University of Alberta

Decaying Oscillatory Solutions of Kadomtsev-Petviashvili Equation II

by

Ion Bica ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

in

Mathematics

Department of Mathematical and Statistical Sciences

Edmonton, Alberta

Fall 2002

0-612-81164-6

University of Alberta

Library Release Form

Name of Author: Ion Bica

Title of Thesis: Decaying Oscillatory Solutions of Kadomtsev-Petviashvili Equation II

Degree: Doctor of Philosophy

Year this Degree Granted: 2002

Ion Bica
#1502, 10020-103rd Ave.
Edmonton, Alberta
T5J 0G8, Canada.

August 27, 2002.

UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Decaying Oscillatory Solutions of Kadomtsev-Petviashvili Equation II** submitted by **Ion Bica** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Martin Legaré

Dr. Martin Legaré (Chair)

Mikhail Kovalyov

Dr. Mikhail Kovalyov (Supervisor)

H. Van Roessel

Dr. Henry Van Roessel

Dr. Jacek A. Tuszynski (Physics)

Dr. Yau Shu Wong

Martin Legaré for

Dr. Oleg Bogoyavlenskij (Queen's University)

August 12, 2002

Abstract

Starting from the N-soliton wall solution of KPII equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

we derive a new class of solutions of KPII (oscillatory solutions called *harmonic breathers*). The formula derived (called *N-harmonic breather solution*) will define the motion and interaction of N harmonic breathers. We show that in certain conditions, the N-harmonic breather solution can be approximated to a Fourier-like expansion and miscellaneous profile-solutions are obtained.

Acknowledgments

The work in this thesis was done in collaboration with my supervisor Mikhail Kovalyov to whom I wish to thank very much for all the help he provided for me during the time of my research on the topic of this thesis.

Dedicated to my mother, my sister and her family, and in the memory of my father and my brother.

Contents

Chapter 1

Introduction

In 1895 Korteweg and de Vries derived an equation (called KdV equation) equivalent to

$$u_t + 6uu_x + u_{xxx} = 0, \tag{I.1}$$

to describe one-dimensional, small-amplitude, long surface gravity waves propagating in shallow water of uniform depth. KdV equation has applications in several other physical settings, as for example: internal solitons in the ocean, nonlinear acoustics of bubbly liquids, and more.

The restriction in the application of the KdV equation as a practical model is that the KdV equation is strictly one-dimensional (one spatial dimension plus time). In 1970 Kadomtsev and Petviashvili derived a two-dimensional generalization of the KdV equation (called KPI and KPII equations) equivalent to

$$(u_t + 6uu_x + u_{xxx})_x + 3\gamma^2 u_{yy} = 0, \qquad \gamma^2 = \pm 1, \tag{I.2}$$

by relaxing the restriction that the waves be strictly one-dimensional. For $\gamma^2 = -1$, the equation (I.2) is known as the KPI equation and for $\gamma^2 = 1$, the equation (I.2) is known as the KPII equation.

In the derivation of the KPI and KPII equations one may start by considering waves of the form

$$u = e^{i(k_x x + k_y y - \omega t)}, \qquad k_y \ll k_x,\ k_x \text{ small} \tag{I.3}$$

that satisfy the corresponding dispersion relationships

$$\omega = \sqrt{k_x^2 + k_y^2 - \frac{2}{3\gamma^2}(k_x^2 + k_y^2)^2} \approx k_x - \frac{1}{3\gamma^2}k_x^3 + \frac{k_y^2}{2k_x} + \cdots, \quad \gamma^2 = \pm 1. \tag{I.4}$$

The latter suggests that $u(x, y, t)$ satisfies the linear equations

$$(3\gamma^2 u_t + 3\gamma^2 u_x + u_{xxx})_x + \frac{3\gamma^2}{2}u_{yy} = 0, \tag{I.5}$$

which under the change of variables

$$t_{old} = 3\gamma^2 t_{new}, \ x_{old} = x_{new} + 3\gamma^2 t_{new}, \ y_{old} = \frac{1}{\sqrt{2}}y_{new},$$

become

$$(u_t + u_{xxx})_x + 3\gamma^2 u_{yy} = 0 \tag{I.6}$$

For $\gamma^2 = -1$, the equation (I.6) is known as the linear KPI equation (lKPI) and for $\gamma^2 = 1$, the equation (I.6) is known as the linear KPII equation. In order to cause a variation in the amplitude in both space and time to the sinusoidal oscillations of $u(x, y, t)$, a nonlinear term $(6uu_x)_x$ is added to the left hand side of (I.6) which turns the equation into the KPI equation (for $\gamma^2 = -1$) and the KPII equation (for $\gamma^2 = 1$). The functions $u(x, y, t)$ given by (I.3) and satisfying the corresponding dispersion relationships (I.4) are solutions of (lKPI) and (lKPII) respectively. By adding nonlinearity into (lKPI)/(lKPII), the functions $u(x, y, t)$ are no longer solutions of the nonlinear KPI/KPII equations respectively obtained. Since KPI (KPII, respectively) can be viewed as a perturbation of (lKPI) ((lKPII), respectively) by the nonlinear term $(6uu_x)_x$, it is only natural to study how such a perturbation affects the solutions (I.3). As well, further on in the thesis it will be shown that the addition of the term $(6uu_x)_x$ to (lKPI) ((lKPII), respectively) transforms the solutions (I.3) into solutions of KPI (KPII, respectively) which possess all important physical properties of the linear waves (I.3). In this thesis only the case of KPII equation is considered.

In the first chapter we present some important results obtained by different authors for both KPI and KPII equations. We show how the KPI/KPII equations came in discussion as a model for surface waves and as well we show the derivation of a special type of solution of the KPI/KPII equations, called the N-soliton wall solution. The material presented in this chapter was gathered together from different references mentioned within the chapter.

The main results of this thesis are presented in Chapter 2. We construct oscillatory solutions of the KPII equation using the formula for the N-soliton wall solution. The simplest oscillatory solutions of the KPII equation that are nonlinear analogue of the linear waves (I.3) are of the form:

$$u(x,y,t) =$$

$$2\frac{\partial^2}{\partial x^2}\ln\left[-\varrho_1 - \mathcal{V}(x,y) - 12t(\lambda_1^2-\mu_1^2) + \frac{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8t\lambda_1(\lambda_1^2 - 3\mu_1^2))}{2\lambda_1}\right] =$$
$$-8\lambda_1^2\frac{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t)}{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t) - 2\lambda_1(\varrho_1 + \mathcal{V}(x,y) + 12(\lambda_1^2-\mu_1^2)t)} -$$
$$8\lambda_1^2\left[\frac{1+\sin(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t)}{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t) - 2\lambda_1(\varrho_1 + \mathcal{V}(x,y) + 12(\lambda_1^2-\mu_1^2)t)}\right]^2, \quad \text{(I.7)}$$

where λ_1, μ_1, ϱ_1 and γ_1 are real parameters and $\mathcal{V}(x,y) = x - 2\mu_1 y$.

One can notice that unlike the solutions (I.3) of (lKPII) satisfying the corresponding dispersion relation (I.4) (the case $\gamma^2 = 1$ in (I.4)), the solutions (I.7) of KPII are singular. Singular solutions were studied for different other partial differential equations as: KdV equation ([Nov 2], [Kov 1-4], [Matveev 1], [Stahlhofen 1], [Jaworski 1]), Sine-Gordon equation ([Jaworski 1]), nonlinear Schrödinger equation ([Kopell 1]) and other. However, since all applied problems are considered in finite space-time physical domains, one may adjust the parameters of (I.7) to assure that within the physical domain the solutions (I.7) stay regular. Moreover, within the physical domain, the solutions (I.7) have oscillatory behavior similar to that of the functions (I.3).

For $\mu_1 = 0$ the functions defined by (I.7) are solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

These solutions have been previously derived and studied by a number of authors ([Nov 2], [Kov 4], [Matveev 1-2], [Stahlhofen 1]) and they have been correspondingly referred to as *harmonic breathers* or *positons.* The functions defined by (I.7) will be referred to as *harmonic breathers* of the KPII equation.

We derive a formula for nonlinear superposition of solutions of the KPII equation that describes the nonlinear interaction of the KPII harmonic breathers. This formula will be referred to as the N-harmonic breather solution of the KPII equation.

We show that in a finite space-time physical domain we can use the N-harmonic breather solution to construct more complicated exact solutions of the KPII equation. In this thesis it is shown that within a finite space-time physical domain we can use the N-harmonic breather solution to construct wave packet-like solutions and δ function-like solutions. We can do this due to a phenomenon that can be viewed as a nonlinear analog of interference from the linear theory of waves.

1.1 History of the Kadomtsev-Petviashvili Equations

In 1844 John Scott Russell reported to the British Association for the Advancement of Science, [Russ 1], a very interesting discovery which in his own words was:

'...I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a round, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overlooked it still rolling at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon...'.

He called this 'great wave of translation' *solitary wave* and described it as wave consisting of a single elevation which, if properly started, may travel for a considerable distance along a uniform channel with little or no change. Amongst his results on this observed type of wave we find the following:

1. The solitary wave is a long, shallow water wave of permanent form.

2. The speed of propagation, c, of a solitary wave in a channel of uniform depth h is given by

$$c^2 = g(h + a), \tag{1.1}$$

   where a is the amplitude of the wave as measured from the undisturbed free surface and g the gravitational acceleration. The solitary wave is therefore a gravity wave.

3. Higher solitary waves travel faster as a consequence of the formula (1.1).

At the time, Russell's observations came in conflict with Airy's shallow-water theory prediction that a wave of finite amplitude cannot propagate without change of profile.

In 1870 Russell's experimentally work on the solitary wave phenomenon was mathematically explained by Boussinesq and Rayleigh who separately studied the phenomenon using the equations of motion for an inviscid incompressible fluid. The two eminent scientists assumed that a solitary wave has a length scale (λ_0)

much greater than the depth of water (h), i.e

$$\delta^2 := \left(\frac{h}{\lambda_0}\right)^2 = O(\varepsilon), \quad \varepsilon \ll 1 \quad \text{(square of the frequency dispersion parameter).} \tag{1.2}$$

Rayleigh treated the problem as one of steady motion and derived the equation [Lamb 1]:

$$\left(\frac{dy}{dx}\right)^2 = 3\frac{(y-h)^2}{h^2}\left(1 - \frac{gy}{c^2}\right), \tag{R1}$$

which governs long one-dimensional, small amplitude, surface gravity waves in a channel of water of uniform depth h, where c is the uniform velocity in the parts of the fluid at a distance from the wave, whether in front or behind. In the equation (R1) we have $y' = 0$ only when $y = h$ or $y = c^2/g$, and since $(1 - gy/c^2)$ must be positive, we obtain that c^2/g is a maximum value of y. Therefore the wave is necessarily one of elevation only, and, denoting by a the maximum height above the undisturbed level, we have

$$c^2 = g(h + a)$$

which is exactly the empirical formula for the wave velocity adopted by Russell (1.1). If in the equation (R1) we use the formula (1.1) and we denote $\eta = y - h$, then the equation becomes:

$$\eta' = \pm\eta\sqrt{\frac{3a}{h^2(h+a)}}\left(1 - \frac{\eta}{a}\right)^{1/2}. \tag{R1$'$}$$

The integration of the equation (R1)$'$ gives

$$\eta = a\ \mathrm{sech}^2\left(\sqrt{\frac{3a}{4h^2(h+a)}}x\right) \Rightarrow y(x) = h + a\ \mathrm{sech}^2\left(\sqrt{\frac{3a}{4h^2(h+a)}}x\right), \quad \text{(SV)}$$

if the x-axis is placed beneath the summit of the wave.

The solitary wave profile (SV) for the ratio $a/h = .1$ is shown in Figure 1.0.

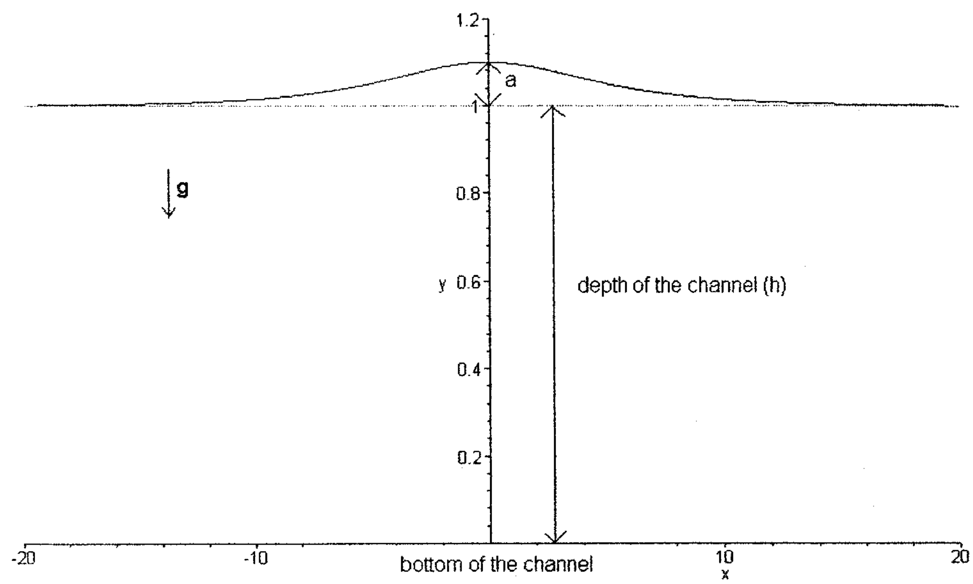


Figure 1.0

In his work, Boussinesq showed that appropriate allowance for the vertical acceleration (which is responsible for dispersion and is neglected in Airy's shallow-water theory) as well as for the finite amplitude, leads to the following solution for the profile $z = \eta(x,t)$ of the free surface elevation

$$\eta(x,t) = a\ \mathrm{sech}^2[\beta(x-ct)], \quad \beta^2 = \frac{1}{\lambda_0^2} = \frac{3a}{4h^2(h+a)} \quad \text{for any } a > 0, \qquad (1.3)$$

where c is given by Russell's formula (1.1). He found as well that the profile (1.3) is correct only if

$$\frac{a}{h} = \varepsilon \ll 1 \quad \text{(ratio of nonlinearity parameter)}. \tag{1.4}$$

From (1.2) and (1.3) we can find the Ursell number

$$U = \frac{3\varepsilon}{4\delta^2} = 1 + \varepsilon, \tag{1.5}$$

which tells us that the solitary waves have the essential quality of balance between nonlinearity and dispersion.

In 1895 Korteweg and de Vries, who apparently did not know of the work of Boussinesq and Rayleigh and who were still trying to answer the objections of Airy [Zeytounian 1], derived a nonlinear evolution partial differential equation (named after them: KdV equation), [KdV 1], governing long one dimensional, small-amplitude, surface gravity waves propagating in shallow water of uniform depth. The KdV equation[1] is

$$\eta_t + \frac{3}{2}c_0\left(\frac{1}{h}\eta\eta_x + \frac{2}{3}\alpha\eta_x + \frac{1}{3}\sigma\eta_{xxx}\right) = 0, \quad \sigma = \frac{1}{3}h^2 - \frac{T}{\rho g}, \quad c_0 = \sqrt{gh}, \tag{1.6}$$

[1]the KdV equation can be obtained from the Boussinesq equation

$$\eta_{tt} = c_0^2\left(\eta_{xx} + \frac{3}{2h}(\eta^2)_{xx} + \frac{1}{3}h^2\eta_{xxxx}\right), \quad c_0 = \sqrt{gh},$$

which describes one-dimensional weakly nonlinear dispersive water waves, for waves propagating in both directions in water of uniform depth h [Zeytounian 1].

where η is the surface elevation of the wave above the equilibrium level h, α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant and σ is a parameter depending on the surface tension T of the liquid (of constant density ρ).

1.1.1 Solitary Waves and KdV Equation

Korteweg and de Vries found a family of periodic solutions for the equation (1.6) which they called *cnoidal waves*, expressible in terms of Jacobian elliptic function 'cn' which is obtained as follows[2]: First we define the integral

$$v = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{J.1}$$

where m is a real parameter such that $m \in [0,\ 1]$. We may compare the integral (J.1) with the elementary integral

$$w = \int_0^{\psi} \frac{dt}{\sqrt{1 - t^2}}, \tag{J.2}$$

where we use $t = \sin\theta$ so that $w = \sin^{-1}\psi$ or $\sin w = \psi$, and so we observe that the integral (J.2) defines the inverse of the trigonometric function 'sin'. This led Jacobi (and also Abel) to define a new pair of inverse functions from (J.1)

$$\text{sn } v = \sin\phi, \qquad \text{cn } v = \cos\phi. \tag{J.3}$$

[2]the brief description of the Jacobian elliptic functions 'cn' and 'sn' is reproduced from [Drazin 1].

These are two of the *Jacobian elliptic functions*; they are usually written $\mathrm{sn}(v|m)$, $\mathrm{cn}(v|m)$ to denote the dependence on the parameter.

The two special cases $m = 0, 1$ enable the integrals (J.1) and functions (J.3) to be reduced to elementary functions: if $m = 0$ then,

$$v = \phi \quad \text{and so} \quad \mathrm{cn}(v|0) = \cos\phi = \cos v, \; \mathrm{sn}(v|0) = \sin\phi = \sin v,$$

and if $m = 1$ the integral (J.1) can be evaluated to yield

$$v = \mathrm{sech}^{-1}(\cos\phi) \quad \text{and so} \quad \mathrm{cn}(v|1) = \mathrm{sech}\ v$$

and

$$v = \tanh^{-1}(\sin\phi) \quad \text{and so} \quad \mathrm{sn}(v|1) = \tanh v$$

Let us show that $\mathrm{cn}(v|1) = \mathrm{sech}\ v$. Consider $\phi \in (0,\ \pi/2)$ in (J.1). For $m = 1$ the integral (J.1) is then

$$v = \ln|\sec\phi + \tan\phi|, \tag{J.4}$$

$$\Downarrow$$

$$\cos\phi\ [(1 + e^{2v})\cos\phi - 2e^{v}] = 0,$$

$$\Downarrow$$

$$\mathrm{cn}(v|1)\ [(1 + e^{2v})\mathrm{cn}(v|1) - 2e^{v}] = 0. \tag{J.5}$$

For nontrivial solutions of (J.5) we obtain the conclusion:

$$\mathrm{cn}(v|1) = \mathrm{sech}\, v. \tag{J.6}$$

The Russell's solitary wave can be recovered from the cnoidal waves in the special case $m = 1$, as it is going to be seen in this section. Following the derivations in [Whitham 1] and [Abl1] we show how the cnoidal waves can be derived from KdV equation (1.6) and that the solitary wave is indeed a solution of the KdV equation (1.6) as well.

Because both the periodic and the solitary waves described by the equation (1.6) are found as solutions of constant shape moving with constant velocity (V), we may describe them in the following form

$$\eta(x,t) = h\zeta(\chi), \quad \chi = x - Vt. \tag{1.7}$$

Under the substitution (1.7), the equation (1.6) becomes

$$\zeta\zeta' - \frac{2}{3}\left(\frac{V}{c_0} - \alpha\right)\zeta' + \frac{\sigma}{3}\zeta''' = 0. \tag{1.8}$$

If in the equation (1.8) we neglect the surface tension (so that the parameter σ from (1.6) becomes $\sigma = h^2/3$), then we obtain the equation:

$$\zeta\zeta' - \frac{2}{3}\left(\frac{V}{c_0} - \alpha\right)\zeta' + \frac{h^2}{9}\zeta''' = 0. \tag{1.9}$$

Integrating the equation (1.9) twice, we obtain

$$\frac{h^2}{3}\zeta'^2 = \mathcal{E}(\zeta), \tag{1.10}$$

where

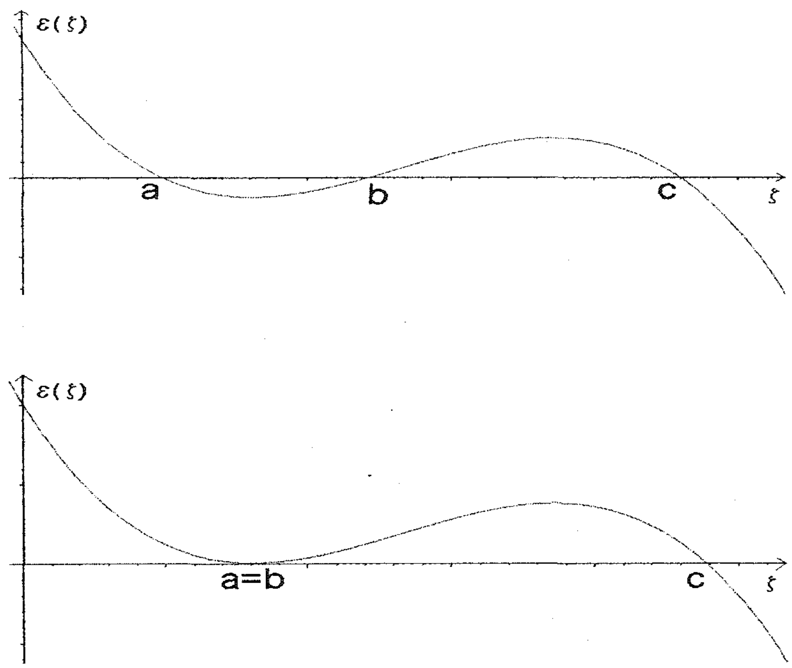
$$\mathcal{E}(\zeta) = -\zeta^3 + 2\left(\frac{V}{c_0} - \alpha\right)\zeta^2 + 6C_1\zeta + C_2, \tag{1.11}$$

with C_1, C_2 being constants of integration.

For practical applications we are interested only in real bounded solutions $\zeta(\chi)$ of (1.10). Thus we require $(\zeta')^2 \geq 0$, and the form of $\mathcal{E}(\zeta)$ shows that ζ will vary monotonically until ζ' vanishes (i.e. $\mathcal{E}(\zeta)$ has at least one real zero). In other words, we can anticipate that the zeros of $\mathcal{E}(\zeta)$ are important. From the form of $\mathcal{E}(\zeta)$ we see that $\mathcal{E}(\zeta)$ can have either one simple real zero or three real zeros. In the case that $\mathcal{E}(\zeta)$ has three real zeros a, b and c (we may assume that $a \leq b \leq c$), we can have one of the following subcases: $a \leq b \leq c$, $a = b \leq c$, $a \leq b = c$ or $a = b = c$. All the situations are depicted in detail in [Abl 1], and here we shall reproduce only the situations of interest for us, i.e. the situations in which we have real bounded solutions of (1.10). These situations are when $\mathcal{E}(\zeta)$ has three real zeros a, b and c $(a \leq b \leq c)$ and we have either the distinct roots case or the $a = b < c$ case. We can write

$$\mathcal{E}(\zeta) = -(\zeta - a)(\zeta - b)(\zeta - c) \tag{1.12}$$

A rough sketch of the behaviors of the function $\mathcal{E}(\zeta)$ versus ζ in the two special situations mentioned above is shown in the figure below:



Rough sketches of $\mathcal{E}(\zeta)$ against ζ.

When all the roots are distinct ($a < b < c$), the real bounded solution is nonlinear and oscillatory between b and c. The case when $a = b < c$ corresponds to the solitary wave solution.

Distinct roots case If $a,\ b,\ c$ are distinct ($a < b < c$), then the solution of (1.10) between b and c is given implicitly by

$$\chi = c \mp \frac{h}{\sqrt{3}} \int_{\zeta}^{c} \frac{dg}{\sqrt{(a-g)(g-b)(g-c)}}. \tag{1.13}$$

The solution (1.13) is transformed into a standard elliptic integral by using the substitution

$$g = c + (b - c)\sin^2\theta. \tag{1.14}$$

We obtain

$$\mp(\chi - c)\frac{\sqrt{3(c-a)}}{2h} = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{1.15}$$

where $m = (c-b)/(c-a) \in (0,\ 1)$ and

$$\zeta = c + (b-c)\sin^2\theta = b + (c-b)\cos^2\theta. \tag{1.16}$$

Thus we have

$$\mathrm{cn}\left[(\chi - c)\frac{\sqrt{3(c-a)}}{2h}|m\right] = \cos\phi, \tag{1.17}$$

where the $\mp$ is suppressed since cn is an even function, and so

$$\zeta(\chi) = b + (c-b)\ \mathrm{cn}^2\left[(\chi - c)\frac{\sqrt{3(c-a)}}{2h}|m\right], \tag{1.18}$$

$$\Downarrow$$

$$\eta(x,t) = h\left\{b + (c-b)\ \mathrm{cn}^2\left[\frac{\sqrt{3(c-a)}}{2h}(x - Vt - c)|m\right]\right\} \tag{1.19}$$

$$\frac{V}{c_0} = \alpha + \frac{a+b+c}{2}, \tag{1.20}$$

the cnoidal-wave solution as Korteweg and de Vries called it.

In the limit case $m \to 1$ (i.e. $b \to a$) we have cn $\to$ sech (as it was shown at the beginning of this section), i.e. the solitary wave is recovered.

The case $a = b < c$ If in (1.19) we take the limit $b \to a$ (i.e. $m \to 1$), then $\eta(x,t)$ reduces to the solitary wave solution of KdV equation (1.6):

$$\eta(x,t) = h\left\{a + (c-a)\,\mathrm{sech}^2\left[\frac{\sqrt{3(c-a)}}{2h}(x - Vt - c)\right]\right\}, \tag{1.21}$$

$$\frac{V}{c_0} = \alpha + \frac{2a+c}{2}.$$

In 1965 Kruskal and Zabusky [KZ 1] discovered that the solitary wave solutions of the KdV equation asymptotically preserve their shape and velocity upon nonlinear interaction with each other, i.e. the interaction is elastic. They called these special solutions *solitons*.

An important problem with the KdV equation is that it cannot be applied when the nonlinear surface gravity waves in weakly dispersing shallow water are not strictly one-dimensional. Therefore an approximate model equation for this case was needed. In 1970 Kadomtsev and Petviashvili obtained in their paper [KP 1] a generalization of the KdV equation to the case (2+1) (two spatial dimensions plus time) under the assumption that locally the waves are almost one-dimensional.

1.1.2 Intuitive Grounds in Deriving the KPI/KPII Equations

In this section we present an intuitive derivation of the KPI/KPII equations. We follow the ideas of Kadomtsev and Petviashvili ([KP 1]) but this derivation is not a reproduction from the article [KP 1]. In the next section we shall present the derivation of the KPI/KPII equations in the physical context of water surface gravity waves.

Recall that the classical wave equation in $\mathbf{R}^n$, $n \geq 1$ positive integer,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \quad u = u(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n, \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \tag{1.22}$$

has solutions[3]

$$u(x) = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{1.23}$$

that satisfy the dispersion relation

$$\omega^2 = c^2 |\mathbf{k}|^2, \quad |\mathbf{k}|^2 = k_1^2 + k_2^2 + \cdots + k_n^2. \tag{D1}$$

We can generalize the right hand side of the dispersion relation (D1) as an arbitrary function of $|\mathbf{k}|^2$:

$$\omega^2 = f(|\mathbf{k}|^2). \tag{D2}$$

[3] c is the speed of wave propagation, $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is the wavenumber vector and $\mathbf{k} \cdot \mathbf{x}$ is the inner product between $\mathbf{k}$ and $\mathbf{x}$ in $\mathbf{R}^n$.

Under the assumption that f is well approximated by its Taylor expansion about $|\mathbf{k}|^2 = 0$ and $f(0) = 0$, $f'(0) = A^2$, $A > 0$, $f''(0) = -B^2$, $B > 0$, we have

$$f(|\mathbf{k}|^2) = A^2|\mathbf{k}|^2 - B^2|\mathbf{k}|^4 + O(|\mathbf{k}|^6) \tag{D3}$$

for small $|\mathbf{k}|^2$. For $|\mathbf{k}|^2$ sufficiently small, in the asymptotic expansion (D3) we can drop the terms $O(|\mathbf{k}|^6)$, so that

$$f(|\mathbf{k}|^2) = A^2|\mathbf{k}|^2 - B^2|\mathbf{k}|^4 \tag{D4}$$

From (D2) and (D4) we obtain

$$\omega^2 = A^2|\mathbf{k}|^2 - B^2|\mathbf{k}|^4, \tag{1.24}$$

for sufficiently small $|\mathbf{k}|^2$.

Restricting ourself to the case $n = 2$, we can write (1.24) as follows:

$$\omega^2 = A^2(k_x^2 + k_y^2) - B^2(k_x^2 + k_y^2)^2. \tag{1.25}$$

Extracting the positive root in (1.25) and assuming k_x small but $k_y/k_x \ll 1$, we obtain

$$\omega = Ak_x - \frac{B^2}{2A}k_x^3 + \frac{A}{2}k_y^2 k_x^{-1} + \cdots. \tag{1.26}$$

Multiplying (1.26) by k_x, we obtain

$$\omega k_x = Ak_x^2 - \frac{B^2}{2A}k_x^4 + \frac{A}{2}k_y^2 + \cdots, \tag{1.27}$$

which suggests that $u(x,y,t)$ (the function (1.23) for the case $n=2$) satisfies the linear equation

$$(u_t + Au_x + \frac{B^2}{2A}u_{xxx})_x + \frac{A}{2}u_{yy} = 0. \tag{1.28}$$

If in the equation (1.28) we make the following change of variables (we take two cases):

$$t_{old} = \sqrt[4]{\frac{2A}{B^2}}t_{new}, \quad x_{old} = \sqrt[4]{\frac{B^2}{2A}}x_{new} + A\sqrt[4]{\frac{2A}{B^2}}t_{new},$$

$$y_{old} = \begin{cases} i\sqrt{\frac{A}{6}}y_{new}, \ i=\sqrt{-1}, & \text{Case I} \\ \sqrt{\frac{A}{6}}y_{new}, & \text{Case II,} \end{cases}$$

then the equation becomes correspondingly

$$(u_t + u_{xxx})_x \mp 3u_{yy} = 0, \tag{1.29}$$

where the $'-'$ case corresponds to the Case I and the $'+'$ case corresponds to the Case II. The equation

$$(u_t + u_{xxx})_x - 3u_{yy} = 0$$

is known as the linear KPI equation (lKPI equation or simply lKPI) and the equation

$$(u_t + u_{xxx})_x + 3u_{yy} = 0$$

is known as the linear KPII equation (lKPII equation or simply lKPII). In order to cause a variation in the amplitude in both space and time to the sinusoidal oscillations of $u(x,y,t)$, a nonlinear term $(6uu_x)_x$ is added to the left hand side of

(1.29) which turns the corresponding linear equations into:

$$(u_t + 6uu_x + u_{xxx})_x \mp 3u_{yy} = 0. \tag{1.30}$$

The equation

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0$$

is known as the KPI equation (KPI equation or simply KPI) and the equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

is known as the KPII equation (KPII equation or simply KPII).

1.2 Physical Derivation of the KPI/KPII Equations

First we are looking to find a mathematical model for describing the surface gravitational waves propagating in the xy-plane (the motion is three-dimensional in the xyz-space).

Consider a fluid which has the following features:

1. irrotational
2. inviscid
3. incompressible

4. homogeneous

5. it is subject to a constant gravitational acceleration g.

Consider that the fluid is lying on an impermeable bed of infinite extent at $z = -h$ and the free surface displacement is $z = \eta(x, y, t)$ (the vertical coordinate z is measured upward from the undisturbed free surface) (Figure 1.1).

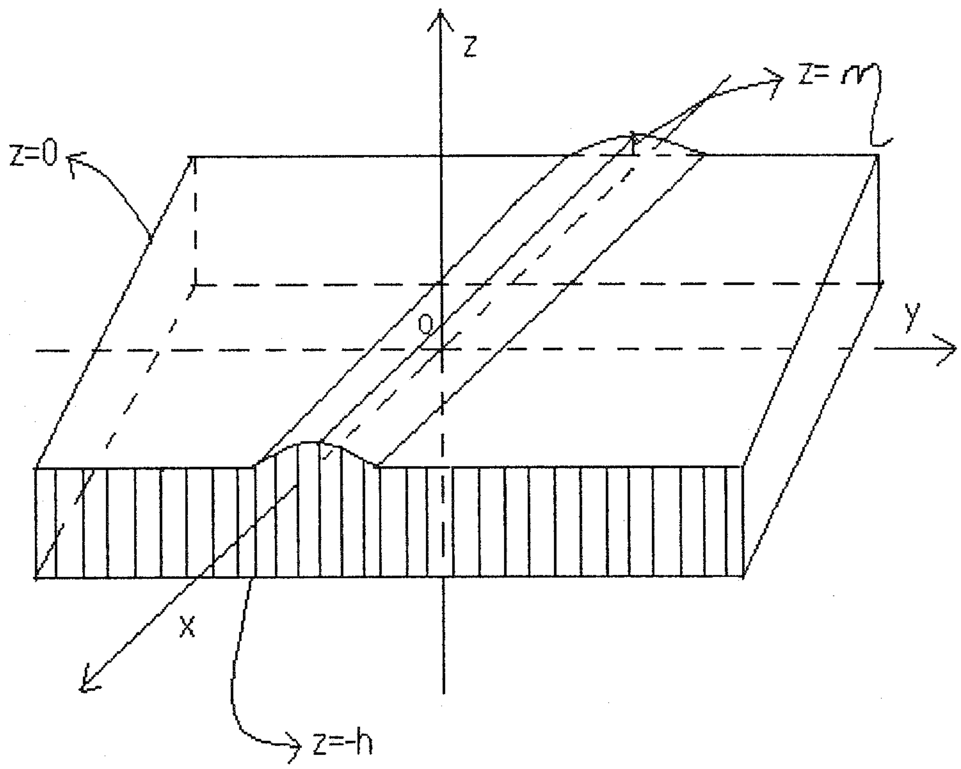


Figure 1.1

Consider $u = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$ the velocity vector of a particle. Because the fluid is irrotational then we have

$$\nabla \times u = 0.^4 \tag{1.31}$$

Thus we can define a velocity potential ϕ of the particle such that

$$u = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right). \tag{1.32}$$

The differential form of the law of conservation of mass (called continuity equation because it assumes that the fluid flow has no voids in it) is

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla u = 0^5, \tag{1.33}$$

where $D\rho/Dt$ is the total rate of change of density following the fluid particle. The density of fluid particles does not change appreciably along the path of the fluid under certain conditions, the most important of which is that the flow speed should be less than the speed of sound in the medium. This is called the Boussinesq approximation and holds in most flows of liquids which include water as well. In these flows the term $(1/\rho) \cdot (D\rho/Dt)$ is much less than any of the derivatives in ∇u, under which condition the continuity equation (1.33) becomes

$$\nabla u = 0, \tag{1.34}$$

[4] $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

[5] $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho$

irrespective of whether the flow is steady or not. From (1.31), (1.32) and (1.34) we obtain that the velocity potential ϕ of the particle satisfies the Laplace equation

$$\nabla^2\phi = 0 \qquad -h < z < \eta(x,y,t). \tag{1.35}$$

Boundary conditions are to be satisfied at the free surface and at the bottom. Because the bed is impermeable, then the condition at the bottom is zero normal velocity, i.e.

$$u_3 = \frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = -h. \tag{1.36}$$

At the free surface, a kinematic boundary condition is that the fluid particle never leaves the surface, that is:

$$\frac{D\eta}{Dt} \equiv \frac{\partial \eta}{\partial t} + \phi_x\eta_x + \phi_y\eta_y = \phi_z \equiv u_3 \qquad \text{at } z = \eta. \tag{1.37}$$

For small-amplitude waves, the quantities ϕ_x, η_x, ϕ_y, η_y are small, so the quadratic terms $\phi_x\eta_x$ and $\phi_y\eta_y$ are one order smaller then the other terms in (1.37), which then simplifies to

$$\frac{\partial \eta}{\partial t} = \phi_z \qquad \text{at } z = \eta. \tag{1.38}$$

Linearizing the condition (1.38) about $\eta = 0$ we obtain

$$\frac{\partial \eta}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} + \eta \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} + \ldots \tag{1.39}$$

which in the first approximation is

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \qquad \text{at } z = 0. \tag{1.40}$$

At $z = \eta$ a dynamic boundary condition arises which involves the interaction at the interface between water and air. A density discontinuity exists at the interface which makes the latter behaves as if it were under tension. Near the interface all the liquid molecules are trying to pool the molecules on the interface inward and in consequence this effect is making the interface to contract. The magnitude of the tensile force per unit length of a line on the interface is known as surface tension, denoted here by T (Figure 1.2). In the Fluids Mechanics literature, [Kundu 1], one can find that the pressure jump across the interface (Figure 1.2) is given by

$$p_a - p = T\left(\frac{1}{R} + \frac{1}{r}\right) \qquad \text{at } z = \eta, \tag{1.41}$$

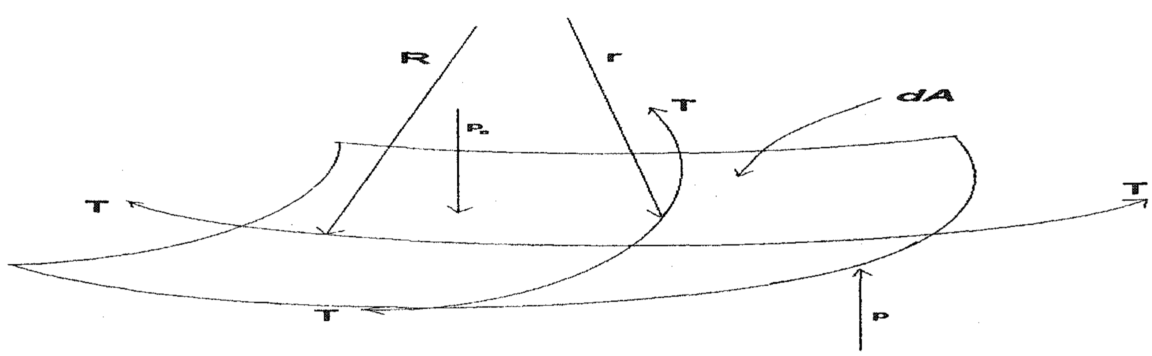


Figure 1.2

where the curvatures $1/R$ and $1/r$ are given by

$$\frac{1}{R} = \frac{\eta_{xx}(1+\eta_y^2)}{(1+\eta_x^2+\eta_y^2)^{3/2}} \approx \eta_{xx} \tag{1.42}$$

and

$$\frac{1}{r} = \frac{\eta_{yy}(1+\eta_x^2)}{(1+\eta_x^2+\eta_y^2)^{3/2}} \approx \eta_{yy}. \tag{1.43}$$

the approximations made in relations (1.42-43) being valid for small slopes only.

Thus, for small slopes and considering the atmospheric pressure $p_a = 0$, the dynamic boundary condition (1.41) becomes

$$p \approx -T(\eta_{xx}+\eta_{yy}) \qquad \text{at } z=\eta. \tag{1.44}$$

For small-amplitude waves, the relation (1.44) can be further simplified. Because the flow is inviscid, irrotational and incompressible, the unsteady Bernoulli's equation for small-amplitude waves is applicable:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = F(t), \tag{1.45}$$

where the integrating function $F(t)$ is independent of location. Redefining ϕ by $\phi - \int F(t)dt$, equation (1.45) becomes

$$\frac{\partial \phi}{\partial t} = -g\eta + \frac{T}{\rho}(\eta_{xx}+\eta_{yy}) \qquad \text{at } z=\eta, \tag{1.46}$$

where the dynamic boundary condition (1.44) has been applied. For small-amplitude waves we can evaluate the term $\partial\phi/\partial t$ in (1.46) at $z=0$ rather than at $z=\eta$.

Hence the linearized mathematical model for describing surface small-amplitude gravitational waves in the three dimensional space is

$$\nabla^2\phi = 0 \qquad -h < z < \eta(x,y,t) \tag{1.47}$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z = -h \tag{1.48}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \qquad \text{at } z = 0 \tag{1.49}$$

$$\frac{\partial\phi}{\partial t} = -g\eta + \frac{T}{\rho}(\eta_{xx} + \eta_{yy}) \qquad \text{at } z = 0. \tag{1.50}$$

From Fourier Analysis we know that an arbitrary disturbance can be decomposed into sinusoidal wave components of different wave lengths and amplitudes. We shall seek then solutions of the linearized problem (1.47-50) of the following form

$$\eta(x,y,t) = e^{i(k_x x + k_y y - \omega t)}. \tag{1.51}$$

Relation (1.51) and boundary conditions (1.49-50) suggest us to look for solutions of (1.47-50) of the following form

$$\phi(x,y,z,t) = Z(z)e^{i(k_x x + k_y y - \omega t)}. \tag{1.52}$$

Substituting (1.52) into the equation (1.47) we obtain a second order differential equation for Z

$$-(k_x^2 + k_y^2)Z + Z'' = 0, \tag{1.53}$$

which has the general solution:

$$Z(z) = Ae^{\sqrt{k_x^2+k_y^2}z} + Be^{-\sqrt{k_x^2+k_y^2}z}, \quad A, B = \text{real constants} \tag{1.54}$$

The velocity potential ϕ becomes then

$$\phi(x,y,z,t) = (Ae^{\sqrt{k_x^2+k_y^2}z} + Be^{-\sqrt{k_x^2+k_y^2}z})e^{i(k_x x + k_y y - \omega t)}. \tag{1.55}$$

Using the boundary conditions (1.48-49) the constants A and B can be found:

$$A = \frac{-i\omega}{\sqrt{k_x^2+k_y^2}(1 - e^{-2h\sqrt{k_x^2+k_y^2}})}, \tag{1.56}$$

$$B = \frac{-i\omega e^{-2h\sqrt{k_x^2+k_y^2}}}{\sqrt{k_x^2+k_y^2}(1 - e^{-2h\sqrt{k_x^2+k_y^2}})}. \tag{1.57}$$

The velocity potential ϕ becomes then

$$\phi(x,y,z,t) = \frac{-i\omega}{\sqrt{k_x^2+k_y^2}} \frac{\cosh[\sqrt{k_x^2+k_y^2}(z+h)]}{\sinh(\sqrt{k_x^2+k_y^2}h)} e^{i(k_x x + k_y y - \omega t)}. \tag{1.58}$$

So far the Laplace equation (1.47) was solved using kinematic boundary conditions only (1.48-49), which is typical of irrotational flows. We want to find now the application of the dynamic free surface condition (1.50) to the problem. This will give us a relationship between $k = (k_x, k_y)$ (the horizontal (x, y) wavenumber characteristic of disturbances) and ω.

Substituting (1.58) and (1.51) into (1.50) we obtain

$$\omega^2 = \left(\kappa g + \frac{T}{\rho}\kappa^3\right)\tanh(\kappa h), \tag{1.59}$$

where $\kappa = \sqrt{k_x^2 + k_y^2}$.

Relation (1.59) is the *dispersion relation* associated with the problem (1.47-50) (it expresses the nature of the dispersion process).

Let us consider the positive root of (1.59)

$$\omega = \sqrt{\left(\kappa g + \frac{T}{\rho}\kappa^3\right)\tanh(\kappa h)}, \tag{1.60}$$

The Taylor's series expansion of ω from (1.60) about $\kappa = 0$ is

$$\omega = c_0\kappa\left[1 - \frac{1}{6}h^2(1-\bar{T})\kappa^2\right] + O(\kappa^5), \tag{1.61}$$

where

$$c_0 = \sqrt{gh}, \quad \bar{T} = \frac{3T}{\rho g h^2} \tag{1.62}$$

represent the non-dispersive phase speed and the dimensionless surface tension respectively.

From (1.61) we can obtain the linearized phase speed

$$c = \frac{\omega}{\kappa} = c_0\left[1 - \frac{1}{6}h^2(1-\bar{T})\kappa^2\right] + O(\kappa^4), \tag{1.63}$$

which shows us that as $\kappa \to 0$ (i.e. long waves, or shallow water waves) the linearized problem (1.47-50) is weakly dispersive. KdV and KP equations arise as models of the water wave problem in this weakly dispersive limit $\kappa h \ll 1$. Indeed, let us consider the dispersion relation (1.61). For κ sufficiently small, in the asymptotic expansion (1.61) we can drop the terms $O(\kappa^5)$, so that

$$\omega = c_0\sqrt{k_x^2 + k_y^2}\left[1 - \frac{1}{6}h^2(1-\bar{T})(k_x^2 + k_y^2)\right] \tag{1.64}$$

Assuming k_x small but $k_y/k_x \ll 1$, from (1.64) we can obtain

$$\frac{1}{c_0}\omega k_x - k_x^2 - \frac{1}{2}k_y^2 + \frac{1}{6}h^2(1-\bar{T})k_x^4 = 0 \tag{1.65}$$

Let us recall that we sought solutions for the linearized problem (1.47-50) of the form

$$\eta(x,y,t) = e^{i(k_x x + k_y y - \omega t)}. \tag{1.66}$$

Equation (1.65) suggests us that $\eta(x,y,t)$ satisfies the equation

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\eta_t + \eta_x + \frac{1}{6}h^2(1-\bar{T})\eta_{xxx}\right] + \frac{1}{2}\eta_{yy} = 0 \tag{1.67}$$

In order to cause a variation in the amplitude in both space and time to the sinusoidal oscillations of $\eta(x,y,t)$, a nonlinear term $(\eta\eta_x)_x$ is added to the equation (1.67), yielding

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\eta_t + \eta_x + \eta\eta_x + \frac{1}{6}h^2(1-\bar{T})\eta_{xxx}\right] + \frac{1}{2}\eta_{yy} = 0 \tag{1.68}$$

Under the transformation

$$r = \frac{x - c_0 t}{h}, \quad \xi = \frac{y}{h}, \quad \tau = \frac{c_0 t}{6h}, \quad \eta = u(r,\xi,\tau) \tag{1.69}$$

we obtain the dimensionless form of the equation (1.68)

$$(u_\tau + 6uu_r + (1-\bar{T})u_{rrr})_r + 3u_{\xi\xi} = 0 \tag{1.70}$$

Equation (1.70) is equivalent to the equation

$$(u_\tau + 6uu_r + u_{rrr})_r + 3\gamma^2 u_{\xi\xi} = 0, \tag{1.71}$$

where $\gamma^2 = \pm 1$, the choice of sign depending on the relevant magnitude of gravity and surface tension, i.e.

1. If $\bar{T} \ll 1$ (gravity dominates surface tension) then equation (1.71) (with $\gamma^2 = 1$) is equivalent to the equation (1.70)

$$(u_\tau + 6uu_r + u_{rrr})_r + 3u_{\xi\xi} = 0. \tag{1.72}$$

   Equation (1.72) is known as KPII and is of most interest in water waves studies.

2. If $\bar{T} > 1$ (surface tension dominates gravity) then equation (1.71) (with $\gamma^2 = -1$) is equivalent to the equation (1.70)

$$(u_\tau + 6uu_r + u_{rrr})_r - 3u_{\xi\xi} = 0. \tag{1.73}$$

   Equation (1.73) is known as KPI and is of interest for very thin sheets of water.

1.3 N-Soliton Wall Solution of the KPI/KPII Equations

In this section we reproduce the derivation of the N-soliton wall solution of the KPI/KPII equations respectively as given by Zakharov and Shabat [ZS1] in 1974.

To derive these solutions Zakharov and Shabat used the Dressing Method devised by them. The Dressing Method yields wide classes of new solutions from already known solutions for many problems - the soliton solutions in particular. These solutions were derived by other authors as well. They used different methods in their derivations, as for example:

1. In 1976 Satsuma, [Satsuma 1], derived the N-soliton wall solution of the KPI/KPII equations respectively by using the Hirota's method.

2. In 1983 Okhuma and Wadati, [Okhuma 1], derived the N-soliton wall solution of the KPII equation by using the trace method.

1.3.1 Lax's Representation

In order to use the Dressing Method we need to introduce the notion of *Lax's representation*. The general concept of Lax's representation is reproduced from [Abl 1].

Consider two differential operators L and A where L is the operator of the spectral problem

$$L\phi = \lambda\phi, \tag{1.74}$$

and A is the operator governing the associated time evolution of the eigenfunctions

$\phi(\mathbf{x}, t)$

$$\phi_t = A\phi. \tag{1.75}$$

Taking $\partial/\partial t$ of (1.74) and using (1.75) we obtain

$$[L_t + (LA - AL)]\phi = \lambda_t \phi, \tag{1.76}$$

and hence in order to solve for nontrivial eigenfunctions $\phi(\mathbf{x}, t)$

$$L_t + [L, A] = 0, \qquad [L, A] = LA - AL, \tag{1.77}$$

if and only if $\lambda_t = 0$.

Equation (1.77) is called *Lax's equation* and contains an evolution equation for suitably chosen differential operators L and A. For example if we consider the spectral equation (1.74) given by the Schrödinger operator

$$L = -\frac{\partial^2}{\partial x^2} + u, \tag{1.78}$$

and the operator A of the simplest form

$$A = c\frac{\partial}{\partial x}, \quad c = \text{ const},$$

then we have

$$L_t + [L, A] = u_t - cu_x,$$

and so we can see that L and A satisfy (1.77) if u satisfies the one-dimensional wave equation

$$u_t - cu_x = 0.$$

We can see then that the one-dimensional wave equation has an associate spectral problem with eigenvalues which are constants of motion and may be thought as the compatibility condition of the above two linear operators L and A.

If a partial differential equation arises as the compatibility condition of two such operators L and A, then (1.77) is called the *Lax representation* of the partial differential equation and the pair (L, A) is called *Lax's pair*.

In 1974 Zakharov and Shabat, [ZS1], and also Dryuma, [Dryuma 1], derived the Lax's pair (L, A) for the KP equations (1.71). They found that

$$L = \gamma \frac{\partial}{\partial y} - M, \tag{1.79}$$

$$M = -\frac{\partial^2}{\partial x^2} - u(x, y, t) \tag{1.80}$$

and

$$A = -4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x + 3\gamma w(x, y, t), \quad w_x = u_y, \quad \gamma^2 = \pm 1. \tag{1.81}$$

The Lax's pair (L, A) given in (1.79-81) allows us to write the evolution forms of the KP equations:

$$u_t + 6uu_x + u_{xxx} + 3\gamma^2 w_y = 0, \; w_x = u_y.$$

This is an infinite family of different equations according to the choice one can make for w. However, once the boundary conditions are selected, then these define w uniquely ([Abl1]).

1.3.2 The Dressing Method Applied to the KPII Equation

Steps in deriving the N-soliton wall solution of the KPII equation (1.71) ([ZS 1]):

Step 1. Consider the Lax's pair (1.79-81) of the KPII equation (1.71)

$$L = \frac{\partial}{\partial y} - M, \tag{1.82}$$

$$M = -\frac{\partial^2}{\partial x^2} - u(x,y,t) \tag{1.83}$$

$$A = -4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x + 3w(x,y,t), \quad w_x = u_y. \tag{1.84}$$

Consider as well the operators L_0 and A_0 corresponding to $u = 0$ and $w = 0$

$$L_0 = \frac{\partial}{\partial y} - M_0, \tag{1.85}$$

$$M_0 = -\frac{\partial^2}{\partial x^2}, \tag{1.86}$$

$$A_0 = -4\frac{\partial^3}{\partial x^3}. \tag{1.87}$$

Step 2. We find a Fredholm operator $\hat{F}$ that commutes with the operators L_0 and $\partial/\partial t - A_0$.

Comments

Consider the Fredholm operator

$$\hat{F}\psi(x,y,t) = \int_{-\infty}^{\infty} F(x,z,y,t)\psi(z,y,t)dz, \tag{1.88}$$

which possesses a sufficiently well-behaved kernel and admits the triangular factorization

$$1 + \hat{F} = (1 + \hat{K}^-)^{-1}(1 + \hat{K}^+), \tag{1.89}$$

where $\hat{K}^-$ and $\hat{K}^+$ are the Volterra operators

$$\hat{K}^-\psi(x) = \int_{-\infty}^{x} K^-(x,z,y,t)\psi(z,y,t)dz, \tag{1.90}$$

$$\hat{K}^+\psi(x) = \int_{x}^{\infty} K^+(x,z,y,t)\psi(z,y,t)dz. \tag{1.91}$$

Supposing that $\hat{F}$ is chosen to commute with L_0 and $\partial/\partial t - A_0$

$$[L_0, \hat{F}] = 0, \quad \left[\frac{\partial}{\partial t} - A_0, \hat{F}\right] = 0, \tag{1.92}$$

then we have that when the space of $\psi(x)$ is transformed by $1+\hat{F}$, the operators L_0 and $\partial/\partial t - A_0$ remain invariant, i.e.

$$(1+\hat{F})^{-1}L_0(1+\hat{F}) = L_0$$

$$\Updownarrow$$

$$(1+\hat{K}^+)L_0(1+\hat{K}^+)^{-1} = (1+\hat{K}^-)L_0(1+\hat{K}^-)^{-1}, \tag{1.93}$$

and

$$(1+\hat{F})^{-1}\left(\frac{\partial}{\partial t} - A_0\right)(1+\hat{F}) = \frac{\partial}{\partial t} - A_0$$

$$\Updownarrow$$

$$(1+\hat{K}^+)(\frac{\partial}{\partial t} - A_0)(1+\hat{K}^+)^{-1} = (1+\hat{K}^-)(\frac{\partial}{\partial t} - A_0)(1+\hat{K}^-)^{-1}. \tag{1.94}$$

Step 3. Once we have obtained $\hat{F}$, we find K^- by solving the GLM equation (1.96).

Comment

Applying $1 + \hat{K}^-$ to the left of (1.89) we obtain

$$(1 + \hat{K}^-)(1 + \hat{F}) = (1 + \hat{K}^+), \tag{1.95}$$

which for $z < x$ leads us to the following equation in terms of the kernels F and K^-

$$F(x, z, t, y) + K^-(x, z, t, y) + \int_{-\infty}^{x} K^-(x, s, t, y) F(s, z, t, y) ds = 0, \tag{1.96}$$

called the Gel'fand-Levitan-Marchenko equation (GLM equation).

Step 4. The new class of solutions of the KPII equation will be generated from

$$u(x, y, t) = -2\frac{\partial}{\partial x} K^-(x, x, t, y). \tag{1.97}$$

Remarks

1. For $u(x, y, t)$ found in Step 4, the following identities hold

$$L = (1 + \hat{K}^-) L_0 (1 + \hat{K}^-)^{-1},$$
$$\frac{\partial}{\partial t} - A = (1 + \hat{K}^-)(\frac{\partial}{\partial t} - A_0)(1 + \hat{K}^-)^{-1}, \tag{1.98}$$

which finalize the Dressing Method (a new class of solutions was obtained from an already known solution).

2. By making the substitution $y \to iy$, $i = \sqrt{-1}$, the formula (1.97) becomes an exact solution of the KPI equation.

1.3.3 Derivation of the N-Soliton Wall Solutions of the KPII Equation.

Consider the Lax's pairs (L, A) and (L_0, A_0) defined by (1.82-84) and (1.85-87) respectively.

We find a Fredholm operator $\hat{F}$ that commutes with the operators L_0 and $\partial/\partial t - A_0$. We shall have

$$[L_0, \hat{F}] = 0$$

$$\Updownarrow$$

$$\left(\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2}\right) \int_{-\infty}^{\infty} F(x,z,y,t)\psi(z,y,t)dz - \int_{-\infty}^{\infty} F(x,z,y,t) \left(\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2}\right) \psi(z,y,t)dz = 0$$

$$\Updownarrow$$

$$\int_{-\infty}^{\infty} \left[\frac{\partial F(x,z,y,t)}{\partial y} + \frac{\partial^2 F(x,z,y,t)}{\partial x^2} - \frac{\partial^2 F(x,z,y,t)}{\partial z^2}\right] \psi(z,y,t)\ dz = 0$$

$$\Updownarrow$$

$$\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0 \tag{1.99}$$

Similarly

$$\left[\frac{\partial}{\partial t} - A_0, \hat{F}\right] = 0$$

$$\Updownarrow$$

$$\left(\frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3}\right) \int_{-\infty}^{\infty} F(x,z,y,t)\psi(z,y,t)dz - \int_{-\infty}^{\infty} F(x,z,y,t) \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial z^3}\right) \psi(z,y,t)dz = 0$$

$$\Updownarrow$$

$$\int_{-\infty}^{\infty}\left[\frac{\partial F(x,z,y,t)}{\partial t}+4\frac{\partial^3 F(x,z,y,t)}{\partial x^3}+4\frac{\partial^3 F(x,z,y,t)}{\partial z^3}\right]\psi(z,y,t)\ dz=0$$

$$\Updownarrow$$

$$\frac{\partial F}{\partial t}+4\frac{\partial^3 F}{\partial x^3}+4\frac{\partial^3 F}{\partial z^3}=0 \tag{1.100}$$

One of the simplest solution of the differential system (1.99-100) is

$$F(x,z,t,y)=\sum_{n=1}^{N} d_n(y,t)e^{(p_n x+q_n z)}, \tag{1.101}$$

with

$$d_n(y,t)=c_n e^{[(q_n^2-p_n^2)y-4(p_n^3+q_n^3)t]}, \tag{1.102}$$

$p_n, q_n \in \mathbf{R}, c_n>0,\ p_n+q_m\neq 0,\ n,m=1\dots N.$

The solution K^- of the GLM equation (1.96) is sought of the following form:

$$K^-(x,z,t,y)=\sum_{n=1}^{N} K_n(x,t,y)e^{q_n z}. \tag{1.103}$$

Substituting (1.101-103) into the GLM equation (1.96) we obtain the following linear system for $(K_n(x,t,y))_{n=1\dots N}$

$$K_n(x,t,y)+\sum_{m=1}^{N} K_m(x,t,y)d_n(y,t)\frac{e^{(p_n+q_m)x}}{p_n+q_m}=-d_n(y,t)e^{p_n x},\qquad n=1,\dots,N. \tag{1.104}$$

The matrix of the system (1.104) is:

$$\mathcal{A}_{mn}=\delta_{mn}+d_n(y,t)\frac{e^{(p_n+q_m)x}}{p_n+q_m},\qquad m,n=1\dots N, \tag{1.105}$$

$$\delta_{mn} = \begin{cases} 1 & \text{if } \; m = n \\ 0 & \text{if } \; m \neq n. \end{cases}$$

Using the Cramer's rule for solving linear systems, the solution of the system (1.104) is:

$$K_n(x,t,y) = \frac{\det \mathcal{A}^{(n)}}{\det \mathcal{A}}, \qquad n = 1 \ldots N, \tag{1.106}$$

where $\mathcal{A}^{(n)}$ is the matrix obtained from the matrix $\mathcal{A}$ in which the n-th column was replaced by the free column of the system (1.104).

Hence the solution K^- of the GLM equation (1.96) is

$$K^-(x,z,t,y) = \frac{1}{\det \mathcal{A}} \sum_{n=1}^{N} \det \mathcal{A}^{(n)} e^{q_n z}. \tag{1.107}$$

Proposition. We have

$$K^-(x,z,t,y)|_{z=x} = -\frac{\partial}{\partial x} \ln \det \mathcal{A} \tag{1.108}$$

Proof

For the matrix $\mathcal{A}$ given in (1.105) we have

$$\det \mathcal{A} = \sum_{\sigma \in \mathcal{P}_n} (-1)^{\iota(\sigma)} \mathcal{A}_{1\sigma(1)} \mathcal{A}_{2\sigma(2)} \ldots \mathcal{A}_{n\sigma(n)} \ldots \mathcal{A}_{N\sigma(N)} \tag{1.109}$$

where $\mathcal{P}_n$ is the group of permutations of $\{1, 2, ..., N\}$ and $\iota(\sigma)$ $(\sigma \in \mathcal{P}_n)$ is the total number of inversions in the permutation σ. Then

$$\frac{\partial}{\partial x} \det \mathcal{A} = \sum_{\sigma \in \mathcal{P}_n} (-1)^{\iota(\sigma)} \sum_{n=1}^{N} \mathcal{A}_{1\sigma(1)} \mathcal{A}_{2\sigma(2)} \ldots \frac{\partial}{\partial x} \mathcal{A}_{n\sigma(n)} \ldots \mathcal{A}_{N\sigma(N)} =$$

$$= -\sum_{n=1}^{N} e^{q_n x} \sum_{\sigma \in \mathcal{P}_n} (-1)^{\iota(\sigma)} \mathcal{A}_{1\sigma(1)} \mathcal{A}_{2\sigma(2)} \ldots (-c_{\sigma(n)}(y,t) e^{p_{\sigma(n)} x}) \ldots \mathcal{A}_{N\sigma(N)} =$$

$$= -\sum_{n=1}^{N} \det \mathcal{A}^{(n)} e^{q_n x}. \tag{1.110}$$

Hence, from (1.107) we obtain

$$K^{-}(x,z,t,y)|_{z=x} = -\frac{\frac{\partial}{\partial x} \det \mathcal{A}}{\det \mathcal{A}} = -\frac{\partial}{\partial x} \ln \det \mathcal{A}.$$

which ends the proof of the proposition.

Finally, from (1.97) and (1.108) we find the new solution of the KPII equation

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{A}, \tag{1.111}$$

which is called the N-soliton wall solution. The new class of solutions obtained is the one formed by soliton walls. A soliton wall is obtained for $N = 1$ in the formula (1.111).

Remark

By making the substitution $y \to iy$, $i = \sqrt{-1}$, the formula (1.111) becomes an exact solution of the KPI equation called the N-soliton wall solution of KPI.

Chapter 2

Basic Oscillatory Solutions of the Kadomtsev-Petviashvili II Equation.

2.1 Motivation.

The KPII equation as derived in Chapter 1 is

$$(u_t + 6uu_x + u_{xxx})_x = -3u_{yy}. \tag{2.1}$$

Since the nonlinear term $(6uu_x)_x$ is small, in applications at least, in comparison with the linear terms, due to the smallness of $u(t,x,y)$ itself, KPII can be viewed as a nonlinear perturbation of its linearization (lKPII)

$$(u_t + u_{xxx})_x = -3u_{yy} \tag{2.2}$$

by the term $(6uu_x)_x$. The (lKPII) equation possesses special solutions of the form

$$\cos[2\gamma(\lambda,\mu) + 2\lambda x - 4\lambda\mu y + 8\lambda(\lambda^2 - 3\mu^2)t], \tag{2.3}$$

akin to the functions (I.3). A fairly large class of solutions of (lKPII), due to the fact that (lKPII) is linear, can be written as

$$u(x,y,t) = \int\int_{\mathcal{L}} \hat{u}(\lambda,\mu)\cos[2\gamma(\lambda,\mu)+2\lambda x-4\lambda\mu y+8\lambda(\lambda^2-3\mu^2)t]\, d\lambda d(\lambda\mu), \tag{2.4}$$

where $\mathcal{L} \subseteq \mathbf{R}^2$ and the two functions $\gamma(\lambda,\mu)$ and $\hat{u}(\lambda,\mu)$ are essentially arbitrary with restrictions imposed only to guarantee convergence in certain sense. The formula (2.4) is nothing else but the Fourier decomposition of solutions of (lKPII) into much simpler basic motions (2.3).

If KPII is viewed as a perturbation of (lKPII) by a small term $(6uu_x)_x$, then such perturbation will affect the basic solutions (2.3) and the decomposition formula (2.4).

2.2 N-Soliton Wall Solution of the KPII Equation.

Recall that nonlinear superposition is already defined and well known for the class of solutions of KPII known as soliton walls. The N-soliton wall solution of the KPII is given by the formula (1.111)

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\det\mathcal{A} \tag{2.5}$$

where $\mathcal{A}$ is a $N \times N$ matrix with entries

$$\mathcal{A}_{mn} = \delta_{mn} + \frac{c_n}{p_n + q_m} e^{(p_n+q_m)x+(q_n^2-p_n^2)y-4(p_n^3+q_n^3)t} \qquad m,n = 1,\ldots,N \tag{2.6}$$

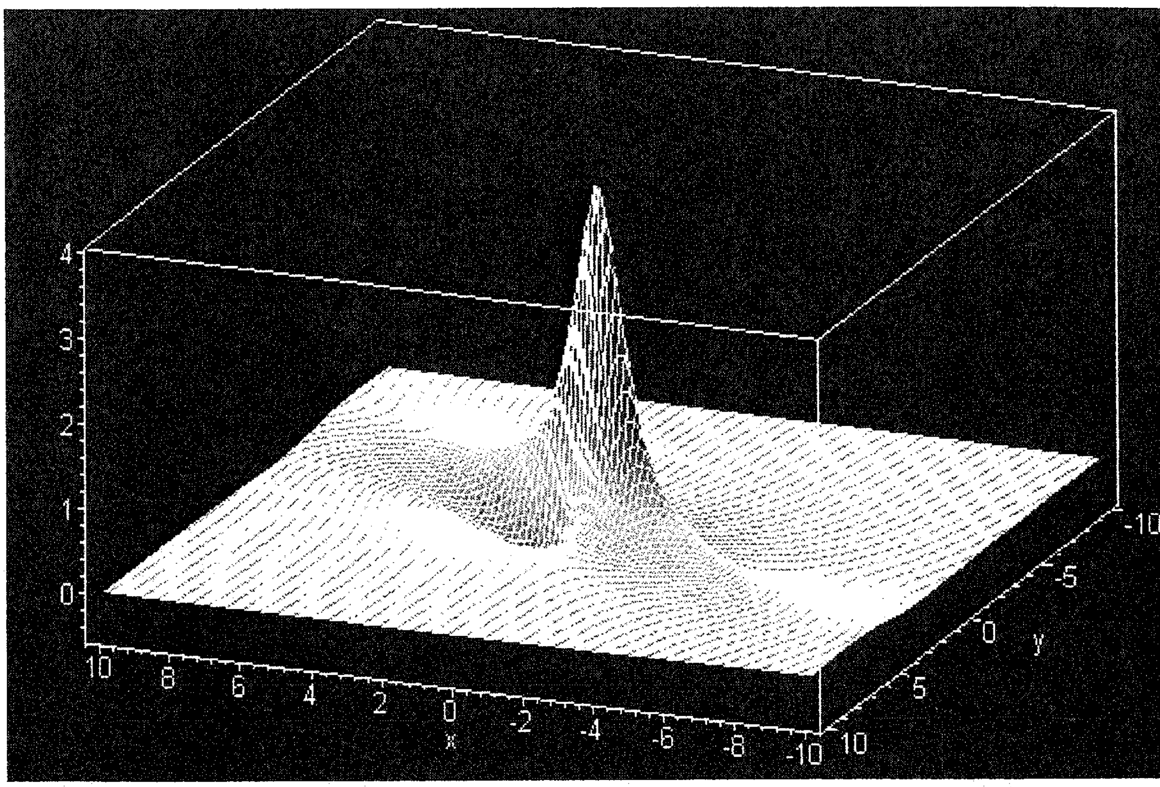
$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

with $p_n, q_n \in \mathbf{R}, c_n > 0,\ p_n + q_m \neq 0,\ n,m = 1\ldots N$.

The formula (2.5-6) represents a *nonlinear superposition principle* for the construction of the N-soliton wall solution of KPII equation.

Remark

By making the substitution $y \to iy$, $i = \sqrt{-1}$, the formula (2.5-6) becomes an exact solution of KPI equation. From them there were obtained rational solutions of KPI, called lump solutions, which decay algebraically as $\sqrt{x^2+y^2} \to +\infty$. Among scientific literature about these solutions of KPI equation we can mention [Manakov 1], [Abl 1], [Nov 2], [Satsuma 2] and [Boiti 1-3].



Lump solution of the KPI equation $(u_t + 6uu_x + u_{xxx})_x = 3u_{yy}$.

Some properties of the soliton walls of the KPII equation.

1. Do not decrease in the directions $x/y = p_n - q_n$, $n = 1..N$.

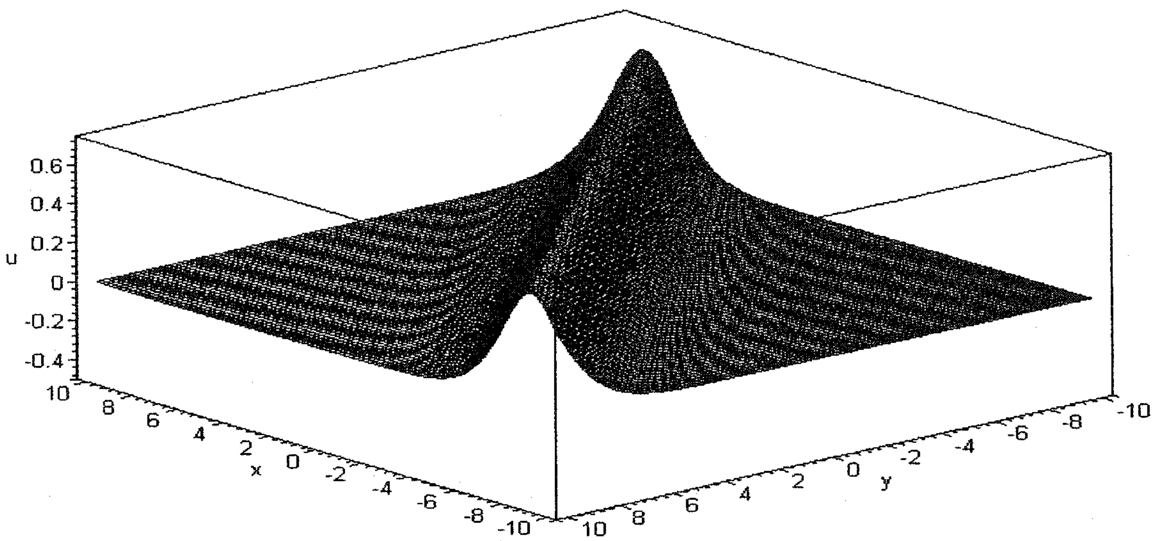
2. Move at a certain angle to the x-axis.

3. Never propagate along the y-axis.

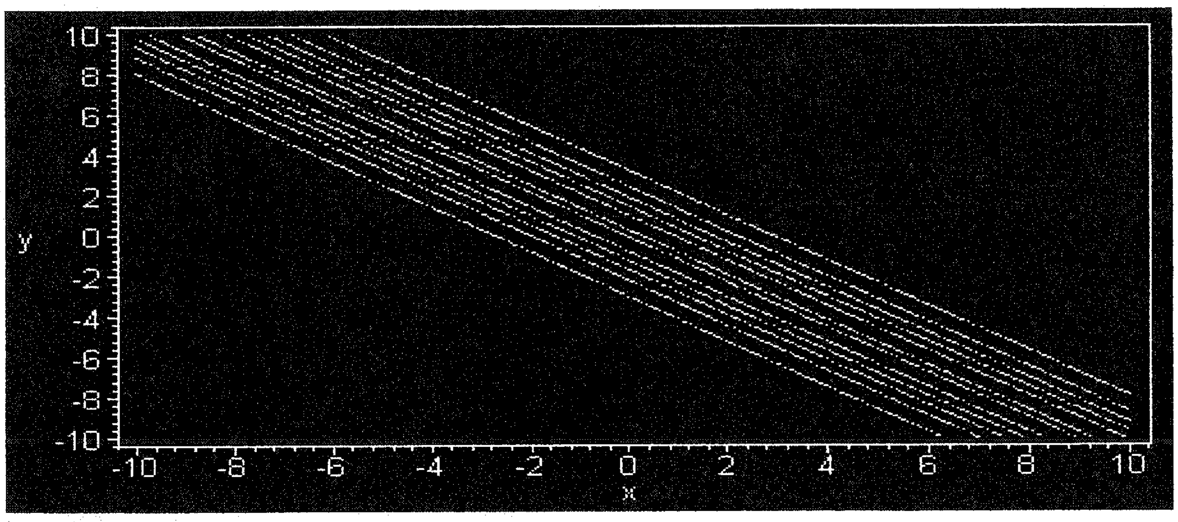
4. If $p_n = q_n$ for some $n \in \{1, \dots, N\}$ in (2.5-6), the corresponding soliton walls are converted into KdV solitons.

5. Miles [Miles 1] noted that the phase shift of two soliton solution tends to infinity when the parameters describing the soliton walls satisfy certain linear-like resonance conditions.

Figure 2.1 a-b: KPII Soliton Wall with $p = 0.1,\ q = 1,\ c = 1$ at $t = 0$ (moves at the angle $\arctan[1/(p-q)] \approx 131.9°$ to the x-axis).



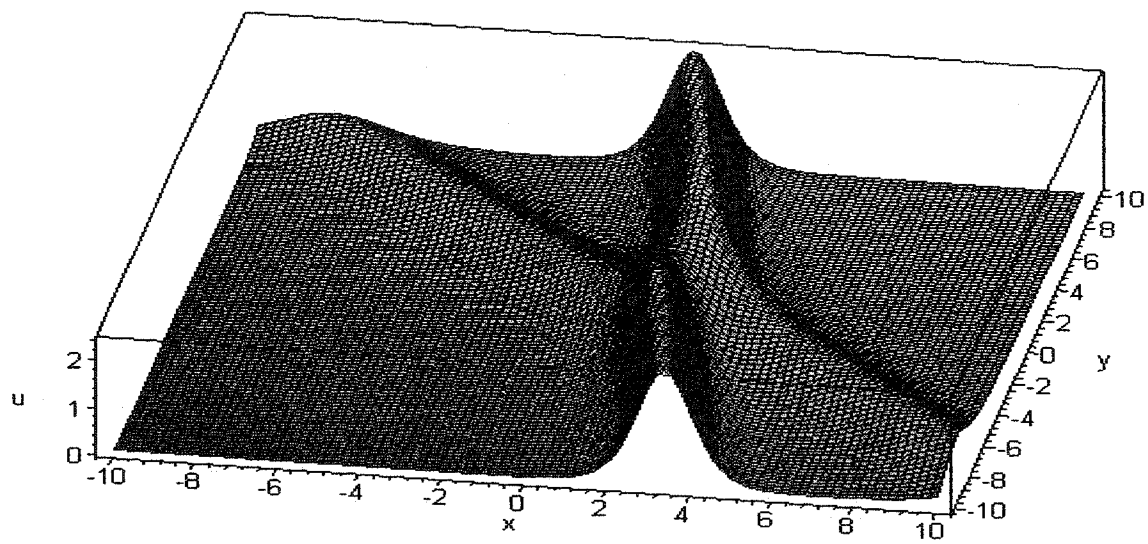
a) 3D plot.



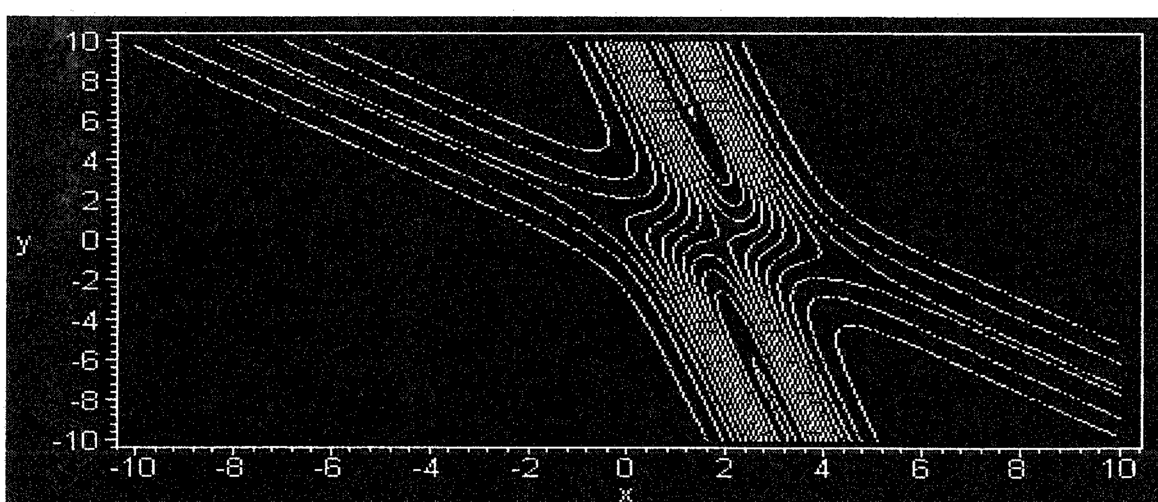
b) Contour plot in projection to xy-plane.

Figure 2.2 a-b: Nonlinear superposition of two KPII Soliton Walls with

$p = [1, 0.1],\ q = [1.2, 1],\ c = [1, 1]$ at $t = 0.2$.

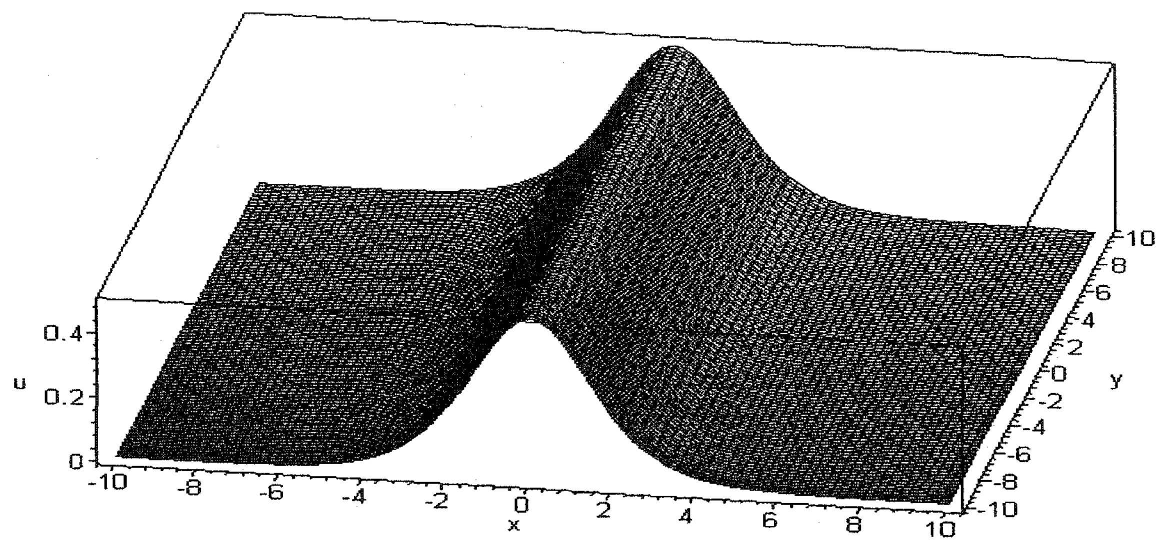


a) 3D plot.

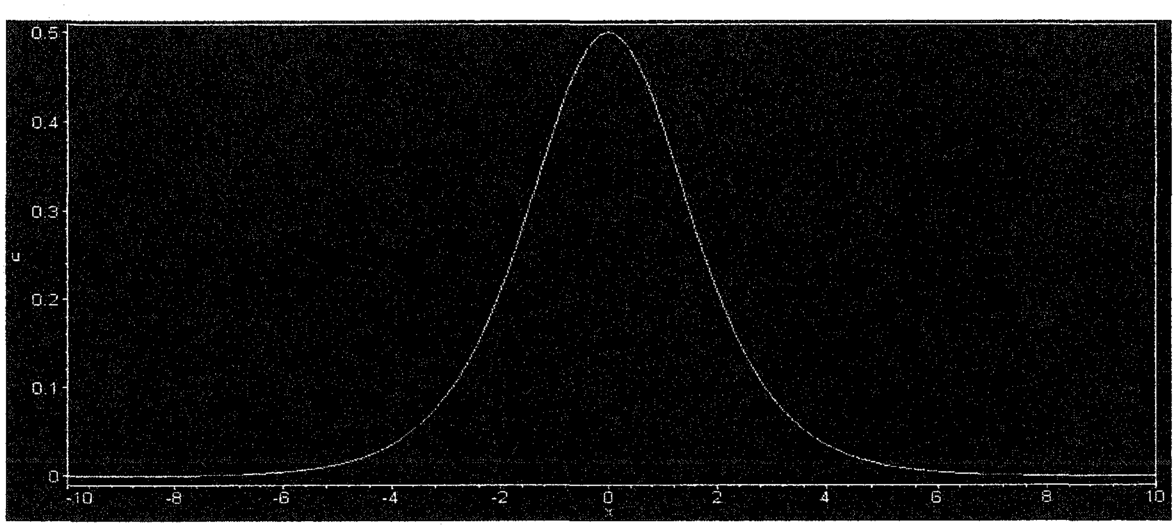


b) Contour plot in projection to xy-plane.

Figure 2.3 a-b: KdV soliton obtained from the formula (2.2-3) for the parameters $p = 0.5,\ q = 0.5,\ c = 1$ at $t = 0$.

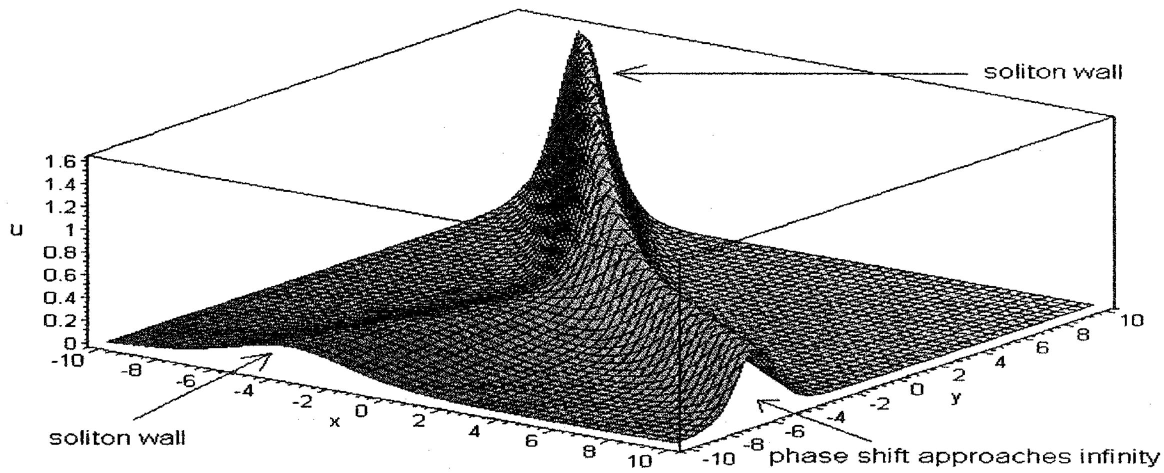


a) 3D plot.

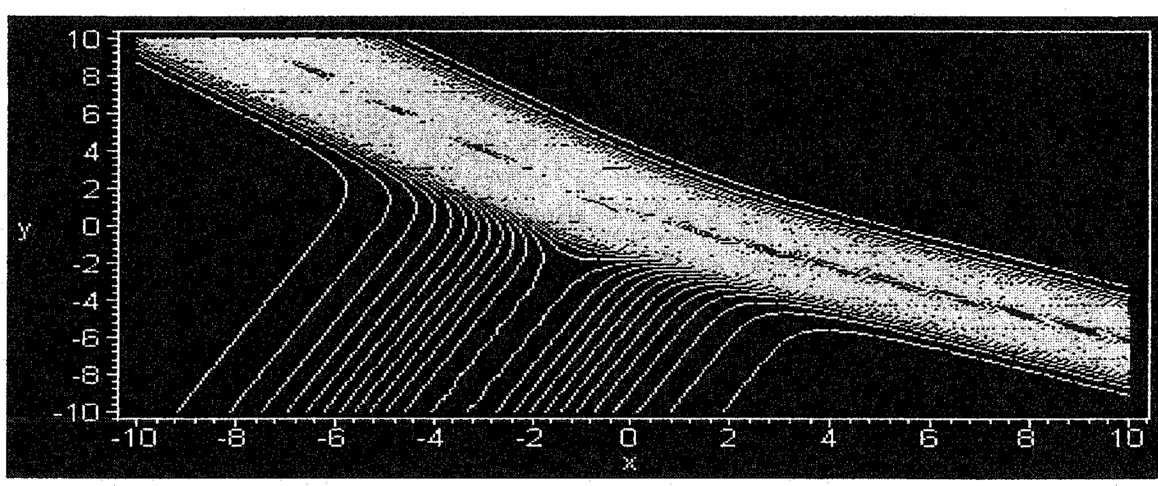


b) Projection to xu-plane.

Figure 2.4 a-b: Miles' observation (Property 5 from page 40). The formula (2.2-3) was used for the parameters $p = [0.5, 0.5]$, $q = [0.2, 1.3]$, $c = [1.1, 1.3]$ at $t = 0$.



a) 3D plot.



b) Contour plot in projection to xy-plane.

2.3 Construction of Oscillatory Solutions of the KPII Equation.

Theorem 1

Let $u(x, y, t)$ be defined by

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{K}, \tag{2.7}$$

where $\mathcal{K}$ is a $N \times N$ matrix with the entries:

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} & \dots & \dots & K_{1N} \\ K_{21} & K_{22} & \dots & \dots & K_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ K_{N1} & K_{N2} & \dots & \dots & K_{NN} \end{pmatrix} \tag{2.8}$$

$$K_{nn} = -\rho_n + \frac{\cos 2\Gamma_n}{2\lambda_n} \tag{2.9}$$

$$K_{mn} = \left[\frac{(\lambda_m + \lambda_n)\cos(\Gamma_m + \Gamma_n)}{(\mu_m - \mu_n)^2 + (\lambda_m + \lambda_n)^2} - \frac{(\lambda_m - \lambda_n)\sin(\Gamma_m - \Gamma_n)}{(\mu_m - \mu_n)^2 + (\lambda_m - \lambda_n)^2}\right] -$$
$$- \left[\frac{(\mu_m - \mu_n)\sin(\Gamma_m + \Gamma_n)}{(\mu_m - \mu_n)^2 + (\lambda_m + \lambda_n)^2} + \frac{(\mu_m - \mu_n)\cos(\Gamma_m - \Gamma_n)}{(\mu_m - \mu_n)^2 + (\lambda_m - \lambda_n)^2}\right], \tag{2.10}$$

$$\rho_n = \varrho_n + x - 2\mu_n y + 12(\lambda_n^2 - \mu_n^2)t, \tag{2.11}$$

$$\Gamma_n = \gamma_n + \lambda_n x - 2\lambda_n\mu_n y + 4\lambda_n(\lambda_n^2 - 3\mu_n^2)t, \tag{2.12}$$

where λ_n's, μ_n's, ϱ_n's and γ_n's are real parameters. Then $u(x,y,t)$ satisfies the KPII equation.

Proof

We follow the ideas from [Kov 1-3]. Consider the N-soliton wall formula (2.5-6) of Zakharov and Shabat. Replacing N with $2N$ and taking

$$\begin{aligned}
&p_{2k-1} = \bar{p}_{2k} = i\lambda_k + \mu_k + \varepsilon \\
&q_{2k-1} = \bar{q}_{2k} = i\lambda_k - \mu_k + \varepsilon \\
&c_{2k-1} = \bar{c}_{2k} = 2\varepsilon e^{i\pi/4 + i\gamma_k + \varrho_k \varepsilon} \\
&\lambda_k, \mu_k, \gamma_k, \varrho_k \in \mathbf{R} \qquad k = 1 \ldots N
\end{aligned} \tag{2.13}$$

the formula (2.5-6) transforms into

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \det \mathcal{A}^{\varepsilon} \tag{2.14}$$

where

$$\mathcal{A}^{\epsilon} = \begin{pmatrix}
A^{\varepsilon}_{11} & A^{\varepsilon}_{12} & \ldots & \ldots & A^{\varepsilon}_{1N} \\
A^{\varepsilon}_{21} & A^{\varepsilon}_{22} & \ldots & \ldots & A^{\varepsilon}_{2N} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
A^{\varepsilon}_{N1} & A^{\varepsilon}_{N2} & \ldots & \ldots & A^{\varepsilon}_{NN}
\end{pmatrix} \tag{2.15}$$

with 2×2 blocks

$$\mathcal{A}^{\varepsilon}_{mn} = \left(\begin{array}{cc} \delta_{mn} + \frac{c_{2n-1} e^{(p_{2n-1}+q_{2m-1})x + (q_{2n-1}^2 - p_{2n-1}^2)y - 4(p_{2n-1}^3 + q_{2n-1}^3)t}}{p_{2n-1}+q_{2m-1}} & \frac{c_{2n} e^{(p_{2n}+q_{2m})x + (q_{2n}^2 - p_{2n}^2)y - 4(p_{2n}^3 + q_{2n}^3)t}}{p_{2n}+q_{2m-1}} \\ \frac{c_{2n-1} e^{(p_{2n-1}+q_{2m-1})x + (q_{2n-1}^2 - p_{2n-1}^2)y - 4(p_{2n-1}^3 + q_{2n-1}^3)t}}{p_{2n-1}+q_{2m}} & \delta_{mn} + \frac{c_{2n} e^{(p_{2n}+q_{2m})x + (q_{2n}^2 - p_{2n}^2)y - 4(p_{2n}^3 + q_{2n}^3)t}}{p_{2n}+q_{2m}} \end{array} \right). \tag{2.15a}$$

where δ_{mn} are the regular Kronecker symbols. Then

$$\det \mathcal{A}^{\varepsilon} = \sum_{\sigma \in \mathcal{P}_{2N}} \iota(\sigma) \prod_{j=1}^{N} \det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}}, \tag{2.15b}$$

where $\mathcal{P}_{2N}$ is the group of permutations of $\{1, 2, \ldots, 2N\}$, $\sigma = \{k_{j1}, k_{j2}, \cdots, k_{j\ 2N}\} \in \mathcal{P}_{2N}$, $\iota(\sigma)$ is the sign of a permutation σ and $\mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}}$ are 2×2 matrices obtained by taking the elements of $\mathcal{A}^{\varepsilon}$ at the intersection of the $(2j-1)$-th and $(2j)$-th rows and k_{j1}-th and k_{j2}-th columns, $k_{j1} < k_{j2}$. The 2×2 determinants $\det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}}$ have the following asymptotic behavior as $\varepsilon \to 0$:

1) if $k_{j1} = 2j-1,\ k_{j2} = 2j$, then

$$\det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}} = 4\varepsilon \left[-\rho_j + \frac{\cos 2\Gamma_j}{2\lambda_j} \right] + O(\varepsilon^2). \tag{2.16}$$

2) if $k_{j1} = 2j-1,\ k_{j2} \neq 2j$, then

$$\det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}} = 2 \left[-\frac{(\mu_{k_{j1}} - \mu_{k_{j2}}) \cos(\Gamma_{k_{j1}} - \Gamma_{k_{j2}})}{(\mu_{k_{j1}} - \mu_{k_{j2}})^2 + (\lambda_{k_{j1}} - \lambda_{k_{j2}})^2} + \frac{(\lambda_{k_{j1}} - \lambda_{k_{j2}}) \sin(\Gamma_{k_{j1}} - \Gamma_{k_{j2}})}{(\mu_{k_{j1}} - \mu_{k_{j2}})^2 + (\lambda_{k_{j1}} - \lambda_{k_{j2}})^2} - \right.$$

$$\left. - \frac{(\mu_{k_{j1}} - \mu_{k_{j2}}) \sin(\Gamma_{k_{j1}} + \Gamma_{k_{j2}})}{(\mu_{k_{j1}} - \mu_{k_{j2}})^2 + (\lambda_{k_{j1}} + \lambda_{k_{j2}})^2} - \frac{(\lambda_{k_{j1}} + \lambda_{k_{j2}}) \cos(\Gamma_{k_{j1}} + \Gamma_{k_{j2}})}{(\mu_{k_{j1}} - \mu_{k_{j2}})^2 + (\lambda_{k_{j1}} + \lambda_{k_{j2}})^2} \right] \varepsilon +$$

$$+2i\varepsilon\left[\frac{(\lambda_{k_{j1}}-\lambda_{k_{j2}})\cos(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}+\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\sin(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}+\right.$$
$$\left.+\frac{(\lambda_{k_{j1}}+\lambda_{k_{j2}})\sin(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}-\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\cos(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}\right]+O(\varepsilon^2). \tag{2.17}$$

3) if $k_{j1}\neq 2j-1,\ k_{j2}=2j$, then

$$\det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}}=2\left[-\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\cos(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}+\frac{(\lambda_{k_{j1}}-\lambda_{k_{j2}})\sin(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}-\right.$$
$$\left.-\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\sin(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}-\frac{(\lambda_{k_{j1}}+\lambda_{k_{j2}})\cos(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}\right]\varepsilon-$$
$$-2i\varepsilon\left[\frac{(\lambda_{k_{j1}}-\lambda_{k_{j2}})\cos(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}+\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\sin(\Gamma_{k_{j1}}-\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}-\lambda_{k_{j2}})^2}+\right.$$
$$\left.+\frac{(\lambda_{k_{j1}}+\lambda_{k_{j2}})\sin(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}-\frac{(\mu_{k_{j1}}-\mu_{k_{j2}})\cos(\Gamma_{k_{j1}}+\Gamma_{k_{j2}})}{(\mu_{k_{j1}}-\mu_{k_{j2}})^2+(\lambda_{k_{j1}}+\lambda_{k_{j2}})^2}\right]+O(\varepsilon^2). \tag{2.18}$$

4) if $k_{j1}\neq 2j-1,\ k_{j2}\neq 2j$, then

$$\det \mathcal{D}^{\varepsilon}_{jk_{j1}k_{j2}}=O(\varepsilon^2), \tag{2.19}$$

where ρ_k's and Γ_k's are given by (2.11-12). For small ε we have

$$\det \mathcal{A}^{\varepsilon}=(4\varepsilon)^N\sum_{\sigma\in\mathcal{P}_n}(-1)^{\iota(\sigma)}\prod_{j=1}^{N}K_{j\sigma(j)}+\text{terms of order } o(\varepsilon^N), \tag{2.20}$$

where K_{jk} are as defined by (2.9-10).

Passing to the limit as $\varepsilon\to 0$, we obtain that $u(x,y,t)$ defined in (2.7-12) satisfies KPII for all values of parameters λ_n, μ_n, ϱ_n and γ_n. *q.e.d.*

Let us consider the simplest case of the formula (2.7-12) when $N = 1$. In this case the formula becomes:

$$u(x,y,t) =$$

$$2\frac{\partial^2}{\partial x^2}\ln\left[-\varrho_1 - \mathcal{V}(x,y) - 12t(\lambda_1^2-\mu_1^2) + \frac{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8t\lambda_1(\lambda_1^2-3\mu_1^2))}{2\lambda_1}\right] =$$

$$-8\lambda_1^2\frac{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t)}{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t) - 2\lambda_1(\varrho_1 + \mathcal{V}(x,y) + 12(\lambda_1^2-\mu_1^2)t)} -$$

$$8\lambda_1^2\left[\frac{1+\sin(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t)}{\cos(2\gamma_1 + 2\lambda_1\mathcal{V}(x,y) + 8\lambda_1(\lambda_1^2-3\mu_1^2)t) - 2\lambda_1(\varrho_1 + \mathcal{V}(x,y) + 12(\lambda_1^2-\mu_1^2)t)}\right]^2, \tag{2.21}$$

where

$$\mathcal{V}(x,y) = x - 2\mu_1 y. \tag{2.22}$$

The functions defined by (2.21) will be referred to here as *harmonic breathers.* Each harmonic breather is determined by a spectral pair (two spectral parameters) $(\lambda_1,\ \mu_1)$. For $\mu_1 = 0$ the functions defined by (2.21) are solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.23}$$

These solutions have been previously derived by a number of authors ([Nov 2], [Kov 4], [Matveev 1-2], [Stahlhofen 1]) and they have been correspondingly referred to as *harmonic breathers* or *positons.*

We are interested now to see whether the harmonic breathers defined by (2.21) are nonlinear analogues of the solutions (2.3) of (lKPII) as a result of nonlinear perturbation of (lKPII) by the term $(6uu_x)_x$. We notice that if $\varrho_1 + x - 2\mu_1 y + 12(\lambda_1^2 - \mu_1^2)t$ is large, the harmonic breathers defined by (2.21) have the following asymptotic behavior:

$$u(x,y,t) \approx 4\lambda_1 \frac{\cos(2\Gamma_1)}{\varrho_1 + x - 2\mu_1 y + 12(\lambda_1^2 - \mu_1^2)t} + O\left(\frac{1}{(\varrho_1 + x - 2\mu_1 y + 12(\lambda_1^2 - \mu_1^2)t)^2}\right). \tag{2.24}$$

The asymptotic behavior (2.24) shows that, in physical domains in which $\varrho_1 + x - 2\mu_1 y + 12(\lambda_1^2 - \mu_1^2)t$ is large, the harmonic breathers defined by (2.21) have oscillatory behavior similar to that of the functions (2.3). Therefore one can think that the harmonic breathers defined by (2.21) are nonlinear analogues of the solutions (2.3) of (lKPII) as result of nonlinear perturbation of (lKPII) by the term $(6uu_x)_x$. Indeed, let us consider in the formula (2.21) the following scaling:

$$x \to \frac{x^{(s)}}{s},\ y \to \frac{y^{(s)}}{s^2},\ t \to \frac{t^{(s)}}{s^3},$$

$$\lambda_1 \to s\lambda_1,\ \mu_1 \to s\mu_1,\ \varrho_1 \to s^2\varrho_1,\ u(x,y,t) \to \frac{u^{(s)}(x^{(s)}, y^{(s)}, t^{(s)})}{s}.$$

The functions $u^{(s)}(x^{(s)}, y^{(s)}, t^{(s)})$ satisfy the scaled KPII equation

$$\left[u^{(s)}_{t^{(s)}} + \frac{6}{s^3} u^{(s)} u^{(s)}_{x^{(s)}} + u^{(s)}_{x^{(s)}x^{(s)}x^{(s)}}\right]_{x^{(s)}} = -3u^{(s)}_{y^{(s)}y^{(s)}}, \tag{2.25}$$

and their asymptotic behavior follows from (2.24):

$$u^{(s)}(x^{(s)}, y^{(s)}, t^{(s)}) = 4\lambda_1 \frac{\cos(2\gamma_1 + 2\lambda_1 \mathcal{V}(x^{(s)}, y^{(s)}) + 8\lambda_1(\lambda_1^2 - 3\mu_1^2)t^{(s)})}{\varrho_1 + \frac{x^{(s)} - 2\mu_1 y^{(s)} + 12(\lambda_1^2 - \mu_1^2)t^{(s)}}{s^3}} +$$

$$O\left(\frac{s^3}{(s^3\varrho_1 + x^{(s)} - 2\mu_1 y^{(s)} + 12(\lambda_1^2 - \mu_1^2)t^{(s)})^2}\right). \tag{2.25a}$$

As $s \to 1$, the equation (2.25) and the formula (2.25a) become respectively the KPII equation and (2.24), whereas as $s \to +\infty$, the equation (2.25) and the formula (2.25a) become respectively the (lKPII) equation and (2.3).

From the asymptotic (2.24) we notice that the harmonic breathers have decaying oscillatory behavior whereas the solutions (2.3) of (lKPII) are not decaying. The price to pay for the decaying behavior of the harmonic breathers is the presence of the singularities within them. The singularities are movable in time for each harmonic breather regardless the spectral pair describing it. The Figure 2.5a-d illustrates a harmonic breather at two different times within the same space domain for both times.

Since all applied problems are considered in finite space-time physical domains, one may adjust the parameters of (2.21) to assure that within the physical domain, which can be considered as large as we wish but we must keep it finite, a harmonic breather stays regular. We may consider the following finite space-time physical domain

$$\Omega_{X_{1,2},Y_{1,2},T_{1,2}} = \{(t,x,y)|X_1 \leq x \leq X_2,\ Y_1 \leq y \leq Y_2,\ T_1 \leq t \leq T_2,\ X_{1,2}, Y_{1,2}, T_{1,2} \text{ finite}\}, \tag{2.26}$$

in which

$$|\varrho_1| \gg |X_{1,2}|, |Y_{1,2}|, |T_{1,2}|, \frac{1}{2\lambda_1}. \tag{2.26a}$$

Then, within $\Omega_{X_{1,2},Y_{1,2},T_{1,2}}$, we have a non-singular part of the harmonic (2.21) (we shall call it tail of the harmonic). For example, in the domain

$$\Omega_{10,10,0..T} = \{(t,x,y) | |x| \leq 10,\ |y| \leq 10,\ 0 \leq t < T,\ T \text{ finite}\}, \tag{2.27}$$

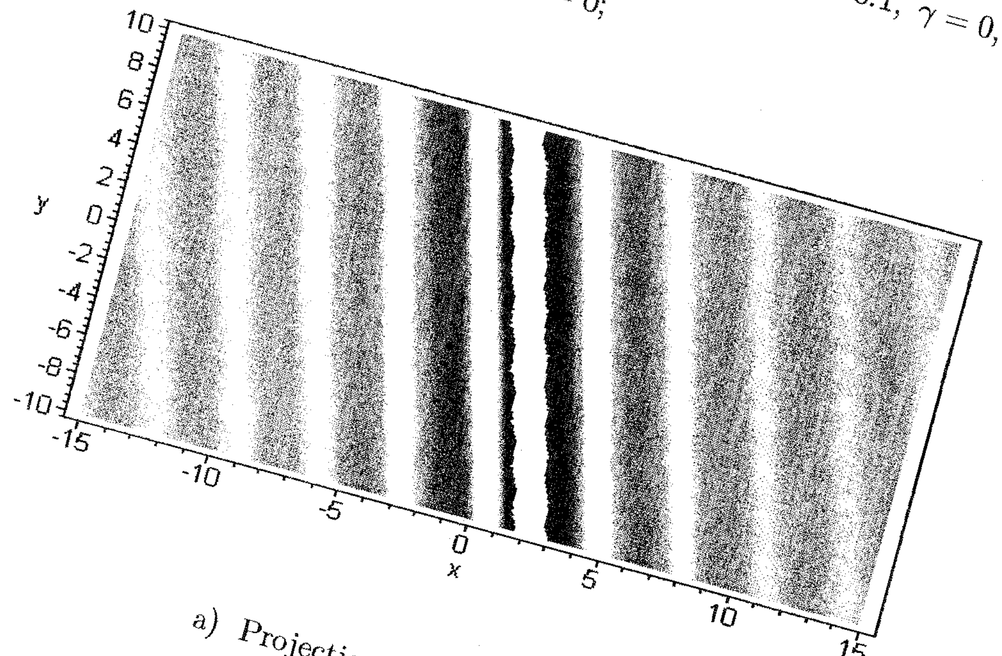
we have a non-singular part (tail) of the harmonic breather (2.21) with $\lambda_1 = 1$, $\mu = 0.1$ and $\varrho = 15$ (Figure 2.6a-d).

The decaying behavior of a harmonic breather defined by (2.21), away from its singularities, is shown in the Figure 2.7a-d.

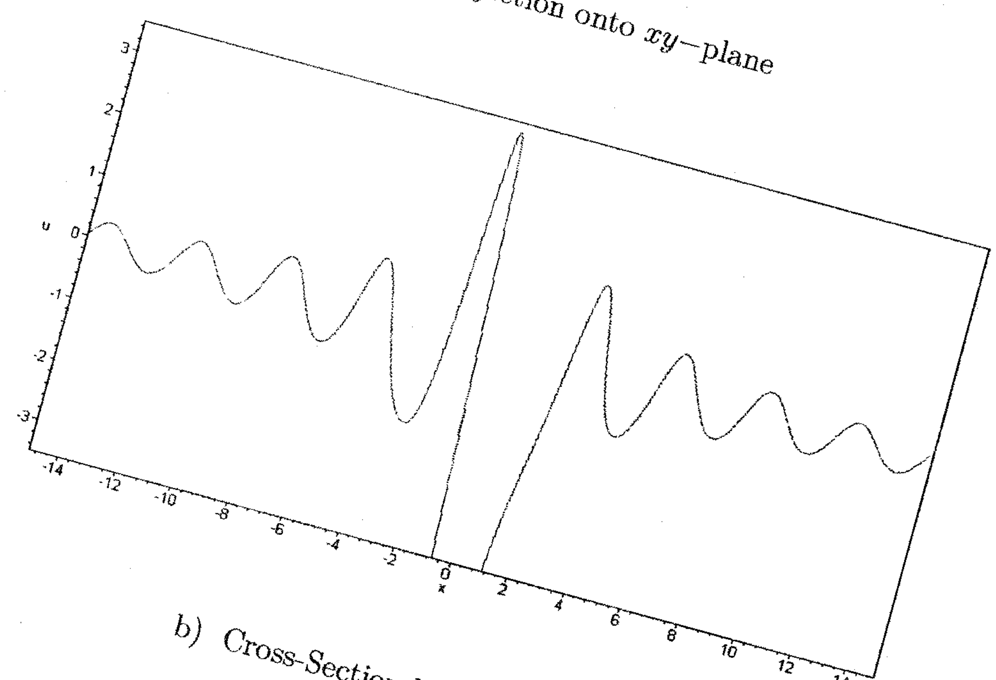
The asymptotic behavior (2.24) of the harmonic breathers defined by (2.21) and the nonlinear analogy of the harmonics with the solutions (2.3) of the (lKPII) equation suggest that we may use the harmonic breathers to construct more complex solutions of the KPII equation as the solutions (2.3) may be used in constructing more complex solutions of the (lKPII) equation. In the next section we shall present how we can do this by using the formula (2.7-12) for $N > 1$.

For $N > 1$, the formula (2.7-12) defines a nonlinear superposition of solutions of the KPII equation that describes the nonlinear interaction of the KPII harmonic breathers. This formula will be referred to as the N-harmonic breather solution of the KPII equation. The case $N = 1$ in the formula (2.7-12) was also obtained in [Cho 1]. Superposition of two harmonic breathers with distinct spectral pairs is shown in Figure 2.8a-d.

Figure 2.5a-b: KPII Harmonic Breather with $\lambda = 1$, $\mu = -0.1$, $\gamma = 0$, $\varrho = 0$ at $t = 0$;



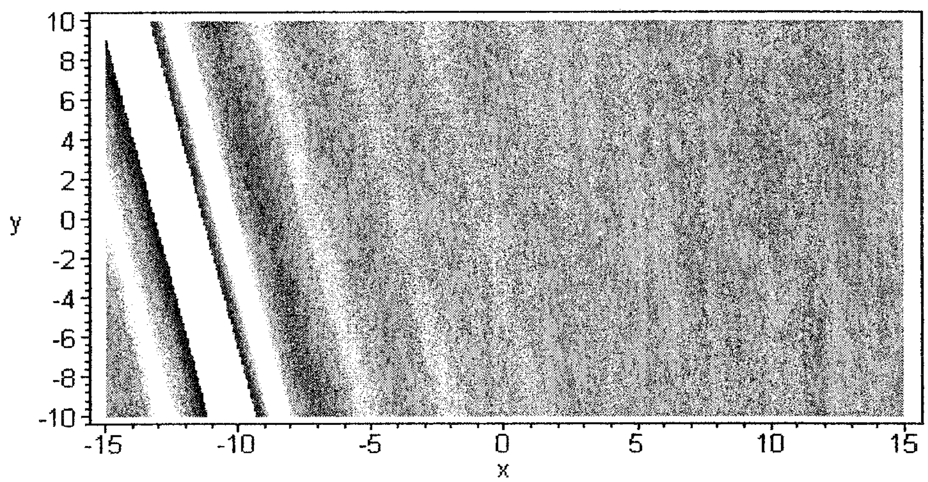
a) Projection onto xy–plane



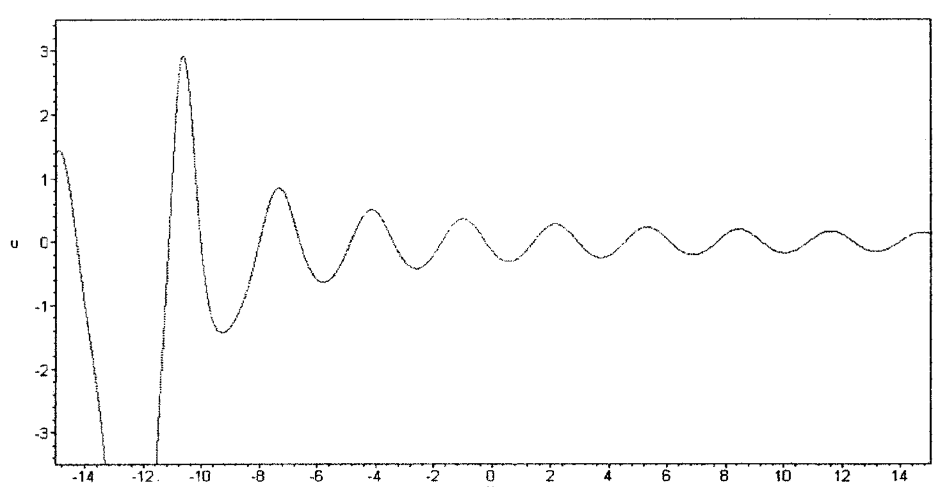
b) Cross-Section by plane $y = 1$.

Figure 2.5c-d: KPII Harmonic Breather with $\lambda = 1,\ \mu = -0.1,\ \gamma = 0,\ \varrho = 0$ at

$t = 1;$



a) Projection onto xy–plane



b) Cross-Section by plane $y = 1$.

Figure 2.6a-b: KPII Harmonic Breather in $\Omega_{10,10,0..T}$ with $\lambda = 1$,

$\mu = 0.1$, $\gamma = 0$, $\varrho = 15$ at (a) at $t = 0$ and b) at $t = 5$);

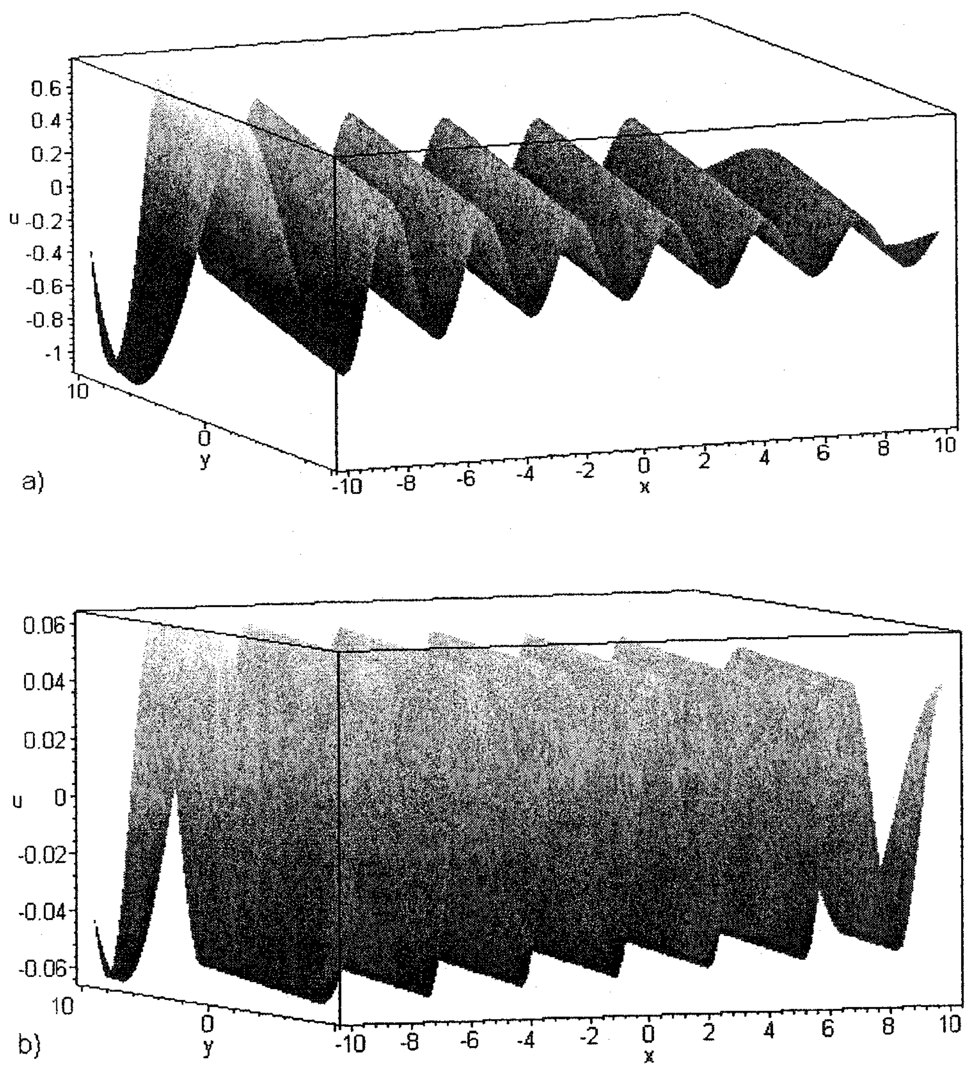


Figure 2.6c-d: KPII Harmonic Breather in $\Omega_{10,10,0..T}$ with $\lambda = 1$, $\mu = 0.1$, $\gamma = 0$, $\varrho = 15$ at (a) at $t = 0$ and b) at $t = 5$). Cross-Section by plane $y = 1.$;

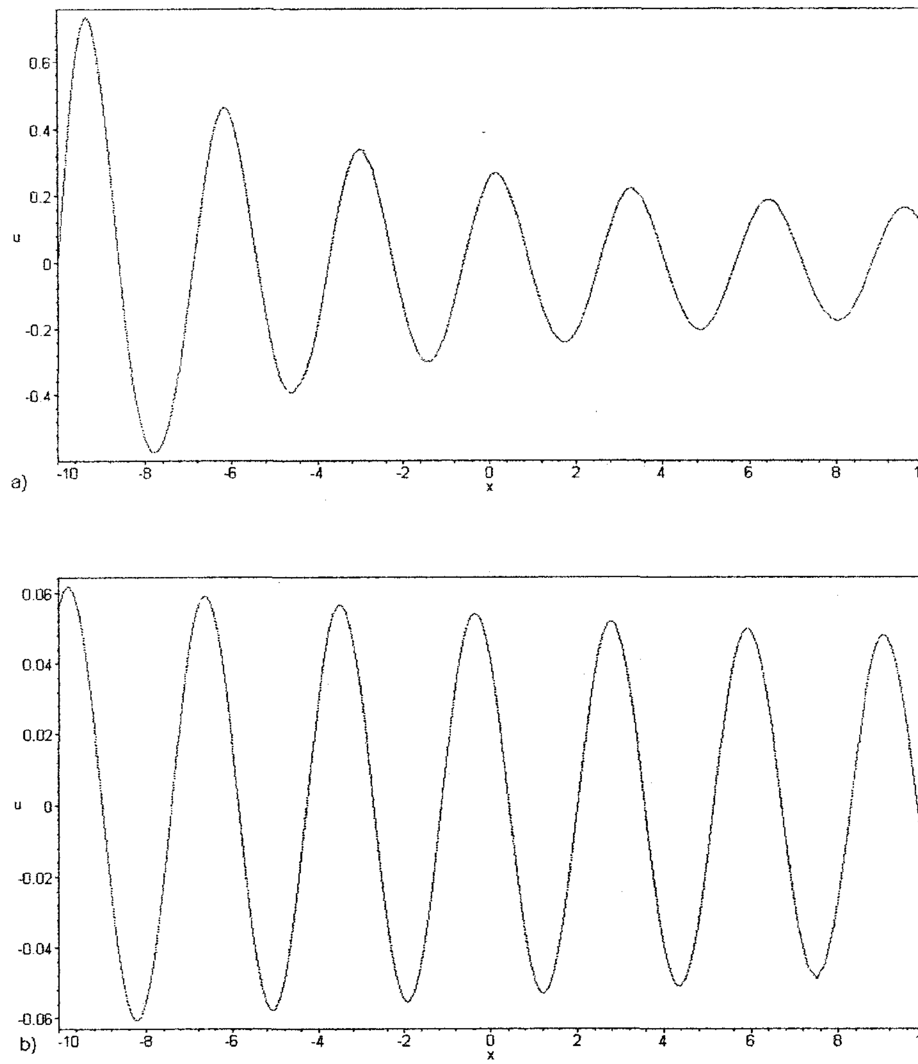
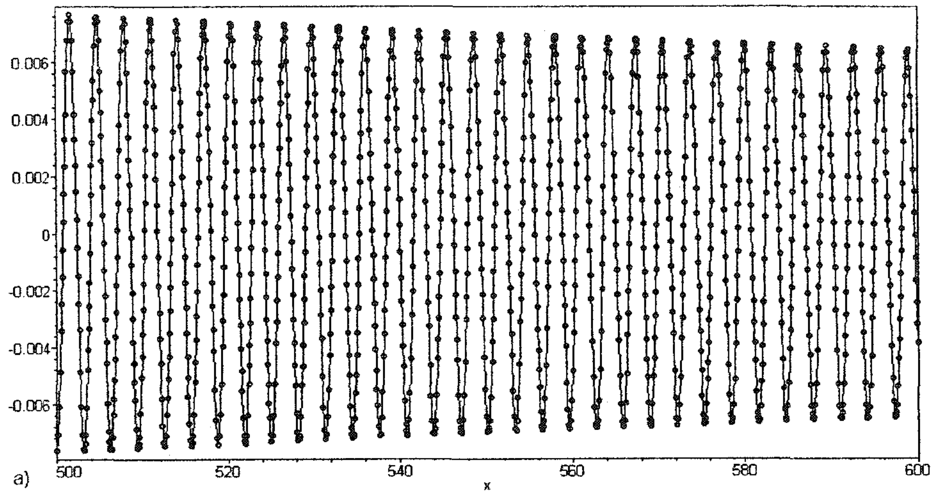


Figure 2.7a-b: Large distance behavior of the KPII Harmonic Breather with $\lambda = 1$, $\mu = -0.1$, $\gamma = 0$, $\varrho = 0$ (snapshots at $t = 0$). Cross-Section by plane $y = 100$. The solid line is the harmonic and the dot line is the asymptotic (2.24).



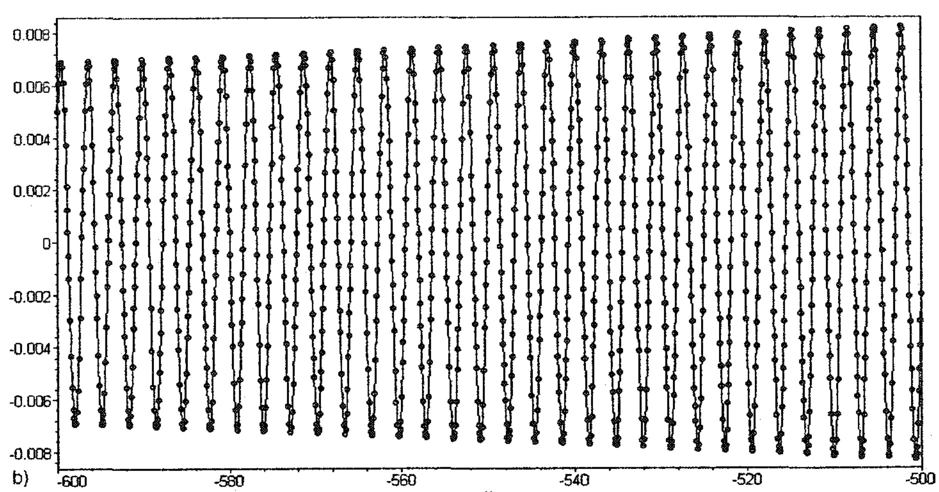
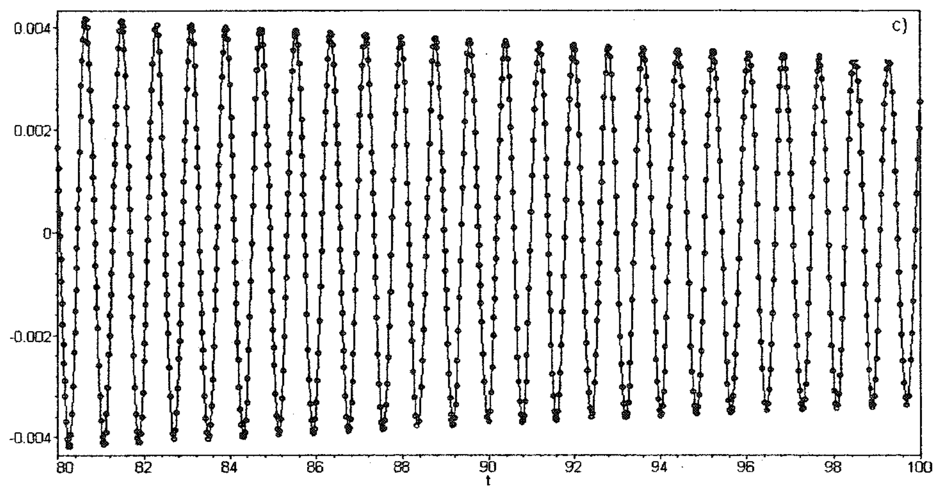


Figure 2.7c-d: Large time behavior of the KPII Harmonic Breather with $\lambda = 1$, $\mu = -0.1$, $\gamma = 0$, $\varrho = 0$. The solid line represents the decaying oscillations of the harmonic as it passes through $x = 1$, $y = 1$ and the dot line is the asymptotic (2.24).



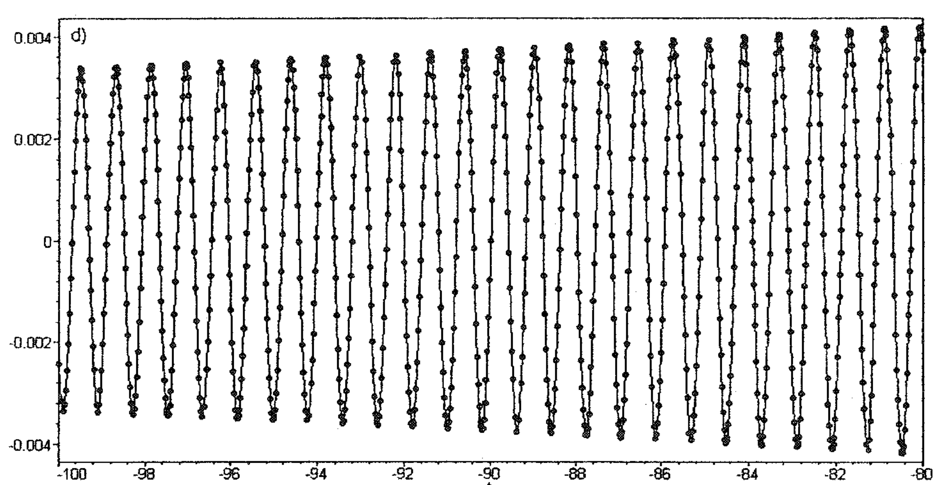


Figure 2.8a-b: Superposition of two KPII Harmonic Breathers with $\lambda = [1, 1]$, $\mu = [0.2, -0.2]$, $\gamma = [0, 0]$, $\varrho = [0, 0]$ at $t = 0$ (a)-projection onto xy-plane and b)-cross-section by plane $y = 10$).

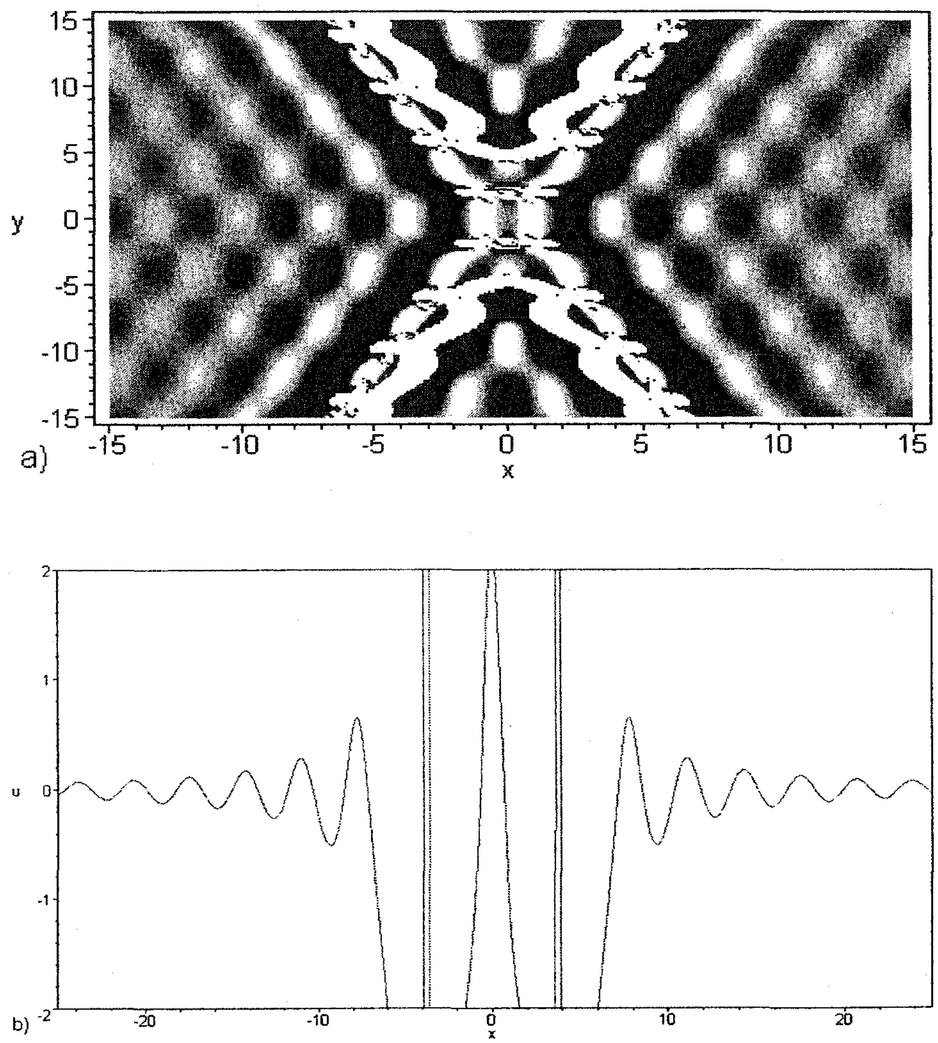
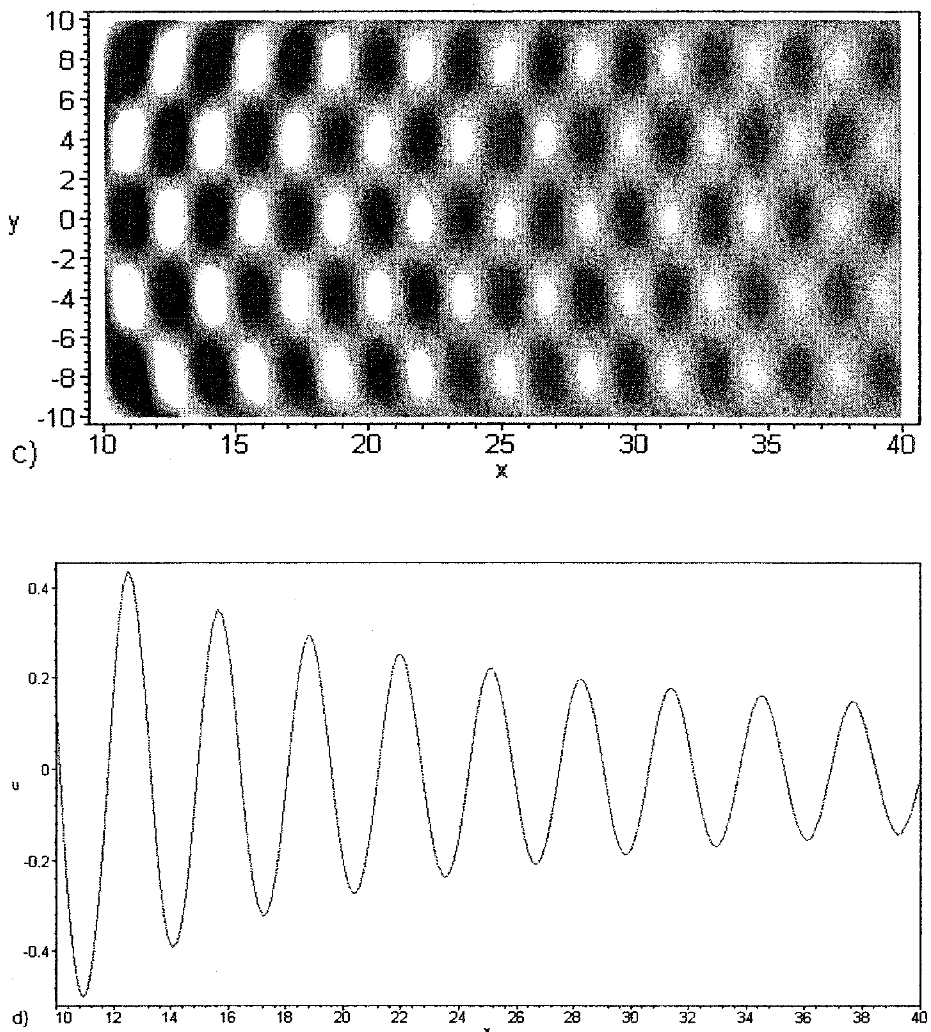


Figure 2.8c-d: Non-singular part of the superposition of two KPII Harmonic Breathers with $\lambda = [1, 1]$, $\mu = [0.2, -0.2]$, $\gamma = [0, 0]$, $\varrho = [0, 0]$ at $t = 0$ (c)-projection onto $xy-$plane and d)-cross-section by plane $y = 1$).



2.4 Nonlinear Interference of Harmonic Breathers of KPII Equation.

Let us consider the N-harmonic breather solution (2.7-12) in the physical domain

$$\Omega_{X,Y,T} = \{(t,x,y) \| x| \leq X,\ |y| \leq Y,\ |t| \leq T,\ X,Y,T \text{ finite}\}, \tag{2.28}$$

such that the ϱ_n's satisfy

$$|\varrho_n| \gg X, Y, T, \frac{1}{2\lambda_n},\ n = 1 \dots N \tag{2.29}$$

$$|\varrho_n| \gg \text{all } K_{nm},\ n \neq m,\ n,m = 1 \dots N^1 \tag{2.30}$$

i.e. the ϱ_n's are dominant elements of the matrix $\mathcal{K}$. Then in $\Omega_{X,Y,T}$ the function $u(x,y,t)$ can be represented by

$$u(x,y,t) \sim \sum_{n=1}^{N} \frac{4\lambda_n}{\varrho_n + x - 2\mu_n y + 12(\lambda_n^2 - \mu_n^2)t} \cos(2\Gamma_n) + O\left(\frac{1}{(\varrho_n + x - 2\mu_n y + 12(\lambda_n^2 - \mu_n^2)t)^2}\right). \tag{2.31}$$

which, due to the condition (2.29), can be further approximated by

$$u(x,y,t) \sim \sum_{n=1}^{N} \frac{4\lambda_n}{\varrho_n} \cos(2\Gamma_n) + O\left(\frac{1}{\varrho_n^2}\right). \tag{2.32}$$

The formula (2.32) resembles the interference case from the linear theory of superposition and interference of waves (Figure 2.9) ([Feynman 1]). It suggests

[1]Observation: Under the conditions (2.29-30), in the domain (2.28) we have only tails of harmonic breathers.

that the tails of the harmonic breathers form some sort of nonlinear Fourier series, and under appropriate conditions can lead to some sort of nonlinear analogue of the Fourier integral. For this we need to see what is happening if $\lambda_n - \lambda_m$ and $\mu_n - \mu_m$ (in the formula (2.7-12)) are small, because in this case the condition (2.30) may no longer hold. We shall show that if $\lambda_n - \lambda_m$ and $\mu_n - \mu_m$ are small enough then a nonlinear analogue of interference from the linear theory of superposition and interference of waves will happen.

For the beginning let us consider the superposition of two harmonic breathers ($N = 2$ in the formula (2.7-12))

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[\left(-\rho_1 + \frac{\cos 2\Gamma_1}{2\lambda_1}\right)\left(-\rho_2 + \frac{\cos 2\Gamma_2}{2\lambda_2}\right) + \right.$$
$$+\left(\frac{(\mu_1-\mu_2)\sin(\Gamma_1+\Gamma_2)}{(\mu_1-\mu_2)^2+(\lambda_1+\lambda_2)^2} + \frac{(\mu_1-\mu_2)\cos(\Gamma_1-\Gamma_2)}{(\mu_1-\mu_2)^2+(\lambda_1-\lambda_2)^2}\right)^2 -$$
$$\left. - \left(\frac{(\lambda_1+\lambda_2)\cos(\Gamma_1+\Gamma_2)}{(\mu_1-\mu_2)^2+(\lambda_1+\lambda_2)^2} - \frac{(\lambda_1-\lambda_2)\sin(\Gamma_1-\Gamma_2)}{(\mu_1-\mu_2)^2+(\lambda_1-\lambda_2)^2}\right)^2\right], \tag{2.33}$$

and let us consider the following asymptotic behavior as $\varepsilon \to 0$:

$$\lambda_2 = \lambda_1 + \varepsilon,\ \mu_2 = \mu_1 + a\varepsilon^l,\ \gamma_2 = \gamma_1 + b\varepsilon^r \quad a, b \in \mathbf{R}\backslash\{0\},\ 1 \geq 2,\ \mathrm{r} \geq 1. \tag{2.33a}$$

Then, the harmonic breathers experience a different kind of interaction, i.e. from their interaction results a single harmonic breather given by the formula:

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[\left(-\rho_1 + \frac{\cos 2\Gamma_1}{2\lambda_1}\right)(2b - \varrho_1 - \varrho_2) + a^2 - (b - \varrho_1)^2\right], \tag{2.33b}$$

which can be simplified to

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[-\varrho_{12} - x + 2\mu_1 y - 12t(\lambda_1^2 - \mu_1^2) + \frac{\cos 2\Gamma_1}{2\lambda_1}\right], \tag{2.33c}$$

where ϱ_{12} is given by

$$\frac{1}{\varrho_{12} - b} = \frac{(\varrho_1 - b) + (\varrho_2 - b)}{(\varrho_1 - b)(\varrho_2 - b) + a^2} \quad \text{if } l = 2,\ r = 1, \tag{2.33d}$$

which, if $\varrho_1, \varrho_2 \gg a, b$, simplifies to

$$\frac{1}{\varrho_{12}} \approx \frac{1}{\varrho_1} + \frac{1}{\varrho_2}. \tag{2.33e}$$

and

$$\frac{1}{\varrho_{12}} = \frac{1}{\varrho_1} + \frac{1}{\varrho_2} \quad \text{if } l > 2,\ r > 1. \tag{2.33f}$$

In the physical domain (2.28) under the condition (2.29), the formula (2.33c) can be approximated by

$$u(x,y,t) \sim \frac{4\lambda_1}{\varrho_{12}}\cos(2\Gamma_1) + O\left(\frac{1}{\varrho_{12}^2}\right). \tag{2.34}$$

Remark

The formula (2.34) resembles the interference case from the linear theory of superposition and interference of waves (Figure 2.10).

For the general case we have the following result:

Theorem 2

If in the N-harmonic breather solution (2.7-12) we have the following asymptotic behavior as $\varepsilon \to 0$

$$\lambda_N = \lambda_{N-1} + \varepsilon,\ \mu_N = \mu_{N-1} + a\varepsilon^l,\ \gamma_N = \gamma_{N-1} + b\varepsilon^r \quad a, b \in \mathbf{R}\backslash\{0\},\ \mathrm{l} \geq 2,\ \mathrm{r} \geq 1, \tag{2.35}$$

then

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \det \begin{pmatrix} K_{11} & K_{12} & \cdots & \cdots & K_{1\ N-1} \\ K_{21} & K_{22} & \cdots & \cdots & K_{2\ N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{N-1\ 1} & K_{N-1\ 2} & \cdots & \cdots & \tilde{K}_{N-1\ N-1} \end{pmatrix}, \tag{2.36}$$

where

$$\tilde{K}_{N-1\ N-1} = -\varrho_{N-1,N} - x + 2\mu_{N-1}y - 12t(\lambda_{N-1}^2 - \mu_{N-1}^2) + \frac{\cos 2\Gamma_{N-1}}{2\lambda_{N-1}}, \tag{2.37}$$

$$\Gamma_{N-1} = \gamma_{N-1} + \lambda_{N-1}x - 2\lambda_{N-1}\mu_{N-1}y + 4t\lambda_{N-1}(\lambda_{N-1}^2 - 3\mu_{N-1}^2), \tag{2.38}$$

$$\frac{1}{\varrho_{N-1,N} - b} = \frac{(\varrho_{N-1} - b) + (\varrho_N - b)}{(\varrho_{N-1} - b)(\varrho_N - b) + a^2} \quad \text{if } l = 2,\ r = 1, \tag{2.39}$$

which, if $\varrho_{N-1}, \varrho_N \gg a, b$, simplifies to

$$\frac{1}{\varrho_{N-1,N}} \approx \frac{1}{\varrho_{N-1}} + \frac{1}{\varrho_N}. \tag{2.40}$$

and

$$\frac{1}{\varrho_{N-1,N}} = \frac{1}{\varrho_{N-1}} + \frac{1}{\varrho_N} \quad \text{if } l > 2,\ r > 1. \tag{2.41}$$

Proof

Using properties of determinants, $\det \mathcal{K}$ from the N-harmonic breather solution (2.7-12) becomes

$\det \mathcal{K} =$

$$= \det \begin{pmatrix} K_{11} & K_{12} & \dots & K_{1\ N-1} & K_{1\ N} - K_{1\ N-1} \\ K_{21} & K_{22} & \dots & K_{2\ N-1} & K_{2\ N} - K_{2\ N-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ K_{N-1\ 1} & K_{N-1\ 2} & \dots & K_{N-1\ N-1} & K_{N-1\ N} - K_{N-1\ N-1} \\ K_{N\ 1} - K_{N-1\ 1} & K_{N\ 2} - K_{N-1\ 2} & \dots & K_{N\ N-1} - K_{N-1\ N-1} & K_{N\ N} + K_{N-1\ N-1} - \\ & & & & K_{N-1\ N} - K_{N\ N-1} \end{pmatrix} =$$

$$= \det \begin{pmatrix} K_{11} & K_{12} & \dots & K_{1\ N-1} & K_{1\ N} - K_{1\ N-1} \\ K_{21} & K_{22} & \dots & K_{2\ N-1} & K_{2\ N} - K_{2\ N-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ K_{N-1\ 1} - Q_1 & K_{N-1\ 2} - Q_2 & \dots & K_{N-1\ N-1} - Q_{N-1} & 0 \\ K_{N\ 1} - K_{N-1\ 1} & K_{N\ 2} - K_{N-1\ 2} & \dots & K_{N\ N-1} - K_{N-1\ N-1} & K_{N\ N} + K_{N-1\ N-1} - \\ & & & & K_{N-1\ N} - K_{N\ N-1} \end{pmatrix}, \tag{2.42}$$

where

$$Q_i = \frac{(K_{N\ i} - K_{N-1\ i})(K_{N-1\ N} - K_{N-1\ N-1})}{K_{N\ N} + K_{N-1\ N-1} - K_{N-1\ N} - K_{N\ N-1}}, \qquad i = 1, \dots, N-1. \tag{2.43}$$

As $\varepsilon \to 0$ we have

$$\lim_{\varepsilon -> 0} (K_{i\ N} - K_{i\ N-1}) = 0, \qquad i = 1, \dots, N-2. \tag{2.44}$$

$$\lim_{\varepsilon -> 0} (K_{N\ i} - K_{N-1\ i}) = 0, \qquad i = 1, \dots, N-2. \tag{2.45}$$

$$\lim_{\varepsilon -> 0} (K_{N\ N} + K_{N-1\ N-1} - K_{N-1\ N} - K_{N\ N-1}) = A - \varrho_{N-1} - \varrho_N, \tag{2.46}$$

where $A = 2b$ if $l = 2,\ r = 1$, and $A = 0$ if $l > 2,\ r > 1$. From (2.45) and (2.46) we have

$$\lim_{\varepsilon -> 0} Q_i = 0, \qquad i = 1, \dots, N-2. \tag{2.47}$$

As $\varepsilon \to 0$ we have as well

$$\lim_{\varepsilon->0}(K_{N-1\ N-1} - Q_{N-1}) = \lim_{\varepsilon->0} \frac{K_{N-1\ N-1}K_{N\ N} - K_{N\ N-1}K_{N-1\ N}}{K_{N\ N} + K_{N-1\ N-1} - K_{N-1\ N} - K_{N\ N-1}} =$$

$$= \begin{cases} -\rho_{N-1} + \frac{\cos 2\Gamma_{N-1}}{2\lambda_{N-1}} + \frac{a^2-(b-\varrho_{N-1})^2}{2b-\varrho_{N-1}-\varrho_N}, & \text{if } l=2,\ r=1 \\ \\ -\rho_{N-1} + \frac{\cos 2\Gamma_{N-1}}{2\lambda_{N-1}} + \frac{\varrho_{N-1}^2}{\varrho_{N-1}+\varrho_N}, & \text{if } l>2,\ r>1 \end{cases} =$$

$$= -\varrho_{N-1,N} - x + 2\mu_{N-1}y - 12t(\lambda_{N-1}^2 - \mu_{N-1}^2) + \frac{\cos 2\Gamma_{N-1}}{2\lambda_{N-1}}, \tag{2.48}$$

$$\frac{1}{\varrho_{N-1,N} - B} = \frac{(\varrho_{N-1} - B) + (\varrho_N - B)}{(\varrho_{N-1} - B)(\varrho_N - B) + C}, \tag{2.49}$$

where $B = b$ if $l = 2,\ r = 1$, and $B = 0$ if $l > 2,\ r > 1$ and as well $C = a^2$ if $l = 2,\ r = 1$, and $C = 0$ if $l > 2,\ r > 1$. From (2.42),(2.44) and (2.46-48) we have

$$\lim_{\varepsilon->0} \det \mathcal{K} =$$

$$= (A - \varrho_{N-1} - \varrho_N) \det \begin{pmatrix} K_{11} & K_{12} & \dots & K_{1\ N-2} & K_{1\ N-1} \\ K_{21} & K_{22} & \dots & K_{2\ N-2} & K_{2\ N-1} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ K_{N-1\ 1} & K_{N-1\ 2} & \dots & K_{N-1\ N-2} & \tilde{K}_{N-1\ N-1} \end{pmatrix} \tag{2.50}$$

where

$$\tilde{K}_{N-1\ N-1} = -\varrho_{N-1,N} - x + 2\mu_{N-1}y - 12t(\lambda_{N-1}^2 - \mu_{N-1}^2) + \frac{\cos 2\Gamma_{N-1}}{2\lambda_{N-1}}. \tag{2.51}$$

Hence, from (2.50) we have that as $\varepsilon \to 0$ the N-harmonic breather solution (2.7-12) will become

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\det\begin{pmatrix} K_{11} & K_{12} & \dots\ \dots & K_{1\ N-1} \\ K_{21} & K_{22} & \dots\ \dots & K_{2\ N-1} \\ \dots & \dots & \dots\ \dots & \dots \\ \dots & \dots & \dots\ \dots & \dots \\ \dots & \dots & \dots\ \dots & \dots \\ K_{N-1\ 1} & K_{N-1\ 2} & \dots\ \dots & \tilde{K}_{N-1\ N-1} \end{pmatrix}, \tag{2.52}$$

because $A - \varrho_{N-1} - \varrho_N$ is a constant and will be canceled by the logarithmic differentiation. The formula (2.52) is exactly (2.36). *q.e.d.*

In the physical domain (2.28) under the condition (2.29), the formula (2.36) is a nonlinear analogue of interference case from the linear theory of superposition and interference of waves, and we shall call the phenomenon happening here *the nonlinear interference property* of the harmonic breathers.

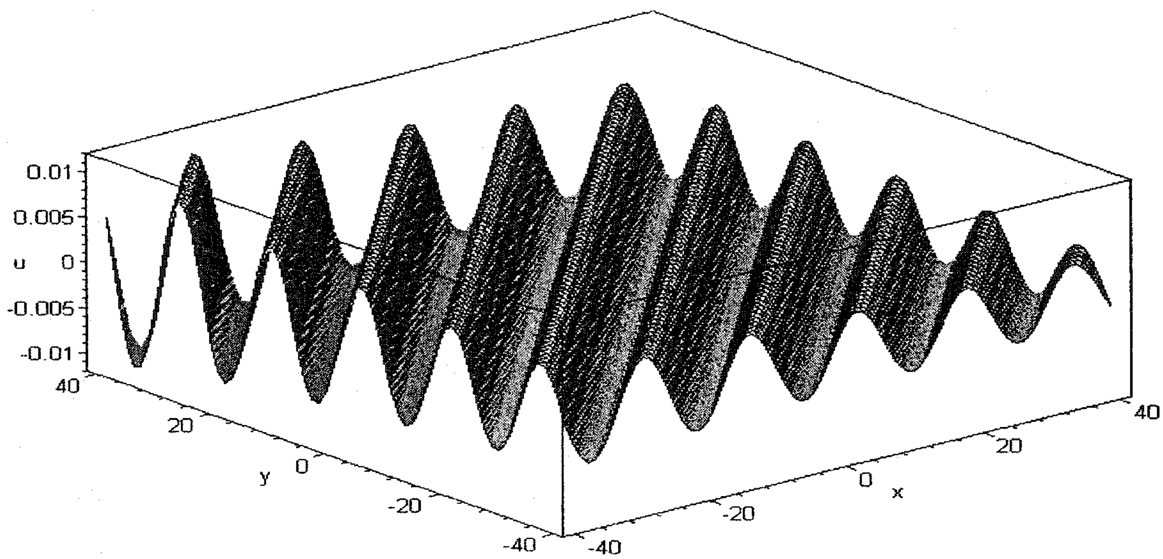
Conclusion of the section

The nonlinear interference property of the harmonic breathers suggests that the formula (2.7-12) can be used to construct solutions of KPII possessing certain properties, such as localization within the physical domain (2.28), $\Omega_{X,Y,T}$. Due to the presence of singularities, the formula (2.7-12) cannot always be used to construct solutions of KPII with required properties in all of $\Omega_{\infty,\infty,\infty} = \{(t,x,y)||x| \leq$

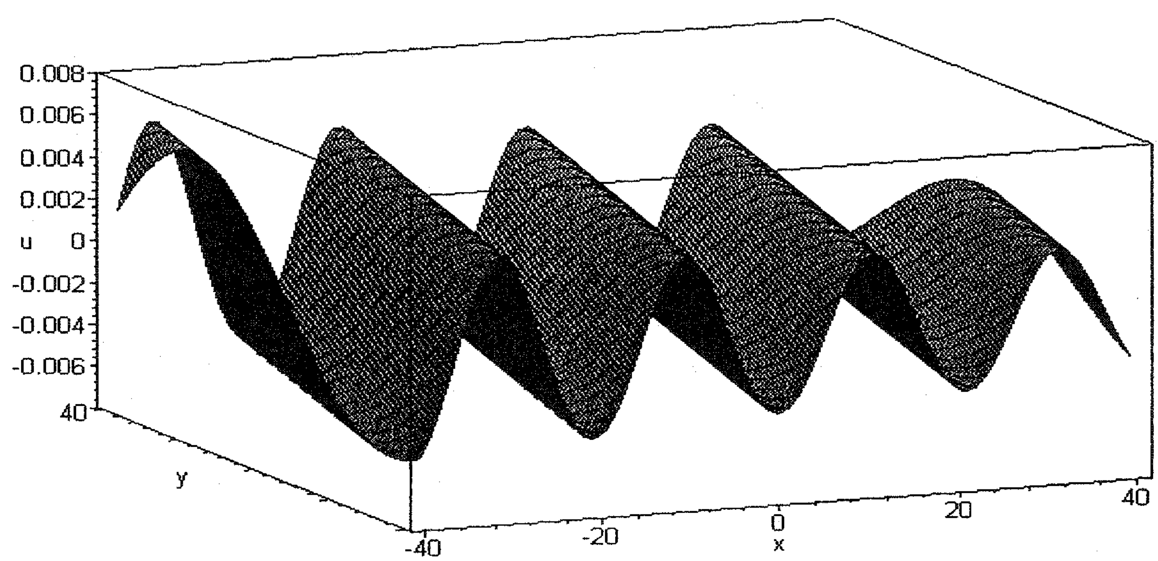
$\infty,\ |y| \leq \infty,\ |t| \leq \infty\}$. Instead, for a given triplet (X, Y, T), the formula (2.7-12) can be used to construct solutions with given properties restricted to a bounded domain $\Omega_{X,Y,T}$.

Figure 2.9a-c Nonlinear interference of two KPII harmonic breathers with

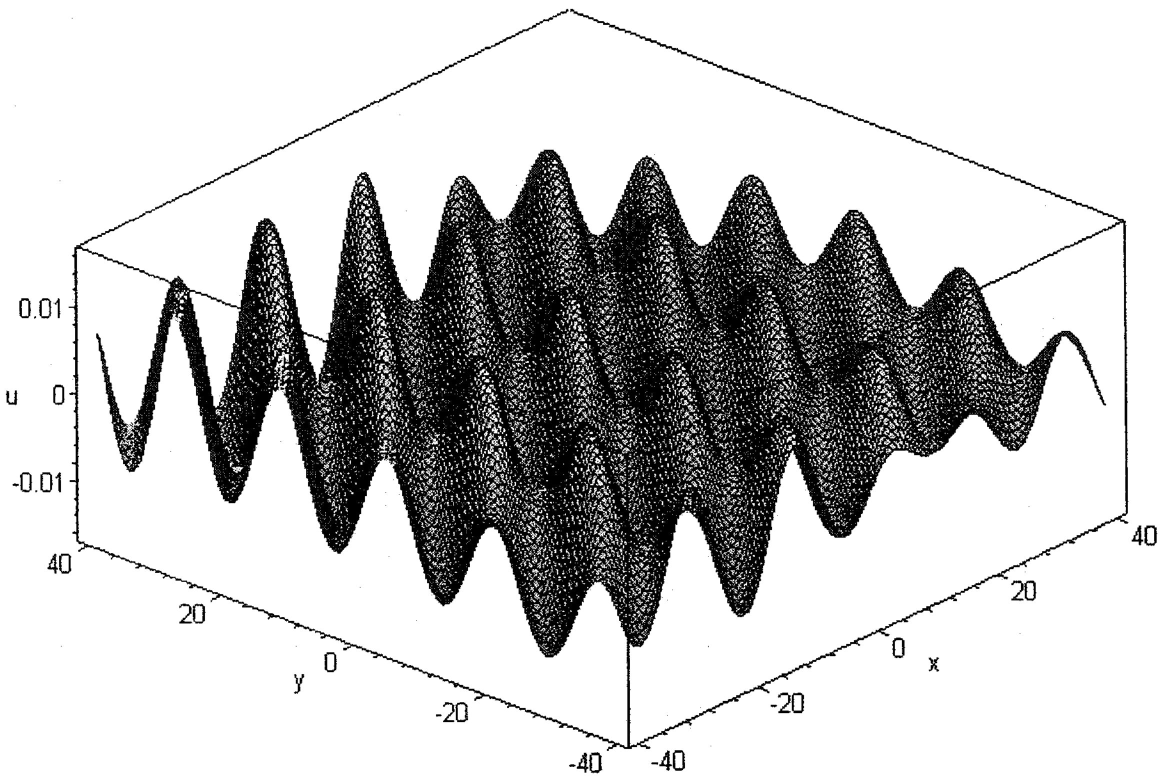
$\lambda = [0.2, 0.15]$, $\mu = [0.5, 0.1]$, $\gamma = [\pi/4, -\pi/3]$, $\varrho = [150, 150]$ at $t = 0$.



a) KPII harmonic breather 1

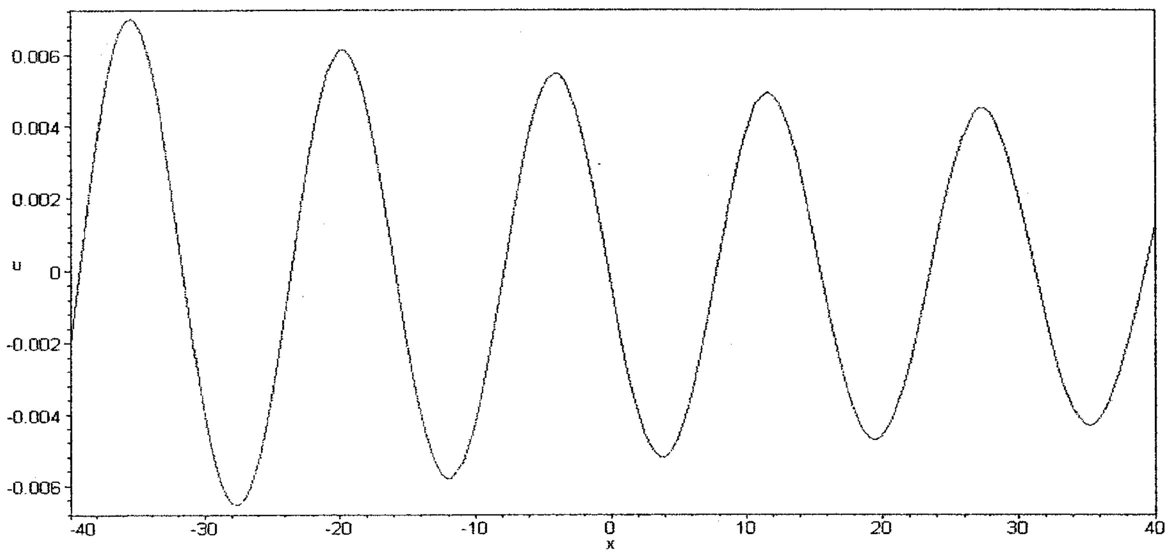


b) KPII harmonic breather 2

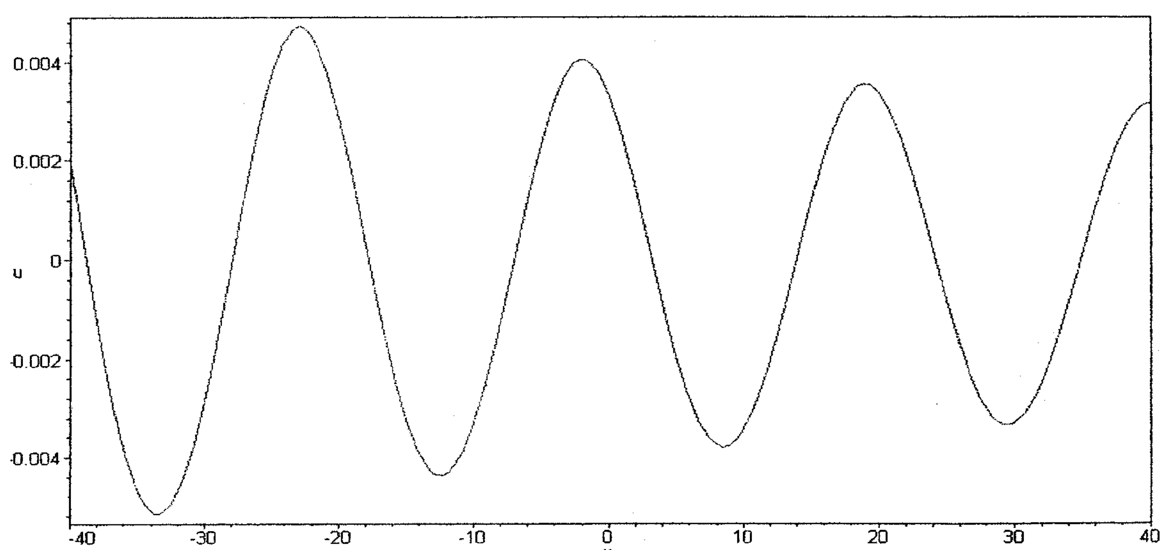


c) Nonlinear interference of KPII harmonic breathers 1 and 2 given by the formula (2.7-12) for $N = 2$.

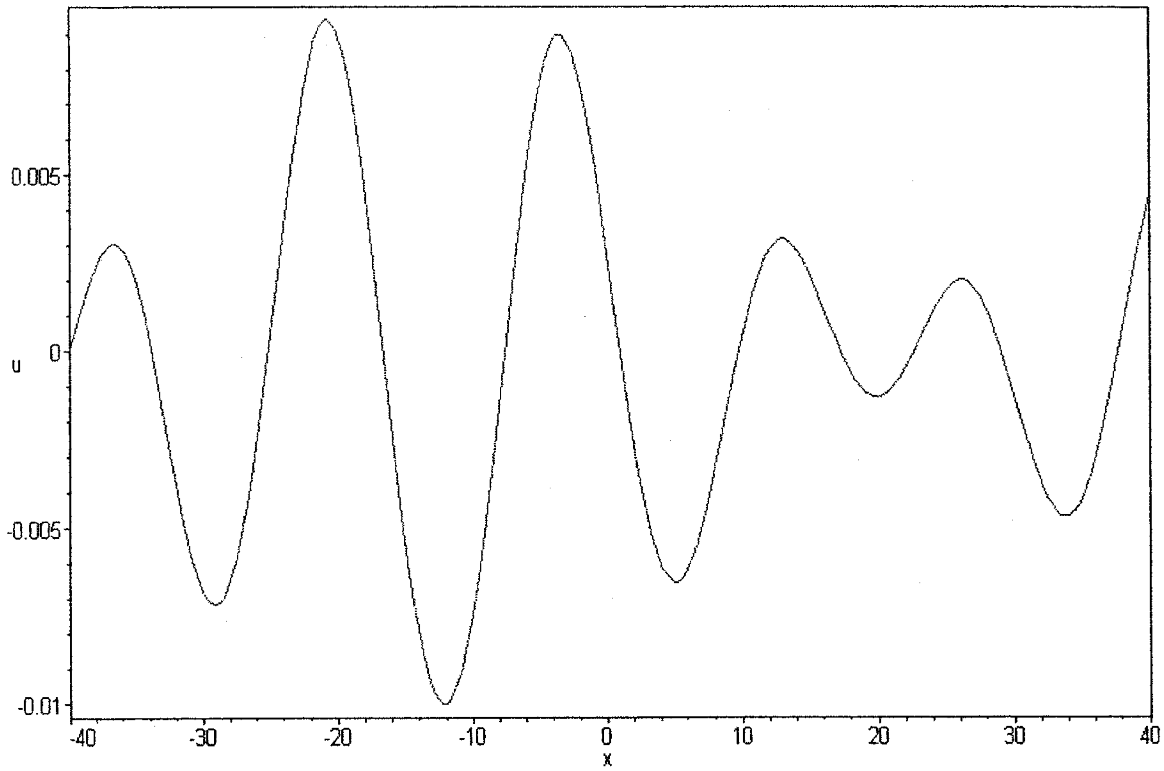
Figure 2.9d-f Nonlinear interference of two KPII harmonic breathers with $\lambda = [0.2, 0.15]$, $\mu = [0.5, 0.1]$, $\gamma = [\pi/4, -\pi/3]$, $\varrho = [150, 150]$ at $t = 0$. Cross section by plane $y = 0$.



d) KPII harmonic breather 1

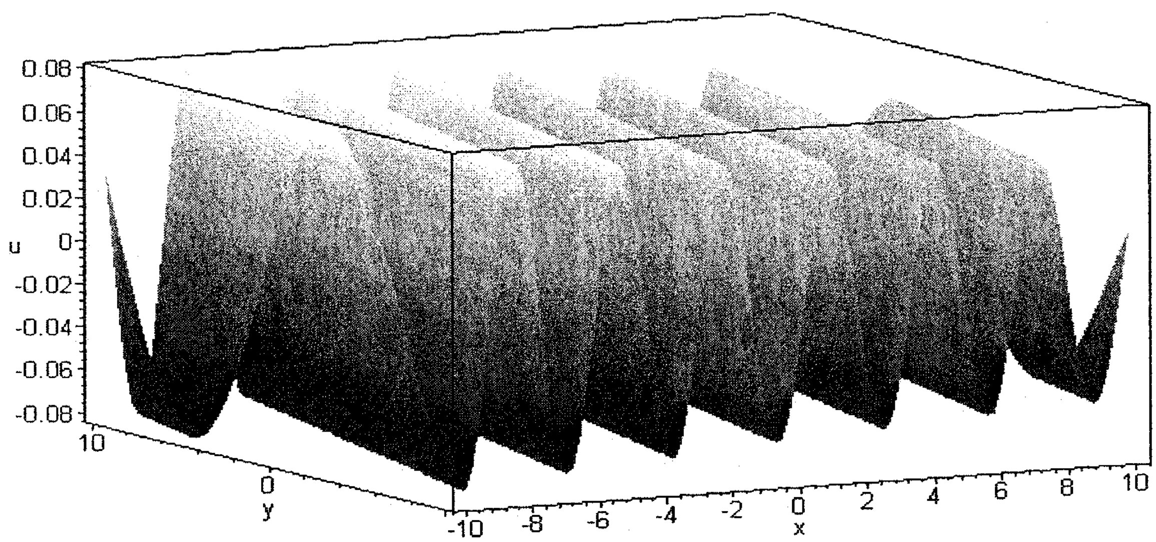


e) KPII harmonic breather 2

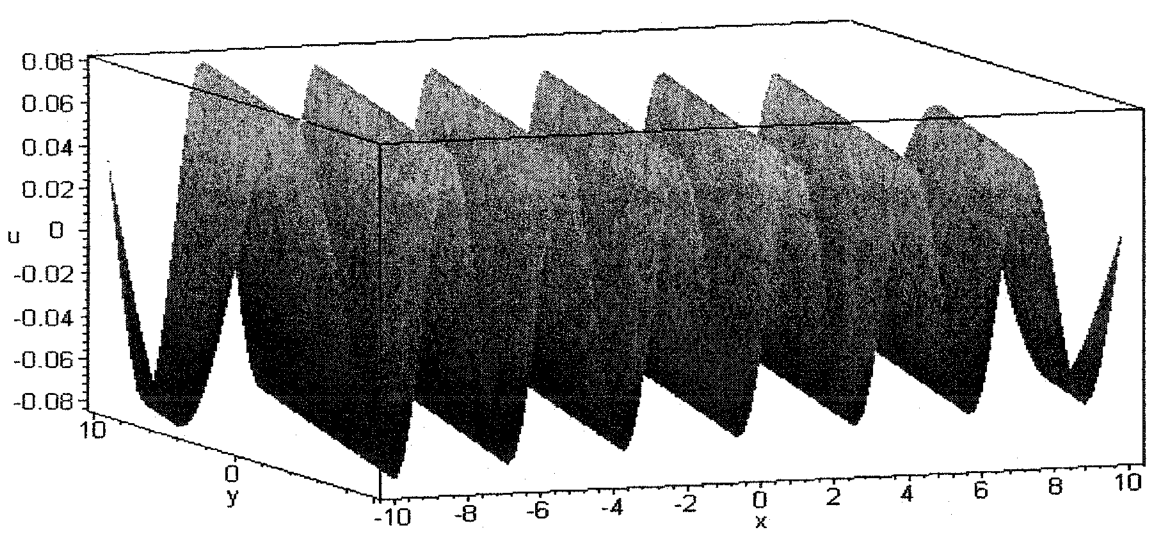


f) Nonlinear interference of KPII harmonic breathers 1 and 2 given by the formula (2.7-12) for $N = 2$. Cross section by plane $y = 0$.

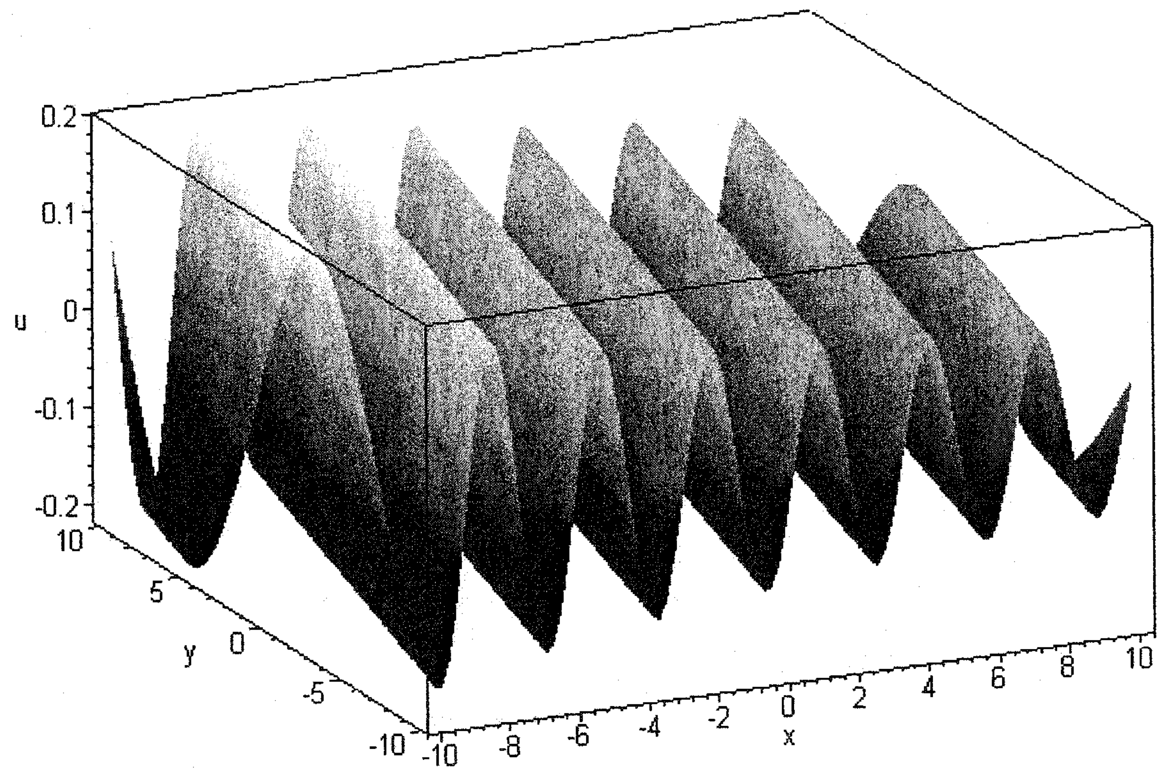
Figure 2.10a-c Nonlinear interference of two KPII harmonic breathers in $\Omega_{10,10,0..T}$ with $\lambda = [1, 1+\varepsilon]$, $\mu = [0.1, 0.1+\varepsilon^2]$, $\gamma = [0, \varepsilon]$, $\varrho = [60, 60]$, $\varepsilon = 10^{-3}$ at $t = 0$.



a) KPII harmonic breather 1

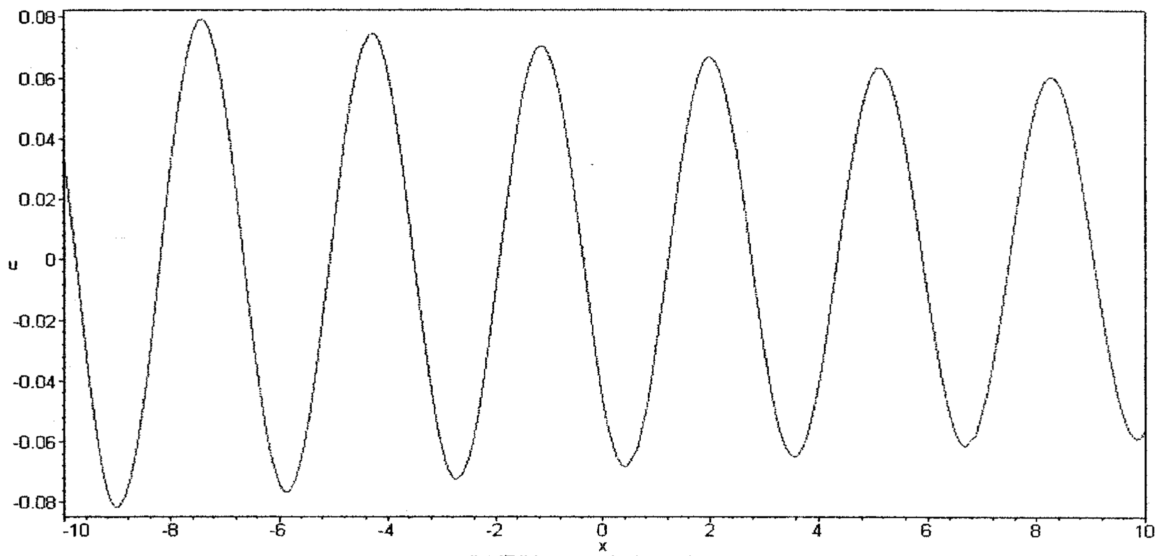


b) KPII harmonic breather 2

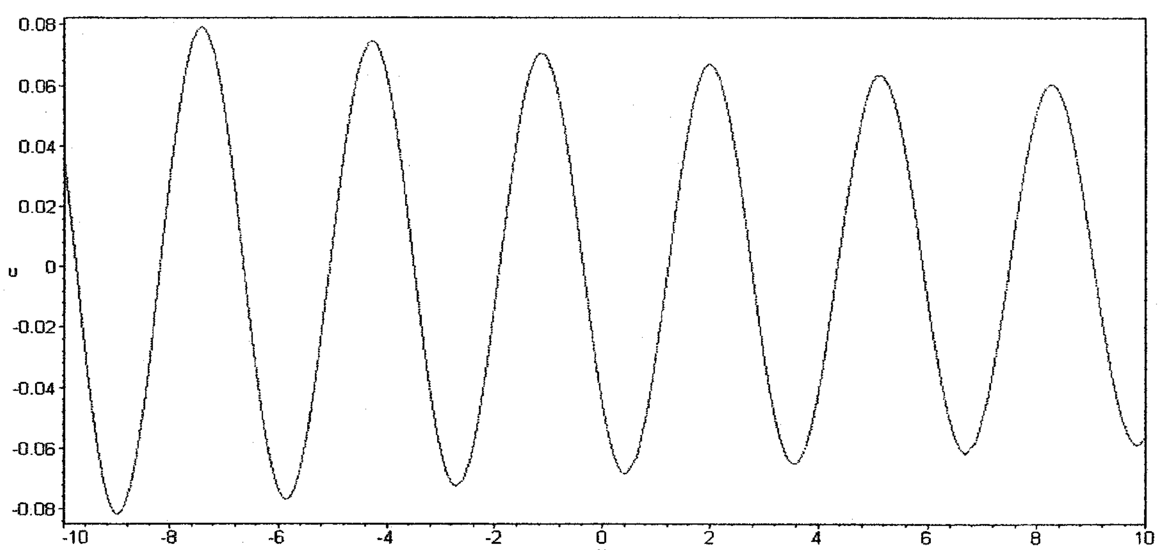


c) Nonlinear interference of KPII harmonic breathers 1 and 2 given by the formula (2.7-12) for $N = 2$.

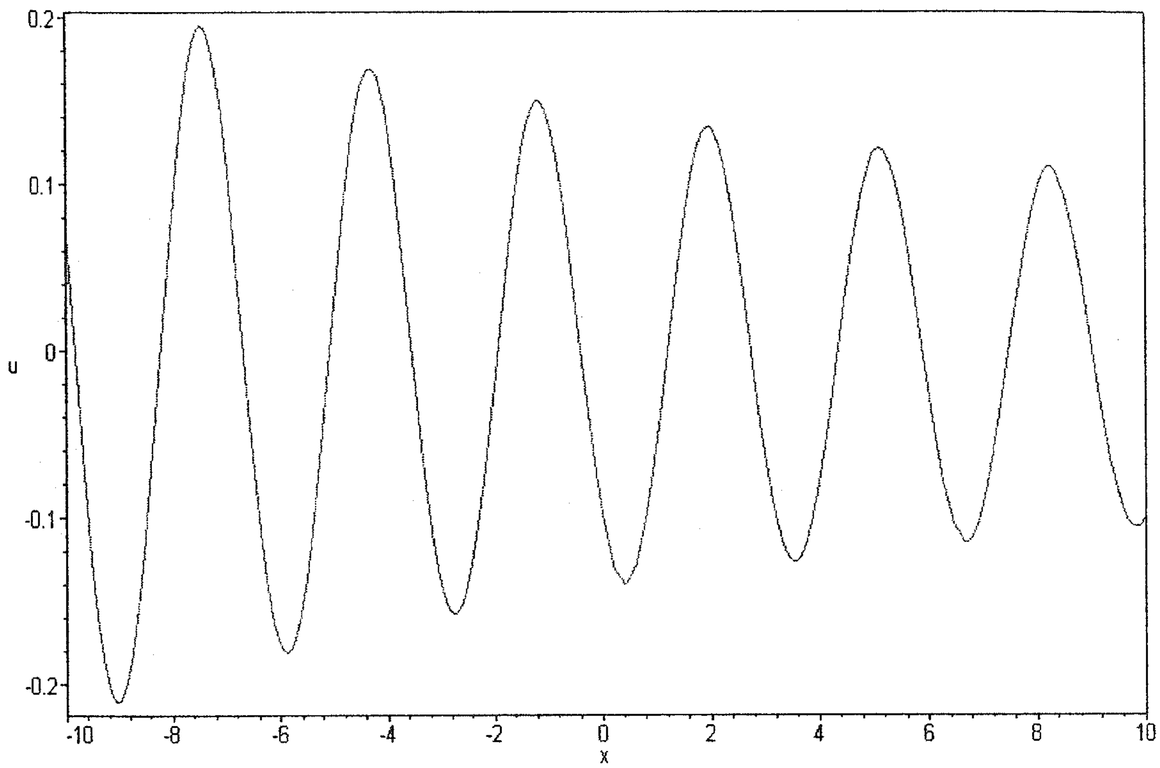
Figure 2.10d-f Nonlinear interference of two KPII harmonic breathers in $\Omega_{10,10,0..T}$ with $\lambda = [1, 1+\varepsilon]$, $\mu = [0.1, 0.1+\varepsilon^2]$, $\gamma = [0, \varepsilon]$, $\varrho = [60, 60]$, $\varepsilon = 10^{-3}$ at $t = 0$. Cross section by plane $y = 10$.



d) KPII harmonic breather 1



e) KPII harmonic breather 2



f) Nonlinear interference of KPII harmonic breathers 1 and 2 given by the formula (2.7-12) for $N = 2$. Cross section by plane $y = 10$.

2.5 Nonlinear Analogues of Phase and Group Velocities.

Let us consider the nonlinear superposition of two harmonic breathers in the physical domain (2.28) under the conditions (2.29-30), described by the parameters $(\lambda_1, \mu_1, \gamma_1 = r\pi, \varrho_1)$, $r \in \mathbf{Z}$, and $(\lambda_2 \to \lambda_1, \mu_2 \to \mu_1, \gamma_2 = \gamma_1, \varrho_2)$. Then the combination of the two harmonic breathers will have approximatively the wave form:

$$w :\approx A_{\text{average}}[\cos(k_{1_x}x + k_{1_y}y - \omega_1 t) + \cos(k_{2_x}x + k_{2_y}y - \omega_2 t)] \tag{2.53}$$

where

$$A_{\text{average}} = \frac{1}{2}\left(\frac{4\lambda_1}{\varrho_1} + \frac{4\lambda_2}{\varrho_2}\right) \tag{2.54}$$

$$k_{i_x} = 2\lambda_i, \quad k_{i_y} = -4\lambda_i\mu_i, \quad \omega_i = 2(12\lambda_i\mu_i^2 - 4\lambda_i^3), \quad i = 1,\ 2. \tag{2.55}$$

Using the trigonometric identity for $\cos a + \cos b$, the combination (2.53) will become:

$$w \approx 2A_{\text{average}} \cos\left(\frac{1}{2}(k_{2_x} - k_{1_x})x + \frac{1}{2}(k_{2_y} - k_{1_y})y - \frac{1}{2}(\omega_1 - \omega_2)t\right) \times$$
$$\cos\left(\frac{1}{2}(k_{2_x} + k_{1_x})x + \frac{1}{2}(k_{2_y} + k_{1_y})y - \frac{1}{2}(\omega_1 + \omega_2)t\right).$$

Writing $\partial k_x = k_{2_x} - k_{1_x}$, $\partial k_y = k_{2_y} - k_{1_y}$, $\partial\omega = \omega_2 - \omega_1$, $k_x = \frac{1}{2}(k_{2_x} + k_{1_x})$, $k_y = \frac{1}{2}(k_{2_y} + k_{1_y})$, $\omega = \frac{1}{2}(\omega_1 + \omega_2)$, we get

$$w \approx 2A_{\text{average}} \cos\left(\frac{1}{2}\partial k_x x + \frac{1}{2}\partial k_y y - \frac{1}{2}\partial \omega t\right) \cos(k_x x + k_y y - \omega t). \qquad (2.56)$$

Here $\cos(k_x x + k_y y - \omega t)$ is a progressive wave with a phase speed (along the x-axis and y-axis) of:

$$c_x = \frac{\omega}{k_x}, \quad c_y = \frac{\omega}{k_y}. \qquad (2.57)$$

Its amplitude $2A_{\text{average}}$, however, is modulated by the slowly varying function $\cos\left(\frac{1}{2}\partial k_x x + \frac{1}{2}\partial k_y y - \frac{1}{2}\partial \omega t\right)$ which propagates along the x-axis and along the y-axis at a speed of:

$$c_{g_x} = \frac{\partial \omega}{\partial k_x}, \quad c_{g_y} = \frac{\partial \omega}{\partial k_y}. \qquad (2.58)$$

Multiplication of a rapidly varying sinusoid and a slowly varying sinusoid, as in (2.56), generates repeating wave groups. The individual wave components propagate with the speed ($c_x = \omega/k_x$, $c_y = \omega/k_y$), but the envelope of the wave groups travel with the speed ($c_{g_x} = \partial\omega/\partial k_x$, $c_{g_y} = \partial\omega/\partial k_y$), which is therefore called the group velocity.

From the description given in this section, we can see that a nonlinear concept of phase and group velocity can be developed for harmonic breathers in the physical domain (2.28) under the conditions (2.29-30). This nonlinear concept may be viewed as a nonlinear analog of the linear concept of phase and group velocity for linear waves.

2.6 Modulating Properties of Harmonic Breathers in the physical domain (2.28), $\Omega_{X,Y,T}$.

2.6.1 Wave Packet-Like Solutions of KPII.

We construct nonlinear wave packet solutions of KPII equation by taking sets: $\mathcal{L} = \{(\lambda_n)_{n=1..N}\}$, $\mathcal{M} = \{(\mu_n)_{n=1..N}\}$, $\mathcal{G} = \{(\gamma_n)_{n=1..N}\}$, $\mathcal{R} = \{(\varrho_n)_{n=1..N}\}$, in the formula (2.7-12).

From the linear theory of waves we know that a two-dimensional linear wave packet can be expressed by the following Fourier integral

$$W(x,y) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(k_1-k_{1_0})^2+(k_2-k_{2_0})^2]}\cos[(k_1-k_{1_0})x+(k_2-k_{2_0})y]\ dk_1\ dk_2. \qquad (2.59)$$

The Fourier integral (2.59) can be approximated in a certain sense by

$$W(x,y) \approx$$

$$\sum_{m=1}^{L}\sum_{n=1}^{L} e^{-[(n\Delta k_1-k_{1_0})^2+(m\Delta k_2-k_{2_0})^2]}\cos[(n\Delta k_1-k_{1_0})x+(m\Delta k_2-k_{2_0})y]\Delta k_1\Delta k_2, \qquad (2.60)$$

for L large enough.

From numerical computation (we cannot prove analytically) we have discovered that in the finite space-time physical domain (2.28), $\Omega_{X,Y,T}$, the formula

(2.7-12) under the conditions (2.29-30) may be viewed as a nonlinear analogue of the Fourier integral (2.59) and the asymptotic (2.32) may be viewed as a nonlinear analogue of the Fourier sum (2.60). Using the formula (2.7-12) such that the conditions (2.29-30) to be satisfied, we have constructed a wave packet-like solution of KPII.

In Figure 2.11a-b (Figure 2.11a shows the $3D$ picture and Figure 2.11b shows the projection on the xy-plane of the 3D picture), a wave packet-like solution of KPII at $t=0$ is presented. The solution was obtained by considering $N=2M^2$, with $M=6$, in the formula (2.7-12) and the following sets of parameters

$$\left((\lambda_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\lambda_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} =$$

$$= ((\lambda_i = 15 + (-M/2 + i - 0.5))_{j=1\ldots M})_{i=1\ldots M},$$

$$\left((\mu_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left(\left(\frac{\mu_i}{\lambda_j}\right)_{j=1\ldots M}\right)_{i=1\ldots M},$$

$$\text{where } \mu_i = 10 + (-M/2 + i - 0.5),\ i = 1\ldots M,$$

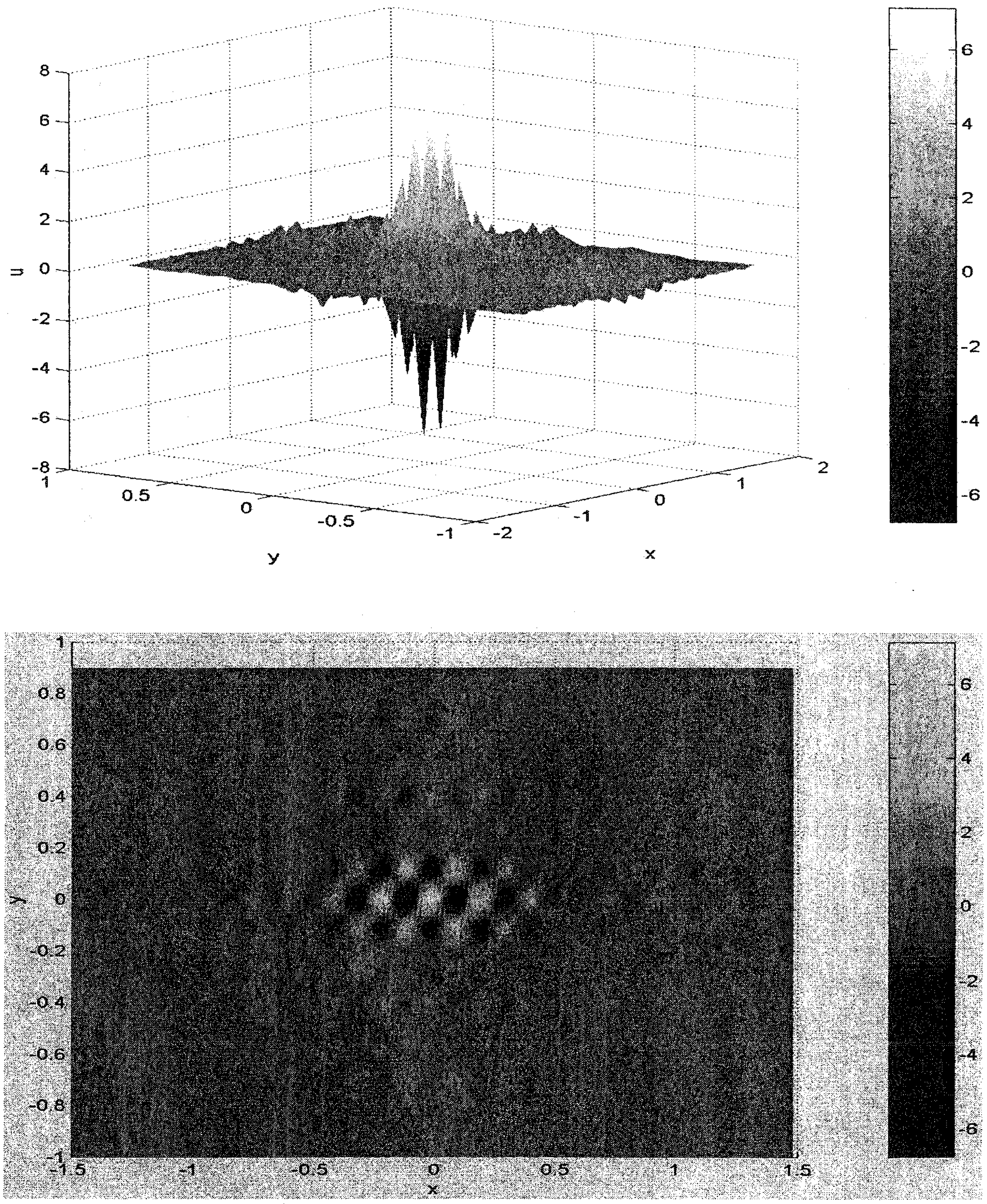
$$\left((\mu_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left(\left(-\frac{\mu_i}{\lambda_j}\right)_{j=1\ldots M}\right)_{i=1\ldots M},$$

$$\left((\gamma_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\gamma_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} = ((\gamma_i = 0)_{j=1\ldots M})_{i=1\ldots M},$$

$$\left((\varrho_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\varrho_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} =$$

$$= \left(\left(\varrho_i = 500e^{0.05(\lambda_i-15)^2+0.008(\mu_i-10)^2}\right)_{j=1\ldots M}\right)_{i=1\ldots M}.$$

Figure 2.11a-b



Properties of a nonlinear wave packet solution of KPII equation

1. It propagates along the x-axis (never along the y-axis); the KP physical model refers to nearly one-dimensional wave solutions with small transverse perturbation to the principal direction of motion.

2. Its leading edge is moving (along the x-axis) with the nonlinear analogue of the group velocity:

$$c_{g_{x\max}} = \max_{(\lambda,\mu)} \left(\frac{\partial\omega}{\partial\lambda}\right) = \max_{(\lambda,\mu)}[12(\mu^2 - \lambda^2)] \tag{2.61}$$

3. Its trailing edge is moving (along the x-axis) with the nonlinear analogue of the group velocity:

$$c_{g_{x\min}} = \min_{(\lambda,\mu)} \left(\frac{\partial\omega}{\partial\lambda}\right) = \min_{(\lambda,\mu)}[12(\mu^2 - \lambda^2)] \tag{2.62}$$

4. As $t \to +\infty$ the wave packet is moving to the left ($c_{g_x} < 0$) and disappears due to dispersion of the harmonic breathers comprising it.

2.6.2 δ Function-Like Solutions of KPII.

Other different exact solutions of KPII (in the physical domain (2.28)) can be constructed by using the formula (2.7-12). One of them is the mimic of the δ function

$$\delta(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(k_1 x + k_2 y)\, dk_1\, dk_2. \tag{2.63}$$

The integral (2.62) can be approximated in a certain sense by

$$\sum_{m=1}^{L}\sum_{n=1}^{L}\cos(n\ \Delta k_1\ x + m\ \Delta k_2\ y)\Delta k_1 \Delta k_2, \tag{2.64}$$

for L large enough.

Numerically we have discovered as well that in the finite space-time physical domain (2.28), $\Omega_{X,Y,T}$, the formula (2.7-12) under the conditions (2.29-30) may be viewed as a nonlinear analogue of the Fourier integral (2.63) and the asymptotic (2.32) may be viewed as a nonlinear analogue of the Fourier sum (2.64). Using the formula (2.7-12) such that the conditions (2.29-30) to be satisfied, we have constructed a delta function-like solution of KPII.

In Figure 2.12 is presented a δ-like solutions of KPII ($t = 0$). The solution was obtained by considering $N = 2M^2$, with $M = 6$, in the formula (2.7-12) and the following sets of parameters

$$\left((\lambda_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\lambda_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} =$$

$$= ((\lambda_i = 0.15i)_{j=1\ldots M})_{i=1\ldots M},$$

$$\left((\mu_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left(\left(\frac{\mu_i}{\lambda_j}\right)_{j=1\ldots M}\right)_{i=1\ldots M}, \text{ where } \mu_i = .051i,\ i = 1\ldots M,$$

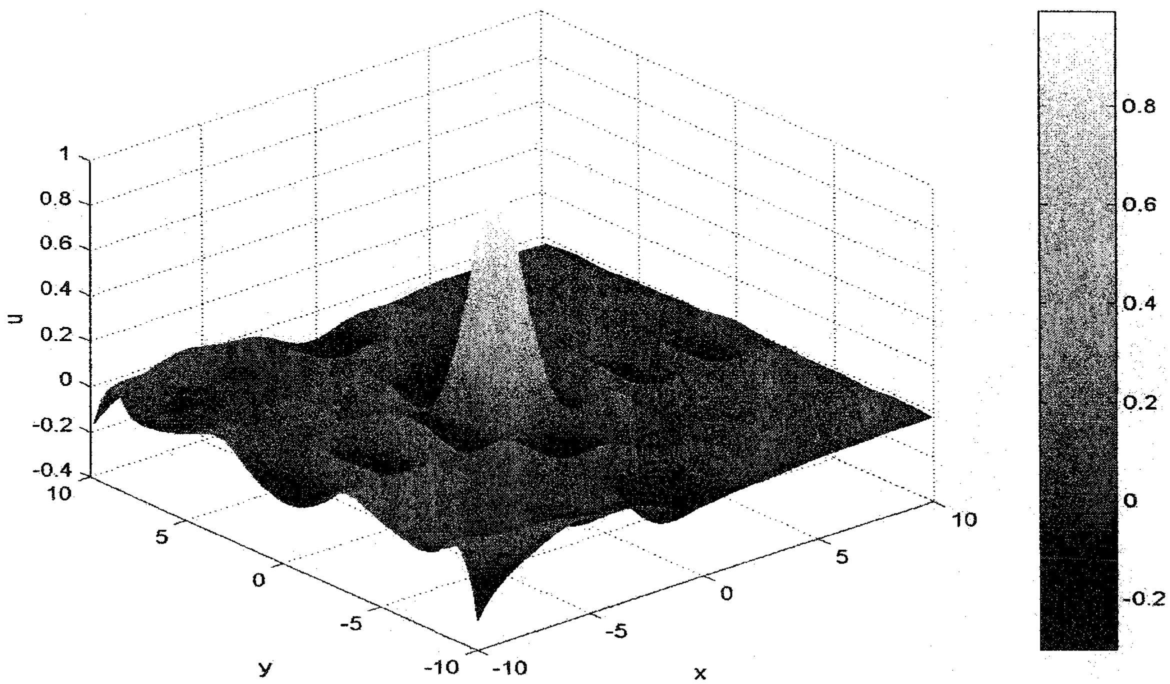
$$\left((\mu_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left(\left(-\frac{\mu_i}{\lambda_j}\right)_{j=1\ldots M}\right)_{i=1\ldots M},$$

$$\left((\gamma_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\gamma_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} = ((\gamma_i = 0)_{j=1\ldots M})_{i=1\ldots M},$$

$$\left((\varrho_{2M(i-1)+M+j})_{j=1\ldots M}\right)_{i=1\ldots M} = \left((\varrho_{2M(i-1)+j})_{j=1\ldots M}\right)_{i=1\ldots M} = ((\varrho_i = 50i)_{j=1\ldots M})_{i=1\ldots M}.$$

In Figure 2.12 we can see the localization of the solution within the chosen domain $\Omega_{X,Y,T}$.

Figure 2.12



2.7 Concluding Remarks

In this thesis we have obtained a new class of oscillatory solutions of the KPII equation that we referred to as harmonic breathers.

We showed how the harmonic breathers are a result of the nonlinear perturbation of (lKPII) by the term $(6uu_x)_x$, i.e. the harmonic breathers of KPII are the nonlinear analogue of the solutions (2.3) of (lKPII) obtained by perturbing (lKPII) with the nonlinear term $(6uu_x)_x$. Unlike the solutions (2.3) the harmonic breathers are singular. However, since all the practical problems are in finite space-time physical domains, one can adjust the parameters involved in the harmonic breathers such that they stay regular within the physical domain. The regular parts of the harmonic breathers within a finite space-time physical domain were referred to as tails of the harmonic breathers.

We derived a formula for nonlinear superposition of solutions of the KPII equation that described the nonlinear interaction of the KPII harmonic breathers. This formula was referred to as the N-harmonic breather solution of the KPII equation. The N-harmonic breather solution of the KPII equation revealed, and it was proven in the section 2.4, that the harmonic breathers possess a property similar in analogy with the interference property of the linear waves. This led us to the idea that we may use the N-harmonic breather solution to construct more complicated exact solutions of the KPII equation.

From numerical computation (we could not prove it analytically) we discovered that, within a finite space-time physical domain in which we have only tails of the harmonics, the N-harmonic breather solution of KPII may be viewed as a nonlinear analogue of the Fourier integral, and so the harmonic breathers may be viewed as nonlinear versions of the Fourier modes. In the section 2.6 we used the N-harmonic breather solution of KPII to construct a wave packet like solution and a δ-like solution of the KPII equation in finite space-time physical domains.

Bibliography

[Abl 1] Ablowitz, M.J., Clarkson, P.A., Solitons, Non-Linear Evolution Equations and Inverse Scattering, *London Math. Society Lecture Notes*, Cambridge University Press 1991.

[Abl 2] Ablowitz, M.J., Segur, H., On the evolution of packets of water waves, *J. Fluid Mechanics*, v.**92**, 691-715, 1979.

[Abl 3] Ablowitz, M.J., Segur, H., Solitons and the Inverse Scattering Transform. SIAM, Philadelphia, 1981.

[Arkadiev 1] Arkadiev, V.A., Pogrebkov, A.K., Polivanov, M.K., The method of the inverse scattering problem in application to singular solutions of nonlinear equations, I,*Theoret. Mat. Fiz.*, **53**, no.**2**,163-180,1982.

[Arkadiev 2] Arkadiev, V.A., Pogrebkov, A.K., Polivanov, M.K., The method of the inverse scattering problem in application to singular solutions of nonlinear equations, II,*Theoret. Mat. Fiz.*, **54**, no.**1**,23-37,1982.

[Bobenko 1] Bobenko, A.I., Bordag, L.A., Periodic multi-phase solutions of the Kadomtsev-Petviashvili equation, *J. Phys. A*, **22**, no.**9**, 1259-1274, 1989.

[Boiti 1] Boiti, M., Pempinelli, F., Pogrebkov, A.K., Properties of solutions of the Kadomtsev-Petviashvili I equation, *J. Math. Phys.*, v. **35**, no. **9**, 4683-4718, 1994.

[Boiti 2] Boiti, M., Pempinelli, F., Pogrebkov, A.K., Solutions of the KPI equation with smooth initial data. *Inverse Problems*, v. **10**, no. **3**, 505-519, 1994.

[Boiti 3] Boiti, M., Leon, J.Jp., Pempinelli, F., Pogrebkov, A.K., Solitons and spectral transform for the DSI and KPI equations. *Solitons and applications (Dubna, 1989)*, 19-30, World Sci. Publishing, River Edge, NJ, 1990.

[Boiti 4] Boiti, M., Pempinelli, F., Pogrebkov, A.K., Some new methods and results in the theory of (2+1)-dimensional integrable equations, *Theoret. and Math. Phys.*, **99**, no.**2**, 511-522, 1994.

[Boiti 5] Boiti, M., Pempinelli, F., Pogrebkov, A.K., Solving the Kadomtsev-Petviashvili equation with initial data not vanishing at large distances. *Inverse Problems*, **13**, no.**3**, L7-L10, 1997.

[Boiti 6] Boiti, M., Pempinelli, F., Pogrebkov, A.K., Prinari, B., Wave soliton solutions on the generic background for KPI equation. *SIDE, Sabaudia*, 47-52, CRM Proc. Lecture Notes, v.**25**, AMS, RI, 2000.

[Chow 1] Chow, K.W., Lai, W.C., Shek, C.K., Tso, K., Positon-like solutions of nonlinear evolution equations in $(2+1)$ dimensions, *Chaos, Solitons and Fractals*, 1998, v.**9**,no.**11**, 1901-1912, 1998.

[Deconinck 1] Deconinck, B. The KP equation with quasiperiodic initial data, Phys. D., v.**123**, 123-152, 1998.

[Dodd 1] Dodd, R.K., Eilbeck, J.C., Gibbon, J.D., Morris, H.C. Solitons and Nonlinear Wave Equations, Academic Press, 1982.

[Dryuma 1] Dryuma, V.S., Analytic solution of the two-dimensional Korteweg-de Vries equation, *Sov. Phys. JETP Lett.*, **19**, 381-388, 1974.

[Drazin 1] Drazin, P.G., Johnson, R.S., Solitons: an introduction, *Cambridge University Press*, 1989.

[Dubrovin 1] Dubrovin, B.A., Flickinger, R., Segur, H., Three-phase solutions of the Kadomtsev-Petviashvili equation, Stud. Appl. Math., v.**99**, 137-203,1997.

[Feynman 1] Feynman, R.P., Lectures on Physics, Addison-Wesley Publishing Company, 1966.

[Hirota 1] Hirota, R., Ohta, Y., Satsuma, J., Solutions of the Kadomtsev-Petviashvili equation and the two-dimensional Toda equations, *J. Phys. Soc. Japan*, **57**, no.**6**, 1901-1904, 1988.

[Hirota 2] Hirota, R., Satsuma, J., N-soliton solutions of model equations for shallow water waves. *J. Phys. Soc. Japan*, **40**, no. **2**, 611-612, 1976.

[Jaworski 1] Jaworski, M., Zagrodziński, J., Positon and Positon-like Solutions of the Korteweg-de Vries and Sine-Gordon Equations, *Chaos, Solitons & Fractals*, Vol. **5**, No. **12**, 2229-2233, 1995.

[Kopell 1] Kopell, N.J., Landman, M., Spatial Structure of the Focusing Singularity of the Nonlinear Schrödinger Equation: A Geometrical Analysis, *SIAM Journal on Applied Mathematics*, Vol. **55**, No. **5**, 1297-1323, 1995.

[KdV 1] Korteweg, D.J., de Vries, G., On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag. (5)*, **39**, 422-43, 1895.

[Kov 1] Kovalyov, M., Slowly Decaying Solutions of KdV, *Contemporary Mathematics*, 2000, Vol. **255**, p.163-180.

[Kov 2] Kovalyov, M., Nonlinear Interference and the Korteweg-de Vries equation, *Applied Mathematics Letters*, 1996, Vol. **9**, No **5**, p.89-92. Erratum, 1997, Vol. **10**, No **5**, p.133.

[Kov 3] Kovalyov, M., Hosseini Ali Abadi Mohammad, An explicit formula for a class of solutions of KdV, *Phys. Lett., A*, **254**, no. **1-2**, p.47-52, 1999.

[Kov 4] Kovalyov, M., Basic motions of the Korteweg-de Vries equation. *Nonlinear Anal. Theory Methods Appl.*, **31**, p.599-619, 1998.

[KP 1] Kadomtsev, B.B., Petviashvili, V.I., On the stability of solitary waves in weakly dispersive media, *Sov. Phys. Dok.*, **15**, p.539-41, 1970.

[Kundu 1] Kundu, P.K., Fluid Mechanics, *Academic Press*, 1990.

[KZ 1] Zabusky, N.S., Kruskal, M.D., Interaction of solitons in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.*, **15**, p240-43.

[Lamb 1] Sir Lamb Horace, Hydrodynamics, New York, Dover Publications, 1945, 252.

[Manakov 1] Manakov, S.V., Zakharov, V.E., Bordag, L.A., Its, A.R., and Matveev, V.B., Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Lett.*, **63A**, 205-206, 1977.

[Matveev 1] Matveev, V.B., Asymptotics of the multipositon-soliton $\tau-$ function of the Korteweg-de Vries equation and the supertransparency. *J. Math. Phys.*, **35**, 2955, 1994.

[Matveev 2] Matveev, V.B., *Phys. Lett.* A, **135**, 2009-2012, 1992.

[Miles 1] Miles, J.W., Resonantly interacting solitary waves, *J. Fluid Mech.*, **79**, p.171-79, 1977.

[Nov 1] Novikov, S.P., Manakov, V.S., Pitaevskii, L.P., Zakharov, V.E., Theory of Solitons, The Inverse Scattering Method, *Contemporary Soviet Mathematics, Consultants Bureau, Plenum Publishing Corporation*, 1984.

[Nov 2] Novikov, R.G., Khenkin, G.M., Oscillatory weakly localized solutions of the KdV equation, *Theoret. Mat. Fiz.*, v.**61**, 199-213, 1984.

[Okhuma 1] Okhuma, K., Wadati, M., The Kadomtsev-Petviashvili equation: the trace Method and the soliton resonances, **J. Phys. Soc. Jpn.**, **52**, 749, 1983.

[Russ 1] Russell, J.S., Report on waves, *Report of the Fourteenth Meeting of the British Associations for the Advancement of Science*, York, 1844, p.311.

[Satsuma 1] Satsuma, J., N-Soliton Solution of the Two-Dimensional Korteweg-de Vries Equation, *J. Phys. Soc. Jpn.*, **40**, p.286-90, 1976.

[Satsuma 2] Satsuma, J., and Ablowitz, M.J., Two-dimensional lumps in nonlinear dispersive systems, *J. Math. Phys.*, **20**, 1496-1503, 1979.

[Sharipov 1] Sharipov, R.A., Multiplet solutions of the Kadomtsev-Petviashvili equation on a finite-gap background, *Uspekhi Mat. Nauk*, **42**, no.**5**, 221-222, 1987.

[Stahlhofen 1] Stahlhofen, A.A., Maisch, H., Dynamical properties of positons. *Phys. Scr.*, **52**, 228, 1995.

[Zeytounian 1] Zeytounian, R.Kh., Nonlinear long waves on water and solitons, *Physics-Uspekhi*, **38**(12), p.1333-1382, 1985.

[ZS 1] Zakharov, V.E., Shabat, A.B., A scheme for integrating the nonlinear equations of Mathematical Physics by the method of the inverse scattering problem, I. *Funct. Anal. Appl.*, **8**, p.226-35, 1974.

[Whitham 1] Whitham, G.B., Linear And Nonlinear Waves, *A Wiley-Interscience Publication*, John Wiley & Sons, 1974, p.467.