k_1, \ldots, k_s \in I\} \\
&= \{\tilde{f}_{k_1} \cdots \tilde{f}_{k_s}\beta(u_\infty \otimes \overline{0}) : k_1, \ldots, k_s \in I\} \\
&= \{\beta\tilde{f}_{k_1} \cdots \tilde{f}_{k_s}(u_\infty \otimes \overline{0}) : k_1, \ldots, k_s \in I\} \\
&= \beta(\mathrm{Im}\Psi_\iota).
\end{aligned}$$

□

CHAPTER 3

$B(N\Lambda_0)$ and sequences of Young Tableaux for $A_{n-1}^{(1)}$

In [**MM90**], Misra and Miwa give a description of $B(\Lambda_0)$, for a fundamental weight Λ_0 of $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$, as a subset of Young Tableaux. Using this description, we can then view $B(N\Lambda_0)$, for $N \in \mathbb{N}$, as a subset of N-tuples of Young Tableaux by using the fact that $B(N\Lambda_0)$ is the connected component of $B(\Lambda_0)\otimes \ldots \otimes B(\Lambda_0)$ (N-factors) containing $u_{\Lambda_0} \otimes \ldots \otimes u_{\Lambda_0}$. A different description of $B(N\Lambda_0)$ as a subset of N-tuples of Young Tableaux is given in [**JMMO91**], where the action of $\tilde{e}_i$ and $\tilde{f}_i$, for $i \in I$, is determined by a given total order on $\{1, \ldots, N\} \times \mathbb{N} \times (-\mathbb{N})$ (see 3.11). In this chapter, we will show that for an arbitrary total order on $\{1, \ldots, N\} \times \mathbb{N} \times (-\mathbb{N})$, we can define a crystal structure on $\mathcal{Y}_N$, the set of N-tuples of Young Tableaux, which

(i) coincides with that defined in [**JMMO91**] if the order is as defined in 3.11,

(ii) coincides with that of $\mathcal{Y}_1 \otimes \ldots \otimes \mathcal{Y}_1$ (here we identify $\mathcal{Y}_N$ with $\mathcal{Y}_1 \otimes \ldots \otimes \mathcal{Y}_1$) if the order is as defined in 3.12, and

(iii) gives us a new description of $B(N\Lambda_0)$ as a subset of $\mathcal{Y}_N$ if the order is as defined in 3.14.

With this third description, the image of an element of $B(N\Lambda_0)$ under the map $(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$ is easily computed if ι and ι' are as in 3.17 and we use this in Chapter 4 to describe the image of $(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$. Also if $w = r_{i_l} \ldots r_{i_1}$ where $\iota = (\ldots, i_2, i_1)$ is as in 3.17, then the elements of $B_w(N\Lambda_0)$ are those elements of

$B(N\Lambda_0)$ which are N-tuples of subtableaux of a single Young Tableau which we call $S(-l)$.

3.1. In this and the next chapter, $\mathfrak{g} = \hat{\mathfrak{sl}}(n)$, and the notation is as in section 2.1 of [**JMMO91**]. Hence $I = \{0, \dots, n-1\}$ and $\Lambda_0 \in P$ is such that $\langle \Lambda_0, h_i \rangle = \delta_{0i}$ for all $i \in I$.

3.2. Definition. *A* ***Young diagram (or tableau)*** *Y is a sequence $\{y_k\}_{k\geq 0}$ such that*

(i) $y_k \in \mathbb{Z}$,

(ii) $y_k \leq y_{k+1}$ *for all k, and*

(iii) $y_k = 0$ *for all* $k >> 0$.

(In [**JMMO91**], Y is called an extended Young diagram of charge 0.)

The empty Young diagram will be denoted by ϕ. i.e. $\phi = (0, 0, \dots)$.

We colour the (x, y)-plane as follows: the "box" $\{(x, y) : k, k' \in \mathbb{Z},\ k-1 < x \leq k,\ k' < y \leq k'+1\}$ is coloured i where $i \in \{0, \dots, n-1\}$ and $k + k' \equiv i \bmod n$. Then the diagram $Y = \{y_k\}_{k\geq 0}$ is represented in the coloured (x, y)-plane by the coloured region defined by $\{(x, y) :\ k \leq x \leq k+1,\ 0 \geq y \geq y_k \text{ for some } k \in \mathbb{N}\}$.

For $N \in \mathbb{N}$, define

$$\mathcal{Y}_N := \{\mathbf{Y} = (Y_1, \dots, Y_N) : \text{ for } 1 \leq r \leq N,\ Y_r \text{ is a Young diagram } \}.$$

(In [**JMMO91**], $\{\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{Y}_N : Y_1 \supseteq \dots \supseteq Y_N\}$ is denoted by $\mathcal{Y}(N\Lambda_0)$.)

3.2.1. **Example.** Let $n = 3,\ N = 2,$ and

$\mathbf{Y} = ((-4, -2, -1, 0, 0, \dots), (-2, -1, -1, 0, 0, \dots)).$

$$\mathbf{Y} : \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} 0 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \ , \quad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right).$$

3.3. Let $\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{Y}_N$ and $Y_r = \{y_{rk}\}_{k \geq 0}$ for $1 \leq r \leq N$. If $y_{rk} \neq y_{r(k+1)}$ for some $r \in \{1, 2, \dots, N\}$ and $k \in \mathbb{N}$, $\mathbf{Y}$ is said to have a **concave (convex) corner** at site $(r, k+1, y_{r(k+1)})$ ($(r, k+1, y_{rk})$, resp.). Also for $r \in \{1, 2, \dots, N\}$, $\mathbf{Y}$ is said to have a **concave corner** at site $(r, 0, y_{r0})$. A corner at site (r, k, y) is called an $i-$**coloured corner** if $i \in I$ with $i \equiv k + y \bmod n$.

3.3.1. **Example.** If $\mathbf{Y}$ is as in 3.2.1, $\mathbf{Y}$ has 0-coloured concave corners at sites $(1, 3, 0)$, $(2, 3, 0)$, and $(2, 1, -1)$, 0-coloured convex corners at sites $(1, 2, -2)$ and $(1, 1, -4)$, 1-coloured concave corners at sites $(1, 2, -1)$ and $(2, 0, -2)$, no 1-coloured convex corners, 2-coloured concave corners at sites $(1, 1, -2)$ and $(1, 0, -4)$, and 2-coloured convex corners at sites $(1, 3, -1)$, $(2, 3, -1)$ and $(2, 1, -2)$.

3.4. Let $\sigma = (\sigma_1, \dots, \sigma_m)$ where $m \in \mathbb{N}$ and for $1 \leq l \leq m$, $\sigma_l \in \{0, 1\}$. Define $J(\sigma)$ as follows: let $J = \{1, \dots, m\}$.

(i) If there exists $r < s$ such that $(\sigma_r, \sigma_s) = (0, 1)$ and $r' \notin J$ for $r < r' < s$, replace J by $J \backslash \{r, s\}$ and repeat this step;

(ii) otherwise let $J(\sigma) = J$.

Let $J(\sigma) = \{j_1, \dots, j_t\}$ with $j_1 < \cdots < j_t$. Define $\sigma_{J(\sigma)} := (\sigma_{j_1}, \dots, \sigma_{j_t})$ $(= (1 \cdots 1\, 0 \cdots 0))$. Let $t_0(\sigma) \in J(\sigma) \cup \{m+1\}$ be the largest element of $J(\sigma) \cup \{m+1\}$

such that $\sigma_l = 1$ for all $l \in J(\sigma)$ with $l < t_0(\sigma)$, and let $t_1(\sigma) \in J(\sigma) \cup \{0\}$ be the smallest element of $J(\sigma) \cup \{0\}$ such that $\sigma_l = 0$ for all $l \in J(\sigma)$ with $l > t_1(\sigma)$.

Now define

$$\tilde{f}(\sigma) := \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_{t_0(\sigma)-1}, 1, \sigma_{t_0(\sigma)+1}, \dots, \sigma_m) & \text{if } t_0(\sigma) \neq m+1, \\ 0 & \text{if } t_0(\sigma) = m+1 \end{cases}$$

$$\tilde{e}(\sigma) := \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_{t_1(\sigma)-1}, 0, \sigma_{t_1(\sigma)+1}, \dots, \sigma_m) & \text{if } t_1(\sigma) \neq 0, \\ 0 & \text{if } t_1(\sigma) = 0 \end{cases}$$

3.4.1. **Example.** Let $\sigma = (1,1,0,0,1,1,0,0,1)$. Then $J(\sigma) = \{1,2,7\}$, $\sigma_{J(\sigma)} = (1,1,0)$, $t_0(\sigma) = 7$, $t_1(\sigma) = 2, \tilde{f}(\sigma) = (1,1,0,0,1,1,1,0,1)$, and $\tilde{e}(\sigma) = (1,0,0,0,1,1,0,0,1)$.

3.5. Lemma. *Let $\sigma = (\sigma_1, \dots, \sigma_m)$ where $m \in \mathbb{N}$ and for $1 \leq t \leq m$, $\sigma_t \in \{0,1\}$. For $1 \leq t \leq m$, define $\omega_t := \begin{cases} 1 & \text{if } \sigma_t = 1 \\ -1 & \text{if } \sigma_t = 0 \end{cases}$. Then*

(i) $\sigma_{J(\sigma)} = (1, \dots, 1)$ *(possibly empty) if and only if $\sum_{j=t}^{m} \omega_j \geq 0$ for all $1 \leq t \leq m$. (Note: in this case $\sum_{j=1}^{m} \omega_j = \#$ of 1's in $\sigma_{J(\sigma)}$), and*

(ii) $\sigma_{J(\sigma)} = (0, \dots, 0)$ *(possibly empty) if and only if $\sum_{j=1}^{t} \omega_j \leq 0$ for all $1 \leq t \leq m$. (Note: in this case $-\sum_{j=1}^{m} \omega_j = \#$ of 0's in $\sigma_{J(\sigma)}$).*

Proof. We use induction on m.

If $m = 0$, $\sigma = (\ )$, $\sigma_{J(\sigma)} = (\ )$ and $\sum_{j=1}^{0} \omega_j = 0$. Assume $m > 0$.

(i) Then $\sigma_{J(\sigma)} = (1, \dots, 1)$

$$\Longleftrightarrow \sigma_{J((\sigma_2, \dots, \sigma_m))} = (1, \dots, 1) \text{ and } \left\{ \sigma_1 = 1 \text{ or } \left\{ \sigma_1 = 0 \text{ and } \sum_{j=2}^{m} \omega_j > 0 \right\} \right\}$$

$$\iff \sum_{j=t}^{m} \omega_j \geq 0 \text{ for all } 2 \leq t \leq m \text{ and } \left\{\sigma_1 = 1 \text{ or } \left\{\sigma_1 = 0 \text{ and } \sum_{j=2}^{m} \omega_j > 0\right\}\right\}$$

$$\iff \sum_{j=t}^{m} \omega_j \geq 0 \text{ for all } 1 \leq t \leq m, \text{ and}$$

(ii) $\sigma_{J(\sigma)} = (0, \dots, 0)$

$$\iff \sigma_{J((\sigma_1,\dots,\sigma_{m-1}))} = (0, \dots, 0) \text{ and } \left\{\sigma_m = 0 \text{ or } \left\{\sigma_m = 1 \text{ and } \sum_{j=1}^{m-1} \omega_j < 0\right\}\right\}$$

$$\iff \sum_{j=1}^{t} \omega_j \leq 0 \text{ for all } 1 \leq t \leq m-1 \text{ and } \left\{\sigma_m = 0 \text{ or } \left\{\sigma_m = 1 \text{ and } \sum_{j=1}^{m-1} \omega_j < 0\right\}\right\}$$

$$\iff \sum_{j=1}^{t} \omega_j \leq 0 \text{ for all } 1 \leq t \leq m,$$

□

3.6. Lemma. *Let $\sigma = (\sigma_1, \dots, \sigma_m)$ where $m \in \mathbb{N}$ and for $1 \leq t \leq m$, $\sigma_t \in \{0, 1\}$. For $1 \leq t \leq m$, define $\omega_t := \begin{cases} 1 & \text{if } \sigma_t = 1 \\ -1 & \text{if } \sigma_t = 0 \end{cases}$. Define $\sum_0 := 0$ and $\sum_t := \sum_{j=1}^{t} \omega_j$, for $1 \leq t \leq m$. Then*

(i) *$\sum_{t_0(\sigma)-1} \geq \sum_t$ for all $0 \leq t < t_0(\sigma) - 1$ and $\sum_{t_0(\sigma)-1} > \sum_t$ for all $t_0(\sigma) \leq t \leq m$, and*

(ii) *$\sum_{t_1(\sigma)} > \sum_t$ for all $0 \leq t \leq t_1(\sigma) - 1$ and $\sum_{t_1(\sigma)} \geq \sum_t$ for all $t_1(\sigma) \leq t \leq m$.*

Proof.

(i) $\sigma_{J((\sigma_1,\dots,\sigma_{t_0(\sigma)-1}))} = (1, \dots, 1)$ and $\sigma_{J((\sigma_{t_0(\sigma)+1},\dots,\sigma_m))} = (0, \dots, 0)$. So by Lemma 3.5, $\sum_{j=t}^{t_0(\sigma)-1} \omega_j \geq 0$ for all $1 \leq t \leq t_0(\sigma) - 1$, and $\sum_{j=t_0(\sigma)+1}^{t} \omega_j \leq 0$ for all $t_0(\sigma) + 1 \leq t \leq m$. Hence for $0 \leq t < t_0(\sigma) - 1$, $\sum_{t_0(\sigma)-1} - \sum_t =$

$\sum_{j=t+1}^{t_0(\sigma)-1}\omega_j \geq 0$ and for $t_0(\sigma) \leq t \leq m$, $\sum_t - \sum_{t_0(\sigma)-1} = \sum_{j=t_0(\sigma)}^{t}\omega_j = \sum_{j=t_0(\sigma)+1}^{t}\omega_j + (-1) < 0$.

(ii) $\sigma_{J((\sigma_1,\dots,\sigma_{t_1(\sigma)-1}))} = (1,\dots,1)$ and $\sigma_{J((\sigma_{t_1(\sigma)+1},\dots,\sigma_m))} = (0,\dots,0)$. So by Lemma 3.5, $\sum_{j=t}^{t_1(\sigma)-1}\omega_j \geq 0$ for all $1 \leq t \leq t_1(\sigma)-1$, and $\sum_{j=t_1(\sigma)+1}^{t}\omega_j \leq 0$ for all $t_1(\sigma)+1 \leq t \leq m$. Hence for all $t_1(\sigma) \leq t \leq m$, $\sum_t - \sum_{t_1(\sigma)} = \sum_{j=t_1(\sigma)+1}^{t}\omega_j \leq 0$ and for all $0 \leq t \leq t_1(\sigma)-1$, $\sum_{t_1(\sigma)} - \sum_t = \sum_{j=t+1}^{t_1(\sigma)}\omega_j = \sum_{j=t+1}^{t_1(\sigma)-1}\omega_j + 1 > 0$.

□

3.6.1. **Example.** Let σ be as in 3.4.1. Then $\sum_0 = 0$, $\sum_1 = 1$, $\sum_2 = 2$, $\sum_3 = 1$, $\sum_4 = 0$, $\sum_5 = 1$, $\sum_6 = 2$, $\sum_7 = 1$, $\sum_8 = 0$, $\sum_9 = 1$; hence by the above lemma, $t_0(\sigma) = 7$ and $t_1(\sigma) = 2$.

3.7. For $i \in I$ and $\mathbf{Y} = (Y_1,\dots,Y_N) \in \mathcal{Y}_N$ with $Y_r = \{y_{rk}\}_{k\geq 0}$, for $1 \leq r \leq N$, let $\mathbf{Y}(i) = (Y'_1,\dots,Y'_N)$, where for $1 \leq r \leq N$, $Y'_r = \{y'_{rk}\}_{k\geq 0}$, and for $k \geq 0$,

$$y'_{rk} = \begin{cases} y_{rk}+1 & \text{if } y_{rk} < y_{r(k+1)} \text{ and } y_{rk}+k+1 \equiv i \bmod n \\ y_{rk} & \text{otherwise} \end{cases}.$$

i.e. $\mathbf{Y}(i)$ is obtained from $\mathbf{Y}$ by removing all of its i-coloured convex corners.

3.8. Let $>$ be a total order on $\{1,\dots,N\} \times \mathbb{N} \times (-\mathbb{N})$ (see 3.11, 3.12, and 3.14 for examples). For $i \in I$, we now define operators $\tilde{e}_i$, $\tilde{f}_i : \mathcal{Y}_N \to \mathcal{Y}_N \cup \{0\}$ which, if $>$ is as in 3.11, coincide with the $\tilde{e}_i$ and $\tilde{f}_i$ acting on $\mathcal{Y}(N\Lambda_0) \subseteq \mathcal{Y}_N$ given in **[JMMO91]**.

Let $\mathbf{Y} = (Y_1,\dots,Y_N) \in \mathcal{Y}_N$ with $Y_r = \{y_{rk}\}_{k\geq 0}$ for $1 \leq r \leq N$.

Suppose $\mathbf{Y}(i)$ has m concave i-coloured corners and let their sites be $(r_1, k_1, y'_{r_1k_1}) > (r_2, k_2, y'_{r_2k_2}) > \cdots > (r_m, k_m, y'_{r_mk_m})$. Define $\sigma_i(\mathbf{Y})$ or simply $\sigma(\mathbf{Y}) = (\sigma_1,\dots,\sigma_m)$, where for $1 \le l \le m$,

$$\sigma_l = \begin{cases} 0 & \text{if } \mathbf{Y} \text{ has a concave corner at site } (r_l, k_l, y'_{r_l k_l}) \\ 1 & \text{if } \mathbf{Y} \text{ has a convex corner at site } (r_l, k_l+1, y_{r_l k_l}) = (r_l, k_l+1, y'_{r_l k_l} - 1). \end{cases}$$

(Here $\sigma(\mathbf{Y}) = (\ )$ if $m = 0$). Then $\mathbf{Y}$ is uniquely determined by $\mathbf{Y}(i)$ and $\sigma(\mathbf{Y})$ and we write $\mathbf{Y} = (\mathbf{Y}(i), \sigma(\mathbf{Y}))$. Define

$$\tilde{f}_i(\mathbf{Y}) = \begin{cases} (\mathbf{Y}(i), \tilde{f}(\sigma(\mathbf{Y}))) & \text{if } \tilde{f}(\sigma(\mathbf{Y})) \neq 0 \\ 0 & \text{if } \tilde{f}(\sigma(\mathbf{Y})) = 0 \end{cases}$$

and

$$\tilde{e}_i(\mathbf{Y}) = \begin{cases} (\mathbf{Y}(i), \tilde{e}(\sigma(\mathbf{Y}))) & \text{if } \tilde{e}(\sigma(\mathbf{Y})) \neq 0 \\ 0 & \text{if } \tilde{e}(\sigma(\mathbf{Y})) = 0 \end{cases}.$$

So if $\tilde{f}_i(\mathbf{Y}) \neq 0$ (resp. $\tilde{e}_i(\mathbf{Y}) \neq 0$), then $\tilde{f}_i(\mathbf{Y})$ (resp. $\tilde{e}_i(\mathbf{Y})$) is obtained from $\mathbf{Y}$ by adding (resp. removing) an i-coloured box. (See examples in 3.11.1, 3.12.1, and 3.14.1.)

3.9. Define maps wt $: \mathcal{Y}_N \to P$, for $i \in I$, $\varepsilon_i : \mathcal{Y}_N \to \mathbb{Z}$ and $\varphi_i : \mathcal{Y}_N \to \mathbb{Z}$ as follows: for $\mathbf{Y} \in \mathcal{Y}_N$ as in 3.7, $\mathrm{wt}(\mathbf{Y}) = N\Lambda_0 - \sum_{j=0}^{n-1} w_j \alpha_j$, where $w_j = \#\{p \in \mathbb{Z} :$ for some $r \in \{1, \dots, N\}$ and $k \in \mathbb{N}$, $y_{rk} < p \le 0$ and $p + k \equiv j \bmod n\}$. i.e. $w_j = \#$ of j-coloured boxes in $\mathbf{Y}$, $\varepsilon_i(\mathbf{Y}) = \max\{p \in \mathbb{N} : \tilde{e}_i^p(\mathbf{Y}) \neq 0\}$, and $\varphi_i(\mathbf{Y}) = \varepsilon_i(\mathbf{Y}) + \langle h_i, \mathrm{wt}(\mathbf{Y})\rangle$.

3.10. Proposition. *$\mathcal{Y}_N$ with wt, ε_i, $\tilde{e}_i$, $\tilde{f}_i$ for $i \in I$ as defined in 3.8 and 3.9 is a crystal.*

Proof. Let $\mathbf{Y} \in \mathcal{Y}_N$. If $\tilde{e}_i(\mathbf{Y}) \neq 0$, then $\mathrm{wt}(\tilde{e}_i(\mathbf{Y})) = \mathrm{wt}(\mathbf{Y}) + \alpha_i$, and if $\tilde{f}_i(\mathbf{Y}) \neq 0$, then $\mathrm{wt}(\tilde{f}_i(\mathbf{Y})) = \mathrm{wt}(\mathbf{Y}) - \alpha_i$. (See comment at the end of 3.8.)

Then (i)-(v) in 1.6 follow from the definitions.

□

Note: $\max\{p \in \mathbb{N} : \tilde{f}_i^p(\mathbf{Y}) \neq 0\} = \#$ of 0's in $J(\sigma(\mathbf{Y})) = \#$ of 1's in $J(\sigma(\mathbf{Y}))$ $+ < h_i, \mathrm{wt}\,(\mathbf{Y}) >= \varepsilon_i(\mathbf{Y}) + < h_i, \mathrm{wt}\,(\mathbf{Y}) >= \varphi_i(\mathbf{Y})$. (The second equality can be shown by induction on the height of the weight of $\mathbf{Y}$.)

3.11. Example. [**JMMO91**]. For $N \in \mathbb{N}$, define the total order $>$ on $\{1, \ldots, N\} \times \mathbb{N} \times (-\mathbb{N})$ as follows:

$$\begin{array}{lll} (r,k,y) > (r',k',y') & \text{iff} & k+y > k'+y' \\ & \text{or} & k+y = k'+y' \text{ and } r < r' \\ & \text{or} & k+y = k'+y', r = r', \text{ and } k > k'. \end{array}$$

(Note: we do not need to order (r,k,y) and (r',k',y') if $k+y = k'+y', r = r'$, and $k > k'$).

Then, for $i \in I$, the operators $\tilde{e}_i|_{\mathcal{Y}(N\Lambda_0)}$ and $\tilde{f}_i|_{\mathcal{Y}(N\Lambda_0)}$ defined in 3.8 coincide with those defined in [**JMMO91**].

3.11.1. **Example.** $(n = 3,\ N = 2)$.

$$\text{Let } \mathbf{Y} = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 & 0 \\ \cline{1-3} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right) \text{ and } > \text{ be as in 3.11.}$$

$$\text{Then } \mathbf{Y}(0) = \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right) \text{ and the sites of the } 0 - \text{coloured}$$

corners of $\mathbf{Y}(0)$ are: $(1,3,0) > (2,3,0) > (1,2,-2) > (2,1,-1) > (1,0,-3)$. So $\sigma_0(\mathbf{Y}) = (1,0,1,0,0)$,

$$\tilde{f}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 & 0 \\ \cline{1-3} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 \\ \cline{1-2} \end{array} \right), \text{ and}$$

$$\tilde{e}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right).$$

3.12. Example. (Tensor product) For $N \in \mathbb{N}$, define the total order $>$ on $\{1, \ldots, N\} \times \mathbb{N} \times (-\mathbb{N})$ as follows:

$$(r, k, y) > (r', k', y') \quad \text{iff} \quad r < r' \quad \text{or} \quad r = r' \text{ and } k \geq k'.$$

(Note: we do not need to order (r, k, y) and (r', k', y') if $r = r'$ and $k = k'$).
Note that if $N = 1$, $i \in I$ and $\mathbf{Y} \in \mathcal{Y}_1$, the ordering of the sites of the i-concave corners of $\mathbf{Y}$ given in 3.11 and in this section is the same.

Proposition. *The map defined by*

$$\begin{aligned} \mathcal{Y}_N \cup \{0\} : &\to \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_1 \cup \{0\} \\ (Y_1, \ldots, Y_N) : &\mapsto Y_1 \otimes \cdots \otimes Y_N \\ \textit{and} \quad 0 : &\mapsto 0 \end{aligned}$$

is an isomorphism of crystals.

(Here the crystal structure of $\mathcal{Y}_N$ and $\mathcal{Y}_1$ is as defined in 3.8 and 3.9 with $>$ as defined at the beginning of this section; and the crystal structure on $\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_1$ is as defined in 1.9.)

Proof. Let $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{Y}$. Then $\text{wt}(\mathbf{Y}) = \text{wt}(Y_1 \otimes \cdots \otimes Y_N)$. To show that this map preserves ε_i, φ_i, and that it commutes with $\tilde{f}_i$ and $\tilde{e}_i$, we use induction on N. If $N = 1$, we are done. So assume $N > 1$. Let $i \in I$. Then

$$\begin{aligned} \varepsilon_i(Y_1, \ldots, Y_N) &= \#\text{ of 1's in } J(\sigma(Y_1)) \\ &\quad + \max\{0, \#\text{ of 1's in } J(\sigma(Y_2, \ldots, Y_N)) - \#\text{ of 0's in } J(\sigma(Y_1))\} \\ &= \varepsilon_i(Y_1) + \max\{0, \varepsilon_i((Y_2, \ldots, Y_N)) - \varphi_i(Y_1)\} \\ &\qquad \text{(see note at the end of 3.10)} \\ &= \max\{\varepsilon_i(Y_1), \varepsilon_i((Y_2, \ldots, Y_N)) - < h_i, \text{wt}(Y_1) >\} \end{aligned}$$

$$= \max\{\varepsilon_i(Y_1), \varepsilon_i(Y_2 \otimes \cdots \otimes Y_N) - < h_i, \mathrm{wt}\,(Y_1) >\}, \quad \text{by induction}$$

$$= \varepsilon_i(Y_1 \otimes \cdots \otimes Y_N).$$

$$\begin{aligned}\varphi_i(Y_1, \dots, Y_N) &= \varepsilon_i(Y_1, \dots, Y_N) + < h_i, \mathrm{wt}\,(Y_1, \dots, Y_N) > \\ &= \varepsilon_i(Y_1 \otimes \cdots \otimes Y_N) + < h_i, \mathrm{wt}\,(Y_1 \otimes \cdots \otimes Y_N) > \\ &= \varphi_i(Y_1 \otimes \cdots \otimes Y_N).\end{aligned}$$

$$\tilde{f}_i(Y_1, \dots, Y_N) = \begin{cases} (\tilde{f}_i(Y_1), Y_2, \dots, Y_N) & \text{if \# of 0's in } J(\sigma(Y_1)) > \\ & \quad \text{\# of 1's in } J(\sigma(Y_2, \dots, Y_N)) \\ (Y_1, \tilde{f}_i(Y_2, \dots, Y_N)) & \text{if \# of 0's in } J(\sigma(Y_1)) \leq \\ & \quad \text{\# of 1's in } J(\sigma(Y_2, \dots, Y_N)) \\ & \quad \text{and \# of 0's in } J(\sigma(Y_2, \dots, Y_N)) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\tilde{f}_i(Y_1), Y_2, \dots, Y_N) & \text{if } \varphi_i(Y_1) > \varepsilon_i(Y_2, \dots, Y_N) \\ (Y_1, \tilde{f}_i(Y_2, \dots, Y_N)) & \text{if } \varphi_i(Y_1) \leq \varepsilon_i(Y_2, \dots, Y_N) \text{ and } \varphi_i(Y_2, \dots, Y_N) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{f}_i(Y_1 \otimes \cdots \otimes Y_N) = \begin{cases} \tilde{f}_i(Y_1) \otimes Y_2 \cdots \otimes Y_N & \text{if } \varphi_i(Y_1) > \varepsilon_i(Y_2, \dots, Y_N) \\ Y_1 \otimes \tilde{f}_i(Y_2 \otimes \cdots \otimes Y_N) & \text{if } \varphi_i(Y_1) \leq \varepsilon_i(Y_2, \dots, Y_N) \\ & \quad \text{and } \varphi_i(Y_2, \dots, Y_N) > 0 \\ 0 & \text{otherwise} \end{cases}$$

So by induction, $\tilde{f}_i(Y_1, \dots, Y_N) \mapsto \tilde{f}_i(Y_1 \otimes \cdots \otimes Y_N)$.

$$\tilde{e}_i(Y_1,\dots,Y_N) = \begin{cases} (\tilde{e}_i(Y_1), Y_2, \dots, Y_N) & \text{if } \varphi_i(Y_1) \geq \varepsilon_i(Y_2,\dots,Y_N) \text{ and } \varepsilon_i(Y_1) > 0 \\ (Y_1, \tilde{e}_i(Y_2,\dots,Y_N)) & \text{if } \varphi_i(Y_1) < \varepsilon_i(Y_2,\dots,Y_N) \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{e}_i(Y_1 \otimes \cdots \otimes Y_N) = \begin{cases} \tilde{e}_i(Y_1) \otimes Y_2 \cdots \otimes Y_N & \text{if } \varphi_i(Y_1) \geq \varepsilon_i(Y_2,\dots,Y_N) \text{ and } \varepsilon_i(Y_1) > 0 \\ Y_1 \otimes \tilde{e}_i(Y_2 \otimes \cdots \otimes Y_N) & \text{if } \varphi_i(Y_1) < \varepsilon_i(Y_2,\dots,Y_N) \\ 0 & \text{otherwise} \end{cases}$$

So by induction, $\tilde{e}_i(Y_1,\dots,Y_N) \mapsto \tilde{e}_i(Y_1 \otimes \cdots \otimes Y_N)$.

□

3.12.1. **Example.** Let $\mathbf{Y}$ be as in 3.11.1 and $>$ be as in 3.12. Then the sites of the 0-coloured corners of $\mathbf{Y}(0)$ are: $(1,3,0) > (1,2,-2) > (1,0,-3) > (2,3,0) > (2,1,-1)$. So $\sigma_0(\mathbf{Y}) = (1,1,0,0,0)$,

$$\tilde{f}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \\ \cline{1-3} 1 & 2 & 0 & \\ \cline{1-3} 0 & & & \\ \cline{1-1} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & & \\ \cline{1-1} \end{array} \right), \text{ and}$$

$$\tilde{e}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \\ \cline{1-3} 1 & 2 & & \\ \cline{1-2} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & & \\ \cline{1-1} \end{array} \right).$$

3.13. So $B(N\Lambda_0)$ can be described as a subset of $\mathcal{Y}_N$ (see **[JMMO91]**, Prop. 3.12) with the crystal structure given by the ordering $>$ in 3.11, and it can also be viewed as the connected component of $B(\Lambda_0) \otimes \cdots \otimes B(\Lambda_0) \subseteq \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_1 \simeq \mathcal{Y}_N$ ($\mathcal{Y}_N$ with the crystal structure given by $>$ in 3.12) containing the highest weight vector of weight $N\Lambda_0$. In 3.14 below we will define a third order $>$ on

$\{1,\dots,N\}\times\mathbb{N}\times(-\mathbb{N})$ which we will show will give us a third description of $B(N\Lambda_0)$ as a subset of $\mathcal{Y}_N$.

3.13.1. **Example.** Let $n=3$. If $>$ is as in 3.11, then

$$(\phi,\ \phi)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \phi\right)\xrightarrow{\tilde f_2}\left(\begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}\end{array}\,,\ \phi\right)$$
$$\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}1\\\cline{1-1}\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1}\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1}\end{array}\,,\ \boxed{0}\right),$$

and

$$(\phi,\ \phi)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \boxed{0}\right)\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\,,\ \boxed{0}\right)\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right)$$
$$\xrightarrow{\tilde f_2}\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right)\xrightarrow{\tilde f_2}\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\right).$$

3.13.2. **Example.** Let $n=3$. If $>$ is as in 3.12, then

$$(\phi,\ \phi)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \phi\right)\xrightarrow{\tilde f_2}\left(\begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}\end{array}\,,\ \phi\right)$$
$$\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2\\\cline{1-1}1\\\cline{1-1}\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1}\end{array}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1}0\\\cline{1-1}\end{array}\,,\ \phi\right),$$

and

$$(\phi,\ \phi)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \phi\right)\xrightarrow{\tilde f_0}\left(\boxed{0}\,,\ \boxed{0}\right)\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\,,\ \boxed{0}\right)$$
$$\xrightarrow{\tilde f_1}\left(\begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right)\xrightarrow{\tilde f_2}\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right)\xrightarrow{\tilde f_2}\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right).$$

$$\begin{array}{llll} \bullet 0 & \bullet 2 & \bullet 4 & \bullet 6 \\ \bullet 1 & \bullet 3 & \bullet 5 & \\ \bullet 2 & \bullet 4 & \bullet 6 & \\ \bullet 3 & \bullet 5 & & \\ \bullet 4 & \bullet 6 & & \\ \bullet 5 & & & \\ \bullet 6 & & & \end{array} \quad , \quad \begin{array}{llll} \bullet 0 & \bullet 2 & \bullet 4 & \bullet 6 \\ \bullet 1 & \bullet 3 & \bullet 5 & \\ \bullet 2 & \bullet 4 & \bullet 6 & \\ \bullet 3 & \bullet 5 & & \\ \bullet 4 & \bullet 6 & & \\ \bullet 5 & & & \\ \bullet 6 & & & \end{array}$$

FIGURE 1. ($n = 3$, $N = 2$). The number beside the dot representing $(r, k, y) \in \{1, \dots, N\} \times \mathbb{N} \times (-\mathbb{N})$ is $(n-1)k - y$.

3.13.3. **Example.** Let $n = 3$. If $>$ is as in 3.14 below, then

$$(\phi,\ \phi) \xrightarrow{\tilde f_0} \left(\boxed{0},\ \phi\right) \xrightarrow{\tilde f_2} \left(\begin{array}{|c|}\hline 0 \\ \hline 2 \\ \hline\end{array},\ \phi\right) \xrightarrow{\tilde f_1} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array},\ \phi\right)$$

$$\xrightarrow{\tilde f_1} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & \\ \cline{1-1} 1 & \\ \cline{1-1}\end{array},\ \phi\right) \xrightarrow{\tilde f_0} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & \\ \cline{1-1}\end{array},\ \phi\right) \xrightarrow{\tilde f_0} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & \\ \cline{1-1} 0 & \\ \cline{1-1}\end{array},\ \phi\right), \text{ and}$$

$$(\phi,\ \phi) \xrightarrow{\tilde f_0} \left(\boxed{0},\ \phi\right) \xrightarrow{\tilde f_0} \left(\boxed{0},\ \boxed{0}\right) \xrightarrow{\tilde f_1} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array},\ \boxed{0}\right)$$

$$\xrightarrow{\tilde f_1} \left(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}\right) \xrightarrow{\tilde f_2} \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}\right) \xrightarrow{\tilde f_2} \left(\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline\end{array},\ \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline\end{array}\right).$$

3.14. For $N \in \mathbb{N}$, define the total order on $\{1, \dots, N\} \times \mathbb{N} \times (-\mathbb{N})$ as follows:

$$\begin{array}{lll} (r, k, y) > (r', k', y') & \text{iff} & (n-1)(k-k') - (y-y') > 0 \\ & \text{or} & (n-1)(k-k') - (y-y') = 0 \text{ and } r < r' \\ & \text{or} & (n-1)(k-k') - (y-y') = 0, r = r' \text{ and } k > k'. \end{array}$$

(See Fig. 1)

For the rest of this Chapter, $\mathcal{Y}_N$ will denote the crystal defined in 3.10 with this order.

3.14.1. **Example.** Let $\mathbf{Y}$ be as in 3.11.1 and $>$ be as in 3.14. Then the sites of the 0-coloured corners of $\mathbf{Y}(0)$ are: $(1,3,0) > (1,2,-2) > (2,3,0) > (1,0,-3) > (2,1,-1)$. So $\sigma_0(\mathbf{Y}) = (1,1,0,0,0)$,

$$\tilde{f}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 2 & 0 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \; , \; \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \right), \text{ and}$$

$$\tilde{e}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \; , \; \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right).$$

3.15. Let $i \in I$ and $j \in \mathbb{N}$. We say that $(r,k,-j+(n-1)k)$ lies on the $\mathbf{j^{th}}$**-stair** for any $r \in \{1,\dots,N\}$ and $0 \le k \le \lfloor \frac{j}{n-1} \rfloor$ (see Fig.1 - the dots labelled j lie on the j^{th} stair), and we say say that $(r,k,i-jn+(n-1)k)$ lies on the $\mathbf{j^{th}}$ **i−stair** for any $r \in \{1,\dots,N\}$ and $0 \le k \le \lfloor \frac{jn-i}{n-1} \rfloor$, (see Fig.1 - the dots labelled by 3 lie on the 1^{st} 0−stair and those labelled by 5 lie on the 2^{nd} 1−stair.)

Let $\mathbf{Y} \in \mathcal{Y}_N$ and $i \in I$. Then $\sigma(\mathbf{Y}) = (\dots,\lambda_1,\lambda_0)$ where for $j \ge 0$, λ_j is the part of $\sigma(\mathbf{Y})$ coming from the concave i-coloured corners of $\mathbf{Y}(i)$ which have sites lying on the j^{th} i−stair. (Note: λ_0 is empty if $i \neq 0$; and for some l, λ_j is empty for all $j > l$.)

For $y \in -\mathbb{N}$, define

$$S(y) := (y, y+(n-1), y+2(n-1), \dots, y + \lfloor \frac{-y}{n-1} \rfloor (n-1), 0, 0, \dots)$$

and

$$\overline{S}(y) := (y+(n-2), y+(n-2), y+2(n-2), y+2(n-2),$$

$$\ldots, y + \lfloor \frac{-y}{n-2} \rfloor (n-2), y + \lfloor \frac{-y}{n-2} \rfloor (n-2), 0, 0, \ldots).$$

3.16. Let $\mathcal{B}_N = \{\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{Y}_N : Y_j = \{y_{jk}\}_{k \geq 0},\ 1 \leq j \leq N$ and

$\mathbf{Y}$ satisfies (i), (ii) and (iii) below $\}$.

(i) $y_{jk} + (n-1) \geq y_{j(k+1)} \geq y_{jk}$ for all $k \geq 0$ and for all $1 \leq j \leq N-1$;

(ii) $Y_1 \supseteq \cdots \supseteq Y_N$ (i.e. $y_{rk} \leq y_{r+1\,k}$ for all $k \in \mathbb{N}$ and $1 \leq r < N$);

(iii) for each $r, s \in \{1, \ldots, N\}$ with $r < s$ and $k \in \mathbb{N}$, there exists an $a(r,s,k,\mathbf{Y}) \in \mathbb{N}_{\geq k}$ (we write $a(r,s,k)$ for $a(r,s,k,\mathbf{Y})$ if $\mathbf{Y}$ is understood) such that

(a) for $k \leq b \leq a(r,s,k)$, $y_{rk} + (n-1)(b-k) \leq y_{sb}$; i.e. the subtableaux of Y_s formed from its k^{th} to $a(r,s,k)^{th}$ columns is contained in $S(y_{rk})$ - (see 3.15).

(b) for $b \geq 1$, $y_{rk} + (n-1)(a(r,s,k) - k) + (n-2)b \geq y_{s(a(r,s,k)+2b)}$; i.e., the subtableaux of Y_s formed from its $(a(r,s,k)+1)^{st}$ to last columns contains $\overline{S}(y_{rk} + (n-1)(a(r,s,k) - k))$.

We will show that $\mathcal{B}_N$ is isomorphic to $B(N\Lambda_0)$.

3.16.1. **Example.**

Let $\mathbf{Y}_1 = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 & 0 \\ \cline{1-3} \end{array} \ , \ \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right)$, $\mathbf{Y}_2 = \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right)$,

$\mathbf{Y}_3 = \left(\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \ \ \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} \right)$, and $\mathbf{Y}_4 = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array} \ , \ \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} \end{array} \right)$.

Then $\mathbf{Y}_1 \in \mathcal{B}_2$, but 3.16 (i), (ii), and (iii) are not satisfied by $\mathbf{Y}_2$, $\mathbf{Y}_3$, and $\mathbf{Y}_4$, respectively.

3.17. Let $\iota' = (j_m, \dots, j_2, j_1)$ be a sequence of elements of $I\backslash\{0\}$ such that $r_{j_m}\cdots r_{j_1}$ is a reduced expression of the longest word of the Weyl group of $\mathfrak{sl}(n)$. Define

$B_{\iota'} := B_{j_m} \otimes \cdots \otimes B_{j_1}$. We will denote by $\overline{0}$ the element $b_{j_m}(0) \otimes \cdots \otimes b_{j_1}(0)$ of $B_{\iota'}$. Let $\iota = (\dots, 1, \dots, n-1, 0, 1, \dots, n-1, 0)$ and $B_\iota = \cdots \otimes B_0 \otimes B_1 \otimes \cdots \otimes B_{n-1} \otimes B_0 \otimes B_1 \otimes \cdots \otimes B_{n-1} \otimes B_0$.

Define the map $\Phi : \mathcal{B}_N \cup \{0\} \to \{u_\infty\} \otimes B_\iota \otimes B_{\iota'} \otimes T_{N\Lambda_0} \cup \{0\}$ as follows:

For $\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{B}_N$ with $Y_r = \{y_{rk}\}_{k\geq 0}$, $1 \leq r \leq N$,

$$\Phi(\mathbf{Y}) := u_\infty \otimes \cdots \otimes b_0(-a_{2n}) \otimes \cdots \otimes b_{n-1}(-a_{n+1}) \otimes b_0(-a_n) \otimes b_1(-a_{n-1}) \otimes \cdots$$
$$\cdots \otimes b_{n-1}(-a_1) \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0}$$

where for $s \in \mathbb{N}$,

$$a_s := a_s(\mathbf{Y}) := \#\{(r,k) : 0 \leq k \leq \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} < -s + k(n-1)\}$$
$$= \#\text{ of boxes in } \mathbf{Y} \text{ whose upper left hand corner lies on the } s^{th} \text{ stair}$$

(and $\Phi(0) = 0$).

To show that Φ is a crystal morphism we first show how the action of $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbf{Y}$ can be described by the a_s's, $s \in \mathbb{N}$ (see 3.21). We first need the following Lemmas.

3.17.1. **Example.** Let $n = 3$. Then

$$\Phi\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1} 0\\\cline{1-1}\end{array}\ ,\ \phi\right) = u_\infty \otimes (\dots,0,0,2,2,1,1)\otimes \bar{0}\otimes t_{2\Lambda_0}$$

$$\Phi\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array}\right) = u_\infty \otimes (\dots,0,0,2,0,2,0,2)\otimes \bar{0}\otimes t_{2\Lambda_0}$$

$$\Phi\left(\begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline 1\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 0\\\hline\end{array}\right) = u_\infty \otimes (\dots,0,0,1,2,1,2)\otimes \bar{0}\otimes t_{2\Lambda_0}$$

$$\Phi\left(\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 2\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\right) = u_\infty \otimes (\dots,0,0,1,0,2,1,2)\otimes \bar{0}\otimes t_{2\Lambda_0}$$

$$\begin{aligned}
\tilde{f}_2^2\,\tilde{f}_1^2\,\tilde{f}_0^2(u_\infty\otimes\bar{0}\otimes\bar{0}\otimes t_{2\Lambda_0}) &= u_\infty\otimes(\dots,0,0,2,0,2,0,2)\otimes\bar{0}\otimes t_{2\Lambda_0}\\
\tilde{f}_0^2\,\tilde{f}_1^2\,\tilde{f}_2\tilde{f}_0(u_\infty\otimes\bar{0}\otimes\bar{0}\otimes t_{2\Lambda_0}) &= u_\infty\otimes(\dots,0,0,2,2,1,1)\otimes\bar{0}\otimes t_{2\Lambda_0}
\end{aligned}$$

So if $>$ is as in 3.11 or 3.12, Φ is not a crystal morphism. (See Examples 3.13.1, 3.13.2 and 3.13.3.)

3.18. Lemma. *Let $i \in I$ and $\mathbf{Y} \in \mathcal{B}_N$. If $\sigma(\mathbf{Y}) = (\dots,\lambda_1,\lambda_0)$ where for $j \geq 0$, λ_j is the part of $\sigma(\mathbf{Y})$ coming from the j^{th} i−stair, then $\lambda_j = (1,\dots,1,0,\dots,0)$ (possibly empty).*

Proof. For $j \geq 0$, let $\lambda_j = (\lambda_{j1},\dots,\lambda_{jN})$, where for $1 \leq r \leq N$, λ_{jr} = part of λ_j coming from the j^{th} i−stair of Y_r. By 3.16 (i), $\lambda_{jr} = (1,\dots,1,0,\dots,0)$ (possibly empty). We now show that if for some $r \in \{1,\dots,N\}$, λ_{jr} has a zero, then for all $s > r$, $\lambda_{js} = (0,\dots,0)$ (possibly empty). This will prove the Lemma.

Suppose that for some $s > r$, λ_{js} has a one. Let (t, y_{rt}) be the site of the concave i–coloured corner corresponding to a zero in λ_{jr}, and $(t'+1, y_{st'})$ the site of the convex i– coloured corner corresponding to a one in λ_{js}. By 3.16 (i) and (ii), $t < t'$ and by 3.16 (iii(a)), since $y_{st'} = y_{rt} + (n-1)(t'-t) - 1 < y_{rt} + (n-1)(t'-t)$, we have $t' > a(r,s,t)$. Let $b = t' - a(r,s,t)$. Then

$$\begin{aligned} y_{s(t'+1)} &\leq y_{s(2t'-a(r,s,t))} \quad \text{by 3.16 (i), since } t'+1 \leq 2t' - a(r,s,t) \\ &= y_{s(a(r,s,t)+2b)} \\ &\leq y_{rt} + (n-1)(a(r,s,t)-t) + (n-2)b \quad \text{by 3.16 (iii(b))} \\ &= (y_{rt}+(n-1)(t'-t)-1)+(n-1)(a(r,s,t)-t')+1-(n-2)(a(r,s,t)-t') \\ &= y_{st'} + a(r,s,t) - t' + 1 \\ &\leq y_{st'}. \end{aligned}$$

This is a contradiction since Y_s was assumed to have a convex corner at site $(t', y_{st'})$, so $y_{st'} < y_{st'+1}$.

□

3.18.1. **Example.** Let $i = 0$ and $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, and $\mathbf{Y}_4$ be as in 3.16.1. If $\mathbf{Y} = \mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, or $\mathbf{Y}_4$, and λ_2 is the part of $\sigma(\mathbf{Y})$ coming from the 2^{nd} 0-stair, then $\lambda_2 = (1,1,0), (0,1,0), (0,0,1)$ or $(1,0,1)$, respectively.

3.19. Lemma. *Let $i \in I$, $\mathbf{Y} \in \mathcal{B}_N$ and $\sigma(\mathbf{Y}) = (\lambda_l, \ldots, \lambda_1, \lambda_0)$ where for $0 \leq j \leq l$, λ_j is defined as in 3.15. For $j \geq 0$, let $1_j = \#$ of 1's in λ_j, $1_{-1} := 0$, and $0_j = \#$ of 0's in λ_j; and for $k \geq -1$, define $A_k := \sum_{j=k}^{l} 1_j - 0_{j+1}$ (Note: $0_{l+1} := 0$).*

Then for $t \in [-1, l]$,

$$\tilde{f}_i(\mathbf{Y}) = \begin{cases} (\mathbf{Y}(i), (\lambda_l, \ldots, \tilde{f}(\lambda_t), \ldots, \lambda_0)) & \text{if } t > -1, \\ 0 & \text{if } t = -1 \end{cases}$$

$$\iff A_t \geq A_k \text{ for all } k \geq t \text{ and } A_t > A_k \text{ for all } -1 \leq k < t,$$

and for $t \in [0, l+1]$

$$\tilde{e}_i(\mathbf{Y}) = \begin{cases} (\mathbf{Y}(i), (\lambda_l, \ldots, \tilde{e}(\lambda_t), \ldots, \lambda_0)) & \text{if } t < l+1, \\ 0 & \text{if } t = l+1 \end{cases}$$

$$\iff A_t > A_k \text{ for all } k > t \text{ and } A_t \geq A_k \text{ for all } 0 \leq k \leq t.$$

Proof. By Lemma 3.18, for $0 \leq j \leq l$, $\lambda_j = (1, \ldots, 1, 0, \ldots, 0)$ (possibly empty). Applying Lemma 3.6 to $\sigma(\mathbf{Y})$, we see that $t_1(\sigma(\mathbf{Y}))$ (as well as $t_0(\sigma(\mathbf{Y})) - 1$) corresponds to the rightmost 1 appearing in one of the λ_j 's. So by this Lemma,

$$\tilde{f}(\sigma(\mathbf{Y})) = \begin{cases} (\lambda_l, \ldots, \tilde{f}(\lambda_t), \ldots, \lambda_0) & \text{if } t > -1, \\ 0 & \text{if } t = -1 \end{cases}$$

$$\iff A_t \geq A_k \text{ for all } k \geq t \text{ and } A_t > A_k \text{ for all } -1 \leq k < t,$$

and

$$\tilde{e}(\sigma(\mathbf{Y})) = \begin{cases} (\lambda_l, \ldots, \tilde{e}(\lambda_t), \ldots, \lambda_0) & \text{if } t < l+1, \\ 0 & \text{if } t = l+1 \end{cases}$$

$$\iff A_t > A_k \text{ for all } k > t \text{ and } A_t \geq A_k \text{ for all } 0 \leq k \leq t.$$

So we have our result.

□

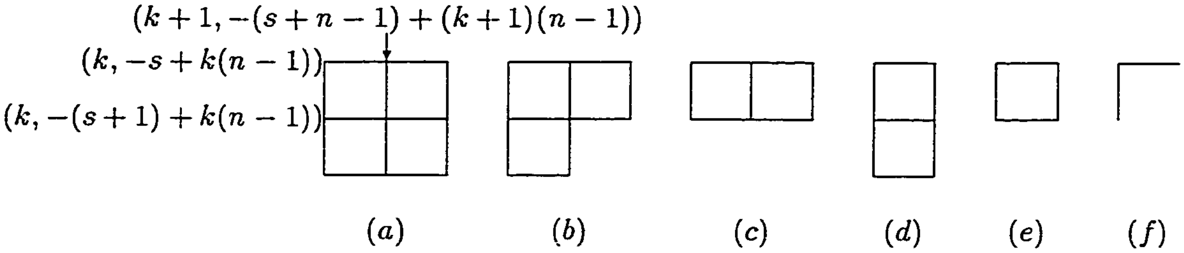

FIGURE 2. If $Y \in \mathcal{B}_1$ intersected with figure (a) above equals figure (a), (b), (c), (d), (e), or (f), this part of Y contributes a $0, -1, 0, 0, 1$, or 0, resp. to $a_s - a_{s+1} - a_{s+n-1} + a_{s+n}$.

3.19.1. **Example.** Let $n = 3$, $N = 2$ and

$$\mathbf{Y} = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 & 0 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array} \, , \quad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right).$$

If $i = 0$, then $\sigma(\mathbf{Y}) = (\lambda_2, \lambda_1, \lambda_0)$ where $\lambda_0 = (\ )$, $\lambda_1 = (1, 0)$, and $\lambda_2 = (1, 1, 0)$; and $A_{-1} = 1$, $A_0 = 1$, $A_1 = 2$, $A_2 = 2$, $A_3 = 0$,

$$\tilde{f}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 & 0 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array} \, , \quad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 \\ \cline{1-2} \end{array} \right) \text{ and}$$

$$\tilde{e}_0(\mathbf{Y}) = \left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array} \, , \quad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \right).$$

3.20. Lemma. *Let the notation be as in Lemma 3.19 and a_s, $s \in \mathbb{N}$ as in 3.17. For $s \in \mathbb{N}$, let $i \in I$ and $j \in \mathbb{N}$ be such that $s = -i + jn$. Then*

$$a_s - a_{s+1} - a_{s+n-1} + a_{s+n} = 1_j - 0_{j+1}.$$

(See Figure 2.)

Proof.

$$
\begin{aligned}
& a_s - a_{s+1} - a_{s+n-1} + a_{s+n} \\
& = \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} < -s + k(n-1)\} \\
& \quad -\#\{(r,k) : 0 \le k \le \lfloor \frac{s+1}{n-1} \rfloor \text{ and } y_{rk} < -(s+1) + k(n-1)\} \\
& \quad -\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor \text{ and } y_{rk} < -(s+n-1) + k(n-1)\} \\
& \quad +\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n}{n-1} \rfloor \text{ and } y_{rk} < -(s+n) + k(n-1)\}
\end{aligned}
$$

$$
\begin{aligned}
& = \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} = -s - 1 + k(n-1)\} \\
& \quad +\#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} \le -s - 2 + k(n-1)\} \\
& \quad -\#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} \le -(s+1) - 1 + k(n-1)\} \\
& \quad -\#\{(r,k) : k = \frac{s+1}{n-1} \in \mathbb{N}, y_{rk} \le -1\} \\
& \quad -\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor \text{ and } y_{rk} = -(s+n-1) - 1 + k(n-1)\} \\
& \quad -\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor \text{ and } y_{rk} \le -(s+n-1) - 2 + k(n-1)\} \\
& \quad +\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor \text{ and } y_{rk} \le -(s+n) - 1 + k(n-1)\} \\
& \quad +\#\{(r,k) : k = \frac{s+n}{n-1} \in \mathbb{N},\ y_{rk} \le -1\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor, y_{rk} = -s-1+k(n-1) \text{ and } y_{rk} < y_{rk+1}\} \\
& + \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} = y_{rk+1} = -s-1+k(n-1)\} \\
& - \#\{(r,k) : k = \frac{s+1}{n-1} \in \mathbb{N} \text{ and } y_{rk} \le -1\} \\
& - \#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor, y_{rk} = -(s+n)+k(n-1), \\
& \qquad \text{and } y_{r(k-1)} < y_{rk} \text{ or } k = 0\} \\
& - \#\{(r,k) : 1 \le k \le \lfloor \frac{s}{n-1} \rfloor + 1, y_{r(k-1)} = y_{rk} = -(s+n)+k(n-1)\} \\
& + \#\{(r,k) : k = \frac{s+n}{n-1} \in \mathbb{N}, \; y_{rk} \le -1\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor, y_{rk} = -s-1+k(n-1) \text{ and } y_{rk} < y_{rk+1}\} \\
& + \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor \text{ and } y_{rk} = y_{rk+1} = -s-1+k(n-1)\} \\
& - \#\{(r,k) : 0 \le k \le \lfloor \frac{s+n-1}{n-1} \rfloor, \; y_{rk} = -(s+n)+k(n-1), \\
& \qquad \text{and } y_{rk-1} < y_{rk} \text{ or } k = 0\} \\
& - \#\{(r,k) : 1 \le k \le \lfloor \frac{s}{n-1} \rfloor + 1, y_{r(k-1)} = y_{rk} = -s-1+(k-1)(n-1)\} \\
& - \#\{(r,k) : k = \frac{s+n}{n-1} \in \mathbb{N}, \; y_{rk} \le -1 \text{ and } y_{r(k+1)} = 0\}
\end{aligned}
$$

$$= \#\{(r,k) : 0 \le k \le \lfloor \frac{s}{n-1} \rfloor,\ y_{rk} = -s-1+k(n-1) \text{ and } y_{rk} < y_{rk+1}\}$$
$$-\#\{(r,k) : 0 \le k \le \lfloor \frac{s+n}{n-1} \rfloor,\ y_{rk} = -(s+n)+k(n-1)$$
$$\text{and } y_{rk-1} < y_{rk} \text{ or } k = 0\}$$
$$= 1_j - 0_{j+1} \qquad (\text{Recall } s = -i+jn.)$$

□

3.21. Proposition. *Let $i \in I$, $\mathbf{Y} \in \mathcal{B}_N$ and $\sigma(\mathbf{Y}) = (\lambda_l, \dots, \lambda_1, \lambda_0)$ where for $0 \le j \le l$, λ_j is defined as in 3.15. For $j \in \mathbb{N}$, let a_j be as in 3.17. Define $a_j := 0$ if $0 > j > -n$. Then for $t \in [0,l]$,*

$$\tilde{f}_i(\mathbf{Y}) = (\mathbf{Y}(i), (\lambda_l, \dots, \tilde{f}(\lambda_t), \dots, \lambda_0))$$

if and only if

$$\sum_{j=t}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \ge$$
$$\sum_{j=k}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \text{ for all } k \ge t$$

and

$$\sum_{j=t}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} >$$
$$\sum_{j=k}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \text{ for all } 0 \le k < t,$$

and for $t \in [0, l+1]$,

$$\tilde{e}_i(\mathbf{Y}) = \begin{cases} (\mathbf{Y}(i), (\lambda_l, \dots, \tilde{f}(\lambda_t), \dots, \lambda_0)) & \text{if } t < l+1 \\ 0 & \text{if } t = l+1 \end{cases}$$

if and only if

$$\sum_{j=t}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} >$$

$$\sum_{j=k}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \text{ for all } k > t$$

and

$$\sum_{j=t}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \geq$$

$$\sum_{j=k}^{l} a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n} \text{ for all } 0 \leq k \leq t.$$

Proof.

$$1_0 - 0_1 = \begin{cases} a_0 - a_1 - a_{n-1} + a_n & \text{if } i = 0 \quad \text{(by Lemma 3.20)} \\ -0_1 & \text{if } i \neq 0 \quad \text{(see Note in 3.15)} \end{cases}$$

$$= \begin{cases} a_0 - a_1 - a_{n-1} + a_n & \text{if } i = 0 \\ a_{n-i} - a_{n-i-1} & \text{if } i \neq 0 \text{ and } i \neq 1 \\ a_{n-1} - a_{n-2} - a_0 & \text{if } i = 1 \end{cases}$$

$$= a_{-i} - a_{-i+1} - a_{-i+(n-1)} + a_{-i+n}.$$

So by Lemma 3.20 and 3.19, we have our result.

□

3.22. Lemma. *Let $i \in I$, B_ι and $B_{\iota'}$ be as in 3.17, and $(\dots, a_2, a_1, a_0) \in B_\iota$. If $i = 0$ let $j \in \mathbb{N}$ and if $i \neq 0$ let $j \in \mathbb{N}_{\geq 1}$, then define*

$$d_j := a_{-i+jn} - a_{-i+jn+1} - a_{-i+jn+n-1} + a_{-i+jn+n},$$

$$d_0 = \begin{cases} -a_{-i+(n-1)} + a_{-i+n} & \text{if } i \neq 1 \text{ and } i \neq 0 \\ -a_0 - a_{n-2} + a_{n-1} & \text{if } i = 1, \end{cases}$$

$A_\infty := 0$, and for $k \in \mathbb{N}$, $A_k := \sum_{j \geq k} d_j$. Then

$$\tilde{e}_i(u_\infty \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0})$$
$$= \begin{cases} \tilde{e}_i(u_\infty) \otimes \cdots = 0 \\ u_\infty \otimes \cdots \otimes \tilde{e}_i(b_i(-a_t)) \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0} \end{cases}$$

$$\iff \begin{cases} A_\infty = 0 \geq A_k \text{ for all } k \geq 0 \\ A_t > A_k \text{ for all } k > t \text{ and } A_t \geq A_k \text{ for all } 0 \leq k \leq t, \end{cases}$$

and

$$\tilde{f}_i(u_\infty \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0})$$
$$= \begin{cases} (u_\infty) \otimes \cdots \otimes \tilde{f}_i(b_i(-a_t)) \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0} \\ u_\infty \otimes \cdots \otimes b_0(-a_0) \otimes \tilde{f}_i(\overline{0}) \otimes t_{N\Lambda_0} \end{cases}$$

$$\iff \begin{cases} A_t \geq A_k \text{ for all } k \geq t \text{ and } A_t > A_k \text{ for all } 0 \leq k < t \\ A_0 \geq A_k \text{ for all } k \geq 1 \qquad (\text{Note: in this case } i \neq 0) \end{cases}$$

Proof. For $k \geq 1$ (or for $k \geq 0$ if $i = 0$),

$$A_k = a_{-i+kn} - \sum_{l \geq k} a_{-i+ln+1} - \sum_{l \geq k} a_{-i+ln+n-1} + \sum_{l \geq k} 2a_{-i+(l+1)n}$$
$$= \varepsilon_i(b_i(-a_{-i+kn})) + \sum_{l \geq k} \langle h_i, a_{-i+ln+1}\, \alpha_{i-1 \bmod (n)} \rangle$$

$$+\begin{cases}\sum_{l\geq k}\langle h_i, a_{-i+ln+n-1}\ \alpha_{i+1 \bmod(n)}\rangle & \text{if } n\neq 2\\ 0 & \text{if } n=2\end{cases}$$

$$+\sum_{l\geq k}\langle h_i, a_{-i+(l+1)n}\ \alpha_i\rangle$$

$$=\varepsilon_i(b_i(-a_{-i+kn}))-\sum_{j>-i+kn}\langle h_i, \mathrm{wt}(b_{-j \bmod(n)}(-a_j))\rangle.$$

In the last equality we used that for $l\in I$, $\mathrm{wt}(b_l(-a))=-a\alpha_l$ and that $\langle h_i,\alpha_l\rangle=0$ if $l\neq i-1$, i or $i+1$.

Similarly, we can show that $A_0=\varepsilon_i(\bar{0})-\sum_{j\geq 0}\langle h_i, \mathrm{wt}(b_{-j \bmod(n)}(-a_j))\rangle$.

Therefore, since $A_\infty=0=\varepsilon_i(u_\infty)$, $\varepsilon_i(t_{N\Lambda_0})=-\infty$ and $\varepsilon_i(b_l(-a))=-\infty$ if $l\neq i$, then this Lemma follows from the definition of $\{u_\infty\}\otimes B_\iota\otimes B_{\iota'}\otimes T_{N\Lambda_0}$ in 1.16.

□

3.23. Theorem. *The map $\Phi:\mathcal{B}_N\cup\{0\}\to\{u_\infty\}\otimes B_\iota\otimes B_{\iota'}\otimes T_{N\Lambda_0}\cup\{0\}$ defined in 3.17 is a full embedding of crystals. Furthermore, if $\mathbf{Y}\in\mathcal{B}_N$, $\Phi(\mathbf{Y})=u_\infty\otimes\cdots\otimes b_0(-a_0)\otimes\bar{0}\otimes t_{N\Lambda_0}$ and $i\in I$,*

$$\tilde{f}_i(\Phi(\mathbf{Y}))=u_\infty\otimes\cdots\otimes\tilde{f}_i(b_i(-a_t))\otimes\cdots\otimes\bar{0}\otimes t_{N\Lambda_0}\iff\tilde{f}_i(\mathbf{Y})\neq 0,$$

and in this case $\Phi(\tilde{f}_i(\mathbf{Y}))=\tilde{f}_i(\Phi(\mathbf{Y}))$.

Proof. Let $\mathbf{Y}\in\mathcal{B}_N$. We first show that Φ and $\tilde{e}_i$, for $i\in I$, commute. Let $\Phi(\mathbf{Y})=u_\infty\otimes\cdots\otimes b_0(-a_0)\otimes\bar{0}\otimes t_{N\Lambda_0}$. Then by Prop. 3.21 and Lemma 3.22, if $(\lambda_l,\ldots,\lambda_0)$ is as in Prop. 3.21,

$$\tilde{e}_i(\Phi(\mathbf{Y}))=\begin{cases}0\\ u_\infty\otimes\cdots\otimes b_i(-a_t+1)\otimes\cdots\otimes b_0(-a_0)\otimes\bar{0}\otimes t_{N\Lambda_0}\end{cases}$$

$$\iff \qquad \tilde{e}_i(\mathbf{Y}) = \begin{cases} 0 \\ (\mathbf{Y}(i), (\lambda_l, \ldots, \tilde{e}(\lambda_t), \ldots, \lambda_0)). \end{cases}$$

So $\Phi_i(\tilde{e}_i(\mathbf{Y})) = \tilde{e}_i(\Phi(\mathbf{Y}))$.

Now assume that $\tilde{f}_i(\Phi(\mathbf{Y})) = u_\infty \otimes \cdots \otimes \tilde{f}_i(b_i(-a_t)) \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0}$. Then again by Prop. 3.21 and Lemma 3.22, if $(\lambda_l, \ldots, \lambda_0)$ is as in Prop. 3.21,

$$\tilde{f}_i(\Phi(\mathbf{Y})) = u_\infty \otimes \cdots \otimes b_i(-a_t - 1) \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0}$$

$$\iff \tilde{f}_i(\mathbf{Y}) = (\mathbf{Y}(i), (\lambda_l, \ldots, \tilde{f}_i(\lambda_t), \ldots, \lambda_0))$$

$$\iff \tilde{f}_i(\mathbf{Y}) \neq 0.$$

So $\Phi(\tilde{f}_i(\mathbf{Y})) = \tilde{f}_i(\Phi(\mathbf{Y}))$.

We now show that Φ is injective. Let $\mathbf{Y}_1$ and $\mathbf{Y}_2 \in \mathcal{B}_N$ such that $\Phi(\mathbf{Y}_1) = \Phi(\mathbf{Y}_2) = u_\infty \otimes \cdots \otimes b_0(-a_0) \otimes \overline{0} \otimes t_{N\Lambda_0}$. If $\mathrm{wt}\,(\Phi(\mathbf{Y}_1)) = N\Lambda_0$, then $\Phi(\mathbf{Y}_1) = u_\infty \otimes \cdots \otimes b_0(0) \otimes \overline{0} \otimes t_{N\Lambda_0}$. So $\mathbf{Y}_1 = \mathbf{Y}_2 = (\phi, \phi, \ldots, \phi)$. If $\mathrm{wt}\,(\Phi(\mathbf{Y}_1)) \neq N\Lambda_0$, there exists an $a_k > 0$ such that $\Phi(\mathbf{Y}_1) = u_\infty \otimes \cdots \otimes b_j(-a_k) \otimes \cdots \otimes \overline{0} \otimes t_{N\Lambda_0}$. Choose the biggest such k. Then

$$\varepsilon_j(\cdots \otimes b_j(-a_k) \otimes \cdots \otimes \overline{0} \otimes t_{N\Lambda_0}) \geq \varepsilon_j(b_j(-a_k)) = a_k > 0 = \phi_j(u_\infty).$$

So $\tilde{e}_j(\Phi(\mathbf{Y}_1)) \neq 0$, (recall that $\varepsilon_j(t_{N\Lambda_0}) = -\infty$) and

$$\Phi(\tilde{e}_j(\mathbf{Y}_1)) = \tilde{e}_j(\Phi(\mathbf{Y}_1)) = \tilde{e}_j(\Phi(\mathbf{Y}_2)) = \Phi(\tilde{e}_j(\mathbf{Y}_2)) \neq 0.$$

So since $\mathrm{wt}\,(\tilde{e}_j(\mathbf{Y}_1)) = \mathrm{wt}\,(\mathbf{Y}_1) + \alpha_j$, by induction we have that $\tilde{e}_j(\mathbf{Y}_1) = \tilde{e}_j(\mathbf{Y}_2) \neq 0$. Hence $\mathbf{Y}_1 = \mathbf{Y}_2$.

Now let $\mathbf{Y} \in \mathcal{B}_n$. Then $\mathrm{wt}(\mathbf{Y}) = \mathrm{wt}(\Phi(\mathbf{Y}))$ by definition (see 3.9, 3.17 and 1.7). Let $i \in I$. Then

$$\varepsilon_i(\mathbf{Y}) := \max\{p \in \mathbb{N} : \tilde{e}_i^p(\mathbf{Y}) \neq 0\} \qquad \text{(by definition)}$$

$$= \max\{p \in \mathbb{N} : \tilde{e}_i^p(\Phi(\mathbf{Y})) \neq 0\} \text{(since } \tilde{e}_i \text{ commutes with } \Phi \text{ and } \Phi \text{ is injective)}$$

$$= \varepsilon_i(\Phi(\mathbf{Y})) \qquad \text{(see 1.16)}$$

From (i) in 1.6, $\varphi_i(\mathbf{Y}) = \varphi_i(\Phi(\mathbf{Y}))$.

□

3.24. We will now show that the image of Φ equals the image of $(\psi_{\iota,\iota'} \otimes id_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$ (see 2.5). To do this we first show that $\mathcal{B}_N = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m}(\phi, \dots, \phi) \neq 0 : i_1, \dots, i_m \in I\}$.

3.25. Lemma. *Let $T = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m}(\phi, \dots, \phi) \neq 0 : i_1, \dots, i_m \in I\}$ and $S \subseteq \{(Y_1, \dots, Y_N) : Y_i$ are Young diagrams such that $Y_i \subseteq Y_{i+1}$ for $1 \leq i \leq N-1\}$. If*

(a) $(\phi, \dots, \phi) \in S$,

(b) $\tilde{f}_i(S) \subseteq S \cup \{0\}$, *and*

(c) $\tilde{e}_i(S) \subseteq S \cup \{0\}$,

then $S = T$.

Proof. Assume that (a), (b), and (c) are true. Then by (a) and (b), $T \subseteq S$. To show that $S \subseteq T$, we use induction on the height of the weight of an element of S. Let $\mathbf{Y} = (Y_1, \dots, Y_N) \in S$. If $\mathrm{wt}(\mathbf{Y}) = 0$, then $(Y_1, \dots, Y_N) = (\phi, \phi, \dots, \phi) \in T$. If $\mathrm{wt}(\mathbf{Y}) \neq 0$, let $Y_1 = (y_{10}, \dots, y_{1l}, 0, 0, \dots)$ with $y_{1l} \neq 0$. Then Y_1 has a convex corner at site $(l+1, y_{1l})$ of some colour, say i. Then $\sigma_i(\mathbf{Y}) = (1, \dots)$. So $\tilde{e}_i(\mathbf{Y}) \neq 0$. By (c), $\tilde{e}_i(\mathbf{Y}) \in S$. By induction, $\tilde{e}_i(\mathbf{Y}) \in T$, and $0 \neq \mathbf{Y} = \tilde{f}_i(\tilde{e}_i(\mathbf{Y})) \in T$.

□

3.26. Lemma. *Let $\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{B}_N$ with $Y_j = \{y_{jk}\}_{k\geq 0}$. Let $r, s \in \{1, 2, \dots, N\}$ with $r < s$. If for $k, t \in \mathbb{N}$ we have $y_{rk}+(n-1)(t-k)+(n-2) < y_{s(t+2)}$, then $y_{rk} + (n-1)(t+1-k) \leq y_{s(t+1)}$.*

Proof. We need to show that $a := a(r, s, k, \mathbf{Y}) \geq t+1$.

If not,

$$\begin{aligned} y_{s(t+2)} \leq y_{s(t+2+t-a)} &= y_{s(a+2b)}, \qquad \text{where } b = t-a+1 \geq 1, \\ &\leq y_{rk} + (n-1)(a-k) + (n-2)b \\ &= y_{rk} + (n-1)(t-k) + (n-2) + (n-1)(a-t) \\ &\qquad +(n-2)(b-1) \\ &< y_{s(t+2)} + a - t \leq y_{s(t+2)}, \qquad \text{a contradiction.} \end{aligned}$$

□

3.27. Definition. *For $m \geq 1$, let $\mathcal{S}_m = \{\mathbf{Y} = (Y_1, \dots, Y_N) : Y_j = \{y_{jk}\}_{k\geq 0},\ 1 \leq j \leq N$ and $\mathbf{Y}$ satisfies 3.16(i), (ii), (iii)(a) and (iii)(b′) below. $\}$*

(iii)(b′) For $1 \leq b \leq m$, $y_{rk} + (n-1)(a(r, s, k) - k) + (n-2)b \geq y_{s(a+2b)}$.

3.28. Lemma. *Let $\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{S}_m$ (or $\mathcal{B}_N$) with $Y_j = \{y_{jk}\}_{\geq 0}$. Let $r, s \in \{1, \dots, N\}$ with $r < s$. If for some $k \in \mathbb{N}$ and $t \geq k-1$, $y_{rk}+(n-1)(t-k) \leq y_{st}$*

and $y_{rk}+(n-1)(t-k+2) > y_{s(t+2)}$ *then* $y_{rk}+(n-1)(t-k+1)+(n-2)b \geq y_{s(t+1+2b)}$ *for* $1 \leq b \leq m$ *(or* $b \geq 1$ *if* $\mathbf{Y} \in \mathcal{B}_n$*).*

Proof. If $t = k-1$, then $t \leq k \leq a(r,s,k)$ and if $t \geq k$, the first inequality above implies $t \leq a(r,s,k)$. Also the second inequality above implies $a(r,s,k) \leq t+1$.

If $a(r,s,k) = t+1$, we are done.

If $a(r,s,k) = t$, $y_{rk} + (n-1)(t-k) + (n-2)b \geq y_{s(t+2b)}$, $1 \leq b \leq m$, and $y_{rk} + (n-1)(t+1-k) + (n-2)b \geq y_{s(t+2b)} + (n-1) \geq y_{s(t+1+2b)}$ by 3.16(i).

□

3.29. Lemma. $\mathcal{B}_N = \mathcal{S}_1$.

Proof. Since $\mathcal{S}_1 \supseteq \cdots \supseteq \mathcal{S}_m \supseteq \mathcal{S}_{m+1} \cdots$ and $\mathcal{B}_N = \cap_{m \geq 1} \mathcal{S}_m$, it suffices to show that $\mathcal{S}_{m-1} \subseteq \mathcal{S}_m$ for all $m \geq 2$. Let $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{S}_{m-1}$ with $Y_j = \{y_{sk}\}_{k \geq 0}$, $1 \leq j \leq N$. Let r, $s \in \{1, \ldots, N\}$ with $r < s$, and $k \in \mathbb{N}$. Set $a := a(r,s,k)$.

Suppose there exists an integer $c \in \{k+1, \ldots, 1+a\}$ such that $y_{rc} < y_{rk} + (n-1)(c-k-1)$. Choose the smallest such c. Then $y_{r(c-1)} \geq y_{rk} + (n-1)(c-k-2)$.

$$\begin{aligned} y_{r(c-1)} + (n-1)(a-(c-1)) &\leq y_{rc} + (n-1)(a-(c-1)) \\ &< y_{rk} + (n-1)(c-k-1) + (n-1)(a-(c-1)) \\ &= y_{rk} + (n-1)(a-k) \leq y_{sa}, \text{ and} \end{aligned}$$

$$\begin{aligned} y_{r(c-1)} + (n-1)(a-(c-1)+2) &= y_{r(c-1)} + (n-1)(a-c+3) \\ &\geq y_{rk} + (n-1)(c-k-2) + (n-1)(a-c+3) \end{aligned}$$

$$\begin{aligned} &= y_{rk} + (n-1)(a-k+1) \\ &> y_{rk} + (n-1)(a-k) + (n-2) \geq y_{s(a+2)} \end{aligned}$$

So by Lemma 3.28, for $1 \leq b \leq m-1$,

$$y_{r(c-1)} + (n-1)(a-(c-1)+1) + (n-2)b \geq y_{s(a+1+2b)}. \tag{10}$$

$$\begin{aligned} y_{rc} + (n-1)(a+1-c) &< y_{rk} + (n-1)(c-k-1) + (n-1)(a+1-c) \\ &= y_{rk} + (n-1)(a-k) \leq y_{sa} \leq y_{sa+1}, \text{ and} \end{aligned}$$

$$\begin{aligned} y_{rc} + (n-1)(a+3-c) &\geq y_{r(c-1)} + (n-1)(a+3-c) \\ &= y_{r(c-1)} + (n-1)(a-(c-1)+1) + (n-2) + 1 \\ &\geq y_{s(a+3)} + 1 \qquad \text{(by 10)} \\ &> y_{s(a+3)}. \end{aligned}$$

So by Lemma 3.28,

$$y_{rc} + (n-1)(a-c+2) + (n-2)(m-1) \geq y_{s(a+2+2(m-1))} = y_{s(a+2m)}.$$

Hence

$$\begin{aligned} y_{rk} + (n-1)(a-k) + (n-2)m &> y_{rc} - (n-1)(c-k-1) + (n-1)(a-k) + (n-2)m \\ &= y_{rc} + (n-1)(a-c+1) + (n-2)m \\ &= y_{rc} + (n-1)(a-c+2) + (n-2)(m-2) - 1 \\ &\geq y_{s(a+2m)} - 1 \end{aligned}$$

So $y_{rk} + (n-1)(a-k) + (n-2)m \geq y_{s(a+2m)}$ and $\mathbf{Y} \in \mathcal{S}_m$.

Now assume that for all $c \in \{k+1, \ldots, a+1\}$, $y_{rc} \geq y_{rk} + (n-1)(c-k-1)$.

In particular, $y_{r(a+1)} \geq y_{rk} + (n-1)(a-k)$, and $y_{r(a+1)} + (n-1) > y_{rk} + (n-1)(a-k) + (n-2) \geq y_{s(a+2)}$. So $y_{r(a+1)} + (n-2)b \geq y_{s(a+1+2b)}$ for all $1 \leq b \leq m-1$. Hence

$$y_{r(a+2)} + (n-1) \geq y_{r(a+1)} + (n-1) > y_{r(a+1)} + (n-2) \geq y_{s(a+3)}.$$

So $y_{r(a+2)} + (n-2)b \geq y_{s(a+2+2b)}$ for all $1 \leq b \leq m-1$. Therefore

$$\begin{aligned} y_{rk} + (n-1)(a-k) + (n-2)m &\geq y_{s(a+2)} + (n-2)(m-1) \\ &\geq y_{r(a+2)} + (n-2)(m-1) \\ &\geq y_{s(a+2m)}, \end{aligned}$$

and $\mathbf{Y} \in \mathcal{S}_m$.

□

3.30. Lemma. *Let $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{B}_N$, $i \in I$ and $\sigma(\mathbf{Y}) = (\lambda_l, \ldots, \lambda_0)$ where for $0 \leq j \leq l$, $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jN})$ and for $1 \leq r \leq N$, λ_{jr} is the part of $\sigma(\mathbf{Y})$ coming from the j^{th} i-stair of Y_r. If there exists an $r > 1$ and a $j \geq 1$ such that λ_{jr} contains a zero, then for all $1 \leq s < r$, $\lambda_{(j-1)s} = (0, \ldots, 0)$ (possibly empty). (Hence if $\lambda_{(j-1)s}$ is not empty, $\lambda_{j-1} = (0, \ldots, 0)$ or $\lambda_j = (1, \ldots, 1)$ (possibly empty).*

Proof. Suppose that $\lambda_{(j-1)s}$ contains a 1 for some $1 \leq s < r$. Let (r, k, y_{rk}) be the site of the concave corner of $\mathbf{Y}_r$ corresponding to a zero in λ_{jr} and $(s, h+1, y_{sh})$ be the site of the convex corner of Y_s corresponding to a 1 in $\lambda_{(j-1)s}$. Then $y_{rk} = y_{sh} + (n-1)(k-h) - (n-1)$.

By 3.16(i) and (ii), for $1 \leq h' \leq h$, $y_{rh} \geq y_{sh} \geq y_{sh'} \geq y_{sh} - (n-1)(h-h')$, and hence $k > h \geq 0$.

Since $y_{sh} + (n-1)(k-h-1) = y_{rk} > y_{r(k-1)}$, $k-1 > a(s,r,h)$.

Let $a := a(s,r,h)$.

$$\begin{aligned}
y_{rk} &\leq y_{r(2k-a-2)} && \text{(by 3.16(i), since } k \geq a+2) \\
&= y_{r(a+2b)}, && \text{where } b = k-(a+1) \geq 1 \\
&\leq y_{sh} + (n-1)(a-h) + (n-2)b && \text{(by 3.16(iii)(b))} \\
&= y_{sh} + (n-1)(k-h-1) + (n-1)(a-k+1) + (n-2)b \\
&= y_{sh} + (n-1)(k-h-1) - b \\
&< y_{sh} + (n-1)(k-h-1) = y_{rk}.
\end{aligned}$$

This is a contradiction.

□

3.31. Theorem. *Let $T = \{\tilde{f}_{i_1} \ldots \tilde{f}_{i_m}(\phi, \ldots, \phi) \neq 0 : i_1, \ldots, i_m \in I\}$. Then $T = \mathcal{B}_N$.*

Proof. By Lemma 3.25, it suffices to show (a), (b), and (c) of that lemma.

(a) $(\phi, \ldots, \phi) \in \mathcal{B}_N$ since for $r, s \in \{1, 2, \ldots, N\}$ with $r < s$ and $k \in \mathbb{N}$, we can choose $a(r,s,k) = k$ and (i),(ii), and (iii) of 3.16 are satisfied.

(b) Let $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{B}_N$, $Y_j = \{y_{jk}\}_{k \geq 0}$ for $1 \leq j \leq N$, and $i \in I$. Suppose $\tilde{f}_i(\mathbf{Y}) \neq 0$. Then there exists an $s \in \{1, \ldots, N\}$ such that $\tilde{f}_i(\mathbf{Y}) = (Y_1, \ldots, \tilde{f}_i(Y_s), \ldots, Y_N)$ and a $k \in \mathbb{N}$ such that Y_s has a concave i-coloured corner at site (s, k, y_{sk}) and $\tilde{f}_i(Y_s)$ has a convex i-coloured corner at site $(s, k+1, y_{sk}-1)$. Note that $k = 0$ or $y_{s(k-1)} < y_{sk}$. Then there exists a j such that the concave corner at site (s, k, y_{sk}) is on the j^{th} i - stair of Y_s, and the first 0 in $\lambda_j = (1, \dots, 1, 0, \dots, 0)$ (λ_j as defined in 3.15) corresponds to this corner.

We now show (i) of 3.16 is satisfied by $\tilde{f}_i(\mathbf{Y})$.

If $y_{sk} + (n-1) = y_{s(k+1)}$, there would be a concave corner in Y_s at site $(s, k+1, y_{s(k+1)})$ and this corner would contribute a 0 to λ_j appearing before the 0 corresponding to the concave corner at site (s, k, y_{sk}). This is a contradiction. So $y_{sk} + (n-1) \neq y_{s(k+1)}$ and (i) in 3.16 imply that $y_{sk} - 1 + (n-1) \geq y_{s(k+1)}$. Also if $k > 0$, $y_{sk} - 1 \geq y_{s(k-1)}$. So (i) of 3.16 is satisfied by $\tilde{f}_i(\mathbf{Y})$.

To show that (ii) of 3.16 is satisfied by $\tilde{f}_i(\mathbf{Y})$, we need to show that if $s > 1$, $\tilde{f}_i(Y_s) \subseteq Y_{s-1}$. If $y_{(s-1)k} = y_{sk}$, then either $k = 0$ or $y_{(s-1)(k-1)} \leq y_{s(k-1)} < y_{sk} = y_{(s-1)k}$. Hence Y_{s-1} would have a concave i-coloured corner at site $(s-1, k, y_{sk}) = (s-1, k, y_{(s-1)k})$ and this corner would contribute a 0 to λ_j appearing before the 0 corresponding to the concave corner at site (s, k, y_{sk}). This is a contradiction. So $y_{(s-1)k} \neq y_{sk}$ and (ii) of 3.16 imply $y_{sk} - 1 \geq y_{(s-1)k}$. Thus $\tilde{f}_i(Y_s) \subseteq Y_{s-1}$.

We now show (iii) of 3.16 is satisfied by $\tilde{f}_i(\mathbf{Y})$.

Let $r \in \{1, \dots, N\}$ be such that $r > s$, and let $a := a(s, r, k, \mathbf{Y})$. Then by (iii) of 3.16,

(11) $$y_{sk} + (n-1)(b-k) \leq y_{rb} \text{ for all } k \leq b \leq a \text{ and}$$

(12) $$y_{sk} + (n-1)(a-k) + (n-2)b \geq y_{r(a+2b)} \text{ for all } b \geq 1$$

So we have for all $b \geq 1$,

(13)

$$
\begin{aligned}
y_{sk} - 1 + (n-1)(a+1-k) + (n-2)b &= y_{sk} + (n-1)(a-k) + (n-2)(b+1) \\
&\geq y_{r(a+2(b+1))}, \qquad \text{by (11)} \\
&\geq y_{r(a+1+2b)}, \qquad \text{by (i) of 3.16}
\end{aligned}
$$

If $(y_{sk} - 1) + (n-1)(a+1-k) = y_{r(a+1)}$, this together with (11) and (13) imply $a(s, r, k, \tilde{f}_i(\mathbf{Y})) := a + 1$. Otherwise,

(14) $$(y_{sk} - 1) + (n-1)(a+1-k) > y_{r(a+1)},$$

since

$$
\begin{aligned}
y_{sk} - 1 + (n-1)(a+1-k) &= y_{sk} + (n-1)(a-k) + (n-2) \\
&\geq y_{r(a+2)}, \qquad \text{by (11)} \\
&\geq y_{r(a+1)}.
\end{aligned}
$$

If $y_{r(a+2)} = y_{sk} + (n-1)(a-k) + (n-2)$, using (14) we get that $y_{r(a+2)} > y_{r(a+1)}$. So Y_r would have an i^{th}-coloured concave corner at site $(r, a+2, y_{r(a+2)})$ which would contribute a 0 to $\lambda_{(j+1)r}$ (see the definition in Lemma 3.30). By 3.18, $\lambda_{j+1} = (1, \ldots, 1, 0, \ldots, 0)$, and by Lemma 3.30, there are no 1's in $\sigma(\mathbf{Y})$ between the 0's corresponding to the concave corners of $\mathbf{Y}$ at sites $(r, a+2, y_{r(a+2)})$ and (s, k, y_{sk}). This is a contradiction. Hence, using (12), we get that $y_{r(a+2)} \leq (y_{sk} - 1) + (n-1)(a-k) + (n-2)$. This, together with (11) and Lemma 3.29, imply $a(s, r, k, \tilde{f}_i(\mathbf{Y})) := a$.

Now let $r \in \{1, \ldots, N\}$ be such that $r < s$, $k' \in \mathbb{N}$ and $a := a(r, s, k')$. Then by 3.16 (iii),

$$y_{rk'} + (n-1)(b-k') \leq y_{sb} \text{ for all } k' \leq b \leq a \text{ and}$$

$$y_{rk'} + (n-1)(a-k') + (n-2)b \geq y_{s(a+2b)} \text{ for all } b \geq 1.$$

So if $k > a$ or $k < k'$, $a(r, s, k', \tilde{f}_i(\mathbf{Y})) = a$. So assume $k' \leq k \leq a$. We suppose that $y_{rk'} + (n-1)(k-k') = y_{sk}$ and obtain a contradiction. If $k' > 0$, then

$$y_{r(k'-1)} + (n-1)(k-2-(k'-1)) + (n-2) = y_{r(k'-1)} + (n-1)(k-k') - 1$$

$$\leq y_{rk'} + (n-1)(k-k') - 1 < y_{sk}.$$

So by Lemma 3.26,

$$y_{r(k'-1)} + (n-1)(k-k') = y_{r(k'-1)} + (n-1)(k-1-(k'-1))$$

$$\leq y_{s(k-1)} < y_{sk} = y_{rk'} + (n-1)(k-k').$$

So either $k' = 0$ or $y_{r(k'-1)} < y_{rk'}$. In either case, there is a concave i–coloured corner in Y_r at site $(r, k', y_{rk'})$, and this corner contributes a 0 to λ_j which appears to the left of the 0 corresponding to (s, k, y_{sk}) in λ_j. This is a contradiction; therefore $y_{rk'} + (n-1)(k-k') \leq y_{sk} - 1$ and $a(r, s, k', \tilde{f}_i(\mathbf{Y})) = a$. So 3.16 (iii) is satisfied by $\tilde{f}_i(\mathbf{Y})$ and therefore $\tilde{f}_i(\mathbf{Y}) \in \mathcal{B}_N$.

(c) Let $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{B}_N$, $Y_j = \{y_{jk}\}_{k \geq 0}$ for $1 \leq j \leq N$, and $i \in I$. Suppose $\tilde{e}_i(\mathbf{Y}) \neq 0$. Then there exists an $s \in \{1, \ldots, N\}$ such that $\tilde{e}_i(\mathbf{Y}) = (Y_1, \ldots, \tilde{e}_i(Y_s), \ldots, Y_N)$ and a $k \in \mathbb{N}$ such that Y_s has a convex i-coloured corner at site $(s, k+1, y_{sk})$ and $\tilde{e}_i(Y_s)$ has a concave i-coloured corner at site $(s, k, y_{sk}+1)$. Note that $y_{s(k+1)} > y_{sk}$. Then there exists a j such that the convex corner at site

$(s,k+1,y_{sk})$ is on the $(j+1)^{st}$ i - stair of Y_s, and the last 1 in $\lambda_j = (1,\dots,1,0,\dots,0)$ (λ_j as defined in 3.15) corresponds to this corner.

3.16 (i) is satisfied by $\tilde{e}_i(\mathbf{Y})$ since $y_{sk}+1 \leq y_{s(k+1)}$ and if $k>0, y_{sk}+1 \leq y_{s(k-1)}+(n-1)$, otherwise $y_{sk} = y_{s(k-1)}+(n-1)$ and there would be a convex corner in Y_s at site $(s,k,y_{s(k-1)})$ contributing a 1 to λ_j appearing to the right of the 1 corresponding to the convex corner at site $(s,k+1,y_{sk})$.

To show 3.16 (ii) is satisfied by $\tilde{e}_i(\mathbf{Y})$, we need to show that if $s<N$, $\tilde{e}_i(Y_s) \supseteq Y_{s+1}$. If $y_{sk} = y_{(s+1)k}$, then $y_{(s+1)(k+1)} \geq y_{s(k+1)} > y_{sk} = y_{(s+1)k}$. So there is a convex corner at site $((s+1),k+1,y_{(s+1)k})$on the $(j+1)^{st}$ i–stair which contributes a 1 to λ_j appearing to the right of the 1 from the convex corner of Y_s at site $(s,k+1,y_{sk})$. This is not possible, so by 3.16 (ii) and $y_{sk} \neq y_{(s+1)k}$, $y_{sk}+1 \leq y_{(s+1)k}$ and $\tilde{e}_i(Y_s) \supseteq Y_{s+1}$.

We now show 3.16 (iii) is satisfied by $\tilde{e}_i(\mathbf{Y})$.

Let $r \in \{1,\dots,N\}$ be such that $r>s$. Let $a := a(s,r,k,\mathbf{Y})$. Then for $k \leq b \leq a$, $y_{sk}+(n-1)(b-k) \leq y_{rb}$ and for $b \geq 1$, $y_{sk}+(n-1)(a-k)+(n-2)b \geq y_{r(a+2b)}$. If for $k \leq b \leq a$, $y_{sk}+(n-1)(b-k) < y_{rb}$, $a(s,r,k,\tilde{e}_i(\mathbf{Y})) = a$. So assume there exists a $k \leq b_0 \leq a$ such that $y_{sk}+(n-1)(b_0-k) = y_{rb_0}$. Choose the smallest such b_0. Then $y_{sk}+1+(n-1)(b-k) \leq y_{rb}$ for all $k \leq b \leq b_0-1$ and $y_{sk}+1+(n-1)(b_0-1-k)+(n-2) = y_{rb_0} \geq y_{r(b_0+1)}$, otherwise $y_{rb_0} < y_{r(b_0+1)}$, and there would be a convex corner on the j^{th} i–stair of Y_r at site (r,b_0+1,y_{rb_0}) corresponding to a 1 appearing to the right of the 1 from the corner at site $(s,k+1,y_{sk})$. So by Lemma 3.29, $y_{sk}+1+(n-1)(b_0-1-k)+(n-2)b \geq y_{r(b_0-1+2b)}$ for all $b \geq 1$. Hence $a(s,r,k,\tilde{e}_i(\mathbf{Y})) := b_0-1$.

Now let $r \in \{1,\dots,N\}$ be such that $r<s$, $k' \in \mathbb{N}$ and $a := a(r,s,k',\mathbf{Y})$. Then

$$y_{rk'}+(n-1)(b-k') \leq y_{sb} \text{ for } k' \leq b \leq a \text{ and} \tag{15}$$

(16) $$y_{rk'} + (n-1)(a-k') + (n-2)b \;\geq\; y_{s(a+2b)} \text{ for } b \geq 1.$$

So if $k \leq a$, $a(r,s,k',\tilde{e}_i(\mathbf{Y})) = a$. So assume $k > a$. If $k \neq a+2$, by Lemma 3.29, $a(r,s,k',\tilde{e}_i(\mathbf{Y})) = a$. So let $k = a+2$. If $y_{rk'} + (n-1)(a-k') + (n-2) > y_{sk}$, $a(r,s,k',\tilde{e}_i(\mathbf{Y})) = a$ (use Lemma 3.29). So assume $y_{rk'}+(n-1)(a-k')+(n-2) = y_{sk}$.

$$\begin{aligned} y_{r(k'+1)} + (a-(k'+1))(n-1) &\leq y_{rk'} + (n-1) + (a-k'-1)(n-1) \text{ (by 3.16(i))} \\ &= y_{rk'} + (a-k')(n-1) \\ &\leq y_{sa} \quad \text{by (15).} \end{aligned}$$

$$\begin{aligned} y_{r(k'+1)} + (a+2-(k'+1))(n-1) &= y_{r(k'+1)} + (a+1-k')(n-1) \\ &> y_{rk'} + (a-k')(n-1) + (n-2) \\ &= y_{sk} \\ &= y_{s(a+2)}. \end{aligned}$$

By Lemma 3.28, $y_{r(k'+1)} + (n-1)(a+1-(k'+1)) + (n-2) \geq y_{s(a+3)} = y_{s(k+1)} > y_{sk} = y_{rk'} + (n-1)(a-k') + (n-2)$.

So $y_{r(k'+1)} > y_{rk'}$, and there is a convex corner in Y_r at site $(r, k'+1, y_{rk'}) = (r, k'+1, y_{sk} + (n-1)(a-k') + (n-2))$ which contributes a 1 in $\lambda_{(j-1)r}$. By Lemma 3.30, there are no zeroes in $\sigma(\mathbf{Y})$ between the 1's from the convex corners at sites $(s, k+1, y_{sk})$ and $(r, k'+1, y_{rk'})$. This is a contradiction. Hence 3.16 (iii) is satisfied by $\tilde{e}_i(\mathbf{Y})$ and $\tilde{e}_i(\mathbf{Y}) \in \mathcal{B}_N$.

□

3.32. Theorem. *If ι and ι' are as in 3.17, then* $\operatorname{Im}\Phi = \operatorname{Im}(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$.

Proof. Recall (see 2.5) that

$$\begin{aligned}\operatorname{Im}(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0} &= \{u_\infty \otimes (\dots, a) \otimes \overline{0} \otimes t_{N\Lambda_0} : a \le N \text{ and} \\ &\qquad u_\infty \otimes (\dots, a) \otimes \overline{0} \otimes t_{N\Lambda_0} = \tilde{f}_{i_1} \cdots \tilde{f}_{i_m}(u_\infty \otimes \overline{0} \otimes \overline{0} \otimes t_{N\Lambda_0}) \\ &\qquad \text{for some } i_1, \dots, i_m \in I\}.\end{aligned}$$

By Theorem 3.31, the definition of Φ, and the fact that Φ is a morphism of crystals, $\operatorname{Im}\Phi \subseteq \operatorname{Im}(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$.

Now let $b \in \operatorname{Im}(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$. If $\mathrm{wt}(b) = N\Lambda_0$, $b = u_\infty \otimes \overline{0} \otimes \overline{0} \otimes t_{N\Lambda_0} = \Phi((\phi, \dots, \phi))$. So assume $\mathrm{wt}(b) \neq N\Lambda_0$. Then there exists $i \in I$ such that $0 \neq \tilde{e}_i b \in \operatorname{Im}(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}$. By induction, there exists $b' \in \mathcal{B}_N$ such that $\Phi(b') = \tilde{e}_i b$. Since $\tilde{f}_i(\tilde{e}_i b) = b = u_\infty \otimes \cdots \otimes \overline{0} \otimes t_{N\Lambda_0}$, by Theorem 3.23, $b = \tilde{f}_i(\Phi(b')) = \Phi(\tilde{f}_i b') \in \operatorname{Im}\Phi$.

□

3.33. Theorem. $\mathcal{B}_N \simeq B(N\Lambda_0)$ *as crystals.*

Proof. Let $\tau := (\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0} : B(N\Lambda_0) \to \{u_\infty\} \otimes B_{\iota,\iota'} \otimes T_{N\Lambda_0}$, and $\gamma = \tau^{-1}|_{\operatorname{Im}\Phi} \circ \Phi : \mathcal{B}_N \to B(N\Lambda_0)$. (Note: Theorem 3.32 says that $\operatorname{Im}\tau = \operatorname{Im}\Phi$.) Then γ is 1-1 and onto and it preserves ε_i, ϕ_i, and wt. To show that γ is an isomorphism of crystals, it suffices to show that γ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$. Since $\tau_{N\Lambda_0}$ is a full embedding of crystals and $\phi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}$ is a strict embedding of crystals, τ commutes with all $\tilde{e}_i$'s, $i \in I$. By Theorem 3.23, Φ commutes with all $\tilde{e}_i$'s, $i \in I$. Hence for all $i \in I$, $\tilde{e}_i \circ \gamma = \tilde{e}_i \circ \tau^{-1} \circ \Phi = \tau^{-1} \circ \tilde{e}_i \circ \Phi = \tau^{-1} \circ \Phi \circ \tilde{e}_i = \gamma \circ \tilde{e}_i$.

Now we show that γ commutes with $\tilde{f}_i$ for $i \in I$. Let $b \in \mathcal{B}_N$. We consider two cases.

If $\tilde{f}_i(b) \neq 0$,

$$\begin{aligned} \tilde{f}_i(\gamma(b)) &= \tilde{f}_i(\gamma(\tilde{e}_i(\tilde{f}_i b))) \qquad \text{since } \tilde{f}_i(b) \neq 0 \\ &= \tilde{f}_i \tilde{e}_i \gamma(\tilde{f}_i(b)), \\ &= \gamma(\tilde{f}_i(b)), \qquad \text{since } \gamma(\tilde{f}_i(b)) \neq 0. \end{aligned}$$

Now suppose that $\tilde{f}_i(b) = 0$. Since Φ is a full embedding (see Theorem 3.23), by Lemma 1.14, $\tilde{f}_i\Phi(b) \notin \operatorname{Im}\Phi\backslash\{0\}$. By Theorem 3.32, $\operatorname{Im}\Phi = \operatorname{Im}\tau$, hence $\tilde{f}_i(\tau(\tau^{-1}(\Phi(b)))) = \tilde{f}_i\Phi(b) \notin \operatorname{Im}\tau$. Since τ is a full embedding, again by Lemma 1.14, $\tilde{f}_i\tau^{-1}\Phi(b) = 0$. Hence $\tilde{f}_i(\gamma(b)) = 0 = \gamma\tilde{f}_i(b)$.

□

3.34. Corollary. *If $w = r_{i_l}\dots r_{i_1}$ is a subword of $\dots r_{n-1}r_0r_1\dots r_{n-1}r_0$, then*

$$B_w(N\Lambda_0) \simeq \{\mathbf{Y} = (Y_1, \dots, Y_N) \in \mathcal{B}_N : Y_r \subseteq S(-l), \text{ for } 1 \leq r \leq N\}.$$

See 3.15 for the definition of $S(-l)$.

Proof. Let ι and ι' be as in 3.17, τ and γ be as in Theorem 3.33, $w' := r_{j_m}\dots r_{j_1}$ and $b \in B(\infty)$ such that $\bar{\pi}_{N\Lambda_0}(b) \neq 0$.

$$\begin{aligned} \bar{\pi}_{N\Lambda_0}(b) \in B_w(N\Lambda_0) &\iff \Psi_{\iota,\iota'}(b) = u_\infty \otimes \dots a_1 a_0 \bar{0} \text{ and } b \in B_w(\infty) \\ &\qquad \text{(see Proposition 3.3.1 in [Kas93])} \\ &\iff \Psi_{\iota,\iota'}(b) = u_\infty \otimes \dots a_1 a_0 \bar{0} \text{ and } b^* \in B_{w^{-1}}(\infty) \end{aligned}$$

(see Proposition 3.2.5 in [**Kas93**])

$$\Longleftrightarrow \Psi_{\iota,\iota'}(b) = u_\infty \otimes \dots a_1 a_0 \bar{0} \text{ and } b^* \in B_{(ww')^{-1}}(\infty)$$

(see Proposition 3.2.5 in [**Kas93**])

$$\Longleftrightarrow \Psi_{\iota,\iota'}(b) = u_\infty \otimes \dots a_1 a_0 \bar{0} \text{ and } a_k = 0 \text{ if } k > l$$

$$\Longleftrightarrow \tau(\bar{\pi}_{N\Lambda_0}(b)) = u_\infty \otimes \dots a_1 a_0 \bar{0} \otimes t_{N\Lambda_0} \text{ and } a_k = 0 \text{ if } k > l$$

$$\Longleftrightarrow \Phi(\gamma^{-1}((\bar{\pi}_{N\Lambda_0}(b)))) = u_\infty \otimes \dots a_1 a_0 \bar{0} \otimes t_{N\Lambda_0} \text{ and } a_k = 0 \text{ if } k > l$$

$$\Longleftrightarrow \gamma^{-1}((\bar{\pi}_{N\Lambda_0}(b))) = (Y_1, \dots, Y_N) \in \mathcal{B}_N \text{ such that } Y_r \subseteq S(-l),$$

for $1 \leq r \leq N$

□

3.35. Corollary. *Let γ be as in Theorem 3.33. Then*

$$\Phi \circ \gamma^{-1} = (\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}.$$

CHAPTER 4

A set of inequalities describing $B(N\Lambda_0)$ and $B(\infty)$.

In [**NZ97**] and [**Nak99**], the authors prove that for a sequence ι of elements of I satisfying certain conditions, the images of Ψ_ι and $\Psi_\iota \otimes \mathrm{id}_\lambda \circ \tau_\lambda$, for $\lambda \in P_+$, can be described by a set of inequalities generated by applying certain operators to a given set of inequalities.

In this Chapter, we use our results from Chapter 3 to explicitly find the inequalities defining the image of $\Psi_\iota \otimes \mathrm{id}_{N\Lambda_0} \circ \tau_{N\Lambda_0}$ for a particular sequence ι (see Theorem 4.7). This together with our results from Chapter 2 and a result from [**Cli98**] (or [**Lit98**]) (see Appendix A), gives us a description of the image of Ψ_ι for a particular ι (see Theorem 4.8).

4.1. For $N \in \mathbb{N}$, let

$$S_N := \{\{a_l\}_{l\geq 0} : \text{ for all } l \in \mathbb{N}\ a_l \in \mathbb{N},\ \text{for } l >> 0\ a_l = 0,$$
$$\text{and } \{a_l\} \text{ satisfies (17)} - \text{(20) below}\}$$

(17) $a_0 \leq N$,

(18) $a_{k(n-1)+i} \leq a_{k(n-1)+i-1}$ if $k \in \mathbb{N}$ and $i \in I \backslash \{0, n-1\}$,

(19) $a_{k(n-1)+i} \leq \frac{1}{k} a_{k(n-1)-1} + a_{(k-1)(n-1)+i}$ if $k \in \mathbb{N}_{\geq 1}$ and $i \in I \backslash \{n-1\}$,

(20) $$a_{k(n-1)+i} \leq a_{k(n-1)-1} + \sum_{s=1}^{r} (a_{(k-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k-s)(n-1)+\sum_{j=0}^{s} i_j})$$
$$+ (k-r) a_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k-r-1) a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j}$$

if $k \in \mathbb{N}_{\geq 2}$, $r \in \mathbb{N}$ such that $1 \leq r \leq k-1$, $i = i_0, i_1, \ldots, i_r \in \mathbb{N}$,

$i_0 + i_1 < (n-2)$, and $i_j + i_{j+1} < (n-1)$ for $1 \leq j \leq r-1$.

Let ι, ι', B_ι, and $B_{\iota'}$ be as in 3.17 and define

$$\mathbb{B}_N := \{u_\infty \otimes \{a_l\}_{l\geq 0} \otimes \bar{0} \otimes t_{N\Lambda_0} \in \{u_\infty\} \otimes B_\iota \otimes B_{\iota'} \otimes T_{N\Lambda_0} : \{a_l\}_{l\geq 0} \in S_N\}.$$

We will show that $\Phi(\mathcal{B}_N) = \mathbb{B}_N$ where $\mathcal{B}_N$ and Φ are as defined in 3.16 and 3.17.

4.2. Lemma. *Let $\mathbb{B}_N$ be as in 4.1, then for all $j \in I$, $\tilde{e}_j(\mathbb{B}_N) \subset \mathbb{B}_N \cup \{0\}$.*

Proof. Let $\boldsymbol{a} = \{a_l\}_{l\geq 0} \in S_N$, $b = u_\infty \otimes \{a_l\}_{l\geq 0} \otimes \bar{0} \otimes t_{N\Lambda_0}$, and $j \in I$. Assume $\tilde{e}_j(b) \neq 0$. Then $\tilde{e}_j(b) = u_\infty \otimes \tilde{e}_j(\{a_l\}_{l\geq 0} \otimes \bar{0}) \otimes t_{N\Lambda_0}$ (see Note 2 at the end of 1.17 for the definition of $\tilde{e}_j(\{a_l\}_{l\geq 0} \otimes \bar{0})$ and recall that $\varepsilon_j(t_{N\Lambda_0}) = -\infty$).

If $j = 0$, $\tilde{e}_j(\{a_l\}_{l\geq 0} \otimes \bar{0}) = \tilde{e}_j(\{a_l\}_{l\geq 0}) \otimes \bar{0}$ since $\varepsilon_0(\bar{0}) = -\infty$.

If $1 \leq j \leq n-2$, by 4.1(18) $a_j \leq a_{j-1}$ and if $j = n-1$ by 4.1(19), $a_j \leq a_{j-1} + a_0$. In either case, $\tilde{e}_j(\{a_l\}_{l\geq 0} \otimes \bar{0}) = (\tilde{e}_j(\{a_l\}_{l\geq 0})) \otimes \bar{0}$.

Let $l \in \mathbb{N}$ be such that $\tilde{e}_j((\dots, a_l, \dots, a_1, a_0)) = (\dots, a_l - 1, \dots, a_1, a_0)$. Then for all $t \in \mathbb{N}$,

$$\sum_{s=0}^{t}(-a_{l+ns} + a_{l+1+ns} + a_{l+(n-1)+ns} - a_{l+n+ns}) < 0 \tag{21}$$

(See Note 2 at the end of 1.17 and the definition in 1.16.)

In particular

$$a_l > a_{l+1} + a_{l+(n-1)} - a_{l+n} \tag{22}$$

We now show that $a' := (\dots, a_l - 1, \dots, a_1, a_0) \in S_N$ i.e. $a_l - 1 \in \mathbb{N}$ and a' satisfies 4.1(17)-(20). Let $k \in \mathbb{N}$ and $i \in I \backslash \{n-1\}$ be such that $l = i + k(n-1)$.

To show that $a_l - 1 \in \mathbb{N}$, we consider two cases.

If $i \neq n-2$,

$$\begin{aligned} a_l - 1 &\geq a_{l+1} + a_{l+(n-1)} - a_{l+n} && \text{by (22)} \\ &= a_{l+1} + a_{(k+1)(n-1)+i} - a_{(k+1)(n-1)+i+1} \\ &\geq a_{l+1} && \text{by 4.1(18) since } i \neq n-2 \\ &\geq 0. \end{aligned} \tag{23}$$

If $i = n-2$,

$$a_l - 1 \geq a_{l+(n-1)} + a_{l+1} - a_{l+n} \qquad \text{by (22)} \tag{24}$$

$$= a_{(k+1)(n-1)+i} + a_{k(n-1)+i+1} - a_{(k+1)(n-1)+i+1}$$

$$= a_{(k+1)(n-1)+(n-2)} + a_{(k+1)(n-1)} - a_{(k+2)(n-1)}$$

$$\geq a_{(k+1)(n-1)+(n-2)} - \frac{1}{k+2} a_{(k+2)(n-1)-1} \qquad \text{by 4.1(19)}$$

$$= \frac{k+1}{k+2} a_{(k+1)(n-1)+(n-2)}$$

$$\geq 0.$$

So in either case $a_l - 1 \in \mathbb{N}$.

To show that 4.1(18) is satisfied by $\boldsymbol{a}'$ all we need is to show that if $i \neq n-2$, $a_{l+1} = a_{k(n-1)+i+1} \leq a_l - 1$. This was done above (see (23)).

We now show that 4.1(19) is satisfied by $\boldsymbol{a}'$. Again we consider two cases. If $i \neq n-2$, we need to show that $a_{l+(n-1)} \leq \frac{1}{k+1} a_{(k+1)(n-1)-1} + (a_l - 1)$.

$$a_{l+(n-1)} \leq a_{l+n} - a_{l+1} + (a_l - 1) \qquad \text{by (22)}$$

$$= a_{(k+1)(n-1)+i+1} - a_{k(n-1)+i+1} + (a_l - 1)$$

$$\leq \frac{1}{k+1} a_{(k+1)(n-1)-1} + (a_l - 1) \qquad \text{since } \boldsymbol{a} \text{ satisfies 4.1(19).}$$

If $i = n-2$, we need to show that $a_{l+(n-1)} \leq (\frac{1}{k+1} + 1)(a_l - 1)$ which was done above (see (24)), and that if $0 \leq j \leq n-3$, $a_{(k+1)(n-1)+j} \leq \frac{1}{k+1}(a_{k(n-1)+(n-2)} - 1) + a_{k(n-1)+j}$.

$$a_{k(n-1)+(n-2)} - 1 = a_l - 1 \geq a_{l+1} + a_{l+(n-1)} - a_{l+n} \qquad \text{by(22)}$$

$$= a_{(k+1)(n-1)} + a_{(k+1)(n-1)+(n-2)} - a_{(k+2)(n-1)}$$

$$\geq (k+1) a_{(k+1)(n-1)+j} - (k+1) a_{k(n-1)+j}$$

by 4.1 (20) with k replaced by $k+2$, $i = i_0 = 0$, $i_1 = j$, and $r = 1$.

To show that $\boldsymbol{a}'$ satisfies 4.1(20) we first show that if $i = n-2$, $r \in \mathbb{N}$ such that $1 \le r \le k$, $i_0 = i'$, $i_1, \dots, i_r \in \mathbb{N}$, $i_0 + i_1 < (n-2)$, and $i_s + i_{s+1} < (n-1)$ for $1 \le s \le r-1$, then

(25)

$$a_{(k+1)(n-1)+i'} \le (a_{k(n-1)+(n-2)} - 1) + \sum_{s=1}^{r}(a_{(k+1-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k+1-s)(n-1)+\sum_{j=0}^{s} i_j}) + (k+1-r)a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j} - (k-r)a_{(k+1-r)(n-1)+\sum_{j=0}^{r} i_j}.$$

Let $i'_0 := 0$, $i'_1 := i'$, and for $1 \le s \le r$, $i'_{s+1} := i_s$. Then $k+2 \in \mathbb{N}_{\ge 2}$, $1 \le r+1 \le k+1$, $i'_0 + i'_1 = i' < (n-2)$ and for $1 \le s \le r$, $i'_s + i'_{s+1} < (n-1)$. Thus since $\boldsymbol{a}$ satisfies 4.1(20),

(26)

$$a_{(k+2)(n-1)} \le a_{(k+1)(n-1)+(n-2)} + \sum_{s=1}^{r+1}(a_{(k+2-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k+2-s)(n-1)+\sum_{j=0}^{s} i'_j}) + (k+2-(r+1))a_{(k+2-(r+1)-1)(n-1)+\sum_{j=0}^{r+1} i'_j} - (k+2-(r+1)-1)a_{(k+2-(r+1))(n-1)+\sum_{j=0}^{r+1} i'_j}.$$

So we have

$$\begin{aligned}(a_{k(n-1)+(n-2)} - 1) &+ \sum_{s=1}^{r}(a_{(k+1-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k+1-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k+1-r)a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j} - (k-r)a_{(k+1-r)(n-1)+\sum_{j=0}^{r} i_j} \\ &\ge a_{(k+1)(n-1)} + a_{(k+1)(n-1)+(n-2)} - a_{(k+2)(n-1)} \\ &+ \sum_{s=1}^{r}(a_{(k+1-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k+1-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k+1-r)a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j} - (k-r)a_{(k+1-r)(n-1)+\sum_{j=0}^{r} i_j} \text{ by (22)}\end{aligned}$$

$$
\begin{aligned}
&= a_{(k+1)(n-1)+i'} - a_{(k+2)(n-1)} + a_{(k+1)(n-1)+(n-2)} \\
&+ \sum_{s=1}^{r+1}(a_{(k+2-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k+2-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\
&+ (k+2-(r+1))a_{(k+2-(r+1)-1)(n-1)+\sum_{j=0}^{r+1} i'_j} \\
&- (k+2-(r+1)-1)a_{(k+2-(r+1))(n-1)+\sum_{j=0}^{r+1} i'_j} \\
&\geq a_{(k+1)(n-1)+i'}. \qquad \text{by (26)}
\end{aligned}
$$

And hence inequality (25) is satisfied.

Secondly, we need to show that if for some $k' \in \mathbb{N}_{\geq 2}$, $1 \leq r \leq k'-1$, $i_0 = i', i_1, \dots, i_r \in \mathbb{N}$ such that $i_0 + i_1 < (n-2)$ and $i_j + i_{j+1} < (n-1)$ for $1 \leq j \leq r-1$, there exists an s_0 such that $1 \leq s_0 \leq r$, $i_{s_0} \geq 1$ and $l = k(n-1)+i = (k'-s_0)(n-1) + \sum_{j=0}^{s_0-1} i_j$, then

$$
\begin{aligned}
(27) \qquad a_{k'(n-1)+i'} \leq\ & a_{k'(n-1)-1} \\
&+ ((a_{(k'-s_0)(n-1)+\sum_{j=0}^{s_0-1} i_j} - 1) - a_{(k'-s_0)(n-1)+\sum_{j=0}^{s_0} i_j}) \\
&+ \sum_{\substack{1 \leq s \leq r \\ s \neq s_0}} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}.
\end{aligned}
$$

We consider three cases:

Case A. First assume that $s_0 > 1$ and that there exists $t \in \mathbb{N}$ such that $2 \leq s_0 - 2t$ ($\leq s_0$) and

$$
i_{(s_0-2t)-2} + i_{(s_0-2t)-1} + 1 < \begin{cases} n-1 & \text{if } s_0 - 2t > 2 \\ n-2 & \text{if } s_0 - 2t = 2. \end{cases}
$$

Let t_1 be the smallest such t. Below we will need that

$$i_{s_0-2t} \geq 1 \text{ for all } 0 \leq t \leq t_1. \tag{28}$$

We prove this by induction on t. If $t = 0$, $i_{s_0} \geq 1$ was assumed. So assume that $t > 0$. Then $i_{s_0-2(t-1)} \geq 1$ and since $i_{s_0-2(t-1)} + i_{s_0-2(t-1)-1} < n-1$, we have that $i_{s_0-2(t-1)-1} < n-2$. So $i_{s_0-2(t-1)-2} + i_{s_0-2(t-1)-1} = n-2$ implies $i_{s_0-2t} \geq 1$. And we have (28).

We will also need that

$$(k' - (s_0 - 2t))(n-1) + \sum_{j=0}^{s_0-2t-1} i_j = l + tn \text{ forall } 0 \leq t \leq t_1 \tag{29}$$

Again we prove this by induction on t. If $t = 0$, we are done since $l = (k' - s_0)(n-1) + \sum_{j=0}^{s_0-1} i_j$. So assume that $t > 0$. Then we have that

$$\begin{aligned}
&(k' - (s_0 - 2t))(n-1) + \sum_{j=0}^{s_0-2t-1} i_j \\
&= (k' - (s_0 - 2(t-1)))(n-1) + 2(n-1) + \sum_{j=0}^{s_0-2(t-1)-1} i_j - (i_{s_0-2t} + i_{s_0-2t+1}) \\
&= l + (t-1)n + 2(n-1) - (i_{s_0-2(t-1)-2} + i_{s_0-2(t-1)-1}), \text{ by induction} \\
&= l + (t-1)n + 2(n-1) - (n-2), \text{ by the minimality of } t_1 \\
&= l + tn.
\end{aligned}$$

So we have (29).

Now for $0 \leq j \leq r$, define

$$i'_j = \begin{cases} i_j - 1 & \text{if } j = s_0 - 2t \text{ and } 0 \leq t \leq t_1 \\ i_j + 1 & \text{if } j = s_0 - 2t - 1 \text{ and } 0 \leq t \leq t_1 \\ i_j & \text{otherwise.} \end{cases}$$

Then $i'_j \in \mathbb{N}$ for all $0 \le j \le r$ (see (28)), $i'_0 + i'_1 < n-2$ and $i'_j + i'_{j+1} < n-1$ for $1 \le j \le r-1$. Note that $i'_{s_0-2t_1-2} + i'_{s_0-2t_1-1} < \begin{cases} n-1 & \text{if } s_0 - 2t_1 > 2 \\ n-2 & \text{if } s_0 - 2t_1 = 2 \end{cases}$.

We have

$$\begin{aligned}
(30) \qquad & \sum_{s=s_0-2t_1-1}^{s_0} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\
&= \sum_{t=0}^{t_1} (a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j+1} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} \\
&+ a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j+1}) \\
&= \sum_{t=0}^{t_1} ((a_{l+tn+1} - a_{l+(t+1)n}) \\
&+ (-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j})) \\
&< \sum_{t=0}^{t_1} ((a_{l+tn} - a_{l+(n-1)+tn}) \\
&+ (-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j})) \\
&= \sum_{t=0}^{t_1} (a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} \\
&\quad + a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j}) \\
&= \sum_{s=s_0-2t_1-1}^{s_0} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})
\end{aligned}$$

In the last inequality, we used (21).

By 4.1(20), we have the first inequality below and by (30), we have the last inequality below. Hence,

$$a_{k'(n-1)+i'} \le a_{k'(n-1)-1} + \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j})$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j}$$

$$= a_{k'(n-1)-1} + \sum_{s=1}^{s_0-2t_1-2} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right)$$

$$+ \sum_{s=s_0-2t_1-1}^{s_0} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}\right)$$

$$+ \sum_{s=s_0+1}^{r} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right)$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

$$< a_{k'(n-1)-1} + \sum_{s=1}^{r} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right)$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

and (27) is satisfied.

CASE B. Now assume that $s_0 > 1$ and that for all $t \in \mathbb{N}$ such that $2 \le s_0 - 2t$ $(\le s_0)$,

$$i_{(s_0-2t)-2} + i_{(s_0-2t)-1} = \begin{cases} n-2 & \text{if } s_0 - 2t > 2 \\ n-3 & \text{if } s_0 - 2t = 2. \end{cases}$$

Let t_1 be the largest t such that $2 \le s_0 - 2t$. So $s_0 - 2t_1 = 2$ or 3.

SUBCASE 1. $s_0 - 2t_1 = 3$

As before (see proofs of (28) and (29)), it can be shown that

(31) $$i_{s_0-2t} \ge 1 \text{ for all } 0 \le t \le t_1 + 1 \text{ and that}$$

(32) $$(k' - (s_0 - 2t))(n-1) + \sum_{j=0}^{s_0-2t-1} i_j = l + tn \text{ for all } 0 \le t \le t_1 + 1.$$

Now for $0 \le j \le r$, define

$$i'_j = \begin{cases} i_j - 1 & \text{if } j = s_0 - 2t \text{ and } 0 \le t \le t_1 + 1 \\ i_j + 1 & \text{if } j = s_0 - 2t - 1 \text{ and } 0 \le t \le t_1 + 1 \\ i_j & \text{otherwise.} \end{cases}$$

(Note: $i'_0 = i_0 + 1 = i' + 1$, $i'_1 = i_1 - 1, \dots$). Then $i'_j \in \mathbb{N}$ for all $0 \le j \le r$ (see (31)), $i'_0 + i'_1 < n - 2$ and $i'_j + i'_{j+1} < n - 1$ for $1 \le j \le r - 1$.

We have,

(33)

$$a_{k'(n-1)+i'} - a_{k'(n-1)+i'+1} + \sum_{s=1}^{s_0} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}\right)$$

$$= \sum_{t=0}^{t_1+1} \Big(a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j+1} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j}$$

$$+ a_{(k'-(s_0-2t-1))(n-1)+i_0+\sum_{j=1}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j+1}\Big)$$

$$= \sum_{t=0}^{t_1+1} \Big(\big(a_{l+tn+1} - a_{l+(t+1)n}\big)$$

$$+ \big(-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+i_0+\sum_{j=1}^{(s_0-2t)-2} i_j}\big)\Big)$$

$$< \sum_{t=0}^{t_1+1} \Big(\big(a_{l+tn} - a_{l+(n-1)+tn}\big)$$

$$+ \big(-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+i_0+\sum_{j=1}^{(s_0-2t)-2} i_j}\big)\Big)$$

$$= \sum_{t=0}^{t_1+1} \Big(a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j}$$

$$+ a_{(k'-(s_0-2t-1))(n-1)+i_0+\sum_{j=1}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j}\Big)$$

$$= \sum_{s=1}^{s_0} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right) \text{(Note: If } t = t_1 + 1,\ s_0 - 2t - 1 = 0\text{)}$$

In the last inequality, we used (21).

By 4.1(20), we have

$$a_{k'(n-1)+i'+1} \leq a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}) + (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j}$$

Hence by this inequality and (33),

$$\begin{aligned} a_{k'(n-1)+i'} &\leq a_{k'(n-1)-1} + a_{k'(n-1)+i'} - a_{k'(n-1)+i'+1} \\ &+ \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\ &+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j} \\ &< a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \end{aligned}$$

and (27) is satisfied.

SUBCASE 2. $s_0 - 2t_1 = 2$

As before (see proofs of (28) and (29)), it can be shown that

(34) $$i_{s_0-2t} \geq 1 \text{ for all } 0 \leq t \leq t_1,$$

(35) $$(k' - (s_0 - 2t))(n-1) + \sum_{j=0}^{s_0-2t-1} i_j = l + tn \text{ for all } 0 \leq t \leq t_1$$

(36) $$\text{and that } k'(n-1) = l + (t_1+1)n + 1.$$

Now for $0 \le j \le r+1$, define

$$i'_j = \begin{cases} 0 & \text{if } j = 0 \\ i' & \text{if } j = 1 \\ i_{j-1} - 1 & \text{if } j-1 = s_0 - 2t \text{ and } 0 \le t \le t_1 \\ i_{j-1} + 1 & \text{if } j-1 = s_0 - 2t - 1 \text{ and } 0 \le t \le t_1 \\ i_{j-1} & \text{otherwise.} \end{cases}$$

Then $i'_j \in \mathbb{N}$ for all $0 \le j \le r$ (see (34)), $i'_0 + i'_1 = i' < n-2$, $i'_1 + i'_2 = i' + i_1 + 1 < n-1$ and $i'_j + i'_{j+1} < n-1$ for $2 \le j \le r$.

We have,

$$(37) \qquad \sum_{s=1}^{s_0}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_{j+1}} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_{j+1}})$$

$$= \sum_{t=0}^{t_1}(a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j+1} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j}$$

$$+a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j+1})$$

$$= \sum_{t=0}^{t_1}((a_{l+tn+1} - a_{l+(t+1)n})$$

$$+(-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j}))$$

$$< \sum_{t=0}^{t_1}((a_{l+tn} - a_{l+(n-1)+tn})$$

$$+(-a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j} + a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j}))$$

$$+ a_{l+(t_1+1)n} - a_{l+(t_1+1)n+1} - a_{l+(t_1+1)n+(n-1)} + a_{l+(t_1+2)n}$$

$$= \sum_{t=0}^{t_1}(a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t-1} i_j} - a_{(k'-(s_0-2t))(n-1)+\sum_{j=0}^{s_0-2t} i_j}$$

$$+ a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-2} i_j} - a_{(k'-(s_0-2t-1))(n-1)+\sum_{j=0}^{(s_0-2t)-1} i_j})$$

$$+ a_{k'(n-1)-1} - a_{k'(n-1)} - a_{(k'+1)(n-1)-1} + a_{(k'+1)(n-1)}$$

$$= \sum_{s=1}^{s_0} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+ a_{k'(n-1)-1} - a_{k'(n-1)} - a_{(k'+1)(n-1)-1} + a_{(k'+1)(n-1)}$$

In the last inequality we used (21) and in the second to last equality we used (36). By 4.1(20), we have

$$a_{(k'+1)(n-1)} \leq a_{(k'+1)(n-1)-1} + a_{k'(n-1)} - a_{k'(n-1)+i'}$$

$$+ \sum_{s=2}^{r+1} (a_{(k'+1-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'+1-s)(n-1)+\sum_{j=0}^{s} i'_j})$$

$$+ ((k'+1)-(r+1))a_{((k'+1)-(r+1)-1)(n-1)+\sum_{j=0}^{r+1} i'_j}$$

$$+ ((k'+1)-(r+1)-1)a_{((k'+1)-(r+1))(n-1)+\sum_{j=0}^{r+1} i'_j}$$

Hence by this inequality and (37),

$$a_{k'(n-1)+i'} \leq a_{k'(n-1)-1} - a_{k'(n-1)-1} + a_{k'(n-1)} + a_{(k'+1)(n-1)-1} - a_{(k'+1)(n-1)}$$

$$+ \sum_{s=1}^{s_0} (a_{(k'-s)(n-1)+\sum_{j=1}^{s} i'_j} - a_{(k'-s)(n-1)+\sum_{j=1}^{s+1} i'_j})$$

$$+ \sum_{s=s_0+1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

$$< a_{k'(n-1)-1} + \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

and (27) is satisfied.

CASE C. Now assume that $s_0 = 1$, then $i'+1 = i_0+1 \le i_0+i_1 < n-2$. Hence by (22) and 4.1(20),

$$\begin{aligned} a_{k'(n-1)+i'} &\le (a_{(k'-1)(n-1)+i'} - 1) - a_{(k'-1)(n-1)+i'+1} + a_{k'(n-1)+i'+1} \\ &\le (a_{(k'-1)(n-1)+i'} - 1) + a_{k'(n-1)-1} - a_{(k'-1)(n-1)+i'+i_1} \\ &+ \sum_{s=2}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &(k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \end{aligned}$$

and (27) is satisfied.

Finally to finish the proof that $\boldsymbol{a}'$ satisfies 4.1(20), we need to show that if $k' \in \mathbb{N}_{\ge 2}$, $1 \le r \le k'-1$, $i_0 = i', i_1, \ldots, i_r \in \mathbb{N}$ such that $i_0 + i_1 < (n-2)$ and $i_j + i_{j+1} < (n-1)$ for $1 \le j \le r-1$, and $l = k(n-1)+i = (k'-r-1)(n-1) + \sum_{j=0}^{r} i_j$, then

(38)

$$\begin{aligned} a_{k'(n-1)+i'} \le a_{k'(n-1)-1} &+ \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}. \end{aligned}$$

We consider five cases:

CASE A. $i_r = n-2$. Note: $r > 1$ since $i_0 + i_1 < n-2$. Then by 4.1(20),

$$\begin{aligned} a_{k'(n-1)+i'} \le a_{k'(n-1)-1} &+ \sum_{s=1}^{r-1} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k'-(r-1))a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} - (k'-r)a_{(k'-(r-1))(n-1)+\sum_{j=0}^{r-1} i_j} \\ = a_{k'(n-1)-1} &+ \sum_{s=1}^{r-1} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \end{aligned}$$

$$+\, a_{(k'-r)(n-1)+\sum_{j=0}^{r-1}}$$

$$+\, (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1})$$

$$< a_{k'(n-1)-1} + \sum_{s=1}^{r-1}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+\, a_{(k'-r)(n-1)+\sum_{j=0}^{r-1}}$$

$$+\, (k'-r)((a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}), \text{ by (22)}$$

$$= a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+\, (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

and (38) is satisfied.

CASE B. $r = 1$ and $i_r = n-3$. Note that $i_0 = i' = 0$ and $l = (k'-2)(n-1)+(n-3)$. By 4.1(20),

$$a_{(k'+1)(n-1)} \leq a_{(k'+1)(n-1)-1} + a_{k'(n-1)} - a_{k'(n-1)} + a_{(k'-1)(n-1)} - a_{(k'-1)(n-1)+i_1+1}$$

$$+\, (k'-1)a_{(k'-2)(n-1)+i_1+1} - (k'-2)a_{(k'-1)(n-1)+i_1+1}.$$

So we have

$$a_{k'(n-1)} \leq a_{k'(n-1)-1} + a_{(k'-1)(n-1)} + a_{l+1} - 2a_{l+n} + a_{l+n+1} + a_{l+2n-1} - a_{l+2n}$$

$$+\, (k'-2)(a_{l+1} - a_{l+n})$$

$$\leq a_{k'(n-1)-1} + a_{(k'-1)(n-1)} + (a_l - 1) - a_{l+(n-1)}$$

$$+\, (k'-2)((a_l - 1) - a_{l+(n-1)}), \text{ by (21) and (22)}$$

$$
\begin{aligned}
&= a_{k'(n-1)-1} + a_{(k'-1)(n-1)} - a_{(k'-1)(n-1)+(n-3)} \\
&\quad + (k'-1)(a_l - 1) - (k'-2)a_{(k'-1)(n-1)+(n-3)}
\end{aligned}
$$

and (38) is satisfied.

For the rest of the proof, we assume that

$$
i_r < \begin{cases} n-2 & \text{if } r > 1 \\ n-3 & \text{if } r = 1 \end{cases} \tag{39}
$$

CASE C. $i_r + i_{r-1} < \begin{cases} n-2 & \text{if } r > 1 \\ n-3 & \text{if } r = 1 \end{cases}$. By 4.1(20),

$$
\begin{aligned}
a_{k'(n-1)+i'} \le\ & a_{k'(n-1)-1} + \sum_{s=1}^{r-1}\left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right) \\
&+ a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \\
&+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \\
&< a_{k'(n-1)-1} + \sum_{s=1}^{r}\left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}\right) \\
&+ (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \quad \text{by (22)}
\end{aligned}
$$

and (38) is satisfied.

So for the rest of the proof, we will assume that

$$
i_r + i_{r-1} = \begin{cases} n-2 & \text{if } r > 1 \\ n-3 & \text{if } r = 1. \end{cases} \tag{40}
$$

CASE D. Assume (39), (40) and that there exists $t \in \mathbb{N}$ such that $2 \le (r+1) - 2t$ ($\le (r+1)$) and

$$
i_{((r+1)-2t)-2} + i_{((r+1)-2t)-1} + 1 < \begin{cases} n-1 & \text{if } (r+1) - 2t > 2 \\ n-2 & \text{if } (r+1) - 2t = 2. \end{cases}
$$

Let t_1 be the smallest such t. Below we will need that

$$i_{(r+1)-2t} \geq 1 \text{ for all } 1 \leq t \leq t_1. \tag{41}$$

Note that (39) and (40) imply that $i_{(r+1)-2} = i_{r-1} \geq 1$. The rest of the proof is similar to the proof of (28).

We will also need that

$$(k' - ((r+1) - 2t))(n-1) + \sum_{j=0}^{(r+1)-2t-1} i_j = l + tn \text{ for all } 0 \leq t \leq t_1 \tag{42}$$

The proof of this is identical to the proof of (29) with s_0 replaced by $r+1$.

Now for $0 \leq j \leq r$, define

$$i'_j = \begin{cases} i_j - 1 & \text{if } j = (r+1) - 2t \text{ and } 1 \leq t \leq t_1 \\ i_j + 1 & \text{if } j = (r+1) - 2t - 1 \text{ and } 0 \leq t \leq t_1 \\ i_j & \text{otherwise.} \end{cases}$$

Then $i'_j \in \mathbb{N}$ for all $0 \leq j \leq r$ (see (41)), $i'_0 + i'_1 < n - 2$ and $i'_j + i'_{j+1} < n - 1$ for $1 \leq j \leq r - 1$. Note that $i'_{(r+1)-2t_1-2} + i'_{(r+1)-2t_1-1} < \begin{cases} n-1 & \text{if } (r+1) - 2t_1 > 2 \\ n-2 & \text{if } (r+1) - 2t_1 = 2 \end{cases}$.
We have

$$\begin{aligned} &\sum_{s=(r+1)-2t_1-1}^{r} \left(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}\right) \\ &+(k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j} \\ &= \sum_{t=1}^{t_1}\Big(a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{((r+1)-2t)-1} i_j+1} - a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} \\ &+a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j} - a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{(r+1)-2t-1} i_j+1}\Big) \\ &+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \\ &+(k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \end{aligned} \tag{43}$$

$$
\begin{aligned}
&= \sum_{t=1}^{t_1}(a_{l+tn+1} - a_{l+(t+1)n}) - a_{l+n} + a_{l+1} \\
&+ \sum_{t=1}^{t_1}(-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j}) \\
&+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + (k'-r-1)(a_{l+1} - a_{l+n}) \\
&< \sum_{t=1}^{t_1}(a_{l+tn} - a_{l+(n-1)+tn}) - a_{l+n-1} + (a_l - 1) \\
&+ \sum_{t=1}^{t_1}(-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j}) \\
&+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + (k'-r-1)((a_l - 1) - a_{l+(n-1)}) \\
&= \sum_{s=(r+1)-2t_1-1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&= (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}
\end{aligned}
$$

In the last inequality, we used (21).

By 4.1(20), we have the first inequality below and by (43), we have the last inequality below. Hence,

$$
\begin{aligned}
a_{k'(n-1)+i'} \leq & a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\
&+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j} \\
=& a_{k'(n-1)-1} + \sum_{s=1}^{(r+1)-2t_1-2} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&+ \sum_{s=(r+1)-2t_1-1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j})
\end{aligned}
$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j}$$

$$\leq a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+ (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

and (38) is satisfied.

CASE E. Now assume (39), (40) and that for all $t \in \mathbb{N}$ such that $2 \leq (r+1)-2t$ ($\leq (r+1)$),

$$i_{((r+1)-2t)-2} + i_{((r+1)-2t)-1} = \begin{cases} n-2 & \text{if } (r+1)-2t > 2 \\ n-3 & \text{if } (r+1)-2t = 2. \end{cases}$$

Let t_1 be the largest t such that $2 \leq (r+1)-2t$. So $(r+1)-2t_1 = 2$ or 3.

SUBCASE 1. $(r+1)-2t_1 = 3$

As before (see proofs of (41) and (29)), it can be shown that

(44) $$i_{(r+1)-2t} \geq 1 \text{ for all } 1 \leq t \leq t_1+1 \text{ and that}$$

(45) $$(k'-((r+1)-2t))(n-1) + \sum_{j=0}^{(r+1)-2t-1} i_j = l + tn \text{ for all } 1 \leq t \leq t_1+1.$$

Now for $0 \leq j \leq r$, define

$$i'_j = \begin{cases} i_j - 1 & \text{if } j = (r+1)-2t \text{ and } 1 \leq t \leq t_1+1 \\ i_j + 1 & \text{if } j = (r+1)-2t-1 \text{ and } 0 \leq t \leq t_1+1 \end{cases}$$

(Note: $i'_0 = i_0 + 1 = i' + 1$, $i'_1 = i_1 - 1, \ldots$). Then $i'_j \in \mathbb{N}$ for all $0 \leq j \leq r$ (see (44)), $i'_0 + i'_1 < n-2$ and $i'_j + i'_{j+1} < n-1$ for $1 \leq j \leq r-1$.

We have,

(46)

$$a_{k'(n-1)+i'} - a_{k'(n-1)+i'+1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j})$$

$$+(k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j}$$

$$= \sum_{t=1}^{t_1+1} (a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{((r+1)-2t)-1} i_j+1} - a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j}$$

$$+a_{(k'-((r+1)-2t-1))(n-1)+i_0+\sum_{j=1}^{((r+1)-2t)-2} i_j} - a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{(r+1)-2t-1} i_j+1})$$

$$+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} + a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1}$$

$$+(k'-r-1)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1})$$

$$= \sum_{t=1}^{t_1+1} (a_{l+tn+1} - a_{l+(t+1)n}) - a_{l+n} + a_{l+1}$$

$$+ \sum_{t=1}^{t_1+1} (-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+i_0+\sum_{j=1}^{((r+1)-2t)-2} i_j})$$

$$+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + (k'-r-1)(a_{l+1} - a_{l+n})$$

$$\le \sum_{t=1}^{t_1+1} (a_{l+tn} - a_{l+(n-1)+tn}) - a_{l+n-1} + (a_l - 1)$$

$$+ \sum_{t=1}^{t_1+1} (-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+i_0+\sum_{j=1}^{((r+1)-2t)-2} i_j})$$

$$+a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + (k'-r-1)((a_l - 1) - a_{l+n-1})$$

$$= \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j})$$

$$+(k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}$$

In the last inequality, we used (21). By 4.1(20), we have

$$a_{k'(n-1)+i'+1} \le a_{k'(n-1)-1} + \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j})$$

$$+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j}$$

Hence by this inequality and (46),

$$\begin{aligned}
a_{k'(n-1)+i'} &\leq a_{k'(n-1)-1} + a_{k'(n-1)+i'} - a_{k'(n-1)+i'+1} \\
&+ \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\
&+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i'_j} + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i'_j} \\
&\leq a_{k'(n-1)-1} + \sum_{s=1}^{r} (a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&+ (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}
\end{aligned}$$

and (38) is satisfied.

SUBCASE 2. $(r+1) - 2t_1 = 2$

As before (see proofs of (41) and (29)), it can be shown that

(47) $$i_{(r+1)-2t} \geq 1 \text{ for all } 1 \leq t \leq t_1,$$

(48) $$(k' - ((r+1) - 2t))(n-1) + \sum_{j=0}^{(r+1)-2t-1} i_j = l + tn \text{ for all } 0 \leq t \leq t_1$$

(49) $$\text{and that } k'(n-1) = l + (t_1+1)n + 1.$$

Now for $0 \leq j \leq r+1$, define

$$i'_j = \begin{cases} 0 & \text{if } j = 0 \\ i' & \text{if } j = 1 \\ i_{j-1} - 1 & \text{if } j - 1 = (r+1) - 2t \text{ and } 1 \leq t \leq t_1 \\ i_{j-1} + 1 & \text{if } j - 1 = (r+1) - 2t - 1 \text{ and } 0 \leq t \leq t_1. \end{cases}$$

Then $i'_j \in \mathbb{N}$ for all $0 \leq j \leq r$ (see (47)), $i'_0 + i'_1 = i' < n-2$, $i'_1 + i'_2 = i' + i_1 + 1 < n-1$ and $i'_j + i'_{j+1} < n-1$ for $2 \leq j \leq r$.

We have,

$$
\begin{aligned}
(50)\quad & \sum_{s=1}^{r}\left(a_{(k'-s)(n-1)+\sum_{j=0}^{s} i'_{j+1}} - a_{(k'-s)(n-1)+\sum_{j=0}^{s+1} i'_{j+1}}\right) \\
& \quad + (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \\
& = \sum_{t=1}^{t_1}\left(a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{((r+1)-2t)-1} i_j+1} - a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j}\right. \\
& \quad \left. + a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j} - a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{(r+1)-2t-1} i_j+1}\right) \\
& \quad + a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} + a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} \\
& \quad + (k'-r-1)\left(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1}\right) \\
& = \sum_{t=1}^{t_1}\left(-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j}\right) \\
& \quad + a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + \sum_{t=1}^{t_1}\left(a_{l+tn+1} - a_{l+(t+1)n}\right) - a_{l+n} + a_{l+1} \\
& \quad + (k'-r-1)(a_{l+1} - a_{l+n}) \\
& \leq \sum_{t=1}^{t_1}\left(-a_{(k'-((r+1)-2t))(n-1)+\sum_{j=0}^{(r+1)-2t} i_j} + a_{(k'-((r+1)-2t-1))(n-1)+\sum_{j=0}^{((r+1)-2t)-2} i_j}\right) \\
& \quad + a_{(k'-r)(n-1)+\sum_{j=0}^{r-1} i_j} + \sum_{t=1}^{t_1}\left(a_{l+tn} - a_{l+(n-1)+tn}\right) - a_{l+n-1} + (a_l - 1) \\
& \quad + a_{l+(t_1+1)n} - a_{l+(t_1+1)n+1} - a_{l+(t_1+1)n+(n-1)} + a_{l+(t_1+2)n} \\
& \quad + (k'-r-1)((a_l - 1) - a_{l+n-1}) \text{ by (21) and (22).}
\end{aligned}
$$

In the last inequality we used (21) and in the last equality we used (49).

By 4.1(20), we have

$$\begin{aligned}
a_{(k'+1)(n-1)} \le\ & a_{(k'+1)(n-1)-1} + a_{k'(n-1)} - a_{k'(n-1)+i'} \\
& + \sum_{s=2}^{r+1}(a_{(k'+1-s)(n-1)+\sum_{j=0}^{s-1} i'_j} - a_{(k'+1-s)(n-1)+\sum_{j=0}^{s} i'_j}) \\
& + ((k'+1)-(r+1))a_{((k'+1)-(r+1)-1)(n-1)+\sum_{j=0}^{r+1} i'_j} \\
& + ((k'+1)-(r+1)-1)a_{((k'+1)-(r+1))(n-1)+\sum_{j=0}^{r+1} i'_j}
\end{aligned}$$

Hence by this inequality and (50),

$$\begin{aligned}
a_{k'(n-1)+i'} \le\ & a_{k'(n-1)-1} - a_{k'(n-1)-1} + a_{k'(n-1)} + a_{(k'+1)(n-1)-1} - a_{(k'+1)(n-1)} \\
& + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=1}^{s} i'_j} - a_{(k'-s)(n-1)+\sum_{j=1}^{s+1} i'_j}) \\
& + (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j+1} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j+1} \\
\le\ & a_{k'(n-1)-1} + \sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
& + (k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - 1) + (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}
\end{aligned}$$

and (38) is satisfied.

□

4.3. Corollary. $\mathbb{B}_N \subseteq \Phi(\mathcal{B}_N)$

Proof. (The proof is similar to that of Lemma 2.2).

Let $\boldsymbol{a} = \{a_s\}_{s\ge0} \in S_N$. We use induction on $\sum_{s\ge0} a_s$ to show that $\boldsymbol{b} = u_\infty \otimes \boldsymbol{a} \otimes \bar{0} \otimes t_{N\Lambda_0} \in \Phi(\mathcal{B}_N)$. If $\sum_{s\ge0} a_s = 0$, since $a_s \in \mathbb{N}$ for all s, then $a_s = 0$ for all s, so $\boldsymbol{b} = \Phi((\phi, \dots, \phi)) \in \Phi(\mathcal{B}_N)$. If $\sum_{s\ge0} a_s \ne 0$, let $\boldsymbol{a} = (\dots, 0, a_l, \dots, a_0)$ with $a_l \ne 0$

then as in the second part of the proof of Lemma 2.2, there exists a $k \in I$ such that $\tilde{e}_k(\boldsymbol{b}) = \tilde{e}_k(u_\infty \otimes \boldsymbol{a} \otimes \bar{0}) \otimes t_{N\lambda_0} = u_\infty \otimes \tilde{e}_k(\boldsymbol{a} \otimes \bar{0}) \otimes t_{N\Lambda_0} \neq 0$. $\tilde{e}_k(\boldsymbol{a} \otimes \bar{0}) = \tilde{e}_k(\boldsymbol{a}) \otimes \bar{0}$ (see the beginning of the proof of Lemma 4.2). By Lemma 4.2, $\tilde{e}_k(\boldsymbol{b}) \in \mathbb{B}_N$. By induction, $\tilde{e}_k(\boldsymbol{b}) \in \Phi(\mathcal{B}_N)$. So $\boldsymbol{b} = \tilde{f}_k(\tilde{e}_k(\boldsymbol{b})) \in \Phi(\mathcal{B}_N)$.

□

4.4. Lemma. *Let $k \in \mathbb{N}_{\geq 1}$, $r \in \mathbb{N}$ such that $1 \leq r \leq k-1$, $k_0 = k$, $k_1, \dots, k_r \in \mathbb{N}$, $i = i_0, \dots, i_r \in I \backslash \{n-1\}$ such that for $1 \leq s \leq r$, $(k_{s-1} - 1)(n-1) + i_{s-1} \leq k_s(n-1) + i_s$, and $c \in \mathbb{N}_{\geq 1}$ such that $c \leq k - r$.*

If $Y = \{y_t\}_{t \geq 0} \in \mathcal{B}_1$ and for $s \geq 0$, a_s is defined as in 3.17, then

$$\sum_{s=1}^{r}(a_{(k_{s-1}-1)(n-1)+i_{s-1}} - a_{k_s(n-1)+i_s}) + ca_{(k_r-1)(n-1)+i_r} - (c-1)a_{k_r(n-1)+i_r}$$
$$\geq \#\{t : t = k-1 \text{ and } y_{k-1} < -i\}.$$

Proof. Note 1: For all $1 \leq s \leq r$, $k_{s-1} - 1 \leq k_s$.

Note 2: For all $1 \leq s \leq r$, $(k-1) - s < k_{s-1} - 1$. (Proof: If $s = 1$, $k-2 < k-1$. Assume $s > 1$; then $(k-1) - s = (k-1) - (s-1) - 1 < k_{s-2} - 1 - 1 \leq k_{s-1} - 1$.)

$$\sum_{s=1}^{r} \left(a_{(k_{s-1}-1)(n-1)+i_{s-1}} - a_{k_s(n-1)+i_s}\right)$$
$$+ca_{(c-1)(n-1)+i_r} - (c-1)a_{k_r(n-1)+i_r}$$
$$= \sum_{s=1}^{r}(\#\{t : 0 \leq t \leq (k_{s-1} - 1) \text{ and } y_t < (t - (k_{s-1} - 1))(n-1) - i_{s-1}\}$$
$$-\#\{t : 0 \leq t \leq k_s \text{ and } y_t < (t - k_s)(n-1) - i_s\})$$

$$
\begin{aligned}
&+c\,a_{(k_r-1)(n-1)+i_r} - (c-1)\,a_{k_r(n-1)+i_r}\\
=\ & \sum_{s=1}^{r-1}(\#\{t : 0 \le t \le k-1-s \text{ and } y_t < (t-(k_{s-1}-1))(n-1)-i_{s-1}\}\\
&\quad -\#\{t : 0 \le t \le k-1-s \text{ and } y_t < (t-k_s)(n-1)-i_s\})\\
&\quad +\#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1)-i_{r-1}\}\\
&\quad -\#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-k_r)(n-1)-i_r\}\\
&\quad +\sum_{s=1}^{r}(\#\{t : k-s \le t \le (k_{s-1}-1) \text{ and } y_t < (t-(k_{s-1}-1))(n-1)-i_{s-1}\}\\
&\qquad\qquad -\#\{t : k-s \le t \le k_s \quad\text{and}\quad y_t < (t-k_s)(n-1)-i_s\})\\
&\quad +c\,a_{(k_r-1)(n-1)+i_r} - (c-1)\,a_{k_r(n-1)+i_r}\\
\ge\ & \#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1)-i_{r-1}\}\\
&\quad -\#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-k_r)(n-1)-i_r\}\\
&\quad +\#\{t : t = k-1 \text{ and } y_t < -i\}\\
&\quad +\sum_{s=1}^{r-1}(\#\{t : k-s-1 \le t \le k_s-1 \text{ and } y_t < (t-(k_s-1))(n-1)-i_s\}\\
&\qquad -\#\{t : k-s \le t \le k_s \text{ and } y_t < (t-k_s)(n-1)-i_s\})\\
&\quad -\#\{t : k-r \le t \le k_r \text{ and } y_t < (t-k_r)(n-1)-i_r\}\\
&\quad +c\,\#\{t : 0 \le t \le k_r-1 \text{ and } y_t < (t-(k_r-1))(n-1)-i_r\}\\
&\quad -(c-1)\,\#\{t : 0 \le t \le k_r \text{ and } y_t < (t-k_r)(n-1)-i_r\}
\end{aligned}
$$

(Here we have used the fact that $(k_{s-1}-1)(n-1)+i_{s-1} \le k_s(n-1)+i_s$.)

$$
\begin{aligned}
= \quad & \#\{t : t = k-1 \text{ and } y_t < -i\} \\
& + \sum_{s=1}^{r-1} (\#\{t : k-s \le t \le k_s \text{ and } y_{t-1} < (t-k_s)(n-1) - i_s\} \\
& \qquad - \#\{t : k-s \le t \le k_s \text{ and } y_t < (t-k_s)(n-1) - i_s\}) \\
& + c\#\{t : 0 \le t \le k_r - 1 \text{ and } y_t < (t-(k_r-1))(n-1) - i_r\} \\
& - c\#\{t : 1 \le t \le k_r \text{ and } y_t < (t-k_r)(n-1) - i_r\} \\
& - c\#\{t : t = 0 \text{ and } y_0 < (t-k_r)(n-1) - i_r\} \\
& + \#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1) - i_{r-1}\} \\
\ge \quad & \#\{t : t = k-1 \text{ and } y_t < -i\} \\
& + c(\#\{t : 1 \le t \le k_r \text{ and } y_{t-1} < (t-k_r)(n-1) - i_r\} \\
& - \#\{t : 1 \le t \le k_r \text{ and } y_t < (t-k_r)(n-1) - i_r\}) \\
& - c\,\#\{t : t = 0 \text{ and } y_0 < (t-k_r)(n-1) - i_r\} \\
& + \#\ \{t : 0 \le t \le k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1) - i_{r-1}\} \\
& \qquad\qquad \text{(by 3.16(i))} \\
\ge \quad & \#\{t : t = k-1 \text{ and } y_t < -i\} \\
& - c\,\#\{t : t = 0 \text{ and } y_0 < (t-k_r)(n-1) - i_r\} \\
& + \#\{t : 0 \le t \le k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1) - i_{r-1}\} \\
& \qquad\qquad \text{(by 3.16(i))}
\end{aligned}
$$

$$\geq \#\{t : t = k-1 \text{ and } y_{k-1} < -i\},$$

since by 3.16(i), if $y_0 < -k_r(n-1) - i_r$, then for all $t \in \mathbb{N}$,

$$\begin{aligned} y_t &\leq t(n-1) + y_0 \\ &< (t-k_r)(n-1) - i_r \\ &\leq (t-(k_{r-1}-1)(n-1) - i_{r-1} \qquad \text{by assumption.} \end{aligned}$$

Hence, in this case

$$\#\{t : 0 \leq t \leq k-1-r \text{ and } y_t < (t-(k_{r-1}-1))(n-1) - i_{r-1}\} = k - r \geq c.$$

□

4.5. Lemma. *If* $\mathbf{Y} \in \mathcal{B}_1$ *and for* $s \geq 0$, a_s *is as defined in 3.17, then* $\{a_s\}_{s\geq 0} \in S_1$.

Proof. Let $\mathbf{Y} = \{y_k\}_{k\geq 0}$. For $k \in \mathbb{N}$ and $i \in I\backslash\{n-1\}$,

$$a_{k(n-1)+i} = \#\{s : 0 \leq s \leq k \text{ and } y_s < (s-k)(n-1) - i\}.$$

(1) $a_0 \leq 1$.

(2) For $k \in \mathbb{N}$ and $0 \leq i \leq n-3$,

$$\begin{aligned} a_{k(n-1)+i+1} &:= \#\{s : 0 \leq s \leq k \text{ and } y_s < (s-k)(n-1) - i - 1\} \\ &= \#\{s : 0 \leq s \leq k \text{ and } y_s < (s-k)(n-1) - i\} \\ &\qquad -\#\{s : 0 \leq s \leq k \text{ and } y_s = (s-k)(n-1) - i - 1\} \\ &\leq a_{k(n-1)+i}. \end{aligned}$$

(3) For $k \in \mathbb{N}_{\geq 1}$ and $i \in I \backslash \{n-1\}$,

$$\begin{aligned}
a_{k(n-1)+i} &:= \#\{s : 0 \leq s \leq k \text{ and } y_s < (s-k)(n-1)-i\} \\
&= \#\{s : 1 \leq s \leq k \text{ and } y_s < (s-k)(n-1)-i\} \\
&\qquad + \#\{s : s = 0 \text{ and } y_0 < -k(n-1)-i\} \\
&= \#\{s : 0 \leq s \leq k-1 \text{ and } y_{s+1} < (s-(k-1))(n-1)-i\} \\
&\qquad + \#\{s : s = 0 \text{ and } y_0 < -k(n-1)-i\} \\
&\leq a_{(k-1)(n-1)+i} + \#\{s : s = 0 \text{ and } y_0 < -k(n-1)-i\}, \qquad \text{by 3.16(i)} \\
&\leq a_{(k-1)(n-1)+i} + \frac{1}{k} a_{k(n-1)-1},
\end{aligned}$$

since if $y_0 < -k(n-1)-i$, by 3.16(i),

$$\begin{aligned}
y_s \leq s(n-1) + y_0 &< (s-k)(n-1) - i \qquad \text{for all } s \in \mathbb{N}; \\
&\leq (s-k)(n-1) + 1.
\end{aligned}$$

Hence
$$\begin{aligned}
a_{k(n-1)-1} &= a_{(k-1)(n-1)+(n+2)} \\
&= \#\{s : 0 \leq s \leq k-1 \text{ and } y_s < (s-(k-1))(n-1)-(n-2)\} \\
&= \#\{s : 0 \leq s \leq k-1 \text{ and } y_s < (s-k)(n-1)+1\} \\
&= k.
\end{aligned}$$

(4) For $k \in \mathbb{N}_{\geq 1}$, $1 \leq r \leq k-1$, $i_0 = i \in I \backslash \{n-1\}$, $i_1, \ldots, i_r \in \mathbb{N}$,

$$a_{k(n-1)+i} = \#\{s : 0 \leq s \leq k \text{ and } y_s < (s-k)(n-1)-i\}$$

$$
\begin{aligned}
&= \#\{s : 0 \le s \le (k-1) \text{ and } y_s < (s-k)(n-1) - i\} \\
&\quad + \#\{s : s = k \text{ and } y_k < -i\} \\
&\le a_{(k-1)(n-1)+(n-2)} + \#\{s : s = k \text{ and } y_k < -i\} \\
&\quad \text{since } -(n-2) \ge -(n-1) - i \\
&\le a_{k(n-1)-1} + \#\{s : s = k-1 \text{ and } y_{k-1} < -i\} \qquad \text{by 3.16(i)} \\
&\le a_{k(n-1)-1} + \sum_{s=1}^{r} \left(a_{(k-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k-s)(n-1)+\sum_{j=0}^{s} i_j} \right) \\
&\quad + (k-r) a_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} \\
&\quad - (k-r-1) a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j},
\end{aligned}
$$

by Lemma 4.4.

□

4.6. Corollary. *$\Phi(\mathcal{B}_N) \subseteq \mathbb{B}_N$ for $N \in \mathbb{N}$.*

Proof. If $\mathbf{Y} = (Y_1, \ldots, Y_N) \in \mathcal{B}_N$, then for $1 \le j \le N$, $Y_j \in \mathcal{B}_1$, and for $s \in \mathbb{N}$, $a_s(\mathbf{Y}) = \sum_{j=1}^{N} a_s(Y_j)$. By Lemma 4.5, for $1 \le j \le N$, $\{a_s(Y_j)\}_{s \ge 0} \in S_1$. Hence $\{a_s(\mathbf{Y})\}_{s \ge 0} \in S_N$. The Corollary now follows from the definition of Φ and the definition of $\mathbb{B}_N$.

□

4.7. Theorem. *$\Phi(\mathcal{B}_N) = \mathbb{B}_N$ for $N \in \mathbb{N}$. Hence if ι and ι' are as in 3.17, then $(\Psi_{\iota,\iota'} \otimes \mathrm{id}_{N\Lambda_0}) \circ \tau_{N\Lambda_0}(B(N\Lambda_0)) = \mathbb{B}_N$ for $N \in \mathbb{N}$.*

Proof. (Corollaries 4.3 and 4.6 and Theorem 3.32.) □

4.8. Theorem. *Let* $\iota = (\dots,0,1,\dots,n-1,0,1,\dots,n-1,0)$ *and* $\iota' = (n-1,\dots,2,1,n-1,\dots,2,\dots,n-1,n-2,n-1)$. *Then the image of* $\Psi_{\iota,\iota'}$ *is the set of all elements of the form*

$$u_\infty \otimes \cdots \otimes b_{n-1}(-a_1) \otimes b_0(-a_0)$$
$$\otimes(b_{n-1}(-a_{1(n-1)}) \otimes \dots \otimes b_2(-a_{12}) \otimes b_1(-a_{11})) \otimes (b_{n-1}(-a_{2(n-1)}) \otimes \dots \otimes b_2(-a_{22}))$$
$$\otimes \dots \otimes (b_{n-1}(-a_{(n-2)(n-1)}) \otimes b_{n-2}(-a_{(n-2)(n-2)}) \otimes b_{n-1}(-a_{(n-1)(n-1)})$$

such that

$$0 \le a_{k(n-1)} \le a_{k(n-2)} \le \dots \le a_{kk} \text{ for } 1 \le k \le n-1,$$

$$a_{k(n-1)+i} \le a_{k(n-1)+i-1} \text{ if } k \in \mathbb{N} \text{ and } i \in I\backslash\{0,n-1\},$$

$$a_{k(n-1)+i} \le \frac{1}{k}a_{k(n-1)-1} + a_{(k-1)(n-1)+i} \text{ if } k \in \mathbb{N}_{\ge 1} \text{ and } i \in I\backslash\{n-1\},$$

$$a_{k(n-1)+i} \le a_{k(n-1)-1} + \sum_{s=1}^{r}(a_{(k-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k-s)(n-1)+\sum_{j=0}^{s} i_j})$$
$$+ (k-r)a_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k-r-1)a_{(k-r)(n-1)+\sum_{j=0}^{r} i_j}$$

if $k \in \mathbb{N}_{\ge 2}$, $r \in \mathbb{N}$ *such that* $1 \le r \le k-1$, $i = i_0, i_1, \dots, i_r \in \mathbb{N}$, $i_0 + i_1 < (n-2)$, *and* $i_j + i_{j+1} < (n-1)$ *for* $1 \le j \le r-1$.

Proof. (Theorem 2.3, Lemma 2.5, Theorem 4.7 and Corollary A.4).

□

4.9. Here we show that every inequality in 4.1(18) is needed to define S_N (for $N \ge 1$).

Let $k \in \mathbb{N}$ and $i \in I\backslash\{0, n-1\}$. For $s \in \mathbb{N}$ define

$$y_s := \begin{cases} -i-1 & \text{if } 0 \leq s \leq k \\ 0 & \text{otherwise} \end{cases}.$$

Then $Y := \{y_s\}_{s\geq 0} \in \mathcal{B}_1$. For $t \in \mathbb{N}$, let $a_t := a_t(Y)$ be as defined in 3.17. Then

$$a_t = \begin{cases} 1 & \text{if } t = k''(n-1)+j \text{ for } 0 \leq k'' \leq k \text{ and } 0 \leq j \leq i \\ 0 & \text{otherwise} \end{cases}.$$

By Lemma 4.5, $\{a_t\}_{t\geq 0} \in S_1 \subseteq S_N$, i.e. $\{a_t\}_{t\geq 0}$ satisfies 4.1(17)-(20).

For $t \in \mathbb{N}$, define $a'_t := \begin{cases} a_t - 1 & \text{if } t = k(n-1)+i-1 \\ a_t & \text{otherwise} \end{cases}$. Note that since $a_{k(n-1)+i-1} = 1$, $a'_t \in \mathbb{N}$ for all $t \in \mathbb{N}$. We will show that $\{a'_t\}_{t\geq 0}$ satisfies all of the inequalities in 4.1(18)-(20) except for

$$(51) \qquad a'_{k(n-1)+i} \leq a'_{k(n-1)+i-1}.$$

The only inequality in 4.1(18) in which $a'_{k(n-1)+i-1}$ appears in the right hand side is (51), so all of the other inequalities are satisfied by $\{a'_t\}_{t\geq 0}$. Since $a'_{k(n-1)+i} = 1$ and $a'_{k(n-1)+i-1} = 0$, 51 is not satisfied.

Since $a'_{(k+1)(n-1)+i-1} = 0$, then

$$a'_{(k+1)(n-1)+i-1} \leq \frac{1}{k+1} a'_{(k+1)(n-1)-1} + a'_{k(n-1)+i-1}.$$

This is the only inequality in 4.1(19) in which $a'_{k(n-1)+i-1}$ appears in the right hand side; hence all of the inequalities in 4.1(19) are satisfied by $\{a'_t\}_{t\geq 0}$.

Now let $k' \in \mathbb{N}_{\geq 2}$, $1 \leq r \leq k'-1$ and $i_0, i_1, \ldots, i_r \in \mathbb{N}$ with $i_0 + i_1 < n-2$ and $i_j + i_{j+1} < n-1$ if $1 \leq j \leq r-1$. Suppose $k(n-1)+i-1 = (k'-s_0)(n-1) + \sum_{j=0}^{s_0-1} i_j$ for some $1 \leq s_0 \leq r+1$. Then

$$k(n-1)+i = (k'-s_0)(n-1) + \sum_{j=0}^{s_0-1} i_j + 1$$

$$
\begin{aligned}
&\leq (k'-s_0)(n-1)+(s_0-1)(n-1)+i_0+1\\
&= (k'-1)(n-1)+i_0+1\\
&\leq (k'-1)(n-1)+(n-2)\\
&< k'(n-1)\\
&\leq k'(n-1)+i_0.
\end{aligned}
$$

Hence $a'_{k'(n-1)+i_0}=a_{k'(n-1)+i_0}=0$.

If $s_0=1$, $a_{(k'-1)(n-1)+i_0}=1$, and if $2\leq s_0\leq r+1$, $a_{(k'-s_0)(n-1)+\sum_{j=0}^{s_0-1}i_j}-a_{(k'-s_0+1)(n-1)+\sum_{j=0}^{s_0-1}i_j}=1-0=1$. Also, for all $1\leq s\leq r$, $-a_{(k'-s)(n-1)+\sum_{j=0}^{s}i_j}+a_{(k'-(s+1))(n-1)+\sum_{j=0}^{s}i_j}\geq 0$. So in either case,

$$
\begin{aligned}
&\textstyle\sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1}i_j}-a_{(k'-s)(n-1)+\sum_{j=0}^{s}i_j})+(k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r}i_j}\\
&\qquad -(k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r}i_j}\\
&= a_{(k'-1)(n-1)+i_0}+\textstyle\sum_{s=1}^{r-1}(-a_{(k'-s)(n-1)+\sum_{j=0}^{s}i_j}+a_{(k'-(s+1))(n-1)+\sum_{j=0}^{s}i_j})\\
&\qquad +(k'-r)(a_{(k'-r-1)(n-1)+\sum_{j=0}^{r}i_j}-a_{(k'-r)(n-1)+\sum_{j=0}^{r}i_j})\\
&\geq 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
a'_{k'(n-1)+i_0}=0\leq a'_{k'(n-1)-1} &+ \sum_{s=1}^{r}(a'_{(k'-s)(n-1)+\sum_{j=0}^{s-1}i_j}-a'_{(k'-s)(n-1)+\sum_{j=0}^{s}i_j}\\
&+ (k'-r)a'_{(k'-r-1)(n-1)+\sum_{j=0}^{r}i_j}\\
&- (k'-r-1)a'_{(k'-r)(n-1)+\sum_{j=0}^{r}i_j},
\end{aligned}
$$

and $\{a'_t\}_{t\geq 0}$ satisfies 4.1(20).

4.10. Here we show that every inequality in 4.1(19) is needed to define S_N ($N \geq 1$). Let $k \in \mathbb{N}_{\geq 1}$ and $i \in I\backslash\{n-1\}$. For $s \in \mathbb{N}$, define

$$y_s = \begin{cases} -(k-1)(n-1)-i-1 & \text{if } s=0 \\ (s-k)(n-1)-i-1 & \text{if } 1 \leq s \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y := \{y_s\}_{s\geq 0} \in \mathcal{B}_1$. If $a_t = a_t(Y)$, $t \in \mathbb{N}$, is as defined in 3.17, then

$$a_t = \begin{cases} \lfloor \frac{t}{n-1} \rfloor + 1 & \text{if } 0 \leq t \leq (k-1)(n-1)+i \\ \lfloor \frac{t}{n-1} \rfloor & \text{if } (k-1)(n-1)+i < t \leq k(n-1)+i \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.5, $\{a_t\}_{t\geq 0} \in S_1$, i.e. $\{a_t\}_{t\geq 0}$ satisfies 4.1(17) - (20). For $t \in \mathbb{N}$, define

$$a'_t = \begin{cases} a_t - 1 & \text{if } t = (k-1)(n-1)+i \\ a_t & \text{otherwise.} \end{cases}$$

Note that $a'_t \in \mathbb{N}$ for all t, since $a_{(k-1)(n-1)+i} = k \geq 1$.

We will show that $\{a'_t\}_{t\geq 0}$ satisfies all of the inequalities in 4.1(18) - (20) except for

$$(52) \qquad a'_{k(n-1)+i} \leq \frac{1}{k} a'_{k(n-1)-1} + a'_{(k-1)(n-1)+i}.$$

If $i \neq n-2$, then $a'_{k(n-1)-1} = a_{(k-1)(n-1)+(n-2)} = k-1$. If $i = n-2$, then $a'_{k(n-1)-1} = a'_{(k-1)(n-1)+(n-2)} = a_{(k-1)(n-1)+(n-2)} - 1 = k-1$. Also, $a'_{k(n-1)+i} = k$ and $a'_{(k-1)(n-1)+i} = k-1$. So in either case, 52 is not satisfied.

Now if $j \in I\backslash\{n-1\}$, $i = n-2$ and $j \neq i$, then $a'_{k(n-1)+j} = k$ and $a'_{(k-1)(n-1)+j} = k$. Hence $a'_{k(n-1)+j} \leq \frac{1}{k} a_{k(n-1)-1} + a'_{(k-1)(n-1)+j}$. So the only inequality in 4.1(19) which is not satisfied by $\{a'_t\}_{t\geq 0}$ is (52).

Now if $i \leq n-3$, $a'_{(k-1)(n-1)+i+1} = k-1$ and $a'_{(k-1)(n-1)+i} = k-1$. So all of the inequalities in 4.1(18) are satisfied by $\{a'_t\}_{t\geq 0}$.

Now let $k' \in \mathbb{N}_{\geq 2}$, $1 \leq r \leq k'-1$ and $i_0, i_1, \ldots, i_r \in \mathbb{N}$ with $i_0 + i_1 < n-2$ and $i_j + i_{j+1} < (n-1)$ if $1 \leq j \leq r-1$. We want to show that

$$(53)\quad a'_{k'(n-1)+i} \leq a'_{k'(n-1)-1} + \sum_{s=1}^{r}(a'_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a'_{(k'-s)(n-1)+\sum_{j=1}^{s} i_j}) + (k'-r)a'_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} + (k'-r-1)a'_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j}.$$

We consider three cases.

CASE 1 $(k-1)(n-1)+i = k'(n-1)-1$.

Then $k' = k$ and $i = n-2$. Hence $a'_{k(n-1)+i_0} = a_{k(n-1)+i_0} = k$ and $a'_{k'(n-1)-1} = k-1$. By Lemma 4.4,

$$\begin{aligned}
&\textstyle\sum_{s=1}^{r}(a'_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a'_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&\qquad + (k'-r)a'_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k'-r-1)a'_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \\
&= \textstyle\sum_{s=1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&\qquad + (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \\
&\geq \#\{t : t = k-1 \text{ and } y_{k-1} < -i\} = 1.
\end{aligned}$$

Hence (53) is satisfied.

CASE 2 $(k-1)(n-1)+i = (k'-1)(n-1)+i_0$.

Then $k = k'$ and $i_0 = i$. So $a_{k'(n-1)+i_0} = k$ and $a'_{k'(n-1)-1} = k-1$. So to get (53), we need to show that

$$\begin{aligned}
(54)\quad &\sum_{s=1}^{r} (a'_{(k-s)(n-1)+\sum_{j=0}^{s-1} i_j} - a'_{(k-s)(n-1)+\sum_{j=1}^{s} i_j}) \\
&+ (k-r)a'_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k-r-1)a'_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} \\
&\geq 1.
\end{aligned}$$

Let Y' be the tableau obtained from Y by removing the last two boxes of the first two columns of Y. So $Y' = \{y'_s\}_{s\geq 0}$, where

$$y'_s = \begin{cases} -(k-1)(n-1)-i & \text{if } s = 0 \text{ or } 1 \\ (s-k)(n-1)-i-1 & \text{if } 2 \leq s \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y' \in \mathcal{B}_1$ and if $a''_t := a_t(Y')$ for $t \in \mathbb{N}$ is as defined in 3.17,

$$a''_t = \begin{cases} \lfloor \frac{t}{n-1} \rfloor + 1 & \text{if } 0 \leq t < (k-1)(n-1)+i \\ \lfloor \frac{t}{n-1} \rfloor & \text{if } (k-1)(n-1)+i \leq t < k(n-1)+i \\ k-1 & \text{if } t = k(n-1)+i \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $0 \leq t < k(n-1)+i$, $a''_t = a'_t$, and that for all $1 \leq s \leq r$,

$$(k-s)(n-1) + \sum_{j=0}^{s-1} i_j < k(n-1)+i.$$

Hence by Lemma 4.4,

$$\begin{aligned} \sum_{s=1}^{r}(a'_{(k-s)(n-1)+\sum_{j=0}^{s-1} i_j} &- a'_{(k-s)(n-1)+\sum_{j=0}^{s} i_j}) \\ &+ (k-r)a'_{(k-r-1)(n-1)+\sum_{j=0}^{r} i_j} - (k-r-1)a'_{(k-1)(n-1)+\sum_{j=0}^{r} i_j} \\ &\geq \#\{t : t = k-1 \text{ and } y'_{k-1} < -i\} \\ &\geq 1. \end{aligned}$$

Hence (54) and (53) are satisfied.

Case 3 $(k-1)(n-1)+i = (k'-s_0)(n-1) + \sum_{j=0}^{s_0-1} i_j$ for some $2 \leq s_0 \leq r+1$.

If $0 \leq s \leq s_0 - 2$,

$$k(n-1)+i \;=\; (k'-s_0+1)(n-1) + \sum_{j=0}^{s_0-1} i_j$$

$$
\begin{aligned}
&= (k'-s)(n-1) + \sum_{j=0}^{s-1} i_j - (s_0 - s - 1)(n-1) + \sum_{j=s}^{s_0-1} i_j \\
&< (k'-s)(n-1) + \sum_{j=0}^{s-1} i_j \\
&\leq (k'-s)(n-1) + \sum_{j=0}^{s} i_j.
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 = a'_{k'(n-1)+i_0} = a'_{k'(n-1)-1} &= a'_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} \\
&= a'_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j} \text{ for all } 1 \leq s \leq s_0 - 2.
\end{aligned}
$$

So we need to show that

$$
\begin{aligned}
\text{(55)} \qquad \sum_{s=s_0-1}^{r} (a'_{(k'-s)(n-1)+\sum_{j=0}^{s-1} i_j} &- a'_{(k'-s)(n-1)+\sum_{j=0}^{s} i_j}) \\
&+ (k'-r)a'_{(k'-r-1)(n-1)+\sum_{j=0}^{r} i_j} \\
&- (k'-r-1)a'_{(k'-r)(n-1)+\sum_{j=0}^{r} i_j} \\
&\geq 0.
\end{aligned}
$$

SUBCASE A Suppose $s_0 \leq r$.

Let $k'' \in \mathbb{N}$ and $i'' \in I\backslash\{n-1\}$ be such that $(k'-(s_0-1))(n-1) + \sum_{j=0}^{s_0-2} i_j = k(n-1) + i - i_{s_0-1} = k''(n-1) + i''$. (Note: $k'' \geq k' - s_0 + 1 \geq k' - r + 1 \geq 2$.)

If $i_{s_0-1} \leq i$, then $k'' = k$, $i'' = i - i_{s_0-1}$, and $y_{k''} = y_k = -i - 1 < -i''$.

If $i_{s_0-1} > i$, then $k'' = k-1$, $i'' = (n-1) + i - i_{s_0-1}$, and $y_{k''} = y_{k-1} = -(n-1) - i - 1 < -i''$.

So in either case, $\#\{t : t = k'' \text{ and } y_{k''-1} < -i''\} = 1$.

By Lemma 4.4,

$$\begin{aligned}\sum_{s=s_0-1}^{r}(a_{(k'-s)(n-1)+\sum_{j=0}^{s-1}i_j} &- a_{(k'-s)(n-1)+\sum_{j=0}^{s}i_j})\\ &+ (k'-r)a_{(k'-r-1)(n-1)+\sum_{j=0}^{r}i_j}\\ &- (k'-r-1)a_{(k'-r)(n-1)+\sum_{j=0}^{r}i_j}\\ &\geq 1,\end{aligned}$$

and (55) and (53) are satisfied.

SUBCASE B Suppose $s_0 = r+1$.

Since $(k-1)(n-1)+i < k(n-1)+i-i_r \leq k(n-1)+i$,

$$a'_{(k'-r)(n-1)+\sum_{j=0}^{r-1}i_j} = a'_{k(n-1)+i-i_r} = \begin{cases} k-1 & \text{if } i_r > i \\ k & \text{if } i_r \leq i. \end{cases}$$

Then

$$\begin{aligned}a'_{(k'-r)(n-1)+\sum_{j=0}^{r-1}i_j} &+ (k'-r)(a'_{(k'-r-1)(n-1)+\sum_{j=0}^{r}i_j} - a'_{(k'-r)(n-1)+\sum_{j=0}^{r}i_j})\\ &= \begin{cases} k-1+(k'-r)(k-1-k) & \text{if } i_r > i \\ k+(k'-r)(k-1-k) & \text{if } i_r \leq i \end{cases}\\ &= \begin{cases} k-1-(k'-r) & \text{if } i_r > i \\ k-(k'-r) & \text{if } i_r \leq i \end{cases}\\ &\geq 0,\end{aligned}$$

since $(k'-r)(n-1)+\sum_{j=0}^{r}i_j = k(n-1)+i$ implies $k'-r \leq k$ and if $i_r > i$, $k'-r < k$. So (55) and (53) are satisfied.

Bibliography

[Cli98] G. Cliff. Crystal bases and young tableaux. *J. Algebra*, 202:10–35, 1998.

[JMMO91] M. Jimbo, K. Misra, T. Miwa, and M. Okado. Combinatorics of representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$ at $q = 0$. *Commun. Math. Phys*, 136:543–566, 1991.

[Jos95] A. Joseph. *Quantum groups and their primitive ideals*. Springer-Verlag, 1995.

[Kac85] V. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 1985.

[Kas91] M. Kashiwara. On crystal bases of the q- analogue of universal enveloping algebras. *Duke Math. J.*, 63:465–516, 1991.

[Kas93] M. Kashiwara. The crystal base and Littelmann's refined Demazure character formula. *Duke Math. J.*, 71:839–858, 1993.

[Kas94] M. Kashiwara. On crystal bases. *CMS Conference Proceedings*, 16:155–197, 1994.

[KN94] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical lie algebras. *J. Algebra*, 165:295–345, 1994.

[Lit95] P. Littelmann. Crystal groups and young tableaux. *J. Algebra*, 1:65–87, 1995.

[Lit98] P. Littelmann. Cones, crystals, and patterns. *Transform. Groups*, 3:145–179, 1998.

[MM90] K. Misra and T. Miwa. Crystal base for the basic representation of $\mathcal{U}_q(\hat{\mathfrak{sl}}(n))$. *Commun. Math. Phys*, 134:79–88, 1990.

[Nak99] T. Nakashima. Polyhedral realizations of crystal bases for integrable highest weight modules. *J. Algebra*, 219:571–597, 1999.

[NZ97] T. Nakashima and A. Zelevinsky. Polyhedral realizations of crystal bases for quantized Kac-Moody algebras. *Advances in Mathematics*, 131:253–278, 1997.

Appendix A

In **[Cli98]** and **[Lit98]**, the authors prove that if $\mathfrak{g}$ is of type A_{n-1} and $\iota = (n-1, \ldots, 2, 1, n-1, \ldots, 2, \ldots n-1, n-2, n-1)$, the image of Ψ_ι is the set of all elements of the form

(56)

$$u_\infty \otimes (b_{n-1}(-a_{1(n-1)}) \otimes \ldots \otimes b_2(-a_{12}) \otimes b_1(-a_{11})) \otimes (b_{n-1}(-a_{2(n-1)}) \otimes \ldots \otimes b_2(-a_{22}))$$

$$\otimes \ldots \otimes (b_{n-1}(-a_{(n-2)(n-1)}) \otimes b_{n-2}(-a_{(n-2)(n-2)})) \otimes b_{n-1}(-a_{(n-1)(n-1)})$$

such that

(57) $$0 \leq a_{k(n-1)} \leq a_{k(n-2)} \leq \ldots \leq a_{kk} \text{ for } 1 \leq k \leq n-1.$$

In this appendix, we use our results from Chapter 2 to give an alternative proof of this result.

A.1. Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 2$ and $I = \{1, 2, \ldots, n-1\}$. In **[KN94]**, it is shown that the crystal graph of $B(\Lambda_1)$ is given by $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \ldots \xrightarrow{n-2} \boxed{\text{n-1}} \xrightarrow{n-1} \boxed{\text{n}}$ and that if we view $B(N\Lambda_1)$ as a subset of $B(\Lambda_1) \otimes \cdots \otimes B(\Lambda_1)$ ($N-$ times), then

$$B(N\Lambda_1) = \{\boxed{a_1} \otimes \cdots \otimes \boxed{a_N} : \text{for } 1 \leq j \leq N,\ a_j \in \mathbb{N} \text{ and } n \geq a_1 \geq \ldots \geq a_N \geq 1\}.$$

A.2. Lemma. *Define the map* $\Phi : B(N\Lambda_1) \to B(\infty) \otimes B_\iota \otimes T_{N\Lambda_1}$ *by, for* $n \geq a_1 \geq \ldots \geq a_N \geq 1$,

$$\Phi(\boxed{a_1} \otimes \cdots \otimes \boxed{a_N}) = u_\infty \otimes b_{n-1}(-c_{n-1}) \otimes \ldots \otimes b_1(-c_1) \otimes \bar{0} \otimes t_{N\Lambda_1}$$

where for $1 \leq k \leq n-1$, $c_k := \#\{t : a_t > k\}$, *and* $\bar{0}$ *is as defined in the last paragraph of 2.1.*

Then $\Phi = (\Psi_\iota \otimes \mathrm{id}_{N\Lambda_1}) \circ \tau_{N\Lambda_1}$.

Proof. Let $b = \boxed{a_1} \otimes \cdots \otimes \boxed{a_N} \in B(N\Lambda_1)$. Then $\mathrm{wt}(b) = \sum_{j=1}^{N} \mathrm{wt}(\boxed{a_j}) = \sum_{j=1}^{N}(\Lambda_1 - \sum_{1 \leq k < a_j} \alpha_k = N\Lambda_1 - \sum_{j=1}^{N} \sum_{1 \leq k < a_j} \alpha_k = N\Lambda_1 - \sum_{k=1}^{n-1} c_k \alpha_k$. Hence $(\Psi_\iota \otimes \mathrm{id}_{N\Lambda_1}) \circ \tau_{N\Lambda_1}(b) = \Phi(b)$, since $\Phi(b)$ is the only element of $\{u_\infty\} \otimes B_{n-1} \otimes \cdots \otimes B_1 \otimes \{\bar{0}\} \otimes T_{N\Lambda_1}$ whose weight is equal to the weight of b.

□

A.3. Lemma. $\operatorname{Im}\Phi = \{u_\infty \otimes b_{n-1}(-c_{n-1}) \otimes \ldots \otimes b_1(-c_1) \otimes \bar{0} \otimes t_{N\Lambda_1} :$

for $1 \leq j \leq n-1$, $c_j \in \mathbb{N}$ *and* $N \geq c_1 \geq \ldots \geq c_{n-1} \geq 0\}$.

Proof. Let S be the set in the right hand side of the above equality. Let $n \geq a_1 \geq \ldots \geq a_N \geq 1$. If T is the Tableau with a_1 boxes in the first column, a_2 boxes in the second column, ... , and a_N boxes in the N^{th} column, then for $1 \leq j \leq n-1$,

$$c_j := \#\{t : a_t > j\} = \#\text{ of columns in } T \text{ which have } > j \text{ boxes}$$

$$= \#\text{ of boxes in the } (j+1)^{st} \text{ row of } T.$$

Hence $N \geq c_1 \geq c_2 \geq \ldots \geq c_{n-1} \geq 0$, and $\mathrm{Im}\Phi \subseteq S$.

Now let $N \geq c_1 \geq c_2 \geq \ldots \geq c_{n-1} \geq 0$. If T is the Tableau with N boxes in the 1^{st} row, c_1 boxes in the 2^{nd} row, ... , and c_{n-1} boxes in the n^{th} row, and if for $1 \leq t \leq N$,

a_t :=# of boxes in the t^{th} column of T, then $n \geq a_1 \geq \ldots \geq a_N \geq 1$ and for $1 \leq j \leq n-1$,

$$c_j = \# \text{ of boxes in the } (j+1)^{st} \text{ row of } T$$

$$= \# \text{ of columns in } T \text{ which have} > j \text{ boxes} = \#\{t : a_{t'} > j\}.$$

Hence $\Phi(\boxed{a_1}\otimes\cdots\otimes\boxed{a_N}) = u_\infty \otimes b_{n-1}(-c_{n-1})\otimes\ldots\otimes b_1(-c_1)\otimes\bar{0}\otimes t_{N\Lambda_1}$ and $S \subseteq \mathrm{Im}\Phi$.

□

A.4. Corollary.[Cli98] $\mathrm{Im}\Psi_\iota$ *is the set of all elements of the form in (56) which satisfy (57).*

Proof. (Induction, Theorem 2.3, Lemma 2.5, Lemma A.3 and Lemma A.2.)

□

Index