Developing Mathematics For Teaching Through Concept Study: A Case Of Pre-Service Teachers In Tanzania

BY

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Abstract

This descriptive qualitative case study investigates the question "*In what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?"* The concept studies were conducted in a classroom driven by five conditions: internal diversity, decentralized control, redundancy, neighbour interactions, and organized randomness (Davis & Simmt, 2003) that underpin complex system. Data was analysed using Davis and Renert's (2014) concept study model that includes realizations, landscapes, entailments, blending, and pedagogical problem solving, and Ball and colleagues (Ball, Thames, & Phelps, 2008) model categories of mathematical knowledge for teaching (MKT).

The study involved a group of ten pre-service teachers who were in their second year of the diploma in secondary education mathematics in a teacher college in Tanzania. They were majors taking one of the combinations: mathematics and physics, mathematics and chemistry, or mathematics and geography. Prior to the research, these pre-service teachers had only eight weeks teaching experience in their first year Block Teaching Practice (BTP). The pre-service teachers participated in 4 full day concept study sessions which involved three phases: pre-questionnaire, concept study workshop, and post questionnaire. Prior to the first concept study, the face-to-face interviews of all ten participants individually was conducted for the research to begin to shape a holistic understanding of them and their context. Four different concept studies of ratio, rate, proportion, and linear function were conducted for the group at an interval of a month from one concept study to another, with the researcher's facilitation as an emphatic second-person observer (Metz & Simmt, 2015). An empathic second-person observer is an observer that becomes a member of the social group while acting as a facilitator as he or she knows the kind of experience the participant is talking about.

The study reveals pre-service teachers' deep understanding of mathematics concepts from the school curriculum, and mathematics for teaching (MFT) was built through group interactions in the concept study. Explicitly, concept study provided the pre-service teachers the opportunities to learn the meaning of the targeted mathematics concepts, their symbolic and iconic representations, their applications outside the school environment, how they are related to other mathematics concepts, and it served to correct participants misconceptions of the mathematics concepts at hand. The study demonstrates the value of mistakes in understanding the mathematics concept and the value of collaboration in pre-service teacher education programs. The study illustrates how through collective work the participants enhanced their mathematics for teaching across Ball's MKT categories of common content knowledge and specialized content knowledge as subject content knowledge, and knowledge of content and teaching and knowledge of content and curriculum which fall under pedagogical content knowledge, with the exception of horizon content knowledge and knowledge of content and student. Although concept study reveals different emphases with pre-service teachers from in-service teachers the researcher illustrates concept study is a potentially viable strategy to use with pre-service teachers in Tanzanian teacher colleges/universities to enhance pre-service teachers' mathematics for teaching both knowledge and skills.

Key words: Concept study, Mathematics for Teaching, Mathematics teachers' professional knowledge, Teacher education

Preface

This thesis is the original research by Ratera Safiel Mayar. This teacher education research project, under the name "Developing mathematics for teaching through concept study. A case of ordinary level secondary school pre-service teachers in Tanzania" and its respective data collection, received ethics approval from the University of Alberta Research Ethics Board number Pro00066888 on August 25, 2016, and from Ministry of Education, Science, and Technology in Tanzania on August 28, 2016 (the third letter in Appendix A).

Dedication

My PhD work is dedicated to:

The love of my life Mr Chrisostom Mumena, for all the trouble he went through taking me to the Muhimbili Orthopaedic Institute hospital to attend the clinic and physiotherapy after the car accident during my data collection in 2016. And my lovely daughter Miss Maria C. Mumena, for encouraging me in every challenging situations of my life and for visiting in 2017 and spend time with me which was a great motivation to me. And the other members of my family and my relatives for always calling me and giving kind words of encouragement.

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List of Abbreviations

In this research, the following abbreviations have been used. The meanings shown to the right are the same as in the reviewed literature and few of these are of my own construction to suit the intended objectives.

ACSEE	Advanced Certificate of Secondary Education Examination
ACK	Academic Content Knowledge
BTP	Block Teaching Practice
CCK	Common Content Knowledge
CK	Content Knowledge
CT	Curriculum Teaching
CSEE	Certificate of Secondary Education Examinations
DED	District Executive Director
DEO	District Education Officer
DSE	Diploma in Secondary Education
DSEE	Diploma in Secondary Education Examinations
HCK	Horizon Content Knowledge
ICT	Information and Communication Technology
KCT	Knowledge of Content and Teaching
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Student
KDU	Key Developmental Understanding
MAT	Mathematics Association of Tanzania
ME	Mathematics Education
MFT	Mathematics for Teaching
MKT	Mathematical Knowledge for Teaching
MOEC	Ministry of Education and Culture
MoEST	Ministry of Education Science and Technology

MoEVT	Ministry of Education and Vocational Training
MSTHE	Ministry of Science, Technology, and Higher Education
MTE	Mathematics Teacher Education
NECTA	National Examination Council of Tanzania
PCK	Pedagogical Content Knowledge
PGC	Psychology, Guidance and Counselling
PUFM	Profound Understanding of Fundamental Mathematics
PUEM	Profound Understanding of Emergent Mathematics
RED	Regional Executive Director
REO	Regional Education Officer
REM	Research, Evaluation and Measurement
SCK	Specialized Content Knowledge
SLTP	Single Lesson Teaching Practice
SMCK	Subject Matter Content Knowledge
TCU	Tanzania Commission for Universities
TDMS	Teacher Development and Management Strategy
TEAMS	Teacher Education Assistance in Mathematics and Science
TIE	Tanzania Institute of Education
UDSM	University of Dar-es Salaam

1 Introduction to the Study

1.1 Background

Mathematics is an essential discipline that is applicable to a wide range of fields. However, to a large extent the success of mathematics education (ME) in any country and at any level depends on the qualities and effectiveness of its mathematics teachers. The qualities and effectiveness of the mathematics teachers depends on the quality of mathematics teacher education (MTE). The quality of mathematics teacher education depends on the pre-service mathematics teachers' preparation, including the programs they take, the courses within those programs, and kind of approaches used in the courses. With appropriate and effective approaches in pre-service teacher program courses could provide opportunities for teachers to develop their teaching of mathematics. Tanzania's Ministry of Education, Science, and Technology (MoEST) through local institutions such as the Tanzania Institute of Education (TIE) and University of Dar es Salaam (UDSM), and organizations such as the Mathematics Association of Tanzania (MAT) organise mathematics teachers' in-service professional development programs for mathematics in secondary schools to enable the teachers to receive high quality MTE. Sometimes the secondary school mathematics teachers also get support from donor-funded projects such as the Netherlands funded Teacher Education Assistance in Mathematics and Science (TEAMS) to help mathematics teachers improve the mathematics teaching and learning in secondary schools. Despite all these efforts in both pre-service and in-service education to support mathematics teachers in secondary schools, the percentage of students who pass mathematics in the Certificate of Secondary Education Examination (CSEE) has declined since 2008. The average percentage pass (Grade A to D) in CSEE for eight consecutive years from 2008 to 2015 as collated from

National Examination Council of Tanzania (NECTA) CSEE statistics of 2009, 2011, 2013, 2014, and 2015 is as illustrated in figure 1. Thus, mathematics education in ordinary level secondary schools in Tanzania is of great concern to the government.

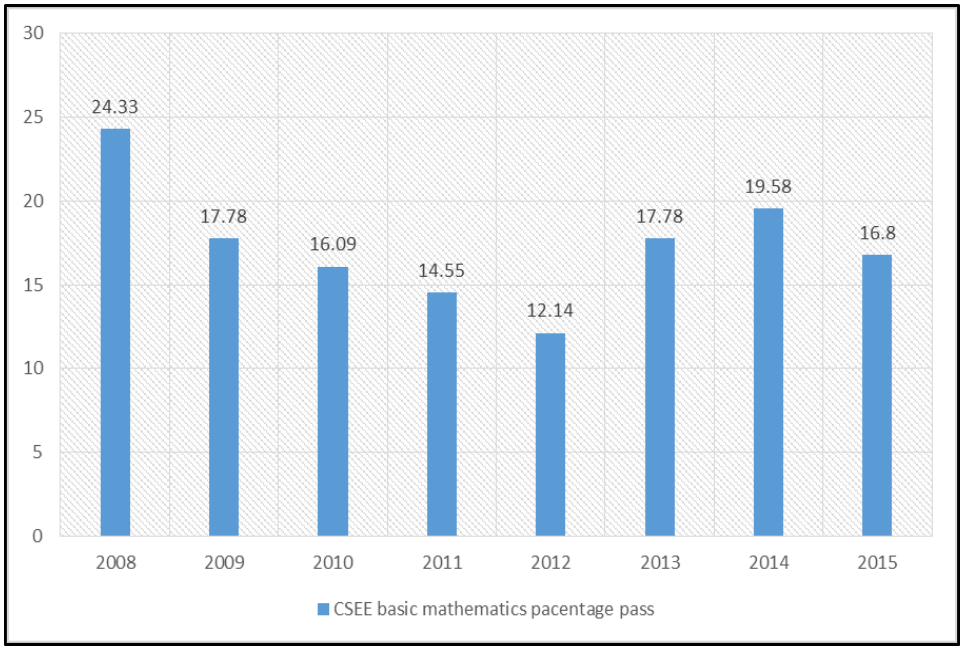


Figure 1: Student average percentage pass (grade A to D) of Basic mathematics in CSEE results from 2008 to 2015

To qualify for teaching at the secondary school teachers must present (at minimum) a diploma level teaching certificate. Most of the CSEE candidates are taught by teachers who have education diplomas, not education degrees (Table 1). For every 2 teachers with a degree there are 3 teachers with a diploma.

Qualification type	**PhD**	**Master**	**Degree**	**Diploma**	**Licence**	**Others**	**Grand Total**
Total Teachers	117	1070	24195	33149	4697	1858	65086
Total Qualified Teachers	90	676	21240	33149			55155*

Source: MoEVT (2012) National Basic Education Statistics in Tanzania (BEST), 2012

Note: Total qualified teachers are teachers who went through teaching training and have diploma level to PhD level certificates. Total teachers include both qualified teachers and non-qualified teachers—teachers who do not do any teaching training but are teaching in secondary schools. Twenty seven PhD holders in other fields have no teacher training at any level of their education but are teaching. Ninety PhD holders have a qualification in teaching.

Table 1: Government and non-government secondary school teachers in 2012

The reliance on teachers who have only a diploma is believed to be one of the reasons for the failure rates; it is conjectured that those teachers receive insufficient training in college (Chonjo, Osaki, Possi, & Mrutu, 1996). The majority of these teachers lack sufficient knowledge of the subject matter, what to teach, and how to teach the subject matter effectively (Chonjo et al., 1996; Sichizya, 1997). Kitta (2004), in a needs assessment about the quality of mathematics teachers in secondary schools, asserts that school inspectors (currently known as school quality assurers) acknowledged that "the quality of most mathematics teachers is poor, as they are both deficient in terms of subject matter knowledge and teaching skills" (p. 27). Kitta argues that the school inspectors correlate the teachers' insufficient training to poor educational backgrounds. According to Kitta, one of the factors associated with the low quality of teaching in the secondary school which all interviewed teacher educators acknowledged is "lack of regular in-service education programmes" (p. 27). One can speculate that these teachers lack "pedagogical content knowledge" (PCK); that is "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). In other words, they lack the knowledge of how to make mathematical ideas understandable to learners, knowledge of what learners find difficult in mathematics, and knowledge of learners' typical insights and misconceptions (Shulman). It is critical to find an approach to work with diploma in secondary education pre-service teachers and the respective in-service teachers in a way that will improve mathematics teaching in ordinary level secondary schools.

Another issue concerning MTE is the criticism about mathematics teaching not only in schools but also in teacher colleges in Tanzania, as articulated in the Teacher Development and Management Strategy (TDMS) 2008 to 2013 report (MoEVT, 2013). That report suggests that there exist challenges such as: an acute shortage of teachers in the sciences and mathematics (Table 2); poor quality of teaching; and curriculum content that lacks a coherent structure for all teacher education programmes. Regarding the issue of the quality of teaching in schools and in teacher training colleges, the teacher educators, tutors, curriculum developers, and the government in general, should all ask what kind of training do we want for teachers. Do we want to train teachers to imitate the poor teaching they experienced as students in school and teacher colleges—teacher centred instruction with little or no attention conceptual understanding relating and applying their knowledge to their everyday life? Or do we want to prepare teachers to be active facilitators in trying to change the way mathematics is taught (meaning, content, and methods), or teachers who will facilitate conceptual understanding and will be able to develop their students' competences? In other words, to prepare teachers for *effective mathematics teaching*. But, what is teaching? Teaching has several meanings, but I consider the one offered by Hiebert and Grouws (2007). Hiebert and Grouws defined teaching as "classroom interactions among teachers and students around content directed toward facilitating students' achievement of learning goals" (p. 372). With this definition, the major role of teaching is to shape students' learning opportunities.

Status of teacher needs	**Biology**	**Chemistry**	**Mathematics**	**Physics**
Number available	5111	4303	4416	2649
Required	10292	9676	11707	9522
Shortage	5181	5373	7291	6873
Percentage shortage	50.34	55.53	62.28	72.18
Source: MoEST (2016), National Basic Education Statistics in Tanzania (BEST), 2012-2016				

Table 2: Teacher's shortages in science-related subjects and mathematics in government secondary schools.

Similarly, the TDMS could be criticized in terms of the dramatic declines between 2012 and 2016 in the number of pre-service teachers enrolled to become science teachers (this includes mathematics) at diploma level for both government and non-government teacher education colleges in Tanzania (Table 3). The percentage of pre-service science teachers enrolled in diploma courses in 2012 dropped to 1.7% of the annual cohort compared to 2008 when it was 32.3%. The fact which correlates with only 86 graduates in diploma in secondary education science and mathematics subjects between 1999 and 2014 (MoEST, 2017). Out of 86 graduates, 31 were mathematics majors. Among 31 mathematics major graduates, 13 specialized in physics and mathematics, 10 in chemistry and mathematics, and 8 in geography and mathematics. There were 1459 diploma in secondary education graduate of science and mathematics in 2015 (MoEST, 2017). This indicates a dramatic increase in enrolment, but the shortage remains high.

Enrolment type		**2008**	**2009**	**2010**	**2011**	**2012**
Science pre-service teachers	Female	2849	5748	6772	1439	279
	Male	4217	7443	9388	2574	438
	Total	7066	13191	16160	4013	717
Total pre-service teachers	Female	10427	17277	16597	17313	18898
	Male	11441	18256	20051	20385	24360
	Total	21868	35533	36648	37698	43258
Percentage of science pre-service teachers		32.3	37.1	44.1	10.6	1.7
Source: MoEVT (2012) National Basic Education Statistics in Tanzania (BEST), 2012						

Table 3: Enrolment of pre-service teachers at the diploma level in science combinations compared to total enrolment for both government and non-government teacher's colleges from 2008-2012

The TDMS criticism is also reflected in pre-service teacher admissions to mathematics teacher education programs at the diploma level from 2006-2012 in government and non-government diploma teacher's colleges (Table 4). This dramatic decline in enrolment of diploma pre-service teachers in science is evidence that pre-service teachers do not opt for science subject combinations that include mathematics. It might also be a signal of student lack of interest in mathematics as their future career, including mathematics teaching. This is highly problematic since mathematics at ordinary level secondary school is a compulsory subject for all students.

What possible approach might be employed so that pre-service teachers can learn the math that they need to know in order to teach at ordinary level secondary schools?

Mathematics	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Female	57	69	110	77	53	63	67
Male	381	507	636	538	285	418	594
Total	438	576	746	615	338	481	661
Source: Collated from NECTA-Statistics (Teacher education, Diploma in Secondary Education Examination (DSEE)), 2008-2014.							

Table 4: Pre-service teacher admissions to mathematics teacher education at diploma level 2006-2012 in government and non-government diploma teacher's colleges in Tanzania

The variations in teacher educators' conceptions about MTE in Tanzania is another signal that calls for our attention. Binde (2010) found that teacher educators have different conceptions about what MTE is ranging from MTE as a process of an individual to become a mathematics teacher, to MTE as a merger between subject matter and pedagogical knowledge, and to understanding MTE as learning about teaching. Binde found that teacher educators' dominant conception of MTE is that it is the integration of subject matter and pedagogical knowledge. He also found teacher educators' thoughts on professional development centre mostly around pedagogical knowledge and skills. Further, Binde asserts that teacher educators' conceptual variations about MTE are a "result of their diverse historical background; ...possible grounds for differences in making pedagogical decisions, while at the same time telling about MTE as it is;

and some mathematics teacher educators see mathematics as an unquestionable field of knowledge” (p. 232-233).

Finally, in terms of MTE internationally, the complexities of mathematics teachers’ disciplinary knowledge (Davis, Sumara & Luce-Capler, 2008) needed for the effective teaching of mathematics (Hiebert & Grouws, 2007) is also a challenging problem and it has put significant pressure on researchers, teacher educators and practitioners around the world to imagine and develop curriculum and pedagogy for the teacher education classroom (Ball, 2017). The research has been focusing on the kinds of mathematical knowledge that pre-service and in-service teachers need to *know,* as well as the *know how* to teach mathematics effectively curtailed as “Mathematics for Teaching (MFT)” (Ma, 1999; Ball & Bass, 2003; Ball, Hill & Bass, 2005; Davis & Simmt, 2006; Adler & Davis, 2006; Ball, Thames, & Phelps, 2008; Davis & Renert, 2014).

Every person has the ability to learn mathematics; though some learn and make connections more quickly than others. In this research, the know-how of MFT (Davis & Renert, 2014) is the skills the teachers need. For example, the teachers need the skills to

- unpack the curriculum,
- select good applications of the mathematical concepts/topics,
- provide multiple explanations of the mathematical concepts/topics,
- provide good examples of the mathematical concepts/topics, and
- plan the lesson.

There is a need to explore suitable teaching strategies for teacher training colleges, to build on the mathematics studied by diploma in secondary education pre-service teachers so that their education is much more suited to teaching than current strategies. In this research, I ask, ***in what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?***

1.2 Rationale

Based on Binde's (2010) argument about the possibilities of teacher educators making different pedagogical decisions due to conceptual variations about MTE, this could be another signal that there exists a problem of mathematics teacher education regarding what pre-service teachers *need to know* and *know how* as future mathematics teachers in ordinary level secondary schools in Tanzania. Adler and Davis (2006) also raise a similar question on what kind and how much mathematics middle school and senior school teachers need to know and know how to use in order to successfully teach mathematics. These questions are important in order to at least conjecture what knowledge teachers need to effectively teach mathematics (Hiebert & Grouws, 2007).

The background of this research suggests that ordinary secondary school teachers in Tanzania lack PCK, skills, and the complexities of disciplinary knowledge needed to teach mathematics effectively. It is very difficult to have a clear picture of how and to what extent these pre-service teachers demonstrate knowledge and know-how regarding PCK unless we employ an approach that might answer this question. I propose the ***concept study approach*** (Davis & Renert, 2014) to explore this question. Davis (2013) defined concept study as "a participatory, collaborative structure for teachers to engage with one another in the examination and elaboration of mathematical understandings" (p. 5), "a structure that is intended to provide the teachers with the sorts of experiences and attitudes that might cultivate disciplinary knowledge founded on conceptual diversity" (Davis & Renert, 2014, p. 38). Concept studies are deliberately structured to foreground teachers' knowing and knowledge and teachers' knowing and knowledge of how mathematics is learned. Davis and Renert demonstrated how a concept study could be used to enhance in-service teachers' deep understanding of mathematics needed for teaching. I have been unable to find research that concentrates on the use of concept study for pre-service teachers' MFT and to my knowledge there is no research on this topic specific to Tanzania or East Africa. My research addresses this gap as it focuses on pre-service teachers' mathematics for teaching using a concept study approach in a Tanzanian context.

1.3 The General Aim of the Study

In view of the background and the rationale, the aim of this study is to investigate the contribution of concept study on the MFT (professional knowledge and skills) of Tanzanian pre-service teachers who are studying for the diploma in secondary education. The research question guiding this work is ***"In what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?"*** The results of this study are expected to contribute to a rich discussion of teacher education specific to mathematics for teaching in Tanzania, and inform curriculum design and instruction for teacher education.

1.4 Significance of the Study

In my view, this study has some practical significance. First, it is my expectation that the results of this study could be used to enhance pre-service teachers' deep understanding of mathematics needed for teaching. Second, this study might serve as a major source in designing the pre-service curriculum for MTE in Tanzania as well as a useful source for an in-service mathematics teachers' professional development program. Third, it might contribute to solutions for the problems of negative attitudes in mathematics learning for teachers and students. Additionally, apart from practical applications, the study has the potential to contribute to the theorization of mathematics for teaching, especially the use of concept study in teacher preparation programs.

1.5 Context of Teacher Education in Tanzania

1.5.1 Introduction

Before December 2015, the management and provision of teacher education programmes in Tanzania were the responsibility of two ministries: Ministry of Education and Vocational Training (MoEVT) and Ministry of Science, Technology, and Higher Education (MSTHE). The MoEVT was responsible for preparing certificate and diploma teachers (generally primary school and ordinary level (junior high) secondary school teachers), and vocational education teachers. The MSTHE was responsible for the preparation of undergraduate and postgraduate teachers (advanced level secondary school and college tutors). The curricula for undergraduate pre-service teachers and postgraduate teachers were decentralised, and it was the responsibility of the respective university. The curricula for diploma and certificate teachers were centralised, while co-ordination and monitoring remained under the government (MoEVT) through the Tanzania Institute of Education (TIE). On December 10, 2015, the new president of the Republic of Tanzania, the Honourable Dr. John Pombe Joseph Magufuli, merged the two ministries and renamed the new entity the Ministry of Education, Science, and Technology (MoEST). Since 2015, the new ministry is responsible for managing all levels of the teacher education. However, the responsibility for the teacher education curriculum for both the certificate and diploma in education in teacher colleges still is centralised, while co-ordination and monitoring remained under the government (MoEST) through the TIE. For the degree level in education, the curriculum remained solely the responsibility of the university. The University of Dodoma has offered some special programs for diploma courses in education under MoEST instructions but at the time of this publication this is exceptional.

1.5.2 The general aims and objectives of teacher education in Tanzania

The new Tanzanian education and training policy of 2014 document from the MoEST do not indicate the objectives that are specific to the teacher education. Hence curriculum for the teacher education programme studied by the diploma in secondary education participants involved in this research uses the centralised curriculum for the diploma in teacher education programme in Tanzania (MoEVT, 2007), as designed by the TIE. This curriculum was prepared using the former education and training policy document of Ministry of Education and Culture (MOEC) (1995). It would be wise to provide its objectives for teacher education to allow the reader to see how they connected to the curriculum. According to an education and policy training document from the MOEC (1995), the aims and objectives of teacher education are:

- to impart to teacher trainee theories and principles of education, psychology, guidance, and counselling;
- to impart to teacher trainee principles and skills of pedagogy, creativity, and innovation;
- to promote an understanding of the foundations of school curriculum;
- to sharpen the knowledge of teacher trainees and ensure their mastery of selected subjects, skills, and technologies;

- to impart skills and techniques of research, assessment, and evaluation in education (p. 20)

1.5.3 Entry qualifications for teacher education in Tanzania

The entry qualification for each level of teacher education in Tanzania relies on the performance of the national school-leaving examinations. MoEVT's (2007, 2012) minimum admission requirements for the certificate in teacher education is Division III of the Certificate of Secondary Education Examination (CSEE) while for the diploma is Division III of the Advanced Certificate of Secondary Education Examination (ACSEE). Additionally, for the diploma program, pre-service teachers must principally have passed two teaching subjects at ACSEE. The minimum principal pass grade is an 'E'. Certificate teachers are assigned to teach primary schools, while diploma teachers are assigned to teach secondary school (Form One and Form Two). However, diploma teachers have been teaching up to Form IV. Kitta (2004) stated that before 1995, the minimum entry qualification for pre-service certificate and diploma teachers was Division IV of the CSEE and Advanced Certificate of Secondary Education Examination (ACSEE) respectively. Kitta noted that in 1995 the government raised this requirement for pre-service teachers to Division III with a view to improving the "quality and competence of secondary school teachers" (p. 21). Degree teacher education programs are offered in universities. Each university sets its own minimum entry qualifications regarding the central one sets by the Tanzania Commission for Universities (TCU) yearly. The focus of my study is diploma in secondary education teacher program taught at teacher's colleges, therefore I explain in details this program in the next section 1.5.4.

1.5.4 Diploma in secondary education teacher education program

Since the pre-service teacher participants in this study are in the diploma in secondary education program, more detail is important for an advanced holistic understanding of the participants. I explain in detail in the next six sub-sections. The first sub-section describes the specific objectives of the diploma in secondary education and the pre-service teacher competences to be developed while the second sub-section describes the structure of the programme. The third sub-section describes the tutors' qualification and their tutoring loads while the fourth describes the teaching and learning materials used in the program. The fifth sub-section describes the recommended model of the teaching methodology while the sixth sub-section describes how the pre-service teacher achievements are assessed.

1.5.4.1 Specific objectives

The Diploma in secondary education pre-service teacher education programs use the centralised curriculum for Diploma in teacher education programs in Tanzania (MoEVT, 2007), as designed by the TIE. The medium of instruction for Diploma level programs is English. According to this curriculum, the specific objectives of the Diploma in secondary education are focused to enables the pre-service teachers to:

- acquire a basic understanding of the nature, purpose and philosophy of secondary education;
- develop a basic understanding of the psychology of children and adolescents;
- understand the process of socialization of learners;
- make a content and pedagogical analysis of the subject they will teach in secondary schools;

- develop guidance and counselling skills;
- develop communication skills and the use of modern information technology;
- acquire competencies in curriculum implementation, classroom presentation, use of educational media and technology, assessment and evaluation;
- acquire basic research skills in education including action research;
- promote creative and critical thinking skills among learners;
- develop an understanding of factors and forces affecting society including crosscutting issues;
- acquire entrepreneurial skills and attitudes;
- develop an understanding of the professional character of teaching; and
- promote student teachers' awareness of teacher ethics. (MoEVT, 2007, p. 6)

1.5.4.2 Structure of the program

According to the curriculum for teacher education programs in Tanzania (MoEVT, 2007), the diploma in secondary education has been packaged in four terms within two years of schooling. Each term has five months. The year of schooling begins in July in accordance with the government financial year. The number of days in schooling is 194 per year which is equivalent to 48.5 weeks and 64 hours in classes per year respectively. The terms of schooling in a year differs in hours. For each academic year, the first term requires 40 hours while the second term requires 24 hours. For each academic year, in the second term, the pre-service teachers have eight weeks (or 24 hours) Block Teaching Practice (BTP). The duration of each subject period for classroom instructions is one hour.

The diploma in secondary education programme has three learning areas: professional studies, academic courses and teaching methods, and general courses. The professional studies include:

- Curriculum and Teaching for secondary education (CT)
- Foundations of Secondary Education (FOE)
- Psychology, Guidance and Counselling for secondary education (PGC)
- Research, Evaluation and Measurement (REM).

Academic courses and teaching methods include: Academic Content Knowledge (ACK) and Pedagogical Content Knowledge (PCK). Academic courses deal with both the courses of two core subject majors for instance Mathematics and Physics. According to MoEVT (2007), ACK improves pre-service teachers' academic competences of the teaching subject such as mathematics while PCK develops in the pre-service teacher the methods, strategies, and techniques used in the teaching and learning of the academic subjects, such as mathematics.

- Core Subject I (Content and Teaching Methods).
- Core Subject II (Content and Teaching Methods).

For example, if the two core subjects are Mathematics and Physics then the pre-service teacher will learn

- Mathematics (content and teaching methods)
- Physics (content and teaching methods)

The general courses for all teacher's colleges include:

- Development Studies

- Information and Communication Technology (ICT)
- Educational Media and Technology
- Communication Skills
- Project Work
- Religion

1.5.4.3 Tutor qualifications

Tutors are college level instructors. MoEVT (2007), identified four major qualities a competent and an effective tutor should have: First, the tutor should have minimum academic skills which, is a Master degree in teacher education. Second, the tutor should have some experiences in teaching at secondary schools at least three years and has attended related short courses/seminars. Third, the tutor should have adequate participation in 'teaching practice'. And fourth, the tutor needs to have the professional development which includes:

- Short courses
- Long courses such as PhD
- Seminars
- Workshops
- Symposia
- Conducting educational research

1.5.4.4 Teaching methodology recommended model

The MoEVT (2007) diploma curriculum recommends the model for the methods to be used in teaching in the diploma in secondary education as student centred and interactive. That means the tutors are facilitators for student learning activities and not the sole sources of knowledge production. The pre-service teachers are encouraged to undertake responsibility for their own individual learning. The academic parts of the course are implemented by "interactive lectures, self-study, seminars, media supported teaching, and practical activities" (p. 22). These strategies are expected to enable the pre-service teachers "to acquire the stated competencies that are critical for making reflective practice and committed teaching" (p. 22). The pre-service teachers are responsible for conducting pedagogic analysis in school curriculum teaching subjects. This analysis is executed using a variety of learning experiences that include "micro-teaching, demonstrations, peer group teaching, single lesson teaching practice and materials production workshops and portfolios" (p. 22). The learner centred model, emphasises the following practices:

- engage students in active learning experiences;
- set high and meaningful student learning expectations;
- provide, regular and timely feedback;
- recognize and respond to different student learning styles and promote the development of multiple intelligences;
- real life applications;
- understand and apply different techniques of student assessment; and
- create opportunities for student- tutor interactions and student - student interactions. (MoEVT, 2007, p. 22)

1.5.4.5 Assessing pre-service teacher achievement

According to the MoEVT (2007), the pre-service teachers are assessed through continuous and final assessments. In the continuous assessment process, tutors administer assessments in various ways over time "to allow them to observe multiple tasks and to collect information about what pre-service teachers know, understand, and can do" (p. 23). Block Teaching Practice (BTP), Single Lesson Teaching Practice (SLTP), and microteaching are also assessed. The setting of assessment exercises focuses on what pre-service teachers know, understand, and can do. Continuous assessment contributes 50 per cent in the final assessment of the pre-service teachers. The following continuous and final assessments procedure are employed.

First, the continuous assessment includes:

- written exercises, tests and examination;
- self-assessment portfolio;
- seminar presentation;
- SLTP;
- essay/report writing;
- micro-teaching;
- practical sessions and projects; and
- BTP. (p. 23)

The BTP have minimum of five assessments which, three are conducted for the first year and two during the second year. During BTP, the tutors, headmasters/headmistress, and subject academic officers are responsible in submitting the assessment records. However, the tutors are responsible to compile all assessment records conducted by headmasters/headmistress, and subject academic officers. The pre-service teacher BTP assessment records are submitted to the college for compilation and submission to the NECTA.

Second, the final assessment includes 50 per cent each for both continuous assessment and final examination respectively for any course programme as illustrated in table 5. According to MoEVT, (2007), NECTA, is responsible for the administration of the final examinations for the course programme. NECTA is also responsible for the accreditation and certification in teacher education Diploma in Secondary Education Examinations (DSEE). In order to qualify for the award of a diploma in secondary education certificate offered by NECTA, a candidate is required to achieve a pass in all the theoretical courses and teaching practice. And, NECTA sets the minimum pass mark and the classification of the certificate in this course (MoEVT). The exit conditions include the following;

- a candidate shall be required to pass all examinations including teaching practice;
- the passes shall be classified as first class, second, third class and fail; and
- any candidate who fails in Teaching Practice shall be considered a total failure. (p. 24)

Table 5 shows a structure of continuous assessment and final examinations of mathematics as one of the teaching subject's courses in the diploma in secondary education.

Name of the course	Type of assessment					
	Continuous assessment			Final exams		Total
		Frequency	%	Frequency	%	%
Teaching subject 1 (Mathematics)	Tests	4	5	1	25	50
	Seminar presentation	2	5			
	Portfolio	2	5			
	Terminal examinations	3	10			
Teaching Methods 1 (Mathematics teaching methods)	Tests	4	5	1	25	50
	Microteaching	3	10			
	Terminal examinations	3	10			
Source: Collated from structure of the continuous assessment and final examination table (MoEVT, 2007, p. 24)						

Table 5: The structure of the continuous assessment and final examination of the mathematics as the teaching subject

1.5.5 Degree level teacher education overview

Some of the CSEE candidates in Tanzania are taught basic mathematics by teachers with degree qualifications, as explained in the background section. It is wise to give readers an overview of the degree teacher education programme for better holistic understanding of the teacher education system in Tanzania. The curriculum for the degree teacher programs offered in Tanzanian universities is their responsibility. Not surprising then is that the curricula for undergraduate pre-service teachers varies from one university to another, though all follow the objectives of the teacher education policy. They share the following unique characteristics: duration (three years), degree specialization offered, and length of teaching practice (eight weeks at end of year one and year two respectively). The pre-service teachers are enrolled in degree programmes such as Bachelor of Science with Education (BSc. with Education) or Bachelor of Arts with Education (B.A. with Education) or Bachelor of Education in Science (B.Ed. in Science) or Bachelor of Education in Arts (B.Ed. in Arts).

To earn a B.Ed. in science or a B.Ed. in arts, pre-service teachers specialise in education majors and one teaching subject, for example, mathematics, physics, chemistry, biology, geography, Kiswahili, English or history. Different from the Canadian case, these candidates are prepared as tutors or teacher educators for teacher colleges. Their programs leave more room for optional courses in teaching subjects and education than do programs for a B.Sc. and B.A. with Education (the degree for teaching in secondary schools). Those pre-service teachers are required to specialise in education as well as in two teaching subjects for secondary schools: for example, physics and mathematics, chemistry and mathematics, biology and chemistry, geography and mathematics, or geography and history. However, due to a shortage of mathematics and science teachers with degrees, diploma teachers have been teaching at ordinary level secondary schools from Form I to Form IV for many years.

1.6 Dissertation Outline

In this research, I worked with pre-service teachers who are prospective mathematics teachers of ordinary level secondary schools in Tanzania's teacher colleges, studying for the diploma in secondary education science. I investigated the contribution of concept study on their MFT (professional knowledge and skills).

Against the background, rationale, general aim, significance and the context of this study, this section outlines the content of the nine chapters. Chapter 2 is the literature review about

MFT and concept study. The literature review describes mathematics for teaching, how it has evolved, and its categories by elaborating the way different researchers assessed, measured, and developed it with some examples of the models used. Also, it elaborates on the meaning of concept study, its origin, and how researchers used it as an approach in facilitating in-service teachers' development of MFT.

Chapter 3 positions the study within a theoretical framework. In this chapter the complexities of mathematics teachers' professional knowledge is described by exploring how mathematics teachers come to know the mathematics they teach, the idea of knowing (Davis, Sumara & Luce-Kapler, 2008), and a view that mathematics teachers' professional knowledge as a complex system itself; as well as, considerations of a group of pre-service teachers in concept study as collective learner rather than a collection of learners (Davis & Renert, 2009; Davis & Renert 2014). In this chapter Davis and Renert's (2014) concept study model, is explored as a frame to analyse the data of pre-service teachers' MFT (professional knowledge and skills). A systematic exploration of the meaning of concept study (Davis & Renert, 2014), its focus, and the assumptions that guided this study are offered, as well as, descriptions of the concept study emphases—realizations, landscape, entailments, blending, and pedagogical problem-solving (Davis & Renert, 2014).

Chapter 4 describes the methodology undertaken in this study which is guided by the research question and based on the nature of the study. It describes the appropriateness for both the chosen paradigm, constructivism, and the research design method, qualitative group case study, as well the strength and weaknesses of qualitative case studies, and the criteria for judging quality of qualitative case study. The chapter also describes how and why complexity science was considered in the classroom design to create a learning environment for the pre-service teachers to engage in concept studies. Finally, the chapter describes the role of a researcher, the research site, and participants, as well the methods used for data collection and its analysis.

Chapter 5 describes the collective pre-service teacher participants' prior mathematics learning experiences for the purpose of a contextual backdrop from which the case can be read.

Chapter 6 describes the findings of how the five emphases of the concept study (Davis & Renert, 2014) helped the pre-service teachers to access and develop their tacit MFT—the professional knowledge and skills of the ratio proportion, rate, and linear functions concepts.

Chapter 7 describes the findings of pre-service teachers' development of explicit MFT—professional knowledge and skills during the concept studies of ratio, proportion, rate, and linear functions concepts as analysed by using Ball, Thames, and Phelps, (2008) categories of Mathematical Knowledge for Teaching (MKT).

Chapter 8 describes the findings on the professional knowledge and skills teachers need for teaching mathematics and the contribution of the concept study method on pre-service teachers' professional knowledge as reflected by the pre-service teacher participants. Also, it discusses how the concept study method: contributed to the pre-service teacher participants' deep understanding of mathematics; gave pre-service teachers the opportunities to learn the mathematics concepts in the school curriculum for the level they are prepared to teach with their colleagues; and contributed in building pre-service teachers' conceptual understanding. The themes emerged from the categories of key ideas originating from pre-service teachers' responses analysis of the pre and post-questionnaires that were used before and after each concept study session of ratio, proportion, rate, and linear function are discussed.

Chapter 9 is a detailed discussion of the research findings, focusing on how the concept study contributes to the development of the pre-service teachers MFT. It is in this chapter that research results from chapter 5, Chapter 6, Chapter 7, and chapter 8 are connected, and implications for mathematics teacher education in the Tanzanian context are offered.

2 Literature Review

2.1 Mathematics for Teaching (MFT)

The complexities in identifying and understanding the knowledge that mathematics teachers need to be effective at teaching mathematics is a challenging problem. Addressing this problem has put significant pressure on researchers, teacher educators, and practitioners around the world to imagine and develop curriculum and pedagogy for the teacher-education classroom. That research has become known as "Mathematics for Teaching (MFT)" (Ball & Bass, 2003; Ball, Hill & Bass, 2005; Davis & Simmt, 2006; Adler & Davis, 2006; Ball et al., 2008; Davis & Renert, 2014). Scholars of teacher education have been focusing on the kinds of mathematical knowledge that pre-service and in-service teachers need to "know," as well as the "know-how" required to teach mathematics effectively (Adler & Davis). Davis and Renert define MFT as "the mathematics knowledge that enables a teacher to structure learning situations, interpret students' actions mindfully and respond flexibly in ways that enable learners to extend understanding and expand the range of their interpretive possibilities through access to powerful connections" (p. 4). In this research, mathematics for teaching refers to mathematics teachers' professional knowledge and skills. The mathematics teachers' professional knowledge and skills suggests a complex phenomenon. What teachers need to know and to know how might also be considered complex because teachers are individual human beings (Davis & Renert). I would say *Mathematics for Teaching* (MFT) is the mathematics knowledge and skills that allows a teacher to have the deep understanding of mathematics for facilitating learning for conceptual understanding to the learners in making the connections of the learned concept with other concepts in mathematics and other subjects, and its applications in everyday life. Before an elaboration of MFT, I begin by discussing how I understand effective mathematics teaching.

According to Hiebert and Grouws (2007), the places where teachers put emphases influence the opportunities for students to learn. For example:

> different learning goals and different topics, the expectations for learning that they set, the time they allocate for particular topics, the kinds of tasks they pose, the kinds of questions they ask and responses they accept, [and] the nature of the discussions they lead. (p. 379)

Methods should align with learning goals. For example, if conceptual understanding is a valued learning goal, then students will need opportunities to develop conceptual understanding, or if problem-solving is a valued learning goal, then students will need opportunities to develop problem-solving skills. Resnick and Ford (1981) suggested the two most valued learning goals in school mathematics are skill efficiency and conceptual understanding (cited in Hiebert & Grouws, 2007, p. 380). Hiebert and Grouws defined skill efficiency as “accurate, smooth, and rapid execution of mathematical procedures. [But] do not include the flexible use of skills or their adaptation to fit new situations” (p. 380). And conceptual understanding as “mental connections among mathematical facts, procedures, and ideas” (p. 380), an understanding that can be seen as an activity of “participating in a community of people who are becoming adept at doing and making sense of mathematics as well as coming to value such activity” (p. 382).

According to Hiebert and Grouws (2007), the teaching that facilitates skill efficiency includes “teacher modeling with many teacher-directed product-type questions, and displays a smooth transition from demonstration to substantial amounts of error free practice” (p. 382).

That means it includes the teachers' skills in organizing, pacing, and presenting information to meet well-defined learning goals. Hiebert and Grouws assert that there are two features of teaching for conceptual understanding: teachers and students attend explicitly to concepts, and students struggle with important mathematics. Hiebert and Grouws explain that students can acquire conceptual understandings of mathematics "if teaching attends explicitly to concepts—to connections among mathematical facts, procedures, and ideas" (p. 383). As they explain further this includes:

> discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas. (p. 383)

Hiebert and Grouws explain that the second feature of teaching for conceptual understanding is "the teaching that consistently facilitates students' conceptual understanding: the engagement of students in struggling or wrestling with important mathematical ideas" (p. 387). As they explain further that the use of the word struggle means "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent... [and not] needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems" (p. 387). Hiebert and Grouws assert that teaching features that promote conceptual understanding also promote skills fluency. As they explain, the skills learning under the 'teaching that promotes skills efficiency' condition the "instruction is quickly paced, teachers ask short-answer targeted questions, and students complete relatively large numbers of problems during the lesson with high success rates" (p. 391). In contrast, skills learning under the 'teaching that promotes conceptual understanding' condition, "instruction is more slowly paced, teachers ask questions that require longer responses, and students complete relatively few problems per lesson" (p. 391). In this research the effective mathematics teaching is the teaching that facilitates conceptual understanding. That means the teaching that has Hiebert & Grouws features of classroom mathematics teaching for conceptual understanding: teachers and students attending explicitly to concepts, and students struggling with important mathematics.

2.1.1 Evolution of mathematics for teaching

Looking back to the teacher education research of the 20th century, studies about teachers' disciplinary knowledge of mathematics attempted to relate between constructs such as teachers' knowledge of mathematics (total number of mathematics courses taken in post-secondary, credit earned in those courses, specific course content, and performance on standardized tests of formal mathematics) and their students' understanding of mathematics (performance on standardized tests, capability to identify connections among topics, and ability to explain procedures or concepts) (Ball et al., 2008; Davis & Renert 2014). Ball et al., (2008) assert that these measures have been shown to be unreliable, and do not reveal the true nature of a teacher's mathematical knowledge. Davis and Renert suggest no significant correlation between teachers' understanding of advanced mathematics and students' achievement. Ball (1989), in her research with pre-service mathematics teachers, found that teachers with advanced degrees in mathematics (or a related field) were not necessarily any better at teaching mathematics. She argued that knowledge of subject content only will not necessarily enable an individual to teach that knowledge to another.

Shulman was the first scholar to provide a clear classification of teachers' non-disciplinary professional knowledge (Ball, Thames, & Phelps, 2008). He provided a particular model of teachers' professional knowledge that divided the content knowledge (CK) into three categories: "subject matter content knowledge" (SMCK), "pedagogical content knowledge" (PCK), and "curricular knowledge" (CK) (Shulman, 1986, p. 9-10). According to Shulman SMCK includes knowledge of the subject matter per se, and its organizing structures. While the CK is knowledge about the programs, methods, and corresponding instructional materials for teaching particular subject/topic at a given level of study. According to Shulman, PCK "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9). The way the teacher represents and formulates the subject so that it can be understandable to the learners includes the capability of mathematics teacher to provide relevant examples, explanations, illustrations, and demonstration of mathematical concepts and topics. Furthermore, according to Shulman, PCK also comprises an understanding of what contributes to the ease or difficulty of students' learning particular topics. This is knowledge about what the learners at different ages and experiences bring with them in learning, i.e. their "conceptions and preconceptions" (p. 9). For Shulman, PCK comprises both the established content and the process by which the content was established. This entails an understanding of both the "substantive and the syntactic" (p. 9) structures of subject matter. The substantive structures are the different ways in which the basic concepts and principles of the domain are structured while the syntactic structures are those that help us determine the validity or invalidity of the material under study. For example, according to Shulman, the teacher must be able to define the mathematical concept, explain why it is deemed warranted, why it is worth knowing and how it's related to other concepts within mathematics and other subjects, both in theory and in practice. Huillet (2009) argues that the distinction between substantive and syntactic structures of knowledge is an important aspect in mathematics education because in many countries teaching mathematics to teachers in colleges does not take into consideration these two structures. As she notes, facts and procedures are taught without linking them, a consequence that teachers understand “that something is so, but not why it is so” (p. 4). Shulman (1987), explains further that PCK as “the category most likely to distinguish the understanding of the content specialist from that of the pedagogue” (p. 8). In other words, PCK is the knowledge that is specific to subject teachers (e.g. mathematics teacher) which is difference from that of a teacher who is addressing the more general learning needs of the child. Thus, Shulman’s conceptualization of PCK, is among the first offered by scholars that suggest the rich source of distinction between types of knowledge teachers need for teaching. His work has had a tremendous impacts on other research for three decades. Some researchers have concentrated on building on Shulman and colleagues’ notion of PCK endeavouring to link it to the effectiveness of the teacher (Ma, 1999; Ball 2003; Ball, Hill, & Bass, 2005; Ball & Hill, 2008; Ball et al., 2008; Hill, Sleep, Lewis, & Ball, 2008). While others have focused on designing teacher knowledge models constructed from learning theories; these offer a foundation for making such claims regarding the nature and the development of MFT (Davis & Simmt, 2006; Davis & Renert, 2009, 2014; Simmt, 2011).

2.1.2 Categories of mathematics for teaching

Some of MFT research describe MFT as explicit knowledge (Ma, 1999; Ball & Bass, 2003; Ball, Hill & Bass, 2005; Hill, Rowan, Ball, 2005; Hill, et al., 2008; Izsak, Orrill, Cohen, & Brown, 2010; Schmidt, Houng, & Cogan, 2011; Izsak & Araujo, 2012) while others consider MFT as a tacit-emergent knowledge (Adler & Davis, 2006; Davis & Simmt, 2006; Simmt, 2011; Davis, 2011, 2012; Davis & Renert, 2009, 2014). The difference between these two conceptions is evidenced in the way the researchers define, assess, and facilitate the development of mathematical knowledge that is specific for teaching. The explicit knowledge defines the teachers' mathematical knowledge as owned by expert teachers assessed through "interviews, observations, or written tests" (Davis, 2012, p. 2). Whereas the tacit-emergent knowledge is the kind of mathematical knowledge that is "highly personal and can be hard to symbolize...knitted into ones being—enacted, embodied, performed, [and] taken for granted" (Davis & Renert, 2014, p. 26). Simmt (2011) further described that the tacit-emergent knowledge is "not a set of skills stored in one's head but rather an emergent phenomenon that is enacted in the context of teaching mathematics" (p.153). In other words, I can say it is a type of teachers' mathematical knowledge that is not conscious to an individual teacher but, it is activated when teachers are engaged in the collective learning in a concept study. The tacit-emergent knowledge of teachers' mathematical knowledge has been described as 'collective' (Davis & Simmt, 2003, 2006; Davis & Renert, 2014), 'tacit' (Davis, 2011, 2012; Davis & Renert; 2009, 2014) and 'complex' (Davis & Simmt, 2003; 2006; Davis & Renert, 2014). Davis and Simmt (2003, 2006) described it further as embodied both biologically and culturally. As Davis and Simmt (2006) assert mathematical knowing is "grounded in biological predispositions that are knitted together with bodily experiences through cultural tools including language and logic that were developed in shared efforts to make sense of the world" (p. 315). The assumption of two types of MFT (tacit-emergent and explicit) guides this doctoral research. What follows is a chronological discussion of four (Ma, 1999; Ball, Thames & Phelps, 2008; Davis & Simmt 2006; Davis & Renert, 2014) up-to-date research programs that have extended upon Shulman's (1986) original research for teacher knowledge. These research programs have developed models that account for both the explicit and tacit-emergent mathematics teacher disciplinary knowledge.

2.1.2.1 Ma's profound understanding of fundamental mathematics research program

Ma (1999) followed in the footsteps of Shulman's PCK with her work that examines the contrasts between mathematical content knowledge possessed by elementary school teachers in China and the United States. Ma found that teachers needed "profound understanding of fundamental mathematics" (PUFM) and, more specifically, that they needed to know specialized mathematics to teach mathematics. Profound understanding means a teacher's "deep, vast, and thorough" (p. 120) understanding of mathematics. While the fundamental is defined as having three "related meanings: foundational, primary, and elementary" (p. 116). Ma found evidence of highly specialized teachers' content knowledge of elementary mathematics in the practice of Chinese teachers but not in the practice of American teachers. She claimed that teachers need to have PUFM as the mathematical knowledge for teaching, to:

- Make connections among mathematical concepts and procedures from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and subdomains.
- Appreciate different facets of an idea and various approaches to a solution as well as their advantages and disadvantages... [And] provide mathematical explanations of these various facets and approaches.
- Display mathematical attitudes and are particularly aware of the 'simple but powerful basic concepts and principles of mathematics' (e.g. the idea of an equation).
- [Make them] not limited to the knowledge that should be taught in a certain grade; rather, they have achieved the fundamental understanding of the whole [specific] mathematics curriculum. (p. 122)

She conceptualized mathematical knowledge for teaching as much more than a procedural fluency with mathematics. According to Ma, PUFM comprises four key properties: connectedness, multiple perspectives, basic ideas and longitudinal coherence amongst the concepts that encompass grade school mathematics curricula. As she explained:

> A teacher with PUFM is aware of the simple but powerful ideas of mathematics and tends to re-visit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the ground work for a concept to be studied later (p. 124).

Ma's work also contributed to what mathematics is worth knowing by teachers as fundamental for success in mathematics teacher education as well as mathematics education. Ma's work motivated other researchers with her assertion of specialized mathematics for teachers.

2.1.2.2 Ball and colleague's mathematical knowledge for teaching research program

Ball and colleagues (Ball, Thames, & Phelps, 2008; Thames & Ball, 2010) built/developed a model of teacher knowledge curtailed as Mathematics Knowledge for Teaching (MKT). Ball's MKT model also built on Shulman's categories of teachers' knowledge exploring further the nature of mathematics content knowledge needed by teachers. The MKT model intends to show all of the categories of teacher mathematical knowledge that is essential for work of teaching (Ball et al., 2008; Thames & Ball, 2010). Ball et al. were the first to coin the term Mathematical Knowledge for Teaching (MKT), describing it as "mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). Ball et al.'s, practice-based studies highlighted the important distinction between the mathematical knowledge that teachers need, and the mathematical knowledge that other specialists like engineers, mathematicians, physicists and chemists need. Thames and Ball's (2010) practice-based study analysed the tasks of teaching in order to define the mathematical skill essential for handling these tasks.

Ball and her colleagues found that MKT consisted of distinguishable, distinct domains each defined in relationship to the work of teaching (Ball, Hill, & Bass, 2005: Ball, Thames, & Phelps, 2008; Thames & Ball, 2010). Thames and Ball (2010) explain further that each of these domains correlates to the different tasks of their distinctive definition of teaching well such as:

- Posing mathematical questions
- Giving and appraising explanations
- Choosing or designing tasks
- Using and choosing representations
- Recording mathematical work on the board
- Selecting and sequencing examples
- Analyzing students' errors
- Appraising students' unconventional ideas
- Mediating a discussion
- Attending to and using math language
- Defining terms mathematically and accessibly
- Choosing or using math notation. (p. 223)

They suggest "teaching well requires an abundance of mathematical skill and of usable mathematical knowledge—the mathematical knowledge in and for teaching" (p. 223). Ball et al.,

(2008) found that teachers' mathematical knowledge is enacted in their daily work and involves unpacking or decompressing content. Figure 2 illustrates Ball et al., (2008) MKT model that shows categories of teacher mathematical knowledge that is essential for work of teaching.

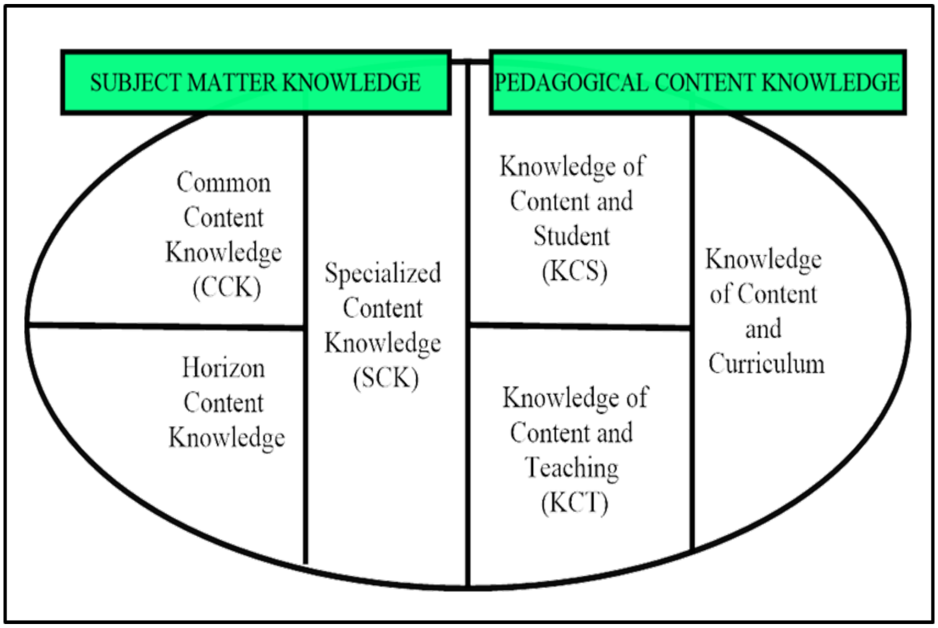


Figure 2: Ball' et al.'s MKT model (Ball, Thames, & Phelps, 2008, p. 403) used with permission

The left hand side of the MKT model deals with subject matter knowledge and is comprised of common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. *The Common Content Knowledge (CCK)* is defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball, Thames, & Phelps, 2008, p. 399). It is the essential mathematical knowledge that the teaching requires but is used in settings apart from the classroom as well. That means a mathematical knowledge other people use in different day to day contexts, but the teacher uses it in his/her practice as well. For example, the knowledge that enhances a teacher to identify whether student gave the wrong answer or textbook provided an incorrect definition to common mathematics such as operations or common place formulas, writing correct mathematical notation, using terms correctly, providing students with the definition of a concept or an object, or demonstrating how to carry out a procedure, etc. *Specialized Content Knowledge (SCK)* — defined as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). It is the mathematical knowledge typically entailed in the work of teaching and not used within other settings (Hill et al. 2008; Hill, Ball, & Schilling, 2008; Thames & Ball, 2010). This type of mathematical knowledge involves "an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching" (Ball et al., p. 400). It is the mathematical knowledge that teachers use in teaching that goes beyond the mathematics topics of the specific school curriculum itself. For example, a teacher needs to be able to define terms "in mathematically correct but accessible ways" (p. 224) or the way the teacher could make sense of solutions other than the one he/she comes to him or herself (Thames & Ball, 2010). *Horizon content knowledge* (HCK) is defined as "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., p. 403). It "includes the vision useful in seeing connections to much later mathematical ideas" (p. 403). For example, how a mathematics topic in school curriculum relates to a topic in college or university mathematics or how a college or university mathematics topic might relate to technical or professional mathematics used in a person's work.

The right hand side of the MKT model deals with the knowledge that merges content knowledge with the pedagogical knowledge, or PCK. Ball, Bass, Sleep, and Thames, (2005) assert PCK is "the unique blend of knowledge of mathematics and its pedagogy" (p. 3). Ball and colleagues divide PCK into sub-domains that combine the knowledge of content with the knowledge of students, teaching, and curriculum. *Knowledge of Content and Students (KCS)* — is one of the domains of PCK defined as "the knowledge that combines knowing about students and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). It is the knowledge that allows teachers to anticipate possible and perplexing things done by students, and be able to hear and construe students' emergent and incomplete thoughtful as voiced in the ways that pupils use language. It is knowledge of how students learn mathematics. The task of teaching here "requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking" (Ball et al., 2008, p. 401). For example, teachers need to familiarize with the most likely error student can make. *Knowledge of Content and Teaching (KCT)* is defined as the knowledge that "combines knowing about teaching and knowing about mathematics" (Ball et al., p. 401). It is the mathematical knowledge that deals with the teachers' design of classroom instruction such as planning the sequence of the specific content, choosing which example to start with for deep content understanding, and assessing the instructional merits and detriments of particular strategies or approaches. For the teacher to perform these tasks he or she "requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al., p. 401). For example, in mediating discussion the teacher should know the time to pause, time to use student remarks to make mathematical point, time of asking questions and time posing tasks to further students' learning. *Knowledge of content and curriculum* (KCC) is the knowledge about the programs, methods and instructional materials for teaching mathematics at a given level of study.

Ball and her colleagues' (Ball, Bass, Sleep &Thames, 2005; Ball & Bass, 2003; Hill & Ball, 2004) innovative work has succeeded in recognising different examples of special ways in which one must know mathematical concepts, procedures and representations to intermingle them effectively when working with students in the context of teaching. They have also demonstrated that there is knowledge specific to the work of teaching (Ball et al., 2005) and that the conceptual demands of teaching mathematics are different than those needed by other mathematics practitioners (Ball & Bass, 2003; Ball et al.). Ball et al.'s research indicates that the teachers' performance on their measure-of-knowledge instrument, which includes both common and specialized content knowledge, was a significant predictor of students' achievement. They suggested that even though mathematical knowledge is not sufficient for reducing the achievement gap, it is certainly necessary to prevent it from growing (Ball et al.). Adler and Zain Davis (2006) assume "… there is a specificity to the mathematics that teachers need to know and know how to use" (p. 271). In particular, the "unpacking" that is done in the mathematical work of teachers (Ball & Bass, 2003) is needed because mathematics itself involves the compression of the information into abstract forms, whose compressed symbolic form allows structures to be more evident. The unpacking of mathematical ideas is an important component of the knowledge that mathematics teachers need to enact as they do their work of teaching (Ball & Bass, 2003; Ball, Bass, & Hill, 2004). Therefore, mathematics courses in teacher-training programmes should be taught in "a way that allows student-teachers to 'unpack' the mathematical knowledge, apply this knowledge to solve real problems from everyday life or from other sciences, using as much as possible active methods of discovery" (Huillet, 2009, p. 9-10).

Ball and her colleagues study their research questions primarily using MKT model and explicitly link particular aspects of teacher mathematical knowledge to student achievement (Hill, 2010; Hill, Rowan, & Ball, 2005). The MKT model has been also used by researchers to understand teacher topic-specific knowledge of students (Hill, Ball & Shilling, 2008; Delaney, Ball, Hill, Schilling, & Zopf, 2008), and to measure teacher quality (Ball, Hill, & Bass, 2005; Hill, et al. 2008; Ball & Hill, 2008). The research of Hill, et al., has shown positive empirical results linking of MKT knowledge domains to the student achievement. These research findings could serve as a source of research studies that explored ways to support, develop and reinforce the type of teachers' mathematical knowledge that could assist the effective teaching of mathematics. Ball and colleagues have made major contributions in the mathematics education community in exploring explicit teacher knowledge, especially by specifying the nature of teachers' content knowledge from the developed model of MKT. This extensive effort allowed the linking of specific aspects of explicit MKT to student achievement. However, her work could be interpreted as to lack the idea that teachers' knowledge might be conceived more than the explicit knowledge (Davis & Renert, 2014). That is, their work lacks exploration of tacit-emergent teacher knowledge (Davis & Simmt, 2006; Davis, 2012; Davis & Renert).

Tacit knowledge is not easily accessible to the consciousness (Davis, 2012) but emerges from the interaction with others. As Davis asserts this type of knowledge is related to "expert webs of associations" (p. 3) that trigger the 'conceptual fluency' of professional teachers which activate the explicit knowledge. According to Davis, in unfamiliar circumstances, when the professional teachers are asked to explain their 'interpretations or actions' about their choice, they have difficulty explaining or justifying them: they simply recognize their 'interpretations or actions' as suitable in the situations encountered. Tacit knowledge originated from Polanyi's work of teacher disciplinary knowledge (as cited in Davis, 2012). It is knowledge that is "neither easily identified nor readily measured" (Davis, 2012, p. 3). Other mathematics education research has been built from Polanyi's work (Adler & Davis, 2006; Davis, 2011; Davis & Renert, 2009; Davis and Renert, 2014).

2.1.2.3 Davis and Simmt's mathematics for teaching research programs

Davis and Simmt (2006), using a complexity science framework to work with in-service teachers, conceptualize mathematics for teaching through theoretical discussions of teachers' MFT. Davis and Simmt view the relationship between teaching and learning as inherently nested and as collective—that is emergent when teachers do mathematics with others. Figure 3 is a representation of Davis and Simmt's MFT model. This model recognizes knowledge as enactive, emergent, and embodied (Davis & Simmt, 2006; Davis 2012; Davis and Renert, 2014). There are four nested intertwining aspects of MFT in the model: subjective understanding, classroom collectivity, curriculum structures, and mathematical objects. Subjective understanding is the embodied innermost layer of a complex system dynamic on multiple levels, as it represents teachers' harmonization of their own emerging mathematical knowledge with their interpretations of evolving student mathematical knowledge (p. 312). For example, in a mathematical environment such as concept study, the teachers' subjective understanding is manifested in their experiences, images, examples, and interpretations, as characterised by their actions, Subjective understanding is observed and understood within a learning environment. The first learning environmental system in Davis and Simmt model in which the subjective learner is embedded is the classroom. This is a collective context made up of others doing mathematics. Davis and Simmt argue: "the 'learning system' that the teacher can most directly influence is not the individual student, but the classroom collective" (p. 309). The classroom layer is the first environmental embedded layer of a complex system that involves the teacher's knowledge of how to participate in collective mathematical action and knowledge of how to facilitate the productive engagement of students (p. 309).

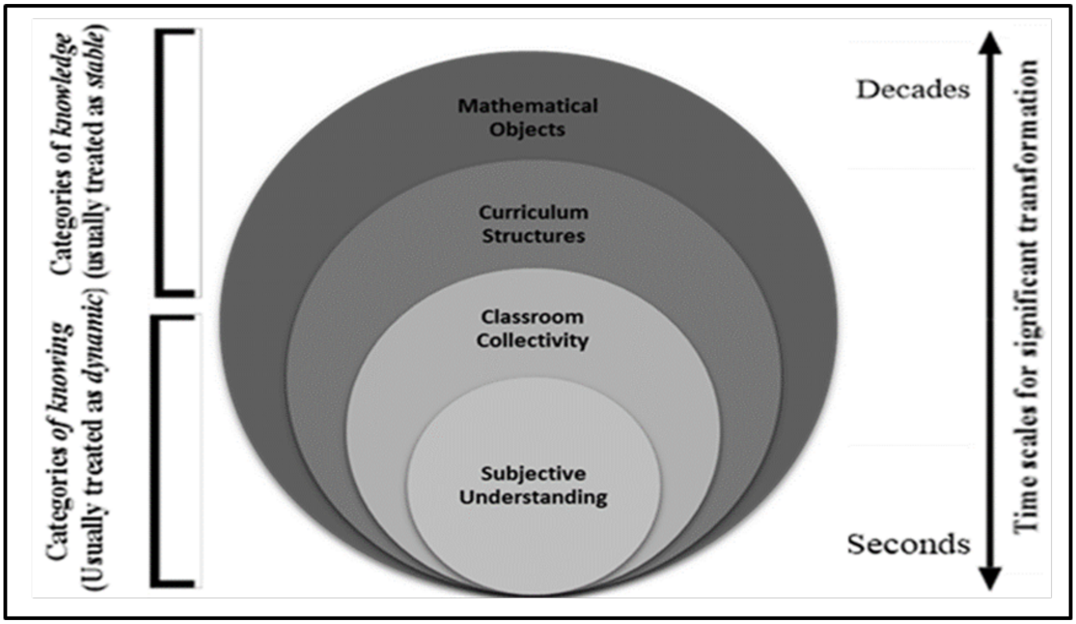


Figure 3: Davis & Simmt M_4T model, some nested complex phenomena of concern to the mathematics teacher (from Davis & Simmt, 2006, p. 296), used with permission

The first two layers, subjective understanding and the classroom collectivity, are the knowledge producing systems which teachers can observe transform. The curriculum structures layer is the third inner layer of a complex system concerned with the teacher's knowledge of the shared cultural interpretations of the structure of mathematics for schooling (p. 303). It includes teacher's knowledge of the curriculum resource materials as well as approved programs of study. The mathematical objects layer is the outermost layer of a complex system of teacher's knowledge of the broad system of discipline as it has evolved through the participation of all humanity over time (p. 300). It includes the individual, the social context, and the curriculum structures binding those the social contexts and the discipline of mathematics as a whole. The time scale located at the right of figure 3 represents Davis and Simmt's attention to competing evolutionary difference of each embedded system. For example, Davis and Simmt asset that the individual understanding of mathematics is volatile as it can change and easily adapt to the new mathematical environments whereas formal mathematics is relatively stable. In a comparatively short duration of time, one can observe the change of an individual's understanding of a certain concept of mathematics, parallel changes to the mathematics itself takes considerably longer time (p. 297). Size of the ellipses in this model is meant to imply the level of embedded complexity as well as address the different time scales required to see significant evolutions in the system. For example, the large size of the ellipse signifies the relatively inert nature of mathematics as a subject. Davis and Simmt assert that it took centuries the introduction of zero to the Western number system to occur, compared to the "pace at which a young learner comes to appreciate a number system that already includes zero" (p. 297).

Davis and Simmt understand MFT "as an emergent phenomenon that is enacted in the context of teaching mathematics" (Simmt, 2011, p. 153). Simmt (2011), found that teachers' expertise in mathematics depends on interaction with others and the "fact that they must understand mathematics as at once well-established knowledge and as enacted knowing…— engage in mathematics as both a cultural product and as personal constructing" (p. 163). She also argues that teachers' expertise includes the "collective we call a class and within a structure we understand as the curriculum" (p. 163). Davis (2012) described mathematics teachers' disciplinary knowledge as "vast, intricate, and evolving" (p. 3) which accounts for explicit, and tacit-emergent knowledge. Davis' argument is that no individual teacher could have all possible interpretations (of a mathematics concept, for example) invoked in teaching of a specific level of school mathematics. Davis suggests, rather than thinking of teacher professional knowledge of mathematics as distinct foundational knowledge held by the individual teacher, it might be productive viewing "it as a flexible, vibrant category of knowing that is distributed across a body of professionals" (p. 3).

2.1.2.4 Davis and Renert's profound understanding of emergent mathematics

Davis and Renert, (2014) claim mathematics for teaching is profound understanding of emergent mathematics (PUEM). PUEM is "a category of knowing...a way of being-with mathematics that includes but elaborates formal content knowledge, specialized pedagogical content knowledge, and content knowledge entailed in the work of teaching" (Davis & Renert, 2014, p. 118). Davis and Renert PUEM embrace Ma's PUFM by considering teachers' disciplinary knowledge as deep and vast (Ma, 1999, p. 120). However, Davis and Renert do not demand the teacher's disciplinary knowledge be fundamental and thorough, arguing that due to the vastness and evolving nature of teachers' disciplinary knowledge it is not the best choice to be considered thorough. Davis and Renert also argue consideration of teachers' disciplinary knowledge as 'fundamental' characterised by Ma as "foundational, primary, and elementary" (Ma, p. 116) suggest a "closed set of insight and understanding that might be catalogued and assessed" (Davis & Renert, p. 118) that might reduce the effort researching the MFT. Davis and Renert suggest the term 'emergent' instead of 'fundamental' to indicate the complexity nature of mathematics teacher disciplinary knowledge as adaptive and evolving. Figure 4 illustrates the comparison of the MFT models Ma (1999; 2010); Balls, Thames, and Phelps (2008); Davis and Simmt (2006); and Davis and Renert (2014). It elaborates how these models differ and similar, how they are connected, whether developed using a practice based approach or complexity theory, whether they used interview instruments or concept studies, and what was their outcomes regarding teachers' PCK and their MFT. Despite the existing theoretical differences between the explicit, and the tacit-emergent knowledge research scholars, they all agreed that there is mathematical knowledge specific to teaching and the knowledge as part of teacher proficiency (Baumert et al., 2010). That is, they all agree the need for researchers to continue exploring MFT

(Davis & Renert, 2014; Ball, Thames, & Phelps, 2008; Ball and Thames, 2010) to give the teachers the opportunities to teaching mathematics effectively.

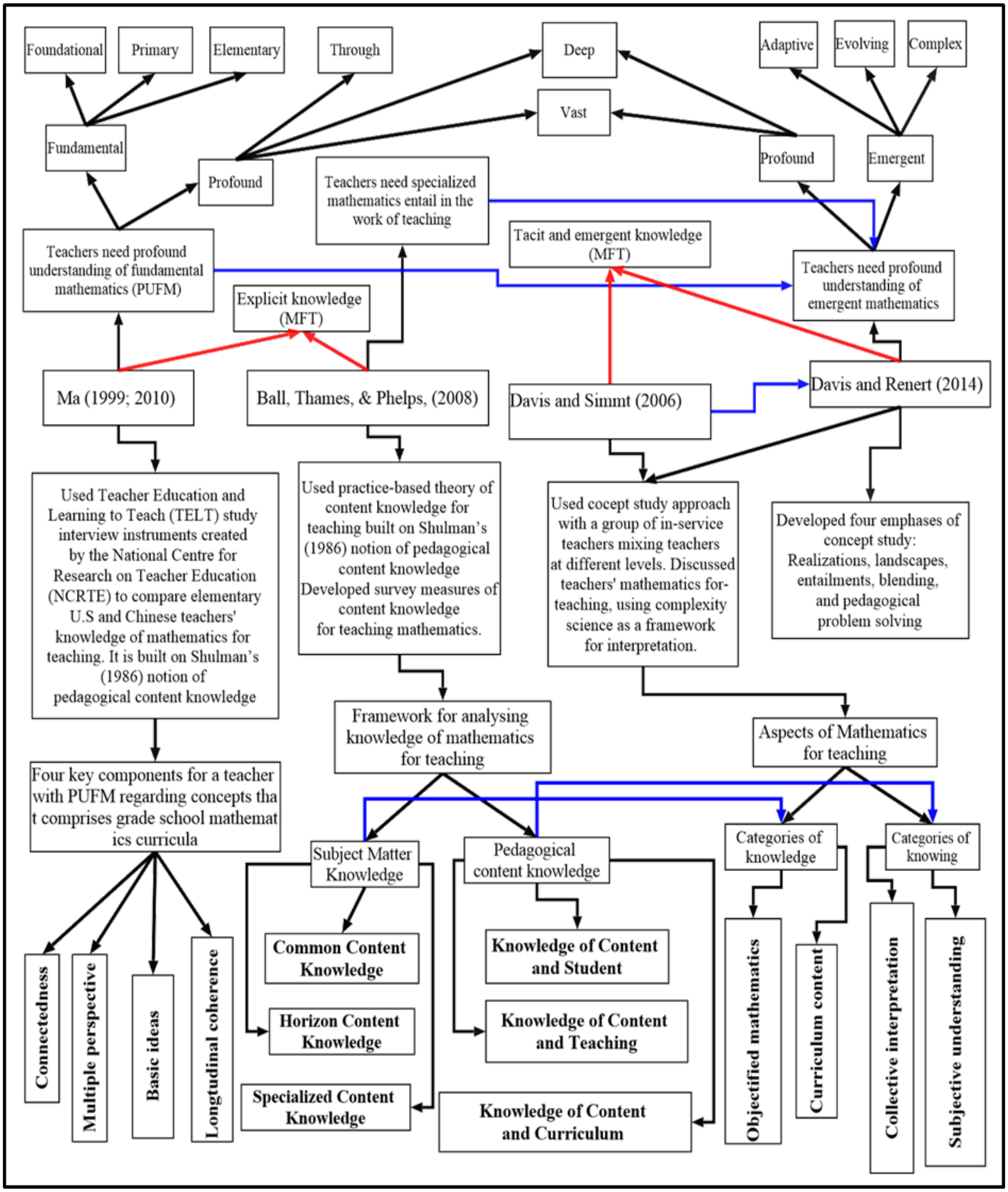


Figure 4: The figure that compares MFT models

Given the research into MFT and the theories developed about MFT, my research considers both explicit and tacit-emergent knowledge, as important kinds of mathematical knowledge that pre-service and in-service teachers need to improve as they develop professional knowledge and skills for teaching mathematics effectively. However, for the purpose of my research, I do not focus on developing a model of mathematics for teaching. Rather, this research concerns investigating the contribution of concept study on the MFT of Tanzanian pre-service teachers who are studying for the diploma in secondary education.

Theorization about MFT has been an important enterprise in making sense of the complexities involved in effective teaching of mathematics. Using the outcomes of such theorizations could contribute to improving teacher education in Tanzania, especially the use of focused collaborative, and participatory dispositions of concept study approach (Adler & Davis 2006; Davis & Simmt, 2006; Simmt, 2011; Davis, 2012; Davis & Renert, 2009; 2014) for teacher preparation programs and teacher professional development programs. The concept study approach is described in the next section. Tanzania is among the countries that might consider using concept study as an approach to enhance mathematics teacher education. The possibility and appropriateness for this, is the central motive in my study.

2.2 Concept Study Approach

Researchers have used concept study (Ma, 1999; Davis & Simmt, 2006; Simmt, 2011; Davis 2012; Davis & Renert 2009; 2014) to develop teachers' MFT by working collaboratively with groups of in-service teachers that are interested in better understanding mathematics (Davis & Simmt, 2003; 2006) and enhancing their teaching of mathematics. The concepts teachers explore through concept study might subsequently be used in facilitating learning for conceptual understandings (Davis, 2008; Kilpatrick, Swafford & Findell, 2001) among students. As Kilpatrick et al. (2001) remind us, knowledge acquired through understanding are the core knowledge for creating new knowledge and resolving new unfamiliar problems. According to Davis and Renert, concept study merges two foci—the mathematical knowledge focusing on the concept analysis (Lakoff & Nunez, 2000; Usiskin, Peressini, Marchisotto, & Stanley, 2003) with the collaborative work of lesson studies (Fernandez & Yoshida, 2004; Chokshi & Fernandez, 2004). The two foci are described hereafter.

Concept analysis (Lakoff & Nunez, 2000; Usiskin, Peressini, Marchisotto, & Stanley, 2003) is a cross-examination of individual mathematical concepts or ideas in answering the question what and why a concept means what they do. Lakoff and Nunez described it as a cognitive analysis of mathematical ideas in the real understanding of mathematical concept or equation. According to Lakoff & Nunez, as one seeks to develop the concept analysis, the question about, "what theorems mean and why they are true on the basis of what they mean" (p. xv) must be asked. They believe it is important to focus mathematics teaching "more toward understanding mathematical ideas and understanding why theorems are true" (p. xv). Usiskin, et al. (2003) describe concept analysis as the analysis which "involves tracing the origins and applications of a concept, looking at the different ways in which it appears both within and outside mathematics, and examining the various representations and definitions used to describe it and their consequences" (p.1). Davis and Renert (2014) paraphrased Usiskin et al.'s (2003) description of a concept analysis as examining the historical roots, applications, representations, definitions, and uses of a mathematical concept. For example, Usiskin et al. explore the question "what does parallel mean?" They assert that choosing to use only one definition of a mathematical concept without considering other possibilities might cause loss of sight of the other possibilities. Thus, “knowledge of the variety of possibilities can assist teachers in knowing why students have trouble both in using their intuition and in applying the abstractions” (p. 2). As Usiskin et al. explore the meaning of each *James and James‘s Mathematics Dictionary* descriptions of four different characterizations of parallel objects as: “are equidistant apart, do not intersect, go in the same direction, and can be obtained from each other by translation” (p. 2) for lines in the plane is up-front though they are not logically equivalent. They explain, the line is parallel to itself under the last two characterizations but not under the second and only under the first if the lines are allowed to have zero distance between them. Usiskin et al. also assert that different instances of a mathematical concept might involve different intuitions.

Lesson study (Fernandez & Yoshida, 2004; Chokshi & Fernandez, 2004) is defined as a unique collaborative activity of teachers with colleagues in planning, observing, and discussing lessons. Engaging teachers in lesson study helps to enhance the “quality of their teaching and to improve their students’ learning experiences” (Fernandez & Yoshida, p. 2). Lesson study helps the teachers “to learn in and from their practices” (p. 3). Learning in practice happens as the teachers collaboratively plan the lesson together, one of them teaching it to his or her students while others are observing. The observing teachers each come with their group lesson plan as a guide in their observations. The discussion of their lesson observations conducted after the lesson helps the group to learn from their practice. That means the teachers get opportunities to share what they observed in the lesson, provide feedback of what they think worked from the plan, needs for improvement and suggest how to improve it. According to Fernandez and Yoshida, the discussion of the lesson observations can be the end of teachers’ lesson study but, others could choose to re-plan the lesson, another teacher from the group teach while others observe and discuss their second observations. As Davis and Renert (2014) explained further that lesson studies are concerned with “new pedagogical possibilities through participatory, collective, and on-going engagements” (p. 39).

Informed by concept analysis and lesson study, Davis and his colleagues have provided insight in accessing, developing and studying the MFT through concept studies with in-service teachers; they characterize concept study as involving two forms of ‘collective’ activity. First, MFT is not a domain of knowledge to be mastered by persons (Davis & Simmt, 2006). MFT is used and develops in a context that involves others. Thus, an awareness of how individuals might be involved in productive collectivity is the central aspect of MFT in concept studies. The second dimension of ‘collective’ in concept studies is that MFT involves collectively invoking and developing tacit or unconscious knowledge (Davis & Simmt, 2006; Martin, Towers, Pirie, 2006; Martin & Towers, 2009a; Davis & Renert, 2009, 2014). MFT as a distributed collective body of knowledge that is shared culturally amongst teachers.

Davis and Renert (2014) described a classroom collective “is not merely as a collection of learners, but as a collective learner” (p. 32). This notion of ‘collective learner’ is also elaborated in other Davis studies address a group of teachers working together as emergent cognitive bodies (Davis & Simmt 2006; Davis 2012) in concept studies. The focus of the concept study is on the collective in supporting the development of resilient, flexible personal understandings. Simmt and Davis argue that the individuals within a collective contribute a variety of interpretations and tactics in making sense of the concept at hand. A diversity of interpretations is at the centre of the interactions of the individuals in concept study. Therefore, in this research, a group of pre-service teachers is observed as a collective learner. I focus on identifying and emphasizing the collective learning of the pre-service mathematics teachers through concept studies while keeping the commitment of helping an individual pre-service mathematics teacher in developing his/her MFT.

The concept study method is described as a “structure intended to provide teachers with the sorts of experiences and attitudes that might develop disciplinary knowledge originating from conceptual diversity” (Davis & Renert, 2014, p. 38). Concept studies focus on content of MFT as they are envisioned to prompt teachers to go beyond the structures of the mathematical concepts of the planned school curriculum. In conducting concept studies, it is not a matter about the rightness or adequacy of concepts rather, what matters is the individual understanding of a mathematical concept as “an emergent form, arising in complex weaves of such experiential and conceptual elements … the objects or agents of the complex system of mathematics for teaching” (p. 58). The development of MFT through concept study draws on the tacit knowledge and specific experiences of the participants. It is “often accidental knowledge” (p. 41) and it is not a mastered “domain of mathematics” (p. 42). Teachers’ mathematics can be understood as a way of being that is enacted when the teacher deals with a new topic, makes sense of a learner’s error, or reconciles individual interpretations and acknowledges that each concept study generates its own results. The variation, context, and broad range of interpretations of mathematics concepts that exist in any given pedagogical moment are among the complexities inherent in teachers’ disciplinary knowledge (p. 24). These complexities demand critical analysis in interpreting and presenting the meaning of mathematical concepts, rooted in previous formal mathematics, that the teachers chose and modified to make it further accessible.

Five nested emergent emphases are generated through concept study: realizations, landscape, entailments, blending, and pedagogical problem-solving (Davis & Renert, 2014)—the realizations as the innermost and the pedagogical problem solving the outermost layers of the nested model. Concept realization refers to “meanings, interpretations, and instantiations” (Davis & Renert, 2014, p. 58) such as metaphors, analogies, images, algorithms, and applications a teacher or learner might associate with a mathematical concept. Landscape is an awareness of how the realizations relate to one another across and within grade levels. For example, in kindergarten, multiplication can be interpreted as skip counting; in grade three as number line hopping; and in grade ten as scaling. For example, a ‘landscape’ for ratio could include a macro-level map of collectively created and then organized ‘realizations’ for the ratio in the school curriculum that would involve in part multiplication. Davis and Renert suggest that “each realization of a concept carries a set of logical implications and entailments” (p. 66). For example, entailments of multiplier, multiplicand, and product in the realization of multiplication as grouping, means the count of groups, the count of objects per groups and the total count of objects, respectively (p. 66). The realizations of multiplication as slope means the slope is the multiplier, the position on the x-axis is the multiplicand, and the associated position on the y-axis is the product respectively. Davis and Renert propose a fourth emphasis called blending. Blending refers to the process that involves the generating, combining, and collapsing varied realizations of a mathematical concept. For example, speed given in kilometre per hour is the rate which is the blends of ratio (distance to time) and measurement (length and time). The fifth emphasis is pedagogical problem solving. Here the emphasis is on the mathematical problems’ teachers encounter in their daily work, specifically in their professional work in classroom instruction. What Davis and Renert call the “*real mathematical work of teachers*” (p. 79) [author’s emphasis]. The problem solving aspect of MFT is “developed around the actual questions that learners ask around meaning seeking” (p. 79). This model of realizations, landscape, entailments, blending, and pedagogical problem solving, illustrates how, concept study promotes teachers' deep understanding of MFT (Davis and Simmt, 2006, Davis & Renert, 2014). In other words, concept studies are intentionally structured to focus on teachers tacit and explicit knowing/knowledge of mathematics as they work through realizations, landscape, entailments, blending, and pedagogical problem-solving. Because of how central concept study is to this research these five emphases will be discussed more in detail theoretical framework detailed in chapter three.

This literature review described what mathematics for teaching means, how it has evolved, and its categories. It elaborated the way different researchers assessed, measured, and developing it with some examples of the models used, and how researchers all focused on finding the professional knowledge and skills the teachers need to know and know-how for effective teaching of mathematics. Also, this chapter elaborated the meaning of the concept study, its origin, and how researchers used it as an approach in facilitating in-service teachers' development of MFT. However, the literature about MFT in the context of secondary school teachers is scant. Although some researchers demonstrated how concept study could be used to enhance in-service teachers' deep understanding of MFT (Davis & Simmt, 2006; Davis & Renert, 2009; Davis & Renert, 2014), so far, I cannot locate any studies that concentrate on the use of concept study for pre-service teachers' MFT and have found no research specific to Tanzania or East Africa. My research addresses this gap as it focuses on diploma in secondary education (for teaching in secondary schools) pre-service teachers' MFT using a concept study approach, in Tanzanian context. Therefore, in this research, I ask, *in what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?*

3 Theoretical Framework

Mathematics teachers' professional knowledge is a complex phenomenon. What teachers need *to know* and *to know how* (how to do) might also be considered complex because teachers are individual human beings (Davis & Renert, 2014). This chapter describes how the frame of concept study can be used to explore the development of MFT with Tanzanian's diploma in secondary education pre-service mathematics teachers. The chapter is divided into three sections.

In the first section, the complexity thinking is described by providing a theoretical basis for understanding the collective (Davis & Sumara; 2006, Davis & Renert, 2006; Davis & Renert, 2014) as a learning system.

In the second section, the complexities of mathematics teachers' professional knowledge is described by exploring how mathematics teachers come to know the mathematics they teach, the idea of knowing (Davis, Sumara & Luce-Kapler, 2008), a view of mathematics teachers' professional knowledge as a complex learning system and the consideration group of pre-service teachers in concept study as a collective learner rather than a collection of learners (Davis & Renert, 2006; Davis & Renert 2014).

In the third section, Davis and Renert's (2014) concept study model is explored, which is used to analyse the data collected in terms of pre-service teachers' MFT (professional knowledge and skills). A systematic exploration of the meaning of concept study and its focus. As well the assumptions that guided this study. A description of the concept study emphases—realizations, landscape, entailments, blending, and pedagogical problem-solving is elaborated. The reasons as to why these emphases Davis and Renert illustrated as a nested visual metaphor and not a linear in the collective learning of mathematical concept in concept studies is provided. Each of the concept study emphases is described in five separate sub-sections to introduce the reader to its meaning and how each is used in this study.

3.1 Complexity Thinking

Complexity thinking provides a theoretical basis for understanding the collective as a learning system. According to Davis and Sumara (2006), the complex system emerges from interactions between the parts (or the agents) and not the sum of its parts (or the agents). The collective learning systems in concept studies emerge from the interactions of individual pre-service teachers as agents of the collective learning system. Their learning is adaptive in such a way that the whole learning system maintains its dynamic coherence environment (Davis & Sumara, 2006). Davis and Simmt (2003) assert that for any complex system, the parts and the whole of the complex system depend on one another. Observing the collective as a complex learning system in concept studies requires an understanding of the individual pre-service teachers and the collective, how the individual pre-service teachers and the collective are related, and how their relationship generate new possibilities for the learning system. The relationship between the individual pre-service teachers and the collective generates useful interdependency that creates a strong learning environment for the collective learning system of the concept studies. Thus, the new understandings of the mathematical concept in concept studies is a result of pre-service teachers' interactions and not the sum of their individual understandings.

3.2 The Complexities of Mathematics Teachers' Professional Knowledge

To better understand the complexities of mathematics teachers' professional knowledge, a researcher needs to know how mathematics teachers come to know the mathematics they teach and know about the how students learn that mathematics. For example, Davis and Renert (2014) suggest the primary concern of the teacher is to know how students learn mathematics, whereas Skemp (1978) claims it is a conceptual understanding of mathematics itself. The effective teaching of mathematics needs competent teachers—teachers with conceptual understanding of mathematics (Skemp, 1978; Davis, 2008; Kilpatrick, Swafford, & Findell, 2001). Taking account that the primary concern of teachers is learning mathematics leads to the following questions: what is learning, and how do teachers learn mathematics? Davis, Sumara, and Luce-Kapler (2008) suggest that learning "is about transforming what is known" (p. 4), the changes that are inseparable from the act of doing and that of being. Teachers need knowledge of how mathematical concepts are connected, how mathematical ideas anticipate others, and so on. But, what is knowing? I answer this question based on my experience as a teacher: the teacher can do what he or she knows. This is similar to the idea of Davis et al., (2008) that there is no difference between knowing and doing phenomena. Davis et al. assert that "knowing always spills over the perceived boundaries of the knower" (p. 7). Teachers are not isolated human beings, but, are "situated in grander social, cultural, and ecological systems" (p. 7). As Davis et al. argue, one needs to realize that each act of knowing is "partial– in the twofold sense of incomplete and biased. Knowing entails a selection and by consequence, a discarding of other interpretive possibilities" (p. 7). Thus, based on the offered explanation of teachers' mathematics learning, an emphasis on learning is to be focused towards "knowing differently" (p. 8). A critical assumption in the theory that Davis et al. posit is that the "knower, knowledge, and the phenomena known can't be separated" (p. 8). Thus, the pre-service teacher's participants are assumed to be the knower, whose knowledge acquired in schools as students and in teachers college as student teachers and their professional knowledge and skills cannot be separated. Teachers are expected to be learning even after completion of their training while teaching in schools through in-service professional development learning and through teaching itself. Professional learning with others is very important to in-service teachers because it is appropriate and valuable to listen to other views concerning the mathematics teaching and learning. As Davis argues, what matters is "knowing differently, not merely knowing more" (p. 8).

Davis and Renert (2014) view mathematics teachers' professional knowledge as a complex system— "a system that knows (i.e., perceives, acts, engages, interprets, etc.) and learns (adapts, evolves, maintains self-coherence)" (p. 20). Mathematics teachers' professional knowledge is described further by Davis (2012) as "vast, intricate, and evolving" (p. 3) accounting for both the explicit, and tacit-emergent knowledge. Davis' insistence for attention on a group of teachers (or the collective) is because no individual teacher could have all possible interpretations invoked in specific teaching level of school mathematics. Rather than thinking of teacher professional knowledge of mathematics as distinct foundational knowledge held by an individual teacher, he suggests it might be productive viewing "it as a flexible, vibrant category of knowing that is distributed across a body of professionals" (p. 3). For example, the meaning of the mathematical concept or idea the teacher has keeps on changing with time as a result of the teachers' collaboration or participatory learning with students and with the other mathematics teachers within evolving knowledge of mathematics. For the purposes of this research, it is assumed that teachers' mathematical understanding is activated when sharing ideas with colleague mathematics teachers in the participatory disposition of concept study and hence, the teachers understanding of the concept can evolve. I viewed the pre-service teachers' professional knowledge and skills as the relationship between their former experience as students in schools and present experience as student teachers developed in the moment of working collectively in learning the mathematical concept or performing any other task in the concept studies. Davis et al. (2008) argue that "knowledge is shared perception and vice versa" (p. 22), and as they explain, perception is more the matter of integrating the relationship between present and former experiences. That means, taking pre-service teachers as an example, an individual pre-service teacher perceives a certain mathematical concept differently as former experience—knowledge the pre-service teacher has: and, when they share these perceptions as part of a the concept study group, the pre-service teacher understanding is both elaborated and transformed as different ideas from others that change how the pre-service teacher formerly perceived it and taken as a new knowledge. Davis et al. demonstrate how teachers come to know something when working together in concept study. They observe a group of teachers learning mathematics together as a collective learner— the complex system of teachers' professional knowledge (Davis & Renert, 2014).

Teachers' professional knowing viewed as a complex system is a system that knows and learns (Davis & Renert, 2014). Within that system teachers are expert knowers who are able to think like novice knowers. This is critical because teachers need to be aware of various interpretations of mathematical ideas/concepts that circulate in the classroom. The consideration of a group of teachers working together not as a collection of individual learners, but as a collective learner, makes possible for the individual teacher' professional knowing being activated by the collective. For example, a group of teachers doing concept study is considered as a collective learner the complex system within which an individual teacher is an agent. According to Davis and Simmt, (2006), "a complex system is bottom-up" (p. 295); its emergence is not dependant of essential organizers or governing structures. In other words, the agents of the complex system "embody their experiences through the continuous modifications in the relationships among agents" (p. 296). As Davis and Simmt explain, complex systems are often nested with several transitional layers of organization, any of which might well be identified as:

> complex and all of which influence (both enabling and constraining) one another. Complexity science prompts attention toward several dynamic, co-implicated, and integrated levels– including the neurological, the experiential, the contextual/material, the social, the symbolic, the cultural, and the ecological. …individual understanding might be seen as enfolded in and unfolding from the broader phenomenon of collective dynamics (p. 296).

Work with teachers on their professional knowledge of mathematics is not as simple as helping the teachers to know what they do not know; rather, it is about detecting what they have not noticed in themselves, as "much of teachers' mathematics for-teaching is tacit" (Davis & Simmt, 2006, p. 295). Davis and Simmt assert that "for teachers, knowledge of established mathematics is inseparable from knowledge of how mathematics is established" (p. 297). What is important are insights into the "historical emergence of core concepts, interconnections among ideas, and the analogies and images that have come to be associated with different principles" (p. 297). Therefore, teachers' mathematics learning and knowing is not about duplicating pre-existing knowledge about mathematics, learning, or the child. Rather teachers' professional learning of mathematics for teaching is about constructing new understandings through reflective acts of linking and relinking their former experiences with experiences in the new environment. As described in the literature review, concept study involves the collective work of teachers learning mathematics with an effort to make sense of the meanings of mathematical concepts. In chapter 4 in section 4.3, I will describe why complexity science is used in this study and how the use of conditions of complexity in the classroom setting in the concept study motivates the development of teachers' MFT.

3.3 Concept Study as the Framework of this Research

In this study, concept study (Davis & Simmt, 2006; Davis & Renert, 2014) that is organised around mathematics teachers' "meaning making and definitions" (Davis & Renert, 2014, p. 38) is used to investigate pre-service teachers' mathematics in developing MFT. For the purpose of this study, *concept study* means an instructional structure envisioned to offer teachers with various experiences and approaches with mathematics and colleagues that might develop "disciplinary knowledge founded on conceptual diversity" (Davis & Renert, 2014, p. 38). It focuses on the actual mathematics content needed for teaching, school mathematics. Thus, concept study gives teachers opportunities to work collectively to re-construct mathematics they have experienced as students or pre-service teachers for pre-service teachers and as students or pre-service teachers or in-service teachers for in-service teachers in ways that express the concepts in ways more manageable for the learners they teach. Furthermore, concept study is not intended as a strategy "to impose a fixed set of procedures in order to generate a uniform product" (p. 89). The purpose of doing concept study is for the teacher to understand explicitly and more deeply the subtle complexities of a concept or process so they can teach it better (Davis & Renert, 2014).

Based on an understanding of mathematics teachers' professional knowledge as an "open disposition" (Davis & Renert, 2014, p. 47) and not merely as a mastery of a distinct body of knowledge, Davis and Renert articulate three assumptions that I will use to guide this work: First, teachers' mathematical knowledge is understood as a distributed collective work of knowers, the teachers. Second, knowing happens inside the individual teacher but is activated with the use of the collective. Third, mathematical concepts are emergent forms arising in complex layers. Thus, all activities in concept studies are organised based on the assumption of the "collective as a cognizing agent" (p. 53). Collective cognition in concept study allows and makes possible the observation of the thinking of the cognizing agent by detecting the interactions and cues that trigger new possibilities and insights for the collective. Building the logic of a complex mathematical idea, it is necessary to generate:

> lists of metaphors, analogies and images that might be associated with that idea. The process of generating such a list both renders explicit the principally tacit nature of human knowing and the principally analogical nature of human learning. ...the aim of unpacking activities within concept study is to recall the figurative aspect of understanding which the expert knower might have forgotten they know (p. 42).

The second assumption is similar to Manouchehri and Enderson's (2003) explanation that in any social interaction, though individuals bring their own constructs and meanings, for the "meaning of knowledge to be viable to the individual, it must be the product of social understandings" (p. 115). Thus, in this study, the pre-service teachers have used concept studies as an approach to re-construct meanings of mathematical concepts among colleagues. As Manouchehri and Enderson argue that "by facilitating communication among the future teachers, and by orchestrating situations in which they must exchange ideas, articulate their thinking, and attempt to solve conflicting views, they develop the capacity to see new perspectives and build new understandings about mathematics and teaching" (p. 116). Thus, the concept study acts as a means for pre-service teachers to communicate and share ideas of the mathematics they need for teaching.

3.3.1 Concept study emphases

Concept study takes the form of five nested emergent emphases (figure 5) — realizations, landscape, entailments, blending, and pedagogical problem-solving (Davis & Renert, 2014). The five emphases could be understood as potentials that are always present in a concept study that repeatedly unfolds, guided by participants' collective re-construction of the mathematical concept at hand in the concept study session (Davis, 2012; Davis & Renert, 2009; 2014). Davis and Renert used a visual metaphor (see figure 5) of the five emphases of concept study as nested circles to indicate that they are non-linear. These emphases are nonlinear as they point to "interpretive strategies that are always simultaneously present alongside others that are yet to be noticed or made explicit" (p. 57). That means each layer is dependent on the prior inner layers but, the movement is simultaneously interpretive from one another or across all.

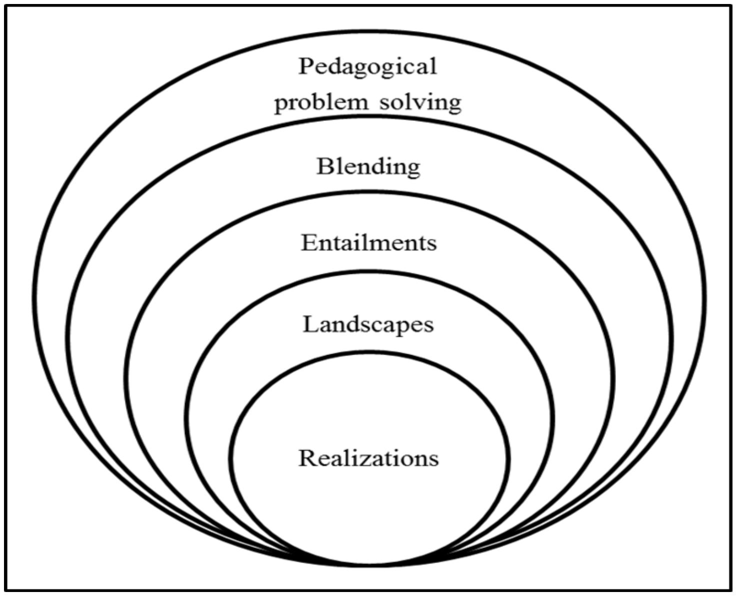


Figure 5: Davis and Renert's visual metaphor depicts the relationship of concept study emphases (Adapted from Davis & Renert, 2014)

More importantly, Davis and Renert note that in concept study, teachers' mathematical sense-making is not about producing accurate interpretations, right answers, optimum images, or specific metaphors. Rather it is about the development of sufficient, suitable, and worthwhile interpretations for the concept or task at hand. Davis and Simmt (2006), explain that with concept study the "collective supports the development of robust, flexible individual understandings" (p. 309) arguing the individuals contribute a variety of interpretations and tactics in making sense of the concept at hand. They note that the variety of interpretations is the centre of the interactions. As Davis and Renert (2014) indicate, a deep understanding of a concept entails more than tearing apart its components; it requires investigation of how these parts are tied together and fall apart in different contexts and circumstances. According to Davis and Renert, four of concept study's five emphases realizations, landscape, entailments, and blending have proven to be successful in professional learning with different groups of teachers.

The model allows for collective elaboration of mathematical concepts. Pedagogical problem solving, the fifth emphasis in concept study rests in everyday complexities of mathematics teaching and problem-solving; it necessarily involves more than one concept at play at a time. For the purpose of this study, the five emphases of concept study are fostered throughout the design of the concept study sessions and are described hereafter to provide terminology for readers to understand a concept study as a context for examining collective development of MFT (professional knowledge and skills) of the ratio/proportion/rate/linear function concept. The five emphases form the framework for the data analysis.

3.3.1.1 First emphasis: Realizations

Davis and Renert (2014) borrowed the term 'realizations' from Sfard (2008). It refers to a set of associations—such as the formal definitions, metaphors, images, algorithms, gestures, and applications—a learner might draw on and connect in an effort to understand a mathematical concept. It is a "micro-level snapshot" (p. 62) of a mathematical concept. The learners or teachers (in this case) build their conceptual understanding when exposed to a learning environment that provides varied interpretations. A group of teachers as a collective learner in concept study makes possible the emergence of varied interpretations of the mathematical concept within a session. Collective identifications of the realizations of a mathematical concept in concept study is a process that is non-linear and not obvious: each individual teacher (as a knower) embraces and utilizes a common or distinctive list of realizations which when they exist in a collective evolve throughout the concept study session. To avoid having teachers in learning situations provide only well-rehearsed definitions, a loose set of definitions could be activated by inviting them to explore how the mathematical concept is "introduced, taken up, applied, and elaborated at different levels [in school curriculum] ... and the problem that learners encounter as they study" (p. 60) it. For example, realizations of mathematical concepts such as ratio could include formal definitions (e.g., ratio is a comparison of two quantities); applications (e.g., ratio is used to find the slope/gradient of a linear functions); algorithms (e.g., find ratio by dividing equally the whole quantity to be shared); images (e.g., slope as a ratio illustrated on the two dimensional plane/x and y plane); and metaphor (e.g. ratio as a part to whole comparison or ratio as a fraction). Davis and Renert offer the concept of multiplication to illustrate the possible elements that the teachers' realizations might draw on:

- formal definitions (e.g. multiplication is repeated grouping)
- algorithms (e.g., perform multiplication by adding repeatedly)
- metaphors (e.g. multiplication as scaling)
- images (e.g., multiplication illustrated as hopping along a number line)
- applications (e.g., multiplication used to calculate area)
- gestures (e.g., multiplication gestured in a step-wise upward motion). (p. 58)

Teachers investigating their realizations of mathematical concepts creates an environment where the development of their MFT (professional knowledge and skills) is possible. To repeat, in concept study there is no claim about the rightness or adequacy of any particular realization, rather what matters is that the process of generating the realizations will allow for further development of the teacher's knowledge about the concept at hand (Davis & Renert).

3.3.1.2 Second emphasis: Landscapes

The term 'landscapes' refers to the organization of 'realizations' of a mathematical concept. A landscape is the outcome of providing an awareness of how the 'realizations' relate within a grade level (horizontal) and across grade levels (vertical) the school mathematics curriculum. It is "a macro-level map" (Davis & Renert, 2014, p. 62) of a mathematical concept. For example, a 'landscape' for ratio would be a macro-level map created out of organized realizations for ratio in the school curriculum (for any particular educational jurisdiction). Understanding the landscape provides the teacher with an awareness of the value and the viability of their realizations for the mathematical concept across the curriculum. Davis and Renert illustrate how developing a landscape can lead a teacher or group of teachers to note a difference in the usefulness of various realizations for the multiplication concept. As they describe, some realizations for the multiplication concept remain viable in most contexts in which teachers encounter it, while others are "situation-specific or perhaps learner-specific" (p. 61). They offer an example, the realization of multiplication as "repeated addition" is limited as it varies in viability depending on its applications. It works when the domain is restricted to whole numbers but fails as an explanation for the multiplication of fractions or vectors. The creative reworking of the 'realizations' for a mathematical concept such as ratio/proportion/rate/linear function provides a context where the pre-service teachers examine how the 'realizations' "hold together and fall apart in different contexts and circumstances" (p. 43) as they come to understand them as constituting a landscape.

3.3.1.3 Third emphasis: Entailments

The term 'entailments' refers to the logical implications each realization of a mathematical concept carries that help to shape understanding of the related mathematical concepts. Again, discussing multiplication, Davis and Renert (2014) describe entailments of the 1) multiplier, 2) multiplicand, and 3) product for the realization of multiplication as grouping. A set of entailments point to: 1) the count of groups in a set, 2) the count of objects per groups, and 3) the total count of objects in a set. While entailments of the multiplier, multiplicand, and product for the realization of multiplication as array/area making means the 1) first dimension, 2) the second dimension, and 3) the total count of cells (area) respectively. Davis and Renert describe that through the process of investigating various entailments the teachers gain a new and a novel strategies that move beyond mere well-rehearsed realizations for the mathematical concept. They explained further that investigating these entailments can sometimes be "tedious and frustrating" (p. 67) for teachers because of positioning them as novices that activate the difficulties and resistances during this emphasis.

3.3.1.4 Fourth emphasis: Blending

The term *blending* refers to the process that involves the activity of generating combining and collapsing varied realizations of a mathematical concept (Davis & Renert, 2014). Davis and Simmt (2006, cited in Davis and Renert 2014) illustrated the blend of area-based image and a grid-based algorithm which shows the connections existing in multiplying multi-digit whole, decimal fractions, mixed numbers, and binomial expressions. Blending is an emphasis that is emphatically different from the first three—realizations, landscapes, and entailments that focus on generating distinctions among realizations and their consequences for a mathematical concept. In this emphasis, the teachers were asked to seek out "meta-level coherences by exploring the deep connections among realizations" of the mathematical concept that might produce further emergent interpretive possibilities (Davis & Renert, 2014, p. 70).

3.3.1.5 Fifth emphasis: Pedagogical problem solving

The term *pedagogical problem solving* refers to work that teachers do when they work on mathematics that emerge from learners' questions in the process of learning. For example, questions such as "Is 1 a prime number?" "Is ∞ a number? What does mean to divide by zero?

And, what is the difference between undefined and infinite?" (Davis & Renert, 2014, p. 78). Teachers as communities of experts deepen their understanding when they discuss these questions, the difficulties students might encounter in solving them, the logical mistakes students might encounter, and the way the teacher could help the transitions in students' understanding. Davis and Renert describe that this emphasis "is developed around the actual questions that meaning-seeking learners ask" (p. 79). It is tied to the work of teaching and investigates the questions raised by students that teachers have encountered in their own teaching experience. Pedagogical problem solving "capitalizes on the interpretive potentials that arise collectively when teachers draw on various instances of individual expertise in order to broach perplexing problems of shared interest" (Davis & Renert, 2014, p. 79-80) of the concepts taught in ordinary level secondary schools.

To summarize, the chapter describes the complexity of mathematics teachers' professional knowledge, and how concept study (Davis & Renert, 2014) can be used to explore the development of MFT, in the case of this study with Tanzanian's diploma in secondary education pre-service mathematics teachers. I described the complexities of mathematics teachers' professional knowledge by exploring how mathematics teachers come to know the mathematics they need to teach, the idea of knowing (Davis, Sumara, & Luce-Kapler, 2008). The view of mathematics teachers' professional knowledge as a complex learning system and the consideration of group of pre-service teachers in concept study as a collective learner rather than a collection of learners (Davis & Renert, 2006; Davis & Renert 2014). I described an exploration of the meaning of concept study as it is used in this study and its focus, and the assumptions that guided this study. I provided a systematic description of the concept study emphases—realizations, landscape, entailments, blending, and pedagogical problem-solving (Davis &

Renert, 2014). Also, I described the reason to why Davis and Renert concept study emphases visual metaphor is nested and not a linear in the collective learning of mathematical concept in concept studies. In the next chapter I will use the notions discussed to justify the methodology of the study.

4. Methodology

This chapter aims to describe the methodology undertaken in this study. In it I discuss the appropriateness for both the chosen paradigm, constructivism, and the research design method, qualitative group case study. As well I address the strength and weaknesses, and the criteria for judging the quality of a qualitative case study. The chapter also describes how complexity science is considered in the design of the concept study sessions rather than as a theoretical frame for analysing the data. A description of how the five conditions of complex systems, the "internal diversity, internal redundancy, decentralized control, enabling interactions, and neighbouring interactions" (Davis & Simmt, 2003; 2006) is used in concept studies of ratio/proportions/rate/linear functions to ensure pre-service teachers' full contributions in the collective learning of the concept studies. Each of the conditions is described in five separate sub-sections to give the reader the understanding of how each contributes to the classroom design for the concept study sessions. The chapter also describes the role of a researcher, the research site, and participants, as well the methods used for data collection and its analysis.

4.1 Why Qualitative Case study as a Research design

A qualitative case study can be defined as "the process of conducting the inquiry (that is, a case study research), the bounded system, or unit of analysis selected for study (that is the case), or the product, the end report of the case investigation" (Merriam, 1998, p. 43). In this case study research is designed to investigate the contribution of concept study on Tanzanian pre-service teachers' professional knowledge and skills (mathematics for teaching (MFT)). Such work calls for the use of a qualitative group case study approach because it provides the possibility of "insight, discovery, and interpretations" (p. 28). This research explores the value of concept studies (Davis & Renert, 2014) as an instructional approach for developing pre-service teachers MFT. Yin (2014) elaborates "the actual outcomes of interest and therefore, the appropriate unit of analysis may be at the community or collective level and not the individual level" (p. 13). In this case, this study investigated a group of pre-service teachers collectively rather than as individuals because the basis of concept study is collective learning (Davis & Simmt, 2006; Davis & Renert, 2014).

Case studies are not all alike. Based on overall intent, Merriam (1998) classifies case studies as descriptive, interpretive, or evaluative. As she explains, descriptive case studies "present detailed accounts of the phenomenon" (p. 38); an interpretive case study "contains thick, rich descriptions" (p. 38); and an evaluative case study "involves description, explanations, and judgement" (p. 38). Merriam argues that because the qualitative case study "provides thick descriptions, is grounded, is holistic, and life like, simplifies data to be considered by reader, illuminates meaning, and communicate[s] tacit knowledge" (p. 39), it is good for evaluations. In this case, this study is a descriptive qualitative case study that draws on a sensibility in which knowledge and meaning making are understood as co-constructed in the collective activity of the group. At the same time it integrates evaluative perspectives, as the research question asks, is "*In what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?"* Case study is used to describe the understandings of the mathematical meaning of pre-service teachers' interpretations of mathematical concepts through concept studies while investigating their MFT and the ways in which concept study contributed to the development of their MFT.

Different scholars suggest looking at a case holistically when dealing with applications of case study designs (Yin, 2014; Merriam, 1998). According to Merriam (1998), qualitative case study “focuses on holistic description and explanations” (p. 29), designed to suit situations that are impossible to separate the phenomenon’s variable from its context. Yin (2014) asserts that “the case study allows investigators to focus on a ‘case’ and retain a holistic and real-world perspective-such as in studying …organizational… processes, school performance…” (p. 4). In this case, developing teachers’ MFT through concept studies with pre-service teachers is of special interest to this research due to the uniqueness of the poor performance of the Tanzanian students in secondary education examination (CSEE) described in Chapter 1. This study investigated concept studies with a particular group of pre-service teachers as a “bounded system or unit of analysis” (Merriam, 1998, p. 43).

This study is an intrinsic descriptive qualitative case study. Stake (1995) describes an intrinsic case study as one in which the focus is on the case. The overall intent is to investigate the contribution of concept study on Tanzanian’ pre-service teachers’ MFT—professional knowledge and skills. To better understand the conditions that influenced the change to take place in concept studies, I investigated the process and outcomes. In other words, how the process involved in concept study as a professional learning approach influences changes in pre-service teachers’ knowledge of mathematics they need as teachers and the use of the pre-service teachers’ interpretations of the ratio, proportion, rate and linear function as the outcomes to illustrate their knowledge. Stake also emphasizes that in the intrinsic case study, “our first obligation is to understand this one case” (p. 4). In this study instead of investigating the contribution on concept studies on pre-service teachers’ MFT—professional knowledge and skills at all levels in Tanzania, this study intention is only on diploma in secondary education pre-service teachers —prospective teachers at ordinary level secondary schools.

4.1.1 Strengths and weaknesses of qualitative case studies

All research designs have strengths and weaknesses. Merriam (1998, p. 41) identifies several strengths of qualitative case studies the potential to: understand complex phenomenon and explore innovations: evaluate programs and inform policies; build on readers' previous experiences; provide a rich holistic account of a phenomenon; and play an important role in advancing a field's knowledge base. Thus, this study has the potential to contribute to our understanding of the multiple aspects involved in pre-service teachers' professional knowledge and skills. This case study was developed from my previous experience as a secondary school mathematics teacher, a mathematics curriculum developer, and currently as a researcher. The study intends to offer a rich holistic description of the contribution of a concept study on Tanzanian' diploma in secondary education pre-service teachers' MFT—their professional knowledge and skills.

Merriam (pp. 42-43) describes a couple of the limitations of qualitative case studies. First, she notes that they have lengthy findings that can be so detailed and so involved that they may not be read by the very people they intend to inform, e.g., busy people like policy makers. Second, the researcher is the primary instrument in data collection and analysis and as a result personal qualities and biases may affect the end product as the author may pick and choose what to present. Third, there is also the possibility that the researcher will oversimplify or exaggerate situations which might lead to wrong conclusions or interpretations. Finally, there are no guidelines on how to write a report and limited guidelines for analysis. With these limitations in mind, I, as a researcher in this study, have tried my level best to write a report that includes just enough information to warrant claims, minimize and acknowledge any biases in analyzing data, get feedback on the interpretations of observations I made of situations, and worked to write a report that respects the case and the reader.

4.1.2 Criteria for judging quality of qualitative case study

This study would be worthwhile if the findings can be trusted. As a qualitative researcher dealing with a case study, I have been honest with the claimed results, validating them, and making reasonable interpretations when reaching conclusions. Merriam (1998) suggested three important aspects in validating the findings: Internal validity, Reliability, and External validity (p. 198-219).

4.1.2.1 Internal validity

Merriam (1998) asserts that the internal validity "deals with questions of how the research findings match the reality. How congruent are the findings with reality? Do the findings capture what is really there? Are the investigators observing or measuring what they think they are measuring?" (p. 201). As she explains, the internal validity deals with the researchers' observations and how the researcher assesses those observations. Further, the researcher observes the construction of reality by the people who are being observed (p. 203). The reality is assumed to be "holistic, multidimensional and ever-changing; it is not a single, fixed, objective phenomenon waiting to be discovered, observed, and measured" (p. 202). Therefore, the internal validity of findings in this research deals with what I observed as professional knowledge and skills the pre-service teachers developed while they were constructing their MFT through series of concept studies; and how I assessed my observations to arrive to the findings of professional knowledge and skills they developed. Merriam (pp 204-205) suggested six basic categories that can enhance internal validity: triangulation, member checks, long-term observation, peer examination, participatory or collaborative modes of research, and researcher's biases.

For triangulation, internal validity requires the use of "multiple investigators, multiple resources of data or multiple methods to confirm the data" (p. 204). So, the findings of this research were based on triangulation of multiple sources of data collected from pre-interviews, my observations in concept studies sessions, and pre and post questionnaires conducted before and after concept studies. That includes, video recordings, audio recordings, field notes, and participants' notes. And I used two methods in analysing the data Davis and Renert (2014), and Ball, Thames, and Phelps, (2008) models of mathematics for teaching. For member checks, internal validity requires "taking the data and tentative interpretations back to people from whom they were derived and asking them if the results are plausible" (p. 204). In this research, I did not do this with the facts that I was analysing data collectively rather than individuals. However, during data collections I used prompts such as 'what do you mean' 'can you elaborate', and how to confirm their meanings. For long-term observations, internal validity requires collecting data at the research site or repeated observations of the same phenomenon. In this research, I collected data in the same teacher college for six months, and I observed four concept studies sessions to ensure internal validity. For peer examinations, internal validity requires "asking colleagues to comments on the findings as they emerge" (p. 204). In this research, I used peer examinations of my supervisors who constantly commented on findings as they were unfolding. For participatory or collaborative modes of research, internal validity requires "involving participants in all phases of research from conceptualizing the study to writing the findings" (p. 205). So, in this research, I involved the participants directly from recruitments of participants to collecting data. However, in data analysis, I involved them indirectly through their notes, and video and audio recordings conversation. In analysis, I constantly went back to video recording and audio recording to confirm anything not clear. For researcher's biases, internal validity requires, clarifying the researcher's assumptions, worldview, and theoretical orientation at the onset of the study" (p. 205). I explained my belief in social constructivism worldview with an assumption that there is always multiple reality, and knowledge is socially constructed. I took a group of ten pre-service teachers as a social group trying to construct and reconstruct their MFT through collective learning in concept studies. I believe each individual pre-service teacher has personal meaning about the mathematics concept at hand in concept studies but, with interaction nature of concept studies the sharing of individual meanings with colleagues influences the construction of new meaning and reconstructing the previous understanding of mathematics concept at hand.

4.1.2.2 Reliability

Merriam (1998) asserts that reliability "refers to the existent to which research findings can be replicated" (p. 205). In other words, suppose the same research is repeated by another researcher under the same conditions will it yield the same result? In this regard, I believe if another researcher repeated this study using concept study with the same pre-service teachers under same assumptions, the findings would resonate with my findings. Merriam asserts that reliability is ensured through explanation of researchers' position with regards to the study, triangulation, and the use of an audit trail (pp. 206-207). In regard to the investigators' position, I explained the assumptions undertaken in this research in chapter 3, explained the group being studied and the criteria used for their selection, their description, and the context under the research site and participants section 4.5. For triangulation, I used multiple data evidence and explained all the methods I used in the data collection and analysis. For audit trial, I explained every step I used in arriving to the results/findings. That means I explained in detail how I collected the data, how categories were derived, and how I made the decision throughout the inquiry (pp. 206-207).

4.1.2.3 External validity

Merriam (1998) asserts that external validity "is concerned with the extent to which the finding of one study can be applied to other situations" (p. 207). In other words, can the results of this research generalizable? With regard to the collective nature of concept study, I believe the findings of this research could be considered in another situations. Merriam suggested three techniques to ensure external validity: use of thick description, typicality or modal categories, and multi-site designs (p. 211). The use of thick description, in this regard I have tried to offer descriptions that could help the reader determine how closely their situations match with this research situation and whether the findings can be transferable. The use of typicality or modal categories requires ones need to describe "how typical the program, event, or individual is compared with others in the same class so that users can make comparisons with their own situations" (p. 211). Thus, the use of typicality or modal categories, I compared the concept studies conducted in this research with the concept study conducted by Davis and Simmt (2006) and Davis and Renert (2014) to allow the reader to make comparisons. The use of multisite designs requires "the use of several sites, cases situations, especially those that maximize diversity in the phenomenon of interest" (p. 112) which gives the reader opportunities to more ranges of other situations. Therefore, I have used four mathematical concepts to help the reader to see how the pre-service teachers developed their MFT.

4.2 Why a Constructivist Paradigm

As an experienced secondary school mathematics teacher, I have come to believe that there is no way that what I know can be considered the only truth or reality. This belief developed through experience. When discussing ideas or concepts in mathematics with my colleague teachers in schools I was teaching I came to realize that what I believed to be right was just one of a number of existing alternatives. My experience aligns with a constructivist paradigm. A paradigm that its core idea is "all reality and interpretation are socially constructed" (Given, 2008, p. 466). Such a view of the nature of knowledge and learning falls within complexity thinking (Davis & Simmt, 2015). The purpose of constructivist paradigm is in understanding the world in which they live and work by construction and reconstruction of knowledge (Guba & Lincoln, 1994). The construction and reconstruction of an individual's new understandings or knowledge are accomplished through the interactions of prior experiences and beliefs, and the ideas, events and activities they come in contact and merging around consensus (Guba and Lincoln, 1994; Cannella & Reiff, 1994). In this study, which uses the concept study approach allows the pre-service teacher participants to learn the mathematical concepts collectively from their experiences as members in the society, as students in schools or as pre-service teachers in teacher colleges in trying to improve their MFT. Guba and Lincoln (1994) assert that constructivism describes knowledge as being in flux, as an individual internally constructs knowledge through social and cultural intervention. In this study, the social activity such as concept study with pre-service teachers' play an important role in the collective construction and reconstruction of the mathematical knowledge they need to *know* and *know-how* (Adler & Davis, 2005; Davis & Simmt (2006) for teaching. The pre-service teacher participants are understood to learn the mathematical concepts from their experiences as members of the Tanzanian society at large, as previous students in schools at all levels, and as pre-service teachers in teacher college. They are considered to embrace different multiple individual meanings that with the collective learning of the concept study the construction of new meanings or understanding of the mathematical concept at hand occurs while constructing their MFT.

Guba and Lincoln (1994) further asset that constructivism "sees the inquirer as orchestrator and facilitator of the inquiry process" (p. 114). In the concept studies with the pre-service teachers, I facilitated the "multivoice reconstructions" (Guba & Lincoln, 1994, p. 115) of mathematical meaning of the concepts on what they need know-what and know-how as knowledge was "created in [the] interaction among investigator and respondents" (p. 111). I have learnt about teachers' mathematical understanding and their attitudes towards mathematics learning that was informed by my experience as a high school teacher, a mathematics curriculum developer, and a researcher. According to Creswell (2014), a constructivist's research goal relies more on the participants' view of the condition being investigated, and the researcher's "focus on the specific contexts in which people live and work in order to understand the historical and cultural settings of the participants" (p. 8). In this case, the collective mathematical understanding and development of their MFT has been driven by social interactions among pre-service teacher's participants as a science group in secondary education majoring mathematics with either physics, chemistry, or geography. In those social interactions, cultural meanings are shared by the group and eventually internalized by the individual (Richardson, 1997).

4.3 Use of the Conditions of Complexity to structure the Classroom Setting for Concept Studies

Davis and Renert (2014) consider classroom collective, in this case a group of mathematics pre-service teachers, not as a "collection of learners but as a collective learner" (p. 32). In complexity science terms, they identify the 'collective' as a cognizing agent, and not as a collection of cognizing agents. As they describe, it is an understanding that permitted them to observe the thinking of the agent by spotting the interactions and prompts that activated new potentials and insight for the collective. Mathematics ideas seem to "emerge as the collective practices of the classroom community evolve" (p. 55), an idea that Davis & Renert attribute to Cobb (1999). Complexity science can inform classroom design. In this classroom design, the pre-service teachers are learning through their discussions, reflections, contemplations, and investigations. Through the interactions of the participants with the mathematical concepts a collective learner emerges. From participation in the collective the teacher can develop the professional knowledge and skills they need for teaching, in other words, their MFT. For example, in concept study each individual teacher provides ideas about the concept at hand as definitions, examples, metaphors, analogies, and applications. The teachers discuss the shared ideas with each other regardless of their adequacy. In doing so, the teachers have the opportunities to correct any misconceptions about the concept at hand and gain new insights, hence the development of their MFT.

Bearing in mind the complexity of teachers' professional knowledge and skills as explained in the previous chapter section 3.2 and the explanation offered in the previous paragraph, the classroom settings of the group of pre-service teachers for all four concept studies was designed to prompt the collective learner rather than treating the group as a collection of learners (Davis & Simmt, 2006, p. 309). The concept study "classroom" in all four concept studies of this research project was designed to ensure the pre-service teachers made full contributions in the collective learning. This was done by designing the class based on the features of a complex system: the internal diversity, internal redundancy, decentralized control, enabling interactions, and neighbouring interactions (Davis & Simmt, 2003), as described thoroughly hereafter.

4.3.1 Internal diversity

Within a complex learning system internal diversity of agents and roles of agents contributes to the vibrancy of the *system*. This condition is a result of the lack of ability to predict what will be necessary for learners working on a novel mathematical task. In work with mathematics learners (Davis & Simmt, 2003) and with teachers (Davis & Simmt, 2006) internal diversity included elements such as differences in approaches to mathematics, different roles in group work, different ideas expressed within the group, all of which contribute to the collective intelligence—levels of mathematical understanding and different ways of knowing (Davis & Renert 2014). The condition of internal diversity in this study was addressed explicitly by having mathematics pre-service teachers from different majoring combinations—either mathematics and physics or chemistry or geography, and their previous ordinary level and advanced level secondary school experiences. It was believed that such differences could lead participants to generate diverse contributions to the concept study. Although certain forms of diversity can be put in place, in concept studies, the diverse contributions of both the individuals and the collective are assumed to emerge in the context of activity. As Davis and Simmt (2003) explain diversity "cannot be assigned or legislated, it must be assumed-and it must be flexible" (p. 143). In this case, the individuals' consciousness of diversity in concept study emerges through the collaborative learning nature of the method.

4.3.2 Internal redundancy

A group of teachers working together professionally in the collective learning of the concept studies to develop their MFT depends as much upon their similarities as their differences (Davis & Simmt, 2003). Redundancy refers to similarities of elements of the classroom collective that ensure mutual understanding and interactions (Davis & Renert, 2014). Sameness among the individual pre-service teachers allows for the interaction among agents in the mathematical environments because they can make sense of contributions from others. In the context of the collective learners in the concept study, internal redundancy is understood as the commonalities shared among individuals such as mathematical vocabularies, mathematical knowledge learning experiences (Davis & Simmt, 2006) as secondary school students, and expectations for MFT as prospective teacher of the ordinary level secondary schools—diploma in secondary education pre-service teachers in Tanzania. Internal redundancy as Davis and Simmt (2003) explain enables "moment-to-moment interactivity" (p. 150) among agents of the collective. The potential for the internal redundancy in this study resulted from selection of the pre-service teacher's participants in the planning phase of this research. The selection of pre-service teachers included the similar level of study—second year diploma, the same professional expertise— mathematics majors, and similar learning environments—same teacher college. The selection resulted in commonalities among pre-service teacher's participants in all four concept studies. These commonalities enabled learners to contribute to the mathematical environments in learning the concept at hand in the concept studies.

4.3.3 Decentralized control

Though there is normally a facilitator in a concept study, the teachers and participants make the moment by moment decisions within the concept study session as possibilities emerge and subside in their interactions with colleagues. Davis and Simmt (2003) found that with the complexity that emerged in the teachers' collective activity there was no single entity that acted as a controller in the teachers' professional learning community. Decision making is distributed, adaptively and democratically among the individual teachers and the facilitators participating in the activity of the professional learning community (Davis & Simmt, 2006). In this case, decentralized control was achieved by the facilitator providing tasks for the teachers at the same time as giving them freedom of contributing ideas of the mathematical concept at hand in the concept studies sessions without an evaluation. That means the participant who wished to contribute ideas in the collective learning was given a chance, and all contributions were respected regardless of length or accuracy. The decision to continue the discussion for the ideas raised by the pre-service teachers during the concept study session, solely remained with them; however, the facilitator offered questions and tasks throughout the sessions when conversations subsided, and it seemed new prompts could be helpful to encourage more meaning making. It is important to state that the facilitator is also a member of the collective and one that has a special function that includes initial prompts for activity within the expectations of the group. For example, the tasks used in the concept studies were all selected from the prospective curriculum the diploma in secondary education pre-service teachers will be teaching. Davis and Simmt (2006) explain that decentralized control is an important aspect in complexity emergence that "it is only possible if the phenomenon is framed by constraints that enable unanticipated possibilities" (p. 311). In this case, considering the group of pre-service teachers as a collective learner, the decision-making of who and what to contribute is distributed in a democratic manner and adapted to the individual pre-service teacher's participant in the mathematical environment of concept studies (Davis & Simmt, 2003).

4.3.4 Organized randomness

The complex systems are governed more by boundaries than rules (Davis & Simmt, 2006). The boundaries are observed as organized randomness that balances the internal diversity and internal redundancy. They are the constraints that are determined by the type of activity provided. These constraints lesser the possibilities of not degeneration into aimlessness (Davis & Renert, 2014). Constraints direct the types of activity by providing an environment that allows the higher level of innovations (Davis & Simmt, 2003). For example, teachers working together on concept studies as a collective face some constraints but, within the context of these constraints the teachers might collectively produce an environment that could be rich with possibilities and innovation (Davis & Simmt, 2003). It is possible for the researcher/facilitator to enable the organized randomness by limiting the collective mathematical focus at the same time supporting complete innovation within the constraints. The pre-service teachers who participated in this research worked collaboratively as a collective in the concept studies under constraints of time, the teacher college course requirements, the availability of the technology (especially internet), and the well-established mathematics reflected in textbooks and curriculum with which they were familiar. So, as a researcher, I enabled *organized randomness* by carefully limiting the tasks to the mathematical focuses of the selected concepts in concept studies while simultaneously supporting complete innovations in developing their MFT within those boundaries.

4.3.5 Neighbour interactions

The collective learner in concept studies is the site of knowledge production. Davis and Simmt (2006) assert the neighbours in any knowledge producing system “are not physical bodies or social groupings. Rather, the neighbo[u]rs that must 'bump' against one another are ideas, hunches, queries, and other manners of representation" (p. 312). Thus, the condition neighbour interactions refer to individual pre-service teachers' important thoughts and insights interacting with each other, the facilitator, and the mathematical concept. In this case within the collective learner in concept studies, the ideas, metaphors, images, algorithms, and applications of the mathematical concept at hand of individual pre-service teachers are exposed and examined by the others. As individual pre-service teachers' ideas and insights collide with one another, a space is provided for the collective development of their MFT. As a researcher and a facilitator in the concept studies, I did not order neighbouring interactions but, I led the ideas and insights of individual pre-service teachers to spill across one another effectively in promoting it. In the next section, I will discuss how concept study was used in developing teachers' MFT.

4.4 Researcher Role

As an experienced high school mathematics teacher, curriculum designer and a researcher, my understanding of the decisions I make and the phenomena I observe has been informed by my history of interactions, making me an emphatic second-person observer (Metz & Simmt, 2015). According to Metz and Simmt, an empathic second-person observer is an observer that "becomes part of the social group" (p. 199) and acts as a coach as he or she knows the kind of experience the first person is talking about. An empathic second-person observer "assumes the role of mediating participants' access to their own awareness" (p. 199). In this case study, I had three primary roles, all of which (I believe) were possible because of being a second person empathic observer. I created a curriculum for the series of concept studies for the research; I intervened in the collective learning when I believed my contribution of a question or new tasks might trigger further meaning making of the mathematics; and I observed with the ears of a mathematics teacher who has had the experience of not only learning mathematics but also teaching mathematics. Finally, as a researcher I study the data analysing it and interpreting it in ways that I anticipate will make sense to other researchers and teacher educators.

4.5 Research Site and Participants

Creswell (2014) suggested five stages to follow in collecting the qualitative data or evidence: 1) "setting the boundaries," identifying "purposefully selected participants or sites" to be studied, 2) accessing the "number of sites and participants" to be involved in the study, 3) indicating the "type of data to be collected" (p. 189), 4) "designing tools for collecting data," and 5) designing "ethical considerations in data collection and handling" (p. 193). This section focuses on identifying the purposive sites and participants respectively. Yin (2014) suggest that for the qualitative case study, in screening participants for single case study, for example, in this research a case, the researcher should select the participants that best fit the researcher's "(literal or theoretical) replication design" (p. 95). In this research, participants need to be pre-service teachers that are mathematics majors and prospective mathematics teachers of ordinary level secondary school students.

I recruited ten second-year diploma in secondary education pre-service mathematics teachers to participate in the research. All ten recruited pre-service teachers participated in interviews and in each of the first three concept studies; nine participants participated in the fourth concept study. All interviews, pre and post tests and concept studies were conducted in English, as this is the medium of instruction for diploma in secondary education pre-service teachers in the teacher college. The pre-service teachers were in the first term of their second year of their teacher education program. They had completed eight weeks of their first year Block Teaching Practice (BTP). I had no reason to believe this group of college students would not be a suitable group for the concept studies. As I wanted to explore, understand, and gain insight about the contribution of concept study on pre-service teachers' MFT and this group was as suitable as any who were studying to be secondary mathematics teachers in Tanzania (Merriam, 1998, Stake 1995).

All of the pre-service teacher participants were from the same teacher college located in the North-east zone in Tanzania. At the teacher college, I met the principal who gave me the permission to involve any second year mathematics pre-service teachers that might consent to participate in this study. He referred me to the academic master who happens to be a mathematics teacher educator. The academic master introduced me to the second-year pre-service mathematics teachers. I introduced my research, to the pre-service teachers, gave them the invitation letters and asked them for their consent to participate in this study. I was pleased that all ten students that I had been introduced to agreed to serve as participants and signed the consent forms. Both the invitation letter and the consent form are included in the research consent documents as explained under section 4.6.1 and can be found in Appendix A. According to Merriam (1988), the central process participant selection is to get good informants "who can express thoughts, feelings, opinions, and perspectives, on the topic being studied" (p. 75). In this research, it was important for me to include in the study: pre-service teachers with experiences in learning mathematics as students; participants who are mathematics majors' pre-service teachers in teacher college: and people would be able to participate in the concept studies by sharing their thoughts, perspectives, and ideas of mathematical concept at hand (e.g. ratio, proportion, rate, and linear function) to develop their MFT. The academic master assured me of the potential of this group to contribute positively to the study from his experience as the mathematics teacher educator, as well as their academic master. I began this research believing that each participant would have the potential to be a good participant based on the contribution of their thoughts, perspective, and ideas in the concept studies to allow me investigates the development of their MFT.

I chose the respective teacher college because it offers a Diploma in Secondary Education (DSE)-science mathematics majors. The DSE-science mathematics major pre-service teachers were studying mathematics with either physics, chemistry, or geography prepared specifically for teaching at ordinary level secondary schools (junior high school). Another reason for choosing this college is that I expected it to have some teaching resources that I could use in conducting the concept studies such as the classroom: the Tanzanian mathematics syllabus for ordinary level secondary school and a white board. Access to the chosen research site was enabled by the existing professional relationship between a colleague of a capacity building development project of which I was a part with the Tanzania Ministry of Education, Science, and Technology (MoEST) research and planning department. Organizational structures in Tanzania are hierarchical. Hence approval was secured from: The Regional Executive Director (RED), the Regional Education Officer (REO), the District Executive Director (DED), the District Education Officer (DEO), the Teacher College Principal, and the Teacher College Academic Officer.

4.6 Data Collection Methods

As described in the previous section, another essential step in data collection is the richness of the participants' contributions in the study in relation to the type of data to be collected (Creswell, 2014). To achieve this goal, I gathered multiple sources of evidence (Yin, 2014): face to face pre-study interviews, concept study workshops, and pre and post questionnaires. I collected video recordings, audio recordings, working papers, field notes and written questionnaires. This section is divided into four sub-sections. The first sub-section describes ethical considerations for this study while the second sub-section describes how I conducted face to face pre-study interviews, why is it used as a data collection method and its significance. The third sub-section describes how I conducted the concept studies workshops whereas the last sub-section describes how the pre and post questionnaires were conducted.

4.6.1 Ethical considerations

Before conducting this research, I secured the approval of the University of Alberta's Research Ethics Board. All ten pre-service teachers that started the study were fully informed of the project and completed consent forms; I had no one withdraw. Confidentiality was assured through the assignment of pseudonyms to pre-service teachers, as well as their teacher college. I reported comments from pre-service teacher participants in such a way that any characteristics or information that might have identified the pre-service teacher were excluded or modified. All pre-service teachers had the ability to have material related to their experiences removed by requesting its removal from the transcript (Costley, Elliott, & Gibbs, 2010). I stored all electronic data on a password protected personal computer and hardcopies never left my room. In every aspect for the data collection in this research, the individual face to face pre-study interviews, pre and post concept study questionnaires, and in concept studies, I abide by all ethical considerations as per the University of Alberta ethics policy.

After the ethical approval of this study by the Research Ethics Board (REB) at the University of Alberta, I sent a letter to the Ministry of Education, Science, and Technology (MoEST) in Tanzania asking for the permission to conduct this research in one of the teacher college as explained under section 4.5. The MoEST research and planning department approved the request for conducting the research and gave a letter which, I took to the Regional Education Officer through Regional Educational Director in the respective region who offered an introduction letter which, I took to the District Educational Officer (DEO) through District Educational Director in the respective district. The DEO gave another introduction letter which, I took to the principal of the teacher college. After research approvals from all the necessary offices and introduced to the pre-service teacher participants, I explained to them what I intended to do and gave each a letter for invitation to participate in this research as well as the consent forms. Upon receiving their signed consent forms, I then agreed with each of them to schedule the pre-concept studies face to face interviews for a location, time, and date that were convenient for them, as well the agreement of the date to conduct the first concept study. The concept study agreement dates for the other three were negotiated in the preceding concept study day. Specifically, I explained to the pre-service teacher participants what I intended to do, how I recorded the data, and the choice of pseudonyms. All the documents mentioned in this paragraph are collected as research consent documents (Appendix A). I requested pre-service teacher participants to fill and sign the same consent forms once again, during face to face pre-study individual interviews and each day of the concept study workshops as described in the next three sections 4.6.2, 4.6.3 and 4.6.4.

4.6.2 Face to face pre-study individual interviews.

Two weeks before the first concept study workshop, the pre-service teacher participants were individually interviewed (and audio recorded) for one hour. I agreed to meet each participant individually at a location that was convenient to them, but in each case the interview took place in teacher college. The interviews were intended to develop an understanding of each participant's experience in learning mathematics. The interview is one of the tools used to collect data in qualitative case studies (Brenner, 2006; Creswell, 2014; Merriam, 1988; 1998; Yin, 2014). Merriam (1998) describes, interviewing is essential "when we cannot observe behavio[u]r, feelings, or how people interpret the world around them" (p. 72). Brenner (2006) describes further the goal of interviewing as to attain more in-depth direct verbal access to participants' meaning rather than to depend on the "the interpretation of surveys, tests, participant observations, or naturally occurring conversations" (Ellis, Hetherington, Lovell, McConaghy & Viczko, 2013, p. 489), which requires less inference. I conducted the face-to-face pre-study interviews to gain a holistic understanding by asking the question "how does the diploma in secondary education (science) pre-service teachers experience learning mathematics?" The open-ended interviews were employed to "give an informant the space to express meaning in his or her own words and to give direction to the interview process" (Brenner, 2006, p. 357). An interview schedule (Appendix B) was prepared in advance, in keeping with the suggestion by Stake (1995) that a researcher should have "a strong advance plan" (p. 64) and setting relevant time and space in conducting the face-to-face interviews with all participants.

The interviews were semi-structured to give the pre-service teachers the opportunities to "define their world in unique ways" (Merriam 1998, p. 74). Therefore, a set of semi-structured interview questions also were prepared in advance (Appendix C). Additionally, one week before the face-to-face pre-study interviews, research topic-related diagrams, drawing, or visual representation (Ellis, 2006) were provided as pre-interview activities (PIAs) (Appendix D) to all participants. The purpose of PIAs was to create an active relationship between the researcher—myself and the interviewees. The PIAs motivate the interviewee to speak more deeply about the topics (Brenner 2006; Ellis et al., 2013). At the beginning of each interview, I began by asking the participant to share his or her chosen PIAs. The interviewees' PIA explanations helped in learning the interviewees' personal and cultural vocabulary (Brenner, 2006). Despite the audio recording the interviews, I took some notes while interviewing. As Brenner argues that taking notes offers the interviewer "an opportunity to note directions that emerge in the interview that warrant further questions" (p. 365). A good interview needs the interviewer to be a good listener. Merriam asserts that "the good qualitative researcher looks and listen everywhere" (p. 23). Rather than interrupting or making comments based on judgements from what the participants were saying or their feeling, I simply listened to their stories and actively demonstrated interested in it and accepting their point of views. The purpose of most of the interview time was not to agree or disagree with the answers but, to hear the description of their stories. For example, I asked for clarifications or elaborations such as could you give me an example, and I am not sure if I followed you well, could you explain more, could you give an example of when it happened? To be able to get holistic of the stories. After the two days of interviews, I listened to each of the participant's one hour audio recorded interviews made calls asking for more clarification to anything that was not understood, and took few notes (Stake, 1995). I transcribed all the interviews.

4.6.3 Concept studies workshops

In exploring mathematics pre-service teachers' development of MFT, I structured a series of concept study workshops. Each concept study workshop focused on generating mathematical meaning of a concept at hand by creating five nested emergent concept study emphases (Figure 5) realizations, landscape, entailments, blending and pedagogical problem-solving as described in the previous chapter. For the purpose of this study, I conducted four concept study workshops using each of the concepts of ratio, rate, proportion, and linear functions. I conducted these studies consecutively, approximately one per month. The concept study workshops started 27 September 2016 and ended on 15 January 2017. The first concept study workshop about ratio was conducted on 27 September 2016, the second workshop about rate on 30 October 2016, the third workshop about proportion on 16 November 2016, and the fourth workshop about linear functions on 15 January 2017. The concept studies were officially completed two weeks after the final concept study workshop that means at the end of January 2017 as no participants suggested to remove their contributions to this research. The research data collection for concept studies was based on the entrance question *"Could you tell me what you know about the___ concept in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."* Other tasks related to the concept as well as the questions emerging from the participants, were given during the four different concept study workshops. For each concept study workshop, I held three sessions (pre-session, concept study (main session), and post-session) with the group of ten pre-service teacher participants. I explain the detail of each concept study workshop sessions hereafter.

For each pre-session, I divided into two sub-sections pre-session 1 and pre-session 2. The pre-session1, I used to distribute materials to be used in concept study session and discuss the reminders of the ethical issues and consent forms to participate in the concept study of that day, and the pre-concept study questionnaires. That means I used 40 minutes to remind the participants that they are voluntary to participate in the concept study workshop and how their anonymity will be protected in any public presentations and publications. I indicated even the name of their college will not be used in any public presentations and publications, and I asked the participants to sign the consent forms of the day. The collected data for pre session1was signed consent forms of the day. For pre-session 2, I provided the participants with the pre-questionnaires to do for forty minutes. The collected data for pre-session 2 was the filled pre-questionnaires.

In each of the concept study which, was the main activity, I conducted a concept study of a chosen concept for three hours with the ten pre-service teachers. The data collected in the concept study session were field notes, video recording and three groups' audio recordings of the concept study session, participants' working sheets, and photographs of the white board. In the concept study session, I asked the ten pre-service teachers to sit in a group of three of their choice though the group was varying in each concept study workshop.

In the post-session, I divided it into two sub-sections post-session1 and post-session 2. The post-session1, I included the post-concept study questionnaires which the participants did for one hour, and the collected data was the filled post questionnaires. For post-session 2, we used to discuss the plan for the next concept study for half an hour except in the last concept study workshop I used this time to remind the participants their freedom to ask for the removal of their contributions in this research within two weeks after that day. I gave the participants 30 minutes break between the pre-concept study questionnaires and concept study, one hour between the concept study and the post-concept study questionnaires, and 20 minutes between the post-concept study questionnaires and planning for the next concept study to give the participants a chance to relax. The pre-session, concept study, and post-session of each concept study workshop are summarised in Table 6.

S/No	Time	Sessions	Description of the sessions	Collected data type
1	8.40 am-9.20am	Pre-session 1	Signing Consent forms (Ethical issues) and distribution of materials to be used in concept study session	Signed consent forms
2	9.20am-10.00am	Pre-session 2	Pre-Concept Study Questionnaires	Filled pre-questionnaires
3	10.00am-10.30am	Break	Tea Break	
4	10.30am- 1.30pm	Main session	Concept Study	• Field notes • Video recording of the session • Three audio recordings for small groups • Participants working sheets (written documents and notes) • White board photograph
5	1.30pm-2.30pm	Break	Lunch Time	
6	2.30pm-3.30pm	Post session 1	Post-Concept Study Questionnaires	Filled post-questionnaires
7	3.30pm-3.50pm	Break	Evening Break	
8	3.50pm-4.20pm	Post session 2	Plan for the next concept study	The dates of the next concept study and name of the concept

Table 6: Concept study workshop sessions

As an instructional structure, the concept study is intended to provide teachers with sorts of experiences to deepen the mathematics teachers need for effective teaching. I facilitated the concept study sessions with the group of ten pre-service teachers as an emphatic second-person observer (Metz & Simmt, 2015) as I described my role as a researcher under section 4.4. I used the curriculum I created for each of the series of concept studies for this research which I termed them scripted questions (see Appendices G, H, I, and J for ratio, proportion, rate, and linear function concepts respectively). In the concept study session, I set the video recording in front of the room where it was possible to record the session appropriately. I asked the participants to sit in a group of 3-4, and I started with a prompt of what the participants know about the mathematics concept at hand. I gave the participants about ten minutes to think about it individually and then share in a small group. Each group had an audio recorder on their table to record their discussion. I monitored the small group discussions. I facilitated their discussions and the large group sharing by inserting myself in the conversations with phrases such as why, how, could elaborate more, what do you mean, and so on. I observed with the ears of an experienced mathematics teacher in learning and teaching mathematics. I described the process in detail in the next two paragraphs.

In each of the four concept study sessions, I posed a guiding question related to the concept. Pre-service teacher participants responded from their experience and knowledge as students and student teachers, using the skills they have. The pre-service teacher participants explored a variety of realizations associated with each concept as it will be explained later in chapter 5. For example, in a concept study session of "ratio", I started with a question, *"Could you tell me what you know about the ratio concept in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."* Although each pre-service teacher as an individual knower embraced and utilized his or her own individual set of realizations, the value of such collective sessions is that they work with the initial set of realizations and expand them through group activity. Even though the opening question for each concept study was well scripted, the session itself depended on the realizations that the pre-service teachers developed. I also took into consideration that the participants are pre-service teachers and developed alternative scripted questions. These were used to make the session more productive and provide the pre-service teacher access to more realizations of the concept at hand.

I also observed the way pre-service teacher participants worked collectively to create the meaning of the concepts. Additionally, I also observed individual pre-service teacher participants' response in relation to the groups' responses. I observed the pre-service teacher participants' realizations such as meanings, explanations, examples, algorithms, images, and mathematical expressions of the concept at hand, and the way they relate the curriculum content of the concept across and within grades levels. The white board served as a public display of participants' realizations and demonstrations. My observations helped in identification of the mathematics teachers' professional knowledge and skills. I used a notebook for writing my comments and observations as a researcher (field notes). My field notes helped in identification of the mathematics teachers' professional knowledge and skills. At the end of each concept study session, I took a photograph of the board and collected any documents written by participants throughout the entire concept study session. I established rapport with the pre-service teacher participants while continuing to be passive so as not to hinder the concept study. I video and audio recorded all concept studies to allow recurrent examinations and then transcribe some sessions verbatim. The video recording was prepared in a way that the video camera placed in front of the room was directly facing pre-service teachers attempting to maximize the interactions that could be examined. The audio recordings were collected by audio recorders placed in the middle of each table arrangements. These recordings were used to help in creating pre-service teachers' transcripts for concept studies activities and collective moments. Those transcripts were central to the analysis of pre-service teachers' MFT.

4.6.4 Concept study pre and post questionnaires

In this study, I prepared questions well in advance, bearing in mind the purpose of this study for pre and post sessions in each concept studies (Appendices E and F respectively). I asked each pre-service teacher participant to fill in the concept study pre and post questionnaires individually. As shown in Table 6, I used pre and post concept study-questionnaires in the concept studies pre and post sessions. Before each of the concept studies, I conducted the pre-concept study questionnaires with the purpose of assessing the prior knowledge of the concept at hand the pre-service teachers have. Each pre-service teacher participants were given forty minutes to complete the pre-questionnaires. In contrast, after each of the concept studies, I conducted the post concept study questionnaires to determine how the participants can make sense of what they have learned in the concept study. Each pre-service teacher participants were given one hour to complete the post-questionnaires.

4.7 Data Analysis

One of the distinguishing features of the qualitative case study research is the bulk of the collected data (Merriam, 1998; Creswell, 2014; Yin, 2014). Patton (1990) comments that at the analysis stage "the challenge is to make sense of massive amounts of data, reduce volume of information, identify significant patterns and construct a framework for communication of what the data reveal" (pp. 371–372). Thus, I did some data analysis while collecting data (Ellis, 2009; Merriam, 1998; Creswell, 2014; Patterson & Williams, 2002). Yin (2014) reminds us that case study analysis "depends on researchers own style of rigorous empirical thinking along with sufficient presentation of evidence and careful consideration of alternative interpretations" (p.133). However, Yin suggested four general strategies for analysing data for qualitative case studies: relying on theoretical propositions, working your data from the ground up, developing a case description, and examining plausible rival explanations (p. 136-142). The goal of the data analysis was to identify what the data was telling me about how the use of concept study in developing MFT contributes to the pre-service teachers' professional knowledge and skills. Keeping my research question at the centre helped me to provide focus on my data analysis especially when I felt overwhelmed by this responsibility. Patton asserts that there is no one right way in organizing, analysing, and interpreting qualitative data but suggested methods and not prescriptions. Instead, the researcher must do his or her best, relying on his or her intellect, experience, and judgment. The process which took an enormous investment of my time and energy to grasp. At various stages in the data analysis, I developed the fear that I was not doing the right thing. Though it took me a while to feel comfortable believing I did not have to be perfect in whatever I do in data analysis with the fact that doctoral study is a learning process. However, I do have an obligation, to report on the analytic procedures that I used.

As a first step, I created a verbatim transcription of all the 10 interviews using a transcribe wreally software (2016). I listened to the transcripts twice over. These transcripts became my texts for analysis of the prior mathematics education learning experiences of the pre-service teacher participants. Before beginning my analysis, however, I called the participants for clarification. Similarly, I created verbatim transcriptions of all the audio and video recordings of all four concept studies sessions using the same transcribe wreally software. The concept studies sessions transcriptions included the lesson notes from the participants and my field notes. Then I listened to the recording a third time. Pre and post questionnaires conducted before and after each concept study session respectively were compiled in a single electronic tabular form of all participants for each concept study. Before beginning the process of coding, I read each transcript in its entirety twice over as I felt it is a necessary thing to do. This reading helped me to acquire a solid understanding of the wholeness of the content of the transcripts and familiarize with it.

4.7.1 Analysis of transcribed interviews

Starting with the transcribed interviews, I used Ellis (2009) general strategies for analysing qualitative case studies. Ellis suggested three general strategies to analyse qualitative case studies: (1) working from all transcripts and field notes to write narrative analyses which “are explanatory stories crafted through the gathering and analysis of events and happenings to form a plot” (p. 484); (2) using written narrative case studies “to analyse for patterns, themes, or insights that can be expressed abstractly” (p. 485); and (3) writing an interpretive account. Thus, I crafted stories from all pre-service teacher participants as my coding from the transcribed interviews for each question concerning mathematics learning experiences. Typically, however, the average size of the units of text for a story I coded were groups of two or three sentences that cohered together as an entire thought. Doing this coding required a careful and attentive reading of the text which, helped me pull together the big picture of participant’s experience. Then, I used the stories from all pre-service teacher participants from each mathematics learning experience questions collectively to analyse for patterns and key words expressed with the collective story and finally, then looked for the themes that connect, or cut across the various topics. Then, I used the themes to write an interpretive account for their collective prior mathematics learning experiences of the pre-service teacher participants. These experiences are described in more detail in chapter 5.

4.7.2 Analysis of data from concept studies sessions

For the created verbatim transcriptions of all four concept studies sessions, I analysed the coded transcripts for each concept study session. First, I looked on how the five emphases of the concept study realizations, landscapes, entailments, blending, and pedagogical problem solving helped the pre-service teachers to access and develop what I observed and believed was their tacit MFT of each concept. I looked for the realizations that participants expressed in concept study sessions from each of the concepts; the ratio, proportion, rate, and the linear function, as well examples that illustrate how collective work on the concept study coalesces into what appears to be collective understanding. Also, I looked for the created landscapes, entailments, blends, and pedagogical problem-solving questions and explorations for each concept study. Sometimes, I was forced to go back and listen to the video and audio recording to get a sense of any transcript that I see some confusion. All was to make sure that I did not move away from my research question. I collected their first list of the realizations of the mathematical concepts for each concept studies of the ratio, proportion, rate, and linear function and compiled them in different tables for each mathematical concepts. And, I summarized their first list of the realizations for each of the mathematical concepts according to my understanding and displayed them in different figures as the summary of the realizations for each of the mathematical concepts. I summarized their created landscapes, entailments, blends, and pedagogical problem-solving questions of each mathematical concepts in different figures as generated from each of concept studies. The examples that illustrate collective moments or how collective works on the concept study coalesces into what appears to be collective understandings were discussed in the findings along with the emphases. This analysis will be described in detail in Chapter 6.

4.7.3 Analysis of realizations, supplementary tasks and post-questionnaire

Second, I analysed pre-service teachers' development of explicit MFT during the concept studies sessions of ratio, proportion, rate, and linear functions concepts using Ball, Thames, and Phelps, (2008) categories of Mathematical Knowledge for Teaching (MKT). I analysed the coded transcripts for all concept studies session definitions, examples, images, algorithms, and applications from life outside of school of each concept first list of realizations. The coded transcripts from the entry question of what they do know about the mathematical concept at hand and any other extra questions used other than the entry question collectively. Also, I analysed the participants' post-questionnaires responses that were used after each concept study session of ratio, proportion, rate, and linear function for responses of two questions: what they do know about the mathematics concept and how it is learned with elaborations. I looked for their definitions, images, illustrations, examples, and applications of each concept in relation to Balls' categories of MKT. As Yin (2014), argues, when researchers "have really triangulated the data, the case study's findings will have been supported by more than a single source of evidence" (p. 121). The finding for this analysis will be described in detail in Chapter 7.

4.7.4 Analysis of session pre and post questionnaires

For the pre and post questionnaires that were conducted before and after each concept study session, I coded single electronic document in tabular form of all participants for each concept of ratio, proportion, rate and linear function. I analysed the pre-service teacher reflections about the professional knowledge teachers need for teaching mathematics and the contribution of the concept study method on pre-service teachers' professional knowledge for each concept. Using Ellis's (2009) strategies for analysing qualitative case studies as explained earlier, I looked for key ideas originating from each participant's coded responses. To uncover the themes that emerged from the analysis is considered a creative process that requires the researcher to make the judgement about what is meaningful in the data set (Patton, 1990). So, I looked for the categories of these key ideas that were shared by participants to form themes for each concept, Ellis (2013) argues that "one can only understand a whole in terms of its parts. Further, one can only understand a part in terms of its relationship to the whole" (p. 491). I took the themes that were reflected in all four concepts as the findings. I did the same procedure for both reflections about the professional knowledge teachers need for teaching mathematics and the contribution of the concept study method on pre-service teachers' professional knowledge. This analysis will be described in detail in Chapter 8.

5 Prior Mathematics Learning Experiences of the Pre-service Teacher Participants

The chapter is used to describe the pre-service teacher participants' in terms of their prior mathematics learning experiences; it is done for the purpose of providing a contextual backdrop from which the case is presented.

As described in chapter 4, two weeks before the first concept study workshop, the pre-service teacher participants were individually interviewed. The interviews were intended to develop an understanding of the participants' prior mathematics learning experiences. Figure 6 shows the pre-service teacher participant mathematics grades in CSEE and ACSEE national results. For CSEE national results in Basic Mathematics, seven participants scored 'C' grade, one scored 'B' grade, and only two scored 'A' grade. While for ACSEE national results in Advanced Mathematics, one participant scored 'E' grade, four scored 'D' grade, four scored 'C' grade, only one participant scored 'B' grade, and none scored A. The pre-service teachers' mathematics grades in CSEE and ACSEE national results help the reader to have an idea of the pre-service teachers' prior performances in mathematics in secondary schools and more importantly their entrance qualifications to the diploma in secondary education teacher program. Figure 7 shows the number of pre-service teacher participants corresponding to the science subject combinations in Teacher College. Five participants take physics and mathematics, four chemistry and mathematics, and one geography and mathematics. Each pre-service teacher participant was given pseudonym P_i FFI with $1 \leq i \leq 10$, which represent ith pre-service teacher face to face interview. For example, P_2FFI represent second pre-service teacher face to face interview.

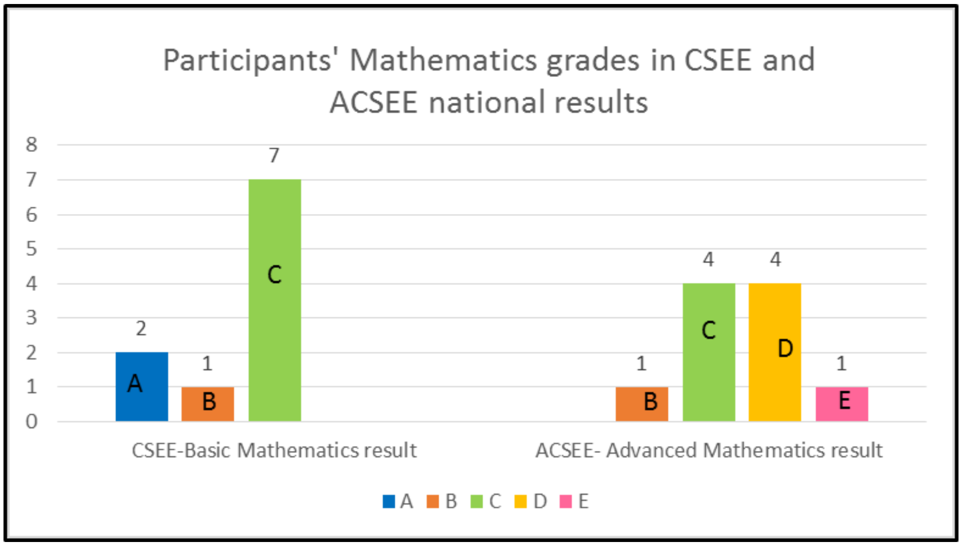


Figure 6: Pre-service teacher participants' mathematics grades in CSEE and ACSEE national results

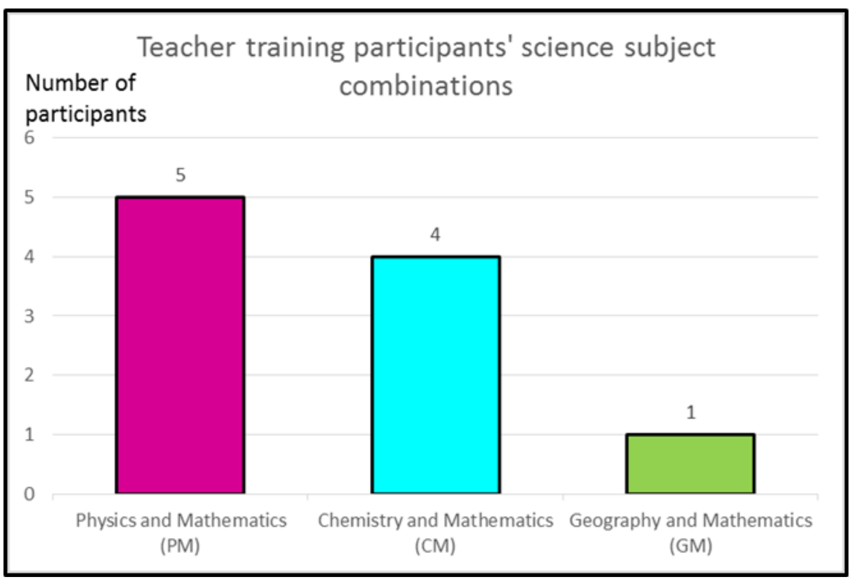


Figure 7: Pre-service teacher participants' teacher training science subject combinations

The analysis of the face to face interviews for all ten participants described in the third paragraph in chapter 4 section 4.6 revealed four themes that explain their prior experiences in learning mathematics. The themes explain the pre-service teachers experience in learning mathematics in various aspects such as the use of teaching and learning aids, teaching and learning strategies, teachers' explanation, motivation from teachers, and the language of instruction. The four themes are:

- The use of varieties of teaching and learning strategies and local material teaching aids, and teachers' better understanding of mathematics concepts motivated these people when they were school students in learning mathematics;
- Inadequate school mathematics teachers and insufficient explanations from teachers in secondary schools discouraged these students in learning mathematics and were attributed as the cause of their low performance on examinations;
- Corporal punishments in mathematics classes discouraged these students in learning mathematics;
- Change of language from Kiswahili to English as the medium of instruction became a barrier in these students' understanding of mathematics in secondary schools.

The use of a variety of teaching and learning strategies and local material teaching aids, and teachers' better understanding of mathematics concepts motivated the students in learning mathematics. Seven out of ten pre-service teachers explained that they experience learning mathematics by the teachers using strategies such as the use of songs and local material (sticks, bottle tops, and seeds) in lower grades especially grade I up to III, and the use of examples and questions from grade IV to VII. However, their experience in secondary schools for both Ordinary and Advanced levels were not the same as in primary school. The pre-service teachers explained that learning mathematics using the local materials as teaching aids helped them to understand and build more interest in learning mathematics. The use of songs enabled them to understand basic operations in numbers as well motivated them to like mathematics. One participant said, “mathematics to me was good because of the kind of teacher which I had...He gave us the concept of addition take us outside the class to do it practically like the use of ‘Mkanturuturu’ seeds [as counters] ...” P_7FFI and he drew the Mkanturuturu tree as his pre-interview activity (PIA) (figure 8). Other participants also pointed to positive experiences created by teachers’ choices of instructional activities:

> [W]e used bottle tops to count and made different basic operation of mathematics, addition or subtraction or multiplication. We also used sticks to count. P_3FFI.

> [W]ith our previous teacher (name removed) in early childhood we learned mathematics through singing. For example, there was a song called mathematics is good ‘hesabu ni nzuri sana in Kiswahili’ ...we used participatory learning, our Madam (The name hidden for anonymity) divided us into different groups. P_2FF1

> For example, in standard VI and Standard VII, I liked the activities given in those classes because the teacher tried to give us a lot of examples while teaching in the class... and a lot of questions or tasks to do as a homework. P_9FFI

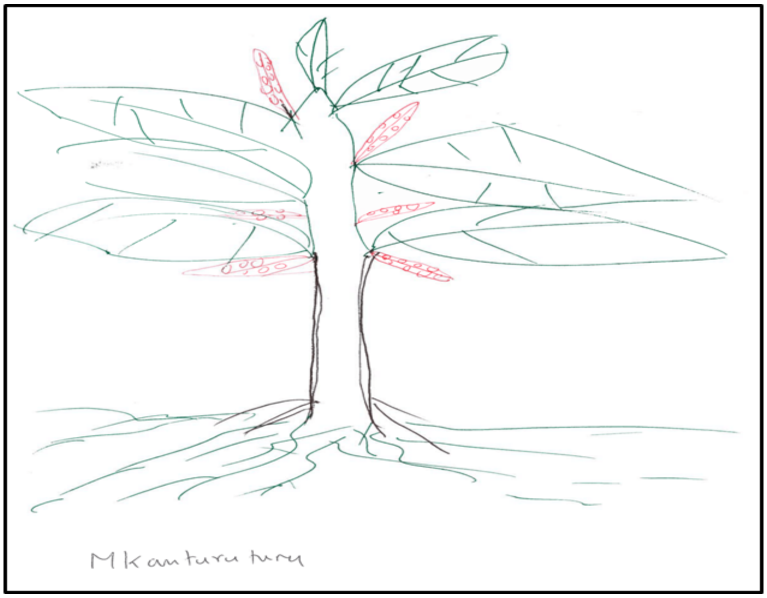


Figure 8: A Mkanturuturu tree that provides its seeds for counting as local materials teaching aids (P_7FFI)

Three out of ten pre-service teachers connected the better understanding of mathematics concepts with teachers who explained well the mathematics concepts and who used examples that demonstrated well the concepts, this motivated the students to learn mathematics. However, three participants described they had opposite experiences in grade I to III explaining that teachers were not able to explain better the mathematics concepts. The teachers who explained better their mathematics concepts (especially in primary schools) enabled the students to build the better foundation for learning more complex concepts in the secondary schools and choose mathematics as their career subject for further studies. At the same time, when teachers focused teaching in the better explanation of the mathematical concept to the students with proper selection, preparation, and better use of the teaching aids they motivate and built students interests in learning mathematics.

> From grade IV to grade VII, I liked mathematics because the teacher was teaching well. He explained the concept and demonstrate examples on the blackboard...the teacher tried to motivate us. From that time, I developed the habit of concentrating much in mathematics till standard VII. P_{10}FFI

> In primary school from standard I to IV, I felt that mathematics is very difficult subject because the teachers who taught me were not competent in mathematics. But, after grade IV onward I met with good teachers which I realised that mathematics is easy. P_6FFI

Inadequately qualified mathematics teachers and insufficient explanations from teachers in secondary schools discouraged the students in learning mathematics and (they believe) resulted in their low performance on examinations. Seven out of ten participants explained there were times when they did not have mathematics teachers (for some up to a whole year) in both ordinary and advanced levels secondary school. The participants explained that sometimes for the ordinary level secondary school they were taught by unqualified teachers such as form VI graduates which instead of teaching the concepts in most cases they were solving questions. This type of teaching encouraged rote learning, and because of that if the question was twisted a bit, the students found difficult for them to tackle the question which, again result in poor performance in mathematics, and discouraging them from learning mathematics and choosing it as their future careers. Also, in the advanced level, the lack of qualified mathematics teachers resulted in either learning mathematics by themselves through discussion or taught by the undergraduate students from university who were unqualified teachers.

> In ordinary level secondary school, the school I joined... I found that the school was having only two mathematics teacher teaching Form I to Form IV. However, Sir [name hidden for anonymity] in form II helped me and my colleague who were interested in mathematics to learn all Form II topics including form one topics which we didn't cover in Form I. In Form III we got a part time teacher just a Form VI lever he was trying his level best but, he was not good in teaching. In Form V, I went to [name hidden for anonymity] secondary school. In this school also, we were not having the advanced mathematics teachers but, since the students were interested in learning mathematics, we were doing discussions and also get help from the teacher who was teaching basic applied mathematics to PCB students. Also, we were

> taught tuition [private tutoring] by student from university of [name hidden for anonymity]. P_8FFI

> In high school ...I concentrated much but, based on self-motivation that I have to pass in order to join university. We were not having advanced mathematics teacher in high school [name of school hidden for anonymity], I remember in Form V the second master taught us only the topic of set in three months and until the end of Form V only that topic we were taught by government teacher, but, other topics we students were organising ourselves and hire a graduated Form VI leaver and come to teach us... In Form VI the school hired a graduated Form VI leaver but, he managed to teach us only one topic of vectors and he was selected to join university and he left, and we struggled again by ourselves. P_{10}FFI

Three out of ten pre-service teacher participants experienced insufficient explanations about particular mathematics concepts in classroom instruction in their ordinary level secondary schooling. They explained that this problem contributed to poor performance in mathematics on their CSEE national results. They described that a teacher needs to provide sufficient explanations to the students' queries about mathematics learning. Failing to respond well to these queries could discourage student in learning mathematics and also select it as their carrier subject for further studies. It might also cause the student to lose faith to the teacher hence lose interest in learning mathematics.

> I can say up to now I don't have a complete reason why 2 is a prime number and at the same time 2 is an even number. ... Unfortunately, no one has given me the reason. I recall when I was in Form I, the teacher explained that 2 is a prime number because it divides itself and not otherwise. But, someone [student] tried to ask him that if we take 2 divides by 1 can't we get an answer? But, he [the teacher] failed to give us[student] a complete reason to why he said two divide itself and not otherwise. P_1FFI

> Mathematics in my ordinary level secondary school, Form I up to Form IV it was not good compared to primary level...coming to form II my graph started to drop down, this is due to the teacher we were having in form II, he was not able to participate well in the classroom...Also this was the same in form III. Generally, the reason which made me not to do well in Ordinary level secondary school was having insufficient or lack of teachers who are well trained in mathematics. For example, our form II mathematics teacher he was not good in solving questions and also the method he used to teach in the classroom was not good. P_4FFI

Corporal punishments in mathematics classes discourages student learning. Five out of ten participants spoke of their experience of corporal punishment in mathematics classes in primary schools. One participant experienced the same situation in ordinary level secondary school, especially in Form II mathematics lessons. The participants spoke of teachers that punished students for not completing or failing to do mathematics task without listening to their reasons. The participants said they felt very bad and sometimes hated the teacher and the subject as well. And others consider mathematics as a difficult subject. One participant described to receiving help at home for whatever he did not understand in the class from his mother, as she was also the primary school teacher. The use of corporal punishment with the student affects the student in many ways such as psychologically and physically that will result in affecting his/her mathematics learning and other subjects as well (Ali, Mirza, & Rauf, 2015). Ali et al. (2015) argue that with corporal punishment "students' learning is influenced and retarded by fear. Physically and emotionally abused children develop anxiety that causes loss of concentration and poor learning. Such students do not take risks even being creative" (Introduction section, para. 3). Student might lose interest in learning the subject and even coming to school in general (Morrel, 2000, 2001). Corporal punishment was illustrated in more than one participant's illustrations as the pre-interview activities (PIA) which they used to explain the way teachers administered the punishments figure 9 as their experiences of not so good day for learning mathematics. In figure 10 the participant explained that the male teacher administered punishments to the student for getting the wrong answer while the female teacher explained to the student how he could do it after getting it wrong. Hereafter are few responses from the participants:

For me it [mathematics] was very bad because ... the teachers were not encouraging the student to learn mathematics. Most of the teachers from standard I up to standard V when student fail to complete the task, they rush in punishing the student instead of sitting down and ask student what happened that made him/her not to complete the task. P_5FFI

In standard IV, I was having a teacher called [the name removed for anonymity] he used to give us corporal punishment when you fail to answer a question or say a multiplication table may be table 2 this made the class not to be good in mathematics, but to my side that was not a problem because my mother was also a primary school mathematics teacher from standard 1 up to VII. So, when I come back home, I get more explanation and teaching from my mother. But, in the class most student didn't like mathematics...those were real bad time for me in learning mathematics. P_8FFI

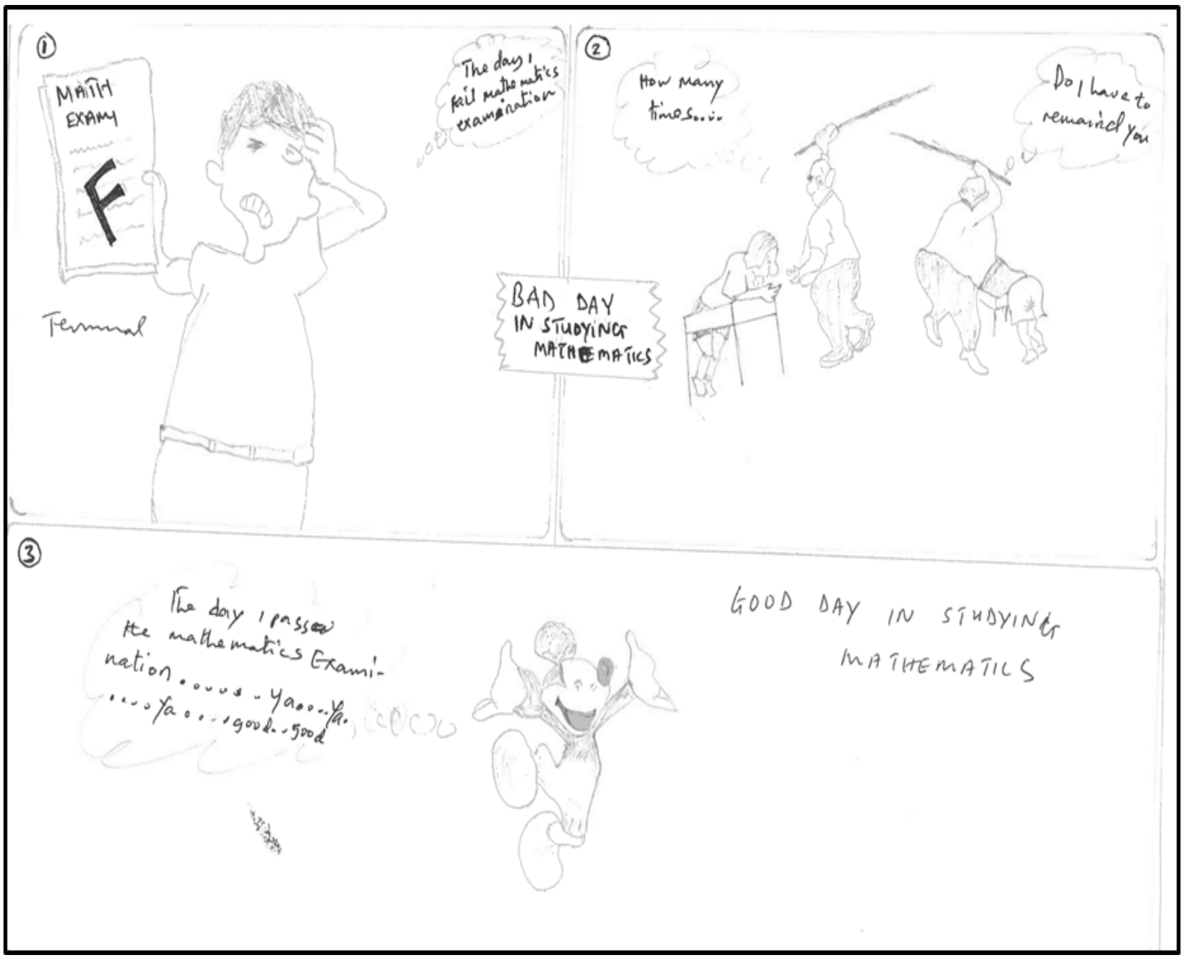


Figure 9: Participant (P_5FFI) illustration drawn as PIA to show good and not so good day for learning mathematics

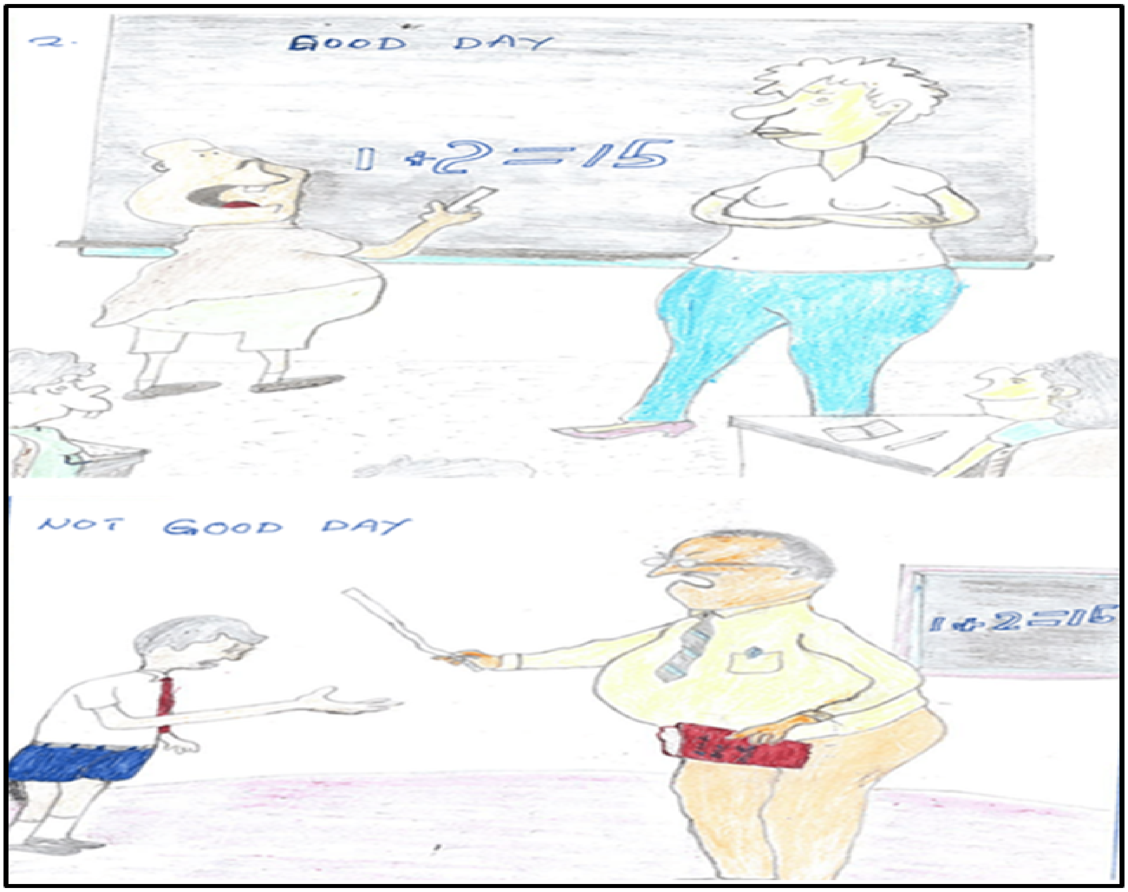


Figure 10: Participant (P_4FFI) illustration drawn as PIA to show good and not so good day for learning mathematics

Change of language from Kiswahili to English as the medium of instruction became a barrier in students' understanding mathematics in secondary schools. Though this was raised by only one participant, it might be important to talk about it. The participant explained that in secondary school he faced challenges for the change of the language used as a medium of instructions in primary school which was Kiswahili to English in secondary schools. He explained that he faced some difficulty in understanding mathematics teacher explanations and even to interpret the meaning of the given questions which as a result he was not able to solve it. However, in Form I, he explained he did not face many challenges because most of the topics were related to that of primary schools. He elaborated that mathematics teachers did not bother to take time to help them in some vocabularies or interpret some explanation with the reason that it is not his/her duty to do so.

> It was a little bit hectic process to me because I met with some strange kind of mathematics which I have never seen in primary level but, I tried my best level to cope with the situation. For example, ...the teacher considered that the learners know a little bit of English, so interpretation was not his task. So, sometimes I faced difficulties to interpret what the question needs or the teachers' explanation because of the words used and some vocabulary as it was our first time to learn mathematics in English. The fact that in primary we were doing mathematics in Kiswahili, we didn't understand some of his explanation in English because of the new words and the teacher didn't bother to interpret to us because he considered it as not his duty. However, in form I, I didn't get much difficult like other forms because many topics were related to that in primary school though in English. P_9FFI

To summarize, the chapter described the qualitative group case study as the method used in this research, its strength and weaknesses, and the criteria for judging the quality of the qualitative case study. I described the reasons why I chose the constructivism as a paradigm for this research based on my belief that the sharing of multiple ideas of the mathematical concept from pre-service teachers contribute to their understanding of the mathematical concept in the collective learning of the concept studies. I described the reasons for using the complexity science as the classroom design with descriptions of how the complexity conditions (Davis & Simmt, 2003) was used in this research, the group of pre-service teachers working in concept studies is considered a collective leaner of the complex learning system. Also, I described my role as a researcher, the research site, and the participants involved in this study. I was a facilitator in the concept studies, a participant observer, an emphatic second-observer (Metz & Simmt, 2015). The research site was one of the teacher colleges in northern zone in Tanzania with second year diploma in secondary education mathematics majors' pre-service teachers. I further described the data collection and data analysis processes in this research. That means the type of data collected, when collected, how collected and how analysed. I employed multiple data collection techniques such as face to face pre-study interviews, concept study workshops, and pre and post questionnaires. In the concept studies workshops, I collected video recordings, audio recordings, working papers, and field notes. Lastly, I described the collective prior pre-service teacher mathematics learning experiences to give the reader the backdrop understanding of the case. I described the pre-service teachers experience in learning mathematics in various aspects such as the use of teaching and learning aids, teaching and learning strategies, teachers' explanation, motivation from teachers, and the language of instruction. Seventy percent of the pre-service teachers' experienced learning mathematics with the use of local teaching aids such as bottle tops, sticks, and seeds and varieties of teaching strategies such as songs especially in grade I to III. However, this was not the case in secondary schools for both ordinary and advanced levels. Some of the pre-service teachers experienced insufficient explanations from the teachers especially, in ordinary level secondary school. Half of the pre-service teachers experienced corporal punishments in primary schools that discouraged learning mathematics and one of them experienced the same in secondary school. The change of language of instruction from primary to ordinary level secondary school especially, in Form I indicated by one participant as a barrier in learning mathematics.

6 How the Use of the Five Emphases of Concept Study help the Pre-Service Teachers to Access and Develop Their Tacit MFT

This chapter describes the findings of how the five emphases of concept study described by Davis and Renert (2014) helped the pre-service teachers to access and develop their tacit MFT of ratio and related concepts of the proportion, rate, and linear functions. MFT emerged through the pre-service teachers' engagements with the collective learning designed activities. The four mathematics concepts were addressed as separate concept studies because in Tanzanian curriculum context they are treated as different topics.

This chapter is divided into five sections. The first section presents the realizations that were expressed in the various concept study sessions. It describes the realizations from each of the concepts: ratio, proportion, rate, and the linear function. As well, examples are offered that illustrate how the pre-service teachers collaboratively working on the concept study coalesced and developed what appears to be collective understanding. The second through fifth sections describe the landscapes, entailments, blends and the pedagogical problem solving that emerged in the collective understanding. The realizations emphasis section is more deeply explored because it is the basis for the other emphases.

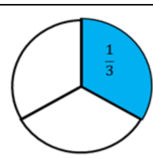
6.1 Realizations

In concept study the realizations of a mathematical concept include associations such as formal definitions, metaphors, images, algorithms, gestures, and applications pre-service teachers might draw on and connect in an effort to understand it (Davis & Renert, 2014). As Davis and Renert add, concept study is not about the rightness or adequacy of the realization but the understanding of the mathematical concept that emerges from the collective learning environment. For that reason, any contribution from pre-service teacher participants is worth sharing collectively because those contributions might activate something that other participants know but have forgotten that they know, or something that they could know.

The concept study session for each concept (ratio, rate, proportion, and linear functions) began with the question *"Could you tell me what you know about the ratio/rate/proportion/linear function concept in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."* The participants were asked to think about the prompt individually, then share in their table group, and finally to share in a whole group discussion. Not surprising, participants come up with different definitions, examples, images, algorithms, and applications of the concepts in everyday life. Some examples of the realizations for each of the mathematical concepts the ratio, proportion, rate, and linear function and some collective moments observed in the particular concept study session are presented hereafter consecutively.

6.1.1 Realizations for ratio concept

The participants first provided definitions with corresponding examples and added illustrations and applications. These are represented in table form with two columns. The first column represents the participants' definitions and the second column represents the examples and illustrations provided along with the definition (Table 7). Some of the applications of the ratio concept that pre-service teachers provided are listed immediately after Table 7.

<table>
<tr><th>Participants' definitions</th><th>Participants' examples and illustrations</th></tr>
<tr><td rowspan="7">Ratio is the comparison of two or more things</td><td>1. In the class there are 20 students. 5 study Mathematics, 6 study Chemistry, 4 study Physics and 5 study Biology. Hence their ratio in the same order will be 5:6:4:5</td></tr>
<tr><td>2. You can get ratio by dividing your monthly salary let say 230,000Tshs into food, clothes, and transport. You can decide to spend 100,000 for food, 80,000 for clothes and 50,000 for transport. In ratio 100,000:80,000:50,000 as the ratio of food to clothes to transport. Also, we can express the ratio of amount of money spent on clothes to the total salary which is 80,000:230,000 into fraction as 80,000 over 230,000 or $\frac{80,000}{230,000}$. Also, we can also express this into percentage when we take 80,000 over 230,000 times 100%, we can express it into percentage.</td></tr>
<tr><td>3. The school have 1000 students which 600 students are girls and 400 students are boys. Hence the ratio of girls to boys is 600:400 =3:2</td></tr>
<tr><td>4. 75% of pupils in the class are girls, so in ratio we write $\frac{75}{100} = \frac{3}{4} = 3 : 4$ as ratio of girls to all students in the class.</td></tr>
<tr><td>5. In the basket there are 5 fruits, 2 are apples and 3 are oranges. So, ratio of apples to oranges is 2:3</td></tr>
<tr><td>6. If someone have got his salary he can divide his salary according to his demand in life. For example, a person X spend his salary as follows: 25% in his luxury, 50% in meals and accommodation, 20% in transport and the rest deposit as saving in bank. If his salary is 1,000,000/= Tanzanian shillings. Find the amount and ratio of all his needs and expenditure.
$\frac{25}{100} \times 1,000,000 = 250,000/=$
$\frac{50}{100} \times 1,000,000 = 500,000/=$
$\frac{20}{100} \times 1,000,000 = 200,000/=$
$\frac{5}{100} \times 1,000,000 = 50,000/=$
In ratio form $2.5 : 5 : 2 : 0.5$</td></tr>
<tr><td>7.  The shaded part is 1 out of 3. Then, the ratio the ratio</td></tr>
</table>

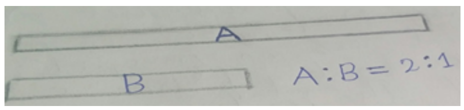
	of shaded to whole circle is written as $\frac{1}{3}$ or 1:3 or ratio of shaded to unshaded part as 1: 2
Ratio is the comparison between two things with the same or identical unit.	8. For example, the ratio of 3kg of potatoes to 4kg of oranges. Therefore, the ratio is 3kg of potatoes: 4kg of oranges or $\frac{\text{3kg of potatoes}}{\text{4kg of oranges}}$
Ratio is the mathematics which divides things in relationship.	9. The length of ruler A is twice that of ruler B. The ratio of the length of ruler A to that of ruler B is 2:1 
Ratio is a tool which is used to compare two things which share one resource.	10. An orange as a resource and two people are going to share this resource. If you have an orange and you want to share it equally. Maybe you are two people. That orange can be divided into two equal parts and the ratio will be 1:1, one person can take 1 part and another one 1 part. You can divide it to more than two parts let say four equal parts and the ratio will be 2:2
Ratio is the comparison of two quantities.	11. The distance travelled compared to the time taken.
Ratio can originate from a mathematical way or method of writing numbers in numerator and denominator.	12. $\frac{a}{b} = a:b$ and $\frac{3}{5} = 3:5$
Note: '/=' is used to represent Tanzanian shillings with no cents.	

Table 7: Pre-service teacher participants' initial lists of realizations of ratio concept

Some of the participants offered applications of ratio. These included:

In industries. For example, chemical industries use ratio in diluting concentrated acids.

Building constructions, for example mixing sand with cement

> In cooking, one need to know the proper amount of each type of ingredients needed for the food.
>
> In industry they use ratio concept. Yeah, in mixing products like making soda how much sugar, water, and flavour do you need.
>
> In making bricks you need to match the amount of sand, cement, and water in order to have bricks with equal ratios
>
> Ratio is used in mixing of two different things. For example: In making bricks, 1 bag of cement [50kg] required 8 buckets [@10 litres] of sand. In ratio form 1:8

The participants' realizations of ratio (table 7), along with their definitions and examples align with Lamon's (2012) definition as a ratio: "the comparison between any two quantities" (p. 225). Ratio might be used to convey ideas that cannot be expressed as a single number and ratio compares the measures of different types and sometimes of the same type (Lamon). The eleventh example in Table 7 is among examples of ratio that compare measures of different types--the length (distance) and time. Lamon explains further that:

> There are two types of ratio that compare measures of the same type: the part-whole comparison and part-part comparison. Part-whole comparisons are ratios that compare the measure of part a set to the measure of the whole set. Part-part comparison compare the measure of part of a set to the measure of another part of the set. (p. 125)

Participant's second and seventh examples in table 7 illustrate examples of ratio as part-whole comparison. While the third example in table 7 is among those that align with the part-part comparison considering the class as a set with two groups, one with 400 boys and another with 600 girls. So, the ratio of girls to boys which is 600:400 is a part -part comparison that compares the number of girls with that of boys in the class. However, taking the ratio of girls or boys to the total number of students in the class will be part-whole comparison. The applications of the ratio concept participants provided indicate their awareness of how the ratio is applied in the environment outside school that means in daily activities. Lobato and Ellis (2010) defined ratio as “a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit” (p. 18). The ninth and second examples in table 7 are among those that align with what Lobato and Ellis refer as the multiplicative comparison of two quantities. The length of ruler A is twice that of ruler B means how many times longer is ruler A than ruler B or the length of ruler B is what part of the length of ruler A. Lobato and Ellis explain that multiplicative comparison is a ratio, but an additive comparison is not. The participant’s tenth example in table 7 is among those that align with what Lobato and Ellis refer as joining of two quantities in a composed unit. The orange is shared by two people equally so, dividing the orange into equal even number will always result into multiples of the first ratio. For example, dividing the orange into two equal parts, the ratio the two people will share is 1:1. So, dividing into four and six equal parts the ratio the two people will share is 2:2 and 3:3 respectively and so on. That means 2:2 is equal to 2(1:1) and 3:3 is equal to 3(1:1) and so on.

I facilitated the group discussion with the intention of focusing on the participants’ awareness of the variety of realizations of the mathematical concept that they provided. The use of “how”, “why”, and “could you elaborate more,” helped the participants access their MFT of the ratio concept. The example below shows an exchange between the researcher and the pre-service teachers that facilitated their awareness of one of the realizations of ratio, the part-to-part and part to whole comparisons.

> Facilitator: Consider one of your examples “in the basket there are 5 fruits, 2 are apples and 3 are oranges. So ratio of apples to oranges is 2:3” If we look closely, what type of relationship or comparison do you see here?

P1 (from group 3): I think it is a comparison by different things.

P2 (first member group 2): Yes, part-part relationship.

Facilitator: Could you elaborate more what you mean by part-to-part relationship?

P3 (second member from group 2): I can help, five fruits is a set and 2 apples, 3 oranges are subsets.

P4 (from group 1): So, you mean the parts are apples and oranges

P2 (first member group 2): Absolutely.

P5 (second member group 3): So, it is part-part relationship ratios.

P3 (second member from group 2): Yes

Facilitator: What if you take the ratio of apples to fruits, which is 2:5, what comparison could it be?

P6 (a second member from group 1): Of course, it will be part to whole relationship.

Facilitator: Why?

P6 (a second member from 1): You compare one subset to the whole set

The pre-service teacher participants conjectured aloud that, if one does not consider teaching ratio with the use of examples that show the way mathematical concepts behave differently in different contexts there are possibilities of the pedagogical consequences such as misconceptions. For example, the participants' use of the explanations such as *"When considering the ratio, we must have the denominator and numerator"* could be problematic to the learners in differentiating ratios and quotients with the fact that ratio is the comparison of two or more things. When you talk about numerator and denominator you are dealing with both terms as numbers, but, that is not the case to ratios. At the same time when you talk about the numerator and denominator you are dealing with fraction as symbol $\frac{a}{b}$ where 'a' is a numerator and 'b' is a denominator (Lamon, 1999). At the same time, the ratio can be considered a quotient when comparing two things. Lobato and Ellis (2010) note that “ratio can be meaningfully reinterpreted as quotients” (p. 31). For example, suppose 4 biscuits are shared between 2 people, the ratio of the number of biscuits to the number of people is 4:2. So, 8 biscuits would be shared between 4 people, 16 biscuits between 8 people and so on. According to Lobato and Ellis, the ratio 4:2 can be reinterpreted as the quotient 4÷2 as “one meaning of division is sharing” (p. 31). Interestingly, the realizations of the ratio ‘as a part-part and a part-whole comparisons, and as a fraction emerged naturally several times in the discussion compared to other realizations. The participants identified ratio as ‘a comparison of two quantities’ as the most common realization of ratio that cut across the Tanzanian’ ordinary level secondary school curriculum—the syllabus. Figure 11 displays the summary of what I interpreted as realizations of ratio from the participants’ first list of realizations and follow up questions that emerged from participants after a considerable amount of discussion. As evidenced in the engagements of the collective, the consideration for the fraction as a symbol and as a part-whole have shown implications with the collective for the participant’s realizations for the ratio concept. Lamon, (2012) insisted that if fraction notation is chosen to be used in ratios care must be taken to avoid confusion between ratio and fractions. For example, the ratio of 3 girls to 5 boys should not be written as $\frac{3}{5}$ but, rather as $\frac{3\text{ girls}}{5\text{boys}}$.

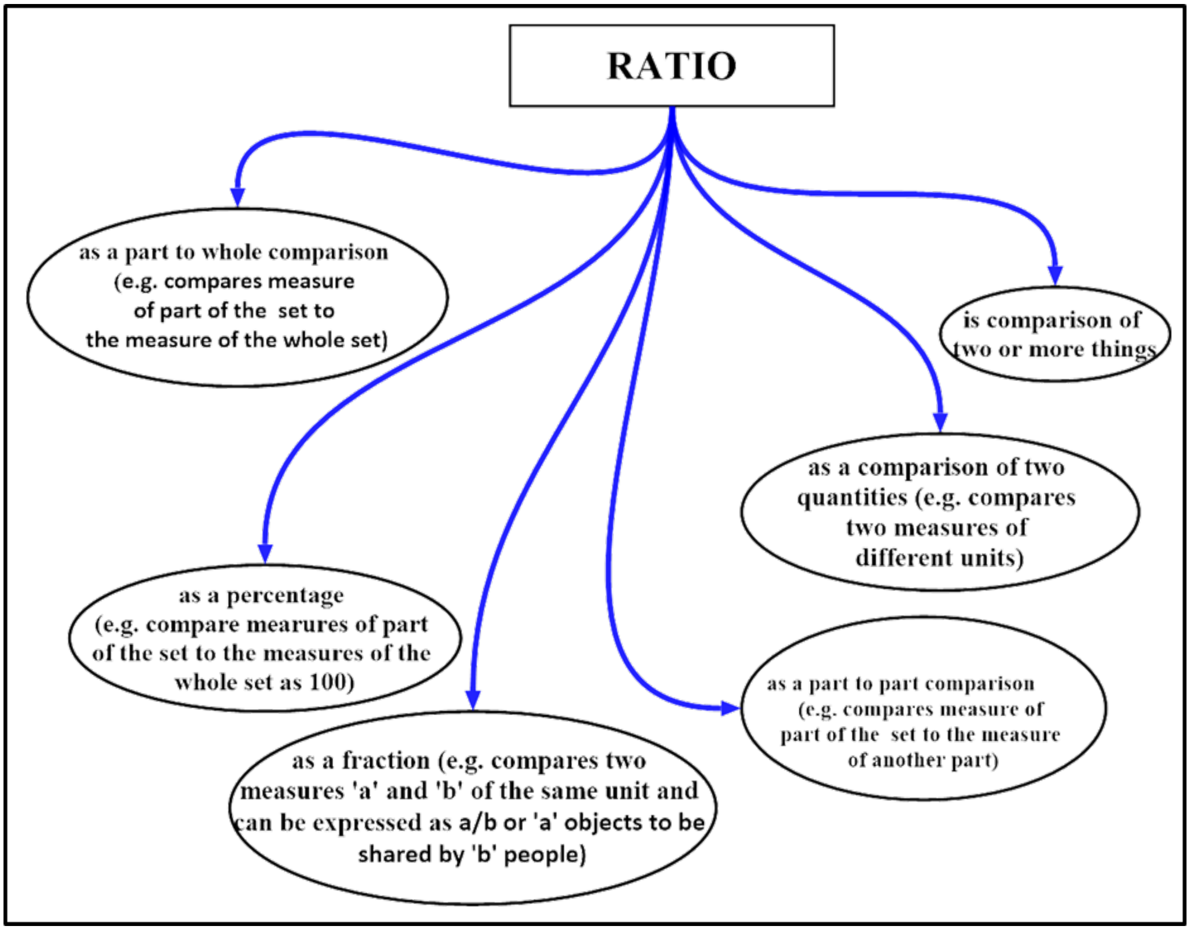


Figure 11: Summary of the realizations of ratio concept

6.1.2 Realizations for the proportion concept

The pre-service teachers provided a number of realizations that evidence their understanding of the meaning of proportion and how it could be used in other mathematical concepts and other subjects but did not provide many examples with its practical applications in everyday life. They provided more examples of the symbolic and iconic representations of proportions. The immediate responses from almost all participants is that proportion can be understood as "two equal ratios." The consideration for the proportion as two equal ratios, and the use of the symbol

$\frac{a}{b} = \frac{c}{d}$ were proposed. Table 8 displays participants' first list of realizations of the proportion concept.

Participants' definitions	**Participants' examples and illustrations provided with that definition**
Proportion is the mathematical statement that refer to two equal ratios.	It can be represented in two ways: $$\frac{a}{b} = \frac{c}{d}$$ or $$a{:}b = c{:}d$$
Proportion is the comparative relation between two ratios or is the equality between two ratios.	If you have a, b, c and d and you need to express it in proportion it will be $$a{:}b = c{:}d$$ or $$\frac{a}{b} = \frac{c}{d}$$ or $$a{:}b :: c{:}d$$
Proportion refers to the equivalent of two given ratios. It is an expression of ratio on either side.	If x varies directly as y then $$x \propto y \rightarrow x = ky$$ $$\frac{x}{y} = k \rightarrow x{:}y = k$$ Increase in kg of wheat flour for making Chapati, also increase amount of required salt.
Proportion is the way in which one quantity increases with increase in another quantity or decreases with decrease in another.	Scientists use proportion to describe the variation of different physical quantities that is in physics. The heat quantity of the body is proportional to the change in temperature of the body. Also, expansion of material is proportional to the change in temperature $$a \propto b \rightarrow a = kb$$ $$\frac{a}{b} = k \therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} \text{ or } \ldots \frac{a_{n-1}}{b_{n-1}} = \frac{a_n}{b_n}$$

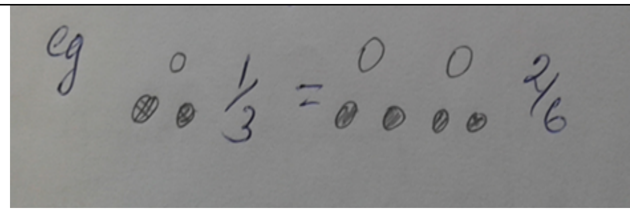
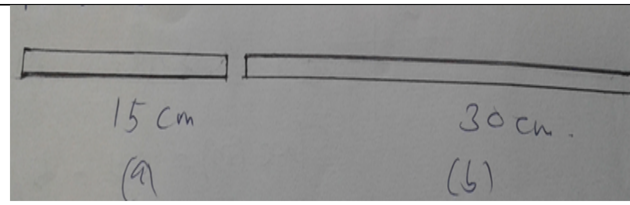
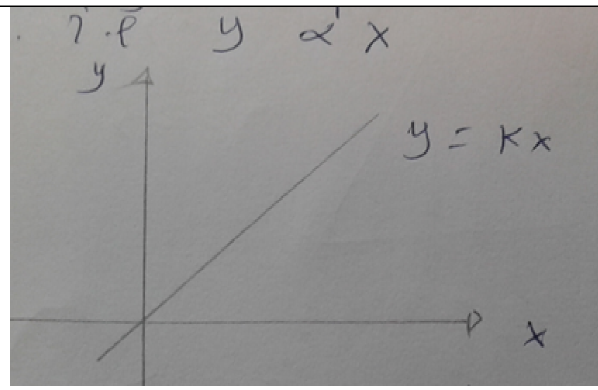
Proportion can be defined as relationship that exists between size, numbers or amount of two things.	
Proportion can be defined as when two quantities have the same shape but different in size in relation to each other. In other words have the same ratio. Example, length of two rulers, the length of the first is 15 cm and that of the second is 30cm	
For instance, you have the quantities a, b, c and d. Taking the ratio $\frac{a}{b} = \frac{c}{d}$ then a, d are called extremes and b, c are called means, So, in order to become proportion the product of means must be equal to product of extreme i.e. $\mathrm{ad} = \mathrm{bc}$	 Increase in y is direct proportional to x. Example of direct proportional, the number of students is direct proportional to chairs in the class. Means that when student increases also chair increases.
Proportion is the mathematical way of showing that two fractions are equal to each other.	That means $\frac{a}{b} = \frac{c}{d}$

Table 8: The pre-service teacher participants' initial lists of realizations of proportion concept

Some of the participants offered applications of proportion. These included:

Proportion is used in physics to determine the resistance of a materials (i.e. to verify Ohm's law) →The current passing though the conductor is proportional to the potential difference between the ends of the conductor $I \propto V \rightarrow I = kV \;\therefore\; V = \frac{I}{k}$ or $V = \frac{1}{k}I$ but $\frac{1}{k} = R$ (Resistance of the conductor) so, $V = IR$

In Newton's law of cooling, verification of Ohm's law, linear graph and derivation of formulae.

Proportion can be used to formulate linear equation, i.e. gradient is equal to change in y over change in x which is equal to tangent of theta, the angle the line incline to the right (figure 12).

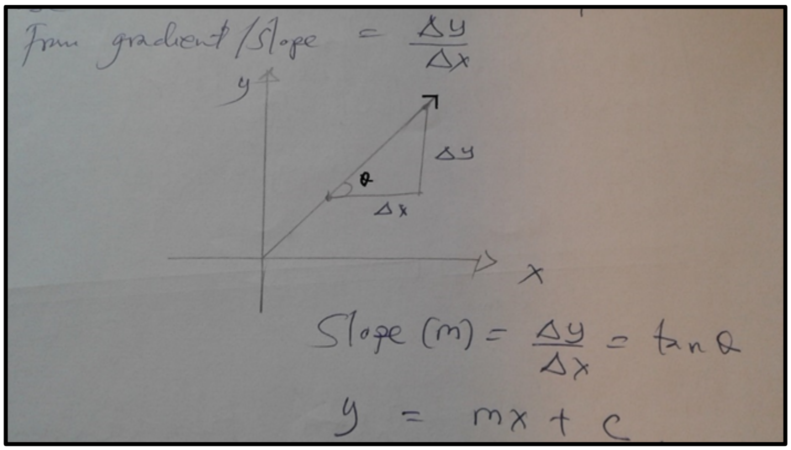


Figure 12: Participants' illustration of the use of proportion to formulate linear equation

The participants elaborated each of their realizations with the use of the proportional reasoning for particular examples. For example, the decrease in the number of days used to cultivate the same piece of land if the number of people increases, the increase in one of the ingredients for cooking certain dish causes an increase of the other ingredients. Also, the increase in the number of students in the class requires an increase in the number of desks to maintain the student: chair ratio. Also, consider one of the observed collective scenarios (as described below in figure 13) that illustrates the participants' understanding of how to connect the symbolic and iconic representation of proportion with the use of idea of enlargement that results into similar figures:

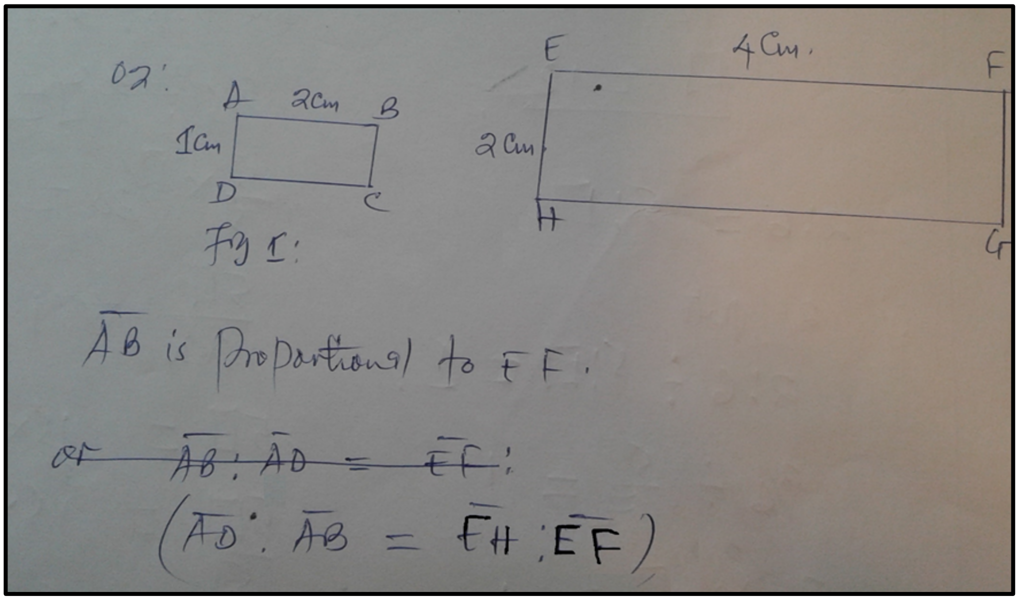


Figure 13: Participants' examples of similar figure that illustrates proportional sides

P1 (A member from group 3): Side $\overline{AB}$ is proportional to side$\overline{EF}$.

P2 (Second member from group 3): Why do you say so?

P3 (Third member from group 3): You can see $\overline{EF}$ is twice $\overline{AB}$

P1 (A member from group 3): They are both the lengths of the two triangles but $\overline{EF}$ is twice $\overline{AB}$

P2 (Second member from group 3): So, $\overline{EF}$ and $\overline{AB}$ are corresponding sides.

P1 (A member from group 3): Side $\overline{AB}$ is proportional to side$\overline{EF}$.

P3 (Third member from group 3): Yes, also $\overline{EH}$ is twice $\overline{AD}$ and they are proportional.

P1 (A member from group 3): The larger figure is an enlargement of the smaller figure.

Figure 14 displays the summary of what I interpreted as realizations of proportion from the participants' first list of realizations and follow up questions after a considerable amount of time in the collective discussion of the proportion concept.

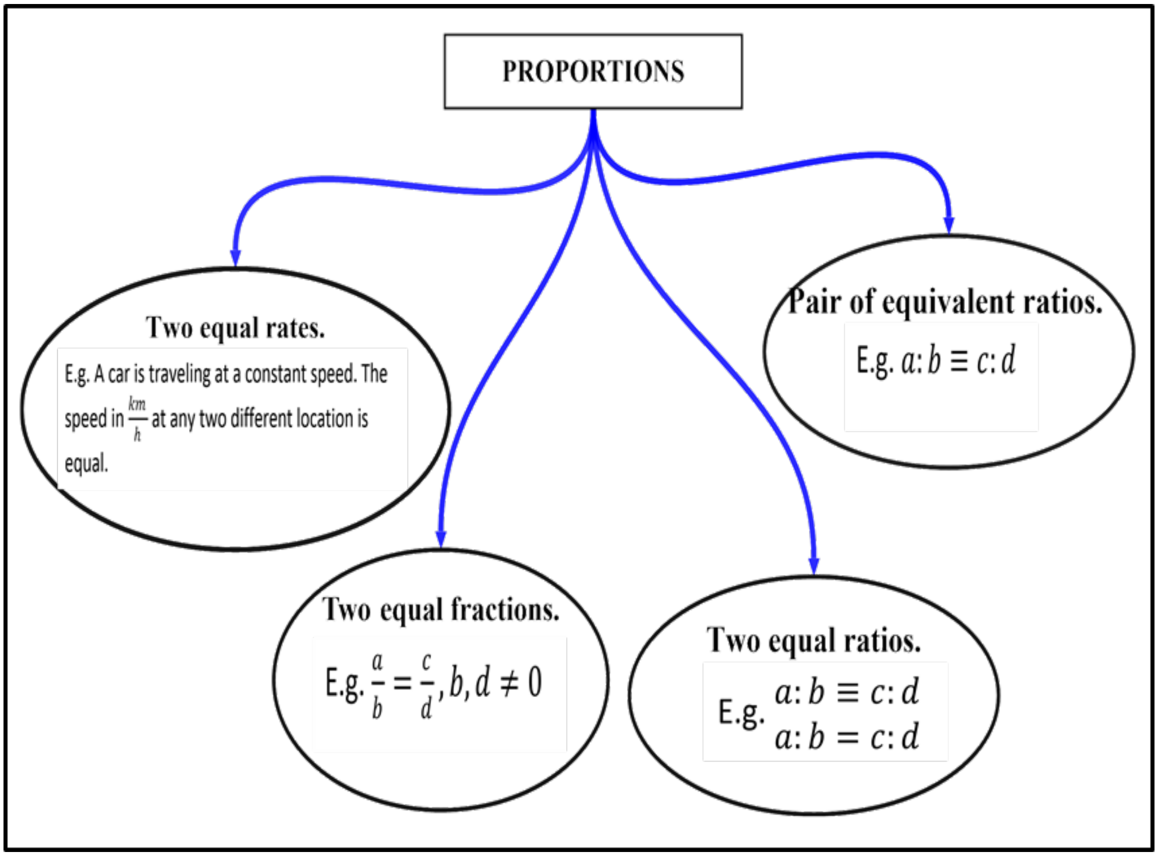


Figure 14: Summary of the realizations of proportion concept

The participants' first list of realizations of proportion (table 8), align to what Lobato and Ellis (2010) defined as a proportion, "a relationship of equality between two ratios" (p. 33). Lobato and Ellis explain that in proportion, even if the values of the quantities change the corresponding ratio of the two quantities remains constant: "if one quantity is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportionality relationship" (p. 36). That means to maintain the proportionality relationship any change in one quantity must be accompanied by the same changes to the other quantity.

6.1.3 Realizations for the rate concept

The participants provided definitions and a variety of examples that express rate as the change in quantity with time, and that express unit rate such as payee rate, cost rate, exchange rate, etc. Bearing in mind that they are science group taking Physics or Chemistry or Geography, a lot of examples that express rate as the change in quantity with time was offered. Table 9 illustrates first list of the realizations of the rate concept, as reported by the participants during concept study of rate. The first column of table 9 represents the participants definitions of rate and the second column represent the examples and illustrations provided along with that definition.

Participants' definitions	**Participants' examples and illustrations provided with that definition**
Rate is the ratio between two quantities which have different units.	A person can run 60 miles for 2 hours. The rate is 30 miles per hour, which is called his speed. If 1 dozen of eggs cost 2400Tsh so, what will be the cost of one egg? To get the answer is what we call it a rate, the cost per egg.
In many cases the concept of rate in physics is used to describe the amount of quantity with respect to time.	Power is the rate doing work. That is Power = $\frac{\text{workdone}}{\text{time}}$, this is the application of rate as applied in physics
A rate is found by dividing one quantity by another i.e. the rate is the change which can be specified per unit time.	The rate of pay consists of money paid divided by the time worked. If a man receives 1000Tshillings for two hours work, his rate of pay is 1000÷2=500Shillings per hour.
Rate in mathematics can be defined as special ratio in which different units of two quantities are considered.	Consider if you walk 70 yards in 10 seconds, 7 yards in 1 second both of them are rate.
Most common type of rate is per unit time such as speed, heat rate and flux.	The rate that have non-time denominator are like exchange rates, cost rates, literacy rate and electric field (volts/meter).

Rate is the ratio describing the relationship existing between two related currencies.	The rate of exchange of 1US$ to Tsh. 1US$=2013.85Tsh. Meaning for each 1US$ is the same as 2013.85 Tanzanian shillings.
Rate is the value describing on how one quantity is related to another quantity of different units.	A man works 20 hours and paid Tsh 200,000, then the rate of payment will be $\frac{200{,}000\text{Tsh}}{20 \text{ hours}} = 1000\text{Tsh}/_{\text{hour}}$

Table 9: The pre-service teacher participants' first lists of realizations of rate concept

Some of the participants offered applications of rate. These included:

It is applied in the Newton's law of cooling and Newton's law of motion.

Rate is used to compare a certain quantity with time.

Also it can be used to compare goods with (money) or currencies.

It is used to choose better price of a certain goods

We use rate to calculate how much money you earn in a week

It is applied in physics to determine the rate of cooling. Example rate of cooling is change in temperature per change in time.

Figure 15 below display a summary of what I interpreted as the realizations of the rate from the participants' first list of realizations and follow up questions after a considerable amount of time in the discussion of rate concept. The meaning of the rate as the ratio of quantities with different units designated with the collective for the participant's realizations for the rate concept. However, during the collective discussion, one of the participants uttered "it is very difficult for the student to understand the rate concept unless the teacher uses different examples". The fact that the rate varies depending on the context is important. Rate as "a set of infinitely many equivalent ratios" (Thompson, cited in Lobato & Ellis, 2010, p. 42). Lobato and Ellis assert that despite Thompson's meaning of rate, two other meanings of rate commonly used are: rate is often defined as “a comparison of two quantities of different units” (p. 42) or rate as “a ratio in which one of the quantities is time” (p. 42). Lamon, (2012) defined a rate as “an extended ratio, a ratio that applies not just to the situation at hand but to a while range of situations in which two quantities are related in the same way” (p. 235). For example, 8 people eat 2kg of rice, 16 people eat 4kg of rice, and 32 people eat 8 kg of rice and so on. Lamon asserts that also rate can be taken “as descriptions of the way quantities changes with time” (p. 236) and these rates are identified by using the word ‘per’ in their names and can be reduced to represent a relationship between one quantity and 1unit of the other quantity. For example, 120 kilometres per 2 hours can be expressed as 60 kilometres per 1 hour what she referred as a unit rate. Lamon explains further that rate can be constant or varying. Thus, looking at participants first list of realizations of rate (table 9), their definitions and examples align with both Lamon (2012) and Lobato and Ellis (2010) definitions and descriptions of rate.

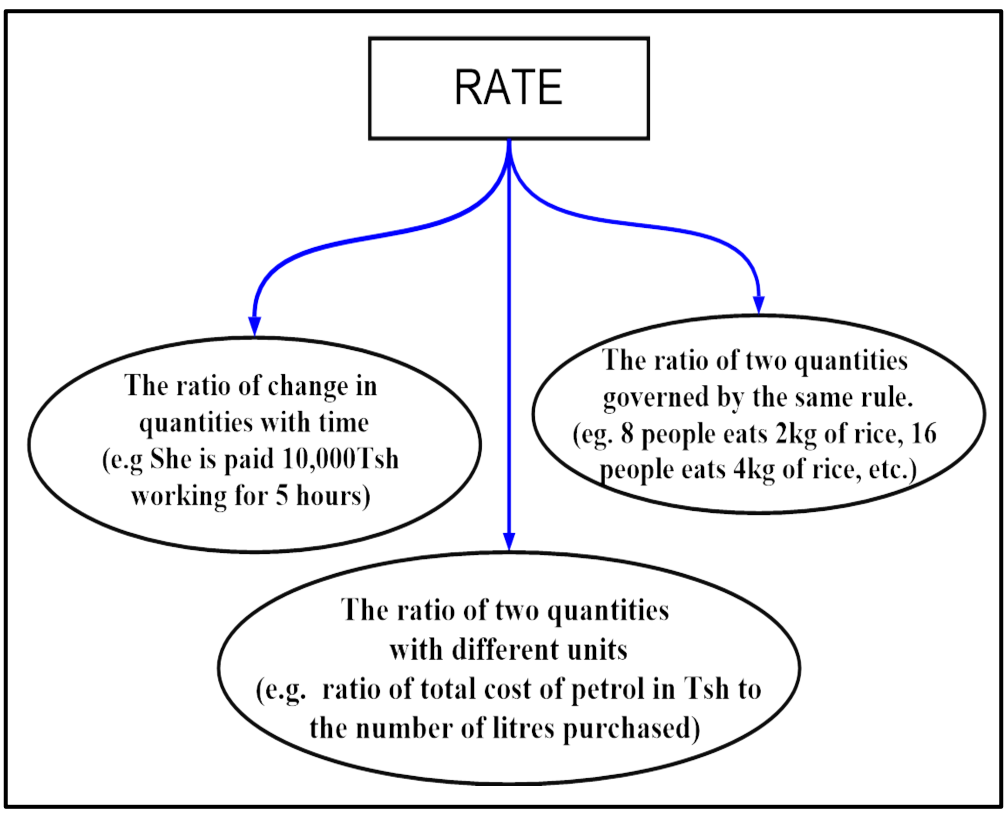


Figure 15: Summary of the realizations of rate concept

6.1.4 Realizations for linear function concept

The participants gave a variety of definitions with more illustrations and some examples and applications to elaborate the concept of linear function. Table 10 shows the first list of realizations of linear function.

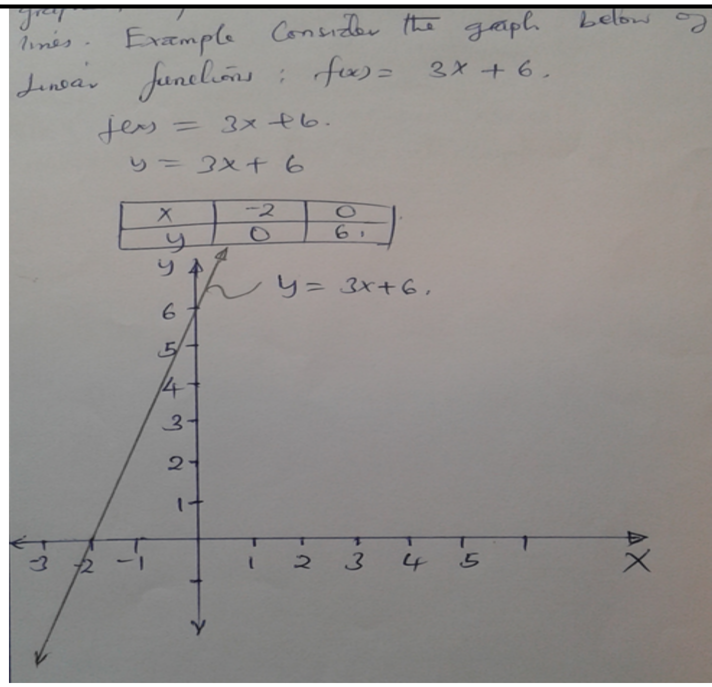
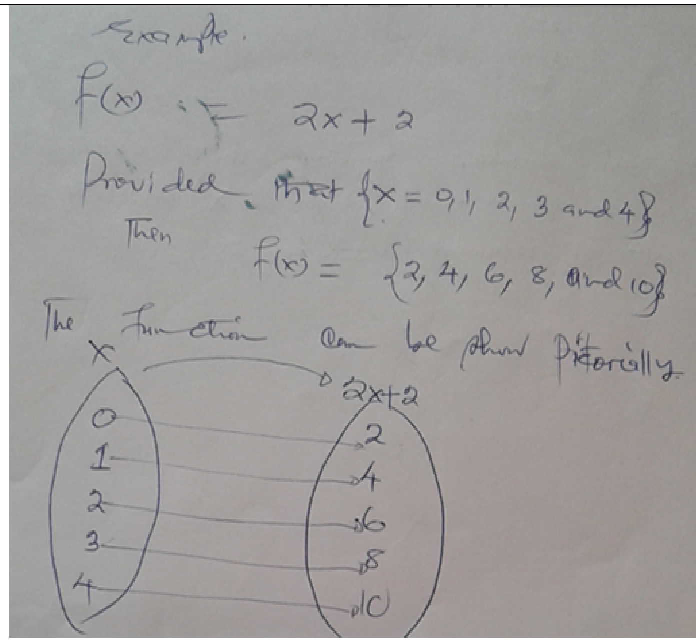
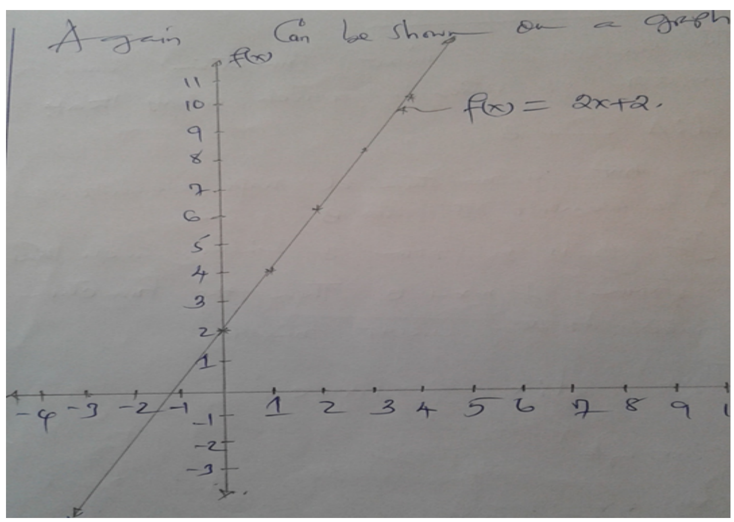
<table>
<tr><th>Participants’ definitions</th><th>Participants’ examples and illustrations provided with that definition</th></tr>
<tr><td>Linear function refers to the mathematical function with only first degree to its variables. The variables can be x and y. It is called linear function simply because the graphical representation of these functions are straight lines.</td><td>

Figure 16: Participants example of linear function graph</td></tr>
<tr><td>Linear function can be defined as function which consist of two variables in the form $f(x) = ax + b$, where ‘a’ and ‘b’ are arbitrary constants. In linear function the graph is a straight line.</td><td></td></tr>
</table>

	 Figure 17: Participants pictorial representation of linear function $f(x) = 2x + 2$ And (0, 2), (1, 4), (2, 6), (3, 8) and (4, 10) as ordered pairs.  Figure 18: Participants graphical representation of linear function $f(x) = 2x + 2$

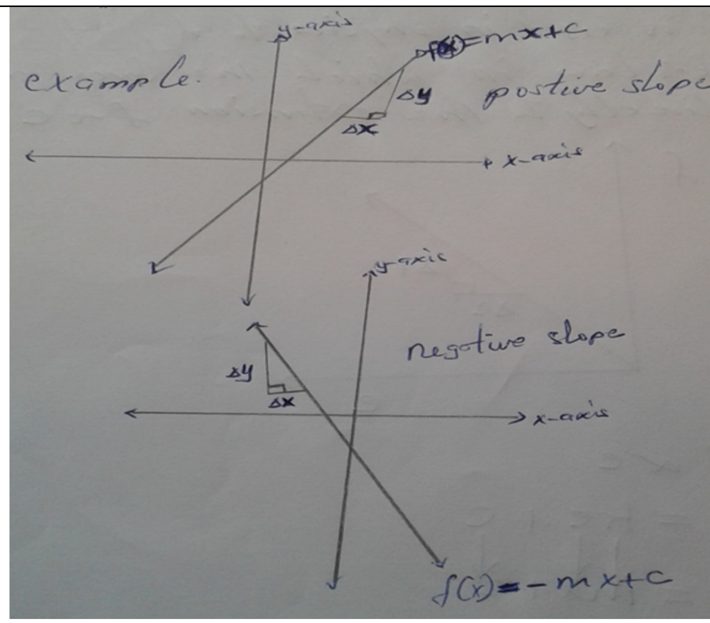
A linear function $f : A \rightarrow B$ means that a function 'f' maps element of set A to element of set B. Where A is independent variables set and B is dependent variables set. Set B must satisfy $mx + c$ and set A must be variables that are independent the x, i.e. $f: x \rightarrow mx + c$	The graph of $f(x) = 25x + 5$ x-intercept, y=0 $(x, 0) = \left(-\frac{1}{5}, 0\right)$ y-intercept, x=0 $(0, y) = (0,5)$
Is the polynomial function whose degree of x in the equation is one, example $f(x) = mx + c$. The graph is straight line cut [crosses] the axes at any point with different slopes—negative or positive.	 Figure 19: Participants illustrations of linear function with positive and negative gradients
Linear function is the first-degree polynomial function of one variable. A linear function is a function that makes a straight line when graphed. A linear function is the function in the form $f(x) = ax + b$ where a $\neq 0$	$f(x) = 2x + 1$. Another relationship between variables given by linear function is the relationship between velocity and time given by equation $v = u + at$ where *v*-final velocity, *u*-initial velocity, *a*- acceleration, and *t*-time. This means that a given body moves with a constant acceleration then the velocity varies directly proportional to the time taken. If a body starts with a velocity 10m/s and moves with acceleration of $2m/s^2$ then its equation is given by $v = 2t + 10$

Table 10: The pre-service teacher participants' first lists of realizations of linear function concept

Some of the participants offered applications of linear function. These included:

> Travel, when an individual move from one point to another —certain distance 'D' with a speed 20km/h at a time *t*. Then *D=20t.*
>
> The application of linear function is determination of the slopes of different areas.
>
> It is used in Physics to determine Hooke's law in which the applied force is directly proportional to the extension i.e.

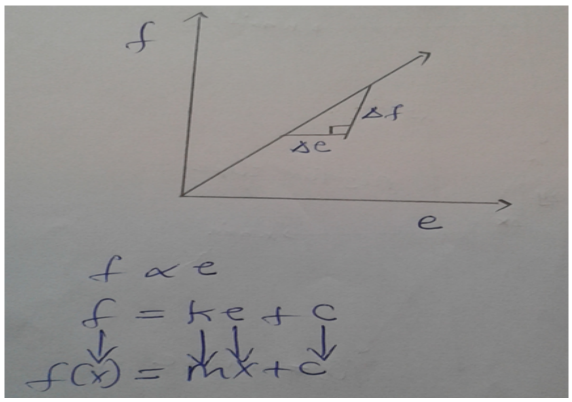


Figure 20: Participants illustration of Hooke's law as an application of linear function

Looking at participants' realizations of linear function provides evidence pre-service teachers' awareness of the mathematical definition of the linear function, its representations: symbolically, graphically, and pictorially. The "linear function in the form of $f(x) = mx$ is a statement of proportionality with *m* as an invariant ratio, also called the *constant of proportionality"* (Lobato & Ellis, 2010, p. 49). Some of the participant's definitions and illustrations are the same as Lobato and Ellis's explanation that "a linear function can be expressed in the form $y = mx + b$. Furthermore $y = mx + b$ is a statement of proportionality, represented by $y = mx$, combined with vertical translation represented by the addition of *b*" (p. 51). The use of follow up questions in the large group discussion helped participants collectively access their MFT of the linear functions concept. The example below illustrates how the facilitator promoted collective awareness of behaviour of the graphs of linear functions in relation to their respective gradients/slopes.

Facilitator: “Could you tell the different between the graph of part a) and the graph of part d)?” (Figure 21).

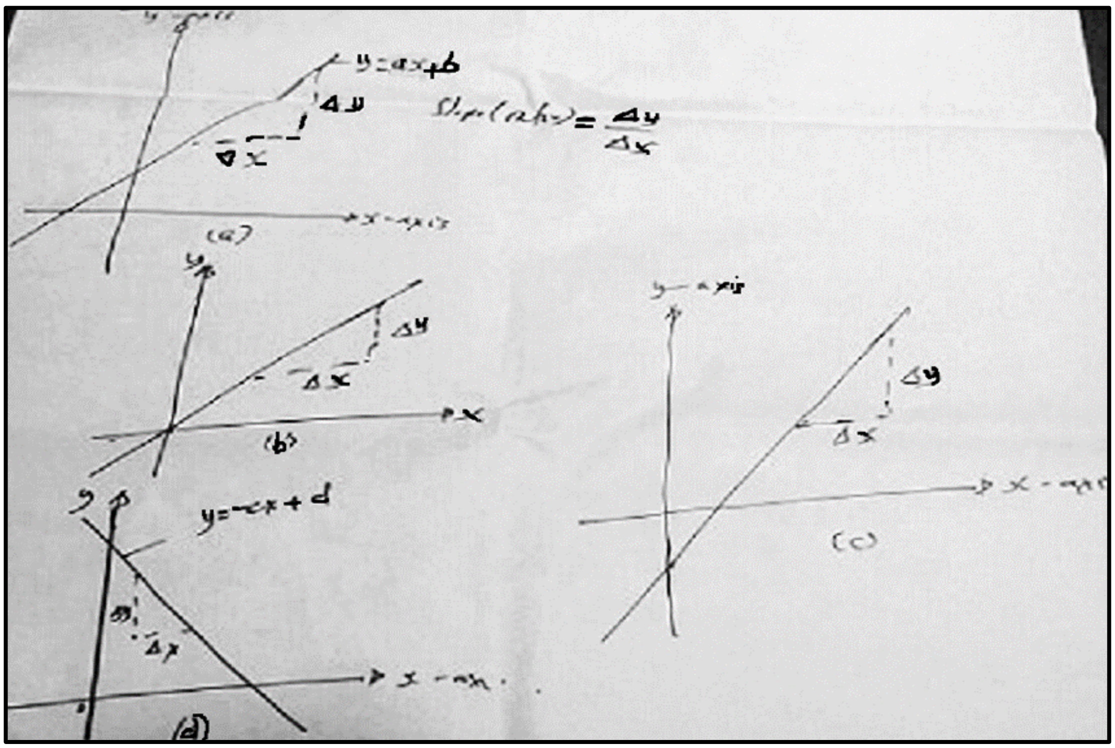


Figure 21: Pre-realizations illustrations of graphs of linear function

P1 (a member from the group 2): The first one (referring to the graph of part a) has the positive gradient and the second one (referring to the graph of part d) has the negative gradient.

Facilitator: How do you know that a given linear function graph have a negative or positive gradient?

P2 (first member from group 1): A graph of linear function inclined to the left have the negative gradient/slope and the one inclined to the right have positive slope/gradient.

P3 (second member from the group 1): The graph of part a) have the positive gradient because when x increases y increases or when x decreases y decreases while the graph of part d) have the negative gradient because when x increases y decreases or when x decreases y increases.

P4 (a member from group 3): In part d) the coefficient of x is negative $y = -cx + d$ while in part a) the coefficient of x is positive $y = ax + b$

Another interesting collective scenario occurred when one of the participants from group 2 defined linear function as a polynomial function.

P1 (a member from group 2): Linear function is the polynomial function whose graph is a straight line and which, normally must obey the condition f(x) = m x + c, where m is the gradient and c is the fixed or constant number. The degree of x must not exceed 1

P2 (a second member from group 2): If the degree of x is zero, is it a linear function?

P3 (a third member from group 2): No, it is not a linear function.

P2 (a second member from group 2): But, we get a straight line.

P3 (a third member from group 2): It is a constant function, and not a linear function.

Facilitator: Why a constant function and not a linear function?

P3 (a third member from group 2): The graph is a straight line but, it is a horizontal line parallel to horizontal axis showing that the value of dependent variable does not change no matter how much the independent variables is changed.

Facilitator: The facilitator wrote '$f(x) = ax + b$ where 'a' and 'b' are arbitrary constants and 'x' is a variable' on the whiteboard and asked the question that follows. 'It is one of your definition so, when a=0, what do we get?'

The group: Shouted $f(x) = b$.

Facilitator: If the exponent of x is zero, what do we get?

P4 (a member from group 1): $f(x) = a + b$

Facilitator: What can you say about both $f(x) = b$ and $f(x) = a + b$?

P5 (a member from group 3): They are both constant functions.

P6 (a second member from group 3): So, does it mean that constant function has the value of 'a' zero and exponent of 'x' zero?

P3 (a third member from group 2): Yes, the gradient of constant function is zero

P7 (a member from group1): Yeah, the constant function is a polynomial function of degree zero.

> P1 (a member from group 2): So, linear function is a polynomial function of degree one and a constant function a polynomial function of degree zero.
>
> P8 (a second member from group 1): What we say is the gradient of linear function is positive or negative and that of constant function is zero.

This is interesting in the sense that the participants thought about the use of the word ‘must not exceed 1’ implies less or equal to one. That means even negative exponents are included which will be contradiction because the lowest degree of polynomial function is zero. And, also how the participants were able to connect the ideas of degree of polynomial functions to constant functions and linear functions.

The collective discussion generating the first list of the realizations and the follow up questions enabled the participants to come up with more explanations to access other realizations of the linear function concept. Figure 22 below display the summary of what I interpreted as realizations of the linear function from the participants’ first list of realizations and follow up questions after a considerable amount of time in the collective discussion of linear function concept.

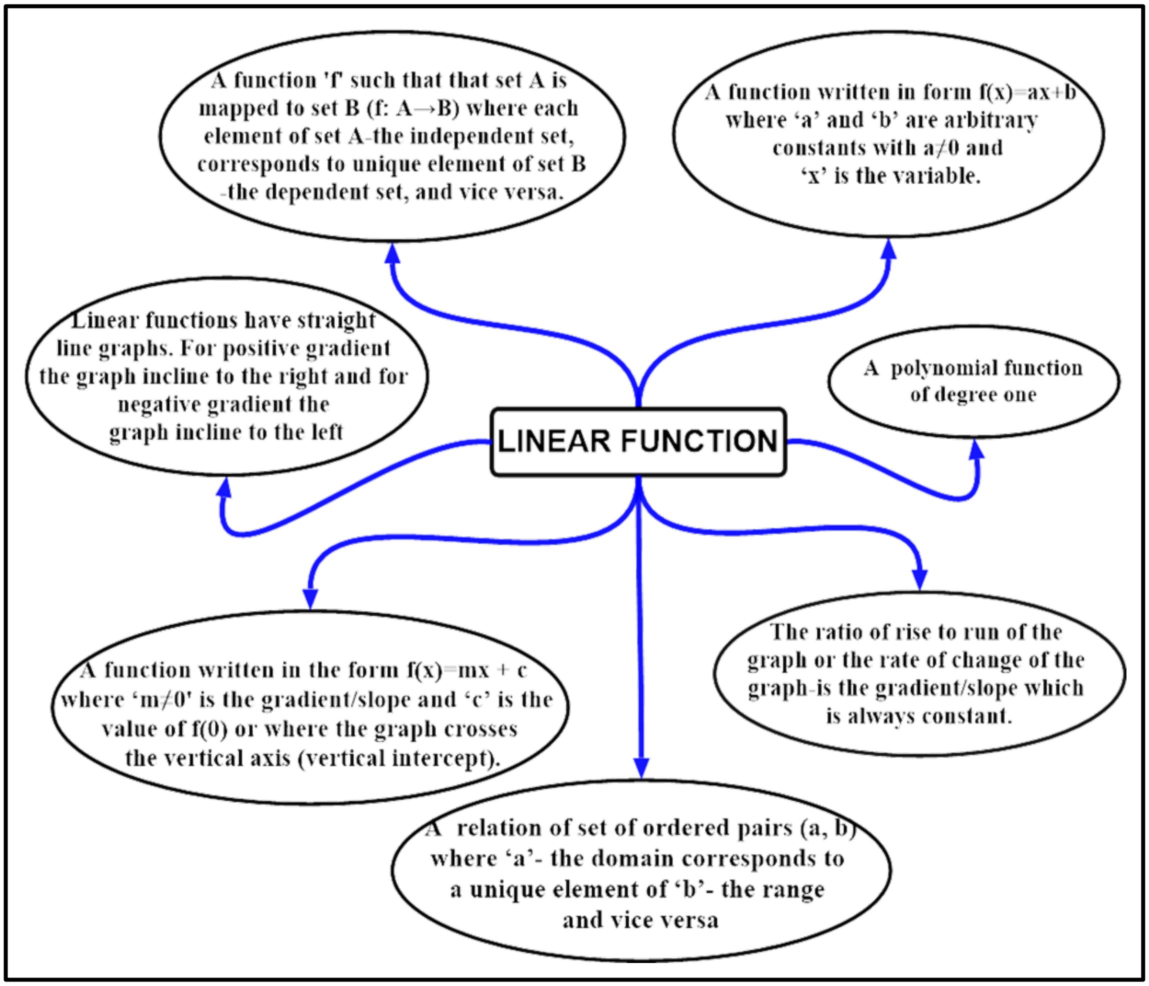


Figure 22: Summary of realizations of the linear functions concept

6.2 The landscapes emphasis in concept studies

In this work with pre-service teachers the landscapes emphasis emerged with planned activity in the concept studies. The participants were asked to examine the Tanzania ordinary level secondary school mathematics curriculum and organize the realizations that emerged from their interactions with each concept. This provided them an awareness of how these realizations relate within a grade level (horizontal awareness) and across grade levels (vertical awareness). The activity resulted in their created grade level map of each concept which I organised in the table charts described in the next paragraph. The group identified the level the concept is taught in the ordinary level secondary school mathematics curriculum, as well the concepts the student needs to know before engaging learning it. For example, the participants identified concepts such as fractions, whole number operations, percentage, decimal, similarities, and measurements as the basic concepts the students need before engaging them in learning the ratio concept. While for the proportion concept the students need the basic concepts such as ratio (especially equivalent ratios), fraction (especially equivalent fractions), measurement, geometry (specifically drawing figures having same shape but different dimensions), whole numbers operations (addition, subtraction, division, and multiplication), and multiples of numbers. Also, they identified mathematics concepts taught within the level (Form) and those taught across the ordinary level secondary school mathematics curriculum which relate to it. The participants discussed how the mathematical concept relates to others and how it is used in other mathematical concepts within the grade level or across the grade levels to create the landscape of each concept.

Ball and Bass (2003) assert teachers must be skilful at construing concepts for learners. But, to do so, they need knowledge of how mathematical concepts are connected, how mathematical ideas anticipate others, and so on. Figures 23, 24, 25, and 26 below represent the landscapes for ratio, proportion, rate, and linear function respectively created collectively by the participants as part of this activity. The table organizes concepts by form (or grade) from the bottom to the top— Form I to Form IV. The interaction with the curriculum enabled the pre-service teachers to access and develop deeper understanding of how the four concepts ratio, proportion, rate, and linear function relate among themselves and with other concepts within the grade level and across the ordinary level mathematics curriculum.

However, the unfolding of the concepts were not organized in a coherent manner. One of the participants commented not having an idea that ratio is related to linear function because in his experience no teacher had ever pointed to that relationship in class. The participant explained further that he did not experience teachers linking such topics/concepts together or encouraging students to see the connections. This student recognized that teachers could use proportional reasoning to show the students how ratio, proportion, rate, and linear function relate (Lobato & Ellis, 2010). Reasoning proportionally is facilitated by a teacher who has many understandings that include "the meaning of a ratio as a multiplicative comparison and as a composed unit; making connections among ratios, fractions, and quotients; and understanding from basic to more sophisticated levels of proportional reasoning" (Lobato & Ellis, p. 48). Understanding that "the linear function in the form of $f(x) = mx$ is a statement of proportionality with m as an invariant ratio" (p. 49) can lead to proportional reasoning rather than the execution of a procedure without understanding why.

Class level	Topics	Based on its uses/related concepts
Form IV	Trigonometry	Finding the trigonometrical ratio of sines, cosines and tangents for angles in the four quadrants
	Probabilities	Finding the probability of an event
	Areas	Finding condition for areas of similar figures
	Coordinate geometry	Computation of gradient/slopes to determine the condition for lines to be parallel or perpendicular
Form III	Variations	• Direct variations • Use of direct variations in writing proportions
	Rates	• Relate quantities of the same kind • Relate quantities of different kinds
	Linear functions	Finding the gradient/slope of linear function
Form II	Trigonometry	Finding the trigonometrical ratios of sines, cosine and tangents
	Geometrical transformations	Enlargement-the use of scale factor
	Congruency and Similarities	Finding the condition for similar figures
Form I	Coordinate geometry	Compute the gradient/slope of linear equation
	Profit and Loss	• Compute percentage loss • Compute the percentage profit
	Rational numbers	To represent a rational numbers
	Geometry	Bisecting lines
	Percentages	Percentages that involves proper fraction
	Decimals	Decimals that involves proper fractions
	Fractions	• Proper fractions • Equivalent fractions
	Whole numbers	Multiples of numbers

Figure 23: A landscape of ratio based on its uses/related concepts—Ordinary level secondary school mathematics in Tanzania.

Class level	Topics	Based on its uses/related concepts
Form IV	Areas	Condition for areas of similar figures
	Coordinate geometry	Condition for lines to be parallel
Form III	Variations	• Direct variations • Use of direct variations in writing proportions
	Rates	Relate quantities of different kinds
	Linear functions	Finding the gradient/slope of linear function
Form II	Geometrical transformations	Enlargement- the use of scale factor for corresponding dimensions
	Congruency and Similarities	Condition for similar figures- equal ratios of corresponding sides/areas/volumes
Form I	Ratio	Equivalent ratios
	Geometry	Drawing figures having same shape but different dimensions
	Fractions	Equivalent fractions
	Whole numbers	Multiples of numbers

Figure 24: A landscape of proportion based on its uses/related concepts—Ordinary level secondary school mathematics in Tanzania.

Class level	Topics	Based on its uses / related concepts
Form III	Variations	• Use of direct variations in writing proportions which are rates
	Rates	• Relate quantities of different kinds • Relate quantities of same kinds
	Accounts	• Interest rates
	Linear functions	Gradient/slope of linear function to compare rates
Form II	Similarity	Enlargement
Form I	Coordinate geometry	• Gradients of the graphs of linear equations
	Ratio	Ratio of quantities with different units
	Proportions	Two equal rates
	Fractions	Proper fraction
	Units	Units of length, mass, time, and volume
	Whole numbers	Basic operations on numbers

Figure 25: A landscape of rate based on its uses/related concepts—Ordinary level secondary school mathematics in Tanzania.

Grade level	Topics	Based on its uses / related concepts
Form IV	Linear programming	• Solving simultaneous equations graphically • Drawing Inequalities
	Coordinate geometry	• Equation of a line • Computation of gradient/slopes • Drawing parallel and perpendicular lines
Form III	Variations	• Direct variations graphs are straight lines graphs • Gradient/slope as a constant of proportionality
	Rates	• Graphs of rates are straight lines • Gradient/slopes of the graphs are the constant rates
	Functions	Linear function is a polynomial function of degree 1
	Relations	Graphs of linear inequalities
Form II	Statistics	• Frequency polygon • Line graphs
	Geometrical transformations	• Translations of lines and • Enlargement of dimensions, length, width, heights, radius, diameters
	Similarities	Similar figures, enlargement of figures
Form I	Coordinate geometry	• Locating coordinate points • Drawing graphs of linear equation • Finding gradient/slope of linear equation
	Proportions	Comparison of line segments
	Ratio	Finding gradient/ slope of a line
	Geometry	Point, line, line segment and a ray
	Units	Unit of length
	Whole numbers	Basic operation on numbers (addition, subtractions, multiplication and division

Figure 26: A landscape of linear function based on its uses/related concepts—Ordinary level secondary school mathematics in Tanzania.

6.3 The entailments emphasis in concept studies

An emphasis on the logical implications of each realization of a mathematical concept that helps to shape the teachers' understanding of the concept, is reported as entailments in this section. As discussed in the theoretical framework the manifestation and expression of the five emphases are not linear. They are placed in this order at the writer's discretion. Unlike what was done to prompt thinking about the landscapes, the entailments, were prompted by a facilitator's question to think about why. Starting with the ratio concept, entailments first emerged from the collective production during the time participants were responding to what they do know about the ratio concept and one of the participants uttered *"...so we say all ratios are fractions."*

P1 (A member in group 3): ...so we say all ratios are fractions

P2 (A member in group 2): Yes, but I am not much sure

P3 (A member in group 3): But, ratios are expressed as 'a' over 'b' and as a part to whole when an object is divided into equal parts to be shared equally or a set with groups of different kinds of quantities or things.

P4 (A member in group 1): It is a bit confusing because fraction is just a number but, ratio specify context, example ratio of orange to mangoes is 2/3.

Facilitator: I think in your group, you can discuss this by finding the reasons to why ratios are fraction? That means 'ratios are fraction because...'

The progress of the collective learning paralleled the emergence of the entailments emphasis from such comment and the realization of the ratio as part-whole relationships. I asked them to respond to, "ratios are fraction because...." In other words, to find the logical implications of the ratio as a fraction. Here is an example from the responses of two groups.

Group 2	Group 3
Ratios are fraction because • It is written as $\frac{a}{b}$ where a is the first quantity and b is the second quantity • It is a part to whole relationship • Compares two quantity only. For example, a farm is planted maize, beans, and potatoes. A farmer harvested 10 sacks of maize, 2 sacks of beans, and 3 sacks of potatoes. The ratio of sacks of potatoes to sacks of maize can be written as 3/10 but, the ratio of sacks of the three crops cannot be written as a fraction.	Ratios are fraction because • It can be written as $\frac{a}{b}$ where a comes from the amount/number of things from the first set and b is the number of things from the second set • It compares two quantities, for example, a basket contains 5 oranges, 7 mangoes, and 2 apples, the ratio of oranges to mangoes is $\frac{5}{7}$ but the ratio of oranges to mangoes to apples cannot be written as a fraction but only in the ratio form 5:7:2 • It represents part to whole relationship, for example, the ratio of orange to fruits in the basket is $\frac{5}{14}$

Table 11: Two groups' responses on reasons to why ratios are fractions

Though there might be more entailments of ratio, from the discussions among the pre-service teachers they were able to create the logical implications of realization of the ratio as a fraction (summarized in figure 27) as entailments of ratios as fractions. Generally speaking the most dominating realization of the ratio concept by this group was the ratio as a comparison of two quantities.

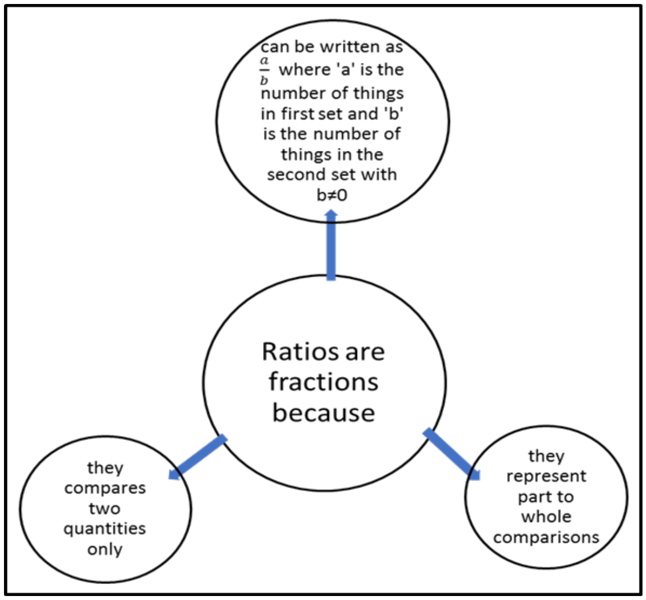


Figure 27: Entailments diagram for realization of ratios as fractions

For the proportion concept, entailments emerged as activities for the participants to find the logical implications of two out of the four realizations that had emerged for the proportion concept. The participants collectively agreed on two realizations of the ratio: as a pair of equivalent ratios; and as two equal fractions. In the course of discussion of what they do know about proportion concept a participant from group 1 voiced "I think we all now agree that we can consider proportion as a pair of equivalent ratio or as two equal fractions" and the member of the group some shouted "of course" and others "yes". However, one participant from group 3 aired "you all said yes, but do you know why?" Then a sadden moment of silence and a sound 'mmh' came up. At that point, I decided to offer two activities to help the participants build the deeper understanding of the two realizations by providing their logical implications. First, I asked, "Proportion is two equal fractions because..." and second, "Proportion is a pair of equivalent ratios because..." The participants gave different responses which they shared within the group and aired the reasons for each in the larger group and summarized five logical implications for proportion as a pair of equivalent ratios I represented by figure 28 and four logical implications for proportion as two equal fractions I represented by figure 29. Generally speaking, the most dominating realization of the proportion concept by this group was two equal ratios.

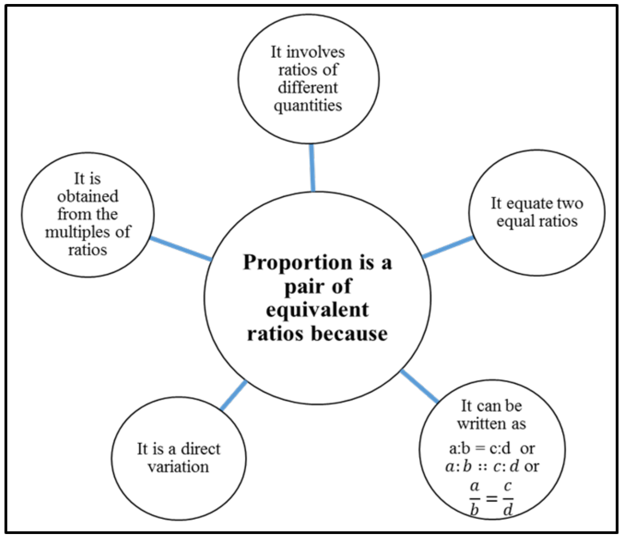


Figure 28: Entailments diagram for realization of proportion as a pair equivalent ratios

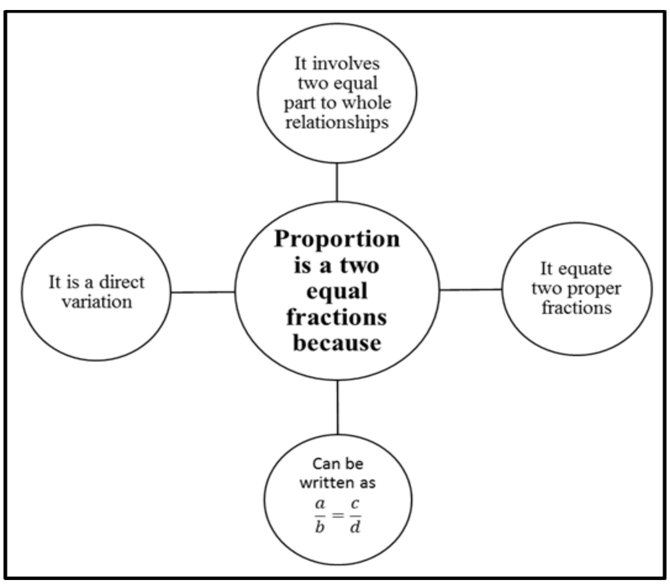


Figure 29: Entailments diagram for realization of proportion as two equal fractions

For the rate concept, the entailments emphasis was neither evident from the collective production brought during the time participants were responding to what they do know about the rate concept nor in the discussion of follow up questions. Even when I prompted “Is there any implications you can think of from the list you provided as the realizations of rate? This challenge might be caused by participants’ insufficient knowledge of rate in school mathematics.

For the linear function, the entailments emphasis first emerged from the collective production that came about during the time the participants were discussing the relationship between the graphs of the linear functions with their gradient/slope and its notation. The effort to relate the three realizations “linear function as a function written in form $f(x) = ax + b$ where ‘a’ and ‘b’ are arbitrary constants, with ‘$a \neq 0$’ and ‘x’ is the variable”; “linear function have straight line graphs”; and “the ratio of the rise to run of the graph or the rate of change of the graph is the gradient/slope which, is always constant” that were appreciated by the collective discussion of the participants. I asked them to do two activities, first, “A linear function $f(x) = ax + b$ has positive gradients if ...” and second “A linear function $f(x) = ax + b$ has negative gradients if ...” There were four common responses of logical implications of linear function $f(x) = ax + b$ to have positive gradients shared by the group, which made the entailments of the linear functions to have positive gradients, I represented in figure 30. And four logical implications of linear function $f(x) = ax + b$ to have negative gradients which made the entailments of the linear function to have the negative gradients, I represented in figure 31. Here are some quotes and images from the participants to illustrate how they were thinking about the entailments. First, a member of group 2 used an image that was among the examples provided as illustrations of the first list of realizations of linear function to illustrate his explanation “a graph of linear function inclined to the left have the negative gradient/slope and the one inclined to the right have positive slope/gradient". Second, a member of group 1 used the image (figure 21) to illustrate his explanation "the graph of part a) have the positive gradient because when x increases y increases or when x decreases y decreases while the graph of part d) have the negative gradient because when x increases y decreases or when x decreases y increases." Some participants demonstrated frustration when prompted to reason about the conditions for the linear function to have the positive gradient, and the negative gradient. For example, a participant voiced, "I find it hard of what to say here, because I have never thought about this before...and we need at least four reasons for each. What I know is that a line with a negative slope incline to the left and the one with negative slope incline to the right. That is all. In most cases, I experienced the teacher pointing about it. Can any of you tell me the other reasons?"

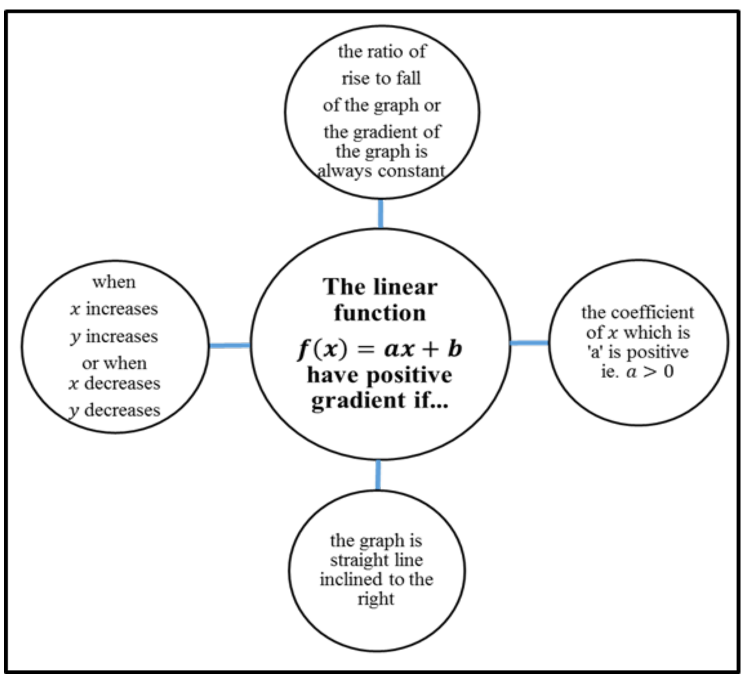


Figure 30: Entailments diagram for the condition of the linear functions to have positive gradients

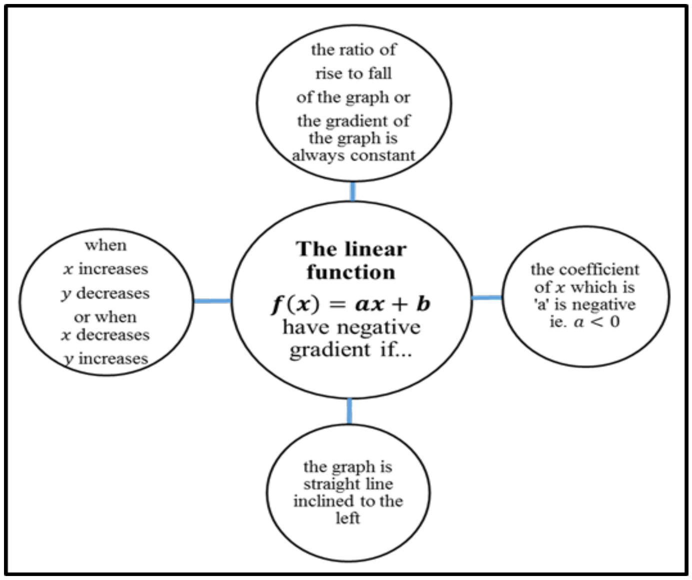


Figure 31: Entailments diagram for the condition of the linear functions to have negative gradients

6.4 The blending emphasis in concept studies

The blending emphasis in concept studies involved the activity of generating, combining, and collapsing varied realizations of each mathematical concept by exploring the deep connections among these realizations that might produce further emergent interpretive possibilities (Davis & Renert, 2014). The participants were prompted to look for the realizations of each mathematical concept and see whether they could generate, combine, or collapse them and get a meaningful result. For example, in ratio concept, I prompted, ‘From your list of realizations can you try to find the connections among any of these realizations?’ or ‘Can you collapse them to something else? This activity was challenging for the participants for the first two concepts the ratio and proportion. Then I asked them to generate the blending even between concepts. For the rate concept, the blending emphasis first occurred from the collective production brought about during the time participants were responding to what they do know about the rate concept. It emerged when one of the participants voiced "...so even rates are ratios" and another participant in the course of discussion uttered "...yes when you have two rates that are equal, we also get proportions". I suggested an activity to discuss collectively how some realizations of the three concepts the ratio, proportion, and rates are relating as a blending activity. I asked them: recall what you have obtained as the realizations of the ratio, proportions, and rate concept. Could you discuss and summarize how these realizations of the three concepts ratio, proportions, and rate are related? The blending activity helped the participants build the understanding of how the realizations of the three concepts are connected despite the fact that they are taught at different levels in the school curriculum. This work provided participants with an opportunity that could help them tackle the questions that might be raised by their future learners. The participants gave diverse responses. There were six common responses offered by the group which made the blending of how some realizations of ratio, rate, and proportions are related as I represented in figure 32.

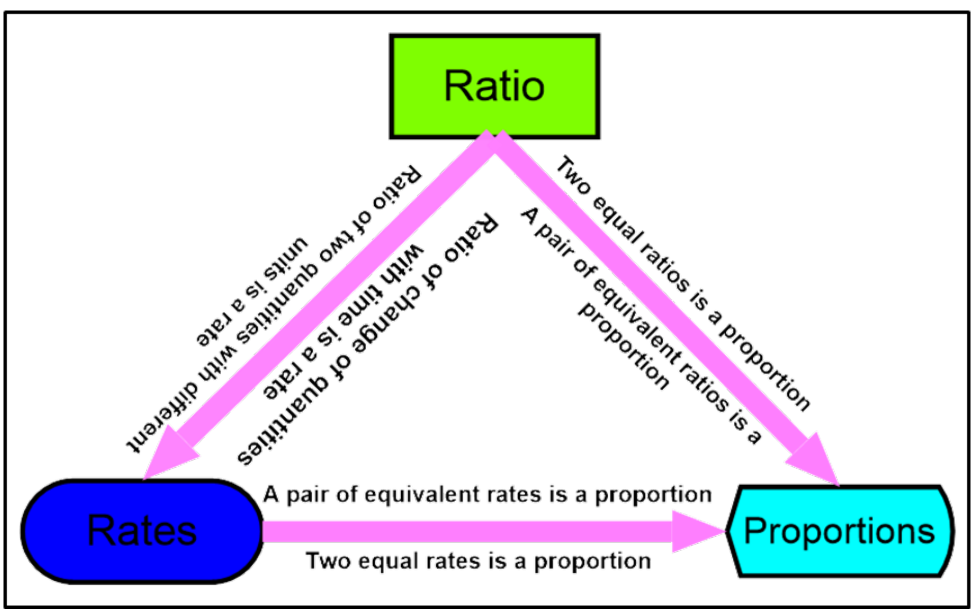


Figure 32: Blending diagram for how some realizations of ratio, proportion, and rate relate.

Also, with the use of examples in the rate concept study they were able to combine the realization of the ratio as the comparison of two quantities and that of the rate as the ratio that compares two quantities with different units. So, speed as a rate is a blend of the ratio (distance to time) concept and measurement concept (length and time). Specifically, the speed given in kilometre per hour is the rate which is the blends of the ratio (distance to time) and measurement (length and time).

$$\text{Speed } \left(\frac{\text{km}}{\text{h}}\right) = \frac{\text{Distance (in km, a measure of length)}}{\text{Time (in hours, a measure of time)}}$$

The second example of a blend that emerged was observed in the concept study of the linear function. The participants drew illustrations that showed a line inclined to the right having positive slope and inclined to the left having negative slope. They also wrote the formula for change in y over change in x. The combining of the realizations of the linear function as the ratio of the rise to run which is the slope/gradient and the linear functions have straight line graphs. The slope as the rate is the blend of the ratio (rise to run of the linear graph) and coordinate geometry (change in y over the change in x, for the linear graph crosses two coordinate points,(x_1, y_1) and (x_2, y_2)). The rise is the change in ordinate values and the run is the change in abscissa values (figure 33).

$$\text{Slope(the inclination of the line)} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{Change in y}}{\text{Change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

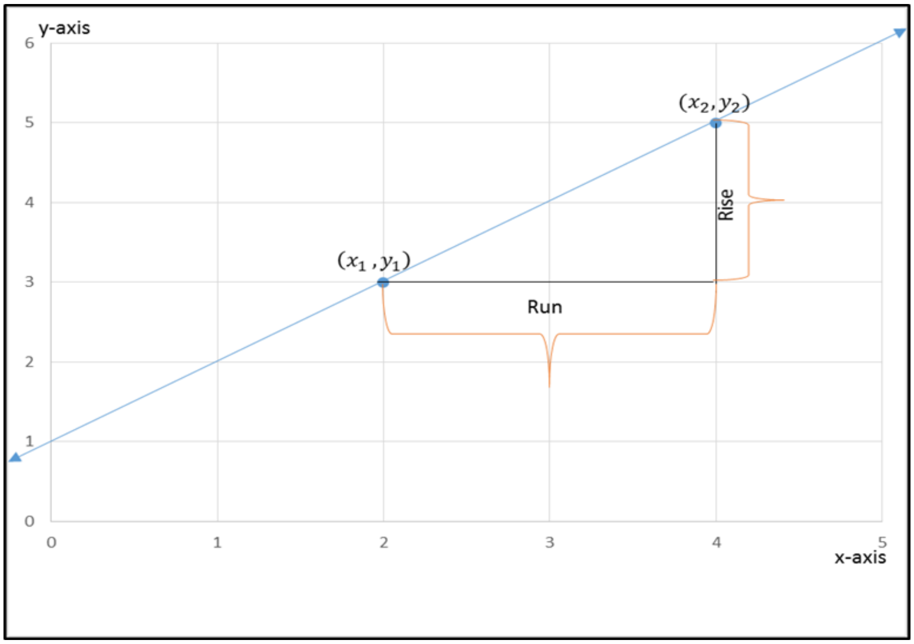


Figure 33: Slope as blend of ratio and coordinate geometry

6.5 Pedagogical problem solving emphasis in concept studies

One of the important aspects in the teacher's mathematics lesson preparation, for promoting student understanding of the concept is speculating what type of questions could be raised by the student during the classroom instruction. These questions are labelled as 'pedagogical problem solving' in the Davis and Renert (2014) framework. They refer to questions which, might be asked by learners around the meaning-seeking that specifically falls within the mathematical concept under exploration. Examples of pedagogical problem-solving did not emerge independently of the facilitator within the discussions. This was anticipated in the planning stage and the facilitator was prepared for promoting pedagogical problem solving in all four concept studies of the ratio, proportion, rate, and linear functions. The participants were provided with a question *"What questions do you expect the students could ask during classroom instruction when learning the concept of ratio? Give examples from your experience as a student or* *prospective teacher of ordinary level secondary school and possibly what the teacher needs to do to help the student"*. The same question was asked in each of the other concept studies of proportion, rate, and linear function as well. Thus, in the last fifteen to twenty minutes of each of the concept studies, the pre-service teacher participants spent trying to raise questions that they thought the student might ask when learning the concept at hand, and they offered what the teacher needs to do. Here are the questions that seem to be a concern to the group collectively as well their suggestions of what teachers need to do to help the students in responding to the anticipated raised student questions and develop the better understanding of the concept in each concept study of ratio, proportion, rate, and linear function respectively. The participants raised questions for each concept, as illustrated in table 12.

Concepts	Participants raised questions for each concept
Ratio	1. Where do we use ratio in real life? 2. What is the difference between ratio and fraction?
Proportion	1. What is the difference between ratio and proportion? 2. Where the proportion concept is used in real life? 3. What is the relationship between proportion and ratio? 4. What are the differences and similarities between ratio and proportion? 5. What are the differences and similarities between proportion and variations? 6. How do you divide a given quantity into proportion parts?
Rate	1. What is the difference between ratio, proportions, and rates? 2. What are the differences and similarities between rate and proportion? 3. What are the differences and similarities between ratio and rate? 4. Does rate relate with time? 5. Where is rate concept used in real life?
Linear function	1. What is the difference between linear equation and linear function? 2. Difference between constant function and linear function? 3. Why slope/gradient of linear function represented as vertical increase over horizontal increase? 4. How is rate related to linear function? 5. Where is linear function applicable in everyday life?

Table 12: Participants' raised questions through 'pedagogical problem solving' emphasis

For the teachers, I think these questions (Table 12) are significant questions as they point to the importance of the teachers need to know how the mathematics concepts in the school curriculum are related and their applications in everyday life to be able to respond better to the student needs. For the students, these questions are the key to their development of the deep understanding of the mathematical concepts as they could be able to see the connections of the mathematics, they learn across the school curriculum and how the mathematics they learn in school as concepts are applicable outside the school environment.

6.5.1 Anticipating what the teacher could do

In responding to the anticipated student questions, the pre-service teachers suggested general things the teacher could do in classroom instructions. For proportion concept, the participant anticipated: proper use of the teaching aids, relating the proportion concept with other concepts in mathematics curriculum and other school curriculum subjects, and learning its applications in real life outside the school environment. For the rate concept, the participants anticipated the teacher needs to use practical examples found in their environments and their everyday activities outside the school to help the student understanding of how the ratio, rates, and proportions concepts relate. Further the teacher could use the ratio table to illustrate the ratio, rates, unit rate, and proportions. For the linear function concept, the teacher needs to use practical examples found in their environments and their everyday activities outside the school to help the student understanding of how the linear equation and linear function concepts relate. Finally, the preservice teachers believed that to use examples which show the notation, graphical, and the gradient differences between linear functions and constant functions is important when teaching.

6.6 Summary of the Chapter

To summarize, the chapter describes the findings of how the five emphases of the concept study (Davis & Renert, 2014) realizations, landscapes, entailments, blending, and pedagogical problem solving helped the pre-service teachers to access and develop their tacit MFT—the professional knowledge and skills of ratio, proportion, rate, and linear functions concepts. MFT emerged through the pre-service teachers' participant engagements with the collective learning designed activities. In the concept studies, the realization reflected what they do know about the mathematics concept but at the same time it gave each of the participants the opportunities to learn different definitions, examples, images, illustrations, and applications of each of the four mathematical concepts through sharing the ideas originating from the members of the group. Their capacity to engage in the discussion and build on each other's ideas suggest that their knowing was developing in the context of the session itself. The collective construction and unpacking of realizations of each of the mathematical concepts gave the individual pre-service teacher the opportunities to correct their misconceptions and recall what they have forgotten about each of the four mathematical concepts. Also, as a facilitator, the use of how, why, and could you elaborate more, encouraged the participants access and develop their MFT. The participants' development of the tacit MFT was elaborated as they were able to come up with varieties of realizations of each of the four mathematical concepts. And as shared activities, each pre-service teacher was able to acquire more than what he knows from the colleagues' contributions and even reminded of what he might have forgotten he knows.

The participants examined the Tanzania ordinary level secondary school mathematics curriculum and organize the realizations that emerged from their interactions of each of the concept study of ratio, proportion, rate, and linear function as landscape activities. The activities provided them with an opportunity to develop awareness of how the realizations relate within a grade level (horizontal awareness) and across grade levels (vertical awareness). The pre-service teachers' collective learning how these realizations relate seemed to help them think about how their prospective students might make connections among concepts they learn in one particular level to another higher grade level, making it easier for the understanding of the new concept and building more interest in learning mathematics.

For some realizations of the ratio, proportion, rate, and linear function, the participants were able to find the logical implications each realization carries as entailments that helps to shape the mathematical concepts understanding. This task was more frustrating for them; this may be an indication of them working with new ideas. Starting with the ratio concept, the participants' expressed logical implications of the realization of the ratio as a fraction. For the proportion, the participants' expressed logical implications of the realization of the proportion as a pair of equivalent ratios and that of the proportion as two equal fractions. For the rate, the entailments emphasis was challenging to the participants. This might be because of the nature of rate concept or the participant's insufficient knowledge of rate in school mathematics. For the linear function, the participants came up with the conditions of the linear functions to have the positive gradient and the negative gradients.

The blending emphasis was the most challenging of Davis and Renert's (2014) emphases for the participants in the concept studies. This challenge could be due to the fact that they are pre-service teachers who joined teachers' college straight from high school and have little experience teaching mathematics (eight weeks block teaching practice in their first year). Or it could be that they have little experience exploring the relations among seemingly unrelated ideas in mathematics. Blending involved the activity of generating, combining, and collapsing varied realizations of mathematical concept at hand by exploring the deep connections among these realizations that might produce further emergent interpretive possibilities. For the rate concept study, the participants generated, combined, and collapsed varied realizations of the three concepts ratio, proportion, and rate and were able to come up with the relationship of some realizations of these three concepts as a blending activity. The participants also, learned that the speed as a rate is a blend of the ratio (distance to time) concept and measurement concept (length and time). Specifically, the speed given in kilometre per hour is the rate which is the blends of the ratio (distance to time) and measurement (length and time). For the linear function, the participants learned the slope as the rate is the blend of the ratio (rise to run of the linear graph) and coordinate geometry (change in y over the change in x, for the linear graph crosses two coordinate points,(x_1, y_1) and (x_2, y_2)).

For the ‘pedagogical problem solving’ emphasis, the participants speculated about the type of questions could be raised by the students during the classroom instruction in learning the ratio, proportion, rate, and linear function concept. This seemed quite new to them but it it did interest them. They suggested general things the teacher could do in classroom instructions in responding to the speculated students’ questions and the better understanding of the concept. However, this might be different if participants could be in-service teachers as each of the teachers could have brought their experiences of what they did after encountered each of the questions separately. The questions the participants speculated in each of the four concepts evidenced the values of the concept studies because these questions built their awareness of the need for the teachers to have an understanding of all concepts of the school curriculum and how they are similar and different for helping students better understanding of these concepts and mathematics in general. For example, they indicated there is a need for the teacher to be able to differentiate ratios and fractions, to know how ratio, proportions, and rate similar and differences and their applications in everyday life and so on.

7 The Pre-service Teachers' Explicit MFT

This chapter describes the findings of pre-service teachers' development of explicit MFT during the concept studies of ratio, proportion, rate, and linear functions as analysed by using Ball, Thames, and Phelps, (2008) categories of Mathematical Knowledge for Teaching (MKT). The analysis was from the definitions, examples, images, algorithms, and applications in everyday life of the concepts from the participants' first list of realizations, and from extra activities that I provided apart from the entrance questions in each of the concept studies. Also, analysed was the data from the post-questionnaires that were used after each concept study session in response to two questions: what you do know about the mathematics concept and how it is learned. For the participants' first list of realizations of each of the mathematical concepts, extra activities, and the response of the post-questionnaires question of 'what you do know about the mathematics concept', I first analysed the list of realizations and response of the question: to see whether they provided correct definitions, wrote correct mathematical notations and used mathematical terms correctly. Though the correctness is not a primary concern in concept studies, correctness is considered with regard to assessing the explicit knowledge within Ball's categories of MKT. I considered if their illustrations fit with each of the mathematical concepts and whether they explained the concepts properly. I also checked how they solved the mathematical tasks I offered to see if attempted correctly, the type of method used and whether the participants could demonstrate the procedures they used. I further analysed the examples they provided by looking at their relevance in building the understanding of each of the mathematical concepts. Finally, I analysed the applications of each of the mathematical concepts they provided to see their relevance in understanding each of the mathematical concepts. Then I associated all of the above with Ball and colleague's categories of MKT. For the question of 'how each of the mathematical concepts is learned' I looked on their explanations and associated with the Ball and colleague's category of MKT.

Ball and colleagues proposed a practice-based model of Mathematical Knowledge for Teaching that was built on Shulman's (1986) categories of teachers' knowledge the subject-specific content knowledge and pedagogical content knowledge. Ball, Thames, and Phelps, (2008) MKT model consist of distinguishable, distinct categories of teachers' mathematical knowledge that is necessary for the work of teaching. They divided the model into six knowledge domains as illustrated in Figure 34.

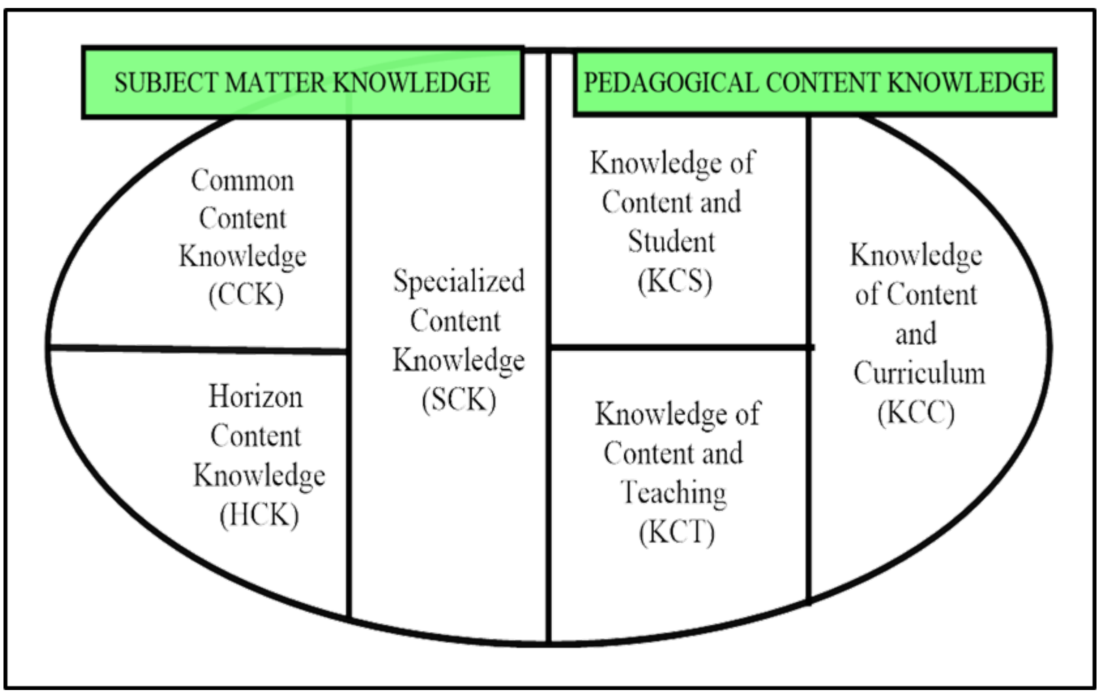


Figure 34: Ball' et al.'s MKT model (Ball, Thames, & Phelps, 2008, p. 403)

In my analysis using Ball's teachers' knowledge categories of the evidence collected in this research in the concept studies and post questionnaires, I looked at different ideas to identify the type of knowledge the participants developed. For the CCK, I looked whether they provided correct mathematical definitions, terms, and notations; the demonstrations of how they carried out procedures of the given tasks; and whether identified misconceptions in their collective discussion from colleagues. For the SCK, I looked for explanations and examples provided by the participants that are unique to the work of teaching. For the HCK, I looked for the explanations provided by the participants that indicate how the mathematical concept/topic in school curriculum relates to a concept/topic in college or university mathematics or how a college or university mathematics topic might relate to technical or professional mathematics used in a person's work. For the KCS, I looked for the explanations, examples, and questions provided by the participants that evidence on how students learn mathematics or the teachers anticipating the possible and confounding things by students. For the KCT, I looked for the explanations and examples provided by the participants on how the concept is learned specifically the different approaches and techniques that are affordable instructional in each of the mathematics concepts of ratio, proportions, rate and linear functions. For the KCC, I looked on the participants' awareness of how the realizations of each of the four mathematical concepts relate within the grade level and across the grade levels of the school curriculum. As well how the four mathematical concepts ratio, proportions, rate, and linear functions are related and their relationship with other concepts within grade level and across grade levels of the school curriculum, and the mathematical concepts they came up with as basic concepts before learning each of the four concepts.

The analysis revealed pre-service teachers' five categories of Balls' MKT: common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK) both under subject matter knowledge, and the knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) both under pedagogical content knowledge

(PCK). However, the HCK was much less observed compared to CCK, SCK, KCT, and KCC. The knowledge of content and student (KCS) under pedagogical content knowledge were more difficult to observe in the activities of the pre-service teachers except in the 'pedagogical problem solving' emphasis. The chapter is organised in five sections that each describe CCK, SCK, KCT, KCC and HCK consecutively.

7.1 Pre-service Teachers' Common Content Knowledge

Common Content Knowledge (CCK) is the domains of mathematical knowledge under subject matter knowledge category (Ball, Thames & Phelps, 2008) that is necessary for the teachers but not specialized; it is also used by other professionals such as engineering, mathematicians, chemists, physicist, biologist, etc. It is described as referring to "questions that typically would be answerable by others who know mathematics" (Ball et al. 2008, p. 399). For example, teachers need to be able to make simple representations, do calculations, know the definition of the mathematical concept, demonstrating how to carry out a procedure, assess whether the student provided correct answer or not, the accuracy of textbook definitions, and to use mathematical notation and terms correctly. What follows is the evidence of pre-service teachers' development of CCK revealed in the concept studies of ratio, proportion, rate, and linear functions respectively.

7.1.1 Ratio

The pre-service teacher participants demonstrated knowledge of representations of the concept by providing correct mathematical definitions of the ratio and using its symbolic form. In other words, both the language based and notation typology of representations (Bruner, 1996). Further, the participants gave correct examples that illustrated their understanding of the meaning of ratio. Language based definitions included: "*Ratio is the* *comparison of two or more things*" and *"Ratio is the comparison of two quantities"*. Notation based examples include *'a' ratio 'b' is a: b*. Examples such as the one below was also offered.

> For example, you can get ratio by dividing your monthly salary let say 230,000Tshs into food, clothes, and transport. You can decide to spend 100,000 for food, 80,000 for clothes and 50,000 for transport. In ratio, 100,000:80,000:50,000 as the ratio of food to clothes to transport.

In the group discussion some participants demonstrated misconceptions in the definitions they offered. For example, "ratio can be originating from mathematical way or method of writing numbers in numerator and denominator. Example, $\frac{a}{b} = a\colon b$ and $\frac{3}{5} = 3\colon 5$" This definition is inadequate and inappropriate since it could be problematic to use it as a definition in secondary schools leading to a situation where students may not be able to distinguish between the quotients and ratios. However, this misconception was cleared in the collective discussion of the ratio concept study session because the same participant who revealed this misconception evidenced to have the clear understandings of the ratio in the post questionnaires when responded to the question 'what he does know about the ratio concept'. As he responded as follows:

> Ratio in mathematics can be defined as a comparison of two or more things, can be length, area, width, figures and years of two people etc.
>
> Ratio in mathematics is the method of writing or comparing two or more given objects. It can be written as fraction, decimal or percentage.
>
> It is written in the form a: b when comparing two things and a: b: c when comparing three things and so on. It can also be written in fractions especially when comparing part to whole relationship
>
> The order of the ratio is very important. For example, the ratio of boys to girls in the class is 3:2 therefore the ratio of girls to boys will be 2:3.

When you talk about numerator and denominator you are dealing with both terms as numbers, but that is not the case for ratios. In ratios, context is central; it involves comparing two situations. For example, given the ratio of oranges to students is 1:2, it is important to know that the 1 concerns the oranges and the 2 concerns the students that can eat the oranges. But, in a quotient, the two values can be changed for one value, in this case $\frac{1}{2}$ or 0.5. At the same time when you talk about the numerator and denominator you are dealing with fraction as symbol $\frac{a}{b}$ where 'a' is a numerator and 'b' is a denominator and $b \neq 0$ or as a rational number (Lamon, 1999). A fraction is currently used two different ways: as a 'numeral' and as a 'number' (Lamon, 2012). According to Lamon (2012), as numerals "fractions are bipartite symbols, a certain form for writing the numbers: $\frac{a}{b}$" (p. 29) where 'a' is a numerator and 'b' a denominator and b≠0, and as numbers, "fractions are non-negative rational numbers" (p. 29). The use of ratio as a fraction needs to be taken very seriously in the sense that not every ratio can be written as a fraction. A ratio could have the second component as zero but, the fraction could never have the denominator as zero (Lamon, 2012). For example, if you report the ratio of girls to boys in a birthday party attended by 5 girls and no boys, you could write 5:0 but, you cannot write it as $\frac{5}{0}$. The pre-service teachers' awareness of the realizations of the ratio as a part to part and a part to whole comparisons in the concept study of the ratio (refer chapter 6, under section 6.1.1) is also an example of CCK.

7.1.2 Proportion

Similarly, for the proportion concept, participants provided examples of language based definitions such as: "Proportion is the mathematical statement that refers to two equal ratios." and "Proportion is the way in which one quantity increases with [an] increase in another quantity or decreases with [the] decrease in another". Their corresponding notation based examples for these definitions respectively included:

$$\frac{a}{b} = \frac{c}{d} \quad or \quad a{:}b = c{:}d \quad or \quad a{:}b :: c{:}d \quad \text{and}$$

$$a \propto b \rightarrow a = kb$$

$$\frac{a}{b} = k \ \therefore \ \frac{a_1}{b_1} = \frac{a_2}{b_2} \ or \ \ldots \frac{a_{n-1}}{b_{n-1}} = \frac{a_n}{b_n}.$$

and an iconic image based examples for the first definition:

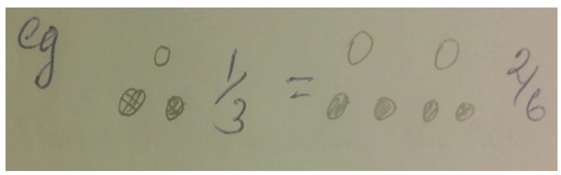


Figure 35: Participants' iconic based example of proportions

The participants were able to provide the correct mathematical definition of the proportion concept symbolically (language based, and notation based) and iconic based is one of the evidences of having CCK.

The proportion concept study elaborated the power of the concept study in correcting the pre-service teachers' misconceptions. The pre-service teachers' misconceptions claiming that "the decrease in one variable with the decrease in another variable is regarded as indirect proportional" might have originated from prior mathematics learning experiences in secondary schools because the concept elaborated the misconception is taught at Form III according to ordinary level secondary school Tanzanian mathematics curriculum. Misconceptions can originate from students' prior learning in their interaction with the world or in the classroom (Smith, diSessa, & Roschelle, 1993). Through discussing about the proportion concept, the pre-service teachers were able to correct the misconception of another concept—variation. Hence, enhancing pre-service teachers' conceptual understanding resulted in improving their professional knowledge and skills. Some misconceptions occurred in the collective discussion. One of the pre-service teacher's participants revealed having the misconception about the meaning of direct and indirect proportional. In his explanation, he used table 2 (see figure 36) as the reference, he explained the increase in one variable with an increase in another variable as the direct proportional but, decrease in one variable with a decrease in another variable as indirectly proportional. He was sure of his explanation until another group member challenged him about his conception of the indirect proportion.

2. In a group of 3-4, study the table 1, table 2, table 3, table 4 and table 5.

Table 1

Student	1	2	3	4	5	30
Apples	3	6	9			

Table 2

Student	1	30	15
Apples	3	90	

Table 5

14	35
18	

Table 3

Student	1	2	3	4	5	11
Apples	3	6	9	12		

Table 4

Student	1	2	3	4	14	9
Apples	3	6	9	12	42	

a. Find the missing number and explain how you found it?
b. How will you explain to your student the procedures in finding the missing numbers in each table 1 to 5 above?
c. What concept/s can be introduced by the use of these tables?
d. Write all ratios obtained from each table
e. Write any two pairs of equivalent ratios in each table above?
f. What does each pair of equivalent ratios in part 'e' represent?

Figure 36: Proportion concept study question 2

This raised a concern for the whole group when he insisted, he was correct. The other participants tried to use different examples for the direct and the indirect proportional to further elaborate; finally, he came to an understanding of the difference between the two. Here is part of the collective discussion in which the participants worked through the misconception:

> P1 (the first member of group 2): In table 2, when student increases the number of apples increases which is direct proportional and when the number of students increases the number of apples decreases which is indirect proportional.
>
> P2 (the second member of group 2): I think you confuse the two, you are talking about only direct proportional.
>
> P1 (the first member of group 2): I am not confusing the two, I am right.
>
> P3 (a third member of group 2): No, you are wrong.
>
> P1 (the first member of group 2): Why do you say that I am wrong. I know, I am right.
>
> P2 (a third member of group 2): Look here, when one variable increase with an increase in another or vice versa is direct proportional and when a variable increase with a decrease in another variable is indirect proportional.
>
> P1 (the first member of group 2): So, it means table 2 is only direct proportional.
>
> Facilitator: This is very interesting can we share this discussion with other groups. Your colleague here were discussing about direct and indirect proportional. How can we differentiate the two?
>
> P4 (the first member of group 3): What you need to know is that whenever one variable increase with an increase in another or vice versa is direct proportional, and whenever one variable increase with a decrease in the other is indirect proportional. For example, all tables represent direct proportional showing that if the number of students increases the number of apples increases and vice versa.
>
> P5 (the first member of group 1): And all direct proportional are straight lines while indirect are not. And we can get the constant of proportionality from these tables as the simplest ratio.
>
> P1 (the first member of group 2): Now, I got it. For direct proportional, increase goes with increase and decrease goes with decrease. While for indirect increase goes with a decrease and a decrease goes with an increase.
>
> P6 (the second member of group 3): Look here, the example of indirect proportional can be the number of people needed to cultivate a hectare of land. If the number of people increases the number of days needed to cultivate the land decreases but if the number of people decreases the number of days needed to finish the same hectare of land increases.

The collective concern about this misconception enabled enhancement of the participants' conceptual understanding of proportion, as well as variation, resulting in improving their professional knowledge and skills. The participants being able to identify this type of misconception is one of the aspects of teacher knowledge the teachers need as CCK. They need to be able to identify student' misconceptions and help to correct them: this was made possible through the concept study.

7.1.3 Rate

For the rate concept study session, some of the pre-service teachers provided correct language-based definitions accompanied with their respective correct examples that illustrate their understanding of the meaning of rate which is an important CCK the teachers need. They provided examples of rates and their corresponding unit rates figure 37.

Rate is the ratio between two quantities which have different units. Example, a person drives 60 miles in 2 hours. The rate is 30 miles per hour which is called his speed.

Rate is the value describing on how one quantity is related to another quantity of different units. For example, a man works 20 hours and paid Tsh 20,000, then the rate of payment will be

$$\frac{20{,}000\text{Tsh}}{20\text{ hours}} = 1000\text{Tsh}/_{\text{hour}}$$

Figure 37: Participant's examples of CCK for rate

Figure 38 is a task that involved decision making during shopping as an application of unit rates in everyday life. It is an example of CCK important for pre-service teacher participants in the learning of rate concept for helping their students understand it and build their interest in learning mathematics. As Laurens, F. A. Batlolona, Batlolona, and Leasa, (2018) assert "it is necessary for the teachers to develop more appropriate learning media, strategies, or model which are more suitable with learning materials or with the contexts that their students are dealing with" (p. 576). The question was discussed collectively using the whiteboard by some individuals who volunteered to demonstrate how they got the solution. The pre-service teachers being able to demonstrate the procedures in solving a particular problem is an important CCK they will need. The pre-service teacher participants who volunteered to demonstrate the solution of this question on the whiteboard both were able to explain properly and make the right decision however they differed in their working. The first demonstrator went straight to the cost of cooking oil per litre before comparing which is the better price while the second demonstrator first wrote the relationship between two quantities as ratios and then found the cost of cooking oil per litre as a unit cost rates before comparing which is the better price.

4. Which is a better price for Korie cooking oil: 29,000.00 Tanzanian shillings for 10litres or 54,000.00 Tanzanian Shillings for 20 litres? Explain how you obtained your answer

Figure 38: Rate concept study question 4

The two demonstrations from the volunteered participants are presented in figure 39 hereafter:

First demonstrator

i. $^{29,000\text{Tsh}}/_{10\text{ litres}}$

$$= {}^{2900\text{Tsh}}/_{\text{litre}}$$

ii. $^{54,000\text{Tsh}}/_{20\text{ litres}}$

$$= {}^{2700\text{Tsh}}/_{\text{litre}}$$

54,000Tsh/20litre is the better price for Korie cooking oil since it gives the lowest price per litre

Second demonstrator

I found the cost rate for each ratio

$$\text{Cost rate} = \frac{\text{Cost [of Korie cooking oil] in Tanzania shillings}}{\text{Total number of litres}}$$

10litres:29,000Tsh

$$\text{Cost rate} = {}^{29,000\text{Tsh}}/_{10\text{l}}$$

$$\text{Cost rate} = {}^{2900\text{Tsh}}/_{\text{l}}$$

Also, 20litres:54,000Tsh

$$\text{Cost rate} = {}^{54000\text{Tsh}}/_{20\text{l}}$$

$$\text{Cost rate} = {}^{2700\text{Tsh}}/_{\text{l}}$$

The better price of Korie cooking oil is 54,000Tshs because 1 litre cost 2,700Tsh, while for 29,000 1 litre cost 2,900Tsh

Figure 39: Participant's demonstrations CCK for unit rate

Though both demonstrators got the final answer right, the CCK they demonstrated differed in the sense that for the second demonstrator it is easier for the students to see the mathematics behind it as a rate. However, showing the rate is the ratio of two different quantities with different units which help the students to see the connection between the rate and ratio and the end product as a unit rate which help them to determine the best price per litre. Alternatively, it could be better for him to be specific of the cost he is talking about at the beginning. For the first demonstrator it could be hard for the students to see the mathematics behind it the students might see it as the division of two quantities.

7.1.4 Linear function

Similarly, in the linear function concept study session, the participants provided correct definitions mathematically in symbolic form with logical examples. The teachers being able to use the mathematical notation and terms correctly and define the mathematical concept is part of CCK the teachers need. In figure 40, all the three definitions represent the notation base (equation form), but the third definition represents the mapping form based as well. The participants also provided graphical and pictorial representations examples of linear function which will be described in the section 7.2.4 of KCT because the examples they offered satisfies for both CCK and KCT.

Linear function is the polynomial function whose degree of x in the equation $f(x) = mx + c$ is one.

A linear function can be defined as a function which consists of two variables in form $f(x) = ax + b$, where 'a' and 'b' are arbitrary constants with $a \neq 0$" represents notation (equation form) based.

A linear function $f: A \rightarrow B$ means that a function 'f' maps element of set A to element of set B. Where A is independent variables set and B is dependent variables set. Set B must satisfy $f(x) = mx + c$ and set A must be variables that are independent— the x, i.e. $f: x \rightarrow mx + c$.

Figure 40: Participants' examples of CCK for the linear function

The provided examples are illustrative of the important CCK the pre-service teachers need as MFT for ratio, proportion, rate, and linear function concepts. These examples would help in facilitating their future students' understanding. These examples were developed in the collective group work done in the concept studies where all participants were able to share and access the different definitions that motivate the deep understanding of these concepts.

7.1.5 Post questionnaires reflected CCK

After the concept study session for each concept of ratio, proportion, rate, and linear function, participants' responses about what they do know about the concept at hand helps to determine how the participants made sense of what they had learned in the concept study session. The pre-service teachers elaborated on their conceptual understanding by providing the meaning of the concept with examples, symbols, and diagrams. For example, they reported ratio to be dealing with the comparison of two or more things, and having the possibility of representing ratio as fractions, decimals, or percentages. Hereafter are some of the pre-service teacher participants' responses that were elaborated in their language and notation-based definitions as well in their iconic representations. All three examples in figure 41 give the notation-based definitions while the second and third include representation as fractions the participants provided at the conclusion of the session. These were much more elaborated CCK than any that were provided in the pre-session questionnaire.

Ratio is mathematics which we compare two or more things in relationship. For example, we can show part to part relationship ratios in this, in a bag there are 15 fruits among them 9 are oranges and 6 are bananas, then the ratio of oranges to bananas is 9:6

Ratio is the comparison between two or more things. Sometimes ratio can be written in fraction way. Example, a teacher has 10 books, out of those books 6 are mathematics books and 4 are history books. Then the ratio of the mathematics to history books is 6:4 and in fraction as

$$\frac{6\ \text{Mathematcs books}}{4\ \text{History books}}$$

Ratio is the mathematical way of comparing two or more quantities. They can be either of the same quantity or different quantity. It should be separated by semi colon between them. For example, a: b also can be written in fraction as $\frac{a}{b}$ where $b \neq 0$

Figure 41: Participants' post questionnaires CCK ratio examples

For example, in a pre-questionnaire a participant responded to the question of what he knows about ratio as:

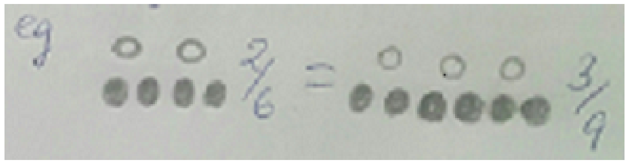
> Ratio in mathematics we can say is just a concept that is used in mathematics to compare two or more parts.

While in the post-questionnaires he responded as

> Ratio is mathematics that refers to the comparison of two or more things which related to generate one meaning. The comparison can be part to part or part to whole. It is very important to consider the order of the ratio. For example, the ratio of men to women in the village is 3:4, therefore the ratio of women to men will be 4:3.

Similarly, participants described the proportion to be equality of two ratios and having the possibility of representing it as two equal fractions. The first response figure 42 gives both the language based and iconic representation definitions while second gives the notation based. All are important CCK reflected by prospective teachers of mathematics in secondary school.

Proportion also is the relationship that exist between size, numbers or amount of two things



Proportion refers to as the mathematical statement that shows the equality between two ratios. Proportion can be expressed mainly in two ways namely by equal fraction such as $\frac{2}{3} = \frac{10}{15}$ also by use of colon example $2: 3 = 10: 15$

Figure 42: Participants' post questionnaires CCK proportion examples

The participants described the rate to be the ratio of two related quantities with different units, and all the rates are ratios but, not all ratios are the rates also noting that not all rates' denominators involve time: for example, *density, exchange rates, literacy rate, electric field,* etc. For example, if person working at the petrol station is paid 2000TShs an hour then the ratio $\frac{2000TShs}{1\ hour}$ is the rate because it applies whether a person works for 1 hour, 5 hours, or 11 hours, etc. But, if a person drives for 2 hours to visit a relative at a speed of 90 km per hour and drives back home at a speed of 60 km per hour the average speed $= \frac{360\ km}{5\ hours} =$ 72 km per hour. Seventy-two km per hour is a ratio but not a rate as it applies only to this particular situation (Lamon, 2012). Figure 43 are some of the pre-service teacher participants' responses that elaborated their notation-based definitions for the rate concept.

Rate is the ratio between two quantities of different units. For example, a man works 20 hours per week and being paid 40000Tsh, then the

$$\text{Rate of payment} = \frac{40000Tsh}{20hours}$$

$$= \frac{2000Tsh}{hour}$$

Then a man paid Tsh2000 per hour. Another example: A car travelled 200km in 2 hours then

$$\text{The speed of moving car} = \text{Rate}$$

$$= \frac{200km}{2hours}$$

$$= 100km/h$$

Therefore, a car travelled 100km per hour

The ratio of two related quantities of different units is called the rate. The rate can be determined by dividing one quantity by another. For example, a rate of pay consists the money paid divided by the time worked. If a man receives 1000Shillings for two hours work, his rate of pay is 1000Tshs ÷ 2hours = 500Shillings per hour.

Figure 43: Participants' post-questionnaires CCK rate examples

Similarly, the participants elaborated their understanding by providing the meaning of linear function with examples and diagrams. They gave its symbolic form (formula and mapping), and graphical representations. Figure 44 are some of the pre-service teacher participants' responses they provided as definitions of the linear function concept the symbolic form (formula and mapping) and its graphical representations (figure 45) that elaborate their important reflected CCK of the linear function the teachers need. All three definitions (figure 44) represent notation-based definitions.

Linear function is any function in the form $f(x) = ax + b$. It has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y or f(x), 'b' is the constant term or the y-intercept—the value of the dependent variable when x=0, 'a' is the coefficient of the independent variable which is constant—it is also known as the slope and give the rate of change of the independent variable. Also, it is a polynomial function with degree 1.

Linear function is the association of elements of set A to the elements of the other set B or...the association of independent variables to dependent variables. Thus, mathematically can be written as $f: A \rightarrow B$ where d is the function, A is independent variable and B is dependent variable. For a linear function dependent variable should be in form of linear such as $B = mx + c$ or $ax + b$.

By definition linear function is the function that relates two distinct things or relate two variables in the form of $y = ax + b$, 'a' is the ratio of change of the two variables $\frac{\Delta y}{\Delta x}$ which is called slope. For the linear function, the slope 'a' is constant, it is graph is a straight line...Example of the graphs [figure 45].

Figure 44: Participants' post questionnaires CCK linear function examples

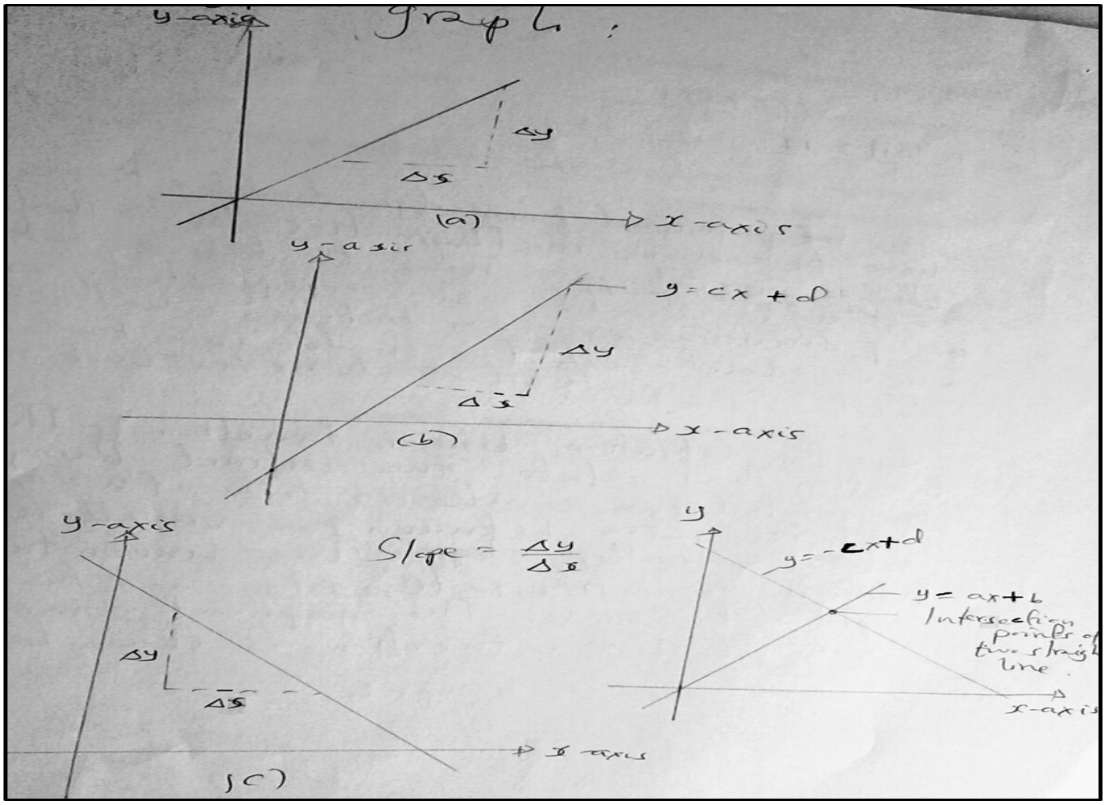


Figure 45: Participant's samples of linear graphs

7.2 Pre-service Teachers' Specialized Content Knowledge

The *Specialized Content Knowledge (SCK)* is the domain of the subject matter knowledge which is unique to teaching professionals and it is not used by other professionals (Ball, Thames & Phelps, 2008; Hill et al. 2008; Hill, Ball, & Schilling, 2008; Thames & Ball, 2010). It is the mathematical knowledge that teachers use in teaching that goes beyond the mathematics topics of the specific school curriculum itself. For example, to make sense of the solution provided by others independent of the particular students, teaching, or curriculum (Thames & Ball, 2010).

7.2.1 Ratio

In the ratio concept study session, the pre-service teacher participants provided examples elaborating the knowledge of the importance of the part to whole comparison context in ratios.

They elaborated the understanding of the part to whole comparisons as fractions and explained how it could be changed into the percentages. Changing the fraction into the percentage might be better understood as a CCK. However, it is important for teachers to know the type of examples he/she should select to demonstrate the part to whole comparison (figure 46) when teaching the ratios as SCK. As was asserted in the concept study,

> Also, we can express the ratio of amount of money spent on clothes to the total salary which is 80,000:230,000 into fraction as 80,000 over 230,000 or $\frac{80000}{230000}$. Also, we can also express this into percentage when we take 80,000 over 230,000 times 100%.

Figure 46: Participants' examples that demonstrate part to whole comparison

The participants provided examples that elaborate the knowledge of the part to part comparison context in ratios. They understood the parts as the subsets of the given set. Understanding the parts as the subsets of the given set is the knowledge for understanding the context to write ratios as important SCK the teachers need but might not be needed by other professionals. The discussion of context is an important aspect to be considered in the ratio concept, and there is no way one could do it without considering the part to whole comparisons and part to part comparisons ratios (Lamon, 2012). The participants demonstrated evidence of the knowledge about representing a given ratio in other ways without changing its meaning. They showed an understanding of writing the given ratio as a proportion using its multiple as well as a fraction of the part to whole comparisons. It is an important aspect of knowledge for the pre-service teachers to have for assessing their future student conceptions of the equivalent ratios as well as the ratio as a fraction. For example, their response of part c) of the second question (figure 47) in writing ratio in other ways without changing its meaning is as provided as

"$i)3:2 = 6:4$ $ii)2:3 = 6:9$ $iii)3:5 = 6:10$ $iv)2:5 = 8:20$. Can be written as equivalent ratios". That means 6:4 is the first multiple of the ratio 3:2 or $2(3:2) = 6:4$; 6:9 is the second multiple of the ratio 2:3 or $3(2:3) = 6:9$; 6:10 is the first multiple of the ratio 3:5 or $2(3:5) = 6:10$; and 8:20 is the fourth multiple of the ratio 2:5 or $4(2:5) = 8:20$. And "$iii)\ 3:5 \rightarrow \frac{3}{5}$ $iv)\ 2:5 \rightarrow \frac{2}{5}$. Can be written as fractions". That means 3:5 and 2:5 can be represented in fraction form as $\frac{3}{5}$ and $\frac{2}{5}$ respectively.

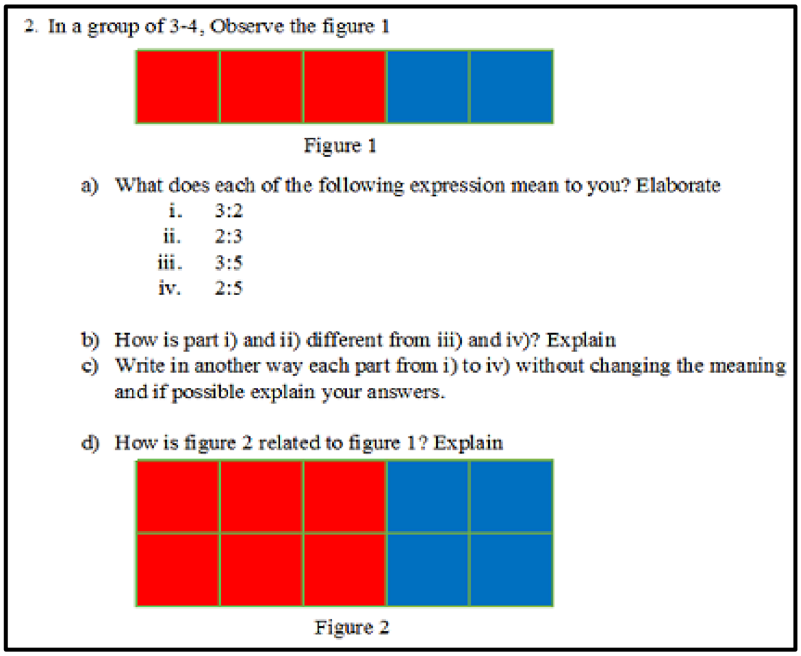

2. In a group of 3-4, Observe the figure 1

Figure 1

a) What does each of the following expression mean to you? Elaborate
   i. 3:2
   ii. 2:3
   iii. 3:5
   iv. 2:5

b) How is part i) and ii) different from iii) and iv)? Explain
c) Write in another way each part from i) to iv) without changing the meaning and if possible explain your answers.

d) How is figure 2 related to figure 1? Explain

Figure 2

Figure 47: Ratio concept study Question 2

Though writing this way $iii)\ 3:5 \rightarrow \frac{3}{5}$ and $iv)\ 2:5 \rightarrow \frac{2}{5}$, is right with the fact that the context of the unit whole is represented by the coloured squares in the figure; it is difficult to differentiate the conceptual understanding of ratios and the part to whole fractions (Lamon, 2012). So, with the idea of Lamon, I think for more clarity and to foster student conceptual understanding the pre-service teachers need to write the given ratios in fraction form as:

$$iii)\ 3:5 \rightarrow \frac{\text{3 red squares}}{\text{5 coloured squares}} \qquad \text{iv)}\ 2:5 \rightarrow \frac{\text{2 blue squares}}{\text{5 coloured squares}}$$

However, some of the participants demonstrated misconceptions in writing ratios as fractions. They represented the part to part comparison ratios of the coloured squares of the first diagram of question 2 (see figure 47) as part to whole fractions that deviates the real meaning in the given context. They wrote $i)\ 3:2 \rightarrow \frac{3}{2}$ and $ii)\ 2:3 \rightarrow \frac{2}{3}$. I think the participants might have seen a ratio rather than a fraction. The context of the question represented the comparison between red and blue coloured squares and the unit whole as five coloured squares in the given rectangle. The way they wrote the two ratios one might confuse it with the part to whole fractions (Lamon, 2012). To remove the confusion with the part to whole fractions, they could write as a fraction as:

$$\text{i)}\ 3:2 \rightarrow \frac{\text{3 red squares}}{\text{2 blue squares}} \qquad \text{and} \qquad \text{ii)}\ 2:3 = \frac{\text{2 blue squares}}{\text{3 red squares}}$$

7.2.2 Proportion

In the proportion concept study, some participants faced challenges in finding the missing values to fill table 5 (refer figure 34) but one member of group 3 elaborated the importance of teachers to have the mathematical knowledge that will help them to make sense of students' solutions. This type of knowledge that the teachers elaborate is the SCK. The teachers need to be able to make sense of these solutions and not rely only on what they know or aware. Thus, the participants built the explicit mathematical knowledge of the use of factor method in ratio tables, group 3 elaborated on how they found the missing number in table 5 (see figure 48) using the factors when the given ratio table that has only two columns and two rows. Group 3 explained 14 in the first row and 18 in the second row both from the first column are divided with their highest common factor which is 2, to get 7 and 9 respectively (figure 48). Then writing 7 and 9 on the left- and right-hand side of the table 5 (see figure 48) and take the 35 from the second column divide by 7 on the right-hand side to get 5 and write on both the top and the lower part of table 5 along the same column of the 35. Then multiply 9 by 5 on the lower part to get the 45. The participants' collective learning of the use of the factor method without mentioning anything about ratios reveals the role of concept study in helping pre-service teachers access their tacit knowledge and develop their MFT-the professional knowledge and skills. The fact that majority of the participants were thinking the only possibility is to use the equivalent ratio or proportion to get the missing number. This could be regarded as the development of the SCK the teachers need. Thames and Ball (2010) assert that teachers need to make sense of solutions provided by others whether student, textbook or other curriculum materials. However, somehow it could also be regarded as KCT if we considered the use of factor method as an instructional approach that could be used in proportion concept.

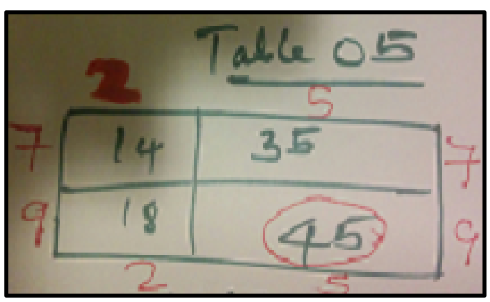


Figure 48: The use of factor method in ratio/proportion tables

7.2.3 Rate

In the rate concept study, the participants elaborated some evidence of development of SCK. This was explicitly revealed in the discussion of the second question (see figure 49). For example, looking at the different strategies as demonstrated by the participants, they were able to fill both tables correctly, however, the working papers from small group 2 show well connected conceptual understandings compared to the other groups. The participants appreciated this group arguing that they were able to provide their working procedures correctly, by showing the units of the rate quantities. This is an important aspect in the understanding of the rate concept; that is indicating exactly the units of the rate quantities related and showing which one is the numerator and the denominator, especially for the case of the unit rates. It is important to write the units used in rate because the pre-service teacher's need to know what it is and how to represent it in a given context.

2. Table 1 represent the tap water that fills the tank at constant rate and Table 2 represent the car traveling at constant speed.

Table 1

Litres of water filled in the tank	1	2	3	10	50	1000
Time taken to fill the tank in minutes	3	6				

Table 2

Distance travelled by a car in Kilometres	450	350	250	150	100	50
Time taken in hours	5					

In a group of 2 student teachers,

a. Find the missing number in table 1 and table 2 and explain how you found it?
b. How will you explain to your student the procedures in finding the missing number?
c. What concept/s can be introduced by the use of these tables?
d. Write all ratios obtained from the table 1
e. Write all ratios obtained from the table 2
f. What does the ratios in part 'd' represent?
g. What does the ratio in part 'e' represent?
h. What is the constant rate the tap is filling the tank?
i. What is the constant speed of the car?

Figure 49: Rate concept study Question 2

Figure 50 represents the examples from their working papers for part a) (refer figure 49), and some explanations that were offered when they were asked for some clarification.

Before finding the missing numbers, we found the rate first which is constant. In table 1, we were given litres of water filled the tank and the time. And in each column, we have been given both the litres and the time. So, we used the first and second column to find the rate which was constant, and we found that the rate constant is one third litres per minute

Table1

$$\text{Rate} = \frac{\text{Litres of water filled}}{\text{Time taken}}$$

$$\text{Rate constant} = \frac{1 litre}{3\ minutes}\ or = \frac{2\ litre}{6\ minutes} = \frac{1}{3} {}^{litres}/_{minute}$$

From the

$$\text{Rate constant} = \frac{1}{3} {}^{litres}/_{minute}$$

$$\text{Time} = \frac{\text{Litres of water}}{\text{Rate constant}} = \frac{3}{{}^{1}/_{3}} = 9 minutes$$

$$\text{Time} = \frac{10}{{}^{1}/_{3}} = 30 minutes$$

$$\text{Time} = \frac{50}{{}^{1}/_{3}} = 150 minutes$$

$$\text{Time} = \frac{1000}{{}^{1}/_{3}} = 3000 minutes$$

Similarly, we did the same procedure in table 2 by finding the rate constant which we used first column and got the rate constant or the speed of the car

$$\text{Rate constant/speed} = {}^{90km}/_{hr}$$

Table 2

$$\text{Rate} = \frac{\text{Distance travelled}}{\text{time taken}} = \frac{450km}{5hours}$$

$$\text{Rate constant/speed} = {}^{90km}/_{hr}$$

$$\text{Time} = \frac{\text{Distance travelled}}{\text{Rate constant}}$$

$$\text{Time} = \frac{350km}{90km/_{hr}} = \frac{35}{9}\,hours$$

$$\text{Time} = \frac{250km}{90km/_{hr}} = \frac{25}{9}\,hour$$

Figure 50: Participants' examples of SCK for the rate concept

The group discussion suggests these types of ratio tables helped the pre-service teachers in understanding how the rate concept is related to the ratio and the proportion concepts. Which in turn will help them facilitate their future student's understanding of how the three concepts are related. The discussion made it obvious that some pre-service teacher participants were not taking seriously the importance of the units in ratios that are the rates but parts f) and g) of question 2 (see figure 39) gave them the opportunity in identifying its importance. The participants believed specifying the units when writing ratios which are rates will make it easier for students in understanding the rate concept. The two parts of the question also, allowed the participants to identify the differences between the rates and unit rate when comparing the responses for parts f) and h) and that of parts g) and i) respectively. The discussions of two given ratio tables 1 and 2 and their corresponding parts questions is an important SCK the teachers need because it helps them to see the connection between rates and unit rate, and how the rate concept is related to the ratio and the proportion concepts. Therefore, I conclude the pre-service teachers developed explicit MFT—the knowledge and skills for the rate concept in the concept study.

7.2.4 Linear function

In the linear function concept study, the discussion of question 2 (see figure 51) that deals with the rate of change, unit rate, and slope/gradient of linear function graphs the SCK was also revealed. SCK was revealed in the whole group discussion for part e), f), h), and *i*).

2. Table 1 represent the recorded cost of different litres of petrol consumed by 7 different cars in one of the petrol station in Dar es Salaam.

Table 1

Cost of Petrol in Tanzanian Shillings (TShs)	2000	4000	8000				
Number of litres filled the cars	1	2	4	8	12	15	20

a) What is the constant rate of change/cost rate per litre of petrol? How did you find it?
b) Fill the table
c) Draw the graph of table 1
d) What type of graph did you get?
e) What is the rise of the graph for 1 litre increase of petrol?
f) Explain different ways of how to get the cost of
   i. 15 litres of petrol?
   ii. 3 litres of petrol?
g) Write the function corresponding to table 1 and define your variables
h) How is part a) and part e) related?
i) From your knowledge of linear function, what is the slope of the graph and how is it related to part a) and part e)?

Figure 51: The linear function concept study question 2

In part e) the groups were able to find the rise of the graph for one litre increase in petrol correctly. For example, I asked group 1 "How did you get the rise for one litre increase in petrol?" One of the participants from group 1 responded, "We found 1 litre on the horizontal axis and draw the dotted line vertically parallel to the vertical axis to meet the graph and draw the dotted line parallel to the horizontal axis to meet the vertical axis to get the rise*"*. The participants' responses for part e) indicates the pre-service teachers' conceptual understanding of the meaning of the slope/gradient of the linear functions that involve rates. This is SCK that the pre-service teachers need to know how: that is how to get the gradient of the linear functions that involve rates by interpreting the meaning of the rise and run of the graph. For part f), the groups demonstrated their understanding of the relationship between the slope/gradient of the graph of the linear function with the rate of change. For example, the groups were able to illustrate how to use the knowledge of the relationship between the unit rate and gradient of the linear function and the use of the graph in finding the cost of certain litres of petrol. This is an important SCK the teachers need because it helps them to see the connection between the ratio, proportion, rate, and linear function concepts that could help their future students to use the 'proportional reasoning' by considering a ratio as a multiplicative comparison of two quantities (Lobato & Ellis, 2010). For example, the pre-service teacher's graph of part c) figure 52 represents infinitely many pairs of the number of litres and the cost of petrol which express the same cost 2000Tsh per litre. The participants might conceive these infinitely many pairs of numbers as ordered pairs or in turn as ratios. They can form ratios as multiplicative comparisons by considering how many times greater each vertical axis value (the cost of petrol in Tanzanian shillings) is than the corresponding horizontal axis value (the number of litres) (Lobato & Ellis, 2010).

For parts h) and i), all groups were able to identify that part a), part e) and the gradient of the linear graph is the same thing but, it is ways of describing the rate of change. Thus, understanding the connection existing between the rate of change, unit rate, and the gradient of the linear function, which is an important SCK the pre-service teachers need to have. This is the knowledge that the other professionals might not need but it is specific to the teachers as they need to see and understands the connections so, that they can help the students to understand the meaning of the gradient/slope in the real world outside the school environment. Figure 52 represents the working of all the parts for one of the groups that demonstrated the participants' SCK.

a)

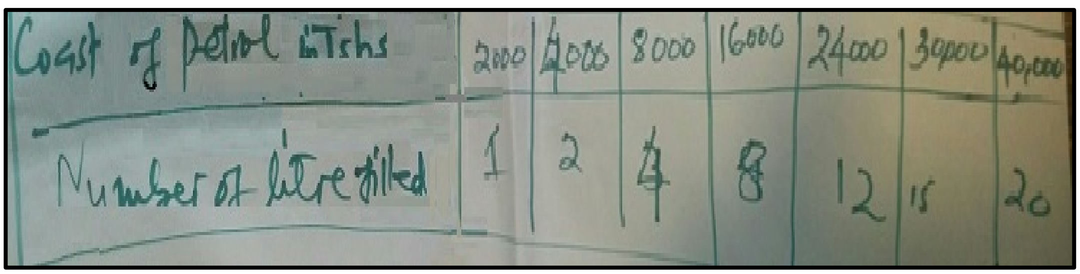
$$\text{Constant rate of change or Cost rate of petrol per litre } = \frac{Cost\ of\ petrol\ in\ Tsh}{Number\ of\ litres}$$

$$\text{Constant rate of change } = \frac{2000Tsh}{1litre} = \frac{4000Tsh}{2litres} = \frac{8000Tsh}{4litres}$$

$$= \frac{2000Tsh}{1litre}$$

$$\therefore \text{ Constant rate of change} = 2000Tsh/L$$

b)



Coast of Petrol in Tshs	2000	4000	8000	16000	24000	30000	40,000
Number of litre filled	1	2	4	8	12	15	20

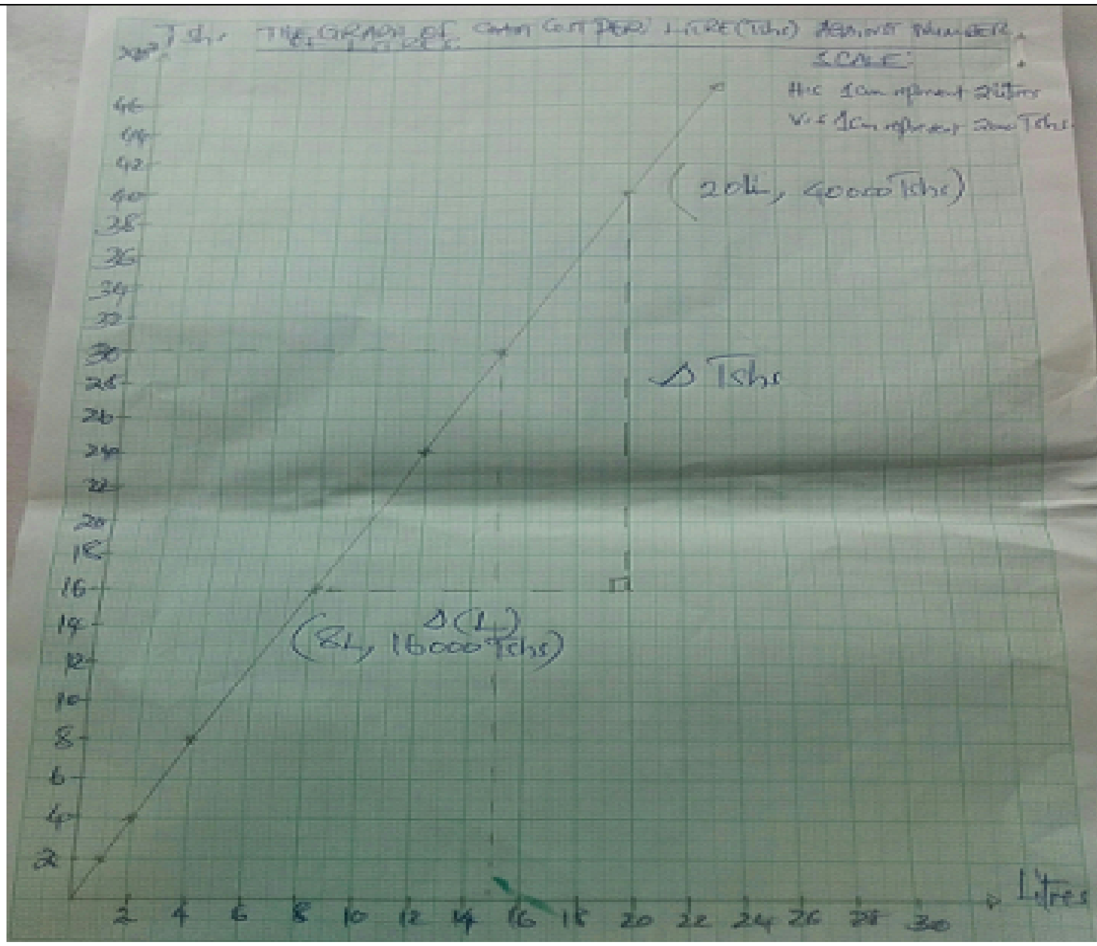
c) 

Figure 52: Participants' response to question 2 part c) (refer figure 40)

d) It is straight line graph

e) The rise for one litre increase of petrol is 2000Tsh

f)(i) The cost of 15 litres can be obtained by:

(a) From constant rate =2000Tsh/L

$$\text{For the cost of 15litres} = \frac{2000\text{Tsh}}{\text{L}} \times 15\text{L}$$
$$= 30{,}000\text{Tsh}$$

(b) From the graph

Draw vertical straight line to the graph from 15 litres, then horizontal straight line is drawn to meet the drawn vertical axis which corresponds to 30,000Tsh

(ii) The cost of 3 litres can be obtained by:

(a) From constant rate =2000Tsh/L

$$\text{For the cost of 3litres} = \frac{2000\text{Tsh}}{\text{L}} \times 3\text{L}$$
$$= 6{,}000\text{Tsh}$$

(b) From the graph

Draw vertical straight line to the graph from 3 litres, then horizontal straight line is drawn to meet the drawn vertical axis which corresponds to 6,000Tsh

g) Let L be the number of litres

$\text{f(L)} = 2000\text{L}$, where L is independent variable and f (L) is dependent variable

h) Part (a) and (e) related in such a way that one litre costs 2000Tsh.

i) From the graph

$$\text{The slope of the graph (m)} = \frac{\Delta y}{\Delta x} = \frac{\Delta \text{Tsh}}{\Delta \text{L}}$$
$$\text{Slope} = \frac{40{,}000\text{Tsh} - 16{,}000\text{Tsh}}{20\text{L} - 8\text{L}}$$
$$\text{Slope} = \frac{24{,}000\text{Tsh}}{12\text{L}}$$
$$\text{Slope} = 2000\text{Tsh/L}$$

Part (a) and (e) related in such a way that the cost of one litre is 2000Tsh, which is the same as the slope of the graph.

Figure 53: Participants' demonstrated SCK for Rate concept

The work demonstrated on figure 53 is the knowledge specific to the work to the teaching professionals as SCK. This is SCK to the extent that there are multiple representations. A person working on this as CCK would not need the multiple representations. Only one to solve the problem. But the teacher needs many. The SCK was revealed in the analysis of each of the four concept studies and post-questionnaires of the responses of the questions. It is what they do know and how they learn each of the four mathematics concepts.

7.3 Pre-service Teachers' Knowledge of Content and Teaching

The *Knowledge of Content and Teaching (KCT)* is the domain of pedagogical content knowledge (PCK) that "combines knowing about teaching and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). For example, though the concept study is dealing with meaning making and not teaching, pre-service teachers need to understand how mathematical concept could be learned, how to choose which example to start with for deep content understanding and identify different approaches and techniques that are affordable instructional for the specific mathematical concept. Apart from that teachers need to have multiple ways to solve mathematical problems because they need to expose students to the different methods in solving the mathematical problems. Further, the teachers need to know how to find the logical implications of realizations of mathematical concept. This is important KCT because it will help them to build logical explanations of mathematical concept in responding to the students' why and how questions. Thus, the logical implications the pre-service teachers determined in each of the four concepts of ratio, proportion, rate, and linear function as entailment activities under section 5.3 is the part of the evidence of developed KCT. What follows is evidence of pre-service teachers' development of KCT apart from logical implications of realizations of each of the mathematical concepts that were revealed in the concept study of ratio, proportion, rate, and linear function respectively. As well analysis of the post-questionnaires when responding to the question of how each of these mathematics concepts could be learned respectively.

7.3.1 Ratio

For the ratio concept study, there is evidence of pre-service teachers' development of KCT as they elaborated awareness of multiple ways to solve mathematical problems because they need to expose their future students to the different methods in solving the mathematical problems and as well assess and understands the different ways the students approached the specific problem. For example, the pre-service teacher participants demonstrated an understanding of how to solve a word problem of ratios relating three things figure 54 by using an idea of equivalent ratios. However, some of the participants used the ratios between the same quantities while the others use different quantities.

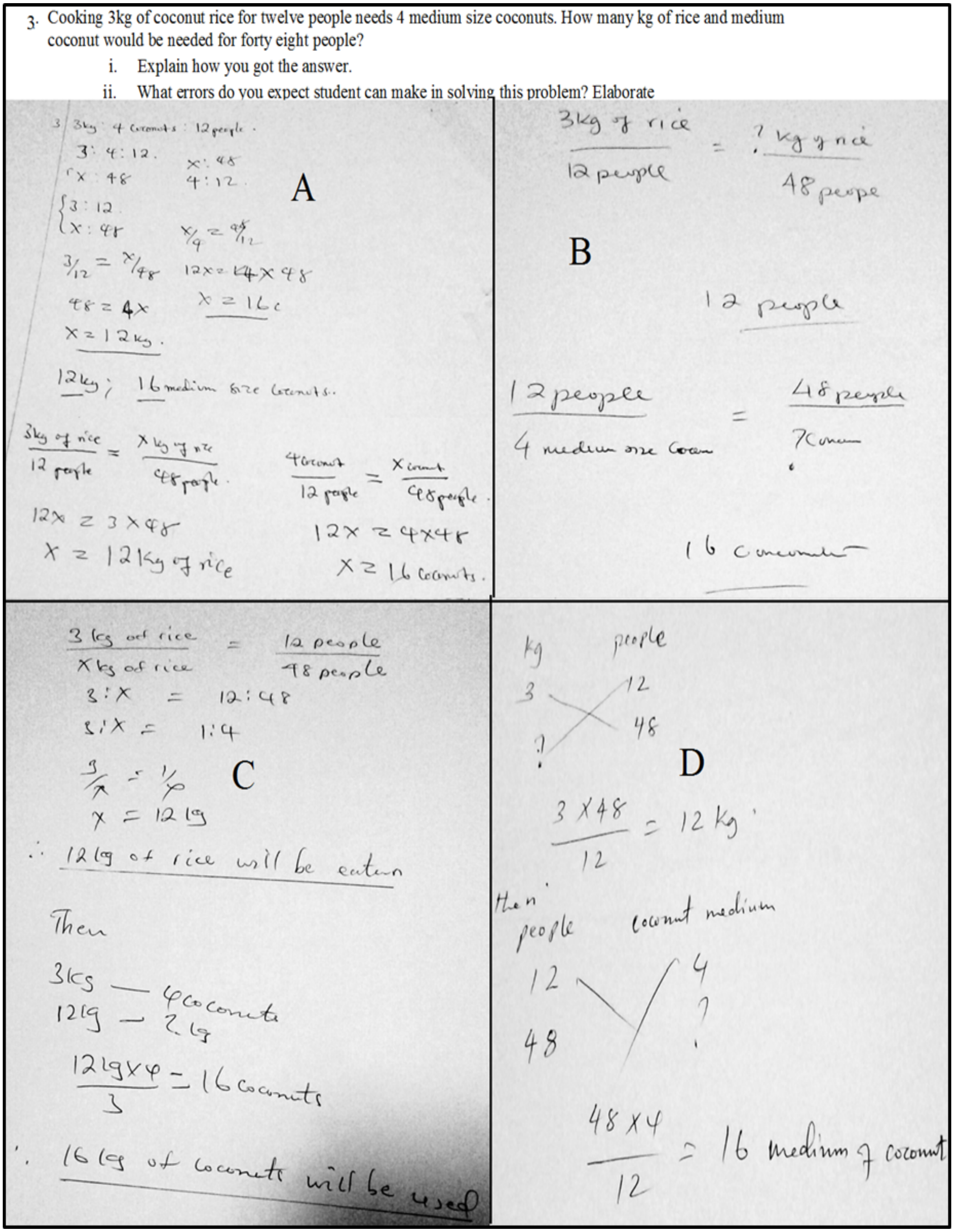


Figure 54: Ratio concept study Question 3 and demonstrated sample solutions

From figure 54, the demonstrated work ‘A’ and ’B’ used the idea of the equivalent ratios, taking the ratios of the different quantities/objects. The demonstrated work ‘C’ used the idea of the equivalent ratios taking the ratios between the same quantities/objects. However, he mixed with the cross multiplication. The demonstrated work ‘D’ used cross multiplication. These few illustrated working examples allowed the participants to learn various ways of solving the word problem involving ratios of three quantities as important KCT with the fact that they shared it in the whole group.

Figure 55 is an application of a word problem in everyday life involving ratios that are the rates. In general, the participants collectively demonstrated how to solve the word problem of ratios relating two things/quantities (figure 55) using the idea of equivalent ratios and unit rate. The whole group discussion demonstrated different ways that the pre-service teacher participants used equivalent ratios and unit rates in solving, as well as the cross-multiplication methods that have shown participants better understanding of the ratio concept. Most of the participants solved the problem using the knowledge of equivalent ratios, taking the ratios of the different quantities. The demonstrated work ‘F’ and ‘E’ used the idea of unit rate, finding the unit cost rate of sweet potatoes — the cost of 1 kg of sweet potatoes first and then used it to find the cost for 16kg. The use of the unit cost rate gives the learners opportunities of having a clear understanding of the context of the given ratio. The demonstrated work ‘G’ and ‘H’ used the idea of equivalent ratio. They use the ratios of two different quantities, the kg of sweet potatoes to the cost in TShs. See figure 55 for more details of the demonstrated working of participants. The few demonstrated works of the participants in figure 55 which shows various methods to solve the word problem involving ratios that are the rates is the KCT that participants shared. Understanding the different ways on how to solve these types of the word problems is an important KCT for pre-service teachers.

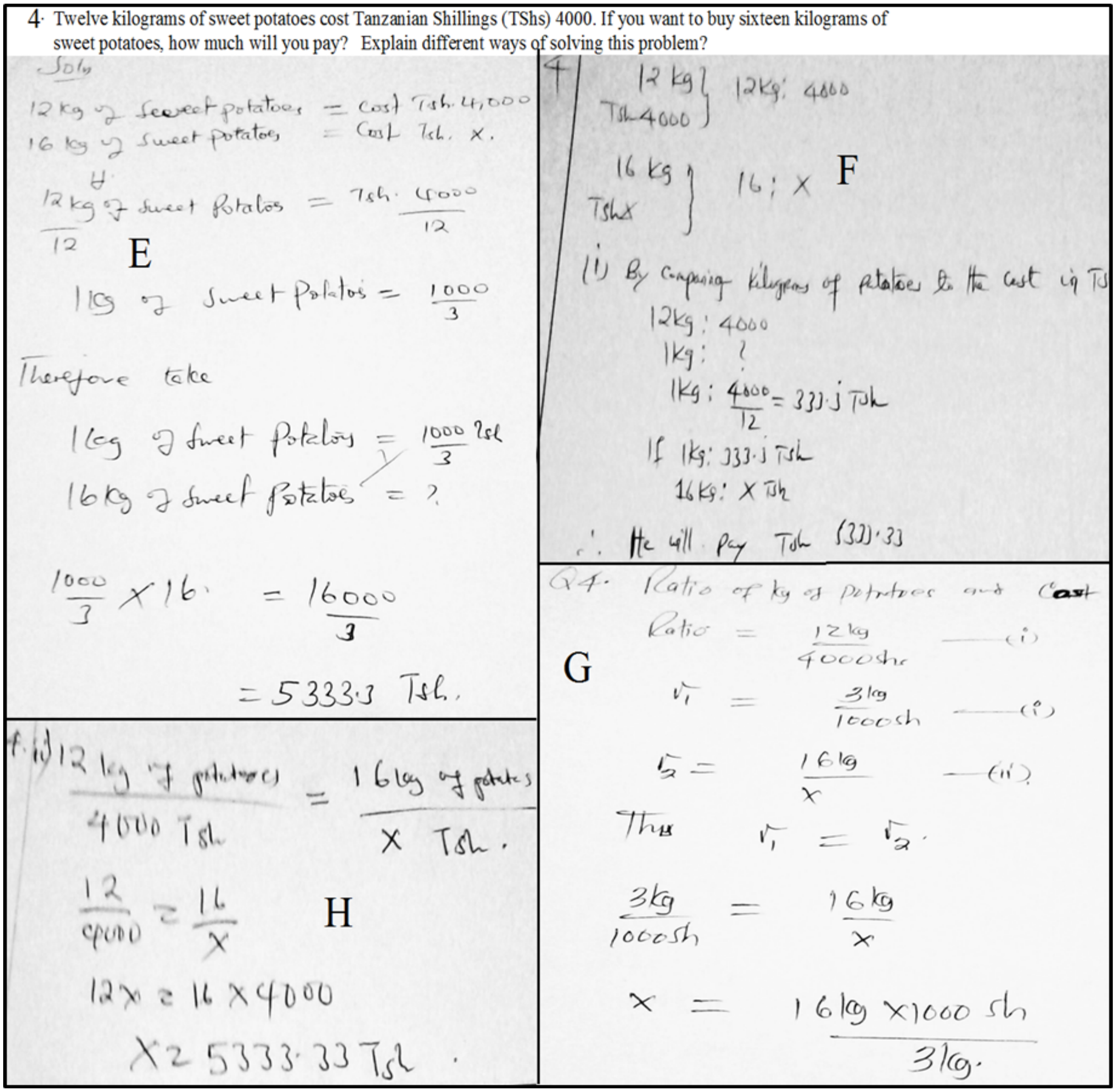


Figure 55: Ratio concept study Question 4 and demonstrated sample solutions

7.3.2 Proportion

For the proportion concept study, there is also evidence of pre-service teachers' development of KCT. This was explicitly revealed in the discussion of the second question (see figure 35). For example, the participants elaborated three ways in completing the ratio or proportion table. They used equivalent ratios, proportional constant, and factor method. Participants' awareness of different methods in solving a particular mathematical problem is an important KCT the teachers' need that could make them assess and understands the different ways the students approached the specific problem. Here is one of the explanations offered that elaborates the use of proportional constant (figure 56).

> First, we considered the relationship between the number of students and the number of apples. We realise that as we increase the number of students the number of apple increases. 1 student need 3 apples, 2 students need 6 apples. We realise that in all relationships we can get the first ratio. The number of students varies direct proportional to the number of apples
>
> $\#\text{Student} \propto \#\text{apples}$
>
> $\#\text{Student} = \text{k}\ \#\text{apples}$
>
> $\text{k} = \frac{\#\text{Student}}{\#\text{apples}} \quad \therefore \text{k} = \frac{1}{3}$
>
> 4^{th} column of table 1, $\frac{1}{3} = \frac{4}{\text{Number of apples}}$. Therefore, the number of apples = 12.

Figure 56: Participants' examples of KCT for proportion

In the proportion concept study, the pre-service teachers were also asked *"With examples, explain how you can facilitate the students' learning of the applications of the proportion concept outside school environment or real-life situations."* The participants' elaborated different ways they could facilitate their future students learning of the applications of the proportion concept outside the school environment. The participants explained the use of map making by considering the map scale and ground distance, the use of the cooking activities that show the material used is proportional to each other, the use of rate of cooling according to Newton's law of cooling. They also, explained the use of verification of Ohm's law, the use of the distribution of fruits to the group of students, and the use of the balancing of nutrition in facilitating the student learning applications of the proportion concept outside the school environment or real-life situations. Identifying the different approaches and techniques that are affordable instructional for the proportion concept is part of the KCT the teachers need. Thus, these examples are evidence of pre-service teacher participants' collective development of explicit MFT—the professional knowledge and skills as KCT of the proportion concept applications outside the school environment which, will help them in facilitating student learning of mathematics by relating with its application in everyday life.

7.3.3 Rate

Being able to use different methods in solving the mathematical problem is important professional knowledge and skills the pre-service teachers need to have as the KCT. In the rate concept study, the participants elaborated evidence of pre-service teachers' development of KCT in the discussion of the third question (see figure 57). After an appropriate time for discussion of question 3 (see figure 57) in small groups, each group was then given an opportunity to present what they had been discussing in whole group discussion. The collective discussion demonstrated different ways that the pre-service teacher participants used in solving the word problem with rate and proportion concepts. All participants solved this question by using the knowledge of unit rate, however some of the participants used proportions that are the rates. The participants indicated they prefer using both ways introducing the concept of rate to the ordinary level secondary school students. Their explanations given were the use of unit rate gives the students the opportunities in learning and understanding the rate concept, while the use of proportions gives the students the opportunities in learning how the rate concept is related to proportion concept.

3. Four kilograms of onions cost Tanzanian Shillings (TSH) 5000.00. If you want to buy twenty seven kilograms of onions, how much will you pay?

a) Explain different ways of solving this problem?

b) Which way in part a) do you prefer in introducing the concept of rate to an ordinary level secondary school student and why?

Figure 57: Rate concept study question 3

Thus, pre-service teacher participants revealed development of KCT also as shown on the demonstrated workings and explanations for two groups (figure 58).

Demonstration 1

In our discussion we found there are two ways in solving the problem. The first way is the one we used to find the unit rate which we obtained by taking the ratio of the cost to kilogram of onions which is 5000Tshs divided by 4 kg of onions. Rate will be 5000Tshs/4kg which gave us a cost rate of 1250Tshs per kg. Then we multiplied the unit rate 1250Tshs per kg by the 27 kg to get its cost.

$$Rate = \frac{Cost\ in\ Tshs}{Kilogram\ of\ onions} = \frac{5000Tshs}{4kg}$$

$$Cost\ rate = \frac{1250Tshs}{1kg}$$

$$Cost\ in\ Tshs = Cost\ rate \times Kilograms\ of\ onions$$

$$= \frac{1250Tsh}{kg} \times 27kg$$

$$= 33{,}750Tsh$$

Demonstration 2

We used the concept of proportion. We know we have two ratios.

4kg of Onions: 5000Tsh and 27kg of Onion: ? Tsh. When comparing the two equal ratios we can get the unknown cost in Tanzanian shillings.

$$\frac{4kg}{5000Tsh} = \frac{27kg}{x\ Tsh}$$

$$x = \frac{27kg \times 5000Tsh}{4kg}$$

$$x = 33750\ Tsh$$

Figure 58: Participants' demonstrated workings and explanations as KCT for the rate concept

Similarly, in the rate concept study, the pre-service teachers were asked *"With examples, explain how you can facilitate student learning of the applications of the rate concept outside school environment or everyday life situations."* The pre-service teachers provided some examples they could use such as finding the speed, payee rate, rate of liquid flow, cost rate, and exchange rates. The participants provided examples such as:

> By relating the distance covered by student from home to school and the time taken and ask them to find their speed
>
> The rate of payee. For example, if a man is paid 1000Tsh do a work for 2 hours. The rate of pay is 500Tsh/hour
>
> For example, the experiment showing the volume of water flowing through the pipe per second
>
> Purchasing goods. E.g. given the cost in Tanzania shillings for a certain number of kilogram of sugars bought and asking the student to find the cost per one kg of sugar in Tanzanian shillings.
>
> Using the rate of exchange, example the rate of exchange for 1US$=2000Tsh
>
> We can use rate in purchasing equipment. For example, you want to purchase a tractor. Normally tractors are labelled according to their power which is the rate of doing work. The one with greater power is the one which can do a lot of work.

These examples are evidence of pre-service teacher participants' collective development of explicit MFT—the professional knowledge and skills as KCT of the rate concept applications outside the school environment which will help them in facilitating student learning of mathematics by relating with its application in everyday life.

7.3.4 Linear function

The choice of which examples to start with for deep content understanding is one of important KCT the teachers need. For the linear function concept study, KCT that elaborate deep content understanding was revealed in different circumstances. For example, the participants offered explanations such as “It is called linear function simply because the graphical representation of these functions are straight lines.” And they gave logical examples such as:

> Consider the graph [refer figure 16] of a linear function: $f(x) = 3x + 6$ which we drew with the use of ‘x’ and ‘y’-intercepts. The x-intercept was obtained by substituting the value of ‘y’ as zero in the equation $y = 3x + 6$ while for y-intercept we used the value of x as zero in the same equation. We located the two coordinate points and then join them with a ruler to form a straight line.

Also, the participants gave an example that represent three forms of the same linear function, in equation form, pictorial form, and as well in graphical form such as “...$f(x) = 2x + 2$, provided that {x=0,1,2,3,and 4}, then, f(x)={2,4,6,8,and 10} the function can be shown pictorially [refer figure 17]. And (0, 2), (1, 4), (2, 6), (3, 8), and (4, 10) as ordered pairs. Again, can be shown on the graph [refer figure 18]”. At the same time, they were able to give graphical examples of the linear function that illustrates the behaviour of the graphs with positive and negative slopes (refer figure 19). These examples that represent three forms of the same linear function: in equation form, pictorial form, and as well in graphical form revealed their KCT because they help in in deep understanding of the content about linear function.

The KCT in this concept study also was revealed in the scenario that happened after one of the participants uttered that the linear function is a polynomial function with degree not exceeding one. The rest of the participants raised the concern that the use of statement ‘not exceeding one’ instructionally might bring confusion to the students in understanding the linear function concept. This made me think twice about what they came up as a linear function and a constant function. That is $f(x) = ax + b$ is a linear function if $a \neq 0$ and 'a' and 'b' are arbitrary constants. While when a=0 or the exponent of x is zero we get a constant function or a polynomial of degree zero $f(x) = k$ where k is a constant. In other words, considering a linear function as a polynomial function of degree 1 and a constant function as a polynomial function of degree 0. As an experienced mathematics teacher and as a researcher still I have a concern which is bothering me that when $b \neq 0$ for the expression $f(x) = ax + b$ do not agree with linearity property of linear mappings when dealing with linear transformations. That means, one can prove that $f(x) = ax + b,$ for either a $\neq 0$ and $b = 0$ or both $a, b = 0, x \in \mathbb{R}$ agrees with the linearity property of linear mappings in linear transformation while $f(x) = ax + b,$ for both $a, b \neq 0\ , x \in \mathbb{R}$ do not agree with linearity property of linear mappings in linear transformation. For example, a function $f(x) = ax + b, for\ all\ x \in \mathbb{R}$ is a linear mapping if $f(cx) = cf(x)\ for\ all\ c \in \mathbb{R}$. Given $a, b \in \mathbb{R} \neq 0,$ for example a=2 and b=1, then $f(x) = 2x + 1, for\ all\ \ x \in \mathbb{R}$.

If $x = 3$ and $c = 2$,

$$\text{Using } f(x) = 2x + 1, \text{for all } \text{x} \in \text{R}, \text{when x} = 3$$

$$f(3) = 2\ (3) + 1$$

$$f(3) = 7 \text{..........................(1)}$$

$$\text{Using } f(cx) = cf(x)\ for\ all\ c \in \mathbb{R}, when\ x = 3\ and\ c = 2$$

$$f(2\times3) = 2f(3)$$

$$f(6) = 2f(3) = 2\times7 = 14 \text{......................(2)}$$

But using $f(x) = 2x + 1$

$$f(6) = 2(6) + 1 = 12 + 1 = 13 \text{....................(3)}$$

Thus, $f(6) = f(2\times3) \neq 2f(3)$ so, $f(x) = ax + b,\ a, b \in \mathbb{R} \neq 0, x \in \mathbb{R}$ does not obey the linearity property of linear mappings $f(cx) = cf(x), for\ all\ x \in \mathbb{R}$ in linear transformation. But, given a ≠0 and b=0 then $f(x) = 2x$

If x=3, $f(3) = 6, but\ f(2\times3) = 2f(3) = 2\times6 = 12 = f(6)$.

Thus, $f(6) = f(2\times3) = 2f(3)$ so, $f(x) = ax + b, a \neq 0, b = 0, x \in \mathbb{R}$ obeys the linearity property of linear mappings $f(cx) = cf(x), for\ all\ x \in \mathbb{R}$ in linear transformation. But, $f(x) = 0, for\ a, b = 0, x \in \mathbb{R}$

If x=3, $f(3) = 0, but\ f(2\times3) = 2f(3) = 2\times0 = 0 = f(6)$. Thus, $f(x) = 0, x \in \mathbb{R}$ obeys the linearity property of linear mappings $f(cx) = cf(x), for\ all\ x \in \mathbb{R}$ in linear transformation. Therefore, the linear function $f(x) = ax + b, for\ all\ x \in \mathbb{R}$ not agreeing with linearity property of linear mappings of linear transformation remain a challenge to me. The linear transformation is the landscape of the linear functions with the blending that requires a linear function to agree with the linearity property. In this case the only expression of linear function that agrees with both having linear graph and agree with property of linearity is $f(x) = ax,\ a \neq 0, x \in R$.

In the whole group discussion of the question that asked them to write the function corresponding to the given ratio table (which they were given some data and asked to fill the rest) and define the variable used, the KCT was revealed. One of the techniques or approaches in making affordable instructions of the linear function to the students is the modelling tasks. Thus, the pre-service teachers need to be able to formulate or model the linear function from particular mathematical problems. In part g) the groups have shown their understanding in formulating or modelling the linear function from the given table. Modelling the given information to linear function is another important KCT for pre-service teachers to help develop the conceptual understanding of their future students. For example, the whole group discussion benefitted all pre-service teacher participants with the fact that group1 was able to describe the variables used and the function while group 2 described the gradient and group 3 described which, one is the dependent and independent variables in the given function.

The KCT was also made evident in the group discussion for the question 3 (figure 59) where the participants were able to draw the graph of the tables according to the given domain the second rows of the tables and range as the first rows of the table for the given functions. The pre-service teacher participants identified the concepts that could be introduced using the given tables as the proportion, rate, variations, and linear functions concepts. Also, in the course of discussion one participant uttered “it is very simple, now I have many alternatives in teaching the linear function, rate, and ratio” which demonstrates the KCT.

3. Table 2 represent the tap water that fills the tank at constant rate and Table 3 represent the car traveling at constant speed.

Table 2

Litres of water filled in the tank	1	2	3	10
Time taken to fill the tank in minutes	3	6		

Table 3

Distance travelled by a car in Kilometres (Km)	450	350	250	150	100	50
Time taken in hours	5					

a. Fill table 2 and table 3 and draw their graphs using different axes
b. What concept/s can be introduced by the use of these tables?
c. What is the constant rate the tap is filling the tank?
d. What is the constant speed of the car?
e. Compare the slopes/gradients of the graphs in part a) with the answers in part c) and part d)

Figure 59: Linear function concept study question 3

In the whole group discussion of question 4 (figure 60), questions iii) and iv) seem difficult for some participants but, as a collective group, the explanations of some helped others understand. Knowing how the graph of the linear function with positive and negative slopes behave is one of the instructional approaches that help students in understanding this concept. This was another important KCT pre-service teacher participants needed for building conceptual understanding of the linear function concept that will help them facilitate their future student in learning this concept.

4. From your knowledge of linear function f(x) = mx + c or linear equation y = mx + c
   i. What is 'm' and 'c'?
   ii. Fill in the blanks
      a) Numerically c is ....when x=0.
      b) Graphically c is ....of the graph.
      c) Numerically f(x) or y increases by ....units for every.......unit increase of...........
   iii. For positive ...., the graph rises .... units for every 1 unit of ...... moved to the right, and rises f(x) =..... for every .... units moved to the right
   iv. For negative ...., the graph drops |......| units for every 1 unit move of *x* to the right, and drops f(x) = |.....| *x* units for every ....units moved to the right.

Figure 60: Linear function concept study question 4

7.3.5 Post questionnaires reflected KCT

After the ratio session, the pre-service teacher participants elaborated their understanding of the ratio concept when responding to the question of how it could be learned. Being able to explain how the mathematical concept could be learned is essential KCT for prospective teachers of mathematics in secondary schools. They elaborated the use of the participatory methods, concrete objects, images, pictures, diagrams, figures, and examples related to the real-life applications in reinforcing the understanding of comparison of two or more things or objects. They also pointed to learning the ratio concept by relating with its other concepts or topics in mathematics and explained learning it by using the real examples that show comparing part to part or part to whole relationships. The consideration of the order of the ratio as an important element in learning the ratio concept. Hereafter are some of the pre-service teacher participants' responses that elaborated the reflected KCT for the ratio concept.

The concept of ratio can be learned through:

i. Its meaning and representations

ii. Use examples to show part to part ratios such as the ratio of number of boys to girls in the class or vice versa

iii. Use examples to show part to whole ratios such as dividing a certain amount of money to three people and show amount each one takes out of total amount of money

Ratio concept can be learned in different ways such as

i. Its meaning and use of examples, diagrams, and various ways in which a ratio can be represented. Part to part or part to whole ratios.

ii. Example, of part to part relationship by comparing number of girls to boys in class which is15 to 12 or 15 boys: 12 girls. This means to every 15 girls there are 12 boys.

The participants explained the use of ratio concept as the fundamental concept in learning the concept of proportion. They also elaborated the use of the participatory methods, concrete objects, drawings, figures, illustrations, and examples, related to the real-life applications in reinforcing the understanding of proportion as two equal ratios. Hereafter are some of the responses from the pre-service teacher participants that elaborated development of the KCT:

> I learn proportion through using the real examples which are used in the daily life. Example, if a cup of tea needs two spoons of sugar then four cups of tea will need 8 spoons of sugar. So, through such examples, it builds a knowledge concerned with proportion because you can show two equal ratios of cups to spoons of sugar.
>
> I can learn proportion concept through the concept of ratio. For example, when two pairs of ratios are equal, they express proportion. Example $2:4 = 8:16,\ 32:16 = 16:8$. Secondly, I can learn proportion using ratio tables to show equal ratios relationship of two quantities.
>
> In order to learn the proportion concept first, we should familiarize with the concept of ratio. Also, use examples that show proportions. For example, if two people use a quarter kilogram of maize flour in making ugali, four people will use a half kilogram, eight people one kilograms and so on. Then, it is a proportion.

Similarly, for the rate, the participants explained the use ratio as the basic concepts in learning the rate concept. However, some of the participants insisted on learners learning other concepts such as proportion, fraction, and percentage before engaging in learning rate. They also explained that it is useful for teachers and learners use of examples found in surrounding environments and for teachers to use illustrations and diagrams related to the real-life applications in reinforcing the understanding of the rate concept. For example, the participants suggested the use of examples such as calculating the speed and cost rate in buying commodities, finding the payee rates and exchange rates, etc. arguing that learning this way facilitates the learners' easy understanding of the concept. Hereafter are some of the pre-service teacher participants' responses that show more understanding of how the rate concept could be learned as the evidence of the development of the KCT:

> Good way of learning rate concept is by using real examples which is applied in our normal environments. For example, the payee rate, that is if a man receives 1000shillings for two hours work, his rate of pay is $\frac{1000\text{Tsh}}{2\text{hours}} = 500$ shillings per hour...including sharing different views given by individuals in a group so that we can come up with the real meaning of the given concept.
>
> First, we have to learn the meaning of rate. But, before studying concept of rate you should learn first some concepts like ratio, fraction, proportion, percentage. Also, you have to know the relationship between rate concept with other concept in mathematics and other subjects. Also, you should use the real examples of rate concept in daily life.
>
> A teacher, in advance, needs to prepare questions to guide the discussion and also provide the real-life examples on the concept of rate...also a teacher has to provide to the learners supporting illustrations and diagrams in order to facilitate the learning process and finally the teacher needs to wind up the study by looking on the collective learning.
>
> Rate is learned in our daily life. It is not only in the class. Example, I can learn the concept of rate in our environment surrounding us, for instance if a student walk from home to school he or she travel a certain distance at a given time so, in order to get the rate = distance per unit time taken. Also, I learn the rate concept when I want to buy certain commodities in the market, for example, 4kg of rice cost 20000Tsh then, find the cost of 1kg. 4kg:20000Tsh→1kg:5000Tsh then 1kg of rice cost 5000Tsh.

For the linear function, the participants elaborated more understanding on how it can be learned. They explained learning its meaning, representations, and its applications in daily life activities. Also, the teachers suggested using examples found in surroundings and the use of illustrations and diagrams related to the real-life applications which, could be modelled to the linear functions in reinforcing its understanding and solve numerical problems involving the concept. The participants argued that learning this way could help to facilitate the learners' easy understanding of the linear functions concept. However, some of the participants insisted the learners in learning the concepts such as the linear equations, ratio, rate, proportion, relations, and coordinate geometry as the basic concepts before engaging them in learning the linear function. The participants also insisted teachers should learn using concept studies, reading various textbooks and literature, and listening and observing video presentations concerning linear functions through the internets. Some of the pre-service teacher responses show more understanding of how the rate concept can be learned, the evidence of the development of the essential KCT:

> I can learn linear function concept by using related variables in real life. For example, given the cost of petrol per litre in Tanzanian shillings (TShs) one can find the total cost in relations to the number of litres filled in any petrol station. In this relation, the total costs in (TShs) is the dependent variable, and the number of litres is the independent variable. So, using the real-life examples which, is applicable in our daily life one can easily learn the linear function and get the real meaning.
>
> I learn the linear function concept through learning its meaning, its representation, and the use of examples of how it is applicable in our daily life...Also, the linear function should be compared with the other topic in mathematics, and with other subjects.
>
> The concept of linear function is learned by combining with the knowledge of other concepts such as the linear equation, ratio, rate, proportion, relation, and coordinate geometry that help an individual to transform that knowledge to the linear function.
>
> Linear function can be learnt in various ways including the following
>
> (i) Through concept study with colleague mathematics teachers before going to teach the learners /students.
>
> (ii) Through reading various literatures about linear functions documented by various experts.

(iii) By using the internet access whereby knowledge and skills can be gained through watching online video presentations, written documents example PDF that provide explanation in various mathematics concepts.

Therefore, KCT the pre-service teachers revealed in each of the four concept studies and the post-questionnaires as PCK that deals with *knowing about teaching and knowing about mathematics* (Ball, Thames, & Phelps, 2008). Such as the need of pre-service teachers to understand how mathematical concepts could be learned, how to choose which examples to start with for deep content understanding and identify different approaches and techniques that are instructional affordable for the specific mathematical concept. There is need of the pre-service teachers to have multiple ways to solve mathematical problems because they need to expose their future prospective students to the different methods in solving the mathematical problems. There is also a need of knowing the logical implications of realizations of each of the mathematical concepts in the concept studies as entailments activities they did as described in section 6.3.

7.4 Pre-service Teachers' Knowledge of Content and Curriculum

The *knowledge of content and curriculum* (KCC) is one of the domains of the pedagogical content knowledge (PCK). It is represented by an understanding of "the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p. 10). For example, the knowledge available in instructional materials such as curriculum, syllabus, text books, reference books, teacher's guides etc. Also, the collective interactions of organizing the realizations of the mathematical concept to find how they relate within the grade level and across the grade levels referred as landscapes (Davis & Renert, 2014) is another knowledge of content and curriculum the pre-service teachers need. The landscapes activities as described in chapter 6 in section 6.2 are knowledge of content and curriculum because it allows the pre-service teachers to understand the connections between the concepts within the grade level and across the grade levels, for example, the ordinary level secondary school mathematics curricula.

Based on my experiences in teaching mathematics in both ordinary and advanced levels secondary schools, knowing the basic concepts the students need before engaging them in learning a particular mathematics concept is an important curricular content knowledge the teachers need for a number of reasons. It helps the teachers to know how the concepts/topics are arranged in a specific level of the school curriculum and select which one to start with regardless of their arrangements in the textbooks that would help the students' better understanding of the mathematics concept/topics. It helps the teachers to be aware of the types of instructional materials for teaching that particular mathematics concept/topic and select the best to use depending on the level of the learners. It helps the teacher to select the appropriate teaching aids to be used for that particular mathematics concept/topic.

What follows is the evidence of pre-service teachers' development of knowledge of content and curriculum revealed in the concept studies of ratio, proportion, rate, and linear functions respectively. I asked the pre-service teachers to use the Tanzania mathematics syllabus of ordinary level secondary school, to identify the basic concept/s they thought the student needs to know before engaging them in learning the concept of ratio. The group identified the concepts the student needs to know before engaging them in learning the ratio concept as the whole number, basic operations in numbers, fractions, percentages, decimals, rational numbers and measurements-units concepts. Similarly, the same question was asked in the concept studies of proportion, rate, and linear function respectively. The question I asked is close to one suggested by the Dreher, Lindmeier, Heinze, and Niemand, (2018), Which concepts and ideas will be picked up in further grades? They considered as part of the school-related content knowledge (SRCK) the secondary school teachers need.

For proportion, the group identified concepts related to as the basic operation in numbers (addition, subtraction, division, and multiplication), ratio, fractions, rates, and variations. While for the rate, the group identified concepts of ratio, proportion, fraction, basic operations, and units and measurement as needed. For the linear function concept, the group identified concepts the student needs to know included: ratio, rate, coordinate geometry, proportion, and similarities. These lists were generated in the group discussion where the participants used the Tanzania mathematics syllabus of ordinary level secondary school discussed in a small group and then shared in the group discussion.

Knowing how mathematical concepts connect (Ball & Bass, 2003) within the school mathematics curricular is an important curricular knowledge the teachers need. Because as experts the teachers need to help the students to see how the concept/topic they learned at a particular level is related to another concept/s/topic/s within the grade level or across the grade levels for better conceptual understanding. Further the teacher will build lessons based on what the student has already encountered in school and what the student will encounter later in school. For example, the landscapes activities enabled the pre-service teachers to identify how each of the mathematical concepts of ratio, proportion, rate, and linear function, relate to others or how it is used in other mathematical concepts taught within the grade level (Form) or across the grade levels (Forms) of ordinary level secondary school mathematics curriculum (Refer back to figures

23, 24, 25 and 26 respectively). As Dreher, Lindmeier, Heinze, and Niemand, (2018) assert, secondary school teachers "should not only know school mathematics, but they should also know about its structure in the sense of meta-knowledge. This is, in the first instance, factual knowledge about the curricular order of contents and their interdependencies" (p. 326).

Therefore, knowing the basic concepts the students need before engaging them in learning the new mathematics concept/topic, knowing how realizations of mathematics concept relates within grade level and across the grade levels, and knowing how the mathematics concepts relates for a particular school curricular is important knowledge of the content and curriculum the pre-service teachers need. This is knowledge that would help their prospective students build the conceptual understanding. Teachers need to know the structure of school mathematics and the reasons for this curricular structure, which are partly rooted in the structure of academic mathematics (Dreher et al., 2018). The design of curriculum materials such as the syllabus, textbooks, and teachers guide the teachers use matters in helping the teachers with knowledge of content and curriculum. The teachers could be able to achieve this better if the designers of curriculum materials would carefully design the curriculum materials that support teacher curriculum relationship (Remillard, 2005).

7.5 Pre-service Teachers' Horizon Content Knowledge

Horizon content knowledge (HCK) is the third category of the subject matter knowledge and is defined as "awareness of how [the] mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Speer, King, and Howell, (2015) defined it as the "knowledge of the mathematics that follows or could follow the mathematics being taught" (p. 108). For example, how the school mathematics of ordinary level is related to the mathematics of the high school or college or university. It "includes the vision useful in seeing connections to much later mathematical ideas" (p. 403). I could find only one example of HCK that I observed in the course of the concept studies. The participants demonstrated a vision of how the ratio could be used in much later mathematical ideas such as in linear function, in rates, and trigonometry respectively. For example, the use of explanations such as: "The ratio is used to find the gradient of linear function in form three", "Yes, also in rate topic in form three to find the ratios of quantities of the same kind and different kinds", and "In form two, ratio is used to find the trigonometric ratios of sines, cosines, and tangents". This lack of HCK throughout the concept study sessions is not immediately explicable.

7.6 Summary of the Chapter

In summary, the analysis revealed aspects of the pre-service teachers' development of Balls' Mathematical Knowledge for Teaching (MKT): the common content knowledge (CCK), specialized content knowledge (SCK), and the horizon content knowledge (HCK) all of which fall under the subject matter knowledge, as well as, the knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) which fall under pedagogical content knowledge (PCK). The CCK and KCT were more obvious in all concept study sessions, as well in participants' reflections of what they do know about the concepts and how the concept is learned. The SCK was obvious in some parts of all concept study sessions but not in the participants' reflections of what they do know about the concepts and how the concept is learned. I think the SCK was not much obvious to the participants because the participants involved in concept studies are pre-service teachers with limited teaching experience and SCK requires teaching tasks that could be used to assess and experience in teaching help in access this. I did not use many teaching tasks that involve “examining, evaluating, and formulating a response to a student-generated solution[s]” (Speer et al., 2015, p. 116); with the fact that concept studies concerned is teachers learning mathematics and not teaching it was hard to differentiate the SCK from the CCK in the analysis. Speer et al asserts that these are central aspects of the SCK according to Ball explanation of SCK but they are also part of day-to-day lives of the mathematicians and other mathematician outside the academic work in other fields such as engineering, physicist, chemists when they evaluate their colleagues’/peers’ work and provide feedback: “In both the teaching and research contexts, the mathematician needs to make sense of the mathematical ideas and reasoning presented by someone else and determine whether the reasoning is correct” (Speer et al., 2015, p. 116). I disagree with the Speer and colleagues because although both the teachers and mathematicians do these tasks but for the teacher this is a primary focus and ends with the student learning. For others who use mathematics in their professional work checking the work of others is typically not their primary work nor is the goal of their work. The development of KCC was more obvious in the concept study landscapes emphasis and when the participants were responding the question of identifying the basic concept/s do think the student needs to know before engaging them in learning the concept of ratio/proportion/rate/linear function. The lack of HCK might be caused by the nature of the concept study and considering the limited experience in teaching the pre-service teacher participants have. The knowledge of content and student (KCS) was more difficult to observe with this group of pre-service teachers. It might also be better observed with mathematics teachers who have worked in classrooms teaching because it deals more with students. KCS was observed only in the ‘pedagogical problem solving’ emphasis when the participants were anticipating the questions students might ask when learning each of the four mathematics concepts of ratio, proportions and linear function (refer chapter 6 section 6.5.1). I think both HCK and KCS are more relevant (or easier to be developed and accessed) in classroom instruction and not in concept studies with pre-service teachers with only eight weeks of experience in teaching mathematics.

8. Contributions of Concept Studies on Pre-service Teachers' Professional Knowledge and Skills

This chapter describes the findings about the contribution of the concept study method on pre-service teachers' professional knowledge as reflected by the pre-service teacher participants. Three themes emerged from the analysis of the pre and post-questionnaires. The chapter is divided into four sections. The first section describes topics as reflected by the pre-service teacher participants that contributed to the theme about the professional knowledge the teacher needs for teaching mathematics. The second and the third sections discuss two themes that describe the contribution of the concept study method on pre-service teachers' professional knowledge and skills, as reflected by the pre-service teacher participants. Specifically, the second section discusses how the concept study method contributed to the pre-service teacher participants' deep understanding of mathematics as the first theme while the third section describes how the concept study method gave pre-service teachers opportunities to learn how to collaborate to learn MFT together as the second theme. The fourth section provides a summary of the chapter.

8.1 The Professional Knowledge Teachers Need for Teaching Mathematics

This section discusses pre-service teacher participants' views about the professional knowledge a teacher needs for teaching mathematics. The topics for this theme were generated from the key ideas originating from participants' responses to the question, "Thinking about teaching mathematics, what professional knowledge should a teacher have for teaching mathematics?" Table 13 provides a quantitative summary of pre-service teacher participants' responses. The values in table 13 represent the number of pre-service teachers that contributed to the key ideas that led to the respective topic in the pre and post questionnaires. More valuable to the research are the comments that the participants made in relation to the questions posed. The table is offered simply to point to the fact that participants change their responses from the beginning of the study to the end of it only in relation the first and second topics. One could speculate that the emphasis on mathematical concepts in the workshops the participants had an impact on their views about the importance of teachers knowing mathematics concepts, and knowing techniques and strategies for teaching concepts. However, it appears to me that participation in the concept study did not have impact of participants' views of the third, fourth, and fifth topics. This could be due to the nature of the concept study as it is dealing with meaning making and not explicitly about teaching strategies.

Concept study/Topic	Pre-questionnaires	Post questionnaires
Topic 1	**Teachers need to have knowledge of mathematical concepts specific to the curriculum**	
Ratio	4/10	10/10
Proportion	10/10	10/10
Rate	10/10	10/10
Linear function	8/9	8/9
Topic 2	**Teachers need to know the techniques and strategies in teaching specific mathematical concepts**	
Ratio	3/10	7/10
Proportion	7/10	8/10
Rate	8/10	8/10
Linear function	7/9	7/9
Topic 3	**Teachers need to have skills on how to prepare and proper use of teaching and learning materials specific for the concept**	
Ratio	3/10	4/10
Proportion	4/10	4/10
Rate	4/10	4/10
Linear function	4/9	4/9
Topic 4	**Knowledge of Evaluations and Classroom Management**	
Ratio	2/10	3/10
Proportion	3/10	4/10
Rate	4/10	4/10
Linear function	4/9	4/9
Topic 5	**Teachers need to have credential certificate for mathematics teacher education for the level the teacher is going to teach**	
Ratio	4/10	6/10
Proportion	6/10	7/10
Rate	7/10	7/10
Linear function	7/9	7/9

Table 13: Pre-service teachers' responses contributing to the theme of professional knowledge and skills teachers need for teaching mathematics

8.1.1 Knowledge of mathematics concepts specific to the curriculum

The pre-service teachers place importance on the knowledge of the mathematics concepts specific to the school curriculum level the teacher will be teaching. Three key ideas were expressed: understanding mathematics concepts for the school curriculum level the pre-service teacher will be teaching, understanding how those concepts relate, and understanding specific examples and applications of the mathematics concept in everyday life.

8.1.1.1 Understanding mathematics concepts for the school curriculum level the teacher will be teaching

The pre-service teacher participants appreciated a teachers' need to learn the mathematics concepts specific to the level he or she will be teaching in the school curriculum. The teachers' abilities in making correct definitions of mathematics concept, its meaning in different contexts, writing its notation, and in making its representations are all critical elements of knowledge teachers need. The pre-service teachers were concerned that a teacher needs to know more than what a mathematics concept is. The need for the teacher is to have conceptual understanding. Three of the participants in the post-questionnaires said this with different nuances.

> He or she [the teacher] should have the knowledge of mathematics concepts of the level he or she is going to teach.
>
> He/she [the teacher] should have enough knowledge in all concepts/topics he/she is going to teach **and their application in everyday life.** [Emphasis added]
>
> A teacher must have a knowledge of mathematics concepts or skills required to teach **according to the level of knowledge of the learners.** [Emphasis added]

The teachers' conceptual understanding (Byerley & Thompson, 2017; Kilpatrick, Swafford, & Findell, 2001; Skemp, 1978) of the mathematics concepts they are going to teach in the curriculum helps him or her build self-confidence in facilitating the teaching and learning process for conceptual understanding for their future students. Byerley and Thompson assert that "teachers who understand an idea they teach coherently provide greater opportunities for students' to learn that idea coherently. Inversely, the less coherently teachers understand an idea they teach, the fewer are students' opportunities to learn that idea coherently" (p. 168). Conceptual understanding of the mathematics concept gives the teacher an opportunity to facilitate student learning, to have the long-time retention of the mathematics concept and apply it in solving different mathematics problems understanding (Byerley & Thompson, 2017; Cummings, 2015; Kilpatrick, Swafford, & Findell, 2001; Skemp, 1978). Conceptual understanding is "an extremely important skill to have in not only mathematics but also all subjects in school. ... Building students' conceptual understanding throughout their education, will ensure that they retain their understanding throughout their lifetime" (Cummings, 2015, p. 18). The concept study emphases of realizations, landscapes, entailments, blending, and pedagogical problem-solving helped the pre-service teachers build their understanding of mathematics concepts. This understanding of mathematics concepts may be the reason why post-questionnaire their responses reflect this as important.

8.1.1.2 Understanding how mathematics concepts relate

The pre-service teacher participants expressed the belief that a teacher needs to know how specific mathematics concepts relate to other concepts. The understanding of the relationship between concepts within the grade level, and across the levels of the mathematics curriculum the teacher is preparing to teach, enable them to explain better the concept to the students who are learning mathematics, and to help the students develop relationships among concepts. This aspect of the mathematics for teaching was elaborated in the concept study sessions that focused on the landscapes. In the post-questionnaires, some pre-service teacher's commented:

> He or she [the teacher] should have the knowledge of mathematics concept...and how the concept is related to other concepts in mathematics and other subjects.

> The teacher should know the real meaning of mathematics concepts, their relationships, [and] the teaching and learning approaches.

> He or she [the teacher] will acquire various skills, knowledge...such as learning the real meaning of mathematical concept, [and] how it relates with other concept in mathematics.

Knowing how mathematics concepts relate helps the teacher in responding appropriately to the questions students ask during classroom instructions. Such teacher knowledge can be used to help students see how the learning of the mathematics concept at a certain grade level is important in learning other concepts within their grade level or across their study level in the school curriculum. Further, knowledge of how mathematics concepts are related across the curriculum could be used to motivate students in building interest in learning mathematics (Skemp, 1978). Specifically, the teacher needs to know how a Form I mathematics concept relates to other concepts taught in same grade level or other concepts in different grade level such as in Form II, III, and IV at Tanzanian's ordinary level secondary school curriculum. Knowing how mathematics concepts relate also includes knowing the basic concepts that the student must have acquired before engaging them in learning the mathematics concept at hand. One pre-service teacher participant wrote "The teacher should know the basic concepts in mathematics that are needed before teaching specific concept/topic." The teacher knowing the basic concepts needed before learning particular mathematics concept gives him/her the opportunity in selecting the examples and questions to be used that would help in facilitating the students' learning for conceptual understanding (Byerley & Thompson, 2017; Cummings, 2015;

Skemp, 1978) and enable them to see the connection of these concepts and build interests in learning mathematics (Skemp). The emphasis on concept study landscapes (Davis & Renert, 2014) provided the pre-service teachers with opportunities to learn how mathematics concepts relate in the school curriculum of the context. As evidence by their comments the pre-service teacher participants found this valuable.

8.1.1.3 Understanding specific examples of applications of the mathematics concept in everyday life

The pre-service teacher participants believe a teacher needs to know the specific examples of the applications of the mathematics concept to everyday life. A pre-service teacher participant wrote “a teacher should know the real examples of specific concept/topic and how they are applicable in our daily life.” The teachers’ understanding of the applications of the mathematics concepts outside the school environments or in everyday life gives the teacher the opportunity to facilitate better student understanding and help students see the connection of the mathematics concept to their everyday activities (Sawyer, 2014), as well building their interest in the learning of mathematics in their future careers (Skemp, 1978). Sawyer, (2014) asserts that “students learn deeper knowledge when they engage in activities that are similar to the everyday activities of professionals who work in a discipline.” (p. 4). The pre-service teachers’ awareness of the examples of applications of mathematics concept in everyd1ay life outside of school could also build their self-confidence in facilitating the student learning. Some pre-service teachers commented:

> Mathematics teacher should have the knowledge for teaching mathematics by concept and teaching mathematics using real examples in daily life related with the specific concept/topic.
>
> The teacher should know...applications of the concept and example specific for the concept or topic.

The concept study gave the pre-service teachers opportunities to identify, learn, and understand some applications of the concepts outside the school environments in the curriculum they are prepared to teach. The concept study realizations emphasis helped the pre-service teachers learn the applications of the mathematics concept outside the school environments.

8.1.2 Knowledge of teaching and learning strategies specific for each mathematics concept of the level the teacher is going to teach

Despite the teachers' understanding well the mathematics concept for the level he/she is going to teach, how it is related to other mathematics concepts, and the examples of its applications in everyday life, the pre-service teachers believe it also is essential for the teacher to understand the specific strategies for teaching and learning that concept. The understanding of the specific strategies for teaching and learning mathematics concept the teacher is going to teach emerged as the second theme from the pre and post questionnaires. One participant in the post-questionnaires commented, "for a teacher to teach mathematics he/she must be...equipped with different methods, techniques, and strategies that can be used to explain the topic/concept."

The teachers' understanding of the specific strategies that could be used for teaching and learning particular mathematics concept enables the teachers to select the best strategy to be used depending on the level of the learners and the size of the class. "A teacher should have a lot of teaching strategies and techniques in order to be flexible in teaching, i.e. he/she can choose to switch from one technique to another," one participant said in the post-questionnaires. It also gives teacher opportunities to choose specific examples that could be used for better students' understanding mathematics concept at hand. Teachers' understanding teaching and learning strategies specific for particular mathematics concept could also give him/her opportunities to select appropriate teaching aids relevant to the selected strategy. Another participant in the post-questionnaires wrote, “a teacher should know how the concept is developed, have a teaching skills, techniques, and methodologies (pedagogical) in order to facilitate well teaching and learning in the classroom.” However, believing that a teacher needs multiple strategies is not the same as knowing multiple strategies. In the post-questionnaires a participant responding to the question ‘how is ratio learned’ wrote:

> Ratio concept can be learned through group discussion that is guided by a facilitator/teacher who is going to introduce the concept and let the student in groups discuss. In this discussion, the useful parts of the concept are noted in order to have a clear understanding of the concept. For example, the teacher could guide the student to learn the type of comparison in ratios using examples, diagrams, and images to reinforce the understanding. The teacher can use examples that demonstrate part to whole comparison and part to part comparison.

8.1.3 Knowledge of specific mathematics teaching and learning aids for each mathematics concepts for the level the teacher is going to teach

The pre-service teacher participants appreciate a teacher’s need to understand teaching and learning aids specific to particular mathematics concept. That is their need that includes knowledge for selecting, preparing, and proper use of specific teaching and learning aids for particular mathematics concepts. One of the pre-service teacher participants wrote, “the teacher should know and be able to prepare the specific teaching aids for a certain topic/concept and how to use it in teaching.” An understanding of the specific teaching and learning aids required for teaching and learning particular mathematics concept gives the pre-service teacher opportunities in proper selection and preparation of the concept that is suitable for the learners depending on their level of mathematics, the size of the class, and the particular strategy and its proper use. “[The teacher] should know how to use the teaching aids and materials properly,” one participant wrote. Knowledge of specific teaching and learning aids also gives teachers the opportunity to choose specific examples that could be used based on choice of teaching and learning aids.

Another pre-service teacher participant wrote "[the teacher] has the ability to use proper teaching aids and materials so as to facilitate students' learning." In spite of the participants seeing concept study as a way of learning about teaching aids and their value, there was only one clear example of a teaching aid offered in the concept studies (factor table).

8.1.4 Knowledge of evaluations and classroom management

A small number of the pre-service teachers (3 to 4) also indicated that the professional knowledge for teachers needs to include knowledge of evaluation and classroom management. The knowledge that includes: evaluations of the mathematics teaching and learning process of the curriculum for the level the teacher is prepared to teach, and classroom management corresponding to the knowledge of how to manage the class depending on the need of the students.

Knowledge in evaluation

A small number of the pre-service teachers (3 to 4) believe the teacher needs to know how to evaluate mathematics teaching and learning processes related to the level he/she is going to teach. More importantly, the figures did not change from pre-test to post test. Thus, it appears that the concept studies did not impact this aspect of their professional knowledge. For example, one pre-service teacher participant wrote, the teacher needs to have "knowledge to make reflection and measure the ability that is possessed by a learner after and before teaching." Having ideas about what the students know at the beginning of the lesson helps the teacher prevent addressing concepts that the students already know, as helps the teacher select proper examples and exercises that will help in facilitating students' better understanding of the lesson. The evaluation at the end of the lesson helps the teacher to understand the extent to which the specific objectives of the lesson were met and make amendments in the next lesson for anything that needed it.

Knowledge of classroom management

A small number of the pre-service teachers (3 to 4) believe the teacher needs to know all about classroom management. They need to have the knowledge in managing the class which in most cases depend on teachers understanding of the student needs. One pre-service teacher participant insightfully wrote, “a teacher should be able to organise the classroom so that students can engage more effectively in the subject. Also, a teacher must be able to know how to consider the fast learners and slow learners in the classroom.” The teacher in knowing what the student needs provides an opportunity to understand why a student may behave differently at a certain point during classroom instruction, whether positively or negatively. The teacher is expected to choose the teaching and learning strategies that would accommodate both the slow and fast learners so, that he/she could meet the needs of both groups. The failure to choose the strategies that would accommodate both slow and fast learners could prevent the teacher from being able to manage the class, and this could cause serious discipline issues in the class. Thus, a mathematics teacher needs to have an understanding of which strategy fits for small and larger class size and the level of the students for easier classroom management. At the same time, the teacher needs to involve all students in asking and answering questions without bias. That means asking both girls and boys, and if forming groups during classroom instruction to mix boys and girls, and slow and fast learners. Also, from the pre-service teachers’ point of view, a teacher’s failure to explain a concept properly to learners could cause learners to lose interests in learning the subject which in turn could motivate the student to misbehave. In this research, the ten pre-service teachers had only eight weeks teaching experience in their BTP. The concept studies did not have an impact on helping them develop further understanding about the issue of the classroom management, though a few pointed out its significance in motivating the students learning of mathematics. The participatory nature of the concept study led them to believe that such methods in the classes they would be teaching could help their students learn mathematics. It could motivate the students to contribute more and help each other learn and remember things they have forgotten. For example, a participant in the post-questionnaire commented: "It [concept study] helps to know what your students need to know about what you need to teach. And learn how to collaborate that will help us use it to help the students learn the mathematical concept well."

8.1.5 A credential (the diploma) for the level the teacher is going to teach

The pre-service teacher participants all felt a teacher needs formal education leading to certification for the level the teacher is going to teach. This notion was as strong at the beginning of the sessions as at the end. The participants agreed that the teacher needs to undergo the teacher training in mathematics either in teacher colleges or universities and be certified as a mathematics teacher. For example, a pre-service teacher participant wrote "a teacher must undergo teacher training in mathematics and acquire a Certificate, Diploma, or Bachelor degree showing his competence." Further, the teacher needs to have passed the mathematics subject for both certificate of secondary education examination (CSEE) and the advanced certificate of secondary education examination (ACSEE). The pre-service teachers seem to believe that the teacher having a certificate means that mathematics teacher will have acquired the professional knowledge as explained in the sub-sections 8.1.1 to 8.1.4. Another pre-service teacher participant wrote "the professional knowledge should a teacher have for teaching mathematics is through attending teacher trainings in college or university where she/he will be taught mathematics concepts, skills and will be awarded certificate to certify his/her performance." The teacher candidates believe the credential certificate is at the core of professional training.

Thus, the pre-service teachers believed the teachers need a certificate for mathematics teacher education for the level the teacher is going to teach. Further, they insisted that the teacher with the credential certificate of mathematics teacher education is expected to have acquired the following professional knowledge: the knowledge of mathematics concepts specific to the curriculum; and the knowledge of teaching and learning strategies specific for each mathematics concepts of the level the teacher is going to teach. As well the teacher is expected to have the knowledge of specific mathematics teaching and learning aids for each mathematics concepts for the level the teacher is going to teach, and the knowledge of both evaluations and classroom management. Concept study, focused as it is on concepts, is limited in its scope as a method for the development of professional knowledge.

8.2 Concept Study Supports the Development of Deep Understanding of Mathematics in Pre-service Teachers

This section discusses how the concept study method contributed to pre-service teacher participants' deep understanding of the mathematics they will teach. As Davis and Renert, (2014) assert that "deep understanding of a concept requires more than pulling apart its constituent parts; it entails examinations of how these parts hold together and fall apart in different contexts and circumstances" (p. 43). The findings revealed the concept study method helped pre-service teachers deepen their understanding of a set of mathematics concepts related to ratio, proportion, rate, and linear function. Within the concept studies, meanings of the mathematics concepts originated from the pre-service teachers' themselves in their collective work, as I prompted them both with the selected topics, selected related mathematics questions and more general prompts such as why, how, and could you elaborate more. In the participants' collective work, they were learning the meaning of the mathematics concepts, how those mathematics concepts relate, the applications of the mathematics concept in everyday life, and possibilities for correcting misconceptions about the mathematics concept at hand. The four key ideas are described hereafter in the next four sub-sections respectively.

8.2.1 Meanings of the mathematics concepts originated from pre-service teachers' themselves in their collective work in the concept study

First, concept study contributed to pre-service teachers' deep understanding of mathematics because through it pre-service teachers had the opportunity to share many ideas concerning the meaning of mathematics concepts under exploration. Pre-service teachers came to understand the meaning of the mathematics concepts, their representations, and how they apply in diverse situations/contexts. Most of those meanings originated from the pre-service teachers' themselves and were activated through their collective learning. For example, the pre-service teachers created various realizations of the mathematical concept at hand, such as formal definitions, metaphors, images, algorithms, gestures, and applications, in responding to a simple question of what they know about the mathematics concept. Together they come up with a summary of the realizations of the mathematical concept. This collective work and collective understanding helped each pre-service teacher develops their individual understandings. Also, it gave them opportunities to extend what they knew as experienced students about the mathematics concept and integrating their ideas with that of their colleagues. Concept study uses open discussion that remains focused on a concept from the perspective of a teacher. The discussions motivating the pre-service teachers' need to know more about the mathematics concepts and to investigate different ideas originate from the mathematics concept or concepts related to it. The collective understandings of the meaning of the mathematics concepts attained through the open discussion and sharing of ideas originating from pre-service teachers' themselves made room for them to have a common understanding of the mathematics concepts. The pre-service teacher participants commented about how doing the concept studies contributed to their mathematics understandings:

> A concept study helps the teachers to understand mathematics deeply because it gives a wide range of the alternative [to the teacher] in expressing his or her ideas about the concept where all the participants participate in the discussion.
>
> In concept study, a teacher is given enough time to explain what he/she knows about the concept and shares with others his/her ideas...[which], makes it easy for a teacher to participate in the discussion ...[and] most of the knowledge comes from their colleague teachers.
>
> The concept study...compares different ideas of given subject for example, in the concept study of ratio more people contributed their ideas about the ratio concept in which a student teacher can make a comparison of different ideas in order to know what the real meaning of the concept is.
>
> It [concept study] allows a teacher to think critically and deeply about the mathematics concept because we learn the real meaning of the concept ...which, makes a teacher to have a better knowledge or the deeper understanding of mathematics. Also, it allows [the]collection of...ideas about the concept from all members...which, gives a teacher a wide range of thinking in his or her mathematics he/she is learning.
>
> In the concept study everyone is participating in the discussion, so, it opens the door for having different views from the members, in so doing to me concept study is the best way to [the] deep understanding of mathematics.

8.2.2 Participants learned how mathematics concepts relate within the curriculum

Second, the concept study contributes to the teachers' deep understanding of mathematics (Ma, 1999; Davis & Renert 2014) in the sense that it provides the pre-service teachers the opportunities to learn how the mathematics concepts are related to other concepts (Skemp, 1978) for the level the teacher is going to teach. In other words, the ways the mathematics concept could be interpreted within the grade level and across grade levels of the school curriculum (Davis & Renert). It includes identifying the fundamental mathematics concepts that are essential in learning the mathematics concept at hand. The landscapes, entailments, and blending emphases in concept study helped the pre-service teachers see how the mathematics concepts relate in different ways. In the landscape emphasis, the pre-service teachers collectively cross-examined the school mathematics curriculum. They organized their realizations of the mathematics concept at hand in relation to the curriculum. Their awareness of how these realizations relate within a grade level (horizontal awareness) and across grade levels (vertical awareness) developed. In regard to the entailment emphasis, the pre-service teachers found the logical implications a realization of a mathematical concept carries that help to shape it. In the blending emphasis, the pre-service teachers generated, combined, and collapsed varied realizations of the mathematical concepts they were working on to explore the deep connections among these realizations that might produce further emergent interpretive possibilities. The pre-service teachers in learning how the mathematics concept relates to other concepts had an opportunity to understand well how to develop the concept at hand at the same time as updating their knowledge about the mathematics concepts related to it. Knowing how mathematics concepts relate helps teachers deepen their understanding of the concept and builds the foundation for explaining to their future learners the connections among the concepts within their grade level and across the school curriculum. Skemp (1978) argued that:

> If people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring a new territory in search of nourishment. (p. 13)

The pre-service teacher participants saw these benefits.

> I think concept study...helps the teacher to be deep and competent because... [it] helps teachers learn the relationship between the concept with other concepts in different topics for [the] better understanding of the concept.
>
> Concept study gives the teacher more update of various concepts in mathematics in relation to the concept they are dealing with at that time.
>
> Concept study helps to come up with many ideas about the concept and how other concepts/topics in mathematics are related to it.
>
> In concept study, you learn...the relationship between the concept and other concepts in mathematics and other subjects...It [concept study] makes a teacher to be active in thinking and creating mathematical ideas (concept) which are related to the real-life.

8.2.3 Participants learned some applications of the mathematics concept in everyday life

Third, the concept study contributed to the teachers' deep understanding of mathematics because it provides pre-service teachers with the opportunities to learn some of the applications of mathematics concepts in everyday life. For example, a pre-service teacher participant commented "concept study makes a [mathematics] teacher to be active in thinking, creating mathematical ideas about the concept which, are related to the real-life activities." In the concept study, the pre-service teachers identified examples that could be used to show how mathematics concept might be applied in life outside of school. "[The] concept study helps the teachers' deep understanding of the mathematics...simply because [it] helps [the] teacher to study more about the concept, example learning its applications in daily life...and make her/him...competent." In doing so, the pre-service teachers learned more about mathematics concept which contributed to the deepening of their understanding of mathematics. "[C]oncept study also enables the teacher to learn the applications of the concept in real life and ...which facilitate easy understanding to the learner because he or she will teach things which the teacher is aware with it." The learning of applications for mathematics concepts the pre-service teacher is going to teach provides them with opportunities to identify and select examples that could be used at the introduction of the mathematics concept, in its development, and in the review of the concept. Learning these applications create possibilities for pre-service teachers to have examples of applications of the mathematics concept that could help in motivating their future learners and raises their interests in learning mathematics as well selecting their future career that deals with mathematics.

8.2.4 Participants corrected some misconceptions

Concept study contributes to the deep understanding of mathematics because it gave the pre-service teachers opportunities to correct misconceptions (Ball & Bass, 2003; Ball, Thames & Phelps; 2008) about the mathematics concept at hand within their collective learning. A pre-service teacher participant wrote "the concept study helps teachers' deep understanding of mathematics...It helps the teacher to correct what he or she understood wrongly about anything related to the concept." The open discussion in the collective learning of the concept study motivated the pre-service teachers to contribute more on what they knew about the concept which, help their colleagues to identify any misconceptions from the explanations which provides the opportunities for others in correcting it with their more elaborations about the idea. The elaboration on how colleagues think it is wrong and could be corrected enables pre-service teachers to deepen their understanding of the concept. "[T]he concept study helps [to] correct or reconstruct the knowledge stored or possesed by one's individual to the other." The correction of the misconceptions during the concept study helped the pre-service teachers to develop deeper understanding of the mathematics concept and mathematics in general. Which then could lead to them better explanations to offer their prospective students. Also, it contributes to the pre-service teachers' competence in teaching mathematics for conceptual understanding to their prospective students. A pre-service teacher commented, "[I]f a teacher is not having a clear understanding of the meaning of the mathematics concept he or she is likely to mislead the learner...and may lead to learners' misconceptions of the meaning of the concept."

These pre-service teachers had had experiences of learning mathematics as students from kindergarten to high school, attended different schools, and were taught by teachers with diversity in the understanding about mathematics concepts they were taught. Their schooling left some of them with misconceptions about certain mathematics concepts that were explored in the concept study. Because there is no special course in their training programs which has them explore the mathematics concepts they will be teaching, they might finish their teacher training with the same mathematics misconceptions and limited understandings that they arrived with. This is problematic since then they might transfer those to their future learners. The concept study provided the pre-service teacher with opportunities to correct any misconceptions he/she had. In sharing ideas about the mathematics concept at hand collectively his/her contributions give others the opportunity to identify his/her misconceptions correct it with explanations that would allow him/her to have the proper understanding of the mathematics concept. For example, pre-service teacher participant wrote

> [W]e normally know everyone has his/her views and understanding on something, and such views or understanding can be right or wrong so, through collaboration with others in concept study, it is where you can know it is wrong or right by listening from different ideas of your fellow teacher(s). Also, it makes you remember something concerning such matters when someone talks about it...that makes you know the mathematics which you need to know [as a teacher].

8.2.5 The pre-service teachers learned specific techniques and strategies that to facilitate student learning of particular mathematics concepts.

Learning the mathematics concepts they will teach in the school curriculum, through concept study, helped pre-service teachers participants identify and understand specific techniques and strategies that facilitate learning and teaching of particular mathematics concepts. They identified the importance of having a range of techniques and strategies for teaching. “A teacher should learn with colleague teachers through concept study before teaching mathematics because ... it gives techniques and strategies that can facilitate teaching/learning specific mathematics concept...” It is common in the teacher colleges for pre-service teachers to learn general techniques and strategies for teaching and learning process, but not know specifically which one is better for facilitating teaching and learning particular mathematics concepts. Identifying good strategies and teaching techniques to use is possible only if the pre-service teachers have the opportunities to learn specific techniques and strategies for particular mathematics concepts. This is something for which concept study is particularly good. For example, the use of manipulatives in learning ratios.

Understanding the specific techniques and strategies that could be used in facilitating teaching and learning particular mathematics concept also helped the pre-service teachers to build self-confidence that they would be able to select which one to use when preparing lesson plans and in facilitating in the classroom instructions. Therefore, understanding specific techniques and strategies for the teaching of particular mathematics concepts makes pre-service teachers more competent in facilitating the teaching and learning mathematics for understanding. In other words, the pre-service teachers’ confidence grew that they might be more effective in teaching mathematics. A pre-service teacher participant confirmed this, “...we need to learn with other colleague teachers in concept study mathematics we need to know in order teach because we get the knowledge, skills, techniques, and strategies in teaching certain mathematics concept that can enable us to teach it effectively." The pre-service teachers understand well the specific techniques and strategies for particular mathematics concepts gives them the choices during classroom instructions to switch in the moment it does not work for the group of learners. The switching of the strategies could help not discourage the students and save the instruction time.

In summary, in the concept study, the pre-service teachers collectively learned the meaning of mathematics concept originated among themselves, its symbolic and iconic representations, its applications in everyday life, and how it is related to other concepts in mathematics (as well in other subjects). This helped them build deep understanding of the mathematics concept at hand. It provided them with the opportunities to discuss how to solve mathematics problems related to the mathematics concept at hand. The pre-service teachers had opportunities to review approaches and strategies and identify the best for teaching and learning particular mathematics concepts. At the same time, they identified useful teaching aids for teaching and ways to learn with them. In post questionnaires, the pre-service teacher participants' responses of the three questions: what they do know about the mathematics concept, how it is learned and what they did learn in the concept study of it, revealed the pre-service teachers felt they understood better the meaning of the mathematics concepts they have learned in the concept studies. The pre-service teachers elaborated their understanding of the meaning of mathematics concepts which is an essential professional knowledge and skills that could enhance teaching and learning mathematics for conceptual understanding in secondary schools. The conceptual understanding is the knowledge that is beyond knowing facts and procedures (Kilpatrick, Swafford, & Findell, 2001; Skemp, 1978). It is the understanding of the meaning of the mathematics concepts, its representations, and how it applies in diverse situations/contexts. Figure 61 represents dimensions of the pre-service teachers understanding of the mathematics concepts developed from the concept study and the type of professional knowledge developed with reference to Ball's MKT model. It is a visual representation of the relationships of how I interpreted the deep understanding of mathematics contribution of the concept study and the type of professional knowledge developed with reference to Ball's MKT model in one figure.

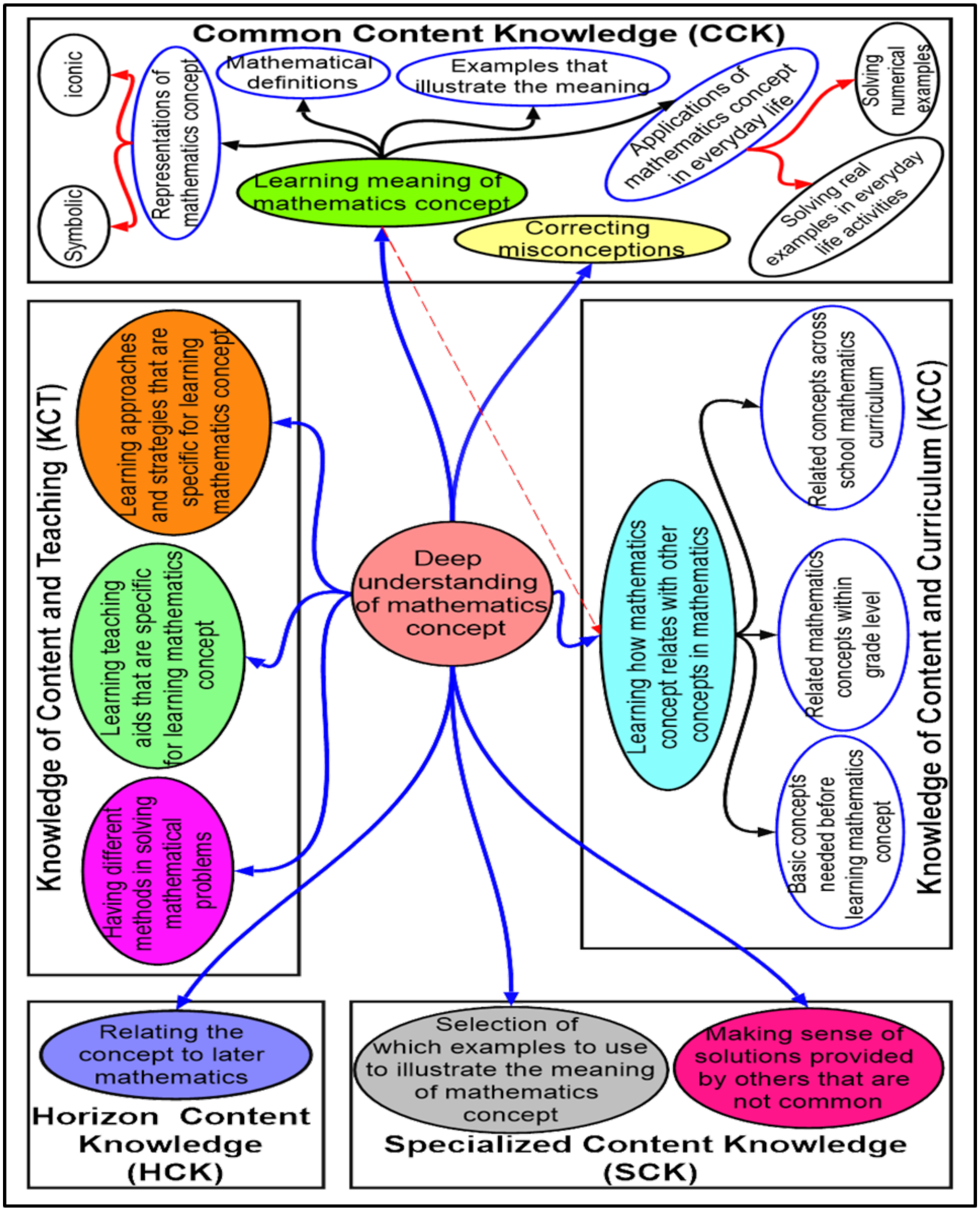


Figure 61: Diagram that illustrates the pre-service teachers understanding of mathematics concept in concept study and the type of professional knowledge developed with reference to Ball’s MKT model

8.3 Concept Study gives the Pre-service Teachers the Opportunities to Learn how to Collaborate to learn MFT together

This section discusses how the concept study method contributed to pre-service teacher participants' learning how to collaborate to learn the MFT together. That means how the pre-service teachers learned MFT through collaborative learning environment of concept study with their colleagues. The collaborative works, and shared understanding through concept study provided the pre-service teachers with more or even common mathematics education regardless of where they attended the school (their background experiences). The concept study enabled the pre-service teachers' full participation to develop the shared understanding of the mathematics concepts of ratio, proportion, rate and linear function. They collectively learned the meaning of these mathematics concepts when they were responding to what they do know about particular mathematics concepts. The discussion of their first list of realizations of each of these mathematics concepts and as well their elaboration of the follow-up prompt of 'why' and 'how' were important for the collective work. Their discussions of the landscapes, entailments, and blending activities (Davis & Renert, 2014) allowed for emergences of new knowledge and skills about the mathematics concept at hand, which benefited all the pre-service teacher participants. The concept study provided them with the opportunities to learn the ways in helping their future students how to collaborate in learning the mathematics with their colleague. They learned how each individual contribution is respected regardless of its adequacy or correctness with the fact that the mistake they made gave them the chance to discuss more about the ideas using the words how and why. The collaboration helped them to foresee to their future students. It also gave the pre-service teachers the opportunities to discuss possible questions their future learners could ask when learning the mathematics concepts which could be of help in their preparations of lesson plans. Learning with the other colleague(s) in concept study helps to add a new knowledge that previously they didn't have. Also, it helps to make the interaction among themselves that help them to contribute the different ideas about the concept. For example, the pre-service teacher participants wrote:

> With my colleague teacher(s) I could learn the mathematics I need to know in order to teach. This can be achieved through the concept study whereby with my colleague teacher(s) we can select mathematics concept and start to make a critical discussion on the concept and every colleague is given chance to contribute what he/she knows about the concept and other members comment on his/her ideas about the concept and after every one is participated in the discussion then we can wind up the discussion by making corrections on the fault/mistakes during discussion and provide an overview of the important things to consider about the concept.
>
> I think I could learn with my colleague teachers because through concept study everyone has the right to speak and no one is right or wrong so, through the contribution of everyone it's where I can learn the mathematics that I need to know in order to teach. Also, through the collaboration different ideas from my fellow teachers could help me to understand what they have and connect with what I have to get the knowledge which I need in order to teach mathematics. So, my colleague teachers are very important in order to know a mathematics which, I need to know.
>
> [T]hrough the interaction with my fellow teachers in concept study I can be able to develop and generate different new ideas, knowledge, and skills about the concept from them...it modifies and increases efficiency in teaching the mathematics concept to our future classes.

These responses reflected the participants benefit in participating in concept studies and they could definitely contribute to their future mathematics classroom instructions.

8.4 Summary of the Chapter

In summary, this chapter has described the findings on the professional knowledge teachers need for teaching mathematics and on the contribution of concept study method on teachers' professional knowledge, both as reflected by the pre-service teachers. For the professional knowledge teachers need for teaching mathematics, the pre-service teachers believed teachers need the certificate for mathematics teacher education for the level the teacher is going to teach. They believed that the teacher, having certified as the mathematics teacher, is expected to have acquired knowledge of mathematics concepts specific for the level the teacher is going to teach and teaching and learning strategies specific for each of these mathematics concepts. Also, they believed the teacher needs to acquire the knowledge of specific mathematics teaching and learning aids for each mathematics concepts for the level the teacher is going to teach and have the knowledge for both evaluation and classroom management. The findings revealed the concept study method gives the pre-service teachers the opportunities to learn the mathematics concepts in the school curriculum for the level they are prepared to teach and supports their development of deep understanding of mathematics. It also provided the pre-service teachers with the opportunities to learn how to collaborate to learn the MFT together.

Learning the mathematics concepts, they are going to teach in concept study build the pre-service teachers' common understanding about the mathematics concepts they explored, it gave them opportunities to correct the misconceptions related to the mathematics concepts, and it enabled them to have opportunities in learning the specific techniques and strategies for particular mathematics concepts. The concept study supports pre-service teachers' development of deep understanding of mathematics because the meanings of the mathematics concepts originate from pre-service teachers' themselves in their collective work in the concept study. They learn how mathematics concepts relate and the applications of the mathematics concept in everyday life. And they correct their own misconceptions about the mathematics concept at hand.

9 Discussion and Conclusions

This chapter comprises a discussion of the research findings and provides recommendations for teacher education in Tanzania and further research. The chapter is divided into three sections. In the first section is a summary of research findings about the contributions of concept study on pre-service teachers' MFT (professional knowledge and skills). The second section includes reflections about the research findings and their implications for mathematics teacher education and future research. The third section provides the conclusion.

9.1. Discussion

My study explored the question, *"in what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?"* It was important to investigate how concept study with its emphases on realizations, landscapes, entailments, blending, and pedagogical problem solving (Davis & Renert, 2014), would function in the context of Tanzanian pre-service teacher education to develop both tacit and explicit mathematics for teaching (MFT). In investigating this, the first concern was to design a classroom setting in which the pre-service teachers came together to work on their mathematics for teaching. Through the work of Davis and his colleagues (Davis & Simmt, 2006; Davis & Renert, 2014) I saw how they understood teachers working collectively in the concept studies as a complex system. They considered a group of teachers as a collective learner rather than a collection of learners. Hence it was important to respect the features that support the complex system in the concept study sessions. Therefore, the research method was to have a group of pre-service teachers learn the mathematics they need to know as teachers by doing concept studies in a classroom driven by five complexity conditions to support a complex system: *internal diversity, decentralized control*, *redundancy, neighbour interactions,* and *organized randomness* (Davis & Simmt, 2003*)*. I believe that acknowledging these conditions and maintaining a supportive environment for the collective meaning making was crucial in the concept study sessions and obtaining the reported findings.

In other words, the purpose of the research was to illustrate how concept study supported the development of pre-service teachers MFT (in terms of professional knowledge and skills). Explicit expressions of the MFT of 10 pre-service teachers was analysed by the use of Ball, Thames, and Phelps, (2008) categories of mathematical knowledge for teaching (MKT). The researched involved doing 4 concept study sessions with 10 pre-service teachers from a teacher training college in Tanzania. Data was collected in the form of a pre-study interview, pre and post session questionnaires, video and audio recording of the sessions, and working papers from each of the concept study sessions. A thematic analysis of the mathematics that emerged in the settings was done using Davis and Renert's (2014) concept study model of MFT. That was followed by an analysis of the data using Ball and colleagues (Ball, Thames, and Phelps, 2008) model of MKT. Although Davis and Renert have found success using concept study with in-service teachers, little research had been conducted with pre-service teachers and no research had been done with pre-service teachers in Tanzania. The findings from the study suggest concept study is an appropriate format for instruction for pre-service teacher education in Tanzania, specifically for the mathematics methods courses.

The findings indicate that the concept study provides opportunities for pre-service teachers to develop deep understanding of the school mathematics they will teach. Specifically, the pre-service teachers had opportunities to learn the meaning of the mathematics concepts, their symbolic and iconic representations, their application in everyday life activities, how they are related to other mathematics concepts of the school curriculum, and to recognize and correct various mathematical misconceptions that they or their colleagues expressed. The connections they were able to build related to the concepts studied reflect deepen of their understanding of the mathematics concepts related to the school mathematics they were preparing to teach. The development of professional knowledge and skills in terms of Ball's MKT model was evident for most but not all of the categories in her model. Specifically, common content knowledge (CCK), specialized content knowledge (SCK), as subject matter knowledge were manifested in the participants individual contributions and in the emergent knowledge of the collective. The pedagogical content knowledge Ball calls knowledge of content and teaching (KCT) also emerged in the collective actions of the participants. Knowledge of content and curriculum (KCC) (also an aspect of PCK) was prompted by a specific task assigned by the facilitator. The CCK and KCT were observed to be developed more fully in the concept studies compared to SCK. The KCC was developed more during the concept study landscapes emphasis and when the participants were responding the facilitator's question of identifying the basic concept/s they thought a student would need to know before engaging them in learning the concept of ratio/proportion/rate/linear function. Horizon content knowledge (HCK) was less evident than CCK, KCT, SCK, and KCC. This may be related to participants' academic knowledge of mathematics, in part due to their poor experience with mathematics in school. The knowledge of content and student (KCS) was also difficult to observe with this group of pre-service teachers: however, it was observed in the 'pedagogical problem solving' emphasis. This emphasis though was highly speculative. I think both HCK and KCS are more explicit (or easier to be developed and accessed) in pre-service educational instruction modes, such as lectures, compared with concept studies. This being especially true with pre-service teachers because of their limited classroom teaching experience.

My observations in this research illustrated the power of collective learning within concept study for pre-service teachers' understanding of the school curriculum and the mathematics of the school curriculum. For example, in the rate concept study session, the collective discussion in one of the tasks that involve writing the ratios from ratio tables and what these ratios represent it was obvious that the pre-service teacher participants were unaware of the importance of the units in ratios that are rates. The collective work on this task gave them the opportunity to identify and discuss the role of units. The use of concept study helped the pre-service teachers to access both their explicit and tacit MFT.

This study also was able to access the pre-service teachers' views of the professional knowledge that they think the mathematics teacher needs. It is interesting that they all believe a certificate is necessary to be a teacher. However, the certificate itself is not professional knowledge. It may be that the pre-service teachers believe that if a person achieves the certificate, they have the MFT. Concept studies in themselves do not result in a certificate but this study demonstrated it did result in the pre-service teachers developing professional knowledge the pre-service teachers believe they need.

What follows is a discussion of my observations of how the concept study contributes to the pre-service teacher participants' deep understanding of school mathematics they will teach—the *know what* and *know how* (Adler, 2005; Davis & Simmt, 2006; Davis & Renert, 2014).

The concept studies helped the pre-service teachers-built skills such as: unpacking the curriculum, select good applications of the mathematical concepts, provide multiple explanations of the mathematical concepts, and provide good examples of the mathematical concepts. In the concept studies of the ratio, proportion, rate, and linear function, the pre-service teachers unpacked the curriculum by providing the landscapes of the ratio, proportion, rate, and linear function, and found the basic concepts the students need before they are engaging in learning each of these concepts. The pre-service teachers provided the multiple explanations and the good examples of the mathematical concepts of ratio, proportion, rate, and linear function. Also, the pre-service teachers selected good applications of the mathematical concepts of the ratio, proportion, rate and linear function in their collective learning in the concept studies of these concepts.

9.1.1 How concept study contributed to development of the mathematics for teaching

The pre-service teachers developed deeper understanding of the school mathematics (related to ratio, proportion, rates, linear functions) they will need to know for teaching because meanings originated from the pre-service teachers' themselves and were activated through collective learning. Collectively, pre-service teachers explored meanings of the mathematical concepts of the school curriculum they will teach, their symbolic and iconic representations, how mathematics concepts relate, their applications in everyday life, and how to correct misconceptions about the mathematics concepts. Further, concept study created space for preservice teachers to discuss how to solve mathematics problems related to the mathematics concepts. They had opportunities to review approaches and strategies for teaching and learning particular mathematics concepts, and to identify useful teaching aids for teaching them.

This collective work, and the collective understanding that emerged from it, helped each pre-service teacher develop their individual understandings. Also, it gave the pre-service teachers opportunities to extend what they knew about the mathematics concept as experienced students in different levels of schooling integrated their ideas with that of their colleagues. One participant commented the "[l]earning with the other colleague(s) in concept study helps to add a new knowledge that previously you didn't have. Also, it helps to make the interaction among ourselves that help us to contribute the different ideas about the concept." The open discussion in concept studies motivates the teachers' need to know more about the mathematics concept, and to learn different ideas originating from the mathematics concept or related to it. The collective understandings of the meaning of the mathematics concepts attained through the open discussion and sharing of ideas originating from pre-service teachers' themselves made room for them to have the deep understanding of the mathematics concepts.

The study demonstrated that the facilitator has a role for pushing forward the discussion in the concept studies by using prompts and questions apart from the entrance question. This was critical because the limited experience teaching that pre-service teachers had. Thus, for all concept studies of ratio, proportion, rate and linear functions, the use of prompts and prepared questions related to the mathematics concepts are critical to further the discussion. The facilitator focusing on the participants' awareness of the variety of realizations of the mathematical concept that they provided, the use of "how", "why", and "could you elaborate more", helps the participants access their MFT of the mathematics concept at hand.

Concept study contributes to the teachers' deep understanding of mathematics in the sense that it provides pre-service teachers opportunities to learn how mathematics concepts are related to others for the level the teacher is going to teach, and the ways the mathematics concept could be interpreted across the school curriculum. Concept study includes identifying the fundamental mathematics concepts that are essential in learning the mathematics concept at hand. The landscapes, entailments, and blending emphases in concept study helps pre-service teachers see how the mathematics concepts relate in different ways. Bearing in mind that these are pre-service teachers with only eight weeks teaching experience engaged in the concept study, the landscapes emphasis which is an illustration of the mathematics as it relates to scope and sequence of curriculum emerged only as a planned activity in the concept studies. To plan the landscapes activity in advance was an important task for the facilitator when it comes to the pre-service teachers with similar characteristics with the group used in this study, before the commencement of the concept study, however, this might be different with the pre-service teachers in other contexts. The entailments, which are logical implications of each realization of a mathematics concept, help to shape the mathematics concept understanding for the pre-service teacher. Working on understanding blends of the concepts was challenging to the pre-service teachers in this research. The challenge might be caused by the rote learning and the poor performance in mathematics they experienced in primary and secondary schooling.

One of the important aspects in the teacher's lesson preparation, is speculating what type of questions could be raised by the student during classroom instruction. The 'pedagogical problem solving' emphasis emerged as a planned activity because the participants are pre-service teachers with limited teaching practice. In contrast, 'pedagogical problem solving' activity is quite different with in-service teachers who have long experiences in teaching mathematics. For in-service teachers, the 'pedagogical problem solving' activity gives the in-service teachers opportunity to contribute the questions that have proven to be a challenge to their students that are related to the concepts they are investigating.

The concept study also contributed to the participants' deep understanding of mathematics because it provided the pre-service teachers with opportunities to identify and learn applications of mathematics concept outside the school environments for the school curriculum they are preparing to teach. A pre-service teacher commented the "concept study makes a teacher (mathematics teacher) ...active in thinking, creating mathematical ideas about the concept which, are related to the real-life activities."

Concept study contributes to the deep understanding of mathematics because it gives the pre-service teachers opportunities to correct any misconceptions about the mathematics concept at hand from their collective learning. The open discussion in the collective learning of the concept study motivated the pre-service teacher to contribute more on what he/she knows about the concept which, his/her contributions give others the opportunity to identify his/her misconceptions correct it with explanations and more elaborations that would allow him/her to develop a deeper understanding of the mathematics concept. The elaboration on how the colleagues think a concept is wrong and could be corrected enables the pre-service teacher to deepen their understanding. The mistakes do not simply result in a negative way as they do in a test, but instead are an opportunity for further learning. Concept study enabled the pre-service teachers to learn the value of a mistake.

I argue that because pre-service teachers need deep understanding of the school mathematics they are preparing to teach, concept study is a useful instructional strategy to use with them. The need to know mathematics for teaching includes: the meaning of the mathematics concept, its symbolic and iconic representation, its application in everyday life activities, how they are related to other mathematics concepts of the school curriculum and to correct misconceptions, hence concept study is a very useful approach for pre-service teachers in Tanzania.

9.2 Recommendations for Teacher education

This section provides recommendations for teacher education in Tanzania and further research. It describes the strength and weakness of this research and how it could be improved; recommendations for general future research, and research specific to Tanzania contexts. It also offers a discussion of a proposed mathematics method course design for Tanzania pre-service teacher education with recommendations for Tanzania teacher educators and curriculum developers.

9.2.1 Strengths and weaknesses for this research

The research method had strengths and weaknesses. This research shows that the uses of concept study approach to develop pre-service teachers' professional knowledge and skills has the strength in motivating collective learning among teachers which helped in building the collective understanding of the mathematics concept at hand. The collective work and collective understanding in concept studies helped each pre-service teacher to develop their individual understandings of the mathematics concepts of ratio, proportion, rate, and linear function. However, this research intervention is a challenging one given that it requires pre-service teachers developing MFT by the use of concept study with limited experience in teaching mathematics. The intention of using concept study with pre-service mathematics teachers is to construct their professional knowledge and skills, instead of accepting knowledge from their tutors/teacher educators. Different challenges can be encountered in implementing the concept study emphases, especially pedagogical problem solving and landscapes because of the fact that the pre-service teachers have limited experience in teaching mathematics. Another challenge could be the ability of the facilitator to prompt the continuation of the discussion in concept study sessions when the pre-service teachers do not have experiences to draw from. The continuation of the discussion in a concept study session depends much on the depth and breadth of teachers' mathematics experience and knowledge. In this sense, in conducting concept study with pre-service teachers the facilitator has the responsibility to trigger the continuity of the discussion of the mathematics concept at hand.

As Chapters 6, 7, and 8 describe, the research findings point to concept study as a potential approach that can be used with pre-service teachers for learning mathematics concepts to develop their professional knowledge and skills. However, the implementation of concept study in this research could have been improved. For example, prior to the concept study session, pre-service teachers could have been asked to prepare teaching aids to be used in learning mathematics concept at hand and bring them in concept study session. During concept study session the pre-service teacher could have been asked to explain how he/she could use the aid/s in learning the mathematics concept. Making time for reflection on a concept study session day or weeks later for the pre-service teachers to discuss the reflections of the first session as a group would have been valuable. These alterations to the study probably it would have provided more quality data and insights about pre-service teachers' mathematical professional knowledge and skills.

9.2.2 Recommendations for future research

Future research that explores the impact of using concept study in pre-service teacher education is recommended. In particular, how the concept study experience helps pre-service teachers teach differently than they were taught in school? How does it teach them to value the mistakes their learners make and use them positively to develop a concept instead of punishing the student for them? Will teachers be better prepared to select the strategies that best fit with the new mathematical concept to be taught and select the corresponding teaching aids to be used? Will they be able to anticipate questions that students could possibly ask and the possible responses that can be offered to help them solve their problems? Having experienced reflections in the concept study session, will teachers learn to guide the students to reflect about the meaning of the new mathematical concept, its applications in everyday life and note any misconceptions that still exist for its consideration in the next lesson? Does concept study contribute to the pre-service teachers' competence in teaching mathematics for conceptual understanding to their prospective students? As a participant commented, "if a teacher is not having a clear understanding of the meaning of the mathematics concept, he or she is likely to mislead the learner...and may lead to learners' misconceptions of the meaning of the concept."

Future research related to this would strengthen the benefits of conducting concept study with pre-service teachers and open up other significant perspectives concerning developing mathematics teachers' professional knowledge and skills. Future research development might include the extension of the research intervention to other levels of mathematics teacher education programs in Tanzania such as: degree level pre-service teachers, instead of diploma level in secondary education pre-service teachers; and in private teacher's college, instead of in public teacher's college. Another possible future research can be implementing diploma in secondary education mathematics methods course by means of concept studies for all mathematics concepts in the school curriculum they are prepared to teach. As well, the same course in degree level teacher programs offered by universities could also be implemented by the same means. The idea is to be less concerned about specific mathematics methods curriculum requirements, presuming that, by finishing the course, the entire mathematics methods course curriculum would be discussed.

9.2.3 Recommendation for Tanzania teacher educators and curriculum developers

There would be value in Tanzanian mathematics teacher educators and mathematics curriculum developers under Ministry of Education, Science, and Technology (MoEST) to consider the use of concept study as an approach for mathematics teacher education both for pre-service and in-service teachers. Thus, from the results of this study, it is recommended that:

(1) The curriculum developers use the concept study approach in two different ways:

   a) First, in collaborations with tutors/teacher educators, to think about how to incorporate the use of concept study approach in the diploma in secondary education teacher education curriculum specifically for mathematics methods course as well as for certificate in teacher education level to give pre-service teachers opportunity to learn how to collaborate to learn MFT.

   b) Second, in collaborations with tutors/teacher educators, to use the concept study approach in conducting in-service teacher professional development for both secondary and primary school's mathematics teachers to give the teachers opportunity to learn the mathematics concepts they need to teach collaboratively.

(2) The tutors/teacher educators in teacher colleges as well as in universities to consider the use of concept study approach in their mathematics methods course for learning the mathematics concepts the teacher is prepared to teach and as well for other concepts that are necessary from the lower levels.

Figure 62 represents a proposed design of mathematics methods course by means of concept studies in Tanzanian context. The design starts from the central rectangle outwards following the direction of arrows. An additional possible future research project could be conducting concept study with a mixed group of pre-service teachers from diploma in secondary education level and degree level.

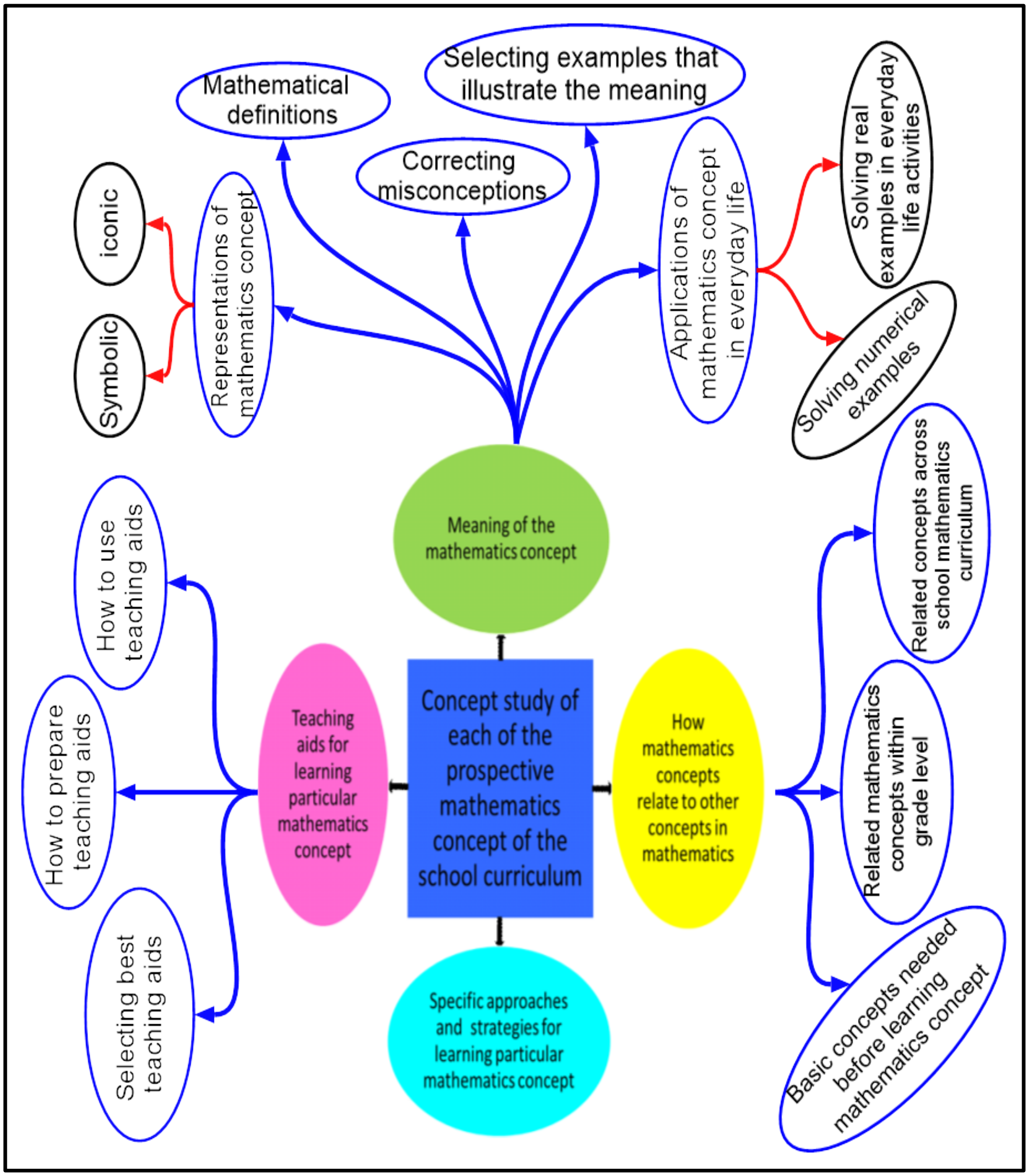


Figure 62: Proposed design of mathematics methods course by means of concept studies in Tanzanian context

9.3 Conclusion

The research in this thesis produced some insights into the question "in what ways does developing mathematics for teaching through concept study contribute to the professional knowledge and skills of pre-service teachers?" The study provides insight on how to incorporate concept study in a mathematics teacher education program in order to develop pre-service teachers' both tacit and explicit mathematics for teaching (MFT) in terms of professional knowledge and skills.

To conclude, it is worth stressing that the research findings in this work have significance for the mathematics teacher education research community worldwide and specifically in Tanzania. The results of the study could inform tutors/teacher educators and researchers to the supportive related challenging task of using concept study in developing of pre-service teachers' professional knowledge and skills. The concept study is confirmed as a productive approach of collective mathematics learning while developing the pre-service teachers' professional knowledge and skills. It develops pre-service teachers' deep understanding of mathematics concepts of the school mathematics curriculum they will teach. The research intervention reinforces the possibility of conducting concept studies in a mathematics method course, without hindering teacher education curriculum goals or wasting allocated instruction time. By accessing pre-service teachers' professional knowledge and skills, this study reveals pre-service teachers' deep understanding of mathematics concept of the school curriculum. Explicitly, concept study gives the pre-service teachers opportunities to learn the meaning of mathematics concept, its symbolic and iconic representations, its applications outside the school environment, and how it is related to other mathematics concepts and as well correcting misconceptions of the mathematics concept at hand. It has shown the value of mistakes in understanding the mathematics concept and the value of collaboration in pre-service teacher education programs. This research provides insights for the tutors/teacher educator better way of helping pre-service mathematics teachers to develop their professional knowledge and skills and correcting misconceptions. The use of Davis and Renert's (2014) concept study emphases and the use of Ball, Thames, and Phelps, (2008) categories of MKT was significant to this study and to the mathematics teacher education community in general.

References

Adler, J. (2005). Research and maths teacher education in ten years of SAARMSTE: Trends and challenges. In R. Vithal, J. Adler, & C. Keitel, *Researching mathematics education in South Africa: Perspectives, practices and possibilities.* Pretoria: HRC Press.

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education, 37*(4), 270-296. doi:10.2307/30034851

Adler, J., Hossain, S., Stevenson, M., Clarke, J., Archer, R., Grantham, B. (2014). Mathematics for teaching and deep subject knowledge: Voices of Mathematics Enhancement Course students in England. *Journal of Mathematics Teacher Education, 17*, 129-148.

Ali, A., Mirza, M. S., & Rauf, M. (2015). *The impact of positive and negative attitude of teachers towards corporal punishment on students' achievement in mathematics.* The dialogue. Retrieved from https://www.thefreelibrary.com/The+Impact+of+Positive+and+Negative+Attitude+of+Teachers+towards...-a0427352006

Ball, D. L. (1989). *Breaking with experience in learning to teach mathematics: The role of a preservice methods course.* San Francisco, CA: Paper presented at the American Educational Research Association. Retrieved from http://flm-journal.org/Articles/18CB972A1A2DFA8E98A93079AAF8B.pdf

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Ball, D. L. (2017). Uncovering the Special Mathematical Work of Teaching. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp. 11-34). Humburg: Springer . doi:10.1007/978-3-319-62597-3_1

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(3), 14-17, 20-22, 43-46. Retrieved from http://deepblue.lib.umich.edu/bitstream/handle/2027.42/65072/Ball_F05.pdf?sequence=4

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59*(5), 389–407. doi:10.1177/0022487108324554

Ball, D., & Hill, H. (2008). Measuring teacher quality in practice. In D. Gitomer (Ed.), *Measurement issues and assessment for teaching quality* (pp. 80-98). Thousand Oaks, CA: Sage Publications.

Ball, D.L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. *In E. Simmt & B. Davis (Eds), Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM. Retrieved from https://eric.ed.gov/?id=ED529557

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., Tsai, Y. (2010). Teachers mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*, 133-180. doi:10.3102/0002831209345157

Binde, A. L. (2010). *Conceptions of mathematics teacher education: thoughts among teacher educators in Tanzania.* Retrieved from https://www.doria.fi/bitstream/handle/10024/88736/binde_andrew.pdf?sequence=1

Brenner, M. (2006). Interviewing in educational research. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in educational research* (pp. 357-370). Mahwah, NJ: Lawrence Erlbaum. Retrieved from University of California Santa Barbara institutional archive website: http://www.history.ucsb.edu/faculty/marcuse/projects/oralhistory/2006MEBrennerInterviewInEducResearchOCR.pdf

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior, 48*, 168–193. doi: http://dx.doi.org/10.1016/j.jmathb.2017.09.003

Cannella, G. S., & Reiff, J. C. (1994). Individual constructivist teacher education: Teachers as empowered learners. *Teacher Education Quarterly, 21*(3), 27-38.

Chonjo, P.N., Osaki. K.M., Possi, & Mrutu, M. (1996). Improving science education in secondary schools: A situational analysis of science teaching in selected government secondary schools in Tanzania Mainland. Dar es Salaam, Tanzania: Ministry of Education and Culture.

Costley, C., Elliott, G. & Gibbs, P. (2010). *Doing Work based Research; approaches to enquiry for insider-researchers.* London: Sage.

Creswell, W. J. (2014). *Research design: qualitative, quantitative and mixed methods approaches.* Thousand Oaks: Sage.

Davis, B. & Simmt, E. (2015). Perspectives on complex systems in mathematics learning. In L. English, & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (pp. 416-432). Oxon, UK: Taylor and Francis.

Davis, B. (2008a). Is 1 a prime number? Developing teacher knowledge through concept. *Mathematics Teaching in the Middle School, 14*(2), 86-91.

Davis, B. (2011). Mathematics teachers' subtle, complex disciplinary knowledge. *Science, 332*, 1506-1507.

Davis, B. (2012). Subtlety and complexity of mathematics teachers' disciplinary. *Paper presented at the 12th International Congress on Mathematical education*. Seoul, Korea.

Davis, B. (2013). Teachers' mathematics knowledge building communities. In A. P. Babb, A. S. Rojas, I. T. Cáceres, & C. B. Zarzar (Ed.), *Proceedings of the First Meeting between the National Pedagogic University (Mexico) and the Faculty of Education of the University of Calgary (Canada).*

Davis, B., & Renert, M. (2009). Mathematics-for-teaching as shared dynamic participation. *For the Learning of Mathematics, 29*(3), 37-43. Retrieved from http://flm-journal.org/Articles/5A3DDD1FB61059E0E6311363AF55D.pdf

Davis, B., & Renert, M. (2014). *The math teachers know: profound understanding of emergent mathematics.* New York, USA: Routledge.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137-167. doi:10.2307/30034903

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: an ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*(3), 293-319. doi:10.1007/s 10649-006-2372-4

Davis, B., Sumara, D., & Luce-Kapler, R. (2008). *Engaging minds:changing teaching in complex times. second edition.* New York, USA: Routledge.

Delaney, S., Ball, D. L, Hill, H. C., Schilling, S. G., & Zopf, D. (2008). Mathematical knowledge for teaching: Adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education, 11*(3), 171–197.

Denzin, N. K., & Lincoln, Y. S. (Eds.). (1994). *Handbook of qualitative research.* Thousand Oaks: Sage .

diSessa, A. A. (2004). Metarepresentation: native competence and targets for instruction. *Cognition and Instruction, 22*(3), 293-331. doi:10.1207/s1532690xci2203_2

Dreher, A., Lindmeier, A., Heinze, A., & Niemand, C. (2018). What kind of content knowledge do secondary mathematics teachers need? A conceptualization taking into account academic and school mathematics. *J Math Didakt, 39*, 319–341. doi:https://doi.org/10.1007/s13138-018-0127-2

Ellis, J. (2006). Researching children’s experience hermeneutically and holistically. *The Alberta Journal of Educational Reseach, 52*(3), 111-126.

Ellis, J. (2009). Interpreting Results. In A. J. Mills, Durepos, G, & Wiebe, E (Eds.), *Encyclopedia of Case study Research.* Thousand Oaks: Sage.

Ellis, J., Hetherington, R., Lovell, M., McConaghy, J., & Vickzo, M . (2013). Draw me a picture, tell me a story: Invoking memory and supporting analysis through pre-interview drawing activities. *Alberta Journal of Educational Research, 58*(4), 488-508.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: a Japanese approach to improving mathematics teaching and learning* . Mahwah, N J: Erlbaum.

Furner, J. M, Yahya, N, & Lou Duffy, M. (2005). Teach mathematics: Strategies to reach all students. *Intervention in School and Clinic, 41*(1), 16–23. Retrieved from http://wp.cune.org/jordanstirtz/files/2014/03/17997132.pdf

Given, M. L. (Ed.). (2008). *The encyclopaedia of qualitative research methods, volume 1&2.* Thousand Oaks, CA: Sage. Retrieved from http://www.stiba-malang.com/uploadbank/pustaka/RM/QUALITATIVE%20METHOD%20SAGE%20ENCY.pdf

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth generation evaluation.* Newbury Park, CA: Sage.

Guba, E.G., & Lincoln, Y.S. (1994). Competing paradigms in qualitative research. In N. Denzin, & Y.S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 105-117). Thousand Oaks, CA: Sage.

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In J. Frank K Lester, *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Charllorte NC: National Teachers Council of Mathematics Education.

Hill, H, Rowan, B. and Ball, D . (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371- 406.

Hill, H. (2007). Mathematical knowledge of middle-school teachers: implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis, 29*(2), 95-104.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's Mathematics Professional Development Institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351. Retrieved from http://www.umich.edu/~lmtweb/files/hillball.pdf

Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teacher's topic specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H., Sleep, L., Lewis, J., & Ball, D. (2008). Assessing teachers' mathematical knowledge. In F. K. Lester (Ed.), *Handbook of research on teaching and learning.* Charlotte, NC: Information Age Publishing.

Huillet, D. (2009). Mathematics for Teaching: An Anthropological Approach and Its Use in Teacher Training. *For the Learning of Mathematics, 29*(3), 4–10.

Izsak, A., Jacobson, E., & Araujo, Z. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics, 43*(4), 391-427.

Izsak, A., Orrill, C., Cohen, A., & Brown, R. (2010). Measuring middle grades teachers' understanding of rational numbers with the mixture rasch model. *The Elementary School Journal, 110*(3), 279-300.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. Adding it up [eletronic resource]: Helping children learn mathematics. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Washington, DC: National Academy.

Kitta, S. (2004). *Enhancing mathematics teachers' pedagogical content knowledge and skills in Tanzania.* Enschede: University of Twente. Retrieved from https://ris.utwente.nl/ws/files/6120699/thesis_Kitta.pdf

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being. Basic books.*

Lamon, J. S. (1999). *Teaching fractions and ratios for understanding: Essential content knowlege and instructional strategies for teachers.* Mahwah: NJ: Lawrence.

Lamon, J. S. (2012). *Teaching fractions and ratios for understanding: Essential content knowlege and instructional strategies for teachers.* New York: Routledge: Taylor and Fransis.

Laurens, T., Batlolona, F. A., Batlolona, J. R., & Leasa, M. (2018). How Does Realistic Mathematics Education (RME) Improve Students' Mathematics Cognitive Achievement? *Eurasia Journal of Mathematics, Science, and Technology Education, 14*(2), 569-578. doi:10.12973/ejmste/76959

Le Grange, L. (2007). Integrating western and indigenous knowledge systems: The basis for effective science education in South Africa? *International Review of Education, 53*, 577–591. doi:10.1007/s11159-007-9056-x

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* Beverly Hills, CA: Sage.

Lobato, J. E, & Amy B.E. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics: grades 6−8.* Reston, VA: National Council of Teachers of Mathematics.

Ma, L. (1999). *Knowing and teaching, elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States.* Mahwah: NJ: Erlbaum.

Manouchehri, A., & Enderson, M. C. (2003). The utility of case study methodology in mathematics teacher preparation. *Teacher Education Quarterly, 30*(1), 113-135.

Merriam, S. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey-Bass Publishers.

Merriam, S. B. (1998). *Qualitative research and case study applications in education. "Revised and Expanded from Case Study Research in Education".* San Francisco: Jossey-Brass Publishers.

Metz, L. M., & Simmt, S. M. E. (2015). Researching mathematical experience from the perspective of an empathic second-person observer. *ZDM Mathematics Education, 47*, 197–209. doi:10.1007/s11858-014-062-z

MOEC. (1995). Ministry of Education and Culture: The education and training policy. Dar es salaam, Tanzania: MoEVT.

MoEST. (2016). *National Basic Education Statistics in Tanzania (BEST) 2012-2016.* Ministry of Education, Science, and Technology, Dar-er-salaam. Retrieved May 2018, from http://www.moe.go.tz/sw/publications/category/30-statistics-takwimu.html

MoEST. (2017). *Orodha ya Walimu wa Stashahada wa Masomo ya Sayansi na Hisabati Waliohitimu Miaka ya Nyuma 1999-2014.* Dar es Salaam: Ministry of Education, Science, and Technology. Retrieved May 2018, from http://www.moe.go.tz/sw/publications/send/34-selections-uchaguzi/331-orodha-ya-wahitimu-wa-shahada-ya-ualimu-kabla-ya-2015.html

MoEST. (2017). *Orodha ya Walimu wa Stashahada wa Masomo ya Sayansi na Hisabati Waliohitimu Mwaka 2015.* Ministry of Education, Science, and Technology. Retrieved May 2018, from http://www.moe.go.tz/sw/publications/send/34-selections-uchaguzi/326-orodha-ya-wahitimu-wa-stashahada-ya-ualimu-wenye-sifa-mwaka-2015.html

MoEVT. (2007). *Curriculum for Diploma in Teacher Education.* Dar es Salaam: Ministry of Education and Vocational Training.

MoEVT. (2008). *Education sector development programme: Teacher development and management strategy (TDMS) 2008-2013.* Dar es Salaam: Ministry of Education and Vocational Training.

MoEVT. (2012). *National Basic Education Statistics in Tanzania (BEST), 2012.* Dar es Salaam: Ministry of Education and Vocation Training. Retrieved November 2015

MoEVT. (2014). *Curriculum information report for higher diploma in secondary education (NTA 7) (science).* Dar es Salaam: Ministry of Education and Vocational Training.

Morgan, D. L. (1988). *Focus groups as qualitative research .* Newbury Park, CA: Sage.

Morrel, R. (2000). Corporal Punishment. *Education Monitor, 11*(1), 42.

Morrel, R. (2001). Corporal punishment in South African schools: a neglected explanation for its persistent. *South African Journal of Education, 21*(4), 292-299.

NECTA. (2009). *The National Examination Council of Tanzania: Certificate of secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2009). *The National Examination Council of Tanzania: Diploma in secondary education examinations.* Statistics department. Dar-es salaam: NECTA.

NECTA. (2011). *The National Examination Council of Tanzania: Certificate of secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2011). *The National Examination Council of Tanzania: Diploma in secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2012). *The National Examination Council of Tanzania: Diploma in secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2013). *The National Examination Council of Tanzania: Certificate of secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2014). *The National Examination Council of Tanzania: Certificate of secondary education examinations.* Statistics department. Dar es salaam: NECTA.

NECTA. (2014). *The National Examination Council of Tanzania: Diploma in secondary education examinations.* Statistics Department. Dar es Salaam: NECTA.

NECTA. (2015). *The National Examination Council of Tanzania: Certificate of secondary education examinations.* Statistics department. Dar es salaam: NECTA.

Patterson, M.E., & Williams,D.R. (2002). *Collecting and Analysing Qualitative Data: Hermeneutic Principles, Methods, and Case Examples.* Champaign, IL: Sagamore Publishing.

Patton, M. (1990). *Qualitative evaluations and Research method* (Second edition ed.). Newbury Park: SAGE.

Polanyi, M. (1996). *The tacit dimension.* New York: Doubleday.

Polkinghorne, E. D. (1995). Narrative configuration in qualitative analysis. *International Journal of Qualitative Studies in Education, 8*(1), 5-23. doi:10.1080/0951839950080103

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of educational research, 75*(2), 211–246.

Richardson, V. (1997). Constructivist teaching and teacher education: Theory and practice. In V. Richardson (Ed.), *Construction of teacher education: building a world with new understanding* (pp. 3-14). London, UK: Falmer press.

Sawyer, R. K. (2014). *The Cambridge handbook of the learning sciences.* Cambridge: Cambridge University Press.

Schiefele, U. (1991). Interest, Learning, and Motivation. *Educational Psychologist, 26*(3-4), 299-323. doi:10.1080/00461520.1991.9653136

Schmidt, W.H., Houang, R., & Cogan, L.S. (2011). Preparing future math teachers. *Science, 332*, 1266–1267. Retrieved from http://news.msu.edu/media/documents/2011/06/0d939cfe-4ff2-49ed-9ab2-e41b3a2afb16.pdf

Sfard, A. (2008). *Thinking as communicating:human development, the growth of discourses, and mathematizing.* New York: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14. doi:10.2307/1175860

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22. Retrieved from http://hepgjournals.org/doi/pdf/10.17763/haer.57.1.j463w79r56455411

Sichizya, F. (1997). Teaching and learning mathematics in Tanzania secondary schools. A paper presented in the Symposium in Mathematics Modelling Workshop on mathematics education. Arusha.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499-511. Retrieved from http://pat-thompson.net/PDFversions/2008SilvermanThompsonMKT.pdf

Simmt, E. (2011). Teacher expertise explored as mathematics for teaching. In Y. Li, & G. Kaiser (Eds.), *Expertise in Mathematics Instruction* (pp. 151-164). New York: Springer Science+Business Media.

Skemp. (1978). Relational Understanding and Instrumental Understanding. *The Arithmetic Teacher, 26*(3), 9-15. Retrieved from http://www.jstor.org/stable/41187667

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconception reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Science, 3*, 115-163.

Speer, N., King, K., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education, 18*(2), 105–122.

Stake, R. E. (1995). *The art of case study research .* Thousand Oaks,CA: Sage.

Thames, M. H. & Ball, D. L. (2010). What mathematical knowledge does teaching require? *Teaching Children Mathematics, 17*(4), 220–229.

Usiskin, Z., Peressin, A., Marchsotto, E.A., & Stanley, D. (2003). *Mathematics for high school teachers.an advanced perspective.* Upper Saddle River, NJ: Pearson.

Weber, S. J. (1986). The nature of interviewing. *Phenomenology and pedagogy, 4*(2), 65-72. Retrieved from https://ejournals.library.ualberta.ca/index.php/pandp/article/view/15020

Yin, K. R. (2014). *Case study research: Design and methods (5th Ed.).* Thousand Oaks: Sage.

Appendix A: Research Consent Documents

[redacted]

[redacted]

COLLEGE OF EDUCATION

SCHOOL OF CURRICULUM AND TEACHER EDUCATION

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Ref: T/UDOM/COED/CIDA/86 11th August, 2016

The Permanent Secretary
Ministry of Education Science and Technology
P.O. Box 9121
Dar es Salaam **Attention**: Director, Policy and Planning

Ref: Research Permit for Ratera Safiel Mayar

Please, this is to inform you that **Ratera Safiel Mayar** is a doctoral student under UDOM/University of Alberta agreement. The agreement focuses on *Capacity Development for Mathematics Teachers in Rural and Remote areas in Tanzania*. The collaboration between the University of Dodoma (UDOM) and the University of Alberta (UAlberta) started in 2012 and is funded by Global Affairs Canada (formally Cida).

Ratera is to conduct a study titled, *Developing Mathematics for Teaching through Concept Study: A Case of Ordinary Level Secondary School Pre-service Teachers in Tanzania –* [redacted] Teacher College. The research is part of the fulfillment of PhD study accredited by the University of Alberta. The time frame for the research work is between 1st September to 25th January, 2017. Research participants will be [redacted] Diploma in Secondary Education (science - maths majors) student teachers who are in their second year of study. Given the importance of the study, especially its focus on mathematics education, I kindly request research permit be given to the said doctoral student to carry out the study in order to fulfill one of the necessary conditions for graduation.

Thank you in advance for your permission and, the continued support in Mathematics education.

Dr. Andrew L. Binde/Project Director, UDOM

cc: VC
DVC-ARC
Director TIE

[redacted]

FACULTY OF EDUCATION
DEPARTMENT OF SECONDARY EDUCATION

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August 14, 2016

The General Secretary
Ministry of Education, Science and Technology
P.O BOX 9121
Dar es Salaam
Tanzania

Dear Sir/Madam,

RE: REQUEST OF A RESEARCH PERMIT

The heading above refers. My name is **Ratera S. Mayar** a Tanzanian PhD candidate in the Faculty of Education, Department of Secondary Education in the Faculty of Education at the **University of Alberta, Canada** and a ~~Senior~~ Principal Curriculum Coordinator (mathematics) at Tanzania Institute of Education (TIE) in the department of Research, Information and Publication (RIP).

I am writing to kindly request permission to carry out a PhD research entitled**Developing Mathematics for Teaching through Concept Study: A case of Ordinary Level Secondary School Pre-service Teachers in Tanzania.**This study is a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Secondary Education at the Department of Secondary Education of theFaculty of Education at the University of Alberta, Canada.

The purpose of the study is to investigate the contribution of concept study on Tanzanian ordinary level secondary school pre-service teachers' mathematics for teaching (professional knowledge and skills). It is a PhD research study of a professional development activity investigating mathematics for teaching through a professional development strategy called concept study. The participants will be Higher Diploma in Education (science-mathematics majors) pre-service teachers at [redacted] Teachers College. The participants who will agree to participate in this study will participate in face to face open ended interview, complete pre and post questionnaires and participate in four concept study sessions. This study will be conducted from September, 2016 to January, 2017.

Please, find the attached letter from the Director of the Project titled, Capacity Development for Mathematics Teachers in Rural and Remote Communities in Tanzania.

I would be pleased if my letter will meet warm response from you.Thank you in advance.

Sincerely,

Ratera S Mayar
A PhD Candidate (Department of Secondary Education, University of Alberta, Canada).

The plan for this study has been reviewed for its adherence to ethical guidelines by the Research Ethic Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office, at 1 780 492-2615. Other question regarding this research contact Ratera S Mayar (ratera@ualberta.ca or 1 780 554-4137 or +255 658 216 227) or Prof. Elaine Simmt (esimmt@ualberta.ca or 1 780 492-0998).

THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION, SCIENCE AND TECHNOLOGY

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P.O.Box 9121,
7 Magogoni Street,
11479 DAR ES SALAAM.

In reply please quote:

28th August, 2016.

Ref. No. ED/EP/ERC/VOL.VII/36

The: Regional Administrative Secretary [redacted]

ATT: Regional Education Officer

RE: RESEARCH CLEARANCE FOR MS. RATERA SAFIEL MAYAR

The captioned matter above refers. The mentioned is bonafide student of The University of Alberta-Canada who is conducting research on the topic titled **"Developing Mathematics for Teaching through Concept Study: A case of Ordinary Level Secondary School Pre-service Teachers in Tanzania"** as part of her course program for the award of PhD Degree of Mathematics Education.

For the purpose of accomplishing this study, a researcher will therefore need to collect data and necessary information related to the research topic from [redacted] Teachers College.

In line with the above information you are being requested to provide the needed assistance that will enable her to complete this study successfully.

The period by which this permission has been granted is from **September 2016 to January, 2017.**

By copy of this letter, Ms. Ratera S. Mayar is required to submit a copy of the report (or part of it) to *the Permanent Secretary, Ministry of Education and Vocational Training* for documentation and reference.

Yours truly,

Gerald Mweli
For: PERMANENT SECRETARY

CC: Ms. Ratera Safiel Mayar - University of Alberta- Canada

THE UNITED REPUBLIC OF TANZANIA
PRESIDENT'S OFFICE
REGIONAL ADMINISTRATION AND LOCAL GOVERNMENT

[redacted] REGION:
ADDRESS TEL. "REGCOM"
TEL: NO [redacted]
Fax No. [redacted]
E-Mail: [redacted]@gmail.com
E-Mail: [redacted]@yahoo.com

REGIONAL COMMISSIONER'S OFFICE
EDUCATION DEPARTMENT
P.O. BOX [redacted]

In Reply Please Quote

RC/[redacted]/ED/R.20/VOL.11/306

05TH September, 2016

District Executive Director,
[redacted] District,
[redacted]

RE: INTRODUCTION TO MS RATERA SAFIEL MAYAR

Please refer to the above subject.

I would like to introduce to you Ms Ratera Safiel Mayar a Student at the University of Alberta - Canada. She wants to conduct a research titled **"Developing Mathematics for Teaching through Concept Study. A case of Ordinary Level Secondary School Pre –Service Teachers in Tanzania in [redacted] District"**. The research will be conducted From September, 2016 to January, 2017.

Kindly assist her with the necessary cooperation.

Thank you for cooperation.

[redacted]

For:- REGIONAL ADMINISTRATIVE SECRETARY
[redacted]

C.C: Miss Ratera S. Mayar- Student (University of Alberta - Canada)

[redacted] DISTRICT COUNCIL

[redacted] REGION
All Correspondences to be addressed [redacted]
The District Executive Director,
Tel. No. [redacted] G.L
[redacted] D.L
Fax No. [redacted]/361
E-mail: [redacted]district.go.tz

COUNCIL HALL,
P.O. BOX [redacted]

In reply please quote:
Ref. No. HW/[redacted]/R5/1/183

05th September, 2016

University of Alberta,
347 Education South,
11210 – 87 Avenue,
Edmonton, Alberta, Canada T6G 2G5.

RE: RESEARCH PERMIT.

Refer your letter with reference no. RC/[redacted]ED/R.20/VOL.11/306 dated **05th September, 2016.**

This is to confirm that the permit has been granted to **Ms.*Ratera Safiel Mayar,*** a PhD student at the University of Alberta, Canada.

The title of the research is "Developing Mathematics for Teaching through Concept Study. A case of Ordinary Level Secondary School Pre- Service Teachers in Tanzania" beginning **September 2016** to **January 2017.**

It's our expectation that the said data collection will enable her towards achieving her objective.

In due course, there will be no financial implications to the Council.

[redacted]
For: DISTRICT EXECUTIVE DIRECTOR,
[redacted] DISTRICT COUNCIL.

Kny: Mkurugenzi Mtendaji
Halmashauri ya Wilaya [redacted]

Copy to:

- **Principal –** [redacted] **Teachers College,** you are requested to accept **Ms.*Ratera Safiel Mayar*** and assist her during her research work.
- **Ms.*Ratera Safiel Mayar,***
  A PhD Student of University of Alberta.

[redacted]

FACULTY OF EDUCATION
DEPARTMENT OF SECONDARY EDUCATION

347 Education South
11210 - 87 Avenue
Edmonton, Alberta, Canada T6G 2G5
Tel: 780.492.3674
Fax: 780.492.9402
e-mail: educ.sec@ualberta.ca
http://www.secondaryed.ualberta.ca

Dear Pre-service teachers:

My name is Ratera S. Mayar and I am a PhD candidate in the Faculty of Education, Department of Secondary Education at the University of Alberta. I am interested in conducting a PhD research study of a professional development activity investigating mathematics for teaching through a professional development strategy called concept study. The quality of mathematics teacher education (MTE) is among the factors that contributes to the quality of mathematics education (ME) in secondary schools. Mathematics education in ordinary secondary schools in Tanzania is concerning to the federal government because of the dismal failure rates on the certificate of secondary education examination (CSEE). The reliance on teachers who have only a diploma is believed to be one of the reasons for to cause the failure rates; it is conjectured that the teachers received insufficient training preparation in college (Chonjo, Osaki, Possi & Mrutu, 1996). The majority of these teachers lack sufficient knowledge of the subject matter, what to teach, and how to teach the subject matter effectively (Chonjo et al., 1996; Sichizya, 1997). The purpose of the study is to investigate the contribution of concept study on Tanzanian ordinary level secondary school pre-service teachers' mathematics for teaching (professional knowledge and skills). This research is entitled **Developing Mathematics for Teaching through Concept Study: A Case of Ordinary Level Secondary School Pre-service Teachers in Tanzania.**

Purpose and Procedure

The purpose of this letter is to invite mathematics major pre-service teachers to participate in my PhD research study. Your participation would involve a face to face interview and participating in four concept study sessions. The face to face interview will be 1 hour and will be done before commencing of concept study sessions. One week before face to face interview you will be provided with pre-interview activities (PIA) which, we will discuss during face to face interview. The PIA and face to face interview will be done in September 2016. The concept study sessions will be spread across four consecutive months from October 2016 to January 2017. Each session will include a 40 minutes pre-session questionnaire, a 3 hour workshop in which participants will work together to explore and unpack a mathematics concept taught in school mathematics and a 1 hour post session individual questionnaire. The concept studies workshop will be conducted on one Saturday [actual dates] each month for four consecutive months at [redacted]eachers College.

As a PhD candidate and a researcher I will conduct all aspects of this research. I will audio record the face to face interviews; collect the pre session questionnaires and post session questionnaires; video and audio record the concept study workshop, collect your working papers, take photographs of the board and any concrete materials used in the workshops; and take notes during the workshop.

Voluntary Participation
You are under no obligation to participate in this research, but if you choose to participate you must be aware that your contributions to the concept study will be used for my PhD research. Further, I may rework the research for future research or publication or presentation to the wider mathematics education community in a professional or research journal or monograph or at a conference. In the event you participate and then change your mind about having your contributions used for my PhD research, future research or publication you must inform me within 2 weeks of the final concept study workshop. This short time frame is necessary because I will be writing my report immediately upon completion of data collection.

Confidentiality and Anonymity
Your anonymity will be protected in my research proposal and in any public presentations and publications through the use of pseudonyms. Any names of participants will be removed from the data and all data will be treated collectively rather than individually. The name of the teacher college will not be used in any publications or presentations. Video cards and working papers will be kept in secure cabinets. Electronic word and jpeg files will be kept on password-protected computers. Because this session involves other participants you are asked (as are all participants) to respect the privacy of others and keep their particular comments and contributions to the session anonymous.

Risk and Benefits
Given the collaborative nature of the concept study workshop we do not foresee any risks for you. Indeed the research literature (Davis & Renert, 2014) suggests this form of professional development shows great promise for enhancing the mathematics knowledge teachers use in their classrooms. Also of benefit to participants will be copies of any papers published from the workshops. For participants who want a copy of papers published from this work, please contact Ratera S Mayar or Prof Elaine Simmt.

Further Information
For any questions you may have about this study you may contact the facilitator Ratera S Mayar (ratera@ualberta.ca or +255 658 216 227 or 780 554-4137) or Prof. Elaine Simmt (esimmt@ualberta.ca or 1 780 492-0998). Thank you for your consideration. The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at (780) 492-2615.

Sincerely,

Ratera S Mayar

[redacted]

FACULTY OF EDUCATION
DEPARTMENT OF SECONDARY EDUCATION

347 Education South
11210 - 87 Avenue
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http://www.secondaryed.ualberta.ca

Consent Form

Study Title: Developing Mathematics for Teaching through Concept Study: A Case of Ordinary Level Secondary School Pre-service Teachers in Tanzania

I have read and understand the attached information letter and hereby agree to participate in the 1 hour pre concept studies face to face interview, four concept studies each (40 minutes pre questionnaires, 3 hours concept study workshop [Ratio, Proportion, Rate, and Linear functions] and 1 hour post questionnaires) facilitated by Ratera S Mayar on [.................date...............] in [redacted] Teachers College, [redacted] in Tanzania.

In the event I wish to opt out of the study I will step out of the workshop and/or request no later than two weeks post workshop any contributions that can be attributed to me be removed from the data set.

In the event I want copies of publications that result from this research I will contact Ratera S Mayar (ratera@ualberta.ca or +255 658 216 227or +1 780 554-4137) or Prof. Elaine Simmt (esimmt@ualberta.ca + 1 780-492-0998).

Print Participant Name | Signature | Date

Appendix B: Interview Schedules for Pre-service teachers

VENUE	(Name of the college hidden for anonymity) TEACHER’S COLLEGE	
TIME	**FIRST DAY**	**SECOND DAY**
8.00AM-9.00AM	First interviewee	Seventh interviewee
9.00AM-9.20AM	Break	Break
9.20AM- 10.20AM	Second interviewee	Eighth interviewee
10.20AM-10.50AM	Tea Break	Tea Break
10.50AM-11.50AM	Third Interviewee	Ninths Interviewee
11.50PM-12.10PM	Break	Break
12.10PM- 1.10PM	Fourth Interviewee	Tenth Interviewee
1.10PM-2.10PM	Lunch Break	Lunch Break
2.10PM.3.10PM	Fifth Interviewee	Eleventh Interviewee
3.10PM-3.30PM	Break	Break
3.30PM-4.30PM	Sixth Interviewee	Twelfths Interviewee

Appendix C: Open-ended interview questions for pre-service teachers

Group1: Questions about pre-service teachers' experience in learning and doing mathematics over her/his life-time

1. Do you recall what mathematics was like for you as a young child in school? Do you remember any routine activities in math classes from those years? What were those activities like for you?
2. What was mathematics like for you as a student in lower grades?
3. What was mathematics like for you as a student in higher grades?
4. How would you compare mathematics to other subjects in school?
5. When you look back over your own years as a student do you recall any mathematics teachers that you appreciated or had special admiration for? (Follow up if necessary) What was special about that math teacher?
6. When you felt like you were doing the best work or your best learning in mathematics, what was the topic of study and what teaching/learning approach being used?

Group 2: Questions about pre-service teacher being a student teacher

7. As you were studying and preparing to join a teacher-training program, what did you expect teacher training to be like? What parts did you think would be great and what parts of learning were you perhaps concerned about?
8. Were there any big surprises after you started you teacher-training program? Were some parts of training greatly different from what you expected?
9. When you think about learning to teach mathematics to ordinary secondary school students, what are some of the parts you like best? And what are some of the parts that you wish were better?

10. Think about mathematics teaching. What professional knowledge should a teacher have for teaching mathematics?
11. If you think about learning mathematics, how could you learn, with your colleague student teacher(s), the mathematics you need to know as a student teacher|?
12. What advice would you offer to teacher educators or tutors who prepare people to be mathematics teachers?
13. If you could have asked to change one thing about your own teacher preparation program what would you change to make it better?

Appendix D: Pre-service Teacher's Pre-Interview Activities (PIA)

Dear (Name…………………………………………)

Thank you for agreeing to participate in an interview with me for my PhD research.

Purpose of the interview:

My research interest is in the area of mathematics learning and teaching. More specifically, I am interested in the way that diploma in secondary education- mathematics majors pre-service teachers in Tanzania experience learning mathematics. In our interview I hope to learn something about how you experience/have experienced learning mathematics.

There are two parts to the interview:

- Pre-Interview Activities (PIAs) (adapted from the work of Ellis, 2006) and
- Open-ended Questions

PIAs -Research topic (Learning and teaching Mathematics)

Please complete one of the following diagrams, drawings, or visual representation activities and bring it to our interview. Please use pens, pencils and preferably coloured markers on blank paper. We will begin our interview by having you show me and tell me about the diagram, drawing, or visual representation that you completed. There are a number of purposes for using the PIAs. By completing these PIAs in a quiet time you may have a better chance to remember more ideas or details to include. And for another, looking at the diagrams or drawings while you talk about them may help me to see how your ideas fit together and what you mean by some of the words you use. The PIAs can give us a better chance for you to tell me about your experience.

1. Use three colours to make a diagram or an abstract drawing that shows the way you experience learning mathematics.
2. Make two drawings, one showing a good day learning mathematics and another showing a not-so-good day learning mathematics. Feel free to use colours, symbols, and words. Also feel free to use thought bubbles or speech bubbles.
3. Think of an event or idea that changed the way you learn mathematics. Make two drawings to show what learning mathematics was like for you before and after the change. Feel free to use thought bubbles or speech bubbles.
4. Make a list of 20 important words that come to your mind when you think about learning mathematics. Then divide the list into two groups in any way that makes sense to you and copy the words into two separate lists. Please bring all three lists to the interview.
5. Make a timeline of your career as a mathematics student showing key events or ideas that have changed the way you approach or experience learning mathematics
6. Use colours to make three drawings that symbolize how your experience of learning mathematics has changed over time.

Open-ended Questions

After we finish chatting about the Pre-Interview Activities you bring, I will ask some open-ended questions that may help you think of other memories or stories you might be able to share. The questions are about the research topic or larger experiences leading up to the research topic.

Appendix E: Pre-concept study questionnaires

Pre-concept study questionnaire for pre-service teachers

Given Research Name: __________________________ Female___ Male ___

Date__________________________ Age: ______________

1. Thinking about teaching mathematics, what professional knowledge should a teacher have for teaching mathematics?

   ……………………………………………………………………………………

2. Tell me what you know about **(Concept to be studied)** in mathematics?

Note: The concept will be either ratio, rate, proportions or linear functions, depending on the concept that will be involved in the main activity of the concept study.

   …………………………………………………………………………………….

3. How do you learn **(Concept to be studied)**? Elaborate.

   ……………………………………………………………………………………..

4. Do you think you could learn with your colleague teacher(s) the mathematics you need to know in order to teach? Elaborate.

   ………………………………………………………………………………………

Note: Enough space will be provided for each question

Appendix F: Post-concept study questionnaires

Post-concept study questionnaire for pre-service teachers

Given Research Name: __________________________ Female___ Male ___

Date__________________________ Age: ______________

1. Thinking about teaching mathematics, what professional knowledge should a teacher have for teaching mathematics?

   ………………………………………………………………………………………

2. Tell me what you know about **(Concept studied)** in mathematics?

Note: The concept will be either ratio, rate, proportions or linear functions depending on the concept involved in the main activity of concept study.

   ……………………………………………………………………………………….

3. How do you learn **(Concept studied)**? Elaborate

   ……………………………………………………………………………………..

4. Do you think you could learn with your colleague teacher(s) the mathematics you need to know in order to teach? Elaborate.

   ……………………………………………………………………………………………

5. Do you think a concept study helps teachers' deep understanding of mathematics? How? Elaborate

   ……………………………………………………………………………………………

6. What did you learn from the concept study of **(concept studied)**? Elaborate.

………………………………………………………………………………………

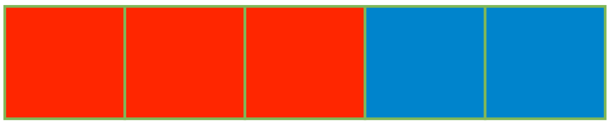
Appendix G: Concept study of ratio scripted questions

Concept study of ratio plan

1. Thinking about the concept of ratio in mathematics, "Could you tell me what you know about the ratio concept in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."

   The scripted extra questions.

2. In a group of 3-4, Observe the figure 1



   a) What does each of the following expression mean to you? Elaborate
      i. 3:2
      ii. 2:3
      iii. 3:5
      iv. 2:5

   b) How is part i) and ii) different from iii) and iv)? Explain

   c) Write in another way each part from i) to iv) without changing the meaning and if possible, explain your answers.

   d) How is figure 2 related to figure 1? Explain

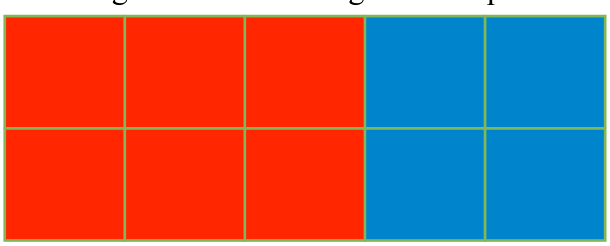


Figure 2

3. Cooking 3kg of coconut rice for twelve people needs 4 medium size coconuts. How many kg of rice and medium coconut would be needed for forty-eight people?

   i. Explain how you got the answer.

   ii. What errors do you expect student can make in solving this problem? Elaborate

4. Twelve kilograms of sweet potatoes cost Tanzanian Shillings (TShs) 4000. If you want to buy sixteen kilograms of sweet potatoes, how much will you pay? Explain different ways of solving this problem?

5. In Tanzania Mathematics syllabus of ordinary level secondary school which class (form) the concept of ratio is taught?

6. What basic concept/s do you think the student needs to know before engaging them in learning the concept of ratio?

7. Within the class (form) in which the concept of ratio is taught which other concept/s are taught in mathematics are related to ratio concept?

8. Which concept/s taught across an ordinary level secondary school in mathematics syllabus are related to the concept of ratio?

9. Which concept/s in other subjects taught across an ordinary level secondary school subjects are related to the concept of ratio?

10. With examples, explain how you can facilitate the student learn the applications of the ratio concept outside school environment or real-life situations.

11. What questions do you expect the students could ask during classroom instruction when learning the concept of ratio? Give examples from your experience and possibly what teachers need to do to help the student.

Appendix H: Concept study of proportion scripted questions

Concept study of proportion plan

1. Thinking about the concept of proportion in mathematics, "Could you tell me what you know about the concept of proportion in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications".

The scripted extra questions.

2. In a group of 3-4, study the table 1, table 2, table 3, table 4 and table 5.

Table 1

Student	1	2	3	4	5	30
Apples	3	6	9			

Table 2

Student	1	30	15
Apples	3	90	

Table 3

Student	1	2	3	4	5	11
Apples	3	6	9	12		

Table 4

Student	1	2	3	4	14	9
Apples	3	6	9	12	42	

Table 5

14	35
18	

a. Find the missing number and explain how you found it?

b. How will you explain to your student the procedures in finding the missing numbers in each table 1 to 5 above?

c. What concept/s can be introduced by the use of these tables?

d. Write all ratios obtained from each table

e. Write any two pairs of equivalent ratios in each table above?

f. What does each pair of equivalent ratios in part ‘e’ represent?

3. In Tanzania Mathematics syllabus of ordinary level secondary school which class (form) the concept of proportion is taught?

4. What basic concept/s do you think the student needs to know before engaging them in learning the concept of proportion?

5. Within the class (form) in which the concept of proportion is taught which other concept/s are taught in mathematics are related to proportion concept?

6. Which concept/s taught across an ordinary level secondary school in mathematics syllabus are related to the concept of proportion?

7. Which concept/s in other subjects taught across an ordinary level secondary school in mathematics syllabus are related to the concept of proportion?

8. With examples, explain how you can facilitate the student learn the applications of the proportion concept outside school environment or real-life situations.

9. What questions do you expect the students could ask during classroom instruction when learning the concept of proportion? Give examples from your experience as a student or a prospective teacher of ordinary level secondary school and possibly what teachers need to do to help the student.

Appendix I: Concept study of rate scripted questions

Concept study of rate plan

1. Thinking about the concept of Rate in mathematics, "Could you tell me what you know about the concept of Rate in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."

   The scripted extra questions.

2. Table 1 represent the tap water that fills the tank at constant rate and Table 2 represent the car traveling at constant speed.

Table 1

Litres of water filled in the tank	1	2	3	10	50	1000
Time taken to fill the tank in minutes	3	6				

Table 2

Distance travelled by a car in Kilometres (Km)	450	350	250	150	100	50
Time taken in hours	5					

In a pair of student teachers,

a. Find the missing number in table 1 and table 2 and explain how you found it?

b. How will you explain to your student the procedures in finding the missing number?

c. What concept/s can be introduced by the use of these tables?

d. Write all ratios obtained from the table 1

e. Write all ratios obtained from the table 2

f. What does the ratios in part 'd' represent?

g. What does the ratios in part 'e' represent?

h. What is the constant rate the tap is filling the tank?

i. What is the constant speed of the car?

3. Four kilograms of onions cost Tanzanian Shillings (TShs) 5000.00. If you want to buy twenty-seven kilograms of onions, how much will you pay?

    a) Explain different ways of solving this problem?

    b) Which way in part a) do you prefer in introducing the concept of rate to an ordinary level secondary school student and why?

4. Which is a better price for Korie cooking oil: 29,000.00 Tanzanian shillings for 10litres or 54,000.00 Tanzanian Shillings for 20 litres? Explain how you obtained your answer

5. In Tanzania Mathematics syllabus of ordinary level secondary school which class (form) the concept of rate is taught?

6. What basic concept/s do you think the student needs to know before engaging them in learning the concept of rate?

7. Within the class (form) in which the concept of Rate is taught which other concept/s are taught in mathematics are related to rate concept?

8. Which concept/s taught across an ordinary level secondary school in mathematics syllabus are related to the concept of rate?

9. Which concept/s in other subjects taught across an ordinary level secondary school are related to the concept of rate?

10. With examples, explain how you can facilitate the student learn the applications of the rate concept outside school environment or real-life situations.

11. What questions do you expect the students could ask during classroom instruction when learning the concept of rate? Give examples from your experience as a student or a prospective teacher of ordinary level secondary school and possibly what teachers need to do to help the student.

❖ Asha drove 450 km in 5 hours. At this rate how far could she travel 9 hours? Explain different ways in solving this problem.

❖ Charles read 30 pages of a book in 5minutes. How many pages will he read in 70 minutes if he reads at a constant rate? Explain different ways in solving this problem.

Appendix J: Concept study of linear function scripted questions

Concept study of linear function plan

1. Thinking about the concept of linear function in mathematics, "Could you tell me what you know about the concept of linear function in mathematics? Elaborate and where possible provide supporting examples, images, algorithms, and applications."

The scripted extra questions.

2. Table 1 represent the recorded cost of different litres of petrol consumed by seven different cars in one of the petrol stations in Dar es Salaam.

Cost of Petrol in Tanzanian Shillings (TShs)	2000	4000	8000				
Number of litres filled the cars	1	2	4	8	12	15	20

a) What is the constant rate of change/cost rate per litre of petrol? How did you find it?

b) Fill the table

c) Draw the graph of table 1

d) What type of graph did you get?

e) What is the rise of the graph for 1 litre increase of petrol?

f) Explain different ways of how to get the cost of

i. 15 litres of petrol?

ii. 3 litres of petrol?

g) Write the equation corresponding to table 1 and define your variables

h) How is part a) and part e) related?

i) From your knowledge of linear function, what is the slope of the graph and how is it related to part a) and part e)?

3. Table 1 represent the tap water that fills the tank at constant rate and Table 2 represent the car traveling at constant speed.

Table 2

Litres of water filled in the tank	1	2	3	10
Time taken to fill the tank in minutes	3	6		

Table 3

Distance travelled by a car in Kilometres (Km)	450	350	250	150	100	50
Time taken in hours	5					

a. Fill table 2 and table 3 and draw their graphs using different axes

b. What concept/s can be introduced by the use of these tables?

c. What is the constant rate the tap is filling the tank?

d. What is the constant speed of the car?

e. Compare the slopes/gradients of the graphs in part a) with the answers in part c) and part d)

4. From your knowledge of linear function f(x) = mx + c, OR linear equation y = mx + c

   i. What is 'm' and 'c'?

   ii. Fill in the blanks

      a) Numerically c is ....when x=0.

      b) Graphically c is ....of the graph.

      c) Numerically f(x) or y increases by ....units for every.......unit increase of...........

   iii. For positive ...., the graph rises .... units for every 1 unit of ...... to the right, and rises f(x) =..... for every .... units moved to the right

   iv. For negative ...., the graph drops |......| units for every 1 unit move to the right, and drops f(x) = |.....| x units for every ....units moved to the right.

5. In Tanzania mathematics syllabus of ordinary level secondary school which class (form) the concept of linear function is taught?

6. What basic concept/s do you think the student needs to know before engaging them in learning the concept of linear function?

7. Within the class (form) in which the concept of linear function is taught which other concept/s are taught in mathematics are related to linear function concept?

8. Which concept/s taught across an ordinary level secondary school in mathematics syllabus are related to the concept of linear function?

9. Which concept/s in other subjects taught across an ordinary level secondary school are related to the concept of linear function?

10. With examples, explain how you can facilitate the student learn the applications of the linear function concept outside school environment or real-life situations.

11. What questions do you expect the students could ask during classroom instruction when learning the concept of linear function? Give examples from your experience as a student or prospective teacher of ordinary level secondary school and possibly what teachers need to do to help the student.