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A Graph-Theoretic Approach to the Construction of Lyapunov Functions for Coupled Systems on Networks

by

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To my parents, wife, and daughters

ABSTRACT

For coupled systems of differential equations on networks, a graph-theoretic approach to the construction of Lyapunov functions is systematically developed in this thesis. Kirchhoff's Matrix-Tree Theorem in graph theory plays an essential role in the approach's development. The approach is successfully applied to several coupled systems well-known in the literature to demonstrate its applicability and effectiveness.

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Chapter 1. Introduction

The method of Lyapunov functions or the directed method of Lyapunov, named after Russian mathematician A. M. Lyapunov [85], is a standard tool in the stability theory of nonlinear differential equations. A difficulty in applying the method of Lyapunov functions is the ad hoc nature of the construction of a suitable Lyapunov function for the investigation; there is no general principle to guide such construction. In this thesis, for a large class of coupled systems of differential equations, we are able to develop a systematic approach that guides the construction of Lyapunov functions.

We start with a brief introduction to the method of Lyapunov functions. Let D be an open set in $\mathbb{R}^m$ and $t_0 \in \mathbb{R}$. Consider a nonautonomous system

$$x' = f(t, x), \tag{1.1}$$

where $f : [t_0, \infty) \times D \to \mathbb{R}^m$ is continuous. In the direct method of Lyapunov, real-valued functions V with the same domain as f, $V : [t_0, \infty) \times D \to \mathbb{R}$, are considered. From properties such as the sign of these functions and knowledge of the manner in which they evolve along solutions of (1.1), inferences are drawn about the qualitative behavior of the solutions. If $x(t)$ satisfies (1.1) and $v(t) = V(t, x(t))$ is continuously differentiable and satisfies, for all t in its domain, a differential inequality of the form

$$v'(t) \leq F(t, v(t)), \tag{1.2}$$

with F continuous, then $v(t) \leq u^*(t)$, where $u^*(t)$ is the maximal solution of the differential equation $u' = F(t, u)$, for $t \geq t_0$ as long as both $x(t)$ and $u^*(t)$ exist. An example of this is found when x is a state variable associated with a physical system whose behavior in time t is governed by a differential equation of the form (1.1) and $V(t, x)$ is the energy in the state x at time t. Then $V(t, x(t)) \leq V(t_0, x(t_0)), t \geq t_0$ as long as $x(t)$ exists if it is known that energy dissipates so that $V(t, x(t))$ is non-increasing. In this example $F(t, v) = 0$ and $u^*(t) = u^*(t_0) = V(t_0, x(t_0))$. In particular, the system never achieves a state in which the energy is greater than that in the initial configuration.

We assume that a unique solution $x(t; t_0, x_0)$ of (1.1) through (t_0, x_0) exists for all $t \geq t_0$ and $x_0 \in D$. If $V : [t_0, \infty) \times D \to \mathbb{R}$ is continuously differentiable on its domain and define the derivative $\overset{\bullet}{V}$ along the solutions of (1.1) as

$$\overset{\bullet}{V}(t, x) := \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x), \tag{1.3}$$

then, with $v(t) = V(t, x(t))$,

$$v'(t) = \overset{\bullet}{V}(t, x(t)) \tag{1.4}$$

when $x(t)$ is a solution of (1.1).

The term *Lyapunov function* is not used with great consistency in the standard texts in differential equations. For instance the broadest use is in [61, page 80 et seq] where it denotes any real-valued function V with the same domain as f and enough smoothness to allow differentiation along solutions. For a given choice of V, a function F is sought such that

$$\overset{\bullet}{V}(t, x) \leq F(t, V(t, x))$$

for all (t, x) and thus an inequality (1.2) is satisfied by $v(t) = V(t, x(t))$. Depending on the qualitative property being investigated, various restrictions on V and F are required. This approach is then used to investigate questions such as global existence, boundedness, and stability. For example, for stability of an equilibrium, various sign requirements on V and $F(t, v)$ near equilibrium are imposed. Lyapunov functions that are not necessarily sign definite may be used to establish instability of an equilibrium by a method of Chetaev [21] (or see Theorem 5.1 in [39]). Necessary and sufficient conditions for the existence of a nontrivial exponential dichotomy for linear systems are expressed in terms of Lyapunov functions which are specifically not sign definite, see [26, Chapter 7]. This approach may also be used to establish hyperbolic behavior near an equilibrium of a nonlinear system. Consistent with the usage of [104, page 77] and [100, page 274], the term will be used throughout the thesis as in the following definition.

Definition 1.1. A continuously differentiable function function $V : [t_0, \infty) \times D \to \mathbb{R}$ is a *Lyapunov function* for (1.1) if $\overset{\bullet}{V}(t, x) \leq 0$ for all $(t, x) \in [t_0, \infty) \times D$.

The main objective of the thesis is to investigate the construction of Lyapunov functions for the following class of coupled systems of differential equations

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \qquad i = 1, 2, \ldots, n. \tag{1.5}$$

Here $u_i \in \mathbb{R}^{m_i}$, $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, and $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$. The class of coupled system (1.5) is an abstraction of a wide variety of physical, natural, and artificial complex dynamical systems: from biological and artificial neural networks [3, 17, 23, 55], coupled systems of nonlinear oscillators on lattices [8, 22], to complex ecosystems [92, 107] and the transmission models of infectious diseases in heterogeneous populations [18, 112]. We look for Lyapunov functions in the form

$$V(t, u_1, \cdots, u_n) = \sum_{i=1}^{n} c_i V_i(t, u_i), \tag{1.6}$$

with constants $c_i \geq 0$ and functions $V_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}$. In practice, functions

V_i are commonly chosen as Lyapunov functions for the uncoupled system

$$u_i' = f_i(t, u_i), \qquad 1 \le i \le n. \tag{1.7}$$

The uncoupled system (1.7) is often of low dimension whose Lyapunov functions V_i are relatively easy to construct. The goal of our investigation is to select coefficients c_i so that V becomes a Lyapunov function for the coupled system (1.5). Due to the complexity and large scale of system (1.5), this is a very challenging task.

In the thesis, utilizing results from graph theory, we are able to develop a uniform and systematic approach of selecting coefficients c_i. The determination of c_i in our approach is given explicitly in terms of the coupling structure of system (1.5). Our approach is sufficiently general that it is applicable to systems with arbitrary coupling structure.

We have chosen several well-known classes of mathematical models in Chapters 3-5 to demonstrate the applicability and effectiveness of our graph-theoretic approach. These examples are chosen from different areas of science and engineering and include coupled mechanical or electrical oscillators, spatial ecological models of interacting species, and models of infectious diseases in heterogeneous populations. These models represent a variety of differential equations: ordinary differential equations, differential equations with time delays, and stochastic differential equations. These examples also include several different types of coupling structure: physical coupling in mechanical and electrical engineering, spatial interaction through species dispersal in ecology, and nonlinear coupling through cross-infection in the spread and transmission of infectious diseases. Our approach applies to both coupling with instantaneous connections and those with time-delayed connections. Our approach also works for different types of Lyapunov functions.

For all of the examples in Chapters 3-5, our graph-theoretic approach allows us to significantly improve the best known results in the literature. In particular, our global-stability result for a multi-group epidemic model (Theorem 3.9) contains a complete resolution of a 30-year old open problem in mathematical epidemiology.

Chapter 2. A Graph-Theoretic Approach to the Construction of Lyapunov Functions

In this chapter we develop a general and systematic approach to the construction of Lyapunov functions for coupled systems on networks. Concepts from graph theory related to our development are reviewed in Section 2.1. In Section 2.2 we apply Kirchhoff's Matrix-Tree Theorem to prove several combinatorial identities, which will be used in our development. The mathematical framework of coupled systems on networks along with several examples is given in Section 2.3. The graph-theoretic approach, the key result of my thesis, is systematically developed in Section 2.4. We apply the approach to establish several stability results in Section 2.5. Related discussion are given in Section 2.6.

2.1 Definitions and Notations from Graph Theory

A *directed graph* or *digraph* $\mathcal{G} = (V, E)$ is a pair of two sets: a set $V = \{1, 2, \ldots, n\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be *spanning* if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph and a spanning subgraph are depicted in Figure 2.1. A digraph $\mathcal{G}$ is *weighted* if each arc (j, i) is assigned a positive weight a_{ij}. In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$. The weight $w(\mathcal{H})$ of a subgraph $\mathcal{H}$ is the product of the weights on all its arcs.

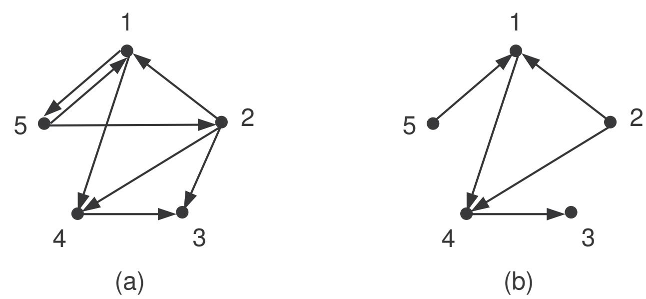

Figure 2.1: (a) A digraph $\mathcal{G}$. (b) A spanning subgraph of $\mathcal{G}$

A *directed path* $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \cdots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, m-1\}$. If $i_m = i_1$, we call $\mathcal{P}$ a *directed cycle*. A subgraph $\mathcal{T}$ in $\mathcal{G}$ is a *rooted tree* if it contains no directed cycles, and there is one vertex called the root that is not a terminal vertex of any arcs while each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph $\mathcal{Q}$ is *unicyclic* if it contains one directed cycle and every vertex of $\mathcal{Q}$ is a terminal vertex of exactly one arc. A unicyclic graph has also been called a contra-functional digraph [44, page 201]. A rooted tree and a unicyclic graph are depicted in Figure 2.2. We refer the reader to [44, 120] for general theory on graphs.

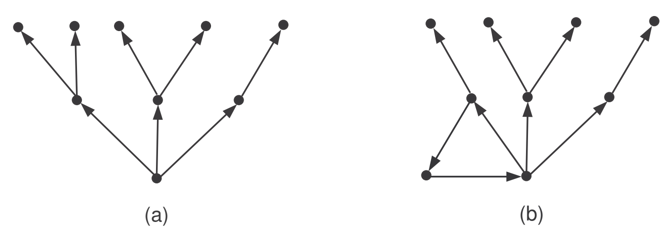

Figure 2.2: (a) A rooted tree. (b) A unicyclic graph

2.2 Matrix-Tree Theorem and Combinatorial Identities

Given a weighted digraph $\mathcal{G}$ with n vertices, define the weight matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where a_{ij} is defined as the weight of arc (j, i) if it exists and 0 otherwise. For our purpose, we denote a weighted digraph as $(\mathcal{G}, A)$. A digraph $\mathcal{G}$ is *strongly connected* if, for any pair of distinct vertices, there exists a directed path from each vertex to the other. A *nonnegative matrix* is a matrix in which all the elements are nonnegative. A nonnegative matrix A is *reducible* if, for some permutation matrix P,

$$PAP^T = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix},$$

and A_1, A_3 are square matrices. Otherwise, A is *irreducible*. A weighted digraph $(\mathcal{G}, A)$ is strongly connected if and only if the weight matrix A is irreducible [15]. The *Laplacian matrix* of $(\mathcal{G}, A)$ is defined as

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}. \tag{2.1}$$

Let c_i denote the cofactor of the i-th diagonal element of L. The following result is standard in graph theory, and customarily called Kirchhoff's Matrix-Tree Theorem. We refer the reader to [64, 95] for its proof.

Proposition 2.1 (Kirchhoff's Matrix-Tree Theorem)**.** *Assume $n \geq 2$. Then*

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n, \tag{2.2}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex i, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for all $1 \leq i \leq n$.

Using Proposition 2.1 we can prove the following combinatorial identity, which is the key step in the development of graph-theoretic approach in Section 2.4.

Theorem 2.2. *Let $(\mathcal{G}, A)$ be a weighted digraph with $n \geq 2$ vertices, and c_i the cofactor of the i-th diagonal element of the associated Laplacian matrix* (2.1)*. Then the following identity holds for arbitrary functions $F_{ij} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}$ and all $x_i \in \mathbb{R}^{m_i}$, $1 \leq i, j \leq n$:*

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s). \tag{2.3}$$

Here $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, $\mathcal{C}_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$, and $E(\mathcal{C}_{\mathcal{Q}})$ is the arc set of $\mathcal{C}_{\mathcal{Q}}$.

Proof. For every spanning tree $\mathcal{T}$ rooted at vertex i,

$$w(\mathcal{T}) \, a_{ij} = w(\mathcal{Q}),$$

where $\mathcal{Q}$ is the unicyclic graph obtained from $\mathcal{T}$ by adding an arc (j, i) from vertex j to the root vertex i, see Figure 2.3. As a consequence,

$$w(\mathcal{T}) \, a_{ij} \, F_{ij}(x_i, x_j) = w(\mathcal{Q}) \, F_{ij}(x_i, x_j), \quad \text{and} \quad (j, i) \in E(\mathcal{C}_{\mathcal{Q}}).$$

When we perform this operation in all possible ways to all rooted trees in $\mathcal{G}$, we obtain all unicyclic graphs in $\mathcal{G}$, and each unicyclic graph $\mathcal{Q}$ is created as

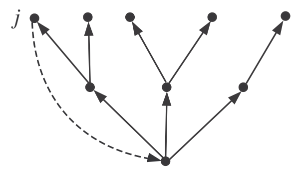

Figure 2.3: A unicyclic graph is formed by adding a directed arc (j, i) to a tree rooted at i.

many times as the number of arcs in its cycle $\mathcal{C}_{\mathcal{Q}}$ (see Theorem 16.5 in [44, page 201]). The identity (2.3) follows from (2.2) if we reorganize the double sum on the left hand side as a sum over all unicyclic graphs in $\mathcal{G}$. □

Corollary 2.3. *Let* $(\mathcal{G}, A)$ *and* c_i *be as in Theorem 2.2. Then the following identity holds for arbitrary functions* $G_i : \mathbb{R}^{m_i} \to \mathbb{R}$ *and all* $x_i \in \mathbb{R}^{m_i}$, $1 \leq i \leq n$*:*

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, G_i(x_i) = \sum_{i,j=1}^{n} c_i \, a_{ij} \, G_j(x_j). \tag{2.4}$$

Proof. Using Theorem 2.2, we know that both sides of (2.4) are equal to

$$\sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{k \in V(\mathcal{C}_{\mathcal{Q}})} G_k(x_k),$$

where $V(\mathcal{C}_{\mathcal{Q}})$ is the vertex set of $\mathcal{C}_{\mathcal{Q}}$. □

A weighted digraph $(\mathcal{G}, A)$ is said to be *balanced* if for any directed cycle $\mathcal{C}$ in $\mathcal{G}$, the reverse $-\mathcal{C}$ is also in $\mathcal{G}$, and $w(\mathcal{C}) = w(-\mathcal{C})$ (see [99]). Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. If the weight matrix A is symmetric, then $(\mathcal{G}, A)$ is balanced. However, the weight matrix of a balanced digraph is not necessarily symmetric. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that $(\mathcal{G}, A)$ is balanced. Then $w(\mathcal{Q}) = w(\tilde{\mathcal{Q}})$.

In the right hand side of identity (2.3), we can further pair $\mathcal{Q}$ with $\tilde{\mathcal{Q}}$ and obtain

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \frac{1}{2} \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} [F_{rs}(x_r, x_s) + F_{sr}(x_s, x_r)], \quad (2.5)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_n}$. We thus have the following result.

Theorem 2.4. *Assume* $(\mathcal{G}, A)$ *is balanced and* $n \geq 2$. *Let* c_i *be as in Theorem 2.2. Then identity* (2.5) *holds for arbitrary functions* $F_{ij} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}, 1 \leq i, j \leq n$.

Using Theorem 2.4 and the same proof as for Corollary 2.3, we obtain the following result.

Corollary 2.5. *Assume* $(\mathcal{G}, A)$ *is balanced and* $n \geq 2$. *Let* c_i *be as in Theorem 2.2. Then the following identity holds for arbitrary functions* $F_{ij} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}$ *and all* $x_i \in \mathbb{R}^{m_i}$, $1 \leq i, j \leq n$*:*

$$\sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ij}(x_i, x_j) = \sum_{i,j=1}^{n} c_i \, a_{ij} \, F_{ji}(x_j, x_i). \quad (2.6)$$

2.3 Coupled Systems on Networks

Given a network represented by digraph $\mathcal{G}$ with n vertices, $n \geq 2$, a coupled system can be built on $\mathcal{G}$ by assigning each vertex its own internal dynamics and then coupling these vertex dynamics based on directed arcs in $\mathcal{G}$. Assume that each vertex dynamics is described by a system of differential equations

$$u_i' = f_i(t, u_i), \quad (2.7)$$

where $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$. Let $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$ represent the influence of u_j from vertex j on u_i from vertex i, and $g_{ij} \equiv 0$ if there exists no arc from j to i in $\mathcal{G}$. See Figure 2.4. Then we obtain the following coupled system on digraph $\mathcal{G}$

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \qquad i = 1, 2, \ldots, n. \tag{2.8}$$

Many large-scale dynamical systems from science and engineering can be represented as coupled systems on networks in the form of (2.8). Several examples are illustrated below. More examples are considered in Chapters 3-5.

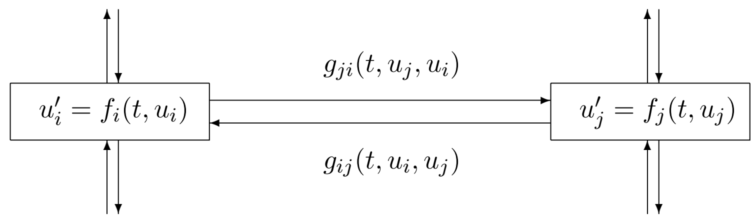

Figure 2.4: A coupled system on a network

Example 1 (Coupled Oscillators). A coupled system of nonlinear oscillators on $\mathcal{G}$ can be built as follows: each vertex i is assigned a nonlinear oscillator described by

$$x_i'' + \alpha_i x_i' + f_i(x_i) = 0, \tag{2.9}$$

where $\alpha_i \geq 0$ is the damping coefficient, $f_i : \mathbb{R} \to \mathbb{R}$ is the nonlinear restoring force, and the influence from vertex j to vertex i is provided in the form $a_{ij}(x_i' - x_j')$ [29, 42]. Here weight constants $a_{ij} \geq 0$, and $a_{ij} = 0$ if and only if no arc exists from j to i in $\mathcal{G}$. See Figure 2.5. We arrive at a coupled system

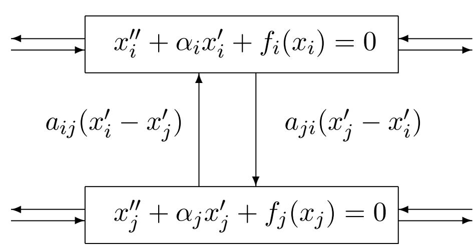

Figure 2.5: Coupled oscillators on a network

of second order differential equations on $\mathcal{G}$

$$x_i'' + \alpha_i x_i' + f_i(x_i) + \sum_{j=1}^{n} a_{ij}(x_i' - x_j') = 0, \qquad i = 1, 2, \ldots, n,$$

or in the form of first order systems

$$\begin{aligned} x_i{}' &= y_i, \\ y_i{}' &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^{n} a_{ij}(y_i - y_j). \end{aligned} \tag{2.10}$$

We will study the global dynamics of (2.10) in Section 3.1.

Example 2 (System Coupled via Dispersal). Assume that the vertex dynamics at each vertex is described by an m-dimensional differential equation

$$u_i' = f_i(t, u_i), \qquad u_i \in \mathbb{R}^m, \quad i = 1, 2, \ldots, n. \tag{2.11}$$

Let $K = \text{diag}\{k_1, k_2, \cdots, k_m\}$ be a diagonal matrix with $k_i \geq 0$ for all $1 \leq i \leq m$. A class of coupled systems of differential equations (2.11) on $\mathcal{G}$ can be given as

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} a_{ij} K(u_j - u_i), \qquad i = 1, 2, \ldots, n. \tag{2.12}$$

The underlying network is described in Figure 2.6. Several mathematical models in the form of (2.12) are investigated in Chapters 3-5.

2.4 Construction of Lyapunov Functions for Coupled Systems

In this section we develop a systematic approach to the construction of Lyapunov function for coupled systems on networks. Our approach allows us to

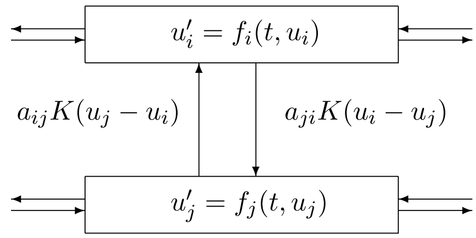

Figure 2.6: System coupled via dispersal

resolve global-stability problems (Chapters 3-5) for many complex systems including coupled oscillators (2.10) and dispersal coupled systems in the form of (2.12).

Consider a coupled system on a digraph $\mathcal{G}$

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \qquad i = 1, 2, \ldots, n. \tag{2.13}$$

The vertex systems after removing all couplings are given as

$$u_i' = f_i(t, u_i), \qquad i = 1, 2, \ldots, n. \tag{2.14}$$

Our objective is to investigate if a Lyapunov function V can be constructed for system (2.13). Such an investigation is significant for the stability and control of large-scale dynamical systems.

Let $U_i \subset \mathbb{R}^{m_i}$ be an open set. For each continuously differentiable function $V_i : \mathbb{R} \times U_i \to \mathbb{R}, 1 \le i \le n$, the derivative $\overset{\bullet}{V}_i$ along the solutions of (2.13), as defined in (1.3), is given as follows

$$\overset{\bullet}{V}_i(t, u_i) = \frac{\partial V_i(t, u_i)}{\partial t} + \frac{\partial V_i(t, u_i)}{\partial u_i}\Big(f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j)\Big). \tag{2.15}$$

In practice, V_i is often chosen as a Lyapunov function for each vertex system (2.14). Let $U = U_1 \times U_2 \times \cdots \times U_n \subset \mathbb{R}^m, m = m_1 + m_2 + \cdots + m_n$, and

$u = (u_1, u_2, \cdots, u_n)$. For a continuously differentiable function $V : \mathbb{R} \times U \to \mathbb{R}$, the derivative $\overset{\bullet}{V}$ along the solutions of (2.13) is given as follows

$$\overset{\bullet}{V}(t,u) = \frac{\partial V(t,u)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t,u)}{\partial u_i} \Big(f_i(t,u_i) + \sum_{j=1}^{n} g_{ij}(t,u_i,u_j) \Big). \tag{2.16}$$

We are particularly interested in constructing Lyapunov functions V for coupled system (2.13) of the form

$$V(t,u) = \sum_{i=1}^{n} c_i V_i(t,u_i), \tag{2.17}$$

where $c_i \geq 0$ are constants. The following result gives a general and systematic approach for selecting suitable coefficients c_i such that V as defined in (2.17) is a Lyapunov function for (2.13).

Theorem 2.6. *Suppose that the following assumptions are satisfied.*

(1) *There exist functions $V_i : \mathbb{R} \times U_i \to \mathbb{R}$, $F_{ij} : \mathbb{R} \times U_i \times U_j \to \mathbb{R}$, and constants $a_{ij} \geq 0$ such that, for every $1 \leq i \leq n$,*

$$\overset{\bullet}{V}_i(t,u_i) \leq \sum_{j=1}^{n} a_{ij} F_{ij}(t,u_i,u_j), \quad \forall\, t > 0,\ u = (u_1, \ldots, u_n) \in U. \tag{2.18}$$

(2) *For each directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G}, A)$, $A = (a_{ij})$,*

$$\sum_{(s,r) \in E(\mathcal{C})} F_{rs}(t,u_r,u_s) \leq 0, \quad \forall\, t > 0,\ u = (u_1, \ldots, u_n) \in U. \tag{2.19}$$

(3) *The weighed digraph $(\mathcal{G}, A)$ is strongly connected.*

Then, the constants c_i given by Proposition 2.1 satisfy $c_i > 0$ for all i and the function

$$V(t,u) = \sum_{i=1}^{n} c_i V_i(t,u_i), \tag{2.20}$$

satisfies $\overset{\bullet}{V}(t,u) \le 0$ *for all* $t > 0$ *and* $u \in U$*, i.e.,* V *is a Lyapunov function for* (2.13).

Proof. Using (2.15), (2.16), and assumption (1), we obtain

$$\overset{\bullet}{V}(t,u) = \sum_{i=1}^{n} c_i \overset{\bullet}{V}_i(t,u_i) \le \sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(t,u_i,u_j).$$

Applying Theorem 2.2 to the weighted digraph $(\mathcal{G}, A)$, we obtain

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(t,u_i,u_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})} F_{rs}(t,u_r,u_s). \tag{2.21}$$

Since $w(\mathcal{Q}) > 0$ and

$$\sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})} F_{rs}(t,u_r,u_s) \le 0,$$

by assumption (2), we arrive at $\overset{\bullet}{V}(t,u) \le 0$, completing the proof of Theorem 2.6. □

If the underlying network $(\mathcal{G}, A)$ has special properties, then Theorem 2.6 holds under a weaker assumption than (2). Suppose that $(\mathcal{G}, A)$ is balanced, as defined in Section 2.2. Using Theorem 2.4, we can further organize the terms on the right hand side of (2.21) and obtain

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(t,u_i,u_j) = \frac{1}{2}\sum_{\mathcal{Q}\in\mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})} [F_{rs}(t,u_r,u_s) + F_{sr}(t,u_s,u_r)]. \tag{2.22}$$

The same proof shows that the conclusion of Theorem 2.6 holds if the assumption (2) is replaced by the following.

(2') Along each directed cycle $\mathcal{C}$

$$\sum_{(s,r)\in E(\mathcal{C})} \Big(F_{rs}(t,u_r,u_s) + F_{sr}(t,u_s,u_r)\Big) \le 0, \quad \forall\, t > 0,\ u \in U. \tag{2.23}$$

We thus have the following result.

Theorem 2.7. *Suppose that $(\mathcal{G}, A)$ is balanced. Then the conclusion of Theorem 2.6 holds if condition* (2.19) *is replaced by* (2.23).

Conditions (2.19) of Theorem 2.6 and (2.23) of Theorem 2.7 can be readily verified if there exist functions $G_i : \mathbb{R} \times U_i \to \mathbb{R}, i = 1, 2, \ldots, n$, such that

$$F_{ij}(t, u_i, u_j) \leq G_i(t, u_i) - G_j(t, u_j), \quad \forall\, 1 \leq i, j \leq n, t > 0, u \in U. \qquad (2.24)$$

Hence the following corollary holds.

Corollary 2.8. *The conclusion of Theorem 2.6 and Theorem 2.7 holds if condition* (2.19) *and* (2.23), *respectively, are replaced by* (2.24).

If V_i satisfies a more restrictive condition

$$\overset{\bullet}{V}_i(t, u_i) \leq -b_i\, V_i(t, u_i) + \sum_{j=1}^{n} a_{ij} F_{ij}(t, u_i, u_j), \quad \forall\, t > 0,\ u \in U,\ 1 \leq i \leq n, \qquad (2.25)$$

for constants $b_i \geq 0$, then a stronger conclusion can be drawn for V. The following result can be proved the same way as Theorem 2.6 and Theorem 2.7.

Theorem 2.9. *Suppose that the following assumptions hold.*

(1) *There exist $V_i : \mathbb{R} \times U_i \to \mathbb{R}$, $F_{ij} : \mathbb{R} \times U_i \times U_j \to \mathbb{R}$, $a_{ij} \geq 0$, and $b_i > 0$ such that* (2.25) *holds.*

(2) *Either* (2.19) *or* (2.24) *holds, or* (2.23) *holds provided that $(\mathcal{G}, A)$ is balanced.*

(3) *The weighted digraph $(\mathcal{G}, A)$ is strongly connected.*

Then, constants c_i given by Proposition 2.1 satisfy $c_i > 0$ for all i and the function V in (2.20) *satisfies*

$$\overset{\bullet}{V}(t, u) \leq -bV(t, u) \quad \text{for all} \ \ t > 0,\ u \in U,$$

where $b = \min\{b_1, b_2, \cdots, b_n\} > 0$.

2.5 An Application to Stability Problems

In this section we use the graph-theoretic approach developed in Section 2.4 to obtain stability results for coupled systems of nonautonomous differential equations, while in Chapters 3-5 we investigate stability problems for autonomous systems.

Let D be an open set in $\mathbb{R}^m$. Consider a nonautonomous system

$$x' = f(t, x), \tag{2.26}$$

where $f : [0, \infty) \times D \to D$ is continuous. We assume that the solution $x(t; t_0, x_0)$ of (1.1) through (t_0, x_0) with $x_0 \in D$ exists for all $t \geq t_0$ and is unique. We say that the origin $x = 0 \in D$ is an *equilibrium* of (2.26) at $t = t_0$ if $f(t, 0) = 0$ for all $t \geq t_0$. The origin $x = 0$ is *uniformly stable* if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \epsilon$ for all $t \geq t_0 > 0$. The origin is *uniformly asymptotically stable* if it is uniformly stable and there exists $\eta > 0$ independent of t_0 such that $|x_0| < \eta$ implies $x(t; t_0, x_0) \to 0$ as $t \to \infty$. The origin is *globally uniformly asymptotically stable* if it is uniformly asymptotically stable and for any $x_0 \in D$ we have $x(t; t_0, x_0) \to 0$ as $t \to \infty$.

Consider the coupled system (2.13) with $U_i = \mathbb{R}^{m_i}, i = 1, 2, \ldots, n$, that is

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \qquad i = 1, 2, \ldots, n, \tag{2.27}$$

where $u_i \in \mathbb{R}^{m_i}$, $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, and $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$. Suppose that the origin is an equilibrium of (2.27). Using the graph-theoretic approach developed in Section 2.4, we have the following stability result.

Theorem 2.10. *Suppose that there exist functions* $V_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}, F_{ij}$:

$\mathbb{R}\times\mathbb{R}^{m_i}\times\mathbb{R}^{m_j}$ and constants $a_{ij}\geq 0, b_i>0, 1\leq i,j\leq n$ such that assumptions (1)-(3) *in Theorem 2.9 are satisfied. In addition, assume that the following conditions hold for every function V_i.*

(1) *(positive definite) $V_i(t,u_i)=0$ for all $t>0$ if and only if $u_i=0$; and there exists function $\Phi_i:\mathbb{R}^{m_i}\to[0,\infty)$ with $\Phi_i(u_i)=0$ iff $u_i=0$ such that $\Phi_i(u_i)\leq V_i(t,u_i)$ for all $t>0$ and $u_i\in\mathbb{R}^{m_i}$.*

(2) *(decrescent) There exists function $\Psi_i:\mathbb{R}^{m_i}\to[0,\infty)$ with $\Psi_i(u_i)=0$ iff $u_i=0$ such that $|V_i(t,u_i)|\leq\Psi_i(u_i)$ for all $t>0$ and $u_i\in\mathbb{R}^{m_i}$.*

(3) *(radially unbounded) $V_i(t,u_i)\to\infty$ uniformly on t as $|u_i|\to\infty$.*

Then the origin is a globally uniformly asymptotically stable equilibrium for system (2.27).

Proof. Applying Theorem 2.9, we find that the function

$$V(t,u)=\sum_{i=1}^{n}c_iV_i(t,u_i),$$

with $c_i>0$ given by Proposition 2.1, is a Lyapunov function for system (2.27), and

$$\overset{\bullet}{V}(t,u)\leq -bV(t,u),\qquad \text{for all } (t,u)\in[0,\infty)\times U,$$

where $b=\min\{b_1,b_2,\ldots,b_n\}>0$. Since V is a linear combination of functions V_i that satisfy conditions (1)-(3), V also satisfies conditions (1)-(3). Therefore, by Theorem 4.3 in [39] (also see [102, Theorem 6.2] or [122, Theorem 9.8]), we conclude that the origin is globally uniformly asymptotically stable. □

Example 3. Consider the following coupled system of nonautonomous differential equations

$$x_i'=\alpha_i(t)x_i+\sum_{j=1}^{n}a_{ij}(x_j-x_i),\qquad i=1,2,\ldots,n, \tag{2.28}$$

where $x_i \in \mathbb{R}$, $\alpha_i : \mathbb{R} \to \mathbb{R}$, and the nonnegative matrix $A = (a_{ij})$ is irreducible. Assume that $\alpha_i(t) \leq -\beta_i < 0$ for all i and $t \geq 0$. Set $V_i(t, x_i) = x_i^2$. It can easily be verified that V_i satisfies conditions (1)-(3) in Theorem 2.10. We also have

$$\dot{V}_i(t, x_i) = 2\alpha_i(t)x_i^2 + \sum_{j=1}^{n} a_{ij}(2x_i x_j - 2x_i^2) \leq -2\beta_i x_i^2 + \sum_{j=1}^{n} a_{ij}(x_j^2 - x_i^2).$$

Let $F_{ij}(x_i, x_j) = V_j(x_j) - V_i(x_i) = x_j^2 - x_i^2$ and $b_i = 2\beta_i > 0$. Hence all assumptions of Theorem 2.9 and Theorem 2.10 can be verified. Therefore, by Theorem 2.10, we conclude that the origin is a globally uniformly asymptotically stable equilibrium.

2.6 Discussion

To demonstrate that mere existence of Lyapunov functions V_i for each vertex system is not sufficient for the existence of V, we consider the following example, which shows that two asymptotically stable linear systems can be linearly coupled as in (2.12) to form an unstable system.

Example 4. Let

$$B = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix},$$

$u_i = (x_i, y_i) \in \mathbb{R}^2$, and $f_i(u_i) = Bu_i$. The vertex systems (2.11) become

$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = B \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2. \tag{2.29}$$

Since the two eigenvalues of B are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, the zero solution of the vertex systems (2.29) is globally asymptotically stable in $\mathbb{R}^2$. Moreover, there exists a 2×2 matrix C such that $V_i = u_i^T C u_i$ is a Lyapunov function for the vertex system (2.29) [41, page 295]. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

denote the weight matrix associated with a digraph $\mathcal{G}$ and $K = \text{diag}\{1, 0\}$. The dispersal coupled system (2.12), interpreted as in Figure 2.7, becomes

$$\begin{aligned} \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= B \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} &= B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.30}$$

whose coefficient matrix

$$\begin{pmatrix} -3 & 3 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 3 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

has a positive eigenvalue $\frac{\sqrt{13}-3}{2}$, and thus the zero solution of the coupled system (2.30) is unstable.

In the rest of the thesis, to demonstrate the applicability and effectiveness of the approach described in Section 2.4, we consider the global-stability problem for various coupled systems. For these coupled systems, the types of vertex systems are ranging from ordinary differential equations, delay differential equations, to stochastic differential equations; the couplings among vertices can be linear or nonlinear, or have delay. We show that Lyapunov functions for these coupled systems can be systematically constructed by our approach.

Chapter 3. Applications to Ordinary Differential Equation Models

In this chapter we choose several well-known mathematical models to demonstrate the applicability and effectiveness of our graph-theoretic approach developed in Chapter 2. These models include coupled oscillators (Section 3.1), spatial ecological models of single species (Section 3.2) and predator-prey (Section 3.4), and heterogeneous epidemic models (Sections 3.5-3.6). These examples have different types of networks, such as mechanical and electrical networks for coupled oscillators, dispersal networks for spatial ecological models, and interaction networks for multi-group epidemic models. Vertex Lyapunov functions for these models are also different, from energy-type functions for coupled oscillators to Volterra-type functions for ecological and epidemiological models. Our graph-theoretic approach allows a unified solution regarding the construction of global Lyapunov functions for all these different type of complex systems.

The graph-theoretic approach also allows us to significantly improve the best-known results in the literature. In particular, the global stability of the endemic equilibrium for multi-group epidemic models has been an open problem for more than thirty years. Our approach resolves this long-standing open problem for a large class of multi-group epidemic models (Theorem 3.9).

3.1 A Model for Coupled Oscillators

Many mechanical, electrical, chemical, and biological systems can be modelled as coupled oscillators. Their dynamical behaviors can be very complicated; bifurcation and chaos have been observed in various model systems (see [8] and references therein). It is also interesting to investigate when and how these coupled oscillators can eventually stop oscillations. In this section we apply the graph-theoretic approach developed in Chapter 2 to investigate the global-stability problem for a class of coupled oscillators. Our approach allows us to construct a global Lyapunov function for coupled oscillators using the well-known energy function for each individual oscillator.

We consider the model of coupled oscillators as constructed in Section 2.3. Recall that, for a given weighted digraph $(\mathcal{G}, A)$ with n vertices, $A = (a_{ij})$, $n \geq 2$, a coupled system of nonlinear oscillators on $(\mathcal{G}, A)$ is

$$\begin{aligned} x_i' &= y_i, \\ y_i' &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^{n} a_{ij}(y_i - y_j). \end{aligned} \tag{3.1}$$

The vertex system at each vertex is described by a nonlinear oscillator as

$$x_i'' + \alpha_i x_i' + f_i(x_i) = 0. \tag{3.2}$$

Here $\alpha_i \geq 0$ is the damping coefficient, $f_i : \mathbb{R} \to \mathbb{R}$ is the nonlinear restoring force.

For each vertex system, we assume that there exists x_i^* such that $f_i(x_i) = 0$ iff $x_i = x_i^*$. We also assume that the potential energy $F_i(x_i) = \int_{x_i^*}^{x_i} f_i(s)ds$ has a global minimum at $x_i = x_i^*$ and $\lim_{x_i \to \infty} F_i(x_i) = \infty$ for all i. Then it is standard that, if $\alpha_i > 0$, the total energy

$$V_i(x_i, y_i) = \frac{y_i^2}{2} + F_i(x_i) \tag{3.3}$$

is a global Lyapunov function for the global asymptotic stability of $x_i = x_i^*$ for vertex system (3.2).

It can be verified that $E^* = (x_1^*, 0, x_2^*, 0, \cdots, x_n^*, 0)$ is an equilibrium of coupled system (3.1). In the following, we apply the graph-theoretic approach to construct a global Lyapunov function for coupled system (3.1), and then establish the global stability of E^*.

Theorem 3.1. *Assume $(\mathcal{G}, A)$ is strongly connected. Suppose that $\alpha_i \geq 0$ for all i and there exists k such that $\alpha_k > 0$. Then equilibrium E^* is globally asymptotically stable in $\mathbb{R}^{2n}$.*

Proof. To apply the graph-theoretic approach, we want to verify that $V_i(x_i, y_i)$ in (3.3) satisfies the assumptions of Theorem 2.6. Differentiating V_i along (3.1) gives

$$\begin{aligned}\dot{V_i} &= -\alpha_i y_i^2 - \sum_{j=1}^{n} a_{ij}(y_i - y_j)y_i \\ &= -\alpha_i y_i^2 + \sum_{j=1}^{n} a_{ij}\Big(-\frac{1}{2}(y_i - y_j)^2 + \frac{1}{2}y_j^2 - \frac{1}{2}y_i^2\Big) \\ &\leq \sum_{j=1}^{n} a_{ij}\Big(\frac{1}{2}y_j^2 - \frac{1}{2}y_i^2\Big).\end{aligned}$$

Let

$$F_{ij}(y_i, y_j) = \frac{1}{2}y_j^2 - \frac{1}{2}y_i^2.$$

We have

$$\dot{V_i} \leq \sum_{j=1}^{n} a_{ij} F_{ij}(y_i, y_j),$$

and along every directed cycle $\mathcal{C}$ of the weighted digraph $(\mathcal{G}, A)$,

$$\sum_{(s,r)\in E(\mathcal{C})} F_{rs}(y_r, y_s) = \sum_{(s,r)\in E(\mathcal{C})} \Big(\frac{1}{2}y_s^2 - \frac{1}{2}y_r^2\Big) = 0.$$

Assumptions (1) and (2) of Theorem 2.6 are satisfied. Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, as given in

Proposition 2.1. Then, by Theorem 2.6,

$$V(x_1, y_1, \cdots, x_n, y_n) = \sum_{i=1}^{n} c_i V_i(x_i, y_i)$$

is a Lyapunov function for (3.1), in fact,

$$\begin{aligned}
\dot{V}(x_1, y_1, \cdots, x_n, y_n) &= \sum_{i=1}^{n} c_i \dot{V}_i(x_i, y_i) \\
&\leq \sum_{i=1}^{n} c_i \sum_{j=1}^{n} a_{ij} F_{ij}(y_i, y_j) \\
&= 0
\end{aligned}$$

for all $(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n}$.

To show E^* is globally asymptotically stable, we examine the largest invariant set where $\dot{V} = 0$. Since $(\mathcal{G}, A)$ is strongly connected, $c_i > 0$ for all $1 \leq i \leq n$. Therefore, $\dot{V} = 0$ implies that $\alpha_i y_i^2 = 0$ and $a_{ij}(y_i - y_j)^2 = 0$ for all $1 \leq i, j \leq n$. As a consequence, $y_i = 0$ if $\alpha_i > 0$; and $y_i = y_j$ if $a_{ij} > 0$, or if there exists an arc from j to i in $(\mathcal{G}, A)$. By our assumption, there exists k such that $\alpha_k > 0$, thus $y_k = 0$. Let $l \neq k$ denote any vertex of $(\mathcal{G}, A)$. Then, by the strong connectivity of $(\mathcal{G}, A)$, there exists a directed path $\mathcal{P}$ from l to k. Applying the relation $y_i = y_j$ to each arc (j, i) of $\mathcal{P}$, we obtain that $y_l = y_k = 0$. Hence, $\dot{V} = 0$ implies $y_i = 0$ for all i. From the second equation of (3.1), we have $0 = y_i' = -f_i(x_i)$, and thus $x_i = x_i^*$ for all i. This implies that the largest invariant subset of

$$\{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} \mid \dot{V} = 0\}$$

is the singleton $\{E^*\}$. Note that V is radially unbounded, namely,

$$V(x_1, y_1, \cdots, x_n, y_n) \to \infty \quad \text{as} \quad |(x_1, y_1, \cdots, x_n, y_n)| \to \infty.$$

Therefore, by the LaSalle Invariance Principle [75], E^* is globally asymptotically stable in $\mathbb{R}^{2n}$. □

Theorem 3.1 shows that in a strongly connected network, the existence of damping in a single oscillator is sufficient to eventually wipe out all oscillations in coupled system (3.1).

3.2 A Single-Species Model in a Patchy Environment

Spatial heterogeneity exists in the natural world and can influence population dynamics [77]. Both continuous reaction-diffusion systems and discrete patchy models are used to study the effect of spatial heterogeneity in the literature (see [98] and references therein). While reaction-diffusion systems are suitable for random spatial dispersal, patchy models are often used to describe directed movement among patches [51, 76].

In this section we consider a general model that describes the dispersal of a single species among n patches ($n \geq 2$)

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \qquad i = 1, 2, \ldots, n. \tag{3.4}$$

Here $x_i \in \mathbb{R}_+$ represents population density of the species in patch i, $f_i \in C^1(\mathbb{R}_+, \mathbb{R})$ represents the intrinsic growth rate in patch i, the constant $d_{ij} \geq 0$ is the dispersal rate from patch j to patch i, and the constant $\alpha_{ij} \geq 0$ can be selected to represent different boundary conditions in the continuous diffusion case [1, 86]. We remark that growth functions of patches can be very different; that is, system (3.4) allows patch-specific population dynamics [2]. To model this specification using continuous space model, one needs to deal with partial differential equations with spatially varying coefficients, which are particularly challenging in stability analysis.

Stability problems for system (3.4) have been investigated by different authors. For example, Hastings [48] studied the local stability of a positive equilibrium of (3.4). Sufficient conditions for uniqueness and global stability of the positive equilibrium were first derived in Beretta and Takeuchi [13] and further generalized in Lu and Takeuchi [86]. In the following, we interpret (3.4) as a coupled system on a network. Using our graph-theoretic approach developed in Chapter 2, we prove a global stability result under weaker restrictions then ones in [13] and [86].

A digraph $\mathcal{G}$ with n vertices can be constructed for system (3.4) as follows: each vertex represents a patch, a directed arc (j, i) is assigned if the dispersal rate d_{ij} from patch j to patch i is positive, and no such arc exists if $d_{ij} = 0$. The dynamics at each vertex are defined by the scalar ordinary differential equation $x_i' = x_i f_i(x_i)$. The coupling among vertices are provided by the dispersal among patches. See Figure 3.1. We remark that the dispersal network $\mathcal{G}$ is strongly connected if and only if the dispersal matrix (d_{ij}) is irreducible.

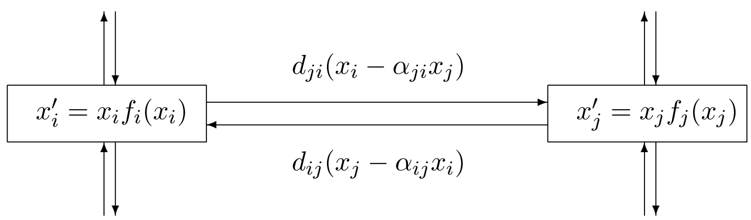

Figure 3.1: A coupled single-species system on a network

Let $\Phi : (0, \infty) \to [0, \infty)$ be such that

$$\Phi(x) = x - 1 - \ln x, \qquad \forall x > 0. \tag{3.5}$$

It can be easily verified that $\Phi(x) \geq 0$ for all $x > 0$ and $\Phi(x) = 0$ if and only if $x = 1$. Denote $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+, i = 1, 2, \ldots, n\}$. Now we are in a position to establish the global stability result for system (3.4). Biologically, our result implies that populations in all patches persist at the unique positive equilibrium level if it exists, irrespective of the initial conditions.

Theorem 3.2. *Suppose that the following assumptions hold:*

(1) *The dispersal matrix* (d_{ij}) *of* (3.4) *is irreducible.*

(2) $f_i'(x_i) \leq 0$, *for all* $x_i > 0, i = 1, 2, \dots, n$, *and there exists* k *such that* $f_k'(u) \not\equiv 0$ *in any open interval of* $\mathbb{R}_+$.

Then, whenever a positive equilibrium $E^* = (x_1^*, x_2^*, \cdots, x_n^*)$ *with* $x_i^* > 0$ *for all* $1 \leq i \leq n$ *exists for system* (3.4), *it is unique and globally asymptotically stable in the positive cone of* $\mathbb{R}_+^n$.

Proof. Let $E^* = (x_1^*, x_2^*, \cdots, x_n^*)$, $x_i^* > 0$, $i = 1, 2, \dots, n$, denote a positive equilibrium of (3.4). Then x_i^* satisfies

$$f_i(x_i^*) = -\sum_{j=1}^{n} d_{ij}\Big(\frac{x_j^*}{x_i^*} - \alpha_{ij}\Big). \tag{3.6}$$

We show that E^* is globally asymptotically stable in the positive cone of $\mathbb{R}_+^n$, and thus is unique.

Set

$$V_i(x_i) = x_i^* \Phi\Big(\frac{x_i}{x_i^*}\Big) = x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}.$$

From the properties of function Φ as defined in (3.5), we obtain that $V_i(x_i) > 0$ for all $x_i > 0$ and $V_i(x_i) = 0$ if and only if $x_i = x_i^*$. Direct calculation and (3.6) yield

$$\begin{aligned}
\overset{\bullet}{V}_i &= (x_i - x_i^*)\Big[f_i(x_i) + \sum_{j=1}^{n} d_{ij}\Big(\frac{x_j}{x_i} - \alpha_{ij}\Big)\Big] \\
&= (x_i - x_i^*)\Big[-\sum_{j=1}^{n} d_{ij}\Big(\frac{x_j^*}{x_i^*} - \alpha_{ij}\Big) + f_i(x_i) - f_i(x_i^*) + \sum_{j=1}^{n} d_{ij}\Big(\frac{x_j}{x_i} - \alpha_{ij}\Big)\Big] \\
&= (x_i - x_i^*)(f_i(x_i) - f_i(x_i^*)) + \sum_{j=1}^{n} d_{ij} x_j^* \Big(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*}\Big).
\end{aligned} \tag{3.7}$$

Let

$$a_{ij} = d_{ij} x_j^* \geq 0,$$

$$F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*},$$

and

$$G_i(x_i) = -\Phi\Big(\frac{x_i}{x_i^*}\Big) = 1 - \frac{x_i}{x_i^*} + \ln\frac{x_i}{x_i^*}.$$

Using the fact $(x_i - x_i^*)(f_i(x_i) - f_i(x_i^*)) \leq 0$ for all $x_i > 0$ and all i, we have

$$\overset{\bullet}{V}_i \leq \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j).$$

On the other hand,

$$\begin{aligned} F_{ij}(x_i, x_j) &= \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*} \\ &= G_i(x_i) - G_j(x_j) + 1 - \frac{x_i^* x_j}{x_i x_j^*} + \ln\frac{x_i^* x_j}{x_i x_j^*} \\ &= G_i(x_i) - G_j(x_j) - \Phi\Big(\frac{x_i^* x_j}{x_i x_j^*}\Big) \\ &\leq G_i(x_i) - G_j(x_j). \end{aligned}$$

We have shown that V_i, F_{ij}, G_i, and a_{ij} satisfy the assumptions of Theorem 2.6 and Corollary 2.8. Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, as given in Proposition 2.1. Therefore,

$$V(x_1, \cdots, x_n) = \sum_{i=1}^{n} c_i V_i(x_i)$$

as defined in Theorem 2.6 is a Lyapunov function for (3.4), namely,

$$\overset{\bullet}{V} \leq 0 \quad \text{for all} \quad (x_1, \cdots, x_n) \in \mathbb{R}_+^n.$$

Using the strong connectivity of $(\mathcal{G}, A)$ and a similar argument as in Section 3.1, we can show that $\overset{\bullet}{V} = 0$ if and only if $x_i = x_i^*$ for all i. Note that V is radially unbounded and $V(x_1, \cdots, x_n) \to \infty$ as $x_i \to 0^+$ for any i. By the classical Lyapunov stability theory, E^* is globally asymptotically stable in the positive cone of $\mathbb{R}_+^n$. This completes the proof of Theorem 3.2. □

The existence requirement for E^* can be satisfied through boundedness and persistence analysis, which only involves dynamics on the boundary of $\mathbb{R}^n_+$. Theorem 3.2 contains an earlier result, as a special case, in Lu and Takeuchi [86], in which the global stability of E^* was proved by the theory of monotone dynamical system under the stricter conditions that $f_i'(x_i) < 0$ in $(0, +\infty)$ for all i. Before our Theorem 3.2, the result of Lu and Takeuchi [86] was the best-known global stability result for system (3.4).

3.3 A Volterra Food Web

Stability and complexity of ecosystems have been studied in the field of mathematical ecology and biology, see [91] and references therein. The interactions of many species within biological communities and/or inter communities result in different complex systems. Global-stability problems for these complex ecosystems can be very challenging due to complexity of ecosystems. In this section and Section 3.4 we demonstrate that our graph-theoretic approach is applicable to different types of complex ecosystems, from food webs (this section) to patchy predator-prey models (Section 3.4).

Food webs [24, 121] are complex ecosystems describing the predator-prey relationships between species. Global-stability problems on food webs can be used to describe the coexistence of species, and thus are interesting and important [31, 46, 99]. In this section, we apply our graph-theoretic approach to investigate the global stability of a Volterra food web

$$x_i' = x_i\Big(e_i + \sum_{j=1}^{n} p_{ij}x_j\Big), \qquad i = 1, 2, \ldots, n. \tag{3.8}$$

Here $x_i \in \mathbb{R}_+$ represents the population density of the i-th species, $e_i \in \mathbb{R}$, $p_{ii} \leq 0$, and $p_{ij}p_{ji} < 0$ if $p_{ij} \neq 0, i \neq j$. Biologically, $p_{ij} > 0$ means that x_i is predator and x_j is prey. We describe (3.8) as a coupled system on a network. Let $\mathcal{G}$ be a digraph with n vertices, in which each vertex represents one species. An arc (j, i) exists if and only if $p_{ij} \neq 0, i \neq j$. The dynamics at each vertex are defined by the scalar ordinary differential equation

$$x_i' = e_i x_i + p_{ii} x_i^2.$$

The coupling among vertices are provided by the interaction among species, in the bilinear form of $p_{ij}x_ix_j, i \neq j$. System (3.8) thus can be regarded as a coupled system on $\mathcal{G}$. See Figure 3.2.

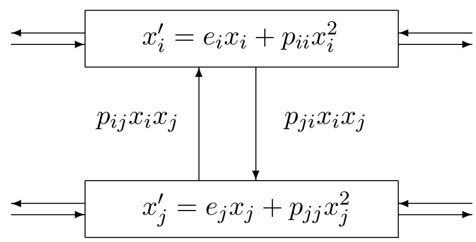

Figure 3.2: A Volterra food web

Suppose that (3.8) admits a positive equilibrium $E^* = (x_1^*, x_2^*, \cdots, x_n^*)$, where $x_i^* > 0, i = 1, 2, \ldots, n$, satisfy the equilibrium equations

$$e_i + \sum_{j=1}^{n} p_{ij} x_j^* = 0, \qquad \forall\, i = 1, 2, \ldots, n. \tag{3.9}$$

Let

$$V_i(x_i) = x_i^* \Phi\Big(\frac{x_i}{x_i^*}\Big) = x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}.$$

Differentiating V_i along with (3.9) gives

$$\begin{aligned} \dot{V}_i &= e_i x_i + \sum_{j=1}^{n} p_{ij} x_i x_j - e_i x_i^* - \sum_{j=1}^{n} p_{ij} x_i^* x_j = \sum_{j=1}^{n} p_{ij} (x_i - x_i^*)(x_j - x_j^*) \\ &= p_{ii}(x_i - x_i^*)^2 + \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j), \end{aligned}$$

where $a_{ij} = |p_{ij}|$ when $i \neq j$, $a_{ii} = 0$, and $F_{ij}(x_i, x_j) = \mathrm{sgn}(p_{ij})(x_i - x_i^*)(x_j - x_j^*)$. Let $(\mathcal{G}, A)$ denote the weighted digraph with the weight matrix (a_{ij}). Suppose that $(\mathcal{G}, A)$ is balanced. Thus condition (2.23) of Theorem 2.7 is satisfied since $F_{ij}(x_i, x_j) = -F_{ji}(x_j, x_i), i \neq j$. Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, as given in Proposition 2.1. Then, by Theorem 2.7,

$$V(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} c_i V_i(x_i) \tag{3.10}$$

is a Lyapunov function for (3.8) provided that $(\mathcal{G}, A)$ is balanced. In particular, we have

$$\dot{V} = \sum_{i=1}^{n} c_i p_{ii}(x_i - x_i^*)^2 \leq 0. \tag{3.11}$$

Therefore, we have the following result that extends an earlier result on global Lyapunov functions for (3.8) in [99].

Theorem 3.3. *Suppose that system* (3.8) *admits a positive equilibrium* E^**. Assume that* $(\mathcal{G}, A)$ *is balanced with weights* $a_{ij} = |p_{ij}|$ *if* $p_{ij} \neq 0$ *and* 0 *otherwise. Then* V *as given in* (3.10) *is a Lyapunov function for* (3.8)*. Furthermore, if* $p_{ii} < 0$ *for all* i*, then* E^* *is globally asymptotically stable in the positive cone of* $\mathbb{R}^n_+$*.*

3.4 A Patchy Predator-Prey Model

Patchy predator-prey models can be used to model complex ecosystems of predator-prey interactions in a heterogeneous environment. Assume that the heterogeneous environment can be divided into several homogeneous regions, called patches, prey and predator populations interact in each patch, and only prey population can dispersal among patches. Then we obtain the following predator-prey model in which prey disperse among n patches ($n \geq 2$).

$$\begin{aligned} x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \\ y_i' &= y_i(-\gamma_i - \mu_i y_i + \epsilon_i x_i), \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{3.12}$$

Here, x_i, y_i denote the densities of preys and predators on the patch i, respectively. The parameters in the model are nonnegative constants, and e_i, ϵ_i are positive. The dispersal constants d_{ij}, α_{ij} are similarly defined as in Section 3.2. We refer the reader to [30, 71] for detailed interpretations of predator-prey models and parameters.

In this section we interpret (3.12) as a coupled system on a network. Utilizing a well-known Lyapunov function for single-patch predator-prey models [56] as V_i and using our graph-theoretic approach, we establish that a positive equilibrium of the n-patch model (3.12) is globally asymptotically stable in $\mathbb{R}^{2n}_+$ as long as it exists. We remark that, for a special case of system (3.12) when patch number $n = 2$, Kuang and Takeuchi [71] proved the global stability of the positive equilibrium by constructing a Lyapunov function. Our graph-theoretic approach allows us to extend such a construction of Lyapunov functions for a two-patchy model to an arbitrarily n-patch model.

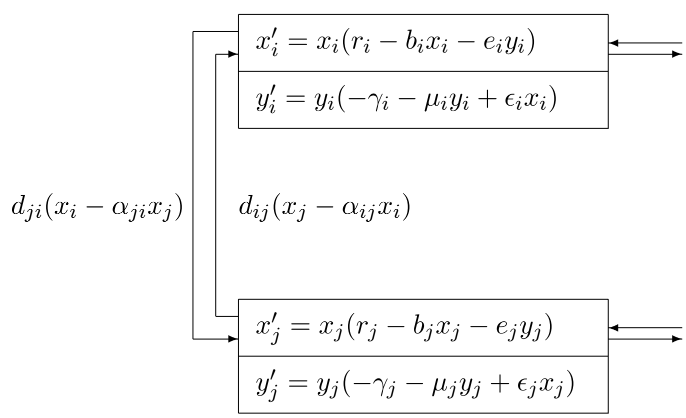

Figure 3.3: A coupled predator-prey system on a network

A digraph $\mathcal{G}$ with n vertices for system (3.12) can be constructed similarly as in Section 3.2. Each vertex represents a patch and $(j, i) \in E(\mathcal{G})$ if and only if $d_{ij} > 0$. At each vertex of $\mathcal{G}$, the vertex dynamics is described by a predator-prey system. See Figure 3.3. The coupling among these predator-prey systems are provided by dispersal among prey populations.

Now we are ready to establish the global stability of system (3.12).

Theorem 3.4. *Suppose that the dispersal matrix* (d_{ij}) *is irreducible. Assume that there exists* k *such that* $b_k > 0$ *or* $\mu_k > 0$. *Then, whenever a positive equilibrium* E^* *exists, it is unique and globally asymptotically stable in the positive cone of* $\mathbb{R}^{2n}_+$.

Proof. Let $E^* = (x_1^*, y_1^*, \cdots, x_n^*, y_n^*)$, with $x_i^*, y_i^* > 0$ for all $1 \le i \le n$, denote the positive equilibrium. Here x^*, y^* satisfy the equilibrium equations

$$\begin{aligned} r_i &= b_i x_i^* + e_i y_i^* - \sum_{j=1}^{n} d_{ij}\Big(\frac{x_j^*}{x_i^*} - \alpha_{ij}\Big), \\ \gamma_i &= -\mu_i y_i^* + \epsilon_i x_i^*. \end{aligned} \tag{3.13}$$

Consider a vertex Lyapunov function in [56] for a single-patch predator-prey model,

$$V_i(x_i, y_i) = \epsilon_i x_i^* \Phi\Big(\frac{x_i}{x_i^*}\Big) + e_i y_i^* \Phi\Big(\frac{y_i}{y_i^*}\Big) = \epsilon_i\Big(x_i - x_i^* - x_i^* \ln\frac{x_i}{x_i^*}\Big) + e_i\Big(y_i - y_i^* - y_i^* \ln\frac{y_i}{y_i^*}\Big).$$

We show that V_i satisfies assumptions of Theorem 2.6. Following similar steps as in (3.7), and using (3.13), we can verify that

$$\begin{aligned} \dot{V}_i &= \epsilon_i(x_i - x_i^*)(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} \epsilon_i d_{ij} \frac{x_i - x_i^*}{x_i}(x_j - \alpha_{ij} x_i) \\ &\quad + e_i(y_i - y_i^*)(-\gamma_i - \mu_i y_i + \epsilon_i x_i) \\ &= \epsilon_i(x_i - x_i^*)\Big(b_i x_i^* + e_i y_i^* - \sum_{j=1}^{n} d_{ij}\Big(\frac{x_j^*}{x_i^*} - \alpha_{ij}\Big) - b_i x_i - e_i y_i\Big) \\ &\quad + \sum_{j=1}^{n} \epsilon_i d_{ij} \frac{x_i - x_i^*}{x_i}(x_j - \alpha_{ij} x_i) + e_i(y_i - y_i^*)(\mu_i y_i^* - \epsilon_i x_i^* - \mu_i y_i + \epsilon_i x_i) \\ &= -\epsilon_i b_i (x_i - x_i^*)^2 - e_i \mu_i (y_i - y_i^*)^2 + \sum_{j=1}^{n} d_{ij} \epsilon_i x_j^* \Big(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_j x_i^*}{x_j^* x_i}\Big). \end{aligned}$$

Set

$$a_{ij} = d_{ij}\epsilon_i x_j^*, \qquad F_{ij}(x_i, x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*},$$

and

$$G_i(x_i) = -\frac{x_i}{x_i^*} + \ln \frac{x_i}{x_i^*}.$$

Then, as in Section 3.2, V_i, F_{ij}, G_i, and a_{ij} satisfy the assumptions of Theorem 2.6 and Corollary 2.8. Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, as given in Proposition 2.1. Therefore, the function

$$V(x_1, y_1, \cdots, x_n, y_n) = \sum_{i=1}^{n} c_i V_i(x_i, y_i)$$

as defined in Theorem 2.6 is a Lyapunov function for (3.12), and

$$\dot{V} \leq 0 \quad \text{for all} \quad (x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}_+^{2n}.$$

Using a similar argument as in Section 3.1, we can show that the largest invariant set on which $\dot{V} = 0$ is the singleton $\{E^*\}$. Notice that V is radially unbounded and for all i we have

$$V(x_1, y_1, \cdots, x_n, y_n) \to \infty \quad \text{as} \quad x_i \to 0^+ \quad \text{or} \quad y_i \to 0^+.$$

The LaSalle Invariance Principle [75] implies that E^* is globally asymptotically stable in the positive cone of $\mathbb{R}_+^{2n}$. This also implies that E^* is unique, completing the proof of Theorem 3.4. □

Theorem 3.4 generalizes a global-stability result in [71] from 2 patches to any number of patches. Biologically, our result indicates that arbitrary prey dispersal among patches never changes the global stability as long as the system persists.

3.5 An SIR Epidemic Model in a Patchy Environment

Heterogeneity exists in many aspects of disease transmission processes [20, 84]: heterogeneous spatial distribution of host populations, heterogeneous susceptibility among age groups, heterogeneous social behaviors among groups for sexually transmitted diseases, multi-hosts for many diseases such as West Nile virus and Avian flu. Heterogeneity produces complexity in disease transmission. Due to extremely large scales of the resulting models, rigorously establishing their global dynamics poses a great mathematical challenge. In this section (spatial heterogeneity) and Section 3.6 (host heterogeneity), our graph-theoretic approach allows us to completely determine the global dynamics of several classes of heterogeneous epidemiological models.

Discrete spatial epidemic models in patchy environments have been proposed in the literature to model the spread of infectious disease in spatially heterogeneous host populations [7, 116]. In the proposed models, a patch can be a city or a country; and directed movement can be migration among countries and regions or travel among cities. Arino and van den Driessche [6] formulated n-city epidemic models to investigate the effects of inter-city travel on the spatial spread of infectious diseases among cities. The basic reproduction number R_0 was derived and numerical simulations were carried out to show that R_0 determines whether the disease dies out ($R_0 < 1$) or becomes endemic ($R_0 > 1$). Wang and Zhao [117] studied an n-patch SIS model with bilinear incidence. In the case that both susceptible and infectious individuals on each patch have the same dispersal rates, they proved that the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$. They also proved that the system is uniformly persistent and admits an endemic equilibrium if $R_0 > 1$. Under the same assumption that the dispersal rates of susceptible and infectious individuals are the same, Jin and Wang [60] showed that the n-patch SIS model can be reduced to a monotone system. Using the theory of monotone dynamical systems, they proved the uniqueness and global stability of the endemic equilibrium when $R_0 > 1$. Salmani and van den Driessche [103] studied an SEIRS model with standard incidence in a patchy environment and proved that, if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, regardless of travel rates. Uniqueness and global stability of endemic equilibria when $R_0 > 1$ is unresolved for many patchy epidemic models.

In this section, we consider an SIR epidemic model in a patchy environment in which the couplings are provided by individual travel among patches

$$\begin{aligned}
S_i' &= \Lambda_i - \beta_i S_i I_i - \mu_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i, \\
I_i' &= \beta_i S_i I_i - (\mu_i^I + \gamma_i) I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i, \\
R_i' &= \gamma_i I_i - \mu_i^R R_i + \sum_{j=1}^n c_{ij} R_j - \sum_{j=1}^n c_{ji} R_i, \qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{3.14}$$

Here S_i, I_i, R_i represent the susceptible, infectious, and removed populations in the i-th patch, respectively, Λ_i is the influx of individuals into the i-th patch, β_i is the transmission coefficient between susceptible and infectious individuals in the i-th patch, μ_i^S, μ_i^I, and μ_i^R represent death rates of S, I, R populations in the i-th patch, respectively, and γ_i represents the recovery rate of infectious individuals in the i-th patch. The travel rates of susceptible, infectious, and removed individuals from the j-th patch to the i-th patch are given by a_{ij}, b_{ij}, c_{ij}, respectively. All parameter values are assumed to be nonnegative and $\Lambda_i, \beta_i, \mu_i^S, \mu_i^I > 0$ for all i. The travel matrices $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ are not required to be symmetric, namely, the travel rate from the i-th patch to the j-th patch may not be the same as that from the j-th to the i-th. A typical assumption we impose on these matrices is that they are irreducible. In biological terms, this means individuals in each compartment can travel between any two patches directly or indirectly through other patches. For detailed discussions of epidemic model with patches, we refer the reader to the articles [7, 116] and the references therein. Model

(3.14) includes as special cases several earlier models in the literature. A two-patch SIS model [115] and a two-patch SIRS model [19] become special cases of model (3.14) if we assume that the disease has permanent immunity. An n-patch model similar to (3.14) was proposed in [87] without global-stability analysis. We remark that model (3.14) differs from those in [103] in that bilinear incidence is used in (3.14) while standard incidences are assumed in [103].

Since the variable R_i does not appear in the first two equations of (3.14), we can first study the reduced system

$$\begin{aligned} S_i' &= \Lambda_i - \beta_i S_i I_i - \mu_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i, \\ I_i' &= \beta_i S_i I_i - (\mu_i^I + \gamma_i) I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i, \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{3.15}$$

with initial conditions $S_i(0) \geq 0$ and $I_i(0) \geq 0$. The behavior of R_i can then be determined from the last equation of (3.14). Our results will be stated for system (3.15) and can be translated straightforwardly to system (3.14).

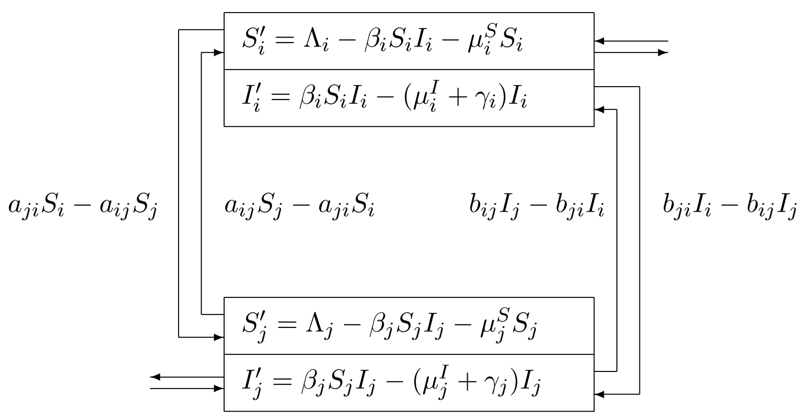

Figure 3.4: A coupled SIR model on a network

System (3.15) can be regarded as a coupled system on a digraph $\mathcal{G}$. See Figure 3.4. For $\mathcal{G}$, each vertex represents a patch and $(j, i) \in E(\mathcal{G})$ if either

$a_{ij} > 0$ or $b_{ij} > 0$. At each vertex, the vertex dynamics is described by a standard epidemic model. The coupling among patches are provided by travel of individuals.

To find the disease-free equilibrium of (3.15), we consider the following linear system

$$\Lambda_i - \mu_i^S S_i + \sum_{j=1}^{n} a_{ij} S_j - \sum_{j=1}^{n} a_{ji} S_i = 0, \quad i = 1, 2, \ldots, n, \tag{3.16}$$

or in the form of matrix system

$$DS = \Lambda,$$

where

$$D = \begin{pmatrix} \mu_1^S + \sum_{j \neq 1} a_{j1} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \mu_2^S + \sum_{j \neq 2} a_{j2} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \mu_n^S + \sum_{j \neq n} a_{jn} \end{pmatrix}, \tag{3.17}$$

$S = (S_1, S_2, \cdots, S_n)^T$, and $\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n)^T$. Since all off-diagonal entries of D are nonpositive and the sum of the entries in each column of D is positive, D is a non-singular M-matrix and $D^{-1} \geq 0$ [15, p.137]. A square matrix is said to be an *M-matrix* if all off-diagonal entries are nonpositive and all eigenvalues have positive real parts. Hence, the linear system (3.16) has a unique positive solution $S^0 = (S_1^0, S_2^0, \cdots, S_n^0)^T = D^{-1}\Lambda$, with $S_i^0 > 0$ for all i. As a consequence, system (3.15) has a unique disease-free equilibrium $P_0 = (S_1^0, 0, S_2^0, 0, \cdots, S_n^0, 0)$. We thus have the following result.

Proposition 3.5. *System* (3.15) *always has a unique disease-free equilibrium* P_0.

Let

$$\bar{\Lambda} = \sum_{i=1}^{n} \Lambda_i,$$

$$\mu^* = \min\{\mu_i^S,\ \mu_i^I + \gamma_i \mid i = 1, 2, \ldots, n\},$$

and

$$N = \sum_{i=1}^{n} (S_i + I_i).$$

Adding all equations of (3.15) gives $N' \leq \bar{\Lambda} - \mu^* N$, which implies that $\limsup_{t\to\infty} N \leq \frac{\bar{\Lambda}}{\mu^*}$. Since all off-diagonal entries of D are nonpositive, it follows from the first equation of (3.15) that

$$S_i' \leq \Lambda_i - \mu_i^S S_i + \sum_{j=1}^{n} a_{ij} S_j - \sum_{j=1}^{n} a_{ji} S_i = (DS^0 - DS)_i \leq 0,$$

when $S_i = S_i^0$ and $S_j \leq S_j^0$ for $j \neq i$. Thus a feasible region of (3.15) can be chosen as

$$\Gamma = \left\{ (S_1, I_1, \cdots, S_n, I_n) \in \mathbb{R}_+^{2n} \;\middle|\; N = \sum_{i=1}^{n} (S_i + I_i) \leq \frac{\bar{\Lambda}}{\mu^*},\ S_i \leq S_i^0,\ 1 \leq i \leq n \right\}.$$

It can be verified that Γ is positively invariant with respect to (3.15). Let $\overset{\circ}{\Gamma}$ denote the interior of Γ, and $\partial\Gamma$ the boundary of Γ.

An *endemic equilibrium* $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ of (3.15) belongs to $\overset{\circ}{\Gamma}$, namely, $S_i^* > 0, I_i^* > 0$ for all $i = 1, 2, \ldots, n$. System (3.15) is said to be *uniformly persistent* [16, 114] in $\overset{\circ}{\Gamma}$ if there exists constant $c > 0$ such that

$$\liminf_{t\to\infty} S_i(t) > c \quad \text{and} \quad \liminf_{t\to\infty} I_i(t) > c, \quad i = 1, \ldots, n,$$

provided $(S_1(0), I_1(0), \ldots, S_n(0), I_n(0)) \in \overset{\circ}{\Gamma}$.

Define

$$F = \begin{pmatrix} \beta_1 S_1^0 & 0 & \cdots & 0 \\ 0 & \beta_2 S_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n S_n^0 \end{pmatrix} \tag{3.18}$$

and

$$V = \begin{pmatrix} \mu_1^I + \gamma_1 + \sum\limits_{j\neq i} b_{j1} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & \mu_2^I + \gamma_2 + \sum\limits_{j\neq 2} b_{j2} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & \mu_n^I + \gamma_n + \sum\limits_{j\neq n} b_{jn} \end{pmatrix}. \tag{3.19}$$

Using the method of van den Driessche and Watmough [113], the basic reproduction number can be calculated as

$$R_0 = \rho(FV^{-1}), \tag{3.20}$$

where ρ represents the spectral radius and FV^{-1} is the so called next generation matrix. The following result for system (3.15) can be proved the same way as in [19, 103].

Proposition 3.6. *Suppose that $B = (b_{ij})$ is either irreducible or equal to 0.*

(1) *If $R_0 \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable in Γ.*

(2) *If $R_0 > 1$, then P_0 is unstable.*

(3) *If $R_0 > 1$ and $B = (b_{ij})$ is irreducible, then system (3.15) is uniformly persistent and there exists an endemic equilibrium P^* in $\mathring{\Gamma}$.*

We remark that when the travel matrix $B = (b_{ij})$ is reducible, system (3.15) can have multiple boundary equilibria and the dynamics of (3.15) can be complicated. In fact, when $B = 0$, system (3.15) can have an asymptotically stable boundary equilibrium when $R_0 > 1$ and thus is not persistent [19, 115]. It is also possible that, if $B = 0$, no endemic equilibrium exists when $R_0 > 1$ [19]. We refer the reader to [5, 6, 19, 115] for discussions on this issue.

The uniqueness and global stability of the endemic equilibrium, if it exists, are established in the following result.

Theorem 3.7. *Assume that $R_0 > 1$ and an endemic equilibrium $P^* = (S_1^*, I_1^*, \cdots, S_n^*, I_n^*)$ exists. Suppose that one of the following assumptions is satisfied.*

(1) *$A = 0$ and B is irreducible;*

(2) *$B = 0$ and A is irreducible;*

(3) *A and B are irreducible, and there exists $\lambda > 0$ such that $a_{ij}S_j^* = \lambda b_{ij} I_j^*$ for all $1 \le i, j \le n$.*

Then P^* is unique and globally asymptotically stable in $\overset{\circ}{\Gamma}$.

Proof. We prove the result when assumption (3) is satisfied. The other two cases can be proved similarly. Under assumption (3) we show that P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. In particular, this implies that P^* is necessarily unique. Set

$$V_i(S_i, I_i) = S_i^* \Phi\Big(\frac{S_i}{S_i^*}\Big) + I_i^* \Phi\Big(\frac{I_i}{I_i^*}\Big) = S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*},$$

which is a Lyapunov function for a one-patch SIR model [67, 69]. From the equilibrium equations of (3.15), we obtain

$$\mu_i^S S_i^* = \Lambda_i - \beta_i S_i^* I_i^* + \sum_{j=1}^{n} a_{ij} S_j^* - \sum_{j=1}^{n} a_{ji} S_i^*, \tag{3.21}$$

and

$$(\mu_i^I + \gamma_i) I_i^* = \beta_i S_i^* I_i^* + \sum_{j=1}^{n} b_{ij} I_j^* - \sum_{j=1}^{n} b_{ji} I_i^*. \tag{3.22}$$

Recall that $\Phi(x) = x - 1 - \ln x \geq 0$ for all $x > 0$ and $\Phi(x) = 0$ if and only if $x = 1$. Differentiating V_i along the solution of system (3.15), and using (3.21), (3.22), we obtain

$$
\begin{aligned}
\overset{\bullet}{V_i} &= \Lambda_i - \mu_i^S S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i - \Lambda_i \frac{S_i^*}{S_i} + \beta_i S_i^* I_i + \mu_i^S S_i^* \\
&- \sum_{j=1}^n a_{ij} S_j \frac{S_i^*}{S_i} + \sum_{j=1}^n a_{ji} S_i^* - (\mu_i^I + \gamma_i) I_i + \sum_{j=1}^n b_{ij} I_j - \sum_{j=1}^n b_{ji} I_i \\
&- \beta_i S_i I_i^* + (\mu_i^I + \gamma_i) I_i^* - \sum_{j=1}^n b_{ij} I_j \frac{I_i^*}{I_i} + \sum_{j=1}^n b_{ji} I_i^* \\
&= \Lambda_i \Big(2 - \frac{S_i}{S_i^*} - \frac{S_i^*}{S_i}\Big) + \sum_{j=1}^n a_{ij} S_j^* \Big(1 - \frac{S_i^* S_j}{S_i S_j^*} + \frac{S_j}{S_j^*} - \frac{S_i}{S_i^*}\Big) \\
&+ \sum_{j=1}^n b_{ij} I_j^* \Big(1 - \frac{I_i^* I_j}{I_i I_j^*} + \frac{I_j}{I_j^*} - \frac{I_i}{I_i^*}\Big) \\
&= -\Lambda_i \Phi\Big(\frac{S_i}{S_i^*}\Big) - \Lambda_i \Phi\Big(\frac{S_i^*}{S_i}\Big) - \sum_{j=1}^n a_{ij} S_j^* \Phi\Big(\frac{S_i^* S_j}{S_i S_j^*}\Big) - \sum_{j=1}^n b_{ij} I_j^* \Phi\Big(\frac{I_i^* I_j}{I_i I_j^*}\Big) \\
&+ \sum_{j=1}^n a_{ij} S_j^* \Big(\frac{S_j}{S_j^*} + \ln \frac{S_j^*}{S_j} - \frac{S_i}{S_i^*} - \ln \frac{S_i^*}{S_i}\Big) + \sum_{j=1}^n b_{ij} I_j^* \Big(\frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j} - \frac{I_i}{I_i^*} - \ln \frac{I_i^*}{I_i}\Big) \\
&\leq \sum_{j=1}^n a_{ij} S_j^* \Big(\frac{S_j}{S_j^*} + \ln \frac{S_j^*}{S_j} - \frac{S_i}{S_i^*} - \ln \frac{S_i^*}{S_i}\Big) + \sum_{j=1}^n b_{ij} I_j^* \Big(\frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j} - \frac{I_i}{I_i^*} - \ln \frac{I_i^*}{I_i}\Big) \\
&= \sum_{j=1}^n b_{ij} I_j^* \Big[\Big(\lambda \frac{S_j}{S_j^*} + \lambda \ln \frac{S_j^*}{S_j} + \frac{I_j}{I_j^*} + \ln \frac{I_j^*}{I_j}\Big) - \Big(\lambda \frac{S_i}{S_i^*} + \lambda \ln \frac{S_i^*}{S_i} + \frac{I_i}{I_i^*} + \ln \frac{I_i^*}{I_i}\Big)\Big].
\end{aligned}
\tag{3.23}
$$

Let $\omega_{ij} = b_{ij} I_j^*$ and

$$G_i(S_i, I_i) = -\Big(\lambda \frac{S_i}{S_i^*} + \lambda \ln \frac{S_i^*}{S_i} + \frac{I_i}{I_i^*} + \ln \frac{I_i^*}{I_i}\Big).$$

Then we have

$$\overset{\bullet}{V_i} \leq \sum_{j=1}^n \omega_{ij} [G_i(S_i, I_i) - G_j(S_j, I_j)],$$

namely, V_i, G_i and ω_{ij} satisfy the assumptions of Theorem 2.6 and Corollary 2.8. Hence,

$$V = \sum_{i=1}^n c_i V_i(S_i, I_i)$$

as defined in Theorem 2.6, is a Lyapunov function for system (3.15). Since B is irreducible, we know that $c_i > 0$ for all i (see Proposition 2.1), and thus $\overset{\bullet}{V} = 0$ implies that $S_i = S_i^*$ for all i. From the first equation of (3.15), we obtain

$$0 = (S_i^*)' = \Lambda_i - \beta_i S_i^* I_i - \mu_i^S S_i^* + \sum_{j=1}^{n} a_{ij} S_j^* - \sum_{j=1}^{n} a_{ji} S_i^*, \quad i = 1, 2, \ldots, n,$$

which implies that $I_i = I_i^*$ for all i. We have verified that the largest invariant set on which $\overset{\bullet}{V} = 0$ is the singleton $\{P^*\}$. Note that $\overset{\circ}{\Gamma}$ is positively invariant and system (3.15) is uniformly persistent. Therefore, by the LaSalle Invariance Principle [75], P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. □

Theorem 3.7 can be readily applied to the n-patch epidemic model in [87] and yield global-stability analysis. When the disease has permanent immunity, the global stability of the endemic equilibrium for two-patch epidemic models in [19, 115] is resolved as a special case of Theorem 3.7.

3.6 A Multi-Group SEIR Epidemic Model

Multi-group epidemic models have been proposed in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations. Heterogeneity in host population can result from many factors. Individual hosts can be divided into groups according to different contact patterns such as those among children and adults for Measles and Mumps, or to distinct number of sexual partners for sexually transmitted diseases and HIV/AIDS. Groups can be geographical such as communities, cities, and countries, or epidemiological, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. Multi-group models can also be used to investigate infectious diseases with multiple hosts such as West-Nile virus and vector borne diseases such as Malaria.

In this section, we consider a multi-group SEIR epidemic model in which inter-group cross infections are described by nonlinear functions. The model

$$
\begin{aligned}
S_i' &= \Lambda_i - \mu_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j), \\
E_i' &= \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j) - (\mu_i^E + \epsilon_i) E_i, \\
I_i' &= \epsilon_i E_i - (\mu_i^I + \gamma_i) I_i, \qquad\qquad i = 1, 2, \cdots, n,
\end{aligned}
\tag{3.24}
$$

describes the spread of an infectious disease in a heterogeneous population, which is partitioned into n homogeneous groups. Each group i is further compartmentalized into S_i, E_i, and I_i, which denote the subpopulations that are susceptible to the disease, infected but non-infectious, and infectious, respectively. The nonlinear coupling term $\beta_{ij} f_{ij}(S_i, I_j)$ represents the cross infection from group j to group i. The parameter ϵ_i represents the rate of becoming infectious after latent period in the i-th group. All other parameters in (3.24) are similarly defined as in Section 3.5. For detailed discussions of the multi-group model and interpretations of parameters, we refer the reader to [37, 112].

Assume that $\epsilon_i > 0$ and $\mu_i^* > 0$, where $\mu_i^* = \min\{\mu_i^S, \mu_i^E, \mu_i^I + \gamma_i\}$. Based on biological considerations, we assume that $f_{ij}(0, I_j) = 0, f_{ij}(S_i, 0) = 0$, and $f_{ij}(S_i, I_j) > 0$ for $S_i > 0, I_j > 0$. We also assume that f_{ij} is sufficiently smooth. For each i, adding the three equations in (3.24) gives

$$
(S_i + E_i + I_i)' \le \Lambda_i - \mu_i^*(S_i + E_i + I_i).
$$

Hence

$$
\limsup_{t\to\infty} (S_i + E_i + I_i) \le \Lambda_i / \mu_i^*.
$$

Similarly, from the S_i equation we obtain

$$
\limsup_{t\to\infty} S_i \le \Lambda_i / \mu_i^S.
$$

Therefore, all ω-limit sets of system (3.24) are contained in the following bounded region in the nonnegative cone of $\mathbb{R}^{3n}$,

$$\Gamma = \left\{(S_1, E_1, I_1, \cdots, S_n, E_n, I_n) \in \mathbb{R}^{3n}_+ \mid S_i \leq \frac{\Lambda_i}{\mu_i^S},\ S_i+E_i+I_i \leq \frac{\Lambda_i}{\mu_i^*},\ 1 \leq i \leq n\right\}. \tag{3.25}$$

It can be verified that region Γ is positively invariant. System (3.24) always has the *disease-free equilibrium* $P_0 = (S_1^0, 0, 0, \cdots, S_n^0, 0, 0)$, on the boundary of Γ, where $S_i^0 = \Lambda_i/\mu_i^S$. An equilibrium $P^* = (S_1^*, E_1^*, I_1^*, \cdots, S_n^*, E_n^*, I_n^*)$ in the interior $\mathring{\Gamma}$ of Γ is called an *endemic equilibrium*, where $S_i^*, E_i^*, I_i^* > 0$ satisfy the equilibrium equations

$$\Lambda_i = \mu_i^S S_i^* + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*), \tag{3.26}$$

$$(\mu_i^E + \epsilon_i)E_i^* = \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*), \tag{3.27}$$

$$\epsilon_i E_i^* = (\mu_i^I + \gamma_i)I_i^*. \tag{3.28}$$

One of the earliest results on multi-group models is by Lajmanovich and Yorke [73] on a class of n-group SIS models for gonorrhea. The global stability of the unique endemic equilibrium is proved using a quadratic global Lyapunov function. Global stability results also exist for other types of multi-group models; see e.g., [10, 49, 50, 83, 111]. Results in the opposite direction also exist in the literature. For a class of n-group SIR models with proportionate incidence, uniqueness of endemic equilibria may not hold when $R_0 > 1$ [58, 112]. Due to the large scale and complexity of multi-group models, the global stability of the endemic equilibrium of (3.24) has been a 30-year open problem in epidemiology until Guo, Li, and Shuai in [36, 37] applied the graph-theoretic approach to construct a global Lyapunov function for (3.24) with bilinear incidence $f_{ij}(S_i, I_j) = I_j S_i$. For general nonlinear incidence, the global stability of the endemic equilibrium remains unsolved.

Let $\mathcal{G}$ be a digraph with n vertices, in which each vertex represents a group. An arc (j, i) exists if and only if $\beta_{ij} > 0$, namely, if the disease can be transmitted from group j to group i. System (3.24) can thus be regarded as a coupled system on $\mathcal{G}$. See Figure 3.5. We note that $\mathcal{G}$ is strongly connected if and only if transmission matrix (β_{ij}) is irreducible.

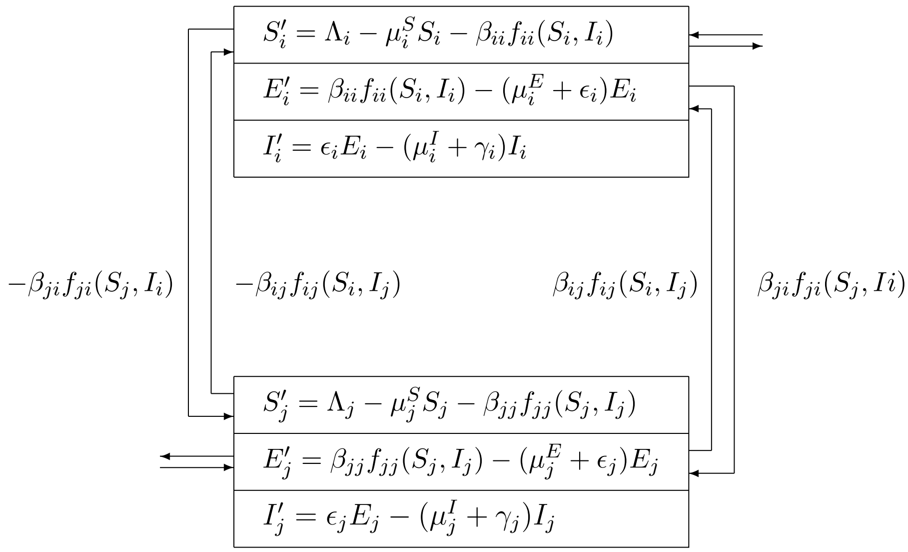

Figure 3.5: A multi-group SEIR model on a network

In the rest of this section we consider the following basic assumptions on functions $f_{ij}(S_i, I_j)$:

(H_1) $0 < \lim\limits_{I_j \to 0^+} \frac{f_{ij}(S_i, I_j)}{I_j} =: C_{ij}(S_i) \leq +\infty$ for all $0 < S_i \leq S_i^0$;

(H_2) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i) I_j$ for sufficiently small I_j;

(H_3) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i) I_j$ for all $I_j > 0$;

(H_4) $C_{ij}(S_i) < C_{ij}(S_i^0)$ for all $0 < S_i < S_i^0$.

Classes of $f_{ij}(S_i, I_j)$ satisfying some assumptions of (H_1)-(H_4) include many common incidence functions such as the bilinear function $f_{ij}(S_i, I_j) = I_j S_i$, the nonlinear function $f_{ij}(S_i, I_j) = I_j^{p_j} S_i^{q_i}$, and the saturated incidence $f_{ij}(S_i, I_j) = \frac{I_j^{p_j}}{I_j + A_j} \frac{S_i^{q_i}}{S_i + B_i}$.

Assume that $f_{ij}(S_i, I_j)$ satisfies (H_1), and let

$$R_0 = \rho(M_0) \tag{3.29}$$

denote the spectral radius of the matrix

$$M_0 = M(S_1^0, S_2^0, \ldots, S_n^0) = \left(\frac{\beta_{ij}\, \epsilon_i\, C_{ij}(S_i^0)}{(\mu_i^E + \epsilon_i)(\mu_i^I + \gamma_i)} \right)_{1 \le i,j \le n}.$$

If $C_{ij}(S_i^0) = +\infty$ for some i and j, we set $R_0 = +\infty$. The parameter R_0 is referred to as the basic reproduction number. Its biological significance is that if $R_0 < 1$ the disease dies out while if $R_0 > 1$ the disease becomes endemic [27, 113]. The following results for system (3.24) can be proved the same way as in [10, 36, 49, 50, 83, 111, 112].

Proposition 3.8. *Assume that $B = (\beta_{ij})$ is irreducible and $f_{ij}(S_i, I_j)$ satisfies (H_1) for all i, j.*

(1) *If $R_0 \le 1$ and assumptions (H_2) and (H_4) hold, then for system* (3.24), *P_0 is locally asymptotically stable.*

(2) *If $R_0 \le 1$ and assumptions (H_3) and (H_4) hold, then P_0 is the unique equilibrium and it is globally asymptotically stable in Γ.*

(3) *If $R_0 > 1$, then P_0 is unstable and system* (3.24) *is uniformly persistent. Furthermore, there exists an endemic equilibrium P^* for system* (3.24).

When $R_0 > 1$, an endemic equilibrium P^* exists by Proposition 3.8. A long-standing open question in mathematical epidemiology is whether a multi-group epidemic model such as (3.24) had a unique endemic equilibrium P^* when $R_0 > 1$, and whether P^* is globally asymptotically stable when it is unique [112]. We prove the following theorem, which answers this open problems for system (3.24).

Theorem 3.9. *Suppose that $R_0 > 1$ and thus an endemic equilibrium $P^* = (S_1^*, E_1^*, I_1^*, \cdots, S_n^*, E_n^*, I_n^*)$ exists. Assume that $B = (\beta_{ij})$ is irreducible and all f_{ij} satisfy (H_1). If f_{ij} satisfy, for every $1 \le i, j \le n$, the following conditions*

$$(S_i - S_i^*)(f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)) > 0, \qquad \forall\, S_i \ne S_i^*, \tag{3.30}$$

$$\Big(f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*) - f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)\Big) \cdot \Big(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{I_j} - \frac{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^*}\Big) \leq 0, \qquad \forall\, S_i, I_j > 0, \tag{3.31}$$

then P^* is unique and globally asymptotically stable in $\overset{\circ}{\Gamma}$.

Proof. The case $n = 1$ is proved in [66]. We only consider $n \geq 2$. Let $P^* = (S_1^*, E_1^*, I_1^*, \cdots, S_n^*, E_n^*, I_n^*)$, where all $S_i^*, E_i^*, I_i^* > 0$ for all $1 \leq i \leq n$, denote an endemic equilibrium which exists from Proposition 3.8-(3). We prove that P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. In particular, this implies that the endemic equilibrium is unique. Let

$$V_i(S_i, E_i, I_i) = \int_{S_i^*}^{S_i} \frac{f_{ii}(\xi, I_i^*) - f_{ii}(S_i^*, I_i^*)}{f_{ii}(\xi, I_i^*)} d\xi + E_i^* \Phi\Big(\frac{E_i}{E_i^*}\Big) + \frac{\mu_i^E + \epsilon_i}{\epsilon_i} I_i^* \Phi\Big(\frac{I_i}{I_i^*}\Big),$$

be the Lyapunov function for a single-group model as considered in [66]. We verify that V_i satisfies the assumptions of Theorem 2.6. Using the equilibrium equations (3.26)-(3.28), we obtain

$$\begin{aligned} \overset{\bullet}{V_i} \;=\; & \Big(1 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\Big)\Big(\Lambda_i - \mu_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j)\Big) + \Big(\sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) \\ & -(\mu_i^E + \epsilon_i) E_i\Big)\Big(1 - \frac{E_i^*}{E_i}\Big) + \frac{\mu_i^E + \epsilon_i}{\epsilon_i}\Big(1 - \frac{I_i^*}{I_i}\Big)\Big(\epsilon_i E_i - (\mu_i^I + \gamma_i) I_i\Big) \\ =\; & \Big(1 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\Big)\Big(\mu_i^S S_i^* + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) - \mu_i^S S_i - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j)\Big) \\ & + \Big(1 - \frac{E_i^*}{E_i}\Big)\Big(\sum_{j=1}^n \beta_{ij} f_{ij}(S_i, I_j) - \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) \frac{E_i}{E_i^*}\Big) \\ & + \sum_{j=1}^n \beta_{ij} \frac{f_{ij}(S_i^*, I_j^*)}{\epsilon_i E_i^*}\Big(1 - \frac{I_i^*}{I_i}\Big)\Big(\epsilon_i E_i - \frac{\epsilon_i E_i^* I_i}{I_i^*}\Big) \\ =\; & -\frac{\mu_i^S}{f_{ii}(S_i, I_i^*)}(S_i - S_i^*)(f_{ii}(S_i, I_i^*) - f_{ii}(S_i^*, I_i^*)) + \sum_{j=1}^n \beta_{ij} f_{ij}(S_i^*, I_j^*) \cdot \\ & \Big(3 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)} + \frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - \frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}\Big). \end{aligned} \tag{3.32}$$

Let

$$a_{ij} = \beta_{ij} f_{ij}(S_i^*, I_j^*),$$

$$G_i(I_i) = -\Phi\Big(\frac{I_i}{I_i^*}\Big) = 1 - \frac{I_i}{I_i^*} + \ln\frac{I_i}{I_i^*},$$

and

$$F_{ij}(S_i, E_i, I_i, I_j) = 3 - \frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)} + \frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - \frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i} - \frac{I_i}{I_i^*} - \frac{E_i I_i^*}{E_i^* I_i}.$$

Then, using condition (3.30) and above notations, we have

$$\overset{\bullet}{V_i} \leq \sum_{i,j=1}^{n} a_{ij} F_{ij}(S_i, E_i, I_i, I_j).$$

Recall that $\Phi(x) = 1 - x + \ln x \leq 0$ for $x > 0$ and equality holds only at $x = 1$. Furthermore,

$$\begin{aligned}
F_{ij} &= G_i(x_i) - G_j(x_j) + \Phi\Big(\frac{f_{ii}(S_i^*, I_i^*)}{f_{ii}(S_i, I_i^*)}\Big) + \Phi\Big(\frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\Big) \\
&\quad + \Phi\Big(\frac{E_i I_i^*}{E_i^* I_i}\Big) + \Phi\Big(\frac{f_{ij}(S_i, I_j) E_i^*}{f_{ij}(S_i^*, I_j^*) E_i}\Big) + \Big(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - 1\Big) \cdot \\
&\quad \Big(1 - \frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\Big) \\
&\leq G_i(x_i) - G_j(x_j) + \Big(\frac{f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}{f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)} - 1\Big) \cdot \\
&\quad \Big(1 - \frac{I_j f_{ij}(S_i^*, I_j^*) f_{ii}(S_i, I_i^*)}{I_j^* f_{ij}(S_i, I_j) f_{ii}(S_i^*, I_i^*)}\Big),
\end{aligned}$$

Under condition (3.31), we can show that V_i, F_{ij}, G_i, a_{ij} satisfy the assumptions of Theorem 2.6 and Corollary 2.8. Therefore, the function

$$V = \sum_{i=1}^{n} c_i V_i(S_i, E_i, I_i)$$

as defined in Theorem 2.6 is a Lyapunov function for (3.24), namely,

$$\overset{\bullet}{V} \leq 0 \quad \text{for all} \quad (S_1, E_1, I_1, \cdots, S_n, E_n, I_n) \in \overset{\circ}{\Gamma}.$$

It can be verified similarly as in Section 3.1 that the largest invariant set where $\overset{\bullet}{V} = 0$ is the singleton $\{P^*\}$. Since $\overset{\circ}{\Gamma}$ is positively invariant and system (3.24) is uniformly persistent, by the LaSalle Invariance Principle [75], P^* is globally asymptotically stable in $\mathring{\Gamma}$. This completes the proof of Theorem 3.9. $\square$

Remarks

1. Condition (3.30) holds if $f_{ii}(S_i, I_i^*)$ is strictly monotonically increasing with respect to S_i.

2. In the special case $f_{ij}(S_i, I_j) = h_i(S_i)g_j(I_j)$, condition (3.31) becomes

$$(g_j(I_j) - g_j(I_j^*))\Big(\frac{g_j(I_j)}{I_j} - \frac{g_j(I_j^*)}{I_j^*}\Big) \le 0. \tag{3.33}$$

   If $g_j(I_j)$ is C^1 for $I_j > 0$, then a sufficient condition for (3.33) is

$$0 \le g_j'(I_j) \le \frac{g_j(I_j)}{I_j}, \quad \forall\, I_j > 0. \tag{3.34}$$

   Furthermore, if $g_j(I_j)$ is monotonically increasing and concave down, then (3.34) holds, and so does (3.33).

3. In the special case $f_{ij}(S_i, I_j) = S_i I_j$, system (3.24) becomes the standard multi-group SEIR model studied in [37]. Theorem 3.9 generalizes Theorem 1.1 in [37], which contains a complete resolution of a well-known open problem on the global-stability of the endemic equilibrium for multi-group epidemic models [112].

4. When $n = 1$, Theorem 3.9 contains earlier results on single-group SEIR models, see [65, 66, 67, 68, 78, 82] and references therein.

Chapter 4. Applications to Delay Differential Equation Models

In this chapter we demonstrate that the graph-theoretic approach developed in Chapter 2 can be also applied to complex systems with time delays. An ecological model on a time-delayed spatial dispersal network is studied in Section 4.1. The delay occurs in connections of the network. We investigate a multi-group epidemic model with age structure in Section 4.2. The delay is of distributed type and occurs in both the network connections and the vertex systems. Our approach allows us to prove global stability for these complex systems.

4.1 A Patchy Single-Species Model with Finite Delays

In Chapter 3 we have applied the graph-theoretic approach to several spatial heterogeneous models in ecology and epidemiology. We assume that dispersal and travel among different patches happen instantaneously. A time delay, however, is natural to include in these models to incorporate the travel time from one patch to the other. The resulting models are systems of delay differential equations. Various complicated dynamical behaviors have been observed for these delay systems [70], such as delay induced instability and oscillations. In this section we demonstrate that our graph-theoretic approach can be applied to establish the global stability of these delay systems. As an example, a patchy single-species model with dispersal delay is investigated in the following.

In Section 3.2 we have studied the patchy single-species model

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \qquad i = 1, 2, \ldots, n. \tag{4.1}$$

The couplings in system (4.1) are provided by dispersal among different patches and the dispersal is assumed to happen instantaneously. That is, the influence from patch j to patch i takes the form:

$$d_{ij} x_j - d_{ij} \alpha_{ij} x_i.$$

In this section, we assume that the populations require some time to travel among patches and thus the influence from patch j to patch i is given as follows:

$$d_{ij} e^{-\lambda_{ij}\tau_{ij}} x_j(t - \tau_{ij}) - \delta_{ij} x_i(t).$$

Here $\tau_{ij} \geq 0$ is the time which population takes to travel from patch j to patch i, $\lambda_{ij} \geq 0$ represents the death rate during the travel, and $\delta_{ij} \geq 0$ is the rate of population in patch i traveling to patch j. Therefore, system (4.1) can be generalized to the following coupled system on a network given by Figure 4.1:

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} x_j(t - \tau_{ij}) - \sum_{j=1}^{n} \delta_{ij} x_i, \quad i = 1, 2, \ldots, n. \tag{4.2}$$

Takeuchi et al. [110] studied a special case of system (4.2) when $f_i(x_i) = a_i - b_i x_i$ and proved the global stability of a positive equilibrium. Using the method of Lyapunov functionals and our graph-theoretic approach, we prove the global stability for system (4.2) with general functions f_i.

Denote $\tau = \max\{\tau_{ij} : i, j = 1, 2, \ldots, n\}$. Let C be the Banach space of

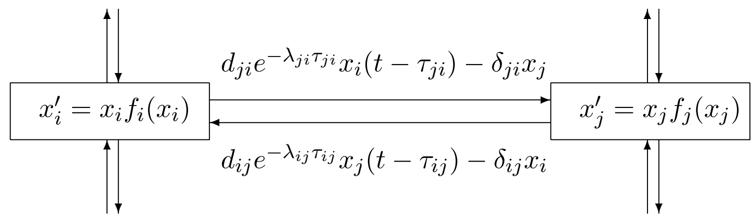

Figure 4.1: A coupled single-species system with delays

continuous functions on $[-\tau, 0]$ with uniform norm. We consider system (4.2) in the phase space

$$X = \prod_{i=1}^{n} C. \tag{4.3}$$

We consider positive initial conditions for system (4.2)

$$x_{i\,0} = \phi^i, \quad i = 1, 2, \ldots, n, \tag{4.4}$$

where $\phi^i \in C$ satisfies $\phi^i(s) > 0$ for $-\tau \le s \le 0$. Let

$$\Delta = \{(\phi^1(\cdot), \phi^2(\cdot), \ldots, \phi^n(\cdot)) \in X \mid \phi^i(s) > 0, \forall s \in [-\tau, 0], i = 1, 2, \ldots, n\}.$$

It can be verified that Δ is positively invariant. We have the following global-stability result.

Theorem 4.1. *Suppose that the following assumptions hold.*

(1) *The dispersal matrix (d_{ij}) of (4.2) is irreducible.*

(2) *$f_i'(x_i) \le 0$ for all $x_i > 0, i = 1, 2, \ldots, n$, and there exists k such that $f_k'(u) \not\equiv 0$ in any open interval of $\mathbb{R}_+$.*

Then, whenever a positive equilibrium E^* exists, it is unique and globally asymptotically stable in Δ.

Proof. Let $E^* = (x_1^*, x_2^*, \cdots, x_n^*)$, $x_i^* > 0$, $i = 1, 2, \ldots, n$, denote a positive equilibrium of (4.2). Then x_i^* satisfies

$$\sum_{j=1}^{n} \delta_{ij} = f_i(x_i^*) + \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} \frac{x_j^*}{x_i^*}. \tag{4.5}$$

We show that E^* is globally asymptotically stable in Δ, and thus is unique. Set $V_i : X \to \mathbb{R}_+$ defined as

$$\begin{aligned} V_i(\phi) &:= \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} \int_0^{\tau_{ij}} x_j^* \Phi\Big(\frac{\phi^j(-r)}{x_j^*}\Big)\, dr + x_i^* \Phi\Big(\frac{\phi^i(0)}{x_i^*}\Big) \\ &= \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} \int_0^{\tau_{ij}} \Big(\phi^j(-r) - x_j^* - x_j^* \ln \frac{\phi^j(-r)}{x_j^*}\Big)\, dr \\ &\quad + \phi^i(0) - x_i^* - x_i^* \ln \frac{\phi^i(0)}{x_i^*}. \end{aligned}$$

Recall that $\Phi(x) = x - 1 - \ln x \geq 0$ for all $x > 0$ and $\Phi(x) = 0$ if and only if $x = 1$. Using integration by parts, we have

$$\begin{aligned} & \int_0^{\tau_{ij}} \frac{\partial}{\partial t}\Big(x_j(t-r) - x_j^* - x_j^* \ln \frac{x_j(t-r)}{x_j^*}\Big)\, dr \\ = & -\int_0^{\tau_{ij}} \frac{\partial}{\partial r}\Big(x_j(t-r) - x_j^* - x_j^* \ln \frac{x_j(t-r)}{x_j^*}\Big)\, dr \\ = & -\Big(x_j(t-\tau_{ij}) - x_j(t) + x_j^* \ln \frac{x_j(t)}{x_j(t-\tau_{ij})}\Big). \end{aligned} \tag{4.6}$$

Direct calculation along with (4.5) and (4.6) yields

$$\begin{aligned} \overset{\bullet}{V_i} &= \frac{d}{dt} V_i(x(t+s)_{s\in[-\tau,0]}) \\ &= (x_i - x_i^*)\Big[f_i(x_i) + \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} \frac{x_j(\cdot - \tau_{ij})}{x_i} - \sum_{j=1}^{n} \delta_{ij}\Big] \\ &\quad - \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} \Big(x_j(\cdot - \tau_{ij}) - x_j + x_j^* \ln \frac{x_j}{x_j(\cdot - \tau_{ij})}\Big) \\ &= (x_i - x_i^*)(f(x_i) - f(x_i^*)) \\ &\quad + \sum_{j=1}^{n} d_{ij} e^{-\lambda_{ij}\tau_{ij}} x_j^* \Big(\Phi\Big(\frac{x_j}{x_j^*}\Big) - \Phi\Big(\frac{x_i}{x_i^*}\Big) - \Phi\Big(\frac{x_i^* x_j(\cdot - \tau_{ij})}{x_i x_j^*}\Big)\Big). \end{aligned} \tag{4.7}$$

Let

$$a_{ij} = d_{ij} e^{-\lambda_{ij}\tau_{ij}} x_j^*,$$

$$F_{ij}(x_i, x_j) = \Phi\Big(\frac{x_j}{x_j^*}\Big) - \Phi\Big(\frac{x_i}{x_i^*}\Big) - \Phi\Big(\frac{x_i^* x_j(\cdot - \tau_{ij})}{x_i x_j^*}\Big),$$

and

$$G_i(x_i) = -\Phi\Big(\frac{x_i}{x_i^*}\Big).$$

Then we have

$$\overset{\bullet}{V_i} \leq \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j),$$

and

$$\begin{aligned} F_{ij}(x_i, x_j) &= G_i(x_i) - G_j(x_j) - \Phi\Big(\frac{x_i^* x_j(\cdot - \tau_{ij})}{x_i x_j^*}\Big) \\ &\leq G_i(x_i) - G_j(x_j). \end{aligned}$$

Here we use the fact: $(x_i - x_i^*)(f(x_i) - f(x_i^*)) \leq 0$. We have shown that V_i, F_{ij}, G_i, and a_{ij} satisfy the assumptions of Theorem 2.6 and Corollary 2.8. Therefore,

$$V(x_1(\cdot), \cdots, x_n(\cdot)) = \sum_{i=1}^{n} c_i V_i(x_i(\cdot))$$

as defined in Theorem 2.6 is a Lyapunov functional for (4.2), namely,

$$\overset{\bullet}{V} \leq 0 \quad \text{for all} \quad (x_1(\cdot), \cdots, x_n(\cdot)) \in \Delta.$$

Using a similar argument as in Section 3.1, we can show that $\overset{\bullet}{V} = 0$ if and only if $x_i = x_i^*$ for all i. By the LaSalle-Lyapunov Theorem (see [75, Theorem 3.4.7] or [43, Theorem 5.3.1]), we conclude that E^* is globally attractive in Δ. Furthermore, it can be verified that E^* is locally stable using the same proof as one for Corollary 5.3.1 in [43]. Therefore, E^* is globally asymptotically stable in Δ. This completes the proof of Theorem 4.1.

□

When $f_i(x_i) = a_i - b_i x_i$ for all i, Theorem 4.1 contains an earlier result in [110] where the global stability of E^* was proved under the condition that $b_i > 0$ for all i while Theorem 4.1 only requires $b_k > 0$ for some k.

When dispersal delays are incorporated into ordinary differential equation models (3.12) and (3.14), we obtain a patchy predator-prey model with dispersal delays and a patchy SIR epidemic model with dispersal delays. We remark that the same proof of Theorem 4.1 can be carried out to prove the global-stability of these delay systems.

4.2 A Multi-Group Epidemic Model with Infinite Distributed Delay

In this section, we consider a multi-group epidemic model that describes the disease spread in a heterogeneous host population with general age-structure and varying infectivity. The host population is divided into several homogeneous groups. Let S_i, E_i, I_i and R_i denote the susceptible, infected but non-infectious, infectious, and recovered populations in the i-th group, respectively. Let $a_i(t,r)$ denote the population of infectious individuals in the i-th group with respect to the age of infection r at time t, and $I_i(t) = \int_{r=0}^{\infty} a_i(t,r)dr$. Let $h_i(r) \geq 0$ be a continuous kernel function that represents the infectivity at the age of infection r. The disease incidence in the i-th group, assuming a bilinear incidence form, can be calculated as

$$\sum_{j=1}^{n} \beta_{ij} S_i(t) \int_{r=0}^{\infty} h_j(r) a_j(t,r)dr, \tag{4.8}$$

where the sum takes into account of cross-infections from all groups. In the special case $h_i(r) \equiv 1$, the incidence in (4.8) becomes $\sum_{j=1}^{n} \beta_{ij} S_i(t) I_j(t)$. Therefore, we consider the following system of differential equations

$$\begin{aligned}
S_i' &= \Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} h_j(r) a_j(t,r)dr - \mu_i^S S_i, \\
E_i' &= \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} h_j(r) a_j(t,r)dr - (\mu_i^E + \epsilon_i) E_i, \\
I_i' &= \epsilon_i E_i - (\mu_i^I + \gamma_i) I_i, \\
R_i' &= \gamma_i I_i - \mu_i^R R_i, \qquad\qquad i = 1, 2, \cdots, n.
\end{aligned} \tag{4.9}$$

Here all parameter values are assumed to be nonnegative and $\Lambda_i, \mu_i^S, \mu_i^E > 0$ for all i. For detailed discussions of the model, we refer the reader to [37, 101] and references therein. Note that

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\Big) a_i(t,r) &= -(\mu_i^I + \gamma_i) a_i(t,r), \\ a_i(t,0) &= \epsilon_i E_i(t), \end{aligned}$$

whose solution is

$$a_i(t,r) = a_i(t-r,0) e^{-(\mu_i^I+\gamma_i)r} = \epsilon_i E_i(t-r) e^{-(\mu_i^I+\gamma_i)r}. \tag{4.10}$$

Substituting (4.10) into (4.9) we obtain

$$\begin{aligned} S_i' &= \Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} h_j(r) \epsilon_j E_j(t-r) e^{-(\mu_j^I+\gamma_j)r} dr - \mu_i^S S_i, \\ E_i' &= \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} h_j(r) \epsilon_j E_j(t-r) e^{-(\mu_j^I+\gamma_j)r} dr - (\mu_i^E + \epsilon_i) E_i, \\ I_i' &= \epsilon_i E_i - (\mu_i^I + \gamma_i) I_i, \\ R_i' &= \gamma_i I_i - \mu_i^R R_i, \qquad\qquad i = 1, 2, \cdots, n. \end{aligned} \tag{4.11}$$

Since the variables I_i and R_i do not appear in the first two equations of (4.11), we can consider the following reduced system with distributed time delays and general kernel functions

$$\begin{aligned} S_i' &= \Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} f_j(r) E_j(t-r) dr - \mu_i^S S_i, \\ E_i' &= \sum_{j=1}^{n} \beta_{ij} S_i \int_{r=0}^{\infty} f_j(r) E_j(t-r) dr - (\mu_i^E + \epsilon_i) E_i, \qquad i = 1, 2, \cdots, n. \end{aligned} \tag{4.12}$$

Here the kernel function $f_i(r) \geq 0$ is continuous and $\int_{r=0}^{\infty} f_i(r) dr = \xi_i > 0$. While this system is derived from a general age-structured model (4.9), it can also be interpreted as a multi-group model for an infectious disease whose latent period r in hosts has a general probability density function $\frac{1}{\xi_i} f_i(r)$, for the i-th group.

System (4.12) can be regarded as a coupled system of delay differential equations on a network; see Figure 4.2 for more details. We will establish the global dynamics of system (4.12).

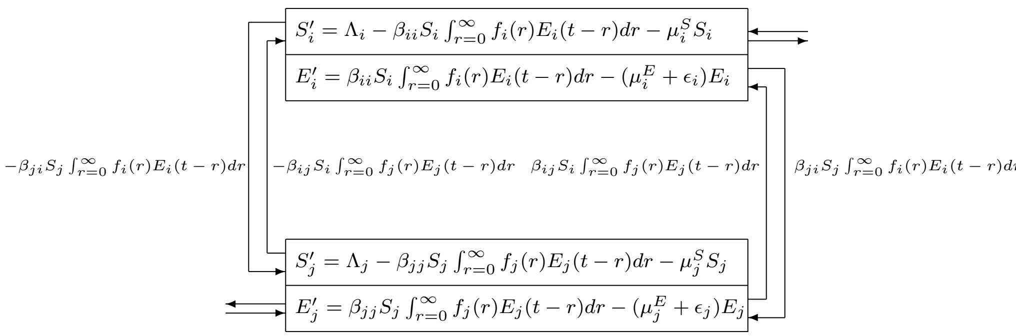

Figure 4.2: A multi-group model with delays

The basic reproduction number $\mathcal{R}_0$ is defined as the expected number of secondary cases produced in an entirely susceptible population by a typical infected individual during its entire infectious period [27]. Intuitively, if $\mathcal{R}_0 < 1$, the disease dies out from the host population, and if $\mathcal{R}_0 > 1$, the disease will persist. Let $S_i^0 = \frac{\Lambda_i}{\mu_i^S}$ for all i. The next generation matrix for system (4.12) is

$$M_0 = \Big(\frac{\beta_{ij} S_i^0 \xi_j}{\mu_i^E + \epsilon_i}\Big)_{n\times n}. \tag{4.13}$$

Motivated by [27, 113, 118], we define the basic reproduction number as the spectral radius of M_0,

$$\mathcal{R}_0 = \rho(M_0). \tag{4.14}$$

We make the following assumption on the kernel function f_i in (4.12)

$$\int_{r=0}^{\infty} f_i(r) e^{\lambda_i r} dr < \infty, \tag{4.15}$$

where λ_i is a positive number, $i = 1, 2, \ldots, n$. Define the following Banach space of fading memory type (see e.g. [9] and references therein)

$$C_i = \Big\{\phi \in C((-\infty,0],\mathbb{R}) \mid s \mapsto \phi(s)e^{\lambda_i s} \text{ is uniformly continuous on } (-\infty,0], \text{ and } \sup_{s\le 0} |\phi(s)|e^{\lambda_i s} < \infty\Big\}, \tag{4.16}$$

with norm $||\phi||_i = \sup_{s\le 0} |\phi(s)|e^{\lambda_i s}$. For $\psi \in C(\mathbb{R},\mathbb{R})$, let $\psi_t \in C_i$ be such that $\psi_t(s) = \psi(t+s)$, $s \in (-\infty,0]$. Let $S_{i,0} \in \mathbb{R}_+$ and $\phi_i \in C_i$ such that $\phi_i(s) \ge 0, s \in (-\infty,0]$. We consider solutions of system (4.12), $(S_1(t), E_{1\,t}, \cdots, S_n(t), E_{n\,t})$ with initial conditions

$$S_i(0) = S_{i,0}, \quad E_{i\,0} = \phi^i, \quad i = 1,2,\ldots,n. \tag{4.17}$$

Standard results of functional differential equations (see [9, Theorem 2.1]) imply that $E_{i\,t} \in C_i$ for all $t > 0$. We consider system (4.12) in the phase space

$$X = \prod_{i=1}^{n} \Big(\mathbb{R} \times C_i\Big). \tag{4.18}$$

It can be verified that solutions of (4.12) in X with initial conditions (4.17) remain nonnegative. In particular, $S_i(t) > 0$ for all $t > 0$. From the first equation of (4.12), we obtain

$$S_i'(t) \le \Lambda_i - \mu_i^S S_i(t).$$

Hence,

$$\limsup_{t\to\infty} S_i(t) \le \frac{\Lambda_i}{\mu_i^S}.$$

For each i, adding the two equations in (4.12) gives

$$(S_i(t) + E_{i\,t}(0))' \le \Lambda_i - \mu_i^*(S_i(t) + E_{i\,t}(0)),$$

where $\mu_i^* = \min\{\mu_i^S, \mu_i^E + \epsilon_i\}$. Hence, we have

$$\limsup_{t\to\infty}(S_i(t) + E_{i\,t}(0)) \le \frac{\Lambda_i}{\mu_i^*}.$$

Therefore, the following set is positively invariant for system (4.12).

$$\Theta = \Big\{(S_1, E_1(\cdot), \cdots, S_n, E_n(\cdot)) \in X \;\Big|\; 0 \le S_i \le \frac{\Lambda_i}{\mu_i^S},\ 0 \le S_i + E_i(0) \le \frac{\Lambda_i}{\mu_i^*}, \\ E_i(s) \ge 0, \forall s \in (-\infty, 0], i = 1, \dots, n\Big\}. \tag{4.19}$$

Let

$$\overset{\circ}{\Theta} = \Big\{(S_1, E_1(\cdot), \cdots, S_n, E_n(\cdot)) \in X \;\Big|\; 0 < S_i < \frac{\Lambda_i}{\mu_i^S},\ 0 < S_i + E_i(0) < \frac{\Lambda_i}{\mu_i^*}, \\ E_i(s) > 0, \forall s \in (-\infty, 0], i = 1, \dots, n\Big\}. \tag{4.20}$$

It can be shown that $\overset{\circ}{\Theta}$ is the interior of Θ.

The equilibria of (4.12) are the same as those of the associated ODE system

$$\begin{aligned} S_i' &= \Lambda_i - \sum_{j=1}^{n} \beta_{ij}\xi_j S_i E_j - \mu_i^S S_i, \\ E_i' &= \sum_{j=1}^{n} \beta_{ij}\xi_j S_i E_j - (\mu_i^E + \epsilon_i)E_i, \qquad i = 1, 2, \dots, n. \end{aligned} \tag{4.21}$$

System (4.21) is similar to a multi-group SIR model considered in [36] with E_i relabeled as I_i. Results established in [36] can be readily applied to system (4.21). In the positively invariant region

$$\Gamma = \Big\{(S_1, E_1, \cdots, S_n, E_n) \in \mathbb{R}_+^{2n} \;\Big|\; S_i \le \frac{\Lambda_i}{\mu_i^S},\ S_i + E_i \le \frac{\Lambda_i}{\mu_i^*},\ 1 \le i \le n\Big\}, \tag{4.22}$$

system (4.21) has two possible equilibria: the disease-free equilibrium $P_0 = (S_1^0, 0, \cdots, S_n^0, 0)$, where $S_i^0 = \frac{\Lambda_i}{\mu_i^S}$, and the endemic equilibrium $P^* = (S_1^*, E_1^*,$

$\cdots, S_n^*, E_n^*)$, where $S_i^*, E_i^* > 0$ and satisfy

$$\Lambda_i = \sum_{j=1}^{n} \beta_{ij}\xi_j S_i^* E_j^* + \mu_i^S S_i^*, \tag{4.23}$$

$$\sum_{j=1}^{n} \beta_{ij}\xi_j S_i^* E_j^* = (\mu_i^E + \epsilon_i) E_i^*, \qquad i = 1, 2, \ldots, n. \tag{4.24}$$

We assume that the transmission matrix $B = (\beta_{ij})$ is irreducible. This is equivalent to assuming that for any two distinct groups i and j, individuals in E_j can infect those in S_i directly or indirectly. The following result is proved in [36].

Proposition 4.2 (Guo, Li, Shuai [36]). *Assume that $B = (\beta_{ij})$ is irreducible.*

(1) *If $\mathcal{R}_0 \leq 1$, then P_0 is the only equilibrium for system* (4.21) *and it is globally asymptotically stable in Γ.*

(2) *If $\mathcal{R}_0 > 1$, then P_0 is unstable and there exists a unique endemic equilibrium P^* for system* (4.21). *Furthermore, P^* is globally asymptotically stable in the interior of Γ.*

Since the delay system (4.12) and the ODE system (4.21) share the same equilibria, the following result follows from Proposition 4.2.

Proposition 4.3. *Assume that $B = (\beta_{ij})$ is irreducible.*

(1) *If $\mathcal{R}_0 \leq 1$, then P_0 is the only equilibrium for system* (4.12) *in Θ.*

(2) *If $\mathcal{R}_0 > 1$, then there exist two equilibria for system* (4.12) *in Θ: the disease-free equilibrium P_0 and a unique endemic equilibrium P^* defined by equations* (4.23) *and* (4.24).

The global dynamics of system (4.12) are completely established in the following result.

Theorem 4.4. *Assume that $B = (\beta_{ij})$ is irreducible.*

(1) *If* $\mathcal{R}_0 \leq 1$*, then the disease-free equilibrium* P_0 *of system* (4.12) *is globally asymptotically stable in* Θ*. If* $\mathcal{R}_0 > 1$*, then* P_0 *is unstable.*

(2) *If* $\mathcal{R}_0 > 1$*, then the endemic equilibrium* P^* *of system* (4.12) *is globally asymptotically stable in* $\overset{\circ}{\Theta}$*.*

Biologically, Theorem 4.4 implies that, if the basic reproduction number $\mathcal{R}_0 \leq 1$, then the disease always dies out from all groups; if $\mathcal{R}_0 > 1$, then the disease always persists in all groups at the unique endemic equilibrium level, irrespective of the initial conditions.

Theorem 4.4 includes several previous results. Choose the kernel function as

$$f_i(r) = \epsilon_i e^{-(\mu_i^I + \gamma_i)r}$$

and let $\tilde{I}_i = \int_{r=0}^{\infty} f_i(r) E_i(t-r) dr$. Then (4.12) takes the form

$$S_i' = \Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i \tilde{I}_j - \mu_i^S S_i,$$

$$E_i' = \sum_{j=1}^{n} \beta_{ij} S_i \tilde{I}_j - (\mu_i^E + \epsilon_i) E_i.$$

Using integration by parts we obtain

$$\tilde{I}_i' = \int_{r=0}^{\infty} f_i(r) \frac{\partial E_i(t-r)}{\partial t} dr = -\int_{r=0}^{\infty} f_i(r) \frac{\partial E_i(t-r)}{\partial r} dr = \epsilon_i E_i - (\mu_i^I + \gamma_i) \tilde{I}_i.$$

System (4.12) is thus reduced to a multi-group SEIR model governed by the system of ordinary differential equations considered in [37]. Note that

$$\int_{r=0}^{\infty} f_i(r) dr = \frac{\epsilon_i}{\mu_i^I + \gamma_i},$$

and the basic reproduction number in (4.14) becomes

$$\mathcal{R}_0 = \rho\Big(\frac{\beta_{ij} \epsilon_i \Lambda_i}{(\mu_i^E + \epsilon_i)(\mu_i^I + \gamma_i) \mu_i^S} \Big)_{1 \leq i,j \leq n},$$

which agrees with that given in [36, 37]. Thus the global stability results in [36, 37] are special cases of Theorem 4.4.

In the case $n = 1$, system (4.12) reduces to a single-group SEIR or SIR model with distributed delays studied in [11, 12, 14, 88, 93, 94, 101]. Theorem 4.4 generalizes the global stability results in [93, 94] to multi-group models.

Proof of Theorem 4.4-(1):

Since B is irreducible, we know that M_0, as defined in (4.13), is also irreducible, and has a positive left eigenvector $(\omega_1, \omega_2, \cdots, \omega_n)$ corresponding to the spectral radius $\rho(M_0) > 0$. Let

$$a_i = \frac{\omega_i}{\mu_i^E + \epsilon_i} \quad \text{and} \quad \alpha_i(r) = \int_{\sigma=r}^{\infty} f_i(\sigma) d\sigma.$$

Consider a Lyapunov functional

$$L = \sum_{i=1}^{n} a_i \Big(S_i^0 \Phi\Big(\frac{S_i}{S_i^0}\Big) + E_i + \sum_{j=1}^{n} \beta_{ij} S_i^0 \int_{r=0}^{\infty} \alpha_j(r) E_j(\cdot - r) dr \Big). \tag{4.25}$$

Note that $\Lambda_i = \mu_i^S S_i^0$, $\alpha_i(0) = \int_{\sigma=0}^{\infty} f_i(\sigma) d\sigma = \xi_i$, and $\Phi(x) \geq 0$ for all $x > 0$ with equality holding if and only if $x = 1$. Differentiating L along the solution of system (4.12) and using integration by parts, we obtain

$$\begin{aligned}
\overset{\bullet}{L} &= \sum_{i=1}^{n} a_i\Big(\Lambda_i - \mu_i^S S_i - \Lambda_i \frac{S_i^0}{S_i} + \sum_{j=1}^{n}\beta_{ij} S_i^0 \int_{r=0}^{\infty} f_j(r)E_j(t-r)\,dr + \mu_i^S S_i^0 \\
&\quad -(\mu_i^E+\epsilon_i)E_i + \sum_{j=1}^{n}\beta_{ij}S_i^0\int_{r=0}^{\infty}\alpha_j(r)\frac{\partial E_j(t-r)}{\partial t}\,dr\Big) \\
&= \sum_{i=1}^{n} a_i\Big[\mu_i^S S_i^0\Big(2-\frac{S_i}{S_i^0}-\frac{S_i^0}{S_i}\Big) + \sum_{j=1}^{n}\beta_{ij}S_i^0\int_{r=0}^{\infty} f_j(r)E_j(t-r)\,dr \\
&\quad -(\mu_i^E+\epsilon_i)E_i + \sum_{j=1}^{n}\beta_{ij}S_i^0\int_{r=0}^{\infty}\alpha_j(r)\Big(-\frac{\partial E_j(t-r)}{\partial r}\Big)\,dr\Big] \\
&= \sum_{i=1}^{n} a_i\Big[\mu_i^S S_i^0\Big(2-\frac{S_i}{S_i^0}-\frac{S_i^0}{S_i}\Big) + \sum_{j=1}^{n}\beta_{ij}S_i^0\int_{r=0}^{\infty} f_j(r)E_j(t-r)\,dr \\
&\quad -(\mu_i^E+\epsilon_i)E_i + \sum_{j=1}^{n}\beta_{ij}S_i^0\Big(\xi_j\,E_j - \int_{r=0}^{\infty} f_j(r)E_j(t-r)\,dr\Big)\Big] \\
&\le \sum_{i=1}^{n}\frac{\omega_i}{\mu_i^E+\epsilon_i}\Big(\sum_{j=1}^{n}\beta_{ij}\xi_j S_i^0 E_j - (\mu_i^E+\epsilon_i)E_i\Big) \\
&= (\omega_1,\omega_2,\cdots,\omega_n)(M_0E - E) \\
&= (\rho(M_0)-1)(\omega_1,\omega_2,\cdots,\omega_n)E \;\le\; 0, \qquad \text{if } \mathcal{R}_0 \le 1.
\end{aligned} \tag{4.26}$$

Here $E(t) = (E_1(t), E_2(t), \cdots, E_n(t))^T$. Denote

$$Y = \{(S_1, E_1(\cdot), \cdots, S_n, E_n(\cdot)) \in \Theta \quad | \quad \overset{\bullet}{L} = 0\},$$

and Z be the largest compact invariant set in Y. We will show $Z = \{P_0\}$. From (4.26), $\overset{\bullet}{L} = 0$ implies $S_i(t) \equiv S_i^0 = \frac{\Lambda_i}{\mu_i^S}$ for each i. Hence, from the first equation of (4.12), we obtain

$$\sum_{j=1}^{n}\beta_{ij}\int_{r=0}^{\infty} f_j(r)E_j(t-r)dr = 0,$$

and thus

$$\beta_{ij}\int_{r=0}^{\infty} f_j(r)E_j(t-r)dr = 0,$$

for all $t > 0$ and $1 \le i, j \le n$. Then, by irreducibility of B, for each j, there exists $i \neq j$ such that $\beta_{ij} \neq 0$, thus for all $t > 0$

$$\int_{r=0}^{\infty} f_j(r) E_j(t-r) dr = 0.$$

This implies that in Z, $E_{j\,t}(s) = 0$ for all $s \in (-\infty, 0]$, $j = 1, 2, \ldots, n$. Therefore, $Z = \{P_0\}$.

Using the LaSalle-Lyapunov Theorem [38, 43, 75] and a similar argument as in Section 4.1, we conclude that P_0 is globally asymptotically stable in Θ if $\mathcal{R}_0 \leq 1$. On the other hand, if $\mathcal{R}_0 > 1$, then $-L$ serves as a Lyapunov functional for system (4.12). The same proof as in Theorem 5.3.3 of [43] can be used to show that P_0 is unstable. This establishes Theorem 4.4-(1).

Proof of Theorem 4.4-(2):

The global stability of the endemic equilibrium of the single-group model with delays has been proved in [93, 94]. In the following, we consider the case $n \geq 2$. Let $P^* = (S_1^*, E_1^*, \cdots, S_n^*, E_n^*)$ denote the unique endemic equilibrium of system (4.12). Set $V_i : X \to \mathbb{R}_+$ defined as

$$\begin{aligned} V_i(S_1, \phi_1, \cdots, S_n, \phi_n) = & \sum_{j=1}^{n} \beta_{ij} S_i^* \int_{r=0}^{\infty} \alpha_j(r) E_j^* \Phi\Big(\frac{\phi_j(-r)}{E_j^*}\Big)\, dr \\ & + S_i^* \Phi\Big(\frac{S_i}{S_i^*}\Big) + E_i^* \Phi\Big(\frac{\phi_i(0)}{E_i^*}\Big) \\ = & \sum_{j=1}^{n} \beta_{ij} S_i^* \int_{r=0}^{\infty} \alpha_j(r) \Big(\phi_j(-r) - E_j^* - E_j^* \ln \frac{\phi_j(-r)}{E_j^*}\Big)\, dr \\ & + S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + \phi_i(0) - E_i^* - E_i^* \ln \frac{\phi_i(0)}{E_i^*}, \end{aligned} \tag{4.27}$$

and

$$\alpha_j(r) = \int_{\sigma=r}^{\infty} f_j(\sigma) d\sigma.$$

Differentiating V_i along the solution of system (4.12), and using the equilibrium equations (4.23), (4.24) and integration by parts, we obtain

$$
\begin{aligned}
\overset{\bullet}{V_i} =&\mu_i^S S_i^*\Big(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*}\Big) + \sum_{j=1}^n \beta_{ij} S_i^* E_j^* \Big[a_j\Big(2 - \frac{S_i^*}{S_i} - \frac{E_i}{E_i^*} + \frac{E_j}{E_j^*}\Big) \\
& - \frac{S_i E_i^*}{S_i^* E_i E_j^*}\int_{r=0}^{\infty} f_j(r) E_j(t-r)dr - \int_{r=0}^{\infty} f_j(r)\ln\frac{E_j(t)}{E_j(t-r)}dr\Big] \\
=&\mu_i^S S_i^*\Big(2 - \frac{S_i^*}{S_i} - \frac{S_i}{S_i^*}\Big) + \sum_{j=1}^n \beta_{ij} S_i^* E_j^* \int_{r=0}^{\infty} f_j(r)\Big[\Phi\Big(\frac{E_j}{E_j^*}\Big) - \Phi\Big(\frac{E_i}{E_i^*}\Big) \qquad (4.28)\\
& - \Phi\Big(\frac{S_i^*}{S_i}\Big) - \Phi\Big(\frac{S_i E_i^* E_j(t-r)}{S_i^* E_i E_j^*}\Big)\Big]\, dr \\
\leq& \sum_{j=1}^n \beta_{ij} S_i^* E_j^* \Big(\Phi\Big(\frac{E_j}{E_j^*}\Big) - \Phi\Big(\frac{E_i}{E_i^*}\Big)\Big).
\end{aligned}
$$

In the above derivation, we have used two facts: $\frac{S_i^*}{S_i} + \frac{S_i}{S_i^*} \geq 2$ with equality holding if and only if $S_i = S_i^*$, and $1 - x + \ln x \leq 0$ for all $x > 0$ with equality holding if and only if $x = 1$. Let

$$a_{ij} = \beta_{ij} S_i^* E_j^*$$

and

$$G_i(I_i) = -\Phi\Big(\frac{E_i}{E_i^*}\Big).$$

Then

$$\overset{\bullet}{V_i} \leq \sum_{j=1}^n a_j (G_i(I_i) - G_j(I_j)).$$

Therefore, V_i, G_i, a_{ij} satisfy the assumptions of Corollary 2.8, and the functional

$$V = \sum_{i=1}^n c_i V_i(S_i, E_i(\cdot))$$

as defined in Theorem 2.6 is a Lyapunov functional for (4.12), namely,

$$\overset{\bullet}{V} \leq 0 \quad \text{for all} \quad (S_1, I_1(\cdot), \cdots, S_n, I_n(\cdot)) \in \overset{\circ}{\Theta}.$$

Using a similar argument as in Section 3.1, we can show that the only compact invariant set where $\dot{V} = 0$ is the singleton $\{P^*\}$. By the LaSalle-Lyapunov Theorem for delayed systems [38, 43, 75] and a similar argument as in Section 4.1, we conclude that P^* is globally asymptotically stable in $\mathring{\Theta}$ if $R_0 > 1$. This establishes Theorem 4.4-(2).

Chapter 5. Applications to Stochastic Differential Equation Models

In this chapter we show that the graph-theoretic approach is applicable to complex systems that incorporate stochastic disturbances. Stochastic differential equations are used to describe vertex dynamics under stochastic perturbations. The network is kept as deterministic. The resulting models are coupled systems of stochastic differential equations on deterministic networks. We investigate how large random perturbations can be allowed in a stable system so that the perturbed system remains stable.

5.1 Preliminaries

In this section, we recall some results from basic theory of stochastic differential equations which we will use in later sections. For more detailed discussions, we refer to [4, 32, 33, 62, 72, 89, 97].

Consider an autonomous n-dimensional stochastic differential equation

$$dx(t) = f(x(t))dt + g(x(t))dW(t) \tag{5.1}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in \mathbb{R}^m$. Here both $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are locally Lipschitz continuous functions and W is a d-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$. Customarily, $\frac{dW(t)}{dt}$ is called *white noise*. It is known that system (5.1) always has a unique continuous solution $x(t;x_0), 0 \le t \le T$, for some $T > 0$ (see [33, page 76], [89, pages 56 and 58], or [109, Theorem 1]).

Assume that for some $x^* \in \mathbb{R}^m$ we have $f(x^*) = 0$ and $g(x^*) = 0$ so that $x = x^*$ is an *equilibrium* or a *trivial solution* of (5.1). The equilibrium $x = x^*$ is called *stochastically stable* if for each pair of $\epsilon > 0$ and $r > 0$, there exists a $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies

$$P\left\{|x(t;x_0)| < r \text{ for } \text{ all } t \ge 0\right\} \ge 1 - \epsilon. \tag{5.2}$$

The equilibrium $x = x^*$ is called *stochastically globally asymptotically stable* in $\mathbb{R}^m$ if it is stochastically stable and for $x_0 \in \mathbb{R}^m$, the solution $x(t;x_0) \to x^*$ a.s. as $t \to \infty$, namely, $P\{\lim_{t\to\infty} x(t;x_0) = x^*\} = 1$.

The method of Lyapunov functions has been developed to establish stability for stochastic differential equations. Let $V : \mathbb{R}^m \to \mathbb{R}_+$ be a continuously twice differentiable function. Define an operator associated with system (5.1) as

$$\mathcal{L}V(x) = V_x(x)f(x) + \frac{1}{2}trace[g^T(x)V_{xx}(x)g(x)]. \tag{5.3}$$

Then Itô's formula [59] states

$$dV(x(t)) = \mathcal{L}V(x(t))dt + V_x(x(t))g(x(t))dW(t). \tag{5.4}$$

Theorem 5.1. *Suppose that $V(x) \ge 0$ for all $x \in \mathbb{R}^m$, $V(x) = 0$ if and only if $x = x^*$, and $V(x) \to \infty$ as $|x| \to \infty$. If $\mathcal{L}V(x) \le 0$ for all $x \in \mathbb{R}^n$ and $\mathcal{L}V(x) = 0$ if and only if $x = x^*$, then the equilibrium $x = x^*$ is stochastically globally asymptotically stable in $\mathbb{R}^m$.*

Theorem 5.1 is a special case of Theorem 4.4 in [62] and Theorem 11.2.8 in [89]. We refer the reader to [62, 89] for its proof.

5.2 A Patchy Predator-Prey Model with Random Perturbations

In this section, we investigate the effect of the random perturbations to the stability of the positive equilibrium of the following patchy predator-prey model

$$
\begin{aligned}
x_i' &= (r_i - b_i x_i - e_i y_i)x_i + \sum_{j=1}^{n} d_{ij}x_j - \sum_{j=1}^{n} \delta_{ij}x_i, \\
y_i' &= (-\gamma_i - \mu_i y_i + \epsilon_i x_i)y_i, \qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{5.5}
$$

System (5.5) has been investigated in Section 3.4. Here $r_i, b_i, e_i, \gamma_i, \mu_i, \epsilon_i, d_{ij}, \delta_{ij}$ are nonnegative parameters. Suppose that there exists a positive equilibrium $E^* = (x_1^*, y_1^*, \ldots, x_n^*, y_n^*)$ to system (5.5), where $x_i^*, y_i^*, 1 \le i \le n$, satisfy the equilibrium equations

$$
\begin{aligned}
r_i - b_i x_i^* - e_i y_i^* + \sum_{j=1}^{n} d_{ij}\frac{x_j^*}{x_i^*} - \sum_{j=1}^{n} \delta_{ij} = 0, \\
-\gamma_i - \mu_i y_i^* + \epsilon_i x_i^* = 0, \qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{5.6}
$$

By Theorem 3.4, E^* is globally asymptotically stable provided that (d_{ij}) is irreducible and for some k either $b_k > 0$ or $\mu_k > 0$. Assume that system (5.5) endures the random perturbations in the form of

$$
\sigma_i x_i (x_i - x_i^*)\frac{dW_i(t)}{dt}, \qquad i = 1, 2, \ldots, n, \tag{5.7}
$$

and

$$
\eta_i y_i (y_i - y_i^*)\frac{dZ_i(t)}{dt}, \qquad i = 1, 2, \ldots, n. \tag{5.8}
$$

Here $W_i, Z_i, 1 \le i \le n$, are independent 1-dimensional Wiener processes. The perturbation terms (5.7), (5.8), are chosen such that random perturbations disappear at the positive equilibrium E^* as we are particularly interested in the randomly perturbed dynamical behavior near E^*. Thus the perturbed

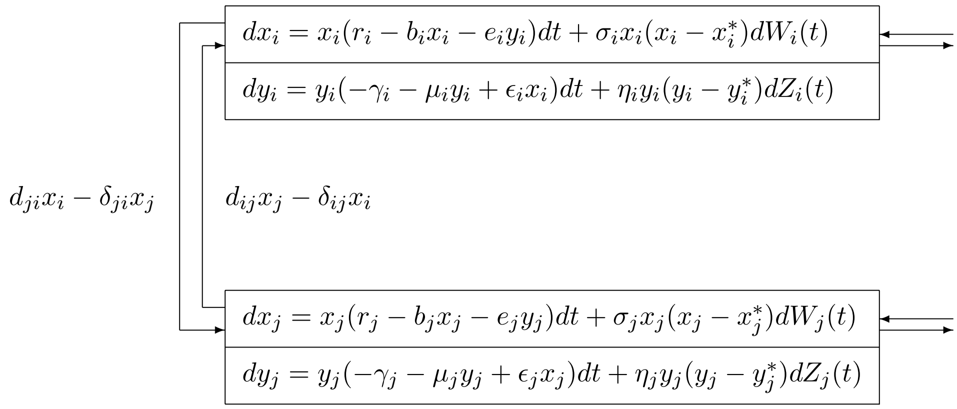

Figure 5.1: A coupled predator-prey system with random perturbations

system can be written as follows

$$\begin{aligned} dx_i &= \Big((r_i - b_i x_i - e_i y_i)x_i + \sum_{j=1}^{n} d_{ij}x_j - \sum_{j=1}^{n} \delta_{ij}x_i\Big)dt + \sigma_i x_i(x_i - x_i^*)dW_i(t), \\ dy_i &= (-\gamma_i - \mu_i y_i + \epsilon_i x_i)y_i dt + \eta_i y_i(y_i - y_i^*)dZ_i(t), \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{5.9}$$

For any given initial conditions $(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)) \in \mathbb{R}_+^{2n}$ to system (5.9), there is a unique solution $(x_1(t), y_1(t), \cdots, x_n(t), y_n(t)), 0 < t < T$, for some $T > 0$. Note that E^* is also an equilibrium of (5.9).

System (5.9) can be regarded as a coupled system of stochastic differential equations on a network; see Figure 5.1. Each vertex represents one patch, vertex dynamics is given by a scalar stochastic differential equation, and the couplings among vertices are provided by the travel among patches. In the following result, using the graph-theoretic approach developed in Chapter 2, we are able to build a Lyapunov function for system (5.9). We prove that the solution to system (5.9) exists for all $t > 0$ and the positive equilibrium is stochastically globally asymptotically stable.

Theorem 5.2. *Assume that (d_{ij}) is irreducible and $b_i > \frac{1}{2}\sigma_i^2 x_i^*$ and $\mu_i > \frac{1}{2}\eta_i^2 y_i^*$ for all $i = 1, 2, \ldots, n$. Then E^* is stochastically globally asymptotically stable in $\mathbb{R}_+^{2n}$.*

Proof. We first show that for any initial value problem to system (5.9) with initial value $(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)) \in \mathbb{R}_+^{2n}$, the unique solution $(x_1(t), y_1(t), \cdots, x_n(t), y_n(t))$ remains in $\mathbb{R}_+^{2n}$ for all $t \geq 0$ with probability 1. The following proof is based on the combination of the graph-theoretic approach developed in Chapter 2 and a stopping time method conducted by Mao and etc [90].

Let τ_e denote the explosion time [32] of the solution $(x_1(t), y_1(t), \cdots, x_n(t), y_n(t))$. We are going to show that $\tau_e = \infty$ with probability 1. Without loss of generality, assume that $x_i(0), y_i(0) \in [1/m_0, m_0]$, for all $1 \leq i \leq n$, for some given positive integer m_0. For each integer $m > m_0$, define the stopping time [32]

$$\tau_m = \inf\{t \in [0, \tau_e] \mid \min_i\{x_i(t), y_i(t)\} \leq \frac{1}{m} \text{ or } \max_i\{x_i(t), y_i(t)\} \geq m\}.$$

For the empty set $\emptyset$, we use the convention that $\inf \emptyset = \infty$. Note that τ_m is an increasing function of m, set $\tau_\infty = \lim_{m\to\infty} \tau_m$ and $\tau_\infty \leq \tau_e$. We are going to show that $\tau_\infty = \infty$ with probability 1, which consequently implies that $\tau_e = \infty$ with probability 1 and thus the solution $(x_1(t), y_1(t), \cdots, x_n(t), y_n(t))$ stays in $\mathbb{R}_+^{2n}$ for all $t \geq 0$ with probability 1.

Suppose there exist $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Since $\tau_m \leq \tau_\infty$, we obtain

$$P\{\tau_m \leq T\} \geq P\{\tau_\infty \leq T\} > \epsilon,$$

for all $m > m_0$. Let $\tau_m^* = \min\{T, \tau_m\}$. For $t \in [0, \tau_m^*]$, define

$$V_i(x_i, y_i) = \epsilon_i x_i^* \Phi\Big(\frac{x_i}{x_i^*}\Big) + e_i y_i^* \Phi\Big(\frac{y_i}{y_i^*}\Big) = \epsilon_i\Big(x_i - x_i^* - x_i^* \ln\frac{x_i}{x_i^*}\Big) + e_i\Big(y_i - y_i^* - y_i^* \ln\frac{y_i}{y_i^*}\Big).$$

By Itô's formula, we obtain

$$\begin{aligned}dV_i &= \epsilon_i\Big(1-\frac{x_i^*}{x_i}\Big)dx_i + \epsilon_i\frac{x_i^*}{2x_i^2}(dx_i)^2 + e_i\Big(1-\frac{y_i^*}{y_i}\Big)dy_i + e_i\frac{y_i^*}{2y_i^2}(dy_i)^2 \\ &= \Big[-\epsilon_i(b_i-\frac{1}{2}\sigma_i^2x_i^*)(x_i-x_i^*)^2 - e_i(\mu_i-\frac{1}{2}\eta_i^2y_i^*)(y_i-y_i^*)^2 \\ &\quad +\sum_{j=1}^n d_{ij}\epsilon_i x_j^*\Big(\frac{x_j}{x_j^*}-\frac{x_i}{x_i^*}+1-\frac{x_i^*x_j}{x_ix_j^*}\Big)\Big]dt \\ &\quad +\epsilon_i\sigma_i(x_i-x_i^*)^2dW_i(t) + e_i\eta_i(y_i-y_i^*)^2dZ_i(t).\end{aligned} \tag{5.10}$$

Let

$$a_{ij} = d_{ij}x_j^*, \qquad F_{ij}(x_i,x_j) = \frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^*x_j}{x_ix_j^*},$$

and

$$G_i(x_i) = -\Phi\Big(\frac{x_i}{x_i^*}\Big).$$

Recall that $\Phi(x) = x-1-\ln x \geq 0$ for all $x>0$ and $\Phi(x)=0$ if and only if $x=1$. It can be easily verified that F_{ij} and G_i satisfy the following relation:

$$F_{ij}(x_i,x_j) = G_i(x_i) - G_j(x_j) - \Phi\Big(\frac{x_i^*x_j}{x_ix_j^*}\Big) \leq G_i(x_i) - G_j(x_j).$$

Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G}, A)$, as given in Proposition 2.1. Then, by Theorem 2.3, we have

$$\sum_{i,j=1}^n c_ia_{ij}F_{ij}(x_i,x_j) \leq \sum_{i,j=1}^n c_ia_{ij}(G_i(x_i)-G_j(x_j)) = 0. \tag{5.11}$$

Consider a Lyapunov function

$$V = \sum_{i=1}^n c_iV_i(x_i(t), y_i(t)).$$

Using (5.11) and Itô's formula, we obtain

$$
\begin{aligned}
dV &= \sum_{i=1}^{n} c_i dV_i \\
&= \sum_{i=1}^{n} c_i \Big[-\epsilon_i (b_i - \frac{1}{2}\sigma_i^2 x_i^*)(x_i - x_i^*)^2 - e_i(\mu_i - \frac{1}{2}\eta_i^2 y_i^*)(y_i - y_i^*)^2 \\
&\quad + \sum_{j=1}^{n} d_{ij} x_j^* \Big(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_i^* x_j}{x_i x_j^*} \Big) + \frac{1}{2}\epsilon_i \sigma_i^2 (x_i - x_i^*)^2 \\
&\quad + \frac{1}{2} e_i \eta_i^2 (y_i - y_i^*)^2 \Big] dt + \sum_{i=1}^{n} c_i \epsilon_i \sigma_i (x_i - x_i^*)^2 dW_i(t) \\
&\quad + \sum_{i=1}^{n} c_i e_i \eta_i (y_i - y_i^*)^2 dZ_i(t) \\
&\le -\sum_{i=1}^{n} c_i \epsilon_i (b_i - \frac{1}{2}\sigma_i^2 x_i^*)(x_i - x_i^*)^2 dt - \sum_{i=1}^{n} c_i e_i (\mu_i - \frac{1}{2}\eta_i^2 y_i^*)(y_i - y_i^*)^2 dt \\
&\quad + \sum_{i=1}^{n} c_i \epsilon_i \sigma_i (x_i - x_i^*)^2 dW_i(t) + \sum_{i=1}^{n} c_i e_i \eta_i (y_i - y_i^*)^2 dZ_i(t).
\end{aligned} \tag{5.12}
$$

Since $b_i > \frac{1}{2}\sigma_i^2 x_i^*$ and $\mu_i > \frac{1}{2}\eta_i^2 y_i^*$, we have

$$dV \le \sum_{i=1}^{n} c_i \epsilon_i \sigma_i (x_i - x_i^*)^2 dW_i(t) + \sum_{i=1}^{n} c_i e_i \eta_i (y_i - y_i^*)^2 dZ_i(t).$$

Integration from 0 to τ_m^* yields

$$
\begin{aligned}
& V(x_1(\tau_m^*), y_1(\tau_m^*), \cdots, x_n(\tau_m^*), y_n(\tau_m^*)) - V(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)) \\
\le \; & \sum_{i=1}^{n} c_i \epsilon_i \sigma_i \int_0^{\tau_m^*} (x_i - x_i^*)^2 dW_i(t) + \sum_{i=1}^{n} c_i e_i \eta_i \int_0^{\tau_m^*} (y_i - y_i^*)^2 dZ_i(t),
\end{aligned}
$$

and thus

$$\mathbb{E}\{V(x_1(\tau_m^*), y_1(\tau_m^*), \cdots, x_n(\tau_m^*), y_n(\tau_m^*))\} \le V(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)). \tag{5.13}$$

Since $P\{\tau_m \leq T\} > \epsilon$ and $\tau_m^* = \min\{T, \tau_m\}$,

$$\begin{aligned} & \mathbb{E}\{V(x_1(\tau_m^*), y_1(\tau_m^*), \cdots, x_n(\tau_m^*), y_n(\tau_m^*))\} \\ \geq \; & P\{\tau_m \leq T\}\mathbb{E}\{V(x_1(\tau_m), y_1(\tau_m), \cdots, x_n(\tau_m), y_n(\tau_m))\} \\ > \; & \epsilon\mathbb{E}\{V(x_1(\tau_m), y_1(\tau_m), \cdots, x_n(\tau_m), y_n(\tau_m))\}. \end{aligned} \tag{5.14}$$

Notice that from the definition of τ_m, there exists k such that one of following identities holds:

$$x_k(\tau_m) = m, \quad x_k(\tau_m) = \frac{1}{m}, \quad y_k(\tau_m) = m, \quad y_k(\tau_m) = \frac{1}{m}.$$

Hence, we have

$$\begin{aligned} & \mathbb{E}\{V(x_1(\tau_m), y_1(\tau_m), \cdots, x_n(\tau_m), y_n(\tau_m))\} \\ \geq \; & \min\left\{c_k e_k V_k(m), c_k \epsilon_k V_k(m), c_k e_k V_k\Big(\frac{1}{m}\Big), c_k \epsilon_k V_k\Big(\frac{1}{m}\Big)\right\}. \end{aligned} \tag{5.15}$$

Combining (5.13), (5.14), and (5.15) yields

$$\begin{aligned} & V(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)) \\ \geq \; & E\{V(x_1(\tau_m^*), y_1(\tau_m^*), \cdots, x_n(\tau_m^*), y_n(\tau_m^*))\} \\ > \; & \epsilon \min\left\{c_k e_k V_k(m), c_k \epsilon_k V_k(m), c_k e_k V_k\Big(\tfrac{1}{m}\Big), c_k \epsilon_k V_k\Big(\tfrac{1}{m}\Big)\right\}. \end{aligned}$$

Letting $m \to \infty$, we obtain

$$V(x_1(0), y_1(0), \cdots, x_n(0), y_n(0)) > \infty$$

since $V_k(m) \to \infty$ and $V_k(\frac{1}{m}) \to \infty$. This is a contradiction. Therefore, $P\{\tau_e < \infty\} = 0$, and the solution $(x_1(t), y_1(t), \cdots, x_n(t), y_n(t))$ stays in $\mathbb{R}_+^{2n}$ for all $t \geq 0$ with probability 1.

From (5.3), (5.4), and (5.12), we obtain

$$\mathcal{L}V \leq -\sum_{i=1}^{n} c_i \epsilon (b_i - \frac{1}{2}\sigma_i^2 x_i^*)(x_i - x_i^*)^2 - \sum_{i=1}^{n} c_i e_i (\mu_i - \frac{1}{2}\eta_i^2 y_i^*)(y_i - y_i^*)^2 \leq 0$$

and $\mathcal{L}V = 0$ if and only if $x_i = x_i^*, y_i = y_i^*$ for all i. Therefore, by Theorem 5.1, E^* is stochastically globally asymptotically stable in $\mathbb{R}_+^{2n}$.. □

When $\sigma_i = \eta_i = 0$ for all i, the noisy system (5.9) becomes system (3.12) discussed in Section 3.4. Biologically, our result indicates that the global-stability result holds as long as the noise is small.

5.3 A Multi-Group SIR Epidemic Model with Random Perturbations

In this section, we apply our graph-theoretic approach to a randomly perturbed multi-group epidemic model. We regard such a model as a coupled system of stochastic differential equations on a deterministic network. Each vertex represents a particular group of individuals, the vertex dynamics is given by a stochastic SIR model, and the coupling among vertices is provided by cross infections. See Figure 5.2. The resulting coupled system is given as follows:

$$\begin{aligned} dS_i &= \Big(\Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i I_j - \mu_i^S S_i\Big)dt + \sigma_i \sqrt{S_i}(S_i - S_i^*)dW_i(t), \\ dI_i &= (\Pi_i + \sum_{j=1}^{n} \beta_{ij} S_i I_j - (\mu_i^I + \gamma_i) I_i)dt + \eta_i \sqrt{I_i}(I_i - I_i^*)dZ_i(t), \\ &\quad i = 1, 2, \ldots, n. \end{aligned} \tag{5.16}$$

When ignoring the random perturbations, i.e., $\sigma_i = \eta_i = 0$ for all i, the noisy system (5.16) reduces to the deterministic multi-group SIR model

$$\begin{aligned} S_i' &= \Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i I_j - \mu_i^S S_i, \\ I_i' &= \Pi_i + \sum_{j=1}^{n} \beta_{ij} S_i I_j - (\mu_i^I + \gamma_i) I_i, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{5.17}$$

Suppose that there exists an endemic equilibrium $P^* = (S_1^*, I_1^*, \ldots, S_n^*, I_n^*)$ to system (5.17), where $S_i^*, I_i^*, 1 \le i \le n$, satisfy the equilibrium equations

$$\begin{aligned}
&\Lambda_i - \sum_{j=1}^{n} \beta_{ij} S_i^* I_j^* - \mu_i^S S_i^* = 0, \\
&\Pi_i + \sum_{j=1}^{n} \beta_{ij} S_i^* I_j^* - (\mu_i^I + \gamma_i) I_i^* = 0, \qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{5.18}$$

Using the same method as in Section 3.6 and Section 4.2, we can prove that P^* is globally asymptotically stable as long as it exists. Note that P^* is also an equilibrium of (5.16).

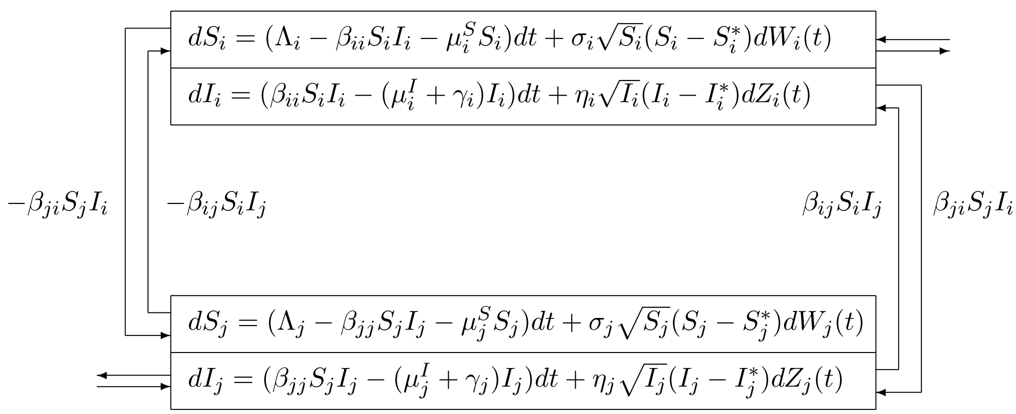

Figure 5.2: A multi-group SIR model with random perturbations

In this section, using the graph-theoretic approach developed in Chapter 2, we are able to build a Lyapunov function for system (5.16) and thus establish global stability of system (5.16).

Theorem 5.3. *Assume that* (β_{ij}) *is irreducible and* $\mu_i^S > \frac{1}{2}\sigma_i^2 S_i^*$, $\Pi_i > \frac{1}{2}\eta_i^2 (I_i^*)^2$, $i = 1, 2, \ldots, n$. *Then* P^* *is stochastically globally asymptotically stable in* $\mathbb{R}_+^{2n}$.

Proof. Set

$$V_i = S_i^* \Phi\Big(\frac{S_i}{S_i^*}\Big) + I_i^* \Phi\Big(\frac{I_i}{I_i^*}\Big) = S_i - S_i^* - S_i^* \ln \frac{S_i}{S_i^*} + I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}.$$

By Itô's formula, we obtain

$$\begin{aligned} dV_i &= \Big(1-\frac{S_i^*}{S_i}\Big)dS_i+\frac{S_i^*}{2S_i^2}(dS_i)^2+\Big(1-\frac{I_i^*}{I_i}\Big)dI_i+\frac{I_i^*}{2I_i^2}(dI_i)^2 \\ &= \Big[-(\mu_i^S-\frac{1}{2}\sigma_i^2S_i^*)\frac{(S_i-S_i^*)^2}{S_i}-(\Pi_i-\frac{1}{2}\eta_i^2(I_i^*)^2)\frac{(I_i-I_i^*)^2}{I_i^*I_i} \\ &\quad+\sum_{j=1}^n\beta_{ij}S_i^*I_j^*\Big(2-\frac{S_i}{S_i^*}-\frac{S_iI_jI_i^*}{S_i^*I_j^*I_i}+\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}\Big)\Big]dt+ \\ &\quad+\sigma_i\frac{(S_i-S_i^*)^2}{\sqrt{S_i}}dW_i(t)+\eta_i\frac{(I_i-I_i^*)^2}{\sqrt{I_i}}dZ_i(t). \end{aligned} \tag{5.19}$$

Let

$$a_{ij}=\beta_{ij}S_I^*I_j^*,$$

$$F_{ij}(S_i,I_i,I_j)=2-\frac{S_i}{S_i^*}-\frac{S_iI_jI_i^*}{S_i^*I_j^*I_i}+\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*},$$

and

$$G_i(I_i)=-\Phi\Big(\frac{I_i}{I_i^*}\Big).$$

Then we have

$$\begin{aligned} F_{ij}(S_i,I_i,I_j) &= G_i(I_i)-G_j(I_j)-\Phi\Big(\frac{S_i^*}{S_i}\Big)-\Phi\Big(\frac{S_iI_jI_i^*}{S_i^*I_j^*I_i}\Big) \\ &\le G_i(I_i)-G_j(I_j). \end{aligned}$$

Let c_i be the cofactor of the i-th diagonal element in the Laplacian matrix of $(\mathcal{G},A)$, as given in Proposition 2.1. Then, by Theorem 2.3, we have

$$\sum_{i,j=1}^n c_ia_{ij}F_{ij}(S_i,S_j)\le\sum_{i,j=1}^n c_ia_{ij}(G_i(S_i)-G_j(S_j))=0.$$

Let

$$V=\sum_{i=1}^n c_iV_i(S_i,I_i). \tag{5.20}$$

We thus have

$$
\begin{aligned}
dV &= \sum_{i=1}^{n} c_i dV_i \\
&\leq -\sum_{i=1}^{n} c_i(\mu_i^S - \frac{1}{2}\sigma_i^2 S_i^*)\frac{(S_i - S_i^*)^2}{S_i}dt \\
&\quad -\sum_{i=1}^{n} c_i(\Pi_i - \frac{1}{2}\eta_i^2 (I_i^*)^2)\frac{(I_i - I_i^*)^2}{I_i^* I_i}dt \\
&\quad +\sum_{i=1}^{n} c_i\sigma_i \frac{(S_i - S_i^*)^2}{\sqrt{S_i}}dW_i(t) + \sum_{i=1}^{n} c_i\eta_i \frac{(I_i - I_i^*)^2}{\sqrt{I_i}}dX_i(t).
\end{aligned}
\tag{5.21}
$$

Since $\mu_i^S > \frac{1}{2}\sigma_i^2 S_i^*$ and $\Pi_i > \frac{1}{2}\eta_i^2(I_i^*)^2$ for all i, we have

$$
dV \leq \sum_{i=1}^{n} c_i\sigma_i \frac{(S_i - S_i^*)^2}{\sqrt{S_i}}dW_i(t) + \sum_{i=1}^{n} c_i\eta_i \frac{(I_i - I_i^*)^2}{\sqrt{I_i}}dX_i(t).
$$

By similar arguments as in Section 5.2, we can show that for any initial value problem to system (5.16) with initial value $(S_1(0), I_1(0), \cdots, S_n(0), I_n(0)) \in \mathbb{R}_+^{2n}$, there is a unique solution $(S_1(t), I_1(t), \cdots, S_n(t), I_n(t))$ which almost surely remains in $\mathbb{R}_+^{2n}$ for all $t \geq 0$. Moreover, we know

$$
\mathcal{L}V \leq -\sum_{i=1}^{n} c_i(\mu_i^S - \frac{1}{2}\sigma_i^2 S_i^*)\frac{(S_i - S_i^*)^2}{S_i} - \sum_{i=1}^{n} c_i(\Pi_i - \frac{1}{2}\eta_i^2 (I_i^*)^2)\frac{(I_i - I_i^*)^2}{I_i^* I_i} \leq 0
$$

and $\mathcal{L}V = 0$ if and only if $S_i = S_i^*, I_i = I_i^*$ for all i. Therefore, by Theorem 5.1, P^* is stochastically globally asymptotically stable in $\mathbb{R}_+^{2n}$.

□

Chapter 6. Future Research Directions

Our graph-theoretic approach to the construction of Lyapunov functions for coupled systems on networks has several advantages.

1. **Our approach is independent of specific forms of the vertex system.** We have shown that our approach is applicable to vertex systems including second-order differential equations for mechanical or electrical oscillators, multi-species interacting models in ecology, and epidemic models for the spread and transmission of infectious diseases.

2. **Our approach is independent of special structures of networks and specific forms of interactions.** We show that our approach is applicable to mechanical or electrical networks in engineering, spatial dispersal networks in ecology, and disease-transmission and spatial spread networks in epidemiology. We have also shown that our approach is applicable to different forms of coupling including physical or electrical connections among oscillators, dispersal of species among patches or communities, and cross infections among different host groups in disease transmission.

3. **Our approach is independent of particular forms of vertex Lyapunov functions.** We show that our approach can work with vertex Lyapunov functions that are energy-type functions for electrical or me-

chanical oscillators or Volterra-type functions for ecological and epidemiological models.

We expect that our graph-theoretic approach can be further applied to much wider classes of mathematical models from many other areas of science and engineering.

Many mathematical questions regarding coupled systems on networks can be further investigated. As future research, I plan to investigate the following questions:

- Apply the graph-theoretic approach to investigate the global-stability problem in neural networks, chemical reaction networks, and control theory.

- Apply our approach to study synchronization problems for coupled oscillators.

- Extend our approach to coupled systems on multiple graphs.

- Investigate stability and bifurcation problems for coupled systems on random networks.

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