Hybrid Model Chambers of Toric Geometric Invariant Theory Quotients

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in Mathematics

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Abstract

We give an explicit criterion for when a toric GIT quotient is a stacky vector bundle over a projective base. That is given a charge matrix satisfying a certain property, we construct a projective base such that the semi-stable locus of the original GIT quotient is a G-equivariant vector bundle over the semi-stable locus of this base. We also relax this criterion to classify toric GIT quotients which differ from a stacky vector bundle by a finite map. As an application, we recover the Herbst Criterion established by Guffin and Clarke. In addition, we prove that when the G-action is quasisymmetic, there is a finite toric morphism from a product of projective spaces to the base.

Acknowledgements

I would like to thank some people without whom this thesis would have been impossible. First and foremost, I would like to thank my supervisor David Favero for his kindness and understanding, for always taking the time to answer my questions, and for teaching me an incredible amount of mathematics. It's hard to fathom how much I have learned from him in the past few years. Secondly, I would like to thank Tara Schuetz for always being willing to help and to answer any questions I might have. Without her kind support and stellar organization, this whole process would be a lot more intimidating than it already is. Finally I would like to thank my fellow graduate students whose moral support helped give me the energy to write this thesis.

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1 Introduction

Toric Geometry is a branch of Algebraic Geometry where roughly we assume the varieties contain a torus. This simple assumption has wonderful implications, most importantly that the variety itself is encoded in a combinatorial object called a fan. The setting for this thesis will be Toric Geometry. An introduction to the subject will be given in Section 2.

The varieties we study will be those obtained as Geometric Invariant Theory (GIT) quotients. Geometric Invariant Theory is a tool used to create and study quotients of algebraic varieties by certain groups. A problem is that this quotient might not be well-defined or well-behaved as a variety. Geometric Invariant Theory corrects this by taking the quotient on certain maximal open subsets. Through this process we end up with several different quotients which, in the toric setting, depend on a choice of character of the group [CLS11, Section 14; Dol03, Introduction]. This variation can be described by what is called the secondary fan, originally described in [GKZ94]. We treat the secondary fan in Section 3.

One of the motivations for our study comes from considerations in string theory. Namely, in physics there is a well-known duality called the Landau-Ginzburg/Calabi-Yau correspondence (See e.g. [FJR15]). In the process of generalising this correspondence to toric varieties, Edward Witten invented the Gauged Linear Sigma Model (GLSM). A full mathematical description of GLSMs was also described in [FJR15].

[FJR15, Section 7] gives a number of examples of their mathematical GLSM, one being what they call the hybrid model. The hybrid model consists of a toric GIT quotient and a so-called R-charge with special properties. In this document, we will consider only the GIT quotient and the properties it needs to be a hybrid model. In Section 4, we discuss the toric construction of a vector bundle, a generalisation called a stacky vector bundle, and a relaxation of the stacky vector bundle description to what we call an almost stacky vector bundle. As a simplification of [FJR15, Definition 7.1.1], we will say a toric GIT quotient is a hybrid model if it is a stacky vector bundle over a projective base. The goal of this paper is to investigate when a given GIT quotient is a hybrid model.

In Section 5.1, we determine a criterion for when a GIT quotient is a hybrid model. As described in Section 3, a toric GIT quotient can be described completely by a closed subgroup $G \subseteq (\mathbb{C}^*)^n$ and a choice of character $\chi \in \widehat{G}$. Equivalent to a choice of a group G is a choice of a matrix $Q : \mathbb{Z}^n \to \widehat{G}$, called the charge matrix. Given the matrix Q, we describe a set called β whose elements are $\beta_i = Q_{\mathbb{R}}(e_i)$. Similarly we define $\beta_i^{\mathbb{Z}} = Q(e_i)$. These sets will be needed to define our condition. In Section 3.2, we describe a combinatorial object called a secondary fan, composed of "chambers" which lie in $\widehat{G}_{\mathbb{R}}$. We use the notation $\Gamma_{\Sigma,I_\emptyset}$ to denote a chamber of the secondary fan. The choice of a chamber $\Gamma_{\Sigma,I_\emptyset}$ is equivalent to a choice of a character $\chi \in \widehat{G}$. Then in Theorem 5.11 and Corollary 5.12 we show that the choice of Q and $\Gamma_{\Sigma,I_\emptyset}$ gives a hybrid model of rank r if and only if the following conditions on β and $\Gamma_{\Sigma,I_\emptyset}$ are satisfied.

1. There exists $\beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ and $a_{ji} \in \mathbb{Z}$ such that $|\beta_\Gamma| = r$ and for each $\beta_i \in \beta_\Gamma$ we have $-\beta_i^{\mathbb{Z}} = \sum_{\beta_j \notin \beta_\Gamma} a_{ji}\beta_j^{\mathbb{Z}}$.
2. $\beta_i \neq 0$ for each $\beta_i \in \beta^0 := \beta \setminus \beta_\Gamma$
3. $\mathrm{Cone}(\beta^0) = \{\sum c_i\beta_i \mid \beta_i \in \beta^0 \text{ and } c_i \geq 0\}$ is strongly convex.

Furthermore if we reduce part 1 of the condition to be "there exists $\emptyset \neq \beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$", we show in Theorem 5.15, that the GIT quotient is an almost stacky vector bundle. Then parts 2 and 3 are equivalent to the almost stacky vector bundle being over a projective base.

In Section 5.2, we compare our results to two other conditions from the literature. We determine that the Herbst condition from [CG15] is a special case of being an almost stacky vector bundle. We also show that in the hybrid model case, if the G-action $G \times \mathbb{C}^n \to \mathbb{C}^n$, given by group multiplication, is quasisymmetric in the sense of [ŠV15], there is a finite toric morphism from a product of projective spaces to the base.

2 Toric Geometry

We will assume a basic knowledge of algebraic geometry, such as that can be found in Chapters I and II of [Har77].

2.1 The Algebraic Torus

Toric Geometry is the study of special types of algebraic varieties called toric varieties. The "toric" in toric variety refers to what is called the algebraic torus.

Definition 2.1. $(\mathbb{C}^*)^n$*, with the structure of an algebraic variety and a multiplicative group, is called the **algebraic torus**. We similarly call any algebraic varieties isomorphic to* $(\mathbb{C}^*)^n$ *a torus, as it is endowed with a group structure through the isomorphism* [CLS11, Page 10].

Every torus comes equipped with a group action $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ given by group multiplication.

Definition 2.2. *We call an algebraic variety X a **toric variety** if it satisfies the following three conditions.*

1. *X is irreducible*
2. *X contains a torus as a Zariski open subset*
3. *The natural action of the torus on itself extends to an algebraic group action $(\mathbb{C}^*)^n \times X \to X$.*

[CLS11, Definition 3.1.1]

Notation 2.3. *We will use the standard notation $\boldsymbol{\mathcal{O}_X}$ to denote the structure sheaf of X.*

Notation 2.4. *We will use the standard notation $\mathbf{C}(\boldsymbol{X})$ to denote the field of rational functions on X.*

2.2 Cones and Fans

Every sufficiently "nice" toric variety can be expressed using a combinatorial object called a fan. In this section we will provide the necessary information to define a fan, as well as some facts and definitions related to fans.

Definition 2.5. *A **lattice** is any group isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}$.* [CLS11, Page 13]

It is often useful to consider a lattice as a subset of a vector space.

Notation 2.6. *Let L be a lattice. Then $\boldsymbol{L_{\mathbb{R}}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space containing L. Note that for L isomorphic to $\mathbb{Z}^n$, $L_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^n$ and L is a subgroup of $L_{\mathbb{R}}$ via the inclusion $l \mapsto l \otimes 1$. This construction also works for arbitrary $\mathbb{Z}$-modules L, but if L is not a lattice then L is not a subgroup of $L_{\mathbb{R}}$.*

Following [CLS11], unless otherwise stated we will be working in the lattice N, with dual lattice M. Since N and M are assumed to be dual lattices there is a natural inner product $\langle M, N \rangle$.

The basic building blocks of fans are called cones, which are positive linear spans of vectors.

Definition 2.7. *Let $\{v_1, ..., v_n\}$ be a set of points in a vector space V. Then* $\textbf{Cone}\,(\boldsymbol{v_1, ..., v_n}) = \{x \in V \mid x = \sum a_i v_i \text{ for some } a_i \geq 0\}$. [CLS11, Definition 1.2.1]

In particular, we will study the following class of cones.

Definition 2.8. *A* ***rational convex polyhedral cone*** *in $N_{\mathbb{R}}$ (referred to as a* ***cone*** *for the rest of this document) is any set of the form* $\mathrm{Cone}(S)$ *for some finite set $S \subseteq N$.* [CLS11, Definitions 1.2.1, 1.2.4]

Definition 2.9. *The* ***dimension*** *of a cone is the dimension of the vector space it spans.* [CLS11, Page 24]

Definition 2.10. *The* ***relative interior*** *of a cone σ, is the interior of $\sigma \cap V$ where $V =$* $\mathrm{span}(\sigma)$*. This is denoted* $\textbf{Relint}\,(\boldsymbol{\sigma})$*.* [CLS11, Page 27]

For each $m \in M_{\mathbb{R}}$ we can define the following hyperplanes. [CLS11, Page 25]

Notation 2.11.

$\boldsymbol{H_m} = \{u \in N_{\mathbb{R}} \mid \langle u, m \rangle = 0\}$

$\boldsymbol{H_m^+} = \{u \in N_{\mathbb{R}} \mid \langle u, m \rangle \geq 0\}$

Roughly speaking a face of a cone is a smaller dimensional subcone. More formally we can define a face as followed.

Definition 2.12. *Let σ be a cone of dimension n in $N_{\mathbb{R}}$.*

- *A* ***face*** *of σ is a cone of the form $\tau = \sigma \cap H_m$ for any $m \in M_{\mathbb{R}}$.*
- *A* ***facet*** *of σ is a face of dimension $n - 1$.*
- *An* ***edge*** *of σ is a face of dimension* 1*.*

[CLS11, 25]

The following proposition highlights the use of the term facet.

Proposition 2.13. *Every cone σ can be written as $H^+_{m_1} \cap \ldots \cap H^+_{m_n}$, where $\sigma \cap H_{m_i}$ are the facets of σ.* [CLS11, Proposition 1.2.8]

An important class of cones is those which are strongly convex. To define the four equivalent definitions of strong convexity, we first introduce the notion of a dual cone.

Definition 2.14. *Let σ be a cone in $N_{\mathbb{R}}$. Then the **dual cone** of σ is a cone in $M_{\mathbb{R}}$ given by $\boldsymbol{\sigma}^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0$ for all $u \in \sigma\}$.* [CLS11, Definition 1.2.3]

Definition 2.15. *A cone in $N_{\mathbb{R}} \cong \mathbb{R}^n$ is called **strongly convex** if it satisfies one of the following four equivalent conditions.*

1. *$\{0\}$ is a face of σ.*

2. *σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$.*

3. *$\sigma \cap -\sigma = 0$.*

4. *$\dim \sigma^{\vee} = n$.*

[CLS11, Proposition 1.2.12]

As mentioned at the beginning of this section, every sufficiently "nice" toric variety can be expressed via a combinatorial object called a fan. Thus the following definition is central to the study of Toric Geometry.

Definition 2.16. *A **fan** Σ in $N_{\mathbb{R}}$ is a finite set of cones in $N_{\mathbb{R}}$ satisfying the following conditions.*

1. *Each $\sigma \in \Sigma$ is strongly convex.*

2. *If $\sigma \in \Sigma$, then $\tau \in \Sigma$ for every τ that is a face of σ.*

3. *If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is in Σ and a face of both σ_1 and σ_2.*

*Σ is called a **generalized fan** if we drop the assumption that the cones are strongly convex.* [CLS11, Definition 3.1.2, Definition 6.2.2]

The following theorem due to Sumihiro provides the correspondence between toric varieties and fans, making it the fundamental theorem in Toric Geometry.

Theorem 2.17. *Let Σ be a fan, then there exists a normal, separated toric variety $\boldsymbol{X_\Sigma}$ constructed from Σ, as below. Similarly every normal, separated toric variety can be constructed this way.* [Sum74; Sum75; CLS11, Page 107-109]

Construction of $\boldsymbol{X_\Sigma}$

Since X_Σ is a variety, it is locally affine. Therefore to construct X_Σ, it is sufficient to provide the affine cover and associated gluing data. To build the cover, we need the concept of an affine semigroup.

Definition 2.18. *An* ***affine semigroup*** *is a set S with an associative, commutative, binary operation $+$, and an identity element 0. Unlike in groups, we do not assume elements in affine semigroups have inverses. S is called* ***finitely generated*** *if every $s \in S$ can be written as $\sum_{i=1}^{n} a_i s_i$ for some finite set $\{s_1, ..., s_n\} \subseteq S$ and $a_i \in \mathbb{N}$.* [CLS11, Page 16]

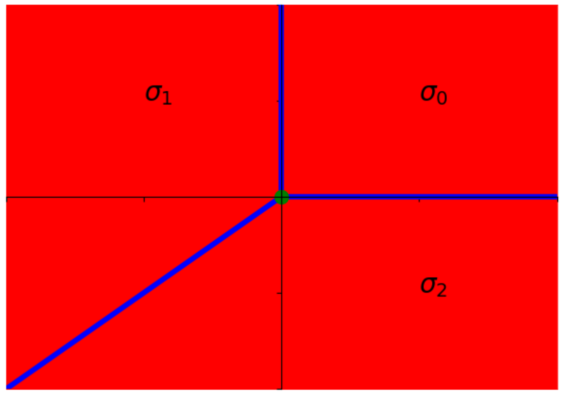
Definition 2.19. *Let S be an affine semigroup. Then we define the* ***semigroup algebra of S*** *as followed. Let $\mathbb{C}[S]$ be the $\mathbb{C}$-vector space with basis elements $\{x^s \mid s \in S\}$. We define multiplication in this ring by saying $x^{s_1}x^{s_2} = x^{s_1+s_2}$. Therefore for S finitely generated, the ring $\mathbb{C}[S]$ is generated by $\{x^s \mid s$ is a generator of $S\}$.* [CLS11, Page 16-17]

We use semigroup algebras to define the affine cover for X_Σ. For each $\sigma \in \Sigma$ we define a semigroup $\boldsymbol{S_\sigma} = \sigma^\vee \cap M$. Then $\boldsymbol{U_\sigma} = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is an affine variety, and $\{U_\sigma \mid \sigma \in \Sigma\}$ provides the affine cover for X_Σ.

One can show that if τ is a face $\sigma_1 \cap \sigma_2$, then there exists $m \in (\sigma_1^\vee \cap (-\sigma_2)^\vee \cap M)$ such that $\mathbb{C}[S_\tau]$ is a localization given by $\mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_1}]_{x^m} = \mathbb{C}[S_{\sigma_2}]_{x^{-m}}$. Taking the spectrum of these rings reverses the diagram and provides the gluing data for X_Σ. For more details, see [CLS11, Page 106].

Example 2.20. The fan Σ in Figure 1 gives the variety $X_\Sigma = \mathbb{P}^2$. We will show this by calculating U_σ for each $\sigma \in \Sigma_{\max}$ and showing that this provides the standard affine open cover $U_i = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x_i \neq 0\} = \mathrm{Spec}(\mathbb{C}[\frac{x_j}{x_i} \mid j \in \{0, 1, 2\} \setminus i])$.

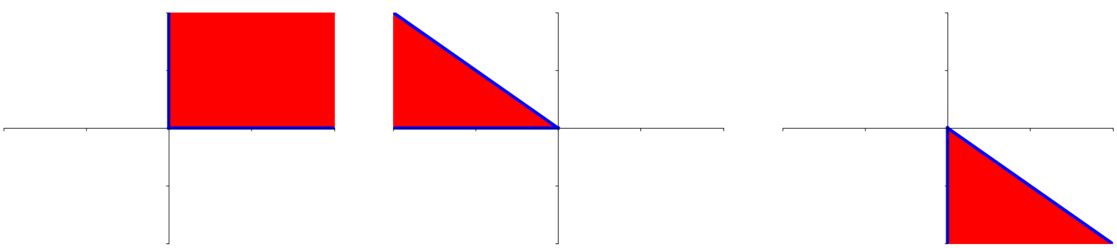
Figure 1: The Fan for $\mathbb{P}^2$.



In order to calculated U_{σ_i}, we must calculate the dual cones. Recall for any cone $\sigma \in N_{\mathbb{R}}$, we have $\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, s \rangle \geq 0 \text{ for all } s \in \sigma\}$. Therefore as in Figure 2 we have

- $\sigma_0^\vee = \text{Cone}([1,0],[0,1])$
- $\sigma_1^\vee = \text{Cone}([-1,0],[-1,1])$
- $\sigma_2^\vee = \text{Cone}([0,-1],[1,-1])$.

Figure 2: Dual Cones for the Fan of $\mathbb{P}^2$



(a) $\sigma_0^\vee$ (b) $\sigma_1^\vee$ (c) $\sigma_2^\vee$

If $[a,b]$ is a semigroup generator of S_σ, we will denote the associated semigroup algebra generator by $x^a y^b$. Therefore

- $\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[x,y]$

- $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x^{-1}, yx^{-1}]$

- $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[y^{-1}, xy^{-1}]$.

To show that this affine cover corresponds to the standard affine cover for $\mathbb{P}^2$, we will use the change of basis $x \to \frac{x_1}{x_0}, y \to \frac{x_2}{x_0}$.

Therefore

- $\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[\frac{x_1}{x_0}, \frac{x_2}{x_0}]$

- $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\frac{x_0}{x_1}, \frac{x_2}{x_0}\frac{x_0}{x_1}] = \mathbb{C}[\frac{x_0}{x_1}, \frac{x_2}{x_1}]$

- $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[\frac{x_0}{x_2}, \frac{x_1}{x_0}\frac{x_0}{x_2}] = \mathbb{C}[\frac{x_0}{x_2}, \frac{x_1}{x_2}]$

Therefore $\mathrm{Spec}(\mathbb{C}[S_{\sigma_i}]) = U_i = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x_i \neq 0\}$, the standard affine cover for $\mathbb{P}^2$. □

Standing Assumption 2.21. *For the rest of this document we will assume that every toric variety is normal and separated, and thus has the form X_Σ.*

The remainder of this section will be dedicated to providing additional facts and definitions about fans that will be used later in the thesis.

It is often not necessary to consider every cone in a fan.

Notation 2.22. *Let Σ be a generalized fan. Then $\boldsymbol{\Sigma_{\max}}$ is the set of cones in Σ that aren't proper subsets of another cone in Σ.* [CLS11, Page 180].

Just as we can completely describe a toric variety from just its fan, we can completely describe the structure of a fan with known rays, using only what is called its irrelevant ideal.

Definition 2.23. *Let Σ be a fan. Then the* ***irrelevant ideal*** *of Σ, denoted $\boldsymbol{B(\Sigma)}$, is given by $B(\Sigma) = \langle \prod_{\rho \notin \sigma(1)} x_\rho \mid \sigma \in \Sigma_{\max} \rangle \subseteq \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$.* [CLS11, Page 207]

Example 2.24. As in Example 2.20, the fan for $\mathbb{P}^2$ has three maximal cones, each generated by two of the three rays. Therefore for $X_\Sigma = \mathbb{P}^2$, $B(\Sigma) = \langle x_1, x_2, x_3 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3]$. □

Often when studying fans, it is useful to veiw them as a subset of $N_\mathbb{R}$ as opposed to just a collection of cones. This yeilds the following definition.

Definition 2.25. *If Σ is a fan, then the* ***support*** *of Σ, denoted* $|\boldsymbol{\Sigma}|$ *is the union of all of its cones.* [CLS11, Page 106]

Definition 2.26. *A* ***support function*** *is a map $\phi : |\Sigma| \to \mathbb{R}$ such that ϕ is linear on each cone $\sigma \in \Sigma$. ϕ is called integral with respect to the lattice N if $\phi(|\Sigma| \cap N) \subseteq \mathbb{Z}$.* [CLS11, Definition 4.2.11]

Notation 2.27. *If Σ is a fan, then the set of r dimensional cones of Σ is denoted by* $\boldsymbol{\Sigma(r)}$. [CLS11, Page 106]

One of the reasons strongly convex cones are useful, is that they can be described completely in terms of their edges.

Definition 2.28. *Let σ be a strongly convex cone, then we call the edges of σ* ***rays****. For each ray ρ, there is a unique semigroup generator of $\rho \cap N$ which we refer to as* $\boldsymbol{u_\rho}$. *Similarly for a fan Σ we call the set $\Sigma(1)$ the* ***rays*** *of Σ.* [CLS11, Page 29]

Proposition 2.29. *A strongly convex cone can be written as* $\mathrm{Cone}(u_\rho \mid \rho \text{ is a ray of } \sigma)$. [CLS11, Lemma 1.2.15]

We conclude the section with five definitions needed for propositions later in the document.

Definition 2.30. *A toric variety X_Σ is called* ***quasiprojective*** *if it is isomorphic to an open subset of a projective variety. It is called* ***semiprojective*** *if it is quasiprojective such that $|\Sigma|$ is convex and full dimensional in $N_\mathbb{R}$.* [CLS11, Definition 7.0.1, Proposition 7.2.9]

Definition 2.31. *A cone is called* ***simplicial*** *if it has a set of minimal generators that are linearly independent over $\mathbb{R}$.* [CLS11, Definition 1.2.16]

Definition 2.32. *A fan Σ is called* ***simplicial*** *if all of its cones are simplicial. Similarly X_Σ is called simplicial if Σ is simplicial.* [CLS11, Definition 3.1.8] *Note: There is another condition on varieties called being* ***$\mathbb{Q}$-factorial*** *that is equivalent to being simplicial on normal toric varieties.* [CLS11, Page 549] *We use this condition in Proposition 5.18.*

2.3 Divisors

The divisors of a toric variety have some nice properties that are used in the definition of a toric vector bundle. In this section we will review the definition of different types of divisors and discuss their special properties in the toric case.

Definition 2.33. *Let X be an irreducible algebraic variety. A* ***prime divisor*** *of X is subvariety of codimension 1.* [Har77, Page 130; CLS11, Page 157]

Definition 2.34. $\mathbf{Div}\,(\boldsymbol{X})$ *is the free abelian group generated by the prime divisors of X. We call the elements of* $\mathrm{Div}(X)$ ***Weil divisors****.* [Har77, Page 130; CLS11, Definition 4.0.8]

Notation 2.35. *Let X be a variety and D a prime divisor.*
Then $\boldsymbol{\mathcal{O}_{X,D}} = \{f \in C(X) \mid U \cap D \neq \emptyset,$ *where U is the open set on which f is defined.*$\}$ [Har77, Page 130; CLS11, Definition 4.0.5]

Proposition 2.36. *Let X be a normal variety. Then for every prime divisor D, the ring $\mathcal{O}_{X,D}$ is a discrete valuation ring. This gives a corresponding discrete valuation $\nu_D : C(X)^* \to \mathbb{Z}$.* [Har77, Page 130; CLS11, Page 156]

Definition 2.37. *Let $f \in \mathbb{C}(X)^*$. Then the* ***divisor of f*** *is defined as*
$\mathbf{div}\,(\boldsymbol{f}) = \sum_{D \text{ a prime divisor of } X} \nu_D(f) \in \mathrm{Div}(X)$*. We call any divisor of this form a* ***principal divisor*** *and denote the subgroup of principal divisors by* $\mathbf{Div_0(\boldsymbol{X})}$*.* [Har77, Page 131; CLS11, Definition 4.0.10]

Definition 2.38. *A* ***Cartier divisor*** *is a Weil Divisor that is locally principal. That is there exists an open cover U_i of X on which $D|_{U_i}$ is a principal divisor. The group of Cartier divisors is denoted* $\mathbf{CDiv(\boldsymbol{X})}$*.* [Har77, Page 141; CLS11, Definition 4.0.12]

Definition 2.39. *The* ***class group*** *of X is defined as* $\mathbf{Cl}\,(\boldsymbol{X}) = \mathrm{Div}(X)/\mathrm{Div}_0(X)$ *and the* ***Picard group*** *is defined as* $\mathbf{Pic}\,(\boldsymbol{X}) = \mathrm{CDiv}(X)/\mathrm{Div}_0(X)$*.* [Har77, Page 131; CLS11, Definition 4.0.13] *(See also* [Har77, Page 143] *for an equivalent definition of* $\mathrm{Pic}(X)$*.)*

Proposition 2.40. *For each $D \in \mathrm{Div}(X)$, there is a coherent sheaf of $\mathcal{O}_X$ modules* $\boldsymbol{\mathcal{O}_X(D)} := \{f \in C^*(X) \mid (div(f) + D)|_U \geq 0\} \cup \{0\}$*.* [Har77, Page 144; CLS11, Page 167]

Definition 2.41. *Let ρ be a ray of Σ. Then as described in* [CLS11, Section 4.1]*, there is a divisor* $\boldsymbol{D_\rho}$ *corresponding to ρ.*

These divisors are called **torus invariant**. Let T_N be the torus associated with the variety X_Σ. Recall that X_Σ comes equipped with an action $T_N \times X_\Sigma \to X_\Sigma$. Then D_ρ is torus invariant means that $T_N \cdot D_\rho = D_\rho$.

Proposition 2.42. *Any torus invariant divisor of X_Σ can be written as $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$.* [CLS11, Page 172]

Proposition 2.43. *A divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is Cartier if and only if for each $\sigma \in \Sigma$ there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.* [CLS11, Proposition 4.2.8]

Proposition 2.44. *Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a Cartier divisor with m_σ as in Proposition 2.43. Then there is an integral support function $\boldsymbol{\phi_D} : |\Sigma| \to \mathbb{R}$ given by $\phi_D(u) = \langle m_\sigma, u \rangle$ for all $u \in \sigma$. Furthermore $\phi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$.* [CLS11, Theorem 4.2.12]

Definition 2.45. *Let S be a convex set and $f : S \to \mathbb{R}$. Then f is called* ***convex*** *if and only if $f(tu+(1-tv)) \geq tf(u)+(1-t)f(v)$ for all $u, v \in S$ and $t \in [0,1]$.* [CLS11, Defintion 6.1.4]

[CLS11, Section 6.3] defines a special property of a Cartier divisor called being a nef divisor. For the sake of this document we will use the following equivalent definition which holds for fans of convex support. [CLS11, Lemma 9.2.1]

Definition 2.46. *Let X_Σ be a toric variety such that $|\Sigma|$ is convex. A Cartier divisor D is* ***nef*** *if and only if the support function $\phi_D : |\Sigma| \to \mathbb{R}$ is convex. (See* [CLS11, Definition 6.3.10] *for an equivalent definition and* [Zar62, Definition 7.6] *for the original reference.)*

Definition 2.47. *Two Cartier divisors C and D are called* ***numerically equivalent*** *if $C - D$ is a nef divisor.* [CLS11, Definition 6.3.16] *(See* [Har77, 364] *for an equivalent definition.)*

Notation 2.48. $\boldsymbol{N^1(X)}$ *is defined as $(\mathrm{CDiv} / \equiv) \otimes_{\mathbb{Z}} \mathbb{R}$ where $C \equiv D$ if they are numerically equivalent. Note that if Σ has full dimensional convex support, we have $N^1(X_\Sigma) = \mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$.* [CLS11, Definition 6.3.17, Page 294]

Definition 2.49. *We define* $\mathbf{Nef}\,(\boldsymbol{X})$ *to be the cone generated by nef divisors of X in $N^1(X)$. This is called the nef cone.* [CLS11, Defintion 6.3.18]

Definition 2.50. *Let X_Σ be a simplicial, semiprojective toric variety. Then* $\mathbf{Eff}\,(\boldsymbol{X_\Sigma}) = \mathrm{Cone}(D_\rho \mid \rho \in \Sigma(1))$ *is called the pseudoeffective cone and $\mathrm{Nef}(X_\Sigma) \subseteq \mathrm{Eff}(X_\Sigma) \subseteq N^1(X_\Sigma)$.* [CLS11, Lemma 15.1.8]

3 Toric Geometric Invariant Theory

The main setting for this thesis is the world of toric GIT. We will consider different quotients of an algebraic torus by its closed subgroups.

3.1 Setup

As described in Chapter 14 of [CLS11], we can create toric varieties using something called Geometric Invariant Theory. Consider a closed subgroup $G \subseteq (\mathbb{C}^*)^n$. Let $\boldsymbol{\widehat{G}} = \mathrm{Hom}(G, \mathbb{C})$, the set of algebraic group homomorphisms. Then $\widehat{G}$ is called the **group of characters** of G, and $\chi \in \widehat{G}$ is called a character. Through the process described in Chapter 14.1 of [CLS11], for every character $\chi \in \widehat{G}$, we get a toric variety $\boldsymbol{\mathbb{C}^n /\!\!/_{\chi} G}$, and by Corollary 14.2.16 of [CLS11], $\mathbb{C}^n /\!\!/_{\chi} G$ is a semiprojective toric variety. This variety is called a GIT (geometric invariant theory) quotient.

As described in [CLS11, Page 686], from our choice of G, we can construct an exact sequence. Consider the inclusion

$$G \hookrightarrow (\mathbb{C}^*)^n.$$

For any torus $(\mathbb{C}^*)^n$, the dual space $\mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C})$ is $\mathbb{Z}^n$. This gives us a dual map

$$\mathbb{Z}^n \to \widehat{G}.$$

By Lemma 14.2.1 in [CLS11], this map is surjective. Therefore we get an exact sequence

$$0 \to M \to \mathbb{Z}^n \to \widehat{G} \to 0$$

where M is defined as the kernel of the map $\mathbb{Z}^n \to \widehat{G}$. Note that M is also a lattice. Indeed any subgroup of $\mathbb{Z}^n$ is a lattice as any element in the subgroup with torsion would also have torsion in $\mathbb{Z}^n$.

Lemma 3.1. *Inclusions $G \hookrightarrow (\mathbb{C}^*)^n$ are in bijection with surjective matrix maps $Q : \mathbb{Z}^n \to \widehat{G}$.*

Proof. First assume we have an inclusion $G \hookrightarrow (\mathbb{C}^*)^n$. Then as described above, we get an exact sequence of algebraic groups

$$0 \to M \to \mathbb{Z}^n \to \widehat{G} \to 0$$

Any group homomorphism $\mathbb{Z}^n \to \widehat{G}$ is a matrix map. Therefore we have a surjective matrix map $Q : \mathbb{Z}^n \to \widehat{G}$.

Similarly, assume we start with such a map Q. By including the kernel, we get an exact sequence

$$0 \to M \to \mathbb{Z}^n \xrightarrow{Q} \widehat{G} \to 0$$

Since the Hom functor is right to left exact, dualizing gives the exact sequence

$$0 \to G \hookrightarrow (\mathbb{C}^*)^n \to \widehat{M}$$

which gives us the necessary inclusion. $\square$

Since the map $M \to \mathbb{Z}^n$ is also a group homomorphism it can also be described with a matrix map. We will use the convention in [CG15] and refer to the maps as Q and A as in the following exact sequence. In some contexts, such as [CG15], Q is referred to as the charge matrix.

$$0 \longrightarrow M \xrightarrow{\ A\ } \mathbb{Z}^n \xrightarrow{\ Q\ } \widehat{G} \longrightarrow 0$$

We will also use the convention in [CLS11] for referring the image of $\mathbb{Z}^n$ under Q and $A^\vee$. Let $e_1, ..., e_n$ be the generators of $\mathbb{Z}^n$, then we will use the notation $\nu_i = A^\vee(e_i) = A^\vee_{\mathbb{R}}(\nu_i)$, $\beta_i = Q_{\mathbb{R}}(e_i)$, and $\beta_i^{\mathbb{Z}} = Q(e_i)$. We will let ν be the set $\{\nu_1, ..., \nu_n\}$, $\beta = \{\beta_1, ..., \beta_n\}$, and $\beta^{\mathbb{Z}} = (\beta_1^{\mathbb{Z}}, ..., \beta_n^{\mathbb{Z}})$.

Each $\beta_i^{\mathbb{Z}}$ lies in $\widehat{G}$ and β_i lies in $\widehat{G}_{\mathbb{R}}$. We will also define C_β to be the cone generated by β in $\widehat{G}_{\mathbb{R}}$. Similarly each ν_i lies in the dual lattice of M, which we will call N. Then C_ν is the cone generated by ν in $N_{\mathbb{R}}$.

Often we will need to consider special subsets of β.

Definition 3.2. *A **β-basis** is any subset β_J of β such that* $\text{Cone}(\beta_i \mid \beta_i \in \beta_J)$ *is simplicial and has full dimension in $\widehat{G}_{\mathbb{R}}$.* [CLS11, 734]

Definition 3.3. *A **β^k-basis** is a subset β_J of β such that* $\text{Cone}(\beta_i \mid \beta_i \in \beta_J)$ *is simplicial and has dimension k in $\widehat{G}_{\mathbb{R}}$.*

3.2 The Secondary Fan

Associated with every GIT quotient is a fan called the GKZ (Gel'fand, Kapranov, Zelevinsky) decomposition, or the secondary fan. The secondary fan is a fan Σ_{GKZ} in $\widehat{G}_{\mathbb{R}}$ with support $|\Sigma_{GKZ}| = C_\beta$. [CLS11, Theorem 14.4.7] For a detailed description of the construction of this fan see Chapter 14.4 of [CLS11].

By construction in [CLS11], every cone in the secondary fan of $\mathbb{C}^n /\!\!/_\chi G$ comes from a generalized fan Σ and a set $I_\emptyset \subseteq \{1, ..., n\}$ that satisfy the following conditions:

- $|\Sigma| = C_\nu$
- X_Σ is semiprojective
- $\sigma = \mathrm{Cone}(\nu_i \mid \nu_i \in \sigma, i \notin I_0)$ for every $\sigma \in \Sigma$.

[CLS11, Proposition 14.4.1]

Therefore cones will be written as $\Gamma_{\Sigma, I_\emptyset}$, following the convention in [CLS11].

Definition 3.4. *The maximal dimension cones of the secondary fan are called* ***chambers****.* [CLS11, Page 718]

Proposition 3.5. *A cone $\Gamma_{\Sigma, I_\emptyset}$ of the secondary fan is a chamber if and only if Σ is a simplicial fan and there is a bijection $\{1, ..., n\} \setminus I_\emptyset \to \Sigma(1)$ given by $i \mapsto \mathrm{Cone}(\nu_i)$.* [CLS11, Proposition 14.4.9]

Note that Σ must be a fan, not just a generalized fan, for $\Gamma_{\Sigma, I_\emptyset}$ to be a chamber as simplicial cones are always strongly convex.

As described in [CLS11], β-bases, have some nice properties with regards to the secondary fan.

Proposition 3.6. *Let $\Gamma_{\Sigma, I_\emptyset}$ be a chamber of the secondary fan and β_J any β-basis. Then either $\mathrm{Cone}(\beta_J)$ contains the chamber, or the interior of $\Gamma_{\Sigma, I_\emptyset}$ and $\mathrm{Cone}(\beta_J)$ are disjoint.* [CLS11, Page 735]

[CLS11, 735] also describes a more explicit way to consider the secondary fan.

Proposition 3.7. *Let $\Gamma_{\Sigma, I_\emptyset}$ be a chamber of the secondary fan. Then:*

(a) If $\sigma \in \Sigma_{max}$ then $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$ is a β-basis.

(b) if J is a β-basis, then $\Gamma_{\Sigma, I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)$ if and only if $J = J_\sigma$ for some σ in Σ_{max}.

(c) $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma\in\Sigma_{max}} \mathrm{Cone}(\beta_{J_\sigma})$

[CLS11, Proposition 15.2.1]

Corollary 3.8. *The chambers of the secondary fan are exactly the maximal full dimensional intersections of cones over β-bases. That is $\Gamma_{\Sigma,I_\emptyset} = \bigcap_A \mathrm{Cone}(\beta_J)$ for some set A of β-bases such that $\bigcap_{A\cup a} \mathrm{Cone}(\beta_J)$ is not full dimensional for any $a \notin A$.*

Proof. First let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. Then by part (c) in Proposition 3.7, we have that $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma\in\Sigma_{max}} \mathrm{Cone}(\beta_{J_\sigma})$. Therefore $\Gamma_{\Sigma,I_\emptyset}$ is an intersection of cones over β-bases, and for this direction we need only show that this intersection is maximal.

Let β_J be another β-basis that is not β_{J_σ} for some $\sigma \in \Sigma_{max}$. Then by Proposition 3.6, $\mathrm{Cone}(\beta_J)$ either contains $\Gamma_{\Sigma,I_\emptyset}$, or is disjoint with its interior.

If $\mathrm{Cone}(\beta_J)$ contains $\Gamma_{\Sigma,I_\emptyset}$, then by part (b) of Proposition 3.7, $J = J_\sigma$ for some $\sigma \in \Sigma_{max}$, a contradiction. Alternatively if $\mathrm{Cone}(\beta_J)$ is disjoint with the interior of $\Gamma_{\Sigma,I_\emptyset}$, then $\mathrm{Cone}(\beta_J) \cap \Gamma_{\Sigma,I_\emptyset}$ is a subset of $\partial\Gamma_{\Sigma,I_\emptyset}$ and not full dimensional.

Therefore indeed $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma\in\Sigma_{max}} \mathrm{Cone}(\beta_{J_\sigma})$ is a maximal full dimensional intersection of cones over β-bases.

Conversely, let Γ be maximal full dimensional intersection of cones over β-bases, $\Gamma = \bigcap \mathrm{Cone}(\beta_{J_i})$. We want to show that Γ is a chamber of the secondary fan.

Since $|\Sigma_{GKZ}| = C_\beta$, and $\Gamma \subseteq C_\beta$, Γ must have a maximal dimension intersection with one of the chambers of the secondary fan $\Gamma_{\Sigma,I_\emptyset}$. Since $\Gamma = \bigcap \mathrm{Cone}(\beta_{J_i})$, $\mathrm{Cone}(\beta_{J_i}) \cap \Gamma_{\Sigma,I_\emptyset}$ have maximal dimension for each i. Therefore $\mathrm{Cone}(\beta_{J_i})$ intersects the interior of $\Gamma_{\Sigma,I_\emptyset}$, so $\mathrm{Cone}(\beta_{J_i}) \supseteq \Gamma_{\Sigma,I_\emptyset}$ for each i. Therefore $\Gamma \supseteq \Gamma_{\Sigma,I_\emptyset}$.

In fact, $\Gamma = \Gamma_{\Sigma,I_\emptyset}$. Assume for the sake of contradiction that this is not the case. Then $\Gamma \supset \Gamma_{\Sigma,I_\emptyset}$. Then $\Gamma \cap \Gamma_{\Sigma,I_\emptyset} = \Gamma_{\Sigma,I_\emptyset}$ is a maximal, full dimensional intersection of cones of β bases, that is strictly smaller than Γ, a contradiction.

Therefore $\Gamma = \Gamma_{\Sigma,I_\emptyset}$ is a chamber of the secondary fan. $\square$

The preceding corollary shows that the secondary fan depends only on the β's or equivalently the matrix Q, and not on the particular GIT quotient we chose. In fact the secondary fan notifies variations in the GIT quotient as we vary the character χ.

The choice of a GIT quotient depends on a character $\chi \in \widehat{G}$. By Proposition 14.3.5 of [CLS11], the GIT quotient for χ is distinct from the empty set if and only if $\chi \otimes 1$ is in C_β. For simplicity of notation we will use χ to denote both $\chi \in \widehat{G}$ and $\chi \otimes 1 \in \widehat{G}_{\mathbb{R}}$. Then by [CLS11, 14.4.7], for any cone $\Gamma_{\Sigma, I_\emptyset}$ in the secondary fan, if $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$, then $\mathbb{C}^n /\!\!/_\chi G$ is isomorphic to X_Σ. Therefore

Proposition 3.9. *Any non trivial GIT quotient can be determined by $\mathbb{C}^n /\!\!/_\chi G = X_\Sigma$ for $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$. Therefore the GIT quotient changes as we move between cones of the secondary fan, and is constant within the relative interior of a cone.*

We can also describe the GIT quotient using what is called the irrelevant ideal. There are a variety of ways to describe the irrelevant ideal, and we will use the one given in [CLS11, Proposition 14.4.14].

Definition 3.10. *For any character $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$, the **irrelevant ideal** can be defined as $\boldsymbol{B(\chi)} = \langle \prod_{\nu_i \notin \sigma \text{ or } i \in I_\emptyset} x_i \mid \sigma \in \Sigma_{\max} \rangle \subseteq \mathbb{C}[x_i \mid \nu_i \in \nu]$.*

Note the similarity to the irrelevant ideal defined in Definition 2.23. Indeed for any $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$, we have $B(\chi) = B(\Sigma) \times I_\emptyset$.

Definition 3.11. *For any character $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$ we define the **vanishing locus** to be $\boldsymbol{V(\chi)} = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in B(\chi)\}$ and the **semi-stable locus** to be $\boldsymbol{U(\chi)} = \mathbb{C}^n \setminus V(\chi)$.* [CLS11, Page 679, Page 698]

The semi-stable locus defined above is useful as it can be used to demonstrate the quotient nature of GIT quotients.

Proposition 3.12. *The inclusion $G \hookrightarrow (\mathbb{C}^*)^n$ induces a group action $G \times \mathbb{C}^n \to \mathbb{C}^n$, where the action is given by group multiplication in $\mathbb{C}^n$. For each character $\chi \in \operatorname{Relint}(\Gamma_{\Sigma, I_\emptyset})$, there is an isomorphism $\mathbb{C}^n /\!\!/_\chi G \cong U(\chi) /\!\!/ G$, where the group action is as described above and the quotient is as described in* [CLS11, Chapter 5] . [CLS11, Corollary 14.2.22]

Lemma 3.13. *For any chamber $\Gamma_{\Sigma, I_\emptyset}$ of the secondary fan, the rays of Σ are exactly the set $\{\operatorname{Cone}(\nu_i) \mid \nu_i \in \nu \setminus I_\emptyset\}$.*

Note: While the ν_i generate the rays, they may not be minimal ray generators.

Proof. By [CLS11, Proposition 14.4.9], there is a bijection $\nu \setminus I_\emptyset \to \Sigma(1)$; therefore the two sets have the same cardinality

As described in [CLS11, Proposition 14.4.1], the cones of Σ are generated by $\{\nu_i \mid \nu_i \in \sigma, i \notin I_\emptyset\}$. Therefore $\Sigma(1) \subseteq \{\mathrm{Cone}(\nu_i) \mid \nu_i \in \nu \setminus I_\emptyset\}$.

Then by the bijection, we know that every generator is needed, so $\Sigma(1) = \{\mathrm{Cone}(\nu_i) \mid \nu_i \in \nu \setminus I_\emptyset\}$. □

Proposition 3.14. *The GIT quotients coming from Q are projective if and only if each β_i is nonzero and* $\mathrm{Cone}(\beta)$ *is strongly convex.* [CLS11, Proposition 14.3.10]

In the best cases, the set ν corresponds to the rays of Σ, for a chamber $\Gamma_{\Sigma,I_\emptyset}$. By Lemma 3.13, this happens exactly when $I_\emptyset = \emptyset$. The following definition and proposition make this more precise.

Definition 3.15. *The set ν is called* ***geometric*** *if $\nu_i \neq 0$ for each $\nu_i \in \nu$ and the ν_i generate distinct rays in $N_\mathbb{R}$. It is called* ***primitive geometric*** *if in addition each ν_i is primitive.* [CLS11, Page 729-730]

Proposition 3.16. *The set ν is geometric if and only if there is a chamber $\Gamma_{\Sigma,I_\emptyset}$ such that $I_\emptyset = \emptyset$.* [CLS11, Proposition 15.1.6]

In this best case scenario, where ν corresponds directly to $\Sigma(1)$, we can describe the nef and pseudoeffective cones of X_Σ using only the set β and the chamber $\Gamma_{\Sigma,I_\emptyset}$.

Proposition 3.17. *If $\Gamma_{\Sigma,I_\emptyset}$ is a chamber with $I_\emptyset = \emptyset$ then there is an isomorphism $N^1(X) \to \widehat{G}_\mathbb{R}$ that takes* $\mathrm{Eff}(X_\Sigma) \to C_\beta$ *and* $\mathrm{Nef}(X_\Sigma) \to \Gamma_{\Sigma,I_\emptyset}$*. This isomorphism takes $D_\rho \to \beta_i$ where ν_i corresponds to the ray ρ as in Lemma 3.13.* [CLS11, Theorem 15.1.10]

Example 3.18. $\mathbb{P}^2$ can be constructed as a GIT quotient $\mathbb{C}^3 /\!\!/_\chi \mathbb{C}^*$. In this example will show this fact in different ways to demonstrate our theorems.

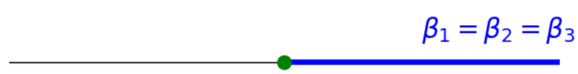
Consider the inclusion $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^3$ given by $x \to \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x$.

Dualizing this map yields $Q = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} : \mathbb{Z}^3 \to Z$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} : \mathbb{Z}^2 \to Z^3$.

We can consider the secondary fan in two different ways. First by Proposition 3.5, we know that every chamber $\Gamma_{\Sigma, I_\emptyset}$ in the secondary fan comes from a simplicial fan Σ such that $|\Sigma| = \text{Cone}(\nu)$ and $\Sigma(1) = \{\text{Cone}(\nu_i) \mid \nu_i \in \nu \setminus I_\emptyset\}$. As in Example 2.20, the fan for $\mathbb{P}^2$ satisfies these conditions. Removing any ray from this fan results in a fan with support strictly less than $\text{Cone}(\nu)$; therefore, no other fan exists that satisfies these properties. Thus the secondary fan must have only one chamber.

By Corollary 3.8, we can calculate the chambers of the secondary fan by taking maximal intersections of β-bases. Since $\beta_1 = \beta_2 = \beta_3$ are all the same, there is only one β-basis. Therefore as shown in Figure 3, $\Gamma_{\Sigma, I_\emptyset} = \text{Cone}(\beta_1) = \text{Cone}(\beta_2) = \text{Cone}(\beta_3)$ is the only chamber of the secondary fan.

Figure 3: The Secondary Fan for $Q = [1, 1, 1]$



Since the only chamber $\Gamma_{\Sigma, I_\emptyset}$ has $X_\Sigma = \mathbb{P}^2$, we know that for $\chi \in \text{Relint}(\Gamma_{\Sigma, I_\emptyset})$, $\mathbb{C}^3 /\!\!/_\chi \mathbb{C}^* = \mathbb{P}^2$ by Proposition 3.9.

One of the standard descriptions of $\mathbb{P}^2$ is as a quotient $(\mathbb{C}^n \setminus 0) /\!\!/ \mathbb{C}^*$. Here the group action is group multiplication, where $\mathbb{C}^*$ is considered a subgroup of $\mathbb{C}^n$ by the inclusion described at the beginning of this example. That is $t \cdot (x_1, x_2, x_3) = (tx_1, tx_2, tx_3)$.

We can use our theorems to recover this description of $\mathbb{P}^2$. By Proposition 3.12, we know that $\mathbb{C}^3 /\!\!/_\chi \mathbb{C}^* = (C^3 \setminus U(\chi)) /\!\!/ \mathbb{C}^*$. As calculated in Example 2.24, we know that for $\chi \in \text{Relint}(\Gamma_{\Sigma, I_\emptyset})$, $B(\Sigma) = B(\chi) = \langle x_1, x_2, x_3 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3]$. Therefore $U(\chi) = 0$. This yields $\mathbb{C}^3 /\!\!/_\chi \mathbb{C}^* = \mathbb{C}^n \setminus \{0\} /\!\!/ \mathbb{C}^*$, where the action is induced by the inclusion as required. $\square$

4 Vector Bundles

Definition 4.1. *In algebraic geometry a* ***vector bundle of rank n*** *is a morphism between varieties $\pi : V \to B$ that satisfies the following conditions.*

1. *There is an open cover U_i of B such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$*
2. *Let $\pi_i : U_i \times \mathbb{C}^n \to U_i$ be the projection onto U_i and $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ the isomorphism. Then $\pi_i \circ \phi_i = \pi|_{\pi^{-1}(U_i)}$.*
3. *For each i, j there are transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1} : U_i \cap U_j \times \mathbb{C}^n \to U_i \cap U_j \times \mathbb{C}^n$. These functions must be of the form $1 \times g_{ij}$ where g_{ij} is an invertible matrix with entries in $\mathcal{O}_B(U_i \cap U_j)$. That is $\phi_{ij}(x, v) = (x, g_{ij}(x)v)$.*

[Har77, 128, Exercise 5.18; CLS11, Definition 6.0.14]

Definition 4.2. *A* ***line bundle*** *is a vector bundle of rank* 1. [CLS11, Page 253]

Proposition 4.3. *There is a one to one correspondence between line bundles and sheaves $\mathcal{O}_X(D)$ for Cartier divisors D. Therefore we will use $\mathcal{O}_X(D)$ to refer to both the sheaf, and the line bundle.* [Har77, Proposition 6.13; CLS11, Proposition 6.0.18, Proposition 6.0.20]

As described in [CLS11, Chapter 7.3, 377], there is a nice description for vector bundles such that the base and bundle are both toric varieties.

Let Σ_0 be the fan of a toric variety in $\mathbb{C}^l$ which will form the base of the bundle. Let u_ρ^0 be the ray generators for rays ρ^0.

Then for any matrix $C = \begin{bmatrix} a_{\rho_1 1} & a_{\rho_1 2} & \cdots & a_{\rho_1 r} \\ \vdots & \vdots & & \vdots \\ a_{\rho_l 1} & a_{\rho_l 2} & \cdots & a_{\rho_l r} \end{bmatrix}$

we can create the fan $\Sigma \subseteq \mathbb{C}^l \oplus \mathbb{C}^r$ of a vector bundle of rank r as followed.

Let $e_1, ..., e_r$ be the standard basis of $\mathbb{C}^r$.

For each ray generator $u_{\rho_i}^0$ of Σ_0 , define $u_{\rho_i} = u_{\rho_i}^0 \oplus \sum_{j=1}^r a_{\rho_i j} e_j$.

For each cone σ_0 of Σ_0, we get a cone $\sigma \in \Sigma$ given by
$\sigma = \mathrm{Cone}(u_\rho \mid \rho^0 \in \sigma_0(1)) + \mathrm{Cone}(e_1, ...e_r)$.

Then Σ is the fan formed from these cones and their faces.

Proposition 4.4. *When each $D_j = \sum_{i=1}^r a_{\rho_i j} D_{\rho_i}$ is a Cartier divisor, the fan Σ is the rank r toric vector bundle $\mathcal{O}_{X_{\Sigma_0}}(-D_{\rho_1}) \oplus ... \oplus \mathcal{O}_{X_{\Sigma_0}}(-D_{\rho_r})$ over X_{Σ_0}.* [CLS11, Page 337]

In our setting, these vector bundles are the only vector bundles possible. This is illustrated by the following theorem of Oda.

Theorem 4.5. *All equivariant vector bundles $\pi : V \to B$, which are themselves toric varieties, are of the form described above.* [Oda78, Page 41]

We can generalize our notion of vector bundle with the following definition.

Definition 4.6. *Regardless of whether the D_j are Cartier, the set of cones Σ constructed above is a fan. We shall call any toric variety of such a fan a **stacky vector bundle** over Σ_0. See Remark 5.13 for comments on the motivation behind this name.*

Lemma 4.7. *For each $\sigma_0 \in \Sigma$ with ray generators $u_1^0, ..., u_l^0$, the ray generators of σ are $u_1, ..., u_l, e_1, ..., e_r$.*

Proof. We already know that these vectors generate the cone, so we just need to show that they are minimal generators. That is we need to show that for any $u_i \in \{u_1, ..., u_l\}$, $u_i \notin \mathrm{Cone}(u_1, ..., \widehat{u_i}, ..., u_l, e_1, ..., e_r)$, and for any $e_i \in \{e_1, ..., e_r\}$, $e_i \notin \mathrm{Cone}(u_1, ..., e_l, e_, ..., \widehat{e_i}, ..., e_r)$. Here $\widehat{\cdot}$ denotes the element has been removed from the set.

First assume that $u_1 = \sum_{i=1}^r b_i e_i + \sum_{i=2}^l c_i u_i$ for $b_i, c_i \geq 0$. Then restricting to coordinates on $\mathbb{C}^l$ we have $u_1^0 = \sum_{i=2}^l c_i u_i^0$. This contradicts the fact that $u_1^0, ..., u_l^0$ are ray generators for σ_0.

An identical argument holds for $u_2, ..., u_l$, so each of these is a ray generator.

Assume for the sake of contraction that $e_1 = \sum_{i=2}^r b_i e_i + \sum_{i=1}^l c_i u_i$. Again restricting to $\mathbb{C}^l$ coordinates, this gives us $\sum_{i=1}^l c_i u_i^0 = 0$. Since all the c_i are non-negative, this contradicts the strong convexity of the cone. Note that all the c_i cannot be zero, otherwise we would have $e_1 = \sum_{i=2}^r b_i e_i$ which is impossible.

An identical proof holds for $e_2, ..., e_r$, so $e_1, ..., e_r$ are all ray generators of the cone. $\square$

Corollary 4.8. *Let Σ_0 have ray generators $u_1^0, ..., u_l^0$. Then if Σ is a stacky vector bundle as described above, it has ray generators $u_1, ..., u_l, e_1, ..., e_r$.*

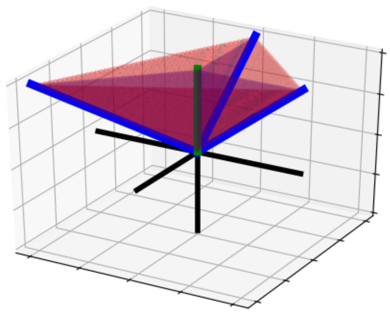
Let N and N' be a pair of lattices such that N is a sublattice of finite index $N \subseteq N'$. Then any fan Σ in $N_{\mathbb{R}}$ can also be seen as a fan in $N'_{\mathbb{R}}$, which we will call Σ'. As described in [CLS11, Proposition 3.3.7], this induces a toric morphism $X_\Sigma \to X_{\Sigma'}$ such that we can view $X_{\Sigma'}$ as $X_\Sigma/(N'/N)$. This motivates the following definition.

Definition 4.9. *Let X_Σ be a toric variety with lattice N. If there exists an overlattice $N' \supseteq N$ such that X_Σ is a stacky vector bundle with respect to N', then we will call X_Σ an* ***almost stacky vector bundle****.*

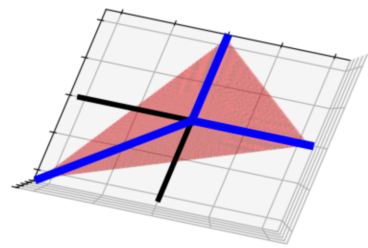
Example 4.10. In this example, we will construct a vector bundle over $\mathbb{P}^2$. We calculated the fan of $\mathbb{P}^2$ in Example 2.20. We will label the rays $\{\rho_1, \rho_2, \rho_3\}$, so that $u_{\rho_1} = [1, 0], u_{\rho_2} = [0, 1]$ and $u_{\rho_3} = [-1, -1]$.

Consider the stacky vector bundle given by the divisor $D = -D_{\rho_1} - D_{\rho_2} - D_{\rho_3}$. Since $\mathbb{P}^2$ is smooth, all Weil divisors are Cartier, which implies all stacky vector bundles over $\mathbb{P}^2$ are in fact vector bundles [CLS11, Theorem 4.0.22]. One can show that D gives the vector bundle $\mathcal{O}(-3)$ over $\mathbb{P}^2$. The fan of this vector bundle is shown in Figure 4. It has three rays lifted from the original rays, which are drawn in blue. These rays are $\rho_1 = [1, 0, 1]$, $\rho_2 = [0, 1, 1]$, $\rho_3 = [-1, -1, 1]$. The additional ray, given by $\rho_4 = [0, 0, 1]$, is drawn in green. Note that if we take the projection of the fan of bundle onto the plane normal to the additional ray, we recover the fan of the base.

Figure 4: The Vector Bundle $\mathcal{O}(-3)$.



(a) The Fan $\widetilde{\Sigma}$ of $\mathcal{O}(-3)$



(b) The Projection of $\widetilde{\Sigma}$ Normal to ρ_4

$\square$

5 Results

5.1 Main Results

Definition 5.1. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. We will call $\Gamma_{\Sigma,I_\emptyset}$ a **hybrid model** if there exists a projective toric variety X_{Σ_0} such that X_Σ is a stacky vector bundle over X_{Σ_0}. Similarly we will call the variety X_Σ a hybrid model if $\Gamma_{\Sigma,I_\emptyset}$ is a hybrid model.*

As described in the introduction, this definition is based on Definition 7.1.1 in [FJR15]. The goal of this project is to determine when $\Gamma_{\Sigma,I_\emptyset}$ is a hybrid model. The following two results give the answer to that question.

Theorem 5.11. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. Then X_Σ is a stacky vector bundle of rank r if and only if there exists a subset $\beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ and $a_{ji} \in \mathbb{Z}$ such that $|\beta_\Gamma| = r$ and for each $\beta_i \in \beta_\Gamma$ we have $-\beta_i^{\mathbb{Z}} = \sum_{\beta_j \notin \beta_\Gamma} a_{ji}\beta_j^{\mathbb{Z}}$.*

Corollary 5.12. *A chamber of the secondary fan is a hybrid model if and only if it satisfies the conditions of the previous theorem, $\beta_i \neq 0$ for each $\beta_i \in \beta^0 := \beta \setminus \beta_\Gamma$, and $\mathrm{Cone}(\beta^0)$ is strongly convex.*

We also extend these theorems to the following result on stacky vector bundles.

Theorem 5.15. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. If $\beta_\Gamma = (-\Gamma_{\Sigma,I_\emptyset} \cap \beta)$ is nonempty with $|\beta_\Gamma| = r$, then X_Σ is an almost stacky vector bundle of rank r.*

In order to prove these results, we will need some background work.

Lemma 5.2. *Let C_1, C_2 be maximal dimensional cones. Then if $C_1 \cap C_2$ is also maximal dimensional, every facet of $C = C_1 \cap C_2$ is contained in a facet of C_1 or C_2.*

Proof. Let W be any facet of C. By Definition 2.12, we know that $W = C \cap H$ for some hyperplane H. By general topology we know that $\partial C = \partial(C_1 \cap C_2) \subseteq \partial C_1 \cup \partial C_2$. Therefore since the boundary of any cone is the union of its facets, we know that W is contained in the union of facets of C_1 and C_2.

Now consider all of W at once. Either W is contained in a single facet, or parts of W are contained in multiple facets. Without loss of generality we can assume that W is contained in facets contained in H. Any parallel facets would not intersect and we would need an infinite number of non-parallel facets, which is nonsense. Therefore W is contained by facets of the form $H \cap C_i$. However any such facet would contain all of W since $W = H \cap C_1 \cap C_2$. This proves the claim. ☐

Condition 5.3. *We will say a cone C in $\widehat{G}_{\mathbb{R}}$ satisfies Condition 5.3 if it has dimension n and every facet in C is contained in* $\mathrm{Cone}(\beta_W)$ *for some β^{n-1}-basis β_W.*

Lemma 5.4. *Let C_1, C_2 be any two cones satisfying Condition 5.3. Then if $C_1 \cap C_2$ has dimension n, it also satisfies Condition 5.3.*

Proof. This follows directly from Lemma 5.2. Let W be a facet of $C_1 \cap C_2$. Then W is contained in a facet of C_1 or C_2, which by assumption is contained in some $\mathrm{Cone}(\beta_W)$. ☐

Corollary 5.5. *By induction, Lemma 5.4 is true for any finite intersection of cones satisfying the condition.*

Lemma 5.6. *Let β_J be a β basis. Condition 5.3 holds for* $\mathrm{Cone}(\beta_J)$.

Proof. Cone structure is preserved under change of basis. Since $\mathrm{Cone}(\beta_J)$ is simplicial, we can do a change of basis so that B_J is the standard basis. Then the facets of the cone are exactly $\mathrm{Cone}(e_{i_1}, ..., e_{i_{n-1}})$. Since under change of basis the e_i are the β_i, the condition is true. ☐

Corollary 5.7. *Condition 5.3 holds for any chamber $\Gamma_{\Sigma,I_\emptyset}$ of the secondary fan.*

Proof. By Proposition 3.7, we have that any chamber of the secondary fan can be written as $\Gamma_{\Sigma,I_\emptyset} = \bigcap \text{Cone}(\beta_J)$ for some β-bases β_J.

Therefore since $\Gamma_{\Sigma,I_\emptyset}$ is an n-dimensional intersection of cones who satisfy the condition, $\Gamma_{\Sigma,I_\emptyset}$ satisfies the condition.

$\square$

Proposition 5.8. *The set* $\beta \cap (\widehat{G}_{\mathbb{R}} \setminus -\Gamma_{\Sigma,I_\emptyset})$ *is equal to the set* $\left\{\beta_i \in \beta \mid \beta_i \text{ extends to a } \beta\text{-basis } \beta_J \text{ with } \text{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}\right\}$

Proof. Let β_i be an element of $\beta \cap (\widehat{G}_{\mathbb{R}} \setminus -\Gamma_{\Sigma,I_\emptyset})$.

Let W be a facet of $\Gamma_{\Sigma,I_\emptyset}$. $\Gamma_{\Sigma,I_\emptyset}$ satisfies Condition 5.3 so we have that W is contained in $\text{Cone}(\beta_W)$ for some β^{n-1}-basis β_W.

Since W is contained in $\text{Cone}(\beta_W)$, the hyperplane spanned by $\text{Cone}(\beta_W)$ must be the hyperplane spanned by W, as they have the same dimension. We will denote this hyperplane by H_W.

H_W separates the ambient space into two half planes. Because it is a facet of $\Gamma_{\Sigma,I_\emptyset}$, only one such half plane will contain $\Gamma_{\Sigma,I_\emptyset}$. Let $H_W^{\geq 0}$ and $H_W^{>0}$ be this halfplane, including and not including H_W respectively. We define $H_W^{\leq 0}$ and $H_W^{<0}$ similarly.

Then as in Proposition 2.13, $\Gamma_{\Sigma,I_\emptyset} = \cap_W H_w^{\geq 0}$, where the intersection is taken over all the facets of $\Gamma_{\Sigma,I_\emptyset}$.

Choose a facet W' such that $\beta_i \in H_{W'}^{>0}$. This is possible because $-\Gamma_{\Sigma,I_\emptyset} = \cap H_W^{\leq 0}$, so $\beta_i \notin -\Gamma_{\Sigma,I_\emptyset}$ implies $\beta_i \notin H_{W'}^{\leq 0}$ for some W'.

Let $\beta_J = \beta_{W'} \cup \beta_i$. It is enough to show that $\text{Cone}(B_J)$ contains $\Gamma_{\Sigma,I_\emptyset}$ as β_J is an extension of β_i.

By Proposition 3.6, $\Gamma_{\Sigma,I_\emptyset} \subseteq \beta_J$ or $\Gamma^\circ_{\Sigma,I_\emptyset} \cap \text{Cone}(\beta_J) = \emptyset$. Therefore we need only show that the interior of $\Gamma_{\Sigma,I_\emptyset}$ intersects $\text{Cone}(\beta_J)$.

Note that $\Gamma^{\circ}_{\Sigma,I_\emptyset} = (\cap(H^{\geq 0}_W))^{\circ} = \cap((H^{\geq 0}_W)^{\circ}) = \cap H^{>0}_W$.

Let w be a point in W'. W' is contained in $\mathrm{Cone}(\beta_{W'}) \subseteq \mathrm{Cone}(\beta_J)$. Up to change of basis we can assume that β_J is the standard basis. Then $\beta_J = \{w = (w_1, ..., w_n) \mid w_i \geq 0, 1 \leq i \leq n\}$ and $\beta_{W'} = \{w = (w_1, ..., w_n) \mid w_i \geq 0, 1 \leq i \leq n-1, w_n = 0\}$.

Therefore $w \in W'$ implies $w_i \geq 0$ and $w_n = 0$. Without loss of generality we can choose w such that the w_i are strictly positive for $i < n$. Otherwise every $w \in W$ would lie in $\cup_{i=1}^{n-1} \{x \mid x_i = 0, x_n = 0\}$ which is not sufficiently high dimensional.

Therefore for sufficiently small ϵ, every point x in the ball of radius ϵ centred at w, will have $x_i > 0$ for $i < n$.

$w \in \partial\Gamma_{\Sigma,I_\emptyset} = \partial\Gamma^{\circ}_{\Sigma,I_\emptyset}$. Therefore there exists $x \in \Gamma^{\circ}_{\Sigma,I_\emptyset}$ such that $||x-w|| < \epsilon$. Therefore x satisfies $x_i > 0$ for $i < n$. Since x is in the interior of $\Gamma_{\Sigma,I_\emptyset}$, we also have $x \in H^{>0}_W$ for all W and in particular $x \in H^{>0}_{W'}$, so $x_n > 0$. Therefore $x \in \mathrm{Cone}(\beta_J)$. Therefore $x \in \mathrm{Cone}(\beta_J) \cap \Gamma^{\circ}_{\Sigma,I_\emptyset}$.

Therefore $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)$ as required.

Conversely let β_i be in $-\Gamma_{\Sigma,I_\emptyset}$ and assume for the sake of contradiction that β_i extends to a β-basis β_J that contains $\Gamma_{\Sigma,I_\emptyset}$. Then $-\beta_i \in \Gamma_{\Sigma,I_\emptyset}$, so $\mathrm{Cone}(\beta_J)$ contains both β_i and $-\beta_i$. However $\mathrm{Cone}(\beta_J)$ is supposed to be strongly convex. A contradiction. □

Proposition 5.9. *The irrelevant ideal $B(\chi)$ can be calculated explicitly by $B(\chi) = \langle \prod_{i \in J} x_i \mid \Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)\rangle$ where $\Gamma_{\Sigma,I_\emptyset}$ is the cone containing χ in its relative interior.*

Proof. By definition, $B(\chi)$ is given by $B(\chi) = \langle \prod_{\nu_i \notin \sigma \text{ or } i \in I_\emptyset} x_i \mid \sigma \in \Sigma_{\max}\rangle$.

In Proposition 3.7, we define $J_\sigma := \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$. Therefore $B(\chi) = \langle \prod_{i \in J_\sigma} x_i \mid \sigma \in \Sigma_{\max}\rangle$.

By part (b) of Proposition 3.7, we know that β-bases β_J are of the form β_{J_σ} for $\sigma \in \Sigma_{\max}$, exactly when $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)$.

Therefore $B(\chi) = \langle \prod_{i \in J} x_i \mid \Gamma_{\Sigma, I_\emptyset} \subseteq \text{Cone}(\beta_J) \rangle$, as required.

$\square$

Lemma 5.10. *If $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, $I_\emptyset = \left\{x_i \mid \beta_i \in \beta_J \text{ for every } \text{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}\right\}$*

Proof. First assume $x_i \in I_\emptyset$. Then if $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$, $x_i \in J_\sigma$ for each $\sigma \in \Sigma_{\max}$. Then by Proposition 3.7, part b), x_i is in β_J for ever $\text{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}$.

Conversely assume x_i is such that $\beta_i \in \beta_J$ for every $\beta_J \supseteq \Gamma_{\Sigma,I_\emptyset}$. Assume for the sake of contradiction that x_i is not in $I_\emptyset$. Then by Lemma 3.13, x_i corresponds to a ray ν_i. Let σ be any maximal cone in Σ containing ν_i. Then $i \notin J_\sigma$ and $\beta_i \notin \beta_{J_\sigma}$.

However by Proposition 3.7 part b) , $\Gamma_{\Sigma,I_\emptyset} \subseteq \beta_{J_\sigma}$.

Therefore by assumption $\beta_i \in \beta_{J_\sigma}$. A contradiction.

$\square$

Using the preceding propositions and lemmas, we arrive at our main results which we restate and prove below.

Theorem 5.11. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. Then X_Σ is a stacky vector bundle of rank r if and only if there exists a subset $\beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ and $a_{ji} \in \mathbb{Z}$ such that $|\beta_\Gamma| = r$ and for each $\beta_i \in \beta_\Gamma$ we have $-\beta_i^{\mathbb{Z}} = \sum_{\beta_j \notin \beta_\Gamma} a_{ji} \beta_j^{\mathbb{Z}}$.*

Proof. First assume β_Γ exists. We will construct a base called X_{Σ_0}, a vector bundle over that base called $X_{\widetilde{\Sigma}}$, and prove that $\Sigma = \widetilde{\Sigma}$, proving the forward direction.

Let $\beta^0 = \beta \setminus \beta_\Gamma$ and $l = n - r$. Without loss of generality we can order β so that $\beta^0 = \{\beta_1, ..., \beta_l\}$.

Recall, the set β comes with an exact sequence as below.

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} \widehat{G} \longrightarrow 0$$

$\mathbb{Z}^n$ can be viewed as $\mathbb{Z}^l \oplus \mathbb{Z}^r$, so with this splitting there is a natural inclusion $g : \mathbb{Z}^l \hookrightarrow \mathbb{Z}^l \oplus \mathbb{Z}^r$, giving us the following diagram.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
 & & \mathbb{Z}^l \arrow[u, "g"] & &
\end{tikzcd}
$$

Let $h : \widehat{G} \to \widehat{G}$, be the identity. Then we can extend the diagram to

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
 & & \mathbb{Z}^l \arrow[u, "g"] & \widehat{G} \arrow[u, "h"] &
\end{tikzcd}
$$

Let $Q_0 = Q \circ g : \mathbb{Z}^l \to \widehat{G}$, giving us the following diagram.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
 & & \mathbb{Z}^l \arrow[u, "g"] \arrow[r, "Q_0"] & \widehat{G} \arrow[u, "h"] &
\end{tikzcd}
$$

We can choose a basis reflecting the splitting $\mathbb{Z}^n = \mathbb{Z}^l \oplus \mathbb{Z}^r$. In this basis Q can be written as $Q = [Q_0, Q_1]$, where $Q_0 : \mathbb{Z}^l \to \widehat{G}$, as in the diagram, and $Q_1 : \mathbb{Z}^r \to \widehat{G}$. By definition $\beta_i^{\mathbb{Z}} = Q(e_i)$. Therefore $\mathrm{Im}(Q_0) = \mathrm{span}(\beta_1^{\mathbb{Z}}, ..., \beta_l^{\mathbb{Z}})$ and $\mathrm{Im}(Q) = \mathrm{span}(\beta_1^{\mathbb{Z}}, ..., \beta_n^{\mathbb{Z}})$. By assumption, for each $\beta_i \in \beta_\Gamma$ we have $-\beta_i^{\mathbb{Z}} = \sum_{i=1}^{l} a_{ji}\beta_j^{\mathbb{Z}}$ for $a_{ij} \in \mathbb{Z}$. Therefore $\mathrm{span}(\beta_1^{\mathbb{Z}}, ..., \beta_l^{\mathbb{Z}}) = \mathrm{span}(\beta_1^{\mathbb{Z}}, ..., \beta_n^{\mathbb{Z}})$ and Q_0 is surjective by the surjectivity of Q.

Therefore the diagram can be written as

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
 & & \mathbb{Z}^l \arrow[u, "g"] \arrow[r, "Q_0"] & \widehat{G} \arrow[u, "h"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

Let M_0 be the kernel of Q_0 and A_0 be any inclusion. This gives us the following diagram.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
0 \arrow[r] & M_0 \arrow[r, "A_0"] & \mathbb{Z}^l \arrow[r, "Q_0"] \arrow[u, "g"] & \widehat{G} \arrow[r] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}
$$

Then there exists an inclusion $f : M_0 \to M$ making the diagram commute. See Lemma A.1 and Lemma A.2 for a proof of this statement.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & M_0 \arrow[r, "A_0"] \arrow[u, "f"] & \mathbb{Z}^l \arrow[r, "Q_0"] \arrow[u, "g"] & \widehat{G} \arrow[r] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}
$$

From the snake lemma, we get the following diagram, where the snake is exact.

$$
\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "A"] \arrow[u] & \mathbb{Z}^n \arrow[r, "Q"] \arrow[u] & \widehat{G} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & M_0 \arrow[r, "A_0"] \arrow[u, "f"] & \mathbb{Z}^l \arrow[r, "Q_0"] \arrow[u, "g"] & \widehat{G} \arrow[r] \arrow[u, "h"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \ker f \arrow[r] \arrow[u] & \ker g \arrow[r] \arrow[u] & \ker h \arrow[u] \arrow[uuull, red, rounded corners]
\end{tikzcd}
$$

Then $\ker f = \ker g = \ker h = \operatorname{coker} h = 0$ and by Lemma A.3 $\operatorname{coker} f \cong \operatorname{coker} g = \mathbb{Z}^r$ giving us the following diagram.

$$
\begin{tikzcd}
 & \mathbb{Z}^r \arrow[r] & \mathbb{Z}^r \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "A"] \arrow[u] & \mathbb{Z}^n \arrow[r, "Q"] \arrow[u] & \widehat{G} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & M_0 \arrow[r, "A_0"] \arrow[u, "f"] & \mathbb{Z}^l \arrow[r, "Q_0"] \arrow[u, "g"] & \widehat{G} \arrow[r] \arrow[u, "h"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \arrow[uuull, rounded corners]
\end{tikzcd}
$$

Define a map $t : \mathbb{Z}^r \to \mathbb{Z}^n$ as follows.

By assumption we can write $-\beta_i^{\mathbb{Z}} = \sum_{\beta_j \in \beta^0} a_{ji}\beta_j^{\mathbb{Z}}$ for $a_{ji} \in \mathbb{Z}$. The we can define t such that for each $\{e_{l+1}, ..., e_n\}$ we have $t(e_i) = \sum_{j=1}^{l} a_{ji}e_j + e_i$. Then the matrix of t is given by

$$\begin{bmatrix} a_{1(l+1)} & a_{1(l+2)} & \cdots & a_{1(l+r)} \\ \vdots & \vdots & & \vdots \\ a_{l(l+1)} & a_{l(l+2)} & \cdots & a_{l(l+r)} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note that since $\beta_i^{\mathbb{Z}}$ is the ith column of Q, this gives us $Q \circ t = 0$.

Then by the universal property of kernels, and the fact that $A : M \to \mathbb{Z}^n$ is the kernel of Q, there is a map $s : \mathbb{Z}^r \to M$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathbb{Z}^r \arrow[dl, "s"'] \arrow[d, "t"] \\ M \arrow[r, "A"] & \mathbb{Z}^n \end{tikzcd}$$

Since the map $\mathbb{Z}^r \to \mathbb{Z}^r$ is an isomorphism between cokernels, we also have that the following square commutes.

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r] \arrow[d, "s"] & \mathbb{Z}^r \arrow[d, "t"] \\ M \arrow[r, "A"] & \mathbb{Z}^n \end{tikzcd}$$

Therefore we have the following diagram where either the blue part or the red part commutes. We will name the cokernel maps of f and g, π_f and π_g respectively.

$$
\begin{tikzcd}
 & \mathbb{Z}^r \arrow[r] \arrow[d, shift left, "s"] & \mathbb{Z}^r \arrow[r] \arrow[d, shift left, "t"] & 0 \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "A"] \arrow[u, shift left, "\pi_f"] & \mathbb{Z}^n \arrow[r, "Q"] \arrow[u, shift left, "\pi_g"] & \widehat{G} \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & M_0 \arrow[r, "A_0"] \arrow[u, "f"] & \mathbb{Z}^l \arrow[r, "Q_0"] \arrow[u, "g"] & \widehat{G} \arrow[r] \arrow[u, "h"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Note that $\pi_g \circ t = \mathrm{id}_{\mathbb{Z}^r}$.

Indeed $\pi_g \circ t(e_i) = \pi_g(\sum_{j=1}^{l} a_{ji}e_j + e_i) = e_i$ for each basis element, so by linearity the claim holds.

Let's consider the following part of the diagram.

$$
\begin{tikzcd}
 & \mathbb{Z}^r \arrow[dl, shift left, "s"] \arrow[dr, shift left, "t"] & \\
M \arrow[ur, shift left, "\pi_f"] \arrow[rr, "A"] & & \mathbb{Z}^n \arrow[ul, shift left, "\pi_g"]
\end{tikzcd}
$$

Then $t = A \circ s$ and $\pi_f = \pi_g \circ A$.
Then $\pi_f \circ s = \pi_g \circ A \circ s = \pi_g \circ t = \mathrm{id}$.

Therefore we have a splitting

$$
\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r, "f"] & M \arrow[r, bend left, "\pi_f"] & \mathbb{Z}^r \arrow[l, bend left, "s"] \arrow[r] & 0
\end{tikzcd}
$$

Therefore, by the splitting lemma we can write $M = M_0 \oplus \mathbb{Z}^r$.

Then we can choose a basis for M to reflect this splitting such that f is the identity $M_0 \to M_0 \subseteq M_0 \oplus \mathbb{Z}^r$ and s is the identity $\mathbb{Z}^r \to \mathbb{Z}^r \subseteq M_0 \oplus \mathbb{Z}^r$.

We can write A in the form $A = [x, y]$ where $x : M_0 \to \mathbb{Z}^n$ and $y : \mathbb{Z}^r \to \mathbb{Z}^n$.

Then $x(m_0) = A \circ f(m_0) = g \circ A_0(m_0)$, so

$$x = \begin{bmatrix} A_0 \\ 0 \end{bmatrix}$$

Similarly $y = A \circ s = t = \begin{bmatrix} a_{1(l+1)} & a_{1(l+2)} & \cdots & a_{1(l+r)} \\ \vdots & \vdots & & \vdots \\ a_{l(l+1)} & a_{l(l+2)} & \cdots & a_{l(l+r)} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

Therefore $A = \begin{bmatrix} A_0 & C \\ 0 & I \end{bmatrix}$

where C is the coefficient matrix $\begin{bmatrix} a_{1(l+1)} & a_{1(l+2)} & \cdots & a_{1(l+r)} \\ \vdots & \vdots & & \vdots \\ a_{l(l+1)} & a_{l(l+2)} & \cdots & a_{l(l+r)} \end{bmatrix}$

Now from β^0 and Q_0 we can construct a new secondary fan. Note that A_0 is by definition the inclusion of the kernel of Q_0, so A_0 is the corresponding "A" matrix.

By Corollary 3.8, any cone of the secondary fan is a maximal intersection of cones over β-bases.

By Proposition 5.8, the β_J such that Cone(β_J) contains $\Gamma_{\Sigma,I_\emptyset}$ are exactly the β-bases that are disjoint from $-\Gamma_{\Sigma,I_\emptyset}$. Since $\beta_\Gamma \subseteq -\Gamma_{\Sigma,I_\emptyset}$ and $\beta^0 = \beta \setminus \beta_\Gamma$, all β-bases forming the chamber are also subsets of β^0. Therefore $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\Gamma_{\Sigma,I_\emptyset} \subseteq \beta_J \subseteq \beta^0} \text{Cone}(\beta_J)$ and is also a chamber in the secondary fan of β^0. We will refer to this chamber as $\Gamma_{\Sigma_0,I_{\emptyset_0}}$, refer to $A_0^\vee(e_i)$ as ν_i^0, and let $\chi_0 \in \Gamma_{\Sigma_0,I_{\emptyset_0}}$ be any character in the relative interior of $\Gamma_{\Sigma_0,I_{\emptyset_0}}$.

We claim that $I_\emptyset = I_{\emptyset_0}$. By Lemma 5.10, $I_\emptyset = \{x_i \mid \beta_i \in \beta_J \text{ for every } \text{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}\}$. Therefore since the β_J forming $\Gamma_{\Sigma,I_\emptyset}$ and $\Gamma_{\Sigma_0,I_{\emptyset_0}}$ are the same by definition, $I_\emptyset = I_{\emptyset_0}$.

By Lemma 3.13, the rays of Σ_0 are generated by $\nu^0 \setminus I_\emptyset$.

Recall the matrix $A = \begin{bmatrix} A_0 & C \\ 0 & I \end{bmatrix}$. We can form a new matrix $A' = \begin{bmatrix} A'_0 & C' \\ 0 & I \end{bmatrix}$ by removing the rows corresponding to ν_i with $i \in I_\emptyset$ and scaling the remaining rows of $\begin{bmatrix} A_0 & C \end{bmatrix}$ to make them minimal generators.

Then the rows of A'_0 are exactly the ray generators of Σ_0 and the rows of A' are exactly the ray generators of Σ. We can build a stacky vector bundle $\widetilde{\Sigma}$ over Σ_0 using the matrix C' and the process in section section 4.

By Lemma 4.7 the rays of $\widetilde{\Sigma}$ are exactly the rows of A', which are exactly the rows of Σ.

By [CLS11, Page 207], if two toric varieties have the same rays and same irrelevant ideals, they are equal. Therefore we need only show $B(\Sigma) = B(\widetilde{\Sigma})$.

The irrelevant ideal of $\widetilde{\Sigma}$ is given by $B(\widetilde{\Sigma}) = \langle \prod_{\rho \notin \sigma(1)} x_\rho \mid \sigma \in \widetilde{\Sigma}_{\max} \rangle$.

Note that by Lemma 4.7, the maximal cones of $\widetilde{\Sigma}$ are generated by rays of cones in Σ_0 and $e_1, ..., e_r$.

Therefore $B(\widetilde{\Sigma}) = \langle \prod_{\rho \notin \sigma_0(1)} x_\rho \mid \sigma_0 \in \Sigma_{0\max} \rangle = B(\Sigma_0)$.

For $\chi \in \Gamma_{\Sigma, I_\emptyset}$, $B(\chi) = \langle \prod_{\nu_i \notin \sigma \text{ or } i \in I_\emptyset} x_i \mid \sigma \in \Sigma_{\max} \rangle = B(\Sigma) \times I_\emptyset$. Similarly $B(\chi_0) = B(\Sigma_0) \times I_\emptyset$.

Alternatively by Proposition 5.9, we have that $B(\chi) = \langle \prod_{i \in J} x_i \mid \Gamma_{\Sigma, I_\emptyset} \subseteq \mathrm{Cone}(\beta_J) \rangle$.

Note that since $\Gamma_{\Sigma, I_\emptyset}$ and $\Gamma_{\Sigma_0, I_{\emptyset 0}}$ are intersections of the same cones by definition $B(\chi) = B(\chi_0)$.

Therefore we have both $B(\chi) = B(\Sigma) \times I_\emptyset = B(\Sigma_0) \times I_\emptyset$.

Therefore $B(\Sigma) = B(\Sigma_0) = B(\widetilde{\Sigma})$.

Since Σ and $\widetilde{\Sigma}$ have the same irrelevant ideals and rays, they are equal.

Therefore X_Σ is indeed a stacky vector bundle of rank r over X_{Σ_0}.

Conversely assume X_Σ is a stacky vector bundle of rank r. Then up to change of basis we can assume the rows of A are scalar multiples of the rows of the matrix $\begin{bmatrix} E_1 & E_2 \\ A_0 & C \\ 0 & I \end{bmatrix}$.

Here I is the $r \times r$ identity, the rows of $\begin{bmatrix} A_0 & C \\ 0 & I \end{bmatrix}$ correspond to the ν_i that are rays of Σ, and $\begin{bmatrix} E_1 & E_2 \end{bmatrix}$ corresponds to the $\nu_i \in I_\emptyset$.

Then we will show that choosing β_Γ to correspond to the rows of $\begin{bmatrix} 0 & I \end{bmatrix}$ satisfies the claim.

First we will show that the β_Γ defined above satisfies $\beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$.

By Proposition 3.7, $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma \in \Sigma_{max}} \text{Cone}(\beta_{J_\sigma})$ where $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$
and by Proposition 5.8,
$(-\Gamma_{\Sigma,I_\emptyset} \cap \beta) = \left\{\beta_i \in \beta \mid \beta_i \text{ does not extend to a } \beta\text{-basis } \beta_J \text{ such that } \text{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}\right\}$.
Therefore $(-\Gamma_{\Sigma,I_\emptyset} \cap \beta) = \{\beta_i \mid i \notin J_\sigma \text{ for any } \sigma \in \Sigma_{\max}\} = \{\beta_i \mid i \notin I_\emptyset \text{ and } i \in \sigma \text{ for all } \sigma \in \Sigma_{\max}\}$.

Since only the rows in $\begin{bmatrix} E_1 & E_2 \end{bmatrix}$ are part of $I_\emptyset$, to prove $\beta_\Gamma \subseteq (-\Gamma_{\Sigma,I_\emptyset} \cap \beta)$ we need only show that the ν_i corresponding to the rows of $\begin{bmatrix} 0 & I \end{bmatrix}$ are in every $\sigma \in \Sigma_{\max}$. This is true by Lemma 4.7 so $\beta_\Gamma \subseteq (-\Gamma_{\Sigma,I_\emptyset} \cap \beta)$.

Now to prove the claim, we need only show that for each $\beta_i \in \beta_\Gamma$, there exists $a_{ji} \in \mathbb{Z}$ such that $-\beta_i^{\mathbb{Z}} = \sum_{\beta_j \in \beta^0} a_{ji}\beta_j^{\mathbb{Z}}$.

By assumption $QA = 0$. Therefore for each for each column of A get a dependence relation between the columns of Q. In particular if we write A as

$$\begin{bmatrix} b_{11} & \cdots & b_{1s} & a_{1(l+1)} & a_{1(l+2)} & \cdots & a_{1(l+r)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{l1} & \cdots & b_{ls} & a_{l(l+1)} & a_{l(l+2)} & \cdots & a_{l(l+r)} \\ 0 & \cdots & 0 & 1 & 0 & ... & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

then the last r columns of A give $\beta_i^{\mathbb{Z}} + \sum_{j=1}^{l} a_{ji}\beta_j^{\mathbb{Z}} = 0$ as required.

$\square$

Corollary 5.12. *A chamber of the secondary fan is a hybrid model if and only if it satisfies the conditions of the previous theorem, $\beta_i \neq 0$ for each $\beta_i \in \beta^0 := \beta \setminus \beta_\Gamma$, and* $\mathrm{Cone}(\beta^0)$ *is strongly convex.*

Proof. Since a hybrid model is a stacky vector bundle over a projective base, this follows directly from Proposition 3.14. $\square$

Remark 5.13. *The proof of the proceeding theorem is also why we use the name stacky vector bundle. In the setup of this proof we construct a base X_{Σ_0} for X_Σ. We show that for any $\chi \in \Gamma_{\Sigma,I_\emptyset}$ and $\chi_0 \in \Gamma_{\Sigma_0,I_{\emptyset 0}}$ we have $B(\chi) = B(\chi_0) = \langle \prod_{i\in J} x_i \mid \Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)\rangle$. The only difference between the ideals is $B(\chi) \subseteq \mathbb{C}[x_i \mid \beta_i \in \beta]$ and $B(\chi_0) \subseteq \mathbb{C}[x_i \mid \beta_i \in \beta^0]$. This implies $U(\chi) = \mathbb{C}^n \times U(\chi_0)$.*

By Proposition 3.12, $X_{\Sigma_0} = U(\chi_0) /\!\!/ G$ and $X_\Sigma = U(\chi) /\!\!/ G$. The G acting on $U(\chi)$ is actually the same G acting on $U(\chi_0)$ since the proof of the theorem shows they share the same $\widehat{G}$. Furthermore the group action $G \times \mathbb{C}^l \to \mathbb{C}^l$ is simply the group action $G \times \mathbb{C}^n \to \mathbb{C}^n$ restricted to $\mathbb{C}^l$. To see this note that the action $G \times \mathbb{C}^n \to \mathbb{C}^n$ is given by $t \to Q^T(g)t$.

Let $\begin{bmatrix} t_1 \\ \vdots \\ t_l \\ s_1 \\ \vdots \\ s_r \end{bmatrix}$ *be an arbitrary element of $\mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^r$ and let $*$ denote group multiplication in either group.*

Then

$$g \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_l \\ s_1 \\ \vdots \\ s_r \end{bmatrix} = Q^T(g) * \begin{bmatrix} t_1 \\ \vdots \\ t_l \\ s_1 \\ \vdots \\ s_r \end{bmatrix} = \begin{bmatrix} \beta_1^{\mathbb{Z}} — \\ \vdots \\ \beta_n^{\mathbb{Z}} — \end{bmatrix} g * \begin{bmatrix} t_1 \\ \vdots \\ t_l \\ s_1 \\ \vdots \\ s_r \end{bmatrix}$$

$$= \begin{bmatrix} \langle \beta_1^{\mathbb{Z}}, g \rangle & \ldots & \langle \beta_n^{\mathbb{Z}}, g \rangle \end{bmatrix} * \begin{bmatrix} t_1 \\ \vdots \\ t_l \\ s_1 \\ \vdots \\ s_r \end{bmatrix} = \begin{bmatrix} \langle \beta_1^{\mathbb{Z}}, g \rangle t_1 \\ \vdots \\ \langle \beta_n^{\mathbb{Z}}, g \rangle s_r \end{bmatrix}$$

Restricting this to $\mathbb{C}^l$ *we get*

$$g \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_l \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \langle \beta_1^{\mathbb{Z}}, g \rangle t_1 \\ \vdots \\ \langle \beta_l^{\mathbb{Z}}, g \rangle t_l \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

.

Similarly the action $G \times \mathbb{C}^l \to \mathbb{C}^l$ *is given by* $t \to Q_0^T(g) * t$*, so we have*

$$g \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_l \end{bmatrix} = \begin{bmatrix} \langle \beta_1^{\mathbb{Z}}, g \rangle t_1 \\ \vdots \\ \langle \beta_l^{\mathbb{Z}}, g \rangle t_l \end{bmatrix}$$

Therefore the action of G *on* $\mathbb{C}^l$ *is just the action of* G *on* $\mathbb{C}^n$ *restricted to* $\mathbb{C}^l$*.*

Since $U(\chi) = \mathbb{C}^r \times U(\chi_0)$*,* $U(\chi)$ *is a* G*-equivariant vector bundle over* $U(\chi_0)$*, that is the* *projection map is compatible with the G-actions. This motivates the terminology "stacky" vector bundle.*

Lemma 5.14. *Let β_i in $(-\Gamma_{\Sigma,I_\emptyset} \cap \beta)$. Without loss of generality we can reorder our β_i into two sets $\beta_\Gamma = (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ and $\beta^0 = \beta \setminus \beta_\Gamma$ such that $\{\beta_1, ..., \beta_l\} \in \beta^0$ and $\{\beta_{l+1}, ..., \beta_n\} \in \beta_\Gamma$. Then there exists $d_i, a_{ji} \in \mathbb{Z}_{\geq 0}$ such that $d_i\beta_i^{\mathbb{Z}} + \sum_{i=1}^{l} a_{ij}\beta_j^{\mathbb{Z}} = 0$ for each $\beta_i \in \beta_\Gamma$.*

Proof. For each $\beta_i \in \beta_\Gamma$, $-\beta_i \in \Gamma_{\Sigma,I_\emptyset}$. By Proposition 5.8, the β_J such that that Cone(β_J) contains $\Gamma_{\Sigma,I_\emptyset}$ are exactly the β-bases that are subsets of β^0. Therefore $\Gamma_{\Sigma,I_\emptyset} \subseteq$ Cone(β_J) for each β-basis $\beta_J \subseteq \beta^0$. Since Cone$(\beta_J)$ is simplicial, elements of β_J are generators of a vector space. Therefore there exist unique $c_{ji} \in \mathbb{R}$ such that $-\beta_i = \sum_{j\in J} c_{ji}\beta_j$. Note that since $-\beta_i$ and each β_j are lattice points, this unique description is also true over $\mathbb{Q}$ and c_{ji} must be rational. Furthermore since $-\beta_i$ is assumed to be in Cone(β_J), the c_{ji} must be positive.

Then multiplying the equation by a suitable integer d_i to clear the denominators, we get $-d_i\beta_i = \sum_{j\in J} a_{ji}\beta_j$ for $a_{ij}, d_i \in \mathbb{Z}_{\geq 0}$.

Note that since we can do this for each $\beta_J \subseteq \beta^0$ this description is not unique.

Our relations so far happen in $\widehat{G}_{\mathbb{R}}$ and not $\widehat{G}$. However, the kernel of the map $\widehat{G} \to \widehat{G}_{\mathbb{R}}$ is a torsion subgroup T. Therefore we have $-d_i\beta_i^{\mathbb{Z}} = \sum_{j\in J} a_{ji}\beta_j^{\mathbb{Z}} + t$ for some $t \in T$. Then multiplying both sides by a suitable integer s will yield $st = 0$, so that we have $-sd_i\beta_i^{\mathbb{Z}} = \sum_{j\in J} sa_{ji}\beta_j^{\mathbb{Z}}$ as required.

$\square$

Theorem 5.15. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. If $\beta_\Gamma = (-\Gamma_{\Sigma,I_\emptyset} \cap \beta)$ is nonempty with $|\beta_\Gamma| = r$, then X_Σ is an almost stacky vector bundle of rank r.*

Proof. First we will assume there exists $\beta_i \in \beta_\Gamma$. Let $\beta^0 = \beta \setminus \beta_\Gamma$ and reorder β so that $\beta^0 = \{\beta_1, ..., \beta_l\}$. Then by Lemma 5.14, there exist $d_{l+1}, ..., d_{l+r}$ and $a_{ij} \in \mathbb{Z}$ such that for each $\beta_i \in \beta_\Gamma$, we have $d_i\beta_i^{\mathbb{Z}} + \sum_{i=1}^{l} a_{ij}\beta_j^{\mathbb{Z}} = 0$.

Let D be the matrix with $d_{l+1}, ..., d_{l+r}$ along the main diagonal and $g : \mathbb{Z}^n \to \mathbb{Z}^n$ be the map given by $\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$ where we consider $\mathbb{Z}^n$ as $\mathbb{Z}^l \oplus \mathbb{Z}^r$.

Let

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} \widehat{G} \longrightarrow 0$$

be the exact sequence associated with β.

Then we can extend the diagram by defining $Q' = Q \circ g$.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
 & & \mathbb{Z}^n \arrow[u, "g"] \arrow[r, "Q'"] & \mathrm{Im}(Q') \arrow[r] & 0
\end{tikzcd}$$

Let M' be the kernel of Q' and A' any inclusion to extend the diagram.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r, "A'"] & \mathbb{Z}^n \arrow[u, "g"] \arrow[r, "Q'"] & \mathrm{Im}(Q') \arrow[r] & 0
\end{tikzcd}$$

Since $g(\mathbb{Z}^n)$ is a subgroup of $\mathbb{Z}^n$, we have $Q'(\mathbb{Z}^n) = Q \circ g(\mathbb{Z}^n)$ is a subgroup of $Q(\mathbb{Z}^n) = \widehat{G}$.

Therefore we can choose an inclusion h so that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r, "A'"] & \mathbb{Z}^n \arrow[u, "g"] \arrow[r, "Q'"] & \mathrm{Im}(Q') \arrow[u, "h"] \arrow[r] & 0
\end{tikzcd}$$

By Lemma A.1 there exists an inclusion $f : A' \to A$ making the diagram commute as below.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "A"] & \mathbb{Z}^n \arrow[r, "Q"] & \widehat{G} \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[u, "f"] \arrow[r, "A'"] & \mathbb{Z}^n \arrow[u, "g"] \arrow[r, "Q'"] & \mathrm{Im}(Q') \arrow[u, "h"] \arrow[r] & 0
\end{tikzcd}$$

We can also consider the diagram where everything is tensored over $\mathbb{Z}$ with $\mathbb{Q}$ as below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\mathbb{Q}} & \xrightarrow{A} & \mathbb{Q}^n & \xrightarrow{Q} & \widehat{G}_{\mathbb{Q}} & \longrightarrow & 0 \\
 & & f\uparrow & & g\uparrow & & h\uparrow & & \\
0 & \longrightarrow & M'_{\mathbb{Q}} & \xrightarrow{A'} & \mathbb{Q}^n & \xrightarrow{Q'} & \mathrm{Im}_{\mathbb{Q}}(Q') & \longrightarrow & 0
\end{array}$$

Then by Lemma A.4, since f and h are inclusions and g is an isomorphism over $\mathbb{Q}$, f and h are also isomorphisms over $\mathbb{Q}$. Therefore by Lemma A.5, as maps of $\mathbb{Z}$-modules, f, g and h have finite index.

Since Q' is a matrix map from $\mathbb{Z}^n$, it is also associated to a collection of GIT quotients and a secondary fan. We will define β' and ν' as the "β" and "ν" of Q'.

In fact the secondary fan for Q' has the same structure as the secondary fan for Q, just in a sparser lattice.

By definition $h(\beta_i'^{\mathbb{Z}}) = h(Q'(e_i)) = Q(g(e_i))$.

$Q(g(e_i)) = Q(e_i)$ for $\beta_i \in \beta^0$ and $Q(g(e_i)) = Q(d_i e_i) = d_i Q(e_i)$ for $\beta_i \in \beta_\Gamma$.

Regardless $h(\beta_i'^{\mathbb{Z}})$ and $\beta_i^{\mathbb{Z}}$ are in the same line. Therefore it makes sense to define $\beta'_\Gamma = \{\beta'_i \mid \beta_i \in \beta_\Gamma\}$ and ${\beta_0}' = \{\beta'_i \mid \beta_i \in \beta^0\}$. Furthermore, since the secondary fan is defined over $\mathbb{Q}$ or $\mathbb{R}$, the sets β and β^0 yield the same secondary fan. Since the secondary fans are the same, it makes sense to talk about the same chambers. Let $\Gamma_{\Sigma', I'_\emptyset}$ be the chamber $\Gamma_{\Sigma, I_\emptyset}$ in the secondary fan of Q'. Since both secondary fans have the same form, $\beta'_i \in \Gamma_{\Sigma', I'_\emptyset}$ exactly when $\beta_i \in \Gamma_{\Sigma, I_\emptyset}$. Therefore $\beta'_\Gamma = (\beta' \cap -\Gamma_{\Sigma', I'_\emptyset})$.

We want to show that $\Gamma_{\Sigma', I'_\emptyset}$ corresponds to a stacky vector bundle of rank r, that N includes into N', and that Σ is just Σ' considered in the sublattice N.

We claim that β'_Γ satisfies the claims of Theorem 5.11, making $X_{\Sigma'}$ a vector bundle of rank r.

We need to show that for any $\beta'_i \in \beta'_\Gamma$ we have $-\beta_i'^{\mathbb{Z}} = \sum_{\beta'_j \in \beta^{0\prime}} a_{ji} \beta_j'^{\mathbb{Z}}$ for $a_{ij} \in \mathbb{Z}$.

As above, if $\beta'_i \in \beta'_\Gamma$ we have $h(\beta_i'^{\mathbb{Z}}) = d_i \beta_i^{\mathbb{Z}}$, and if $\beta'_i \in \beta^{0\prime}$ we have $h(\beta_i'^{\mathbb{Z}}) = \beta_i^{\mathbb{Z}}$

Therefore since $d_i\beta_i^{\mathbb{Z}} + \sum_{i=1}^r a_{ij}\beta_j^{\mathbb{Z}} = 0$, we also have $h(\beta_i'^{\mathbb{Z}}) + \sum_{i=1}^r a_{ij}h(\beta_j'^{\mathbb{Z}}) = 0$ which implies $\beta_i'^{\mathbb{Z}} + \sum_{i=1}^r a_{ij}\beta_j'^{\mathbb{Z}} = 0$ by injectivity of h.

Therefore X'_Σ is indeed a stacky vector bundle of rank r.

Next we will show that N is a sublattice of N'.

Let $\widehat{f} : N \to N'$ be the dual map to f. The dual to an injective lattice map of finite index is always an injective map of finite index. See Lemma A.6 for a proof of this. Therefore $\widehat{f}$ realises N as a sublattice of finite index of N'.

All that remains to be shown is that Σ and Σ' are the same fan in different lattices.

By Lemma 5.10, $I_\emptyset = \left\{x_i \mid \beta_i \in \beta_J \text{ for every } \operatorname{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset}\right\}$. Therefore since $\Gamma_{\Sigma,I_\emptyset}$ and $\Gamma_{\Sigma',I'_\emptyset}$ are constructed from the same β-bases over $\mathbb{Q}$, $I_\emptyset$ is the same for both.

By definition $\nu_i = A^\vee(e_i)$.
Then $f^\vee A^\vee(e_i) = (Af)^\vee e_i = (gA')^\vee e_i = (A')^\vee g^\vee e_i = (A')^\vee d_i e_i = d_i(A')^\vee e_i = d_i\nu_i'$
for $\beta_i \in \beta_\Gamma$
and $f^\vee A^\vee(e_i) = (Af)^\vee e_i = (gA')^\vee e_i = (A')^\vee g^\vee e_i = (A')^\vee e_i = \nu_i'$
for $\beta_i \in \beta^0$.
Note that we used the fact that in dual bases $g^\vee = g^T$.

Therefore $\widehat{f}$ takes the ray generated by ν_i to the ray generated by ν_i'. Therefore when N is viewed as a sublattice of N', the lines generated by ν and ν' are the same.

Since $I_\emptyset$ is the same for both fans, Σ and Σ' have the same rays in different lattices.

Therefore to show that $\widehat{f}$ takes Σ to Σ', it is sufficient to show that Σ and Σ' have the same maximal cones.

By Proposition 3.7, a maximal cone σ of X_Σ gives a β-basis β_{J_σ} such that $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$. Therefore J_σ uniquely determines the rays of the maximal cone σ.

By Proposition 3.7, $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_{J_\sigma})$ if and only if $\sigma \in \Sigma_{\max}$. Therefore the choice of chamber $\Gamma_{\Sigma,I_\emptyset}$ in the secondary fan tells us exactly which maximal cones Σ has. Since $\Gamma_{\Sigma,I_\emptyset}$ and $\Gamma_{\Sigma',I'_\emptyset}$ correspond to the same chamber in the secondary fan, they must have the same maximal cones $\sigma \in \Sigma_{\max} = \Sigma'_{\max}$ which proves the claim.

□

Example 5.16. We calculated the fan of $\mathcal{O}(-3)$ in Example 4.10. This variety can also be realized as a GIT quotient with matrices $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}$.

This gives the secondary fan in Figure 5.

Figure 5: The Secondary Fan for $Q = [1, 1, 1, -3]$

Consider the chamber $\Gamma_{\Sigma,I_\emptyset} = \mathrm{Cone}(\beta_1) = \mathrm{Cone}(\beta_2) = \mathrm{Cone}(\beta_3)$. $-\Gamma_{\Sigma,I_\emptyset} \cap \beta = \beta_4$. By Theorem 5.11, Σ is stacky vector bundle of rank 1, since $\beta_4^{\mathbb{Z}}$ can be written as an integral linear combination of the remaining β via $\beta_4^{\mathbb{Z}} = -3\beta_1^{\mathbb{Z}} = -3\beta_2^{\mathbb{Z}} = -3\beta_2^{\mathbb{Z}}$. By Corollary 5.12, the base of this bundle is projective as $\mathrm{Cone}(\beta^0) = \mathrm{Cone}(\beta_1, \beta_2, \beta_3)$ is strongly convex.

Since by Lemma 5.10, $I_\emptyset$ can be calculated by $I_\emptyset = \left\{ x_i \mid \beta_i \in \beta_J \text{ for every } \mathrm{Cone}(\beta_J) \supseteq \Gamma_{\Sigma,I_\emptyset} \right\}$, we know $I_\emptyset = \emptyset$ for this chamber. Therefore Σ is a simplicial fan with rays $\rho_1, \rho_2, \rho_3, \rho_4$. The only such fan is the fan for $\mathcal{O}(-3)$, which is indeed a rank one vector bundle over $\mathbb{P}^2$.

We can also calculate the GIT quotient for the other chamber of the secondary fan. Since $\beta_J = \{\beta_4\}$ is the only β-basis with $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\beta_J)$ for this chamber, we know that $I_\emptyset = \beta_4$. Therefore Σ is the only simplicial fan with rays $\rho_1 = [1, 0, 1]$, $\rho_2 = [0, 1, 1]$, $\rho_3 = [-1, -1, 1]$, which happens to be the simplicial cone with these rays. One can show that this is the variety $\mathbb{C}^3/\mathbb{Z}_3$. Since $\beta_1^{\mathbb{Z}} = \beta_2^{\mathbb{Z}} = \beta_3^{\mathbb{Z}} = -\frac{1}{3}\beta_4^{\mathbb{Z}}$, this is an almost stacky vector bundle over a point. $\square$

5.2 Comparisons

Toric GIT is a powerful tool because we can construct our quotients from just two matrices. In the literature, there are a variety of conditions on the Q matrix that give useful results about the constructed varieties. In this section we will compare two such results to our results obtained in the previous section.

5.2.1 Quasisymmetry

Recall our setting of starting with an inclusion $G \hookrightarrow (\mathbb{C}^*)^n$ and a dual exact sequence

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} \widehat{G} \longrightarrow 0$$

$(\mathbb{C}^*)^n$ can be seen as the diagonal subgroup of $\mathrm{GL}_n(\mathbb{C})$. Therefore this inclusion can be seen as a group representation $\phi : G \to \mathrm{GL}_n(\mathbb{C})$.

Definition 5.17. *We can quotient β into subsets $C = \beta \cap L$ where L is any line in $\widehat{G}$. Then as in* [ŠV15, Page 8]*, we call the representation **quasisymmetric** if $\sum_{\beta_i \in C} \beta_i = 0$ for each C. We will also call β itself and the action $G \times \mathbb{C}^n \to \mathbb{C}^n$ quasisymmetric if this condition is satisfied.*

The quasisymmetric condition gives some useful results beyond the scope of this thesis that can be found in [HS16]. When we assume the quasisymmetric condition in addition to assuming we have a hybrid model, the base of the stacky vector bundle takes a particular form.

Proposition 5.18. *Let X_Σ be a projective, $\mathbb{Q}$-factorial toric variety such that* $\mathrm{Nef}(X) = \mathrm{Eff}(X)$*. Then there is a finite toric morphism $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \to X$ for some $\sum_{i=1}^{s} n_i =$* $\dim N^1(X)$. [FS09, Proposition 5.3]

Proposition 5.19. *Let β be quasisymmetric, and ν be primitive geometric. Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan such that $I_\emptyset = \emptyset$ and such that X_Σ is a vector bundle over a projective base X_{Σ_0}. Then there is a finite toric morphism $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \to X_{\Sigma_0}$ for some $\sum_{i=1}^{s} n_i = dim N^1(X_{\Sigma_0})$.*

Proof. By Proposition 3.17 $\Gamma_{\Sigma,I_\emptyset}$ is isomorphic to the nef cone. As in the proof of Theorem 5.11, there is a chamber $\Gamma_{\Sigma_0,I_{\emptyset 0}}$ corresponding to this chamber in the secondary fan of Σ_0. Furthermore since $\Gamma_{\Sigma_0,I_{\emptyset 0}}$ has the same $I_\emptyset = \emptyset$, it is isomorphic to the nef cone of X_{Σ_0}.

Similarly since any chamber in the secondary fan of $\Gamma_{\Sigma_0,I_{\emptyset 0}}$ is an intersection of β-bases in $\beta^0 \subseteq \beta$, any chamber in the secondary fan of Σ_0 is also a chamber in the secondary fan of Σ.

We claim that $\mathrm{Eff}(X_{\Sigma_0}) = \mathrm{Nef}(X_{\Sigma_0})$. By definition $\mathrm{Nef}(X_{\Sigma_0}) \subseteq \mathrm{Eff}(X_{\Sigma_0})$ so we only need to show $\mathrm{Eff}(X_{\Sigma_0}) \subseteq \mathrm{Nef}(X_{\Sigma_0})$. Since $\mathrm{Eff}(X_{\Sigma_0}) = \mathrm{Cone}(D_\rho \mid \rho \in \Sigma_0(1))$, we need only show that $D_\rho \in \mathrm{Nef}(X_{\Sigma_0})$ for each D_ρ.

By Proposition 3.17, D_ρ corresponds to some $\beta_i \in \beta^0$ in the isomorphism. Since β is quasisymmetric, we also have $-\beta_i \in \beta$. By Corollary 5.12, we know that $\mathrm{Cone}(\beta^0)$ is strongly convex, therefore $-\beta_i \notin \beta^0$. Therefore by construction in Theorem 5.11, we know that $-\beta_i \in -\Gamma_{\Sigma,I_\emptyset}$. Therefore $\beta_i \in \Gamma_{\Sigma,I_\emptyset}$, and by the isomorphism we have $D_\rho \in \mathrm{Nef}(X_{\Sigma_0})$.

Then the conclusion follows from Proposition 5.18.

□

5.2.2 The Herbst Criterion

Definition 5.20. *The matrix Q is said to satisfy the* ***Herbst Criterion*** *if we can choose* $\operatorname{rank} Q = r$ *linearly independent columns of Q, $\beta_1, ..., \beta_r$, so that the remaining columns lie in* $-\mathrm{Cone}(\beta_1, ..., \beta_r)$. [CG15, Page 2]

[CG15] shows that Q satisfies the Herbst criterion if and only if there is an affine stable point in the image of Q. This is equivalent to the condition that C_ν is simplicial. We will use ideas of their proof [CG15, Lemma 3.7] to show this equivalence.

Proposition 5.21. *Q satisfies the Herbst criterion if and only if C_ν is simplicial.*

Proof. First assume Q satisfies the Herbst criterion. Then change the basis over $\mathbb{Q}$ so that the chosen r linearly independent columns of Q are the first r columns, and that they are $e_1, ..., e_r$. Since by assumption we have $\beta_i \in -\mathrm{Cone}(\beta_1, ..., \beta_r)$ for the remaining columns of Q, we can write Q in the form $Q = \begin{bmatrix} I & -N \end{bmatrix}$ for some matrix N with non-negative entries.

Since A has full rank by assumption, there exists a change of basis over $\mathbb{Q}$ such that $A = \begin{bmatrix} A_1 \\ I \end{bmatrix}$.

Then since $QA = 0$ we get $\begin{bmatrix} I & -N \end{bmatrix} \begin{bmatrix} A_1 \\ I \end{bmatrix} = 0 \implies A_1 - N = 0 \implies A_1 = N$.

Therefore A is of the form $\begin{bmatrix} N \\ I \end{bmatrix}$ for some matrix N with non-negative entries.

Then since each of the rows of N lies inside the cone generated by the rows of I, C_ν is generated by the rows of I and is simplicial as required.

Conversely assume that C_ν is simplicial. Then since the ν_i generated the cone, we can organise it so that $\nu_1, ..., \nu_l$ are the ray generators and $\nu_{l+1}, ..., \nu_n$ lie in the cone generated by $\nu_1, ..., \nu_l$. Then we can change basis over $\mathbb{Q}$ so $\nu_1, ..., \nu_l = e_1, ..., e_l$. Therefore A is of the form $\begin{bmatrix} N \\ I \end{bmatrix}$ for a matrix N with non-negative entries. Since Q has full rank we can do a change of basis over $\mathbb{Q}$ so that Q is of the form $\begin{bmatrix} I & Q_2 \end{bmatrix}$. Then $QA = 0 \implies \begin{bmatrix} I & Q_2 \end{bmatrix} \begin{bmatrix} N \\ I \end{bmatrix} \implies Q_2 = -N$. Therefore $Q_2 = \begin{bmatrix} I & -N \end{bmatrix}$ which satisfies the Herbst criterion. $\square$

The situation where the Herbst criterion is satisfied is a special case of the situation in which Q has a chamber that is an almost stacky vector bundle.

Proposition 5.22. *If Q satisfies the Herbst criterion then there exists a chamber of the secondary fan $\Gamma_{\Sigma, I_\emptyset}$ such that X_Σ is an almost stacky vector bundle.*

Proof. As in the proof of Proposition 5.21, there is a change of basis over $\mathbb{Q}$ such that we can write $Q = \begin{bmatrix} I & -N \end{bmatrix}$ and $A = \begin{bmatrix} N \\ I \end{bmatrix}$.

We claim that $\mathrm{Cone}(e_1, ..., e_r)$ forms a chamber of the secondary fan. By Corollary 3.8, the chambers of the secondary fan are maximal intersections of β-bases. Therefore to show that the $\mathrm{Cone}(e_1, ..., e_r)$ is a chamber of the secondary fan, we need only show that no other β-basis has a full dimensional intersection with it.

Let β_J be another β-basis. Without loss of generality we can assume that the β_i in it are $e_1, ..., e_s$ and some additional β_i that are columns of $-N$. Therefore $\mathrm{Cone}(\beta_J) \subseteq$

$\text{Cone}(e_1, ..., e_s, -e_1, ..., -e_r)$. Therefore it is sufficient to show that this cone has no maximal dimension intersection with $\text{Cone}(e_1, ..., e_r)$. Indeed
$\text{Cone}(e_1, ..., e_s, -e_1, ..., -e_r) \cap \text{Cone}(e_1, ..., e_r) = \text{Cone}(e_1, ..., e_s)$
which is not maximal dimensional. Then since the remaining columns of Q lie in the negative cone, the GIT quotient for this chamber is an almost stacky vector bundle by Theorem 5.15.

□

6 Conclusion

The goal of this thesis has been to investigate when a chamber of the secondary fan is a hybrid model. In Theorem 5.11 and Corollary 5.12, we determine that $\Gamma_{\Sigma,I_\emptyset}$ is a hybrid model or rank r exactly when it satisfies the following conditions.

1. There exists $\beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ and $a_{ji} \in \mathbb{Z}$ such that $|\beta_\Gamma| = r$ and for each $\beta_i^{\mathbb{Z}} \in \beta_\Gamma$ we have $-\beta_i = \sum_{\beta_j \notin \beta_\Gamma} a_{ji}\beta_j^{\mathbb{Z}}$.

2. $\beta_i \neq 0$ for each $\beta_i \in \beta^0 := \beta \setminus \beta_\Gamma$

3. $\text{Cone}(\beta^0)$ is strongly convex.

In Theorem 5.15 we also show that a reduced version of these conditions is sufficient for $\Gamma_{\Sigma,I_\emptyset}$ to be an almost stacky vector bundle. That is if there exists $\emptyset \neq \beta_\Gamma \subseteq (\beta \cap -\Gamma_{\Sigma,I_\emptyset})$ then X_Σ is an almost stacky vector bundle.

To see why these results are relevant, we recall that the hybrid model condition in [FJR15] gives an example of a Gauged Linear Sigma Model. Our condition is slightly weaker but it is nevertheless the first step to finding a condition describing this class of GLSMs. More than this, however, our results are useful in the same sense that Toric Geometry itself is useful. Toric Geometry is powerful because all the varieties can be described completely by fans, with many geometric notions expressed completely using just fan data. In toric GIT we are not just studying a single variety, but a class of varieties, so instead of a fan, our data becomes the charge matrix. The relevance of our results lies in the fact that even though being a hybrid model is a geometric concept, the existence or non-existence of a hybrid model can be entirely described by a condition on the charge matrix.

Math research is of course an ongoing process, and for every answer one finds, one should discover several more questions. Hence in the spirit of continued scientific endeavour, we conclude this thesis with the following four.

1. Can we use the charge matrix to describe when a chamber is a vector bundle, as opposed to a stacky vector bundle?
2. Is the converse to Theorem 5.15 true?
3. For which R-charge actions does an almost stacky vector bundle yield a hybrid model in the sense of [FJR15]?
4. When is a function defined on the semistable locus of a GIT quotient smooth?

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Appendix A Basic Commutative Algebra Proofs

In this section we will prove some basic facts about commutative algebra that are used in the body of the thesis.

Lemma A.1. *Given any commutative diagram with exact rows as below, there exists a unique map $f : A' \to A$ that makes the diagram commute.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}$$

Proof. Consider the map $g \circ \alpha' : A' \to B$. By the commutative diagram we have $\beta \circ g \circ \alpha' = h \circ \beta' \circ \alpha' = h \circ 0 = 0$. Then by the universal property of kernels, this induces a map $f : A' \to A$ such that the diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[u, "f"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}$$

□

Lemma A.2. *Let the following be a commutative diagram with exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[u, "f"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r, "\gamma'"] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}$$

If g and h have trivial kernel, then so does does f.

Proof. We apply the snake lemma to get the following diagram with an exact snake.

$$
\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[u] & B \arrow[r, "\beta"] \arrow[u] & C \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A' \arrow[r, "\alpha'"] \arrow[u, "f"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \ker f \arrow[r] \arrow[u] & \ker g \arrow[r] \arrow[u] & \ker h \arrow[u] \arrow[lluuu, red, rounded corners]
\end{tikzcd}
$$

By assumption we can replace $\ker g$ and $\ker h$ with zero as below.

$$
\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[u] & B \arrow[r, "\beta"] \arrow[u] & C \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & A' \arrow[r, "\alpha'"] \arrow[u, "f"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \ker f \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \arrow[lluuu, red, rounded corners]
\end{tikzcd}
$$

Then by exactness $\ker f = 0$ as well.

$\square$

Lemma A.3. *Consider the following exact sequence with exact rows. If h is an isomorphism, then $\operatorname{coker} f$ is isomorphic to $\operatorname{coker} g$.*

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[u, "f"] & B' \arrow[r, "\beta'"] \arrow[u, "g"] & C' \arrow[r] \arrow[u, "h"] & 0 \arrow[u]
\end{tikzcd}
$$

Proof. The snake lemma gives the following diagram with an exact snake.

$$\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u] \arrow[r, "\alpha"] & B \arrow[u] \arrow[r, "\beta"] & C \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & A' \arrow[u, "f"] \arrow[r, "\alpha'"] & B' \arrow[u, "g"] \arrow[r, "\beta'"] & C' \arrow[u, "h"] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & \ker f \arrow[u] \arrow[r] & \ker g \arrow[u] \arrow[r] & \ker h \arrow[u] \arrow[uuull, red, bend right]
\end{tikzcd}$$

Since h is an isomorphism this gives $\operatorname{coker} h = \ker h = 0$.

$$\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u] \arrow[r, "\alpha"] & B \arrow[u] \arrow[r, "\beta"] & C \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & A' \arrow[u, "f"] \arrow[r, "\alpha'"] & B' \arrow[u, "g"] \arrow[r, "\beta'"] & C' \arrow[u, "h"] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & \ker f \arrow[u] \arrow[r] & \ker g \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[uuull, red, bend right]
\end{tikzcd}$$

Therefore exactness of the snake yields $\operatorname{coker} h \cong \operatorname{coker} g$. $\square$

Lemma A.4. *Consider the following diagram with exact rows and g an isomorphism. If f and h both have trivial kernels, they are also both isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[u, "f"] \arrow[r, "\alpha'"] & B' \arrow[u, "g"] \arrow[r, "\beta'"] & C' \arrow[u, "h"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Proof. Apply the snake lemma to the diagram as follows.

$$\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & 0 \arrow[r] & \operatorname{coker} h \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[u] \arrow[r, "\alpha"] & B \arrow[u] \arrow[r, "\beta"] & C \arrow[u] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & A' \arrow[u, "f"] \arrow[r, "\alpha'"] & B' \arrow[u, "g"] \arrow[r, "\beta'"] & C' \arrow[u, "h"] \arrow[r] & 0 \arrow[u] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[uuull, red, rounded corners] &
\end{tikzcd}$$

Exactness at $\operatorname{coker} f$ and $\operatorname{coker} h$ yields the claim.

$\square$

Lemma A.5. *Let $f : A \to B$ be an inclusion of $\mathbb{Z}$-modules that is an isomorphism over $\mathbb{Q}$. Then A has finite index in B.*

Proof. Consider the exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B/A \longrightarrow 0$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$ module, tensoring by $\otimes_{\mathbb{Z}}\mathbb{Q}$ gives the following exact sequence.

$$0 \longrightarrow A_{\mathbb{Q}} \xrightarrow{f} B_{\mathbb{Q}} \longrightarrow (B/A)_{\mathbb{Q}} \longrightarrow 0$$

However over $\mathbb{Q}$, f is assumed to be an isomorphism, so we also have

$$0 \longrightarrow A_{\mathbb{Q}} \xrightarrow{f} B_{\mathbb{Q}} \longrightarrow 0 \longrightarrow 0$$

$\square$

Therefore $(B/A)_{\mathbb{Q}} \cong 0$. This is only possible if (B/A) is a finite group. Therefore A has finite index in B.

Lemma A.6. *Let $f : A \to B$ be a map of $\mathbb{Z}$ modules that is an inclusion of finite index. Then the dual map is also an inclusion of finite index.*

Proof. Consider the exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow B/A \longrightarrow 0$$

Since $\operatorname{Hom}(-, \mathbb{C})$ is right to left exact, dualizing gives the following exact sequence.

$$0 \longrightarrow \mathrm{Hom}(B/A, \mathbb{C}) \longrightarrow \mathrm{Hom}(B, \mathbb{C}) \xrightarrow{f^\vee} \mathrm{Hom}(A, \mathbb{C})$$

Since B/A is a finite group, $\mathrm{Hom}(B/A, \mathbb{C}) = 0$ as below.

$$0 \longrightarrow 0 \longrightarrow \mathrm{Hom}(B, \mathbb{C}) \xrightarrow{f^\vee} \mathrm{Hom}(A, \mathbb{C})$$

This shows that $f^\vee$ is injective.

Returning to the original exact sequence. We can tensor by $\mathbb{Q}$, a flat module, to get the following exact sequence.

$$0 \longrightarrow A_{\mathbb{Q}} \xrightarrow{f} B_{\mathbb{Q}} \longrightarrow 0 \longrightarrow 0$$

Therefore f is an isomorphism over $\mathbb{Q}$. The modules A and B are isomorphic over $\mathbb{Q}$, which implies $\mathrm{Hom}(A, \mathbb{C})$ and $\mathrm{Hom}(B, \mathbb{C})$ are isomorphic over $\mathbb{Q}$. Therefore they must have the same rank.

Then since rank is additive over short exact sequences, the cokernel of $f^\vee$ must have rank 0 making it a finite group as required.

$\square$