Selected Topics in Valuation of Financial and Insurance Contracts

by

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Abstract

In this thesis, selected topics on valuation and hedging of financial and insurance contracts are studied. First of all, we study the most common in mathematical finance Black-Scholes market and provide an alternative derivation of the famous Black-Scholes formula from the binomial option pricing model. Secondly, we develop a method for pricing and hedging the equity-linked life insurance contracts without switching to a new probability measure, using quadratic risk-minimization criterion. Thirdly, we consider a quantile hedging problem for the Black-Scholes and jump-diffusion markets and extend existing results in this subarea by introducing dividends. Application to pricing and hedging the equity-linked life insurance contracts is demonstrated. Fourthly, we study a market with defaultable securities and develop a quantile hedging methodology for this market, providing insurance applications. Finally, we revisit the Bachelier model – the first model of the financial market in mathematical finance history. We study the modification of the classical Bachelier model by absorbing the stock price at zero and give alternative proofs for the option pricing formulas on this market. Using these results, we develop a quantile hedging methodology and provide insurance applications for both classical and modified Bachelier markets.

Preface

This thesis is based on a collection of published and submitted for publication research articles.

Chapter 1 has been published as Anna Glazyrina and Alexander Melnikov, "Bernstein's inequalities and their extensions for getting the Black-Scholes option pricing formula", *Statistics and Probability Letters* 111, 86–92, 2016, doi:10.1016/j.spl.2016.01.002.

Chapter 2 has been published as Anna Glazyrina and Alexander Melnikov, "Quadratic hedging of equity-linked life insurance contracts under the real-world measure in discrete time", *Risk and Decision Analysis* 6(2), 167-175, 2017, doi:10.3233/RDA-170121. This research was also presented during Alberta Mathematics Dialogue conference at Mount Royal University (Calgary, Canada) in April 2016.

A version of Chapter 3 has been submitted for publication as Anna Glazyrina and Alexander Melnikov, "Quantile hedging in models with dividends and application to equity-linked life insurance contracts".

A version of Chapter 4 has been submitted for publication as Anna Glazyrina and Alexander Melnikov, "Quantile hedging in a defaultable market with life insurance applications".

A version of Chapter 5 has been submitted for publication as Anna Glazyrina and Alexander Melnikov, "Bachelier model with stopping time and its insurance application".

In all the joint papers I was responsible for the proofs of the results and their applications as well as the manuscript composition. Dr. Melnikov was the supervisory author who was involved in concept formation and advised on general approaches for the proofs.

In theory there is no difference between theory and practice,

but in practice there is.

– Attributed to multiple people

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Introduction

The focus of this thesis is to study different aspects of valuation and hedging for financial and insurance contracts. It consists of five self-contained chapters, each investigating its own rather independent problem related to option pricing theory, partial hedging methods, actuarial science, or a combination of these fields. Each chapter is based either on a published paper or on a preprint currently under review in a journal.

In Chapter 1 (Glazyrina and Melnikov [22]) we give an alternative proof of the Black-Scholes option pricing formula using results of Bernstein [4] as well as Zubkov and Serov [70] on the normal approximation to the binomial distribution.

In Chapter 2 (Glazyrina and Melnikov [23]) we consider a quadratic risk-minimization problem in the framework of discrete-time financial market and develop pricing and hedging algorithms by means of finding a discounting portfolio (a numeraire) such that discounted price processes are martingales under the physical probability measure. We demonstrate the applications to pricing and hedging of equity-linked life insurance contracts assuming a binomial financial market.

Chapter 3 (Glazyrina and Melnikov [26]) deals with a problem of quantile hedging in markets with dividends. We derive explicit formulas for pricing and hedging the European contingent claim assuming the Black-Scholes and the jump-diffusion models of the financial market and provide insurance applications.

Chapter 4 (Glazyrina and Melnikov [25]) is devoted to quantile hedging in a market with defaultable securities. Both perfect and quantile hedging strategies are given for a European call option on a vulnerable equity and insurance applications are demonstrated.

In Chapter 5 (Glazyrina and Melnikov [24]) a modification of a classical Bachelier model by letting a stock price absorb at zero is revisited. We give alternative proofs of option pricing formulas under the modified Bachelier model and use these results to develop quantile hedging methodology for both classical and modified Bachelier markets, also providing insurance applications.

Chapter 1

Bernstein's inequalities and their extensions for getting the Black-Scholes option pricing formula

1.1 Introduction

Many models and facts in modern probability theory are based upon a Bernoulli scheme and the corresponding binomial distribution developed under classical stochastic experiments. For this reason, any results in this direction have the potential for further research and extension. One of the key facts usually referred to in this regard is the classical De Moivre-Laplace theorem which gives normal approximation to the binomial distribution when the number of trials, n, grows without bound. This theorem has numerous applications. In particular, it is used for verification of the convergence of option prices in the financial market model with discrete time (binomial market, Cox-Ross-Rubinstein formula) to option prices in the Black-Scholes model (see Cox et al. [1.5]). Application of the De Moivre-Laplace theorem in this case provides both the convergence of corresponding option prices and the rate of convergence of $1/\sqrt{n}$.

The deeper studies of these subjects apply various modifications of the De Moivre-Laplace theorem. The most prominent and to some extent finishing touch is the use of Uspensky's theorem [1.8], which provides for convergence of order $1/n$ (see Chang and Palmer [1.3]; Chung and Shih [1.4]; Leisen and Reimer [1.6]). In a recent article, Zubkov and Serov [1.9] brought into view one more classical modification of the De Moivre-Laplace theorem, given by Bernstein [1.1] in 1943. In this paper we demonstrate how these results can be used to show the convergence of binomial (Cox-Ross-Rubinstein) option prices to the Black-Scholes prices.

1.2 Binomial and Black-Scholes option pricing models

Suppose there are n time periods, each time period a price of a stock can go up with rate of return $(u-1)$ or down with rate $(d-1)$. Let r_f denote one plus a risk-free interest rate over one period. Recall a Cox-Ross-Rubinstein formula [1.5], also known as Binomial Option Pricing formula, for evaluating a price of a European call option, C_{CRR}, with initial stock price S_0, strike price K, and with expiration in n periods:

$$\begin{aligned} C_{CRR} = S_0\left[\sum_{j=k_0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\left(\frac{u^j d^{n-j}}{r_f^n}\right)\right] \\ -Kr_f^{-n}\left[\sum_{j=k_0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\right], \end{aligned} \tag{1.2.1}$$

where j is a number of upward movements of stock price; $(n-j)$ is a number of downward movements over n time periods; k_0 is a minimum number of upward movements the stock price must make for the call option to finish in-the-money, such that $S_0 u^{k_0} d^{n-k_0} > K$; $p = (r_f - d)/(u-d)$ is a risk-neutral probability of an upward movement $(0 < d < r_f < u)$.

The latter bracketed expression in (1.2.1) is a complementary binomial distribution function $B[k_0; n, p]$. The first bracketed expression can also be viewed as a complementary distribution function $B[k_0; n, p']$ with $p' = (u/r_f)p$ and $1 - p' = (d/r_f)(1-p)$.

Thus, we can rewrite formula (1.2.1) more succinctly as:

$$C_{CRR} = S_0 B[k_0; n, p'] - K r_f^{-n} B[k_0; n, p], \tag{1.2.2}$$

where k_0 can be seen as the smallest non-negative integer greater than $\log(\frac{K}{S_0 d^n})/\log(u/d)$; if $k_0 > n$, $C_{CRR} = 0$. It should be noted that the form (1.2.2) of the equation (1.2.1) is common in mathematical finance.

The above binomial model is often considered as a discrete-time version of the famous Black-Scholes-Merton model (after Black and Scholes [1.2], and Merton [1.7]) for a price of a European call, C_{BS}:

$$C_{BS} = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2), \tag{1.2.3}$$

where

$$d_1 = \frac{\log(S_0/K) + T(r + \sigma^2/2)}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\log(S_0/K) + T(r - \sigma^2/2)}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

r is a continuously compounded risk-free rate, σ is a volatility of the underlying stock, T is a time to expiration in years, $\Phi(\cdot)$ denotes a cumulative distribution function of a standard normal distribution.

Comparing the two pricing models, note that in a binomial case option expires in n periods, while in a Black-Scholes case option expires in T years. Basically, T represents the length of the calendar time to option maturity, which consists of n periods of equal length. Then the length of each period is T/n. Continuously compounded risk-free rate, r, is related to the risk-free rate over one period of length T/n as follows: $\exp(rT) = r_f^n$, so $r_f = \exp(rT/n)$. Choosing parameters u and d in a specific way, one can derive the Black-Scholes formula (1.2.3) as a limiting case of a Cox-Ross-Rubinstein formula (1.2.2) with the help of the De Moivre-Laplace theorem which is a special case of the central limit theorem. Specifically, let $u = \exp(\sigma\sqrt{T/n})$ and $d = \exp(-\sigma\sqrt{T/n})$ (see, for instance, Cox et al. [1.5]). Then

$$B[k_0; n, p'] \simeq \Phi\left(\frac{np' - (k_0 - 1)}{\sqrt{np'(1-p')}}\right) \underset{n\to\infty}{\longrightarrow} \Phi(d_1)$$

and (1.2.4)

$$B[k_0; n, p] \simeq \Phi\left(\frac{np - (k_0 - 1)}{\sqrt{np(1-p)}}\right) \underset{n\to\infty}{\longrightarrow} \Phi(d_2),$$

and the rate of such convergence is $1/\sqrt{n}$ as usual in the central limit theorem.

1.3 Bernstein's inequalities and their application to the derivation of the Black-Scholes formula

A natural interest to give alternative proofs for expressions (1.2.4) arises. It turns out that the following notable inequalities of Bernstein S.N. [1.1], one of the prominent figures in modern probability theory, prove to be useful for this purpose:

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{z_1}^{\infty} e^{-z^2}\mathrm{d}z - \frac{e^{-\sqrt[3]{2npq}}}{2\sqrt{\pi}\sqrt[6]{2npq}} &< P(m \le m_0) \\ &< \frac{1}{\sqrt{\pi}}\int_{z_2}^{\infty} e^{-z^2}\mathrm{d}z + e^{-\sqrt[3]{2npq}},\ \text{if } m_0 + \frac{3}{2} \le np,\end{aligned} \tag{1.3.1}$$

and

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{z_3}^{\infty} e^{-z^2}\mathrm{d}z - \frac{e^{-\sqrt[3]{2npq}}}{2\sqrt{\pi}\sqrt[6]{2npq}} &< P(m \ge m_0) \\ &< \frac{1}{\sqrt{\pi}}\int_{z_4}^{\infty} e^{-z^2}\mathrm{d}z + e^{-\sqrt[3]{2npq}},\ \text{if } m_0 - \frac{3}{2} \ge np,\end{aligned} \tag{1.3.2}$$

where m bas a binomial distribution with parameters n and p; $q = 1 - p$ ($npq \ge 62.5$); m_0 is an integer; z_i (i=1,2,3,4) are non-negative roots of the quadratic equations:

$$-z_1\sqrt{2npq} + \frac{q-p}{3}z_1^2 = m_0 - np - \frac{1}{2}, \tag{1.3.3}$$

$$-z_2\sqrt{2npq} + \frac{q-p}{3}z_2^2 = m_0 - np + \frac{3}{2}; \tag{1.3.4}$$

$$z_3\sqrt{2npq} + \frac{q-p}{3}z_3^2 = m_0 - np + \frac{1}{2}, \tag{1.3.5}$$

$$z_4\sqrt{2npq} + \frac{q-p}{3}z_4^2 = m_0 - np - \frac{3}{2}. \tag{1.3.6}$$

There is a deep connection between a proof of this excellent result and a proof of the famous De Moivre-Laplace theorem. The classical proof of the De Moivre-Laplace theorem relies on a Stirling's formula to show that as n grows

$$\binom{n}{m}p^m q^{n-m} \simeq \sqrt{\frac{n}{2\pi m(n-m)}}\left(\frac{np}{m}\right)^m\left(\frac{nq}{n-m}\right)^{n-m}.$$

It is then proven that for $m = np + z\sqrt{2npq}$,

$$\left(\frac{np}{m}\right)^m \left(\frac{nq}{n-m}\right)^{n-m} \underset{n\to\infty}{\longrightarrow} e^{-z^2}.$$

Studying this approximation, Bernstein noticed that assigning probability equal to

$$\sqrt{\frac{n}{2\pi m(n-m)}}e^{-z^2},$$

to the integer closest to $np + z\sqrt{2npq} + z^2(q-p)/3$ would be more accurate. This observation allowed to reevaluate the normal approximation to the binomial distribution. The proof of Bernstein's inequalities employs Stirling's formula, a modification of Chebyshev's inequality, and requires a considerable computational effort.

Let us consider the inequality (1.3.1), when $m_0 + 3/2 \leq np$. This condition guarantees that at least one root of each of the Eqs. (1.3.3) and (1.3.4) is non-negative. It is easy to show that the root

$$z_i = \left(1 - \sqrt{1 - \frac{2}{3}\frac{(q-p)(np - m_0 + \alpha_i)}{npq}}\right)\frac{3\sqrt{2npq}}{2(q-p)}, \quad \text{where } i = 1,2;\ \alpha_1 = 1/2, \alpha_2 = -3/2, \tag{1.3.7}$$

is always non-negative given the condition $m_0 + 3/2 \leq np$ holds.

Thus, we can express binomial probabilities in (1.2.2) through standard normal probabilities by means of Bernstein's result:

$$\Phi(\sqrt{2}z_2') - c_1' < B[k_0; n, p'] < \Phi(\sqrt{2}z_1') + c_0'$$

and

$$\Phi(\sqrt{2}z_2) - c_1 < B[k_0; n, p] < \Phi(\sqrt{2}z_1) + c_0,$$

where z_i $(i = 1, 2)$ is defined as in (1.3.7) with $m_0 = k_0 - 1$; z_i' $(i = 1, 2)$ is defined as in (1.3.7) with $m_0 = k_0 - 1$ and with replacing parameter p with p', parameter q with q' $(q' = 1 - p')$; $c_0 = \exp(-\sqrt[3]{2npq})/(2\sqrt{\pi}\sqrt[6]{2npq})$; $c_1 = \exp(-\sqrt[3]{2npq})$; c_0' and c_1' are defined similarly to c_0 and c_1 by means of replacing p and q with p' and q', respectively.

Letting number of trading periods, n, approach infinity, Black-Scholes formula is obtained. To see it, first recall that $p = (r_f - d)/(u - d)$, $u = \exp(\sigma\sqrt{T/n})$, $d = \exp(-\sigma\sqrt{T/n})$, and $r_f = \exp(rT/n)$.

Taylor series expansion of exponential functions yields:

$$p = \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}} = \frac{\sigma + (r - \frac{\sigma^2}{2})\sqrt{\frac{T}{n}} + \frac{1}{6}\sigma^3\frac{T}{n} + O(\frac{1}{n^{3/2}})}{2\sigma + \frac{1}{3}\sigma^3\frac{T}{n} + O(\frac{1}{n^2})},$$

hence

$$p = \frac{1}{2} + \frac{1}{2}\frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{\frac{T}{n}} + O\left(\frac{1}{n^{3/2}}\right),$$
$$pq = \frac{1}{4} - \frac{1}{4}\frac{(r - \frac{\sigma^2}{2})^2}{\sigma^2}\frac{T}{n} + O\left(\frac{1}{n^{3/2}}\right).$$

From equations (1.3.3) and (1.3.4) with $m_0 = k_0 - 1$ we have:

$$\frac{q-p}{3\sqrt{2npq}}z_i^2 - z_i + \frac{np - (k_0 - 1) + \alpha_i}{\sqrt{2npq}} = 0, \tag{1.3.8}$$

where $i = 1, 2$.

Substituting z_i in (1.3.8) with (1.3.7) gives:

$$z_i = \sqrt{2}\frac{np - (k_0 - 1) + \alpha_i}{\sqrt{npq}} \Bigg/ \left(1 + \sqrt{1 - \frac{2}{3}\frac{(q-p)(np - (k_0 - 1) + \alpha_i)}{npq}}\right),$$

where $i = 1, 2$.

It follows from the binomial option pricing formula that

$$k_0 - 1 = \frac{\log(K/S_0) - n\log(d)}{\log(u/d)} - \epsilon, \tag{1.3.9}$$

where $\epsilon \in [0, 1)$.

Using (1.3.9), we get:

$$\frac{np - (k_0 - 1) + \alpha_i}{\sqrt{npq}} = \frac{\log(S_0/K) + n\big[\log(d) + p\log(u/d)\big] + (\epsilon + \alpha_i)\log(u/d)}{\sqrt{npq}\log(u/d)},$$

where $i = 1, 2$.

Evaluating $\log(u/d)$, $\big[\log(d) + p\log(u/d)\big]n$ and $\sqrt{npq}\log(u/d)$, we obtain

$$\log(u/d) = 2\sigma\sqrt{\frac{T}{n}} = O\left(\frac{1}{\sqrt{n}}\right),$$

$$\big[\log(d) + p\log(u/d)\big]n = (r - \frac{\sigma^2}{2})T + O\left(\frac{1}{n}\right),$$

$$\sqrt{npq}\log(u/d) = \sigma\sqrt{T} + O\left(\frac{1}{n}\right).$$

Thus,

$$\begin{aligned} \frac{np - (k_0 - 1) + \alpha_i}{\sqrt{npq}} &= \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\ &\quad + \frac{2(\epsilon + \alpha_i)}{\sqrt{n}} + O\left(\frac{1}{n}\right), \end{aligned} \tag{1.3.10}$$

where $i = 1, 2$.

Noticing that $(q-p)/\sqrt{npq} = O(1/n)$, we obtain that

$$\sqrt{1 - \frac{2}{3}\frac{(q-p)}{\sqrt{npq}}\frac{(np - (k_0 - 1) + \alpha_i)}{\sqrt{npq}}} = 1 + O\left(\frac{1}{n}\right) \text{ for } i = 1, 2.$$

Therefore,

$$z_i = \frac{\sqrt{2}}{2}\left(\frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} + \frac{2(\epsilon + \alpha_i)}{\sqrt{n}}\right) + O\left(\frac{1}{n}\right) \text{ for } i = 1, 2.$$

Finally, we can write:

$$\begin{aligned} \Phi\left[d_2 + \frac{2(\epsilon + \alpha_2)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right] - c_1 &< B[k_0; n, p] \\ &< \Phi\left[d_2 + \frac{2(\epsilon + \alpha_1)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right] + c_0. \end{aligned} \tag{1.3.11}$$

By similar arguments we can show that

$$\begin{aligned} \Phi\left[d_1 + \frac{2(\epsilon + \alpha_2)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right] - c_1' &< B[k_0; n, p'] \\ &< \Phi\left[d_1 + \frac{2(\epsilon + \alpha_1)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right] + c_0'. \end{aligned} \tag{1.3.12}$$

It is clear that both lower and upper bounds of inequality (1.3.11) converge to the same limit, $\Phi(d_2)$, as n approaches infinity, while both bounds of inequality (1.3.12) converge to $\Phi(d_1)$. We carry over $2(\epsilon + \alpha_i)/\sqrt{n}$ and O-terms for the purpose of establishing the rate of convergence of the binomial option prices to the Black-Scholes prices.

Using inequalities (1.3.11) and (1.3.12) to evaluate binomial option price in formula (1.2.2), we get:

$$
\begin{aligned}
S_0\Big[\Phi\Big(d_1+\frac{2(\epsilon+\alpha_2)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)\Big)-c_1'\Big]& \\
-Ke^{-rT}\Big[\Phi\Big(d_2+\frac{2(\epsilon+\alpha_1)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)\Big)+c_0\Big]& \\
< C_{CRR}& \qquad (1.3.13)\\
< S_0\Big[\Phi\Big(d_1+\frac{2(\epsilon+\alpha_1)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)\Big)+c_0'\Big]& \\
-Ke^{-rT}\Big[\Phi\Big(d_2+\frac{2(\epsilon+\alpha_2)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)\Big)-c_1\Big].&
\end{aligned}
$$

Applying Taylor series expansion to a normal distribution function and noticing that c_0, c_1, c_0', and c_1' are not worse than $O(1/n)$, inequalities (1.3.13) become:

$$
\begin{aligned}
C_{BS}+S_0\phi(d_1)\frac{2(\epsilon+\alpha_2)}{\sqrt{n}}-Ke^{-rT}\phi(d_2)\frac{2(\epsilon+\alpha_1)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)<C_{CRR}\\
<C_{BS}+S_0\phi(d_1)\frac{2(\epsilon+\alpha_1)}{\sqrt{n}}-Ke^{-rT}\phi(d_2)\frac{2(\epsilon+\alpha_2)}{\sqrt{n}}+O\Big(\frac{1}{n}\Big).
\end{aligned}
$$

where $\phi(\cdot)$ denotes probability density function of a standard normal distribution.

Using the fact that $S_0\phi(d_1)=Ke^{-rT}\phi(d_2)$ and recalling that $\alpha_1=1/2$, $\alpha_2=-3/2$, we are left with:

$$
C_{BS}-4S_0\phi(d_1)\frac{1}{\sqrt{n}}+O\Big(\frac{1}{n}\Big)<C_{CRR}<C_{BS}+4S_0\phi(d_1)\frac{1}{\sqrt{n}}+O\Big(\frac{1}{n}\Big).
$$

Thus, application of Bernstein's inequality (1.3.1) allows for convergence to Black-Scholes option prices with the rate $1/\sqrt{n}$, as usual in the central limit theorem.

Consider the inequality (1.3.2), when $m_0-3/2\geq np$. This condition guarantees that at least one root of each of the Eqs. (1.3.5) and (1.3.6) is non-negative and this root is

$$
z_i=\Bigg(-1+\sqrt{1-\frac{2}{3}\frac{(q-p)(np-m_0+\alpha_i)}{npq}}\Bigg)\frac{3\sqrt{2npq}}{2(q-p)}, \qquad (1.3.14)
$$

where $i=3,4$; $\alpha_3=-1/2$, $\alpha_4=3/2$.

Then, binomial probabilities in (1.2.2) can be expressed through standard normal probabilities by means of Bernstein's inequality (1.3.2) as:

$$\Phi(-\sqrt{2}z_3') - c_0' < B[k_0; n, p'] < \Phi(-\sqrt{2}z_4') + c_1'$$

and

$$\Phi(-\sqrt{2}z_3) - c_0 < B[k_0; n, p] < \Phi(-\sqrt{2}z_4) + c_1,$$

where z_i $(i = 3, 4)$ is defined as in (1.3.14); z_i' $(i = 3, 4)$ is defined as in (1.3.14) with replacing parameter p with p', parameter q with q' $(q' = 1 - p')$; $c_0 = \exp(-\sqrt[3]{2npq})/(2\sqrt{\pi}\sqrt[6]{2npq})$; $c_1 = \exp(-\sqrt[3]{2npq})$; c_0' and c_1' are defined similarly to c_0 and c_1 by means of replacing p and q with p' and q'.

The root in (1.3.14) with $m_0 = k_0$ can be rewritten in the form:

$$z_i = -\sqrt{2}\frac{np - k_0 + \alpha_i}{\sqrt{npq}} \bigg/ \left(1 + \sqrt{1 - \frac{2}{3}\frac{(q-p)(np - k_0 + \alpha_i)}{npq}}\right) \text{ for } i = 3, 4.$$

Similarly to the case of inequality (1.3.1), we can show that

$$\frac{np - k_0 + \alpha_i}{\sqrt{npq}} \underset{n\to\infty}{\longrightarrow} \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}},$$

and therefore,

$$z_i \underset{n\to\infty}{\longrightarrow} -\frac{\sqrt{2}}{2}\frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \text{ for } i = 3, 4.$$

Finally,

$$B[k_0; n, p] \underset{n\to\infty}{\longrightarrow} \Phi(-\sqrt{2}z_i) \underset{n\to\infty}{\longrightarrow} \Phi\left(\frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \text{ for } i = 3, 4.$$

By similar arguments,

$$B[k_0; n, p'] \underset{n\to\infty}{\longrightarrow} \Phi(-\sqrt{2}z_i') \underset{n\to\infty}{\longrightarrow} \Phi\left(\frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \text{ for } i = 3, 4.$$

It can be shown that the rate of convergence is similar to the case of inequality (1.3.1), i.e. binomial option prices converge to the Black-Scholes prices with the rate $1/\sqrt{n}$.

This illustrates another way of arriving at Black-Scholes formula from a discrete-time binomial model.

We can also use an interesting extension of Bernstein's inequalities given by Zubkov and Serov [1.9] to derive a Black-Scholes formula from a binomial option pricing model. The proof of this extension mainly relies on a Stirling's formula.

Let $H(x,p) = x\log(x/p) + (1-x)\log\Big((1-x)/(1-p)\Big)$, $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$ and $\operatorname{sgn}(0) = 0$, let $\{C_{n,p}(m_0)\}_{m_0=0}^{n}$ be increasing sequences defined as follows: $C_{n,p}(0) = (1-p)^n$, $C_{n,p}(n) = 1-p^n$,

$$C_{n,p}(m_0) = \Phi\left(\operatorname{sgn}\Big(\frac{m_0}{n} - p\Big)\sqrt{2nH\Big(\frac{m_0}{n}, p\Big)}\right), 1 \leq m_0 < n. \tag{1.3.15}$$

Then for every $m_0 = 0, 1, ..., n-1$ and for every $p \in (0,1)$

$$C_{n,p}(m_0) \leq P(m \leq m_0) \leq C_{n,p}(m_0+1),$$

and equalities may happen for $m_0 = 0$ and $m_0 = n-1$ only.

For our purposes we need a function (1.3.15) to be valid for all possible m_0, i.e. $0 \leq m_0 \leq n$. Provided that we use a convention $0^0 = 1$, a function $H(\frac{m_0}{n}, p)$ is well-defined for $m_0 = 0$ and $m_0 = n$ on $p \in (0,1)$, and therefore $C_{n,p}(m_0)$ in (1.3.15) can be computed for $0 \leq m_0 \leq n$.

Then the following inequalities are true for $0 \leq m_0 \leq n-1$:

$$C_{n,p}(m_0) < P(m \leq m_0) < C_{n,p}(m_0+1) \tag{1.3.16}$$

with $C_{n,p}(m_0)$ defined as in (1.3.15) for $0 \leq m_0 \leq n$.

To see that (1.3.16) holds, consider two functions:

$$\psi(p) = P(m \leq 0) - C_{n,p}(0) = (1-p)^n - \Phi\Big(-\sqrt{-2n\log(1-p)}\Big) \tag{1.3.17}$$

and

$$\delta(p) = P(m \leq n-1) - C_{n,p}(n) = 1 - p^n - \Phi\Big(\sqrt{-2n\log(p)}\Big). \tag{1.3.18}$$

Note that $\psi(0) = 0.5$, $\lim_{p\to 1}\psi(p) = 0$, $\lim_{p\to 0}\delta(p) = 0$, $\delta(1) = -0.5$, and both functions (1.3.17) and (1.3.18) are differentiable with respect to p:

$$\psi'(p) = -n(1-p)^{n-1} + \phi\Big(-\sqrt{-2n\log(1-p)}\Big)\Big(-2n\log(1-p)\Big)^{-1/2}\frac{n}{1-p},$$
$$\delta'(p) = -np^{n-1} + \phi\Big(\sqrt{-2n\log(p)}\Big)\Big(-2n\log(p)\Big)^{-1/2}\frac{n}{p}.$$

Equation $\psi'(p) = 0$ has a unique root $p_0 = 1 - \exp(-\frac{1}{4\pi n})$ on $(0,1)$; $\psi'(p) > 0$ on $(0, p_0)$ and $\psi'(p) < 0$ on $(p_0, 1)$. Therefore, $\psi(p) > 0$.

Equation $\delta'(p) = 0$ has a unique root $p_1 = \exp(-\frac{1}{4\pi n})$ on $(0,1)$; $\delta'(p) < 0$ on $(0, p_1)$ and $\delta'(p) > 0$ on $(p_1, 1)$. Therefore, $\delta(p) < 0$.

This shows that we can use inequalities (1.3.16) for $0 \leq m_0 < n$ with $C_{n,p}(m_0)$ defined as in (1.3.15) for all m_0.

Next, define $x := (m_0 - np)/\sqrt{npq}$. Then we can rewrite the function $H(m_0/n, p)$ as follows:

$$H\Big(\frac{m_0}{n}, p\Big) = \frac{1}{n}\Big[(np + x\sqrt{npq}) \log\Big(1 + x\sqrt{\frac{q}{np}}\Big) + (nq - x\sqrt{npq}) \log\Big(1 - x\sqrt{\frac{p}{nq}}\Big)\Big]. \quad (1.3.19)$$

Consider $m_0 = k_0 - 1$, where $k_0 - 1$ is defined as in (1.3.9). Then $x = -(d_2 + 2\epsilon/\sqrt{n}) + O(1/n)$ as follows from (1.3.10).

Letting n be sufficiently large so that $|x\sqrt{q/np}| < 1$ and $|x\sqrt{p/nq}| < 1$, we can apply Taylor series expansion of logarithmic function. Then (1.3.19) becomes:

$$\begin{aligned} H\Big(\frac{m_0}{n}, p\Big) &= \frac{1}{n}\Big((np + x\sqrt{npq})\Big[x\sqrt{\frac{q}{np}} - \frac{x^2 q}{2np} + \frac{x^3 q^{3/2}}{3(np)^{3/2}} + O\Big(\frac{1}{n^2}\Big)\Big] \\ &\quad + (nq - x\sqrt{npq})\Big[-x\sqrt{\frac{p}{nq}} - \frac{x^2 p}{2nq} - \frac{x^3 p^{3/2}}{3(nq)^{3/2}} + O\Big(\frac{1}{n^2}\Big)\Big]\Big) \\ &= \frac{1}{n}\Big[\frac{x^2}{2} + O\Big(\frac{1}{n}\Big)\Big], \end{aligned}$$

and (1.3.15) becomes:

$$C_{n,p}(m_0) = \Phi\Big(\mathrm{sgn}\Big(\frac{m_0}{n} - p\Big)\sqrt{x^2 + O\Big(\frac{1}{n}\Big)}\Big) = \Phi\Big(x + O\Big(\frac{1}{n}\Big)\Big).$$

Hence,

$$\Phi\Big(\frac{(k_0 - 1) - np}{\sqrt{npq}} + O\Big(\frac{1}{n}\Big)\Big) < P(m \leq k_0 - 1) < \Phi\Big(\frac{k_0 - np}{\sqrt{npq}} + O\Big(\frac{1}{n}\Big)\Big),$$

where $k_0 = 1, 2, ..., n$.

Therefore, binomial probabilities in formula (1.2.2) can be evaluated as follows:

$$\Phi\Big[d_2 + \frac{2(\epsilon - 1)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big)\Big] < B[k_0; n, p] < \Phi\Big[d_2 + \frac{2\epsilon}{\sqrt{n}} + O\Big(\frac{1}{n}\Big)\Big]$$

and

$$\Phi\left[d_1 + \frac{2(\epsilon - 1)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right] < B[k_0; n, p'] < \Phi\left[d_1 + \frac{2\epsilon}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right].$$

Applying Taylor series expansion to a normal distribution function as was done for the case of Bernstein's inequalities, we can show that the convergence of binomial option price to the Black-Scholes price in this case is also of order $1/\sqrt{n}$:

$$C_{BS} - 2S_0\phi(d_1)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right) < C_{CRR} < C_{BS} + 2S_0\phi(d_1)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

1.4 Concluding remarks

Convergence of a binomial option pricing model to its continuous-time limit is typically shown by means of De Moivre-Laplace theorem or remarkable Uspensky's result. We show an alternative derivation of the Black-Scholes formula from its binomial counterpart by means of Bernstein's inequalities as well as Zubkov and Serov inequalities. These less known results allow for a convergence rate $1/\sqrt{n}$.

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Chapter 2

Quadratic hedging of equity-linked life insurance contracts under the real-world measure in discrete time

2.1 Introduction

Pricing and hedging of contingent claims through the so-called martingale approach has been an important topic and a powerful technique. This method suggests finding an equivalent martingale measure such that the discounted asset prices are martingales. In this work, instead of switching to a new probability measure, we perform pricing and hedging by finding a discounting portfolio such that the discounted price processes in a financial market become martingales under the real-world probability measure.

This approach is related to a change of numeraire technique which rapidly gained its popularity after the article by Geman et al. [2.5] was published. It is worth noting that a similar idea appeared in Shiryaev et al. [2.11], where a stock price was chosen as a numeraire and the corresponding martingale measure was called a dual one. This technique is used to simplify many valuation problems by changing a discounting portfolio (a numeraire) and searching for an associated martingale measure. It is in fact possible to find the discounting portfolio so that associated martingale measure is a physical probability measure, that is no substitution of measure is actually needed for valuation.

The idea of fixing the physical probability measure and finding a suitable numeraire was introduced in Long Jr. [2.8] (see also Melnikov [2.9], and Becherer [2.1]). However, in actuarial science this idea is not yet widespread and only few papers attempted to use a numeraire portfolio in actuarial context (e.g., Bühlmann and Platen [2.2], Korn and Schäl [2.6]). In this paper, we explain how the P-discounting portfolio can be constructed for a discrete-time market and how the method can be deployed for pricing and hedging the equity-linked life insurance contracts. We intentionally aim to keep the setting simple by considering only two securities in discrete time as it significantly reduces the complexity while allowing to demonstrate the fundamentals of hedging the equity-linked life insurance contracts under the real-world probability measure. Similar setting is often a preferred choice: some relevant examples include Møller [2.10] and Lamberton et al. [2.7].

The paper is organized as follows. In Section 2.2, the assumed financial market and model are described. Section 2.3 introduces the concept of a P-discounting portfolio and its use for a valuation in a complete market. In Section 2.4 the idea of pricing and hedging under a physical probability measure is studied for the incomplete markets employing a quadratic risk-minimization criterion (Föllmer and Sondermann [2.4]) for finding the optimal hedging strategy. The corresponding application to pricing and hedging of equity-linked life insurance contracts is demonstrated in Section 2.5. Our findings are illustrated by two numerical examples (see Sections 2.3 and 2.5).

Different aspects of a risk-minimization approach to hedging were developed in a series of works. For instance, a recent paper by Du and Platen [2.3] published in July 2016 attempts pricing and hedging under a physical probability measure in a general semimartingale markets under a quadratic criterion. Although there are obvious differences in our approaches (for example, different quadratic criteria were considered), since the key problem is quite similar, we feel necessary to emphasize that this research was performed independently and was presented in April 2016 during Alberta Mathematics Dialogue at Mount Royal University (Calgary, Canada). Moreover, our interest is not only in developing a general methodology for pricing and hedging under a real-world measure, but mostly an application of this methodology in actuarial context, specifically to pricing and hedging the equity-linked life insurance contracts.

2.2 Preliminaries: financial market and model

We consider a financial market consisting of one unit of a risky asset - a stock, and one unit of a risk-free asset - a bond (or, alternatively, a deposit into a savings account). The price processes of stock and bond will be denoted as $S = (S_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$, correspondingly. If an investor holds one share of a stock and one bond from time t to time $t+1$ ($t = 0, 1, 2, ...$), we assume that their values change from S_t and B_t to

$$S_{t+1} = S_t(1 + \rho_{t+1}),\ S_0 > 0, \tag{2.2.1}$$

$$B_{t+1} = B_t(1 + r),\ B_0 > 0, \tag{2.2.2}$$

where ρ_{t+1} is the return per unit of stock during the time interval $(t, t+1]$ and its value is not known prior to time $t+1$; r is the interest rate earned on a bond (or interest rate on a savings account) during the time interval $(t, t+1]$ and its value is known in advance.

The price processes S and B are defined on a stochastic basis $(\Omega,\ \mathcal{F},\ \mathbb{F} = (\mathcal{F}_t)_{t \geq 0},\ P)$ and are adapted to filtration $\mathbb{F}$. A pair $\pi_t = (\xi_t, \eta_t)$ with ξ_t representing the number of shares and η_t representing the number of bonds is called an investment strategy (portfolio) held at time t. The value (capital) of the investor's portfolio at time t will be denoted as V_t. The value process $V = (V_t)_{t \geq 0}$ describing the evolution of the investor's capital is given by:

$$V_t = \xi_t S_t + \eta_t B_t,$$

$$\Delta V_t = \xi_{t-1} \Delta S_t + \eta_{t-1} \Delta B_t + \Delta I_t,$$

where $\Delta I_t = I_t - I_{t-1}$ is the additional investment (consumption) during the time interval $(t-1, t]$.

If the change in the value of capital is due to the trading gains only, that is, no additional investment is required at all times t, then such strategy is called self-financing.

Definition 2.1. *Strategy $\pi = (\xi, \eta)$ is called self-financing if $\Delta I_t = 0$, $t \geq 1$, i.e. the corresponding value process satisfies:*

$$\Delta V_t = \xi_{t-1} \Delta S_t + \eta_{t-1} \Delta B_t. \tag{2.2.3}$$

2.3 Hedging in complete markets

Consider a European-type contingent claim having a single payout, $f(S_T)$, on a maturity date T. The pricing of such contingent claims is usually accomplished by using a bond as a discounting portfolio and finding a new probability measure P^* equivalent to P, under which the discounted asset prices are martingales. If such a measure can be found, then a unique price of a contingent claim is its discounted expected payoff, i.e.

$$C = \mathrm{E}^* \left(\frac{f(S_T)}{B_T} \right), \; B_0 = 1,$$

and the capital of the self-financing hedging strategy, such that $V_T = f(S_T)$, is

$$V_t = B_t \, \mathrm{E}^* \left(\frac{f(S_T)}{B_T} \middle| \mathcal{F}_t \right),$$

that is, the discounted value process $(V_t/B_t)_{t \geq 0}$ is a martingale under P^*.

The discounting portfolio allows for comparison of accumulated wealth at different time periods. From an economic point of view, an arbitrary self-financing portfolio with strictly positive capital can be used as a discounting portfolio. Therefore, if we can find a discounting portfolio with capital $X = (X_t)_{t \geq 0}$ such that the process $V/X = (V_t/X_t)_{t \geq 0}$ is a martingale under a physical probability measure P, then a price of a contingent claim can be found as

$$C = X_0 \, \mathrm{E} \left(\frac{f(S_T)}{X_T} \right),$$

with the value of capital

$$V_t = X_t \, \mathrm{E} \left(\frac{f(S_T)}{X_T} \middle| \mathcal{F}_t \right).$$

The described portfolio with capital X will be called a P-discounting portfolio[1] (see Melnikov [2.9], pp. 68-71, for some insights into the problem).

Definition 2.2. *A self-financing strategy $\varphi = (\gamma, \beta)$ with strictly positive capital $X = (X_t)_{t\geq 0}$, $X_0 = 1$ is called a P-discounting portfolio if for any self-financing portfolio $\pi = (\xi, \eta)$ a discounted value process $V/X = (V_t/X_t)_{t\geq 0}$ is a P-martingale (or, equivalently, discounted price processes $S/X = (S_t/X_t)_{t\geq 0}$ and $B/X = (B_t/X_t)_{t\geq 0}$ are P-martingales).*

Finding a P-discounting portfolio enables us to work exclusively with a real-world, "physical", probability model, describing the true nature of the processes. Moreover, such a discounting portfolio is unique if it exists.

Lemma 2.1. *If a P-discounting portfolio $\varphi = (\gamma, \beta)$ with initial capital $X_0 = 1$ exists, then it is unique.*

Proof. Let $\varphi' = (\gamma', \beta')$ be another P-discounting portfolio with the capital $X' = (X'_t)_{t\geq 0}$ and $X'_0 = 1$. Then by Definition 2.2, $(X_t/X'_t)_{t\geq 0}$ and $(X'_t/X_t)_{t\geq 0}$ are P-martingales. Therefore, if $Y_t := X_t/X'_t$, then $\mathrm{E}(Y_t) = \mathrm{E}(1/Y_t) = 1$. Since $\varphi(y) = 1/y$, $y > 0$, is strictly convex downward function, then by Jensen's inequality[2], $Y_t = \mathrm{E}(Y_t) = 1$. □

Let $\kappa_t = \gamma_t S_t / X_t$ denote a proportion of risky capital in a P-discounting portfolio φ at time t. Then it follows from Eqs. (2.2.3), (2.2.1) and (2.2.2) that

$$\begin{aligned}\Delta X_t &= \gamma_{t-1}\Delta S_t + \beta_{t-1}\Delta B_t \\ &= X_{t-1}\kappa_{t-1}\rho_t + X_{t-1}(1-\kappa_{t-1})r \\ &= X_{t-1}(r + \kappa_{t-1}(\rho_t - r)),\end{aligned}$$

and hence

$$X_t = X_{t-1}(1 + r + \kappa_{t-1}(\rho_t - r)). \tag{2.3.1}$$

[1] We use a term "P-discounting portfolio" instead of a more popular "numeraire portfolio" as the former encompasses an important financial concept of discounting and, thus, is more intuitive, in our opinion.

[2] If φ is a convex downward function and X is a random variable, $\varphi(\mathrm{E}X) \leq \mathrm{E}(\varphi(X))$. Moreover, for a strictly convex downward function φ, equality in Jensen's inequality holds if and only if $X = \mathrm{E}X$ a.s. (i.e. X is a constant).

Thus, the capital $(X_t)_{t\geq 0}$ (with $X_0 = 1$) of a P-discounting portfolio can be defined by a sequence $(\kappa_t)_{t\geq 0}$, as the following lemma shows.

Lemma 2.2. *A P-discounting portfolio $\varphi = (\gamma, \beta)$ with strictly positive capital $X = (X_t)_{t\geq 0}$, $X_0 = 1$, exists if and only if there exists an $\mathbb{F}$-adapted sequence $\kappa = (\kappa_t)_{t\geq 0} = (\gamma_t S_t / X_t)_{t\geq 0}$ such that*
(i) $1 + r + \kappa_t(\rho_{t+1} - r) > 0$ (P-a.s.);
(ii) a process $M = (M_t)_{t\geq 0}$, defined by

$$M_0 = 0,\ M_t = \sum_{i=1}^{t} \frac{\rho_i - r}{1 + r + \kappa_{i-1}(\rho_i - r)}, \tag{2.3.2}$$

is a P-martingale.

Proof.

a) Necessity.

Let $\varphi = (\gamma, \beta)$ be a P-discounting portfolio with strictly positive capital $X = (X_t)_{t\geq 0}$, $X_0 = 1$, and proportion of risky capital $\kappa_t = \gamma_t S_t / X_t$. Since $X_t > 0$, $t \geq 0$, part (i) of the lemma follows from Eq. (2.3.1).

Let $\pi = (\xi, \eta)$ be an arbitrary self-financing portfolio with capital $V = (V_t)_{t\geq 0}$ and proportion of risky capital $\alpha_t = \xi_t S_t / V_t$. Using a well-known fact that self-financing portfolios remain self-financing under a change of numeraire (see Geman et al. [2.5]), we can write

$$\begin{aligned}
\Delta \frac{V_t}{X_t} &= \xi_{t-1} \Delta \frac{S_t}{X_t} + \eta_{t-1} \Delta \frac{B_t}{X_t} = \xi_{t-1} \Delta \frac{S_t}{X_t} + \frac{V_{t-1} - \xi_{t-1} S_{t-1}}{B_{t-1}} \Delta \frac{B_t}{X_t} \\
&= \frac{V_{t-1}}{X_{t-1}} \left[\frac{X_{t-1}}{B_{t-1}} \Delta \frac{B_t}{X_t} + \alpha_{t-1} \left(\frac{X_{t-1}}{S_{t-1}} \frac{S_t}{X_t} - \frac{X_{t-1}}{B_{t-1}} \frac{B_t}{X_t} \right) \right] \\
&= \frac{V_{t-1}}{X_{t-1}} (\alpha_{t-1} - \kappa_{t-1}) \Delta M_t,
\end{aligned} \tag{2.3.3}$$

where

$$\Delta M_t = \frac{X_{t-1}}{S_{t-1}} \frac{S_t}{X_t} - \frac{X_{t-1}}{B_{t-1}} \frac{B_t}{X_t} = \frac{\rho_t - r}{1 + r + \kappa_{t-1}(\rho_t - r)}. \tag{2.3.4}$$

The second equality in (2.3.4) follows from (2.3.1), (2.2.1) and (2.2.2). The processes $(S_t/X_t)_{t\geq 0}$ and $(B_t/X_t)_{t\geq 0}$ are martingales by definition of a P-discounting portfolio (Definition 2.2), hence $(\Delta M_t)_{t\geq 1}$ is a martingale difference.

b) Sufficiency.

Let $\kappa = (\kappa_t)_{t\geq 0} = (\gamma_t S_t/X_t)_{t\geq 0}$ be an $\mathbb{F}$-adapted sequence such that conditions (i) and (ii) hold. Then $X = (X_t)_{t\geq 0}$ with $X_0 = 1$ and $X_t = X_{t-1}(1 + r + \kappa_{t-1}(\rho_t - r))$ is strictly positive $\mathbb{F}$-adapted sequence.

It follows from the definition of a self-financing portfolio (Definition 2.1) and representation

$$\Delta X_t = X_{t-1}\kappa_{t-1}\rho_t + X_{t-1}r(1 - \kappa_{t-1})$$

that X is a capital of self-financing portfolio $\varphi = (\gamma, \beta)$ with $\gamma_t = \kappa_t X_t/S_t$ and $\beta_t = (1 - \kappa_t)X_t/B_t$.

Finally, from condition (ii) and representation (2.3.3) it follows that for any self-financing portfolio $\pi = (\xi, \eta)$ with capital $V = (V_t)_{t\geq 0}$ a sequence $V/X = (V_t/X_t)_{t\geq 0}$ is a martingale. Hence, $\varphi = (\gamma, \beta)$ is a P-discounting portfolio by Definition 2.2. □

Thus, the existence of the sequence $(\kappa_t)_{t\geq 0}$ determines the conditions for existence of a P-discounting portfolio. We further assume that a P-discounting portfolio with capital $X = (X_t)_{t\geq 0}$, $X_0 = 1$, exists.

2.3.1 Numerical example

To illustrate the concept, consider the two-step binomial market. The evolution of prices is given by Eqs. (2.2.1) and (2.2.2) with $B_0 = \$1$, $S_0 = \$100$, with constant interest rate $r = 0.12$, and with return per unit of stock ρ_t taking two values: $b = 0.25$ or $a = -0.1$. Let probability of an up move, p, be 0.4, then probability of a down move is 0.6. Find a price of European call option with payoff $f(S_2) = \max(S_2 - K, 0)$, where strike price $K = \$110$.

Fig. 2.1 shows two-period binomial tree for the evolution of stock prices S. The price of a contingent claim at a terminal time is easily computed. At the next step, a risk-neutral probability is usually found as $p^* = (r - a)/(b - a) = 0.6286$ and the call price is then the discounted expectation with respect to this new probability: $C = E^*[f(S_2)/(1+r)^2] = \$15.50$. The price at time $t = 1$ is either \$26.79 or \$1.40. These prices are shown in Fig. 2.1 in parentheses.

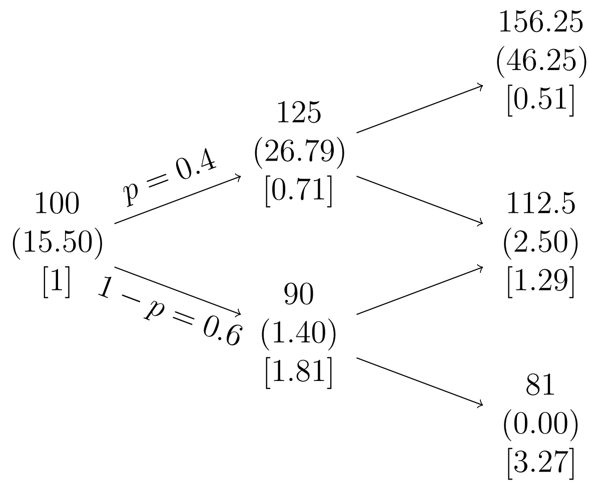

Figure 2.1: Binomial tree for the stock price, price of a call option, and the capital of a P-discounting portfolio.
Note: The upper numbers denote stock prices, the middle numbers in parentheses denote the price of a contingent claim, and the lower numbers in square brackets represent a capital of a P-discounting portfolio.

To solve the same problem without switching to a new probability, find the discounting portfolio X as described in Lemma 2.2. In this binomial model proportion of risky capital is constant ($\kappa_t = \kappa$ for all t) and can be found by the formula:

$$\kappa = \frac{(1+r)(\mu - r)}{(b-r)(r-a)}, \text{ with } \mu = \mathrm{E}(\rho_t|\mathcal{F}_{t-1}),$$

from where we have κ = -3.13. Further calculations give: $X_0 = 1$; $X_1 = 0.71$ in case of an upward move, or $X_1 = 1.81$ in case of a downward move; $X_2 =$ 0.51 for the upper node, 1.29 for the middle mode, and 3.27 for the lower node (shown in square brackets in Fig. 2.1). Now, the price at time $t = 0$ is $C = E[f(S_2)/X_2] = \$15.50$, and at time $t = 1$ it is either \$26.79 or \$1.40.

As expected, both methods lead to the same results.

2.4 Hedging in incomplete markets

In incomplete markets replicating self-financing strategies may not exist. The examples of incomplete markets include markets subject to portfolio constraints, markets with frictions such as transaction costs or taxes, markets where contingent claims depend on an additional source of risk independent of the financial risk (e.g. mortality risk), so that the claim can not be perfectly hedged by trading on a financial market. One possible approach to tackle the problem is to relax the self-financing requirement and to consider non-self-financing strategies allowing for additional cash inflows/outflows. In this case, when the self-financing strategy replicating the payoff cannot be found, but additional inflows and outflows of the capital are allowed, it is desirable to have a strategy with a property $\mathrm{E}(\Delta I_t/X_t|\mathcal{F}_{t-1}) = 0$. Such strategies were introduced in Föllmer and Sondermann [2.4] and are called mean-self-financing.

Definition 2.3. *Strategy $\pi = (\xi, \eta)$ with investment process $I = (I_t)_{t\geq 0}$ is called P-mean-self-financing, or simply mean-self-financing, if*

$$\mathrm{E}\left(\frac{\Delta I_t}{X_t}\middle|\mathcal{F}_{t-1}\right) = 0,$$

i.e. $(\Delta I_t/X_t)_{t\geq 1}$ is a martingale difference.

It is clear that the class of mean-self-financing strategies includes the self-financing portfolios. We can also give a martingale characterization of mean-self-financing strategies.

Lemma 2.3. *Strategy $\pi = (\xi, \eta)$ with capital $V = (V_t)_{t\geq 0}$ is mean-self-financing if and only if $V/X = (V_t/X_t)_{t\geq 0}$ is a martingale.*

Proof.

a) Necessity.

Let strategy $\pi = (\xi, \eta)$ with capital $V = (V_t)_{t\geq 0}$ be mean-self-financing. Then, similarly to (2.3.3) we have:

$$\begin{aligned} \Delta\frac{V_t}{X_t} &= \xi_{t-1}\Delta\frac{S_t}{X_t} + \eta_{t-1}\Delta\frac{B_t}{X_t} + \frac{\Delta I_t}{X_t} \\ &= \frac{V_{t-1}}{X_{t-1}}(\alpha_{t-1} - \kappa_{t-1})\Delta M_t + \frac{\Delta I_t}{X_t}, \end{aligned} \tag{2.4.1}$$

where M_t is defined in (2.3.2).

By Lemma 2.2 and Definition 2.3, a sequence $(\Delta(V_t/X_t))_{t\geq 1}$ is a martingale difference.

b) Sufficiency.

Let $V/X = (V_t/X_t)_{t\geq 0}$ be a martingale and find arbitrary sequences $\xi = (\xi_t)_{t\geq 0}$ and $\eta = (\eta_t)_{t\geq 0} = ((V_t - \xi_t S_t)/B_t)_{t\geq 0}$, so that a strategy $\pi = (\xi, \eta)$ had a capital V. Then by Eq. (2.4.1) π is mean-self-financing. □

Further, we will be interested in finding the "cheapest" replicating mean-self-financing strategy. Note the additional investments at each time, $(\Delta I_t)_{t\geq 1}$, are random and a risk-averse investor will want to minimize uncertainty over the time of a contract. These considerations lead to a concept of risk-minimization introduced by Föllmer and Sondermann [2.4], who suggested to determine the trading strategy by minimizing the variance of the future costs.

We consider a global risk-minimization problem. Among all strategies with $V_T = f_T$, where f_T is the payout of a contingent claim, we seek a mean-self-financing trading strategy minimizing the variance of all the additional investments over the life of a contract:

$$\text{minimize}$$

$$R = \mathrm{Var}\bigg(\sum_{t=1}^{T} \frac{\Delta I_t}{X_t}\bigg), \tag{2.4.2}$$

$$\text{subject to } \mathrm{E}\bigg(\frac{\Delta I_t}{X_t}\bigg|\mathcal{F}_{t-1}\bigg) = 0. \tag{2.4.3}$$

The resulting strategy will be called *risk-minimizing*. Due to condition (2.4.3), the risk-minimization problem (2.4.2) can be simplified to a minimization of $R = \mathrm{E}\sum_{t=1}^{T}(\Delta I_t/X_t)^2$.

Theorem 2.4. *A unique risk-minimizing mean-self-financing hedging strategy π^* for the contract with payout f_T has capital*

$$V_t^* = X_t\,\mathrm{E}\bigg(\frac{f_T}{X_T}\bigg|\mathcal{F}_t\bigg), 0 \leq t \leq T, \tag{2.4.4}$$

and structure

$$\xi_t^* = \gamma_t \frac{V_t^*}{X_t} + \frac{X_t}{S_t}\frac{\mathrm{E}\big[\Delta\frac{V_{t+1}^*}{X_{t+1}}\Delta M_{t+1}|\mathcal{F}_t\big]}{\mathrm{E}\big[(\Delta M_{t+1})^2|\mathcal{F}_t)\big]}, \tag{2.4.5}$$

$$\eta_t^* = \frac{V_t^* - \xi_t^* S_t}{B_t}. \tag{2.4.6}$$

Risk associated with a contingent claim f_T is

$$R^{\pi^*} = \sum_{t=1}^{T} \left[\mathrm{E}\left(\Delta \frac{V_t^*}{X_t}\right)^2 - \mathrm{E}\frac{\mathrm{E}^2\left(\Delta \frac{V_t^*}{X_t} \Delta M_t | \mathcal{F}_{t-1}\right)}{\mathrm{E}\left[(\Delta M_t)^2 | \mathcal{F}_{t-1}\right]} \right]. \tag{2.4.7}$$

Proof. Due to mean-self-financing requirement (2.4.3), $V^*/X = (V_t^*/X_t)_{t \geq 0}$ is a martingale by Lemma 2.3, and we require $V_T^* = f_T$, hence Eq. (2.4.4) is true. The strategy clearly exists as the condition $V_T^* = f_T$ can always be achieved, for example, by the appropriate choice of η_T at terminal time T as it is adapted.

Similarly to Eq. (2.4.1) we have:

$$\Delta \frac{V_t^*}{X_t} = \frac{S_{t-1}}{X_{t-1}} \left(\xi_{t-1}^* - \gamma_{t-1} \frac{V_{t-1}^*}{X_{t-1}} \right) \Delta M_t + \frac{\Delta I_t^*}{X_t}. \tag{2.4.8}$$

It follows from Eqs. (2.4.8) and (2.4.5) that

$$\frac{\Delta I_t^*}{X_t} = \Delta \frac{V_t^*}{X_t} - \frac{\mathrm{E}\left[\Delta \frac{V_t^*}{X_t} \Delta M_t | \mathcal{F}_{t-1}\right]}{\mathrm{E}\left[(\Delta M_t)^2 | \mathcal{F}_{t-1}\right]} \Delta M_t. \tag{2.4.9}$$

From Eq. (2.4.9) we get the expression (2.4.7) for the risk of a contingent claim.

Now, let $\pi = (\xi, \eta)$ be another mean-self-financing hedging strategy with $V_0 = V_0^*$. Hence,

$$V_t = X_t \, \mathrm{E}\left(\frac{f_T}{X_T} \middle| \mathcal{F}_t \right) = V_t^*. \tag{2.4.10}$$

Similarly to (2.4.8) we have

$$\Delta \frac{V_t}{X_t} = \frac{S_{t-1}}{X_{t-1}} \left(\xi_{t-1} - \gamma_{t-1} \frac{V_{t-1}}{X_{t-1}} \right) \Delta M_t + \frac{\Delta I_t}{X_t}. \tag{2.4.11}$$

From (2.4.11), (2.4.10) and (2.4.8) we get

$$\frac{\Delta I_t}{X_t} = \frac{S_{t-1}}{X_{t-1}} (\xi_{t-1}^* - \xi_{t-1}) \Delta M_t + \frac{\Delta I_t^*}{X_t}. \tag{2.4.12}$$

Using Eq. (2.4.12) and noting that $[(\Delta I_t^*/X_t)\Delta M_t]_{t \geq 1}$ is a martingale difference, as follows from (2.4.9), we arrive at:

$$R^{\pi} = \mathrm{E} \sum_{t=1}^{T} \left(\frac{\Delta I_t}{X_t} \right)^2 = \sum_{t=1}^{T} \mathrm{E} \frac{S_{t-1}^2}{X_{t-1}^2} (\xi_{t-1}^* - \xi_{t-1})^2 (\Delta M_t)^2 + R^{\pi^*}.$$

Hence, any other mean-self-financing strategy with the same initial capital will involve a higher risk. □

2.5 Hedging equity-linked life insurance contracts

In equity-linked life insurance contracts benefit the policyholder receives at a terminal time T, provided he or she is still alive at this time, is directly linked to the development of the stocks or stock indices. For instance, the amount paid to the policyholder, $f(S_T)$, can be:

$$f(S_T) = S_T \tag{2.5.1}$$

or

$$f(S_T) = \max(S_T, K). \tag{2.5.2}$$

The contract (2.5.1) is called a *pure equity-linked contract*; in such contracts the financial risk associated with stock prices can be entirely charged to the policyholder. The contract (2.5.2) is known as *equity-linked contract with guarantee* (here K is the guarantee) and represents an example of the contract when the financial risk is shared between a policyholder and an insurance company.

We assume that n policyholders buy the same type of a contract at time 0 at the age of x and their remaining lifetimes $T_1, ..., T_n$ are independent and identically distributed. We also assume that the remaining life times are independent of the discounted stock price process $S/X = (S_t/X_t)_{t\geq 0}$ (or, more generally, that financial market is independent of the insurance risk). The random variable Y_t denotes the number of policyholders who survived to time t. The survival probability of the insured is denoted

$$P(T_i > t) = {}_tp_x.$$

The number of survivors, Y_t, can be seen as the sum of independent Bernoulli random variables, each taking on value 1 with probability ${}_tp_x$, or 0 with probability $1 - {}_tp_x$. Then the expected number of the survivors at terminal time T is

$$\mathrm{E}(Y_T) = \mathrm{E}\Big(\sum_{i=1}^{n} I_{[T_i>T]}\Big) = \sum_{i=1}^{n} \mathrm{E}\big(I_{[T_i>T]}\big) = n\ {}_Tp_x,$$

and the variance is

$$\mathrm{Var}(\mathrm{Y_T}) = \mathrm{Var}\Big(\sum_{i=1}^{n} I_{[T_i>T]}\Big) = \sum_{i=1}^{n} \mathrm{Var}\big(I_{[T_i>T]}\big) = n\ {}_Tp_x(1 - {}_Tp_x).$$

Thus, the insurer's liability, f_T, depends on the number of survivors, Y_T, and the evolution of the stock prices $f(S_T)$: $f_T = Y_T f(S_T)$.

Formally, to construct a model combining insurance risk (uncertainty regarding number of policyholders who will survive to terminal time) and financial risk (uncertainty regarding the development of the stock prices), we need to start with two separate filtered probability spaces. Let $(\Omega_1, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P_1)$ be a space carrying the financial risk, with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ containing information about the evolution of the financial market, i.e. $\mathcal{F}_t = \sigma(S_0, ..., S_t)$. Let $(\Omega_2, \mathcal{H}, \mathbb{H} = (\mathcal{H}_t)_{t\geq 0}, P_2)$ be a space incorporating the insurance risk, with filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ containing information about policyholders, i.e. $\mathcal{H}_t = \sigma(Y_0, ..., Y_t)$. These two models are then embedded into a product space $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}, P)$, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, defined by $\mathcal{G}_t = \sigma(\mathcal{F}_t \cup \mathcal{H}_t)$, contains all the available information up to time t. Thus, it is assumed that the insurance company at any time t has access to the current information about both stock performance and the number of survived policyholders. Filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{H}_t)_{t\geq 0}$ are independent under P.

Market defined on a described product space will always be incomplete (even if the model of the financial market is complete), as contingent claim is allowed to depend on an additional source of risk independent of the financial risk. Hence, we can view the equity-linked life insurance contract as a contingent claim in an incomplete market, and therefore a risk-minimizing trading strategy for such a contract is defined similarly to Section 2.4.

Let $\pi^f = (\xi^f, \eta^f)$ denote a risk-minimizing hedging strategy for the financial contract with payout $f(S_T)$. The capital of π^f is given by
$V_t^f = X_t \mathrm{E}(f(S_T)/X_T|\mathcal{F}_t)$, where $f(S_T) = V_T^f$. Let $\pi^* = (\xi^*, \eta^*)$ be a risk-minimizing trading strategy for the equity-linked life insurance contract with liability $f_T = V_T^* = Y_T V_T^f$. Due to independence of financial and insurance

risks, risk-minimizing mean-self-financing strategy π^* has the value process:

$$\begin{aligned} V_t^* &= X_t \,\mathrm{E}\left(\frac{Y_T f(S_T)}{X_T}\middle|\mathcal{G}_t\right) \\ &= X_t \,\mathrm{E}(Y_T|\mathcal{H}_t)\,\mathrm{E}\left(\frac{f(S_T)}{X_T}\middle|\mathcal{F}_t\right) \\ &= Y_t \;{}_{T-t}p_{x+t}\; V_t^f, \end{aligned} \tag{2.5.3}$$

and structure:

$$\begin{aligned} \xi_t^* &= \gamma_t \frac{V_t^*}{X_t} + \frac{X_t}{S_t}\frac{\mathrm{E}\big[\Delta \frac{V_{t+1}^*}{X_{t+1}}\Delta M_{t+1}|\mathcal{G}_t\big]}{\mathrm{E}\big[(\Delta M_{t+1})^2|\mathcal{G}_t\big]} \\ &= Y_t \;{}_{T-t}p_{x+t}\left(\gamma_t \frac{V_t^f}{X_t} + \frac{X_t}{S_t}\frac{\mathrm{E}\big[\Delta \frac{V_{t+1}^f}{X_{t+1}}\Delta M_{t+1}|\mathcal{F}_t\big]}{\mathrm{E}\big[(\Delta M_{t+1})^2|\mathcal{F}_t\big]}\right) \\ &= Y_t \;{}_{T-t}p_{x+t}\; \xi_t^f, \end{aligned}$$

$$\eta_t^* = \frac{V_t^* - \xi_t^* S_t}{B_t} = Y_t \;{}_{T-t}p_{x+t}\; \eta_t^f,\ t \geq 0.$$

In Eq. (2.5.3), $\mathrm{E}(Y_T|\mathcal{H}_t) = Y_t \;{}_{T-t}p_{x+t}$, as $Y_T|\mathcal{H}_t \sim \mathrm{Binomial}(Y_t, {}_{T-t}p_{x+t})$, where Y_t is actual number of survivors at time t, and ${}_{T-t}p_{x+t}$ is a conditional probability of survival to time T given that the insured is alive at time t.

The risk associated with a contingent claim $f_T = Y_T f(S_T)$ is

$$\begin{aligned} R^{\pi^*} = n \;{}_Tp_x \sum_{t=1}^{T} \Bigg[&{}_{T-t}p_{x+t}\,\mathrm{E}\left(\frac{V_t^f}{X_t}\right)^2 \\ &- {}_{T-(t-1)}p_{x+(t-1)}\left(\mathrm{E}\left(\frac{V_{t-1}^f}{X_{t-1}}\right)^2 + \mathrm{E}\frac{\mathrm{E}^2\big[\Delta M_t \Delta \frac{V_t^f}{X_t}|\mathcal{F}_{t-1}\big]}{\mathrm{E}\big[(\Delta M)_t^2|\mathcal{F}_{t-1}\big]}\right)\Bigg] \\ &+ n(n-1)\;{}_Tp_x^2 \sum_{t=1}^{T}\left[\mathrm{E}\left(\Delta\frac{V_t^f}{X_t}\right)^2 - \mathrm{E}\frac{\mathrm{E}^2\big[\Delta M_t \Delta \frac{V_t^f}{X_t}|\mathcal{F}_{t-1}\big]}{\mathrm{E}\big[(\Delta M)_t^2|\mathcal{F}_{t-1}\big]}\right]. \end{aligned} \tag{2.5.4}$$

In cases when incomplete market arises from a complete financial market by allowing contingent claims to depend on an insurance risk that is independent of the financial risk, the second sum in Eq. (2.5.4) becomes 0.

2.5.1 Numerical example

To illustrate our methodology numerically, we consider the example from Møller [2.10], where he obtained a local risk-minimization strategy with respect to a martingale measure P^* to hedge an equity-linked life insurance contract. We use the same example to demonstrate hedging as a result of a global risk-minimization strategy with respect to a physical measure P.

Consider a four-period model, so that there are 5 trading times: $k = 0, 1, 2, 3, 4$. For example, it can be a year with 4 periods. The length of each period is $\Delta t = 1/4$. Assume that the remaining lifetimes of the policyholders are independent and exponentially distributed with hazard rate μ, and consider $\mu = 1$. Thus, the survival probability is ${}_kp_x = \exp(-\mu k\Delta t) = \exp(-k\Delta t)$.

Amount payable to a policyholder provided he/she is alive at time T is $f(S_T) = \max(S_T, K)$, where $K = \$103$. Assume that ρ can take two values: $a = -0.1$ and $b = 0.15$; $r = 0.015$; $p = 0.5$, $S_0 = \$100$; $B_0 = 1$.

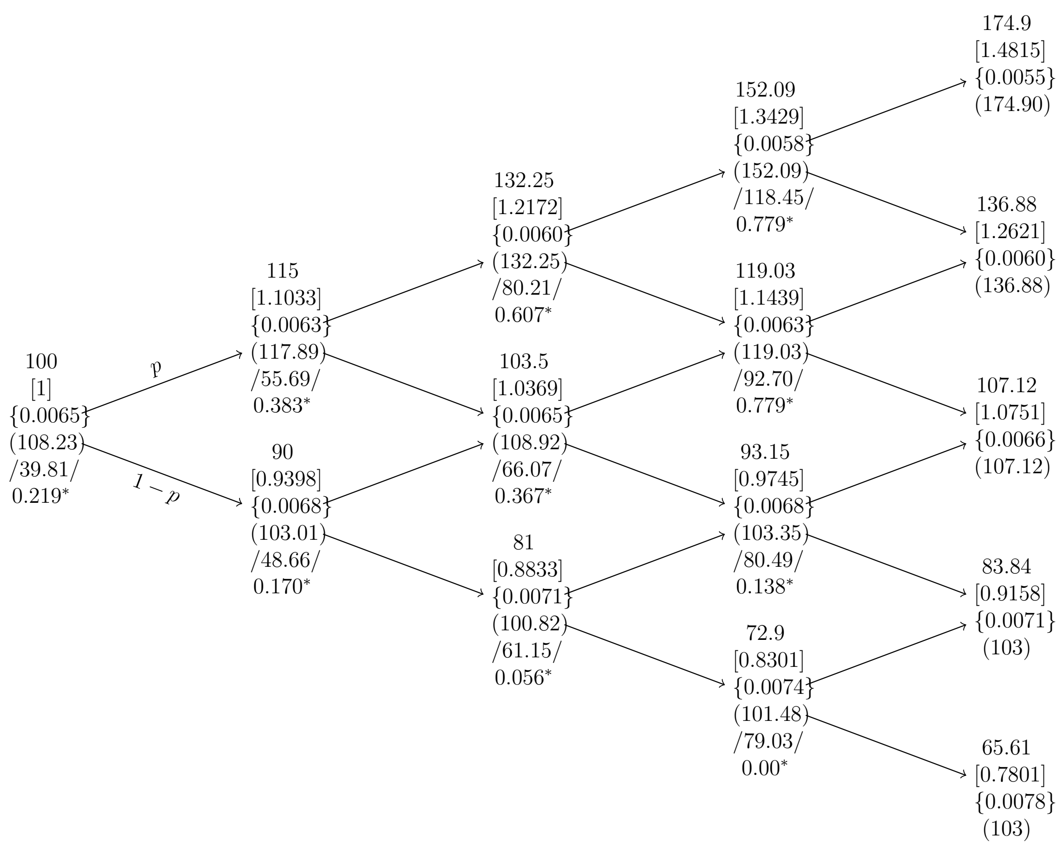

Figure 2.2: Binomial tree for the risk-minimizing hedging portfolio.
Note: The upper numbers denote stock prices; [X] is the capital of a discounting portfolio; $\{\gamma\}$ is the number of stocks in a P-discounting portfolio; (C) is the capital of hedging strategy for the contract $\max(S_T, K)$; $/V/$ is the capital of risk-minimizing trading strategy; and the lower numbers, ξ^*, denote the number of stocks in risk-minimizing hedging portfolio.

Dynamics of the stock prices, the capital of a contract paying $f(S_T)$ and of a risk-minimizing hedging strategy as well as the number of stocks in a risk-minimizing strategy are shown in Fig. 2.2. The optimal capital of a hedging strategy and the optimal number of stocks are given for one policyholder who is still alive at the time of consideration. At time 0, the optimal structure of a hedging strategy is given by $\xi_0^* = 0.219$ and $\eta_0^* = 17.9$. At time 1, if a policyholder is still alive and a stock price moves up, the optimal hedging strategy will consist of $\xi_1^* = 0.383$ and $\eta_1^* = 11.4$. If at time 1, the insured is not alive, $\xi_1^* = 0$ and $\eta_1^* = 0$.

Thus, the strategy obtained by means of finding a P-discounting portfolio coincided with the one obtained by Møller [2.10] by a "traditional" approach.

2.6 Concluding remarks

The paper has demonstrated how the valuation in complete and incomplete markets can be performed under the real-world probability measure using risk-minimization criterion. In particular, hedging and pricing in incomplete markets where the source of incompleteness stems from an additional source of risk, such as in the valuation of equity-linked life insurance contracts, are considered. The proposed approach is related to a change of numeraire technique which is popular in the financial literature, but has not yet gained much attention in the context of actuarial applications. The paper specifically aimed at developing the method for hedging the equity-linked life insurance contracts in discrete time without switching to a new probability measure.

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Chapter 3

Quantile hedging in models with dividends and application to equity-linked life insurance contracts

3.1 Introduction

Quantile hedging, introduced by Föllmer and Leukert Föllmer and Leukert [3.4], is an imperfect form of hedging employed when a contingent claim cannot admit a perfect hedge due to a budget constraint. The objective is to maximize a probability that the terminal wealth will be sufficient to cover the existing obligation. First applications of quantile methodology to pricing and hedging of equity-linked life insurance contracts were offered by Melnikov [3.7], whose ideas were extended over the years by different authors. Numerous publications on this topic cover different types of insurance contracts and different models of a financial market, as well as other types of imperfect hedging. However, when considering a model for a financial market, for simplicity, the dividends the underlying asset(s) may pay, are usually not taken into account. Yet, dividends are of significant importance for the overall performance of investment and hedging strategies. This work aims to cover the existing gap by extending financial models in quantile setting with insurance applications to include dividends.

The motivation for introducing dividends initially arose from encountering a conference paper by Daniliuc and Rozhkova [3.3]. The authors essentially aimed at extending the results of Melnikov et al. [3.9] by including dividends in a quantile hedging problem for a Black-Scholes market (without insurance applications). The paper, however, did not achieve its purpose: the authors did not consider all possible scenarios and did not provide a complete proof for getting the hedging strategy. Moreover, some formulas were not stated correctly. Since introduction of dividends into the model turned out to be not a trivial task and at the same time a problem attracting certain interest, we deem it important to cover this extension.

It is worth noting that some of the famous option pricing formulas were generalized to include dividends. For example, Xu and Wu [3.12] considered a Black-Scholes formula with dividends paid continuously, Yang and Zheng [3.13] studied the problem of pricing European options under the jump-diffusion model with discretely-paid dividends. Our goal is introducing dividends in a quantile hedging setting with insurance applications.

We consider two models of a financial market: a Black-Scholes model and a jump-diffusion model. Quantile hedging and its application to insurance in a Black-Scholes setting are presented in Melnikov [3.7]. Quantile hedging in a jump-diffusion market is given in Krutchenko and Melnikov [3.6] and the insurance context is provided in Kirch and Melnikov [3.5]. Quantile hedging problem in the Black-Scholes and jump-diffusion markets (without insurance application) is also covered in Melnikov et al. [3.9]. All of these publications deal with non-dividend paying assets. In this paper, we give a set of explicit formulas for both perfect and quantile hedges in models with dividends to provide a convenient basis for further generalizations and extensions. Our main focus, however, is to demonstrate the effect the dividends may have on pricing and hedging the equity-linked life insurance contracts.

This paper is organized as follows. In Section 3.2 we describe perfect hedging and pricing in two models with dividends. In Section 3.3 we set out a quantile hedging problem in markets with dividends and provide explicit formulas for pricing and hedging. In Section 3.4 we discuss an application of a quantile methodology in the insurance context, specifically, we focus on pure endowment with fixed guarantee equity-linked life insurance contracts. Finally, in Section 3.5, we provide an illustrative example.

3.2 Models with dividends

3.2.1 Black-Scholes model with dividends

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a standard stochastic basis. We shall consider a Black-Scholes model of financial market with two assets: a bond (or a bank account) and a stock, whose prices, B_t and S_t, evolve according to the equations:

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B_0 = 1, \tag{3.2.1}$$

$$\mathrm{d}S_t = S_t(\mu\mathrm{d}t + \sigma\mathrm{d}W_t), \quad S_0 > 0 \tag{3.2.2}$$

where $r \geq 0$ is a risk-free interest rate, $W_t = (W_t)_{t\geq 0}$ is a standard Wiener process under a market measure P, $\sigma > 0$ is a volatility of a stock, μ is appreciation rate.

Solutions to the Eqs. (3.2.1)-(3.2.2) are known to be (see, for example, Melnikov [3.8]):

$$B_t = e^{rt},$$
$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}.$$

The value of a portfolio, X_t, at time t is given by

$$X_t = \beta_t\, B_t + \gamma_t\, S_t, \tag{3.2.3}$$

where β_t denotes units of riskless bond, and γ_t denotes units of stock at time t.

We further assume that the stock is paying dividends continuously at a constant rate δ. In finance, δ is referred to as a *dividend yield*, i.e. a fraction of a stock price the firm pays to the stockholders as dividends per unit of time. Under the assumption of continuously paid dividend stream, the dividend process D_t satisfies

$$\mathrm{d}D_t = \delta S_t\,\mathrm{d}t.$$

In other words, dividend per share of asset paid within the interval $\mathrm{d}t$ is $\delta S_t\,\mathrm{d}t$, and D_t can be interpreted as total amount of dividends received up to time t.

A trading strategy $\pi_t = (\beta_t, \gamma_t)$ will be called self-financing if the corresponding portfolio value process (3.2.3) satisfies:

$$\mathrm{d}X_t = \beta_t\,\mathrm{d}B_t + \gamma_t\,\mathrm{d}S_t + \gamma_t\,\mathrm{d}D_t = \beta_t\,\mathrm{d}B_t + \gamma_t\,(\mathrm{d}S_t + \delta S_t\,\mathrm{d}t).$$

Inclusion of the term $\delta S_t\,\mathrm{d}t$ in the above equation highlights the fact that for the portfolio to be self-financing the gain from the dividends needs to be fully reinvested in the market.

Denote a discounted value of a portfolio by $V_t = X_t/B_t$. The Itô's lemma yields:

$$\mathrm{d}V_t = -rV_t\ \mathrm{d}t + e^{-rt}\ \mathrm{d}X_t = \gamma_t\frac{S_t}{B_t}\big[(\mu+\delta-r)\ \mathrm{d}t + \sigma\ \mathrm{d}W_t\big] = \gamma_t\,\sigma\frac{S_t}{B_t}\ \mathrm{d}W_t^*,$$

where

$$W_t^* = W_t + \frac{\mu+\delta-r}{\sigma}\,t$$

is a standard Brownian motion with respect to the unique martingale measure P^*, which is defined according to Girsanov theorem by

$$Z_t^* = \frac{\mathrm{d}P_t^*}{\mathrm{d}P_t} = \exp\left\{-\frac{\mu+\delta-r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu+\delta-r}{\sigma}\right)^2 t\right\}. \tag{3.2.4}$$

It is clear that for a dividend-paying asset the discounted stock price S_t/B_t is no longer a P^*-martingale. Instead, a process $S_te^{\delta t}/B_t$ is a P^*-martingale.

The dynamics of the asset price S_t under a measure P^* becomes:

$$\begin{aligned}\mathrm{d}S_t &= S_t(r-\delta)\ \mathrm{d}t + S_t\,\sigma\ \mathrm{d}W_t^*,\\ S_t &= S_0\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)t + \sigma W_t^*\right\}.\end{aligned} \tag{3.2.5}$$

Since V_t is a martingale under P^*, any contingent claim with payoff $f_T = X_T$, where T denotes the exercise time, can be priced via:

$$\mathrm{E}^*\left(\frac{f_T}{B_T}\middle|\mathcal{F}_t\right)B_t = X_t.$$

Consider a European call option with payoff $f_T = (S_T - K)^+$. The option price at inception of the contract, C_0, can be computed as

$$\begin{aligned}
C_0 = X_0 = \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right) &= e^{-rT}\mathrm{E}^*\big(S_0 e^{(r-\delta-\sigma^2/2)T+\sigma W_T^*} - K\big)^+ \\
&= e^{-rT}\left[\int_{-d_-^{(T)}}^{\infty} S_0 e^{(r-\delta-\sigma^2/2)T} e^{\sigma y\sqrt{T}}\phi(y)\ \mathrm{d}y - K\int_{-d_-^{(T)}}^{\infty}\phi(y)\mathrm{d}y\right] \\
&= S_0 e^{(-\delta-\sigma^2/2)T}\int_{-d_-^{(T)}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y-\sigma\sqrt{T})^2}{2}\right\}\exp\left\{\frac{\sigma^2 T}{2}\right\}\mathrm{d}y \\
&\quad - Ke^{-rT}\Phi(d_-^{(T)}) \\
&= S_0 e^{-\delta T}\Phi\big(d_+^{(T)}\big) - Ke^{-rT}\Phi\big(d_-^{(T)}\big), \qquad (3.2.6)
\end{aligned}$$

where $\phi(\cdot)$ denotes a probability density function of a standard normal random variable; $\Phi(\cdot)$ denotes a standard normal cumulative distribution function;

$$d_\pm^{(T-t)} = \frac{\ln(S_t/K) + (r - \delta \pm \sigma^2/2)\ (T-t)}{\sigma\sqrt{T-t}}. \qquad (3.2.7)$$

Similarly, a price of a European call option at any time $t < T$ is given by the value of the self-financing portfolio at time t:

$$\begin{aligned}
X_t^* &= S_t e^{-\delta(T-t)}\Phi\left(\frac{\ln(S_t/K) + (r-\delta+\sigma^2/2)\ (T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - Ke^{-r(T-t)}\Phi\left(\frac{\ln(S_t/K) + (r-\delta-\sigma^2/2)\ (T-t)}{\sigma\sqrt{T-t}}\right) \\
&= S_t e^{-\delta(T-t)}\Phi\big(d_+^{(T-t)}\big) - Ke^{-r(T-t)}\Phi\big(d_-^{(T-t)}\big),
\end{aligned}$$

and the self-financing trading strategy π^* has the following structure:

$$\begin{aligned}
\gamma_t^* &= e^{-\delta(T-t)}\Phi\left(\frac{\ln(S_t/K) + (r-\delta+\sigma^2/2)\ (T-t)}{\sigma\sqrt{T-t}}\right) \\
&= e^{-\delta(T-t)}\Phi\big(d_+^{(T-t)}\big), \qquad (3.2.8) \\
\beta_t^* &= \frac{X_t^* - \gamma_t^* S_t}{B_t} = -Ke^{-rT}\Phi\left(\frac{\ln(S_t/K) + (r-\delta-\sigma^2/2)\ (T-t)}{\sigma\sqrt{T-t}}\right) \\
&= -Ke^{-rT}\Phi\big(d_-^{(T-t)}\big).
\end{aligned}$$

3.2.2 Jump-diffusion model with dividends

Although a traditional Black-Scholes model is suitable for many applications, it does not capture situations in which market crashes or rallies. Adding a jump component to a diffusion model allows to reflect more extreme stock price movements and results in a *jump-diffusion* model of a financial market.

In this section, we consider a jump-diffusion market with three assets, namely, a risk-free asset (a bond or a cash account), and two risky assets with the following dynamics:

$$\begin{aligned} \mathrm{d}B_t &= rB_t\mathrm{d}t, \quad B_0 = 1, \\ \mathrm{d}S_t^{(i)} &= S_{t-}^{(i)}(\mu_i\mathrm{d}t + \sigma_i\mathrm{d}W_t - \nu_i\mathrm{d}\Pi_t), \quad S_0^{(i)} > 0, \quad i = 1, 2, \end{aligned} \tag{3.2.9}$$

where r, $\mu_i \in \mathbb{R}_+^1$, $\sigma_i > 0$, $\nu_i < 1$, $W_t = (W_t)_{t\geq 0}$ is a standard Wiener process under a market measure P, $\Pi = (\Pi_t)_{t\geq 0}$ is a Poisson process with a positive intensity λ.

For each time t, the jump size of the Poisson process is either 0 or 1, i.e. $\Delta\Pi_t = 0$ or $\Delta\Pi_t = 1$ with probability 1. Probability of a random variable Π_t being equal to n ($n \in \mathbb{Z}_{\geq 0}$) at any time t is given by

$$P(\Pi_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

It is assumed that W and Π are mutually independent and generate the filtration $\mathbb{F}$, and that both stocks pay dividends continuously at rates δ_i, $i = 1, 2$.

The solution to the stochastic differential equation (3.2.9) is

$$S_t^{(i)} = S_0^{(i)} \exp\left\{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t + \Pi_t \ln(1 - \nu_i)\right\}, \quad i = 1, 2.$$

To put it into words, the stock price process evolves as a geometric Brownian motion between the jumps, and after each jump the stock price is multiplied by $(1 - \nu_i)$, i.e. the jump in a stock price is given by

$$\Delta S_t^{(i)} = -\nu_i S_{t-}^{(i)} \Delta\Pi_t, \quad i = 1, 2. \tag{3.2.10}$$

The value of a portfolio, X_t, at time t is

$$X_t = \beta_t B_t + \gamma_t S_t^{(1)} + \xi_t S_t^{(2)}. \tag{3.2.11}$$

A strategy $\pi_t = (\beta_t, \gamma_t, \xi_t)$ will be called self-financing if the portfolio value process (3.2.11) satisfies:

$$\mathrm{d}X_t = \beta_t \mathrm{d}B_t + \gamma_t\big(\mathrm{d}S_t^{(1)} + \delta_1 S_{t-}^{(1)}\mathrm{d}t\big) + \xi_t\big(\mathrm{d}S_t^{(2)} + \delta_2 S_{t-}^{(2)}\mathrm{d}t\big). \tag{3.2.12}$$

Denote the total gain from holding the stock i as $G_t^{(i)} = S_t^{(i)} + D_t^{(i)}$, where $D_t^{(i)}$ is the cumulative dividend process. Then the process $G_t^{(i)}$ satisfies:

$$\begin{aligned} \mathrm{d}G_t^{(i)} &= \mathrm{d}S_t^{(i)} + \mathrm{d}D_t^{(i)} = \mathrm{d}S_t^{(i)} + \delta_i S_{t-}^{(i)}\mathrm{d}t \\ &= S_{t-}^{(i)}\Big((\mu_i + \delta_i)\mathrm{d}t + \sigma_i \mathrm{d}W_t - \nu_i \mathrm{d}\Pi_t\Big), \quad i = 1, 2. \end{aligned} \tag{3.2.13}$$

Next, define an auxiliary process $S_t^{(\delta_i)} := e^{\delta_i t} S_t^{(i)}$. Applying Itô formula to the process $\log(S_t^{(\delta_i)})$, we get

$$\mathrm{d}\log(S_t^{(\delta_i)}) = \Big(\delta_i + \mu_i - \frac{\sigma_i^2}{2}\Big)\mathrm{d}t + \sigma_i \mathrm{d}W_t + \ln(1-\nu_i)\mathrm{d}\Pi_t, \quad i = 1, 2.$$

Exponentiating leads to

$$S_t^{(\delta_i)} = S_0^{(i)} \exp\left\{\Big(\delta_i + \mu_i - \frac{\sigma_i^2}{2}\Big)t + \sigma_i W_t + \Pi_t \ln(1-\nu_i)\right\}, \quad i = 1, 2,$$

or in SDE form:

$$\mathrm{d}S_t^{(\delta_i)} = S_{t-}^{(\delta_i)}\Big((\mu_i + \delta_i)\mathrm{d}t + \sigma_i \mathrm{d}W_t - \nu_i \mathrm{d}\Pi_t\Big), \quad i = 1, 2. \tag{3.2.14}$$

Similarly, for the discounted auxiliary process $\tilde{S}_t^{(i)} := e^{-rt} S_t^{(\delta_i)} = e^{(\delta_i - r)t} S_t^{(i)}$, $i = 1, 2$, we have

$$\tilde{S}_t^{(i)} = S_0^{(i)} \exp\left\{\Big(\delta_i - r + \mu_i - \frac{\sigma_i^2}{2}\Big)t + \sigma_i W_t + \Pi_t \ln(1-\nu_i)\right\}, \quad i = 1, 2,$$

or the SDE:

$$\mathrm{d}\tilde{S}_t^{(i)} = \tilde{S}_{t-}^{(i)}((\mu_i + \delta_i - r)\mathrm{d}t + \sigma_i \mathrm{d}W_t - \nu_i \mathrm{d}\Pi_t), \quad i = 1, 2. \tag{3.2.15}$$

By comparing (3.2.15) with (3.2.14), or through a general Itô's lemma applied to $e^{-rt} S_t^{(\delta_i)}$, we may note that

$$\mathrm{d}\tilde{S}_t^{(i)} = e^{-rt}\mathrm{d}S_t^{(\delta_i)} - r e^{(\delta_i - r)t} S_{t-}^{(i)}\mathrm{d}t, \quad i = 1, 2. \tag{3.2.16}$$

From Eqs. (3.2.13) and (3.2.14), it follows that $\mathrm{d}G_t^{(i)}$ can be written as a function of the auxiliary process $S_t^{(\delta_i)}$ as

$$\mathrm{d}G_t^{(i)} = e^{-\delta_i t}\mathrm{d}S_t^{(\delta_i)},$$

and the value process of the self-financing portfolio (3.2.12) can be expressed as

$$\begin{aligned} \mathrm{d}X_t &= \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}G_t^{(1)} + \xi_t \mathrm{d}G_t^{(2)} \\ &= \beta_t \mathrm{d}B_t + \gamma_t e^{-\delta_1 t} \mathrm{d}S_t^{(\delta_1)} + \xi_t e^{-\delta_2 t} \mathrm{d}S_t^{(\delta_2)}. \end{aligned} \tag{3.2.17}$$

Applying a general Itô formula (see, for example, Pascucci [3.10]) to the discounted portfolio value process $V_t = X_t/B_t$ and using relationships (3.2.17) and (3.2.16), we get

$$\begin{aligned} \mathrm{d}V_t &= -re^{-rt} X_{t-} \mathrm{d}t + e^{-rt} \mathrm{d}X_t \qquad (3.2.18) \\ &= e^{-rt}\Big(\beta_t \mathrm{d}B_t + \gamma_t e^{-\delta_1 t} \mathrm{d}S_t^{(\delta_1)} + \xi_t e^{-\delta_2 t} \mathrm{d}S_t^{(\delta_2)}\Big) - \frac{X_{t-}}{B_t^2} \mathrm{d}B_t \\ &= \gamma_t e^{-\delta_1 t} \mathrm{d}\tilde{S}_t^{(1)} + \xi_t e^{-\delta_2 t} \mathrm{d}\tilde{S}_t^{(2)} + \Big(\beta_t B_t + \gamma_t S_{t-}^{(1)} + \xi_t S_{t-}^{(2)}\Big) \frac{\mathrm{d}B_t}{B_t^2} - X_{t-} \frac{\mathrm{d}B_t}{B_t^2} \\ &= \gamma_t e^{-\delta_1 t} \mathrm{d}\tilde{S}_t^{(1)} + \xi_t e^{-\delta_2 t} \mathrm{d}\tilde{S}_t^{(2)}. \qquad (3.2.19) \end{aligned}$$

Hence, we need to find a measure such that the discounted auxiliary processes $\tilde{S}_t^{(i)}$, $i = 1, 2$, are martingales.

From Girsanov theorem for jump-diffusion processes we know that $\mathrm{d}W_t = \varphi \mathrm{d}t + \mathrm{d}W_t^*$, where $W^* = (W_t^*)_{t \geq 0}$ is a standard Brownian motion under a martingale measure P^*, and $\Pi = (\Pi_t)_{t \geq 0}$ is a Poisson process with P^*-intensity $\lambda^* = \lambda(1 + \psi)$, $\psi > -1$. Plugging these back into (3.2.15) and compensating Π under P^*:

$$\begin{aligned} \mathrm{d}\tilde{S}_t^{(i)} = \tilde{S}_{t-}^{(i)}\Big(& (\mu_i + \delta_i - r + \varphi\sigma_i - \nu_i\lambda - \nu_i\lambda\psi)\mathrm{d}t + \sigma_i \mathrm{d}W_t^* \\ & - \nu_i(\mathrm{d}\Pi_t - \lambda(1+\psi)\mathrm{d}t)\Big), \quad i = 1, 2. \end{aligned} \tag{3.2.20}$$

From (3.2.20) it is clear that P^* is a martingale measure if and only if

$$\mu_i + \delta_i - r + \varphi\sigma_i - \nu_i\lambda - \nu_i\lambda\psi = 0 \quad \text{for } i = 1, 2. \tag{3.2.21}$$

The unique solution to (3.2.21) is given by

$$\begin{aligned} \varphi &= \frac{(\mu_1 + \delta_1 - r)\nu_2 - (\mu_2 + \delta_2 - r)\nu_1}{\sigma_2\nu_1 - \sigma_1\nu_2}, \qquad (3.2.22) \\ \psi &= \left(\frac{(\mu_1 + \delta_1 - r)\sigma_2 - (\mu_2 + \delta_2 - r)\sigma_1}{\sigma_2\nu_1 - \sigma_1\nu_2} - \lambda\right)\Big/\lambda, \end{aligned}$$

assuming that $\sigma_2\nu_1 - \sigma_1\nu_2 \neq 0$.

Hence,

$$\lambda^* = \frac{(\mu_1 + \delta_1 - r)\sigma_2 - (\mu_2 + \delta_2 - r)\sigma_1}{\sigma_2\nu_1 - \sigma_1\nu_2}. \tag{3.2.23}$$

The density of the martingale measure P^* can then be uniquely determined from:

$$\begin{aligned} Z_t^* = \frac{\mathrm{d}P_t^*}{\mathrm{d}P_t} &= \exp\Big\{ -\lambda\psi t - \frac{\varphi^2}{2}t + \varphi W_t + \Pi_t \ln(1+\psi)\Big\} \\ &= \exp\Big\{(\lambda - \lambda^*)t - \frac{\varphi^2}{2}t + \varphi W_t + \Pi_t(\ln\lambda^* - \ln\lambda)\Big\}. \end{aligned} \tag{3.2.24}$$

Consequently, the risk-neutral dynamics of the stock price is given by

$$S_t^{(i)} = S_0^{(i)} \exp\Big\{\Big(r - \delta_i + \nu_i\lambda^* - \frac{\sigma_i^2}{2}\Big)t + \sigma_i W_t^* + \Pi_t \ln(1-\nu_i)\Big\}, \quad i = 1, 2, \tag{3.2.25}$$

or in SDE form

$$\mathrm{d}S_t^{(i)} = S_{t-}^{(i)}\big((r - \delta_i)\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_t^* - \nu_i\,\mathrm{d}(\Pi_t - \lambda^* t)\big), \quad i = 1, 2. \tag{3.2.26}$$

Next, we show how to price a European call option with payoff $f_T = (S_T^{(1)} - K)^+$ in a complete jump-diffusion market. The initial price, C_0, of this claim in view of (3.2.25) can be expressed as

$$\begin{aligned} C_0 &= \mathrm{E}^*(e^{-rT} f_T) = \mathrm{E}^* e^{-rT}\big(S_T^{(1)} - K\big)^+ \\ &= \mathrm{E}^*\Big[S_0^{(1)} \exp\Big\{\Big(-\delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)T + \sigma_1 W_T^* + \Pi_T \ln(1-\nu_1)\Big\} - e^{-rT}K\Big]^+. \end{aligned}$$

This expectation can be computed by conditioning on a number of jumps and using the independence of the Wiener process W^* and the Poisson process Π with respect to a measure P^*:

$$\begin{aligned} C_0 &= \mathrm{E}^*\Big[\mathrm{E}^*\Big(\Big(S_0^{(1)} \exp\Big\{\Big(-\delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)T + \sigma_1 W_T^* + \Pi_T \ln(1-\nu_1)\Big\} \\ &\qquad - e^{-rT}K\Big)^+ \Big| \Pi_T\Big)\Big] \\ &= \sum_{n=0}^{\infty} \mathrm{E}^*\Big[\Big(S_0^{(1)} \exp\Big\{\Big(-\delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)T + \sigma_1 W_T^* + \Pi_T \ln(1-\nu_1)\Big\} \\ &\qquad - e^{-rT}K\Big)^+ \Big| \Pi_T = n\Big] e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}^* \sum_{n=0}^{\infty} \Big[S_0^{(1)} (1-\nu_1)^n \exp\Big\{ \Big(-\delta_1 + \nu_1 \lambda^* - \frac{\sigma_1^2}{2} \Big) T + \sigma_1 W_T^* \Big\} - e^{-rT} K \Big]^+ \\
&\quad \times e^{-\lambda^* T} \frac{(\lambda^* T)^n}{n!} \\
&= e^{-\lambda^* T} \sum_{n=0}^{\infty} \Big[\frac{(\lambda^* T)^n}{n!} \mathrm{E}^* \Big(S_0^{(1)} (1-\nu_1)^n e^{\nu_1 \lambda^* T} \\
&\quad \times \exp\Big\{ \Big(-\delta_1 - \frac{\sigma_1^2}{2} \Big) T + \sigma_1 W_T^* \Big\} - e^{-rT} K \Big)^+ \Big].
\end{aligned}
\tag{3.2.27}
$$

Denote the Black-Scholes price in Eq. (3.2.6) as $C^{BS}(S_0, K, T, r, \sigma, \delta)$. Now, comparing the expectation in (3.2.27) with the expression for the Black-Scholes price (3.2.6), we note that the price in (3.2.27) can be written in terms of the Black-Scholes prices as

$$
C_0 = e^{-\lambda^* T} \sum_{n=0}^{\infty} \Big[\frac{(\lambda^* T)^n}{n!} \, C^{BS}\Big(S_0^{(1)} (1-\nu_1)^n e^{\nu_1 \lambda^* T}, K, T, r, \sigma_1, \delta_1 \Big) \Big]. \tag{3.2.28}
$$

The price (and a hedging strategy) of a call option at any time $t < T$ is then given by

$$
\begin{aligned}
&X_t^* = C(S_t^{(1)}, t) \\
&= e^{-\lambda^*(T-t)} \sum_{n=0}^{\infty} \Big[\frac{[\lambda^*(T-t)]^n}{n!} \, C^{BS}\Big(S_t^{(1)} (1-\nu_1)^n e^{\nu_1 \lambda^* (T-t)}, K, T-t, r, \sigma_1, \delta_1 \Big) \Big].
\end{aligned}
$$

To find the components of a hedging strategy, we use relationships (3.2.18)-(3.2.21) to rewrite $\mathrm{d}V_t$ as

$$
\begin{aligned}
\mathrm{d}V_t &= \mathrm{d}\Big(\frac{C(S_t^{(1)}, t)}{B_t} \Big) \\
&= \Big(\frac{\gamma_t S_{t-}^{(1)} \sigma_1 + \xi_t S_{t-}^{(2)} \sigma_2}{B_t} \Big) \mathrm{d}W_t^* - \Big(\frac{\gamma_t S_{t-}^{(1)} \nu_1 + \xi_t S_{t-}^{(2)} \nu_2}{B_t} \Big) \mathrm{d}(\Pi_t - \lambda^* t) && (3.2.29) \\
&= e^{-rt} \mathrm{d}C(S_t^{(1)}, t) - r e^{-rt} C(S_{t-}^{(1)}, t) \mathrm{d}t. && (3.2.30)
\end{aligned}
$$

By Itô formula,

$$
\begin{aligned}
\mathrm{d}C(S_t^{(1)}, t) &= \frac{\partial}{\partial x} C(S_{t-}^{(1)}, t) \mathrm{d}S_t^{(1)} + \frac{\partial}{\partial t} C(S_{t-}^{(1)}, t) \mathrm{d}t \\
&\quad + \frac{1}{2} \frac{\partial^2}{\partial x^2} C(S_{t-}^{(1)}, t) \mathrm{d}\langle S^{(1)} \rangle_t^c \\
&\quad + \Big(C(S_t^{(1)}, t) - C(S_{t-}^{(1)}, t) - \frac{\partial}{\partial x} C(S_{t-}^{(1)}, t) \Delta S_t^{(1)} \Big),
\end{aligned}
\tag{3.2.31}
$$

where $\partial f/\partial x$ is a partial derivative with respect to the first argument of f.

Next, we note that

$$\mathrm{d}\langle S^{(1)}\rangle_t^c = \big(\sigma_1 S_{t-}^{(1)}\big)^2 \mathrm{d}t, \tag{3.2.32}$$

and

$$C(S_t^{(1)}, t) - C(S_{t-}^{(1)}, t) = \Big[C(S_{t-}^{(1)}(1-\nu_1), t) - C(S_{t-}^{(1)}, t)\Big]\Delta\Pi_t. \tag{3.2.33}$$

Using (3.2.32), (3.2.33), and (3.2.10), we can write (3.2.31) as

$$\begin{aligned} \mathrm{d}C(S_t^{(1)}, t) = {} & \frac{\partial}{\partial x} C(S_{t-}^{(1)}, t)\mathrm{d}S_t^{(1)} + \frac{\partial}{\partial t} C(S_{t-}^{(1)}, t)\mathrm{d}t \\ & + \frac{1}{2}\big(\sigma_1 S_{t-}^{(1)}\big)^2 \frac{\partial^2}{\partial x^2} C(S_{t-}^{(1)}, t)\mathrm{d}t \\ & + \Big[C(S_{t-}^{(1)}(1-\nu_1), t) - C(S_{t-}^{(1)}, t)\Big]\mathrm{d}\Pi_t \\ & + \frac{\partial}{\partial x} C(S_{t-}^{(1)}, t)(\nu_1 S_{t-}^{(1)})\mathrm{d}\Pi_t. \end{aligned} \tag{3.2.34}$$

Substituting (3.2.34) into (3.2.30) and using (3.2.26), we obtain

$$\begin{aligned} \mathrm{d}\bigg(\frac{C(S_t^{(1)}, t)}{B_t}\bigg) = {} & \bigg(\sigma_1 \frac{S_{t-}^{(1)}}{B_t}\frac{\partial}{\partial x} C(S_{t-}^{(1)}, t)\bigg)\mathrm{d}W_t^* \\ & + \bigg(C(S_{t-}^{(1)}(1-\nu_1), t) - C(S_{t-}^{(1)}, t)\bigg)e^{-rt}\mathrm{d}(\Pi_t - \lambda^* t) \\ & + \bigg(\big[C(S_{t-}^{(1)}(1-\nu_1), t) - C(S_{t-}^{(1)}, t)\big]\lambda^* \\ & + S_{t-}^{(1)}(r - \delta_1 + \nu_1\lambda^*)\frac{\partial}{\partial x} C(S_{t-}^{(1)}, t) + \frac{\partial}{\partial t} C(S_{t-}^{(1)}, t) \\ & + \frac{1}{2}\big(\sigma_1 S_{t-}^{(1)}\big)^2 \frac{\partial^2}{\partial x^2} C(S_{t-}^{(1)}, t) - rC(S_{t-}^{(1)}, t)\bigg)e^{-rt}\mathrm{d}t. \end{aligned} \tag{3.2.35}$$

Finally, by comparing (3.2.35) with (3.2.29), we find that the components of the hedging strategy π^* satisfy

$$\begin{cases} \gamma_t^* \sigma_1 S_{t-}^{(1)} + \xi_t^* \sigma_2 S_{t-}^{(2)} = \sigma_1 S_{t-}^{(1)} \dfrac{\partial}{\partial x} C(S_{t-}^{(1)}, t), \\ \gamma_t^* \nu_1 S_{t-}^{(1)} + \xi_t^* \nu_2 S_{t-}^{(2)} = C(S_{t-}^{(1)}, t) - C(S_{t-}^{(1)}(1-\nu_1), t). \end{cases}$$

Units of bond, β_t^*, can be found from the balance equation:

$$\beta_t = \frac{C(S_{t-}^{(1)}, t) - \gamma_t^* S_{t-}^{(1)} - \xi_t^* S_{t-}^{(2)}}{B_t}.$$

In addition, the value of the hedging strategy, $X_t^* = C(S_t^{(1)}, t)$, satisfies

$$\big[C(S_{t-}^{(1)}(1-\nu_1), t) - C(S_{t-}^{(1)}, t)\big]\lambda^* + S_{t-}^{(1)}(r - \delta_1 + \nu_1\lambda^*)\frac{\partial}{\partial x}C(S_{t-}^{(1)}, t)$$
$$+\frac{\partial}{\partial t}C(S_{t-}^{(1)}, t) + \frac{1}{2}\big(\sigma_1 S_{t-}^{(1)}\big)^2\frac{\partial^2}{\partial x^2}C(S_{t-}^{(1)}, t) - rC(S_{t-}^{(1)}, t) = 0.$$

3.3 Quantile hedging in models with dividends

The main idea of quantile hedging is constructing a strategy which maximizes the probability of a successful hedge under the physical measure P, given a budget constraint. It was shown by Föllmer and Leukert [3.4] that construction of such hedging strategy can be reduced to determination of the maximal success set using the Neyman-Pearson lemma.

Specifically, let us define a successful hedging set of a claim with payout f_T for a strategy π as

$$A = \{X_T^\pi \geq f_T\} = \{V_T \geq f_T/B_T\}.$$

It is well known that a perfect hedge has an initial capital $X_0 = \mathrm{E}^*(f_T/B_T)$. Probability of successful hedging, $P(A)$, in this case is 1. However, if the investor is not able to dispose the capital X_0 required for a perfect hedge, the probability of a successful hedge is necessarily less than 1. In such a situation, if probability of a successful hedge is chosen as an optimality criterion, an investor would look for an admissible strategy maximizing the probability of generating enough capital to cover the obligation under the budget constraint:

$$\text{maximize } P(A) \tag{3.3.1}$$

$$\text{subject to } x_0 < X_0, \tag{3.3.2}$$

where x_0 is the initial capital which is smaller than the capital required for a perfect hedge.

The problem (3.3.1)-(3.3.2) is known as a *quantile* hedging problem.

Föllmer and Leukert [3.4] showed that the quantile hedging strategy is given by the perfect hedge for the knockout option

$$f_T \mathbb{1}[\mathrm{d}P/\mathrm{d}P^* > const \cdot f_T], \tag{3.3.3}$$

if the condition

$$P\Big(\frac{\mathrm{d}P}{\mathrm{d}P^*} = const \cdot f_T\Big) = 0 \tag{3.3.4}$$

is satisfied. Constant in (3.3.3) is chosen so that $\mathrm{E}^*(f_T\, \mathbb{1}[\mathrm{d}P/\mathrm{d}P^* > const \cdot f_T]/B_T) = x_0$. If the condition (3.3.4) does not hold, it is necessary to use the extended form of the Neyman-Pearson lemma. In our settings, the condition (3.3.4) is satisfied. Therefore, in constructing quantile hedges, we will rely on the following theorem.

Theorem 3.1. *(Melnikov et al. [3.9], p.107). An optimal strategy π^* for the problem (3.3.1)-(3.3.2) coincides with a perfect hedge for the contingent claim $f_T\, \mathbb{1}_A$, where the maximal success set A has the form*

$$A = \Big\{w : \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} > const \cdot f_T\Big\}. \tag{3.3.5}$$

3.3.1 Black-Scholes market with dividends

Using (3.2.4), let us express the maximal success set (3.3.5) in terms of S_T:

$$\begin{aligned}
A &= \Big\{ \exp\Big[\frac{\mu+\delta-r}{\sigma} W_T + \frac{1}{2}\Big(\frac{\mu+\delta-r}{\sigma}\Big)^2 T\Big] > const \cdot (S_T - K)^+ \Big\} \\
&= \Big\{ \exp\Big[\frac{\mu+\delta-r}{\sigma} W_T^* - \frac{1}{2}\Big(\frac{\mu+\delta-r}{\sigma}\Big)^2 T\Big] > const \cdot (S_T - K)^+ \Big\} \\
&= \Big\{ \exp\Big[\frac{\mu+\delta-r}{\sigma^2}\Big(\ln S_0 + \Big(r-\delta-\frac{1}{2}\sigma^2\Big)T + \sigma W_T^*\Big)\Big] \\
&\qquad \times \exp\Big[-\frac{\mu+\delta-r}{\sigma^2}\Big(\ln S_0 + \Big(r-\delta-\frac{1}{2}\sigma^2\Big)T\Big) - \frac{1}{2}\Big(\frac{\mu+\delta-r}{\sigma}\Big)^2 T\Big] \\
&\qquad > const \cdot (S_T - K)^+ \Big\} \\
&= \Big\{ S_T^{\frac{\mu+\delta-r}{\sigma^2}} \exp\Big[-\frac{\mu+\delta-r}{\sigma^2}\Big(\ln S_0 + \frac{\mu-\delta+r-\sigma^2}{2} T\Big)\Big] \\
&\qquad > const \cdot (S_T - K)^+ \Big\} \\
&= \Big\{ S_T^{\frac{\mu+\delta-r}{\sigma^2}} > \tilde{C} \cdot (S_T - K)^+ \Big\},
\end{aligned} \tag{3.3.6}$$

and the positive constant $\tilde{C}$ is chosen so that

$$\mathrm{E}^*\Big(\frac{f_T}{B_T} \mathbb{1}_A\Big) = x_0. \tag{3.3.7}$$

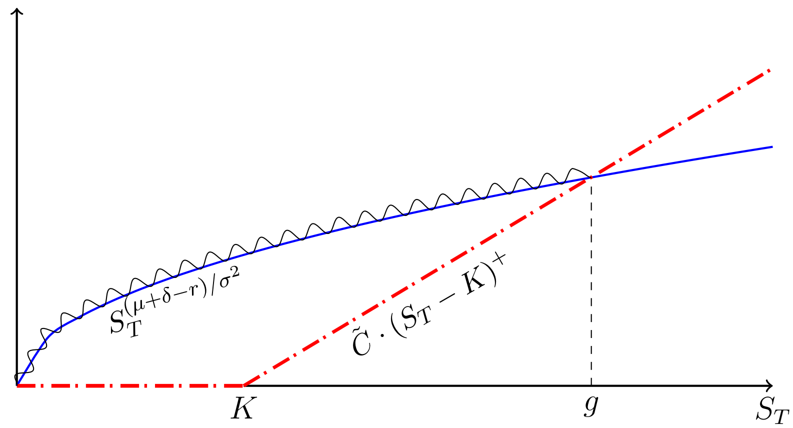


Figure 3.1: Structure of the hedging set for $\frac{\mu+\delta-r}{\sigma^2} \le 1$.

Depending on the relative magnitude of $(\mu + \delta - r)$ and σ^2, the quantile hedging strategy may admit two forms, which we consider further.

Case 1. $(\mu + \delta - r)/\sigma^2 \le 1$

Then the function $x \to x^{(\mu+\delta-r)/\sigma^2}$ is concave and the solution of the inequality (3.3.6) has the form (see Fig. 3.1):

$$A = \{S_T < g\} = \{W_T^* < b^{(T)}\}, \tag{3.3.8}$$

where $g = g(\tilde{C})$ is a solution of the equation:

$$x^{\frac{\mu+\delta-r}{\sigma^2}} = \tilde{C}(x-K)^+. \tag{3.3.9}$$

The relationship between the boundaries g and $b^{(T)}$, as follows from (3.2.5), is given by

$$g = S_0 \exp\left\{b^{(T)}\sigma + \left(r - \delta - \frac{\sigma^2}{2}\right)T\right\}. \tag{3.3.10}$$

It follows from (3.3.8) that the probability of a successful hedge is

$$P(A) = \Phi\left(\frac{b^{(T)} - \frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\right).$$

Note that the modified claim $f_T\,\mathbb{1}_A$ can be written as a combination of two call options and a binary option:

$$f_T\,\mathbb{1}_A = (S_T - K)^+\ \mathbb{1}[S_T < g]$$

$$= (S_T - K)^+ - (S_T - K)^+ \ \mathbb{1}[S_T > g]$$
$$= (S_T - K)^+ - (S_T - g)^+ - (g - K) \ \mathbb{1}[S_T > g]. \qquad (3.3.11)$$

For finding the boundary $b^{(T)}$ (and, hence, constants g and $\tilde{C}$) we use the condition (3.3.7) and representation (3.3.11):

$$x_0 = \mathrm{E}^*(e^{-rT} f_T \, \mathbb{1}_A) = \mathrm{E}^*\Big[e^{-rT}\Big((S_T - K)^+ - (S_T - g)^+ - (g - K)\,\mathbb{1}[S_T > g]\Big)\Big]$$
$$= S_0 e^{-\delta T}\Big[\Phi(d_+^{(T)}) - \Phi\Big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\Big)\Big] - K e^{-rT}\Big[\Phi(d_-^{(T)}) - \Phi\Big(-\frac{b^{(T)}}{\sqrt{T}}\Big)\Big], \qquad (3.3.12)$$

where $d_\pm^{(T)}$ are defined in (3.2.7) with $t = 0$.

Thus, the unknown constants $b^{(T)}$, g, and $\tilde{C}$ can be found from Eqs. (3.3.12), (3.3.10), and (3.3.9).

The capital X_t^* of the hedging strategy $\pi^* = (\gamma_t^*, \beta_t^*)$ at time $t < T$ can then be found as follows:

$$\begin{aligned}
X_t^* &= \mathrm{E}^*(e^{-r(T-t)} f_T \, \mathbb{1}_A | \mathcal{F}_t) \\
&= e^{-r(T-t)}\mathrm{E}^*\bigg(\bigg[S_t \exp\Big\{\Big(r - \delta - \frac{\sigma^2}{2}\Big)(T-t) + \sigma(W_T^* - W_t^*)\Big\} - K\bigg]^+ \\
&\qquad \times \mathbb{1}[S_t \exp\{(r - \delta - \sigma^2/2)(T-t) + \sigma(W_T^* - W_t^*)\} < g]\Big|\mathcal{F}_t\bigg) \\
&= e^{-r(T-t)} \int_{-d_-^{(T-t)}}^{\frac{b^{(T-t)}}{\sqrt{T-t}}} S_t \exp\Big\{\Big(r - \delta - \frac{\sigma^2}{2}\Big)(T-t) + \sigma\sqrt{T-t}\,y\Big\}\phi(y)\mathrm{d}y \\
&\quad - e^{-r(T-t)} \int_{-d_-^{(T-t)}}^{\frac{b^{(T-t)}}{\sqrt{T-t}}} K\phi(y)\mathrm{d}y \\
&= S_t e^{-\delta(T-t)} \int_{-d_-^{(T-t)}}^{\frac{b^{(T-t)}}{\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{(y - \sigma\sqrt{T-t})^2}{2}\Big\}\mathrm{d}y \\
&\quad - K e^{-r(T-t)}\Big[\Phi\big(d_-^{(T-t)}\big) - \Phi\Big(-\frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big] \\
&= S_t e^{-\delta(T-t)}\Big[\Phi\big(d_+^{(T-t)}\big) - \Phi\Big(\sigma\sqrt{T-t} - \frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big] \\
&\quad - K e^{-r(T-t)}\Big[\Phi\big(d_-^{(T-t)}\big) - \Phi\Big(-\frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big],
\end{aligned} \qquad (3.3.13)$$

where $d_\pm^{(T-t)}$ are defined in (3.2.7), and constant g relates to $b^{(T-t)}$ such that

$$g = S_t \exp\Big\{b^{(T-t)}\sigma + \Big(r - \delta - \frac{\sigma^2}{2}\Big)(T-t)\Big\}. \qquad (3.3.14)$$

We can rewrite hedging capital (3.3.13) in terms of a constant g:

$$X_t^* = S_t e^{-\delta(T-t)}\Big[\Phi\big(d_+^{(T-t)}\big) - \Phi\big(d_{+g}^{(T-t)}\big)\Big] - Ke^{-r(T-t)}\Big[\Phi\big(d_-^{(T-t)}\big) - \Phi\big(d_{-g}^{(T-t)}\big)\Big], \tag{3.3.15}$$

where

$$d_{\pm g}^{(T-t)} = \frac{\ln({S_t}/{g}) + (r - \delta \pm {\sigma^2}/{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{3.3.16}$$

The corresponding hedging strategy can be found by differentiating X_t^* with respect to S_t:

$$\gamma_t^* = \frac{\partial X_t^*}{\partial S_t} = \frac{\partial X_t^I}{\partial S_t} - \frac{\partial X_t^{II}}{\partial S_t},$$

where

$$X_t^I = S_t e^{-\delta(T-t)}\,\Phi\big(d_+^{(T-t)}\big) - Ke^{-r(T-t)}\,\Phi\big(d_-^{(T-t)}\big), \tag{3.3.17}$$

$$X_t^{II} = S_t e^{-\delta(T-t)}\,\Phi\big(d_{+g}^{(T-t)}\big) - Ke^{-r(T-t)}\,\Phi\big(d_{-g}^{(T-t)}\big). \tag{3.3.18}$$

The partial derivative of (3.3.17) with respect to the stock price is given by (3.2.8), i.e.

$$\frac{\partial X_t^I}{\partial S_t} = e^{-\delta(T-t)}\,\Phi(d_+^{(T-t)}).$$

The partial derivative of (3.3.18) with respect to the stock price can be computed as follows:

$$\begin{aligned}
\frac{\partial X_t^{II}}{\partial S_t} &= e^{-\delta(T-t)}\Phi\big(d_{+g}^{(T-t)}\big) + \frac{e^{-\delta(T-t)}}{\sigma\sqrt{T-t}}\,\phi\big(d_{+g}^{(T-t)}\big) \\
&\quad - \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}\frac{K}{S_t}\,\phi\big(d_{+g}^{(T-t)} - \sigma\sqrt{T-t}\big) \\
&= e^{-\delta(T-t)}\Phi\big(d_{+g}^{(T-t)}\big) + \frac{e^{-\delta(T-t)}}{\sigma\sqrt{T-t}}\,\phi\big(d_{+g}^{(T-t)}\big) \\
&\quad - \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}\frac{K}{S_t}\,\phi\big(d_{+g}^{(T-t)}\big)\exp\left(d_{+g}^{(T-t)}\sigma\sqrt{T-t} - \frac{\sigma^2}{2}(T-t)\right) \\
&= e^{-\delta(T-t)}\Phi\big(d_{+g}^{(T-t)}\big) + \frac{e^{-\delta(T-t)}}{\sigma\sqrt{T-t}}\,\phi\big(d_{+g}^{(T-t)}\big) - \frac{e^{-\delta(T-t)}}{\sigma\sqrt{T-t}}\frac{K}{g}\,\phi\big(d_{+g}^{(T-t)}\big) = \\
&= e^{-\delta(T-t)}\Phi\big(d_{+g}^{(T-t)}\big) + \frac{e^{-\delta(T-t)}}{\sigma\sqrt{T-t}}\frac{g-K}{g}\,\phi\big(d_{+g}^{(T-t)}\big).
\end{aligned}$$

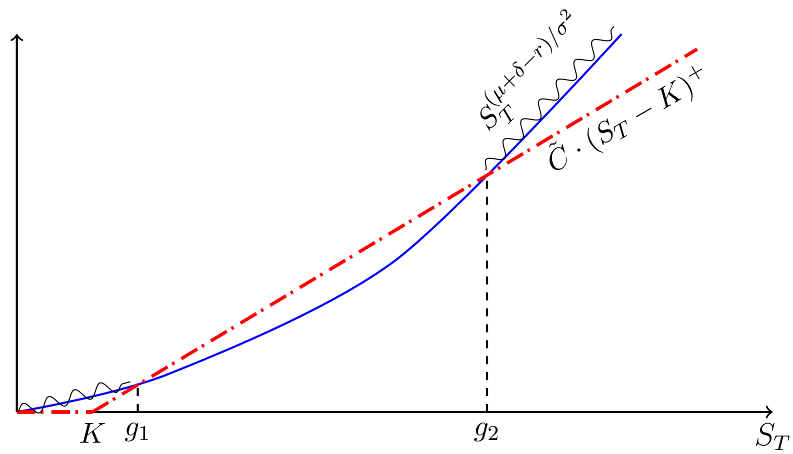


Figure 3.2: Structure of the hedging set for $\frac{\mu+\delta-r}{\sigma^2}>1$.

Hence, the structure of the hedging strategy is given by

$$\gamma_t^* = e^{-\delta(T-t)}\left[\Phi(d_+^{(T-t)}) - \Phi\big(d_{+g}^{(T-t)}\big) - \frac{\phi\big(d_{+g}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\,\frac{g-K}{g}\right], \tag{3.3.19}$$

$$\beta_t^* = -Ke^{-rT}\big[\Phi(d_-^{(T-t)}) - \Phi\big(d_{-g}^{(T-t)}\big)\big] + S_t\,\phi\big(d_{+g}^{(T-t)}\big)\,\frac{e^{-\delta(T-t)-rt}}{\sigma\sqrt{T-t}}\,\frac{g-K}{g}. \tag{3.3.20}$$

Case 2. $(\mu+\delta-r)/\sigma^2>1$

In this case the function $x\to x^{(\mu+\delta-r)/\sigma^2}$ is convex and a success set A has the form (see Fig. 3.2):

$$A=\{S_T\ <g_1\}\cup\{S_T\ >g_2\}=\{W_T^*\ <b_1^{(T)}\}\cup\{W_T^*\ >b_2^{(T)}\},$$

where $g_1=g_1(\tilde{C})<g_2=g_2(\tilde{C})$ are two solutions of the equation

$$x^{(\mu+\delta-r)/\sigma^2}=\tilde{C}(x-K)^+. \tag{3.3.21}$$

The relationships between g_i and $b_i^{(T)}$ for $i=1,2$ are similar to (3.3.10).

The probability of successful hedging is

$$P(A)=\Phi\left(\frac{b_1^{(T)}-\frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\right)+\Phi\left(\frac{-b_2^{(T)}+\frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\right). \tag{3.3.22}$$

Again, the claim $f_T\,\mathbb{1}_A$ can be written as a combination of call options and digital options:

$$f_T\,\mathbb{1}_A=f_T\,\mathbb{1}[S_T<g_1]+f_T\,\mathbb{1}[S_T>g_2]$$

$$= (S_T - K)^+ - (S_T - g_1)^+ - (g_1 - K)\,\mathbb{1}[S_T > g_1]$$
$$+ (S_T - g_2)^+ + (g_2 - K)\,\mathbb{1}[S_T > g_2].$$

Constants $b_1^{(T)}$ and $b_2^{(T)}$ (and consequently, constants g_1, g_2, and $\tilde{C}$) can be determined from the condition:

$$\begin{aligned} x_0 &= \mathrm{E}^*(e^{-rT} f_T\,\mathbb{1}_A) \\ &= S_0 e^{-\delta T}\Big[\Phi\big(d_+^{(T)}\big) - \Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\ &\quad - Ke^{-rT}\Big[\Phi\big(d_-^{(T)}\big) - \Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big], \end{aligned} \tag{3.3.23}$$

where $d_\pm^{(T)}$ are given by (3.2.7), as before.

To summarize, all the necessary constants can be found using (3.3.23), (3.3.10), and (3.3.21):

$$\begin{cases} g_1^{(\mu+\delta-r)/\sigma^2} = \tilde{C}(g_1 - K)^+, \\ g_2^{(\mu+\delta-r)/\sigma^2} = \tilde{C}(g_2 - K)^+, \\ g_1 = S_0 \exp\big\{b_1^{(T)}\sigma + \big(r - \delta - \sigma^2/2\big)\big\}, \\ g_2 = S_0 \exp\big\{b_2^{(T)}\sigma + \big(r - \delta - \sigma^2/2\big)\big\}, \\ x_0 = S_0 e^{-\delta T}\big[\Phi\big(d_+^{(T)}\big) - \Phi\big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\big) + \Phi\big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\big)\big] \\ \qquad -Ke^{-rT}\big[\Phi\big(d_-^{(T)}\big) - \Phi\big(-\frac{b_1^{(T)}}{\sqrt{T}}\big) + \Phi\big(-\frac{b_2^{(T)}}{\sqrt{T}}\big)\big]. \end{cases} \tag{3.3.24}$$

The hedging capital X_t^* and the hedging strategy $\pi^* = (\gamma_t^*, \beta_t^*)$ at time $t < T$ can be found similarly to the previous case. Specifically, the hedging capital is given by:

$$\begin{aligned} X_t^* &= \mathrm{E}^*\big(e^{-r(T-t)} f_T\,\mathbb{1}_A | \mathcal{F}_t\big) \\ &= S_t e^{-\delta(T-t)}\Big[\Phi\big(d_+^{(T-t)}\big) - \Phi\Big(\sigma\sqrt{T-t} - \frac{b_1^{(T-t)}}{\sqrt{T-t}}\Big) \\ &\quad + \Phi\Big(\sigma\sqrt{T-t} - \frac{b_2^{(T-t)}}{\sqrt{T-t}}\Big)\Big] \\ &\quad - Ke^{-r(T-t)}\Big[\Phi\big(d_-^{(T-t)}\big) - \Phi\Big(-\frac{b_1^{(T-t)}}{\sqrt{T-t}}\Big) + \Phi\Big(-\frac{b_2^{(T-t)}}{\sqrt{T-t}}\Big)\Big], \end{aligned} \tag{3.3.25}$$

where $d_\pm^{(T-t)}$ are given by (3.2.7), and the relationships between g_i and $b_i^{(T-t)}$ for $i = 1, 2$ are as in (3.3.14).

Hedging capital in (3.3.25) can be rewritten in terms of the boundaries g_1 and g_2:

$$X_t^* = S_t e^{-\delta(T-t)}\Big[\Phi\big(d_+^{(T-t)}\big) - \Phi\big(d_{+g_1}^{(T-t)}\big) + \Phi\big(d_{+g_2}^{(T-t)}\big)\Big] \\ - Ke^{-r(T-t)}\Big[\Phi\big(d_-^{(T-t)}\big) - \Phi\big(d_{-g_1}^{(T-t)}\big) + \Phi\big(d_{-g_2}^{(T-t)}\big)\Big], \tag{3.3.26}$$

where $d_{\pm g_i}^{(T-t)}$, $i = 1, 2$, are similar to (3.3.16).

The structure of the hedging strategy $\pi^* = (\gamma_t^*, \beta_t^*)$ is then given by

$$\gamma_t^* = e^{-\delta(T-t)}\Bigg[\Phi\big(d_+^{(T-t)}\big) - \Phi(d_{+g_1}^{(T-t)}) + \Phi\big(d_{+g_2}^{(T-t)}\big) \\ - \frac{\phi\big(d_{+g_1}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\frac{g_1 - K}{g_1} + \frac{\phi\big(d_{+g_2}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\frac{g_2 - K}{g_2}\Bigg], \tag{3.3.27}$$

$$\beta_t^* = -Ke^{-rT}\Bigg[\Phi\big(d_-^{(T-t)}\big) - \Phi\big(d_{-g_1}^{(T-t)}\big) + \Phi\big(d_{-g_2}^{(T-t)}\big)\Bigg] \\ + S_t\,\frac{e^{-\delta(T-t)-rt}}{\sigma\sqrt{T-t}}\left[\phi\big(d_{+g_1}^{(T-t)}\big)\,\frac{g_1 - K}{g_1} - \phi\big(d_{+g_2}^{(T-t)}\big)\,\frac{g_2 - K}{g_2}\right]. \tag{3.3.28}$$

3.3.2 Jump-diffusion market with dividends

We consider the same model as in Section 3.2.2, but introduce an important constraint: the hedging capital at the investor's disposal is less than required for a perfect hedge. Thus, we are looking for a self-financing strategy maximizing probability of a successful hedge.

As in Section 3.2.2, we are dealing with a European contingent claim on one of the risky assets: $f_T = (S_T^{(1)} - K)^+$.

Let us rewrite the density process (3.2.24) in terms of $S_T^{(1)}$:

$$\begin{aligned} Z_T^* = \frac{\mathrm{d}P_T^*}{\mathrm{d}P_T} &= \exp\Big\{(\lambda - \lambda^*)T + \frac{\varphi^2}{2}T + \varphi W_T^* + \Pi_T(\ln\lambda^* - \ln\lambda)\Big\} \\ &= \exp\Big\{\frac{\varphi}{\sigma_1}\Big(\ln S_0^{(1)} + \big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\big)T + \sigma_1 W_T^* + \Pi_T\ln(1-\nu_1)\Big)\Big\} \\ &\quad \times \exp\Big\{-\frac{\varphi}{\sigma_1}\Big(\ln S_0^{(1)} + \big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\big)T\Big) + \frac{\varphi^2}{2}T + (\lambda - \lambda^*)T\Big\} \\ &\quad \times \Big[\frac{\lambda^*}{\lambda(1-\nu_1)^{\varphi/\sigma_1}}\Big]^{\Pi_T} \\ &= \big(S_T^{(1)}\big)^{\varphi/\sigma_1}\cdot C_1 \cdot C_2^{\Pi_T}, \end{aligned}$$

where φ and λ^* are given by (3.2.22) and (3.2.23),

$$C_1 = \big(S_0^{(1)}\big)^{-\varphi/\sigma_1} \exp\Big\{ -\frac{\varphi}{\sigma_1}\big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\big)T + \frac{\varphi^2}{2}T + (\lambda - \lambda^*)T\Big\},$$

$$C_2 = \frac{\lambda^*}{\lambda(1-\nu_1)^{\varphi/\sigma_1}}.$$

Then the critical set A is of the form

$$A = \Big\{(S_T^{(1)})^{-\varphi/\sigma_1} > const \cdot C_1 \cdot C_2^{\Pi_T} \cdot f_T\Big\}, \tag{3.3.29}$$

where unknown positive $const$ is chosen so that $\mathrm{E}^*(e^{-rT} f_T\, \mathbb{1}_A) = x_0$.

Using the independence of W^* and Π with respect to P^*,

$$\begin{aligned} \mathrm{E}^*\bigg(\frac{f_T}{B_T}\mathbb{1}_A\bigg) &= \sum_{n=0}^{\infty} \mathrm{E}^*\bigg(\frac{f_T}{B_T}\mathbb{1}_{A|\Pi_T=n}\bigg)P^*(\Pi_T = n) \qquad (3.3.30)\\ &= \sum_{n=0}^{\infty} \mathrm{E}^*\Big[e^{-rT}\big(S_T^{(1)} - K\big)^+ \mathbb{1}\big[\big(S_T^{(1)}\big)^{-\varphi/\sigma_1} > const \cdot C_1 \cdot C_2^n \cdot (S_T^{(1)} - K)^+\big]\Big] \\ &\quad \times e^{-\lambda^* T}\frac{(\lambda^* T)^n}{n!}. \end{aligned}$$

Similarly to Section 3.3.1, we consider two cases.

Case 1. $-\varphi/\sigma_1 \le 1$

In this case the set A conditionally on the number of jumps has the form:

$$A|\{\Pi_T = n\} = \{S_T^{(1)} < g^{(T)}(n)\} = \{W_T^* < b^{(T)}(n)\},$$

where $g^{(T)}(n)$ is a constant dependent on n that can be found by solving

$$x^{-\varphi/\sigma_1} = const \cdot C_1 \cdot C_2^n \cdot (x - K)^+. \tag{3.3.31}$$

The relationship between $g^{(T)}(n)$ and $b^{(T)}(n)$ is as follows:

$$g^{(T)}(n) = S_0^{(1)}(1-\nu_1)^n \exp\Big\{\sigma_1 b^{(T)}(n) + \Big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)T\Big\}. \tag{3.3.32}$$

The maximal probability of a successful hedge is given by

$$P(A) = \sum_n P(A|\Pi_T = n)P(\Pi_T = n) = e^{-\lambda T}\sum_{n=0}^{\infty}\Phi\bigg(\frac{b^{(T)}(n) + \varphi T}{\sqrt{T}}\bigg)\frac{(\lambda T)^n}{n!}, \tag{3.3.33}$$

where φ is defined in (3.2.22).

We note that a modified claim $f_T\,\mathbb{1}_A$ conditional on a number of jumps can be represented as a sum of two call options and a binary option:

$$\begin{aligned} f_T\,\mathbb{1}_{A|\Pi_T=n} &= \big(S_T^{(1)}-K\big)^+\,\mathbb{1}[S_T^{(1)}<g^{(T)}(n)] \\ &= \big(S_T^{(1)}-K\big)^+ - \big(S_T^{(1)}-g^{(T)}(n)\big)^+ - \big(g^{(T)}(n)-K\big)\,\mathbb{1}[S_T^{(1)}>g^{(T)}(n)]. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathrm{E}^*\bigg(\frac{f_T}{B_T}\mathbb{1}_{A|\Pi_T=n}\bigg) &= \mathrm{E}^*e^{-rT}\big(S_T^{(1)}-K\big)^+ - \mathrm{E}^*e^{-rT}\big(S_T^{(1)}-g^{(T)}(n)\big)^+ \\ &\quad - e^{-rT}\big(g^{(T)}(n)-K\big)P^*\big(S_T^{(1)}>g^{(T)}(n)\big), \end{aligned}$$

where

$$P^*\big(S_T^{(1)}>g^{(T)}(n)\big) = P^*\big(W_T^*>b^{(T)}(n)\big) = \Phi\bigg(-\frac{b^{(T)}(n)}{\sqrt{T}}\bigg).$$

Hence, we can write (3.3.30) as

$$\begin{aligned} \mathrm{E}^*\bigg(\frac{f_T}{B_T}\mathbb{1}_A\bigg) &= e^{-\lambda^*T}\sum_{n=0}^{\infty}\Big[C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*T},K,T,r,\sigma_1,\delta_1\big) \\ &\quad - C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*T},g^{(T)}(n),T,r,\sigma_1,\delta_1\big) \\ &\quad - e^{-rT}\big(g^{(T)}(n)-K\big)\Phi\bigg(-\frac{b^{(T)}(n)}{\sqrt{T}}\bigg)\Big]\frac{(\lambda^*T)^n}{n!}. \end{aligned} \tag{3.3.34}$$

The unknown *const* in (3.3.29) is found from the condition $x_0=\mathrm{E}^*(e^{-rT}f_T\,\mathbb{1}_A)$ by means of (3.3.34).

The value of the quantile hedging strategy at time $t<T$ is given by

$$\begin{aligned} X_t^* &= \mathrm{E}^*(e^{-r(T-t)}f_T\,\mathbb{1}_A|\mathcal{F}_t) \\ &= \mathrm{E}^*\Big[e^{-r(T-t)}(S_T^{(1)}-K)^+\,\mathbb{1}\big[\big(S_T^{(1)}\big)^{-\varphi/\sigma_1}> const\cdot C_1\cdot C_2^{\Pi_t}\cdot C_2^{\Pi_{T-t}}\cdot(S_T^{(1)}-K)^+\big]\Big|\mathcal{F}_t\Big] \\ &= \sum_{n=0}^{\infty}\mathrm{E}^*\Big[e^{-r(T-t)}(S_T^{(1)}-K)^+\,\mathbb{1}\big[\big(S_T^{(1)}\big)^{-\varphi/\sigma_1}> const\cdot C_1\cdot C_2^{\Pi_t}\cdot C_2^{n}\cdot(S_T^{(1)}-K)^+\big]\Big|\mathcal{F}_t\Big] \\ &\quad\times e^{-\lambda^*(T-t)}\frac{\big(\lambda^*(T-t)\big)^n}{n!} \\ &= \sum_{n=0}^{\infty}\mathrm{E}^*\Big[e^{-r(T-t)}(S_T^{(1)}-K)^+\,\mathbb{1}[S_T^{(1)}<g^{(T-t)}(n)]\Big|\mathcal{F}_t\Big]e^{-\lambda^*(T-t)}\frac{\big(\lambda^*(T-t)\big)^n}{n!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \mathrm{E}^*\Big[e^{-r(T-t)}\Big((S_T^{(1)} - K)^+ - \big(S_T^{(1)} - g^{(T-t)}(n)\big)^+$$
$$- \big(g^{(T-t)}(n) - K\big)\, \mathbb{1}[S_T^{(1)} > g^{(T-t)}(n)]\Big)\Big|\mathcal{F}_t\Big] e^{-\lambda^*(T-t)} \frac{\big(\lambda^*(T-t)\big)^n}{n!},$$

where $g^{(T-t)}(n)$ is a unique root of the equation

$$x^{-\varphi/\sigma_1} = const \cdot C_1 \cdot C_2^{\Pi_t} \cdot C_2^n \cdot (x-K)^+,$$

and Π_t can be expressed in terms of the observable stock prices $S_t^{(1)}$ and $S_t^{(2)}$, as will be shown further.

From (3.2.25) we get

$$W_t^* = \frac{1}{\sigma_i}\Big[\ln \frac{S_t^{(i)}}{S_0^{(i)}(1-\nu_i)^{\Pi_t}} - \Big(r - \delta_i + \nu_i\lambda^* - \frac{\sigma_i^2}{2}\Big)t\Big],$$

and hence

$$\frac{1}{\sigma_1}\Big[\ln \frac{S_t^{(1)}}{S_0^{(1)}(1-\nu_1)^{\Pi_t}} - \Big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)t\Big]$$
$$= \frac{1}{\sigma_2}\Big[\ln \frac{S_t^{(2)}}{S_0^{(2)}(1-\nu_2)^{\Pi_t}} - \Big(r - \delta_2 + \nu_2\lambda^* - \frac{\sigma_2^2}{2}\Big)t\Big],$$

from where the expression for Π_t in terms of $S_t^{(1)}$ and $S_t^{(2)}$ follows:

$$\Pi_t = \Big[\frac{1}{\sigma_1}\ln\big(S_t^{(1)}/S_0^{(1)}\big) - \frac{1}{\sigma_2}\ln\big(S_t^{(2)}/S_0^{(2)}\big) - \frac{1}{\sigma_1}\big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\big)t + \frac{1}{\sigma_2}\big(r - \delta_2 + \nu_2\lambda^* - \frac{\sigma_2^2}{2}\big)t\Big] \Big/ \Big[\frac{1}{\sigma_1}\ln(1-\nu_1) - \frac{1}{\sigma_2}\ln(1-\nu_2)\Big]. \tag{3.3.35}$$

We further note that

$$\mathrm{E}^*\big(\mathbb{1}[S_T^{(1)} > g^{(T-t)}(n)] \,\big|\, \mathcal{F}_t\big)$$
$$= P^*\Big(S_t^{(1)} \exp\Big\{\Big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)(T-t) + \sigma W_{T-t}^*\Big\}(1-\nu_1)^n$$
$$> g^{(T-t)}(n) \,\big|\, \mathcal{F}_t\Big)$$
$$= P^*\big(W_{T-t}^* > b^{(T-t)}(n)\big) = \Phi\Big(-\frac{b^{(T-t)}(n)}{\sqrt{T-t}}\Big),$$

where constant $g^{(T-t)}(n)$ relates to constant $b^{(T-t)}(n)$ as follows:

$$g^{(T-t)}(n) = S_t^{(1)}(1-\nu_1)^n \exp\Big\{\sigma_1 b^{(T-t)}(n) + \Big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)(T-t)\Big\}.$$

Hence, the final formula for the capital of the quantile hedging strategy is given by

$$\begin{aligned}
X_t^* &= C(S_t^{(1)}, S_t^{(2)}, t) \\
&= e^{-\lambda^*(T-t)} \sum_{n=0}^{\infty} \Big[C^{BS}\Big(S_t^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*(T-t)}, K, T-t, r, \sigma_1, \delta_1\Big) \\
&\quad - C^{BS}\Big(S_t^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*(T-t)}, g^{(T-t)}(n), T-t, r, \sigma_1, \delta_1\Big) \qquad (3.3.36) \\
&\quad - e^{-r(T-t)}\big(g^{(T-t)}(n) - K\big)\Phi\Big(-\frac{b^{(T-t)}(n)}{\sqrt{T-t}}\Big)\Big] \frac{\big(\lambda^*(T-t)\big)^n}{n!}.
\end{aligned}$$

Case 2. $-\varphi/\sigma_1 > 1$

In this case a success set A conditionally on a number of jumps has the form

$$\begin{aligned}
A|\{\Pi_T = n\} &= \{S_T^{(1)} < g_1^{(T)}(n)\} \cup \{S_T^{(1)} > g_2^{(T)}(n)\} \\
&= \{W_T^* < b_1^{(T)}(n)\} \cup \{W_T^* > b_2^{(T)}(n)\},
\end{aligned}$$

where $g_1^{(T)}(n) < g_2^{(T)}(n)$ are two distinct solutions of the equation

$$x^{-\varphi/\sigma_1} = const \cdot C_1 \cdot C_2^n \cdot (x-K)^+, \qquad (3.3.37)$$

and the relationships between $g_i^{(T)}(n)$ and $b_i^{(T)}(n)$ $(i = 1, 2)$ are similar to (3.3.32).

The maximal probability of successful hedging is given by

$$\begin{aligned}
P(A) &= \sum_n P(A|\Pi_T = n)\, P(\Pi_T = n) \\
&= e^{-\lambda T} \sum_{n=0}^{\infty} \Big[\Phi\Big(\frac{b_1^{(T)}(n) + \varphi T}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2^{(T)}(n) + \varphi T}{\sqrt{T}}\Big)\Big] \frac{(\lambda T)^n}{n!}, \qquad (3.3.38)
\end{aligned}$$

where φ is defined in (3.2.22).

As before, a modified claim conditional on a number of jumps can be seen as a combination of call options and digital options:

$$\begin{aligned}
f_T \mathbb{1}_{A|\Pi_T=n} &= f_T\, \mathbb{1}[S_T^{(1)} < g_1^{(T)}(n)] + f_T\, \mathbb{1}[S_T^{(1)} > g_2^{(T)}(n)] \\
&= \big(S_T^{(1)} - K\big)^+ - \big(S_T^{(1)} - g_1^{(T)}(n)\big)^+ - \big(g_1^{(T)}(n) - K\big)\, \mathbb{1}[S_T^{(1)} > g_1^{(T)}(n)] \\
&\quad + \big(S_T^{(1)} - g_2^{(T)}(n)\big)^+ + \big(g_2^{(T)}(n) - K\big)\, \mathbb{1}[S_T^{(1)} > g_2^{(T)}(n)].
\end{aligned}$$

Thus,

$$\mathrm{E}^*\left(\frac{f_T}{B_T}\mathbb{1}_A\right) = e^{-\lambda^* T}\sum_{n=0}^{\infty}\Big[C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\big)$$
$$- C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_1^{(T)}(n), T, r, \sigma_1, \delta_1\big)$$
$$+ C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_2^{(T)}(n), T, r, \sigma_1, \delta_1\big) \qquad (3.3.39)$$
$$- e^{-rT}\big(g_1^{(T)}(n) - K\big)\Phi\left(-\frac{b_1^{(T)}(n)}{\sqrt{T}}\right)$$
$$+ e^{-rT}\big(g_2^{(T)}(n) - K\big)\Phi\left(-\frac{b_2^{(T)}(n)}{\sqrt{T}}\right)\Big]\frac{(\lambda^* T)^n}{n!}.$$

The unknown *const* can be found from the condition $x_0 = \mathrm{E}^*(e^{-rT} f_T\,\mathbb{1}_A)$ by means of Eq. (3.3.39).

The value of the quantile hedging strategy at time $t < T$ is given by

$$X_t^* = C(S_t^{(1)}, S_t^{(2)}, t)$$
$$= e^{-\lambda^*(T-t)}\sum_{n=0}^{\infty}\Big[C^{BS}\Big(S_t^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*(T-t)}, K, T-t, r, \sigma_1, \delta_1\Big)$$
$$- C^{BS}\Big(S_t^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*(T-t)}, g_1^{(T-t)}(n), T-t, r, \sigma_1, \delta_1\Big)$$
$$+ C^{BS}\Big(S_t^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^*(T-t)}, g_2^{(T-t)}(n), T-t, r, \sigma_1, \delta_1\Big) \qquad (3.3.40)$$
$$- e^{-r(T-t)}\big(g_1^{(T-t)}(n) - K\big)\Phi\Big(-\frac{b_1^{(T-t)}(n)}{\sqrt{T-t}}\Big)$$
$$+ e^{-r(T-t)}\big(g_2^{(T-t)}(n) - K\big)\Phi\Big(-\frac{b_2^{(T-t)}(n)}{\sqrt{T-t}}\Big)\Big]\frac{\big(\lambda^*(T-t)\big)^n}{n!},$$

where $g_1^{(T-t)}(n)$ and $g_2^{(T-t)}(n)$ are the roots of the equation

$$x^{-\varphi/\sigma_1} = const\cdot C_1\cdot C_2^{\Pi_t}\cdot C_2^n\cdot(x-K)^+$$

with Π_t being a function of $S_t^{(1)}$ and $S_t^{(2)}$ as in (3.3.35).

Constants $g_i^{(T-t)}(n)$ relate to constants $b_i^{(T-t)}(n)$ for $i = 1, 2$ as follows:

$$g_i^{(T-t)}(n) = S_t^{(1)}(1-\nu_1)^n \exp\Big\{\sigma_1 b_i^{(T-t)}(n) + \Big(r - \delta_1 + \nu_1\lambda^* - \frac{\sigma_1^2}{2}\Big)(T-t)\Big\}.$$

The proof of (3.3.40) is analogous to case 1.

An important difference between the values of a quantile hedge and a perfect hedge, considered in Section 3.2.2, is that the hedging capital in the latter case depends only on the value of $S_t^{(1)}$, while in case of quantile hedging, the value depends on both $S_t^{(1)}$ and $S_t^{(2)}$ due to $\Pi_t = \Pi_t(S_t^{(1)}, S_t^{(2)})$.

The components of a quantile hedging strategy can be found in a similar fashion to the Section 3.2.2. We give a proof here for completeness.

Using (3.2.18)-(3.2.21), we can write $\mathrm{d}V_t$, where $V_t = X_t/B_t$, as:

$$\begin{aligned} \mathrm{d}V_t &= \mathrm{d}\bigg(\frac{C(S_t^{(1)}, S_t^{(2)}, t)}{B_t}\bigg) \\ &= \bigg(\frac{\gamma_t S_{t-}^{(1)}\sigma_1 + \xi_t S_{t-}^{(2)}\sigma_2}{B_t}\bigg)\mathrm{d}W_t^* - \bigg(\frac{\gamma_t S_{t-}^{(1)}\nu_1 + \xi_t S_{t-}^{(2)}\nu_2}{B_t}\bigg)\mathrm{d}(\Pi_t - \lambda^* t) \quad (3.3.41) \\ &= e^{-rt}\mathrm{d}C(S_t^{(1)}, S_t^{(2)}, t) - re^{-rt}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\mathrm{d}t. \quad (3.3.42) \end{aligned}$$

By Itô formula,

$$\begin{aligned} \mathrm{d}C(S_t^{(1)}, S_t^{(2)}, t) = {} & \frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}S_t^{(1)} + \frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}S_t^{(2)} \\ & + \frac{\partial}{\partial t}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}t + \frac{1}{2}\frac{\partial^2}{\partial x^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}\langle S^{(1)}, S^{(1)}\rangle_t^c \\ & + \frac{1}{2}\frac{\partial^2}{\partial y^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}\langle S^{(2)}, S^{(2)}\rangle_t^c \\ & + \frac{\partial^2}{\partial x \partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}\langle S^{(1)}, S^{(2)}\rangle_t^c \\ & + \Big(C(S_t^{(1)}, S_t^{(2)}, t) - C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) - \frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\Delta S_t^{(1)} \\ & - \frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\Delta S_t^{(2)}\Big), \end{aligned} \quad (3.3.43)$$

where $\partial f/\partial x$ and $\partial f/\partial y$ are a partial derivatives with respect to the first and second arguments of f, respectively.

We note that

$$\mathrm{d}\langle S^{(i)}, S^{(j)}\rangle_t^c = \sigma_i\sigma_j S_{t-}^{(i)} S_{t-}^{(j)}\mathrm{d}t, \quad i, j = 1, 2, \quad (3.3.44)$$

and

$$\begin{aligned} & C(S_t^{(1)}, S_t^{(2)}, t) - C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) \\ & = \Big[C\big(S_{t-}^{(1)}(1-\nu_1), S_{t-}^{(2)}(1-\nu_2), t\big) - C\big(S_{t-}^{(1)}, S_{t-}^{(2)}, t\big)\Big]\Delta\Pi_t. \end{aligned} \quad (3.3.45)$$

With (3.3.44), (3.3.45), and (3.2.10), Eq. (3.3.43) can be written as

$$\begin{aligned}
&\mathrm{d}C(S_t^{(1)}, S_t^{(2)}, t) \\
&= \frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}S_t^{(1)} + \frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}S_t^{(2)} \\
&+ \frac{\partial}{\partial t}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}t + \frac{1}{2}\big(\sigma_1 S_{t-}^{(1)}\big)^2\frac{\partial^2}{\partial x^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}t \\
&+ \frac{1}{2}\big(\sigma_2 S_{t-}^{(2)}\big)^2\frac{\partial^2}{\partial y^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}t \\
&+ \sigma_1\sigma_2 S_{t-}^{(1)}S_{t-}^{(2)}\frac{\partial^2}{\partial x\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\,\mathrm{d}t \\
&+ \Big(C(S_{t-}^{(1)}(1-\nu_1), S_{t-}^{(2)}(1-\nu_2), t) - C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\Big)\mathrm{d}\Pi_t \\
&+ \frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)(\nu_1 S_{t-}^{(1)})\,\mathrm{d}\Pi_t + \frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)(\nu_2 S_{t-}^{(2)})\,\mathrm{d}\Pi_t.
\end{aligned} \tag{3.3.46}$$

Substituting (3.3.46) into (3.3.42) and using (3.2.26), we obtain

$$\begin{aligned}
&\mathrm{d}\bigg(\frac{C(S_t^{(1)}, S_t^{(2)}, t)}{B_t}\bigg) \\
&= \bigg(\sigma_1\frac{S_{t-}^{(1)}}{B_t}\frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) + \sigma_2\frac{S_{t-}^{(2)}}{B_t}\frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\bigg)\mathrm{d}W_t^* \\
&\quad + \bigg(C(S_{t-}^{(1)}(1-\nu_1), S_{t-}^{(2)}(1-\nu_2), t) - C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\bigg) \\
&\quad \times e^{-rt}\,\mathrm{d}(\Pi_t - \lambda^* t) \\
&\quad + \bigg(\big[C(S_{t-}^{(1)}(1-\nu_1), S_{t-}^{(2)}(1-\nu_2), t) - C(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\big]\lambda^* \\
&\quad + S_{t-}^{(1)}(r-\delta_1+\nu_1\lambda^*)\frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) \\
&\quad + S_{t-}^{(2)}(r-\delta_2+\nu_2\lambda^*)\frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) \\
&\quad + \frac{\partial}{\partial t}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) + \frac{1}{2}\big(\sigma_1 S_{t-}^{(1)}\big)^2\frac{\partial^2}{\partial x^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) \\
&\quad + \frac{1}{2}\big(\sigma_2 S_{t-}^{(2)}\big)^2\frac{\partial^2}{\partial y^2}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) \\
&\quad + \sigma_1\sigma_2 S_{t-}^{(1)}S_{t-}^{(2)}\frac{\partial^2}{\partial x\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) - rC(S_{t-}^{(1)}, S_{t-}^{(2)}, t)\bigg)e^{-rt}\mathrm{d}t.
\end{aligned} \tag{3.3.47}$$

Finally, by comparing (3.3.47) with (3.3.41), we find that the components of a hedging strategy satisfy

$$\begin{cases}
\gamma_t\sigma_1 S_{t-}^{(1)} + \xi_t\sigma_2 S_{t-}^{(2)} = \sigma_1 S_{t-}^{(1)}\frac{\partial}{\partial x}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) + \sigma_2 S_{t-}^{(2)}\frac{\partial}{\partial y}C(S_{t-}^{(1)}, S_{t-}^{(2)}, t), \\
\gamma_t\nu_1 S_{t-}^{(1)} + \xi_t\nu_2 S_{t-}^{(2)} = C(S_{t-}^{(1)}, S_{t-}^{(2)}, t) - C(S_{t-}^{(1)}(1-\nu_1), S_{t-}^{(2)}(1-\nu_2), t).
\end{cases} \tag{3.3.48}$$

Units of bond, β_t, can be found from the balance equation:

$$\beta_t = \frac{C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) - \gamma_t S^{(1)}_{t-} - \xi_t S^{(2)}_{t-}}{B_t}. \tag{3.3.49}$$

We also observe that the value of the hedging strategy X^*_t satisfies

$$\begin{aligned}
&\big[C(S^{(1)}_{t-}(1-\nu_1), S^{(2)}_{t-}(1-\nu_2), t) - C(S^{(1)}_{t-}, S^{(2)}_{t-}, t)\big]\lambda^* \\
&+ S^{(1)}_{t-}(r - \delta_1 + \nu_1\lambda^*)\frac{\partial}{\partial x}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) \\
&+ S^{(2)}_{t-}(r - \delta_2 + \nu_2\lambda^*)\frac{\partial}{\partial y}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) + \frac{\partial}{\partial t}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) \\
&+ \frac{1}{2}\big(\sigma_1 S^{(1)}_{t-}\big)^2\frac{\partial^2}{\partial x^2}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) + \frac{1}{2}\big(\sigma_2 S^{(2)}_{t-}\big)^2\frac{\partial^2}{\partial y^2}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) \\
&+ \sigma_1\sigma_2 S^{(1)}_{t-}S^{(2)}_{t-}\frac{\partial^2}{\partial x\partial y}C(S^{(1)}_{t-}, S^{(2)}_{t-}, t) - rC(S^{(1)}_{t-}, S^{(2)}_{t-}, t) = 0.
\end{aligned}$$

3.4 Application to equity-linked life insurance contracts

Equity-linked life insurance contract is a contract where insurance benefit depends on the evolution of the financial market and the longevity of the insured. Thus, such contracts have two sources of uncertainty:

- ◇ Market risk of the underlying asset;
- ◇ Mortality risk of the insured person.

It is natural to assume that the two risks are independent of each other and we can consider them on the product of probability spaces: $(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$, where the space $(\Omega_1, \mathcal{F}_1, P_1)$ is a probabilistic base for the financial risk, and $(\Omega_2, \mathcal{F}_2, P_2)$ is a base for insurance risk.

In general, a future obligation of the insurance company to each insured, who have purchased such contract, is given by

$$f_T = f(S_T)\,\mathbb{1}_{[T(x)>T]},$$

where $f(S_T)$ is a payoff function of a contingent claim incorporating financial risk, $T(x)$ is a remaining lifetime of the client who is currently of age x, T is a time of a contract maturity.

In this section, we focus on *pure endowment with fixed guarantee life insurance contracts*. According to this contract, insured, upon survival to time T, receives a payout equal to

$$f(S_T) = \max(S_T, K),$$

where K is a guaranteed amount (a constant).

The main question is how to determine a one-time premium (price) of such a contract, ${}_TU_x$. We remark that the main goal is to find the premium of a *single* contract, as for a group of l_x insureds, all aged x and all subscribed to the same contract, the cumulative future obligation becomes

$$\sum_{i=1}^{l_x} f(S_T)\, \mathbb{1}_{[T_i(x)>T]},$$

where remaining lifetimes, $T_i(x)$, are assumed to be i.i.d., and the total premium is

$$U(T, l_x) = l_x \; {}_TU_x.$$

The fair premium, ${}_TU_x$, can be found according to an equivalence principle with respect to a martingale measure P^* in a corresponding model of a financial market. Equivalence principle postulates that an insurer's income and expenses should balance, on average. Due to the assumed independence between the two sources of risk the fair price of a contract, known as *Brennan-Schwartz* price (Brennan and Schwartz [3.1]), is given by

$$\begin{aligned} {}_TU_x &= \mathrm{E}^* \times \mathrm{E}_2 \left(e^{-rT} f(S_T)\, \mathbb{1}_{[T(x)>T]}\right) \\ &= P_2(T(x) > T)\; E^*\big(e^{-rT} f(S_T)\big) = \; {}_Tp_x \; E^*\big(e^{-rT} f(S_T)\big), \end{aligned} \tag{3.4.1}$$

where $\mathrm{E}^* \times \mathrm{E}_2$ denotes the expectation under the product measure $P^* \times P_2$; ${}_Tp_x := P_2(T(x) > T)$ denotes a survival probability.

Observe that without mortality risk, the fair price of a given "pure financial" claim $f(S_T)$ is determined by

$$\mathrm{E}^*\big(e^{-rT} f(S_T)\big).$$

It is clear that the price of a "mixed" contract $f_T = f(S_T)\,\mathbb{1}_{[T(x)>T]}$, given by ${}_TU_x$, is strictly less than the corresponding fair price of a pure financial contract $f(S_T)$, as a survival probability is always less than 1:

$$ {}_TU_x < \mathrm{E}^*\big(e^{-rT} f(S_T)\big). $$

Therefore, the perfect hedge of $f(S_T)$ starting from the initial capital ${}_TU_x$ is impossible. This consideration suggests that a quantile hedging approach could be efficiently applied to solving this problem. Following this approach, we will look for a strategy with some initial budget constraint whose payoff at maturity will cover a potential obligation $f(S_T)$ with maximal probability.

We further note that since the obligation related to a pure endowment with fixed guarantee equity-linked life insurance contract can be decomposed as

$$ \begin{aligned} f_T &= f(S_T)\,\mathbb{1}_{[T(x)>T]} = \max(S_T, K)\,\mathbb{1}_{[T(x)>T]} \\ &= K\,\mathbb{1}_{[T(x)>T]} + (S_T - K)^+\,\mathbb{1}_{[T(x)>T]}, \end{aligned} \tag{3.4.2} $$

its fair premium, as follows from (3.4.1) and (3.4.2), can be represented as a sum of the two components:

$$ {}_TU_x = {}_Tp_x\; Ke^{-rT} + {}_Tp_x\; \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right). \tag{3.4.3} $$

The above equation shows that in order to find the premium ${}_TU_x$, it is sufficient to find a premium for the second component. The initial capital available for a call option $(S_T - K)^+$, called *embedded option*, as follows from (3.4.3), is given by

$$ {}_TU_x^C = {}_TU_x - {}_Tp_x\; Ke^{-rT} = {}_Tp_x\; \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right). \tag{3.4.4} $$

Therefore, instead of working with the claim $f(S_T) = \max(S_T, K)$ and initial capital ${}_TU_x$, we can work with a more convenient payoff of the embedded call option and initial capital ${}_TU_x^C$. Note that the initial capital ${}_TU_x^C$, available for the embedded option, is strictly less than $\mathrm{E}^*(e^{-rT}(S_T - K)^+)$, meaning a perfect hedging of the call option is not possible under a given budget constraint. Hence, we will employ a quantile methodology to construct a successful hedging set A for a call option $(S_T - K)^+$ starting from the initial capital ${}_TU_x^C$ given in (3.4.4). Recall, that a perfect hedge for a modified contingent claim $(S_T - K)^+ \, \mathbb{1}_A$ will maximize the probability of successful hedging for $(S_T - K)^+$. Hence, we have

$$ {}_TU_x^C = \mathrm{E}^* \left(\frac{(S_T - K)^+}{B_T} \, \mathbb{1}_A \right) = \ {}_Tp_x \ \mathrm{E}^* \left(\frac{(S_T - K)^+}{B_T} \right), $$

from where it follows that

$$ {}_Tp_x = \frac{\mathrm{E}^* \big[(S_T - K)^+ \, \mathbb{1}_A / B_T \big]}{\mathrm{E}^* \big[(S_T - K)^+ / B_T \big]}. \tag{3.4.5} $$

Eq. (3.4.5) is called a *balance equation*. If survival probability is given (derived from some mortality table based on a client's age), the balance equation can be used to calculate maximal probability of successful hedging, $P(A)$. Alternatively, if the insurance company is willing to accept a certain level of risk $\epsilon \in (0,1)$, then the balance equation can be used to determine a value of ${}_Tp_x$ corresponding to the specified level ϵ of a shortfall probability. This actuarial value of ${}_Tp_x$ can then be compared with mortality tables to identify the optimal age of insureds this contract is suitable for.

We further provide explicit formulas for each of the models considered in the previous sections.

3.4.1 Black-Scholes Model

Using (3.4.3) and (3.2.6)-(3.2.7), we can write the initial price of the pure endowment with fixed guarantee life insurance contract, with a financial market evolving according to the Black-Scholes model, as:

$$ \begin{aligned} {}_TU_x &= \ {}_Tp_x \, K e^{-rT} + \ {}_Tp_x \ \big(S_0 e^{-\delta T} \Phi(d_+^{(T)}) - K e^{-rT} \Phi(d_-^{(T)}) \big) \\ &= \ {}_Tp_x \bigg[K e^{-rT} + \ S_0 e^{-\delta T} \Phi\bigg(\frac{\ln(S_0/K) + (r - \delta + \sigma^2/2)\ T}{\sigma\sqrt{T}} \bigg) \\ &\quad - K e^{-rT} \Phi\bigg(\frac{\ln(S_0/K) + (r - \delta - \sigma^2/2)\ T}{\sigma\sqrt{T}} \bigg) \bigg] \\ &= \ {}_Tp_x \big[K e^{-rT} + C^{BS}(S_0, K, T, r, \sigma, \delta) \big], \end{aligned} \tag{3.4.6} $$

where ${}_Tp_x$ is a survival probability.

The hedging capital for a call option is then

$$ {}_TU_x^C = {}_TU_x - {}_Tp_x\, Ke^{-rT} = {}_Tp_x\, C^{BS}(S_0, K, T, r, \sigma, \delta), \tag{3.4.7} $$

and the balance equation is

$$ {}_Tp_x = \frac{\mathrm{E}^*[e^{-rT}(S_T - K)^+ \mathbb{1}_A]}{C^{BS}(S_0, K, T, r, \sigma, \delta)}. \tag{3.4.8} $$

The set A may take two distinct forms, thus, similarly to the Section 3.3, we need to consider two cases separately.

Case 1. $(\mu + \delta - r)/\sigma^2 \leq 1$

Then the set A is of the form

$$ A = \{W_T^* < b^{(T)}\}. \tag{3.4.9} $$

If survival probability is known, Eq. (3.4.8) can be used to identify a constant $b^{(T)}$. It follows from Eq. (3.3.12) that ${}_Tp_x$ can be written as

$$ \begin{aligned} {}_Tp_x &= \frac{S_0e^{-\delta T}\Big[\Phi(d_+^{(T)}) - \Phi\big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\big)\Big] - Ke^{-rT}\Big[\Phi(d_-^{(T)}) - \Phi\big(-\frac{b^{(T)}}{\sqrt{T}}\big)\Big]}{S_0e^{-\delta T}\Phi(d_+) - Ke^{-rT}\Phi(d_-)} \\ &= 1 - \frac{S_0e^{-\delta T}\Phi\big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\big) - Ke^{-rT}\Phi\big(-\frac{b^{(T)}}{\sqrt{T}}\big)}{S_0e^{-\delta T}\Phi(d_+^{(T)}) - Ke^{-rT}\Phi(d_-^{(T)})}, \end{aligned} \tag{3.4.10} $$

where $d_\pm^{(T)}$ are defined in (3.2.7).

We remind that constant $b^{(T)}$ is needed to determine a probability of successful hedging of the call option. Moreover, the optimal hedging strategy and its capital are completely described by Eqs. (3.3.15), (3.3.19), (3.3.20), where constant g is related to $b^{(T)}$ through (3.3.10).

On the other hand, the insurance company may be willing to accept a certain level of risk $\epsilon \in (0, 1)$, such that

$$ 1 - \epsilon = P(A). \tag{3.4.11} $$

If the structure of the set A is described by (3.4.9), then its probability is

$$ P(A) = \Phi\bigg(\frac{b^{(T)} - \frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\bigg). \tag{3.4.12} $$

Combining (3.4.11) and (3.4.12) gives a value for a constant $b^{(T)}$:

$$b^{(T)} = \sqrt{T}\,\Phi^{-1}(1-\epsilon) + \frac{\mu+\delta-r}{\sigma}T. \tag{3.4.13}$$

Plugging (3.4.13) into (3.4.10) gives an actuarial value for ${}_Tp_x$. The insurance company can then use this value to reconstruct the optimal age of the clients who should be targeted for this contract. The premium for a call option is given by Eq. (3.4.7), and the premium that should be charged for the pure endowment with fixed guarantee equity-linked life insurance contract can be determined from (3.4.6).

Case 2. $(\mu+\delta-r)/\sigma^2 > 1$

Then the set A has the form

$$A = \{W_T^* < b_1^{(T)}\} \cup \{W_T^* > b_2^{(T)}\},$$

where constants $b_1^{(T)} < b_2^{(T)}$ (as well as constants g_1, g_2, $\tilde{C}$) can be determined from the first four conditions in the system of equations (3.3.24) and the balance equation, assuming a survival probability is given. The balance equation is

$$\begin{aligned}
{}_Tp_x &= \Bigg(S_0e^{-\delta T}\Big[\Phi(d_+^{(T)}) - \Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&\quad - Ke^{-rT}\Big[\Phi(d_-^{(T)}) - \Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big]\Bigg) \\
&\quad \Big/ \Big(S_0e^{-\delta T}\Phi(d_+^{(T)}) - Ke^{-rT}\Phi(d_-^{(T)})\Big) \\
&= 1 - \Bigg(S_0e^{-\delta T}\Big[\Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) - \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&\quad - Ke^{-rT}\Big[\Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) - \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big]\Bigg) \\
&\quad \Big/ \Big(S_0e^{-\delta T}\Phi(d_+^{(T)}) - Ke^{-rT}\Phi(d_-^{(T)})\Big),
\end{aligned} \tag{3.4.14}$$

where $d_\pm^{(T)}$ are defined in (3.2.7).

With known constants, the probability of successful hedging of a call option can be computed according to Eq. (3.3.22). The capital of the hedging strategy and its structure are given by Eqs. (3.3.26)-(3.3.28).

Alternatively, if there is a set level of risk, $\epsilon \in (0,1)$, that the insurance company is ready to accept, the constants $b_1^{(T)} < b_2^{(T)}$ (along with constants g_1, g_2, $\tilde{C}$) can be found from the condition:

$$\Phi\left(\frac{b_1^{(T)} - \frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\right) + \Phi\left(\frac{-b_2^{(T)} + \frac{\mu+\delta-r}{\sigma}T}{\sqrt{T}}\right) = 1 - \epsilon,$$

and the first four conditions in a system of equations (3.3.24).

Constants $b_1^{(T)}$ and $b_2^{(T)}$ are then used to compute the survival probability given by (3.4.14). Optimal age of the insureds can be derived from mortality tables, using this survival probability. The required premiums for the call option and the contract under consideration can be computed from Eqs. (3.4.7) and (3.4.6), respectively.

3.4.2 Jump-Diffusion Model

In case of a jump-diffusion financial market, the price of a pure endowment with fixed guarantee life insurance contract can be found using (3.2.28) as follows:

$$\begin{aligned} {}_TU_x &= {}_Tp_x\, Ke^{-rT} + {}_Tp_x\, \mathrm{E}^*\left(\frac{(S_T^{(1)} - K)^+}{B_T}\right) \\ &= {}_Tp_x\Bigg(Ke^{-rT} + e^{-\lambda^* T} \\ &\quad \times \sum_{n=0}^{\infty}\left[\frac{(\lambda^* T)^n}{n!}\, C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\right]\Bigg) \end{aligned} \tag{3.4.15}$$

with λ^* given by (3.2.23).

Initial capital available for the embedded option is

$$\begin{aligned} {}_TU_x^C &= {}_TU_x - {}_Tp_x\, Ke^{-rT} \\ &= {}_Tp_x\, e^{-\lambda^* T}\sum_{n=0}^{\infty}\left[\frac{(\lambda^* T)^n}{n!}\, C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\right], \end{aligned} \tag{3.4.16}$$

and the balance equation is

$${}_Tp_x = \frac{\mathrm{E}^*[e^{-rT}(S_T^{(1)} - K)^+\, \mathbb{1}_A]}{e^{-\lambda^* T}\sum_{n=0}^{\infty}\left[\frac{(\lambda^* T)^n}{n!}\, C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\right]}. \tag{3.4.17}$$

Again, we must distinguish between the two cases depending on the form the set A takes.

Case 1. $-\varphi/\sigma_1 \leq 1$

In this case, the success set A conditional on the number of jumps is of the form

$$A|\{\Pi_T = n\} = \{W_T^* < b^{(T)}(n)\}.$$

If survival probability is known, constants $b^{(T)}(n)$ (along with constants $g^{(T)}(n)$ and *const*) can be determined using relationships (3.3.31), (3.3.32), and the balance equation (3.4.17) with the numerator given by (3.3.34):

$$\begin{aligned}
{}_Tp_x = \Bigg(\sum_{n=0}^{\infty} \bigg[& C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\big) \\
& - C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
& - e^{-rT}\big(g^{(T)}(n) - K\big)\Phi\bigg(-\frac{b^{(T)}(n)}{\sqrt{T}}\bigg)\bigg]\frac{(\lambda^* T)^n}{n!}\Bigg) \\
& \Bigg/ \Bigg(\sum_{n=0}^{\infty} \bigg[C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\bigg]\frac{(\lambda^* T)^n}{n!}\Bigg) \\
= 1 - \Bigg(\sum_{n=0}^{\infty} \bigg[& C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
& + e^{-rT}\big(g^{(T)}(n) - K\big)\Phi\bigg(-\frac{b^{(T)}(n)}{\sqrt{T}}\bigg)\bigg]\frac{(\lambda^* T)^n}{n!}\Bigg) \qquad (3.4.18) \\
& \Bigg/ \Bigg(\sum_{n=0}^{\infty} \bigg[C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\bigg]\frac{(\lambda^* T)^n}{n!}\Bigg).
\end{aligned}$$

Probability of successful hedging can then be computed by means of Eq. (3.3.33). The quantile hedging strategy for a call option is described by Eqs. (3.3.36), (3.3.48), (3.3.49).

If, instead of maximizing the probability of successful hedging, the insurance company decides to fix a level of financial risk at some $\epsilon \in (0,1)$, so that the probability of a successful hedge is $P(A) = 1-\epsilon$, then constants $b^{(T)}(n)$ (together with constants $g^{(T)}(n)$ and *const*) can be found from the condition:

$$e^{-\lambda T}\sum_{n=0}^{\infty} \Phi\bigg(\frac{b^{(T)}(n) + \varphi T}{\sqrt{T}}\bigg)\frac{(\lambda T)^n}{n!} = 1-\epsilon,$$

and Eqs. (3.3.31)-(3.3.32).

Constants $b^{(T)}(n)$ are then plugged into (3.4.18) to find the survival probability, which in turn can be used to determine the optimal age of the insureds for this contract. The fair premiums for the call option and the equity-linked life insurance contract can be computed using (3.4.16) and (3.4.15), respectively.

Case 2. $-\varphi/\sigma_1 > 1$

The set A conditional on the number of jumps has the form

$$A|\{\Pi_T = n\} = \{W_T^* < b_1^{(T)}(n)\} \cup \{W_T^* > b_2^{(T)}(n)\}.$$

If survival probability is given, all unknown constants can be found from

$$\begin{aligned}
{}_Tp_x = &\Bigg(\sum_{n=0}^{\infty}\Big[C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\big) \\
&- C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_1^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
&+ C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_2^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
&- e^{-rT}\big(g_1^{(T)}(n) - K\big)\Phi\Big(-\frac{b_1^{(T)}(n)}{\sqrt{T}}\Big) \\
&+ e^{-rT}\big(g_2^{(T)}(n) - K\big)\Phi\Big(-\frac{b_2^{(T)}(n)}{\sqrt{T}}\Big)\Big]\frac{(\lambda^* T)^n}{n!}\Bigg) \\
&\Bigg/\Bigg(\sum_{n=0}^{\infty}\Big[C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\Big]\frac{(\lambda^* T)^n}{n!}\Bigg) \\
= 1 - &\Bigg(\sum_{n=0}^{\infty}\Big[C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_1^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
&- C^{BS}\big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, g_2^{(T)}(n), T, r, \sigma_1, \delta_1\big) \\
&+ e^{-rT}\big(g_1^{(T)}(n) - K\big)\Phi\Big(-\frac{b_1^{(T)}(n)}{\sqrt{T}}\Big) \\
&- e^{-rT}\big(g_2^{(T)}(n) - K\big)\Phi\Big(-\frac{b_2^{(T)}(n)}{\sqrt{T}}\Big)\Big]\frac{(\lambda^* T)^n}{n!}\Bigg) \\
&\Bigg/\Bigg(\sum_{n=0}^{\infty}\Big[C^{BS}\Big(S_0^{(1)}(1-\nu_1)^n e^{\nu_1\lambda^* T}, K, T, r, \sigma_1, \delta_1\Big)\Big]\frac{(\lambda^* T)^n}{n!}\Bigg),
\end{aligned} \tag{3.4.19}$$

and equations (3.3.37) and (3.3.32).

The maximal probability of a successful hedge can be computed using (3.3.38). The capital and the components of the quantile hedging strategy are described by Eqs. (3.3.40) and (3.3.48)-(3.3.49).

Alternatively, if the insurance company decides to fix a maximal shortfall probability at some level $\epsilon \in (0,1)$, so that $P(A) = 1-\epsilon$, then unknown constants can be determined from the condition

$$e^{-\lambda T}\sum_{n=0}^{\infty}\left[\Phi\left(\frac{b_1^{(T)}(n)+\varphi T}{\sqrt{T}}\right)+\Phi\left(-\frac{b_2^{(T)}(n)+\varphi T}{\sqrt{T}}\right)\right]\frac{(\lambda T)^n}{n!}=1-\epsilon,$$

and Eqs. (3.3.37) and (3.3.32).

Constants $b_1^{(T)}(n)$, $b_2^{(T)}(n)$, $g_1^{(T)}(n)$, $g_2^{(T)}(n)$ are then used to compute the survival probability given by (3.4.19). Optimal age of the clients suitable for the contract can be reconstructed from the mortality tables based on the derived survival probability. Finally, the fair premiums for the call option and for the pure endowment with fixed guarantee life insurance contract are given by Eqs. (3.4.16) and (3.4.15), respectively.

3.5 Illustrative example

To illustrate the effect of dividends on a payoff of a quantile hedging strategy, we consider an option in a Black-Scholes market with the same set of parameters as in Föllmer and Leukert [3.4], but assuming non-zero interest and dividend rates (in Föllmer and Leukert [3.4] both r and δ are assumed to be 0):

$$T=0.25, \quad \sigma=0.3, \quad \mu=0.08, \quad S_0=100, \quad K=110, \quad r=0.01,$$

$$\delta=0 \text{ or } \delta=0.07.$$

In case of zero dividends, $\delta=0$, the Black-Scholes price is 2.57; in case of $\delta=0.07$, the Black-Scholes price is 2.09. We further assume that the hedging budget, x_0, is only 1.5.

It can be of interest to compare the probabilities of a successful hedge for the two cases: when $\delta=0$, $P(A)=0.9499$; when $\delta=0.07$, $P(A)=0.9665$. This result coincides with our intuition: for a dividend-paying stock (all else

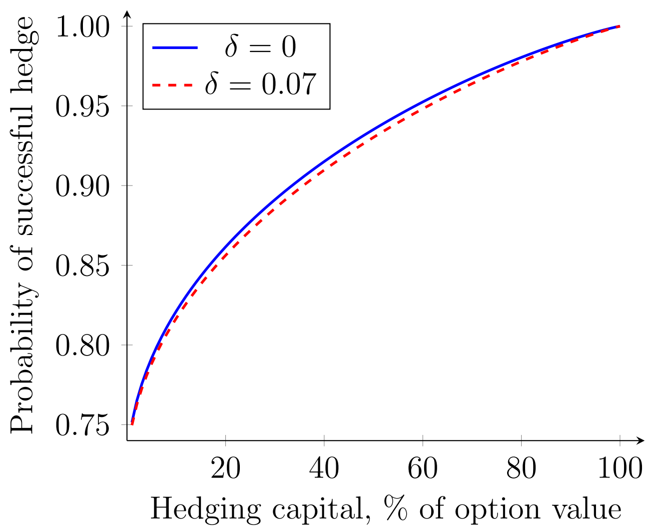
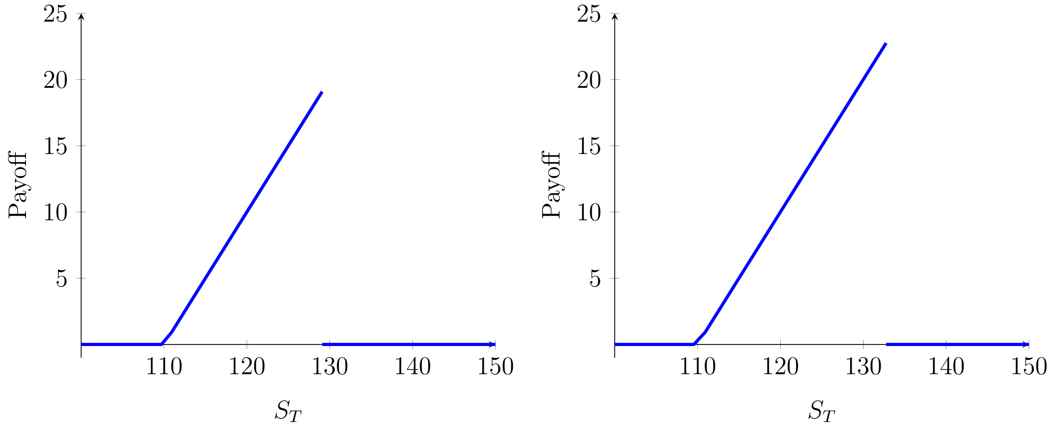


Figure 3.3: Probability of successful hedging as a function of initial capital.

held equal), the probability of a successful hedge is expected to be higher, since the gap between the capital needed for a perfect hedge and available hedging budget ($2.09 - 1.5 = 0.59$) is substantially smaller, than in the non-dividend paying case ($2.57 - 1.5 = 1.07$). In percentage terms, available hedging capital represents 72% of the capital needed for a perfect hedge in a dividend-paying case and 58% in a non-dividend paying case. However, should we have allocated the same percentage in both cases, the probability of a successful hedge would be higher for a non-dividend paying stock (see Fig. 3.3). Say, we choose to allocate 72%, or 1.5 for stock paying dividends and 1.84 for a non-dividend paying stock. Then probabilities of a successful hedge are 0.9665 and 0.9698, respectively.

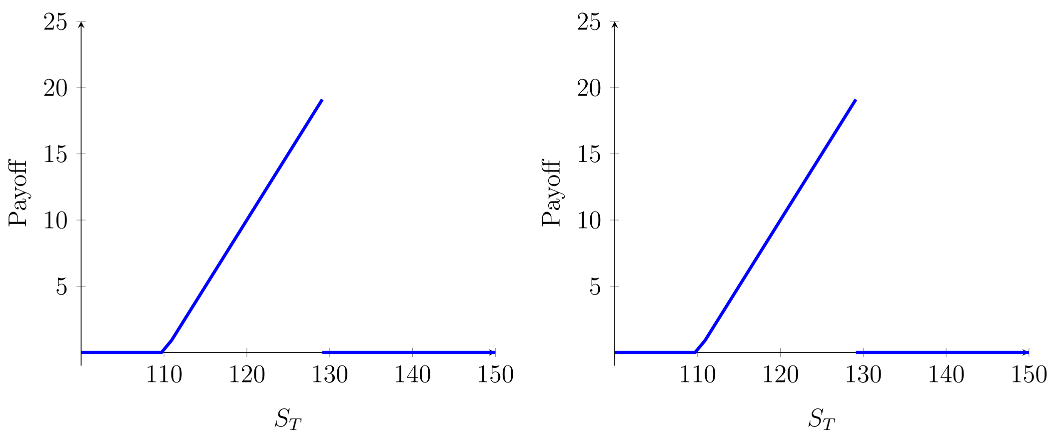
The payoffs of the quantile hedging strategies for different values of δ, assuming initial capital of 1.5, are given in Fig. 3.4. In case of a non-dividend paying stock (Fig. 3.4a), the European call option remains fully hedged for $S_T \leq 129.09$. For $S_T > 129.09$, the hedger becomes fully exposed to a potential obligation of $S_T - 110$, due to a limited hedging capital. The shape of the payoff for the case of $\delta = 0.07$ (Fig. 3.4b) is similar to the first scenario, however, in this case the option remains fully hedged for larger stock values, specifically, for $S_T \leq 132.76$.

Next, let us consider a dual problem. If the hedger is willing to accept a



(a) $\delta = 0$ (b) $\delta = 0.07$

Figure 3.4: Payoff of a quantile hedging portfolio for a hedging capital of 1.5.



(a) $\delta = 0$ (b) $\delta = 0.07$

Figure 3.5: Payoff of a quantile hedging portfolio for a shortfall probability of 5%.

shortfall probability of 5%, how much capital will be needed in both scenarios? In a zero-dividend scenario, the capital needed is 1.50[1]; in the 7%-dividend case, the required capital is 1.28; or 58% and 61% relative to the call prices, respectively. As anticipated, payoff diagrams for the two scenarios are identical (see Fig. 3.5): option remains fully hedged for $S_T \leq 129.11$ and becomes fully unhedged if the stock price increases beyond 129.11.

To put it into insurance context, let us consider a longer maturity, $T = 3$, with other parameters being unchanged. Suppose the survival probability is known to be 0.9400. Then in case of a non-dividend paying stock, the capital needed for a perfect hedge is 17.98, the capital available for hedging of the embedded call is 16.90, and the probability of a successful hedge is 0.9893. When stock pays dividends at a rate $\delta = 0.07$, the capital needed for a perfect hedge is 8.97, initial capital is 8.43, and the probability of a successful hedge is 0.9805.

Next, suppose the insurance company is willing to accept a 3% chance of a shortfall. Then, we can derive survival probabilities using (3.4.13) and (3.4.10): ${}_3p_x = 0.8507$ for a non-dividend paying stock, and ${}_3p_x = 0.9068$ for a stock paying dividends at a rate 7%. Using a mortality table, for example, a Valuation Basic Table from Society of Actuaries [3.11], we can reconstruct the age of the clients suitable for this contract: $x \geq 90$ and $x \geq 87$[2], for $\delta = 0$ and $\delta = 0.07$, respectively. In other words, in case of a dividend paying stock, the insurance company can trade a contract among a wider group of clients. In this example, quantile prices represent about 15% and 10% reduction from the Black-Scholes prices.

The effect of the dividends on pricing and hedging of equity-linked life insurance contracts was not previously discussed in the literature. At the same time, many acknowledge that dividends have important real-life implications. For this reason, the methodology and results presented here may be of high importance to the actuarial practitioners and insurance companies.

[1] Rounded to two decimal places

[2] Using VBT ANB Male Unismoke 2015 mortality table, www.soa.org

3.6 Concluding remarks

We have considered a problem of quantile hedging in financial markets with dividends and provided important insurance applications. These results represent a subject of interest from both theoretical and practical points of view. From a theoretical perspective, this work might be used as a basis for studying more general models with dividends, such as Lèvy model (discussed, for example, in Cont and Tankov [3.2]) and may be expanded to a stochastic dividend case. From a practical perspective, these results could be incorporated into decision-making by insurance companies as they provide a valuable insight into the impact the dividends may have on pricing and hedging for equity-linked life insurance contracts.

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Chapter 4

Quantile hedging in a defaultable market with life insurance applications

4.1 Introduction

Quantile hedging is an imperfect hedging technique proposed by Föllmer and Leukert [4.3] aimed at maximizing probability of a successful hedge under a capital constraint. First applications of the quantile hedging methodology to pricing the equity-linked life insurance contracts were offered by Melnikov [4.10] in 2004, and have been extensively studied thereafter. Melnikov and Skornyakova [4.12] develop quantile hedging in a two-factor jump-diffusion market and apply the results to pure endowment equity-linked life insurance contracts with flexible guarantee. Wang [4.19] shows how quantile hedging can be utilized for guaranteed minimum death benefits. Gao et al. [4.4] examine quantile hedging for equity-linked life insurance contracts in a stochastic interest rate economy. Melnikov and Tong [4.13] analyze the application of quantile hedging on pure endowment contracts in the presense of transaction costs. This list is by no means exhaustive. A useful summary on the developments in the field of imperfect hedging with life insurance in view can be found in Melnikov and Nosrati [4.11].

The focus of our interest is application of quantile hedging in markets that are not default-free.

A general pricing and hedging theory for defaultable claims is given in Bielecki et al. [4.1]. The two primary classes of approaches to credit risk modeling are structural and reduced-form models. In a structural model a default of a company is driven by the value of the company's assets. This approach, also known as a firm-value approach, was originated in a seminal work of Merton [4.14] and was generalized by the numerous authors (see, for example, Hull and White [4.6], Hanke [4.5]). In contrast to the structural models, the reduced-form models are less precise in their description of a mechanism leading to a default and usually assume the existence of a default intensity to model a default event (Merton [4.15], Jarrow and Turnbull [4.7], Linetsky [4.9]). In this paper we use a default intensity approach, which lies within a reduced-form class, to modeling a default.

Application of quantile hedging methodology in markets with default risk is studied in Sekine [4.17] and Nakano [4.16]. In particular, Sekine [4.17] applies quantile hedging for defaultable securities in incomplete market when intensity of the default time is correlated with tradable assets. Nakano [4.16] studies a quantile hedging problem for defaultable claims in incomplete markets using a convex duality approach.

In this paper we consider a complete financial market with two defaultable securities traded (a defaultable equity paying dividends and a defaultable bond) along with a default-free bond and develop a quantile methodology for pricing and hedging a European call option on this market. Developed methodology is applied to pricing the pure endowment equity-linked life insurance contracts. To our knowledge, this model of the defaultable market has not yet been studied in a quantile hedging context with insurance applications.

This paper is organized as follows. In Section 4.2, we describe the market model and give formulas for perfect hedging and pricing. Section 4.3 presents a solution to a quantile hedging problem in a defaultable market. Section 4.4 provides an application of the quantile methodology to pricing the pure endowment with fixed guarantee equity-linked life insurance contracts. A numerical example is given in Section 4.5. Concluding remarks are in Section 4.6.

4.2 Market with defaultable assets

We shall consider a financial market with three assets:

◇ Bank savings account;

◇ Defaultable bond;

◇ Defaultable equity.

Among the three assets, the bank savings account is the only default-free asset. We shall assume that in case of a default both bond and stock prices jump to zero. The time of default, denoted as τ, is a positive random variable on a probability space $(\Omega, \mathcal{G}, P)$, such that $P(\tau = 0) = 0$ and $P(\tau > t) > 0$. The associated default indicator process is defined as

$$H_t = I_{\{\tau \leq t\}},$$

which is equal to 1 if default occurs before time t and equal to 0 otherwise.

Let us specify that $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is a joined filtration, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, i.e. $G_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every $t \in \mathbb{R}_+$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration generated by a standard Brownian motion $(W_t)_{t\geq 0}$, and $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is a filtration generated by the process $(H_t)_{t\geq 0}$. Hence, the filtration $\mathbb{G}$ represents full information available to the participants in the defaultable market. We assume that τ is independent of W (i.e. $\mathbb{F}$ and $\mathbb{H}$ are independent).

We further assume that $P(\tau > t) = e^{-\lambda t}$, where $\lambda > 0$ is a constant default intensity (also known as a hazard rate).

Next, we define a process M_t as

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda \mathrm{d}s = H_t - (t \wedge \tau)\lambda = H_t - \int_0^t \lambda(1 - H_s)\mathrm{d}s.$$

The process $(M_t)_{t\geq 0}$ is both $\mathbb{H}$-martingale and $\mathbb{G}$-martingale.

Mathematically, the prices of the three assets in a defined defaultable market evolve as follows:

1) Savings account:

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B_0 = 1.$$

2) Defaultable zero-coupon bond:

$$\begin{cases} \mathrm{d}J_t = J_{t-}(\alpha\mathrm{d}t - \mathrm{d}M_t), \\ J_T = 1 - H_T. \end{cases} \tag{4.2.1}$$

A defaultable zero-coupon bond delivers the payment of one monetary unit at maturity if default did not occur by maturity time T. If default occurred by time T, the bond delivers nothing.

A solution to the above system is given by

$$J_t = (1 - H_t)e^{-(\alpha+\lambda)(T-t)}, \quad t \in [0, T],$$

where $\alpha \geq r$.

3) Defaultable equity:

$$\mathrm{d}S_t = S_{t-}(\mu \mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}M_t), \quad S_0 > 0. \tag{4.2.2}$$

The price of a defaultable equity after a default time τ is 0, i.e. $S_t = 0$ for $t \geq \tau$.

The solution to Eq. (4.2.2) is given by

$$S_t = S_0 \exp\Big\{\Big(\mu - \frac{\sigma^2}{2} + \lambda\Big)t + \sigma W_t\Big\}(1 - H_t).$$

Both bond and stock described by Eqs. (4.2.1) and (4.2.2) are said to be subject to a *total default*, as in the event of default (at time τ) their prices drop to 0 and never recover. It is sometimes convenient to describe the prices of the defaultable assets as follows:

$$J_t = I_{[t<\tau]}\breve{J}_t,$$
$$S_t = I_{[t<\tau]}\breve{S}_t,$$

where $\breve{J}_t$ and $\breve{S}_t$ denote pre-default prices governed by the SDEs:

$$\mathrm{d}\breve{J}_t = \breve{J}_t(\alpha + \lambda)\mathrm{d}t,$$
$$\mathrm{d}\breve{S}_t = \breve{S}_t\big((\mu + \lambda)\mathrm{d}t + \sigma \mathrm{d}W_t\big). \tag{4.2.3}$$

We further assume that the stock pays a continuous dividend at a rate δ. In case of a default the dividend $D_t - D_{t-}$ due at time $t = \tau$ will not be received.

The value of a portfolio, X_t, at time t is given by

$$X_t = \beta_t B_t + \gamma_t J_t + \xi_t S_t.$$

A strategy $\pi_t = (\beta_t, \gamma_t, \xi_t)$ will be called self-financing if

$$\mathrm{d}X_t = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}J_t + \xi_t(\mathrm{d}S_t + \delta S_{t-}\mathrm{d}t), \tag{4.2.4}$$

i.e. the dividends are fully reinvested in the market.

Defining an auxiliary process $S_t^{(\delta)} := e^{\delta t} S_t$, we can express the self-financing strategy (4.2.4) as

$$\mathrm{d}X_t = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}J_t + \xi_t e^{-\delta t} \mathrm{d}S_t^{(\delta)},$$

where

$$\mathrm{d}S_t^{(\delta)} = S_{t-}^{(\delta)}\big((\mu + \delta)\mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}M_t\big). \tag{4.2.5}$$

For a discounted auxiliary process $\tilde{S}_t := e^{(\delta - r)t} S_t$, we have

$$\mathrm{d}\tilde{S}_t = \tilde{S}_{t-}\big((\mu + \delta - r)\mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}M_t\big), \tag{4.2.6}$$

and by comparing (4.2.5) with (4.2.6), or by applying Itô formula to $\tilde{S}_t = e^{-rt} S_t^{(\delta)}$, we get the relationship

$$\mathrm{d}\tilde{S}_t = e^{-rt} \mathrm{d}S_t^{(\delta)} - r e^{(\delta - r)t} S_{t-}\mathrm{d}t. \tag{4.2.7}$$

Then by Itô formula, the discounted portfolio value, $V_t = X_t / B_t$, becomes:

$$\begin{aligned}
\mathrm{d}V_t &= -r e^{-rt} X_{t-}\mathrm{d}t + e^{-rt}\mathrm{d}X_t \\
&= e^{-rt}\big(\beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}J_t + \xi_t e^{-\delta t}\mathrm{d}S_t^{(\delta)}\big) - \frac{X_{t-}}{B_t^2}\mathrm{d}B_t.
\end{aligned}$$

Using relationship (4.2.7) and noting that the discounted bond price process $\tilde{J}_t := J_t / B_t$ satisfies:

$$\mathrm{d}\tilde{J}_t = -r e^{-rt} J_{t-}\mathrm{d}t + e^{-rt}\mathrm{d}J_t,$$

we can write $\mathrm{d}V_t$ as

$$\begin{aligned}
\mathrm{d}V_t &= \gamma_t \mathrm{d}\tilde{J}_t + \xi_t e^{-\delta t}\mathrm{d}\tilde{S}_t + (\beta_t B_t + \gamma_t J_{t-} + \xi_t S_{t-})\frac{\mathrm{d}B_t}{B_t^2} - X_{t-}\frac{\mathrm{d}B_t}{B_t^2} \\
&= \gamma_t \mathrm{d}\tilde{J}_t + \xi_t e^{-\delta t}\mathrm{d}\tilde{S}_t.
\end{aligned} \tag{4.2.8}$$

It follows from (4.2.8) that we are in need of an equivalent probability measure P^*, such that the discounted bond price process $\tilde{J}_t := J_t / B_t$ and discounted auxiliary stock price process $\tilde{S}_t := e^{\delta t} S_t / B_t$ are martingales under P^*.

By Girsanov theorem, the density of such a measure can be determined from:

$$Z_t^* = \frac{\mathrm{d}P_t^*}{\mathrm{d}P_t} = \exp\Big\{ -\frac{\varphi^2}{2} t + \varphi W_t + H_t \ln(1+\psi) - \psi\lambda(t \wedge \tau)\Big\},$$

where $\psi > -1$.

Let us define two processes:

$$W_t^* := W_t - \int_0^t \varphi \, \mathrm{d}s = W_t - \varphi t, \tag{4.2.9}$$

$$\begin{aligned} M_t^* &:= H_t - \int_0^t (1-H_s)(1+\psi)\lambda \, \mathrm{d}s = H_t - \int_0^{t\wedge\tau} \lambda(1+\psi)\mathrm{d}s \\ &= H_t - (t\wedge\tau)\lambda(1+\psi). \end{aligned} \tag{4.2.10}$$

As shown by Kusuoka [4.8], W_t^* is a Brownian motion under the equivalent martingale measure P^*, and M_t^* is a $\mathbb{G}$-martingale under this measure. Intensity of τ under P^* is $\lambda^* := \lambda(1+\psi)$.

By Itô formula, the price dynamics of the processes $\tilde{S}_t$ and $\tilde{J}_t$ under the measure P are given by

$$\mathrm{d}\tilde{S}_t = \tilde{S}_{t-}\big((\mu+\delta-r)\mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}M_t\big), \tag{4.2.11}$$

$$\mathrm{d}\tilde{J}_t = \tilde{J}_{t-}\big((\alpha-r)\mathrm{d}t - \mathrm{d}M_t\big), \quad \tilde{J}_T = \frac{1-H_T}{B_T}. \tag{4.2.12}$$

Using (4.2.9) and (4.2.10), Eqs. (4.2.11) and (4.2.12) can be written as

$$\begin{aligned} \mathrm{d}\tilde{S}_t &= \tilde{S}_{t-}\big[\big(\mu+\delta-r+\sigma\varphi-\lambda\psi\big)\mathrm{d}t + \sigma\mathrm{d}W_t^* - \mathrm{d}M_t^*\big], \\ \mathrm{d}\tilde{J}_t &= \tilde{J}_{t-}\big[\big(\alpha-r-\lambda\psi\big)\mathrm{d}t - \mathrm{d}M_t^*\big], \quad \tilde{J}_T = \frac{1-H_T}{B_T}. \end{aligned}$$

The processes $\tilde{S}_t$ and $\tilde{J}_t$ are martingales under P^* if the drift terms vanish:

$$\begin{cases} \mu+\delta-r+\sigma\varphi-\lambda\psi = 0, \\ \alpha-r-\lambda\psi = 0. \end{cases}$$

The above system has a unique solution:

$$\varphi = \frac{\alpha-\mu-\delta}{\sigma}, \tag{4.2.13}$$

$$\psi = \frac{\alpha - r}{\lambda} > -1.$$

The existence of the unique solution confirms that the martingale measure P^* is unique and the financial market is complete.

Hence, the stock price SDE (4.2.2) under P^* becomes:

$$\mathrm{d}S_t = S_{t-}\big((r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t^* - \mathrm{d}M_t^*\big),$$

with the solution

$$S_t = S_0 \exp\left\{\Big(\alpha + \lambda - \delta - \frac{\sigma^2}{2}\Big)t + \sigma W_t^*\right\} I_{[\tau > t]}. \tag{4.2.14}$$

Next, we study pricing and hedging of a European call option with payoff $f_T = (S_T - K)^+$, where K is a strike price. Such options can be priced by means of conditioning on the default event, using independence of τ and W^*:

$$\begin{aligned} C_0 = \mathrm{E}^*\left(\frac{(S_T-K)^+}{B_T}\right) &= \mathrm{E}^*\big(e^{-rT}(S_T-K)^+ | \tau > T\big) P^*(\tau > T) \\ &= S_0 e^{-\delta T}\Phi(\tilde{d}_+^{(T)}) - K e^{-(\alpha+\lambda)T}\Phi(\tilde{d}_-^{(T)}), \end{aligned} \tag{4.2.15}$$

where $\Phi(\cdot)$ denotes a standard normal cumulative distribution function;

$$\tilde{d}_\pm^{(T-t)} = \frac{\ln\left({}^{S_t}/_K\right) + \left(\alpha + \lambda - \delta \pm {}^{\sigma^2}/_2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{4.2.16}$$

Recall that a Black-Scholes price of a European call on a dividend-paying stock is given by

$$C^{BS}(S_0, K, T, r, \sigma, \delta) = S_0 e^{-\delta T}\Phi\big(d_+^{(T)}\big) - K e^{-rT}\Phi\big(d_-^{(T)}\big), \tag{4.2.17}$$

where

$$d_\pm^{(T-t)} = \frac{\ln({}^{S_t}/_K) + (r - \delta \pm {}^{\sigma^2}/_2)\ (T-t)}{\sigma\sqrt{T-t}}. \tag{4.2.18}$$

It is apparent from comparing Eqs. (4.2.15)-(4.2.16) with Eqs. (4.2.17)-(4.2.18), that the call option price in (4.2.15) can be written as a Black-Scholes price with $\alpha + \lambda$ replacing the risk-free rate r, i.e. $C_0 = C^{BS}(S_0, K, T, \alpha + \lambda, \sigma, \delta)$.

The price of a call option (and a value of a hedging strategy) at any given time $t < T$ is then

$$X_t^* = C_t = C^{BS}(S_t, K, T-t, \alpha+\lambda, \sigma, \delta). \tag{4.2.19}$$

Note that equation (4.2.19) is used only for $t \in [0, \tau \wedge T)$, and $X_t^* = 0$ on the set $t \in [\tau, T)$, which is empty if the default does not occur before the contract's maturity. Hence,

$$X_t^* = \begin{cases} C^{BS}(S_t, K, T-t, \alpha+\lambda, \sigma, \delta), & t \in [0, \tau \wedge T), \\ 0, & t \in [\tau, T), \end{cases}$$

and the self-financing strategy π_t^* has the following structure

$$\xi_t^* = \begin{cases} e^{-\delta(T-t)}\Phi(\tilde{d}_+^{(T-t)}), & t \in [0, \tau \wedge T), \\ 0, & t \in [\tau, T), \end{cases}$$

$$\gamma_t^* = \begin{cases} -K\Phi(\tilde{d}_-^{(T-t)}), & t \in [0, \tau \wedge T), \\ 0, & t \in [\tau, T), \end{cases}$$

$$\beta_t^* = 0.$$

Thus, a call payoff can be perfectly replicated by dynamic trading in just two assets: defaultable equity and defaultable bond. This is due to the fact that call price becomes 0 once the default occurs. At the same time, the positions in defaultable stock and bond also jump to 0, replicating the call value.

4.3 Quantile hedging in a defaultable market

Let us suppose that a seller of a contingent claim is not in a position to put up a capital required for a perfect hedge. Then what is the optimal partial hedge that can be achieved under a given budget constraint? The answer to this question depends on the chosen optimality criterion expressing the seller's attitude towards risk. In a quantile hedging problem, the investor chooses to maximize the probability of a successful hedge given a constraint on an initial funding, i.e.

$$\text{maximize } P(V_T \geq f_T/B_T), \tag{4.3.1}$$

$$\text{subject to } x_0 < X_0, \tag{4.3.2}$$

where x_0 is a hedging budget that is smaller than a capital X_0 needed to construct a perfect hedging strategy.

Föllmer and Leukert [4.3] showed that replicating strategy for a knockout option $f_T I_A$, where A denotes a success set $\{V_T \geq f_T/B_T\}$, solves an optimization problem (4.3.1)-(4.3.2), and the success set has a form

$$A = \Big\{ w : \frac{\mathrm{d}P}{\mathrm{d}P^*} > const \cdot f_T \Big\}. \tag{4.3.3}$$

A constant in (4.3.3) is chosen so that $\mathrm{E}^*(f_T I_A / B_T) = x_0$.

We remark that this solution relies on the assumption that $P\big(\frac{\mathrm{d}P}{\mathrm{d}P^*} = const \cdot f_T\big) = 0$, which is satisfied in our settings.

Let us find a quantile hedging strategy for the European call with payoff $f_T = (S_T - K)^+$.

First, note that

$$\begin{aligned} \mathrm{E}^*\Big(\frac{f_T}{B_T} I_A\Big) &= \sum_{n=0}^{1} \mathrm{E}^*\Big(\frac{f_T}{B_T} I_A \big| H_T = n\Big) P^*(H_T = n) \\ &= \mathrm{E}^*\Big(\frac{f_T}{B_T} I_A \big| \tau > T\Big) P^*(\tau > T). \end{aligned} \tag{4.3.4}$$

On a set $\{\tau > T\}$, Radon-Nikodym density can be expressed in terms of a stock price as follows:

$$\begin{aligned} Z_T^* = \frac{\mathrm{d}P_T^*}{\mathrm{d}P_T} &= \exp\Big\{(r-\alpha)T + \frac{\varphi^2}{2}T + \varphi W_T^*\Big\} \\ &= \exp\Big\{\frac{\varphi}{\sigma}\Big(\ln S_0 + (\alpha + \lambda - \delta - \frac{\sigma^2}{2})T + \sigma W_T^*\Big)\Big\} \\ &\quad \times \exp\Big\{-\frac{\varphi}{\sigma}\Big(\ln S_0 + (\alpha + \lambda - \delta - \frac{\sigma^2}{2})T\Big) + \frac{\varphi^2}{2}T + (r-\alpha)T\Big\} \\ &= \breve{S}_T^{\varphi/\sigma} \cdot \tilde{C}, \end{aligned}$$

where

$$\tilde{C} = S_0^{-\varphi/\sigma} \exp\Big\{-\frac{\varphi}{\sigma}(\alpha + \lambda - \delta - \frac{\sigma^2}{2})T + \frac{\varphi^2}{2}T + (r-\alpha)T\Big\},$$

φ is given in (4.2.13), and $\breve{S}$ denotes a pre-default stock price governed by the SDE (4.2.3).

Then,

$$A|\{\tau > T\} = \{\breve{S}_T^{-\varphi/\sigma} > const \cdot \tilde{C} \cdot (\breve{S}_T - K)^+\}.$$

Quantile hedging strategy may admit two forms depending on the magnitude of $-\varphi/\sigma$. Thus, we need to consider two cases separately.

Case 1. $-\varphi/\sigma \le 1$

In this case the set A conditionally on a default event is of the form

$$A|\{\tau > T\} = \{\breve{S}_T < g\} = \{W_T^* < b^{(T)}\},$$

where constant g is a solution of the equation:

$$x^{-\varphi/\sigma} = const \cdot \tilde{C} \cdot (x - K)^+.$$

The relationship between constants g and $b^{(T)}$ is given by

$$g = S_0 \exp\Big\{\sigma b^{(T)} + \Big(\alpha + \lambda - \delta - \frac{\sigma^2}{2}\Big)T\Big\}. \tag{4.3.5}$$

A modified claim $f_T I_A$ conditional on no jump to default occurring during the life of a contract can be represented as

$$\begin{aligned} f_T I_A|\{\tau > T\} &= (\breve{S}_T - K)^+ I_{[\breve{S}_T < g]} \\ &= (\breve{S}_T - K)^+ - (\breve{S}_T - g)^+ - (g - K) I_{[\breve{S}_T > g]}. \end{aligned}$$

Using (4.3.4) and recalling that

$$P^*(\tau > T) = e^{-\lambda^* T} = e^{-(\alpha+\lambda-r)T}, \tag{4.3.6}$$

we have

$$\begin{aligned} \mathrm{E}^*\Big(\frac{f_T}{B_T} I_A\Big) &= C^{BS}(S_0, K, T, \alpha+\lambda, \sigma, \delta) - C^{BS}(S_0, g, T, \alpha+\lambda, \sigma, \delta) \\ &- e^{-(\alpha+\lambda)T}\,(g - K)\,\Phi\Big(-\frac{b^{(T)}}{\sqrt{T}}\Big) \\ &= S_0 e^{-\delta T}\Big[\Phi\big(\tilde{d}_+^{(T)}\big) - \Phi\Big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\Big)\Big] - K e^{-(\alpha+\lambda)T}\Big[\Phi\big(\tilde{d}_-^{(T)}\big) - \Phi\Big(-\frac{b^{(T)}}{\sqrt{T}}\Big)\Big], \end{aligned} \tag{4.3.7}$$

where $\tilde{d}_{\pm}^{(T)}$ are given by (4.2.16).

The unknown boundary $b^{(T)}$ (and hence, g and $const$) can be found from the condition $x_0 = \mathrm{E}^*(e^{-rT} f_T I_A)$.

The value of the quantile hedging strategy at time $t \in [0, \tau \wedge T)$ can be computed as follows:

$$
\begin{aligned}
X_t^* &= \mathrm{E}^*(e^{-r(T-t)} f_T I_A | \mathcal{G}_t) = \mathrm{E}^*(e^{-r(T-t)} (\breve{S}_T - K)^+ I_{[\breve{S}_T < g]} | \mathcal{G}_t) e^{-(\alpha+\lambda-r)(T-t)} \\
&= \Big[\mathrm{E}^*\big((\breve{S}_T - K)^+ | \mathcal{G}_t\big) - \mathrm{E}^*\big((\breve{S}_T - g)^+ | \mathcal{G}_t\big) - (g-K) P^*(\breve{S}_T > g | \mathcal{G}_t)\Big] \\
&\quad \times e^{-(\alpha+\lambda)(T-t)} \\
&= C^{BS}(S_t, K, T-t, \alpha+\lambda, \sigma, \delta) - C^{BS}(S_t, g, T-t, \alpha+\lambda, \sigma, \delta) \\
&\quad - e^{-(\alpha+\lambda)(T-t)} (g-K) \Phi\Big(-\frac{b^{(T-t)}}{\sqrt{T-t}}\Big) \\
&= S_t\, e^{-\delta(T-t)} \Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\Big(\sigma\sqrt{T-t} - \frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big] \\
&\quad - K e^{-(\alpha+\lambda)(T-t)} \Big[\Phi\big(\tilde{d}_-^{(T-t)}\big) - \Phi\Big(-\frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big].
\end{aligned}
$$

To summarize, the value of the quantile hedging strategy at any time $t < T$ is given by

$$
X_t^* = \begin{cases} S_t\, e^{-\delta(T-t)} \Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\Big(\sigma\sqrt{T-t} - \frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big] \\ \quad -K e^{-(\alpha+\lambda)(T-t)} \Big[\Phi\big(\tilde{d}_-^{(T-t)}\big) - \Phi\Big(-\frac{b^{(T-t)}}{\sqrt{T-t}}\Big)\Big], \\ \quad t \in [0, \tau \wedge T), \\ 0, \quad t \in [\tau, T). \end{cases} \tag{4.3.8}
$$

For finding the components of the quantile hedging strategy, it is convenient to rewrite the value of the hedging strategy before default in terms of a constant g:

$$
X_t^* = S_t e^{-\delta(T-t)} \Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\big(\tilde{d}_{+g}^{(T-t)}\big)\Big] - K e^{-(\alpha+\lambda)(T-t)} \Big[\Phi\big(\tilde{d}_-^{(T-t)} - \Phi\big(\tilde{d}_{-g}^{(T-t)}\big)\big)\Big],
$$

where

$$
\tilde{d}_{\pm g}^{(T-t)} = \frac{\ln(S_t/g) + (\alpha + \lambda - \delta \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{4.3.9}
$$

and $\tilde{d}_{\pm}^{(T-t)}$ are given by (4.2.16).

The quantile hedging strategy π_t^* is then given by

$$\xi_t^* = \begin{cases} e^{-\delta(T-t)}\Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\big(\tilde{d}_{+g}^{(T-t)}\big) - \dfrac{\phi\big(\tilde{d}_{+g}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\,\dfrac{g-K}{g}\Big], \\ \quad t \in [0, \tau\wedge T), \\ \\ 0, \quad t\in[\tau, T), \end{cases} \tag{4.3.10}$$

$$\gamma_t^* = \begin{cases} -K\Big[\Phi\big(\tilde{d}_-^{(T-t)}\big) - \Phi\big(\tilde{d}_{-g}^{(T-t)}\big)\Big] \\ \qquad + S_t\,\phi\big(\tilde{d}_{+g}^{(T-t)}\big)\,\dfrac{e^{(\alpha+\lambda-\delta)(T-t)}}{\sigma\sqrt{T-t}}\,\dfrac{g-K}{g}, \quad t\in[0,\tau\wedge T), \\ 0, \quad t\in[\tau,T), \end{cases} \tag{4.3.11}$$

$$\beta_t^* = 0. \tag{4.3.12}$$

Case 2. $-\varphi/\sigma > 1$

In this case a set A conditionally on a default event is of the form

$$A|\{\tau > T\} = \{\breve{S}_T < g_1\}\cup\{\breve{S}_T > g_2\} = \{W_T^* < b_1^{(T)}\}\cup\{W_T^* > b_2^{(T)}\},$$

where constants $g_1 < g_2$ are two distinct solutions of the equation

$$x^{-\varphi/\sigma} = const\cdot\tilde{C}\cdot(x-K)^+.$$

Relationships between g_i and $b_i^{(T)}$ for $i = 1, 2$ are similar to (4.3.5).

A modified claim $f_T I_A$ conditional on no default occurring by the contract maturity can be written as a combination of options:

$$\begin{aligned} f_T I_A|\{\tau>T\} &= (\breve{S}_T - K)^+ I_{[\breve{S}_T<g_1]} + (\breve{S}_T-K)^+ I_{[\breve{S}_T>g_2]} \\ &= (\breve{S}_T-K)^+ - (\breve{S}_T-g_1)^+ - (g_1-K)I_{[\breve{S}_T>g_1]} \\ &\quad + (\breve{S}_T-g_2)^+ + (g_2-K)I_{[\breve{S}_T>g_2]}. \end{aligned}$$

Then, using (4.3.4) and (4.3.6), we have

$$\begin{aligned} \mathrm{E}^*\Big(\frac{f_T}{B_T}I_A\Big) &= C^{BS}(S_0,K,T,\alpha+\lambda,\sigma,\delta) - C^{BS}(S_0,g_1,T,\alpha+\lambda,\sigma,\delta) \\ &+ C^{BS}(S_0,g_2,T,\alpha+\lambda,\sigma,\delta) \\ &- e^{-(\alpha+\lambda)T}\Big[(g_1-K)\Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) - (g_2-K)\Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \end{aligned}$$

$$
\begin{aligned}
&= S_0 e^{-\delta T}\Big[\Phi\big(\tilde{d}_+^{(T)}\big) - \Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&\quad - Ke^{-(\alpha+\lambda)T}\Big[\Phi\big(\tilde{d}_-^{(T)}\big) - \Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big],
\end{aligned}
\tag{4.3.13}
$$

where $\tilde{d}_\pm^{(T)}$ are given by (4.2.16).

The unknown constants can be found from the condition $x_0 = \mathrm{E}^*(e^{-rT} f_T I_A)$.

It can be shown that the value of the quantile hedging strategy at time $t < T$ is given by

$$
X_t^* = \begin{cases}
S_t e^{-\delta(T-t)}\Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\Big(\sigma\sqrt{T-t} - \frac{b_1^{(T-t)}}{\sqrt{T-t}}\Big) \\
\quad + \Phi\Big(\sigma\sqrt{T-t} - \frac{b_2^{(T-t)}}{\sqrt{T-t}}\Big)\Big] - Ke^{-(\alpha+\lambda)(T-t)} \\
\quad \times\Big[\Phi\big(\tilde{d}_-^{(T-t)}\big) - \Phi\Big(-\frac{b_1^{(T-t)}}{\sqrt{T-t}}\Big) + \Phi\Big(-\frac{b_2^{(T-t)}}{\sqrt{T-t}}\Big)\Big], \\
\quad t \in [0, \tau \wedge T), \\
0, \quad t \in [\tau, T)
\end{cases}
\tag{4.3.14}
$$

Expressing X_t^* before the default in terms of a constant g gives

$$
\begin{aligned}
X_t^* &= S_t e^{-\delta(T-t)}\Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\big(\tilde{d}_{+g_1}^{(T-t)}\big) + \Phi\big(\tilde{d}_{+g_2}^{(T-t)}\big)\Big] \\
&\quad - Ke^{-(\alpha+\lambda)(T-t)}\Big[\Phi\big(\tilde{d}_-^{(T-t)}\big) - \Phi\big(\tilde{d}_{-g_1}^{(T-t)}\big) + \Phi\big(\tilde{d}_{-g_2}^{(T-t)}\big)\Big],
\end{aligned}
$$

where $\tilde{d}_\pm^{(T-t)}$ and $\tilde{d}_{\pm g_i}^{(T-t)}$ with $i = 1, 2$ are given by (4.2.16) and (4.3.9), respectively.

Finally, the components of the quantile hedging strategy π_t^* are described by

$$
\xi_t^* = \begin{cases}
e^{-\delta(T-t)}\Big[\Phi\big(\tilde{d}_+^{(T-t)}\big) - \Phi\big(\tilde{d}_{+g_1}^{(T-t)}\big) - \frac{\phi\big(\tilde{d}_{+g_1}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\,\frac{g_1 - K}{g_1} \\
\quad + \Phi\big(\tilde{d}_{+g_2}^{(T-t)}\big) + \frac{\phi\big(\tilde{d}_{+g_2}^{(T-t)}\big)}{\sigma\sqrt{T-t}}\,\frac{g_2 - K}{g_2}\Big], \quad t \in [0, \tau \wedge T), \\
0, \quad t \in [\tau, T),
\end{cases}
\tag{4.3.15}
$$

$$\gamma_t^* = \begin{cases} -K\Big[\Phi\big(\tilde{d}_{-}^{(T-t)}\big) - \Phi\big(\tilde{d}_{-g_1}^{(T-t)}\big) + \Phi\big(\tilde{d}_{-g_2}^{(T-t)}\big)\Big] \\ \quad + S_t \dfrac{e^{(\alpha+\lambda-\delta)(T-t)}}{\sigma\sqrt{T-t}} \\ \quad \times \Big[\phi\big(\tilde{d}_{+g_1}\big)\dfrac{g_1 - K}{g_1} - \phi\big(\tilde{d}_{+g_2}\big)\dfrac{g_2 - K}{g_2}\Big], \\ \quad t \in [0, \tau \wedge T), \\ 0, \quad t \in [\tau, T), \end{cases} \tag{4.3.16}$$

$$\beta_t^* = 0. \tag{4.3.17}$$

4.4 Application to the pricing of equity-linked life insurance contracts

Equity-linked life insurance contracts have two sources of uncertainty: market risk of the underlying asset and mortality of the insured person. It is assumed that the two risks are independent of each other, and we can consider them on the product of probability spaces: $(\Omega, \mathcal{G}, P) = (\Omega_1 \times \Omega_2, \mathcal{G}_1 \times \mathcal{G}_2, P_1 \times P_2)$, where the space $(\Omega_1, \mathcal{G}_1, P_1)$ is a probabilistic base for the financial risk, and $(\Omega_2, \mathcal{G}_2, P_2)$ is a probabilistic base for the insurance risk.

We will consider a 'pure endowment' type of contracts, when the payment is exercised only if the insured is still alive at a contract maturity T (the other case can be solved by symmetry). In this case the future obligation of the insurance company, f_T, can be expressed as

$$f_T = f(S_T) I_{[T(x)>T]}, \tag{4.4.1}$$

where $f(S_T)$ is a future payment dependent on the evolution of the financial market, $T(x)$ is the remaining lifetime of the insured who is currently of age x.

If $f(S_T) = \max(S_T, K)$, where K is some guaranteed amount, such contracts are known as 'pure endowment with guarantee' or 'equity-linked with guarantee' life insurance contracts. Such contracts will be a focus of this section, and an important question is how to determine a one-time premium of a single contract, ${}_TU_x$.

A useful observation is that payoff $f(S_T) = \max(S_T, K)$ can be decomposed into two parts:

$$\max(S_T, K) = K + (S_T - K)^+. \tag{4.4.2}$$

The second component of (4.4.2) is called an *embedded (call) option*. Applying option pricing theory to the contract with payoff (4.4.1), where an amount paid to a policyholder if he or she is still alive at time T is given by (4.4.2), we arrive at the *Brennan-Schwartz* price (Brennan and Schwartz [4.2]):

$${}_TU_x = {}_Tp_x\, Ke^{-rT} + {}_Tp_x\, \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right), \tag{4.4.3}$$

where ${}_Tp_x := P_2(T(x) > T)$ denotes a survival probability of the policyholder.

Eq. (4.4.3) shows that in order to find the fair premium for a pure endowment with fixed guarantee life insurance contract, it is sufficient to find a premium for the second component:

$${}_TU_x^C = {}_TU_x - {}_Tp_x\, Ke^{-rT} = {}_Tp_x\, \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right). \tag{4.4.4}$$

Note that the initial capital available for the embedded call option, ${}_TU_x^C$, is strictly less than the capital required for a perfect hedge, since survival probability is always less than 1. In this case quantile hedging technique can be applied effectively to obtain an optimal hedging subject to a capital constraint. In particular, the insurance company can use quantile hedging results from Section 4.3 to find a perfect hedge for a modified contingent claim $(S_T - K)^+ I_A$, under the initial hedging budget up to the amount ${}_TU_x^C$. This strategy will maximize the probability of a successful hedging for $(S_T - K)^+$. With these considerations, we obtain the following equation:

$${}_TU_x^C = \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T} I_A\right) = {}_Tp_x\, \mathrm{E}^*\left(\frac{(S_T - K)^+}{B_T}\right), \tag{4.4.5}$$

which leads to a so-called *balance equation*:

$${}_Tp_x = \frac{\mathrm{E}^*\big[(S_T - K)^+ I_A / B_T\big]}{\mathrm{E}^*\big[(S_T - K)^+ / B_T\big]}. \tag{4.4.6}$$

Relations (4.4.5) and (4.4.6) are key for the risk management analysis. The insurance company can either offer an equity-linked life insurance contract to any client and then maximize the probability of successful hedging based on a premium received, or it can set an acceptable level of risk and, based on it, identify clients suitable for such a contract.

We will further derive explicit formulas for the defaultable market described in Section 4.2.

The premium of a single equity-linked life insurance contract, as follows from (4.2.15) and (4.4.3), is given by

$$\begin{aligned}
{}_TU_x = &\ {}_Tp_x\left(Ke^{-rT} + S_0e^{-\delta T}\Phi(\tilde{d}_+^{(T)}) - Ke^{-(\alpha+\lambda)T}\Phi(\tilde{d}_-^{(T)})\right) \\
= &\ {}_Tp_x\Bigg[Ke^{-rT} + \ S_0e^{-\delta T}\Phi\left(\frac{\ln(S_0/K) + (\alpha+\lambda-\delta+\sigma^2/2)\ T}{\sigma\sqrt{T}}\right) \\
&\qquad - Ke^{-rT}\Phi\left(\frac{\ln(S_0/K) + (\alpha+\lambda-\delta-\sigma^2/2)\ T}{\sigma\sqrt{T}}\right)\Bigg] \\
= &\ {}_Tp_x\big[Ke^{-rT} + C^{BS}(S_0, K, T, \alpha+\lambda, \sigma, \delta)\big].
\end{aligned}$$

Initial capital available for the embedded call options is

$${}_TU_x^C = {}_TU_x - {}_Tp_x\, Ke^{-rT} = {}_Tp_x\, C^{BS}(S_0, K, T, \alpha+\lambda, \sigma, \delta),$$

and the balance equation is

$${}_Tp_x = \frac{\mathrm{E}^*[e^{-rT}(S_T-K)^+ I_A]}{C^{BS}(S_0, K, T, \alpha+\lambda, \sigma, \delta)}. \tag{4.4.7}$$

The set A may take two distinct forms, thus, we need to consider two cases separately.

Case 1. $-\varphi/\sigma \leq 1$

In this case, the success set A conditional on a default event is of the form

$$A|\{\tau > T\} = \{W_T^* < b^{(T)}\}.$$

If the equity-linked life insurance contract is offered to a random client, and hence the survival probability is known (derived from a mortality table based on a client's age, gender, smoking status, etc.), constant $b^{(T)}$ (along with g and *const*) can be found from the balance equation (4.4.7), which can be written using (4.3.7) as:

$${}_Tp_x = \frac{S_0e^{-\delta T}\left[\Phi\big(\tilde{d}_+^{(T)}\big) - \Phi\big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\big)\right] - Ke^{-(\alpha+\lambda)T}\left[\Phi\big(\tilde{d}_-^{(T)}\big) - \Phi\big(-\frac{b^{(T)}}{\sqrt{T}}\big)\right]}{S_0e^{-\delta T}\Phi\big(\tilde{d}_+^{(T)}\big) - Ke^{-(\alpha+\lambda)T}\Phi\big(\tilde{d}_-^{(T)}\big)}$$

$$= 1 - \frac{S_0 e^{-\delta T}\Phi\big(\sigma\sqrt{T} - \frac{b^{(T)}}{\sqrt{T}}\big) - Ke^{-(\alpha+\lambda)T}\Phi\big(-\frac{b^{(T)}}{\sqrt{T}}\big)}{S_0 e^{-\delta T}\Phi\big(\tilde{d}_+^{(T)}\big) - Ke^{-(\alpha+\lambda)T}\Phi\big(\tilde{d}_-^{(T)}\big)}, \tag{4.4.8}$$

where $\tilde{d}_\pm^{(T)}$ are given by (4.2.16).

A probability of successful hedging of a call option in case a vulnerable equity does not default can be found as follows

$$P(A \cap \{\tau > T\}) = P(A|\tau > T)P(\tau > T) = \Phi\bigg(\frac{b^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\bigg)e^{-\lambda T}.$$

We note that in the case of default, obligation related to a call option $(S_T - K)^+$ becomes 0, which is hedged with a void strategy. Hence,

$$P(A \cap \{\tau \le T\}) = P(A|\tau \le T)P(\tau \le T) = 1 - e^{-\lambda T}. \tag{4.4.9}$$

Consequently, the probability to cover the financial obligation becomes

$$P(A) = 1 - e^{-\lambda T} + \Phi\bigg(\frac{b^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\bigg)e^{-\lambda T}. \tag{4.4.10}$$

The capital and the components of the quantile hedging strategy for a call option are described by Eqs. (4.3.8) and (4.3.10)-(4.3.12).

On the other hand, the insurance company may be willing to accept a certain level of the financial risk $\epsilon \in (0, 1)$. If the insurance company aims to achieve $(1 - \epsilon)$ statistical probability of successful hedging, then

$$P(A) = 1 - \epsilon,$$

which in combination with (4.4.10) gives

$$b^{(T)} = \sqrt{T}\ \Phi^{-1}\big[1 - \epsilon \cdot e^{\lambda T}\big] + \frac{\mu + \delta - \alpha}{\sigma}T.$$

Plugging $b^{(T)}$ back into the balance equation (4.4.8) gives an actuarial value for the survival probability. Using this probability and appropriate mortality tables, the insurance company can derive the optimal age of the clients suitable for this contract.

Case 2. $-\varphi/\sigma > 1$

In this case the set A conditional on a default event has the form

$$A|\{\tau > T\} = \{W_T^* < b_1^{(T)}\} \cup \{W_T^* > b_2^{(T)}\}.$$

If survival probability is given, boundaries $b_1^{(T)}$ and $b_2^{(T)}$ can be found using the balance equation (4.4.7) with the numerator given by (4.3.13):

$$\begin{aligned}
{}_Tp_x = &\Bigg(S_0e^{-\delta T}\Big[\Phi\big(\tilde{d}_+^{(T)}\big) - \Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&- Ke^{-(\alpha+\lambda)T}\Big[\Phi\big(\tilde{d}_-^{(T)}\big) - \Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) + \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big]\Bigg) \\
&\Big/\Big(S_0e^{-\delta T}\Phi\big(\tilde{d}_+^{(T)}\big) - Ke^{-(\alpha+\lambda)T}\Phi\big(\tilde{d}_-^{(T)}\big)\Big) \\
= 1 - &\Bigg(S_0e^{-\delta T}\Big[\Phi\Big(\sigma\sqrt{T} - \frac{b_1^{(T)}}{\sqrt{T}}\Big) - \Phi\Big(\sigma\sqrt{T} - \frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&- Ke^{-(\alpha+\lambda)T}\Big[\Phi\Big(-\frac{b_1^{(T)}}{\sqrt{T}}\Big) - \Phi\Big(-\frac{b_2^{(T)}}{\sqrt{T}}\Big)\Big] \\
&\Big/\Big(S_0e^{-\delta T}\Phi\big(\tilde{d}_+^{(T)}\big) - Ke^{-(\alpha+\lambda)T}\Phi\big(\tilde{d}_-^{(T)}\big)\Big), \qquad (4.4.11)
\end{aligned}$$

where $\tilde{d}_\pm^{(T)}$ are given by (4.2.16).

Probability of successful hedging in case the underlying equity does not default can be computed as

$$P(A\cap\{\tau > T\}) = e^{-\lambda T}\Bigg[\Phi\Bigg(\frac{b_1^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg) + \Phi\Bigg(-\frac{b_2^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg)\Bigg].$$

In case of a default of the vulnerable equity the probability of success is similar to (4.4.9), and hence the total probability of having enough capital to cover the liability at maturity is

$$P(A) = 1 - e^{-\lambda T} + e^{-\lambda T}\Bigg[\Phi\Bigg(\frac{b_1^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg) + \Phi\Bigg(-\frac{b_2^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg)\Bigg].$$

The optimal hedging strategy and its capital are described by Eqs. (4.3.14)-(4.3.17).

Alternatively, the insurance company might fix the maximal level of the shortfall probability, so that $P(A) = 1-\epsilon$. In this case constants $b_1^{(T)}$ and $b_2^{(T)}$ can be found from

$$\Phi\Bigg(\frac{b_1^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg) + \Phi\Bigg(-\frac{b_2^{(T)} + \frac{\alpha-\mu-\delta}{\sigma}T}{\sqrt{T}}\Bigg) = 1 - \epsilon\, e^{\lambda T}.$$

Constants $b_1^{(T)}$ and $b_2^{(T)}$ are then used to compute the survival probability given by the balance equation (4.4.11). Optimal age of the clients suitable for this contract can be determined from the appropriate mortality tables based on this survival probability.

4.5 Numerical example

To demonstrate the effect of a default, we consider a call option in the market with the following characteristics:

$$T = 10, \quad \sigma = 0.3, \quad \mu = 0.08, \quad S_0 = \$100, \quad K = \$200, \quad \delta = 0,$$

$$r = 0.01, \quad \alpha = 0.01, \quad \lambda = 0.015.$$

We take $\alpha = r$ to facilitate the comparison with the Black-Scholes model. Under this condition, the case of $\lambda = 0$ leads to a Black-Scholes price.

The price of a call option, and hence the capital needed for a perfect hedging, in a default-free Black-Scholes market is \$19.44, in a defaultable market it is \$23.31. This result might seem counterintuitive as one could expect an option on a defaultable stock to cost less. However, notice from Eq. (4.2.14) that the drift of the risk-neutral stock price process is also affected by the default intensity, leading to a higher probability for an option to finish in-the-money. Specifically, in a default-free market $P^*(S_T > K) = 0.1358$, and in a defaultable market $P^*(S_T > K) = P^*(\breve{S}_T > K)P^*(\tau > T) = 0.1732 \cdot 0.8607 = 0.1491$. In other words, in the risk-neutal setting a stock price tends to move upward with a greater probability when there is a possibility of default. Similar behavior of option prices in the presence of default was described in Merton [4.15]:

> The option price is an increasing function of interest rate, and therefore an option on a stock that has a positive probability of complete ruin is more valuable than an option on a stock that does not.

Next, let us examine the effect of a default on a probability of successful hedging, $P(A)$. First, consider a situation when available hedging capital is \$0. In this case, we can compute probability to cover the financial obligation

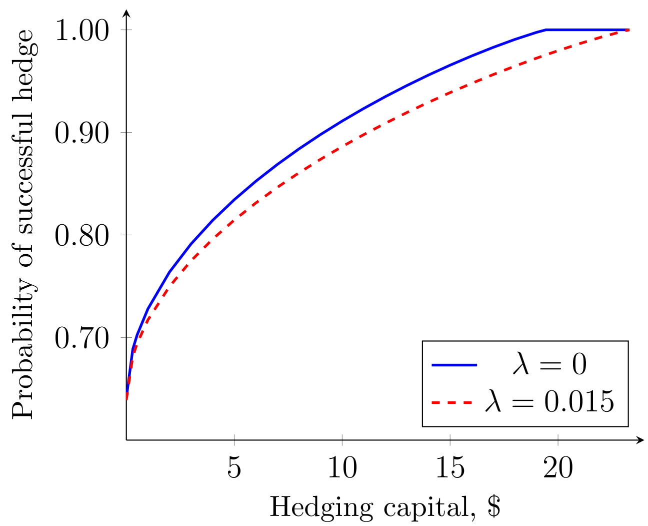

Figure 4.1: Probability of successful hedging as a function of initial capital.

as $P(S_T \leq K) = 0.6412$ in default-free market, and $P(S_T \leq K) = P(\check{S}_T \leq K)P(\tau > T) + P(S_T \leq K|\tau \leq T)P(\tau \leq T) = 0.5807 \cdot 0.8607 + 1 \cdot 0.1392 = 0.6391$ in a defaultable market. In this example, the probability of successful hedging (or, in effect, of successful not-hedging) for a call on a defaultable stock is lower than that for a call on a default-free stock, due to a higher probability of exercising the option when the market is not default-free.

On the other hand, if the hedging budget is \$19.44, equal to an option price in a default-free market, then, naturally, $P(A)$ for an option on a defaul-free stock is 1, and $P(A)$ for an option on a defaultable stock is less than 1, as its price is higher than \$19.44, meaning that a perfect hedge is not possible with this hedging capital.

Based on the analysis of these two extreme cases, we would expect $P(A)$ in this example to be higher in a default-free market for any dollar value of the hedging budget. Fig. 4.1 shows a probability of successful hedging for different values of hedging capital, when a default intensity, λ, is set to 0 and 0.015, and the results are in line with our expectations.

To provide the insurance context, suppose, that the insurance company is willing to accept a certain level of a shortfall risk, ϵ, and let us examine the acceptable survival probability of the clients, age of the clients, and available capital to hedge an embedded call as a function of ϵ (see Tables 4.1-4.2).

Table 4.1: Survival probability, client age, and available initial capital for different levels of a shortfall risk, $\lambda = 0.015$.

Shortfall risk	Acceptable survival probability of the clients	Acceptable age of the clients	Available initial capital, $
0.01	0.9240	≥ 63	21.53
0.03	0.8021	≥ 75	18.69
0.05	0.6972	≥ 79	16.25
0.10	0.4802	≥ 84	11.19

As evident from Table 4.1, an increase in a shortfall risk leads to a decrease in a maximum acceptable survival probability of the clients. Using these survival probabilities as thresholds, an acceptable age of the insureds can be derived from a suitable mortality table. In this example, we used the Valuation Basic Table from the Society of Actuaries [4.18] for a male unismoke population. As expected, a lower survival probability implies older insureds. In other words, as financial risk increases, the insurance company needs to compensate for it by managing a mortality component of risk that can be achieved by attracting older clients. Finally, as the insurance company allows for increased levels of risk, the required hedging capital is reduced. The same patterns can be observed for a default-free market (see Table 4.2).

Table 4.2: Survival probability, client age, and available initial capital for different levels of a shortfall risk, $\lambda = 0$.

Shortfall risk	Acceptable survival probability of the clients	Acceptable age of the clients	Available initial capital, $
0.01	0.9217	≥ 63	17.91
0.03	0.7969	≥ 75	15.49
0.05	0.6901	≥ 79	13.41
0.10	0.4709	≥ 84	9.15

As seen from Tables 4.1 and 4.2, the maximum acceptable survival probabilities in case of a default-free market ($\lambda = 0$) are slightly lower that those in a defaultable market ($\lambda = 0.015$) for any fixed ϵ. In this particular example it does not affect the ages of the insureds, but in general, lower survival probabilities correspond to older clients. Another observation is that the same level of a shortfall risk can be achieved with a smaller initial capital when the default is not possible. This is in line with earlier observations (see Fig. 4.1).

4.6 Concluding remarks

In this paper we consider a complete financial market with three tradable securities, two of which are subject to a risk of default. Default is modeled as an exogenous event using an intensity-based approach under the assumption of a constant hazard rate. Explicit formulas for pricing and hedging of a European call option on a vulnerable equity are derived, in both perfect and quantile settings. We demonstrate how quantile methodology can be effectively used for pricing the equity-linked insurance contracts and provide closed-form solutions for a pure endowment with fixed guarantee type of contract. We also give a numerical example to illustrate the effect of a default on the option price, on the probability of successful hedging, and on the insurance-related variables. Our results are in line with the results of Merton [4.15] in that an option on a defaultable stock is worth more than that on a default-free stock.

As part of a future research, it may be of interest to examine the effect of a default in an efficient hedging setting with insurance in view, as well as to extend these results to a case of non-constant default intensity.

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Chapter 5

Bachelier model with stopping time and its insurance application

5.1 Introduction

Louis Bachelier was the first to initiate the study of continuous-time processes and introduce Brownian motion mathematically, with the goal of developing an option-pricing theory (Bachelier [5.2]). Thus, Bachelier model is known as the first model to describe the evolution of the stock prices by means of the Brownian motion. The work of Bachelier was read and had an influence on many famous scientists including Paul Lévy, Paul Erdös, Mark Kac, William Feller, Kai Lai Chung, Paul Samuelson, Kiyosi Itô, Andrey Kolmogorov (see Taqqu [5.20] for a discussion on the importance of Bachelier's work). However, this undisputably remarkable result possessed a weakness that the stock price could get negative with a positive probability. This weakness was overcome by the widely used nowadays Black-Scholes-Merton model, introduced in 1973.

It is important to emphasize that for smaller values of volatility and time to maturity Black-Scholes and Bachelier formulas closely coincide, as shown by Schachermayer and Teichmann [5.17]. Moreover, in certain situations, especially for short-lived options, Bachelier model appears to be a better choice, compared to the Black-Scholes model, as it provides a better fit to the actual market data (see, for example, Versluis [5.21]).

In this work, we consider a modification of the Bachelier model by introducing a stopping time: once a stock price hits 0, it stays there forever. Economically, hitting of 0 is equivalent to a default event. This adjustment ensures that a stock price is always non-negative and also resolves a problem with limiting behavior of volatility and time to maturity pertaining to a classical Bachelier model. We find that a modified Bachelier model tends to coincide much closer with the Black-Scholes model, when volatility and time to maturity increase. The idea to absorb a stock price at zero, when a stock is modeled as an arithmetic Brownian motion, appears in Cox and Ross [5.5] and later in Goldenberg [5.9]. In this paper, we revisit this concept and provide alternative proofs of these results. Specifically, for a zero interest, Goldenberg [5.9] uses a transition density function of absorbed arithmetic Brownian motion, while we derive the option price by a straighforward application of a reflection principle. For a non-zero interest rate, he uses a time change to arrive at the option pricing formula. We follow an analytical method instead and obtain a different expression for the option price. Thus, our approach is more intuitive and is convenient to follow as a guideline in pricing different types of the barrier options.

Using these findings for the Bachelier model, we consider a quantile hedging problem which became very interesting and valuable in mathematical finance in the last two decades. We develop a quantile hedging methodology and provide life insurance applications, using classical and modified Bachelier models. Although the Bachelier model and its modifications attracted interest in a general option pricing context, they remain a relatively unexplored area for insurance applications (in this regard we may point out a paper by Melnikov and Moliboga [5.12]).

This paper is organized as follows. Next section outlines the standard and modified Bachelier models and applies them to pricing of a European call. Subsection 5.2.1 deals with a zero-interest rate case. Subsection 5.2.2 presents an option pricing in a non-zero interest rate environment. Numerical comparison of both Bachelier models with the Black-Scholes model is provided in Subsection 5.2.3.

In Section 5.3 we derive a quantile hedging strategy, its capital, and a probability of successful hedging for both a standard Bachelier model and a Bachelier model with stopping time. These results are applied to pricing of equity-linked life insurance contracts in Section 5.4. Developed concepts are illustrated with a numerical example in Section 5.5. Concluding remarks are given in Section 5.6.

5.2 Bachelier model and its modification

5.2.1 Zero interest rate

A standard contemporary Bachelier market is (B, S)-market, where the bank account $(B_t)_{0\leq t\leq T}$ remains the same over time ($B_t = 1$ for all t, i.e. risk-free interest rate, r, is 0), and the stock price $(S_t)_{0\leq t\leq T}$ is described by a linear Brownian motion with drift:

$$S_t = S_0 + \mu t + \sigma W_t, \quad S_0 > 0, \tag{5.2.1}$$

where $\sigma > 0$ denotes a volatility in the Bachelier model, μ is a drift parameter, $(W_t)_{t\geq 0}$ is a standard Brownian motion on its natural base $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$.

We note that volatility in a Bachelier model (called 'parameter of nervousness' by Bachelier) is defined differently from a volatility in a famous Black-Scholes model. The former is an absolute standard deviation of stock prices, while the latter is a relative standard deviation.

The stock price process (5.2.1) is a martingale with respect to a unique martingale measure P^* defined by

$$Z_t = \frac{\mathrm{d}P_t^*}{\mathrm{d}P_t} = \exp\Big(-\frac{\mu}{\sigma} W_t - \frac{1}{2}\Big(\frac{\mu}{\sigma}\Big)^2 t\Big).$$

Moreover, by Girsanov theorem, a process

$$W_t^* = W_t + \frac{\mu}{\sigma} t$$

is a standard Brownian motion under the measure P^*.

The fair price of a European call option can then be found as the expected value of the payoff with respect to a measure P^*, $\mathrm{E}^*(S_T - K)^+$, and the result is known as the 'Bachelier's formula' (see, for example, Shiryaev [5.18]):

$$C_0 = C_0(S_0) = (S_0 - K)\,\Phi\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) + \sigma\sqrt{T}\,\phi\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big), \tag{5.2.2}$$

where $K > 0$ is a strike price, T is a time to contract's maturity, $\phi(\cdot)$ denotes a probability density function of a standard normal distribution, $\Phi(\cdot)$ denotes a standard normal cumulative distribution function.

It can be shown that the no-arbitrage price $C(t, S_t)$, or, equivalently, the value of the perfect hedging strategy, $X_t^* = X_t^*(S_t)$, at time $t < T$ is given by

$$C(t, S_t) = X_t^*(S_t) = (S_t - K)\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right), \tag{5.2.3}$$

and the unique self-financing replicating strategy of the call option, π_t^*, has the following structure

$$\gamma_t^* = \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right),$$
$$\beta_t^* = X_t^* - \gamma_t^* S_t = -K\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right).$$

As evident from Eq. (5.2.1), it is possible for a share price to turn negative in a Bachelier market. To overcome this disadvantage, we offer the following modification of the stock price process:

$$S_{t\wedge\tau} = S_0 + \mu(t \wedge \tau) + \sigma W_{t\wedge\tau}, \quad S_0 > 0,$$

where τ is a stopping time,

$$\tau := \inf\{t : S_0 + \mu t + \sigma W_t \leq 0\}. \tag{5.2.4}$$

In a modified model, the stock price is always non-negative. If at some point of time the price becomes 0, it stays at 0 forever. Thus, we may think of τ as of default time of the company issued the stock.

The stopped process $S_{t\wedge\tau}$ remains a martingale under the measure P^*:

$$S_{t\wedge\tau} = S_0 + \sigma W_{t\wedge\tau}^*.$$

This preservation of martingality greatly facilitates the derivation of the option price under a modified Bachelier model. The following theorem gives a fair option price, as well as a perfect hedging strategy and its capital, for an adjusted model.

We remark that formula (5.2.5) appears in Goldenberg [5.9] with $\sigma = 1$. He used a transition density function of absorbed arithmetic Brownian motion derived by Karlin and Taylor [5.10]. We give an alternative proof of this result by a direct application of the reflection principle.

Theorem 5.1. *Consider a European call option with payoff* $(S_{T\wedge\tau} - K)^+$ *on a modified Bachelier market with stopping time, assuming* $r = 0$.

(a) *A fair price of a call option is given by*

$$\begin{aligned}\tilde{C}_0 &= (S_0 - K)\,\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + (S_0 + K)\,\Phi\left(\frac{-S_0 - K}{\sigma\sqrt{T}}\right)\\ &\quad + \sigma\sqrt{T}\left[\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right)\right].\end{aligned} \tag{5.2.5}$$

(b) *The capital and the structure of the replicating (hedging) strategy at any time* $t < T$ *are given by*

$$\tilde{X}_t^* = \begin{cases} (S_t - K)\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + (S_t + K)\,\Phi\left(\frac{-S_t - K}{\sigma\sqrt{T-t}}\right) \\ \quad +\sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, \tau\wedge T), \\ 0, \quad t \in [\tau, T); \end{cases} \tag{5.2.6}$$

$$\tilde{\gamma}_t^* = \begin{cases} \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{-S_t - K}{\sigma\sqrt{T-t}}\right), \quad t \in [0, \tau\wedge T), \\ 0, \quad t \in [\tau, T); \end{cases}$$

$$\tilde{\beta}_t^* = \begin{cases} K\left[\Phi\left(\frac{-S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right)\right] \\ \quad +\sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, \tau\wedge T), \\ 0, \quad t \in [\tau, T). \end{cases}$$

Proof. **(a)** Due to the preservation of martingality by the stopped stock price process, the fair price of an option can be found as an expected value of the payoff with respect to the unique martingale measure P^*:

$$\begin{aligned}\tilde{C}_0 &= \mathrm{E}^*(S_{T\wedge\tau} - K)^+ = \mathrm{E}^*(S_T - K)^+\,\mathbb{1}[\tau > T]\\ &= \mathrm{E}^*(S_T - K)^+ - \mathrm{E}^*(S_T - K)^+\,\mathbb{1}[\tau \le T].\end{aligned} \tag{5.2.7}$$

From the reflection principle for Brownian motion it immediately follows that

$$\mathrm{E}^*(S_0 + \sigma W_T^* - K)^+\,\mathbb{1}[\tau \le T] = \mathrm{E}^*(-S_0 + \sigma W_T^* - K)^+. \tag{5.2.8}$$

Combining (5.2.7) and (5.2.8), we can write the price of the call option as

$$\tilde{C}_0 = \mathrm{E}^*(S_0 + \sigma W_T^* - K)^+ - \mathrm{E}^*(-S_0 + \sigma W_T^* - K)^+ = C_0(S_0) - C_0(-S_0),$$

and (5.2.5) follows.

(b) If default has not occurred by time $t < T$, then the value of the hedging strategy at this time can be computed as:

$$\begin{aligned}\tilde{X}_t^* &= \mathrm{E}^*\big((S_{T\wedge\tau} - K)^+ \,|\mathcal{F}_t\big) = \mathrm{E}^*\big((S_T - K)^+ \,\mathbb{1}[\tau > T]\,|\mathcal{F}_t\big) \\ &= \mathrm{E}^*\big((S_T - K)^+|\mathcal{F}_t\big) - \mathrm{E}^*\big((S_T - K)^+ \,\mathbb{1}[\tau \le T]\,|\mathcal{F}_t\big) \\ &= X_t^*(S_t) - X_t^*(-S_t),\end{aligned}$$

where the last equality is obtained using reflection principle, as previously; and X_t^* is given by (5.2.3).

On a set $t \in [\tau, T)$, which may be empty if default event does not occur by the maturity of the contract, the capital $\tilde{X}_t^*$ is 0, as the payoff is 0. Hence, the value of the hedging strategy at time $t < T$ is given by (5.2.6).

Note that $\tilde{X}_t^* = \tilde{C}(t, S_t)$ and $\mathrm{d}\tilde{X}_t^* = \tilde{\gamma}_t^*\,\mathrm{d}S_t$. Simultaneously, by Itô formula for $\tilde{C}(t, S_t)$:

$$\mathrm{d}\tilde{C}(t, S_t) = \frac{\partial \tilde{C}}{\partial S}\,\mathrm{d}S_t + \left(\frac{\partial \tilde{C}}{\partial t} + \frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial S^2}\sigma^2\right)\mathrm{d}t,$$

where $\mathrm{d}S_t = \sigma\,\mathrm{d}W_t^*$, and we can conclude that

$$\tilde{\gamma}_t^* = \frac{\partial \tilde{C}}{\partial S}(t, S_t).$$

Hence, the components of the unique self-financing strategy, $\tilde{\pi}_t^*$, on a set $t \in [0, \tau\wedge T)$ are given by

$$\begin{aligned}\tilde{\gamma}_t^* &= \frac{\partial \tilde{X}_t^*}{\partial S_t} = \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{-S_t - K}{\sigma\sqrt{T-t}}\right), \\ \tilde{\beta}_t^* &= \tilde{X}_t^* - \tilde{\gamma}_t^* S_t.\end{aligned}$$

On a set $t \in [\tau, T)$ the obligation ceases to exist and the option is hedged with a void strategy, i.e. $\tilde{\gamma}_t^* = 0$ and $\tilde{\beta}_t^* = 0$. □

5.2.2 Non-zero interest rate

The risk-free interest rate does not explicitly appear in a Bachelier's formula, as at his times all payments, including option premium, were done at option's maturity. In his original work, the prices were to be understood as 'true' prices using the terminology from 1900, or 'forward' prices using the modern terminology. Moreover, Bachelier assumed a real-world probability measure coincided with a martingale measure.

To introduce the risk-free interest rate, one possibility is to assume $S_t = e^{rt}(S_0 + \sigma W_t^*)$ (see Musiela and Rutkowski [5.14]). The other possibility is to follow a today's approach to option pricing, which we pursue further.

Consider a discounted stock price process $(S_t/B_t)_{t\in[0,T]}$, where $B_t = e^{rt}$ is a savings account or a bond. An application of Itô formula to $e^{-rt}S_t$ yields:

$$\mathrm{d}(S_t/B_t) = -re^{-rT}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t = e^{-rt}\big((\mu - rS_t)\mathrm{d}t + \sigma\mathrm{d}W_t\big). \qquad (5.2.9)$$

It is apparent from (5.2.9) that for a discounted stock price process to be a martingale the drift of the stock price should be $rS_t dt$ under a martingale measure P^*.

By Girsanov theorem, the process

$$W_t^* = W_t + \int_0^t \frac{\mu - rS_u}{\sigma}\mathrm{d}u$$

is a standard Brownian motion under the unique risk-neutural measure P^* defined by

$$Z_t = \frac{\mathrm{d}P_t^*}{\mathrm{d}P_t} = \exp\bigg(\int_0^t \frac{rS_u - \mu}{\sigma}\mathrm{d}W_u - \frac{1}{2}\int_0^t \frac{(rS_u - \mu)^2}{\sigma^2}\mathrm{d}u\bigg).$$

Thus, after the change of measure we have

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma\mathrm{d}W_t^*. \qquad (5.2.10)$$

The solution of the linear SDE (5.2.10) is given by

$$S_t = S_0e^{rt} + \sigma\int_0^t e^{r(t-u)}\mathrm{d}W_u^*.$$

A price of a call option on this market is given by the formula:

$$C_0^{(r)} = (S_0 - Ke^{-rT})\,\Phi\left(\frac{S_0 - Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\right) + \sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}\,\phi\left(\frac{S_0 - Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\right). \tag{5.2.11}$$

Next, similarly to the Section 5.2.1, we consider a modified Bachelier market with stopping time and find a fair price of a call option on this market. We remark that expression for an option price given in the following theorem is different from the one in Goldenberg [5.9], but naturally, two formulas are equivalent.

In his paper a time change is exploited to arrive at the option pricing formula. We follow an analytical approach instead. No time change is needed in this case. Thus, the approach we take is more intuitive and is convenient to follow as a guideline in pricing different types of the barrier options, as the problem under study can be transformed into the problem of pricing a barrier option.

Theorem 5.2. *A fair price of a European call option with payoff* $(S_{T\wedge\tau} - K)^+$ *assuming* $r > 0$ *is given by*

$$\begin{aligned}
\tilde{C}_0^{(r)} &= (S_0 - Ke^{-rT})\,\Phi\left(\frac{S_0 - Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\right) \\
&+ \sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}\,\phi\left(\frac{S_0 - Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\right) \\
&- e^{-rT} S_0\, r \int_0^T \left(\frac{1}{\sinh(rt)}\right)^{3/2} \exp\left(-\frac{rt}{2}\right)\sqrt{\frac{e^{2r(T-t)}-1}{2}} \\
&\times \phi\left(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\right)\phi\left(\frac{K}{\sigma}\sqrt{\frac{2r}{e^{2r(T-t)}-1}}\right) \mathrm{d}t \\
&+ e^{-rT}\frac{K\,S_0}{\sigma}\int_0^T \left(\frac{r}{\sinh(rt)}\right)^{3/2} \exp\left(-\frac{rt}{2}\right) \\
&\times \phi\left(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\right)\Phi\left(-\frac{K}{\sigma}\sqrt{\frac{2r}{e^{2r(T-t)}-1}}\right) \mathrm{d}t.
\end{aligned} \tag{5.2.12}$$

Proof. The fair option price can be found as the expected discounted payoff of the option:

$$\begin{aligned}\tilde{C}_0^{(r)} &= \mathrm{E}^*\big(e^{-r(T\wedge\tau)}(S_{T\wedge\tau}-K)^+\big) = \mathrm{E}^*\big(e^{-rT}(S_T-K)^+\,\mathbb{1}[\tau>T]\big)\\ &= \mathrm{E}^*\big(e^{-rT}(S_T-K)^+\big) - \mathrm{E}^*\big(e^{-rT}(S_T-K)^+\mathbb{1}[\tau\le T]\big). \qquad (5.2.13)\end{aligned}$$

Note that under P^*,

$$S_T \sim N\Big(S_0\,e^{rT}, \frac{\sigma^2}{2r}(e^{2rT}-1)\Big).$$

Then the first term in (5.2.13) can be found as

$$\begin{aligned}\mathrm{E}^*\big(e^{-rT}(S_T-K)^+\big) &= e^{-rT}\,\mathrm{E}^*\bigg(S_0\,e^{rT}+\sqrt{\frac{\sigma^2}{2r}(e^{2rT}-1)}\;Z-K\bigg)^+\\ &= (S_0-Ke^{-rT})\,\Phi\Bigg(\frac{S_0-Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\Bigg)\\ &\quad+\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}\,\phi\Bigg(\frac{S_0-Ke^{-rT}}{\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}}\Bigg),\end{aligned} \qquad (5.2.14)$$

where Z is a standard normal random variable, and the last equality is obtained using the relationship (see, for example, Shiryaev [5.18])

$$\mathrm{E}(a+bZ)^+ = a\,\Phi\Big(\frac{a}{b}\Big)+b\,\phi\Big(\frac{a}{b}\Big),$$

for $a\in\mathbb{R}$ and $b\ge 0$, by setting $a = S_0\,e^{rT}-K$ and $b=\sqrt{\frac{\sigma^2}{2r}(1-e^{-2rT})}$.

The second term in (5.2.13) can be decomposed as

$$\begin{aligned}\mathrm{E}^*\big(e^{-rT}(S_T-K)^+\,\mathbb{1}[\tau\le T]\big) &= \mathrm{E}^*\big(e^{-rT}(S_T-K)\mathbb{1}[\tau\le T, S_T>K]\big)\\ &= \mathrm{E}^*\big(e^{-rT}S_T\,\mathbb{1}[\tau\le T,S_T>K]\big) - Ke^{-rT}P^*[\tau\le T, S_T>K]. \qquad (5.2.15)\end{aligned}$$

The probability in (5.2.15) can be expressed as

$$P^*(\tau\le T, S_T>K) = \int_K^\infty\int_0^T p_\tau(t)\,f(x,T-t)\,\mathrm{d}t\,\mathrm{d}x, \qquad (5.2.16)$$

where $p_\tau(t)$ is the density of the stopping time (5.2.4);

$$f(x,T-t) = \sqrt{\frac{r}{\pi\sigma^2(e^{2r(T-t)}-1)}}\exp\bigg(-\frac{x^2r}{\sigma^2(e^{2r(T-t)}-1)}\bigg)$$

is a conditional density of the stock price, as follows from the observation that

$$S_T|\{S_t = 0\} \sim N\Big(0, \frac{\sigma^2}{2r}(e^{2r(T-t)} - 1)\Big).$$

To simplify further calculations, let us introduce a new variable: $\dot{S}_t = S_t/\sigma$ for all $t \geq 0$, such that

$$\mathrm{d}\dot{S}_t = r\dot{S}_t\mathrm{d}t + \mathrm{d}W_t^*, \quad \dot{S}_0 = S_0/\sigma. \tag{5.2.17}$$

Then, $\tau = \inf\{t : S_t \leq 0\} = \inf\{t : \dot{S}_t \leq 0\}$.

Next, notice that the process $(\dot{S}_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck (OU) process. The law of hitting time for OU processes such as in (5.2.17) with parameter $-r$ has been studied in a number of papers. The case with parameter r can be derived using absolute continuity relationship between the process $\dot{S}_t$ and Brownian motion W^* started at $\dot{S}_0$.

Denote the law of $(\dot{S}_t)_{t\geq 0}$ starting from $\dot{S}_0$ by $\mathbb{P}^{(r)}_{\dot{S}_0}$, and the law of Brownian motion started at $\dot{S}_0$ by $\mathbb{P}^{(0)}_{\dot{S}_0} = \mathbb{P}_{\dot{S}_0}$. Then absolute continuity relationship (see, for example, Yor [5.22], Chapter 2) is

$$\mathrm{d}\mathbb{P}^{(r)}_{\dot{S}_0|\mathcal{F}_t} = \exp\left(\frac{r}{2}\big((W_t^*)^2 - \dot{S}_0^2 - t\big) - \frac{r^2}{2}\int_0^t (W_s^*)^2\,\mathrm{d}s\right)\mathrm{d}\mathbb{P}_{\dot{S}_0|\mathcal{F}_t}. \tag{5.2.18}$$

From (5.2.18), it can be deduced that

$$\mathrm{d}\mathbb{P}^{(r)}_{\dot{S}_0|\mathcal{F}_t} = \exp\Big(r\big((W_t^*)^2 - \dot{S}_0^2 - t\big)\Big)\mathrm{d}\mathbb{P}^{(-r)}_{\dot{S}_0|\mathcal{F}_t}.$$

The relationship (5.2.18) holds for any stopping time assumed to be finite under both $\mathbb{P}^{(r)}_{\dot{S}_0}$ and $\mathbb{P}_{\dot{S}_0}$, as noted in Göing-Jaeschke and Yor [5.8]. Therefore the following relationship can be established[1]

$$p^{(r)}(t) = \exp(-r(\dot{S}_0^2 + t))\,p^{(-r)}(t),$$

where $p^{(r)}$ and $p^{(-r)}$ denote the densities of the stopping time τ for the OU processes with parameters r and $-r$, respectively, i.e.

$$p^{(\pm r)}(t) = \frac{\mathbb{P}^{(\pm r)}_{\dot{S}_0}(\tau \in \mathrm{d}t)}{\mathrm{d}t},\; t > 0.$$

[1] We point out that this relationship is stated in Alili et al. [5.1], Remark 2.3, with a missing negative sign. Specifically, in their notations, the correct expression is $p^{(\lambda)}_{x\to a}(t) = \exp(-\lambda(a^2 - x^2 - t))p^{(-\lambda)}_{x\to a}(t)$. In our case, $a = 0$ (a barrier), $x = \dot{S}_0$ (initial value), and $p^{(\lambda)}_{x\to a} = p^{(-r)}$.

Finally, the expression for $p^{(-r)}(t)$ for a zero-level barrier[2] is well known and can be found in a number of papers, including Elworthy et al. [5.6], Alili et al. [5.1], Lachaud [5.11]:

$$p^{(-r)}(t) = \frac{|\dot{S}_0|}{\sqrt{2\pi}}\left(\frac{r}{\sinh{(rt)}}\right)^{3/2} \exp\left(- \frac{r\dot{S}_0^2 e^{-rt}}{2\sinh{(rt)}} + \frac{rt}{2}\right). \tag{5.2.19}$$

We remark that, in a general case, we would need to differentiate between the situations when the process (5.2.17) started below or above the hitting barrier. The opposite case could be recovered by replacing the initial value of the process and of a barrier in the density with those of the opposite sign. However, in the case of a zero-level barrier, the densities are identical, as seen from the formula (5.2.19).

It follows that the required first hitting time density, $p_\tau(t) = p^{(r)}(t)$, is given by

$$\begin{aligned} p^{(r)}(t) &= \frac{S_0}{\sigma\sqrt{2\pi}}\left(\frac{r}{\sinh{(rt)}}\right)^{3/2} \exp\left(- \frac{rS_0^2 e^{rt}}{2\sigma^2\sinh(rt)} - \frac{rt}{2}\right) \\ &= \frac{S_0}{\sigma}\left(\frac{r}{\sinh(rt)}\right)^{3/2} \exp\left(- \frac{rt}{2}\right)\phi\left(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\right). \end{aligned}$$

Then (5.2.16) becomes

$$\begin{aligned} P^*(\tau \le T, S_T > K) &= \int_0^T \frac{S_0}{\sigma}\left(\frac{r}{\sinh(rt)}\right)^{3/2} \exp\left(- \frac{rt}{2}\right)\phi\left(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\right) \\ &\times \int_K^\infty \sqrt{\frac{r}{\pi\sigma^2(e^{2r(T-t)}-1)}} \exp\left(- \frac{x^2 r}{\sigma^2(e^{2r(T-t)}-1)}\right) \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \frac{S_0}{\sigma}\left(\frac{r}{\sinh(rt)}\right)^{3/2} \exp\left(- \frac{rt}{2}\right)\phi\left(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\right) \\ &\times \Phi\left(- \frac{K}{\sigma}\sqrt{\frac{2r}{e^{2r(T-t)}-1}}\right)\mathrm{d}t. \end{aligned} \tag{5.2.20}$$

The joint density of τ and S_t is given by

$$g_{\tau,S_t}(t,x) = -\frac{\mathrm{d}P^*(\tau \le t, S_t > x)}{\mathrm{d}t\,\mathrm{d}x} = p_\tau(t) f(x, T-t).$$

[2]For a general level a, the density can be represented by a series expansion in terms of parabolic cylinder functions. See, for example, Alili et al. [5.1] and Novikov et al. [5.15].

Thus, the expectation in (5.2.15) can be found as

$$\begin{aligned}
\mathrm{E}^*\big(e^{-rT}S_T\,\mathbb{1}[\tau\le T, S_T>K]\big) &= \int_0^T\int_K^\infty e^{-rT}x\,p_\tau(t)\,f(x,T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&= e^{-rT}\int_0^T p_\tau(t)\int_K^\infty x\,f(x,T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&= e^{-rT}\int_0^T p_\tau(t)\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}\,\phi\bigg(\frac{K}{\sigma}\sqrt{\frac{2r}{e^{2r(T-t)}-1}}\bigg)\,\mathrm{d}t\\
&= e^{-rT}S_0\,r\int_0^T\bigg(\frac{1}{\sinh(rt)}\bigg)^{3/2}\exp\bigg(-\frac{rt}{2}\bigg)\sqrt{\frac{e^{2r(T-t)}-1}{2}}\\
&\quad\times\phi\bigg(\frac{S_0}{\sigma}\sqrt{\frac{re^{rt}}{\sinh(rt)}}\bigg)\,\phi\bigg(\frac{K}{\sigma}\sqrt{\frac{2r}{e^{2r(T-t)}-1}}\bigg)\,\mathrm{d}t,
\end{aligned}\tag{5.2.21}$$

where $\phi(\cdot)$ denotes a standard normal density, and the third equality follows from the observation that $\phi'(x)=-x\phi(x)$.

Formula (5.2.12) follows from (5.2.13)-(5.2.15) and (5.2.20)-(5.2.21). □

5.2.3 Example: comparison with the Black-Scholes model

Let us recall the Black-Scholes option pricing formula:

$$\begin{aligned}
C_0^{BS} &= S_0\,\Phi\bigg(\frac{\ln(S_0/K)+\big(r+1/2\,(\sigma^{BS})^2\big)\,T}{\sigma^{BS}\sqrt{T}}\bigg)\\
&\quad-Ke^{-rT}\,\Phi\bigg(\frac{\ln(S_0/K)+\big(r-1/2\,(\sigma^{BS})^2\big)\,T}{\sigma^{BS}\sqrt{T}}\bigg).
\end{aligned}\tag{5.2.22}$$

Here, σ^{BS} denotes the volatility in the Black-Scholes model. The rough correspondence with σ in the Bachelier model is $\sigma\approx\sigma^{BS}S_0$.

It is shown by Schachermayer and Teichmann [5.17] that for fixed $\sigma^{BS}>0$, $S_0=K$, and $\sigma=\sigma^{BS}S_0$, Bachelier's option pricing formula (5.2.2) coincides closely with the Black-Scholes formula (5.2.22), assuming $r=0$, when $\sigma^{BS}\sqrt{T}$ is small. However, in the long run, the differences can be spectacular.

In this example we show that under the same assumptions as in Schachermayer and Teichmann [5.17], as σ^{BS} and T increase, the modified Bachelier formulas (5.2.5) and (5.2.12) tend to provide a closer fit to the Black-Scholes formula, compared to the classical Bachelier formulas (5.2.2) and (5.2.11). As we are interested in insurance applications, where time to contract's maturity spans several years, it is important to consider cases with time to maturity larger than a few months, standard in exchange option trading.

Let us first examine the results for the data reported by Bachelier [5.2], assuming zero risk-free interest rate. The relative volatility was of order 2.4% on an annual basis and the time to maturity was of order 1 month or 1/12 years. The difference between Bachelier and Black-Scholes prices, $C_0 - C_0^{BS}$, is $5.527896 \cdot 10^{-9} \cdot S_0$, i.e. the difference is of the order 10^{-9} of the stock price S_0. When using a modified Bachelier formula, we get the same result, i.e. $\tilde{C}_0 - C_0^{BS} = 5.527896 \cdot 10^{-9} \cdot S_0$. Indeed, additional terms in formula (5.2.5) are virtually 0 for such a small value of $\sigma^{BS}\sqrt{T}$ (statistical packages report a straight 0), and, thus, two Bachelier formulas result in the same price.

Next, let us take larger values of volatility and time to maturity: $\sigma^{BS} = 0.3$ and $T = 10$ years. Then, $C_0 - C_0^{BS} = 0.01373 \cdot S_0$, and $\tilde{C}_0 - C_0^{BS} = 0.00773 \cdot S_0$. Here we see that although a difference with the Black-Scholes formula becomes noticeable in both cases as $\sigma^{BS}\sqrt{T}$ increases, this difference is much smaller for a Bachelier model with stopping time.

For a non-zero risk-free interest rate ($r = 2\%$ used in this example), we observe very similar patterns. For small values of volatility and time to maturity, classical and modified Bachelier formulas (5.2.11) and (5.2.12) result in the same option price and the difference with the Black-Scholes price is minimal. As volatility and time to maturity increase, the difference between all three formulas starts to show, but, again, Bachelier model with stopping time produces an option price much closer to that of the Black-Scholes.

These results are summarized in Table 5.1.

Table 5.1: Option prices from Bachelier and Black-Scholes models with $S_0 = K = 100$.

Parameters			Standard Bachelier	Modified Bachelier	Black-Scholes	Difference	Difference
$r, \%$	$\sigma^{BS}, \%$	T, years	(1)	(2)	(3)	(1) − (3)	(2) − (3)
0	2.4	1/12	0.2764	0.2764	0.2764	$5.528 \cdot 10^{-7}$	$5.528 \cdot 10^{-7}$
0	30	10	37.847	37.247	36.474	1.3726	0.7727
2	2.4	1/12	0.3674	0.3674	0.3674	$5.663 \cdot 10^{-7}$	$5.663 \cdot 10^{-7}$
2	30	10	44.181	43.642	42.910	1.2712	0.7318

In the next section we consider a quantile hedging technique and specialize on a case of a zero interest rate, which is a common assumption for a Bachelier model.

5.3 Quantile hedging in standard and modified Bachelier models

Market completeness implies that a contingent claim f_T can be replicated (perfectly hedged) by a unique self-financing trading strategy $\pi_t^* = (\gamma_t^*, \beta_t^*)$. The perfect hedge requires the initial capital of $X_0 = \mathrm{E}^*(f_T)$, assuming zero interest rate. However, a situation of practical interest is when an investor is not able or willing to allocate the capital necessary for a perfect hedge, but still wishes to reduce a risk associated with the liability f_T to a certain degree. In such a case, one may look for a strategy that maximizes the probability that the terminal wealth X_T will exceed the obligation f_T. We will call the set $\{X_T \geq f_T\}$ a *success set*. In other words, we are looking for an admissible strategy such that

$$P(X_T \geq f_T) = \max \tag{5.3.1}$$

under the constraint

$$x_0 < X_0, \tag{5.3.2}$$

where x_0 is an available hedging capital smaller than the capital X_0 needed for a perfect hedge.

The problem (5.3.1)-(5.3.2) is known as a quantile hedging problem. The solution to this problem is given by Föllmer and Leukert [5.7] and formulated in the following theorem, adapted from Melnikov et al. [5.13].

Theorem 5.3. *Assume that*

$$P\Big(\frac{\mathrm{d}P}{\mathrm{d}P^*} = const \cdot f_T\Big) = 0. \tag{5.3.3}$$

Then an optimal strategy π^* for the problem (5.3.1)-(5.3.2) coincides with a perfect hedge for the contingent claim $f_T \, \mathbb{1}[A]$, where the maximal success set A has the following form

$$A = \Big\{w : \frac{\mathrm{d}P}{\mathrm{d}P^*} > const \cdot f_T\Big\},$$

and the constant is determined from

$$\mathrm{E}^*(f_T\,\mathbb{1}[A]) = x_0.$$

Assumption (5.3.3) in the Bachelier model is satisfied by the continuity of the distribution, therefore we will rely on Theorem 5.3 going forward.

Let us express the set A in terms of the stock price.

$$\begin{aligned} A &= \left\{ \exp\left(\frac{\mu}{\sigma}W_T^* - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right) > const \cdot (S_T - K)^+ \right\} \\ &= \left\{ \exp\left(\frac{\mu}{\sigma^2}(S_0 + \sigma W_T^*) - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T - \frac{\mu}{\sigma^2}S_0\right) > const \cdot (S_T - K)^+ \right\} \\ &= \left\{ \exp\left(\frac{\mu}{\sigma^2}S_T\right) > \tilde{C}\,(S_T - K)^+ \right\}, \end{aligned}$$

where a constant $\tilde{C}$ is chosen so that

$$\mathrm{E}^*(f_T\mathbb{1}[A]) = x_0.$$

Thus, the success set A for a standard Bachelier model has the form (see Fig. 5.1)

$$A = \{S_T < g_1\} \cup \{S_T > g_2\}, \tag{5.3.4}$$

where $g_1 = g_1(\tilde{C}) < g_2 = g_2(\tilde{C})$ are two solutions of the equation

$$\exp\left(\frac{\mu}{\sigma^2}x\right) = \tilde{C}\,(x - K)^+. \tag{5.3.5}$$

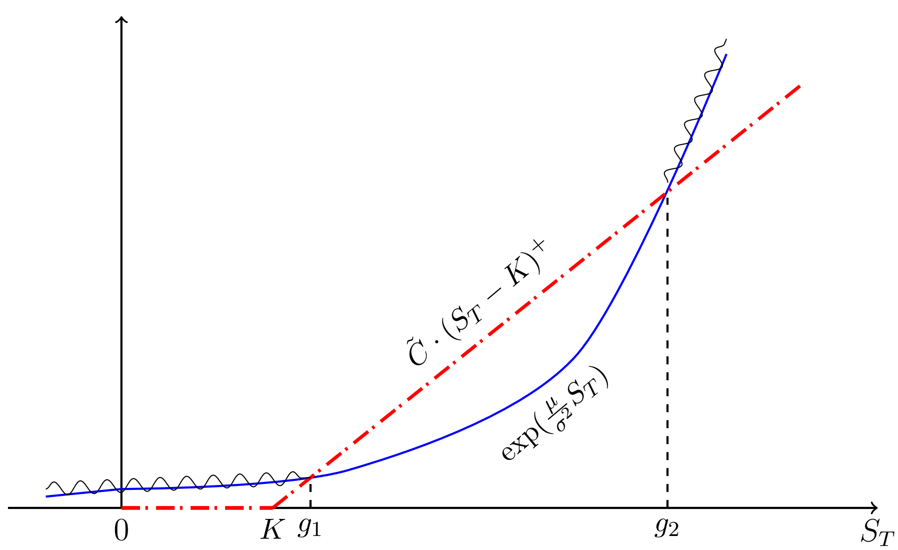

Figure 5.1: Structure of the hedging set.

For a modified Bachelier model with stopping time, success set, denoted by $\tilde{A}$, becomes:

$$\tilde{A} = \begin{cases} A \cap \{S_T > 0\}, \text{ if } \tau > T, \\ \Omega, \text{ if } \tau \le T, \end{cases} \tag{5.3.6}$$

where set A is described by (5.3.4).

The main results regarding a quantile form of hedging in standard and modified Bachelier models are formulated in the following two theorems.

Theorem 5.4. *Consider a European call option with payoff* $(S_T - K)^+$ *on a standard Bachelier market, assuming* $r = 0$. *Under the capital constraint* $x_0 < \mathrm{E}^*(S_T - K)^+$, *the following holds.*

(a) *Quantile price of a call option is given by*

$$\begin{aligned} x_0 =& (S_0 - K)\left[\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right)\right] \\ &+ \sigma\sqrt{T}\left[\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) + \phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right)\right]. \end{aligned} \tag{5.3.7}$$

(b) *The value and the components of the quantile hedging strategy at any time* $t < T$ *are given by*

$$\begin{aligned} X_t^* =& (S_t - K)\left[\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\ &+ \sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right]; \end{aligned} \tag{5.3.8}$$

$$\begin{aligned} \gamma_t^* =& \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) \\ &- \frac{g_1 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \frac{g_2 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right); \end{aligned} \tag{5.3.9}$$

$$\begin{aligned} \beta_t^* =& -K\left[\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\ &+ \sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\ &+ S_t\frac{g_1 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) - S_t\frac{g_2 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right), \end{aligned} \tag{5.3.10}$$

where constants g_1 *and* g_2 *are obtained from* (5.3.5) *and* (5.3.7).

(c) *Maximal probability of successful hedging is given by*

$$P(A) = \Phi\left(\frac{g_1 - S_0 - \mu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-g_2 + S_0 + \mu T}{\sigma\sqrt{T}}\right). \tag{5.3.11}$$

Proof. **(a)** By Theorem 5.3, quantile price is given by $\mathrm{E}^*(f_T\mathbb{1}[A])$ and coincides with available capital x_0.

Note that a modified claim $(S_T - K)^+\,\mathbb{1}[A]$ can be written as a combination of call options and digital options:

$$\begin{aligned}(S_T - K)^+\,\mathbb{1}[A] &= (S_T - K)^+\,\mathbb{1}[S_T < g_1] + (S_T - K)^+\,\mathbb{1}[S_T > g_2] \\ &= (S_T - K)^+ - (S_T - g_1)^+ - (g_1 - K)\,\mathbb{1}[S_T > g_1] + (S_T - g_2)^+ \\ &\quad + (g_2 - K)\,\mathbb{1}[S_T > g_2].\end{aligned} \tag{5.3.12}$$

Then,

$$\begin{aligned}x_0 &= \mathrm{E}^*\big((S_T - K)^+\,\mathbb{1}[A]\big) \\ &= \mathrm{E}^*(S_T - K)^+ - \mathrm{E}^*(S_T - g_1)^+ - (g_1 - K)P^*(S_T > g_1) + \mathrm{E}^*(S_T - g_2)^+ \\ &\quad + (g_2 - K)P^*(S_T > g_2),\end{aligned}$$

and (5.3.7) follows.

(b) By Theorem 5.3, quantile hedging strategy coincides with the perfect hedge for the modified claim $f_T\mathbb{1}[A]$. Thus, the hedging capital of the quantile hedging strategy can be found using (5.3.12) and (5.2.3) as follows:

$$\begin{aligned}X_t^* &= \mathrm{E}^*\big((S_T - K)^+\,\mathbb{1}[A]\,|\mathcal{F}_t\big) \\ &= \mathrm{E}^*\big((S_T - K)^+\,|\mathcal{F}_t\big) - \mathrm{E}^*\big((S_T - g_1)^+\,|\mathcal{F}_t\big) - (g_1 - K)P^*(S_T > g_1\,|\mathcal{F}_t) \\ &\quad + \mathrm{E}^*\big((S_T - g_2)^+\,|\mathcal{F}_t\big) + (g_2 - K)P^*(S_T > g_2\,|\mathcal{F}_t) \\ &= (S_t - K)\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\ &\quad - (S_t - g_1)\,\Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) - \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) \\ &\quad + (S_t - g_2)\,\Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) \\ &\quad - (g_1 - K)\,\Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + (g_2 - K)\,\Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right).\end{aligned} \tag{5.3.13}$$

Eq. (5.3.8) is obtained by rearranging the terms in (5.3.13).

The constants g_1 and g_2 are found from (5.3.5) and (5.3.7).

By the same arguments as in the proof of the Theorem 5.1, part (b), we can deduce that the quantity of the risky asset (5.3.9) needed for hedging can be found by taking a partial derivative of X_t^* with respect to a stock price:

$$\begin{aligned}
\gamma_t^* = \frac{\partial X_t^*}{\partial S_t} &= \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) \\
&+ \frac{S_t - K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\
&- \frac{S_t - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \frac{S_t - g_1}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) \\
&- \frac{S_t - g_2}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) \\
&- \frac{g_1 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \frac{g_2 - K}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

Quantity of the riskless asset (5.3.10) is found from a balance equation

$$\beta_t^* = X_t^* - \gamma_t^* S_t.$$

(c) The maximal success set A has the form (5.3.4). Consequently, maximal probability of successful hedging is given by

$$\begin{aligned}
P(A) &= P(S_T < g_1) + P(S_T > g_2) \\
&= P(S_0 + \mu T + \sigma W_T < g_1) + P(S_0 + \mu T + \sigma W_T > g_2) \\
&= \Phi\left(\frac{g_1 - S_0 - \mu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-g_2 + S_0 + \mu T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

□

Theorem 5.5 describes a quantile hedging strategy for a call option on the Bachelier market with stopping time.

Theorem 5.5. *Consider a European call option with payoff* $(S_{T\wedge\tau} - K)^+$ *on a modified Bachelier market with stopping time, assuming* $r = 0$*. Under the capital constraint* $x_0 < \mathrm{E}^*(S_{T\wedge\tau} - K)^+$*, the following holds.*

(a) *Quantile price of a call option is given by*

$$
\begin{aligned}
x_0 = (S_0 - K)&\left[\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right)\right] \\
&- (S_0 + K)\left[\Phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{-S_0 - g_2}{\sigma\sqrt{T}}\right)\right] \\
&+ \sigma\sqrt{T}\left[\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) + \phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right)\right. \\
&\left. - \phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right) + \phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 + g_2}{\sigma\sqrt{T}}\right)\right].
\end{aligned}
\tag{5.3.14}
$$

(b) *The value and the components of the quantile hedging strategy at any time $t < T$ are given by*

$$
\tilde{X}_t^* = \begin{cases}
(S_t - K)\left[\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\
\quad -(S_t + K)\left[\Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\
\quad +\sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right. \\
\quad \left. -\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, \tau \wedge T), \\
0, \quad t \in [\tau, T);
\end{cases}
\tag{5.3.15}
$$

$$
\tilde{\gamma}_t^* = \begin{cases}
\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) \\
\quad +\Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right) \\
\quad -\frac{g_1 - K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right)\right] \\
\quad +\frac{g_2 - K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, \tau \wedge T), \\
0, \quad t \in [\tau, T);
\end{cases}
\tag{5.3.16}
$$

$$
\tilde{\beta}_t^* = \begin{cases}
-K\left[\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) + \Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right)\right. \\
\quad \left. -\Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\
\quad +\sigma\sqrt{T-t}\left[\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right)\right. \\
\quad \left. -\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right] \\
\quad +S_t\frac{g_1 - K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t - g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right)\right] \\
\quad -S_t\frac{g_2 - K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t - g_2}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, \tau \wedge T), \\
0, \quad t \in [\tau, T),
\end{cases}
\tag{5.3.17}
$$

where constants g_1 *and* g_2 *are obtained from* (5.3.5) *and* (5.3.14).

(c) *Maximal probability of successful hedging is given by*

$$P(\tilde{A}) = \Phi\left(\frac{g_1 - S_0 - \mu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-g_2 + S_0 + \mu T}{\sigma\sqrt{T}}\right) + \exp\Big(-\frac{2S_0\mu}{\sigma^2}\Big)\left[\Phi\left(\frac{-g_1 - S_0 + \mu T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{-g_2 - S_0 + \mu T}{\sigma\sqrt{T}}\right)\right]. \tag{5.3.18}$$

Proof. **(a)** By Theorem 5.3, quantile price is given by $\mathrm{E}^*(S_{T\wedge\tau} - K)^+\,\mathbb{1}[\tilde{A}]$ and coincides with available capital x_0. It follows that

$$\begin{aligned} x_0 &= \mathrm{E}^*\big((S_{T\wedge\tau} - K)^+\,\mathbb{1}[\tilde{A}]\big) = \mathrm{E}^*(S_T - K)^+\,\mathbb{1}[A \cap \{\tau > T\}] \\ &= \mathrm{E}^*(S_T - K)^+\,\mathbb{1}[A] - \mathrm{E}^*(S_T - K)^+\,\mathbb{1}[A \cap \{\tau \le T\}]. \end{aligned} \tag{5.3.19}$$

The first term in (5.3.19) is given by (5.3.7).

The second term in (5.3.19) can be written as

$$\mathrm{E}^*(S_T - K)^+\,\mathbb{1}[A \cap \{\tau \le T\}] = \mathrm{E}^*(S_T - K)\,\mathbb{1}[A \cap \{\tau \le T\} \cap \{S_T > K\}]$$

$$= \mathrm{E}^*(S_T - K)\,\mathbb{1}[K < S_T < g_1]\,\mathbb{1}[\min_{0\le t\le T} S_t \le 0] \tag{5.3.20}$$

$$+ \mathrm{E}^*(S_T - K)\,\mathbb{1}[S_T > g_2]\,\mathbb{1}[\min_{0\le t\le T} S_t \le 0]. \tag{5.3.21}$$

Under P^*, W_t^* is a standard Brownian motion. The joint density of driftless Brownian motion and its minimum (see, for example, Privault [5.16]) is given by

$$f_{W_T^*,\, \min_{0\le t\le T} W_t^*}(x, y) = \mathbb{1}[x \ge y]\,\mathbb{1}[y \le 0]\,\frac{2(x - 2y)}{T\sqrt{2\pi T}}\exp\left\{-\frac{(2y - x)^2}{2T}\right\}. \tag{5.3.22}$$

Using (5.3.22), we can find expectations (5.3.20) and (5.3.21):

$$\begin{aligned} &\mathrm{E}^*(S_T - K)\,\mathbb{1}[K < S_T < g_1]\,\mathbb{1}[\min_{0\le t\le T} S_t \le 0] \\ &\quad = (S_0 + K)\left[\Phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right)\right] \\ &\qquad + \sigma\sqrt{T}\left[\phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right)\right]; \\ &\mathrm{E}^*(S_T - K)\,\mathbb{1}[S_T > g_2]\,\mathbb{1}[\min_{0\le t\le T} S_t \le 0] \\ &\quad = -(S_0 + K)\,\Phi\left(\frac{-S_0 - g_2}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\,\phi\left(\frac{S_0 + g_2}{\sigma\sqrt{T}}\right). \end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{E}^*(S_T-K)^+\,\mathbb{1}[A\cap\{\tau\le T\}]\\
&\quad=(S_0+K)\left[\Phi\left(\frac{S_0+K}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{S_0+g_1}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{-S_0-g_2}{\sigma\sqrt{T}}\right)\right]\\
&\qquad+\sigma\sqrt{T}\left[\phi\left(\frac{S_0+K}{\sigma\sqrt{T}}\right)-\phi\left(\frac{S_0+g_1}{\sigma\sqrt{T}}\right)+\phi\left(\frac{S_0+g_2}{\sigma\sqrt{T}}\right)\right].
\end{aligned} \tag{5.3.23}$$

Finally, quantile price (5.3.14) follows from (5.3.19), (5.3.7), and (5.3.23).

(b) By Theorem 5.3, quantile hedging strategy coincides with the perfect hedge for the modified claim $(S_{T\wedge\tau}-K)^+\,\mathbb{1}[\tilde{A}]$.

Thus, if default has not occurred by time t, the value of the quantile hedging strategy can be computed as

$$\begin{aligned}
\tilde{X}_t^* &= \mathrm{E}^*\big((S_{T\wedge\tau}-K)^+\,\mathbb{1}[\tilde{A}]\,|\mathcal{F}_t\big)=\mathrm{E}^*\big((S_T-K)^+\,\mathbb{1}[A]\,\mathbb{1}[\tau>T]\,|\mathcal{F}_t\big)\\
&=\mathrm{E}^*\big((S_T-K)^+\mathbb{1}[A]\,|\mathcal{F}_t\big)-\mathrm{E}^*\big((S_T-K)^+\,\mathbb{1}[A]\,\mathbb{1}[\tau\le T]\,|\mathcal{F}_t\big).
\end{aligned} \tag{5.3.24}$$

The first term in (5.3.24) is given by (5.3.8). The second term in (5.3.24) can be computed as follows:

$$\begin{aligned}
&\mathrm{E}^*\big((S_T-K)^+\,\mathbb{1}[A]\,\mathbb{1}[\tau\le T]\,|\mathcal{F}_t\big)\\
&\quad=\mathrm{E}^*\Bigg((S_t+\sigma W^*_{T-t}-K)\,\mathbb{1}\left[\frac{K-S_t}{\sigma}<W^*_{T-t}<\frac{g_1-S_t}{\sigma}\right]\\
&\qquad\times\mathbb{1}\left[\min_{0\le u\le T-t}W^*_u\le-\frac{S_t}{\sigma}\right]\Bigg|\mathcal{F}_t\Bigg)\\
&\qquad+\mathrm{E}^*\Bigg((S_t+\sigma W^*_{T-t}-K)\,\mathbb{1}\left[W^*_{T-t}>\frac{g_2-S_t}{\sigma}\right]\\
&\qquad\times\mathbb{1}\left[\min_{0\le u\le T-t}W^*_u\le-\frac{S_t}{\sigma}\right]\Bigg|\mathcal{F}_t\Bigg)\\
&\quad=\int_{(K-S_t)/\sigma}^{(g_1-S_t)/\sigma}\int_{-\infty}^{-S_t/\sigma}(S_t-K+\sigma x)\frac{2(x-2y)}{(T-t)\sqrt{2\pi(T-t)}}\\
&\qquad\times\exp\left(-\frac{(2y-x)^2}{2(T-t)}\right)\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\int_{(g_2-S_t)/\sigma}^{\infty}\int_{-\infty}^{-S_t/\sigma}(S_t-K+\sigma x)\frac{2(x-2y)}{(T-t)\sqrt{2\pi(T-t)}}\\
&\qquad\times\exp\left(-\frac{(2y-x)^2}{2(T-t)}\right)\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= (S_t + K)\left[\Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right)\right] \\
&\quad + \sigma\sqrt{T-t}\left[\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right)\right] \\
&\quad - (S_t + K)\,\Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\,\Phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right).
\end{aligned} \tag{5.3.25}
$$

The capital of the quantile hedging strategy at time $t < T$, if there was no default by time t, is then obtained by subtracting (5.3.25) from (5.3.8). If $\tau < t$, the hedging capital is 0, and the option is hedged with a void strategy.

The constants g_1 and g_2 are found from (5.3.5) and (5.3.14).

Note that a capital $\tilde{X}_t^*$ can be written as

$$
\tilde{X}_t^* = X_t^* - \ddot{X}_t^*,
$$

where X_t^* is given by (5.3.8) and

$$
\begin{aligned}
\ddot{X}_t^* &= (S_t + K)\left[\Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right)\right] \\
&\quad + \sigma\sqrt{T-t}\left[\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right].
\end{aligned}
$$

The quantity of the risky asset, $\tilde{\gamma}_t^*$, if there was no default by time t, can be computed as a partial derivative of $\tilde{X}_t^*$ with respect to S_t (by the same arguments as in the proof of part (b) in Theorem 5.1):

$$
\frac{\partial \tilde{X}_t^*}{\partial S_t} = \frac{\partial X_t^*}{\partial S_t} - \frac{\partial \ddot{X}_t^*}{\partial S_t},
$$

where $\partial X_t^* / \partial S_t$ is given by (5.3.9), and

$$
\begin{aligned}
\frac{\partial \ddot{X}_t^*}{\partial S_t} &= \Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right) \\
&\quad + \frac{S_t + K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right)\right] \\
&\quad - \frac{S_t + K}{\sigma\sqrt{T-t}}\,\phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) + \frac{S_t + g_1}{\sigma\sqrt{T-t}}\,\phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) \\
&\quad - \frac{S_t + g_2}{\sigma\sqrt{T-t}}\,\phi\left(\frac{S_t + g_2}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{S_t + K}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{S_t + g_1}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{-S_t - g_2}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

$$+\frac{g_1-K}{\sigma\sqrt{T-t}}\,\phi\left(\frac{S_t+g_1}{\sigma\sqrt{T-t}}\right)-\frac{g_2-K}{\sigma\sqrt{T-t}}\,\phi\left(\frac{S_t+g_2}{\sigma\sqrt{T-t}}\right).$$

Hence,

$$\begin{aligned}\tilde{\gamma}_t^* = {} & \Phi\left(\frac{S_t-K}{\sigma\sqrt{T-t}}\right)-\Phi\left(\frac{S_t-g_1}{\sigma\sqrt{T-t}}\right)+\Phi\left(\frac{S_t-g_2}{\sigma\sqrt{T-t}}\right)\\ & -\Phi\left(\frac{S_t+K}{\sigma\sqrt{T-t}}\right)+\Phi\left(\frac{S_t+g_1}{\sigma\sqrt{T-t}}\right)+\Phi\left(\frac{-S_t-g_2}{\sigma\sqrt{T-t}}\right)\\ & -\frac{g_1-K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t-g_1}{\sigma\sqrt{T-t}}\right)+\phi\left(\frac{S_t+g_1}{\sigma\sqrt{T-t}}\right)\right]\\ & +\frac{g_2-K}{\sigma\sqrt{T-t}}\left[\phi\left(\frac{S_t-g_2}{\sigma\sqrt{T-t}}\right)+\phi\left(\frac{S_t+g_2}{\sigma\sqrt{T-t}}\right)\right].\end{aligned}$$

The quantity of the riskless asset, $\tilde{\beta}_t^*$ is found from the balance equation:

$$\tilde{\beta}_t^* = \tilde{X}_t^* - \tilde{\gamma}_t^* S_t.$$

(c) The maximal success set $\tilde{A}$ is of the form (5.3.6). Thus, maximal probability of successful hedging is found as follows:

$$\begin{aligned}P(\tilde{A}) &= P(A\cap\{\tau>T\})+P(\Omega\cap\{\tau\le T\})\\ &= P(A\cap \min_{0\le t\le T} S_t>0)+P(\min_{0\le t\le T} S_t\le 0)\\ &= P(S_T<g_1,\min_{0\le t\le T} S_t>0)+P(S_T>g_2,\min_{0\le t\le T} S_t>0)+P(\min_{0\le t\le T} S_t\le 0)\\ &= P(S_T<g_1)-\big(P(\min_{0\le t\le T} S_t\le 0)-P(S_T>g_1,\min_{0\le t\le T} S_t\le 0)\big)\\ &\quad +P(S_T>g_2)-P(S_T>g_2,\min_{0\le t\le T} S_t\le 0)+P(\min_{0\le t\le T} S_t\le 0)\\ &= P(S_T<g_1)+P(S_T>g_2)+P(S_T>g_1,\min_{0\le t\le T} S_t\le 0)\\ &\quad -P(S_T>g_2,\min_{0\le t\le T} S_t\le 0)\\ &= P\left(W_T<\frac{g_1-S_0-\mu T}{\sigma}\right)+P\left(W_T>\frac{g_2-S_0-\mu T}{\sigma}\right)\\ &\quad +P\left(W_T+\frac{\mu}{\sigma}T>\frac{g_1-S_0}{\sigma},\ \min_{0\le t\le T}\left(W_t+\frac{\mu}{\sigma}t\right)\le-\frac{S_0}{\sigma}\right)\\ &\quad -P\left(W_T+\frac{\mu}{\sigma}T>\frac{g_2-S_0}{\sigma},\ \min_{0\le t\le T}\left(W_t+\frac{\mu}{\sigma}t\right)\le-\frac{S_0}{\sigma}\right)\\ &= \Phi\left(\frac{g_1-S_0-\mu T}{\sigma\sqrt{T}}\right)+\Phi\left(\frac{-g_2+S_0+\mu T}{\sigma\sqrt{T}}\right)\\ &\quad +P\left(W_T^*>\frac{g_1-S_0}{\sigma},\ \min_{0\le t\le T} W_t^*\le-\frac{S_0}{\sigma}\right)\end{aligned}$$

$$
\begin{aligned}
&- P\left(W_T^* > \frac{g_2 - S_0}{\sigma},\ \min_{0\le t\le T} W_t^* \le -\frac{S_0}{\sigma}\right)\\
&= \Phi\left(\frac{g_1 - S_0 - \mu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-g_2 + S_0 + \mu T}{\sigma\sqrt{T}}\right)\\
&\quad + \exp\Big(-\frac{2S_0\mu}{\sigma^2}\Big)\left[\Phi\left(\frac{-g_1 - S_0 + \mu T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{-g_2 - S_0 + \mu T}{\sigma\sqrt{T}}\right)\right].
\end{aligned}
$$

To compute the last equality, a joint density of a drifted Brownian motion and its minimum is used (see, for example, Privault [5.16]):

$$
\begin{aligned}
f_{W_T^*, \min_{0\le t\le T} W_t^*}(x,y) &= \mathbb{1}[x \ge y]\, \mathbb{1}[y \le 0]\\
&\times \frac{2(x-2y)}{T\sqrt{2\pi T}} \exp\Big(-\Big(\frac{\mu}{\sigma}\Big)^2\frac{T}{2} + \frac{\mu}{\sigma}x - \frac{(2y-x)^2}{2T}\Big). \quad \square
\end{aligned}
$$

5.4 Application to equity-linked life insurance contracts

We further extend the technique developed in a previous section to the equity-linked life insurance contracts. The payoff in these contracts is linked to both market value of a portfolio and mortality of a client. We assume these two sources of uncertainty – financial risk and mortality risk – are independent of each other, and consider them on a product of probability spaces: $(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$, where the space $(\Omega_1, \mathcal{F}_1, P_1)$ is a probabilistic base for the financial risk, and $(\Omega_2, \mathcal{F}_2, P_2)$ is a probabilistic base for the insurance risk.

We will concentrate on a *pure endowment with fixed guarantee* type of contracts, when the payment is exercised only if the insured is alive at contract's maturity and the payoff represents the maximum of a stock (stock index) price and a fixed (constant) guarantee. Mathematically, the payoff f_T, which represents an obligation for the insurance company, is defined as

$$
f_T = \max(S_T, K)\, \mathbb{1}[T(x) > T],
$$

where $T(x)$ is the remaining lifetime of a client who is currently of age x, K is a guaranteed amount.

The fair premium for such a contract known as a *Brennan-Schwartz* price (Brennan and Schwartz [5.4]) is given by

$$ {}_TU_x = \mathrm{E}^*\big(\max(S_T, K)\big)P_2(T(x) > T) = {}_Tp_x\, K + {}_Tp_x\, \mathrm{E}^*(S_T - K)^+, \quad (5.4.1) $$

where the payoff $\max(S_T, K)$ was decomposed into two components K and $(S_T - K)^+$; the second component, $(S_T - K)^+$, is called an *embedded option*; the quantity ${}_Tp_x = P_2(T(X) > T)$ is called a survival probability of the client.

It is apparent from (5.4.1) that to find a premium for a pure endowment with fixed guarantee life insurance contract, it suffices to find a premium for the second component:

$$ {}_TU_x^C = {}_TU_x - {}_Tp_x\, K = {}_Tp_x\, \mathrm{E}^*(S_T - K)^+. $$

As the survival probability ${}_Tp_x$ is always less than 1, initial capital available for an embedded option is strictly less than the capital required for a perfect hedge, i.e. ${}_TU_x^C < \mathrm{E}^*(S_T - K)^+$. This explains why a quantile methodology can be an effective tool in insurance applications.

By Theorem 5.3, under a budget constraint, a perfect hedge for a modified contingent claim $f_T\,\mathbb{1}[A]$ will maximize the probability of successful hedging for f_T. Hence, for a claim $(S_T - K)^+$ on a standard Bachelier market, we have

$$ {}_TU_x^C = \mathrm{E}^*\big((S_T - K)^+\,\mathbb{1}[A]\big) = {}_Tp_x\, \mathrm{E}^*(S_T - K)^+, \quad (5.4.2) $$

from where the *balance equation* follows

$$ {}_Tp_x = \frac{\mathrm{E}^*\big((S_T - K)^+\,\mathbb{1}[A]\big)}{\mathrm{E}^*(S_T - K)^+}, \quad (5.4.3) $$

with maximal success set A of the form (5.3.4).

Similarly, for a claim $(S_{T\wedge\tau} - K)^+$ on a modified Bachelier market, the balance equation is

$$ {}_Tp_x = \frac{\mathrm{E}^*\big((S_{T\wedge\tau} - K)^+\,\mathbb{1}[\tilde{A}]\big)}{\mathrm{E}^*(S_{T\wedge\tau} - K)^+}, \quad (5.4.4) $$

where maximal success set $\tilde{A}$ is described by (5.3.6).

The balance equation plays a key role in risk management analysis. If survival probability is given, the balance equation can be used to calculate the maximal probability of successful hedging. On the other hand, a company may be willing to accept a certain level of a shortfall risk. In this case, the balance equation can be used to determine the survival probability corresponding to a certain level of risk, which then can be compared with appropriate mortality table to identify the age of the clients suitable for such a contract.

The theoretical results are summarized in the following two theorems.

Theorem 5.6. *Assume a financial market evolves according to a standard Bachelier model. Consider an insurance company that sells a single premium pure endowment with fixed guarantee life insurance contract with payoff* $f_T = \max(S_T, K)$. *The premium of a contract and the survival probability determined from a quantile hedging methodology are as follows:*

$$ {}_TU_x = {}_Tp_x \left[K + (S_0 - K)\,\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\,\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right], \tag{5.4.5} $$

$$ {}_Tp_x = 1 - \frac{(S_0 - K)\big[\Phi\big(\frac{S_0 - g_1}{\sigma\sqrt{T}}\big) - \Phi\big(\frac{S_0 - g_2}{\sigma\sqrt{T}}\big)\big] + \sigma\sqrt{T}\big[\phi\big(\frac{S_0 - g_1}{\sigma\sqrt{T}}\big) - \phi\big(\frac{S_0 - g_2}{\sigma\sqrt{T}}\big)\big]}{(S_0 - K)\,\Phi\big(\frac{S_0 - K}{\sigma\sqrt{T}}\big) + \sigma\sqrt{T}\,\phi\big(\frac{S_0 - K}{\sigma\sqrt{T}}\big)}. \tag{5.4.6} $$

Proof. Expression for a contract's premium (5.4.5) follows directly from plugging Eq. (5.2.2) into Eq. (5.4.1).

The survival probability (5.4.6) is obtained by plugging (5.3.7) and (5.2.2) into the balance equation (5.4.3). □

Theorem 5.7. *Assume a financial market evolves according to a modified Bachelier model with stopping time. Consider an insurance company that sells a single premium pure endowment with fixed guarantee life insurance contract with payoff* $f_T = \max(S_{T\wedge\tau}, K)$. *The premium of a contract and the survival probability determined from a quantile hedging methodology are as follows:*

$$ \begin{aligned} {}_TU_x = {}_Tp_x \Bigg(& K + (S_0 - K)\,\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + (S_0 + K)\,\Phi\left(\frac{-S_0 - K}{\sigma\sqrt{T}}\right) \\ & + \sigma\sqrt{T}\left[\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right)\right]\Bigg), \end{aligned} \tag{5.4.7} $$

$$
\begin{aligned}
{}_T p_x = 1 - \Bigg(&(S_0 - K)\left[\Phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right)\right] \\
&- (S_0 + K)\left[\Phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{S_0 + g_2}{\sigma\sqrt{T}}\right)\right] \\
&- \sigma\sqrt{T}\left[\phi\left(\frac{S_0 + g_1}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 - g_1}{\sigma\sqrt{T}}\right) + \phi\left(\frac{S_0 - g_2}{\sigma\sqrt{T}}\right) \right. \\
&\left. - \phi\left(\frac{S_0 + g_2}{\sigma\sqrt{T}}\right)\right]\Bigg) \\
&\Bigg/ \Bigg((S_0 - K)\,\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + (S_0 + K)\,\Phi\left(\frac{-S_0 - K}{\sigma\sqrt{T}}\right) \\
&+ \sigma\sqrt{T}\left[\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) - \phi\left(\frac{S_0 + K}{\sigma\sqrt{T}}\right)\right]\Bigg).
\end{aligned}
\tag{5.4.8}
$$

Proof. The Brennan-Schwartz price in this case is given by

$$
{}_T U_x = {}_T p_x \, K + {}_T p_x \, \mathrm{E}^*(S_{T\wedge\tau} - K)^+. \tag{5.4.9}
$$

Plugging (5.2.5) into (5.4.9) leads to (5.4.7).

Survival probability (5.4.8) follows from the balance equation (5.4.4), Eq. (5.3.14) and Eq. (5.2.5). □

We note that the optimal hedging strategy and its capital for an embedded call are completely described by equations (5.3.8)-(5.3.10), assuming a standard Bachelier market, and by Eqs. (5.3.15)-(5.3.17), assuming a modified Bachelier market with stopping time.

5.5 Numerical example

Let us suppose the 15-year pure endowment life insurance contract with payoff $\max(S_T, K)$ is offered to the male client who is currently of age 45 as of nearest birthday. Assume $S_0 = K = 100$, $\mu = 4$ and $\sigma = 30$. Then using Valuation Basic Table from the Society of Actuaries [5.19] for male unismoke population, we can find the probability that the client survives to the contract's maturity ${}_{15}p_{45} = 0.9671$. Using this probability, the contract's premium can be computed from (5.4.5) or (5.4.7), depending on the assumptions we make about the development of the financial market. If we assume a standard Bachelier model, the fair premium for a contract is 141.54, including 44.83 premium for an embedded call. Under the assumption of a modified Bachelier model with stopping time, the fair premium for a contract is 139.59, including 42.88 premium for an embedded call. Notice, the premiums are smaller when a possibility of default is present: in a standard Bachelier market, even if the stock price hits 0, it still can bounce back, while in a modified Bachelier market once the stock price hits 0, it stays there forever, thereby lowering the probability of a payout on the embedded option.

To find the probability of successful hedging in each case, we use balance equations (5.4.6) and (5.4.8) along with Eq. (5.3.5) to determine all the necessary constants, and then plug them into formulas (5.3.11) and (5.3.18). For a standard Bachelier model, success probability is 0.9839, and for a modified Bachelier model with stopping time, it is 0.9846.

Alternatively, the insurance company may accept a certain level of a shortfall risk, $\epsilon \in (0,1)$. In this case, we set (5.3.11) and (5.3.18) equal to $1-\epsilon$, i.e. $P(A) = 1-\epsilon$ and $P(\tilde{A}) = 1-\epsilon$, from where all the necessary constants are found. The balance equations (5.4.6) and (5.4.8) are then used to determine a survival probability ${}_Tp_x$, which can be compared with a mortality table to identify the optimal age of the clients that should be targeted for such a contract.

The following table presents the maximum acceptable survival probability of the clients, age of the clients, and the capital available to hedge an embedded call for different levels of a shortfall risk in the two models.

Table 5.2: Survival probability, client age, and available initial capital for different levels of a shortfall risk in standard and modified Bachelier models.

Shortfall risk	Acceptable survival probability of the clients	Acceptable age of the clients	Available initial capital, $
Standard Bachelier model			
0.02	0.9592	≥ 48	44.46
0.04	0.9185	≥ 54	42.57
0.06	0.8778	≥ 58	40.69
0.10	0.7970	≥ 65	36.94
Modified Bachelier model			
0.02	0.9573	≥ 48	42.44
0.04	0.9147	≥ 55	40.56
0.06	0.8723	≥ 59	38.67
0.10	0.7878	≥ 66	34.93

It is easily observable from Table 5.2 that, irrespective of the model, as the shortfall probability increases, the company needs to control the insurance component of risk by offering contracts to older groups of clients. Naturally, the higher the acceptable level of risk, the lower the premium for an embedded call.

When comparing two models with each other, we observe that given the same level of a shortfall probability, the capital available for an embedded call (and for the insurance contract) is smaller for a model with stopping time. In other words, with a smaller capital, the same probability of successful hedging can be achieved if we assume the stock cannot bounce back once its price reaches 0. This assumption is very natural to make and we believe this model is more representative of the reality.

5.6 Concluding remarks

In this work we revisit a modification of a classical Bachelier model that guarantees non-negativity of option prices and allows to adequately reflect a real life. Alternative proofs for getting the option prices under both classical and modified Bachelier models are provided. For a non-zero interest rate case, we analytically obtain an expression for an option price. With obtained expressions, we can demonstrate that the option price from the Bachelier model with stopping time is much closer aligned with the Black-Scholes price, compared to the original model, when volatility and time to expiration increase. Previous empirical studies show that in certain situations the classical Bachelier model may outperform the Black-Scholes model in the sense that it better fits actual market data. It would be interesting to compare the accuracy of a Bachelier model with absorbing barrier and the Black-Scholes model, using actual market data, as a future research.

We also develop the quantile methodology for both classical and modified Bachelier markets and provide an application to pricing of the equity-linked life insurance contracts, assuming a zero interest rate (which is a common assumption for the Bachelier model). For a non-zero interest rate, the maximal success set has a rather complicated structure, making it impossible to use Föllmer and Leukert [5.7] approach directly. A practical solution could be to construct another (simpler) type of set and obtain an estimate from below for the probability of success, as was done in Bratyk and Mishura [5.3] to overcome a problem with overly complicated structure of the success set for the case of fractional Brownian motion. Our model, however, would require developing a totally different procedure to reach the same goal. We, therefore, leave it for a future research.

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Concluding remarks and directions for future research

The results obtained in the course of the present research extend and deepen our understanding of the various aspects related to the valuation and hedging of the financial and insurance contracts. These results also lead to new questions and bring up possible directions of future work.

First of all, we studied the Black-Scholes market and presented a new derivation of the Black-Scholes formula from its binomial counterpart by means of Bernstein's inequalities [4] as well as Zubkov and Serov inequalities [70]. The convergence rate stated in the proof was of order $1/\sqrt{n}$ which is similar to that obtained by means of the central limit theorem. Nevertheless, there is hope that our approach contains a reserve to improve convergence. It would be an interesting direction of future research to develop a proposed technique for obtaining a faster convergence rate.

Secondly, we studied a quadratic risk-minimization approach in the framework of discrete-time financial market (focusing on a binomial case) and provided a pricing and hedging mechanism by finding a discounting portfolio such that asset price processes are martingales under the real-world probability measure. As this approach does not require a change of measure, it is more intuitive in this sense. Adapting this methodology to a continuous-time case could be a next step, especially in the context of life insurance applications, similarly to how it was done in discrete-time setting.

Further, we focused on a quantile hedging methodology and extended existing results by introducing dividends, assuming the Black-Scholes and jump-diffusion models of the financial market. Dividends are often not taken into account, yet they are integral part of the financial market. Future research may consider models with different types of dividends, including a stochastic dividend case.

The last two research topics dealt with defaultable markets in quantile hedging context. One of them addressed a financial market with two defaultable securities, where the default is modeled as an exogenous event assuming a constant hazard rate. These results could be generalized to a case of non-constant hazard rate.

The other defaultable market case arose from a modification of a classical Bachelier model by letting the stock price absorb at zero to guarantee non-negativity of the stock price at all times. We provided alternative proofs of option pricing formulas for a modified Bachelier market and developed quantile hedging methodology for the case of a zero interest rate. The case of a non-zero interest rate requires a different approach due to a rather complicated structure of a success set and represents an exciting direction of future research.

It may be of interest to extend the results obtained in a quantile hedging setting (Chapters 3-5) to an efficient hedging setting and provide insurance applications. As pertains to insurance applications, our focus was on pure endowment with fixed guarantee life insurance contracts (Chapters 2-5). Naturally, other types of insurance products (e.g. with flexible guarantee) may be studied, as a part of future research.

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